1. Observational Projection Operator (clean version)

Let be a 4-dimensional real inner-product space with inner product .

1.1 Observer states and hyperplanes
1.2

An observer state is a unit vector

$$N \in V,\quad \|n\| = 1.$$

It defines a 3-dimensional hyperplane

$$H_n := \{ x \in V : \langle n, x \rangle = 0 \},$$

1.3 Operator definition
1.4

The observational projection associated with is the linear operator

$$P_n : V \to H_n,\qquad$$

$$P_n(x) = x - \langle n,x \rangle\, n.$$

Equivalently, every 4D state is decomposed as

$$X = x_\perp + x_\parallel,\quad$$

$$X_\perp := P_n(x) \in H_n,\quad$$

$$X_\parallel := \langle n,x\rangle\, n \in \text{span}\{n\},$$

1.3 Basic properties

1. Linearity

$$P_n(\alpha x + \beta y) = \alpha P_n(x) + \beta P_n(y)\quad \forall\,\alpha,\beta \in \mathbb{R},\ x,y \in V.$$

2. Idempotence

$$P_n(P_n(x)) = P_n(x)\quad \forall x \in V.$$

3. Self-adjointness

$$\langle P_n(x), y \rangle = \langle x, P_n(y) \rangle\quad \forall x,y \in V.$$

4. Kernel and image

$$\ker P_n = \text{span}\{n\}, \qquad \operatorname{Im} P_n = H_n.$$

5. Norm contraction

$$\|P_n(x)\| \le \|x\|\quad \forall x \in V,$$

1.5 Equivalence classes (observer indistinguishability)
1.6

Define an equivalence relation

$$X \sim_n x' \iff P_n(x) = P_n(x').$$

Because

$$P_n(x + \alpha n) = x + \alpha n - \langle n, x + \alpha n \rangle n$$
$$= x - \langle n,x\rangle n = P_n(x),$$

$$X \sim_n x' \iff \exists\,\alpha \in \mathbb{R} \text{ such that } x' = x + \alpha n.$$

So each equivalence class is a 4D line parallel to :

$$[x]_n = \{ x + \alpha n : \alpha \in \mathbb{R} \}.$$

Interpretation: all points in share the same observable 3D shadow for this observer; they differ only along the hidden axis .

1.5 Observer-relative distance on equivalence classes

Define a distance between classes and by

$$D_n([x]_n,[y]_n) := \big\| P_n(x) - P_n(y) \big\|.$$

This is well-defined (independent of the particular representatives) because adding any to or is annihilated by .

So the quotient space is isometric to the 3D subspace via

$$[x]_n \mapsto P_n(x).$$

You can literally say: “the observer’s reality is the quotient , canonically identified with .”

1.7 Coordinate / matrix form (for implementation)
1.8

Pick an orthonormal basis with .

Write , so .

Then

$$P_n(x) = (x_1, x_2, x_3, 0)^\top.$$

In this basis, the matrix of is

[P_n] =

\begin{pmatrix}

1 & 0 & 0 & 0 \\

0 & 1 & 0 & 0 \\

0 & 0 & 1 & 0 \\

0 & 0 & 0 & 0

\end{pmatrix},