The RSA Algorithm

This Video Covers:

- **Modulo Operation (no video)**
- Euler's Theorem
- Extended Euclidean Algorithm
- RSA Algorithm
- Examples

Modulo Operation

- The RSA algorithm is based on **modulo operations**

modulus

$$\underbrace{a \bmod n}_{} = r$$

remainder/residue

Modulo Operation

- The RSA algorithm is based on **modulo operations**

$$a \bmod \underbrace{n}_{\text{modulus}} = \underbrace{r}_{\text{remainder/residue}}$$

- Examples:
  - $10 \ \mathtt{mod}\ 3 = ?$
  - $15 \ \mathtt{mod}\ 5 = ?$

Modulo Operation

- The RSA algorithm is based on **modulo operations**

modulus

remainder/residue

$$a \bmod n = r$$

- Examples:
  - $10 \;\texttt{mod}\; 3 = 1$
  - $15 \;\texttt{mod}\; 5 = 0$

Modulo Operation

- The RSA algorithm is based on **modulo operations**

modulus

remainder/residue

$$a \bmod n = r$$

- Examples:
  - $10 \mathtt{\ mod\ } 3 = 1$
  - $15 \mathtt{\ mod\ } 5 = 0$

- Modulo operations are ***distributive***:

$$(a + b) \bmod n = [(a \bmod n) + (b \bmod n)] \bmod n$$
$$a * b \bmod n = [(a \bmod n) * (b \bmod n)] \bmod n$$
$$a^x \bmod n = (a \bmod n)^x \bmod n$$

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Euler's Theorem → easily reduce large powers modulo n

- Euler's totient function ϕ(n) counts the positive integers up to a given integer n that are *relatively prime* to n
  - $\phi(n) = n - 1$, if n is a prime number.
- Euler's totient function property:
  - if m and n are relatively prime, $\phi(mn) = \phi(m) * \phi(n)$
- Euler's theorem states:
  - $a^{\phi(n)} = 1 \pmod{n}$

Euler's Theorem *(cont.)* → easily reduce large powers modulo n

- **Example:** Calculate $4^{100003} \bmod 33$

Euler's Theorem *(cont.)* → easily reduce large powers modulo n

- **Example:** Calculate $4^{100003} \bmod 33$

use $a^{\phi(n)} = 1 \bmod n$
to simplify

$a = 4$

$n = 33$

$\phi(n) = \phi(33) = \cdots$

Euler's Theorem *(cont.)* → **easily reduce large powers modulo n**

- **Example:** Calculate $4^{100003} \bmod 33$
  - $\phi(33) = \phi(3) * \phi(11) = (3 - 1) * (11 - 1) = 20$
  - $100003 = 5000\phi(33) + 3$

Euler's Theorem *(cont.)* → **easily reduce large powers modulo n**

- **Example:** Calculate 4^{100003} mod 33
  - $\phi(33) = \phi(3) * \phi(11) = (3 - 1) * (11 - 1) = 20$
  - $100003 = 5000\phi(33) + 3$

$$
\begin{aligned}
4^{100003} \bmod 33 &= 4^{20 \cdot 5000 + 3} \bmod 33 \\
&= \left(4^{20}\right)^{5000} * 4^3 \bmod 33 \\
&= \left[\left(\left(4^{20}\right)^{5000} \bmod 33\right)\right] * 4^3 \bmod 33 \text{ \textbf{(applying distributive rule)}} \\
&= \left[\left(4^{20} \bmod 33\right)\right]^{5000} * 4^3 \bmod 33 \text{ \textbf{(applying distributive rule)}} \\
&= 1^{5000} * 64 \bmod 33 \text{ \textbf{(applying Euler's theorem)}} \\
&= 31
\end{aligned}
$$

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RSA and the Extended Euclidean Algorithm

- **Euclid's algorithm:** an efficient method for computing GCD of two #'s
- **Extended Euclidean algorithm:**
  - computes GCD of integers a and b
  - finds integers x and y, such that: $ax + by = g = \gcd(a, b)$

RSA and the Extended Euclidean Algorithm

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- RSA uses Extended Euclidean algorithm:
  - e and n are components of public key
  - Find solution to equation:

    $e * x + \phi(n) * y = \gcd(e, \phi(n)) = 1$

```
def egcd(a, b):
    if a == 0:
        return (b, 0, 1)
    else:
        g, x, y = egcd(b % a, a)
        return (g, y - (b // a) * x, x)
```

RSA and the Extended Euclidean Algorithm

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- RSA uses Extended Euclidean algorithm:
  - e and n are components of public key
  - Find solution to equation:

    $e * x + \phi(n) * y = \gcd(e, \phi(n)) = 1$
  - x is private key *(also referred as d)*
  - Equation results: $e * d \bmod \phi(n) = 1$

```
def egcd(a, b):
    if a == 0:
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RSA: Key Generation

Key Generation *→ Encryption → Decryption*

- **Need to generate:** modulus n, public key exponent e, private key exponent d

- **Approach:**
  - Choose p, q → *large random prime numbers (secret!)*
  - $n = pq$ → *should be LARGE; computationally hard to factor n → Euler's Theorem*
  - Choose e, $1 < e < \varphi(n)$ and e is *relatively prime* to $\Phi(n)$

    → *e is the "public-key exponent" (e.g., e = 65537)*
  - Find d, $ed \bmod \varphi(n) = 1$

    → *solve using the Extended Euclidean Algorithm; d is the "private-key exponent" (secret!)*

    Can be solved in polynomial time if you know p, q, and e!

- **Result:**
  - (e, n) is public key → *without knowledge of p or q, computationally hard to find d*
  - d is private key

RSA: Encryption & Decryption

Key Generation → **Encryption** → **Decryption**

Encryption

- Treat the plaintext as a number
- Assuming $M < n$
- $C = M^e \bmod n$

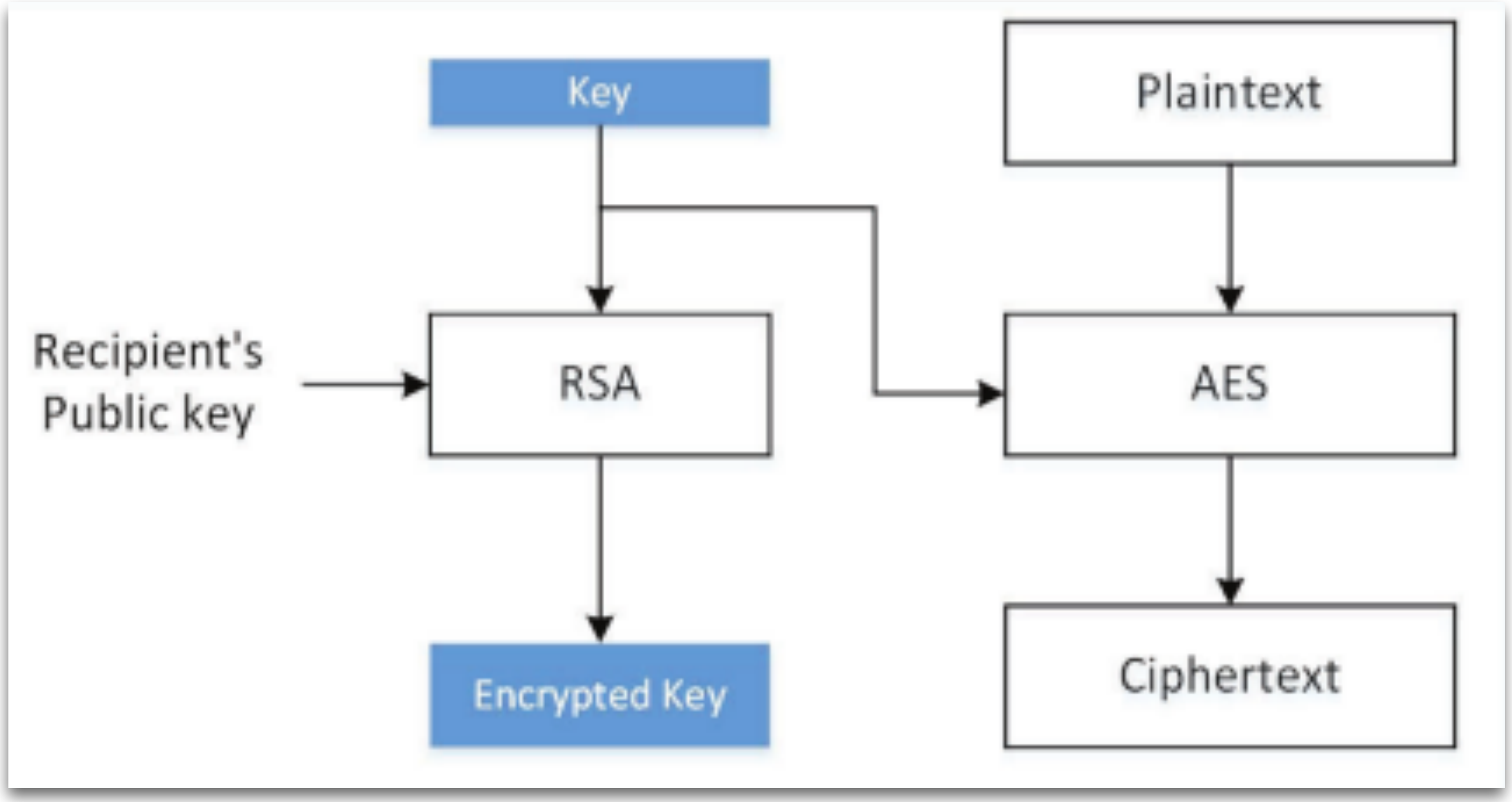
Decryption

- $M = C^d \bmod n$

You can convince yourself (see below) that decryption does indeed yield back the message, M…

$$
\begin{aligned}
M^{ed} \bmod n &= M^{k\phi(n)+1} \bmod n \quad \textbf{(note: } ed = k\phi(n) + 1\textbf{)} \\
&= M^{k\phi(n)} * M \quad \bmod n \\
&= \left(M^{\phi(n)} \bmod n\right)^k * M \bmod n \quad \textbf{(applying distributive rule)} \\
&= 1^k * M \bmod n \quad \textbf{(applying Euler's theorem)} \\
&= M
\end{aligned}
$$

Hybrid Encryption



- Public-key encryption is computationally expensive (e.g., large-number multiplications)
- Use public key algorithms to ***exchange a secret session key***
- The key (data-encryption key) used to encrypt data using a symmetric-key algorithm (e.g., AES-128-CBC)

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RSA: Exercise w/ Small Numbers

- Choose two prime numbers $p = 13$ and $q = 17$
- Find e:
  - $n = pq = 221$
  - $\varphi(n) = (p - 1)(q - 1) = 192$
  - choose $e = 7$ → *7 is relatively prime to* $\varphi(n)$
- Find $\varphi(n)$:
  - $ed \;= 1 \texttt{ mod } \varphi(n)$
- Solving the above equation is equivalent to: $7d + 192y = 1$
- Using Extended Euclidean algorithm, we get $d = 55$ and $y = -2$

RSA: Exercise w/ Small Numbers *(cont.)*

- Encrypt $M = 36$

$$
\begin{aligned}
M^e \bmod n &= 36^7 \bmod 221 \\
&= \left(36^2 \bmod 221\right)^3 * 36 \bmod 221 \\
&= 191^3 * 36 \ \mathbf{mod}\ 221 \\
&= 179 \ \mathbf{mod}\ 221.
\end{aligned}
$$

- Ciphertext $C = 179$

RSA: Exercise w/ Small Numbers *(cont.)*

$$
\begin{aligned}
C^d \textbf{ mod } n &= 179^{55} \textbf{ mod } 221 \\
&= \left(179^2 \textbf{ mod } 221\right)^{27} * 179 \textbf{ mod } 221 \\
&= 217^{27} * 179 \textbf{ mod } 221 \\
&= \left(217^2 \textbf{ mod } 221\right)^{13} * 217 * 179 \textbf{ mod } 221 \\
&= 16^{13} * 217 * 179 \textbf{ mod } 221 \\
&= \left(16^2 \textbf{ mod } 221\right)^{6} * 16 * 217 * 179 \textbf{ mod } 221 \\
&= 35^{6} * 16 * 217 * 179 \textbf{ mod } 221 \\
&= \left(35^2 \textbf{ mod } 221\right)^{3} * 16 * 217 * 179 \textbf{ mod } 221 \\
&= 120^{3} * 16 * 217 * 179 \textbf{ mod } 221 \\
&= \left(120^2 \textbf{ mod } 221\right) * 120 * 16 * 217 * 179 \textbf{ mod } 221 \\
&= 35 * 120 * 16 * 217 * 179 \textbf{ mod } 221 \\
&= 36 \textbf{ mod } 221
\end{aligned}
$$

RSA: Exercise w/ Large Numbers

Example w/ larger numbers discussed in the text

+

`rsa.c`