

Caputo Fractional Derivative

Generalized Differentiation Formula Derivation for Polynomial Functions

Definition

The Caputo fractional derivative of order α for a function $f(x)$ is defined as:

$${}^C D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{(\alpha+1-n)}} dt \quad (1)$$

Where:

- ${}^C D$ specifies a fractional derivative of type Caputo
- $n = \lceil \alpha \rceil$ is the ceiling of α
- Γ is the Gamma function
- $f^{(n)}(t)$ is the n -th derivative of $f(t)$

General Derivative Formula for Polynomial Functions

Consider a generic polynomial function $f(x) = x^k$.

For $f(t) = t^k$, the n -th derivative $f^{(n)}(t)$ is given by:

$$f^{(n)}(t) = \frac{k!}{(k-n)!} t^{k-n}$$

Substituting this expression into (1):

$${}^C D^\alpha [x^k] = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{\left(\frac{k!}{(k-n)!} t^{(k-n)} \right)}{(x-t)^{(\alpha+1-n)}} dt$$

Simplifying the expression:

$${}^C D^\alpha [x^k] = \frac{k!}{\Gamma(n-\alpha)(k-n)!} \int_0^x \frac{t^{(k-n)}}{(x-t)^{(\alpha+1-n)}} dt \quad (2)$$

Evaluation of the integral will require a change of variables from t to u .

Let $u = \frac{t}{x}$ and $du = \frac{dt}{x}$ such that when $t = 0$, $u = 0$, and when $t = x$, $u = 1$:

$$\int_0^1 \frac{(ux)^{(k-n)}}{(x-(ux))^{(\alpha+1-n)}} (x \cdot du)$$

Factoring out x from the numerator and denominator:

$$\int_0^1 \frac{x^{(k-n+1)} \cdot u^{(k-n)}}{x^{(\alpha+1-n)} \cdot (1-u)^{(\alpha+1-n)}} du$$

Pulling the factors of x in front of the integral:

$$\frac{x^{(k-n+1)}}{x^{(\alpha+1-n)}} \int_0^1 \frac{u^{(k-n)}}{(1-u)^{(\alpha+1-n)}} du$$

Simplifying:

$$x^{(k-\alpha)} \int_0^1 \frac{u^{(k-n)}}{(1-u)^{(\alpha+1-n)}} du$$

Rewriting the integrand:

$$x^{(k-\alpha)} \int_0^1 u^{(k-n)} (1-u)^{(-\alpha-1+n)} du \quad (3)$$

This matches the form of a famous expression known as the beta function.

The Beta Function

The beta function has the following definition:

$$\int_0^1 t^{(p-1)} (1-t)^{(q-1)} dt$$

The general solution to the beta function is:

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

The solution of (3) can be expressed in a similar way once p and q are determined:

$$\begin{aligned} p-1 &= k-n \\ \implies p &= k-n+1 \end{aligned}$$

$$\begin{aligned} q-1 &= -\alpha-1+n \\ \implies q &= n-\alpha \end{aligned}$$

Thus,

$$B\left((k-n+1), (n-\alpha)\right) = \frac{\Gamma(k-n+1)\Gamma(n-\alpha)}{\Gamma(k-\alpha+1)} \quad (4)$$

Applying the Result of the Beta Function

The coefficient from (2) and the result from (4) can be utilized in (3):

$${}^C D^\alpha [x^k] = \frac{k!}{\Gamma(n-\alpha)(k-n)!} \cdot \frac{\Gamma(k-n+1)\Gamma(n-\alpha)}{\Gamma(k-\alpha+1)} \cdot x^{(k-\alpha)}$$

Given that $k! = \Gamma(k+1)$ and $(k-n)! = \Gamma(k-n+1)$:

$${}^C D^\alpha [x^k] = \frac{\Gamma(k+1)}{\Gamma(n-\alpha)\Gamma(k-n+1)} \cdot \frac{\Gamma(k-n+1)\Gamma(n-\alpha)}{\Gamma(k-\alpha+1)} \cdot x^{(k-\alpha)}$$

Canceling like terms:

$${}^C D^\alpha [x^k] = \frac{\Gamma(k+1)}{\Gamma(n-\alpha)\Gamma(k-n+1)} \cdot \frac{\Gamma(k-n+1)\Gamma(n-\alpha)}{\Gamma(k-\alpha+1)} \cdot x^{(k-\alpha)}$$

Therefore, the generalized Caputo fractional derivative formula for a generic polynomial is:

$$\boxed{{}^C D^\alpha [x^k] = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{(k-\alpha)}} \quad (5)$$

Example API Calculation

Let $f(x) = 3x^2 + 2x + 1$ and $\alpha = 0.35$.

The API utilizes (5) for each term in $f(x)$ when calculating the fractional derivative.

For $3x^2$:

$$\begin{aligned} {}^C D^{0.35} [3x^2] &= 3 \cdot \frac{\Gamma(2+1)}{\Gamma(2-0.35+1)} x^{(2-0.35)} \\ &= 3 \cdot \frac{\Gamma(3)}{\Gamma(2.65)} x^{1.65} \\ &\approx 3 \cdot \frac{2}{1.485} x^{1.65} \\ &\approx 4.040 x^{1.65} \end{aligned} \quad (6)$$

For $2x$:

$$\begin{aligned} {}^C D^{0.35} [2x] &= 2 \cdot \frac{\Gamma(1+1)}{\Gamma(1-0.35+1)} x^{(1-0.35)} \\ &= 2 \cdot \frac{\Gamma(2)}{\Gamma(1.65)} x^{0.65} \\ &\approx 2 \cdot \frac{1}{0.900} x^{0.65} \\ &\approx 2.222x^{0.65} \end{aligned} \tag{7}$$

For 1:

$${}^C D^{0.35} [1] = 0 \tag{8}$$

Using the results of (6), (7), and (8), the API output for the 0.35th Caputo fractional derivative of $3x^2 + 2x + 1$ is:

$$\boxed{= 4.040x^{1.65} + 2.222x^{0.65}}$$