On the Base Frame Choice in Free-Floating Mechanical Systems and its Connection to Centroidal Dynamics*

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Abstract— **This paper contributes towards the development of a unified standpoint on the equations of motion associated with free-floating systems. In particular, the contribution of the manuscript is twofold. First, we show how to write the system equations of motion for any choice of the base frame, without the need of re-applying numerical algorithms for evaluating the dynamic quantities, such as mass, coriolis, and gravity matrix. A particular attention is paid to the properties associated with the mechanical systems, which are shown to be invariant with respect to the base frame choice. Secondly, we show that the so-called *centroidal dynamics* can be obtained from any expression of the system equations of motion via an appropriate base frame change. In this case, we show that the mass matrix associated with the new base frame is block diagonal, and the new base velocity corresponds to the so-called *average 6D velocity*.**

I. INTRODUCTION

Classical modelling techniques for multi-body mechanical systems have their roots in the fundamental principles of mechanics. For instance, Euler-Lagrange and Euler-Poincaré formalisms are among the main tools for deriving the equations of motion of systems evolving on vector spaces and Lie groups, respectively [1]. The application of these formalisms to common robotics applications assume that the underlying mechanical system is composed of a collection of rigid bodies –called links– interconnected by mechanisms –called joints– constraining the links relative motion. The obtained equations of motions, however, often depend upon the arbitrary representation of the configuration space of the multi-body system. This paper investigates the role of the choice of the *base frame* when defining the configuration space of free-floating robots [2].

The configuration space of a multi-body system composed by $n + 1$ links may be characterised by the positions of all the points of the system. To this inconvenient representation, however, it is always preferred that consisting of the orientations and the center-of-mass positions of all the rigid bodies. When the links are interconnected by n joints, any point of the system can be expressed in terms of the *joint angles* and a *frame* attached to a link of the system. This frame is referred to as *base frame* and its choice is dictated by rule-of-thumb principles, such as attaching the base frame to the heaviest link of the considered robot. Then, the configuration space of the mechanical system is of dimension $n+6$, and the system is usually referred to as *free-floating* or *floating base*.

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If one of the links has a constant position-and-orientation with respect to the inertial frame, the configuration space collapses to joint angles only, and its dimension drops to n. In this case, the system is referred to as *fixed base*, and robotic manipulators attached to ground do fall into this category.

Modelling, estimation, and control of robots, however, were developed first for fixed-base manipulators, and then for floating-base systems. Then, floating-base systems are usually viewed as a special case of fixed base robots, and their modelling was often addressed by considering a six degree of freedom joint between the inertial and the base frame [3]. This solution has the advantage of reusing the fixed base machinery to derive the equations of motion for floating base robots. However, it does hide some properties of floating systems, and it makes the use of minimal representations for the base orientation really tempting, which in turn introduces artificial singularities for this orientation [4].

Attempts to generalize the floating-base systems' equations of motion irrespective from the base frame choice can be found in [5, Chapter 3] and in [6, Chapter 17, Section 3.6]. Unfortunately, these works have been largely ignored by the humanoid whole-body control literature. These two works, however, share the same limitation: they assume the base frame to be fixed to one of the links of the system.

In this paper, we model floating systems without assuming any base frame as the preferred one, and we show how the kinematics and dynamics of the robot arise by choosing different floating base frames. In particular, we show how the equations of motion associated with different "base frames" can be obtained as a nonlinear change of variables of the robot state. We also show that this change of variables can be used to express the robot dynamics as a combination of the "internal" and the "centroidal dynamics", as introduced in [3] and extended in [7]. While most of the results of this paper can be linked to classical Lie Group geometrical mechanics results, we avoided to introduce such a link to avoid to overcomplexify the paper for people not familiar with geometrical mechanics. However, an extended explanation on the connection between the notation employed here and geometrical mechanics theory can be found in [8].

This paper is organized as follows. Section II introduces notation, kinematics, and dynamics for a single rigid body. Section III recalls and complements the free-floating notation and equations of motion. Section IV explains the effects of changing the base frame on the system's kinematics and dynamics. Section V discusses the connection between a change of the base frame and the centroidal dynamics. Remarks and perspectives conclude the paper.

II. BACKGROUND

This section introduces basic notation used for dynamics computations. For further information and details on the notation used throughout this paper, the reader is referred to [8]. Although what follows is partially based on Spatial Vector Algebra notation [2], [9], several specific modifications have been introduced for the purpose of this paper. For instance, the *dot operator* means:

$$\dot{(\cdot)} = \frac{d}{dt}(\cdot),$$

thus avoiding the introduction of an arbitrary (and often hidden) inertial frame w.r.t. which the derivative is computed.

- The set of real numbers is denoted by $\mathbb{R}$. Let u and v be two n-dimensional column vectors of real numbers, i.e. $u, v \in \mathbb{R}^n$, then their inner product is denoted as $u^\top v$, with "$\top$" the transpose operator.
- The identity matrix of dimension n is denoted by $1_{n\times n} \in \mathbb{R}^{n\times n}$; the zero column vector of dimension n is denoted by $0_n \in \mathbb{R}^n$; the zero matrix of dimension $n \times m$ is denoted by $0_{n\times m} \in \mathbb{R}^{n\times m}$.
- Given a vector $w = \begin{bmatrix} x & y & z \end{bmatrix}^\top \in \mathbb{R}^3$, we define $w^\wedge$ (read w *hat*) as the 3×3 *skew-symmetric* matrix associated with the cross product $\times$ in $\mathbb{R}^3$, i.e. $w^\wedge x = w \times x$. Given the *skew-symmetric matrix* $W = w^\wedge$, we define $W^\vee \in \mathbb{R}^3$ (read W *vee*) as $W^\vee := w$.
- The set SO(3) is the set of $\mathbb{R}^{3\times 3}$ orthogonal matrices with determinant equal to one, namely $\mathrm{SO}(3) := \{ R \in \mathbb{R}^{3\times 3} \mid R^\top R = I_3,\ \det(R) = 1 \}$.

A. Frames notation

A *frame* is defined as the pair of a point (called *origin*) and an *orientation frame* in the 3D space [10], and we use capital letters to indicate frames. Given a frame F, we denote with o_F its origin and with $[F]$ its orientation frame. Formally, we write this as $F = (o_F, [F])$. Then, we have:

A	Denote the inertial (*A*bsolute) frame.
p	An arbitrary point.
$B[A]$	Frame with origin o_B and orientation $[A]$.
$\overline{B}$	Shorthand for $B[A]$.

B. Coordinates vectors and transformation matrices notation

${}^A p \in \mathbb{R}^3$	Coordinates of p w.r.t. to A.
${}^A o_B \in \mathbb{R}^3$	Coordinates of o_B w.r.t. to A.
${}^A R_B \in \mathbb{R}^{3\times 3}$	Rotation matrix transforming a 3D vector expressed in $[B]$ into one expressed in $[A]$.
${}^A X_B = \begin{bmatrix} {}^A R_B & {}^A o_B^\wedge {}^A R_B \\ 0_{3\times 3} & {}^A R_B \end{bmatrix}$	Velocity transformation from B to A.
${}^C \mathrm{v}_{A,B} \in \mathbb{R}^6$	Twist representing the velocity of B w.r.t. to A expressed in C.
${}_B \mathrm{f} \in \mathbb{R}^6$	Coordinates of the wrench f w.r.t. B.
${}_A X^B = \begin{bmatrix} {}^A R_B & 0_{3\times 3} \\ {}^A o_B^\wedge {}^A R_B & {}^A R_B \end{bmatrix}$	Wrench transformation from B to A (${}_A X^B = {}^A X_B^{-\top}$).
${}_B I_L \in \mathbb{R}^{6\times 6}$	6D inertia matrix of body L expressed in the frame B .

C. Frame velocity representation

The position and orientation of a frame B w.r.t. an inertial frame A is usually represented with the homogeneous matrix ${}^A H_B \in \mathbb{R}^{4\times 4}$ transforming a position vector expressed in B to a position vector expressed in A, i.e.

$$ {}^A H_B = \begin{bmatrix} {}^A R_B & {}^A o_B \\ 0_{1\times 3} & 1 \end{bmatrix}, \quad {}^A p = {}^A R_B {}^B p + {}^A o_B. \quad (1)$$

Then, the body velocity can be represented by ${}^A \dot{H}_B$. It is more convenient, however, to express the body velocity as a $\mathbb{R}^6$ vector. Common representations of ${}^A \dot{H}_B$ as an element of $\mathbb{R}^6$ are given by:

$$ {}^A \mathrm{v}_{A,B} = \begin{bmatrix} {}^A \dot{o}_B - {}^A \omega_{A,B}^\wedge {}^A o_B \\ {}^A \omega_{A,B} \end{bmatrix}, $$

$$ {}^B \mathrm{v}_{A,B} = \begin{bmatrix} {}^B R_A {}^A \dot{o}_B \\ {}^B R_A {}^A \omega_{A,B} \end{bmatrix}, \quad {}^{B[A]} \mathrm{v}_{A,B} = \begin{bmatrix} {}^A \dot{o}_B \\ {}^A \omega_{A,B} \end{bmatrix}. $$

where ${}^A \omega_{A,B} := ({}^A \dot{R}_B {}^A R_B^\top)^\vee$. The above representations are usually referred to as: ${}^A \mathrm{v}_{A,B}$ the *inertial* representation of the body velocity, ${}^B \mathrm{v}_{A,B}$ the *body-fixed* representation, and ${}^{B[A]} \mathrm{v}_{A,B}$ the *mixed* representation (also known as *hybrid* representation [11]). The *inertial* and *body-fixed* representation are widespread in the literature of Lie group-based geometric mechanics [12] and recursive robot dynamics algorithms [2], [13]. Yet, in this paper we use the *mixed* velocity representation because it is commonly used in multi-task control frameworks [14], [15], [16]. It can be shown, however, that most of the results of the paper hold irrespective of the velocity representation. Furthermore, we use the mixed representation even for body accelerations, wrenches and mementa. To avoid overloading the notation, in the rest of the document we define:

$$ \mathrm{v}_L := {}^{\overline{L}} \mathrm{v}_{A,L} = {}^{L[A]} \mathrm{v}_{A,L}, \quad \dot{\mathrm{v}}_L := {}^{\overline{L}} \dot{\mathrm{v}}_{A,L} = {}^{L[A]} \dot{\mathrm{v}}_{A,L}. \quad (2)$$

D. Rigid Body Dynamics

This section recalls the equations of motion of a single rigid body assuming that a frame L is attached to it.

The *mixed* body momentum $h_L \in \mathbb{R}^6$ is given by [8]:

$$ h_L = {}_{\overline{L}} h_L = {}_{\overline{L}} X^L {}_L I_L {}^L X_{\overline{L}} \mathrm{v}_L = {}_{\overline{L}} I_L \mathrm{v}_L, \quad (3)$$

where ${}_L I_L$ is the 6D body inertia expressed w.r.t. the origin of L and the orientation of A, i.e.

$$ {}_L I_L = \begin{bmatrix} m_L 1_{3\times 3} & m_L {}^L c_L^\wedge \\ -m_L {}^L c_L^\wedge & \overline{I}_L \end{bmatrix}, \quad (4)$$

with $m_L \in \mathbb{R}$ the mass of the body, ${}^L c_L \in \mathbb{R}^3$ the coordinates of the center of mass c_L of the body w.r.t. the frame L, and $\overline{I}_L \in \mathbb{R}^{3\times 3}$ the 3D inertia of the link expressed w.r.t. the orientation and origin of L [2], [17]. Notice that ${}_L I_L$ is constant, while ${}_{\overline{L}} I = {}_{\overline{L}} X^L I_L {}^L X_{\overline{L}}$ (i.e. the inertia expressed in *mixed* coordinates) is not because ${}^L X_L$ depends on the orientation of L w.r.t. A.

The Newton-Euler principle states that the *rate-of-change* of the momentum equals the sum of the external wrenches applied to the body. By computing the time derivative of (3) one obtains the *mixed* Newton-Euler equations of motion:

$$_{\overline{L}}I_L \dot{\mathrm{v}}_L + C(\mathrm{v}_L, {}^A R_L)\mathrm{v}_L = \mathrm{f}^x_L + {}_{\overline{L}}I_L \begin{bmatrix} {}^A R_L^\top {}^A g \\ 0_{3\times 1} \end{bmatrix}, \qquad (5)$$

where ${}^A g \in \mathbb{R}^3$ is the gravitational acceleration expressed in the inertial frame, $C(\mathrm{v}_L, {}^A R_L) := \frac{d}{dt}\left({}_{\overline{L}}I_L\right)$, and $\mathrm{f}^x_L \in \mathbb{R}^6$ the *mixed* external wrench applied to the body, i.e. $\mathrm{f}^x_L := {}_{\overline{L}}\mathrm{f}^x_L = {}_{L[A]}\mathrm{f}^x_L$.

III. Recalls and Complements on Free-Floating Multi-Body Dynamics

This section recalls notation, state definition, and equations of motion associated with a *free-floating* mechanical system. In particular, we assume that one is given with a mechanical system composed of $n+1$ rigid bodies –called *links*– interconnected by n mechanisms –called *joints*– constraining the link relative motion. The set of links is indicated by $\mathfrak{L}$, and the set of joints is indicated by $\mathfrak{J}$. We assume that each link $L \in \mathfrak{L}$ is associated with a link-fixed frame. In the sequel, we often refer to the frame attached to link L simply as L. So, one of these frames is called *base frame* B. We also assume that each joint possess only one degree-of-freedom, and interconnects two links with no internal loop.

A. State definition

Being a mechanical system, the equations of motion governing the free-floating dynamics are a second-order differential system. Hence, we have to define a *state* of the system composed of a properly defined *position* and *velocity*.

1) Free-Floating system position: The process of defining the *system position* aims at determining a set of variables from which the position of each point of the multi-body system can be retrieved in the absolute frame A. Being a composition of rigid bodies, each point of the multi-body system can be retrieved from the position-and-orientation – referred to as *pose* – of each link frame $L \in \mathfrak{L}$. Given the topology of the considered multi-body system, each link pose can be determined from the pose of the base frame B and the *joint configurations*.

We assume that the configuration space of each joint is $\mathbb{R}$, which is the case of rotational and prismatic joints subject to joint limit constraints. Then, the *joint positions* can be identified by a vector of coordinates $\breve{q} \in \mathbb{R}^n$. Contrary to [2], we assume that the *numbering* of the joints remains constant, namely, the joint serialization is independent from the base frame choice B, thus getting rid of the dependence of the joint serialization upon floating base choice [18]. Clearly, the joint positions $\breve{q}$ uniquely characterize the pose of any link w.r.t. the base frame B.

In the light of the above, the configuration set of a free-floating system is given by the combination of the *base pose* $({}^A o_B, {}^A R_B) \in \mathbb{R}^3 \times \mathrm{SO}(3)$ and the *joint* positions $\breve{q} \in \mathbb{R}^n$, i.e.

$$\mathbb{Q} = \mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^n, \qquad (6)$$

$$q^B = ({}^A o_B, {}^A R_B, \breve{q}) \in \mathbb{Q}. \qquad (7)$$

2) Free-Floating system velocity: In view of (7), the derivative of the robot position is given by:

$$\dot{q}^B = ({}^A \dot{o}_B, {}^A \dot{R}_B, \dot{\breve{q}}).$$

As in the case of a single rigid body (see Sec.II-C), it is more convenient to represent the velocity as a column vector. Using the *mixed* body velocity representation, we can define the system velocity vector $\nu^B \in \mathbb{R}^{6+n}$ as follows:

$$\nu^B = \begin{bmatrix} \mathrm{v}_B \\ \dot{\breve{q}} \end{bmatrix} = \begin{bmatrix} {}^A \dot{o}_B \\ {}^A \omega_{A,B} \\ \dot{\breve{q}} \end{bmatrix} = \begin{bmatrix} {}^A \dot{o}_B \\ ({}^A \dot{R}_B {}^A R_B^\top)^\vee \\ \dot{\breve{q}} \end{bmatrix} \in \mathbb{R}^{n+6} \quad (8)$$

with $\mathrm{v}_B \in \mathbb{R}^6$ the *base* velocity, and $\dot{\breve{q}}$ the *joint* velocities. Clearly, the system velocity depends upon the choice of base frame B.

B. Kinematics

We recall below how to relate the base frame pose and velocity with the pose and velocity of an arbitrary link frame $L \in \mathfrak{L}$.

The position of a link L w.r.t to the inertial frame is a function of the system position q^B:

$${}^A H_L(q^B) \,:\, \mathbb{Q} \mapsto \mathbb{R}^3 \times \mathrm{SO}(3), \qquad (9a)$$

$${}^A H_L(q^B) = {}^A H_B {}^B H_L(\breve{q}) = \begin{bmatrix} {}^A R_B & {}^A o_B \\ 0_{1\times 3} & 1 \end{bmatrix} {}^B H_L(q). \qquad (9b)$$

The velocity of a frame L w.r.t. to the inertial frame is the product between a robot position-dependent Jacobian matrix $J_{L,B}(q_B) \in \mathbb{R}^{6\times n+6}$ times the system velocity:

$$\mathrm{v}_L(q^B, \nu^B) = J_{L,B}(q_B)\nu^B, \qquad (10a)$$

$$J_{L,B}(q_B) = \begin{bmatrix} {}^{\overline{L}}X_{\overline{B}} & \breve{J}_{L,B}(\breve{q}) \end{bmatrix}. \qquad (10b)$$

We recall that ${}^{\overline{L}}X_{\overline{B}}$ is given by:

$${}^{\overline{L}}X_{\overline{B}} = \begin{bmatrix} 1_{3\times 3} & ({}^A R_L {}^L o_B)^\wedge \\ 0_{3\times 3} & 1_{3\times 3} \end{bmatrix} = \begin{bmatrix} 1_{3\times 3} & ({}^A o_B - {}^A o_L)^\wedge \\ 0_{3\times 3} & 1_{3\times 3} \end{bmatrix}.$$

The equations (9) (10) represent the so-called *forward kinematics* of the link L using B as the base frame. How to compute the *forward kinematics* and the *Jacobian matrix* is out of the scope of this paper. If the base frame is a link-fixed frame, the algorithms for computing such quantities can be found in [2], [13], [17].

C. Free-floating system equations of motion

Using the Euler-Poincarè equations [1, Chapter 13], it is possible to write the equations of motion for a free-floating system subject to external wrenches $\mathrm{f}^x_L \in \mathbb{R}^6$. These equations write [2], [6]:

$$\dot{q}^B = \left({}^A \dot{o}_B, {}^A \omega^\wedge_{A,B} {}^A R_B, \dot{\breve{q}}\right) \qquad (11a)$$

$$M_B(q^B)\dot{\nu}^B + C_B(q^B, \nu_B)\nu_B + G_B(q^B) = \begin{bmatrix} 0_{6\times 1} \\ \tau \end{bmatrix} + \sum_{L\in\mathfrak{L}} J^\top_{L,B}\mathrm{f}^x_L, \qquad (11b)$$

where $M_B(q^B) \in \mathbb{R}^{n+6\times n+6}$, $C_B(q^B, \nu_B) \in \mathbb{R}^{n+6\times n+6}$, and $G_B(q^B) \in \mathbb{R}^{n+6}$ denote the mass matrix, the Coriolis matrix, and the gravity torques, respectively. The matrix $J_{L,B} \in \mathbb{R}^{6\times n+6}$ denotes the jacobian of the link L on which the external wrench acts. The suffix B in all these symbols highlights the dependence of the quantity upon the choice of the base frame B. The *joint torques* $\tau \in \mathbb{R}^n$, which is the input to the system, clearly do not depend upon this choice.

Let us now recall some important facts on the model (12).

Lemma 1 (Kinetic Energy, [2]). *The kinematic energy of the free-floating system, i.e. the sum of all the kinematic energies of each link of the system, is given by:*

$$K = \frac{1}{2}\sum_{L\in\mathfrak{L}} \mathrm{v}_L^\top {}_{\overline{L}}I_L \mathrm{v}_L = \frac{1}{2}\nu_B^\top M_B \nu_B. \tag{12}$$

Theorem 1 ([19]). *The mass matrix M_B can be expressed as follows:*

$$M_B = \begin{bmatrix} {}_{\overline{B}}I^C & F_B \\ F_B^\top & H_B \end{bmatrix}, \tag{13}$$

where ${}_{\overline{B}}I^C$ is the so-called locked 6D rigid body inertia *of the multi-body system, i.e.*

$${}_{\overline{B}}I^C = \sum_{L\in\mathfrak{L}} {}_{\overline{B}}X^L I_L {}^L X_{\overline{B}}. \tag{14}$$

Furthermore, the first six rows of the mass matrix define the jacobian of the articulated body momentum (i.e. the sum of the linear/angular momentum of all the bodies composing the robot), referred to as the Momentum Matrix.

$${}_{\overline{B}}\mathrm{h} = \begin{bmatrix} {}_{\overline{B}}I^C & F_B \end{bmatrix} \nu_B = \sum_{L\in\mathfrak{L}} {}_{\overline{B}}X^L I_L {}^L\mathrm{v}_L \tag{15}$$

The matrix M_B is known as *Composite Rigid Body Inertia* (CRBA) of the robot in whole body control literature [3]. The following properties hold on the model (12) [12]:

Property 1. *The mass matrix M_B is symmetric and positive definite.*

Property 2. *The matrix $\dot{M}_B - 2C_B$ is skew symmetric*

1) Free-floating generalized force and its power: The effects of an external wrench (i.e. a wrench acting between a robot link and the environment) and of internal wrenches (i.e. acting between two robot links) can be represented with a free-floating generalized force vector:

$$\phi_B = \begin{bmatrix} \mathrm{f}_B^x \\ \tau_B \end{bmatrix} \in \mathbb{R}^{n+6},$$

where $\mathrm{f}_B^x \in \mathbb{R}^6$ (the *base* wrench) represents the net external force-torque exerted by the environment on the robot and expressed with the orientation of the inertial frame and at the origin of the B frame; while $\tau_B \in \mathbb{R}^n$ represents the *joint* torques. Then, the power $P \in \mathbb{R}$ transmitted by a given generalized force on the free-floating robot moving with a given velocity can be defined as follows [2]:

$$P = \phi_B^\top \nu^B. \tag{16}$$

IV. The effects of changing base frame: from the state to the equations of motion

In this section, we answer to the following questions.

Q1) Which state transformation is associated with the base frame change from the base B to the base frame B'?

Q2) How does this transformation affect the equations of motion (11) *and its properties?*

The answer to the first question is stated below.

Lemma 2. *Assume that the base frame is changed from a frame B to a frame B'. Then, the following results hold.*

1) *The system position and velocity are subject to the following transformations*

$$q^{B'} = ({}^A H_B {}^B H_{B'}(\breve{q}), \breve{q}), \tag{17a}$$

$$\nu^{B'} = {}^{B'}T_B \nu^B \tag{17b}$$

where

$${}^{B'}T_B = \begin{bmatrix} {}^{\overline{B'}}X_{\overline{B}} & \breve{J}_{B',B} \\ 0_{n\times 6} & 1_{n\times n} \end{bmatrix}. \tag{18}$$

2) *A free-floating generalized force $\phi_B \in \mathbb{R}^{n+6}$ is subject to the following transformation*

$$\phi_{B'} = {}_{B'}T^B \phi_B, \tag{19}$$

with

$${}_{B'}T^B = {}^{B'}T_B^{-\top} = \begin{bmatrix} {}_{\overline{B'}}X^{\overline{B}} & 0_{6\times n} \\ -(\breve{J}_{B',B})^\top {}_{\overline{B'}}X^{\overline{B}} & 1_{n\times n} \end{bmatrix} \tag{20}$$

3) *The jacobian $J_{L,B} \in \mathbb{R}^{6\times n+6}$ of a link frame L is subject to the following transformation*

$$J_{L,B'} = J_{L,B} {}^B T_{B'}. \tag{21}$$

The proof is given in the Appendix. Item 1) details how the system position and velocity transform when the base frame is moved from B to B'. The relationships (17) follow directly from the system kinematics (9) and (10) between the two frames.

The item 2) defines the transformation of the free-floating generalized force ϕ_B. The key point to define the transformation (19)-(20) is to observe that the power P is independent from choice of the base frame, and thus invariant when moving the base frame from B to B'. Then, the transformation associated with the force ϕ_B can be found.

The item 3) defines how the jacobian of a link frame L changes depending on a base frame link change. The relationship (21) is a direct consequence of (10).

Remark IV.1 (Pure internal joint torques invariance). *Let ϕ_B be a free-floating generalized force acting on the multi-body system. If ϕ_B has a null base wrench, then it is independent of the base frame in which it is expressed, i.e.*

$${}_{B'}T^B \begin{bmatrix} 0_{6\times 1} \\ \tau_B \end{bmatrix} = \begin{bmatrix} 0_{6\times 1} \\ \tau_{B'} \end{bmatrix} = \begin{bmatrix} 0_{6\times 1} \\ \tau \end{bmatrix}.$$

Remark IV.2 (External joint torque base dependency)**.** *Let B be a base frame rigidly attached to a link B. The effect of a pure base wrench f^x_B on a the link B is by definition the generalized force where the first six elements are given by f^x_B and joint part is equal to zero:*

$$\phi_B = \begin{bmatrix} \mathrm{f}^x_B \\ 0_{n\times 1} \end{bmatrix}.$$

The same generalized force, expressed with a different link-fixed base frame B' is given by:

$$\phi_{B'} = {}_{B'}T^B \phi_B = \begin{bmatrix} {}_{\overline{B'}}X^{\overline{B}} & 0_{6\times n} \\ (\breve{J}_{B',B})^\top {}_{\overline{B'}}X^{\overline{B}} & 1_{n\times n} \end{bmatrix} \begin{bmatrix} \mathrm{f}^x_B \\ 0_{n\times 1} \end{bmatrix} =$$
$$= \begin{bmatrix} {}_{\overline{B'}}X^{\overline{B}}\mathrm{f}^x_B \\ (\breve{J}_{B',B})^\top {}_{\overline{B'}}X^{\overline{B}}\mathrm{f}^x_B \end{bmatrix}.$$

The first six rows are still the same external wrench, with the ${}_{\overline{B'}}X^{\overline{B}}$ transform that accounts for the change of reference point from the origin of B to the origin B'. In the joint torques part, there are some "external joint torques", induced only by the change of floating base. In particular this torques follow a precise sparsity pattern induced by the jacobian $\breve{J}^\top_{B',B}$, i.e. there are non-zero only for the joints that connect the links B' and B. It is important to note that while the external joint torques *are uniquely defined in the case of fixed base robots, in the case of floating base robots their value depends on the representation choice (the floating base) and as such they do not described any measurable physical quantity.*

Let us now answer to the question $Q2$).

Theorem 2. *Assume that the base frame is changed from a frame B to a frame B'. Then, the following results hold.*

1) *The equations of motions* (11) *can be written as*

$$\dot{q}^{B'} = \left({}^A\dot{o}_{B'}, {}^A\omega^\wedge_{A,B'}{}^AR_{B'}, \dot{\breve{q}}\right) \tag{22a}$$
$$M_{B'}(q^{B'})\dot{\nu}^{B'} + C_{B'}(q^{B'}, \nu_{B'})\nu_{B'} + G_{B'}(q^{B'}) =$$
$$\begin{bmatrix} 0_{6\times 1} \\ \tau \end{bmatrix} + \sum_{L\in\mathfrak{L}} J^\top_{L,B'}\mathrm{f}^x_L, \tag{22b}$$

with

$$M_{B'} = {}_{B'}T^B M_B {}^BT_{B'}, \tag{23a}$$
$$C_{B'} = {}_{B'}T^B \left(M_B {}^B\dot{T}_{B'} + C_B {}^BT_{B'}\right), \tag{23b}$$
$$G_{B'} = {}_{B'}T^B G_B; \tag{23c}$$

2) *The free-floating system kinetic energy is given by:*

$$K = \frac{1}{2}\nu^\top_{B'}M_{B'}\nu_{B'}; \tag{24}$$

3) *Assume that Property 2 holds, i.e. $\dot{M}_B - 2C_B$ is skew-symmetric. Then also $\dot{M}_{B'} - 2C_{B'}$ is skew-symmetric.*

The proof is given in the Appendix. The above theorem characterizes how the equations of motion of the free-floating system transform when the base frame is moved from B to B'. In particular, it states that the representation of the equations of motion (22)-(23) encompasses the invariance of the fundamental Properties 1 and 2 w.r.t. a base frame change. Note that the item 2) and 3) highlight that the expressions (22)-(23) are particularly useful when designing passivity-based control algorithms [17]. In fact, the proof of these algorithms usually assume that the mass matrix is positive definite, and that the passivity property $\dot{M} - 2C$ holds when defining candidate Lyapunov functions associated with position and velocity tracking errors.

A. Frames not rigidly attached to a link

So far, the base frame transformation was meant to be between a frame B and a frame B', and both these frames were assumed to be attached to a physical link. This assumption, however, is not strict. As a matter of fact, we can assume that the base frame B' is a frame whose origin is that of a frame C, and whose orientation that of a frame D:

$$q^{C[D]} := ({}^A\dot{o}_C, {}^AR_D, \dot{\breve{q}}). \tag{25}$$

Clearly, the relative generalized robot velocity is given by:

$$\nu^{B[C]} = \begin{bmatrix} {}^A\dot{o}_C \\ {}^A\omega_{A,D} \\ \dot{\breve{q}} \end{bmatrix} = \begin{bmatrix} {}^A\dot{o}_C \\ \left({}^A\dot{R}_D {}^AR^\top_D\right)^\vee \\ \dot{\breve{q}} \end{bmatrix} \tag{26}$$

Since we have chosen the mixed velocity representation, then the new *base* velocity reflect the different choices of the *base* position variables. The Jacobian of the velocity of a frame defined in such a compound way is simply given by the combination of the linear part (first three rows) of $J_{C,B}$, indicated with $J^l_{C,B} \in \mathbb{R}^{3\times(6+n)}$ and the angular part of (last three rows) of $J_{D,B}$, indicated with $J^a_{D,B} \in \mathbb{R}^{3\times(6+n)}$:

$$J_{C[D],B} = \begin{bmatrix} J^l_{C,B} \\ J^a_{D,B} \end{bmatrix}. \tag{27}$$

From the above equation, we can easily retrieve the transformation matrix ${}^{C[D]}T_B$, i.e.:

$${}^{C[D]}T_B = \begin{bmatrix} J_{C[D],B} & \\ 0_{6\times n} & 1_{n\times n} \end{bmatrix} \tag{28}$$

V. Change of Base Frame and Centroidal Dynamics

In humanoid robot whole-body control, it is widespread to control as primary task the position of the center of mass [20] and to minize the angular momentum of the robot [21], [3], [22], [23], [24], [25]. For analysis purpose it is then convenient to include the center of mass in the state representing the robot position. Such a transformation can still be expressed as a change of variables using the formalism discussed in the previous sections. Expressing such change of variables in the proposed formalism simplifies the computation of the system dynamics in the new state.

A. Background on centroidal quantities

To properly define the *centroidal dynamics*, it is convenient to define a frame $\overline{G}$ whose origin is the center of mass of the multi body system, and whose orientation is that of the inertial frame A. Note that the use of $\overline{G}$ is an abuse of notation, as we *do not* define an orientation for frame G.

Then, the total momentum and the composite rigid body inertia (CRBA) of the system expressed w.r.t. the $\overline{G}$ frame are given by:

$$
{}_{\overline{G}}\mathrm{h} = {}_{\overline{G}}X^{\overline{B}}{}_{\overline{B}}\mathrm{h}, \qquad {}_{\overline{G}}I^{C} = {}_{\overline{G}}X^{\overline{B}}{}_{\overline{B}}I^{C\overline{B}}X_{\overline{G}}. \tag{29}
$$

This quantities are known in literature [3] as *Centroidal Momentum* and as *Centroidal Composite Rigid Body Inertia (CCRBI)*. The structure of this *centroidal* quantities is [3] :

$$
{}_{\overline{G}}\mathrm{h} = \begin{bmatrix} m^{A}\dot{c} \\ l \end{bmatrix}, \qquad {}_{\overline{G}}I^{C} = \begin{bmatrix} m1_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & L^{C} \end{bmatrix}, \tag{30}
$$

with $m \in \mathbb{R}$ is the total mass of the robot, ${}^{A}c \in R^{3}$ is the center of the mass of the robot expressed in the inertial frame, $l \in R^{3}$ is the total angular momentum and $L^{C} \in \mathbb{R}^{3\times3}$ is the locked 3D inertia of the robot , both expressed in $\overline{G}$.

Given a base frame B, note that the matrix $A_{G,B}$ that multiplied by the generalized robot velocities vector ν^{B} gives the *centroidal momentum* can be easily obtained from the floating base mass matrix, combining (15) and (29):

$$
{}_{\overline{G}}\mathrm{h} = A_{G,B}\nu^{B}, \quad A_{G,B} = {}_{\overline{G}}X^{\overline{B}}\begin{bmatrix}{}_{\overline{B}}I^{C} & F_{B}\end{bmatrix} \tag{31}
$$

The $A_{G,B}$ matrix is the *Centroidal Momentum Matrix* [21], [3].

B. Average 6D Velocity

In humanoids whole-body control literature [3], it is common to define the *average 6D velocity* v_{G} of the robot as:

$$
\mathrm{v}_{G} = ({}_{\overline{G}}I^{C})^{-1}{}_{\overline{G}}\mathrm{h} = \begin{bmatrix} \frac{1}{m}(m^{A}\dot{o}_{G}) \\ (L^{C})^{-1}l \end{bmatrix} = \begin{bmatrix} {}^{A}\dot{o}_{G} \\ \omega_{G} \end{bmatrix} \tag{32}
$$

From the definition, the linear part of the *average 6D velocity* is simply the velocity of the center of mass of the system. On the other hand, its angular part is called the *average angular velocity* ω_{G} [26], [27], even if this name is an abuse of the term *angular velocity*, because the term ω_{G} is not defined as the angular velocity of an orientation frame. In fact, the existence of a rotation matrix $R(\breve{q}) \in SO(3)$ such that $R(\breve{q})R^{\top}(\breve{q}) = (\omega_{G})^{\wedge}$, i.e. the integrability of ω_{G}, is still an open research problem [3]. The *average angular velocity* has a precise physical meaning if a multibody system is evolving without external forces acting on it. If an impulse blocks all its joint motions instantaneously, the resulting rigid body would evolve with an angular velocity ω_{G}. For this reason, it is also known as *locked angular velocity* in geometrical mechanics literature [28].

Assuming that we have a floating base frame B, the relationship between the generalized robot velocities vector ν^{B} and the *centroidal velocity* v_{G} can be easily obtained combining (31) and (32):

$$
\begin{aligned}
\mathrm{v}^{G} &= ({}_{\overline{G}}I^{C})^{-1}A_{G,B}\nu^{B} = \\
&= ({}_{\overline{G}}I^{C})^{-1}{}_{\overline{G}}X^{\overline{B}}\begin{bmatrix}{}_{\overline{B}}I^{C} & F_{B}\end{bmatrix}\nu^{B} = \\
&= \begin{bmatrix}{}^{\overline{G}}X_{\overline{B}} & ({}_{\overline{G}}I^{C})^{-1}{}_{\overline{G}}X^{\overline{B}}F_{B}\end{bmatrix}\nu^{B}
\end{aligned}
$$

Noting that this matrix has the same structure of the usual floating base jacobian (11), we borrow the notation that we use for the Jacobians of links, even if v_{G} is not defined as the mixed velocity of a well defined frame, and we define $J_{G,B}$ and $\breve{J}_{G,B}$ as:

$$
\nu^{G} = J_{G,B}\nu^{B} \tag{33}
$$

$$
J_{G,B} := \begin{bmatrix}{}^{\overline{G}}X_{\overline{B}} & \breve{J}_{G,B}\end{bmatrix} := \begin{bmatrix}{}^{\overline{G}}X_{\overline{B}} & ({}_{\overline{G}}I^{C})^{-1}{}_{\overline{G}}X^{\overline{B}}F_{B}\end{bmatrix} \tag{34}
$$

C. Centroidal changes of variables

1) Include center of mass in the robot state: For the sake of including the center of mass dynamics directly in the multibody dynamics, we can define a new robot position, as follows:

$$
q^{G[B]} = ({}^{A}c, {}^{A}R_{B}, \breve{q}). \tag{35}
$$

And its corresponding generalized velocity:

$$
\nu^{G[B]} = \begin{bmatrix} \mathrm{v}_{G[B]} \\ \dot{\breve{q}} \end{bmatrix} = \begin{bmatrix} {}^{A}\dot{c} \\ ({}^{A}\dot{R}_{B}{}^{A}R_{B})^{\vee} \\ \dot{\breve{q}} \end{bmatrix} \tag{36}
$$

The corresponding transformation matrix is given by:

$$
{}^{G[B]}T_{B} = \begin{bmatrix} \multicolumn{2}{c}{J^{l}_{G[B],B}} \\ 0_{3\times(3+n)} & 1_{(3+n)\times(3+n)} \end{bmatrix} \tag{37}
$$

where $J^{l}_{G[B],B}$ is the center of mass jacobian, i.e. the matrix such that ${}^{A}\dot{c} = J^{l}_{G[B],B}\nu_{B}$. Note that this is also equal to the first three rows of (34), i.e. $J^{l}_{G[B],B} = J^{l}_{G,B}$. The change of base introduced by ${}^{G[B]}T_{B}$ is the one used in [20].

2) Block diagonalization of the mass matrix: Using the definition of *average 6D velocity* of the multibody system, we can define a *centroidal* generalized joint velocities vector, in which we combine the average 6D velocity and the joint velocities :

$$
\nu^{G} = \begin{bmatrix} \mathrm{v}_{G} \\ \dot{\breve{q}} \end{bmatrix} \tag{38}
$$

Note again that there is no such thing as a G frame: the notation ν^{G} is again an abuse of notation. In particular, it does not make sense to write q^{G} : the change of variables induced by the use of the average 6D velocity is only a change of variables in the velocity space, while for the position space it is usually convenient to continue to use the $q^{G[B]}$ position introduced before.

The change of variable from the usual *floating base* generalized velocities vector ν^{B} has the same structure of the change of floating base introduced in (18):

$$
\nu^{G} = {}^{G}T_{B}\nu^{B}, \quad {}^{G}T_{B} = \begin{bmatrix} {}^{\overline{G}}X_{\overline{B}} & \breve{J}_{G,B} \\ 0_{6\times n} & 1_{n\times n} \end{bmatrix}. \tag{39}
$$

This induces a change of variables also on the generalized robot forces as in (19), given by :

$$ {}_G T^B = {}^G T_B^{-\top} \tag{40} $$

Using the transformation ${}^G T_B$, we can obtain an equation of motions in the usual form:

$$ \dot{q}^{G[B]} = \left({}^A\dot{c}, {}^A\omega^{\wedge}_{A,B}(\nu^G){}^A R_B, \dot{\check{q}}\right) \tag{41a} $$

$$ M_G(q^{G[B]})\dot{\nu}^G + C_G(q^{G[B]}, \nu_G)\nu_G + G_G = \begin{bmatrix} 0_{6\times 1} \\ \tau \end{bmatrix} + \sum_{L\in\mathfrak{L}} J_{L,G}^\top \mathrm{f}_L^x, \tag{41b} $$

where ${}^A\omega_{A,B}(\nu^G)$ is the angular velocity of the link B as a function of the *centroidal* generalized joint velocity ν_G.

The equation of motions describing the derivative of the robot generalized velocities incorporates in the first six rows what is usually called the *centroidal dynamics*, together with the joint dynamics in a single set of equations of motions. The specific features of such a representation of the equations of motions are highlighted in the following theorem, and they have been already exploited in [16].

Theorem 3. *The centroidal mass matrix has a block diagonal structure of :*

$$ M_G(q^{G[B]}) = \begin{bmatrix} m1_{3\times 3} & 0_{3\times 3} & 0_{3\times n} \\ 0_{3\times 3} & L^C(q^{G[B]}) & 0_{3\times n} \\ 0_{3\times 3} & 0_{3\times 3} & \check{M}_G(\check{q}) \end{bmatrix} \tag{42} $$

Furthermore, the centroidal gravity term is independent from the robot state and has the following structure:

$$ G_G = \begin{bmatrix} {}^A g \\ 0_{3\times 1} \\ 0_{n\times 1} \end{bmatrix} \tag{43} $$

Proof. The proof is obtained by direct computation applying the centroidal transformation (39) for transforming the floating base equation of motions (12) as described in (23). ☐

VI. CONCLUSIONS

In this paper we introduce a formalism to model the arbitrary of the choice of the base frame in modeling free-floating robot systems. We demonstrate how some fundamental properties are retained by the change of base frame transformations, even for base frames that are not fixed to a link, such as the one emerging when dealing with the so-called *centroidal dynamics*. Future work will be directed in investigating how the specific structure of the base frame change transformation can be used in free-floating robot control and estimation, as already partially done in [16] and [29].

APPENDIX

Proof of Lemma 2

1) The robot position q^B can be transformed in $q^{B'}$ simply by using the *forward kinematics* (9) of B' considering B to be the base frame:

$$ q^{B'} = ({}^A H_{B'}(q^B), \check{q}) = ({}^A H_B {}^B H_{B'}(\check{q}), \check{q}) $$

where for simplicity we write $({}^A o_B, {}^A R_B)$ as ${}^A H_B$. Similarly the transformation between ν^B and $\nu^{B'}$ can be easily obtained using the jacobian (10) of frame B' considering B as the floating base:

$$ \nu^{B'} = \begin{bmatrix} \mathrm{v}_{B'} \\ \dot{\check{q}} \end{bmatrix} = \begin{bmatrix} J_{B',B}\nu^B \\ \dot{\check{q}} \end{bmatrix} = \begin{bmatrix} & J_{B',B} & \\ 0_{n\times 6} & & 1_{n\times n} \end{bmatrix} \nu^B $$
$$ = \begin{bmatrix} {}^{\overline{B'}}X_{\overline{B}} & \check{J}_{B',B} \\ 0_{n\times 6} & 1_{n\times n} \end{bmatrix} \nu^B. $$

From which we get:

$$ {}^{B'}T_B = \begin{bmatrix} {}^{\overline{B'}}X_{\overline{B}} & \check{J}_{B',B} \\ 0_{n\times 6} & 1_{n\times n} \end{bmatrix}. $$

2) The power P transmitted to the robot by a generalized force ϕ_B on a robot moving with velocity ν^B is independent of the arbitrary choice of the floating base :

$$ P = \phi_B^\top \nu^B = \phi_{B'}^\top \nu^{B'}. $$

Substituting $\nu^B = {}^B T_{B'}\nu^{B'}$ in the second term and transposing, we get:

$$ (\nu^{B'})^\top {}^B T_{B'}^\top \phi_B = (\nu^{B'})^\top \phi_{B'}, \quad {}^B T_{B'}^\top \phi_B = \phi_{B'} $$

From which we get:

$$ {}_{B'}T^B = {}^B T_{B'}^\top = {}^{B'}T_B^{-\top} = \begin{bmatrix} {}_{\overline{B'}}X^{\overline{B}} & 0_{6\times n} \\ -(\check{J}_{B',B})^\top {}_{\overline{B'}}X^{\overline{B}} & 1_{n\times n} \end{bmatrix}. $$

3) The *mixed* velocity v_L of a link L has the same value independently of the assumed floating base, so we can write:

$$ \mathrm{v}_L = J_{L,B}\nu^B = J_{L,B'}\nu^{B'}. $$

Substituting $\nu^B = {}^B T_{B'}\nu^{B'}$ in the second term, we get:

$$ J_{L,B}{}^B T_{B'}\nu^{B'} = J_{L,B'}\nu^{B'}. $$

From which we get:

$$ J_{L,B'} = J_{L,B}{}^B T_{B'}. $$

Proof of Theorem 2

1) For transforming the equations of motions of a rigid body system, we must first get the transformation law for accelerations. To do this we just derive the transformation law for velocities (18):

$$ \dot{\nu}^{B'} = {}^{B'}T_B\dot{\nu}^B + {}^{B'}\dot{T}_B\nu^B. \tag{44} $$

We can transform the dynamics of a free-floating from being expressed with respect to a floating base B to being expressed with respect to a floating base B'. In particular we can modify (12) by applying (18), (44) and by multiplying it on the left by ${}_{B'}T^B$ and we obtain the demonstration for the first part of the theorem.

2) Given from (12) that:

$$ K = \frac{1}{2}{\nu^B}^\top M_B \nu^B. $$

By substituting $\nu^B = {}^B T_{B'} \nu^{B'}$ we got :

$$K = \frac{1}{2} (\nu^{B'})^\top {}^B T_{B'}^\top M_B {}^B T_{B'} \nu^B.$$

From which we get that the kinematic energy is given also by:

$$K = \frac{1}{2} (\nu^{B'})^\top M_{B'} \nu^{B'}, \quad M_{B'} {=} {}_{B'}T^B M_B {}^B T_{B'}.$$

3) The condition of $\dot{M}_B - 2C_B$ being skew-symmetric is equivalent to the condition $\dot{M}_B = C_b + C_B^\top$, so we will demonstrate that $\dot{M}_B - C_B - C_B^\top = \mathbf{0}$ implies $\dot{M}_{B'} - C_{B'} - C_{B'}^\top = 0_{(n+6)\times(n+6)}$. Let's write $\dot{M}_{B'} - C_{B'} - C_{B'}^\top$ using (23a) and (24):

$$\begin{aligned} &\dot{M}_{B'} - C_{B'} - C_{B'}^\top = \\ = &\underbrace{{}_{B'}\dot{T}^B M_B {}^B T_{B'} + {}_{B'}T^B \dot{M}_B {}^B T_{B'} + {}_{B'}T^B M_B {}^B \dot{T}_{B'}}_{\dot{\mathbf{M}}_{B'}} \\ &\underbrace{- {}_{B'}T^B M_B {}^B \dot{T}_{B'} - {}_{B'}T^B C_B {}^B T_{B'}}_{-C_{B'}} \\ &\underbrace{- {}^B \dot{T}_{B'}^\top M_B {}_{B'}T^{B^\top} - {}^B T_{B'}^\top C_B{}^\top {}_{B'}T^{B^\top}}_{-C_{B'}^\top} \end{aligned}$$

Noting that from (19) we have that ${}_{B'}T^{B^\top} = {}^B T_{B'}$, we can write:

$$\begin{aligned} &\dot{M}_{B'} - C_{B'} - C_{B'}^\top = \\ &{}_{B'}T^B \left(\dot{M}_B - C_B - C_B^\top \right) {}^B T_{B'} + \\ &+ {}_{B'}\dot{T}^B M_B {}^B T_{B'} - {}_{B'}\dot{T}^B M_B {}^B T_{B'} + \\ &+ {}_{B'}T^B M_B {}^B \dot{T}_{B'} - {}_{B'}T^B M_B {}^B \dot{T}_{B'} \end{aligned}$$

Using the hypothesis $\dot{M}_B - C_B - C_B^\top = 0$ we can then conclude that $\dot{M}_{B'} - C_{B'} - C_{B'}^\top = 0$.

REFERENCES

[1] J. E. Marsden and T. Ratiu, *Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems*. Springer Science & Business Media, 2013, vol. 17.

[2] R. Featherstone, *Rigid body dynamics algorithms*. Springer, 2008.

[3] D. E. Orin, A. Goswami, and S.-H. Lee, "Centroidal dynamics of a humanoid robot," *Auton. Robots*, vol. 35, no. 2-3, pp. 161–176, Jun. 2013.

[4] P.-B. Wieber, R. Tedrake, and S. Kuindersma, "Modeling and control of legged robots," in *Springer Handbook of Robotics, 2nd Ed*, B. Siciliano and O. Khatib, Eds., 2016.

[5] G. Van Oort, "Analysis, control and design of walking robots," Ph.D. dissertation, Queensland University of Technology, 2011.

[6] A. Jain, *Robot and Multibody dynamics: analysis and algorithms*. Springer Science & Business Media, 2010.

[7] G. Garofalo, B. Henze, J. Englsberger, and C. Ott, "On the inertially decoupled structure of the floating base robot dynamics," *IFAC-PapersOnLine*, vol. 48, no. 1, pp. 322–327, 2015.

[8] S. Traversaro and A. Saccon, "Multibody Dynamics Notation," Technische Universiteit Eindhoven, Tech. Rep., 2016. [Online]. Available: https://pure.tue.nl/ws/files/25753352/Traversaro_en_Saccon_DC_2016.064.pdf

[9] R. Featherstone and D. E. Orin, *Springer Handbook of Robotics*, B. Siciliano and O. Khatib, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, 2008.

[10] T. De Laet, S. Bellens, R. Smits, E. Aertbeliën, H. Bruyninckx, and J. De Schutter, "Geometric relations between rigid bodies (part 1): Semantics for standardization," *Robotics & Automation Magazine, IEEE*, vol. 20, no. 1, pp. 84–93, 2013.

[11] H. Bruyninckx and J. De Schutter, "Symbolic differentiation of the velocity mapping for a serial kinematic chain," *Mechanism and machine theory*, vol. 31, no. 2, pp. 135–148, 1996.

[12] R. M. Murray, Z. Li, S. S. Sastry, and S. S. Sastry, *A mathematical introduction to robotic manipulation*. CRC press, 1994.

[13] A. Jain, *Robot and Multibody Dynamics: Analysis and Algorithms*. Springer, 2010.

[14] B. Siciliano, L. Sciavicco, L. Villani, and G. Oriolo, *Robotics: modelling, planning and control*. Springer Science & Business Media, 2010.

[15] S. Chiaverini, G. Oriolo, and I. D. Walker, "Kinematically redundant manipulators," in *Springer handbook of robotics*. Springer, 2008, pp. 245–268.

[16] G. Nava, F. Romano, F. Nori, and D. Pucci, "Stability analysis and design of momentum-based controllers for humanoid robots," *arXiv preprint arXiv:1603.04178*, 2016.

[17] G. Garofalo, C. Ott, and A. Albu-Schaffer, "On the closed form computation of the dynamic matrices and their differentiations," in *Intelligent Robots and Systems (IROS), 2013 IEEE/RSJ International Conference on*. IEEE, 2013, pp. 2364–2359.

[18] P. Wensing, I. Palmer, LutherR., and D. Orin, "Efficient recursive dynamics algorithms for operational-space control with application to legged locomotion," *Autonomous Robots*, vol. 38, no. 4, pp. 363–381, 2015. [Online]. Available: http://dx.doi.org/10.1007/s10514-015-9420-9

[19] P. M. Wensing and D. E. Orin, "Improved computation of the humanoid centroidal dynamics and application for whole-body control," *International Journal of Humanoid Robotics*, vol. 13, no. 01, p. 1550039, 2016. [Online]. Available: http://www.worldscientific.com/doi/abs/10.1142/S0219843615500395

[20] C. Ott, M. A. Roa, and G. Hirzinger, "Posture and balance control for biped robots based on contact force optimization," in *Humanoid Robots (Humanoids), 2011 11th IEEE-RAS International Conference on*. IEEE, 2011, pp. 26–33.

[21] D. E. Orin and A. Goswami, "Centroidal momentum matrix of a humanoid robot: Structure and properties," in *2008 IEEE/RSJ International Conference on Intelligent Robots and Systems*. IEEE, 2008, pp. 653–659.

[22] H. Dai, A. Valenzuela, and R. Tedrake, "Whole-body motion planning with centroidal dynamics and full kinematics," in *2014 IEEE-RAS International Conference on Humanoid Robots*. IEEE, 2014, pp. 295–302.

[23] A. Herzog, N. Rotella, S. Mason, F. Grimminger, S. Schaal, and L. Righetti, "Momentum control with hierarchical inverse dynamics on a torque-controlled humanoid," *Autonomous Robots*, vol. 40, no. 3, pp. 473–491, 2016.

[24] S.-H. Lee and A. Goswami, "A momentum-based balance controller for humanoid robots on non-level and non-stationary ground," *Autonomous Robots*, vol. 33, no. 4, pp. 399–414, 2012.

[25] T. Koolen, S. Bertrand, G. Thomas, T. De Boer, T. Wu, J. Smith, J. Englsberger, and J. Pratt, "Design of a momentum-based control framework and application to the humanoid robot atlas," *International Journal of Humanoid Robotics*, vol. 13, no. 01, p. 1650007, 2016.

[26] J. Jellinek and D. Li, "Separation of the energy of overall rotation in any n-body system," *Physical review letters*, vol. 62, no. 3, p. 241, 1989.

[27] H. Essén, "Average angular velocity," *European journal of physics*, vol. 14, no. 5, p. 201, 1993.

[28] J. E. Marsden and J. Scheurle, "The reduced euler-lagrange equations," *Fields Institute Comm*, vol. 1, pp. 139–164, 1993.

[29] S. Traversaro, A. Del Prete, S. Ivaldi, and F. Nori, "Inertial parameters identification and joint torques estimation with proximal force/torque sensing," in *2015 IEEE International Conference on Robotics and Automation (ICRA)*. IEEE, 2015, pp. 2105–2110.