Statistical and Econometric Models

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Problem Set – Week 3 (for tutorial on May 06, 2024)

Properties of "Hat Matrix" and "Residual Maker"

Problem 3.1:

In the linear regression framework $y = X\beta + \varepsilon$,

- the "Hat Matrix" H yields the fitted values $\hat{y} = X\hat{\beta}$
- and the "Residual Maker" matrix M yields the residuals $e = y - \hat{y}$

when right-multiplied with y, respectively.

(a) Show that $H = X(X'X)^{-1}X'$ and $M = I - H$.

(b) Show that H and M are square and symmetric.

(c) Show that H and M are idempotent.

(d) Show that $MX = 0 \in \mathbb{R}^{N\times(K+1)}$.

(e) Show that the residuals e are orthogonal to all covariates x_j, i.e. $x_j'e = 0$, where x_j is the j-th column of X.

(f) Show that a symmetric and idempotent matrix $A \in \mathbb{N} \times \mathbb{N}$ is positive semi-definite.

Interpreting parameters with logarithm transformed variables

Problem 3.2:

In the following, we want to consider the interpretation of the slope coefficient in models in which dependent and/or independent variables are log-transformed. For the following derivations, use the approximation $f(z) = \exp(az) \approx 1 + az$ for $az \in \mathbb{R}$ close to zero at a suitable point.
In the following, x_0 denotes the initial level, Δx any absolute change and $\frac{\Delta x}{x_0}$ any percentage change in the regressor ($\frac{\Delta x}{x_0} = 0.01$ corresponds to a one per cent change).

(a) Given the *log-level* model

$$\log(y) = \beta_0 + \beta_1 x + \varepsilon$$

with fitted values $\hat{y}(x) = \exp(\hat{\beta}_0 + \hat{\beta}_1 x)$. Show that the following applies

$$\frac{\hat{y}(x_0 + \Delta x) - \hat{y}(x_0)}{\hat{y}(x_0)} \approx \hat{\beta}_1 \Delta x. \tag{1}$$

Interpret $\hat{\beta}_1$ for the case that $\Delta x = 1$ applies. Then transform equation (1) to $\hat{y}(x_0 + \Delta x)$ and check your interpretation using the transformation (continue to use $\Delta x = 1$).

(b) Given the *log-log* model

$$\log(y) = \beta_0 + \beta_1 \log(x) + \varepsilon$$

with fitted values $\hat{y}(x) = \exp(\hat{\beta}_0 + \hat{\beta}_1 \log(x))$. Show that the following applies

$$\frac{\hat{y}(x_0 + \Delta x) - \hat{y}(x_0)}{\hat{y}(x_0)} \approx \hat{\beta}_1 \frac{\Delta x}{x_0}. \tag{2}$$

Interpret $\hat{\beta}_1$ for the case that $\frac{\Delta x}{x_0} = 0.01$ applies. Then solve equation (2) to $\hat{y}(x_0 + \Delta x)$ and check your interpretation (continue to use $\frac{\Delta x}{x_0} = 0.01$).

(c) Given the *level-log* model

$$y = \beta_0 + \beta_1 \log(x) + \varepsilon$$

with fitted values $\hat{y}(x) = \hat{\beta}_0 + \hat{\beta}_1 \log(x)$. Show that the following applies

$$\hat{y}(x_0 + \Delta x) - \hat{y}(x_0) \approx \hat{\beta}_1 \frac{\Delta x}{x_0}. \tag{3}$$

Interpret $\hat{\beta}_1$ for the case that $\frac{\Delta x}{x_0} = 0.01$ applies. Then solve equation (3) to $\hat{y}(x_0 + \Delta x)$ and check your interpretation (continue to use $\frac{\Delta x}{x_0} = 0.01$).