

Definition

The Riemann-Liouville fractional derivative of order α for a function $f(x)$ is defined as:

$${}^{RL}D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left(\int_0^x \frac{f(t)}{(x-t)^{(\alpha+1-n)}} dt \right) \quad (1)$$

Where:

- ${}^{RL}D$ specifies a fractional derivative of type Riemann-Liouville
- $n = \lceil \alpha \rceil$ (the ceiling of α)
- Γ is the Gamma function

General Derivative Formula for Polynomial Functions

Consider a generic polynomial function $f(x) = x^k$.

For $f(t) = t^k$, (1) can be rewritten as:

$${}^{RL}D^\alpha [x^k] = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left(\int_0^x \frac{t^k}{(x-t)^{(\alpha+1-n)}} dt \right) \quad (2)$$

Evaluation of the integral will require a change of variables from t to u .

Let $u = \frac{t}{x}$ and $du = \frac{dt}{x}$ such that when $t = 0$, $u = 0$, and when $t = x$, $u = 1$:

$$\int_0^1 \frac{(ux)^k}{(x-(ux))^{(\alpha+1-n)}} (x \cdot du)$$

Factoring out x from the numerator and denominator:

$$\int_0^1 \frac{x^{(k+1)} \cdot u^k}{x^{(\alpha+1-n)} (1-u)^{(\alpha+1-n)}} du$$

Pulling the factors of x in front of the integral:

$$\frac{x^{(k+1)}}{x^{(\alpha+1-n)}} \int_0^1 \frac{u^k}{(1-u)^{(\alpha+1-n)}} du$$

Simplifying:

$$x^{(k-\alpha+n)} \int_0^1 \frac{u^k}{(1-u)^{(\alpha+1-n)}} du$$

Rewriting the integrand:

$$x^{(k-\alpha+n)} \int_0^1 u^k (1-u)^{(n-\alpha-1)} du \quad (3)$$

This matches the form of a famous expression known as the beta function.

The Beta Function

The beta function has the following definition:

$$\int_0^1 t^{(p-1)} (1-t)^{(q-1)} dt$$

The general solution to the beta function is:

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

The solution of (3) can be expressed in a similar way once p and q are determined:

$$\begin{aligned} p-1 &= k \\ \implies p &= k+1 \end{aligned}$$

$$\begin{aligned} q-1 &= n-\alpha-1 \\ \implies q &= n-\alpha \end{aligned}$$

Thus,

$$B((k+1), (n-\alpha)) = \frac{\Gamma(k+1)\Gamma(n-\alpha)}{\Gamma(k+1+n-\alpha)} \quad (4)$$

Applying the Result of the Beta Function

The expression from (2) and the result from (4) can be utilized in (3):

$${}^{RL}D^\alpha [x^k] = \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dx^n} \left(\frac{\Gamma(k+1)\Gamma(n-\alpha)}{\Gamma(k+1+n-\alpha)} x^{(k-\alpha+n)} \right)$$

The ratio of gamma values is a constant and can be factored out of the derivative:

$${}^{RL}D^\alpha [x^k] = \frac{1}{\Gamma(n-\alpha)} \cdot \frac{\Gamma(k+1)\Gamma(n-\alpha)}{\Gamma(k+1+n-\alpha)} \cdot \frac{d^n}{dx^n} \left(x^{(k-\alpha+n)} \right)$$

Canceling like terms:

$${}^{RL}D^{\alpha} [x^k] = \frac{1}{\cancel{\Gamma(n-\alpha)}} \cdot \frac{\Gamma(k+1)\cancel{\Gamma(n-\alpha)}}{\Gamma(k+1+n-\alpha)} \cdot \frac{d^n}{dx^n} (x^{(k-\alpha+n)})$$

Simplifying:

$${}^{RL}D^{\alpha} [x^k] = \frac{\Gamma(k+1)}{\Gamma(k+1+n-\alpha)} \cdot \frac{d^n}{dx^n} (x^{(k-\alpha+n)})$$

Performing the derivative:

$${}^{RL}D^{\alpha} [x^k] = \frac{\Gamma(k+1)}{\Gamma(k+1+n-\alpha)} \cdot \frac{(k-\alpha+n)!}{(k-\alpha)!} \cdot x^{(k-\alpha)}$$

Given that $(k-\alpha+n)! = \Gamma(k-\alpha+n+1)$ and that $(k-\alpha)! = \Gamma(k-\alpha+1)$:

$${}^{RL}D^{\alpha} [x^k] = \frac{\Gamma(k+1)}{\Gamma(k+1+n-\alpha)} \cdot \frac{\Gamma(k-\alpha+n+1)}{\Gamma(k-\alpha+1)} \cdot x^{(k-\alpha)}$$

Canceling like terms once more:

$${}^{RL}D^{\alpha} [x^k] = \frac{\Gamma(k+1)}{\cancel{\Gamma(k+1+n-\alpha)}} \cdot \frac{\cancel{\Gamma(k-\alpha+n+1)}}{\Gamma(k-\alpha+1)} \cdot x^{(k-\alpha)}$$

Therefore, the generalized Riemann-Liouville fractional derivative formula for a generic polynomial is the following:

$$\boxed{{}^{RL}D^{\alpha} [x^k] = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{(k-\alpha)}} \quad (5)$$

Example API Calculation

Let $f(x) = 3x^2 + 2x + 1$ and $\alpha = 0.35$.

The API utilizes (5) for each term in $f(x)$ when calculating the fractional derivative.

For $3x^2$:

$$\begin{aligned} {}^{RL}D^{0.35} [3x^2] &= 3 \cdot \frac{\Gamma(2+1)}{\Gamma(2-0.35+1)} x^{(2-0.35)} \\ &= 3 \cdot \frac{\Gamma(3)}{\Gamma(2.65)} x^{1.65} \\ &\approx 3 \cdot \frac{2}{1.485} x^{1.65} \\ &\approx 4.040 x^{1.65} \end{aligned} \quad (6)$$

For $2x$:

$$\begin{aligned} {}^{RL}D^{0.35} [2x] &= 2 \cdot \frac{\Gamma(1+1)}{\Gamma(1-0.35+1)} x^{(1-0.35)} \\ &= 2 \cdot \frac{\Gamma(2)}{\Gamma(1.65)} x^{0.65} \\ &\approx 2 \cdot \frac{1}{0.900} x^{0.65} \\ &\approx 2.222x^{0.65} \end{aligned} \tag{7}$$

For 1:

$$\begin{aligned} {}^{RL}D^{0.35} [1x^0] &= \frac{\Gamma(0+1)}{\Gamma(0-0.35+1)} x^{0-0.35} \\ &= \frac{\Gamma(1)}{\Gamma(0.65)} x^{-0.35} \\ &\approx \frac{1}{1.385} x^{-0.35} \\ &\approx 0.722x^{-0.35} \end{aligned} \tag{8}$$

Using the results of (6), (7), and (8), the API output for the 0.35th Riemann-Liouville fractional derivative of $3x^2 + 2x + 1$ is:

$$\boxed{= 4.040x^{1.65} + 2.222x^{0.65} + 0.722x^{-0.35}}$$