Definition

The fractional integral of order α for a function $f(x)$ shares the same definition between the Caputo and Riemann-Liouville methods:

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{(\alpha-1)} f(t)\, dt \tag{1}$$

Where Γ represents the Gamma function.

General Integration Formula for Polynomial Functions

Consider a generic polynomial function $f(x) = x^k$.

For $f(t) = t^k$, (1) can be rewritten as:

$$I^{\alpha}\, [x^k] = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{(\alpha-1)} t^k\, dt \tag{2}$$

Evaluation of the integral will require a change of variables from t to u.

Let $u = \frac{t}{x}$ and $du = \frac{dt}{x}$ such that when $t = 0$, $u = 0$, and when $t = x$, $u = 1$:

$$\int_0^1 \Big(x - (xu)\Big)^{(\alpha-1)} (xu)^k\, (x \cdot du)$$

Factoring out x and rearranging the integrand:

$$x^{(\alpha+k)} \int_0^1 u^k (1-u)^{(\alpha-1)}\, du \tag{3}$$

This matches the form of a famous expression known as the beta function.

The Beta Function

The beta function has the following definition:

$$\int_0^1 t^{(p-1)} (1-t)^{(q-1)}\, dt$$

The general solution to the beta function is:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

The solution of (3) can be expressed in a similar way once p and q are determined:

$$\begin{aligned} p - 1 &= k \\ \implies p &= k + 1 \end{aligned}$$

$$\begin{aligned} q - 1 &= \alpha - 1 \\ \implies q &= \alpha \end{aligned}$$

Thus,

$$B\Big((k+1), (\alpha)\Big) = \frac{\Gamma(k+1)\Gamma(\alpha)}{\Gamma(k+1+\alpha)} \tag{4}$$

Applying the Result of the Beta Function

The coefficient from (2) and the result from (4) can be utilized in (3):

$$I^{\alpha}\ [x^k] = \frac{1}{\Gamma(\alpha)} \cdot \frac{\Gamma(k+1)\Gamma(\alpha)}{\Gamma(k+1+\alpha)} \cdot x^{\alpha+k}$$

Canceling like terms:

$$I^{\alpha}\ [x^k] = \frac{1}{\cancel{\Gamma(\alpha)}} \cdot \frac{\Gamma(k+1)\cancel{\Gamma(\alpha)}}{\Gamma(k+1+\alpha)} \cdot x^{\alpha+k}$$

Therefore, the generalized fractional integration formula for a generic polynomial is:

$$\boxed{I^{\alpha}\ [x^k] = \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} x^{(\alpha+k)}} \tag{5}$$

Example API Calculation

Let $f(x) = 3x^2 + 2x + 1$ and $\alpha = 0.35$.

The API utilizes (5) for each term in $f(x)$ when calculating the fractional derivative.

For $3x^2$:

$$I^{0.35}\ [3x^2] = 3 \cdot \frac{\Gamma(2+1)}{\Gamma(2+1+0.35)} x^{(0.35+2)}$$

$$= 3 \cdot \frac{\Gamma(3)}{\Gamma(3.35)} x^{2.35}$$

$$\approx 3 \cdot \frac{2}{2.827} x^{2.35}$$

$$\approx 2.122 x^{2.35} \tag{6}$$

For 2x:

$$I^{0.35}\ [2x] = 2 \cdot \frac{\Gamma(1+1)}{\Gamma(1+1+0.35)} x^{(0.35+1)}$$

$$= 2 \cdot \frac{\Gamma(2)}{\Gamma(2.35)} x^{1.35}$$

$$\approx 2 \cdot \frac{1}{1.203} x^{1.35}$$

$$\approx 1.662 x^{1.35} \tag{7}$$

For 1:

$$I^{0.35}\ [1] = \frac{\Gamma(0+1)}{\Gamma(0+1+0.35)} x^{0.35+0}$$

$$= \frac{\Gamma(1)}{\Gamma(1.35)} x^{0.35}$$

$$\approx \frac{1}{0.891} x^{0.35}$$

$$\approx 1.122 x^{0.35} \tag{8}$$

Using the results of (6), (7), and (8), the API output for the 0.35th unified fractional integral of $3x^2 + 2x + 1$ is:

$$\boxed{= 2.122x^{2.35} + 1.662x^{1.35} + 1.122x^{0.35} + \mathrm{C}}$$