Monday 2021-10-18 ①

Pset #3 Due Wednesday

Last time

- $\tilde{F}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i \omega t}\, dt$ forward $t \to \omega$

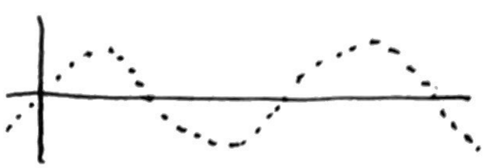
  $f(t) = \int_{-\infty}^{\infty} \tilde{F}(\omega)\, \underbrace{e^{2\pi i \omega t}}\, d\omega$ inverse $\omega \to t$

- Eulers Formula

  $e^{2\pi i \theta} = \cos(2\pi\theta) + i\sin(2\pi\theta)$

- FT → function of freq.; complex number
  - Amplitude (power @ given freq.)
  - phase (L/R shift)

- REAL DATA is Discrete



a vector of numbers.

$\vec{y} = f(\vec{x})$

- CAN FIT w/ LEAST SQUARES

$C = (G' * G) \backslash (f * G)'$;

f is data

G is matrix of models g

$G = [g_1\ g_2\ \dots\ g_m]$

m functions

n data points

$g = e^{i\omega_0 m n\, dt}$; $\omega_0 = \frac{2\pi}{N\, dt}$

$m = 1, 2, \dots$

$n = [0 : N-1]$

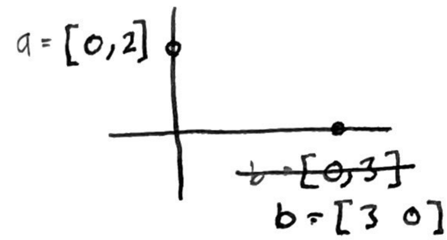
- ORTHOGONALITY

means $\vec{a} \perp \vec{b}$

we can use Dot Product (inner product) $= \vec{0}$ to show orthogonality.

i.e. $\vec{a} * \vec{b}^T = \vec{0}$

$$[0\ 2]\begin{bmatrix}3\\0\end{bmatrix} = (0\cdot 3)+(2\cdot 0) = 0$$



- Orthonormal special case

$|a| = |b| = 1$

- $\sin\left(\frac{2\pi n}{N}\right)$ & $\cos\left(\frac{2\pi n}{N}\right)$ are Orthogonal.

means all non diagonals elements are 0

we worked through algebra to get to

$$\tilde{C}_m = \frac{1}{N}\sum_{n=1}^{N} f(t)\, e^{\left(\frac{-i\omega m n}{N}\right)} \leftarrow \text{DFT}$$

- Today.

DFT Properties

(1) $\tilde{F}(ax+by) = a\tilde{F}(x) + b\tilde{F}(y)$

LINEARITY!

(2) $\tilde{X}_0 = \sum_{n=0}^{N-1} X_n$ ... $C_0 = \frac{1}{N}\sum_{n=0}^{N-1} f(t)\, \cancel{e^0}^{\,1}$

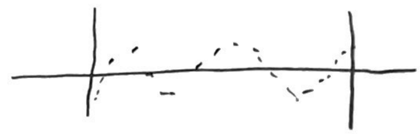
- 0 frequency (DC) measures the mean

fixed component.

3 Linear phase shift

if X_{n-k} is substituted for X_n, $\frac{2\pi k m}{N}$ radians are subtracted from X_m

4 Values beyond the edge of vector are assumed to be 0



- Carollary. Edge effects!
  windowing is crucial.

5 Differentiation

$$\frac{d\tilde{F}(x)}{dx} = \frac{d}{dx}\tilde{F}(x) = \frac{d}{dx}\int_{-\infty}^{\infty} f(x)\, e^{-i\omega t}\, d\omega = \int_{-\infty}^{\infty} f(x)\, d\omega \int_{-\infty}^{\infty} \frac{d}{dx} e^{-i\omega t}\, d\omega$$

$$\frac{d\tilde{F}(x)}{dx} = \int_{-\infty}^{\infty} f(x)\, d\omega \cdot -i\omega\, e^{-i\omega t} = -i\omega \underbrace{\int_{-\infty}^{\infty} f(x)\, e^{-i\omega t}\, d\omega}_{\tilde{F}(x)}$$

$$\boxed{\frac{d\tilde{F}(x)}{dx} = -i\omega\, \tilde{F}(x)}$$

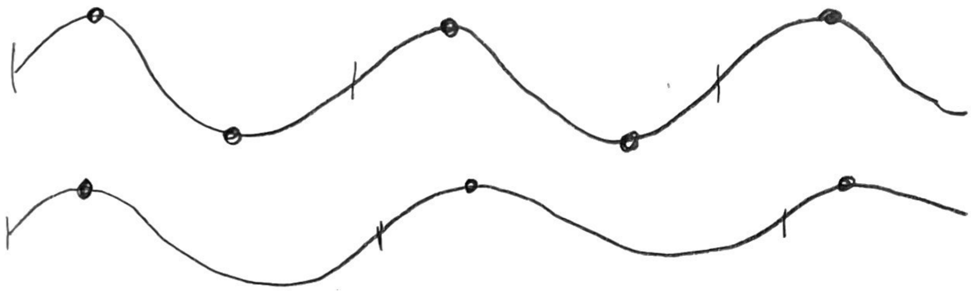
6 Parsevals Theorem

$$\sum_{n=0}^{N-1} X_n^2 = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{X}_m^2$$

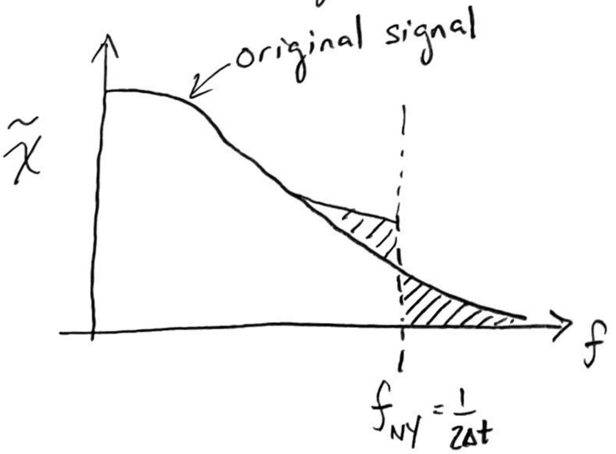
power is preserved.

Sampling Theorem

If a continuous signal is sampled at greater than twice its highest freq. component, then it is possible to recover the signal from the samples



If a signal is under sampled aliasing will occur.
maps higher f to lower f



Nyquist freq. $= \frac{1}{2\Delta t}$ ← samp. interval

Therefore the range of independent freq

$$\vec{f} = \left[\, 0 \quad \frac{1}{N\Delta t} \quad - - - - \quad \frac{1}{2\Delta t} \,\right]$$

DC. ↑ duration of signal ↑ $f_{NY} = \frac{N}{2(N\Delta t)}$

"$\frac{1}{2}$ of freq)

$$\vec{f} = \frac{[0 : \frac{N}{2}]}{N\Delta t}$$

Implication: all actual data is band limited at discrete frequencies

Other Properties of DFT

1. DFT has N pos. freq. comps. $(M = N)$

$$\tilde{X}_{m+N} = \sum_{n=0}^{N-1} x_n e^{-2\pi i (m+N) n/N}$$

$$= \sum_{n=0}^{N-1} x_n e^{-2\pi i m n/N} e^{-2\pi i N n/N} = \sum_{n=0}^{N-1} x_n e^{-2\pi i m n/N} e^{-2\pi i n}$$

1 for any integer, n

So freq. beyond N are redundant.

2. $$\tilde{X}_{m-N} = \sum_{n=0}^{N-1} x_n e^{-2\pi i (m-N) n/N}$$

$$= \underbrace{\sum_{n=0}^{N-1} x_n e^{-(-2\pi i m n/N)}}_{\text{conjugate of } (X_m)} \underbrace{e^{-2\pi i n N/N}}_{=1}$$

$\overline{X}_m$

so $X_{m-N} = \overline{X}_m$

therefore we take $M = N$