Fin 395 4 Lecture 2: Testing Consumption-Based Models

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Preference-based asset-pricing models

Preference-based dynamic asset-pricing models place restrictions on the SDF (agents' intertemporal marginal rate of substitution)

These restrictions potentially provide answers as to:

- What should be the riskless interest rate in equilibrium?
- What should be the risk premium earned by risky assets?
- How predictable should returns be?

Today:

- Methods to assess the empirical performance of such models
- Key findings and issues in this literature

HJ bounds - discussion

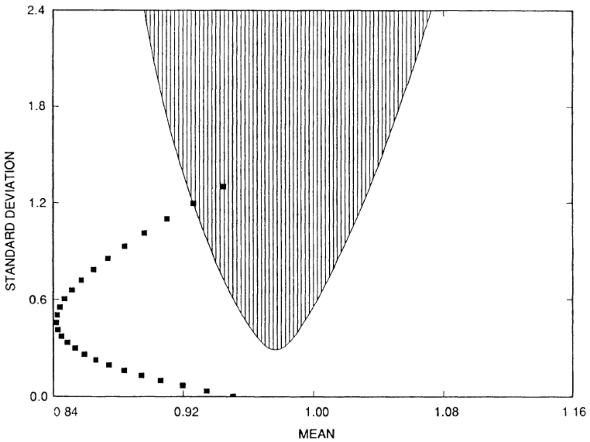


FIG. 1.—IMRS frontier computed using annual data

from HJ (1991), based on annual data on real stock and bond returns from 1891-1985, $\beta = 0.95$, and non-durables and services consumption data

Hansen-Jagannathan bounds

Hansen and Jagnnathan (1991) derive bounds on the volatility of the SDF that every candidate SDF must satisfy to be able to price a given set of assets

- SDF volatile $\Leftrightarrow$ marginal utility risky $\Leftrightarrow$ high max Sharpe Ratio
- So if we observe a SR it sets lower bound on SDF vol

From Lecture 1, using $t+1$ for the next period (we are going to think about many discrete periods), any SDF m_{t+1} satisfies:

$$\mathbb{E}(m_{t+1} R_{i,t+1}) = 1.$$

For any tradeable portfolio with return $R_{i,t+1} = w_i' R_{t+1}$, where R_{t+1} is an $N \times 1$ vector of non-redundant tradeable assets. There a unique SDF in the space of tradeable portfolios $m^*_{t+1} = w_m^{'} R_{t+1}$.

Hansen-Jagannathan bounds

Consider a regression of any SDF m_{t+1} on m^*_{t+1}:

$$m_{t+1} = a + b \cdot m^*_{t+1} + \epsilon_{t+1}.$$

Because m^*_{t+1} is in the space of traded portfolios, we have that:

$$\begin{aligned} \mathbb{E}(m_{t+1} m^*_{t+1}) &= 1, \\ \Rightarrow \mathbb{E}((a + b \cdot m^*_{t+1} + \epsilon_{t+1}) m^*_{t+1}) &= 1, \\ \Rightarrow a \cdot \mathbb{E}(m^*_{t+1}) + b \cdot \mathbb{E}(m^*_{t+1} m^*_{t+1}) &= 1. \end{aligned}$$

Since m^*_{t+1} prices itself and $\mathbb{E}(m_{t+1}) = m^*_{t+1}$, $a = 0$ and $b = 1$, which gives us:

$$\begin{aligned} m_{t+1} &= m^*_{t+1} + \epsilon_{t+1}, \\ \Rightarrow \text{Var}(m_{t+1}) &= \text{Var}(m^*_{t+1}) + \text{Var}(\epsilon_{t+1}) \geq \text{Var}(m^*_{t+1}), \end{aligned}$$

which is the HJ-bound on the variance of m_{t+1}

- m^*_{t+1} often referred to as the "minimum-variance pricing kernel"

HJ bounds w/o riskless payoff

Suppose the portfolio space does not contain a truly riskless payoff (realistic)

Let's augment the portfolio space with a hypothetical riskless asset, posit $R_f = \frac{1}{\nu}$ as the unconditional risk-free rate a candidate stochastic discount factor would assign to the hypothetical risk-less payoff, and include it in the portfolio space

Now the volatility bound becomes:

$$\mathsf{Var}(m_{t+1}) \geq \mathsf{Var}(m^*_{t+1}(\nu))$$

The bound tells us the lower bound for the volatility that a candidate SDF must satisfy, for a given mean of the SDF ν, to be able to correctly price the payoffs in the return space.

Calculation of HJ Bounds

A convenient way of calculating Var$(m^*_{t+1}(\nu))$ is to express it as a projection of any m_{t+1} onto the space onto the vector of non-redundant risky assets R_{t+1} and a constant, and absorb the mean of m_{t+1} ($\mathbb{E}[m_{t+1}] = \nu$) in the intercept:

$$\begin{aligned}
m^*_{t+1}(\nu) &= \nu + \beta_{m,R}(\nu)'(R_{t+1} - \mathbb{E}[R_{t+1}]),\\
\text{where } \beta_{m,R}(\nu) &= \Sigma^{-1}\mathbb{E}[(m_{t+1} - \nu)(R_{t+1} - \mathbb{E}[R_{t+1}])]\\
&= \Sigma^{-1}\mathbb{E}[m_{t+1}(R_{t+1} - \mathbb{E}[R_{t+1}])] = \Sigma^{-1}(1 - \nu\mathbb{E}[R_{t+1}]),\\
\text{and } \Sigma &= \mathbb{E}[(R_{t+1} - \mathbb{E}[R_{t+1}])(R_{t+1} - \mathbb{E}[R_{t+1}])']
\end{aligned}$$

It follows that:

$$\begin{aligned}
\text{Var}[m^*_{t+1}(\nu)] &= (1 - \nu\mathbb{E}[R_{t+1}])'\Sigma^{-1}(1 - \nu\mathbb{E}[R_{t+1}])\\
&= \left(\mathbb{E}[R_{t+1}] - \frac{1}{\nu}\right)'\Sigma^{-1}\left(\mathbb{E}[R_{t+1}] - \frac{1}{\nu}\right)\nu^2
\end{aligned}$$

HJ bounds and Sharpe Ratio bounds

Note also that, for a given ν,

$$\left(\mathbb{E}[R_{t+1}] - \frac{1}{\nu}\right)' \Sigma^{-1} \left(\mathbb{E}[R_{t+1}] - \frac{1}{\nu}\right)$$

is the max. squared Sharpe Ratio that can be obtained from the returns R_{t+1}, which is also equal to the Sharpe Ratio of tangency portfolio with returns $R_{\textsf{msr}}(\nu)$, and so:

$$\mathsf{Var}[R^*_{t+1}(\nu)] = \frac{(\mathbb{E}[R_{\textsf{msr}}(\nu)] - \frac{1}{\nu})^2}{\mathsf{Var}[R_{\textsf{msr}}(\nu)]}\nu^2$$

and the HJ bound becomes

$$\mathsf{Var}[m_{t+1}] \geq \frac{(\mathbb{E}[R_{\textsf{msr}}(\nu)] - \frac{1}{\nu})^2}{\mathsf{Var}[R_{\textsf{msr}}(\nu)]}\nu^2$$

HJ bounds and Sharpe Ratio bounds

Noting that $\mathbb{E}[m_{t+1}] = \mathbb{E}[M^*_{t+1}(\nu)] = \nu = R_f(\nu)^{-1}$ and taking square roots,

$$\frac{\sigma_m}{\mathbb{E}[m_{t+1}]} \geq \frac{\mathbb{E}[R_{\text{msr}}(\nu)] - R_F(\nu)}{\sigma_{\text{msr}}}$$

Hence, we get an intuitive relationship between the mean-variance frontier of stochastic discount factors (the HJ bound) and the mean-variance frontier for returns

- When a portfolio with high Sharpe Ratio can be constructed from returns, this means that there is a high risk premium earned by these assets per unit of risk $\Rightarrow$ the SDF that prices these returns must be highly volatile.
- **Economic interpretation**: When the risk premium is high, this means that SDF must assign very different prices to different states of the world (marginal utilities are very different in different states of the world), i.e., it must be very volatile

Comparison of HJ bounds with candidate pricing kernels

Example: Single representative investor with power utility

$$m_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

$$\gamma > 0$$

The HJ bounds are useful to assess whether a model like this might be able to explain the asset returns we see in the data.

Changing the relative risk aversion parameter γ has two effects

- Higher γ makes m_{t+1} more volatile
- Higher γ lowers (raises) $\mathbb{E}[m_{t+1}]$ for low (high) γ
  - HJ (1991) describes why (convexity of m)

Equity premium puzzle

- We need γ close to 30 so that the power utility SDF can simultaneously price stocks and bonds
  - Important to remember: it's just a bound, doesn't exhaust all the testable implications of the pricing restriction
  - Doesn't mean that power utility model with $\gamma = 30$ can actually price these assets
  - Doesn't take into account the correlation of m_{t+1} and x_{t+1}

- Our first glimpse of the **equity premium puzzle**: equity market returns much higher than predicted by reasonably parameterized CRRA models
  - Stated most forcefully by Mehra and Prescott (1985) (more on this in a bit)

Risk-free rate puzzle

HJ bounds also illustrate connection between equity premium puzzle and a related puzzle: risk-free rates are too low relative to average equity returns

- Moderately high values of γ imply low $\mathbb{E}[m_{t+1}]$, which implies a (too) high risk-free rate. For comparison w/ HJ Figure 1: historical short-term real interest rate is around 2% (annually), implying $\mathbb{E}[m_{t+1}]$ of around $\frac{1}{1.02} \approx 0.98$.
- In our earlier HJ bounds graph, $\gamma \approx 30$ "solves" both the equity premium and risk-free rate puzzles. However, this turns out to be a knife-edge result that is very sensitive to the data sample used to evaluate the bound. (**Homework 1**)
  - Moreover, $\gamma \approx 35$ doesn't square with experimental evidence on risk aversion, which says $\gamma \leq 10$ is plausible range

Euler-equation tests of preference-based models

- The HJ bound is just a bound - it is possible that a candidate SDF satisfies the bound, but yet it doesn't price the assets
- But the HJ bounds are a useful first check, rejecting many preference-based models on their own
- To go beyond the HJ bounds, we now use additional restrictions implied by asset-pricing theory about the joint distribution of asset returns and model-implied aggregates like consumption
  - Need to check pricing relation – do assets with high correlation to SDF have higher Sharpe Ratio – in addition to the volatility of the SDF
- We focus on conditional moment restrictions implied by first-order conditions of agents' inter temporal optimization problem (Euler equations). We can test these restrictions with GMM without making additional auxiliary assumptions about the distribution of asset returns and risk factors.

Euler-equation tests of preference-based models

Assume a representative agent with CRRA preferences, as in Hansen and Singleton (1982), with power utility who maximizes

$$\mathbb{E}\left[\sum_{j=0}^{\infty}\beta^j \frac{C^{1-\gamma}-1}{1-\gamma}\middle| I_t\right], \gamma > 0,$$

First-order conditions yield the Euler equations (with gross returns on N assets)

$$\mathbb{E}\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} R_{t+1} - 1 \middle| I_t\right] = 0$$

These conditional moment restrictions imply that $\forall\,(Q\times 1)\; z_t \in I_t$,

$$g(\beta,\gamma) \equiv E\left[\left(\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} R_{t+1} - 1\right)\otimes z_t\right] = 0$$

I.e., we have NQ population orthogonality conditions.

GMM estimation

To have at least exact identification of the parameter vector $\theta \equiv \begin{bmatrix} \beta & \gamma \end{bmatrix}'$, we need $NQ \geq 2$. The sample counterpart of the population orthogonality conditions is

$$g_T(\theta) \equiv \mathbb{E}_T \left[\left(\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} - 1 \right) \otimes z_t \right],$$

where $\mathbb{E}_T[\cdots] \equiv \frac{1}{T} \sum_{t=1}^{T} (\cdots)$. Under the null hypothesis that the model of asset prices is correctly specified, $g_T(\theta_T)$ should be close to zero in a sufficiently large sample as $\theta_T \to \theta$. HS estimate θ with GMM, which involves choosing θ_T to minimize

$$J_T(\theta) = g_T(\theta)' W_T g_T(\theta),$$

where W_T is a symmetric $NQ \times NQ$, positive definite matrix that can depend on sample information and prescribes how much weight we put on each orthogonality condition when minimizing J_T

GMM estimation

Consistency and asymptotic normality of the estimator θ_T : Under the null that the model is correctly specified (Hansen 1982),

$$
\begin{aligned}
\sqrt{T} g_T(\theta_T) &\overset{a}{\to} N(0, S), \\
S &\equiv \lim_{T\to\infty} \mathbb{E}_T[f(x_{t+1}, z_t, \theta_T) f(x_{t+1}, z_t, \theta_T)'] \\
f(x_{t+1}, z_t, \theta_T) &\equiv \left(\left(\beta \frac{C_{t+1}}{C_t} \right)^{-\gamma} R_{t+1} - 1 \right) \otimes z_t
\end{aligned}
$$

GMM estimation

Let $d' \equiv \frac{\partial g_T(\theta_T)'}{\partial \theta_T}$. Hansen (1982) shows that

$$\sqrt{T}(\theta_T - \theta) \sim N\left(0, (d'Wd)^{-1} d'WSWd(d'Wd)^{-1}\right),$$

with $W = S^{-1}$ this simplifies to

$$\sqrt{T}(\theta_T - \theta) \sim N\left(0, (d'S^{-1}d)^{-1}\right).$$

Hansen (1982) shows that (under the null) the choice $W = S^{-1}$ yields the asymptotically efficient estimator ("optimal GMM")

- In practice we can obtain a first-stage estimate with an essentially arbitrary W (often the identity matrix) and the obtain S_T from the first-stage estimates, and set $W = S_T^{-1}$ in the second stage
- Alternatively, one can iterate this until convergence (this does not make a difference for consistency and efficiency, but may work better in finite samples)

GMM goodness of fit tests / test of the over identifying restrictions

Let J_T be obtained with $W = S^{-1}$. Then, Hansen (1982) shows:

$$T \cdot J_T(\theta_T) \sim \chi^2_{NQ-K}$$

- Low J_T indicates parameters fit the moments well, high J_T poorly $\Rightarrow$ rejecting the null means model rejected (want low J_T, high p-value)
- Note that failure to reject can be a consequence of low power or estimation error, not a 'good' or 'correct' model

Hansen and Singleton (1982)

Estimates the representative agent/CRRA model using:

- x_t: vector of 2-3 different portfolios returns
- z_t: $t, t-1, \ldots$ values of returns and consumption growth
- Their coefficient α is the negative of the RRA coefficient
- Sample period 1959:2 - 1978:12, monthly
- Two-step optimal GMM estimator

This approach is useful in lots of settings because you do not need to solve the model in closed form, all you need are some first-order conditions

Hansen and Singleton (1982) (Erratum, 1984)

- Overidentifying restrictions reject the model at a high level of confidence (p-value is 1-Prob. in Table III).
  - This is a manifestation of the equity premium and risk-free rate puzzles
- Point estimates of α is sometimes positive, (implying convex utility) although not significantly so
- Results are sensitive to the assets included
- How sensitive are the results to the instrument set?
- Is the sample long enough to rely on asymptotics?
  - SE seem small relative to how much estimates depend on choices of returns/instruments

Econometric Issues

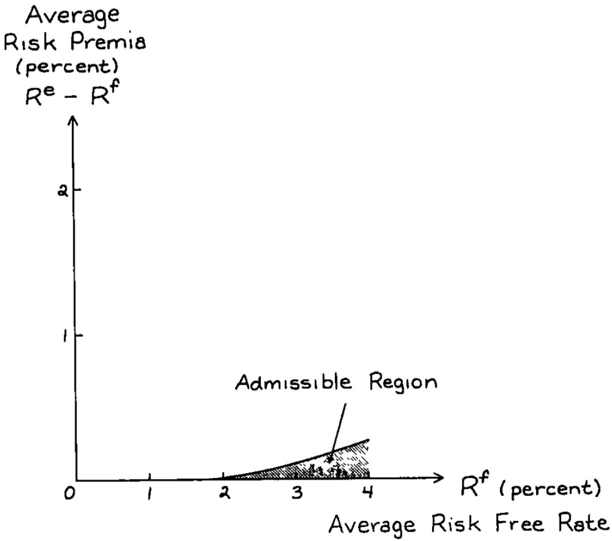
Finite sample performance of GMM (under the null of correct specification)

- Tauchen (1986), Hansen Heaton and Yaron (1996).

Misspecification

- Asymptotic theory is based under the null that the model is perfectly specified.
- Parameter estimates are interpreted as estimates of RRA, and more generally, as estimates of "deep" parameters of an underlying structural model, but this practice is questionable with a misspecified model that is rejected by the data.
- Hall and Inoue (2003) show how misspecification affects the limiting distribution of GMM estimators.

Mehra and Prescott (1985) discussion



Mehra and Prescott (1985) moments

Facts about moments, 1889-1978:

- Short-term real interest rate: 0.8%/year
- Excess stock return: 6.2%/year
- Annual growth rate of agg. per capita consumption: 1.83%
- Standard deviation of this growth rate: 3.57%

Can the CRRA/rep agent consumption-based utility SDF fit just these facts for reasonable risk aversion values?

Mehra and Prescott (1985) assumptions

Show that all calibrations with $\gamma < 10$ fail under assumptions:

- CRRA utility
- Consumption growth dynamics

$$\lambda_{t+1} \equiv \frac{C_{t+1}}{C_t}, \quad \lambda_t \ i.i.d., E[\lambda_t] = \bar{\lambda}, \mathrm{Var}(\lambda_t) = \sigma_\lambda^2$$

- Stochastic Discount Factor

$$m_{t+1} = \beta \lambda_{t+1}^{-\gamma}$$

- Implied risk-free rate

$$R_{f,t+1} = \frac{1}{\mathbb{E}_t\left(m_{t+1}\right)} = \frac{1}{\mathbb{E}(\beta \lambda_t^{-\gamma})}$$

- Stock market is a claim on aggregate consumption
  - Simplifies connection between cons. and stock moments

Mehra and Prescott (1985) results

These assumptions imply:

$$\mathbb{E}(R_{m,t+1}) - \mathbb{E}(R_{f,t+1}) = \frac{\overline{\lambda}}{\mathbb{E}\left[\beta\lambda_{t+1}^{1-\gamma}\right]} - \frac{1}{\mathbb{E}\left[\beta\lambda_{t+1}^{-\gamma}\right]}$$

$$\mathbb{E}(R_{f,t+1}) = \frac{1}{\mathbb{E}\left[\beta\lambda_{t+1}^{-\gamma}\right]}$$

Now, search over all

$$0 \leq \gamma \leq 10$$
$$0 \leq \beta \leq 1$$

to find range of unconditional risk premia and risk-free rates.

- Uses consumption data to compute moments of λ
- Uses stock data to compute 'target' moment $\mathbb{E}(R_{m,t+1} - R_{f,t+1})$

Mehra and Prescott (1985) conclusions

"With real per capita consumption growing at nearly two percent per year on average, the elasticities of substitution (TJ: $-\gamma$) between the year t and year $t+1$ consumption good that are sufficiently small to yield the six percent average equity premium also yield real rates of return far in excess of those observed."

- Similar to conclusion reached more generally in Hansen Jagannathan (1991)
- Started a still-ongoing trend of matching moments rather than estimating models
- Other papers in 1980s had similar results, but MP understand the importance of what they found and explained it well

Summary

- Consumption-based models featuring CRRA representative agents rejected by historical asset pricing data
  - Consumption growth is not sufficiently correlated with risky asset returns
  - With power utility, relative risk aversion is handcuffed to intertemporal elasticity of substitution $\Rightarrow$ hard to simultaneously match risk-free rate and risk premium
- Some of the recent modeling innovations in the frictionless, representative-agent framework seem to have had some success in improving the explanatory power, but how much is still debated
  - Later today: long-run risk, habit formation, etc
- Perhaps the reason for the empirical failures is that most models operate in a world that is too idealized: Rational expectations, no frictions, agents have identical preferences, ...
  - Later in the semester!

BREAK

Consumption-based asset pricing

Central question

- Can consumption-based models explain key moments of the asset pricing data?
- Focus primarily on average returns of short-term bonds and the stock market, less so on the cross-section of stock returns

Rules of the game

- Assume agents derive utility from consumption only
- Show that joint properties of consumption, asset returns are mutually consistent
- Typically assume representative agent, frictionless markets, one consumption good

Summary of equity premium puzzle

Core of equity premium puzzle

- Consumption growth is not sufficiently correlated with risky asset returns
- With power utility, RRA is tied to IES $\Rightarrow$ hard to simultaneously match risk-free rate and risk premium

Proposed "solutions"

- Kreps-Porteus (1978) utility: e.g., Epstein and Zin (1991)
- Long-run risk: e.g., Bansal and Yaron (2004)
- Habit formation: e.g., Campbell and Cochrane (1999)
- Heterogeneous agents: e.g., Brav, Constantinides, and Gezcy (2002)
- Rare disasters: e.g, Barro (2006)
- Measurement error: e.g., Savov (2011)
- ...

Epstein and Zin (1991)

First proposed solution: **recursive utility**

$$U_t = \left\{(1-\beta)C_t^{\frac{1-\gamma}{\theta}} + \beta E_t[U_{t+1}^{1-\gamma}]^{\frac{1}{\theta}}\right\}^{\frac{\theta}{1-\gamma}}$$

where γ is the RRA parameter, $\theta \equiv \frac{1-\gamma}{1-1/\psi}$ and ψ is the elasticity of intertemporal substitution (EIS)

- γ measures the willingness to substitute consumption across states of nature
- ψ measures willingness to substitute over time as interest rate changes
- $\gamma = \frac{1}{\psi}$: power utility (no preference over timing of uncertainty resolution)
- $\gamma > \frac{1}{\psi}$: early resolution of uncertainty is preferred
- $\gamma < \frac{1}{\psi}$: late resolution of uncertainty is preferred

Epstein and Zin (1991)

Epstein and Zin show that this utility specification leads to the Euler equation

$$E_t\left[\beta^{\theta}\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}}\left(\frac{1}{R_{M,t+1}}\right)^{1-\theta}R_{t+1}\right]=1,$$

where $R_{M,t+1}$ is the return on a claim to aggregate consumption (wealth), Epstein and Zin use stock market index return as a proxy.

- By breaking the $\gamma = \frac{1}{\psi}$ link between RRA and EIS in the power utility model, Epstein-Zin preferences have the potential to resolve the asset-pricing puzzles in the sense of allowing the model to fit TBill returns and stock returns simultaneously (not necessarily in the sense of fitting the data with reasonable RRA and EIS parameters)

Epstein and Zin (1991)

GMM RESULTS FOR STOCKS AND NONDURABLES

	1959:4–1986:12			1959:4–1978:12		
	INST1	INST2	INST3	INST1	INST2	INST3
			Nondurables			
δ	.0033	.0033	−.0015	.0006	.0006	−.0040
	(.0018)	(.0018)	(.0036)	(.0031)	(.0031)	(.0031)
γ	−.0108	−.0146	−.0235	−.0297	−.0122	−.0065
	(.0564)	(.0564)	(.0746)	(.0659)	(.0644)	(.0889)
σ	.8652	.8158	.1754	.8499	.7973	.2392
	(.5422)	(.5021)	(.0728)	(.6331)	(.5948)	(.0950)
α	.0016	.0033	.1106	.0052	.0031	.0207
	(.0127)	(.0182)	(.3402)	(.0307)	(.0215)	(.2790)
$J(12)$	19.04	24.20	8.054	18.54	18.69	10.22
	[.088]	[.019]	[.781]	[.100]	[.096]	[.597]

Note: $\delta_{EZ} = 1 - \beta, \gamma_{EZ} = \alpha_{EZ}/(1 - \frac{1}{\psi}), \sigma_{EZ} = \psi, \alpha_{EZ} = 1 - \gamma$

- $\hat{\beta} > 1$ (??)
- RRA $\hat{\gamma} \approx 1$ (reasonable)
- $\hat{\psi} < \frac{1}{\gamma} \Rightarrow$ consumers prefer **late** resolution of uncertainty (??)

Long-run risk models

Epstein and Zin (1991), like Mehra and Prescott (1985), assume equity market proportional to aggregate consumption claim.

- Greatly simplifies estimation
- Empirically dubious given small correlation between equity returns/dividend growth and consumption growth

Long run risk models, starting with Bansal and Yaron (2004), make assumptions about the consumption dynamics to explicitly solve for the return on the consumption claim within a model, e.g.:

$$\Delta c_{t+1} = \mu + x_t + \sigma_t \nu_{t+1}$$
$$x_{t+1} = \rho x_t + \varphi \sigma_t e_{t+1},$$

where $\varphi \sigma_t$ can be quite small, but ρ is close to one, and ν_{t+1} and e_{t+1} are iid normal shocks. e_{t+1} represents "long-run risk"

Bansal and Yaron (2004) intuition

Fluctuations in x_{t+1}:

- Small enough we cannot distinguish observed consumption process from random walk given data
- So persistent they generate large swings in aggregate wealth $=$ NPV of future consumption

Because rep agent has **recursive utility**, they care not only about shocks to $realized$ consumption, but also about shocks to $expected$ consumption

- With $\psi > \frac{1}{\gamma}$, they dislike fluctuations in expected consumption growth

The fluctuations in expected consumption growth enter the SDF and make it more volatile without affecting observed interest rates or consumption volatility, allowing for a high risk premium in the model with a low risk-free rate

Bansal and Yaron (2004) intuition

Additional tricks Bansal and Yaron (2004) use to help their model fit key moments:

- Postulate we don't observe consumption claim, equity market is claim to dividends d_t satisfying:

$$\begin{aligned}\Delta d_{t+1} &= \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1} \\ \Delta c_{t+1} &= \mu + x_t + \sigma_t \nu_{t+1} \\ x_{t+1} &= \rho x_t + \varphi_e \sigma_t e_{t+1},\end{aligned}$$

  where u, ν, and e are iid standard normal, $\phi > 1$ and $\varphi_d > \varphi_e$. This essentially "juices up" anything we observe on the consumption side in the equity market size.
- Add time-varying volatility to help explain return predictability evidence (more on this in Lectures 4–5)

Bansal and Yaron (2004) calibration results – no stochastic vol

γ	ψ	$E(R_m - R_f)$	$E(R_f)$	$\sigma(R_m)$	$\sigma(R_f)$	$\sigma(p-d)$
		Panel A: $\phi = 3.0$, $\rho = 0.979$				
7.5	0.5	0.55	4.80	13.11	1.17	0.07
7.5	1.5	2.71	1.61	16.21	0.39	0.16
10.0	0.5	1.19	4.89	13.11	1.17	0.07
10.0	1.5	4.20	1.34	16.21	0.39	0.16
		Panel B: $\phi = 3.5$, $\rho = 0.979$				
7.5	0.5	1.11	4.80	14.17	1.17	0.10
7.5	1.5	3.29	1.61	18.23	0.39	0.19
10.0	0.5	2.07	4.89	14.17	1.17	0.10
10.0	1.5	5.10	1.34	18.23	0.39	0.19
		Panel C: $\phi = 3.0$, $\rho = \varphi_e = 0$				
7.5	0.5	-0.74	4.02	12.15	0.00	0.00
7.5	1.5	-0.74	1.93	12.15	0.00	0.00
10.0	0.5	-0.74	3.75	12.15	0.00	0.00
10.0	1.5	-0.74	1.78	12.15	0.00	0.00

Bansal and Yaron (2004) calibration results – stochastic vol

	Data		Model	
Variable	Estimate	S.E.	$\gamma = 7.5$	$\gamma = 10$
		Returns		
$E(r_m - r_f)$	6.33	(2.15)	4.01	6.84
$E(r_f)$	0.86	(0.42)	1.44	0.93
$\sigma(r_m)$	19.42	(3.07)	17.81	18.65
$\sigma(r_f)$	0.97	(0.28)	0.44	0.57
		Price Dividend		
$E(exp(p-d))$	26.56	(2.53)	25.02	19.98
$\sigma(p-d)$	0.29	(0.04)	0.18	0.21
$AC1(p-d)$	0.81	(0.09)	0.80	0.82
$AC2(p-d)$	0.64	(0.15)	0.65	0.67

Bansal, Kiku, and Yaron (2016): estimating LRR

Bansal, Kiku, and Yaron (2016) ("Risks For the Long Run: Estimation with Time Aggregation")

- Model has 3 preference parameters and 10 cash flow process parameters
- They estimate model using GMM with 20 different moments from the asset pricing and consumption data
  - Note these are **not** Euler-equation moments as in Hansen and Singleton (1982)
  - In my opinion, calibration-style moments great for estimating cash flow parameters, but should mix with Euler-equation-style moments to estimate preference parameters
  - Think about it as a more-systematic calibration
- They add a 'time aggregation' parameter to reflect a difference between data sampling frequency (one year) and time horizon on which investors make decisions (33 days?)
  - Seems to help model fit a lot!

Bansal, Kiku and Yaron (2016) estimation

Parameters	Empirical Data		Simulated Data			
	Estimate	SE	Population	5%	50%	95%
Preferences						
γ	13.83	3.42	12.93	11.01	14.78	19.64
ψ	1.05	0.61	1.22	0.86	1.17	1.62
δ	0.9944	0.0018	0.994	0.987	0.996	0.999
Cash Flows						
μ_c	0.0141	0.0025	0.015	0.010	0.016	0.021
ρ	0.8741	0.0646	0.859	0.832	0.893	0.952
φ_e	0.1661	0.0345	0.172	0.133	0.177	0.288
σ_0	0.0241	0.0054	0.026	0.001	0.015	0.023
ν	0.9220	0.0415	0.860	0.728	0.901	0.954
σ_w	3.49e-06	4.57e-06	3.5e-06	2.1e-06	3.5e-06	5.3e-06
μ_d	0.0107	0.0052	0.009	0.000	0.009	0.013
ϕ_d	2.51	0.54	2.72	1.872	2.647	3.601
φ_d	5.12	0.56	5.27	2.958	5.162	7.294
ϱ_d	0.60	0.02	0.60	0.560	0.601	0.651
χ^2-test	231.5			18.4	137.9	792.4
p-value	0.00			0.00	0.00	0.01

Bansal, Kiku and Yaron (2016) estimation

Parameters	LRR Model		No-Vol Model	
	Estimate	SE	Estimate	SE
Preferences				
γ	9.67	1.44	7.98	1.94
ψ	2.18	0.21	2.28	0.85
δ	0.9990	0.0001	0.9987	0.0004
Cash Flows				
μ_c	0.0016	0.0005	0.0016	0.0006
ρ	0.9762	0.0035	0.9796	0.0091
φ_e	0.0318	0.0053	0.0399	0.0113
σ_0	0.0070	0.0009	0.0081	0.0014
ν	0.9984	0.0007		
σ_w	2.12e-6	5.32e-7		
μ_d	0.0027	0.0010	0.0025	0.0010
ϕ_d	4.51	0.45	4.77	1.74
φ_d	4.65	0.48	4.44	1.04
ϱ_d	0.51	0.10	0.41	0.18
Aggregation				
h	11	2.16	9	2.69
χ^2-test	10.4		78.5	
p-value	0.11		0.00	

Skeptical view of long-run risk

- Empirical evidence primarily of the moment-matching or calibration variety
  - Euler equation tests not nearly as successful
  - Potential for 'moment mining'
  - Often still rejected
  - Hard to know you've found global optimum with in 13-dimensional numeric optimization
- Is $\psi > 2$ any more plausible than $\gamma > 10$?
  - Epstein, Farhi, and Strzalecki (2014): Bansal Yaron calibrations imply representative agent would give up 25% of expected consumption to have consumption uncertainty resolved immediately rather than later, holding fixed actual consumption volatility
- Very hard to measure a small highly persistent component in consumption growth x_t without an extremely long sample (Hansen, Heaton, and Li 2008)
  - How can investors be expected to include it in their valuations?

Habit formation

Approach in Campbell and Cochrane (1999):

- Retain iid consumption process
- Make risk aversion time varying via **habit formation** utility:
  - Utility of consumption today measured in part relative to past consumption, rather than some absolute benchmark
  - Can explain return predictability evidence
  - Can result in larger risk premia for given risk aversion / consumption volatility because rep agent measures c_t relative to much higher benchmark
- Purely a time-varying price of risk story vs. time varying quantity of risk

Campbell Cochrane (1999)

Identical agents with utility function:

$$U_t = \mathbb{E}_t \sum_{j=1}^{\infty} \beta^j \frac{(C_{t+j} - X_{t+j})^{1-\gamma}}{1-\gamma}.$$

Aggregate log consumption follows random walk:

$$\Delta c_{t+1} = \mu + \sigma_c \epsilon_{c,t+1}, \epsilon_{c,t+1} \sim N(0, \sigma^2)$$

'Habit' or 'minimum consumption' process X_t is only difference from the standard CRRA framework

Campbell Cochrane (1999) calibration

No analytic solutions available for total wealth or equity values/returns, solve numerically over grid of plausible s_t with following parameters

PARAMETER CHOICES

Parameter	Variable	Value
Assumed:		
Mean consumption growth (%)*	g	1.89
Standard deviation of consumption growth (%)*	σ	1.50
Log risk-free rate (%)*	r^f	.94
Persistence coefficient*	ϕ	.87
Utility curvature	γ	2.00
Standard deviation of dividend growth (%)*	σ_w	11.2
Correlation between Δd and Δc	ρ	.2
Implied:		
Subjective discount factor*	δ	.89
Steady-state surplus consumption ratio	$\bar{S}$	.057
Maximum surplus consumption ratio	$S_{\max}$	.094

* Annualized values, e.g., $12g$, $\sqrt{12}\sigma$, $12r^f$, ϕ^{12}, and δ^{12}, since the model is simulated at a monthly frequency.

Campbell Cochrane (1999) moments

Means and Standard Deviations of Simulated and Historical Data

Statistic	Consumption Claim	Dividend Claim	Postwar Sample	Long Sample
$E(\Delta c)$	1.89*		1.89	1.72
$\sigma(\Delta c)$	1.22*		1.22	3.32
$E(r^f)$	.094*		.094	2.92
$E(r - r^f)/\sigma(r - r^f)$	.43*	.33	.43	.22
$E(R - R^f)/\sigma(R - R^f)$	.50		.50	
$E(r - r^f)$	6.64	6.52	6.69	3.90
$\sigma(r - r^f)$	15.2	20.0	15.7	18.0
$\exp[E(p - d)]$	18.3	18.7	24.7	21.1
$\sigma(p - d)$	.27	.29	.26	.27

Note.—The model is simulated at a monthly frequency; statistics are calculated from artificial time-averaged data at an annual frequency. All returns are annual percentages.

* Statistics that model parameters were chosen to replicate.

Campbell Cochrane (1999) conclusions

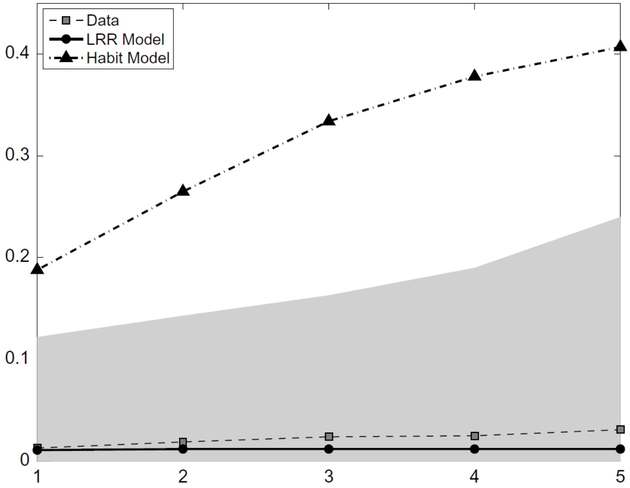
Good news:

- Gets unconditional returns and variation in p/d ratio reasonably close
- Model works by having very high expected excess returns when surplus low
- Matches high Sharpe Ratio and risk premium on equities
- Time-varying risk aversion delivers the predictability related to the dividend-price ratio in the data
- Slowly time-varying discount rates give volatile equity returns

Bad news:

- It implies very high risk aversion (average around 100)
- It is a conditional consumption CAPM: only shock is $\epsilon_{c,t+1}$
  - More recent empirical debate focuses on testing this
- Seem reverse-engineered to fit these moments, none of it feels over-identified

Bansal, Kiku, and Yaron (2012): LRR > Habit



R^2 from regression $p_{t+1} - d_{t+1} = \alpha_0 + \sum_{j=1}^{L} \alpha_j \Delta c_{t+1-j} + u_{t+1}$ for different of L

Campbell and Cochrane (2000)

Panel A: Basic Results				
Model	α/σ	$\alpha(\%)$	$\rho_{Y,M}$	η
(a) CAPM, $Y_{t+1} = a + bR^w_{t+1}$	0.40	7.9	0.77	
(b) Scaled CAPM, $Y_{t+1} = a_0 + a_1(pd_t) + [b_0 + b_1(pd_t)]R_{t+1}$	0.36	7.1	0.82	
(c) Power utility, $Y_{t+1} = \beta(C_{t+1}/C_t)^{-\eta}$	0.52	10.3	0.56	29
(d) Power utility, β, η chosen to price R^w, R^f	1.01	20.2	0.56	78
(e) Risk-neutral, $Y_{t+1} = 1/R^f$	0.62	12.5	0	

Panel B: Variations			
Model	α/σ	$\alpha(\%)$	$\rho_{Y,M}$
(f) CAPM, monthly simulated data	0.13	2.5	0.97
(g) Power utility model, monthly simulated data	0.23	4.7	0.91
(h) Consumption factor model, $Y_{t+1} = a + b(C_{t+1}/C_t)$	0.54	10.8	0.50
(i) Consumption factor model, a, b chosen to price R^w, R^f	0.93	18.5	0.50
(j) CAPM, dividend-claim return $Y_{t+1} = a + bR^d_{t+1}$	0.48	9.5	0.65
(k) Scaled CAPM, dividend-claim return	0.35	7.0	0.83

Campbell and Cochrane (2000) defense of consumption-based asset pricing

"all current asset pricing models are derived as specializations of the consumption-based model rather than as alternatives to it. All current models predict that expected returns should line up against covariances of returns with some function of consumption (possibly including leads and lags)."

In notation of our class,

$$m_{t+1} = \beta \mathbb{E}_t \frac{U'(C_{t+1})}{U'(C_t)}$$

must be true for any agent at an interior solution of consumption and investment decisions. Just need better models of utility function, better data, more focus on right kinds of agents, etc.

Cambpell and Cochrane (2000) discussion

Heterogeneous agents

Income and consumption of households are **more volatile** than aggregates. Can this help explain asset-pricing puzzles?

Households often appear to be at **corner solutions** in asset holdings (no investment in many assets) Why? Can this help explain asset-pricing puzzles?

Investor heterogeneity introduces two problems for researchers

1. Modeling difficulties: keeping track of agents, income, consumption, trading
2. Data difficulties: multiple sources that do not line up, substantial noise

Heterogeneous agents: early results not good

Early theoretical results showed this case often collapses onto rep agent case in terms of asset pricing implications:

- If shocks are transitory, agents can smooth them out using:
  - Borrowing/lending at risk-free rate (Telmer, 1993)
  - Long positions in lending, stock markets (Lucas, 1994)
  - Even with reasonable txn costs (Heaton and Lucas, 1996)

- Only theories that work have permanent, non-diffusive idiosyncratic shocks, e.g. Constantinides and Duffie (1996)
  - Empirical problems: not enough x-sectional variance in consumption growth, this variance appears unrelated to market return (Cogley, 2002)

Brav, Constantinides, and Geczy (2002)

Brav, Constantinides, and Geczy (2002) use an unweighted average of **household-level SDFs** directly in CRRA model:

$$\mathbb{E}_t\left[\left(\frac{1}{N}\sum_{i=1}^{N}\beta\left(\frac{C_{i,t+1}}{C_{i,t}}\right)^{-\gamma}\right)R_{t+1}\right]=1$$

based on household-level consumption data.

Authors face large problem: noise in C_t raised to power $-\gamma$ in empirical SDF

- Use 3rd order Taylor expansion of individual SDF to reduce effect of noise
- Find γ of 3 to 5 fits equity premium, value premium
- Weird result, given rest of literature; maybe driven by Taylor expansion procedure? (2nd order expansion $\Rightarrow$ different result)

Rare disasters

What if consumption-based models are correct for reasonable values of γ, but we are mis-estimating true consumption volatility and/or risk premia?

- Large standard errors on all these point estimates
- What if rare disasters (once in every 50 years?) have a substantial impact on the variance of consumption growth and stock returns, but may be underrepresentated in data
  - Introduced in Rietz (1988)
  - **Concern 1**: in major real consumption disasters, government likely to default as well, meaning this type of risk premia should be reflected in long-term bonds, but empirically there is still a big equity premia incremental to the long-term treasuries
  - **Concern 2**: put-protected equity portfolio still earns large Sharpe ratio
  - Barro (2006, 2008) calibrate a rare-disaster model using large panel of international data, argue that given observed sovereign default and recovery rates, disaster risk resolves the equity premium puzzle with low(ish) long-term government yields

Measurement error

Key magnitudes for consumption-based asset pricing:

- Contemporaneous covariance between asset returns and consumption growth
- Volatility of consumption growth
- Predictability of consumption growth

Method used to measure consumption distorts these moments

- Asset returns measured point to point; consumption smoothed out over time
- Same for other macro variables (GDP, income)
- See Working (1960) for math, Grossman, Melino, and Shiller (1987) for asset pricing implications

Problems with consumption data

Measurement error

- Most consumption data comes surveys of retailers
- Allocation into nondurables, services, durables somewhat arbitrary
- Relative prices also based on surveys
- Adjusting consumption in response to information takes time
- Should consumption-based asset pricing model work for daily returns?

Potential solution in Savov (2011): garbage!

Consumption-based asset pricing summary

Literature is huge, but in most cases use calibration, not Euler-equation tests (which mostly reject models) and cross-sectional tests (for which reduced-form factors outperform)

> *"All models can be rejected, and the more important issue is which approximate models are most useful." – Campbell and Cochrane (2000)*

While I agree, consumption-based models have proven less useful to both academics and practitioners

An empirically-useful, intuitively satisfying, preference-based model of aggregate risk premia and risk-free rates would be revolutionary in asset pricing and macroeconomics, but remains elusive