Caputo Fractional Derivative

Generalized Differentiation Formula Derivation for Polynomial Functions

Definition

The Caputo fractional derivative of order α for a function $f(x)$ is defined as:

$$ {}^{C}D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{(\alpha+1-n)}}\, dt \tag{1} $$

Where:

- ${}^{C}D$ specifies a fractional derivative of type Caputo
- $n = \lceil \alpha \rceil$ (the ceiling of α)
- Γ is the Gamma function
- $f^{(n)}(t)$ is the n-th derivative of $f(t)$

General Derivative Formula for Polynomial Functions

Consider a generic polynomial function $f(x) = x^k$.

For $f(t) = t^k$, the n-th derivative $f^{(n)}(t)$ is given by:

$$ f^{(n)}(t) = \frac{k!}{(k-n)!} t^{k-n} $$

Substituting this expression into (1):

$$ {}^{C}D^{\alpha}\left[x^k\right] = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{\left(\frac{k!}{(k-n)!} t^{(k-n)}\right)}{(x-t)^{(\alpha+1-n)}}\, dt $$

Simplifying the expression:

$$ {}^{C}D^{\alpha}\left[x^k\right] = \frac{k!}{\Gamma(n-\alpha)(k-n)!} \int_0^x \frac{t^{(k-n)}}{(x-t)^{(\alpha+1-n)}}\, dt \tag{2} $$

Evaluation of the integral will require a change of variables from t to u.

Let $u = \frac{t}{x}$ and $du = \frac{dt}{x}$ such that when $t = 0$, $u = 0$, and when $t = x$, $u = 1$:

$$ \int_0^1 \frac{(ux)^{(k-n)}}{\left(x - (ux)\right)^{(\alpha+1-n)}} (x \cdot du) $$

Factoring out x from the numerator and denominator:

$$\int_0^1 \frac{x^{(k-n+1)} \cdot u^{(k-n)}}{x^{(\alpha+1-n)} \cdot (1-u)^{(\alpha+1-n)}} \, du$$

Pulling the factors of x in front of the integral:

$$\frac{x^{(k-n+1)}}{x^{(\alpha+1-n)}} \int_0^1 \frac{u^{(k-n)}}{(1-u)^{(\alpha+1-n)}} \, du$$

Simplifying:

$$x^{(k-\alpha)} \int_0^1 \frac{u^{(k-n)}}{(1-u)^{(\alpha+1-n)}} \, du$$

Rewriting the integrand:

$$x^{(k-\alpha)} \int_0^1 u^{(k-n)} (1-u)^{(-\alpha-1+n)} \, du \tag{3}$$

This matches the form of a famous expression known as the beta function.

The Beta Function

The beta function has the following definition:

$$\int_0^1 t^{(p-1)} (1-t)^{(q-1)} \, dt$$

The general solution to the beta function is:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

The solution of (3) can be expressed in a similar way once p and q are determined:

$$\begin{aligned} p - 1 &= k - n \\ \implies p &= k - n + 1 \end{aligned}$$

$$\begin{aligned} q - 1 &= -\alpha - 1 + n \\ \implies q &= n - \alpha \end{aligned}$$

Thus,

$$B\Big((k-n+1), (n-\alpha)\Big) = \frac{\Gamma(k-n+1)\Gamma(n-\alpha)}{\Gamma(k-\alpha+1)} \tag{4}$$

Applying the Result of the Beta Function

The coefficient from (2) and the result from (4) can be utilized in (3):

$$ {}^{C}D^{\alpha}\left[x^{k}\right] = \frac{k!}{\Gamma(n-\alpha)(k-n)!} \cdot \frac{\Gamma(k-n+1)\Gamma(n-\alpha)}{\Gamma(k-\alpha+1)} \cdot x^{(k-\alpha)} $$

Given that $k! = \Gamma(k+1)$ and $(k-n)! = \Gamma(k-n+1)$:

$$ {}^{C}D^{\alpha}\left[x^{k}\right] = \frac{\Gamma(k+1)}{\Gamma(n-\alpha)\Gamma(k-n+1)} \cdot \frac{\Gamma(k-n+1)\Gamma(n-\alpha)}{\Gamma(k-\alpha+1)} \cdot x^{(k-\alpha)} $$

Canceling like terms:

$$ {}^{C}D^{\alpha}\left[x^{k}\right] = \frac{\Gamma(k+1)}{\cancel{\Gamma(n-\alpha)\Gamma(k-n+1)}} \cdot \frac{\cancel{\Gamma(k-n+1)\Gamma(n-\alpha)}}{\Gamma(k-\alpha+1)} \cdot x^{(k-\alpha)} $$

Therefore, the generalized Caputo fractional derivative formula for a generic polynomial is:

$$ \boxed{{}^{C}D^{\alpha}\left[x^{k}\right] = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} x^{(k-\alpha)}} \tag{5} $$

Example API Calculation

Let $f(x) = 3x^2 + 2x + 1$ and $\alpha = 0.35$.

The API utilizes (5) for each term in $f(x)$ when calculating the fractional derivative.

For $3x^2$:

$$ {}^{C}D^{0.35}\left[3x^{2}\right] = 3 \cdot \frac{\Gamma(2+1)}{\Gamma(2-0.35+1)} x^{(2-0.35)} $$

$$ = 3 \cdot \frac{\Gamma(3)}{\Gamma(2.65)} x^{1.65} $$

$$ \approx 3 \cdot \frac{2}{1.485} x^{1.65} $$

$$ \approx 4.040 x^{1.65} \tag{6} $$

For 2x:

$$
\begin{aligned}
{}^{C}D^{0.35}\left[2x\right] &= 2 \cdot \frac{\Gamma(1+1)}{\Gamma(1-0.35+1)} x^{(1-0.35)} \\
&= 2 \cdot \frac{\Gamma(2)}{\Gamma(1.65)} x^{0.65} \\
&\approx 2 \cdot \frac{1}{0.900} x^{0.65} \\
&\approx 2.222 x^{0.65} \qquad (7)
\end{aligned}
$$

For 1:

$$
{}^{C}D^{0.35}\left[1\right] = 0 \qquad (8)
$$

Using the results of (6), (7), and (8), the API output for the 0.35th Caputo fractional derivative of $3x^2 + 2x + 1$ is:

$$
\boxed{= 4.040x^{1.65} + 2.222x^{0.65}}
$$