DYNAMICS AND ENTROPY OF $\mathcal{S}$-GRAPH SHIFTS

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12 October 2020

Abstract

S-gap shifts are an important and well-studied class of symbolic dynamical systems, and several generalizations of S-gap shifts have been proposed. This paper introduces a new class of symbolic dynamical systems called $\mathcal{S}$-graph shifts which unites these generalizations with several other classes of symbolic dynamical systems using a graph-theoretic framework. The main result is a formula for the entropy of any $\mathcal{S}$-graph shift, which, by specialization, resolves a problem raised by Matson and Sattler. We also show that every entropy value is obtained by uncountably many $\mathcal{S}$-graph shifts and characterize those shifts that possess certain dynamical properties.

1 Introduction

S-gap shifts are a class of symbolic dynamical shift spaces valued for their practical and theoretical applications. Their structure is especially rich, and analyses of S-gap shifts have drawn from many areas, including algebra [1], topology [2], number theory [3, 5], and ergodic theory [6]. Each S-gap shift, denoted $X(S)$, is uniquely defined by a subset S of the natural numbers. S-gap shifts are especially useful in producing simple concrete examples of shift spaces with a variety of dynamical properties. For example, the shift where S is the set of prime numbers is an explicitly defined non-sofic shift space.

Motivated by coding theory, Dastjerdi and Shaldehi [4] introduced (S, S')-gap shifts, a generalization of S-gap shifts, and studied their dynamical properties. Each (S, S')-gap shift is defined by a pair (S, S') of subsets of the natural numbers. Later, Matson and Sattler [8] further generalized (S, S')-gap shifts, in two different ways, to ordered and unordered $\mathcal{S}$-limited shifts; any $\mathcal{S}$-limited shift is specified by a finite tuple of subsets of the natural numbers. They extended many of the dynamical results for S- and (S, S')-gap shifts in [2] and [4] to $\mathcal{S}$-limited shifts.

Entropy is one of the key invariants in symbolic dynamics. Spandl [12] first computed the entropy of S-gap shifts in 2007, obtaining the following formula.

Theorem 1.1. *If $S \subseteq \mathbb{N}$ is nonempty and $x = \lambda$ is the unique positive solution to*

$$\sum_{s \in S} x^{-(s+1)} = 1,$$

then the entropy of $X(S)$ is $\log \lambda^{-1}$.

Unfortunately, Spandl's proof is incomplete, and efforts to fill the gap in the proof have as yet proved unsuccessful. As a result, the canonical proofs of the S-gap entropy formula are the two alternative ones presented on Vaughn Climenhaga's blog.[1] Matson and Sattler extended one of these proofs to produce an entropy formula for ordered $\mathcal{S}$-limited shifts. They note, however, that their approach is insufficient to prove a formula for unordered $\mathcal{S}$-limited shifts.

[1] `https://vaughnclimenhaga.wordpress.com/2014/09/08/entropy-of-s-gap-shifts/`

Finding and proving such a formula is the motivation for this work. This paper introduces a much larger class of shift spaces, called $\mathcal{S}$-graph shifts, that unites ordered and unordered $\mathcal{S}$-limited shifts, as well as vertex and edge shifts, into one framework. Each $\mathcal{S}$-graph shift is defined by a finite set-equipped graph, formed by assigning a subset of the natural numbers to each vertex. As described in Section 4, we can assign a matrix-valued function $B(x)$, called the generating matrix, to each $\mathcal{S}$-graph shift space. Our first main result is an entropy formula for $\mathcal{S}$-graph shifts.

Theorem 1.2. *Let X be an $\mathcal{S}$-graph shift, $B(x)$ its generating matrix, and $\ghost(x)$ the spectral radius of $B(x)$. The equation $\ghost(x) = 1$ has a unique positive solution $x = \lambda$, and the entropy of X is* $\log \lambda^{-1}$.

In particular, Theorem 1.2 provides an entropy formula for unordered $\mathcal{S}$-limited shifts by specialization. Theorem 4.13 is a power series formulation of Theorem 1.2 that simplifies to Theorem 1.1 when X is an S-gap shift.

Our second main theorem is a strong result on the distribution of entropy among $\mathcal{S}$-graph shifts.

Theorem 1.3. *Let $\lambda > 1$. For each $n \geq 2\lceil \lambda \rceil - 1$, there are uncountably many pairwise non-conjugate $\mathcal{S}$-graph shifts on n vertices that have the specification property and entropy $\log \lambda$.*

Proposition 4.2 and Theorem 4.3 in [5] provide similar results that are restricted to S-gap shifts.

The paper is organized as follows. In Section 2, we describe the relevant background from symbolic dynamics and basic definitions for $\mathcal{S}$-graph shifts. Section 3 characterizes the $\mathcal{S}$-graph shifts that have key dynamical properties, extending the work in [5] and [8]. The proof of Theorem 1.2 comprises Section 4. Section 5 provides several operations on $\mathcal{S}$-graph shifts which are leveraged in Section 6 to prove Theorem 1.3. Finally, Section 7 describes how Theorem 1.2 extends to the more general class of labelled $\mathcal{S}$-graph shifts.

2 Background and definitions

Proofs of the assertions made in this section can be found in the introductory symbolic dynamics text by by Lind and Marcus [7]. Examples of the definitions appear following the definition of S-gap shifts.

Throughout this paper, we let $\mathbb{N}_0$ denote the set of nonnegative integers and $\mathbb{N}$ denote the set of positive integers. We start with a finite set $\mathcal{A}$, called the *alphabet*, whose elements are called *letters*. The *full shift on* $\mathcal{A}$, denoted $\mathcal{A}^{\mathbb{Z}}$, is the set of all bi-infinite strings of elements in $\mathcal{A}$. Equivalently, the full shift is the set of all sequences $(a_n)_{n=-\infty}^{\infty}$ in $\mathcal{A}$; this index notation will be used throughout the paper.

The *shift map* $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ sends the point $x \in \mathcal{A}^{\mathbb{Z}}$ to the point $y \in \mathcal{A}^{\mathbb{Z}}$ that satisfies $y_i = x_{i+1}$ for each $i \in \mathbb{Z}$. We equip the full shift with a metric d, setting $d(x, y) = 2^{-(k+1)}$ when k is the greatest integer such that $x_{-k} \cdots x_k = y_{-k} \cdots y_k$ and $d(x, y) = 0$ if no such k exists. (If $x_0 \neq y_0$, then $d(x, y) = 1$.) With respect to this metric, $\mathcal{A}^{\mathbb{Z}}$ is a compact metric space and σ is continuous. A *subshift* or *shift space over* $\mathcal{A}$ is a subset X of $\mathcal{A}^{\mathbb{Z}}$ that is closed and shift-invariant (that is, $\sigma(X) = X$). A point $x \in \mathcal{A}^{\mathbb{Z}}$ is called *periodic* with period n if $\sigma^n(x) = x$; the point x has *least period* n if furthermore $\sigma^k(x) \neq x$ for each $k \in \{1, \dots, n-1\}$. The number of points in X with period n is denoted $p_n(X)$, and the number of points in X with least period n is denoted $q_n(X)$.

A *block* or *word* over $\mathcal{A}$ is a finite sequence of letters. We denote the concatenation of two words by juxtaposition, so if $x = x_1 \dots x_n$ and $y = x_{n+1} \cdots x_{n+k}$, then $xy = x_1 \cdots x_{n+k}$. The n-fold concatenation of $u \in \mathcal{L}(X)$ is denoted by u^n. We say that a block $\alpha = \alpha_1 \cdots \alpha_k$ *appears* in a point $x \in \mathcal{A}^{\mathbb{Z}}$ if α is a consecutive subsequence of x, that is, $\alpha = x_n \cdots x_{n+k-1}$ for some $n \in \mathbb{Z}$.

Definition 2.1. For each set $\mathcal{F}$ of blocks over $\mathcal{A}$, we define

$$X_{\mathcal{F}} = \{x \in \mathcal{A}^{\mathbb{Z}} : \text{no element of } \mathcal{F} \text{ appears in } x.\}$$

It turns out that the subspaces defined by forbidden words are exactly the subshifts of $\mathcal{A}^{\mathbb{Z}}$: Each set $X_{\mathcal{F}}$ is both closed and shift-invariant, and for each subshift X there is a set of forbidden blocks $\mathcal{F}$ (not necessarily unique) such that $X = X_{\mathcal{F}}$. We say that a subshift X is a *subshift of finite type* (or *SFT*) if there is a finite set $\mathcal{F}$ such that $X = X_{\mathcal{F}}$.

The set of *allowable blocks of length* n, denoted $\mathcal{B}_n(X)$, is the set of blocks of n letters that appear in some word in X. The *language* of X is the set $\mathcal{L}(X) = \bigcup_{n=1}^{\infty} \mathcal{B}_n(X)$ of all allowable words.

Fact 2.2. *Two shift spaces are equal if and only if their languages are.*

The entropy of a shift space is a measure of the size of its language, which is, in some sense, a measure of the complexity of the shift space.

Definition 2.3. The (*topological*) *entropy* of a shift space X is

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log|\mathcal{B}_n(X)|.$$

This limit always exists if X is nonempty, in which case $1 \leq |\mathcal{B}_n(X)| \leq |\mathcal{A}|^n$; consequently $0 \leq h(X) \leq \log|\mathcal{A}|$.

Given two shift spaces X and Y, a *homomorphism* from X to Y is a continuous map $\phi\colon X \to Y$ that commutes with the shift map. A *conjugacy* is a bijective homomorphism with continuous inverse; if there is a conjugacy from X to Y, we call X and Y *conjugate* shift spaces and write $X \cong Y$. If X and Y are conjugate, then $h(X) = h(Y)$.

Definition 2.4. Let X be a shift space, $m, n \in \mathbb{N}_0$, and $\Phi\colon \mathcal{B}_{m+n+1}(X) \to \mathcal{A}$. The *sliding block code* with memory m and anticipation n is the map $\phi\colon X \to Y$ defined by

$$\phi(x)_i = \Phi(x_{i-m} \cdots x_{i+n}).$$

Every sliding block code is a homomorphism. The Curtis-Lyndon-Hedlund Theorem is the surprising result that the converse is true: Every homomorphism between two shift spaces is a sliding block code.

A shift space is called *sofic* if it is the image of an SFT under a homomorphism. The *follower set* of a word $\alpha \in \mathcal{L}(X)$ is $F_X(\alpha) = \{\beta \in \mathcal{L}(X) : \alpha\beta \in \mathcal{L}(X)\}$.

Fact 2.5. *A shift space X is sofic if and only if it has a finite number of distinct follower sets.*

For example, the full shift has only one follower set, so $\mathcal{A}^{\mathbb{Z}}$ is sofic.

Definition 2.6. For each $S \subseteq \mathbb{N}_0$, the S-gap shift $X(S)$ is the subshift of $\{0,1\}^{\mathbb{Z}}$ obtained by taking the closure of the set

$$\{\cdots 10^{s_{-1}}10^{s_0}10^{s_1}1\cdots : s_i \in S \text{ for each } i \in \mathbb{N}\}$$

with respect to d.

Taking the closure has no effect when S is finite; when S is infinite, it allows strings to begin or end with infinitely many 0's.

Example 2.7 (Golden mean shift)**.** When $S = \mathbb{N}$, the shift space $X(S)$ is called the *golden mean shift*. The name is derived from the entropy of $X(S)$: Using Theorem 1.1, it is straightforward to show that $h(X(S)) = (1+\sqrt{5})/2$. If we set $\mathcal{F} = \{11\}$, then $X(S) = X_{\mathcal{F}}$, so the golden mean shift is an SFT; since any SFT is sofic, $X(S)$ is sofic. Finally, $\mathcal{B}_3(X(S)) = \{000, 001, 010, 100, 101\}$.

Example 2.8 (Even shift)**.** When $S = 2\mathbb{N}_0$, the shift space $X(S)$ is called the *even shift*. Since no finite set of words can forbid $10^{2n+1}1$ for every $n \in \mathbb{N}_0$ while simultaneously allowing $10^{2n}1$ for every $n \in \mathbb{N}_0$, the even shift is not an SFT. However, every follower set of $X(S)$ is equal to one of $F_{X(S)}(0)$, $F_{X(S)}(1)$, or $F_{X(S)}(10)$, so the even shift is sofic. Alternatively, define $\Phi\colon \{0,1\}^3 \to \{0,1\}$ by

$$\Phi(x_1x_2x_3) = \begin{cases} 0 & \text{if } x_1x_2 = 01 \text{ or } x_2x_3 = 01 \\ 1 & \text{otherwise.} \end{cases}$$

Let $\phi\colon X(\{0,1\}) \to X(S)$ be the sliding block code with memory 1 and anticipation 1 induced by Φ. Since $X(\{0,1\})$ is an SFT (take $\mathcal{F} = \{00\}$) and ϕ is surjective, the even shift is sofic.

(S,S')-gap shifts extend the definition of S-gap shifts to include restrictions on blocks of consecutive 1's [4].

Definition 2.9. Let $S, S' \subseteq \mathbb{N}$. The (S,S')-gap shift $X(S,S')$ is the subshift of $\{0,1\}^{\mathbb{Z}}$ obtained by taking the closure of

$$\{\cdots 1^{s'_{-1}}0^{s_{-1}}1^{s'_0}0^{s_0}1^{s'_1}0^{s_1}\cdots : s_i \in S \text{ and } s'_i \in S' \text{ for each } i \in \mathbb{N}\}.$$

So, for example, $X(S) = X(S,\mathbb{N})$ when $0 \in S$ and $X(S) = X(S,\{1\})$ otherwise. $\mathcal{S}$-limited shifts are extensions of (S,S')-gap shifts to alphabets with more than two letters [8]. We denote by $[n]$ the set $\{1,2,\ldots,n\}$.

Definition 2.10. Let $n \geq 2$ and $\mathcal{S} = (S_1,\ldots,S_n)$ be an n-tuple of subsets of $\mathbb{N}$. We set $G_{\mathcal{S}} = \{1^{m_1}2^{m_2}\cdots n^{m_n} : m_i \in S_i \text{ for each } i \in [n]\}$. The *ordered $\mathcal{S}$-limited shift*, denoted $\overrightarrow{X}(\mathcal{S})$, is the subshift of $\{1,2,\ldots,n\}^{\mathbb{Z}}$ obtained by taking the closure of

$$\{\cdots x_{-1}x_0x_1\cdots : x_i \in G_{\mathcal{S}} \text{ for each } i \in \mathbb{Z}\}.$$

The *unordered $\mathcal{S}$-limited shift*, denoted $X(\mathcal{S})$, is the subshift of $\{1,2,\ldots,n\}^{\mathbb{Z}}$ obtained by taking the closure of

$$\{\cdots a_{-1}^{m_{-1}}a_0^{m_0}a_1^{m_1}\cdots : a_i \in [n],\ m_i \in S_{a_i}, \text{ and } a_{i+1} \neq a_i \text{ for each } i \in \mathbb{Z}\}.$$

Ordered and unordered $\mathcal{S}$-limited shifts coincide when $n = 2$, yielding (S,S')-gap shifts.[2]

To describe $\mathcal{S}$-graph shifts, we need the language of graph theory. Unless otherwise mentioned, every graph in this paper is finite, directed, and simple (no loops and no multiple edges). The edge and vertex sets of a graph G are denoted by $E(G)$ and $V(G)$, respectively. We call G *irreducible* if there is a directed path from u to v for every $u,v \in V(G)$. A vertex of G is *essential* if its in-degree and out-degree are both at least 1; a graph is *essential* if every vertex is. A *walk* on G is a finite sequence of vertices $(v_1,\ldots,v_n)$ such that $(v_i,v_{i+1}) \in E(G)$ for each $i \in \{1,2,\ldots,n-1\}$. The set

[2] In [8], ordered and unordered $\mathcal{S}$-limited shifts are referred to as "$\mathcal{S}$-limited shifts" and "generalized $\mathcal{S}$-limited shifts" and denoted by $X(\mathcal{S})$ and $X_{\mathcal{S}}$, respectively. Since most $\mathcal{S}$-limited shifts are not generalized $\mathcal{S}$-limited shifts, this paper instead uses the modifiers "ordered" and "unordered." The notation was also adjusted to avoid overloading shift space construction using subscripts.

of all walks on G is denoted $W(G)$. A *cycle* is a walk with $(v_n, v_1) \in E(G)$ that does not repeat any vertex.

Given a graph G, the *vertex shift on* G is the set

$$\{\cdots u_{-1}u_0u_1\cdots : u_i \in V(G) \text{ and } (u_i, u_{i+1}) \in E(G) \text{ for each } i \in \mathbb{Z}\}.$$

A more robust construction uses vertex labels.

Definition 2.11. A *labelled graph* is a pair (G, ℓ), where G is a graph, $\mathcal{A}$ is a finite alphabet, and $\ell\colon V(G) \to \mathcal{A}$. The shift space generated by a labelled graph $\mathcal{G}$, denoted $X_{\mathcal{G}}$, is the set

$$\{\cdots \ell(u_{-1})\ell(u_0)\ell(u_1)\cdots : u_i \in V(G) \text{ and } (u_i, u_{i+1}) \in E(G) \text{ for each } i \in \mathbb{Z}\}.$$

Every shift space obtained from a labelled graph is sofic, and for every sofic shift X there is a labelled graph $\mathcal{G}$ so that $X = X_{\mathcal{G}}$.

Having set the scene, we now introduce $\mathcal{S}$-graph shifts.

Definition 2.12. A *set-equipped graph* is a pair $(G, \mathcal{S})$ where G is a directed graph and $\mathcal{S} = (S_v)_{v\in V(G)}$ is a collection of subsets of $\mathbb{N}$. If $\mathcal{G} = (G, \mathcal{S})$ is a set-equipped graph, the $\mathcal{S}$-graph shift $X(\mathcal{G}) = X(G, \mathcal{S})$ is the closure (over the alphabet $\mathcal{A} = V(G)$) of the set

$$\{\cdots u_{-1}^{n_{-1}}u_0^{n_0}u_1^{n_1} : u_i \in V(G),\, n_i \in S_{v_i}, \text{ and } (u_i, u_{i+1}) \in E(G) \text{ for each } i \in \mathbb{Z}\}.$$

If G is a directed cycle, then $X(G, \mathcal{S})$ is the ordered $\mathcal{S}$-limited shift $\overrightarrow{X}(\mathcal{S})$; if G is a complete graph, then $X(G, \mathcal{S}) = X(\mathcal{S})$. Section 7 extends this definition to *labelled $\mathcal{S}$-graph shifts*, which include all sofic shifts.

3 Dynamical properties

We first provide necessary and sufficient conditions for an $\mathcal{S}$-graph shift to be an SFT or sofic. Since an $\mathcal{S}$-graph shift remains unchanged after removing any non-essential vertices, we assume in this section that every graph is essential.

Proposition 3.1. *The shift space $X(G, \mathcal{S})$ is of finite type if and only if each $S_i \in \mathcal{S}$ is either finite or cofinite.*

Proof. Let $X = X(G, \mathcal{S})$. First suppose each S_i is either finite or cofinite and define

$$F_0 = \{ij : i \text{ and } j \text{ are not adjacent in } G\}.$$

For each $i \in \mathcal{A}$, if S_i is finite with maximum element m, we define

$$F_i = \{ai^kb : a, b \in \mathcal{A} \setminus \{i\} \text{ and } k \in \{1, 2, \ldots, m\} \setminus S_i\} \cup \{i^{m+1}\};$$

otherwise S_i is cofinite and we define

$$F_i = \{ai^kb : a, b \in \mathcal{A} \setminus \{i\} \text{ and } k \in \mathbb{N} \setminus S_i\}.$$

Each set F_i is finite. Setting $\mathcal{F} = \bigcup_{i=0}^{|\mathcal{A}|} F_i$, we have $X = X_{\mathcal{F}}$, which shows that X is an SFT.

Now suppose that S_i is neither finite nor cofinite for some $i \in \mathcal{A}$. No finite list of forbidden words will disallow all words in $\{ai^kb : a, b \in \mathcal{A} \setminus \{i\} \text{ and } k \notin S_i\}$ and simultaneously allow all words in $\{ai^kb : (a, i), (i, b) \in E(G) \text{ and } k \in S_i\}$. Therefore X is not an SFT. $\square$

The *difference sequence* of a set $S \subseteq \mathbb{N}$, denoted $\Delta(S)$, is the sequence of differences between consecutive members of S. That is, if S has elements $s_1 < s_2 < \cdots$, then $\Delta(S) = (d_n)_{n=1}^{\infty}$ is given by $d_n = s_{n+1} - s_n$.

Proposition 3.2. *The shift space $X(G, \mathcal{S})$ is sofic if and only if $\Delta(S_i)$ is eventually periodic for every $i \in V(G)$.*

Proof. ($\Rightarrow$) Let $X = X(G, \mathcal{S})$. Assuming that X is sofic, it has only finitely many follower sets by Fact 2.5. Let $(a, b) \in E(G)$. There must be a pair of integers $m < n$ so that $F(ab^m) = F(ab^n)$. This implies that $\Delta(S_b)$ eventually has period $n - m$.

($\Leftarrow$) Let $(a, b) \in E(G)$. If $\alpha ab^n \in \mathcal{L}(X)$ for some word α, then $F(\alpha ab^n) = F(ab^n)$, so we need only consider follower sets of the form $F(ab^n)$ or $F(b^n)$. Assume that $\Delta(S_b)$ eventually has period m_b, that is, there exists an $N_b \in \mathbb{N}$ so that $s \in S_i$ if and only if $s + m_b \in S_i$ for every $s \geq N_b$. Then $F(ab^s) = F(ab^{s+m_b})$ for every $s \geq N_b$. Then every follower set in X has one of the following two forms:
1. $F(b^k)$ with $k \in [0, N_b] \cap S_b$ or
2. $F(ab^k)$ with $k \in [0, N_b + m_b] \cap S_b$ and $(a, b) \in E(G)$.

Since there are only finitely many such follower sets, X is sofic. □

We next turn to the dynamical properties of mixing and (almost) specification.

Definition 3.3. A shift space X is called *mixing* if for every pair of words $\alpha, \beta \in \mathcal{L}(X)$ the following is true: There exists a positive integer N such that for every $n \geq N$ there is a word $\gamma \in \mathcal{B}_n(X)$ such that $\alpha\gamma\beta \in \mathcal{L}(X)$.

Proposition 3.4. *The shift space $X(G, \mathcal{S})$ is mixing if and only if G is irreducible and* $\gcd\left\{\sum_{i \in C} s_i \,:\, C \text{ is a cycle in } G \text{ and } s_i \in S_i\right\} = 1$.

Proof. Let $X = X(G, \mathcal{S})$ and $T = \left\{\sum_{i \in C} s_i \,:\, C \text{ is a cycle in } G \text{ and } s_i \in S_i\right\}$. First, suppose that X is mixing. Since any mixing shift is irreducible, it follows that G is irreducible. Let (a, b) be an edge in G. Because X is mixing, there is an $n \in \mathbb{N}$ with two words $\omega_1 \in \mathcal{B}_n(X)$ and $\omega_2 \in \mathcal{B}_{n+1}(X)$ such that $b\omega_1 a,\ b\omega_2 a \in \mathcal{L}(X)$. The words ω_1 and ω_2 must have the form $b^{k_0 - 1} c_1^{k_1} \cdots c_m^{k_m} a^{k_{m+1} - 1}$ for some $k_i \in S_{c_i}$, where $c_0 = b$, $c_{m+1} = a$, and $(c_i, c_{i+1}) \in E(G)$. Then $b\omega_1 a \in \mathcal{B}_{n+2}(X)$ and $b\omega_2 a \in \mathcal{B}_{n+3}(X)$.

Both $b\omega_1 a$ and $a\omega_2 b$ correspond to closed walks in G since (a, b) is an edge. Any closed walk can be decomposed into cycles. Since $b\omega_1 a \in \mathcal{B}_{n+2}(X)$ and $b\omega_2 a \in \mathcal{B}_{n+3}(X)$, both $n+2$ and $n+3$ can be expressed as sums of elements of T. It follows that the greatest common divisor of the elements of T is 1.

Now suppose that $\gcd(T) = 1$ and G is irreducible. There exists some N such that for all $n \geq N$, we can write n as a sum of elements of T. Let $(a, b),\ (c, a) \in E(G)$ and ω be a word comprised of concatenated full blocks (that is, words of the form a^k with $k \in S_a$) obtained from a walk w in G that begins at b, ends at c, and visits every vertex at least once. We first show that for all $n \geq N + |\omega|$, there is a word in $\mathcal{B}_n(X)$ consisting of concatenated full blocks. We will use this to prove that X is mixing.

If $n \geq N + |\omega|$, then $n - |\omega|$ can be written as a sum of elements in T, which corresponds to a multiset of cycles in G. By inserting these cycles into the walk w, we obtain a walk w_n and its corresponding word ω_n of length n that begins with a full block of b and ends with a full block of c. For every word $\alpha \in \mathcal{L}(X)$, let $\xi_\alpha^{\rightarrow}$ denote a word of shortest length that begins with α and ends with a full block of a; let $\xi_\alpha^{\leftarrow}$ denote a word of shortest length that begins with a full block of a and ends with α

Let $\alpha, \beta \in \mathcal{L}(X)$ and $n \geq N + |\omega| + |\xi_\alpha^{\rightarrow}| + |\xi_\beta^{\leftarrow}|$. Since $\xi_\alpha\, \omega_{n-|\xi_\alpha^{\rightarrow}|-|\xi_\beta^{\leftarrow}|}\, \xi_\beta^{\leftarrow} \in \mathcal{L}(X)$, the shift space X is mixing. □

Definition 3.5. A shift space X has the *specification property* if there exists a positive integer N such that for every pair of words $\alpha, \beta \in \mathcal{L}(X)$ there is a $\gamma \in \mathcal{B}_N(X)$ with $\alpha\gamma\beta \in \mathcal{L}(X)$. The shift space X has the *almost specification property* if there exists a positive integer N such that for every pair of words $\alpha, \beta \in \mathcal{L}(X)$ there is a word $\gamma \in \mathcal{L}(X)$ with $|\gamma| \leq N$ and $\alpha\gamma\beta \in \mathcal{L}(X)$.

Proposition 3.6. *The shift space $X(G, \mathcal{S})$ has the almost specification property if and only if G is irreducible and* $\sup \Delta(S_i) < \infty$ *for each $i \in V(G)$.*

A set $S \subseteq \mathbb{Z}$ with $\sup \Delta(S) < \infty$ is often called *syndetic*.

Proof. ($\Rightarrow$) Let $X = X(G, \mathcal{S})$. First suppose that G is not irreducible, so that there there exist $a, b \in V(G)$ with no path from a to b. Then there is no word $\gamma \in \mathcal{L}(X)$ such that $a\gamma b \in \mathcal{L}(X)$, so X does not have the almost specification property.

Now suppose that G is irreducible but $\sup \Delta(S_a) = \infty$ for some $a \in \mathcal{A}$. Let (a, b) be an edge in G and $N > 0$. Listing the elements of S_a as $s_1 < s_2 < \cdots$, we may choose k so that $n_{k+1} - n_k > N+1$. For any $b \in \mathcal{A} \setminus \{a\}$, there exists no word $\gamma \in \mathcal{B}_N(X)$ such that $a^{n_k+1}\gamma b \in \mathcal{L}(X)$, so X does not have the almost specification property.

($\Leftarrow$) Assume G is irreducible and $\sup \Delta(S_i)$ is finite for each $i \in \mathcal{A}$, and let $d = \max_i\big(\sup \Delta(S_i)\big)$. For each $a, b \in \mathcal{A}$, choose some word $\tau(a, b) \in \mathcal{L}(X)$ that does not begin with a and does not end with b such that $a\,\tau(a, b)\, b \in \mathcal{L}(X)$. We claim that $N = \max\{|\tau(a, b)| : a, b \in V(G)\} + 2d$ satisfies the criterion for the almost specification property as follows.

Let $\alpha, \beta \in \mathcal{L}(X)$, and suppose α ends with the block a^{k_1} and β begins with the block b^{k_2}. There exist positive integers $s_1 \in S_a$ and $s_2 \in S_b$ such that $0 \leq s_1 - k_1 \leq d$ and $0 \leq s_2 - k_2 \leq d$. Then $\alpha a^{s_1-k_1}\, \tau(a, b)\, b^{s_2-k_2}\beta \in \mathcal{L}(X)$. Since α and β were arbitrary, X has the almost specification property. □

Proposition 3.7. *An $\mathcal{S}$-graph shift has the specification property if and only if it is mixing and has the almost specification property.*

Proof. ($\Rightarrow$) Let $X = X(G, \mathcal{S})$. First suppose that X has the specification property. Certainly X has the almost specification property. Let N be the integer guaranteed by the specification property. Given any two words $\alpha, \beta \in \mathcal{L}(X)$ and an integer $n \geq N$, choose some $\tau \in \mathcal{B}_{n-N}(X)$ with $\alpha\tau \in \mathcal{L}(X)$. There is a word $\gamma \in \mathcal{B}_N(X)$ so that $\alpha\tau\gamma\beta \in \mathcal{L}(X)$. Since α and β were arbitrary, X is mixing.

($\Leftarrow$) Now suppose that X is mixing and has the almost specification property. By Proposition 3.6, the integer $d = \max_i\big(\sup \Delta(S_i)\big)$ is finite. For each $a, b \in V(G)$, choose an integer $N(a, b)$ such that for all $n \geq N(a, b)$, there exists some $\tau \in \mathcal{B}_n(X)$ that does not begin with a nor end with b and $a\tau b \in \mathcal{L}(X)$. The integers $N(a, b)$ exist because X is mixing. Then $N = \max\{N(a, b) : a, b \in V(G)\} + 2d$ satisfies the criteria for the specification property as follows.

Let $\alpha, \beta \in \mathcal{L}(X)$. Suppose that α ends with the block a^{k_1} and β begins with the block b^{k_2}. There exist positive integers $s_1 \in S_a$ and $s_2 \in S_b$ such that $0 \leq s_1 - k_1 \leq d$ and $0 \leq s_2 - k_2 \leq d$. We can choose a word $\tau \in \mathcal{L}(X)$ with $|\tau| = N - (s_1 - k_1) - (s_2 - k_2)$ such that $a^{s_1-k_1}\tau b^{s_2-k_2} \in \mathcal{L}(X)$, which implies that $\alpha a^{s_1-k_1}\tau b^{s_2-k_2}\beta \in \mathcal{L}(X)$. □

4 Proof of the entropy formula

As in the previous section, any graph in this section is essential. Moreover, we only consider nonempty $\mathcal{S}$-graph shifts, so every graph contains at least one cycle. For convenience, we also set $V(G) = \{1, 2, \ldots, p\}$.

We will require a handful of combinatorial structures. Recall that the adjacency matrix of G is the $p \times p$ square matrix A_G with $(A_G)_{i,j} = 1$ if $(i, j) \in E(G)$ and $(A_G)_{i,j} = 0$ otherwise.

Definition 4.1. Let $\mathcal{G} = (G, \mathcal{S})$ be a set-equipped graph on the vertex set $\{1, 2, \ldots, p\}$. To each vertex $i \in V(G)$ we assign the generating function $H_i(x) = \sum_{s \in S_i} x^s$. The *generating matrix* of $\mathcal{G}$ is the $p \times p$ matrix function $B_{\mathcal{G}}(x)$ with entries

$$B_{\mathcal{G}}(x)_{i,j} = \begin{cases} H_i(x) & \text{if } (i,j) \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

The *diagonal generating matrix* is $D_{\mathcal{G}}(x) = \operatorname{diag}\big(H_1(x), \ldots, H_n(x)\big)$. (So $B_{\mathcal{G}}(x) = D_{\mathcal{G}}(x) A_G$.)

The next proposition provides the essential link between the graph representation of an $\mathcal{S}$-graph shift and its entropy. If $(G, \mathcal{S})$ is a set-equipped graph and $w = (v_1, \ldots, v_n)$ is a walk on G, we let $H_w(x)$ denote the power series $\prod_{i=1}^n H_i(x)$.

Proposition 4.2. *Let $X = X(G, \mathcal{S})$ be an $\mathcal{S}$-graph shift. If λ is the radius of convergence of the generating function $\sum_{w \in W(G)} H_w(x)$, then the entropy of X is $\log \lambda^{-1}$.*

Proof. Let $f(x) = \sum_{n=1}^\infty \mathcal{B}_n(X) x^n$, which has radius of convergence $e^{-h(X)}$. In the rest of the proof, we show that the radius of convergence of $f(x)$ is λ, which implies that $h(X) = \log \lambda^{-1}$.

Let $W = W(G)$. The coefficient of x^n in $\sum_{w \in W} H_w(x)$ is the number of words of the form $\alpha_1^{k_1} \cdots \alpha_j^{k_j}$ where $k_i \in S_{\alpha_i}$, each $(\alpha_i, \alpha_{i+1}) \in E(G)$, and $k_1 + \cdots + k_j = n$. The number of such words is at most $\mathcal{B}_n(X)$, so

$$\sum_{w \in W} H_w(x) \leq f(x)$$

for every $x \geq 0$. Every allowable word has the form $a^{k_1} \omega b^{k_2}$, where ω is composed of concatenated full blocks, $a, b \in \mathcal{A}$, and $k_1, k_2 \in \mathbb{N}$. The number of such words (ignoring for the moment whether they are in $\mathcal{L}(X)$) has generating function

$$\left(\frac{|\mathcal{A}|}{1-x}\right) \left(\sum_{w \in W} H_w(x)\right) \left(\frac{|\mathcal{A}|}{1-x}\right).$$

This overcounts the number of words in $\mathcal{B}_n(X)$, so

$$\sum_{w \in W} H_w(x) \leq f(x) \leq \left(\frac{|\mathcal{A}|}{1-x}\right)^2 \sum_{w \in W} H_w(x)$$

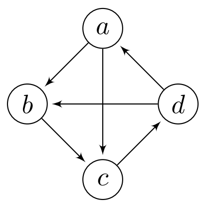
for every $x \geq 0$. Because G contains at least one cycle, $\sum_{w \in W} H_w(x)$ contains arbitrarily large powers of x; all coefficients are nonnegative, so the power series diverges at $x = 1$. The power series $(1-x)^{-1}$ has a radius of convergence of 1, so $f(x)$ will converge exactly when $\sum_{w \in W} H_w(x)$ does. Therefore $e^{-h(X)} = \lambda$. $\square$

Proposition 4.2 is enough to calculate the entropy of some simple $\mathcal{S}$-graph shifts, in particular when each set S_i is the same or G has few edges, but the calculation rapidly becomes intractable in any more complicated situation. To tame the sum $\sum_{w\in W(G)} H_w(x)$, we'll need finer control over the vertices in a walk.

Definition 4.3. Let G be a graph with vertices $v_1, \dots, v_p$, and let A_G denote the adjacency matrix of G. Considering each of the vertices as indeterminates, we define $D_G = \operatorname{diag}(v_1, \dots, v_p)$. The *symbolic adjacency matrix* of G is $B_G = D_G A_G$.

We may omit the subscript if only one graph is under discussion. The symbolic adjacency matrix is the labelled adjacency matrix for the graph obtained from G by labelling each edge with the vertex it is directed from.

Example 4.4. A graph and its symbolic adjacency matrix.



$$\begin{pmatrix} 0 & a & a & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & d & 0 & 0 \end{pmatrix}$$

Lemma 4.5. *Let G be a graph with D_G and B_G as above. The number of walks of length n on G which visit vertex v_i exactly k_i times is the coefficient on $v_1^{k_1} \cdots v_p^{k_p}$ in $\sum_{i,j} (B^n D)_{i,j}$.*

Proof. Induction on n shows that the number of walks that visit v_i exactly k_i times and begin at v_s is the coefficient of $v_1^{k_1} \cdots v_p^{k_p}$ in $\sum_{j=1}^{p} (B^n D)_{s,j}$. Summing over all values of s proves the lemma. $\square$

Corollary 4.6. *If $(G, \mathcal{S})$ is a set-equipped graph, then*

$$\sum_{n=0}^{\infty} \sum_{i,j} \big(B_G^n(x) D(x)\big)_{i,j} = \sum_{w\in W(G)} H_w(x).$$

Proof. The coefficient of $H_1(x)^{k_1} \cdots H_p(x)^{k_p}$ in the sum $\sum_{w\in W(G)} H_w(x)$ is the number of walks in G that visit vertex i exactly k_i times, so it is equal to the coefficient of $v_1^{k_1} \cdots v_p^{k_p}$ in $\sum_{i,j}(B^n D)_{i,j}$. Summing over all compositions $k_1 + \cdots + k_p$ of n shows that the coefficients of x^n in $\sum_{w\in W(G)} H_w(x)$ and $\sum_{n=0}^{\infty} \sum_{i,j} \big(B_G^n(x) D(x)\big)_{i,j}$ are equal. $\square$

We can eliminate the diagonal matrix while maintaining the radius of convergence.

Lemma 4.7. *The power series $\sum_{n=0}^{\infty} \sum_{i,j} \big(B_G^n(x) D(x)\big)_{i,j}$ and $\sum_{n=0}^{\infty} \sum_{i,j} \big(B^n(x)\big)_{i,j}$ have the same radius of convergence.*

Proof. Since every coefficient in both series is positive and the series are centered at 0, the radii of convergence can be determined by examining only nonnegative real numbers. Fix some $x \in \mathbb{R}_{\geq 0}$. If $H_i(x)$ diverges for any $i \in \mathcal{A}$, then both $\sum_{n=0}^{\infty} \sum_{i,j} (B^n(x))_{i,j}$ and $\sum_{n=0}^{\infty} \sum_{i,j} (B^n(x) D(x))_{i,j}$ diverge. So suppose that $H_i(x)$ is finite for each $i \in \mathcal{A}$. If c is the minimum of $\{H_1(x), \dots, H_p(x)\}$ and d is the maximum, then

$$c \sum_{n=0}^{\infty} \sum_{i,j} \big(B^n(x)\big)_{i,j} \leq \sum_{n=0}^{\infty} \sum_{i,j} \big(B^n(x) D(x)\big)_{i,j} \leq d \sum_{n=0}^{\infty} \sum_{i,j} \big(B^n(x)\big)_{i,j}.$$

Since both c and d are nonnegative, it follows that $\sum_{n=0}^{\infty}\sum_{i,j}(B^n(x)D(x))_{i,j}$ converges if and only if $\sum_{n=0}^{\infty}\sum_{i,j}(B^n(x))_{i,j}$ does. $\square$

We need two more ingredients. The first is a consequence of the Perron-Frobenius theorem from linear algebra (see Chapter 4 of [7]). We call a nonnegative square matrix A *irreducible* if for every pair of indices i, j, there is a positive integer n so that $A^n_{i,j} > 0$. The adjacency matrix A_G of G is irreducible if and only if G is.

Theorem 4.8. *If A is an irreducible matrix with spectral radius $\mathghost$,*[3] *then there exist positive real numbers c and d such that*

$$c\,\mathghost^n \leq \sum_{i,j}(A^n)_{i,j} \leq d\,\mathghost^n$$

for every $n \in \mathbb{N}$.

This result can be extended to all nonnegative matrices with only a modest change to the bounds.

Proposition 4.9. *If A is a nonnegative square matrix with spectral radius $\mathghost$, then there exist positive real numbers c and d and an integer k such that*

$$c\,\mathghost^n \leq \sum_{i,j}(A^n)_{i,j} \leq dn^k\,\mathghost^n$$

for every $n \in \mathbb{N}$.

Proof. Suppose that A is an $m \times m$ matrix. By a standard result from linear algebra, we may relabel the indices of A so that A is block upper-triangular and each block in the main diagonal is irreducible. Let $C_1, \ldots, C_k$ denote the blocks on the main diagonal of A and $\mathghost(C_i)$ be the spectral radius of C_i. Let t be an index for which $\mathghost(C_t)$ is maximal; since the spectrum of A is the union of the spectra of the C_i, we have $\mathghost = \mathghost(C_t)$. For each $1 \leq i \leq k$, let c_i and d_i be the constants supplied by Theorem 4.8 when applied to C_i, and set $\bar{d} = \max_i d_i$; also set $\alpha = \max\{1, \max_{i,j} A_{i,j}\}$.

Theorem 4.8 implies that

$$\sum_{i,j}(A^n)_{i,j} \geq \sum_{i,j}(C_t^n)_{i,j} \geq c_t\,\mathghost(C_t)^n = c_t\,\mathghost^n,$$

which proves the first inequality. Let K be the complete graph with loops on the vertex set $\{1, 2, \ldots, m\}$ (that is, $E(K) = V(K) \times V(K)$). Given a walk $w = (v_1 \ldots, v_s)$ on K, we define

$$f(w) = A_{v_1,v_2}A_{v_2,v_3}\cdots A_{v_{s-1},v_s}$$

and note that $\sum_{i,j}(A^n)_{i,j} = \sum_{|w|=n+1} f(w)$. By a minor abuse of notation, we let $V(C_i)$ denote the set of indices that determine the matrix C_i in A and call these the *blocks* of $V(K)$.

If $f(w) \neq 0$, then there are at most $k-1$ indices r so that v_r and v_{r+1} are in different blocks. Therefore, we decompose w as $w_1e_1w_2e_2\cdots e_{j-1}w_j$, where each e_i is an edge between blocks and each w_j is a walk inside a single block. Since there are m^2 possibilities for each e_i and n possible positions, the number of arrangements for the interblock edges is at most $(m^2n)^k$. Fix one such

[3] Spectral, get it?

arrangement R of interblock edges, and let W_R denote the set of all walks of length $n+1$ with arrangement R. By Theorem 4.8, we have

$$\sum_{w \in W_R} f(w) \leq \alpha^k \sum_{w \in W_R} f(w_1) \cdots f(w_j) \leq \alpha^k \bar{d} \ghost^n.$$

Therefore

$$\sum_{i,j} (A^n)_{i,j} = \sum_{|w|=n+1} f(w) = \sum_{R} \sum_{w \in W_R} f(w) \leq (m^2 n)^k \alpha^k \bar{d} \ghost^n.$$

So taking $d = \bar{d}(\alpha m^2)^k$ finishes the proof. □

The second ingredient is a lemma about the spectral radius of a generating matrix.

Lemma 4.10. *If $(G, \mathcal{S})$ is a set-equipped graph and $\ghost(x)$ is the spectral radius of $B_{\mathcal{G}}(x)$, then $\ghost$ is continuous and strictly increasing on the set of nonnegative real numbers with range $[0, +\infty)$.*

Proof. The spectral radius of a matrix is a continuous function of its entries.[4] Since the entries of $B(x)$ vary continuously with x, the function $\ghost(x)$ is continuous.

Let x and y be positive and within the radius of convergence of each power series H_i, with $x < y$. By Proposition 4.9, there exist positive constants c_x, d_x, and k_x such that

$$c_x \ghost(x)^n \leq \sum_{i,j} \big(B^n(x)\big)_{i,j} \leq d_x n^{k_x} \ghost(x)^n$$

and constants c_y, d_y, and k_y for $B(y)^n$. After expanding $\sum_{i,j} \big(B^n(y)\big)_{i,j}$ into a power series, each term has degree at least n. A general term ay^m in the series therefore satisfies

$$ay^m = \left(\frac{y}{x}\right)^n a y^{m-n} x^n \geq \left(\frac{y}{x}\right)^n a x^m.$$

Therefore

$$\sum_{i,j} \big(B^n(y)\big)_{i,j} \geq \left(\frac{y}{x}\right)^n \sum_{i,j} \big(B^n(x)\big)_{i,j} \geq \left(\frac{y}{x}\right)^n c_x \ghost(x)^n.$$

Since $y > x$, we can choose n large enough so that $(\frac{y}{x})^n > \frac{d_y}{c_x} n^{k_y}$. For that value of n,

$$d_y n^{k_y} \ghost(y)^n \geq \sum_{i,j} \big(B(y)^n\big)_{i,j} > d_y n^{k_y} \ghost(x)^n.$$

Dividing by $d_y n^{k_y}$ shows that $\ghost(y) > \ghost(x)$.

Finally, we prove that the range of $\ghost(x)$ is $[0, +\infty)$. Since $\ghost(0) = 0$ and $\ghost$ is continuous, it suffices to show that $\ghost(x) \to \infty$ as $x \to \infty$. If $x \neq 0$, then $B(x)$ is not the zero matrix, so $\ghost(x) > 0$. For every $x \geq 0$, the inequality $B(2x) \geq 2B(x)$ holds entrywise. By Proposition 4.9, there exist positive constants c, d and k so that

$$c\,(2\ghost(x))^n \leq \sum_{i,j} \big((2B(x))^n\big)_{i,j} \leq \sum_{i,j} \big(B(2x)^n\big)_{i,j} \leq dn^k \ghost(2x)^n$$

for every $n \in \mathbb{N}$. Therefore $\ghost(2x) \geq 2\ghost(x)$ for each $x \geq 0$; so $\lim_{x\to\infty} \ghost(x) = \infty$. □

[4] This is a standard result that follows from a chain of continuous maps: the map from the matrix entries to the coefficients of the characteristic polynomial, the map from a complex polynomial to its set of roots (see, for example, [9]), and the map from the set of roots to its maximum complex norm.

We can now prove the Big Theorem.

Theorem 4.11. *Let $(G, \mathcal{S})$ be a set-equipped graph and $\mathghost(x)$ be the spectral radius of $B_{\mathcal{G}}(x)$. The equation $\mathghost(x) = 1$ has a unique positive solution $x = \lambda$, and the entropy of $X(G, \mathcal{S})$ is $\log \lambda^{-1}$.*

Proof. Let $x > 0$ be within the radius of convergence of each $H_i(x)$. By Proposition 4.9 there are positive constants c, d, and k such that

$$c\,\mathghost(x)^n \leq \sum_{i,j} \big(B(x)^n\big)_{i,j} \leq dn^k\,\mathghost(x)^n$$

for every $n \in \mathbb{N}$. Summing over n, we get

$$c\sum_{n=0}^{\infty} \mathghost(x)^n \leq \sum_{n=0}^{\infty}\sum_{i,j} \big(B(x)^n\big)_{i,j} \leq d\sum_{n=0}^{\infty} n^k\,\mathghost(x)^n.$$

The bounding series converge if and only if $|\mathghost(x)| < 1$. Lemma 4.10 guarantees a unique positive solution $x = \lambda$ to the equation $\mathghost(x) = 1$. Then λ is the radius of convergence of $\sum_{n=0}^{\infty}\sum_{i,j}(B(x)^n)_{i,j}$; by Proposition 4.2, the entropy of $X(G, \mathcal{S})$ is $\log \lambda^{-1}$. $\square$

A shift space X is called *almost sofic* if for every $\varepsilon > 0$ there exists a subshift of finite type $Y \subseteq X$ such that $h(Y) \geq h(X) - \varepsilon$. Almost sofic shifts were introduced by Petersen in [10]; they are the class of shift spaces that are particularly interesting from the perspective of data encoding.

Corollary 4.12. *Every $\mathcal{S}$-graph shift is almost sofic.*

Proof. Let $\mathcal{G} = (G, \mathcal{S})$ be a set-equipped graph. Set $\mathcal{S}_n = (S_v \cap [1, n])_{v \in V(G)}$ and $\mathcal{G}_n = (G, \mathcal{S}_n)$. Then $B_{\mathcal{G}_n}(x)$ tends to $B_{\mathcal{G}}(x)$ pointwise as $n \to \infty$, so the spectral radii tend toward the spectral radius of $B_{\mathcal{G}}(x)$. Therefore $h(X(\mathcal{G}_n)) \to h(X(\mathcal{G}))$. Since $X(\mathcal{G}_n)$ is an SFT for every $n \in \mathbb{N}$ by Proposition 3.1, the shift space $X(\mathcal{G})$ is almost sofic. $\square$

Theorem 4.11 can be reformulated with a more pungent power-series flavor. We denote by $\mathcal{C}_G$ the collection of all nonempty sets of vertex-disjoint cycles in a graph G.

Theorem 4.13. *Let $(G, \mathcal{S})$ be a set-equipped graph. If λ is the least positive solution to*

$$\sum_{C \in \mathcal{C}_G} (-1)^{|C|+1} \prod_{c \in C} H_c(x) = 1, \tag{4.1}$$

then the entropy of $X(G, \mathcal{S})$ is $\log \lambda^{-1}$.

Proof. Let $\mathghost(x)$ be the spectral radius of $B_{\mathcal{G}}(x)$. Since $\mathghost(x)$ is increasing, the solution to $\mathghost(x) = 1$ is the least positive solution to $\det(B(x) - I) = 0$. We prove that $\det(B(x) - I) = 0$ if and only if (4.1) holds; appealing to Theorem 4.11 finishes the proof.

Let π be a permutation of $V(G)$. A cycle in the cycle decomposition of π corresponds to a cycle in G if and only if $(i, \pi(i)) \in E(G)$ for each element i in that cycle. But $(i, \pi(i)) \in E(G)$ exactly when $\big(B(x)\big)_{i,\pi(i)} \neq 0$ for every $x > 0$. So either the cycles of π correspond to a set of disjoint cycles in G, or the corresponding term in the permutation expansion of $\det\big(B(x) - I\big)$ is 0. Suppose π corresponds to the set $C = \{c_1, \ldots, c_m\}$ of disjoint cycles in G. The sign of π is $\operatorname{sgn}(\pi) = \prod_{i=1}^{m}(-1)^{|c_i|+1}$. Also, we have

$$(B(x) - I)_{i,\pi(i)} = \begin{cases} H_i(x) & \text{if } i \neq \pi(i) \\ -1 & \text{if } i = \pi(i). \end{cases}$$

If π fixes exactly s vertices, then (recalling that $|V(G)| = p$)

$$\operatorname{sgn}(\pi)\prod_{i=1}^{p} B(x)_{i,\pi(i)} = \left[\prod_{i=1}^{m}(-1)^{|c_i|+1}\right]\left[(-1)^s \prod_{c\in C} H_c(x)\right] = (-1)^{m+p}\prod_{c\in C} H_c(x).$$

Thus

$$\begin{aligned}\det(B(x)-I) &= \sum_{\pi\in\mathfrak{S}_p}\operatorname{sgn}(\pi)\prod_{i=1}^{p} B(x)_{i,\pi(i)} \\ &= (-1)^p + \sum_{C\in\mathcal{C}_G}(-1)^{|C|+p}\prod_{c\in C} H_c(x).\end{aligned}$$

Therefore $\det(B(x)-I) = 0$ if and only if

$$\sum_{C\in\mathcal{C}_G}(-1)^{|C|+1}\prod_{c\in C} H_c(x) = 1. \qquad \square$$

Equation (4.1) generalizes the entropy formula for ordered $\mathcal{S}$-limited shifts. The ordered $\mathcal{S}$-limited shift $\overrightarrow{X}(\mathcal{S})$ can be represented as the $\mathcal{S}$-graph shift $X(G,\mathcal{S})$, where G is a directed cycle. Then $\mathcal{C}_G$ contains only a single cycle (the whole graph), and (4.1) reduces to

$$\prod_{i=1}^{p} H_i(x) = \sum_{\omega\in G_{\mathcal{S}}} x^{|\omega|}.$$

5 Graph operations

5.1 Edge extension

To describe edge extensions, we first need to define sum decomposition of subsets of $\mathbb{N}$.

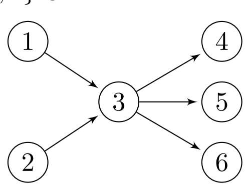
Definition 5.1. Given two subsets $S, T \subseteq \mathbb{N}$, their *sumset* is $S + T = \{s + t : s \in S \text{ and } t \in T\}$. We write $S = S_1 \oplus S_2$ if every element $s \in S$ can be written uniquely as a sum $s_1 + s_2$ with $s_1 \in S_1$ and $s_2 \in S_2$.

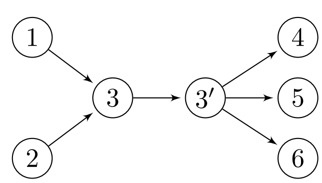
Definition 5.2. Let $(G,\mathcal{S})$ be a set-equipped graph, v be a vertex of G, and suppose that $S_v = T_1 \oplus T_2$. The *edge extension* of G at v with respect to the decomposition $T_1 \oplus T_2$ the set-equipped graph $(G',\mathcal{S}')$ defined as follows: The vertex set of G' is $V(G)\cup\{v'\}$, and $(u_1,u_2)\in E(G')$ if and only if one of the following is true:

1. $u_1, u_2 \notin \{v, v'\}$ and $(u_1,u_2)\in E(G)$,
2. $u_1 = v'$ and $(v,u_2)\in E(G)$,
3. $u_2 = v$ and $(u_1,v)\in E(G)$, or
4. $u_1 = v$ and $u_2 = v'$.

Finally, $S'_u = S_u$ if $u \notin \{v,v'\}$, $S'_v = T_1$, and $S'_{v'} = T_2$.

Example 5.3. The graph on the right is an edge extension of the graph on the left. If $S_3 = \mathbb{N}_0 + 3$, then $S_3 = \{1,2\}\oplus 2\mathbb{N}$ is a valid decomposition.





We have the following two results on edge extension.

Proposition 5.4. *Edge extension preserves entropy and the mixing property.*

Proof. Let $X = X(G,\mathcal{S})$ and $Y = X(G',\mathcal{S}')$ be the shift obtained by an edge extension at $v \in V(G)$ according to the decomposition $S_v = T_1 \oplus T_2$. There is a one-to-one correspondence between cycles in G and cycles in G', since a cycle in G' contains v if and only if it contains the edge (v, v'). Moreover,

$$\left(\sum_{a\in T_1} x^a\right)\left(\sum_{a\in T_2} x^a\right) = \sum_{a\in S_1} x^a,$$

since the representation of $a \in S_v$ as a sum of elements in T_1 and T_2 is unique. So the power series in (4.1) is the same for X and Y. By Theorem 4.13, the entropy of the two shift spaces is equal.

Let $U(G,\mathcal{S}) = \{\sum_{i\in C} s_i : C \text{ is a cycle in } G \text{ and } s_i \in S_i\}$. The bijection between cycles in G and G' shows that $U(G,\mathcal{S}) = U(G',\mathcal{S}')$. By Proposition 3.4, X is mixing if and only if X is. □

Proposition 5.5. *If $S_v = T_1 \oplus T_2$ and one of T_1 or T_2 is a singleton, then edge extension at v is a conjugacy.*

Proof. We prove the case where T_2 is a singleton; the other case is similar. Let $T_2 = \{m\}$ and $X = X(G,\mathcal{S})$, and suppose Y is the shift space obtained by edge extension of X at v. We define the block map $\Phi\colon \mathcal{B}_{m+1}(X) \to V(G')$ by

$$\Phi(x_0 \cdots x_m) = \begin{cases} x_0 & \text{if } x_0 \neq v \\ v' & \text{if } x_0 = v \text{ and } x_k \neq v \text{ for some } k \in \{1, 2 \dots, m\}. \\ v & \text{if } x_i = v \text{ for every } i \in \{0, 1, 2, \dots, m\} \end{cases}$$

To show that $X \cong Y$, we exhibit a sliding block inverse to the sliding block map $\phi\colon X \to Y$ induced by Φ: the code induced by the map $\Psi\colon \mathcal{B}_1(Y) \to V(G)$ defined by

$$\Psi(y_0) = \begin{cases} y_0 & \text{if } y_0 \neq v' \\ v & \text{if } y_0 = v'. \end{cases}$$

□

A similar argument shows that edge extension at v is a conjugacy whenever S_v is finite. Edge extension is not, however, a conjugacy in general: Suppose Y is obtained from X by an edge extension at v using the decomposition $S_v = T_1 \oplus T_2$. If both T_1 and T_2 are infinite, then $p_1(Y) = p_1(X) + 1$. Nevertheless, edge extension does retain almost all periodic point behavior.

Proposition 5.6. *If Y is a shift obtained from an edge extension of the $\mathcal{S}$-graph shift X, then $q_n(X) = q_n(Y)$ for all $n \geq 2$.*

Proof. Any periodic point with least period 2 or greater in the $\mathcal{S}$-graph shift $X(G,\mathcal{S})$ corresponds to a closed walk in G in which each vertex v is labelled with an element from S_v. This bijection between closed walks on the set-equipped graph and periodic points lifts to a bijection between periodic points of X and Y. □

5.2 State splitting

All conjugacies between edge shifts can be obtained through a process known as state splitting [7]. In this section, we introduce state splitting for $\mathcal{S}$-graph shifts.

Definition 5.7. Let G be a graph. The *out-neighborhood* of a set $I \subseteq V(G)$, denoted $\Gamma_G^+(I)$, is the set of vertices $v \in V(G)$ such that there exists a vertex $u \in I$ with $(u, v) \in E(G)$.

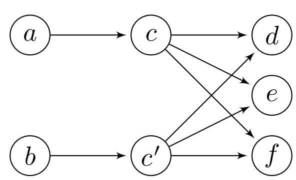
Definition 5.8. Let $(G, \mathcal{S})$ be a set-equipped graph, $v \in V(G)$, and $[E_1, E_2]$ be a bipartition of the out-neighbors of v. The *out-splitting* of $(G, \mathcal{S})$ at v with respect to the partition $[E_1, E_2]$ is the set-equipped graph $(G', \mathcal{S}')$ defined as follows: The vertex set of G' is $V(G) \cup \{v'\}$, and $(u_1, u_2) \in E(G)$ if and only if one of the following is true:

1. $u_1, u_2 \notin \{v, v'\}$ and $(u_1, u_2) \in E(G)$,
2. $u_1 = v$ and $u_2 \in E_1$,
3. $u_1 = v'$ and $u_2 \in E_2$, or
4. $u_2 \in \{v, v'\}$ and $(u_1, v) \in E(G)$.

Finally, $S'_u = S_u$ if $u \neq v'$ and $S'_{v'} = S_v$.

Example 5.9. The out-splitting of the graph on the left at vertex c with $E_1 = \{d, e\}$ and $E_2 = \{f\}$ is shown on the right.

An in-splitting is be defined analogously to an out-splitting, by partitioning the in-neighbors instead. For example, the in-splitting of the graph in Example 5.9 at c with bipartition $[\{a\}, \{b\}]$ is



It turns out that state-splitting gets along quite nicely with many dynamical properties. To prove this, we'll need the following lemma.

Lemma 5.10. *Let A be a $p \times p$ matrix with rows a_i and suppose $a_m = x + y$. If A' is the matrix obtained by duplicating the mth column in the matrix whose rows are $a_1, \ldots, a_{m-1}, x, y, a_{m+1}, \ldots, a_p$, then* $\operatorname{spec}(A') = \operatorname{spec}(A) \cup \{0\}$.

Proof. We calculate the characteristic polynomial of A'. By interchanging rows and columns, we can assume that $m = p$, so that $A' - \lambda I$ has the form

$$\begin{pmatrix} a_{1,1} - \lambda & a_{1,2} & \cdots & a_{1,p} & a_{1,p} \\ a_{2,1} & a_{2,2} - \lambda & \cdots & a_{2,p} & a_{2,p} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_1 a_{p,1} & b_2 a_{p,2} & \cdots & b_p a_{p,p} - \lambda & b_p a_{p,p} \\ c_1 a_{p,1} & c_2 a_{p,2} & \cdots & c_p a_{p,p} & c_p a_{p,p} - \lambda \end{pmatrix},$$

where $b_i + c_i = 1$ for all $1 \leq i \leq p$. By adding the last row to the penultimate one and then subtracting the penultimate column from the final one, we obtain the matrix

$$\begin{pmatrix} a_{1,1} - \mathghost & a_{1,2} & \cdots & a_{1,p} & 0 \\ a_{2,1} & a_{2,2} - \mathghost & \cdots & a_{2,p} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{p,1} & a_{p,2} & \cdots & a_{p,p} - \mathghost & 0 \\ c_1 a_{p,1} & c_2 a_{p,2} & \cdots & c_p a_{p,p} & -\mathghost \end{pmatrix}.$$

Therefore $\det(A' - \mathghost I) = -\mathghost \det(A - \mathghost I)$, and the conclusion follows. □

Proposition 5.11. *State splitting of $\mathcal{S}$-graphs preserves entropy, the mixing property, almost specification, and specification. Further, state splittings of an SFT or sofic shift are SFTs or sofic, respectively.*

Proof. Let $X = X(G, \mathcal{S})$, and let $Y = X(G', \mathcal{S}')$ be a state splitting of X at $v \in V(G)$. We assume Y is obtained as an out-splitting, since the proof for in-splittings is nearly identical. We can take $A = B_{\mathcal{G}}(x)$ and $A' = B_{\mathcal{G}'}(x)$ in Lemma 5.10, so $B_{\mathcal{G}}(x)$ and $B_{\mathcal{G}'}(x)$ have the same spectral radius for all x. By Theorem 4.11, the shift spaces X and Y have the same entropy.

Next, define $U(X) = \left\{\sum_{i \in C} s_i \,:\, C \text{ is a cycle in } G \text{ and } s_i \in S_i\right\}$ and $U(Y)$ similarly. Any cycle in G that does not contain v is also a cycle in G'; if the cycle contains v, then replacing v by exactly one of v or v' yields a cycle in G'. Since $S'_v = S'_{v'} = S_v$, we have $U(X) \subseteq U(Y)$.[5] If X is mixing, then $\gcd(U(X)) = 1$ by Proposition 3.4; thus $\gcd(U(Y)) = 1$, so Y is mixing.

The sets in $\mathcal{S}$ are the same as those in $\mathcal{S}'$, so X is almost specified if and only if Y is by Proposition 3.6. The same holds for SFT and sofic shifts by Propositions 3.1 and 3.2.

Since state splitting preserves the mixing property and almost specification, it preserves specification by Proposition 3.7. □

If S_v is infinite, then state-splitting increases the number of words of period 1. So in general state-splitting is not a conjugacy. It is a conjugacy whenever S_v is finite, however.

Proposition 5.12. *Let $(G, \mathcal{S})$ be a set-equipped graph. If S_v is finite, then an $\mathcal{S}$-graph shift resulting from a state splitting at vertex v is conjugate to $X(G, \mathcal{S})$.*

Proof. Let $X = X(G, \mathcal{S})$ and $[E_1, E_2]$ be a partition of the out-neighbors of v. Let Y be the shift obtained from the out-splitting at v with the bipartition $[E_1, E_2]$. Setting $N = \max S_v$, we define a map $\Phi \colon \mathcal{B}_{N+1}(X) \to V(G')$ by

$$\Phi(x_0 \cdots x_N) = \begin{cases} x_0 & \text{if } x_0 \neq v \\ v & \text{if } x_i = v \text{ for each } i \in \{0, 1, \ldots, m-1\},\ x_m \neq v, \text{ and } x_m \in E_1 \\ v' & \text{if } x_i = v \text{ for each } i \in \{0, 1, \ldots, m-1\},\ x_m \neq v, \text{ and } x_m \in E_2. \end{cases}$$

The map Φ induces a sliding block code $\phi \colon X \to Y$ whose inverse is induced by the map $\Psi \colon \mathcal{B}_1(Y) \to V(G)$ defined by

$$\Psi(y_0) = \begin{cases} y_0 & \text{if } y_0 \neq v' \\ v & \text{if } y_0 = v'. \end{cases}$$

So ϕ is a conjugacy from X to Y. □

[5] It is not true in general that $U(Y) \subseteq U(X)$; a cycle in G' may contain both v and v', in which case it does not correspond to a cycle in G.

Proposition 5.13. *If Y is a shift space obtained from a state splitting of X, then $q_n(X) = q_n(Y)$ for all $n \geq 2$.*

Proof. Suppose $X = X(G, \mathcal{S})$ and $Y = X(G', \mathcal{S}')$. There is a natural bijection between closed walks in G and G'. Since any periodic point with least period 2 or greater corresponds to a closed walk, this bijection lifts to a bijection between periodic points. $\square$

5.3 Vertex cloning

Whereas state splitting partitions edges, vertex cloning partitions sets.

Definition 5.14. Let $(G, \mathcal{S})$ be a set-equipped graph, v be a vertex of G, and suppose that $S_v = S_1 \sqcup S_2$. The *vertex cloning* of G at v with respect to the decomposition $S_1 \sqcup S_2$ the set-equipped graph $(G', \mathcal{S}')$ defined as follows: The vertex set of G' is $V(G) \cup \{v'\}$, and $(u_1, u_2) \in E(G')$ if and only if one of the following is true:

1. $u_1, u_2 \notin \{v, v'\}$ and $(u_1, u_2) \in E(G)$,
2. $u_1 \in \{v, v'\}$ and $(v, u_2) \in E(G)$, or
3. $u_2 \in \{v, v'\}$ and $(u_1, v) \in E(G)$.

Finally, $S'_u = S_u$ if $u \notin \{v, v'\}$, $S'_v = S_1$, and $S'_{v'} = S_2$.

Example 5.15. The graph on the right is a vertex cloning of the graph on the left.

Proposition 5.16. *Vertex cloning preserves entropy and the mixing property.*

Proof. Suppose that $Y = X(\mathcal{G}')$ is obtained from the shift space $X = X(\mathcal{G})$ by cloning vertex v. We can take $B_{\mathcal{G}}(x)$ and $B_{\mathcal{G}'}(x)$ to be A and A' in Lemma 5.10. Therefore their spectral radii are equal for every x, so Theorem 4.11 implies that $h(X) = h(Y)$.

Let $U(X) = \left\{\sum_{i \in C} s_i : C \text{ is a cycle in } G \text{ and } s_i \in S_i\right\}$. As in the proof of Lemma 5.10, $U(X) \subseteq U(Y)$. By Proposition 3.4, if X is mixing, then Y is, as well. $\square$

As in the previous two sections, vertex cloning preserves most periodic points.

Proposition 5.17. *If Y is obtained by a vertex cloning of X, then $q_n(X) = q_n(Y)$ for all $n \geq 2$.*

Vertex cloning is not a conjugacy, as it does not in general preserve almost specification, shifts of finite type, or sofic shifts. In one special case, however, it is a conjugacy.

Proposition 5.18. *Let $(G, \mathcal{S})$ be a set-equipped graph and $v \in V(G)$. If $S_v = S_1 \sqcup S_2$ and either S_1 or S_2 is finite, then the shift space obtained by cloning vertex v is conjugate to $X(G, \mathcal{S})$.*

Proof. Let $X = X(G, \mathcal{S})$ and Y be the shift space obtained by vertex cloning at v with respect to the partition $S_1 \sqcup S_2$. By symmetry, we assume that S_1 is finite and let N be its maximum. Let $\Phi \colon \mathcal{B}_{2N+1}(X) \to V(G')$ be defined by

$$\Phi(x_{-N} \cdots x_N) = \begin{cases} x_0 & \text{if } x_0 \neq v \\ v & \text{if } x_{-s}, \ldots, x_t = v,\ x_{-s-1} \neq v,\ x_{t+1} \neq v, \text{ and } t+s+1 \in S_1 \\ v' & \text{if } x_{-s}, \ldots, x_t = v,\ x_{-s-1} \neq v,\ x_{t+1} \neq v, \text{ and } t+s+1 \in S_2. \end{cases}$$

The map Φ induces a sliding block code $\phi\colon X \to Y$, whose inverse is induced by the map $\Psi\colon \mathcal{B}_1(Y) \to V(G)$ defined by

$$\Psi(y_0) = \begin{cases} y_0 & \text{if } y_0 \neq v' \\ v & \text{if } y_0 = v'. \end{cases}$$

So $X \cong Y$. □

5.4 Lifting $\mathcal{S}$-graph shifts

In this section we characterize the $\mathcal{S}$-graph shifts with entropy 0 and use this to describe the circumstances in which an $\mathcal{S}$-graph shift is conjugate to another $\mathcal{S}$-graph shift on larger number of vertices. The first result states that any $\mathcal{S}$-graph shift with zero entropy consists of finitely many periodic points and their iterates under the shift map.

Proposition 5.19. *An irreducible $\mathcal{S}$-graph shift $X(G, \mathcal{S})$ has entropy 0 if and only if G is a directed cycle and each set in $\mathcal{S}$ is a singleton.*

Proof. If G is not a directed cycle, then some vertex v of G has outdegree at least 2. Set

$$N = \max\Big\{ \sum_{i\in c} \min S_i : c \text{ is a cycle that includes } v \Big\}.$$

We can build words in $\mathcal{L}(X)$ by beginning a walk at v, choosing an edge on which to leave, circling back to v, and repeating. Since returning to v takes at most N letters, there are at least $2^{\lceil n/N \rceil}$ walks of length n in G. So $h(X) \geq (\log 2)/N$.

Now suppose that G is a directed cycle. If some set S_v is not a singleton, then choose some nonempty sets $A, B \subseteq \mathbb{N}$ such that $S_v = A \sqcup B$. By Proposition 5.16, the vertex cloning of X at v preserves entropy. Since the resulting graph is not a directed cycle, $h(X) > 0$.

Finally, suppose that G is a directed cycle and each set in $\mathcal{S}$ is a singleton. Suppose $S_i = \{s_i\}$ set $s = \sum_{i=1}^{p} s_i$. The only positive solution to $x^s = 1$ is $x = 1$, so the entropy of X is $\log 1^{-1} = 0$ by Theorem 4.13. □

Proposition 5.20. *Let p and q be positive integers with $q > p$ and $(G, \mathcal{S})$ be an irreducible set-equipped graph on p vertices. If G is a directed cycle, every S_i is a singleton, and the sum of all elements of the S_i is less than q, then there is no $\mathcal{S}$-graph shift on q letters conjugate to $X(G, \mathcal{S})$. In every other case, there is such an $\mathcal{S}$-graph shift*

Proof. First suppose S_v is infinite for some $v \in V(G)$. Cloning this vertex using a partition that has one finite set is a conjugacy by Proposition 5.18. This leaves at least one vertex whose set is infinite, so we can repeat the cloning step until the resulting graph has q vertices.

Otherwise each set in $\mathcal{S}$ is finite. If G is not a directed cycle, then there is a vertex v that has at least two out-neighbors. State splitting at v is a conjugacy by Proposition 5.12, and any in-neighbor of v has out-degree at least two in the resulting graph. So state splitting can be repeated until the graph has q vertices.

Now suppose that G is a directed cycle. If any set S_v is not a singleton, then cloning it using a partition where one set is finite returns us to the previous case.

Thus, we can assume that each set S_i is a singleton. Suppose $S_i = \{s_i\}$ and $s = \sum_{i=1}^{p} s_i$. If $s_v > 1$, then an edge extension at v according to the decomposition $S_v = \{1\} \oplus \{s_v - 1\}$ is a conjugacy by Proposition 5.5. We can continue extending edges until all sets are $\{1\}$. Thus, if $s \geq q$, we can obtain a conjugate shift on q vertices.

Finally, we suppose that $s < q$. From Proposition 5.19, the entropy of X is 0, and any shift on q letters with entropy 0 must be a directed cycle on q vertices. But any such shift has no words of least period s, since $s < q$, whereas X has exactly one word of least period s. Therefore X is not conjugate to any $\mathcal{S}$-graph shift on q letters. $\square$

We call an $\mathcal{S}$-graph shift *trivial* if it has entropy 0. We can restate Proposition 5.20 as follows.

Corollary 5.21. *Let X be a nontrivial irreducible $\mathcal{S}$-graph shift on p letters. For every $q > p$, there is an $\mathcal{S}$-graph shift on q letters that is conjugate to X.*

6 Shifts with a given entropy

In this section, we answer the question: Does there exist an $\mathcal{S}$-graph shift with entropy h for every $h \geq 0$? Proposition 6.2 answers this question in the affirmative, and Theorem 6.6 strengthens this statement considerably.

Definition 6.1. Let $\beta > 1$ be a real number. A **β-expansion** for the real number x is a sequence $(x_n)_{n=1}^{\infty}$ such that each $x_n \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$ and

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \cdots .$$

In [11], Rényi proved that every real number in the interval $[0, (\lceil \beta \rceil - 1)/(\beta - 1)]$ has at least one β-expansion. (This can also be shown using a greedy algorithm.)

Proposition 6.2. *For every real number $\lambda > 1$, there exists an $\mathcal{S}$-graph shift with entropy* $\log \lambda$.

Proof. Let $(x_i)_{i=1}^{\infty}$ be a λ-expansion of 1 and set $n = \lceil \lambda \rceil - 1$. For each $1 \leq k \leq n$, we define the set

$$T_k = \{m \in \mathbb{N} : x_m \geq k\}.$$

Let G be the complete bipartite graph $K_{n,n}$ with vertex bipartition $[\{1, 2, \ldots, n\}, \{n+1, \ldots, 2n\}]$. For each $1 \leq k \leq n$ we define

$$S_k = S_{k+n} = T_k$$

and set $\mathcal{S} = (S_i)_{i=1}^{2n}$. We claim that $X(G, \mathcal{S})$ has entropy $\log \lambda$.

By Theorem 4.11, it suffices to show that the maximal eigenvalue of $B_{\mathcal{G}}(\lambda^{-1})$ is 1. By repeated application of Lemma 5.10, we see that the spectral radii of

$$B_{\mathcal{G}}(x) = \begin{pmatrix} H_1(x) & & \\ & \ddots & \\ & & H_{2n}(x) \end{pmatrix} \begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ 1_{n\times n} & 0_{n\times n} \end{pmatrix}$$

and

$$\begin{pmatrix} \sum_{i=1}^{n} H_i(x) & 0 \\ 0 & \sum_{i=n+1}^{2n} H_i(x) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are equal. Since $H_{k+n}(x) = H_k(x)$ for each $k \in \{1, 2, \ldots, n\}$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ has spectral radius 1, the spectral radius of of $B(\lambda^{-1})$ is

$$\sum_{i=1}^{n} H_i(\lambda^{-1}) = \sum_{i=1}^{\infty} \frac{x_i}{\lambda^i} = 1. \qquad \square$$

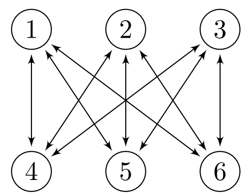
Example 6.3. Let $\lambda = 2 + \sqrt{3}$, and suppose that we want to construct an $\mathcal{S}$-graph shift with entropy $\log \lambda$. We first note that $\lceil \lambda \rceil - 1 = 3$. One λ-expansion of 1 is given by

$$x_m = \begin{cases} 3 & \text{if } i = 1 \\ 2 & \text{if } i > 1. \end{cases}$$

The corresponding sets are

$$T_1 = \mathbb{N}, \qquad T_2 = \mathbb{N}, \qquad \text{and} \qquad T_3 = \{1\}.$$

We set G to be the graph



and define $\mathcal{S} = (S_i)_{i=1}^6$ by $S_1 = S_2 = S_4 = S_5 = \mathbb{N}$ and $S_3 = S_6 = \{1\}$. Then $h(X(G, \mathcal{S})) = \log \lambda$.

We will use two lemmas—one on β-expansions and one on general subshifts—to strengthen Proposition 6.2.

Lemma 6.4. *For every $\lambda > 1$, there is a λ-expansion of 1 that contains a pair of consecutive nonzero digits and in which eventually every other digit is $\lceil \lambda \rceil - 1$.*

Proof. Let $n = \lceil \lambda \rceil - 1$. We want to find a sequence $(x_i)_{i=1}^\infty$ such that

$$1 = \sum_{i=1}^{k} \frac{x_i}{\lambda^i} + \frac{n}{\lambda^{k+1}} + \frac{x_{k+1}}{\lambda^{k+2}} + \frac{n}{\lambda^{k+3}} + \frac{x_{k+2}}{\lambda^{k+4}} + \cdots \tag{6.1}$$

for some $k \in \mathbb{N}$. The sum

$$\frac{n}{\lambda^{k+1}} + \frac{n}{\lambda^{k+3}} + \frac{n}{\lambda^{k+5}} + \cdots = \frac{n/\lambda^{k-1}}{\lambda^2 - 1}$$

is less than 1 if k is large enough. Fix such a k. We compute values of (x_i) according to a greedy algorithm: After choosing values for $x_1, \ldots, x_{m-1}$, we choose x_m to be the largest value in $\{0, 1, \ldots, n\}$ such that, if $x_i = 0$ for every $i > m$, the right-hand side of (6.1) is at most 1. The resulting λ-expansion represents a real number x, which is at most 1 by construction. Since we chose x_i to be maximal at each step, we know that

$$x + \frac{1}{\lambda^{k+2i}} > 1$$

whenever $x_{k+i} < n$. If there are infinitely many values of i for which that inequality holds, then taking the limit as $i \to \infty$ shows that $x \geq 1$, in which case (x_i) is a λ-expansion of 1.

Now suppose there are finitely many i such that $x_i < n$, and let x_m be the last such term. The value of the tail after this point is

$$\sum_{i=m+1}^{\infty} \frac{n}{\lambda^i} = \frac{n}{\lambda^m(\lambda - 1)} \geq \frac{1}{\lambda^m},$$

contradicting the fact that x_m is maximal. This means that $x_i = n$ for every $i \in \mathbb{N}$, so

$$1 \geq \sum_{i=1}^{\infty} \frac{n}{\lambda^i} = \frac{n}{\lambda - 1}$$

by construction. On the other hand, $\lambda - 1 \leq n$, so $n/(\lambda - 1) \geq 1$. Therefore (x_i) is a λ-expansion of 1.

Since we chose (x_i) by the greedy algorithm, we know that $x_1 > 0$. If (x_i) does not contain two consecutive nonzero integers, then define

$$y_i = \begin{cases} x_1 - 1 & \text{if } i = 1 \\ x_i & \text{if } i > 1 \text{ and } x_i > 0 \\ x_{i-1} & \text{if } i > 1 \text{ and } x_i = 0. \end{cases}$$

Otherwise, set $y_i = x_i$ for all $i \in \mathbb{N}$. In either case, the sequence $(y_i)_{i=1}^{\infty}$ is a λ-expansion of 1 that satisfies the conditions. □

Lemma 6.5. *Let X be a shift space on a finite alphabet. There are only countably many shift spaces on the same alphabet that are conjugate to X.*

Proof. Suppose the alphabet of X is $\mathcal{A}$. Each conjugate of X is determined by a sliding block code, which is itself induced by a map $\mathcal{A}^n \to \mathcal{A}$, and there are only countably many such maps. □

Theorem 6.6. *Let $\lambda > 1$. For each $p \geq 2\lceil\lambda\rceil - 1$, there are uncountably many pairwise non-conjugate $\mathcal{S}$-graph shifts on p vertices that have the specification property and entropy $\log\lambda$.*

Proof. Let $n = \lceil\lambda\rceil - 1$. By Lemma 6.4, there is a λ-expansion $(x_i)_{i=1}^{\infty}$ of 1 that contains a pair of consecutive nonzero integers and in which eventually every other digit is n. Let $X = X(G, \mathcal{S})$ be the $\mathcal{S}$-graph shift constructed in the proof of Proposition 6.2 that corresponds to the λ-expansion $(x_i)_{i=1}^{\infty}$. The entropy of X is $\log\lambda$, and G has $2n$ vertices.

Since eventually every other term of (x_i) is n, it follows that eventually every other positive integer is in S_i. Therefore $\sup \Delta(S_i) < \infty$; by Proposition 3.6, X is almost specified. If x_m and x_{m+1} are a pair of consecutive nonzero terms of (x_i), then m and $m + 1$ are both elements of $S_1 = S_{n+1}$. The vertices 1 and $n + 1$ comprise a length 2 cycle in G, so

$$2m,\, 2m + 1 \in \Big\{ \sum_{i \in C} s_i \,:\, C \text{ is a cycle in } G \text{ and } s_i \in S_i \Big\}.$$

Therefore X is mixing by Proposition 3.4, and X has the specification property by Proposition 3.7.

Suppose $\max \Delta(S_1) = N$. There are uncountably many ways to choose infinite sets $A, B \subseteq \mathbb{N}$ such that $S_1 = A \sqcup B$ and both $\Delta(A)$ and $\Delta(B)$ are bounded by $2N$. For any such choice, vertex cloning at vertex 1 with respect to the decomposition $S_1 = A \sqcup B$ produces an $\mathcal{S}$-graph shift on $2\lceil\lambda\rceil - 1$ vertices which is mixing and has entropy $\log\lambda$ by Proposition 5.16. Further, since $\Delta(A)$ and $\Delta(B)$ are bounded by $2N$, the shift space has the almost specification property by Proposition 3.6 and the specification property by Proposition 3.7. Since there are uncountably many partitions $A \sqcup B$, Lemma 6.5 guarantees uncountably many pairwise non-conjugate $\mathcal{S}$-graph shifts on $2\lceil\lambda\rceil - 1$ vertices with entropy $\log\lambda$ and the specification property.

If $p > 2\lceil\lambda\rceil - 1$, then vertex clone as many times as necessary to acquire p vertices, maintaining the specification property and entropy at each step. □

Since the entropy of an $\mathcal{S}$-graph shift on p vertices is at most $\log p$, the number $2\lceil\lambda\rceil - 1$ in Theorem 6.6 can be improved by at most a factor of 2.

7 A more general $\mathcal{S}$-graph shift

We can generalize $\mathcal{S}$-graph shifts in an analogous manner to way in which sofic shifts generalize vertex shifts.

Definition 7.1. A *vertex-labelled graph* (or *labelled graph*) on an alphabet $\mathcal{A}$ is a graph G together with a map $\ell \colon V(G) \to \mathcal{A}$. We call $\ell(v)$ the *label* of v. A *set-equipped labelled graph* is a triple $(G, \mathcal{S}, \ell)$ where $(G, \mathcal{S})$ is a set-equipped graph and (G, ℓ) is a vertex-labelled graph.

Definition 7.2. Let $(G, \mathcal{S}, \ell)$ be a set-equipped labelled graph over a finite alphabet $\mathcal{A}$. The (*labelled*) *$\mathcal{S}$-graph shift* $X(G, \mathcal{S}, \ell)$ is the closure of

$$\{\cdots \ell(v_{i-1})^{k_{i-1}} \ell(v_i)^{k_i} \ell(v_{i+1})^{k_{i+1}} \cdots : k_i \in S_{v_i} \text{ and } (v_i, v_{i+1}) \in E(G) \text{ for every } i \in \mathbb{Z}\}.$$

So $X(G, \mathcal{S}) = X(G, \mathcal{S}, \ell)$ when ℓ is the identity map on $V(G)$. We call an $\mathcal{S}$-graph shift *primitive* if ℓ is the identity.

Definition 7.3. Let $\mathcal{G} = (G, \mathcal{S}, \ell)$ be a set-equipped labelled graph. We say that $\mathcal{G}$ is *letter-separated* if $\ell(u) \neq \ell(v)$ for every $(u, v) \in E(\mathcal{G})$. We say that $\mathcal{G}$ is *right-resolving* if for every vertex $u \in V(G)$ and letter $a \in \mathcal{A}$, the sets in $\{S_v : v \in \Gamma_G^+(u) \text{ and } \ell(v) = a\}$ are pairwise disjoint.

The proof of Theorem 4.11 works as well for any right-resolving, letter-separated $\mathcal{S}$-graph shift as it does for a primitive one, so we get the following result for free.

Theorem 7.4. *Let $\mathcal{G}$ be a set-equipped labelled graph that is both right-resolving and letter-separated, and let $\rho(x)$ denote the spectral radius of $B_{\mathcal{G}}(x)$. The equation $\rho(x) = 1$ has a unique positive solution $x = \lambda$, and the entropy of $X(\mathcal{G})$ is $\log \lambda^{-1}$.*

The entropy of any shift space with a finite number of points is 0. The key result of this section is that the entropy formula in Theorem 7.4 can be applied to any other $\mathcal{S}$-graph shift, provided we first find a suitable presentation.

Theorem 7.5. *For every infinite $\mathcal{S}$-graph shift X there exists a set-equipped labelled graph $(G, \mathcal{S}, \ell)$ that is letter-separated and right-resolving such that $X = X(G, \mathcal{S}, \ell)$.*

The proof of Theorem 7.5 comprises the remainder of this section.

7.1 Proof of Theorem 7.5

Let $\mathcal{G} = (G, \mathcal{S}, \ell)$ be a set-equipped labelled graph such that $X = X(\mathcal{G})$. Since X is not finite, there is a directed cycle in G with at least two distinct labels. We proceed in two stages, first constructing a presentation for X that is letter-separated and, from that, constructing a letter-separated right-resolving presentation.

7.1.1 Letter separation

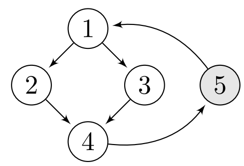
For each $a \in \mathcal{A}$, let V_a denote the set of vertices with label a. For each $u, v \in V(G)$ with $\ell(u) = \ell(v)$, let $\mathbf{W}(u, v)$ denote the set of directed walks from u to v that only use vertices in $V_{\ell(u)}$.

We define a new graph $\mathcal{G}^1 = (G^1, \mathcal{S}^1, \ell^1)$ on the vertex set $\{(u, v) : \ell(u) = \ell(v) \text{ and } \mathbf{W}(u, v) \neq \emptyset\}$. The labelling is given by $\ell^1(u, v) = \ell(u)$, and G^1 contains an edge directed from (u, v) to (x, y) if and only if $\ell(v) \neq \ell(x)$ and $(v, x) \in E(G)$. Finally, the set associated with $(u, v) \in V(G^1)$ is

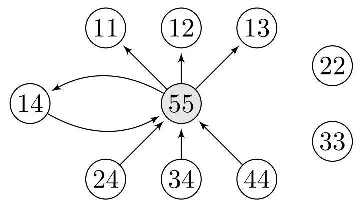
$$S^1_{(u,v)} = \bigcup_{W \in \mathbf{W}(u,v)} \sum_{w \in W} S_w,$$

where the sum is taken over the vertices $w \in W$ with repetition. In essence, G^1 compresses walks along vertices with the same label into a single vertex.

Example 7.6. Suppose $\mathcal{A} = \{0, 1\}$ and G is given by



where $\ell(5) = 1$ and all other labels are 0. The graph G^1 is



where we replace (u, v) by uv for brevity. So the underlying essential graph is a 2-cycle between the vertices $(1, 4)$ and $(5, 5)$. The associated sets are $S^1_{(1,4)} = (S_1 + S_2 + S_4) \cup (S_1 + S_3 + S_4)$ and $S^1_{(5,5)} = S_5$.

Claim 7.7. *The graph $\mathcal{G}^1$ is letter-separated and $X(\mathcal{G}^1) = X(\mathcal{G})$.*

Proof. If there is an edge from (u, v) to (x, y) in G^1, then $\ell^1(u, v) = \ell(v) \neq \ell(x) = \ell^1(x, y)$; so $X(\mathcal{G}^1)$ is letter-separated. We show that $X(\mathcal{G}^1) = X(\mathcal{G})$ by proving that their languages coincide. Let ω be a word in $L(X(\mathcal{G}))$ comprised of full blocks; that is, $\omega = \ell(v_1)^{k_1} \cdots \ell(v_n)^{k_n}$ for some walk $W = (v_1, v_2, \ldots, v_n)$ in G with $k_i \in S_{v_i}$. We denote by $v_{m_1}, v_{m_2}, \ldots, v_{m_r}$ the set of vertices in W such that $\ell(v_{m_i-1}) \neq \ell(v_{m_i})$ and set $m_0 = 1$ and $m_{r+1} = n + 1$. For each $i \in \{0, 1, 2, \ldots, r\}$, the walk $(v_{m_i}, v_{m_i+1}, \ldots, v_{m_{i+1}-1})$ is a maximal mono-labelled subwalk of W. Therefore

$$\omega = \ell(v_1)^{k_1+\cdots+k_{m_1-1}} \ell(v_{m_1})^{k_{m_1}+\cdots+k_{m_2-1}} \cdots \ell(v_{m_r})^{k_{m_r}+\cdots+k_n}.$$

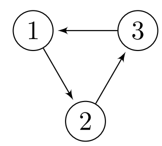
That $\sum_{j=m_i}^{m_{i+1}-1} k_j \in S_{(v_{m_i}, v_{m_{i+1}-1})}$ follows from the fact that $(v_{m_i}, v_{m_i+1}, \ldots, v_{m_{i+1}-1})$ is a member of $W(v_{m_i}, v_{m_{i+1}-1})$. Moreover, $(v_1, v_{m_1-1}), (v_{m_1}, v_{m_2-1}), \ldots, (v_{m_r}, v_n)$ is a walk in G^1, so $\omega \in \mathcal{L}(X(\mathcal{G}^1))$. Since every word in $\mathcal{L}(X(\mathcal{G}))$ or $\mathcal{L}(X(\mathcal{G}^1))$ is a subword of of a word composed of full blocks, this shows that $\mathcal{L}(X(\mathcal{G})) \subseteq \mathcal{L}(X(\mathcal{G}^1))$. A similar argument in the other direction proves the reverse containment. The languages are equal, so $X(\mathcal{G}) = X(\mathcal{G}^1)$ by Fact 2.2. ∎

7.1.2 Set division

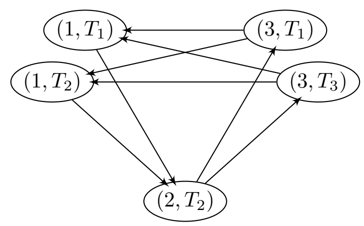
We let $\mathcal{M}$ denote the finite collection of minimal nonempty elements of the Boolean lattice generated by $\mathcal{S}^1$. Each set $S_v \in \mathcal{S}^1$ can be uniquely expressed as the union of the elements of a subset $\mathcal{T}_v \subseteq \mathcal{M}$.

We define a new graph $\mathcal{G}^2 = (G^2, \mathcal{S}^2, \ell^2)$ as follows. The vertex set of G^2 is $\bigcup_{v \in V(G^1)} \{v\} \times \mathcal{T}_v$. We assign the label $\ell^2(v, T) = \ell^1(v)$, and G^2 contains an edge from (u, T) to (v, R) if and only if $(u, v) \in E(G)$. Finally, we set $S^2_{(v,T)} = T$. In short, $\mathcal{G}^2$ is formed by replacing each vertex $v \in V(G)$ with a collection of independent vertices and distributing the set S_v amongst them.

Example 7.8. Suppose G^1 is the triangle graph



with associated sets $S_1 = \{1, 2, 3\}$, $S_2 = \{2, 3\}$, and $S_3 = \{1, 4\}$. The minimal nonempty sets of the Boolean lattice generated by S_1, S_2, and S_3 are $T_1 = \{1\}$, $T_2 = \{2, 3\}$, and $T_3 = \{4\}$. So G^2 is the graph



Claim 7.9. *The graph $\mathcal{G}^2$ is letter-separated, and $X(\mathcal{G}^2) = X(\mathcal{G}^1)$.*

Proof. Let $\omega = \ell^1(v_1)^{k_1} \cdots \ell^1(v_n)^{k_n}$ be a word in $\mathcal{L}(X(\mathcal{G}^1))$ composed of full blocks. For each v_i, let $u_i = (v_i, T)$ be the vertex in G^2 with $k_i \in T$, so that $\omega = \ell^2(u_1)^{k_1} \cdots \ell^2(u_n)^{k_n}$. The sequence $(u_1, \ldots, u_n)$ is a walk in G^2 since $(v_1, \ldots, v_n)$ is a walk in G^1, so ω is a word in $\mathcal{L}(X(\mathcal{G}^2))$. The reverse inclusion is similar, so $X(\mathcal{G}^1) = X(\mathcal{G}^2)$. If G^2 contains an edge from (u, T) to (v, R), then $(u, v) \in G^1$, and $\ell^2(u, T) = \ell^1(u) \neq \ell^1(v) = \ell^2(v, R)$. So $\mathcal{G}^2$ is letter-separated. ■

The resulting graph $\mathcal{G}^2$ is not necessarily right-resolving, however. The trouble arises when two out-neighbors of a vertex in $\mathcal{G}^1$ have the same label and intersecting sets. We address this problem next.

7.1.3 Right resolution

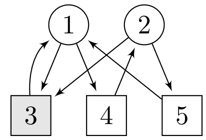
If $u, v \in V(G^2)$, then either $S_u = S_v$ or $S_u \cap S_v = \emptyset$. This is the crucial property for the final step in the construction.

We form one more graph $\mathcal{G}^3 = (G^3, \mathcal{S}^3, \ell^3)$ with $V(G^3) \subseteq \mathcal{P}(V(G^2))$ as follows. A nonempty set $I \in \mathcal{P}(V(G^2))$ is a vertex of G^3 if $\ell^2(u) = \ell^2(v)$ and $S_u = S_v$ for every $u, v \in I$. We set $\ell^3(I) = \ell^3(u)$ and $S^3_I = S_u$ for any $u \in I$. Finally, we describe the edges of G^3. For each vertex $I \in V(G^3)$, label $a \in \mathcal{A}$, and set $S \in \mathcal{S}^2$, we define

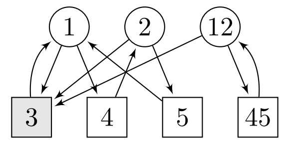
$$J = J(I, a, S) = \{v \in \Gamma^+_{G^2}(I) : \ell^2(v) = a \text{ and } S^2_v = S\}.$$

If J is nonempty, then (I, J) is an edge of G^3; otherwise it is not. (Any pair of subsets of $V(G^2)$ not arising in this manner is also not an edge in G^3.)

Example 7.10. Suppose the graph $\mathcal{G}^2$ is



where the shapes represent different labels and the shadings represent different associated sets. In this case, the graph $\mathcal{G}^3$ is



where sets $\{u, v\}$ are compressed to uv.

Claim 7.11. *The graph $\mathcal{G}^3$ is letter-separated and right-resolving, and $X(\mathcal{G}^3) = X(\mathcal{G})$.*

Proof. For each edge $(I, J) \in E(G^3)$, choose two vertices $u \in I$ and $v \in J$ with $(u, v) \in E(G^2)$. Since $\mathcal{G}^2$ is letter-separated, $\ell^3(I) = \ell^2(u) \neq \ell^2(v) = \ell^3(J)$, so $\mathcal{G}^3$ is letter-separated.

Fix a vertex $I \in \mathcal{G}^3$ and an integer $n \in \bigcup \mathcal{S}^3$; let $S \in \mathcal{S}^2$ be the set that contains n. If $J(I, a, S)$ is nonempty for a given a, then (I, J) is an edge in G^3; moreover, it is the only edge beginning at I whose other incident vertex is labelled a and has set S. If $J(I, a, S)$ is empty, then there is no such edge. So $(G^3, \mathcal{S}^3, \ell^3)$ is right-resolving.

We now prove that $X(\mathcal{G}^3) = X(\mathcal{G})$. By Claim 7.9, we need only check that $X(\mathcal{G}^3) = X(\mathcal{G}^2)$. Let $\omega = \ell^2(v_1)^{k_1} \cdots \ell^2(v_n)^{k_n}$ be a word in $\mathcal{L}(X(\mathcal{G}^2))$ composed of full blocks, so that $k_i \in S_{v_i}$ for each $n \in \{1, 2, \ldots, n\}$. We choose I_1 to be an essential vertex of G^3 that contains v_1 and define

$$\{I_i = u \in \Gamma^+_{G^2}(I_{i-1}) : \ell^2(u) = \ell(v_i) \text{ and } S^2_u = S^2_{v_i}\}$$

for each $i \in \{2, 3, \ldots, n\}$. The sequence $(I_1, \ldots, I_n)$ is a path in the underlying essential graph of G^3, and $k_i \in S^3_{I_i}$ for each i. So $\omega = \ell^3(I_1)^{k_1} \cdots \ell^3(I_n)^{k_n}$ is in $\mathcal{L}(X(\mathcal{G}^3))$.

Conversely, let $\tau = \ell^3(I_1)^{k_1} \cdots \ell^3(I_n)^{k_n}$ be a word in $\mathcal{L}(X(\mathcal{G}^3))$. By the definition of $\mathcal{G}^3$, there is a walk $(v_1, v_2, \ldots, v_n)$ in G^2 so that $v_i \in I_i$ for each i. Then $\ell^2(v_i) = \ell^3(I_i)$ and $S^2_{v_i} = S^3_{I_i}$, so $\tau = \ell^2(v_1)^{k_1} \cdots \ell^2(v_n)^{k_n}$ is in $\mathcal{L}(X(\mathcal{G}^2, Q(\mathcal{S})))$. Therefore $\mathcal{L}(X(\mathcal{G}^3)) = \mathcal{L}(X(\mathcal{G}^2))$. It follows that $X(\mathcal{G}^3) = X(\mathcal{G})$. ■

This proves Theorem 7.5. □

Remark. If G has n vertices, then G^1 has at most n^2 vertices, G^2 has at most $n^2 2^{n^2}$, and G^3 has at most $2^{n^2 2^{n^2}}$.

Although the proof of Theorem 4.11 works for any labelled $\mathcal{S}$-graph shift, the proofs in Section 3 are unable to handle multiple vertices with the same label.

Question 7.12. What conditions guarantee dynamical properties, such as the mixing property and specification, for labelled $\mathcal{S}$-graph shifts?

Acknowledgments

The author thanks Elizabeth Sattler for suggesting an investigation into the entropy of unordered $\mathcal{S}$-limited shifts, for her insightful advice, and for her many helpful suggestions.

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