Stochastic Diffusion

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November 6, 2014

1 Solving a 1D Diffusion Equation With Non-Constant Drift Velocity by Stochastic Sampling

In this problem we found the steady state solution to the following 1D diffusion equation:

$$\frac{\partial n(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 n(x,t)}{\partial x^2} + \frac{\partial [xn(x,t)]}{\partial x}$$

With initial condition:

$$n(x, t = 0) = \exp(-(x + 3)^2/2)$$

This makes the initial distribution a Gaussian centered at -3 with a σ of 1. To Stochastically solve this, I created 1000000 random walkers in an array with positions randomly chosen based on this normal distribution. After this array had been initialized, I iterated through each time step of .01 seconds for a total of 6 seconds. This allowed ample time for a stationary state to converge and a small enough resolution to make the approximation of constant velocity at each time step plausible. The following equation was used to step each walker at each time step:

$$r^{(i)}(t + \tau) = r^{(i)}(t) + v_0\tau + \vec{\eta}^{(i)}$$

Where τ is the time step, v_0 is the constant drift velocity, and η is a random vector whose components are chosen from a normal distribution centered at 0 with $\sigma^2 = 2\tau$

This equation is the solution for a constant v_0 but can be used for a non-constant v_0 when the time step is small enough. In this problem we were tasked with finding a non-constant velocity that resulted in a steady state solution as $t \to \infty$. The velocity I chose was one that was proportional to $-x$. This resulted in particles always tending to flow toward the origin, and hopefully to a steady state. The final equation used to step the particles in this simulation was:

$$r^{(i)}(t + \tau) = r^{(i)}(t) - x\tau + \vec{\eta}^{(i)}$$

After running the simulation, the solution did indeed find a steady state centered at the origin. I thought the final distribution might have a $\sigma < 1$, and it was about 0.7. This came from the balance of the random η function taking the walkers away from the origin and the non-constant drift velocity bringing them in.

The following two histograms show the initial distribution and the final steady state solution of this problem. The final solution is modeled by a Gaussian distribution with $\mu = -0.008$ and $\sigma = 0.71$

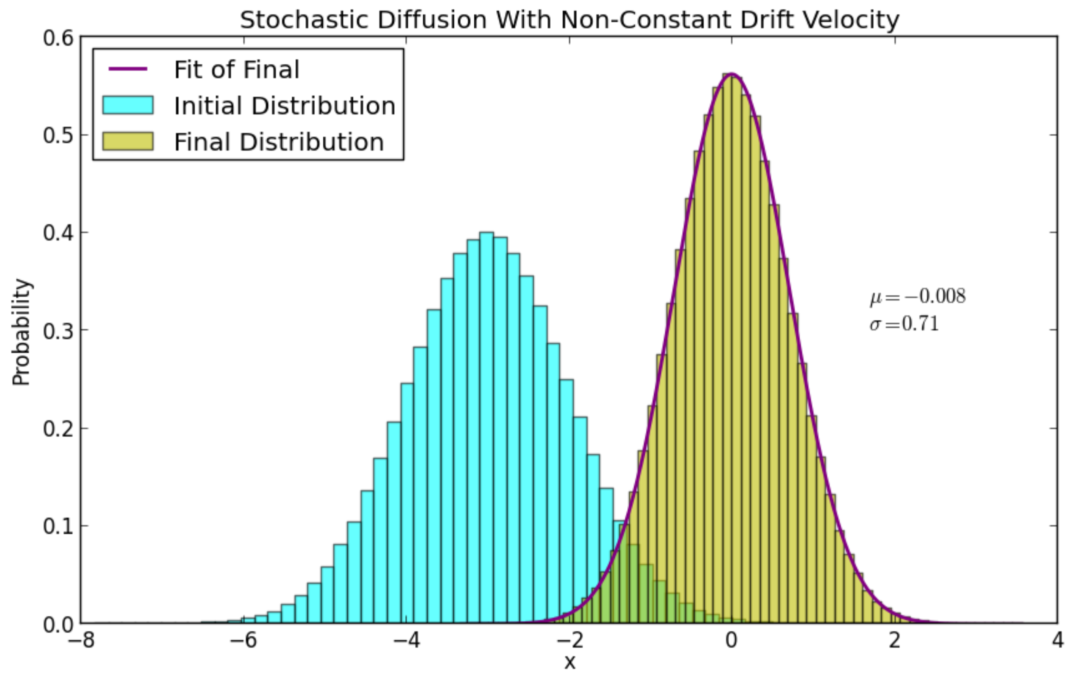

Figure 1: The initial and final distributions of 1000000 random walkers. The final distribution is centered at -0.008 with a σ of 0.71

A Problem Code

```
#!/usr/bin/python
import numpy
import math
import matplotlib.mlab as mlab
import matplotlib.pyplot as plt
from scipy.stats import norm

walkers = 1000000
time = 6.0
timeStep = .01
steps = int(round(time / timeStep))
particles = numpy.random.normal(-3, 1, walkers)
og = numpy.copy(particles)
for i in range(steps):
    particles[:] = particles[:] - particles[:] * timeStep + numpy.random.normal(0, math.sqrt(timeStep),

mu, sigma = norm.fit(particles)
print("mu = " + str(mu) + "\nsigma = " + str(sigma))

p1 = numpy.arange(min(particles), max(particles), 0.01)
plt.hist(og, 60, normed=1, facecolor='cyan', alpha=.6, label="Initial Distribution")
```

```
plt.hist(particles, 60, normed=1, facecolor='y', alpha=.6, label="Final Distribution")
plt.plot(p1, mlab.normpdf(p1, mu, sigma), 'purple', linewidth=2, label="Fit of Final")
plt.legend(loc=2)
plt.title("Stochastic Diffusion With Non-Constant Drift Velocity")
plt.xlabel("x")
plt.ylabel("Probability")
plt.figtext(0.75, 0.5, "$\mu = " + str(round(mu, 3)) + "$\n$\sigma = " + str(round(sigma, 3)) + "$", for

plt.show()
```