Singular values of covariance matrices under localization

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We have two ensembles representing the same system. The ensemble X is a coarse representation of the state, but has many ensemble members. The ensemble Z is a fine representation of the sates, but has few ensemble members.

$$\dim X = N_x, N_{ex}$$

$$\dim Z = N_z, N_{ez}$$

In order to find the eigenvalues and eigenvectors of the sample correlation of X, we take the singular value decomposition of $\tilde{X} = (N_{ex} - 1)^{-1/2}(X - \bar{X})/\operatorname{sd}(X)$, where $\operatorname{sd}(X)$ is the standard deviation of each element of X.

$$U_x S_x V_x^T = \tilde{X}$$

This means that,

$$U_x S_z^2 U_x^T = \hat{C}_x$$

where $\hat{C}_x$ is the sample correlation of X.

We want to then use U_x, or its leading columns, to estimate the leading U_z. To do this, we interpolate U_x to the z space and then use QR factorization to ensure the interpolated u_x's are orthonormal:

$$U_x = \operatorname{interp}(U_x)$$

$$U_{xi}, R = \operatorname{QR}(U_{xi}).$$

We then must choose how many of the columns of U_{xi} should be used. The best way to make this choice is unclear. A few ways that I am considering are:

1. Keep U_{xi} based on the cumulative sum of their corresponding eigenvalues.

2. Keep U_{xi} based on the rate of change of their corresponding eigenvalues.

3. Keep U_{xi} based on some measure of the length scale produced by the low rank approximation of $\hat{C}_x$ and the residual.

4. Keep U_{xi} based on how orthogonal the interpolated vector is in z space before orthogonalization.

It should be noted that keeping all columns can be detrimental to the assimilation process. If some of the columns of U_{xi} are representing small scale structures, then in the following steps they will still be used to represent some of these small scale structures.

After choosing which columns of U_{xi} to keep, we must determine what eigenvalues they should have. We do this by taking $\lambda = u_{xi}^T \hat{C}_z u_{xi}$ as the eigenvalue of uxi for our approximation of C_z. Alternatively, we can take $\lambda = (\tilde{Z}^T u_{xi})^T (\tilde{Z}^T u_{xi})$ where $\tilde{Z}$ is defined similarly as $\tilde{X}$. This will give us the leading eigenvectors and an approximation of their eigenvalues of C_Z.

These eigenvectors and eigenvalues represent the large scale structure of the problem. To then find the small scales of the problem, we take

$$C_z^{\perp} = C_z^{\|} - \hat{C}_z$$

where

$$C_z^{\|} = U_{xi} \Lambda_{xi} U_{xi}^T$$

and Λ_{xi} is the diagonal matrix with the λs described above on the diagonal. We can then localize $C_z^{\perp}$ to get at the small scales that are represented in Z.

We must choose how to localize $C_z^{\perp}$. One reasonable expectation is that the scales in $C_z^{\perp}$ will be shorter or similar to N_z/N_x. This choice will also be affected by the choice of how many columns of U_{xi} to keep. Once the localization matrix L is chosen, we can then generate our localized correlation matrix as:

$$C_z^{loc} = C_z^{\|} + L \circ C_Z^{\perp}$$

and the corresponding covariance matrix as:

$$P_z^{loc} = D_z C_z^{loc} D_z.$$

where D_z is the diagonal matrix with sample standard deviations of Z on the diagonal.

We can then find the leading eigenvectors and eigenvalues of C_z^{loc}:

$$Q \Lambda Q^T = C_z^{loc}$$

and use them to transform the z variable. First, we must calculate a whitening transformation

$$T_w = \Lambda^{-1/2} Q^T$$

and its right inverse

$$T_w^i = Q \Lambda^{1/2}$$

and the singular value decomposition

$$U \Sigma V^T = R^{-1/2} H T_w^i.$$

We can then define transformations for both z and y

$$T_z = V^T T_w$$

$$T_y = U^T R^{-1/2}$$

We then know $z^* = T_z z$ and $y^* = T_y y$ have identity covariance matrices and

$$y^* = \Sigma z^* + \epsilon^*$$

$$U^T R^{-1/2} y = \Sigma V^T \Lambda^{-1/2} Q^T z + \epsilon^*$$

$$y = R^{1/2} U \Sigma V^T \Lambda^{-1/2} Q^T z + \epsilon$$

$$y = R^{1/2} R^{-1/2} H Q \Lambda^{1/2} \Lambda^{-1/2} Q^T z + \epsilon$$

$$y = Hz + \epsilon$$

The question is: can I do this better? Specifically:

1. Can I calculate the eigenvectors and values of C^{loc} without generating $C^{\|}$ or $C^{\perp}$ explicitly?

2. Can I calculate the eigenvectors and values of P from those of C without generating C?

3. Can I calculate the singular vectors and values of $R^{-1/2} H T_w^i$ from the singular value decomposition of R, H, and T_w^i?