A Simple Balanced Search Tree with No Balance Criterion

Abstract

This paper shows that we can maintain a balanced binary search tree by using a simple and intuitive method that does not use any balance criterion. Ultimately, we aimed for maximum simplicity, and we only need to add a simple rebuilding algorithm and minimal extra code to a naive binary search tree. Our method will be suitable as a simple and short solution when amortized logarithmic costs are enough.

1 Introduction

The balanced binary search tree (or balanced BST) is one of the most fundamental data structures. Though there are many ways to implement the dictionary, we use the balanced BST when we need logarithmic costs. Logarithmic cost is the key feature of balanced BSTs, and it distinguishes a balanced BST from a naive BST, where the former uses a tree balancing method to guarantee logarithmic costs. Common examples of a balanced BST include AVL trees [1], binary B-trees [4] or its simplified variant [2] (also known as the AA-tree), weight-balanced trees [6] or its variant [7], and general balanced trees [3]. Note that the splay tree [8] is not a balanced BST since its height can become linear.

However, it is a pity that such a fundamental data structure is infamous for being cumbersome to implement. To cite Munro, Papadakis, and Sedgewick [5], balanced BSTs use a notorious balancing algorithm that needs to consider numerous cases, and hence, implementations are said to be too complicated for average programmers. Not to mention that balanced BSTs are often neglected, or to cite Andersson [2], they are often replaced by poor methods instead. Andersson [3] also said in another paper that many of the commonly used balanced BSTs use a complicated balance criterion to restrict trees and detect imbalance. Also, note that especially nowadays, computer science is needed and used by so many people from all kinds of fields. In many cases, people may not need cutting-edge performance but rather just want to quickly and simply implement their idea. These strongly emphasize the importance of a simple and explicit implementation.

However, we go further than trying to get a simpler balance criterion. We can maintain a balanced BST without using any balance criterion so that implementing the tree is much easier. The ultimate aim is a tree balancing method

with maximum simplicity, and we show this leads to a highly simplified concept and implementation. This is obtained from the following observations.

- The concept of "detecting imbalance and then appropriately rebalancing" can be replaced by "scheduled rebuilding."

- Scheduling rebuilds can be easily done by bookmarking $O(\log n)$ bits on each node and adding simple minimal code to common implementations of tree procedures.

- We can maintain a tree using the exact same balancing method for both inserts and deletes.

- Using tree rebuilds instead of tree rotations, we can offer a more abstract and simple balancing method. Also, rebuilds can be easily implemented only using a basic tree traversal and a simple recursive algorithm.

In this paper, we present a method that offers all the above by using an intuitive method. In Section 3, we explain our method and see how we can implement our method by only adding minimal code and a simple partial-rebuilding algorithm to a naive BST. Next, in Section 4, we show that our tree has a logarithmic height and that an insert/delete operation has an amortized logarithmic cost.

2 Notations and Notes

In this paper, we use the following notations.

For a node *node*,

- *node.key* is the key stored at *node*.

- *node.left* is the left child node of *node*.

- *node.right* is the right child node of *node*.

- *node.timer* is the *timer*[1] stored at *node*.

Note that in the following, when we use the word "tree," we refer to the entire tree, not just a part of it.

3 The Balancing Method

Our tree occasionally does a *partial-rebuilding*, which balances a subtree into a perfectly balanced[2] one. In most cases, a subtree is rebuilt when it does not satisfy the balance criterion. However, in our method, we instead schedule rebuilds in advance.

As a start, we briefly explain our method.

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[1] Definition of a *timer* is explained in Section 3.
[2] A perfectly balanced subtree is a subtree whose left and right subtree's size differ by one at most.

*Every subtree has a **timer**, and we decrease it by 1 each time we insert or delete a node in the subtree. If a subtree's **timer** reaches 0, we rebuild it and reset its **timer** to $\lfloor kn \rfloor$ (or 1 if $\lfloor kn \rfloor = 0$), where $0 < k < 1$ is a constant and n is the subtree's size.*

We will store a subtree's *timer* at its root node. Obviously, the *timer* stores the remaining number of insertions/deletions until the subtree's next rebuild. Additionally, for newly inserted nodes, we set the *timer* to 1. Now, using these *timer* integers, inserts and deletes are done in the following way.

Insert
(1) Insert a leaf node.

(2) Go back up the trail from the leaf node to the root. For each node t, do UPDATE(t).

Delete
(1) Delete a leaf node.[3]

(2) Go back up the trail from the leaf node to the root. For each node t, do UPDATE(t).

UPDATE(t)
(1) Decrease $t.timer$ by 1.

(2) If $t.timer$ is now 0, rebuild the subtree rooted by t.

Rebuild
(1) GETCOPY: Using an inorder traversal, copy all nodes of the subtree to an array.

(2) BUILDTREE: Make a perfectly balanced subtree using this array in a divide-and-conquer sense. We also reset each subtree's *timer*.

Note that for rebuilds, we used a common method that is simple and easy to implement. It only uses a basic tree traversal and a simple recursive algorithm.

The following Fig. 1 and 2 are each an example of an insert operation and a delete operation on a tree. In both figures, the *timer* is abbreviated to t, and the black node denotes the node we want to do UPDATE.

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[3]If the node we want to delete is not a leaf node, we use the common method of exchanging the original problem to a problem of deleting a leaf node.
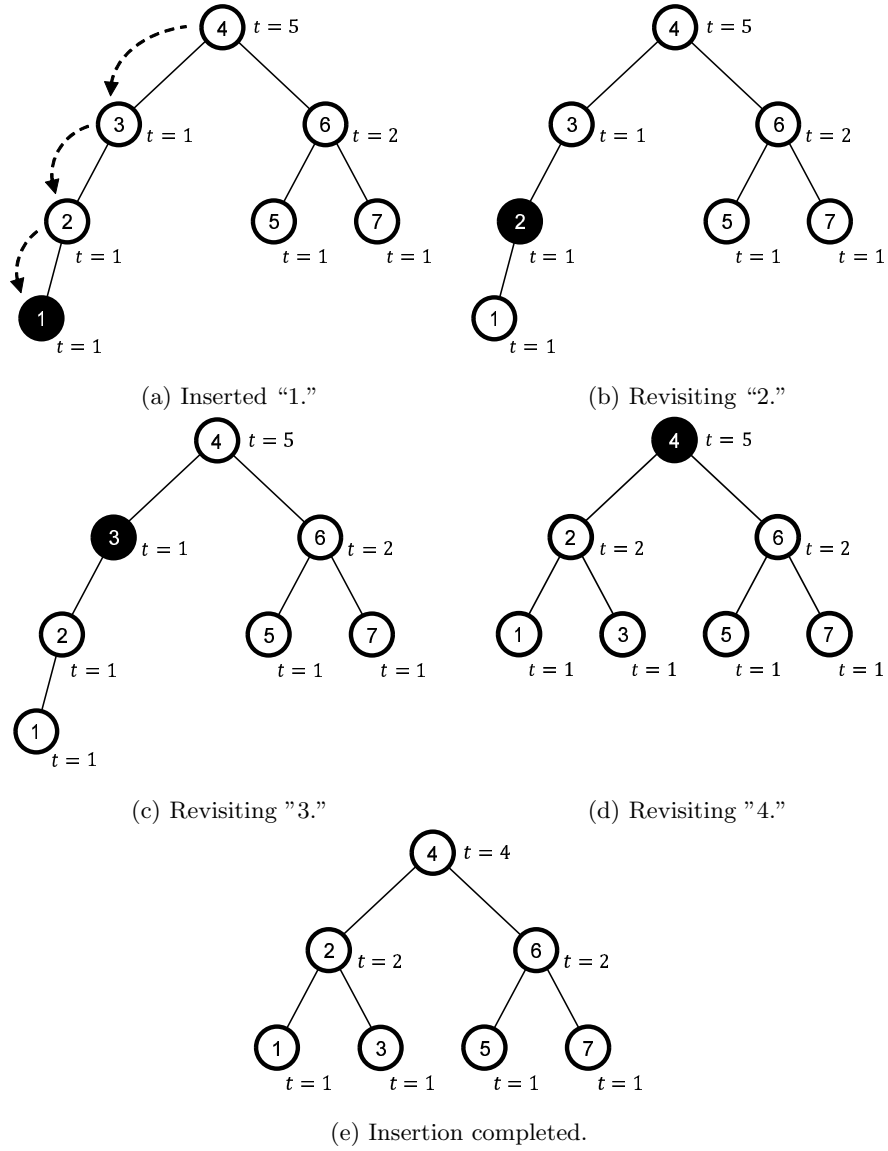
(a) Inserted "1."

(b) Revisiting "2."

(c) Revisiting "3."

(d) Revisiting "4."

(e) Insertion completed.

Figure 1: Example of inserting "1" to a tree.

(a) Deleting "1."

(b) Revisiting "2."

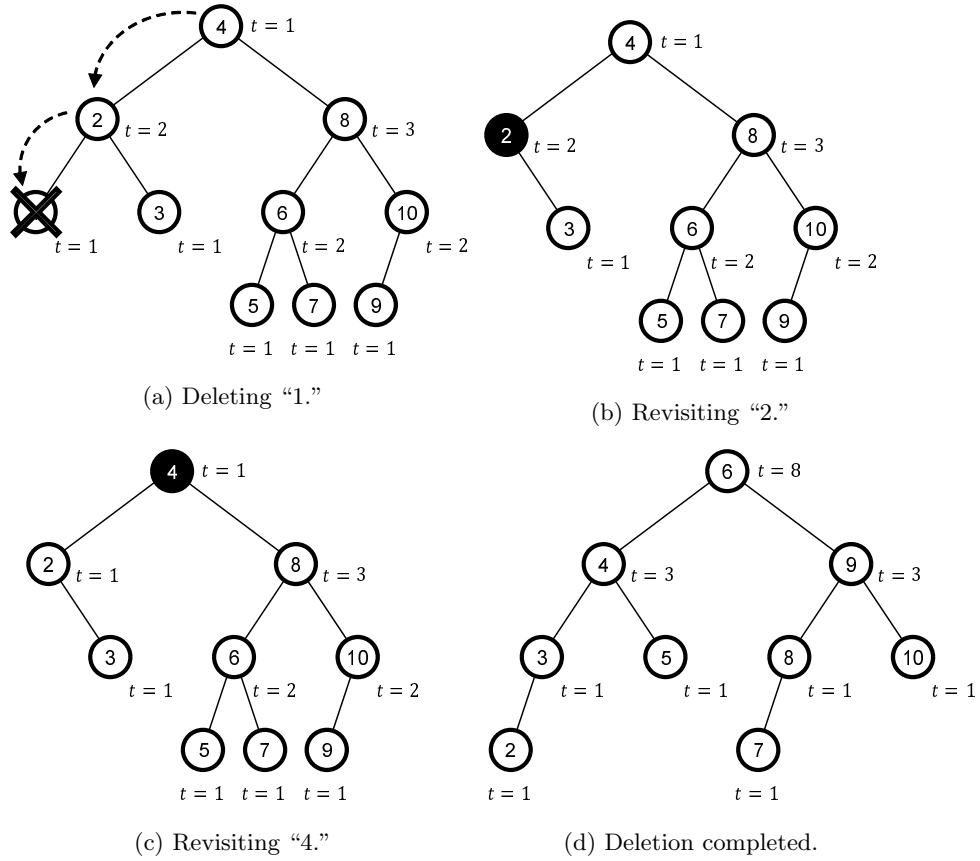(c) Revisiting "4."

(d) Deletion completed.

Figure 2: Example of deleting "1" in a tree.

Implementation

Starting from a common implementation of a naive BST, we can implement our method by adding a rebuilding algorithm and a simple, minimal code. Following is an example of an implementation. We only add UPDATE at the end of INSERT and DELETE.

INSERT(t, key, res)
  if $t = null$ **then**
    $t \leftarrow$ new node
    $res \leftarrow$ true
  else if $key < t.key$ **then** INSERT($t.left, key, res$)
  else if $key > t.key$ **then** INSERT($t.right, key, res$)
  else $res \leftarrow$ false
  UPDATE(t, res)

DELETE(t, key, res)
  if $t = null$ **then** $res \leftarrow$ false
  else if $key < t.key$ **then** DELETE($t.left, key, res$)
  else if $key > t.key$ **then** DELETE($t.right, key, res$)
  else
    if $t.left \neq null$ and $t.right \neq null$ **then**
      $t.key \leftarrow$ GETMAX($t.left$)$.key$
      DELETE($t.left, t.key, res$)
    else
      if $t.left \neq null$ **then** $t \leftarrow t.left$
      else $t \leftarrow t.right$
      $res \leftarrow$ true
      return
  UPDATE(t, res)

GETMAX(t)
  while $t.right \neq null$ **do** $t \leftarrow t.right$
  return t

UPDATE(t, res)
  if $res =$ true **then**
    $t.timer \leftarrow t.timer - 1$
    if $t.timer = 0$ **then**
      $arr \leftarrow$ empty array
      GETCOPY(t, arr)
      $t \leftarrow$ BUILDTREE($arr, 0, arr.size - 1$)

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GETCOPY(t, arr)
    if t ≠ null then
        GETCOPY(t.left, arr)
        arr.push(t)
        GETCOPY(t.right, arr)


BUILDTREE(arr, s, f)
    if s > f then return null
    m ← ⌊(s + f)/2⌋
    arr[m].left ← BUILDTREE(arr, s, m − 1)
    arr[m].right ← BUILDTREE(arr, m + 1, f)
    arr[m].timer ← max (1, ⌊k · (f − s + 1)⌋)
    return arr[m]
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4 Proof of Logarithmic Height and Amortized Logarithmic Costs

In this section, we will show that the tree has logarithmic height, which means that a search operation has a logarithmic worst-case time complexity. Also, we will show that an insert/delete operation has an amortized logarithmic cost.

4.1 Logarithmic height

We first start with the following definition.

Definition 1. *Define $T.size$ as the size of T.*

Many trees that use partial-rebuilding maintain logarithmic height by maintaining a constant lower bound (or upper bound) on the quotient between a subtree's weight and its own left/right subtree's weight. However, this kind of analysis does not work in our tree because our tree is too flexible in shape. It is not hard to see that $\frac{T.left.size+1}{T.size+1}$ or $\frac{T.right.size+1}{T.size+1}$ is larger than $\frac{1-2k}{2-2k}$ when $0 < k < 0.5$, but when $k \geq 0.5$, such quotient is not bounded by a constant in $(0, 1)$. Therefore, we will instead show that the quotient between "upper bounds on sizes" is upper bounded by a constant.

First, lets say that when we create a new subtree (which happens when we insert a new node), the *timer* gets initially "set", and when we rebuild a subtree, the *timer* gets "reset" for all subtrees of it. Based on a subtree's last *timer* (re)set, if we refer to an insertion or deletion of a node as an "update," we can divide a subtree into two cases.

Lemma 1. *All subtrees are always in one of the following two cases.*

1. *No updates happened to it since its last timer (re)set.*

2. At least one update happened to it since its last timer (re)set.

Based on this, we will often divide subtrees into two cases. The following definition will come in handy when we do so.

Definition 2. *Define $T.n_0$ as the size of T after its last timer (re)set.*

Also, note the following.

Lemma 2. *If a subtree of size less than $\frac{2}{k}$ went through no updates since its last timer (re)set, then its timer gets reset after a single update in it, assuming it is still non-empty.*

Lemma 3. *For a perfectly balanced subtree of size n, the size of its subtree is no more than $\frac{n}{2}$, if exists.*

Using these, we can see the following.

Theorem 4. *Suppose we can do any combination of x or less updates to a subtree T whose timer was (re)set. If $T.size \leq n$ in all cases, then T's subtree's size is no more than $\left(\frac{1+2k}{2+2k}\right) n$ in all cases.*

Proof. Suppose not. That is, T's size cannot exceed n in any case, but we can have a case where T's subtree's size is bigger than $\left(\frac{1+2k}{2+2k}\right) n$. Suppose that this happened after doing u updates since T's last *timer* (re)set.

Then, $0 \leq u < \max(1, \lfloor kT.n_0 \rfloor)$. Also, because of Lemma 3, T's subtree's size should be no more than $\frac{T.n_0}{2} + u$, so $\frac{T.n_0}{2} + u > \left(\frac{1+2k}{2+2k}\right) n$ must also be true. Finally, note that $n \geq T.n_0 + u$ because of our assumption.

However, note the following.

(i) If $T.n_0 < \frac{2}{k}$, then $u = 0$, and we can see that

$$\frac{T.n_0}{2} + u \leq \left(\frac{1+2k}{2+2k}\right) T.n_0 \leq \left(\frac{1+2k}{2+2k}\right) n.$$

(ii) If $T.n_0 \geq \frac{2}{k}$, then $u \leq \lfloor kT.n_0 \rfloor - 1 \leq kT.n_0$.
However, we said that

$$\frac{T.n_0}{2} + u > \left(\frac{1+2k}{2+2k}\right) n \geq \left(\frac{1+2k}{2+2k}\right) (T.n_0 + u),$$

and this means $u > kT.n_0$, which contradicts.

Therefore, we conclude that it is impossible. $\qquad\square$

Also, it is not hard to see that $\left(\frac{1+2k}{2+2k}\right)$ is the smallest constant that we can guarantee in Theorem 4.

Now, we finally have the following.

Theorem 5. *The height of a non-empty tree with size n is no more than* $\left\lfloor \log_{\frac{2+2k}{1+2k}} \frac{1+2k}{2-2k} n \right\rfloor + 1$. *That is, the height of the tree is $O(\log n)$.*

Proof. We divide the tree into two cases using Lemma 1.

A tree in the first case of Lemma 1 is perfectly balanced. That is, its height is no more than $\lfloor \log_2 n \rfloor$.

Now, we check for a non-empty tree T in the second case of Lemma 1. Note that $T.n_0 \geq \frac{2}{k}$ by Lemma 2. Also, note that we are considering trees that went through no more than $\lfloor kT.n_0 \rfloor - 1$ any updates since its last *timer* reset.

Then, for both T's left or right subtree, its size cannot exceed $\frac{T.n_0}{2} + \lfloor kT.n_0 \rfloor - 1$ for any combination of no more than $\lfloor kT.n_0 \rfloor - 1$ updates. Also, note that $n \geq T.n_0 - \lfloor kT.n_0 \rfloor + 1$. It is not hard to see that the quotient between the two is less than $\frac{1+2k}{2-2k}$. Finally, using Theorem 4, we can easily see that the size of a subtree rooted by a node at depth d is always less than

$$\frac{(1+2k)^d}{(2+2k)^{d-1}(2-2k)} n.$$

Therefore, we conclude that the height is no more than $\left\lfloor \log_{\frac{2+2k}{1+2k}} \frac{1+2k}{2-2k} n \right\rfloor + 1$. $\qquad\square$

Note that a different k value leads to a different upper bound on height. In our tree, we can say that a smaller k leads to a smaller height but more frequent rebuilds.

4.2 The amortized cost of an insert/delete operation

Theorem 6. *Assume that rebuilding a subtree T costs no more than $T.size$. If a subtree's timer reached 0, then the quotient between its rebuild cost and u, the number of updates it went through since its last timer (re)set, is less than $\left(\frac{2}{k}+1\right)$. That is, the rebuild cost is less than $\left(\frac{2}{k}+1\right) u$.*

Proof. Call $n = T.size$ and $n_0 = T.n_0$. Then, $n \leq n_0 + u$, and $u = \max\left(1, \lfloor kn_0 \rfloor\right)$. Now, we check two cases.

(i) If $n_0 < \frac{1}{k}$, then $u = 1$ and $n < \frac{1}{k} + 1$.

(ii) If $n_0 \geq \frac{1}{k}$, then $u = \lfloor kn_0 \rfloor$ and $n \leq n_0 + \lfloor kn_0 \rfloor$.
   Then, for the quotient between the rebuild cost and u,

$$\frac{n}{u} \leq \frac{n_0 + \lfloor kn_0 \rfloor}{\lfloor kn_0 \rfloor} = \frac{n_0}{\lfloor kn_0 \rfloor} + 1 < \frac{2}{k} + 1.$$

Therefore, in both cases, the quotient is less than $\left(\frac{2}{k}+1\right)$. $\qquad\square$

This means that if we add $\left(\frac{2}{k} + 1\right)$ extra credit at all ancestor nodes of the inserted/deleted node after each insertion/deletion of a node, a subtree's root will always have enough credit to pay the subtree's rebuild cost by the time its *timer* reached 0. Since our tree's height is $O(\log n)$, the number of ancestor nodes of the inserted/deleted node is also $O(\log n)$, and therefore, we conclude with the following.

Theorem 7. *Our tree can be maintained with $O(\log n)$ amortized cost per insert/delete operation, where n is the size of the tree the operation occurred.*

5 Conclusion

It shows that we can maintain a balanced BST with a simple and intuitive method that does not require much detail or consideration of various cases. Starting from a naive BST, we only need to add a simple rebuilding algorithm and minimal code. For novice or when amortized logarithmic costs are enough, our method will be suitable as a simple solution.

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