Definition

The Riemann-Liouville fractional derivative of order α for a function $f(x)$ is defined as:

$$^{RL}D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left(\int_0^x \frac{f(t)}{(x-t)^{(\alpha+1-n)}} \, dt \right) \tag{1}$$

Where:

- ^{RL}D specifies a fractional derivative of type Riemann-Liouville
- $n = \lceil \alpha \rceil$ is the ceiling of α
- Γ is the Gamma function

General Derivative Formula for Polynomial Functions

Consider a generic polynomial function $f(x) = x^k$.

For $f(t) = t^k$, (1) can be rewritten as:

$$^{RL}D^{\alpha} \, [x^k] = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left(\int_0^x \frac{t^k}{(x-t)^{(\alpha+1-n)}} \, dt \right) \tag{2}$$

Evaluation of the integral will require a change of variables from t to u.

Let $u = \dfrac{t}{x}$ and $du = \dfrac{dt}{x}$ such that when $t = 0$, $u = 0$, and when $t = x$, $u = 1$:

$$\int_0^1 \frac{(ux)^k}{\Big(x - (ux)\Big)^{(\alpha+1-n)}} (x \cdot du)$$

Factoring out x from the numerator and denominator:

$$\int_0^1 \frac{x^{(k+1)} \cdot u^k}{x^{(\alpha+1-n)}(1-u)^{(\alpha+1-n)}} \, du$$

Pulling the factors of x in front of the integral:

$$\frac{x^{(k+1)}}{x^{(\alpha+1-n)}} \int_0^1 \frac{u^k}{(1-u)^{(\alpha+1-n)}} \, du$$

Simplifying:

$$x^{(k-\alpha+n)} \int_0^1 \frac{u^k}{(1-u)^{(\alpha+1-n)}} \, du$$

Rewriting the integrand:

$$x^{(k-\alpha+n)} \int_0^1 u^k (1-u)^{(n-\alpha-1)}\, du \tag{3}$$

This matches the form of a famous expression known as the beta function.

The Beta Function

The beta function has the following definition:

$$\int_0^1 t^{(p-1)}(1-t)^{(q-1)}\, dt$$

The general solution to the beta function is:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

The solution of (3) can be expressed in a similar way once p and q are determined:

$$\begin{aligned} &p - 1 = k \\ \implies\ &p = k + 1 \end{aligned}$$

$$\begin{aligned} &q - 1 = n - \alpha - 1 \\ \implies\ &q = n - \alpha \end{aligned}$$

Thus,

$$B\Big((k+1),(n-\alpha)\Big) = \frac{\Gamma(k+1)\Gamma(n-\alpha)}{\Gamma(k+1+n-\alpha)} \tag{4}$$

Applying the Result of the Beta Function

The expression from (2) and the result from (4) can be utilized in (3):

$$^{RL}D^{\alpha}\ [x^k] = \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dx^n}\Big(\frac{\Gamma(k+1)\Gamma(n-\alpha)}{\Gamma(k+1+n-\alpha)} x^{(k-\alpha+n)}\Big)$$

The ratio of gamma values is a constant and can be factored out of the derivative:

$$^{RL}D^{\alpha}\ [x^k] = \frac{1}{\Gamma(n-\alpha)} \cdot \frac{\Gamma(k+1)\Gamma(n-\alpha)}{\Gamma(k+1+n-\alpha)} \cdot \frac{d^n}{dx^n}\Big(x^{(k-\alpha+n)}\Big)$$

Canceling like terms:

$$^{RL}D^{\alpha}\ [x^k] = \frac{1}{\cancel{\Gamma(n-\alpha)}} \cdot \frac{\Gamma(k+1)\cancel{\Gamma(n-\alpha)}}{\Gamma(k+1+n-\alpha)} \cdot \frac{d^n}{dx^n}\left(x^{(k-\alpha+n)}\right)$$

Simplifying:

$$^{RL}D^{\alpha}\ [x^k] = \frac{\Gamma(k+1)}{\Gamma(k+1+n-\alpha)} \cdot \frac{d^n}{dx^n}\left(x^{(k-\alpha+n)}\right)$$

Performing the derivative:

$$^{RL}D^{\alpha}\ [x^k] = \frac{\Gamma(k+1)}{\Gamma(k+1+n-\alpha)} \cdot \frac{(k-\alpha+n)!}{(k-\alpha)!} \cdot x^{(k-\alpha)}$$

Given that $(k-\alpha+n)! = \Gamma(k-\alpha+n+1)$ and that $(k-\alpha)! = \Gamma(k-\alpha+1)$:

$$^{RL}D^{\alpha}\ [x^k] = \frac{\Gamma(k+1)}{\Gamma(k+1+n-\alpha)} \cdot \frac{\Gamma(k-\alpha+n+1)}{\Gamma(k-\alpha+1)} \cdot x^{(k-\alpha)}$$

Canceling like terms once more:

$$^{RL}D^{\alpha}\ [x^k] = \frac{\Gamma(k+1)}{\cancel{\Gamma(k+1+n-\alpha)}} \cdot \frac{\cancel{\Gamma(k-\alpha+n+1)}}{\Gamma(k-\alpha+1)} \cdot x^{(k-\alpha)}$$

Therefore, the generalized Riemann-Liouville fractional derivative formula for a generic polynomial is the following:

$$\boxed{^{RL}D^{\alpha}\ [x^k] = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{(k-\alpha)}} \tag{5}$$

Example API Calculation

Let $f(x) = 3x^2 + 2x + 1$ and $\alpha = 0.35$.

The API utilizes (5) for each term in $f(x)$ when calculating the fractional derivative.

For $3x^2$:

$$\begin{aligned} ^{RL}D^{0.35}\ [3x^2] &= 3 \cdot \frac{\Gamma(2+1)}{\Gamma(2-0.35+1)} x^{(2-0.35)} \\ &= 3 \cdot \frac{\Gamma(3)}{\Gamma(2.65)} x^{1.65} \\ &\approx 3 \cdot \frac{2}{1.485} x^{1.65} \\ &\approx 4.040x^{1.65} \end{aligned} \tag{6}$$

For 2x:

$$
\begin{aligned}
{}^{RL}D^{0.35}\left[2x\right] &= 2 \cdot \frac{\Gamma(1+1)}{\Gamma(1-0.35+1)} x^{(1-0.35)} \\
&= 2 \cdot \frac{\Gamma(2)}{\Gamma(1.65)} x^{0.65} \\
&\approx 2 \cdot \frac{1}{0.900} x^{0.65} \\
&\approx 2.222x^{0.65} \qquad (7)
\end{aligned}
$$

For 1:

$$
\begin{aligned}
{}^{RL}D^{0.35}\left[1x^0\right] &= \frac{\Gamma(0+1)}{\Gamma(0-0.35+1)} x^{0-0.35} \\
&= \frac{\Gamma(1)}{\Gamma(0.65)} x^{-0.35} \\
&\approx \frac{1}{1.385} x^{-0.35} \\
&\approx 0.722x^{-0.35} \qquad (8)
\end{aligned}
$$

Using the results of (6), (7), and (8), the API output for the 0.35th Riemann-Liouville fractional derivative of $3x^2 + 2x + 1$ is:

$$
\boxed{= 4.040x^{1.65} + 2.222x^{0.65} + 0.722x^{-0.35}}
$$