On the choice of base frame in free-floating robot dynamics and its connection to "centroidal" dynamics*

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Abstract— **This paper contributes towards the development of a unified standpoint on the equations of motion associated with free-floating systems. In particular, the contribution of the manuscript is twofold. First, we show how to write the system equations of motion for any choice of the base frame, without the need of re-applying numerical algorithms for evaluating the dynamic quantities, such as mass, coriolis, and gravity matrix. A particular attention is paid to the properties associated with the mechanical systems, which are shown to be invariant with respect to the base frame choice. Secondly, we show that the so-called *centroidal dynamics* can be obtained from any expression of the system equations of motion via an appropriate base frame change. In this case, we show that the mass matrix associated with the new base frame is block diagonal, and the new base velocity corresponds to the so-called *average spatial velocity*.**

I. INTRODUCTION

Classical modelling techniques for multi-body mechanical systems have their roots in the fundamental principles of mechanics. For instance, Euler-Lagrange and Euler-Poincaré formalisms are among the main tools for deriving the equations of motion of systems evolving on vector spaces and Lie groups, respectively [1]. The application of these formalisms to common robotics applications assume that the underlying mechanical system is composed of a collection of rigid bodies –called links– interconnected by mechanisms –called joints– constraining the links relative motion. The obtained equations of motions, however, often depend upon the arbitrary representation of the configuration space of the multi-body system. This paper investigates the role of the choice of the *base frame* when defining the configuration space of free floating robots [2].

The configuration space of a multi-body system composed by $n+1$ links may be characterised by the positions of all the points of the system. To this inconvenient representation, however, it is always preferred that consisting of the orientations and the center-of-mass positions of all the rigid bodies. When the links are interconnected by n joints, any point of the system can be expressed in terms of the *joint angles* and a *frame* attached to a link of the system. This frame is referred to as *base frame* and its choice is dictated by rule-of-thumb principles, such as attaching the base frame to the heaviest link of the considered robot. Then, the configuration space of the mechanical system is of dimension $n+6$, and the system is usually referred to as *free floating* or *floating base*.

*This paper was supported by the FP7 EU projects CoDyCo (No. 600716 ICT 2011.2.1 Cognitive Systems and Robotics) and Koroibot (No. 611909 ICT-2013.2.1 Cognitive Systems and Robotics).

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When one of the links has a constant position-and-orientation with respect to (w.r.t.) the inertial frame, the configuration space collapses to the joint angles only, and its dimension drops to n. In this case, the system is referred to as *fixed base*, and robotic manipulators attached to ground do fall into this category.

Modelling, estimation, and control of robots, however, were developed first for fixed-base manipulators, and then for floating-base systems. As a consequence, floating-base systems are usually viewed as a special case of fixed base robots, and their modelling was often addressed by considering a six degrees of freedom joint connecting the inertial frame to the base frame [3]. This solution has the advantage of reusing the fixed base machinery to derive the equations of motion for floating base robots. However, it does hide some important properties of floating systems, and it makes the use of minimal representations of the base orientation really tempting, which in turn introduces artificial singularities for this orientation [4].

Attempts to generalized the floating-base systems' equations of motion irrespective from the chosen base frame can be found at [5, Chapter 3] and in [6, Chapter 17, Section 3.6]. Unfortunately this works (respectively a PhD thesis and a book) were never published as peer reviewed publication, and have been largely ignored by the humanoids whole-control literature. This two approaches anyway share the same limitation, as they always assume the base frame to be fixed to one of links of model.

In this paper we model floating robot system without assuming any base frame as the preferred one, showing how then the kinematics and dynamics of the robot arise by choosing a different floating base frame. In particular we show how the equation of motion obtained using different "base frames" can be obtained as a nonlinear change of variables of the robot dynamics. We also show that the same structural change of variables can be also used to express the robot dynamics as a combination of the internal dynamics and the "centroidal dynamics", as introduced in [3] and extended in [7]. While most of the results of this paper can be linked to the classical Lie Group geometrical mechanics results, we avoided to insert direct reference to those topic to avoid to overcomplexify the paper for people not familiar with geometrical mechanics. However an extended explanation on the connection the notation employed here and the geometrical mechanics theory can be found in [8].

II. BACKGROUND

This section introduces basic notation used for dynamics computations. For further information and details on the notation used throughout this paper, the reader is referred to [8]. Although what follows is partially based on Spatial Vector Algebra notation [9], [10], several specific modifications have been introduced for the purpose of this paper. For instance, the *dot operator* means:

$$\dot{(\cdot)} = \frac{d}{dt}(\cdot),$$

thus avoiding the introduction of an arbitrary (and often hidden) inertial frame w.r.t. which the derivative is computed.

- The set of real numbers is denoted by $\mathbb{R}$. Let u and v be two n-dimensional column vectors of real numbers, i.e. $u, v \in \mathbb{R}^n$, then their inner product is denoted as $u^\top v$, with "$\top$" the transpose operator.
- The identity matrix of dimension n is denoted by $1_{n\times n} \in \mathbb{R}^{n\times n}$; the zero column vector of dimension n is denoted by $0_n \in \mathbb{R}^n$; the zero matrix of dimension $n \times m$ is denoted by $0_{n\times m} \in \mathbb{R}^{n\times m}$.
- Given a vector $w = \begin{bmatrix} x & y & z \end{bmatrix}^\top \in \mathbb{R}^3$, we define $w^\wedge$ (read w *hat*) as the 3×3 *skew-symmetric* matrix associated with the cross product $\times$ in $\mathbb{R}^3$, i.e. $w^\wedge x = w \times x$. Given the *skew-symmetric matrix* $W = w^\wedge$, we define $W^\vee \in \mathbb{R}^3$ (read W *vee*) as $W^\vee := w$.
- The set SO(3) is the set of $\mathbb{R}^{3\times 3}$ orthogonal matrices with determinant equal to one, namely $\mathrm{SO}(3) := \{ R \in \mathbb{R}^{3\times 3} \mid R^\top R = I_3, \ \det(R) = 1 \}$.

A. Frames notation

A *frame* is defined as the pair of a point (called *origin*) and an *orientation frame* in the 3D space [11], and we use capital letters to indicate frames. Given a frame F, we denote with o_F its origin and with $[F]$ its orientation frame. Formally, we write this as $F = (o_F, [F])$. Then, we have.

A	Denote the inertial (*A*bsolute) frame.
p	An arbitrary point.
$B[A]$	Frame with origin o_B and orientation $[A]$.
$\overline{B}$	Shorthand for $B[A]$.

B. Coordinates vectors and transformation matrices notation

${}^A p \in \mathbb{R}^3$	Coordinates of p w.r.t. to A.
${}^A o_B \in \mathbb{R}^3$	Coordinates of o_B w.r.t. to A.
${}^A R_B \in \mathbb{R}^{3\times 3}$	Rotation matrix transforming a 3D vector expressed in $[B]$ into one expressed in $[A]$.
${}^A X_B = \begin{bmatrix} {}^A R_B & {}^A o_B^\wedge {}^A R_B \\ 0_{3\times 3} & {}^A R_B \end{bmatrix}$	Velocity transformation from B to A.
${}^C \mathrm{v}_{A,B} \in \mathbb{R}^6$	Twist representing the velocity of B w.r.t. to A expressed in C.
${}_B \mathrm{f} \in \mathbb{R}^6$	coordinates of the wrench f w.r.t. B.
${}_A X^B = \begin{bmatrix} {}^A R_B & 0_{3\times 3} \\ {}^A o_B^\wedge {}^A R_B & {}^A R_B \end{bmatrix}$	wrench transformation from B to A.
${}_B I_L \in \mathbb{R}^{6\times 6}$	6D inertia matrix of body L expressed in the frame B .

C. Frame velocity representation

The position and orientation of a frame B w.r.t. an inertial frame A is usually represented with the homogeneous matrix ${}^A H_B \in \mathbb{R}^{4\times 4}$ transforming a position vector expressed in B to a position vector expressed in A, i.e.

$${}^A H_B = \begin{bmatrix} {}^A R_B & {}^A o_B \\ 0_{1\times 3} & 1 \end{bmatrix}, \quad {}^A p = {}^A R_B {}^B p + {}^A o_B. \tag{1}$$

Then, the body velocity can be represented by ${}^A \dot{H}_B$. It is more convenient, however, to express the body velocity as a $\mathbb{R}^6$ vector. Common representations of ${}^A \dot{H}_B$ as an element of $\mathbb{R}^6$ are given by:

$${}^A \mathrm{v}_{A,B} = \begin{bmatrix} {}^A \dot{o}_B - {}^A \omega_{A,B}^\wedge {}^A o_B \\ {}^A \omega_{A,B} \end{bmatrix},$$

$${}^B \mathrm{v}_{A,B} = \begin{bmatrix} {}^B R_A {}^A \dot{o}_B \\ {}^B R_A {}^A \omega_{A,B} \end{bmatrix}, \quad {}^{B[A]} \mathrm{v}_{A,B} = \begin{bmatrix} {}^A \dot{o}_B \\ {}^A \omega_{A,B} \end{bmatrix}.$$

where ${}^A \omega_{A,B} = ({}^A \dot{R}_B {}^A R_B^\top)^\vee$. The above representations are usually referred to as: ${}^A \mathrm{v}_{A,B}$ the *inertial* representation of the body velocity, ${}^B \mathrm{v}_{A,B}$ the *body-fixed* representation, and ${}^{B[A]} \mathrm{v}_{A,B}$ the *mixed* representation (also known as *hybrid* representation [12]). The *inertial* and *body-fixed* representation are widespread in the literature of Lie group-based geometric mechanics [13] and recursive robot dynamics algorithms [9], [14]. Yet, in this paper we use the *mixed* velocity representation because it is commonly used in multi-task control frameworks [15], [16], [17]. It can be shown, however, that most of the results of the paper hold irrespective of the velocity representation. Furthermore, we use the mixed representation even for body accelerations, wrenches and mementa. To avoid overloading the notation, in the rest of the document we define:

$$\mathrm{v}_L := {}^{\overline{L}} \mathrm{v}_{A,L} = {}^{L[A]} \mathrm{v}_{A,L}, \quad \dot{\mathrm{v}}_L := {}^{\overline{L}} \dot{\mathrm{v}}_{A,L} = {}^{L[A]} \dot{\mathrm{v}}_{A,L}. \tag{2}$$

D. Rigid Body Dynamics

This section recalls the equations of motion of a single rigid body assuming that a frame L is attached to it.

The *mixed* body momentum $h_L \in \mathbb{R}^6$ is given by [8]:

$$h_L = {}_{\overline{L}} h_L = {}_{\overline{L}} X^L {}_L I_L {}^L X_{\overline{L}} \mathrm{v}_L = {}_{\overline{L}} I_L \mathrm{v}_L \tag{3}$$

where ${}_L I_L$ is the 6D body inertia expressed w.r.t. the origin of L and the orientation of A, i.e.

$${}_L I_L = \begin{bmatrix} m_L 1_{3\times 3} & m_L {}^L c_L^\wedge \\ -m_L {}^L c_L^\wedge & \overline{I}_L \end{bmatrix}, \tag{4}$$

with $m_L \in \mathbb{R}$ the mass of the body, ${}^L o_G \in \mathbb{R}^3$ the coordinates of the center of mass c_L of the body w.r.t. the frame L, and $\overline{I}_L \in \mathbb{R}^{3\times 3}$ the 3D inertia of the link expressed w.r.t. the orientation and origin of L [9], [18]. Notice that ${}_L I_L$ is constant, while ${}_{\overline{L}} I = {}_{\overline{L}} X^L I_L {}^L X_{\overline{L}}$ (i.e. the inertia expressed in *mixed* coordinates) is not because ${}^L X_L$ depends on the orientation of L w.r.t. A.

The Newton-Euler principle states that the *rate-of-change* of the momentum equals the sum of the external wrenches applied to the body. By computing the time derivative of (3) one obtains the *mixed* Newton-Euler equations of motion:

$$ {}_{\overline{L}}I_L\dot{\mathrm{v}}_L + C(\mathrm{v}_L, {}^A R_L)\mathrm{v}_L = \mathrm{f}_L^x + {}_{\overline{L}}I_L \begin{bmatrix} {}^A R_L^\top {}^A g \\ 0_{3\times 1} \end{bmatrix}. \quad (5) $$

where ${}^A g \in \mathbb{R}^3$ is the gravitational acceleration expressed in the inertial frame, $C(\mathrm{v}_L, {}^A R_L) := \frac{d}{dt}\left({}_{\overline{L}}I_L\right)$, and $\mathrm{f}_L^x \in \mathbb{R}^6$ the *mixed* external wrench applied to the body, i.e. $\mathrm{f}_L^x := {}_{\overline{L}}\mathrm{f}_L^x = {}_{L[A]}\mathrm{f}_L^x$.

III. Recalls and Complements on Free-Floating Multi-Body Dynamics

This section recalls notation, state definition, and equations of motion associated with a *free-floating* mechanical system. In particular, we assume that one is given with a mechanical system composed of $n+1$ rigid bodies –called *links*– interconnected by n mechanisms –called *joints*– constraining the link relative motion. The set of links is indicated by $\mathfrak{L}$, and the set of joints is indicated by $\mathfrak{J}$. We assume that each link $L \in \mathfrak{L}$ is associated with a link-fixed frame. In the rest of the paper, we often refer to the frame attached to link L simply as L. So, one of these frames is called *base-frame* B. We also assume that each joint possess only one degree-of-freedom, and interconnects two links with no internal loop.

A. State definition

Being a mechanical system, the equations of motion governing the free floating dynamics are a second-order differential system. Hence, we have to define a *state* of the system composed of a properly defined *position* and *velocity*.

1) Free-Floating system position: The process of defining the *system position* aims at determining a set of variables from which the position of each point of the multi-body system can be retrieved in the absolute frame A. Being a composition of rigid bodies, each point of the multi-body system can be retrieved from the position-and-orientation – referred to as *pose* – of each link frame $L \in \mathfrak{L}$. Given the topology of the considered multi-body system, each link pose can be determined from the pose of the base frame B and the *joint configurations*.

We assume that the configuration space of each joint is $\mathbb{R}$, which is the case of rotational and prismatic joints subject to joint limit constraints. Then, the *joint positions* can be identified by a vector of coordinates $\breve{q} \in \mathbb{R}^n$. Contrary to [9], we assume that the *numbering* of the joints remains constant, namely, the joint serialization is independent from the base frame choice B, thus getting rid of the dependence of the joint serialization upon floating base choice [19]. Clearly, the joint positions $\breve{q}$ uniquely characterize the pose of any link w.r.t. the base frame B.

Then, the configuration set of a free floating system is given by the combination of the *base pose* $({}^A o_B, {}^A R_B) \in \mathbb{R}^3 \times \mathrm{SO}(3)$ and the *joint* positions $\breve{q} \in \mathbb{R}^n$, i.e.

$$ \mathbb{Q} = \mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^n, \quad (6) $$
$$ q^B = ({}^A o_B, {}^A R_B, \breve{q}) \in \mathbb{Q}. \quad (7) $$

2) Free-Floating system velocity: In view of (6), the derivative of the robot position is given by:

$$ \dot{q}^B = ({}^A\dot{o}_B, {}^A\dot{R}_B, \dot{\breve{q}}). $$

As in the case of a single rigid body (see Sec.II-C), it is more convenient to represent the velocity as a column vector. Using the *mixed* body velocity representation, we can define the system velocity vector $\nu^B \in \mathbb{R}^{6+n}$ as follows:

$$ \nu^B = \begin{bmatrix} \mathrm{v}_B \\ \dot{\breve{q}} \end{bmatrix} = \begin{bmatrix} {}^A\dot{o}_B \\ {}^A\omega_{A,B} \\ \dot{\breve{q}} \end{bmatrix} = \begin{bmatrix} {}^A\dot{o}_B \\ ({}^A\dot{R}_B {}^A R_B^\top)^\vee \\ \dot{\breve{q}} \end{bmatrix} \in \mathbb{R}^{n+6} \quad (8) $$

with $\mathrm{v}_B \in \mathbb{R}^6$ the *base* velocity, and $\dot{\breve{q}}$ the *joint* velocities. Clearly, the system velocity depends upon the choice of the base frame B.

B. Free-floating system equations of motion

Using the Euler-Poincarè equations [1], it is possible to write the equations of motion for a free floating system subject to external wrenches $\mathrm{f}_L^x \in \mathbb{R}^6$. These equations write [9], [6]

$$ \begin{aligned} M_B(q^B)\dot{\nu}^B + C_B\left(q^B, \nu^B\right)\nu_B + G_B(q^B) = \\ \begin{bmatrix} 0_{6\times 1} \\ \tau \end{bmatrix} + \sum_{L\in\mathfrak{L}} J_{L,B}^\top \mathrm{f}_L^x, \end{aligned} \quad (9) $$

where $M_B(q^B) \in \mathbb{R}^{n+6\times n+6}$, $C_B\left(q^B, \nu^B\right) \in \mathbb{R}^{n+6\times n+6}$, and $G_B(q^B) \in \mathbb{R}^{n+6}$ denote the mass matrix, the Coriolis matrix, and the gravity torques, respectively. The matrix $J_{L,B} \in \mathbb{R}^{6\times n+6}$ denotes the jacobian of the frame L on which the external wrench acts. The suffix B in all these symbols highlights the dependence of the quantity upon the choice of the base frame B. The *joint torques* $\tau \in \mathbb{R}^n$, which is the input to the system, clearly do not depend upon this choice.

Let us now recall some important facts on the model (9).

Theorem 1 ([20])**.** *The mass matrix M_B can be expressed as follows:*

$$ M_B = \begin{bmatrix} {}_{\overline{B}}I^C & F_B \\ F_B^\top & H_B \end{bmatrix}, \quad (10) $$

where ${}_{\overline{B}}I^C$ is the so-called locked 6D rigid body inertia *of the multi-body system, i.e.*

$$ {}_{\overline{B}}I^C = \sum_{L\in\mathfrak{L}} {}_{\overline{B}}X^L I_L {}^L X_{\overline{B}}. \quad (11) $$

Furthermore, the first six rows of the mass matrix define the jacobian of the articulated body momentum (i.e. the sum of the linear/angular momentum of all the bodies composing the robot), known as the Momentum Matrix.

$$ {}_{\overline{B}}\mathrm{h} = \begin{bmatrix} {}_{\overline{B}}I^C & F_B \end{bmatrix}\nu_B = \sum_{L\in\mathfrak{L}} {}_{\overline{B}}X^L I_L {}^L\mathrm{v}_L \quad (12) $$

The mass matrix M_B is known as *Composite Rigid Body Inertia* (CRBA) of the robot in whole body control literature [3]. Also, the following properties hold on the model (9) [13]:

Commented out the boundness stuff until it is used

Property 1. *The matrix* $\dot{M}_B - 2C_B\left(q^B, \nu^B\right)\nu_B$ *is skew symmetric*

1) Free-floating generalized force and its power: The effects of an external wrench (i.e. a wrench acting between a robot link and the environment) and of internal wrenches (i.e. acting between two robot links) can be represented with a $6+n$ free-floating generalized force:

$$\phi_B = \begin{bmatrix} \mathrm{f}_B^x \\ \tau_B \end{bmatrix} \in \mathbb{R}^{n+6}$$

where $\mathrm{f}_B^x \in \mathbb{R}^6$ (the *base* wrench) represents the external force-torque exerted by the environment on the robot and expressed with the orientation of the inertial frame and at the origin of the B frame; while $\tau_B \in \mathbb{R}^n$ represents the *joint* torques. Then, the power $P \in \mathbb{R}$ transmitted by a given generalized force on the free floating robot moving with a given velocity can be defined as follows [9]:

$$P = \phi_B^\top \nu^B. \tag{13}$$

C. Kinematics

We briefly recall below how to relate the base frame pose and velocity with the pose and velocity of and arbitrary link frame $L \in \mathfrak{L}$.

The position of a link L w.r.t to the inertial frame is a function of the system position q^B:

$$^AH_L(q^B) \ : \ \mathbb{Q} \mapsto \mathbb{R}^3 \times \mathrm{SO}(3), \tag{14a}$$

$$^AH_L(q^B) = {}^AH_B{}^BH_L(\breve{q}) = \begin{bmatrix} ^AR_B & ^Ao_B \\ 0_{1\times 3} & 1 \end{bmatrix} {}^BH_L(\breve{q}). \tag{14b}$$

The velocity of a frame L w.r.t. to the inertial frame is the product between a robot position-dependent Jacobian matrix $J_{L,B}(q_B) \in \mathbb{R}^{6\times n+6}$ times the system velocity:

$$\mathrm{v}_L(q^B, \nu^B) = J_{L,B}(q_B)\nu^B, \tag{15a}$$

$$J_{L,B}(q_B) = \begin{bmatrix} ^{\overline{L}}X_{\overline{B}} & \breve{J}_{L,B}(\breve{q}) \end{bmatrix}. \tag{15b}$$

The equations (14) (15) represent the so-called *forward kinematics* of the link L using B as the floating base. How to compute the *forward kinematics* and the *Jacobian matrix* is out of the scope of this paper. If the base frame is a link-fixed frame, the algorithms for computing such quantities can be found in [9], [14], [18].

IV. The effects of changing base frame: from the state to the equations of motion

In this section, we answer to the following questions.

What nonlinear state transformation is associated with the base frame change from the base B to the base frame B'?

How does this transformation affect the equations of motion (9) *and its properties?*

The answers to the above questions are stated in the following section.

A. Free-floating state transformation

Given two possible choices of the floating bases B, B', the robot position q^B can be transformed in $q^{B'}$ simply by using the *forward kinematics* of B' considering B to be the floating base :

$$q^{B'}(q^B) = ({}^AH_{B'}(q^B), \breve{q}) = ({}^AH_B{}^BH_{B'}(\breve{q}), \breve{q}) \tag{16}$$

where for compactness we wrote $({}^Ao_B, {}^AR_B)$ as ${}^AH_B$.

Similarly the transformation between ν^B and $\nu^{B'}$ can be easily obtained using the jacobian of frame B' considering B as the floating base:

$$\nu^{B'} = \begin{bmatrix} \mathrm{v}_{B'} \\ \dot{\breve{q}} \end{bmatrix} = \begin{bmatrix} J_{B',B}\nu^B \\ \dot{\breve{q}} \end{bmatrix} = $$

$$= \begin{bmatrix} ^{\overline{B'}}X_{\overline{B}} & \breve{J}_{B',B} \\ 0_{n\times 6} & 1_{n\times n} \end{bmatrix} \nu^B$$

We can then define the *base change* velocity transformation matrix as the matrix ${}^{B'}T_B \in \mathbb{R}^{n+6\times n+6}$ such that:

$$\nu^{B'} = {}^{B'}T_B\nu^B, \quad {}^{B'}T_B = \begin{bmatrix} ^{\overline{B'}}X_{\overline{B}} & \breve{J}_{B',B} \\ 0_{n\times 6} & 1_{n\times n} \end{bmatrix}. \tag{17}$$

B. Free-floating generalized force transformation

The power P is physical quantity independent of the arbitrary choice of the floating base, we then have that given two frames B and B' :

$$\phi_B^\top \nu^B = \phi_{B'}^\top \nu^{B'} = P.$$

We will exploit this property to understand how s generalized forces vector is transformed by choosing another floating base.

As an inner product invariant to the floating base change is defined between generalized velocities and generalized forces, the *base change* velocity transformation matrix induces a *base change* force transformation matrix, defined as:

$$\phi_{B'} = {}_{B'}T^B\phi_B, \tag{18a}$$

$${}_{B'}T^B = {}^{B'}T_B^{-\top} = \begin{bmatrix} {}_{\overline{B'}}X^{\overline{B}} & 0_{6\times n} \\ -(\breve{J}_{B'}^B)^\top {}_{\overline{B'}}X^{\overline{B}} & 1_{n\times n} \end{bmatrix}. \tag{18b}$$

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Remark IV.1 (Effect of the choice of *mixed* representation)**.** *Note that while the* joint velocities *are not affected by a base change, both the base wrench and the joint torques are affected by a base change. This is given by the specific structure of the base change transformation matrices. However it is interesting to note that the base wrench would be invariant to the base change if we choose to represent base quantities (velocities, wrenches) using the* inertial *representation rather then the* mixed *one, as done in [5].*

Remark IV.2 (Pure internal joint torques invariance)**.** *If a generalized force has a null base wrench, then it is independent of the base frame in which it is expressed.*

$${}_{B'}T^B \begin{bmatrix} 0_{6\times 1} \\ \tau_B \end{bmatrix} = \begin{bmatrix} 0_{6\times 1} \\ \tau_{B'} \end{bmatrix} = \begin{bmatrix} 0_{6\times 1} \\ \tau \end{bmatrix}$$

Remark IV.3 (External joint torque base dependency)**.** *The effect of a pure external wrench f^x_B on a the floating base B is by definition the generalized force where the first six elements are given by f^x_B and joint part is equal to zero:*

$$\phi_B = \begin{bmatrix} \mathrm{f}^x_B \\ 0_{n\times 1} \end{bmatrix}$$

the same generalized force, expressed with a different floating-base B is given by:

$$\phi_{B'} = {}_{B'}T^B \phi_B = \begin{bmatrix} {}_{\overline{B'}}X^{\overline{B}} & 0_{6\times n} \\ (\breve{J}_{B',B})^\top {}_{\overline{B'}}X^{\overline{B}} & 1_{n\times n} \end{bmatrix} \begin{bmatrix} \mathrm{f}^x_B \\ 0_{n\times 1} \end{bmatrix} = \tag{19}$$

$$= \begin{bmatrix} {}_{\overline{B'}}X^{\overline{B}} \mathrm{f}^x_B \\ (\breve{J}_{B',B})^\top {}_{\overline{B'}}X^{\overline{B}} \mathrm{f}^x_B \end{bmatrix}. \tag{20}$$

The first 6 rows are still the same external wrench, with the ${}_{\overline{B'}}X^{\overline{B}}$ transform that accounts for the change of reference point from the origin of B to the origin B'. In the joint torques part, there are some "external joint torques", induced only by the change of floating base. In particular this torques follow a precise sparsity pattern induced by the jacobian $\breve{J}^\top_{B',B}$, i.e. there are non-zero only for the joints that connect the links B' and B. It is important to note that while the external joint torques *are uniquely defined in the case of fixed base robots, in the case of floating base robots their value depends on the representation choice (the floating base) and as such they do not described any measurable physical quantity.*

C. Generalized Accelerations Transformation Law

For getting the transformation law for accelerations, we just derive the transformation law for velocities (17):

$$\dot{\nu}^{B'} = {}^{B'}T_B \dot{\nu}^B + {}^{B'}\dot{T}_B \nu^B. \tag{21}$$

1) Dynamics Transformation Law: Considering the transformation laws presented in the previous section, we can transform the dynamics of a free floating from being expressed with respect to a floating base B to being expressed with respect to a floating base B'.

Fix this overlapping/number equation

In particular we can modify (9) by applying (17), (21) and by multiplying it on the left by ${}_{B'}T^B$ and we obtain:

$$M_{B'} = {}_{B'}T^B M_B {}^B T_{B'}, \tag{22a}$$

$$C_{B'}\left(q^B, \nu^{B'}\right) = {}_{B'}T^B \left(M_B {}^B\dot{T}_{B'} + C_B\left(q^B, \nu^B\right) {}^B T_{B'}\right) \tag{22b}$$

$$G_{B'} = {}_{B'}T^B G_B, \tag{22c}$$

$$J_{L,B'} = J_{L,B} {}^B T_{B'}. \tag{22d}$$

D. Properties

1) Passivity Invariance: An useful property of the base frame transformation is that it does not change the passivity properties of the Coriolis matrix, as demonstrated in the following lemma.

Theorem 2. *If $\dot{M}_B - 2C_B$ is skew-symmetric, then also $\dot{M}_{B'} - 2C_{B'}$ is skew-symmetric (where the matrices $M_{b'}$ and $C_{B'}$ have been obtained through a base frame transform)*

Proof. The condition of $\dot{M}_B - 2C_B$ being skew-symmetric is equivalent to the condition $\dot{M}_B = C_b + C_B^\top$, so we will demonstrate that $\dot{M}_B - C_B - C_B^\top = \mathbf{0}$ implies $\dot{M}_{B'} - C_{B'} - C^\top_{B'} = 0_{(n+6)\times(n+6)}$. Let's write $\dot{M}_{B'} - C_{b'} - C^\top_{B'}$ using (22) and (23):

$$\dot{M}_{B'} - C_{B'} - C^\top_{B'} =$$

$$= \underbrace{{}_{B'}\dot{T}^B M_B {}^B T_{B'} + {}_{B'}T^B \dot{M}_B {}^B T_{B'} + {}_{B'}T^B M_B {}^B\dot{T}_{B'}}_{\dot{\mathbf{M}}_{B'}}$$

$$\underbrace{- {}_{B'}T^B M_B {}^B\dot{T}_{B'} - {}_{B'}T^B C_B {}^B T_{B'}}_{-C_{B'}}$$

$$\underbrace{- {}^B\dot{T}^\top_{B'} M_B {}_{B'}T^{B^\top} - {}^B T^\top_{B'} C_B{}^\top {}_{B'}T^{B^\top}}_{-C^\top_{B'}}$$

Noting that from (18) we have that ${}_{B'}T^{B^\top} = {}^B T_{B'}$, we can write:

$$\begin{aligned} \dot{M}_{B'} - C_{B'} - C^\top_{B'} &= \\ {}_{B'}T^B \left(\dot{M}_B - C_B - C_B^\top\right) {}^B T_{B'} &+ \\ + {}_{B'}\dot{T}^B M_B {}^B T_{B'} - {}_{B'}\dot{T}^B M_B {}^B T_{B'} &+ \\ + {}_{B'}T^B M_B {}^B\dot{T}_{B'} - {}_{B'}T^B M_B {}^B\dot{T}_{B'} & \end{aligned}$$

Using the hypothesis $\dot{M}_B - C_B - C_B^\top = 0$ we can then conclude that $\dot{M}_{B'} - C_{B'} - C^\top_{B'} = 0$.

□

E. Frames not rigidly attached to a link

While in the previous section we always assume that the base transformation of a robot is expressed with the pose of link-attached frame with respect to the inertial frame A, this is not a strict requirement. As a matter of fact, we can also assume that the robot base state is expressed with the origin of a frame C and the orientation of a frame D:

$$q^{C[D]} := ({}^A\dot{o}_C, {}^A R_D, \dot{\breve{q}}) \tag{23}$$

Clearly the relative generalized robot velocity is given by:

$$\nu^{B[C]} = \begin{bmatrix} {}^A\dot{o}_C \\ {}^A\omega_{A,D} \\ \dot{\breve{q}} \end{bmatrix} = \begin{bmatrix} {}^A\dot{o}_C \\ \left({}^A\dot{R}_D {}^A R_D^\top\right)^\vee \\ \dot{\breve{q}} \end{bmatrix} \tag{24}$$

Given that we choose the mixed velocity, the two components of the *base* velocity reflect the different choices of the *base* position variables. The Jacobian of the velocity of a frame defined in such a compound way assuming a floating base B' is simply given by the combination of the linear part (first three rows) of $J_{C,B}$, indicated with $J^l_{C,B}$ and the angular part of (last three rows) of $J_{D,B}$, indicated with $J^a_{D,B}$:

$$J_{C[D],B} = \begin{bmatrix} J^l_{C,B} \\ J^a_{D,B} \end{bmatrix}. \tag{25}$$

From which we can easily get that the transformation matrix ${}_{C[D]}T^B$ is:

$$
{}_{C[D]}T^B = \begin{bmatrix} J_{C[D],B} & \\ 0_{6\times n} & 1_{n\times n} \end{bmatrix} \tag{26}
$$

V. CHANGE OF BASE FRAME AND CENTROIDAL DYNAMICS

Cite dai2014whole,wensing2016,koolen2016design

In humanoids robot control it is widespread to control as primary task the position of the center of mass [21] or to minize the angular momentum of the robot [22], [3]. For analysis purpose is then convenient to include the center of mass in the state representing the robot position. Such a transformation can still be expressed as a change of variables using the formalism discuss in the previous section, but considering frames that are not rigidly attached to a given link. Expressing such change of variables in the proposed formalism simplifies the computation of the system dynamics in the new state, using the dynamics transformation laws (22).

A. Background on centroidal quantities

To properly define such concepts, it is convenient to define a frame called $\overline{G}$ whose origin is the center of mass of the multi body system, and whose orientation is the one of the inertial frame A. Note that the use of $\overline{G}$ is an abuse of notation, as we *do not* define an orientation for frame G as we did not fully defined the $\overline{G}$ frame. Then we can define:

We can define the total momentum and the composite rigid body inertia (CRBA) of the system expressed in the $\overline{G}$ frame as:

$$
{}_{\overline{G}}\mathrm{h} = {}_{\overline{G}}X^{\overline{B}}{}_{\overline{B}}\mathrm{h} \qquad {}_{\overline{G}}I^C = {}_{\overline{G}}X^{\overline{B}}{}_{\overline{B}}I^{C\,\overline{B}}X_{\overline{G}} \tag{27}
$$

Note that this quantities are known in literature [3] as the *Centroidal Momentum* and as the *Centroidal Composite Rigid Body Inertia (CCRBI)*. This *centroidal* quantities have an interesting structure [3] (remind that ${}^Ao_G$ are the coordinates of the center of mass in the inertial frame A):

$$
{}_{\overline{G}}\mathrm{h} = \begin{bmatrix} m{}^A\dot{o}_G \\ l \end{bmatrix} \qquad {}_{\overline{G}}I^C = \begin{bmatrix} m1_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & L^C \end{bmatrix} \tag{28}
$$

Where m is the total mass of the robot, l is the total angular momentum and L^C is the locked 3D inertia of the robot , both expressed in the center of mass with the orientation of the inertial frame.

Given a floating base link B, note that the matrix A_G that multiplied for the generalized robot velocities vector ν^B gives the *centroidal momentum* can be easily obtained from the floating base mass matrix, combining (12) and (27):

$$
A_G = {}_{\overline{G}}X^{\overline{B}} \begin{bmatrix} {}_{\overline{B}}I^C & F_B \end{bmatrix} \tag{29}
$$

The A_G matrix is the *Centroidal Momentum Matrix* [22], [3].

B. Average 6D Velocity

In humanoids whole-body control literature, it is common to define the *average 6D velocity* v_G of the robot as:

$$
\mathrm{v}_G = {}_{\overline{G}}I^{-1}{}_{\overline{G}}\mathrm{h} = \begin{bmatrix} \frac{1}{m}(m{}^A\dot{o}_G) \\ (L^C)^{-1}l \end{bmatrix} = \begin{bmatrix} {}^A\dot{o}_G \\ \omega_G \end{bmatrix} \tag{30}
$$

From the definition, the linear part of the *average 6D velocity* is simply the velocity of the center of mass of the system. On the other hand, its angular part is called [23], [24], [25] the *average angular velocity* ω_G, even if this name is a bit of an abuse of the term *angular velocity*, because the term ω_G is not defined as the angular velocity of an orientation frame. In fact, the existence of a rotation matrix $R(\breve{q}) \in SO(3)$ such that $R(\breve{q})R^\top(\breve{q}) = (\omega_C)^\wedge$, i.e. the integrability of ω_C, is still an open research problem [3]. The *average angular velocity* has a precise physical meaning if a multibody system is evolving without external forces acting on it, if an impulse block all its joint motions instantaneously the resulting rigid body would evolve with an angular velocity ω_C. For this reason it is also known as *locked angular velocity* in geometrical mechanics literature [26].

Assuming that we have a floating base link B, the relationship between the generalized robot velocities vector ν^B and the *centroidal velocity* v_G can be easily obtained combining (29) and (30):

$$
\begin{aligned}
({}_{\overline{G}}I^C)^{-1}A_G &= ({}_{\overline{G}}I^C)^{-1}{}_{\overline{G}}X^{\overline{B}} \begin{bmatrix} {}_{\overline{B}}I^C & F_B \end{bmatrix} = && (31)\\
&= \begin{bmatrix} {}^{\overline{G}}X_{\overline{B}} & ({}_{\overline{G}}I^C)^{-1}{}_{\overline{G}}X^{\overline{B}}F_B \end{bmatrix} && (32)
\end{aligned}
$$

Noting that this matrix has the same structure of the usual floating base jacobian (16) we borrow the notation that we use for the Jacobians of links, even if v_G is not defined as the mixed velocity of a well defined frame, and we define $J_{G,B}$ and $\breve{J}_{G,B}$ as:

$$
J^B_G := \begin{bmatrix} {}^{\overline{G}}X_{\overline{B}} & \breve{J}^B_G \end{bmatrix} := \begin{bmatrix} {}^{\overline{G}}X_{\overline{B}} & {}_{\overline{G}}I^{-1}{}_{\overline{G}}X^{\overline{B}}F_B \end{bmatrix} \tag{33}
$$

C. Centroidal changes of variables

1) Include center of mass in the robot state: For the sake of including the center of mass dynamics directly in the multibody dynamics, we can define a new robot state position:

$$
q^{G[B]} = ({}^Ao_G, {}^AR_B, \breve{q}). \tag{34}
$$

And its corresponding generalized velocity:

$$
\nu^{G[B]} = \begin{bmatrix} \mathrm{v}_{G[B]} \\ \dot{\breve{q}} \end{bmatrix} = \begin{bmatrix} {}^A\dot{o}_G \\ ({}^A\dot{R}_B{}^AR_B)^\vee \\ \dot{\breve{q}} \end{bmatrix} \tag{35}
$$

The corresponding transformation matrix is given by:

$$
{}^{G[B]}T_B = \begin{bmatrix} J^l_{G[B],B} & \\ 0_{3\times(3+n)} & 1_{(3+n)\times(3+n)} \end{bmatrix} \tag{36}
$$

where $J^l_{G[B],B}$ is the COM jacobian, i.e. the matrix such that ${}^A\dot{o}_G = J^l_{G[B],B}\nu_B$. Note that this is also equal to the first three rows of (33), i.e. $J^l_{G[B],B} = J^l_{G,B}$.

2) Block diagonalization of the mass matrix: Using the definition of *average 6D velocity* or *centroidal 6D velocity* of the multibody system, we can define a *centroidal* generalized joint velocities vector, in which we combine the average 6D velocity and the joint velocities :

$$\nu^G = \begin{bmatrix} \mathrm{v}_G \\ \dot{\breve{q}} \end{bmatrix} \tag{37}$$

Note again that there is no thing such as a G link: the notation ν^G is again a abuse of notation. In particular, it does not make sense to write q^G : the change of variables induced by the use of the centroidal 6D velocity is only a change of variables in the velocity space. The change of variable from the usual *floating base* generalized velocities vector ν^B has the same structure of the change of floating base introduced in (17):

$$\nu^G = {}^G T_B \nu^B, \quad {}^G T_B = \begin{bmatrix} {}^{\overline{G}}X_{\overline{B}} & \breve{J}_{G,B} \\ 0_{6\times n} & 1_{n\times n} \end{bmatrix}. \tag{38}$$

This induces a change of variables also on the generalized robot forces as in (18), given by :

$$_G T^B = {}^G T_B^{-\top} \tag{39}$$

3) Centroidal Dynamics: Using the transformation ${}^G T_B$, we can obtain an equation of motions in the usual form:

$$\dot{q}^B = \left({}^A\dot{o}_B, {}^A\omega_B^\wedge(\nu^G)^A R_B, \dot{\breve{q}} \right), \tag{40}$$

$$M_G \dot{\nu}^G + C_G\left(q^B, \nu^G\right)\nu_G + G_G = \left[\begin{smallmatrix} 0_{6\times 1} \\ \tau \end{smallmatrix}\right] + \sum_{L \in L} J_{L,G}^\top \mathrm{f}_L^x. \tag{41}$$

Note that differently

While we still assume the existence of the floating base for the position part of the state, the equation of motions describing the derivative of the robot generalized velocities incorporates in the first six rows what is usually called the *centroidal dynamics*, together with the joint dynamics in a single set of equations of motions. The specific features of such a representation of the equations of motions are highlighted in the following theorem, and they have been already exploited in [17].

Theorem 3. *The centroidal mass matrix has a block diagonal structure of :*

$$M_G = \begin{bmatrix} m1_{3\times 3} & 0_{3\times 3} & 0_{3\times n} \\ 0_{3\times 3} & L^C & 0_{3\times n} \\ 0_{3\times 3} & 0_{3\times 3} & \breve{M}_G \end{bmatrix} \tag{42}$$

Furthermore, the centroidal gravity term has the following structure:

$$G_G = \begin{bmatrix} {}^A g \\ 0_{3\times 1} \\ 0_{n\times 1} \end{bmatrix} \tag{43}$$

Proof. The proof is obtained by direct computation applying using the centroidal transformation (**??**) transforming the floating base equation of motions (**??**). □

VI. CONCLUSIONS

A conclusion section is not required, especially if you just ate a pizza nearby Quarter Latin.

APPENDIX

ACKNOWLEDGMENT

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