PHYS 562: Electromagnetic Theory

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Preface

Disclaimer

These notes are a work in progress, and thus typos, inaccuracies, and errors are to be expected. I'd be grateful if you could let me know of mistakes that you find while going through them—I'll happily add your name to the acknowledgements!

Goals and content

The goal of this course is to achieve an understanding of the mathematical description of electromagnetic phenomena and special relativity, and specially of how both these theories can be brought together. We will see how electrodynamics can be constructed as one of the fundamental gauge theories of nature. Although our treatment will be purely classical, toward the end of the course we will appreciate how a quantum mechanical treatment becomes necessary to explain certain phenomena. This course thus paves the way for the study of the fundamental theories of nature in a relativistic and quantum-mechanical framework, thus motivating the introduction of quantum electrodynamics and, more generally, quantum field theories.

Prerequisites

Basics of multivariate calculus, integration, and differentiation will be taken for granted. An elementary knowledge of electromagnetism at the level of D. J. Griffiths' *Introduction to Electrodynamics* will also be assumed. Although the course will be self-contained as we will introduce all the necessary concepts, the review of nonrelativistic electromagnetism will be fairly brief and will focus on more advanced topics like the use of Green functions to solve electrostatic problems.

Recommended books and reading

The main reference for the course is J. D. Jackson, *Classical Electrodynamics*, Third Edition, Wiley. The course and these lecture notes follow closely the structure and the discussions of this textbook.

Other texts that may improve your understanding of the topics included in the course are:

- E. M. Lifshitz and L. D. Landau, *Electrodynamics of Continuous Media* and *The Classical Theory of Fields*. The book treats many of the topics that will be covered in the course at a similar or slightly higher level.
- R. Feynman, *Feynman Lectures in Physics*, Volume two. Good reference for basic magnetostatics and electrostatics to build up intuition on various electromagnetic effects.

Contents

Chapter 1

Introduction to electrostatics

Electrodynamics as a fundamental theory of nature. There are four fundamental known forces in nature:

- Gravity,
- the electromagnetic force,
- the weak force, and
- the strong force.

Furthermore, we know that the weak and electromagnetic forces can be unified in the electroweak force.

Among the forces above, it is the electromagnetic one that is responsible for the vast majority of the phenomena occurring around us and the basic starting point for many different fields of study (e.g. chemistry, electronics, optics,...). Only for astronomical and subatomic length scales do the other forces become relevant.

The reason for this can be understood as a combination of two factors. The first one is that only gravity and electromagnetism have infinite range—the strong and weak forces have a very short (subatomic) range of action. The second one is that the electromagnetic force is roughly 10^{40} times stronger as gravitation. This is the main reason why electrodynamics has historically been such an important field of research.

Force	relative strength	Range of action
Strong force	~ 1	$\sim 10^{-15}$ m
Electromagnetic force	$\sim 1/137$	∞
Weak force	$\sim 10^{-6}$	$\sim 10^{13}$ m
Gravity	$\sim 10^{-39}$	∞

In this course we will deal with *classical* phenomena, for which quantum effects can be neglected. In general, this is a good approximation for situations involving a large number of photons, which is certainly the case for the vast majority of macroscopic effects. As an example, 1 meter away from a 100 watt light bulb, there are of the order of 10^{15} photons per centimeter square per second. However, for many microscopic phenomena like the emission of single photons from atoms or the photoelectric effect, quantum effects cannot be neglected and the classical description is not appropriate.

With this in mind, let us start or discussion following a rough historical order, thus starting with the first manifestations of electromagnetism known to humankind: electrostatic phenomena.

Coulomb's law. Electrostatics is the study of forces generated by time-independent distributions of charges. Coulomb demonstrated experimentally that the force $\vec{F}$ between two point-like charges is proportional to the magnitude of the charges and inversely proportional to the square of the distance between them:

$$\vec{F} = kq_1q_2\frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3} \equiv kq_1q_2\frac{\hat{d}}{d^2}, \tag{1.1}$$

where $\vec{d} = \vec{x}_1 - \vec{x}_2$ and $d = |\vec{d}|$ as shown in Figure 1.1, and k is a proportionality constant (Coulomb's constant) that depends on the units used. In the SI, $k = (4\pi\epsilon_0)^{-1} \simeq 9 \times 10^9\,\mathrm{N\,m^2\,C^{-2}}$, where ϵ_0 is the vacuum electric permittivity (also known as the "electric constant").

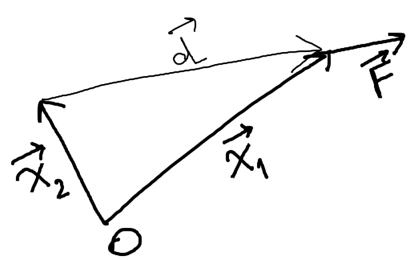

Figure 1.1

It is useful to define the *electric field* $\vec{E}$ as the force per unit charge acting at a given point. Thus, we have that $\vec{F} = q\vec{E}$, which means that

$$\vec{E} = kq_1\frac{\hat{d}}{d^2}. \tag{1.2}$$

The electric field is usually measured in V/m. An important property of electrostatics is the linear superposition principle, by which

$$\vec{E}(\vec{x}) = k\sum_{i=1}^{n} q_i \frac{\vec{x}-\vec{x}_i}{|\vec{x}-\vec{x}_i|^3}, \tag{1.3}$$

for a set of charges q_i located at positions $\vec{x}_i$. Similarly, for a continuous distribution of charge described by a charge density $\rho(\vec{x}')$, we have

$$\vec{E}(\vec{x}) = k\int \mathrm{d}^3x'\, \rho(\vec{x}')\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}, \tag{1.4}$$

where $\mathrm{d}^3x' = \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$ in Cartesian coordinates. The integral expression of the electric field can also describe a discrete distribution of point charges by using delta functions,

$$\rho(\vec{x}) = \sum_{i=1}^{n} q_i\, \delta(\vec{x}-\vec{x}_i). \tag{1.5}$$

Gauss' law. Consider a point charge q inside a closed surface S as shown in Figure 1.2. For a surface element $\mathrm{d}a$ on S with unit normal $\vec{n}$ at a distance r from q, we have

$$\vec{E}\cdot\vec{n}\,\mathrm{d}a = kq\cos\theta\frac{\mathrm{d}a}{r^2}. \tag{1.6}$$

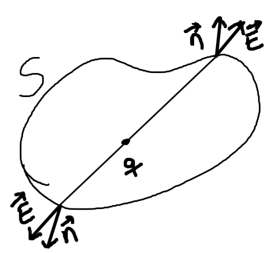

Figure 1.2

If $\mathrm{d}\Omega$ is the solid angle subtended by $\mathrm{d}a$, then $\mathrm{d}a\cos\theta = r^2\mathrm{d}\Omega$ and we can write

$$\vec{E}\cdot\vec{n}\,\mathrm{d}a = kq\,\mathrm{d}\Omega. \tag{1.7}$$

It is then intuitive to see that after integrating over the whole closed surface, we get

$$\oint_S \vec{E}\cdot\vec{n}\,\mathrm{d}a = \frac{q}{\epsilon_0}. \tag{1.8}$$

As before, generalizing to a continuous distribution of charge we arrive to

$$\oint_S \vec{E} \cdot \vec{n}\,\mathrm{d}a = \frac{1}{\epsilon_0} \int_V \mathrm{d}^3x\, \rho(\vec{x}), \tag{1.9}$$

where V is the volume enclosed by the surface S. Using the divergence theorem, we can rewrite Gauss' law as

$$\int_V \mathrm{d}^3x \left(\vec{\nabla} \cdot \vec{E} - \frac{\rho}{\epsilon_0} \right) = 0. \tag{1.10}$$

As this is valid for any arbitrary volume V, the integrand has to vanish locally:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}. \tag{1.11}$$

This is the differential form of Gauss' law.

The scalar potential. Along with its divergence given by Eq. (1.11), we need the curl of $\vec{E}$ in order to specify it. The curl of $\vec{E}$ can be obtained from Coulomb's law by using the identity

$$\frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} = -\vec{\nabla} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right). \tag{1.12}$$

Then, Eq. (1.4) can be rewritten as

$$\vec{E}(\vec{x}) = \frac{-1}{4\pi\epsilon_0} \vec{\nabla} \int \mathrm{d}^3x'\, \rho(\vec{x}') \frac{1}{|\vec{x} - \vec{x}'|}, \tag{1.13}$$

where we have exchanged the order of integration and differentiation. Since the curl of the gradient of a scalar function vanishes, we conclude that

$$\vec{\nabla} \times \vec{E} = 0. \tag{1.14}$$

The electric field is thus pure divergence and can be expressed as the gradient of a scalar function, which denote by $\Phi(\vec{x})$ and call *scalar potential*:

$$\vec{E} = -\vec{\nabla}\Phi, \tag{1.15}$$

with

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3x'\, \rho(\vec{x}') \frac{1}{|\vec{x} - \vec{x}'|}. \tag{1.16}$$

The physical interpretation of the scalar potential is that it corresponds to the potential energy per unit of charge of a test particle in an electrostatic field. This can be seen by the fact that the work done on a test charge q to displace it from point A to point B in the presence of an electrostatic field is

$$\begin{aligned} W &= -\int_A^B \vec{F}\cdot \mathrm{d}\vec{l} = -q\int_A^B \vec{E}\cdot \mathrm{d}\vec{l} \\ &= q\int_A^B \vec{\nabla}\Phi \cdot \mathrm{d}\vec{l} = q\int_A^B \mathrm{d}\Phi \\ &= q\left(\Phi_B - \Phi_A\right). \end{aligned} \tag{1.17}$$

Note that the line integral is independent of the path taken.

The differential form of Gauss' law, Eq. (1.11) can be rewritten in terms of the scalar potential as

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}, \tag{1.18}$$

which is called the *Poisson equation*. In the absence of a charge density at a point in space, it reduces to the *Laplace equation*

$$\nabla^2\Phi = 0. \tag{1.19}$$

Using the identity

$$\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\delta(\vec{x}-\vec{x}'), \tag{1.20}$$

it is easy to show that the integral expression for the scalar potential in Eq. (1.23) indeed satisfies the Poisson equation.

As you will show in the assignments, in the presence of a surface charge density $\sigma(\vec{x})$ the electric field has a discontinuity

$$\left(\vec{E}_2 - \vec{E}_1\right)\cdot\vec{n} = \frac{\sigma}{\epsilon_0}, \tag{1.21}$$

where the unit normal is directed from side 1 to side 2 of the surface. However, the scalar potential is continuous everywhere,

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_S \mathrm{d}a'\,\frac{\sigma(\vec{x}')}{|\vec{x}-\vec{x}'|}, \tag{1.22}$$

where $\mathrm{d}a'$ denotes the area element on the surface S. The scalar potential is however discontinuous in the presence of a dipole layer of strength $D(\vec{x})$. Denoting $\mathrm{d}\Omega$ the element of solid angle subtended by the area element $\mathrm{d}a$ of the dipole surface, we find

$$\Phi(\vec{x}) = -\frac{1}{4\pi\epsilon_0}\int_S \mathrm{d}\Omega\, D(\vec{x}), \tag{1.23}$$

and thus the potential jumps by an amount

$$\Phi_2 - \Phi_1 = \frac{D}{\epsilon_0} \tag{1.24}$$

when crossing from one side of the surface to the other.

Green's Theorem. Eq. (1.23) is useful for solving problems involving charge distributions without boundaries. However, in the practice we often deal with finite regions of space with prescribed boundary conditions. Let us derive a set of identities that are useful in those cases. We start with with the divergence theorem for a vector field $\vec{A}$ defined in a volume V bounded by a surface S with outward pointing normal $\vec{n}$,

$$\int_V \mathrm{d}^3x\, \vec{\nabla}\cdot\vec{A} = \oint_S \mathrm{d}a\, \vec{A}\cdot\vec{n} \tag{1.25}$$

and apply it to the vector field $\vec{A} = \phi\vec{\nabla}\psi$, with ϕ and ψ arbitrary scalar fields. Using

$$\vec{\nabla}\cdot(\vec{A}) = \vec{\nabla}\cdot(\phi\vec{\nabla}\psi) = \phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi, \tag{1.26}$$

$$\vec{A}\cdot\vec{n} = \phi\vec{n}\cdot\vec{\nabla}\psi = \phi n^i\partial_i\psi \equiv \phi\frac{\partial\psi}{\partial n}, \tag{1.27}$$

we arrive to *Green's first identity*,

$$\int_V \mathrm{d}^3x \left(\phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi\right) = \oint_S \mathrm{d}a\, \phi\frac{\partial\psi}{\partial n}. \tag{1.28}$$

Subtracting from Eq. (1.28) the same expression with ϕ and ψ interchanged, we obtain *Green's second identity*,

$$\int_V \mathrm{d}^3x \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) = \oint_S \mathrm{d}a \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right). \tag{1.29}$$

We can now specify to the following particular choice of scalar functions:

$$\psi = \frac{1}{R} \equiv \frac{1}{|\vec{x}-\vec{x}'|} \qquad \Rightarrow \qquad \nabla^2\psi = \nabla^2\left(\frac{1}{R}\right) = -4\pi\delta(\vec{x}-\vec{x}'), \tag{1.30}$$

$$\phi = \Phi \qquad \Rightarrow \qquad \nabla^2\phi = \nabla^2\Phi = -\frac{\rho}{\epsilon_0}, \tag{1.31}$$

where in the second line we have used the Poisson equation. Then, Eq. (1.29) results in

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \mathrm{d}^3x'\,\frac{\rho(\vec{x}')}{R} + \frac{1}{4\pi}\oint_S \mathrm{d}a'\left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right], \tag{1.32}$$

which is commonly known as *Green's theorem.* If the charge on the surface vanishes, we recover the result in Eq. (1.23). On the other hand, if there are no charges inside the volume V, Green's theorem uniquely specifies the potential everywhere in terms of the potential and its normal derivative on the surface of the volume. These specifications are usually referred to as *Cauchy boundary conditions*. But intuitively, it should be enough with specifying just one among the potential or its normal derivative (i.e. the electric field) at the surface, corresponding to *Dirichlet* and *Neumann boundary conditions*, respectively. Let us know show that this intuition is indeed correct.

Dirichlet or Neumann boundary conditions. To show that the potential in a volume V enclosed by a surface S is completely determined by either Dirichlet or Neumann boundary conditions, suppose that there exist two solutions Φ_1 and Φ_2. Given that each of them must separately obey the Poisson equation $\nabla^2\Phi_1 = \nabla^2\Phi_2 = 0$, the difference

$$U = \Phi_2 - \Phi_1 \tag{1.33}$$

satisfies $\nabla^2 U = 0$ inside V. If using Dirichlet boundary conditions, we know that on the boundary surface $U = \Phi_2 - \Phi_1 = 0$. Alternatively, for Neumann boundary conditions we have that $\partial U/\partial n = \partial\Phi_2/\partial n - \partial\Phi_1/\partial n = 0$ on S. In both cases, we can use Green's first identity (1.28) for $\phi = \psi = U$ to conclude that

$$\int_V \mathrm{d}^3x \left(U\nabla^2 U + \vec{\nabla}U \cdot \vec{\nabla}U\right) = \oint_S \mathrm{d}a\, U\frac{\partial U}{\partial n} = 0. \tag{1.34}$$

As U satisfies the Laplace equation, it follows that

$$\int_V \mathrm{d}^3x \left|\vec{\nabla}U\right|^2 = 0, \tag{1.35}$$

meaning that $|\vec{\nabla}U| = 0$ and thus U is constant within V. For Dirichlet boundary conditions, U vanishing on S means that $\Phi_1 = \Phi_2$ everywhere in V. For Neumann boundary conditions, Φ_1 and Φ_2 can only differ by a constant, and they thus lead to the same electric field configuration in V. We thus conclude that the solution to the Poisson equation is uniquely determined by either Dirichlet or Neumann boundary conditions. As a corollary, we deduce that Cauchy boundary conditions constitute an overspecification of the problem and it is in general not consistent to specify both Φ and $\partial\Phi/\partial n$ at the boundary.

Green's functions. When deriving Green's theorem, we chose ψ to be $1/R$ due to the fact that $\nabla'^2(1/R) = -4\pi\delta(\vec{x} - \vec{x}')$. However, this choice is not unique as there is a whole set of functions, called *Green's functions*, that satisfy the property[1]

$$\nabla'^2 G(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}'). \tag{1.36}$$

Indeed, we could have chosen any function

$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + F(\vec{x}, \vec{x}') \tag{1.37}$$

with F satisfying the Laplace equation $\nabla'^2 F(\vec{x}, \vec{x}') = 0$ in V. By choosing $\psi = G(\vec{x}, \vec{x}')$ in (1.29), we can use the freedom that $F(\vec{x}, \vec{x}')$ brings in order to eliminate one of the two surface integrals in the right-hand side. That way, we can obtain an expression for the scalar

[1] You can think of Green's functions as being the inverse of Laplacian operator, taking the delta function to be identity in the space of functions in V.

potential that only involves Dirichlet or Neumann boundary conditions and is therefore well posed to solve problems univocally.

Let us thus evaluate Green's second identity (1.29) for $\phi = \Phi$ and $\psi = G(\vec{x}, \vec{x}')$. After some simple manipulations, we arrive to

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \mathrm{d}^3x'\,\rho(\vec{x}')G(\vec{x}, \vec{x}') + \frac{1}{4\pi}\oint_S \mathrm{d}a'\left[G(\vec{x}, \vec{x}')\frac{\partial\Phi(\vec{x}')}{\partial n'} - \Phi(\vec{x}')\frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'}\right]. \quad (1.38)$$

Now the freedom in determining G allows us to cancel one of the terms in the surface integral. For instance, for Dirichlet boundary conditions we demand

$$G_D(\vec{x}, \vec{x}') = 0 \qquad \text{for } \vec{x}, \text{ on } S, \quad (1.39)$$

which leads to

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \mathrm{d}^3x'\,\rho(\vec{x}')G_D(\vec{x}, \vec{x}') - \frac{1}{4\pi}\oint_S \mathrm{d}a'\,\Phi(\vec{x}')\frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'}. \quad (1.40)$$

For Neumann boundary conditions, one would be tempted to demand $\partial G_N \partial n' = 0$ on S. However, this is not compatible with (1.36), which after using the divergence theorem for $\vec{A} = \vec{\nabla} G$ gives

$$\oint_S \mathrm{d}a'\,\frac{\partial G}{\partial n'} = \int_V \mathrm{d}^3x'\,\nabla^2 G = -4\pi. \quad (1.41)$$

The simplest boundary condition that we can choose compatible with this is

$$\frac{\partial G_N(\vec{x}, \vec{x}')}{\partial n'} = -\frac{4\pi}{A} \qquad \text{for } \vec{x} \text{ on } S, \quad (1.42)$$

where A is the area of S. This choice of Green's function leads to

$$\Phi(\vec{x}) = \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \mathrm{d}^3x'\,\rho(\vec{x}')G_N(\vec{x}, \vec{x}') + \frac{1}{4\pi}\oint_S \mathrm{d}a'\,G_N(\vec{x}, \vec{x}')\frac{\partial\Phi(\vec{x}')}{\partial n'}. \quad (1.43)$$

Here, $\langle\Phi\rangle_S$ denotes the average of the scalar potential over the boundary surface. The Neumann boundary condition is useful for "exterior" problems where V is the volume outside of a finite closed surface S. In that case, the total boundary surface is the union of S and the surface at infinity, which has infinite area and over which the average term vanishes.

Electrostatic energy. Recall our interpretation of the scalar potential in terms of a potential energy per unit charge, Eq. (1.17). Considering a set of charges q_i, $i = 1, \dots N$ located at positions $\vec{x}_i$, the total potential energy of the charges due to the forces acting between them is

$$\begin{aligned} W &= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j<i}\frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \\ &= \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}. \end{aligned} \quad (1.44)$$

The first equality can be arrived at by bringing each charge from infinity to its location in succession[2]. In the second line, we have just rewritten the sum in a symmetric way (the terms with $i = j$ are understood to be omitted from the sum).

In a leap of faith that will be justified *a posteriori*, we write the potential energy of a continuous charge distribution by analogy to (1.44) as

$$W = \frac{1}{8\pi\epsilon_0} \int \int \mathrm{d}^3x \, \mathrm{d}^3x' \, \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}. \tag{1.45}$$

Using the definition of the scalar potential (1.23), we can rewrite this as

$$W = \frac{1}{2} \int \mathrm{d}^3x \, \rho(\vec{x})\Phi(\vec{x}), \tag{1.46}$$

which can be further transformed using the Poisson equation, leading to

$$W = \frac{-\epsilon_0}{2} \int \mathrm{d}^3x \, \Phi(\vec{x})\vec{\nabla}^2\Phi(\vec{x}). \tag{1.47}$$

Integration by parts allows us to express this as

$$W = \frac{\epsilon_0}{2} \int \mathrm{d}^3x \, |\vec{\nabla}\Phi|^2 = \frac{\epsilon_0}{2} \int \mathrm{d}^3x \, |\vec{E}|^2, \tag{1.48}$$

where the integration is carried over a sufficiently large volume so that boundary terms can be neglected. Note that (1.48) does not make any explicit reference to the charges sourcing the electrostatic field—the energy is rather interpreted as being stored in the field itself. We can then talk about an electrostatic energy density permeating space,

$$w = \frac{\epsilon_0}{2}|\vec{E}|^2. \tag{1.49}$$

At this point, a careful reader would get worried about the fact that this expression for the energy density does not match the one that we wrote for the discrete distribution of charges (1.44). In fact, the integral (1.48) diverges when calculated for a single point-like charge charge! The reason for this is that in (1.48) we are integrating over the location of the charges and therefore not explicitly removing the infinite self-energy contributions as we did in (1.44). In practice and given that only *differences* in potential energy are physically relevant, these nominally infinite contributions do not lead to any inconsistency. You can easily check that after subtracting the self-energy terms, the expressions (1.44) and (1.44) agree with each other.

[2]This implies that we are calculating the potential energy with respect to the reference situation of all the charges being infinitely separated from each other

Chapter 2

Methods for boundary-value problems

In this chapter we will introduce some of the basic methods that allow to solve boundary-value problems in electrostatics. These are situations involving static charges surrounded by a surface where the potential or the charge density is specified. Even though the Green's function method the formal general solution of these problems, its practical implementation is sometimes challenging. We will review some setups that are amenable to this method and present some techniques that help in determining suitable Green's functions.

2.1 Method of images

The method of images can be applied to problems involving boundary surfaces and is closely related to the construction of the appropriate Green's function. Given a boundary surface with a fixed potential or charged density, we look for a suitable distribution of point charges that can reproduce the same boundary conditions as the surface. Importantly, these *image charges* have to be placed outside the volume of interest, as their potentials have to satisfy the Laplace equation inside it. The potential inside the volume can then be recovered by replacing the boundary surface by the image charges. Note that the locations and magnitudes of the image charges will in general also depend on the charges present inside the volume of interest.

The most classical example is that of a point charge held at a given distance to an infinite plane conductor at zero potential. The image charge in this case has the same magnitude as and opposite charge to the original one and is located diametrically opposed to it with respect to the conductor plane as shown in Figure 2.1.

Point charge outside of a conducting sphere. Another classical setup involves a point charge located outside[1] of a conducting sphere held at zero potential, see Figure 2.2. We take

[1] The case where the charge is located inside the sphere is completely analogous.

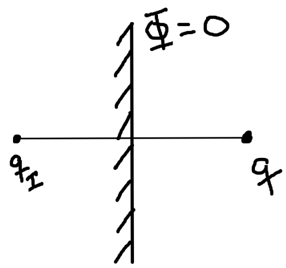

Figure 2.1

the sphere of radius a to be at the origin of coordinates and the charge q to be placed at a position $\vec{y}$. It is easy to see that a single image charge q' located inside the sphere, at a distance y' of the origin along the direction $\hat{y}$,

$$q' = -\frac{a}{y} q \qquad y' = \frac{a^2}{y} \tag{2.1}$$

reproduces the given boundary conditions. Indeed, the potential at the surface of the conducting sphere in an arbitrary direction $\hat{x}$

$$\Phi(x = a) = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{|\vec{x} - \vec{y}|} + \frac{q'}{|\vec{x} - \vec{y}'|} \right) = 0. \tag{2.2}$$

Substituting the boundary sphere for its image charge allows to calculate all relevant quantities. As an example, we can calculate the surface charge density induced on the conducting sphere as

$$\sigma = -\epsilon_0 \left. \frac{\partial \Phi}{\partial x} \right|_{x=a} = -\frac{q}{4\pi a} \frac{y^2 - a^2}{(y^2 + a^2 - 2ay \cos\gamma)^{3/2}}, \tag{2.3}$$

where γ is the angle between the observation point $\vec{x}$ and the location of the charge $\vec{y}$. The total induced charge on the sphere is exactly q', as required by Gauss's law.

This setup can be easily generalized using the linear superposition principle. For a sphere with a fixed charge Q, just add an additional charge with magnitude $Q - q'$ at the origin of coordinates. For a sphere held a fixed nonzero potential, add a charge $4\pi\epsilon_0 V a$ instead.

Green's function for the sphere. We can use the result above to construct the Green's function for the sphere. Indeed, the Green's function of a given geometry is nothing but a solution of the Poisson equation for a unit point charge, which is precisely what we have just calculated for the Dirichlet boundary condition. Adjusting the normalization to comply

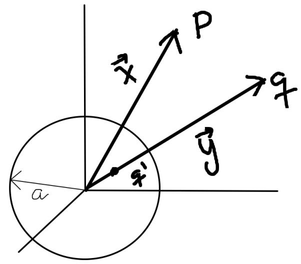

Figure 2.2

with (1.36), we have that

$$G_D(\vec{x},\, \vec{y}) = \frac{1}{|\vec{x}-\vec{y}|} - \frac{a}{y\left|\vec{x}-\frac{a^2}{y^2}\vec{y}\right|} \tag{2.4}$$

$$= \frac{1}{\left(x^2+y^2-2xy\cos\gamma\right)^{1/2}} - \frac{1}{\left(\frac{x^2y^2}{a^2}+a^2-2xy\cos\gamma\right)^{1/2}}. \tag{2.5}$$

The second expression in terms of the angle between the observation point $\vec{x}$ and the source point $\vec{y}$ is helpful to calculate $\partial G_D/\partial n$, where $\vec{n}$ is the outward unit normal in the direction of $\vec{y}$,

$$\left.\frac{\partial G_D}{\partial n}\right|_{y=a} = -\frac{1}{a}\frac{(x^2-a^2)}{\left(x^2+a^2-2ax\cos\gamma\right)^{3/2}}. \tag{2.6}$$

With this, we can use (1.40) to write down the potential generated outside the sphere with an arbitrary potential specified in its surface. In the absence of any charges outside the sphere, it is

$$\Phi(\vec{x}) = \frac{1}{4\pi}\int \mathrm{d}\Omega\, \Phi(a,\, \theta,\, \phi)\frac{a(x^2-a^2)}{\left(x^2+a^2-2ax\cos\gamma\right)^{3/2}}. \tag{2.7}$$

Here, $\mathrm{d}\Omega$ denotes the element of solid angle at each point y in the sphere specified by the spherical coordinates $(a,\, \theta,\, \phi)$. In the presence of a charge distribution external to the sphere, the above expression can be supplemented with the first term of (1.40) using the Green's function (2.4). The interior problem can be solved in a completely analogous way, and the final result for the Green's function only differs in an overall minus sign.

2.2 Separation of variables

The Laplace and Poisson equations can often be solved using the method of separation of variables, which allows to rewrite a partial differential equation as a set of ordinary differential equations. Let us see a few examples where that is the case.

Rectangular coordinates. Let us assume that the solution of the Laplace equation in cartesian coordinates,

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0, \tag{2.8}$$

can be written as a product

$$\Phi(x,\, y,\, z) = X(x)Y(y)Z(z). \tag{2.9}$$

Plugging this ansatz into the above equation and dividing by Φ leads to

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0. \tag{2.10}$$

As each term depends on a single variable, each function must separately satisfy

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\alpha^2, \qquad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -\beta^2, \qquad \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = \gamma^2, \tag{2.11}$$

where α, β, and γ are constants satisfying

$$\alpha^2 + \beta^2 = \gamma^2. \tag{2.12}$$

The ordinary differential equations can be easily solved, leading to a general solution for the potential

$$\Phi = \mathrm{e}^{\pm i\alpha x}\, \mathrm{e}^{\pm i\beta y}\, \mathrm{e}^{\pm\sqrt{\alpha^2+\beta^2}z}, \tag{2.13}$$

where α and β are so far unspecified, and can be narrowed down by imposing boundary conditions. For example, let us consider a rectangular setup with $\Phi = 0$ at $x = 0$, $x = a$, $y = 0$, $y = b$, and $z = 0$, and $\Phi = V(x, y)$ at $z = c$ (see figure 2.3). Then the solution can be expressed as a two-dimensional Fourier series

$$\Phi(x,\, y,\, z) = \sum_{n,m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \sinh\left(\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\pi z\right), \tag{2.14}$$

where the coefficients are linked to the double Fourier expansion of the boundary condition at $z = c$,

$$A_{nm} = \frac{4}{ab \sinh\left(\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\pi c\right)} \int_0^a \mathrm{d}x \int_0^b \mathrm{d}y\, V(x, y) \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \tag{2.15}$$

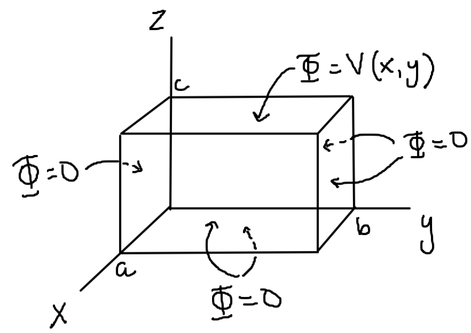

Figure 2.3

Spherical coordinates. The Laplace equation in spherical (r, θ, ϕ) coordinates reads

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0. \tag{2.16}$$

Employing the separation of variables ansatz

$$\Phi = \frac{U(r)}{r}P(\theta)Q(\phi), \tag{2.17}$$

the Laplace equation can be rewritten as

$$r^2\sin^2\theta\left[\frac{1}{U}\frac{\mathrm{d}^2U}{\mathrm{d}r^2} + \frac{1}{Pr^2\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P}{\mathrm{d}\theta}\right)\right] + \frac{1}{Q}\frac{\mathrm{d}^2Q}{\mathrm{d}\phi^2} = 0. \tag{2.18}$$

Similarly to what occurred with the cartesian case, the ϕ dependence is isolated in the last term, which must therefore be equal to a constant. Furthermore, the first term inside the brackets is only a function of r and comparing with the other terms must be proportional to $1/r^2$. Finally, the second term inside the bracket contains the θ dependence. All in all, we recover three separate equations for each of the functions in (2.17):

$$\frac{1}{Q}\frac{\mathrm{d}^2Q}{\mathrm{d}\phi^2} = -m^2, \tag{2.19}$$

$$\frac{\mathrm{d}^2U}{\mathrm{d}r^2} - \frac{l(l+1)}{r^2}U = 0, \tag{2.20}$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P}{\mathrm{d}\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0. \tag{2.21}$$

The ϕ equation can be readily solve giving

$$Q = \mathrm{e}^{\pm im\phi}, \tag{2.22}$$

where we see that m must be an integer if $\phi \in (-\pi,\, \pi)$. The radial equation is also easily solved by

$$U = Ar^{l+1} + Br^{-l}, \tag{2.23}$$

where A and B are integration constants and, as will be shown next, l is always an integer.

Problems with azimuthal symmetry. The solution of the θ equation is simplified if we assume that our problem has azimuthal symmetry, which implies that Φ does not depend on ϕ and thus $m = 0$. In that case, (2.21) reduces to the well-known Legendre differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P}{\mathrm{d}x}\right] + l(l+1)P = 0, \tag{2.24}$$

written here in terms of $x = \cos\theta$. The solutions of this equation are the Legendre polynomials $P_l(x)$, where l must be integer valued. The easiest way to specify them is via Rodriges' formula

$$P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l, \tag{2.25}$$

with which you can check that $P_0 = 1$, $P_1 = x$, $P_2 = (3x^2-1)/2$, and so on. The Legendre polynomials form a complete set of orthonormal functions on the interval $-1 \le x \le 1$. This allows us to construct the general solution for a problem with azimuthal symmetry as

$$\Phi(r,\, \theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta), \tag{2.26}$$

where the coefficients A_l and B_l are fixed by boundary conditions. If said boundary conditions are given at a sphere of radius a and we are interested in the interior problem (assuming no charges present at the origin), then $B_l = 0$ and

$$A_l = \frac{2l+1}{2a^l}\int_0^{\pi} \mathrm{d}\theta \sin\theta V(\theta) P_l(\cos\theta). \tag{2.27}$$

For the exterior problem, we just exchange the roles of A_l and B_l. A particularly important example is that of the potential generated at a point $\vec{x}$ due to a point charge located at a point $\vec{x}'$. Without any loss of generality, take $\vec{x}'$ to be on the z axis. Then, for any point $\vec{x} \neq \vec{x}'$ also located along the z axis, we have

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|r-r'|} = \frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l, \tag{2.28}$$

where $r_> = \max(r,\, r')$, $r_< = \min(r,\, r')$ in the last equality we have just Taylor expanded $(1 - r_</r_>)^{-1}$. For points $\vec{x}$ off axis, it is reasonable to guess that the above expression generalizes to

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma), \tag{2.29}$$

where γ is the angle between $\vec{x}$ and $\vec{x}'$ (see figure 2.4), i.e. θ for $\vec{x}'$ along the z axis. Indeed, note that this expansion is of the form (2.26) and reproduces the required potential on the z axis by construction. Then, using the fact that the expansion (2.26) is unique (which follows from the completeness of the Legendre polynomials), we conclude that the expansion (2.29) must be valid for all points.

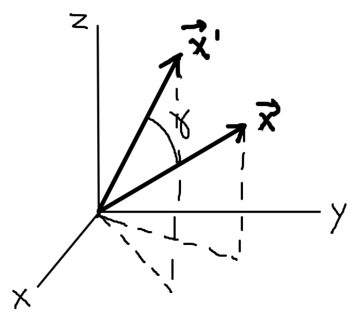

Figure 2.4

Problems without azimuthal symmetry. For charge distributions lacking azimuthal symmetry, we have to solve (2.21) for nonvanishing m. It can be shown that solutions can still only be found for nonnegative integer values of l and for m taking the values $-l$, $-(l-1)$, $\ldots$, 0, $\ldots$, $(l-1)$, l. Then (2.21) is solved by the associated Legendre functions, which can be represented as

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{\frac{m}{2}} \frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}(x^2-1)^l. \tag{2.30}$$

For a fixed m, these functions form a complete orthogonal set in the index l for $-1 \leq x \leq 1$. As the functions $Q_m = \exp(\pm im\phi)$ introduced to solve the ϕ equation also form a complete set in m on the interval $0 \leq \phi \leq 2\pi$, the product $P_l^m Q_m$ form a complete orthogonal set on the surface of the unit sphere. This motivates the introduction of the spherical harmonics

$$Y_{lm}(\theta,\, \phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) \mathrm{e}^{im\phi}, \tag{2.31}$$

which in this form are normalized such that the general solution for a boundary-value problem in spherical harmonics can be written as

$$\Phi(r,\,\theta,\,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}r^{l}+B_{lm}r^{-(l+1)}\right]Y_{lm}(\theta,\,\phi). \tag{2.32}$$

The coefficients A_{lm} and B_{lm} correspond to the expansion in spherical harmonics of the boundary condition $V(\theta,\,\phi)$ on a spherical surface,

$$A_{lm}=\int \mathrm{d}\Omega\, Y^{*}_{lm}(\theta,\,\phi)V(\theta,\,\phi) \tag{2.33}$$

for the interior problem and similarly for B_{lm} for the exterior one. There is a useful relation between Legendre polynomials and spherical harmonics which is usually referred to as the addition theorem. Given two vectors $\vec{x}$ and $\vec{x}'$ with corresponding spherical coordinates and denoting the angle between them as γ, it is easy to show the following relation:

$$P_{l}(\cos\gamma)=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y^{*}_{lm}(\theta',\,\phi')Y_{lm}(\theta,\,\phi). \tag{2.34}$$

With this, we can rewrite the expansion (2.29) of the potential due to a point charge as

$$\frac{1}{|\vec{x}-\vec{x}'|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^{l}}{r_>^{l+1}}Y^{*}_{lm}(\theta',\,\phi')Y_{lm}(\theta,\,\phi). \tag{2.35}$$

Spherical Green's functions The expression of the potential due to a point charge (2.35) can be seen as the spherical harmonics expansion of the Green's function in the absence of any boundary conditions. To solve problems where the boundary conditions are present and are given on spherical surfaces, it is useful to find an expansion of the Green's function in terms of spherical harmonics. Let us go back to our example of the exterior problem with a spherical Dirichlet boundary conditions at $r=a$, for which we obtained the Green's function (2.4) using the image method. We can now expand the two terms in (2.4) using spherical harmonics to find

$$G_D(\vec{x},\vec{x}')=4\pi\sum_{l,\,m}\frac{1}{2l+1}\left[\frac{r_<^{l}}{r_>^{l+1}}-\frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right]Y^{*}_{lm}(\theta',\,\phi')Y_{lm}(\theta,\,\phi). \tag{2.36}$$

This expression is manifestly symmetric in r and r'. You can check that $G_D=0$ if either r or r' is equal to a and that for fixed r' it is a solution of the Laplace equation in r for $r\neq r'$.

To construct the general solution of the potential

$$\Phi(\vec{x})=\frac{1}{4\pi\epsilon_0}\int_V \mathrm{d}^3x'\,\rho(\vec{x}')G_D(\vec{x},\,\vec{x}')-\frac{1}{4\pi}\oint_S \mathrm{d}a'\,\Phi(\vec{x}')\frac{\partial G_D(\vec{x},\,\vec{x}')}{\partial n'}, \tag{2.37}$$

we need to evaluate

$$\frac{\partial G_D}{\partial n'} = -\frac{\partial G_D}{\partial r'} = \frac{4\pi}{a^2}\sum_{l,\,m}\left(\frac{r}{a}\right)^l Y^*_{lm}(\theta',\,\phi')Y_{lm}(\theta,\,\phi). \tag{2.38}$$

In the absence of any charge outside of the sphere of radius a, we can combine the three expressions above to obtain the following expression of the exterior potential:

$$\Phi(\vec{x}) = -\sum_{l,\,m}\left[\int \mathrm{d}\Omega'\, V(\theta',\,\phi')Y^*_{lm}(\theta',\,\phi')\right]\left(\frac{a}{r}\right)^{l+1} Y_{lm}(\theta,\,\phi). \tag{2.39}$$

This coincides with the result that we obtained for a unit sphere in (2.32). The power of the Green's function method is that it allows us to easily generalize this result to situations including charge distributions within the volume of interest.

General expression of the Green function. It is also possible to obtain the spherical harmonic expansion of the Green function by explicitly solving the differential equation

$$\nabla^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}') \tag{2.40}$$

with the desired boundary conditions. In spherical coordinates, the delta function becomes

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\delta(r-r')\delta(\phi-\phi')\delta(\cos\theta-\cos\theta') \tag{2.41}$$

$$= \frac{1}{r^2}\delta(r-r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi), \tag{2.42}$$

Where in the second line we have used the completeness relation for spherical harmonics. Now, the Green function can be expanded in spherical harmonics as

$$G(\vec{x},\vec{x}') = \sum_{l,m} A_{lm}(r,r',\theta',\phi')Y_{lm}(\theta,\phi). \tag{2.43}$$

Acting with the Laplacian on this expression gives

$$\nabla^2 G(\vec{x},\vec{x}') = \sum_{l,m}\left\{\frac{1}{r}\frac{\partial^2(rA_{lm})}{\partial r^2}Y_{lm} + A_{lm}\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_{lm}}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_{lm}}{\partial\phi^2}\right]\right\} \tag{2.44}$$

$$= \sum_{l,m}\left[\frac{1}{r}\frac{\partial^2(rA_{lm})}{\partial r^2} - \frac{l(l+1)}{r^2}A_{lm}\right]Y_{lm}. \tag{2.45}$$

To get to the second line, we have used the fact that the spherical harmonics are a solution of the angular part of the Laplace equation in spherical coordinates. Inserting (2.42) and (2.45) into (2.40), we find that A_{lm} can be written as

$$A_{lm}(r,r',\theta',\phi') = g_l(r,r')Y^*_{lm}(\theta',\phi'), \tag{2.46}$$

with the radial function satisfying

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r^2}\left(rg_l(r,r')\right)-\frac{l(l+1)}{r^2}g_l(r,r')=-\frac{4\pi}{r^2}\delta(r-r'). \tag{2.47}$$

Everywhere except at points with $r=r'$, the homogeneous radial equation can be straightforwardly solved to give

$$g_l(r,r')=\begin{cases}Ar^l+B\frac{1}{r^{l+1}}, & \text{if} \quad r<r'\\ A'r^l+B'\frac{1}{r^{l+1}}, & \text{if} \quad r>r'\end{cases}. \tag{2.48}$$

The four coefficients in this expression are not all independent. In fact, the symmetry of the Green function under the exchange of r and r' forces g_l to be of the form

$$g_l(r,r')=\left(Ar_<^l+B\frac{1}{r_<^{l+1}}\right)\left(A'r_<^l+B'\frac{1}{r_<^{l+1}}\right) \tag{2.49}$$

$$=C\left(r_<^l+D\frac{1}{r_<^{l+1}}\right)\left(D'r_>^l+\frac{1}{r_>^{l+1}}\right), \tag{2.50}$$

where in the second line we have removed one spurious coefficient. As usual, $r_{</>}$ is the smaller/larger between r and r'.

Another one of the coefficients is fixed by considering how the solution behaves at $r=r'$. For that, we multiply (2.47) by r and integrate it over the interval $r\in(r'-\epsilon,\, r+\epsilon)$ with infinitesimal ϵ. Taking into account the fact that the Green function is a continuous function everywhere in the volume of interest, we obtain the requirement

$$\left.\frac{\mathrm{d}(rg_l)}{\mathrm{d}r}\right|_{r=r'+\epsilon}-\left.\frac{\mathrm{d}(rg_l)}{\mathrm{d}r}\right|_{r=r'-\epsilon}=-\frac{4\pi}{r'}. \tag{2.51}$$

We thus see that the delta function induces a discontinuity in the radial derivative of the Green function. It is possible to evaluate the derivatives above explicitly, resulting in

$$\left.\frac{\mathrm{d}(rg_l)}{\mathrm{d}r}\right|_{r=r'+\epsilon}=C\left(r'^l+D\frac{1}{r'^{l+1}}\right)\left((l+1)D'r'^l-l\frac{1}{r'^{l+1}}\right), \tag{2.52}$$

$$\left.\frac{\mathrm{d}(rg_l)}{\mathrm{d}r}\right|_{r=r'-\epsilon}=C\left(D'r'^l+\frac{1}{r'^{l+1}}\right)\left((l+1)r'^l-Dl\frac{1}{r'^{l+1}}\right), \tag{2.53}$$

where the limit $\epsilon\to 0$ has already been taken. Plugging these back into (2.51), we arrive at the simple condition

$$C=\frac{4\pi}{(2l+1)(1-DD')}, \tag{2.54}$$

and thus to the general expression for the radial Green function

$$g_l(r,r')=\frac{4\pi}{(2l+1)(1-DD')}\left(r_<^l+D\frac{1}{r_<^{l+1}}\right)\left(D'r_>^l+\frac{1}{r_>^{l+1}}\right), \tag{2.55}$$

The remaining two integration constants are fixed by the boundary conditions specific to each geometry. Let us discuss a few examples:

- For $V = \mathbb{R}^3$, avoiding the divergences in the Green function at $r = 0$ and $r \to \infty$ is only possible if $D = D' = 0$. We thus recover the result (2.35).

- If the boundary surfaces are concentric spheres of radii $a < b$ (see figure 2.5), we demand that $g_l = 0$ at $r = a$ and $r = b$. It is easy to see that in this case

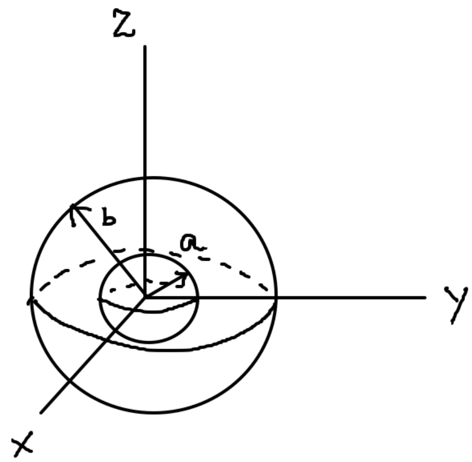

Figure 2.5

$$D = -a^{2l+1} \quad \text{and} \quad D' = -\frac{1}{b^{2l+1}}, \tag{2.56}$$

and the Dirichlet Green function on the spherical shell with $a < r, r' < b$ can be written as

$$G_D(\vec{x}, \vec{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)}{(2l+1)\left[1 - \left(\frac{a}{b}\right)^{2l+1}\right]} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi). \tag{2.57}$$

The Green function (2.36) for the interior problem can be recovered by taking $b \to \infty$.

- The general expression (2.55) also allows to find the Neumann Green function. For that, we would just need to impose Neumann boundary conditions (1.42) instead of Dirichlet ones.

Cylindrical coordinates. A similar analysis to the ones before can be performed in situations that feature cylindrical symmetry. The Laplace equation can be shown to also be separable in that case, and the solutions can be expressed in terms of Bessel functions. We will not discuss them in detail here, since the procedure is qualitatively analogous to the one we just followed in spherical coordinates.

Chapter 3

Multipole expansion and dielectrics

So far we have dealt with the idealized situation of electrostatic fields and charge distributions in vacuum. In practice, we are often interested in problems involving media whose electric responses have to be taken into account. Fortunately, in many situations, the microscopic details of the distribution of charges in media can be described at the macroscopic level using only a few quantities: the multipole moments.

Multipole expansion. Consider a distribution of charge $\rho(\vec{x}')$ that is localized within a spherical region around the origin. We know that the generated electrostatic potential at a point $\vec{x}$ can be written as

$$\Phi(\vec{x}) = \int \mathrm{d}^3x' \, \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}. \tag{3.1}$$

For points outside of that sphere such that $|\vec{x}| \gg |\vec{x}'|$, we can Taylor expand

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r} + \frac{\vec{x}\cdot\vec{x}'}{r^3} + \frac{1}{6}\sum_{k,l=1}^{3} \frac{3x_k x_l - r^2\delta_{kl}}{r^5}(3x_l' x_l' - r'^2\delta_{kl}) + \dots, \tag{3.2}$$

where as usual $r = |\vec{x}|$ and, as is conventional, in the last term we have subtracted a vanishing term (the one proportional to r'^2). With this, we can expand the potential in multipole components as

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2}\sum_{i,j} Q_{ij}\frac{x_i x_j}{r^5} + \dots\right]. \tag{3.3}$$

Here, q is the total charge or monopole moment

$$q = \int \mathrm{d}^3x' \, \rho(\vec{x}'), \tag{3.4}$$

$\vec{p}$ is the electric dipole moment

$$\vec{p} = \int \mathrm{d}^3x' \, \vec{x}'\rho(\vec{x}'), \tag{3.5}$$

and Q_{ij} is the quadrupole moment tensor

$$Q_{ij} = \int \mathrm{d}^3x' (3x'_i x'_j - r'^2 \delta_{ij}) \rho(\vec{x}'). \tag{3.6}$$

Note that the subtraction mentioned above in (3.2) the expansion renders the quadrupole moment traceless. The electric field can be readily calculated by taking the gradient in (3.2)

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \left[q\frac{\vec{x}}{r^3} + \frac{3\vec{x}(\vec{p}\cdot\vec{x}) - r^2\vec{p}}{r^5} + \dots \right], \tag{3.7}$$

where we only show the monopole and dipole terms. For higher-order terms in the expansion, it is more convenient to work in spherical coordinates. We can plug in the expansion (2.35) in terms of spherical harmonics into (3.1) to find

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{1}{r^{l+1}} Y_{lm}(\theta,\, \phi), \tag{3.8}$$

with the multipole moments

$$q_{lm} = \int \mathrm{d}^3x'\, Y^*_{lm}(\theta',\, \phi') r'^l \rho(\vec{x}'). \tag{3.9}$$

You can check that the multipole moments in spherical coordinates can be related to the ones in Cartesian coordinates order by order. For instance, $q_{00} = q/\sqrt{4\pi}$, $q_{10} = p_z\sqrt{3/4\pi}$, etc.

You may wonder why for $l > 1$ there seems to be more Cartesian multipole moments at each order—$(l+1)(l+2)/2$— compared to spherical ones—$(2l+1)$—. The reason is that the Cartesian tensors are reducible while the spherical ones irreducible, meaning that not all the Cartesian ones are actually independent. This is what allowed us to subtract the trace part from the quadrupole moment in (3.6).

Also important is the fact that, in general, the multipole moments depend on the choice of the origin of the coordinate system. You can easily check that for a point source, only the monopole q is invariant under a transformation $\vec{x}' \to \vec{x}' + \vec{a}$, for a constant vector $\vec{a}$. For two point charges of charges $+e$ and $-e$ located at positions x_0 and x_1, we have

$$q_{lm} = e\left[r_0^l Y^*_{lm}(\theta_0,\, \phi_0) - r_1^l Y^*_{lm}(\theta_1,\, \phi_1) \right] \tag{3.10}$$

and the monopole (total charge) vanishes. The dipole $\vec{p} = e(\vec{x}_0 - \vec{x}_1)$ is manifestly independent of the choice of coordinate origin, while higher multipoles are not. This result can be proven to be true in general: only the lowest nonvanishing multipole moment of a charge distribution is independent of the choice of origin of the coordinates.

A quick look at the above equations shows that for large r, the nth term in the multipole expansion of the potential falls as $1/r^{n+1}$, while that of the electric field falls as $1/r^{n+2}$. This justifies truncating the expansion when evaluating the electrostatic field at large distances away from the source charge distribution.

Multipole expansion of the electrostatic energy. We know that the electrostatic energy density of a localized charge distribution $\rho(\vec{x})$ in an external potential $\Phi(\vec{x})$ can be written as

$$W = \int d^3x\, \rho(\vec{x})\Phi(\vec{x}). \tag{3.11}$$

If the potential is slowly varying, we can perform a Taylor expansion

$$\Phi(\vec{x}) = \Phi(0) + \vec{x}\cdot\vec{\nabla}\Phi(0) + \frac{1}{2}\sum_{i,j} x_i x_j \left.\frac{\partial^2\Phi}{\partial x_i \partial x_j}\right|_0 + \dots \tag{3.12}$$

$$= \Phi(0) + \vec{x}\cdot\vec{E}(0) + \frac{1}{2}\sum_{i,j} x_i x_j \left.\frac{\partial E_j}{\partial x_i}\right|_0 + \dots, \tag{3.13}$$

$$= \Phi(0) + \vec{x}\cdot\vec{E}(0) + \frac{1}{6}\sum_{i,j} (3x_i x_j - r^2\delta_{ij}) \left.\frac{\partial E_j}{\partial x_i}\right|_0 + \dots, \tag{3.14}$$

where in the second line we have substituted $\vec{E} = -\vec{\nabla}\Phi$ and in the third line we have subtracted $r^2\vec{\nabla}\cdot\vec{E}(0)/6$ from the last term since for an external field $\vec{\nabla}\cdot\vec{E} = 0$. Now we can insert this expansion together with the definitions of the first multipoles into (3.11), arriving at

$$W = q\Phi(0) - \vec{p}\cdot\vec{E}(0) - \frac{1}{6}\sum_{i,j} Q_{ij} \left.\frac{\partial E_j}{\partial x_i}\right|_0 + \dots \tag{3.15}$$

This expansion demonstrates how the different multipoles interact with an external electric field: the total charge couples to the potential, the dipole to the electric field, the quadrupole to the gradient of the field, etc. This expansion can also be used to calculate the interaction energy between two dipoles, using the dipole field (3.7)

$$W_{12} = \frac{1}{4\pi\epsilon_0}\left[\frac{\vec{p}_1\cdot\vec{p}_2}{r^3} - 3\frac{(\vec{p}_1\cdot x)(\vec{p}_2\cdot x)}{r^5}\right], \tag{3.16}$$

where $\vec{x}$ is the vector pointing from $\vec{p}_2$ to $\vec{p}_1$ and $r = |\vec{x}|$. The sign of the interaction energy determines the sign of the interaction: it depends on both the relative orientation of the dipoles as well as their orientation with respect to the line joining their centers, as depicted in Figure 3.1.

Dielectrics. When dealing with media that are not electrically inert (as opposed to the vacuum or, to a good approximation, air), we have to take into account the electric response of the microscopic charge distribution in the material of interest. We are often interested with the macroscopic behavior of the charges and fields, and we thus would like to average the microscopic fields over a sufficiently large region.

The first thing to note is that the divergenceless condition $\vec{\nabla}\times\vec{E} = 0$ of the microscopic field at every point readily translates into the same condition for the macroscopic field. The

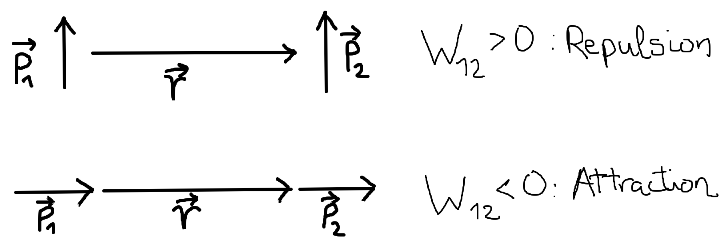

Figure 3.1

averaged electrostatic field can therefore also be derived from a potential $\Phi(\vec{x})$. To obtain this potential in a medium made up of a large number of atoms or molecules, we make use of the multipole expansion.

If the molecules are uncharged on average, the monopole term corresponds to the free charges existing in the material, with a distribution $\rho(\vec{x}')$. The next term, the dipole, is often the dominant molecular multipole that responds to an applied electrostatic field. The averaged dipole moment per unit volume is known as the *electric polarization* $\vec{P}$ of the medium, and is given by

$$\vec{P}(\vec{x}') = \sum_i N_i \langle \vec{p}_i \rangle , \tag{3.17}$$

where $\vec{p}_i$ is the dipole moment of the ith type of molecule in the medium, which has average number per unit volume N_i. The average is taken over a small volume d^3x' around the point $\vec{x}'$. Neglecting higher-order multipoles, the total electrostatic potential at a point $\vec{x}$ can be obtained from (3.3) by integrating over all space,

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3x' \left[\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\vec{P}(\vec{x}')\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} \right] . \tag{3.18}$$

Using $(\vec{x}-\vec{x}')/|\vec{x}-\vec{x}'|^3 = \vec{\nabla}'(1/|\vec{x}-\vec{x}'|)$ and integrating the second term by parts, we arrive at

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3x' \, \frac{1}{|\vec{x}-\vec{x}'|} \left[\rho(\vec{x}') - \vec{\nabla}' \cdot \vec{P}(\vec{x}') \right] . \tag{3.19}$$

Notice that we can interpret this expression as the one for the potential generated by a charge distribution $\rho(\vec{x}') - \vec{\nabla}' \cdot \vec{P}(\vec{x}')$. Combining it with $\vec{E} = -\vec{\nabla}\Phi$, the first Maxwell equation becomes

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \left[\rho - \vec{\nabla} \cdot \vec{P} \right] . \tag{3.20}$$

The second term represents a polarization-charge density that can arise when there exists a spatial variation of the polarization. We can rewrite this in a way that resembles the vacuum

Maxwell equation by defining the *electric displacement* vector

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}, \tag{3.21}$$

which satisfies

$$\vec{\nabla} \cdot \vec{D} = \rho. \tag{3.22}$$

In order to solve this equation we need to characterize the electric response of a medium via a *constitutive relation* connecting $\vec{D}$ and $\vec{E}$. Such relation can in principle be very complex, but we can make a number if restrictive assumptions to study the most simple materials. First, we assume that the response of the system to the applied electric field is linear, meaning that

$$D_i = \sum_j \epsilon_{ij} E_j, \tag{3.23}$$

where ϵ_{ij} is the *electric permittivity* or *dielectric tensor*. Substances other than ferroelectrics have a linear response as long as the applied fields are sufficiently weak. If the medium is isotropic (i.e. it doesn't have any preferred direction), then ϵ_{ij} is diagonal with three equal diagonal elements, and therefore the displacement is parallel to the applied electric field,

$$\vec{D} = \epsilon \vec{E}. \tag{3.24}$$

The proportionality constant ϵ is known as the *dielectric constant* or relative electric permittivity. If the medium is not only isotropic but also homogeneous (i.e. uniform), then ϵ is independent of position and (3.22) can be simply rewritten as

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon}. \tag{3.25}$$

Thus, the solution of problems involving linear, isotropic, and homogeneous media analogous to those in vacuum except that the electric fields produced by given charges are reduced by a factor ϵ_0/ϵ. The fact that $\epsilon > \epsilon_0$ in these materials is a consequence of the fact that the induced polarization is parallel to the electric field,

$$\vec{P} = \epsilon_0 \chi_e \vec{E}, \tag{3.26}$$

where χ_e is the *electric susceptibility* of the medium. From the definition of the electric displacement (3.21), we have that

$$\epsilon = \epsilon_0 (1 + \chi_e), \tag{3.27}$$

and thus $\epsilon > \epsilon_0$ for $\chi_e > 0$. Physically, the reduction of the electric field is due to the induced dipoles, which generate fields opposing the external one, as is exemplified in figure 3.2.

Often, we have to deal with situations where different media with different electric responses are juxtaposed, and we therefore have to consider boundary conditions on $\vec{D}$ and $\vec{E}$ at

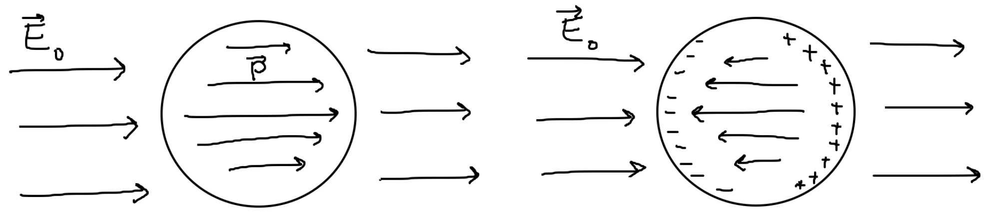

Figure 3.2

the interfaces of media. Using Maxwell's equations, it is possible to show that the normal components of $\vec{D}$ satisfy

$$(\vec{D}_1 - \vec{D}_2) \cdot \vec{n}_{21} = \sigma, \tag{3.28}$$

while the tangential components of $\vec{E}$ satisfy

$$(\vec{E}_2 - \vec{E}_1) \times \vec{n}_{21} = 0. \tag{3.29}$$

Here, $\vec{n}_{21}$ is the unit normal to the interface, directed from region 1 to region 2, and σ is the macroscopic surface charge density on the boundary, which is not included in the polarization charge.

Chapter 4

Magnetostatics

Although magnetic phenomena were known experimentally since early times, the laws of magnetism took longer than those of electrostatics to be derived. The main reason for this can be summarized by the fact that free magnetic charges do not exist. We will understand the reason behind this better when we discuss the unified theory of electromagnetism in the next chapters. For now, let us state the basic equations that allow us to study steady-state magnetic phenomena.

Basic equations of magnetostatics. The basic entity of magnetostatics is the *magnetic induction* or magnetic flux density $\vec{B}$, which obeys a very similar relation to Coulomb's law for electrostatics,

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3x' \, \vec{J}(\vec{x}') \times \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}, \tag{4.1}$$

which is sometimes referred to as the Biot-Savart law. Here, $\vec{J}$ is a current density measured in units of positive charge crossing unit area per unit time (coulombs per square meter-second or amperes per square meter in SI units). The proportionality constant μ_0 is known as the *vacuum permeability* or magnetic constant, and in SI units it is $\mu_0/4\pi = 10^{-7}$ newton per square ampere (note that $\vec{B}$ has dimensions of newtons per ampere-meter). This law was experimentally established by Oersted, Biot and Savart, and Ampère, who deduced it from the study of the force between permanent magnets and wires carrying electric charge.

Using the identity (1.12), it is possible to rewrite (4.1) as

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \vec{\nabla} \times \int \mathrm{d}^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}, \tag{4.2}$$

from which we immediately deduce that $\vec{B}$ is divergence-free,

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{4.3}$$

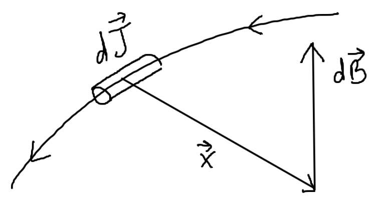

Figure 4.1

which constitutes the first equation of magnetostatics in analogy with $\vec{\nabla} \times \vec{E} = 0$ in electrostatics. Next, we calculate the curl of $\vec{B}$. Using the identity $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$ for an arbitrary vector field $\vec{A}$, we can rewrite (4.2) as

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \vec{\nabla} \int \mathrm{d}^3x' \, \vec{J}(\vec{x}') \cdot \vec{\nabla} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) - \frac{\mu_0}{4\pi} \int \mathrm{d}^3x' \, \vec{J}(\vec{x}') \nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \tag{4.4}$$

$$= -\frac{\mu_0}{4\pi} \vec{\nabla} \int \mathrm{d}^3x' \, \vec{J}(\vec{x}') \cdot \vec{\nabla}' \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) + \mu_0 \vec{J}(\vec{x}) \tag{4.5}$$

$$= \mu_0 \vec{J}(\vec{x}) + \frac{\mu_0}{4\pi} \vec{\nabla} \int \mathrm{d}^3x' \, \frac{\vec{\nabla}' \cdot \vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}. \tag{4.6}$$

To get to the second line, we have used

$$\vec{\nabla} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) = -\vec{\nabla}' \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \quad \text{and} \quad \nabla^2 \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) = -4\pi \delta(\vec{x} - \vec{x}'), \tag{4.7}$$

and we have integrated by parts to get to the third line. Conservation of charge links the gradient of the current density to the time variation of the charge density via the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0. \tag{4.8}$$

In steady-state situations where time derivatives vanish, we simply have $\vec{\nabla} \cdot \vec{J} = 0$, and we can therefore further simplify (4.6) to

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \tag{4.9}$$

This is the second equation of magentostatics and is the analogous of $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ in electrostatics. Of course, the two equations that we just derived are nothing but the two magnetic Maxwell equations for time-independent situations where time derivatives vanish.

Ampère's Law. In electrostatics, we derived Gauss' law as the integral form of $\vec{\nabla}\cdot\vec{E}=\rho/\epsilon_0$. In magnetostatics, we can do the same with (4.9) to arrive to Ampère's law. Consider an open surface S bounded by a closed curve C as in figure 4.2. We start by integrating the component of (4.9) normal to the surface,

$$\int_S \mathrm{d}a\,\vec{\nabla}\times\vec{B}\cdot\vec{n}=\mu_0\int_S \mathrm{d}a\,\vec{J}\times\vec{n}, \tag{4.10}$$

We now apply Stoke's theorem to the left hand-side, and note that the right-hand side is just the total current I crossing through the curve, to arrive at

$$\oint_C \vec{B}\cdot\mathrm{d}\vec{l}=\mu_0 I, \tag{4.11}$$

which is the usual form of Ampère's law.

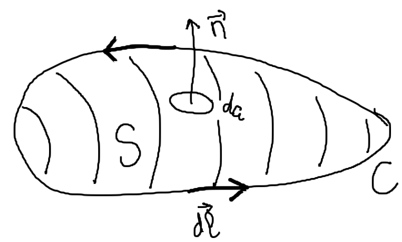

Figure 4.2

Magnetic force and torque. A magnetic induction can be measured by the force that it exerts on charge currents. The total force on given a current density $\vec{J}$ in an external magnetic flux density $\vec{B}$ is

$$\vec{F}=\int \mathrm{d}^3x\,\vec{J}(\vec{x})\times\vec{B}(\vec{x}), \tag{4.12}$$

and the torque is

$$\vec{N}=\int \mathrm{d}^3x\,\vec{x}\times\left[\vec{J}(\vec{x})\times\vec{B}(\vec{x})\right]. \tag{4.13}$$

Using (4.12) together with (4.1) for the specific case of two current loops traversed by currents I_1 and I_2, one can arrive at the following expression for the force acting between them,

$$\vec{F}_{12}=-\frac{\mu_0}{4\pi}I_1I_2\oint\oint\frac{(\mathrm{d}\vec{l}_1\cdot\mathrm{d}\vec{l}_2)\vec{x}_{12}}{|\vec{x}_{12}|^3}. \tag{4.14}$$

Here, the current I is defined as the integral of the current density over the small cross section of the wire, and the geometric quantities in the expression correspond to those shown in figure 4.3. This expression is manifestly symmetric in 1 and 2 and thus explicitly satisfies Newton's third law.

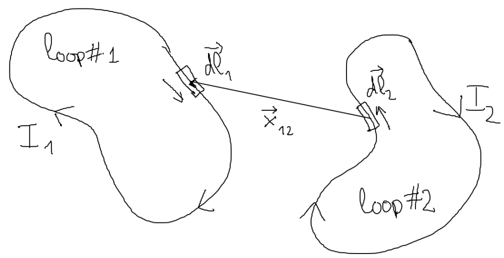

Figure 4.3

Vector potential. In analogy to the scalar potential of electrostatics, the divergencelessness of $\vec{B}$ implies that it can be written as a curl of some vector field $\vec{A}$ which we call the *vector potential*,

$$\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x}). \tag{4.15}$$

Importantly, this condition only specifies $\vec{A}$ up to the gradient of an arbitrary scalar function: the vector potential can be freely transformed by a *gauge transformation*

$$\vec{A} \to \vec{A} + \vec{\nabla}\Psi \tag{4.16}$$

without affecting (4.15). The freedom to perform gauge transformations on the vector potential allows us to make $\vec{\nabla} \cdot \vec{A}$ take any convenient functional form. A choice that is often convenient is to choose Ψ such that $\vec{\nabla} \cdot \vec{A} = 0$, as in that case it is easy to show that each component of $\vec{A}$ satisfies the Poisson equation

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}, \tag{4.17}$$

which follows by taking the curl of (4.15) and applying (4.9). In unbounded space and using this gauge choice, the expression for the vector potential can be directly read off (4.2),

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}. \tag{4.18}$$

It is easy to check that this $\vec{A}$ is divergenceless as a consequence of the continuity equation $\vec{\nabla} \cdot \vec{J} = 0$.

Localized current distributions and magnetic moments. As we did for the electrostatic potential, we can also expand the magnetic vector potential in a multipole expansion. We consider a localized current distribution $\vec{J}(\vec{x}')$ that is nonvanishing only in a small region around the origin as in figure 4.4. Then for points with $|\vec{x}| \gg |\vec{x}'|$, we can Taylor expand (4.18) using (3.2) up to second order to find, for each component of $\vec{A}$,

$$A_i(\vec{x}) = \frac{\mu_0}{4\pi} \left[\frac{1}{|\vec{x}|} \int \mathrm{d}^3x' \, J_i(\vec{x}') + \frac{\vec{x}}{|\vec{x}|^3} \cdot \int \mathrm{d}^3x' \, J_i(\vec{x}')\vec{x}' + \dots \right]. \tag{4.19}$$

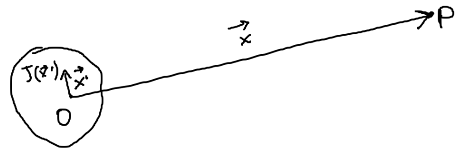

Figure 4.4

We can take advantage of the fact that $\vec{J}$ is a localized function to simplify this expression. For arbitrary scalar functions $f(\vec{x}')$ and $g(\vec{x}')$, we have

$$\int \mathrm{d}^3x' \left(f\vec{J}\cdot\vec{\nabla}'g + g\vec{J}\cdot\vec{\nabla}'f + fg\vec{\nabla}'\cdot\vec{J} \right) = \int \mathrm{d}^3x'\, \vec{\nabla}'\cdot(gf\vec{J}) = 0, \tag{4.20}$$

where the last equality is due to the divergence theorem and $\vec{J}=0$ far from the origin. We can first specify to $f=1$ and $g=x'_i$, noting that $\vec{\nabla}'\vec{J}=0$, to conclude that

$$\int \mathrm{d}^3x'\, J_i(\vec{x}') = 0. \tag{4.21}$$

This means that the first term in the expansion (4.19) is absent, which is another way of saying that there do not exist magnetic monopoles. Taking $f=x'_i$ and $g=x'_j$, (4.20) yields

$$\int \mathrm{d}^3x' \left(x'_iJ_j(\vec{x}') + x'_jJ_i(\vec{x}') \right) = 0, \tag{4.22}$$

and thus the second term in (4.19) can be rewritten using

$$\vec{x}\cdot\int \mathrm{d}^3x'\, J_i\vec{x}' = \sum_j x_j \int \mathrm{d}^3x'\, x'_jJ_i \tag{4.23}$$

$$= -\frac{1}{2}\sum_j x_j \int \mathrm{d}^3x' \left(x'_iJ_j - x'_jJ_i \right) \tag{4.24}$$

$$= -\frac{1}{2}\sum_{j,k} \epsilon_{ijk}x_j \int \mathrm{d}^3x' \left(\vec{x}'\times\vec{J} \right)_k \tag{4.25}$$

$$= -\frac{1}{2}\left[\vec{x}\times\int \mathrm{d}^3x' \left(\vec{x}'\times\vec{J} \right) \right]_i . \tag{4.26}$$

Defining the *magnetization* or *magnetic moment density*

$$\vec{\mathcal{M}}(\vec{x}) = \frac{1}{2}\vec{x}\times\vec{J}(\vec{x}), \tag{4.27}$$

and its integral, the *magnetic moment*

$$\vec{m} = \frac{1}{2}\int \mathrm{d}^3x'\, \vec{x}'\times\vec{J}(\vec{x}'), \tag{4.28}$$

the second term in (4.19) gives the magnetic dipole vector potential

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3}. \tag{4.29}$$

The dipole is thus the first nonvanishing term in the magnetic multipole expansion for a steady-state current distribution. The associated magnetic induction can be found by taking the curl (see exercises),

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{3\vec{n}(\vec{n}\cdot\vec{m})-\vec{m}}{|\vec{x}|^3}\right], \tag{4.30}$$

where $\vec{n}$ the a unit vector in the $\vec{x}$ direction. As higher multipoles correspond to higher-order terms in the $\vec{x}'/\vec{x}$ expansion, far away from any localized current distribution the magnetic induction can be well approximated by that of a magnetic dipole with magnetic moment (4.28).

Magnetic moment of a plane loop current. Let us consider the example of a current I running through a closed loop that is confined to a plane. If $\mathrm{d}\vec{l}$ is the line element of the curve, the magnetic moment is

$$\vec{m} = \frac{I}{2}\oint \vec{x}\times \mathrm{d}\vec{l} \tag{4.31}$$

and is perpendicular to the plane of the loop. Now, $|\vec{x}\times\mathrm{d}\vec{l}\,|/2 = \mathrm{d}a$ is the differential area element in the loop as depicted in figure 4.5, and thus the integration above gives

$$|\vec{m}| = I\cdot(\text{Area of the loop}) \tag{4.32}$$

irrespective of the shape of the closed curve. If the current is generated by particles of charge

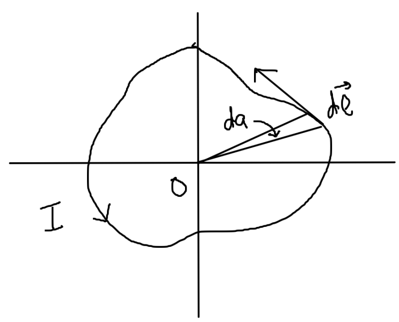

Figure 4.5

q_i and mass M_i moving with velocities $\vec{v}_i$, we have

$$\vec{J} = \sum_i q_i\vec{v}_i\delta(\vec{x}-\vec{x}_i), \tag{4.33}$$

where $\vec{x}_i$ is the position of the ith particle. The corresponding magnetic moment is

$$\vec{m} = \frac{1}{2}\sum_i q_i \vec{x}_i \times \vec{v}_i \tag{4.34}$$

$$= \sum_i \frac{q_i}{2M_i}\vec{L}_i, \tag{4.35}$$

where $\vec{L}_i = M_i \vec{x}_i \times \vec{v}_i$ is the particle's orbital angular momentum. For particles of a single species, we can simply write

$$\vec{m} = \frac{q}{2M}\sum_i \vec{L}_i = \frac{q}{2M}\vec{L}, \tag{4.36}$$

where the proportionality factor between $\vec{m}$ and $\vec{L}$ is usually known as the *gyromagnetic ratio*. We could be tempted to extend this classical relation between orbital angular momentum and magnetic moment to the internal angular momentum (or spin) of elementary particles like the electron, for which we can write

$$\vec{m} = g\frac{q}{2M}\vec{S}, \tag{4.37}$$

where S is the spin angular momentum. Contrary to the classical expectation of $g = 1$, one actually finds that for the electron $g = 2.0023\ldots$, which shows that spin is not just a normal angular momentum. The deviation of this factor from 2 arises from quantum-mechanical effects and can only be calculated within the framework of quantum electrodynamics.

Force and torque on a magnetic dipole. We can see how a magnetic dipole interacts with an external magnetic by expanding (4.12) and (4.13) for a magnetic induction that varies slowly over the region where the dipole is located. Taking the dipole to be located at the origin of coordinates, we can Taylor expand a component of $\vec{B}$ as

$$B_k(\vec{x}) = B_k(0) + \vec{x}\cdot\vec{\nabla}B_k(0) + \ldots \tag{4.38}$$

With this, the force (4.12) becomes

$$F_i = \sum_{jk}\epsilon_{ijk}\left[B_k(0)\int \mathrm{d}^3x'\, J_j(\vec{x}') + \int \mathrm{d}^3x'\, J_j(\vec{x}')\vec{x}'\cdot\vec{\nabla}B_k(0)\right]. \tag{4.39}$$

As we deduced in (4.21), the first integral is vanishing and the second one can be rewritten using (4.23) exchanging $\vec{x} \to \vec{\nabla}B_k(0)$, resulting in

$$F_i = \sum_{jk}\epsilon_{ijk}(\vec{m}\times\vec{\nabla})_j \left. B_k(\vec{x})\right|_0, \tag{4.40}$$

or, in vector notation

$$\vec{F} = (\vec{m}\times\vec{\nabla})\times\vec{B} \tag{4.41}$$

$$= \vec{\nabla}(\vec{m}\cdot\vec{B}) - \vec{m}(\vec{\nabla}\cdot\vec{B}) \tag{4.42}$$

$$= \vec{\nabla}(\vec{m}\cdot\vec{B}). \tag{4.43}$$

Here, it is understood that the derivative is to be evaluated at the origin of coordinates. For a constant $\vec{m}$, we see that there is only a force when the magnetic induction is inhomogeneous. The total torque can be computed in a similar way from (4.13). In this case, the zeroth-order term is already nonvanishing:

$$\vec{N} = \int \mathrm{d}^3x'\, \vec{x}' \times \left[\vec{J} \times \vec{B}(0)\right] \tag{4.44}$$

$$= \int \mathrm{d}^3x' \left[(\vec{x}' \cdot \vec{B}(0))\vec{J} - (\vec{x}' \cdot \vec{J})\vec{B}(0)\right]. \tag{4.45}$$

Here, the second integral can be shown to vanish using (4.20) with $f = g = r'$, while the first one has the same form as (4.39) after exchanging $\vec{\nabla} B_k \to \vec{B}$, and we therefore have

$$\vec{N} = \vec{m} \times \vec{B}(0), \tag{4.46}$$

as we could have expected. We can also compute the potential energy of the dipole moment in an external magnetic field by noting that the force (4.43) is just minus the gradient of the potential energy, so

$$U = -\vec{m} \cdot \vec{B}. \tag{4.47}$$

The potential is minimized when the dipole is parallel to the magnetic induction, and thus the dipoles tend to align with the external field.

Macroscopic magnetostatics. So far we have assumed that the current density $\vec{J}$ is a perfectly known function at each point in space. This may be the case in idealized situations like a current in a thin wire in vacuum, but in real-life problems the situation is more complex. The atoms that make up media have electrons that give rise to microscopic atomic currents, and nuclei and electronic distributions may have intrinsic magnetic moments. These all contribute to generate current densities which are rapidly fluctuating at microscopic distances.

That said, we are often interested in macroscopic descriptions of magnetic materials, and for that purpose it is enough to obtain an averaged description of the currents and induction fields in media. The procedure is analogous to the one that we followed for the case of electrostatics.

First, the averaging of the homogeneous equation $\vec{\nabla} \cdot \vec{B} = 0$ holding at each point is trivially

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{4.48}$$

which means that $\vec{B}$ can be obtained from a vector potential $\vec{A}$ at the macroscopic level too. We can obtain a macroscopic *magnetization* field as an average magnetic moment density,

$$\vec{M}(\vec{x}) = \sum_i N_i \langle \vec{m}_i \rangle . \tag{4.49}$$

Here, N_i is the number density of molecules of type i, which have average magnetic moment $\langle \vec{m}_i \rangle$, calculated in a small volume around the point $\vec{x}$.

Allowing for the existence of a macroscopic current $\vec{J}$ in addition to the bulk magnetization $\vec{M}$, we can express the vector potential as an integral over the material of interest,

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3x' \left[\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{\vec{M}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\right] \tag{4.50}$$

$$= \frac{\mu_0}{4\pi}\int \mathrm{d}^3x' \, \frac{\left[\vec{J}(\vec{x}') + \vec{\nabla}' \times \vec{M}(\vec{x}')\right]}{|\vec{x}-\vec{x}'|}. \tag{4.51}$$

To get to the second line, we have used the identity (1.12) on the second term and integrated it by parts to move the gradient over to the magnetization. A surface term is also generated by the integration by parts, but we have dropped it assuming that the magnetization is sufficiently localized.

We can interpret the above expression as the vector potential generated by an effective current distrubution $\vec{J} + \vec{\nabla}\times\vec{M}$. Then, the second equation of magnetostatics (4.9) can be modified to incorporate the bulk magnetization,

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \vec{\nabla}\times\vec{M}\right). \tag{4.52}$$

We can recast this in a form closer to the vacuum equation by defining a new macroscopic *magnetic field* $\vec{H}$,

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}, \tag{4.53}$$

which satisfies

$$\vec{\nabla}\times\vec{H} = \vec{J}. \tag{4.54}$$

It is worth emphasizing that the fundamental field is still $\vec{B}$, the field $\vec{H}$ is just a convenient way of incorporating the effects of atomic currents when working in magnetic media.

The additional information that we need at this stage is a constitutive relation between $\vec{B}$ and $\vec{H}$. For materials where the response is linear, we can write

$$B_i = \sum_j \mu_{ij} H_j, \tag{4.55}$$

and μ_{ij} is known as the magnetic permeability tensor. For isotropic materials, only the diagonal entries are nonzero (and equal), and thus

$$\vec{B} = \mu\vec{H}. \tag{4.56}$$

Normally μ/μ_0 is very close to unity. Materials with $\mu > \mu_0$ are called paramagnetic, while those with $\mu < \mu_0$ are called diamagnetic. The third kind of substances are ferromagnets, which feature a fully nonlinear response function

$$\vec{B} = \vec{F}(\vec{H}), \tag{4.57}$$

which is not even a single valued function due to the existence of hysteresis, due to which the function $\vec{F}(\vec{H})$ depends on the past history of the material, as is exemplified in figure 4.6. For this kind of materials, $\mu(\vec{H})/\mu_0$ can be as large as 10^5.

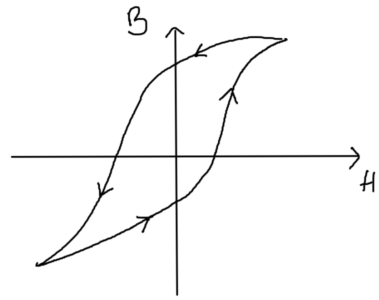

Figure 4.6

Boundary conditions on $\vec{B}$ and $\vec{H}$. The relations between $\vec{B}$ and $\vec{H}$ at both sides of an interface separating materials with different magnetic properties can be derived in the same way as we did for the electric and displacement fields across dielectrics. The result of such exercise are the boundary conditions

$$(\vec{B}_2 - \vec{B}_1) \cdot \vec{n} = 0, \tag{4.58}$$

$$\vec{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K}, \tag{4.59}$$

where the normal $\vec{n}$ points from region 1 into region 2 and $\vec{K}$ is the surface current density. In practice, most materials do not support surface current densities and $\vec{K} = 0$.

Boundary-value problems in magnetostatics. At this point, we have all the necessary ingredients to solve boundary-value problems involving materials with linear magnetic response, such that $\vec{B} = \mu\vec{H}$. In theory, we only need to solve the two basic equations

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{4.60}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}, \tag{4.61}$$

and use the boundary conditions (4.58) and (4.58) to connect the solutions at the interfaces of media with different magnetic permeabilities. In practice, there are two main techniques that we can follow to solve these equations.

- For general situations where $\vec{J} \neq 0$, we can always introduce a vector potential satisfying

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{4.62}$$

With the choice of Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$ and the curl equation (4.61) becomes

$$\vec{\nabla}^2 \vec{A} = -\mu\vec{J}. \tag{4.63}$$

We therefore have to solve three separate Poisson equations for each of the components of $\vec{A}$. In order to do so, we can apply all the techniques that we developed for electrostatic problems, like the expansion in orthogonal functions.

- If $\vec{J} = 0$ and all currents vanish in some region of space, the equation (4.61) becomes simply $\vec{\nabla} \times \vec{H} = 0$. This means that we can introduce a *magnetic scalar potential* Φ_M so that

$$\vec{H} = -\vec{\nabla}\Phi_M, \tag{4.64}$$

just like $\vec{E} = -\vec{\nabla}\Phi$ in electrostatics. Then, as long as we are dealing with materials with constant magnetic permeability μ, the divergence equation (4.60) translates into

$$\vec{\nabla}^2\Phi_M = 0 \tag{4.65}$$

in each region of space with constant μ.

- Some materials can accommodate permanent magnetizations independently of external fields (for moderate enough fields). They are sometimes referred to as *hard ferromagnets* and can be described using a given magnetization $\vec{M}(\vec{x})$, which satisfies

$$0 = \vec{\nabla} \cdot \vec{B} = \mu_0 \vec{\nabla} \cdot (\vec{H} + \vec{M}). \tag{4.66}$$

In the absence of free currents ($\vec{J} = 0$), we can employ the magnetic potential $\vec{H} = -\vec{\nabla}\Phi_M$, which then satisfies

$$\nabla^2\Phi_M = -\rho_M, \tag{4.67}$$

where we have defined the *effective magnetic charge density*

$$\rho_M = -\vec{\nabla} \cdot \vec{M}. \tag{4.68}$$

For finite-size hard ferromagnets, we can also define a *effective magnetic surface charge density*

$$\sigma_M = \vec{n} \cdot \vec{M}, \tag{4.69}$$

where $\vec{n}$ is the outward-pointing normal. This can also be written as an effective surface current

$$\vec{K}_M = \vec{M} \times \vec{n}. \tag{4.70}$$

With this, the magnetic scalar potential can be calculated as

$$\Phi_M = (\vec{x}) = \frac{1}{4\pi}\int_V \mathrm{d}^3x' \, \frac{\rho_M}{|\vec{x} - \vec{x}'|} + \frac{1}{4\pi}\int_S \mathrm{d}a \, \frac{\sigma_M}{|\vec{x} - \vec{x}'|}. \tag{4.71}$$

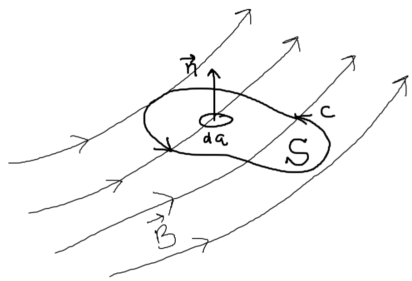

Figure 4.7

Faraday's law of induction. Faraday carried out the first experiments involving currents in the presence of time-varying magnetic fields. He observed that transient currents are generated in a circuit when there is a change in the magnetic flux that traverses the circuit. Mathematically, Faraday's findings can be expressed as follows.

Consider a closed circuit C that bounds a surface S with unit normal $\vec{n}$ (see figure 4.7). The magnetic flux traversing the circuit is given by

$$F = \int_S \mathrm{d}a\, \vec{B} \cdot \vec{n}. \tag{4.72}$$

The induced current in the circuit is caused by an induced magnetic field $\vec{E}'$, which generates an electromotive force

$$\mathcal{E} = \oint_C \vec{E} \cdot \mathrm{d}\vec{l}, \tag{4.73}$$

where $\mathrm{d}\vec{l}$ is the line element around the circuit. Faraday found that there is a relation

$$\mathcal{E} = -\frac{\mathrm{d}F}{\mathrm{d}t} \tag{4.74}$$

between the two quantities defined above. The minus sign indicates that the induced current opposes the change in flux through the circuit. By substituting the definitions above, the relation can be expressed as

$$\oint_C \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathrm{d}a\, \vec{B} \cdot \vec{n}, \tag{4.75}$$

which only involves electric and magnetic fields and thus does not make explicit reference to any current—C can be thought of as a geometric curve in space rather than a physical circuit. If $\vec{B}$ and $\vec{E}$ are defined in the same reference frame, the total time derivative becomes a partial time derivative, and we can use Stoke's theorem on the left-hand side of (4.75) to rewrite it as

$$\int_s \mathrm{d}a \left(\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \right) \cdot \vec{n} = 0. \tag{4.76}$$

Since this relation must hold for any curve C and surface S, the integrand must vanish identically,

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \tag{4.77}$$

which is the differential form of Faraday's law. Note that this generalizes the electrostatic relation $\vec{\nabla} \times \vec{E} = 0$ to situations involving time-dependent fields.

Energy in the magnetic field. Reasoning in an analogous way to the one we followed to obtain the expression for the electrostatic energy (1.48), it is possible to show that the energy stored in a para- or diamagnetic medium can be expressed as

$$W = \frac{1}{2} \int \mathrm{d}^3x \, \vec{H} \cdot \vec{B} \tag{4.78}$$

in terms of the magnetic and induction field, or as

$$W = \frac{1}{2} \int \mathrm{d}^3x \, \vec{J} \cdot \vec{A} \tag{4.79}$$

in terms of the current and vector potential.

Chapter 5

Maxwell equations and electromagnetic waves

From now on we will go beyond static situations to consider fully time-dependent electric and magnetic phenomena. Already in the discussion of Faraday's law we saw that electric and magnetic fields become mixed when their temporal variations are considered. J. C. Maxwell was the first one to fully appreciate this interrelation, which he stablished in his four famous Maxwell equations

$$\text{Coulomb's law} \qquad \vec{\nabla} \cdot \vec{D} = \rho, \tag{5.1}$$

$$\text{Ampère's law} \qquad \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}, \tag{5.2}$$

$$\text{Faraday's law} \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \tag{5.3}$$

$$\text{No magnetic monopoles} \qquad \vec{\nabla} \cdot \vec{B} = 0. \tag{5.4}$$

The main insight of Maxwell was to modify Ampère's law to include time-dependent electric fields. We can understand this necessity from the continuity equation

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0, \tag{5.5}$$

which generalizes $\vec{\nabla} \cdot \vec{J} = 0$ as we considered in magnetostatics. Using Coulomb's equation, the continuity equation can be rewritten as

$$\vec{\nabla} \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) = 0. \tag{5.6}$$

This shows that making Ampère's law consistent with the time-dependent continuity equation requires the replacement

$$\vec{J} \rightarrow \vec{J} + \frac{\partial \vec{D}}{\partial t}, \tag{5.7}$$

where the additional term is known as the *displacement current*. The physical meaning of the modified Ampère's law is that a time-varying electric field induces a magnetic field—conversely to Faraday's law.

The Maxwell equations written above are valid for electromagnetic phenomena occurring in any arbitrary media. As we know, they have to be supplemented with the constitutive relations connecting $\vec{E}$ and $\vec{B}$ to $\vec{D}$ and $\vec{H}$. For now, we will concentrate on phenomena occurring in vacuum, where we have the simple relations $\vec{D} = \epsilon_0 \vec{E}$ and $\vec{B} = \mu_0 \vec{H}$. The corresponding vacuum Maxwell equations are

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \qquad \vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}, \tag{5.8}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \qquad \vec{\nabla} \cdot \vec{B} = 0, \tag{5.9}$$

where $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the *speed of light*. When combined with the Lorentz force equation

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right), \tag{5.10}$$

these equations allow to describe the interactions of charged particles and electromagnetic fields at the classical level. Let us now see how the set of four first-order coupled partial differential equations can be solved.

Vector and scalar potentials Let us first look at the two homogeneous Maxwell equations. Since $\vec{\nabla} \cdot \vec{B} = 0$ is still valid, we can continue to define $\vec{B}$ as the curl of a vector potential,

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{5.11}$$

Then, Faraday's law can be written in the form

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0, \tag{5.12}$$

meaning that the quantity inside the brackets can be written as a gradient of a scalar potential Φ, and therefore

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}. \tag{5.13}$$

In terms of the vector and scalar potentials, the inhomogeneous Maxwell equations read

$$\nabla^2 \Phi + \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{A}\right) = -\frac{\rho}{\epsilon_0}, \tag{5.14}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) = -\mu_0 \vec{J}. \tag{5.15}$$

We have reduced the problem to solving two second-order equations which are still coupled. However, there is some freedom in choosing the vector and scalar potentials that can be exploited to decouple these equations.

Gauge transformations. Given that $\vec{B}$ is defined as the curl of $\vec{A}$, the vector potential can be modified by a total gradient

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda \tag{5.16}$$

without affecting $\vec{B}$. Furthermore, if the scalar potential potential is simultaneously transformed by

$$\Phi \to \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}, \tag{5.17}$$

the electric field $\vec{E}$ remains invariant as well. These kind of transformations are called *gauge transformations*, and make up a group which is called the *gauge group*. The physical (measurable) fields $\vec{E}$ and $\vec{B}$ are independent of the gauge choice and are therefore *gauge invariant*, as opposed to the gauge dependent potentials, which do not have any direct physical meaning (they cannot be directly measured).

By making an appropriate gauge choice, it is possible to simplify (5.14) and (5.15).

Coulomb gauge. Using the gauge freedom, we can demand that

$$\vec{\nabla} \cdot \vec{A} = 0, \tag{5.18}$$

which is known as the *Coulomb*, *radiation*, or *transverse gauge* condition. To see that it is always possible to find $\vec{A}$ satisfying this gauge condition, assume that originally $\vec{\nabla} \cdot \vec{A} = 4\pi\eta(\vec{x}, t) \neq 0$. Then, we can perform a gauge transformation with a function Λ, so that

$$\vec{\nabla} \cdot \vec{A}' = \vec{\nabla} \cdot \vec{A} + \nabla^2 \Lambda = 0, \tag{5.19}$$

which means that we need

$$\nabla^2 \Lambda = -4\pi\eta(\vec{x}, t). \tag{5.20}$$

As we know, this can be solved via

$$\Lambda(\vec{x}, t) = \int \mathrm{d}^3x' \, \frac{\eta(\vec{x}', t)}{|\vec{x} - \vec{x}'|}, \tag{5.21}$$

and we can therefore always find a suitable Λ to transform to Coulomb gauge.

With the condition $\vec{\nabla} \cdot \vec{A} = 0$, the equation (5.14) simplifies to

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0}, \tag{5.22}$$

which is the Poisson equation that we know can be solved through

$$\Phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int \mathrm{d}^3x' \, \frac{\rho(\vec{x}', t)}{|\vec{x} - \vec{x}'|}. \tag{5.23}$$

This is noting but the *instantaneous Coulomb potential* due to the charge density $\rho(\vec{x},t)$ valid at every time t. There is something that at first looks worrying in this equation: changes in the charge distribution cause instantaneous changes of the scalar potential, which we suspect may lead to violation of causality. However, it is possible to show that there is actually no issue: the physical fields $\vec{E}$ and $\vec{B}$ propagate with a finite speed c. It is only the unmeasurable scalar potential Φ that features this unphysical instantaneous propagation.

After solving for the scalar potential, the equation (5.15) for the vector potential becomes

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J} + \frac{\mu_0}{4\pi}\vec{\nabla}\int \mathrm{d}^3x'\,\frac{\frac{\partial}{\partial t}\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|} \tag{5.24}$$

$$= -\mu_0\left(\vec{J} - \frac{1}{4\pi}\vec{\nabla}\int \mathrm{d}^3x'\,\frac{\vec{\nabla}'\cdot\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}\right) \tag{5.25}$$

$$= -\mu_0\vec{J}_t. \tag{5.26}$$

In the second line, we have used the continuity equation $\vec{\nabla}\cdot\vec{J} + \partial\rho/\partial t = 0$ and in the third one we have defined the *transverse current*

$$\vec{J}_t(\vec{x},t) = \frac{1}{4\pi}\vec{\nabla}\times\vec{\nabla}\times\int \mathrm{d}^3x'\,\frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|} \tag{5.27}$$

which satisfies $\vec{\nabla}\cdot\vec{J}_t = 0$. Analogously one can define the *longitudinal current*

$$\vec{J}_l(\vec{x},t) = \frac{1}{4\pi}\vec{\nabla}\int \mathrm{d}^3x'\,\frac{\vec{\nabla}'\cdot\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|}, \tag{5.28}$$

with $\vec{\nabla}\times\vec{J}_l = 0$. It is clear that the total current is the sum of the transverse and longitudinal ones, $\vec{J} = \vec{J}_l + \vec{J}_t$.

The Coulomb gauge is particularly useful in the absence of sources. In that case, $\Phi = 0$ and $\vec{A}$ satisfies a homogeneous wave equation. The fields are then given by

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla}\times\vec{A}. \tag{5.29}$$

Lorenz gauge. Another useful gauge is the Lorenz gauge, for which we require

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0. \tag{5.30}$$

It is easy to see that in this gauge, the Maxwell equations decouple and become two separate wave equations

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \tag{5.31}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}, \tag{5.32}$$

whose solution we will discuss in detail later. For now, let us check that it is always possible to find potentials that satisfy the Lorenz gauge condition. Let Φ and $\vec{A}$ be arbitrary potentials that do not satisfy the Lorenz condition, and let

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 4\pi\eta(\vec{x}, t) \neq 0. \tag{5.33}$$

Perform a gauge transformation with function Λ to get to the Lorenz gauge, so that

$$\begin{aligned} \vec{\nabla} \cdot \vec{A}' + \frac{1}{c^2}\frac{\partial \Phi'}{\partial t} &= \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} + \nabla^2\Phi - \frac{1}{c^2}\frac{\partial \Lambda}{\partial t} && (5.34)\\ &= 4\pi\eta + \nabla^2\Phi - \frac{1}{c^2}\frac{\partial \Lambda}{\partial t} && (5.35)\\ &= 0. && (5.36) \end{aligned}$$

Thus, such a gauge transformation can be found as long as we can solve the equation

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial \Lambda}{\partial t} = -4\pi\eta, \tag{5.37}$$

which is precisely the kind of wave equation that we will solve shortly.

One thing to note is that the Lorenz condition does not completely specify the potentials. Indeed it is still possible to perform *restricted gauge transformations* with function Λ satisfying a homogeneous wave equation

$$\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0, \tag{5.38}$$

without affecting the Lorenz condition.

One of the biggest advantages of the Lorenz gauge is that it is manifestly invariant under the Lorentz transformations of special relativity, which will be extremely useful when we discuss relativistic electrodynamics.

5.1 Electromagnetic waves

The wave equations (5.26), (5.31), and (5.32) that we wish to solve all have the form

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = -4\pi f(\vec{x}, t). \tag{5.39}$$

Here, $f(\vec{x}, t)$ is the source distribution and c is the velocity of propagation in the medium, which here we assume to be the vacuum. Furthermore, in the absence of sources ($\rho = 0$ and $\vec{J} = 0$), the Maxwell equations can be directly rewritten as wave equations for the fields $\vec{E}$ and $\vec{B}$. It is therefore instructive to discuss the solution for the wave equation in detail, starting by the free case $f = 0$.

Solution of the free wave equation The free (or homogeneous) wave equation for an arbitrary scalar field ψ is

$$\Box\psi(\vec{x},t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\vec{x},t) - \nabla^2\psi(\vec{x},t) = 0, \tag{5.40}$$

where we have introduced the *D'Alemebert operator* $\Box$. It is convenient to look for complex solutions of this equation. For real quantities such as the electric and magnetic field, it is enough to take the real part of the complex solution.

We start by making a *plane wave* ansatz

$$\psi(\vec{x},t) = a\,\mathrm{e}^{i(\vec{k}\cdot\vec{x}-\omega t)}, \tag{5.41}$$

where $\vec{k}$ is called the *wave number vector*. Denoting $k = |\vec{k}|$, the *wavelength* of the wave is $\lambda = 2\pi/k$. For fixed t, ψ is a periodic function of position with period λ in the direction of $\vec{k}$,

$$\psi(\vec{x}+\lambda\hat{k},t) = \psi(\vec{x},t). \tag{5.42}$$

Similarly, ω is the *angular frequency* of the wave, $\nu = \omega/(2\pi)$ is its *frequency*, and $T = \frac{1}{\nu} = 2\pi/\omega$ is the *period* of the plane wave. For fixed $\vec{x}$, ψ is a periodic function in time with period T.

With the ansatz (5.41), the wave equation (5.40) reduces to

$$\left(k^2 - \frac{1}{c^2}\omega^2\right)\psi = 0, \tag{5.43}$$

which implies that the frequency and the wave number of the plane wave are related by

$$k^2 = \frac{\omega^2}{c^2} \quad \Rightarrow k = \pm\frac{\omega}{c}. \tag{5.44}$$

Thus, $\lambda\nu = c$. We call $(\vec{k}\cdot\vec{x}-\omega t)$ the *phase* of the wave. Surfaces of constant phase are thus orthogonal to $\vec{k}$, as figure 5.1 shows.

If, for instance, $\vec{k}$ is oriented in the x direction, the points $\vec{x}_0$ and $\vec{x}$ have the same phase at times t_0 and t if

$$x = \vec{x}_0 + \frac{\omega}{k}(t-t_0), \tag{5.45}$$

which corresponds to planes orthogonal to the direction of propagation $\vec{x}$. Surfaces of constant phase thus move with a *phase velocity*

$$v = \frac{\omega}{k} = c. \tag{5.46}$$

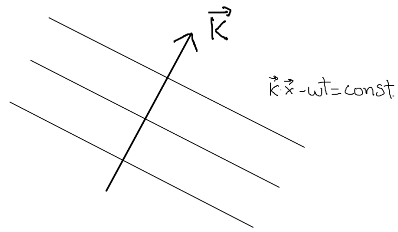

Figure 5.1

Monochromatic plane electromagnetic waves. Let us start by considering *monochromatic* plane waves, that is, plane waves with given values of $\vec{k}$ and ω. In Coulomb (or radiation) gauge $\vec{\nabla}\cdot\vec{A}=0$, and for $\rho=0$ and $\vec{J}=0$, the Maxwell equation (5.14) immediately implies $\Phi=0$. With this, (5.15) reduces to a free wave equation for each component of $\vec{A}$,

$$\Box\vec{A}(\vec{x},t)=0, \tag{5.47}$$

with $c^2=\epsilon\mu$. With the plane wave solution found above, we can write

$$\vec{A}=\vec{A}_0\mathrm{e}^{i(\vec{k}\cdot\vec{x}-\omega t)}=\vec{\epsilon}|\vec{A}_0|\mathrm{e}^{i(\vec{k}\cdot\vec{x}-\omega t)} \tag{5.48}$$

with $\omega=\pm kc$. The phase velocity of the wave is then

$$v=\frac{\omega}{k}=\frac{c_0}{n}, \tag{5.49}$$

where $c_0=\sqrt{\epsilon_0\mu_0}$ is the speed of light in vacuum, and

$$n=\sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \tag{5.50}$$

is called the *index of refraction* and is in general a function of the frequency. In the following we will assume that we are dealing with uniform isotropic linear media with constant ϵ and μ.

The vector $\vec{\epsilon}$ in (5.48) is independent of $\vec{x}$ and t and is complex valued in general (to recover the physical fields we will only keep the real parts in the end). It is referred to as the *polarization vector* of the plane wave. Because $\vec{\nabla}\cdot\vec{A}=0$, the polarization and wave number vectors are perpendicular to each other,

$$\vec{\epsilon}\cdot\vec{k}=0. \tag{5.51}$$

The electric and magnetic fields can be readily calculated from the vector potential,

$$\vec{E}=-\frac{\partial\vec{A}}{\partial t}=i\omega\vec{\epsilon}|A_0|\mathrm{e}^{i(\vec{k}\cdot\vec{x}-\omega t)}, \tag{5.52}$$

$$\vec{B}=\vec{\nabla}\times\vec{A}=i(\vec{k}\times\vec{\epsilon})\vec{\epsilon}|A_0|\mathrm{e}^{i(\vec{k}\cdot\vec{x}-\omega t)}=\frac{n}{c}\hat{k}\times\vec{E}, \tag{5.53}$$

or rather the real parts of the above quantities. From these, we readily conclude that

$$\vec{k}\cdot\vec{E}=0 \quad \text{and} \quad \vec{k}\cdot\vec{B}=0, \tag{5.54}$$

meaning that both fields are perpendicular to the direction of propagation: plane electromagnetic waves are *transverse waves*. Another immediate consequences are

$$|E|=\frac{n}{c}|B| \tag{5.55}$$

and

$$\vec{E}\cdot\vec{B}=0. \tag{5.56}$$

Thus, a plane electromagnetic wave is completely determined by either $\vec{E}$ or $\vec{B}$.

Polarization of plane waves. Let us now look at the polarization vector more closely. For concreteness, we will choose coordinates such that $\vec{k}=k\vec{e}_3$, so that

$$\vec{\epsilon}=\begin{pmatrix}\epsilon_1\\ \epsilon_2\\ 0\end{pmatrix}=\begin{pmatrix}|\epsilon_1|\mathrm{e}^{i\delta_1}\\ |\epsilon_2|\mathrm{e}^{i\delta_2}\\ 0\end{pmatrix}. \tag{5.57}$$

At a fixed point $\vec{x}_0$, the electric and magnetic fields vary with time as

$$\begin{aligned}
E_1(t)=&\ \mathrm{Re}\Big[|\epsilon_1||\vec{A}_0|\omega \mathrm{e}^{i(\vec{k}\cdot\vec{x}_0-\omega t+\frac{\pi}{2}+\delta_1)}\Big] && (5.58)\\
=&\ |\epsilon_1||\vec{A}_0|\omega\cos\left(\omega t-\vec{k}\cdot\vec{x}_0-\frac{\pi}{2}-\delta_1\right) && (5.59)\\
E_2(t)=&\ |\epsilon_2||\vec{A}_0|\omega\cos\left(\omega t-\vec{k}\cdot\vec{x}_0-\frac{\pi}{2}-\delta_2\right) && (5.60)\\
B_1(t)=&\ -|\epsilon_2||\vec{A}_0|\frac{\omega c}{n}\cos\left(\omega t-\vec{k}\cdot\vec{x}_0-\frac{\pi}{2}-\delta_2\right) && (5.61)\\
B_2(t)=&\ |\epsilon_1||\vec{A}_0|\frac{\omega c}{n}\cos\left(\omega t-\vec{k}\cdot\vec{x}_0-\frac{\pi}{2}-\delta_1\right). && (5.62)
\end{aligned}$$

There are a few special cases depending on the relative values of the phases:

- If $\delta_1=\delta_2$ or $\delta_1=\delta_2+\pi$, $\vec{\epsilon}$ is real (up to an overall phase) and the wave is *linearly polarized* (figure 5.2).
- If $\delta_1\neq\delta_2$ and $\delta_1=\neq\delta_2+\pi$, $\vec{\epsilon}$ is complex and the wave is *elliptically polarized* (figure 5.3). The axes of the ellipse coincide with the coordinate axes if $\delta_2-\delta_1=\pm\pi/2$. Two special cases of elliptic polarization are:
- If $\delta_2=\delta_1+\pi/2$ and $|\epsilon_1|=|\epsilon_2|=1/\sqrt{2}$, the wave is *left circularly polarized* or has *positive helicity* (figure 5.4).
- If $\delta_2=\delta_1-\pi/2$ and $|\epsilon_1|=|\epsilon_2|=1/\sqrt{2}$, the wave is *right circularly polarized* or has *negative helicity* (figure 5.5).

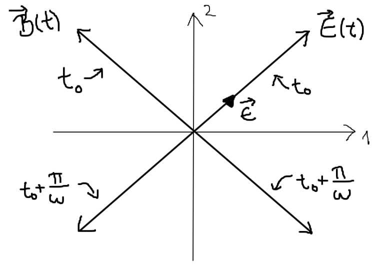

Figure 5.2

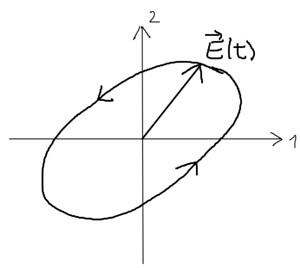

Figure 5.3

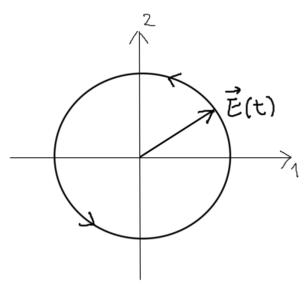

Figure 5.4

General solution of the free wave equation. The free wave equation (5.40) is a linear equation. This means that the linear combination of any number of solutions is also a solution. We can use this property to obtain the general solution of the free wave equation as a superposition of all possible plane waves of the form discussed above. To construct such solution, the first step is to make a Fourier transformation (or spectral decomposition) in the

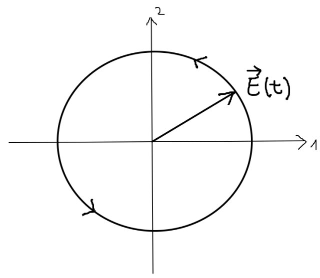

Figure 5.5

spatial and temporal coordinates,

$$\psi(\vec{x},t) = \frac{1}{(2\pi)^4}\int \mathrm{d}^3k\,\mathrm{d}\omega\,\tilde{\psi}(\vec{k},\omega)\mathrm{e}^{i(\vec{k}\cdot\vec{x}-\omega t)}, \tag{5.63}$$

where $\tilde{\psi}(\vec{k},\omega)$ is known as the *spectral function.* Plugging this ansatz into the free wave equation (5.40), we arrive at

$$\frac{1}{(2\pi)^4}\int \mathrm{d}^3k\,\mathrm{d}\omega\left[\vec{k}^2 - \left(\frac{\omega}{c}\right)^2\right]\tilde{\psi}(\vec{k},\omega)\mathrm{e}^{i(\vec{k}\cdot\vec{x}-\omega t)} = 0, \tag{5.64}$$

which means that

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\tilde{\psi}(\vec{k},\omega), \tag{5.65}$$

where, as usual, $k = |\vec{k}|$. This means that $\tilde{\psi} = 0$ for $k^2 \neq \omega^2/c^2$, and $\tilde{\psi}$ is arbitrary for $k^2 = \omega^2/c^2$. Said otherwise, $\tilde{\psi}$ can only be different from zero in three-dimensional hypercone defined through

$$k^2 - \frac{\omega^2}{c^2} = 0 \tag{5.66}$$

within the four-dimensional $(\omega,\vec{k})$ space, as shown in figure 5.6.

This condition can be rewritten using delta functions. For that, we will use that the most general solution of the functional equation

$$xf(x) = 0 \tag{5.67}$$

is given by

$$f(x) = a\,\delta(x), \tag{5.68}$$

with an arbitrary constant a. The general solution of

$$(x - x_0)(x + x_0)f(x) = 0 \tag{5.69}$$

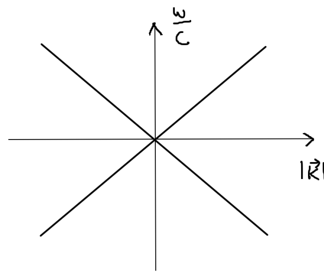

Figure 5.6

is then

$$f(x) = a_+\delta(x - x_0) + a_-\delta(x + x_0). \tag{5.70}$$

If we see $\vec{k}$ as a parameter and define $\omega_0 = c|\vec{k}|$, the condition (5.65) becomes

$$(\omega - \omega_0)(\omega + \omega_0)\tilde{\psi}(\vec{k}, \omega) = 0, \tag{5.71}$$

which is then solved by

$$\tilde{\psi}(\vec{k}, \omega) = a_+(\vec{k})\delta(\omega - \omega_0) + a_-(\vec{k})\delta(\omega + \omega_0). \tag{5.72}$$

Inserting this back into (5.63) and evaluating the delta functions, we get

$$\psi(\vec{x}, t) = \frac{1}{(2\pi)^4}\int \mathrm{d}^3k \left[a_+(\vec{k})\mathrm{e}^{i(\vec{k}\cdot\vec{x}-\omega_0 t)} + a_-(\vec{k})\mathrm{e}^{i(\vec{k}\cdot\vec{x}+\omega_0 t)}\right], \tag{5.73}$$

where the two functions $a_\pm(\vec{k})$ are fixed by boundary conditions. As an example, let's assume that we know $\psi(\vec{x}, t = 0)$ and $\partial_t\psi(\vec{x}, t = 0)$. Then, we have

$$\psi(\vec{x}, 0) = \frac{1}{(2\pi)^4}\int \mathrm{d}^3k \left[a_+(\vec{k}) + a_-(\vec{k})\right]\mathrm{e}^{i\vec{k}\cdot\vec{x}}, \tag{5.74}$$

$$\partial_t\psi(\vec{x}, 0) = \frac{i\omega_0}{(2\pi)^4}\int \mathrm{d}^3k \left[-a_+(\vec{k}) + a_-(\vec{k})\right]\mathrm{e}^{i\vec{k}\cdot\vec{x}}. \tag{5.75}$$

Inverting the Fourier transformation and solving the resulting linear system of equations, we find

$$a_+(\vec{k}) = \pi\int \mathrm{d}^3x' \left[\psi(\vec{x}', 0) - \frac{1}{i\omega_0}\partial_t(\vec{x}', 0)\right]\mathrm{e}^{-i\vec{k}\cdot\vec{x}'}, \tag{5.76}$$

$$a_-(\vec{k}) = \pi\int \mathrm{d}^3x' \left[\psi(\vec{x}', 0) + \frac{1}{i\omega_0}\partial_t(\vec{x}', 0)\right]\mathrm{e}^{-i\vec{k}\cdot\vec{x}'}. \tag{5.77}$$

At this point, it is useful to define the function

$$D(\vec{x},t) = \frac{-i}{2(2\pi)^3}\int \frac{\mathrm{d}^3k}{\omega_0}\mathrm{e}^{i\vec{k}\cdot\vec{x}}\left(\mathrm{e}^{-i\omega_0 t} - \mathrm{e}^{i\omega_0 t}\right), \tag{5.78}$$

so that we can rewrite (5.73) using (5.76) and (5.77) as

$$\psi(\vec{x},t) = -\int \mathrm{d}^3x' \left[\partial_t D(\vec{x}-\vec{x}',t)\right]\psi(\vec{x}',0) - \int \mathrm{d}^3x'\, D(\vec{x}-\vec{x}',t)\,\partial_t\psi(\vec{x}',0). \tag{5.79}$$

Following similar steps, it is easy to generalize this expression for boundary conditions given at an arbitrary time $t = t'$ instead of $t = 0$. Doing so, the general solution to the free wave equation can be recast as

$$\psi(\vec{x},t) = -\int \mathrm{d}^3x'\, D(\vec{x}-\vec{x}',t-t')\,\overset{\leftrightarrow}{\partial_t}\,\psi(\vec{x}',t'), \tag{5.80}$$

where $\overset{\leftrightarrow}{\partial_t}$ is defined through

$$f\,\overset{\leftrightarrow}{\partial_t}\,g = f\,\partial_t g - g\,\partial_t f. \tag{5.81}$$

It is easy to see that $D(\vec{x},t)$ is actually a solution of the free wave equation (as it is of the form (5.73)), and it satisfies the boundary conditions

$$D(\vec{x},0) = 0, \tag{5.82}$$

$$\partial_t D(\vec{x},0) = \delta(\vec{x}). \tag{5.83}$$

Using spherical coordinates for $\vec{k}$, it is possible to find an explicit expression for $D(\vec{x},t)$,

$$D(\vec{x},t) = \frac{-1}{2\pi c}\epsilon(t)\delta(r^2 - c^2t^2), \tag{5.84}$$

with $r = |\vec{x}|$ and

$$\epsilon(t) = \begin{cases} -1 & \text{if} \quad t<0, \\ 0 & \text{if} \quad t=0, \\ +1 & \text{if} \quad t>0. \end{cases} \tag{5.85}$$

Inhomogeneous wave equation. In the presence of sources, the wave equations (5.26), (5.31), and (5.32) are of the form

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi f(\vec{x},t). \tag{5.86}$$

The general solution of the full wave equation can be obtained by combining the previously found solution to the homogeneous equation and the special one for the inhomogeneous equation,

$$\psi(\vec{x},t) = \psi_{\mathrm{spec}}(\vec{x},t) + \psi_{\mathrm{hom}}(\vec{x},t). \tag{5.87}$$

The physical meaning of the special solution is that of a wave propagating from the source $f(\vec{x}, t)$. As before, it is convenient to work in frequency space by performing the Fourier transformations

$$\psi(\vec{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \psi(\vec{x}, \omega) \mathrm{e}^{-i\omega t}, \tag{5.88}$$

$$f(\vec{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, f(\vec{x}, \omega) \mathrm{e}^{-i\omega t}, \tag{5.89}$$

whose inverses are

$$\psi(\vec{x}, \omega) = \int_{-\infty}^{\infty} \mathrm{d}\omega \, \psi(\vec{x}, t) \mathrm{e}^{i\omega t}, \tag{5.90}$$

$$f(\vec{x}, \omega) = \int_{-\infty}^{\infty} \mathrm{d}\omega \, f(\vec{x}, t) \mathrm{e}^{i\omega t}. \tag{5.91}$$

The Fourier transform satisfies the *inhomogeneous Helmholtz wave equation*

$$(\nabla^2 + k^2)\psi(\vec{x}, \omega) = -4\pi f(\vec{x}, \omega) \tag{5.92}$$

for each ω. We have introduced the wave number $k = \omega/c$ in vacuum, in general there can be more complicated dispersion relations between ω and k.

This equation reduces to the Poisson equation for $k = 0$, so we will work similarly in trying to find an appropriate Green function G_k such that

$$\left(\nabla^2 + k^2\right) G_k(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x} - \vec{x}'). \tag{5.93}$$

Due to spherical symmetry, the Green function can only depend on $R = |\vec{x} - \vec{x}'|$. Writing the Laplace operator in spherical coordinates as in (2.16), the equation (5.93) becomes

$$\frac{1}{R}\frac{\mathrm{d}^2}{\mathrm{d}R^2}(RG_k) + k^2 G_k = -4\pi\delta(\vec{R}) \tag{5.94}$$

Away from $R = 0$, this is simply

$$\frac{\mathrm{d}^2}{\mathrm{d}R^2}(RG_k) + k^2(RG_k) = 0, \tag{5.95}$$

which has the two solutions

$$G_k^{(\pm)}(R) = \frac{1}{R}\mathrm{e}^{\pm ikR}. \tag{5.96}$$

We can now inverse-Fourier transform to obtain the Green function that satisfies

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G^{(\pm)}(\vec{x}, t; \vec{x}', t') = -4\pi\delta(\vec{x} - \vec{x}')\delta(t - t'). \tag{5.97}$$

We find

$$G^{(\pm)}(R,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{\mathrm{e}^{\pm ikR}}{R}\mathrm{e}^{-i\omega\tau} \tag{5.98}$$

$$= \frac{1}{R}\delta\left(\tau \mp \frac{R}{c}\right), \tag{5.99}$$

where we have used that $k = \omega/c$ and we have defined $\tau = t - t'$. The Green function $G^{(+)}$ is known as the *retarded Green function*, and $G^{(-)}$ is known as the *advanced Green function*.

It is easy to see that $G^{(+)} \neq 0$ only when $\tau \geq 0$, while $G^{(-)} \neq 0$ only when $\tau \leq 0$. We can thus write

$$\begin{aligned} G^{(+)}(R,\tau) &= \frac{1}{R}\delta\left(\tau \mp \frac{R}{c}\right)\theta(\tau) \\ &= 2c\delta(R^2 - c^2\tau^2)\theta(\tau), \end{aligned} \tag{5.100}$$

meaning that for $\tau > 0$

$$G^{(+)} = -\frac{c^2}{4\pi}D, \tag{5.101}$$

with D as in (5.84). Similarly, for $\tau < 0$ we find that

$$G^{(-)} = \frac{c^2}{4\pi}D. \tag{5.102}$$

This makes it explicit that the difference between the two special solutions of the inhomogeneous wave equation, $G^{(+)}$ and $G^{(-)}$, differ by a solution of the homogeneous one, D.

Following this discussion, we say that $G^{(+)} \neq 0$ only in the *future light cone* L_+ defined by $R = c|\tau|$ with $\tau \geq 0$. We can thus interpret $G^{(+)}$ as an expanding spherical wave pulse, that originates at t', x' and propagates outwards with the speed of light.

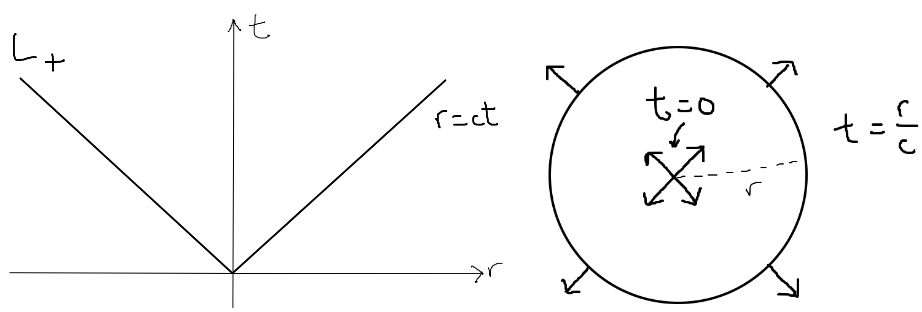

Figure 5.7

Similarly, we say that $G^{(-)} \neq 0$ only in the *past light cone* L_- defined by $R = c|\tau|$ with $\tau \leq 0$. The interpretation in this case is that of a contracting spherical wave pulse, which at time t' is absorbed by the source (or rather the sink) located at position $\vec{x}'$.

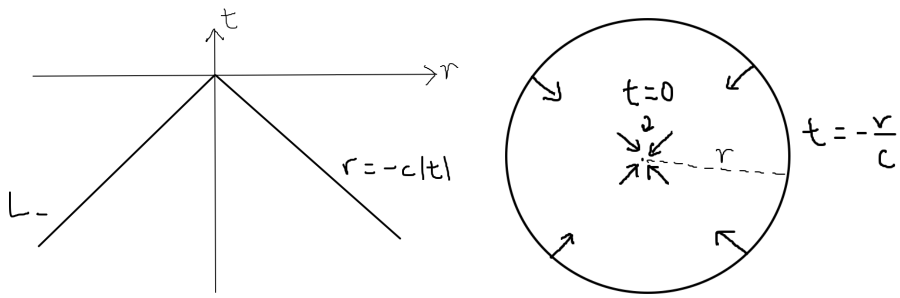

Figure 5.8

These Green functions allow to construct solutions to the inhomogeneous wave equation in the well-known way

$$\Psi^{(\pm)}(\vec{x},t) = \int\int \mathrm{d}^3x'\, \mathrm{d}t'\, G^{(\pm)}(\vec{x},t;\vec{x}',t') f(\vec{x}',t') \tag{5.103}$$

$$= \int\int \mathrm{d}^3x'\, \mathrm{d}t'\, \frac{1}{|\vec{x}-\vec{x}'|}\delta\left(t - t' \mp \frac{1}{c}|\vec{x}-\vec{x}'|\right) f(\vec{x}',t') \tag{5.104}$$

$$= \int \mathrm{d}^3x'\, \frac{1}{|\vec{x}-\vec{x}'|} f(x', t'_{\substack{\mathrm{ret}\\ \mathrm{adv}}}), \tag{5.105}$$

where we have defined

$$t'_{\mathrm{ret}} = t - \frac{|\vec{x}-\vec{x}'|}{c}, \quad \text{and} \quad t'_{\mathrm{adv}} = t + \frac{|\vec{x}-\vec{x}'|}{c}. \tag{5.106}$$

Of course, it is always possible to add to this a solution to the homogeneous wave equation. As we saw above, the difference between the retarded and advanced Green functions is a solution of the homogeneous equation,

$$\left(\nabla^2 + k^2\right)\left(G^{(+)} - G^{(-)}\right) = 0. \tag{5.107}$$

The appropriate addition of homogeneous solutions is determined by the concrete physical problem at hand and its boundary conditions.

The retarded potentials. Most of the physical situations we are interested in involve the retarded Green function. Specifying (5.105) to the wave equations (5.31) and (5.32) for the potentials in Lorenz gauge leads to

$$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x'\, \frac{1}{|\vec{x}-\vec{x}'|}\rho(\vec{x}', t'_{\mathrm{ret}}), \tag{5.108}$$

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3x'\, \frac{1}{|\vec{x}-\vec{x}'|}\vec{J}(\vec{x}', t'_{\mathrm{ret}}). \tag{5.109}$$

Once the potentials are obtained, the fields can be calculated by differentiation,

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}, \tag{5.110}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{5.111}$$

Let us calculate the different terms involved. First, we have

$$-\vec{\nabla}\Phi(\vec{x},t) = -\frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x'\, \vec{\nabla}\left[\frac{1}{|\vec{x}-\vec{x}'|}\rho(\vec{x}',t'_{\mathrm{ret}})\right], \tag{5.112}$$

$$= -\frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \left[\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\rho(\vec{x}',t'_{\mathrm{ret}}) + \frac{1}{|\vec{x}-\vec{x}'|}\left.\frac{\partial\rho}{\partial t}\right|_{(\vec{x}',t'_{\mathrm{ret}})}\vec{\nabla}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\right], \tag{5.113}$$

$$= \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \left[\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\rho(\vec{x}',t'_{\mathrm{ret}}) + \frac{1}{c}\left.\frac{\partial\rho}{\partial t}\right|_{(\vec{x}',t'_{\mathrm{ret}})}\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^2}\right], \tag{5.114}$$

where we have used (1.12) and

$$\vec{\nabla}\left(|\vec{x}-\vec{x}'|\right) = \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|}. \tag{5.115}$$

Next,

$$-\frac{\partial\vec{A}}{\partial t} = -\frac{\mu_0}{4\pi}\int \mathrm{d}^3x'\, \frac{1}{|\vec{x}-\vec{x}'|}\left.\frac{\partial\vec{J}}{\partial t}\right|_{(\vec{x}',t'_{\mathrm{ret}})}. \tag{5.116}$$

Finally, we calculate

$$\vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\int \mathrm{d}^3x'\, \vec{\nabla}\times\left[\frac{1}{|\vec{x}-\vec{x}'|}\vec{J}(\vec{x}',t'_{\mathrm{ret}})\right] \tag{5.117}$$

$$= \frac{\mu_0}{4\pi}\int \mathrm{d}^3x' \left[\vec{\nabla}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\times\vec{J}(\vec{x}',t'_{\mathrm{ret}}) + \frac{1}{|\vec{x}-\vec{x}'|}\vec{\nabla}\times\vec{J}(\vec{x}',t'_{\mathrm{ret}})\right] \tag{5.118}$$

$$= \frac{\mu_0}{4\pi}\int \mathrm{d}^3x' \left[\vec{J}(\vec{x}',t'_{\mathrm{ret}})\times\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} + \frac{1}{c|\vec{x}-\vec{x}'|}\left.\frac{\partial\vec{J}}{\partial t}\right|_{(\vec{x}',t'_{\mathrm{ret}})}\times\frac{\vec{x}-\vec{x}'}{c|\vec{x}-\vec{x}'|}\right], \tag{5.119}$$

$$\tag{5.120}$$

for which we have used the chain rule

$$\vec{\nabla}\times\vec{J}(\vec{x}',t'_{\mathrm{ret}}) = \left.\frac{\partial\vec{J}}{\partial t}\right|_{(\vec{x}',t'_{\mathrm{ret}})}\times\vec{\nabla}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right). \tag{5.121}$$

Putting everything together, we arrive at

$$\vec{E}(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3x' \left[\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\rho(\vec{x}',t'_{\mathrm{ret}}) + \frac{1}{c}\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^2}\left.\frac{\partial\rho}{\partial t}\right|_{(\vec{x}',t'_{\mathrm{ret}})} - \frac{1}{c^2}\frac{1}{|\vec{x}-\vec{x}'|}\left.\frac{\partial\vec{J}}{\partial t}\right|_{(\vec{x}',t'_{\mathrm{ret}})}\right], \tag{5.122}$$

$$\vec{B}(\vec{x},t) = \frac{\mu_0}{4\pi}\int \mathrm{d}^3x' \left[\frac{1}{|\vec{x}-\vec{x}'|^3}\vec{J}(\vec{x}',t'_{\mathrm{ret}}) + \frac{1}{c}\frac{1}{|\vec{x}-\vec{x}'|^2}\left.\frac{\partial\vec{J}}{\partial t}\right|_{(\vec{x}',t'_{\mathrm{ret}})}\right] \times (\vec{x}-\vec{x}')\,. \tag{5.123}$$

Conservation of energy: Poynting's Theorem. A charge q in an external electromagnetic field is subject to the Lorentz force

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right), \tag{5.124}$$

or

$$\vec{F} = \int \mathrm{d}^3x\,\vec{f} = \int \mathrm{d}^3x\left(\rho\vec{E} + \vec{J}\times\vec{B}\right). \tag{5.125}$$

The work done by the field to displace it an infinitesimal amount is

$$\mathrm{d}W = \vec{F}\cdot\mathrm{d}\vec{x} = \vec{F}\cdot\vec{v}\mathrm{d}t = q\vec{E}\cdot\vec{v}\mathrm{d}t = \vec{E}\cdot\vec{J}\mathrm{d}t, \tag{5.126}$$

The magnetic field does no work since the force it exerts is orthogonal to the velocity. Thus, the power acting on a charge distribution in a finite volume V is

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \int_V \mathrm{d}^3x\,\vec{J}\cdot\vec{E}. \tag{5.127}$$

Clearly, this work must come at the cost of a decrease in the energy stored in the electromagnetic field in V. This can be seen by rewriting the above expression using Maxwell's equations,

$$\int_V \mathrm{d}^3x\,\vec{J}\cdot\vec{E} = \int_V \mathrm{d}^3x\left[\vec{E}\cdot(\vec{\nabla}\times\vec{H}) - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right] \tag{5.128}$$

$$= -\int_V \mathrm{d}^3x\left[\vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right]. \tag{5.129}$$

In the first line, we have used the Ampère-Maxwell law to substitute $\vec{J}$, and in the second one we have used the identity

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H}) \tag{5.130}$$

together with Faraday's law. It is now customary to denote

$$u = \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}), \tag{5.131}$$

In general media, the time derivative of u is a complicated function that depends on its electric and magnetic responses. Even if the medium is linear, there is always *dispersion*, that is, dependence of ϵ and μ on the frequency of the electromagnetic radiation, which must be taken into account when computing the time derivative of u.

Poynting's theorem in vacuum We will take care of the general case later, but for now let us assume that we are in vacuum so that $\vec{D} = \epsilon_0 \vec{E}$ and $\vec{B} = \mu_0 \vec{H}$. In this case, u represents the total energy density in the electromagnetic field, so that the total energy in the stored in the fields in the volume V is

$$U_{\text{field}} = \int_V \mathrm{d}^3x\, u(\vec{x}). \tag{5.132}$$

Furthermore, in this case we can write

$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{\partial u}{\partial t}. \tag{5.133}$$

Then, we can rewrite (5.129) as

$$-\int_V \mathrm{d}^3x\, \vec{J} \cdot \vec{E} = \int_V \mathrm{d}^3x \left[\vec{\nabla} \cdot (\vec{E} \times \vec{H}) + \frac{\partial u}{\partial t}\right]. \tag{5.134}$$

As the volume V is arbitrary at this stage, the equality must hold exactly between the integrands,

$$-\frac{\partial u}{\partial t} = \vec{\nabla} \cdot \vec{S} + \vec{J} \cdot \vec{E}. \tag{5.135}$$

In these expressions, $-\partial u/\partial t$ is the rate of change of the energy density and $\vec{J} \cdot \vec{E}$ is the rate at which the fields do work on charge distributions. Thus, the *Poynting vector*

$$\vec{S} = \vec{E} \times \vec{H} \tag{5.136}$$

must represent the directional energy flux in the electromagnetic field, that is, the rate of energy transfer per unit area. At this stage, we could add the curl of any vector field to $\vec{S}$ without any effects, but as we will see, relativistic considerations actually single out its definition above as the only consistent one.

As an application, we can calculate the time-averaged flux of energy that a plane electromagnetic wave represents by calculating the real part of the Poynting vector (5.136) as applied to the fields (5.52) and (5.53),

$$\vec{S} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|\vec{E}|^2 \hat{k}. \tag{5.137}$$

This represents the energy per unit area per unit time carried by the wave, as in this case there are no charges or currents to which work is applied and (5.135) reduces to

$$-\frac{\partial u}{\partial t} = \vec{\nabla} \cdot \vec{S}. \tag{5.138}$$

The time-averaged energy density (5.131) of the wave is

$$u = \frac{\epsilon}{2}|\vec{E}|^2. \tag{5.139}$$

Taking the ratio of $|\vec{S}|$ and u, we explicitly see that the electromagnetic energy flows with speed $v = c$, as expected.

Stress tensor of the electromagnetic field. In the previous example we have focused solely on the electromagnetic fields: let us now consider in more detail what happens in the presence of charged particles. The work done on those, $\vec{J} \cdot \vec{E}$, will ultimately convert into a rate of increase of the kinetic energy of the charged particles within the volume of interest. If we denote the total energy of the particles within V as U_{mech} and assume that no particles escape the volume, we have

$$\frac{\mathrm{d}U_{\text{mech}}}{\mathrm{d}t} = \int_V \mathrm{d}^3x\, \vec{J} \cdot \vec{E}. \tag{5.140}$$

With this, Poynting's theorem (5.135) encloses the conservation of energy in the whole system as

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(U_{\text{mech}} + U_{\text{field}}\right) = -\oint_S \mathrm{d}a\, \vec{n} \cdot \vec{S}, \tag{5.141}$$

where in vacuum we can write

$$U_{\text{field}} = \frac{\epsilon_0}{2}\int_V \mathrm{d}^3x \left(E^2 + c^2B^2\right). \tag{5.142}$$

Let us now look at the conservation of momentum. For the charged particles, we can use Newton's second law as applied to the Lorentz force (5.124), which gives

$$\frac{\mathrm{d}\vec{p}_{\text{mech}}}{\mathrm{d}t} = \int_V \mathrm{d}^3x \left(\rho\vec{E} + \vec{J} \times \vec{B}\right). \tag{5.143}$$

Here, $\vec{p}_{\text{mech}}$ denotes the sum of all the momenta of all the particles in V. Using Maxwell's equations, we can replace

$$\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}, \qquad \vec{J} = \frac{1}{\mu_0}\vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{5.144}$$

to rewrite the integrand in (5.143) as

$$\rho\vec{E} + \vec{J} \times \vec{B} = \epsilon_0 \left[\vec{E}(\vec{\nabla} \cdot \vec{E}) + \vec{B} \times \frac{\partial \vec{E}}{\partial t} - c^2\vec{B} \times (\vec{\nabla} \times \vec{B})\right]. \tag{5.145}$$

We can now perform a few tricks. First, note that

$$\vec{B} \times \frac{\partial \vec{E}}{\partial t} = -\frac{\partial}{\partial t}(\vec{E} \times \vec{B}) + \vec{E} \times \frac{\partial \vec{B}}{\partial t}. \tag{5.146}$$

Second, we can add the term $c^2\vec{B}(\vec{\nabla}\cdot\vec{B})=0$ to the square bracket in (5.145) to arrive at

$$\begin{aligned}\rho\vec{E}+\vec{J}\times\vec{B}=&\epsilon_0\left[\vec{E}(\vec{\nabla}\cdot\vec{E})+c^2\vec{B}(\vec{\nabla}\cdot\vec{B})-\vec{E}\times(\vec{\nabla}\times\vec{E})-c^2\vec{B}\times(\vec{\nabla}\times\vec{B})\right]\\&-\epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}),\end{aligned}\tag{5.147}$$

where we have also used Faraday's law. Then, the rate of change of mechanical momentum (5.143) becomes

$$\begin{aligned}&\frac{\mathrm{d}\vec{p}_{\mathrm{mech}}}{\mathrm{d}t}+\frac{\partial}{\partial t}\int_V \mathrm{d}^3x\,\epsilon_0(\vec{E}\times\vec{B})=\\&\qquad=\epsilon_0\int_V \mathrm{d}^3x\left[\vec{E}(\vec{\nabla}\cdot\vec{E})+c^2\vec{B}(\vec{\nabla}\cdot\vec{B})-\vec{E}\times(\vec{\nabla}\times\vec{E})-c^2\vec{B}\times(\vec{\nabla}\times\vec{B})\right].\end{aligned}\tag{5.148}$$

In analogy with what we did for the energy in Poynting's theorem, we can try to interpret the integral on the left hand side as the momentum of the electromagnetic field in the volume V,

$$\vec{p}_{\mathrm{field}}=\epsilon_0\int_V \mathrm{d}^3x\,\vec{E}\times\vec{B}\tag{5.149}$$

$$=\epsilon_0\mu_0\int_V \mathrm{d}^3x\,\vec{E}\times\vec{H}.\tag{5.150}$$

Note that the electromagnetic momentum density

$$\vec{g}=\frac{1}{c^2}(\vec{E}\times\vec{H})\tag{5.151}$$

is proportional to the Poynting vector $\vec{S}$, that is, to the energy-flux density. The next step is to rewrite the right hand side of (5.148) as a surface integral of the normal component of something that we can interpret as momentum flow. In Cartesian coordinates, we can write the components of the terms involving the electric field as

$$\begin{aligned}\left[\vec{E}(\vec{\nabla}\cdot\vec{E})-\vec{E}\times(\vec{\nabla}\times\vec{E})\right]_i&=E_i\partial_jE_j-\epsilon_{ijk}E_j(\epsilon_{klm}\partial_lE_m)\\&=E_i\partial_jE_j-(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})E_j\partial_lE_m\\&=E_i\partial_jE_j-E_j\partial_iE_j+E_j\partial_jE_i\\&=\partial_j(E_iE_j)-\frac{1}{2}\partial_i(E_kE_k)\\&=\sum_j\frac{\partial}{\partial x_j}\left(E_iE_j-\frac{1}{2}\vec{E}\cdot\vec{E}\,\delta_{ij}\right),\end{aligned}\tag{5.152}$$

and similarly for $\vec{B}$. The expression (5.152) has precisely the form of a divergence of a rank-2 tensor, and we therefore define the *Maxwell stress tensor* T_{ij} as

$$T_{ij}=\epsilon_0\left[E_iE_j+c^2B_iB_j-\frac{1}{2}\left(\vec{E}\cdot\vec{E}+c^2\vec{B}\cdot\vec{B}\right)\delta_{ij}\right].\tag{5.153}$$

The Maxwell stress tensor is symmetric, $T_{ij} = T_{ji}$. Putting everything together, the law of conservation of momentum can be written as

$$\frac{\partial}{\partial t}\left(\vec{p}_{\text{mech}} + \vec{p}_{\text{field}}\right)_i = \sum_j \int_V \mathrm{d}^3x \, \frac{\partial}{\partial x_j} T_{ij} \tag{5.154}$$

$$= \oint_S \mathrm{d}a \sum_j T_{ij} n_j, \tag{5.155}$$

where we have used the divergence theorem to rewrite the volume integral as a surface one (as usual, $\vec{n}$ denotes the outward normal to the surface). As for Poyting's theorem, the volume V in the law of momentum conservation is arbitrary. We can thus also write it in differential form as

$$f_i + \frac{\partial}{\partial t} g_i = \partial_j T_{ij}, \tag{5.156}$$

with $\vec{f}$ and $\vec{g}$ as defined above. We can interpret $\sum_j T_{ij} n_j$ as the flow per unit area of momentum across the surface S into the volume V. Said otherwise, it is the force per unit area transmitted across the surface S and acting on the combined system of particles and fields inside V. As an application, we could use this expression to calculate the forces acting on an object immersed in an electromagnetic field: we would need to enclose the object in a surface S and calculate the electromagnetic force by integrating $\sum_j T_{ij} n_j$ over the surface.

Let us work out a simple example. Consider for simplicity only the electric part of the stress tensor, which we denote T^E. We use a coordinate system such that at a given point the electric field is oriented in the z direction, so that at that point we only have one component $E_z = |\vec{E}|$. The stress tensor in such circumstances is diagonal,

$$T^E = \frac{1}{2}\epsilon_0 E_z^2 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.157}$$

If we now place a small surface of area Δa and normal $\vec{n}$ at that point, it will feel a force

$$\Delta\vec{F} = T^E \vec{n} \, \mathrm{d}a = \frac{1}{2}\epsilon_0 E_z^2 \, \mathrm{d}a \begin{pmatrix} -n_x \\ -n_y \\ n_z \end{pmatrix}. \tag{5.158}$$

We can consider two cases:

- If the surface is perpendicular to the field lines, so that $\vec{n} \parallel \vec{E}$ and $n_x = n_y = 0$, the force acts in the direction of the normal $\vec{n}$.
- If the surface is parallel to the field lines, so that $\vec{n} \perp \vec{E}$ and $n_z = 0$, the force acts in the opposite direction of the normal $\vec{n}$.

A small cube with outward-pointing normal on its sides and located at the point of interest would thus be stretched in the direction of the field lines, but compressed in the direction perpendicular to them. Said otherwise, there is tension in the direction of the field lines, and pressure perpendicular to them.

5.2 Electromagnetic waves in matter.

Let us now consider some particularities of electromagnetic waves propagating in non-vacuum situations. We will assume that we are dealing with linear materials for which we can write $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$, where we allow for the electric and magnetic constants to depend on the frequency of the electromagnetic radiation, $\epsilon = \epsilon(\omega)$ and $\mu = \mu(\omega)$.

Electric onductivity. Let us introduce another constitutive relation motivated by experimental observations. In electric *conductors* containing free charges that can move through the material, an electric field produces an electric current in the direction of the field. In many conductors, the current density induced is proportional to the electric field and this phenomenon can be thus characterized by *Ohm's law*

$$\vec{J}(\vec{x}) = \sigma\vec{E}(\vec{x}). \tag{5.159}$$

Here, σ is known as the *electric conductivity* of the material, while its inverse $\rho = 1/\sigma$ is known as the *electric resistivity.* When $\sigma \to 0$, we have that $\vec{J} = 0$ and the material is said to be an *insulator*, while for $\rho \to 0$ we have $\vec{E} = 0$ and the material is called a *perfect (or ideal) conductor*.

In general, the conductivity σ depends on the frequency of the electromagnetic field. As we will see, this is very important when studying the propagation of electromagnetic waves. Ohm's law can be written in Fourier space as

$$\vec{J}(\omega) = \sigma(\omega)\vec{E}(\omega). \tag{5.160}$$

When referring to the electric conductivity, we will always mean the one defined in Fourier space, $\sigma(\omega)$.

One of the simplest models to explain the microscopic origin of the conductivity and its frequency dependence is the Drude model. In this model, the free electric charges (electrons) in the material are accelerated by the electric field $\vec{E}$, but after a typical time τ (sometimes known as the relaxation time) collide with the atoms in the material and are stopped.

We can use this model to obtain an expression for σ, first in the zero frequency limit ($\vec{E}$ uniform and constant). At any time t, an electron (or any other charge carrier) will on average have travelled for a time τ after its last collision, and will thus have accumulated a momentum

$$\Delta\langle\vec{p}\rangle = e\vec{E}\tau, \tag{5.161}$$

where e is the electron charge. Because right after its last collision the electron was as likely to have bounced backwards as forwards, any prior momentum can be neglected and we thus have

$$\langle \vec{p} \rangle = e\vec{E}\tau. \tag{5.162}$$

Now, using that $\langle \vec{p} \rangle = m \langle \vec{v} \rangle$, where m is the electron mass and $\vec{J} = nq \langle \vec{v} \rangle$ with n_0 the number density of electrons, to write

$$\vec{J} = \sigma_0 \vec{E}, \qquad \text{with} \quad \sigma_0 = \frac{n_o e^2 \tau}{m}. \tag{5.163}$$

As an example, for copper one has $\tau \sim 10^{-14}$ s.

This model also allows to qualitatively analyze the behavior of σ as a function of the frequency of the applied electric field. For low frequencies $\omega \ll 1/\tau$, the response should be very similar to the static one we just discussed. However, for high frequencies $\omega \gg 1/\tau$, the charge carriers change their trajectory very fast, before any collision can occur. We thus expect no damping to occur in that case: the conductor becomes transparent to electromagnetic waves of sufficiently high frequency. A more quantitative analysis shows that the frequency-dependence of the conductivity in this simple model is

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}. \tag{5.164}$$

The imaginary part of σ indicates that the current lags behind the electric field, as the electrons need a time τ to accelerate in response to the changing electric field.

Telegrapher equations. Let us consider homogeneous and isotropic media that are characterized by their values of ϵ, μ, and σ. Thus, they can support nonvanishing currents $\vec{J} = \sigma\vec{E}$, but not free charge densities, i.e. we fix $\rho = 0$. The macroscopic Maxwell equations (5.1)- (5.4) can under these assumptions be recast as the *telegrapher equations*

$$\left(\nabla^2 - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} - \sigma\mu\frac{\partial}{\partial t}\vec{E} = 0, \tag{5.165}$$

$$\left(\nabla^2 - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{B} - \sigma\mu\frac{\partial}{\partial t}\vec{B} = 0, \tag{5.166}$$

where n is the index of refraction of the medium. These are wave equations with a damping term. This term, which is linear in the time derivatives, leads to an energy loss when the wave propagates through a conductor with $\sigma \neq 0$. We can solve the telegrapher equations using plane waves but always keeping in mind that ϵ, ω, and σ are frequency-dependent quantities.

Electromagnetic waves in insulators, dispersion. In an insulator, $\sigma = 0$ and the telegrapher equations reduce to free wave equations. We have already discussed their solutions, which can be constructed from plane waves with a phase velocity c/n. Because $\epsilon = \epsilon(\omega)$ and $\mu = \mu(\omega)$ the relation

$$\omega^2 = \frac{c^2}{n^2}k^2 = \frac{1}{\epsilon(\omega)\mu(\omega)}k^2 \tag{5.167}$$

is not the same for all frequencies. Usually, the relation

$$\omega = \omega(k) \tag{5.168}$$

is known as the *dispersion relation* of the medium. Sometimes, ω cannot be written as a single-valued function of k and the dispersion relation is rather written implicitly as $f(\omega, k) = 0$. If $\epsilon(\omega)$ is not a constant function, we say that the medium is *dispersive* (in most materials $\mu \simeq \mu_0$ and thus μ does not introduce any significant dispersion).

Let us illustrate the consequences of dispersion in the simplified scenario of waves propagating in one spatial dimension. The general solution of the wave equation can be obtained as a superposition of plane waves

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} \mathrm{d}k\, A(k)\mathrm{e}^{i[kx-\omega(k)t]}, \tag{5.169}$$

which is known as the *wave packet*. If the function $A(k)$ is peaked around a central wave number $k = k_0$, we can expand $\omega(k)$ as

$$\begin{aligned}\omega(k) &= \omega(k_0) + (k-k_0)\left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k_0} + \dots \\ &= \omega_0 + (k-k_0)v_g + \dots,\end{aligned} \tag{5.170}$$

where we have defined the *group velocity*

$$v_g = \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k_0}. \tag{5.171}$$

With this, the wave packet solution can be approximated as

$$u(x,t) \simeq \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} \mathrm{d}k\, A(k)\mathrm{e}^{i[kx-\omega_0 t-(k-k_0)v_g t]} \tag{5.172}$$

$$= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-i(\omega_0-k_0v_g)t}\int_{\infty}^{\infty} \mathrm{d}k\, A(k)\mathrm{e}^{ik(x-v_g t)]}. \tag{5.173}$$

We can thus write the *intensity* of the wave as

$$|u(x,t)^2| = |u(x-v_g t)|^2. \tag{5.174}$$

This shows that the wave pulse travels with velocity v_g. In this linear approximation, the shape of the pulse is not distorted. If an energy density is associated with the magnitude of the wave, we see that the transport of energy occurs with the group velocity rather than the phase velocity. For electromagnetic waves, we have

$$\omega(k) = \frac{ck}{n(k)}, \tag{5.175}$$

where c is the speed of light in vacuum and $n(k)$ is the index of refraction express as a function of k. The phase velocity is in this case

$$v_p = \frac{\omega(k)}{k} = \frac{c}{n(k)}, \tag{5.176}$$

and can be greater or smaller than c depending on whether $n(k)$ is smaller or greater than one (usually $n(k) > 1$). The group velocity

$$v_g = \frac{c}{n(\omega) + \omega \frac{\mathrm{d}n}{\mathrm{d}\omega}}, \tag{5.177}$$

where we now regard n as a function of ω, is in general different to the phase velocity. We can have two different kinds of dispersion relations,

- Normal dispersion if $\mathrm{d}n/\mathrm{d}\omega > 0$. As usually $n > 1$, this means that the velocity of energy flow is less than the phase velocity and also less than c.

- Anomalous dispersion if $\mathrm{d}n/\mathrm{d}\omega < 0$. This usually occurs near frequencies that suffer strong absorption in the material, known as resonant frequencies, and for which the function $\epsilon(\omega)$ can exhibit poles in the complex ω plane. In these situations, the group velocity can become larger than the speed of light. This is however no cause of concern, because in these cases the Taylor expansion (5.170) is not a good approximation and the above definition of v_g is not a useful one. Usually, a pulse whose dominant frequencies are close to a strong absorption line is absorbed in the material and distorted as it travels through it. With more advanced mathematical techniques, it is possible to show that the transport of energy in dispersive media never occurs with a velocity greater than the speed of light in vacuum.

To conclude our discussion of insulators, it is worth pointing that beyond the linear approximation in (5.170), v_g can still be interpreted as the velocity of the center of the wave packet. However, for a dispersion relation in which non-linear terms are relevant, as is the case most media, the shape of the pulse is distorted as is propagates. Typically, the narrower the wave packet is, the faster it broadens (or disperses) as it travels along.

Electromagnetic waves in conductors, plasma frequency. In conductors, $\sigma \neq 0$ and the damping term in the telegrapher equations becomes relevant. Let us solve the one for $\vec{E}$ (the solution for $\vec{B}$ is analogous),

$$\left(\nabla^2 - \epsilon\mu\frac{\partial^2}{\partial t^2}\right)\vec{E} - \sigma\mu\frac{\partial}{\partial t}\vec{E} = 0, \tag{5.178}$$

via plane waves. For that, start with the ansatz

$$\vec{E} = \vec{\epsilon}\,\mathrm{e}^{i(\vec{k}\cdot\vec{x}-\omega t)}. \tag{5.179}$$

Inserting it into (5.178) results in

$$k^2 - \epsilon\mu\omega^2 - i\sigma\mu\omega = 0, \tag{5.180}$$

and thus

$$k^2 = \left(\epsilon\mu + i\frac{\sigma\mu}{\omega}\right)\omega^2. \tag{5.181}$$

In analogy to the insulator case, we can define

$$n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0} + i\frac{1}{\omega}\frac{\sigma\mu}{\epsilon_0\mu_0}} \tag{5.182}$$

as a generalized index of refraction with real and imaginary parts

$$n = n_R + in_I. \tag{5.183}$$

The generalized index of refraction is clearly a function of frequency. In many materials, $\mu \simeq \mu_0$ and we ca approximate

$$n \simeq \sqrt{\frac{\epsilon}{\epsilon_0} - \frac{\sigma}{i\omega\epsilon_0}}. \tag{5.184}$$

We can thus translate to the conductor case the known results for insulators by employing a complex dielectric constant

$$\epsilon \to \epsilon(\omega) = \epsilon - \frac{\sigma}{i\omega}, \tag{5.185}$$

where $\sigma = \sigma(\omega)$. With this, the index of refraction can be written as $n = \sqrt{\epsilon(\omega)/\epsilon_0}$. A plane wave with wave number $k = n\omega/c = (n_R + in_I)\omega/c$ propagating in the z direction has

$$\vec{E} = \vec{\epsilon}\mathrm{e}^{i(kz-\omega t)} = \vec{\epsilon}\,\mathrm{e}^{i\omega(n_R\frac{z}{c}-t)}\mathrm{e}^{-n_I\frac{\omega z}{c}}. \tag{5.186}$$

Its real part is

$$\mathrm{Re}\Big[\vec{E}\Big] = \mathrm{Re}\Big[\vec{\epsilon}\,\mathrm{e}^{i\omega(n_R\frac{z}{c}-t)}\Big]\,\mathrm{e}^{-n_I\frac{\omega z}{c}}, \tag{5.187}$$

and analogously for $\vec{B}$. The last factor represents an exponential damping of the amplitude. The electromagnetic wave is thus absorbed after a distance

$$d = \frac{c}{\omega n_I}, \tag{5.188}$$

which is known as the *penetration depth* of the material.

In the Drude model, we argued that the electric conductivity can be written as

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}, \tag{5.189}$$

with σ_0 given in (5.163). This means that the complex electric conductivity becomes

$$\epsilon(\omega) = \epsilon + i\frac{\sigma_0}{\omega(1 - i\omega\tau)}. \tag{5.190}$$

We can distinguish two limiting cases in this expression:

- For small frequencies $\omega \ll 1/\tau$, we have

$$\sqrt{\epsilon(\omega)} \simeq \sqrt{i\frac{\sigma_0}{\omega}} = (1+i)\sqrt{\frac{\sigma_0}{2\omega}}, \tag{5.191}$$

  and thus

$$n_R = n_I = \sqrt{\frac{\sigma_0}{2\epsilon_0\omega}}. \tag{5.192}$$

  The penetration depth is in this case

$$d = \sqrt{\frac{2}{\mu_0\sigma_0\omega}}. \tag{5.193}$$

  For the example of copper, $\sigma_0 \simeq 5.8 \times 10^{17}\,\mathrm{s}^{-1}$ and

$$\omega = 2\pi \cdot 50\,\mathrm{s}^{-1} \quad \Rightarrow \quad d = 0.9\,\mathrm{cm}, \tag{5.194}$$
$$\omega = 2\pi \cdot 10^9\,\mathrm{s}^{-1} \quad \Rightarrow \quad d = 2 \times 10^{-4}\,\mathrm{cm}. \tag{5.195}$$
$$\tag{5.196}$$

  The field and current densities can only penetrate the outermost layer of the conductor. This phenomenon is known as the *skin effect*.

- For high frequencies $\omega \gg 1/\tau$ we have, in contrast,

$$\epsilon(\omega) = \epsilon - \frac{\sigma_0}{\tau\omega^2} = \epsilon\left(1 - \frac{\omega_P^2}{\omega^2}\right), \tag{5.197}$$

where we have defined the *plasma frequency*

$$\omega_P = \sqrt{\frac{n_0 e^2}{\epsilon m}}. \tag{5.198}$$

For frequencies $\omega < \omega_P$, the dielectric constant is negative and thus

$$n_R = 0 \qquad \text{and} \qquad n_I = \sqrt{\frac{\epsilon}{\epsilon_0}\left(\frac{\omega_P^2}{\omega^2} - 1\right)}. \tag{5.199}$$

This leads to an exponential decay: waves incident penetrate only a very short distance into the material and are almost completely reflected.

On the other hand, for frequencies $\omega > \omega_P$, the dielectric constant is positive and

$$n_R = \sqrt{\frac{\epsilon}{\epsilon_0}\left(1 - \frac{\omega_P^2}{\omega^2}\right)} \qquad \text{and} \qquad n_I = 0. \tag{5.200}$$

The conductor can in this regime completely transmit the radiation and becomes *ultraviolet transparent*.

Continuing with our previous examples, copper has a plasma frequency $\omega_P \simeq 1.6 \times 10^{16}\,\mathrm{s}^{-1}$. This means that for visible light in the frequency range $\omega = (2.4{-}5.2) \times 10^{15}\,\mathrm{s}^{-1}$, copper is opaque. Almost all metals have a plasma frequency in the ultraviolet and thus reflect light.

Electrolytes are media in which electrical conductivity occurs through the movement of ions rather than electrons (examples are soluble salts and acids). There, the number density of charge carriers n_0 is typically lower and their mass m is larger than for electrons in a metallic conductor. As a consequence, ω_P is smaller and electrolytes are normally transparent to visible light.

Poynting's theorem in linear dispersive media with losses. The vacuum result above can be straightforwardly extended to linear media (where $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$) as long as there is no dispersion or losses, that is, if ϵ and μ are real and frequency independent. However, most materials do exhibit dispersion and losses. We can take those into account by performing a Fourier decomposition of the fields in time

$$\vec{E}(\vec{x}, t) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \vec{E}(\vec{x}, \omega)\mathrm{e}^{-i\omega t}, \vec{D}(\vec{x}, t) = \int_{-\infty}^{\infty} \mathrm{d}\omega\, \vec{D}(\vec{x}, \omega)\mathrm{e}^{-i\omega t}, \tag{5.201}$$

and similarly for $\vec{B}$ and $\vec{H}$. We still require the medium to have a linear response so that we can write

$$\vec{D}(\vec{x}, \omega) = \epsilon(\omega)\vec{E}(\vec{x}, \omega), \tag{5.202}$$

but we now allow for the susceptibility to be frequency dependent and complex. As the (time-space) fields have to be real, we have that $\vec{E}(\vec{x},-\omega) = \vec{E}^*(\vec{x},\omega)$ and $\vec{D}(\vec{x},-\omega) = \vec{D}^*(\vec{x},\omega)$, which thus implies that $\epsilon(-\omega) = \epsilon^*(\omega)$. The frequency dependence of ω induces nonlocality in the relation between the time-space fields $\vec{D}(\vec{x},t)$ and $\vec{E}(\vec{x},t)$. A similar discussion can be made for the magnetic counterpart with $\vec{B}(\vec{x},\omega) = \mu(\omega)\vec{H}(\vec{x},\omega)$.

In Fourier space, we can calculate the time derivative $\vec{E}\cdot(\partial\vec{D}/\partial t)$ appearing in (5.129) as

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \int \mathrm{d}\omega \int \mathrm{d}\omega'\, E_i^*(\omega')\,[-i\omega\epsilon(\omega)]\cdot E_i(\omega)\mathrm{e}^{-i(\omega-\omega')t} \tag{5.203}$$

$$= \frac{1}{2}\int \mathrm{d}\omega \int \mathrm{d}\omega'\, E_i^*(\omega')\,[-i\omega\epsilon(\omega) + i\omega'\epsilon^*(\omega')]\cdot E_i(\omega)\mathrm{e}^{-i(\omega-\omega')t}. \tag{5.204}$$

In the first line, the complex conjugation comes from the definition of the scalar product in complex vector spaces. The second line can be shown to be equivalent to the first one by splitting the two integrals and seeing that they are equivalent after substituting $\omega \leftrightarrow -\omega'$ and the reality conditions of the fields and the susceptivity.

Now, we assume that the electric field is dominated by a narrow range of frequencies around a characteristic one ω_0, over which $\epsilon(\omega)$ does not change significantly. This allows to expand the term in brackets around ω_0 to find

$$\begin{aligned}[-i\omega\epsilon(\omega) + i\omega'\epsilon^*(\omega')] &= -i\omega_0\epsilon(\omega_0) + i\omega_0\epsilon^*(\omega_0) \\ &\quad - i(\omega-\omega_0)\left.\frac{\mathrm{d}\,(\omega\epsilon^*)}{\mathrm{d}\omega}\right|_{\omega_0} + i(\omega'-\omega_0)\left.\frac{\mathrm{d}\,(\omega\epsilon)}{\mathrm{d}\omega}\right|_{\omega_0} + \dots \qquad (5.205)\\ &= 2\omega_0\,\mathrm{Im}[\epsilon(\omega_0)] - i(\omega-\omega')\,\mathrm{Re}\left[\left.\frac{\mathrm{d}\,(\omega\epsilon)}{\mathrm{d}\omega}\right|_{\omega_0}\right] + \dots \qquad (5.206)\end{aligned}$$

Where we have used the fact that

$$\left(\left.\frac{\mathrm{d}\,(\omega\epsilon)}{\mathrm{d}\omega}\right|_{\omega_0}\right)^* = \left.\frac{\mathrm{d}\,(\omega\epsilon^*(\omega))}{\mathrm{d}\omega}\right|_{\omega_0} = \left.\frac{\mathrm{d}\,(\omega\epsilon(-\omega))}{\mathrm{d}\omega}\right|_{\omega_0} = \left.\frac{\mathrm{d}\,(-\omega\epsilon(\omega))}{\mathrm{d}(-\omega)}\right|_{\omega_0} = \left.\frac{\mathrm{d}\,(\omega\epsilon)}{\mathrm{d}\omega}\right|_{\omega_0}, \tag{5.207}$$

so that this derivative is actually real. With this, we can write

$$\begin{aligned}\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} &= \int \mathrm{d}\omega \int \mathrm{d}\omega'\, E_i^*(\omega')\omega_0\,\mathrm{Im}[\epsilon(\omega_0)]\cdot E_i(\omega)\mathrm{e}^{-i(\omega-\omega')t} \\ &\quad + \frac{\partial}{\partial t}\frac{1}{2}\int \mathrm{d}\omega \int \mathrm{d}\omega'\, E_i^*(\omega')\,\mathrm{Re}\left[\left.\frac{\mathrm{d}\,(\omega\epsilon)}{\mathrm{d}\omega}\right|_{\omega_0}\right]\cdot E_i(\omega)\mathrm{e}^{-i(\omega-\omega')t}. \qquad (5.208)\end{aligned}$$

An analogous procedure can be followed for the magnetic counterpart $\vec{H}\cdot(\partial\vec{B}/\partial t)$.

Although the expression above is not too transparent, we can glimpse that the first term in (5.208) corresponds to some form of loss of electrical energy, and the second term must be some form of effective electrical energy density. With this in mind and adding the magnetic term, we get to

$$\langle\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}+\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\rangle=\omega_0\,\mathrm{Im}[\epsilon(\omega_0)]\,\langle\vec{E}(\vec{x},t)\cdot\vec{E}(\vec{x},t)\rangle+\omega_0\,\mathrm{Im}[\mu(\omega_0)]\,\langle\vec{H}(\vec{x},t)\cdot\vec{H}(\vec{x},t)\rangle + \frac{\partial u_{\mathrm{eff}}}{\partial t}, \tag{5.209}$$

where we have defined the effective electromagnetic energy density as

$$u_{\mathrm{eff}}=\frac{1}{2}\,\mathrm{Re}\left[\left.\frac{\mathrm{d}\,(\omega\epsilon)}{\mathrm{d}\omega}\right|_{\omega_0}\right]\langle\vec{E}(\vec{x},t)\cdot\vec{E}(\vec{x},t)\rangle+\frac{1}{2}\,\mathrm{Re}\left[\left.\frac{\mathrm{d}\,(\omega\mu)}{\mathrm{d}\omega}\right|_{\omega_0}\right]\langle\vec{H}(\vec{x},t)\cdot\vec{H}(\vec{x},t)\rangle\,. \tag{5.210}$$

The brakets in these expressions should be understood as a time average over the period $\sim 1/\omega_0$ of the main frequency in the propagating radiation, as or result is only valid for electromagnetic radiation with frequencies close to ω_0.

We can substitute this result in (5.129) to obtain the following Poyinting's theorem for a dispersive medium with losses:

$$\frac{\partial u_{\mathrm{eff}}}{\partial t}+\vec{\nabla}\cdot\vec{S}=-\vec{J}\cdot\vec{E}-\omega_0\,\mathrm{Im}[\epsilon(\omega_0)]\,\langle\vec{E}(\vec{x},t)\cdot\vec{E}(\vec{x},t)\rangle-\omega_0\,\mathrm{Im}[\mu(\omega_0)]\,\langle\vec{H}(\vec{x},t)\cdot\vec{H}(\vec{x},t)\rangle\,. \tag{5.211}$$

The term $\vec{J}\cdot\vec{E}$ in the right hand side describing ohmic losses due to conduction is sometimes included in the following terms that describe generic dissipation in the medium. This expression encodes the local conservation of electromagnetic energy in realistic situations where, as well as energy flow out the locality ($\vec{\nabla}\cdot\vec{S}\neq 0$), there can be absorption and dissipation of the electromagnetic radiation in the medium, described by the imaginary part of ϵ and μ.

Chapter 6

Special relativity

6.1 Relativity and the laws of nature

Classical mechanics and Galilean transformations. In classical (or Newtonian) mechanics, observers moving relative to each other at a constant speed observe the same laws of nature. This can be formulated as an invariance of the laws of Newtonian mechanics under *Galilean transformations*

$$t' = t + \tau, \tag{6.1}$$
$$\vec{x}' = -\vec{v}t + R\vec{x} + \vec{a}, \tag{6.2}$$

where τ is a constant scalar, $\vec{v}$, and $\vec{a}$ are two constant vectors, and R is an orthogonal 3×3 matrix ($RR^T = 1$). In total, these transformations form a 10-parameter group, called the *Galilean group*. Physically, the invariance under Galilean transformations is a consequence of

- the homogeneity of time (τ),
- the homogeneity of space ($\vec{a}$),
- the isotropy of space (R), and
- the equivalence of observers which differ only by constant relative motion ($\vec{v}$).

The last point motivates introducing the notion of *inertial reference frames*, which are systems that move with respect to each other with constant velocity. The (sometimes referred to as Newtonian) relativity principle of classical mechanics can thus be formulated as the equivalence of all inertial reference frames.

Relativity and electrodynamics. It is easy to see that the Maxwell equations are not invariant under Galilean transformations. Intuitively, the equations explicitly depend on the speed of light, which cannot possibly be the same in two reference frames that move relative to each other. More explicitly, consider a plane wave solution to the free Maxwell equations,

$$\mathrm{e}^{i(\vec{k}\cdot\vec{x}-\omega t)} \qquad \text{with} \quad c|\vec{k}| = \omega. \tag{6.3}$$

Under the Galilean transformation $t' = t$, $\vec{x}' = \vec{x} - \vec{v}t$, the plane wave transforms as

$$\mathrm{e}^{i(\vec{k}\cdot\vec{x}'+\vec{k}\cdot\vec{v}t-\omega t')} = \mathrm{e}^{i(\vec{k}'\cdot\vec{x}'-\omega' t')}, \tag{6.4}$$

with $\vec{k}' = \vec{k}$ and $\omega' = \omega - \vec{k}\cdot\vec{v}$. But, in general, $\omega' = \omega - \vec{k}\cdot\vec{v} \neq c|\vec{k}'|$, meaning that the transformed plane wave is not a solution of the free Maxwell equations. This means that the laws of electrodynamics are not the same in inertial reference frames that are moving with respect to each other. At this stage, there are three possible explanations for this:

- The Maxwell equations are incorrect.
- Classical mechanics are Galilean invariant, but electrodynamics has a preferred reference frame in which the Maxwell equations hold: the *aether* rest frame.
- The Newtonian relativity principle is incorrect, and neither mechanics nor electrodynamics are in fact invariant under Galilean transformations.

While first two possibilities are extremely disfavored by experimental observations, the third one, which was fruitfully pursued by Albert Einstein, has been shown to be correct. In 1905, Einstein formulated the two basic postulates that would reconcile the invariance properties of mechanics and electrodynamics:

1. All physical laws are invariant in all inertial reference frames.
2. The speed of light in vacuum is the same for all observers, regardless of the motion of the light source or observer.

Let us now work out the consequences of accepting these postulates as true.

Special Lorentz transformations. We want to find the transformation laws between two inertial frames respecting the two postulates above. For that, consider a reference frame I' that moves with velocity $\vec{v}$ relative to another frame I. The origins of both systems coincide for $t = 0$, $t' = 0$. If a light pulse is emitted at $t = 0$, its wavefront in the frame I is described by

$$|\vec{x}|^2 - c^2t^2 = 0, \qquad \text{for} \quad t > 0. \tag{6.5}$$

Recall that we called this the future light cone L_+. Because the speed of light has to be the same in all frames, the wavefront must have the same form in the frame I', contrary to the classical expectation,

$$|\vec{x}'|^2 - c^2t'^2 = 0, \qquad \text{for} \quad t' > 0. \tag{6.6}$$

It is easy to see that for this to be possible, the time coordinate t' has to transform together with the spacial coordinates $\vec{x}'$ (aside for rotations and spatial translations). An example of transformations that satisfy this condition are the *Lorentz boosts*. A boost in the x direction with velocity v corresponds to

$$t' = \gamma\left(t - \frac{\beta}{c}x\right), \tag{6.7}$$

$$x' = \gamma(x - \beta ct), \tag{6.8}$$

$$y' = y, \tag{6.9}$$

$$z' = z, \tag{6.10}$$

with

$$\beta = \frac{v}{c} \qquad \text{and} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{6.11}$$

It is convenient to introduce the *rapidity* w via

$$\tanh w = \beta = \frac{v}{c}, \tag{6.12}$$

which means that

$$\sinh w = \gamma\beta, \tag{6.13}$$

$$\cosh w = \gamma. \tag{6.14}$$

You can check that indeed $\cosh^2 w - \sinh^2 w = 1$. With this definition, we can write the Lorentz transformation in matrix form as

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh w & -\sinh w \\ -\sinh w & \cosh w \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}. \tag{6.15}$$

This has a form reminiscent of a rotation. The appearance of hyperbolic functions instead of circular ones can be traced back to the relative minus sign between the space and time terms in (6.5). The parametrization in terms of rapidity makes it easy to check that

$$c^2 t'^2 - \vec{x}'^2 = c^2 t^2 - \vec{x}^2. \tag{6.16}$$

6.2 The kinematics of special relativity

Four-vectors and Minkowski spacetime. We have seen that special Lorentz transformations, which leave the future light cone invariant, can be seen as generalized rotations that mix space and temporal coordinates. It is therefore convenient to combine ct and $\vec{x}$ into the *coordinate 4-vector*

$$(x^\mu) = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}. \tag{6.17}$$

The coordinate 4-vector describes the coordinates of a point (or event) in spacetime. By convention, we use Greek letters ($\mu, \nu, \rho, \dots$) to denote spacetime indices that run from 0 to 3, while we use Latin letters ($i, j, k, \dots$) to denote spatial indices ranging from 1 to 3 in Cartesian coordinates.

Analogously to the Euclidean metric in the 3-dimensional "spatial" space, we can define the *Minkowski metric tensor* $g_{\mu\nu}$ in the 4-dimensional spacetime via

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{6.18}$$

With it, the scalar product of two vectors can be written as

$$x \cdot y = g_{\mu\nu} x^\mu y^\nu = c^2 t_x t_y - \vec{x} \cdot \vec{y}. \tag{6.19}$$

We are using the Einstein summation convention, by which repeated upper and lower indices are to be summed over. Note that this is an indefinite scalar product, as the quantity $x \cdot y$ can be positive, negative or null. The space of coordinate 4-vectors x^μ endowed with the scalar product induced by the Minkowski metric $g_{\mu\nu}$ is known as the *Minkowski space* (or *Minkowski spacetime*).

We can also use the metric to define the *invariant length* of a coordinate 4-vector,

$$x^2 = x \cdot x = g_{\mu\nu} x^\mu x^\nu = c^2 t^2 - \vec{x}^2, \tag{6.20}$$

or sometimes the square root of that, $\sqrt{x^2}$. With this notation, the equation of the wavefront of a light pulse emitted at $t = 0$ from $\vec{x} = 0$ can be simply written as

$$x^2 = 0. \tag{6.21}$$

More generally, we can define the distance (or *invariant interval*) between two spacetime points x and y as

$$s^2 = (x - y)^2 = c^2 (t_x - t_y)^2 - (\vec{x} - \vec{y})^2. \tag{6.22}$$

The interval can be $s^2 = 0$, in which case we say that the separation between the two points is *light-like*; it can be $s^2 < 0$, in which case we say it to be *space-like*; or it can be $s^2 > 0$, in which case the points are said to be *time-like* separated.

Another useful object to introduce is the 4-dimensional Kronecker δ, given by

$$\delta^\mu_\nu = \begin{cases} 1 & \text{if} \quad \mu = \nu, \\ 0 & \text{if} \quad \mu \neq \nu. \end{cases} \tag{6.23}$$

In matrix notation, this corresponds to the 4-dimensional identity matrix. The Kronecker delta allows to define the inverse metric tensor $g^{\mu\nu}$ (note the location of the indices) via

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho. \tag{6.24}$$

Linear transformations of coordinates act on coordinate 4-vectors as

$$x'^{\mu} = A^{\mu}{}_{\nu} x^{\nu}, \tag{6.25}$$

where $(A^{\mu}{}_{\nu})$ is a real 4x4 matrix. A Lorentz boost of velocity v in the x^1 direction thus corresponds to a coordinate transformation $x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$, with

$$(\Lambda^{\mu}{}_{\nu}) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.26}$$

Any 4-vector that transforms in the same way as the coordinate 4-vector under a Lorentz boost is called a *contravariant 4-vector* (or simply a vector), and is written with an upper index:

$$a^{\mu} \to a'^{\mu} = \Lambda^{\mu}{}_{\nu} a^{\nu}. \tag{6.27}$$

The Kronecker δ also allows to define the inverse of a Lorentz boost, Λ^{-1}, via

$$(\Lambda^{-1})^{\mu}{}_{\nu} \Lambda^{\nu}{}_{\rho} = \delta^{\mu}_{\rho}. \tag{6.28}$$

For instance, for the Lorentz boost above, we have

$$((\Lambda^{-1})^{\mu}{}_{\nu}) = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.29}$$

A 4-vector that transforms with Λ^{-1} is the one constructed with the derivatives with respect to the coordinates,

$$\frac{\partial}{\partial x^{\mu}} \to \frac{\partial}{\partial x'^{\mu}} = (\Lambda^{-1})^{\nu}{}_{\mu} \frac{\partial}{\partial x^{\nu}}. \tag{6.30}$$

We often just write ∂_{μ} to refer to this 4-vector. We say that any vector transforming like ∂_{μ} is a *covariant 4-vector* (or covector), and we write it with a lower index:

$$a_{\mu} \to a'_{\mu} = (\Lambda^{-1})^{\nu}{}_{\mu} a_{\nu}. \tag{6.31}$$

This generalizes straightforwardly to higher-order tensors: for each upper index, we add a factor of Λ to the transformation law; for each lower index, a factor of Λ^{-1}. As an example, the metric tensor transforms as

$$g_{\mu\nu} \to g'_{\mu\nu} = (\Lambda^{-1})^{\rho}{}_{\mu} (\Lambda^{-1})^{\sigma}{}_{\nu} g_{\rho\sigma}. \tag{6.32}$$

The metric tensor (or its inverse) allows to “convert” vectors into covectors (and viceversa) via

$$a_{\mu} = g_{\mu\nu} a^{\nu} \qquad \text{and} \qquad a^{\mu} = g^{\mu\nu} a_{\nu}. \tag{6.33}$$

For instance, the coordinate 4-vector is dual to the covector

$$(x_\mu) = (g_{\mu\nu}x^\nu)\begin{pmatrix} ct \\ \vec{x} \end{pmatrix} = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}. \tag{6.34}$$

With this notation, the scalar product of two 4-vectors satisfies

$$a \cdot b = g_{\mu\nu}a^\mu b^\nu = a^\mu b_\mu = a_\mu b^\mu = g^{\mu\nu}a_\mu b_\nu. \tag{6.35}$$

This generalizes to higher-order tensors as well: the metric or its inverse can be used to lower or raise any index. For example, we have that

$$g^\mu{}_\nu = g^{\mu\rho}g_{\rho\nu} = \delta^\mu_\nu. \tag{6.36}$$

Proper time and 4-velocity. Consider a particle that is moving with instantaneous velocity $\vec{u}(t)$ relative to some inertial frame I. In a short time interval $\mathrm{d}t$, the position of the particle changes by $\mathrm{d}\vec{x} = \vec{u}\mathrm{d}t$. The corresponding infinitesimal interval $\mathrm{d}s^2$ (6.22) is

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - |\mathrm{d}\vec{x}|^2 = c^2\mathrm{d}t^2(1-\beta^2), \tag{6.37}$$

with $\beta = u/c$. In the reference frame I' in which the particle is instantaneously at rest, we have $\mathrm{d}t' = \mathrm{d}\tau$ and $\mathrm{d}\vec{x}' = 0$, and thus $\mathrm{d}s^2 = c^2\mathrm{d}\tau^2$. This motivates us to define the *proper time* of the particle as $\mathrm{d}\tau = \mathrm{d}s/c$. The proper time is the time kept by a clock moving with the particle, and is thus a Lorentz invariant quantity. It takes the form

$$\mathrm{d}\tau = \mathrm{d}t\,\sqrt{1-\beta^2(t)} = \frac{\mathrm{d}t}{\gamma(t)}. \tag{6.38}$$

In the frame I, a finite proper time interval $\tau_2 - \tau_1$ is seen as a time interval

$$t_2 - t_1 = \int_{\tau_1}^{\tau_2} \frac{\mathrm{d}\tau}{\sqrt{1-\beta^2(\tau)}} = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\,\gamma(\tau). \tag{6.39}$$

This implies that a moving clock runs more slowly than a stationary clock: the time intervals in the moving frame are a factor of $\gamma > 1$ smaller than in the clock's rest frame. We will investigate this effect further in the next section.

With the notion of proper time at hand, we can define the *4-velocity* of the particle as

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \gamma\begin{pmatrix} c \\ \vec{u} \end{pmatrix}. \tag{6.40}$$

The 4-velocity is a contravariant vector, and it is easy to check that its invariant length is $u^2 = c^2$.

The contravariant nature of 4-velocities means that they are additive, and thus the spatial velocities $\vec{u}_1$ and $\vec{u}_2$ are not. If we denote the resulting 4-velocity as $v^\mu = u_1^\mu + u_2^\mu$, in the simple case that $\vec{u}_1$ and $\vec{u}_2$ are parallel we can write $\vec{u}_1 = u_1\hat{x}^1$, $\vec{u}_2 = u_2\hat{x}^1$, $\vec{v} = v\hat{x}^1$. One then has

$$v = \frac{u_1 + u_2}{1 + \frac{u_1 u_2}{c^2}}. \tag{6.41}$$

We can easily see that $v \leq c$; and if $u_1, u_2 \ll c$, we recover the Galilean result $\vec{v} \simeq \vec{u}_1 + \vec{u}_2$.

Similarly, one can define the *4-acceleration* as

$$a^\mu = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2}, \tag{6.42}$$

so that it is a contravariant vector too.

Relativistic momentum and energy. We would like to define the relativistic version of the momentum and energy of a particle. First, we define its *rest mass* m as the mass measured in a reference frame at rest with the particle. With this, the natural definition of the *4-momentum* is via the 4-velocity,

$$p^\mu = m u^\mu, \tag{6.43}$$

so that it is automatically a contravariant vector. Using the Lagrangian formalism, one can check this is indeed the appropriate definition. Furthermore, one can see that the zeroth component of the 4-momentum is related to the *relativistic energy* of the particle, $E = cp^0$, so that

$$(p^\mu) = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}. \tag{6.44}$$

Here, we have defined the relativistic energy $E = \gamma m c^2$ and momentum $\vec{p} = \gamma m \vec{v}$. With this, and using the fact that $u^2 = c^2$, we arrive at the relation

$$p^2 = \frac{E^2}{c^2} - |\vec{p}|^2 = m^2 c^2 \tag{6.45}$$

Another usual way to write this relation is

$$E = \sqrt{c^2|\vec{p}|^2 + m^2 c^4}. \tag{6.46}$$

6.3 The geometry of Minkowski spacetime

Let us now discuss the most important consequences of the principles of relativity and the constancy of the speed of light.

Causality. Looking at the definition of the relativistic energy, we see that $E \to \infty$ as $v \to c$ for any particle with $m \neq 0$. Furthermore, massless particles necessarily have $v = c$. This hints that the speed of light is a limit velocity for any physical process, or, said otherwise, that physical effects (or signals) cannot propagate faster than the speed of light. This fact is encoded in the geometric structure of the Minkowski spacetime, which for each point separates into different regions depending on the value of

$$x^2 = c^2t^2 - |\vec{x}|^2 \begin{cases} > 0 & \text{time} - \text{like}, \\ < 0 & \text{space} - \text{like}, \\ = 0 & \text{light} - \text{like}. \end{cases} \tag{6.47}$$

This can be showcased using a *Minkowski diagram* like the one in figure 6.1. We can distinguish

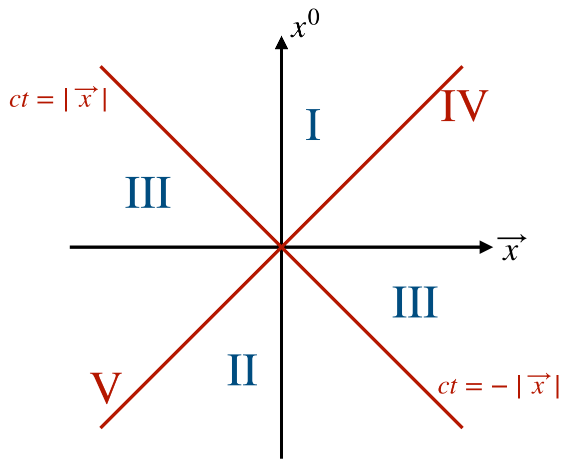

Figure 6.1

five different regions, which are invariant under Lorentz boosts:

I: $(x^2 > 0,\ x^0 > 0)$, time-like future.

II: $(x^2 > 0,\ x^0 < 0)$, time-like past.

III: $(x^2 < 0)$, space-like region.

IV: $(x^2 = 0,\ x^0 > 0)$, light-like future (or forward light cone L_+).

V: $(x^2 = 0,\ x^0 < 0)$, light-like past (or past light cone L_-).

The space like region cannot be classified as past or future. The reason for this is that for any point in region III, we can find an inertial observer for which the point is simultaneous with the origin. We can see that by making a geometric interpretation of the Lorentz boost. In one spatial dimension, for simplicity, we have

$$x'^0 = \gamma(x^0 - \beta x^1), \qquad x'^1 = \gamma(x^1 - \beta x^0). \tag{6.48}$$

The x'^0 and x'^1 axes form an angle in the (x^0, x^1) frame which is bisected by the light cone. One way to see this is that the x'^0 axis, defined by $x'^1 = 0$, satisfies $x^0 = x^1/\beta$. Similarly, the x'^0 axis, defined by $x'^0 = 0$, satisfies $x^0 = \beta x^1$. The angle α between x'^0 and x^0 is thus equal to the one between x'^1 and x^1, and it is given by $\tan\alpha = \beta$. This is illustrated graphically in the Minkwoski diagram of figure 6.2.

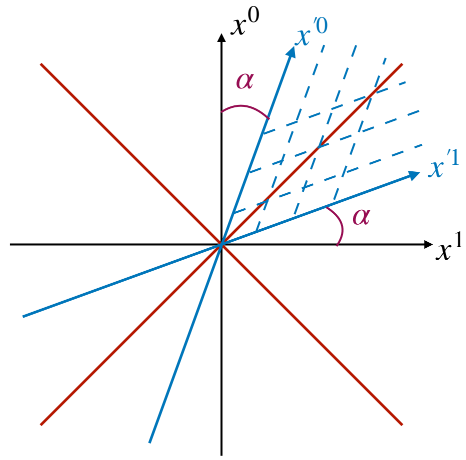

Figure 6.2

By varying β between 1 and -1, the x'^1 sweeps over the whole region III, and thus for any point P in it, we can find an inertial reference frame where both P and the origin lie on the x'^1 axis and have the same $x'^0 = ct'$ coordinate. Lines of simultaneity are parallel to the x'^1 axis, while events occurring at the same location are located in lines parallel to the x'^0 axis.

Let us assume that it were possible for a signal sent from the origin O to reach a space-like separated point P (in region III). We know that there exists an inertial frame I' in which P has a negative time coordinate, while the origin has $x'^0 = 0$. Then, in system I', the signal would reach the point P before it is sent from O. This would clearly be a violation of *macroscopical causality*, which states that a physical consequence must always occur later or at most simultaneously than its cause. This means that the speed of light is an upper limit to the velocity of propagation of physical consequences or signals.

We can summarize our findings about causality in the following sentence: any physical effect emanating from a point can only propagate within its forward light cone (or on the edge of it).

Time dilation. Consider the example of the radioactive decay of a (massive) particle, for instance a muon, which decays as $\mu^- \to e^- + \bar{\nu}_e + \nu_\tau$. The particle is at rest in the reference frame I and decays after a time T as measured in I. Its *worldline* (or spacetime trajectory) corresponds to OP in the diagram 6.3.

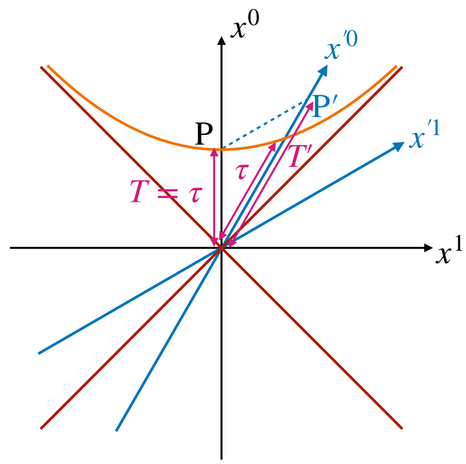

Figure 6.3

In another reference frame I', the point P has a time coordinate T', which can be obtained by tracing a line parallel to the x'^1 passing through P. To compare T and T', we have to transport the unit of time to the system I'. This can be done using the invariance of the proper time,

$$(x^0)^2 - (x^1)^2 = c^2\tau^2 = (x'^0)^2 - (x'^1)^2, \tag{6.49}$$

which defines a hyperbola as shown in figure 6.3. This hyperbola connects points with equal proper time between different inertial observers, and thus allows to translate the "time measuring rod" between inertial frames. In the rest frame, we have that $T = \tau$. In the frame I', it is easy to see geometrically that $T' > \tau$. Quantitatively, from the Lorentz boost

$$x'^0 = \gamma(x^0 + \beta x^1) \tag{6.50}$$

with $x^1 = 0$, $x'^0 = T'$, and $x^0 = T$, we find

$$T' = \gamma\, T = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\, T. \tag{6.51}$$

The lifetime of the particle is thus a factor γ larger in the moving frame: this is what is known as *time dilation*.

Length contraction. Take a measuring rod of length l to be at rest in system I, with its endpoints moving along the straight worldlines OO' and AA' in the Minkowski diagram of figure 6.4.

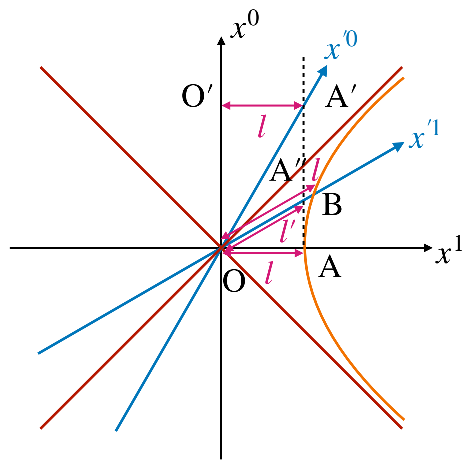

Figure 6.4

In another inertial frame I', the length of the rod is the distance between two points in OO' and AA' that are simultaneous in I': for example, O and A''. The length of the rod in I' is thus $l' = \overline{OA''}$.

As before, to properly compare l and l', we first need to translate the length l to the system I'. To do this, note that

$$-\,l^2 = x^2 = (x^0)^2 - (x^1)^2 = (x'^0)^2 - (x'^1)^2, \tag{6.52}$$

which agains defines a hyperbola. Thus, the point B marks the distance $x'^1 = l$ from O in the system I'. We can visually appreciate that $l' < l$, and from the inverse Lorentz boost

$$x^1 = \gamma(x'^1 + \beta x'^0), \tag{6.53}$$

with $x'^0 = 0$, $x'^1 = l'$, and $x^1 = l$, we find

$$l' = \frac{1}{\gamma}\, l = \sqrt{1 - \frac{v^2}{c^2}}\, l. \tag{6.54}$$

This is what we call *length contraction*, and is ultimately a consequence of the relativity of the concept of simultaneity.

Relativistic Doppler effect. The phase of a wave is a relativistic invariant quantity. It is simply related to the number of wave crests that have passed an observer, and such a counting is an operation that must be the same in all frames. For a plane wave of frequency ω and wave vector $\vec{k}$ in a frame I, and ω' and $\vec{k}'$ in another inertial frame I' moving with respect to I, we must thus have

$$\phi = \omega t - \vec{k}\cdot\vec{x} = \omega' t' - \vec{k}'\cdot\vec{x}'. \tag{6.55}$$

If $\vec{\beta}$ is the parameter of the Lorentz boost that takes us from I to I' and we write $k_0 = \omega/c$ and $k_0' = \omega'/c$, (6.55) implies that

$$k_0' = \gamma(k_0 - \vec{\beta}\cdot\vec{k}), \tag{6.56}$$
$$k_\parallel' = \gamma(k_\parallel - \beta k_0), \tag{6.57}$$
$$\vec{k}_\perp' = \vec{k}_\perp. \tag{6.58}$$

Here, $k_\parallel$ and $\vec{k}_\perp$ denote the components of $\vec{k}$ parallel and perpendicular to $\vec{\beta}$, respectively. We thus see that we can construct the *wave 4-vector*

$$k^\mu = \begin{pmatrix} \omega/c \\ \vec{k} \end{pmatrix}, \tag{6.59}$$

which transforms as a contravariant vector. Thus, the invariance of the phase is nothing but the invariance of the scalar product $k^\mu x_\mu$ under Lorentz boosts.

For electromagnetic waves, the fact that $\omega = c|\vec{k}|$ means that

$$k^2 = k^\mu k_\mu = 0. \tag{6.60}$$

With this, we can express the transformation law of the zeroth component of the wave 4-vector in the form

$$\omega' = \gamma\omega(1 - \beta\cos\theta), \tag{6.61}$$

where θ is the angle between $\vec{k}$ and $\vec{\beta}$. This frequency shift is often know as the *relativistic Doppler effect*. We can distinguish two limiting cases:

- If $\theta = 0$ or π, we have the longitudinal Doppler effect,

$$\omega' = \gamma\omega(1 \mp \beta) = \begin{cases} \omega\sqrt{\frac{1-\beta}{1+\beta}} \\ \omega\sqrt{\frac{1+\beta}{1-\beta}} \end{cases} \tag{6.62}$$

  For $\beta \ll 1$, we recover the classical result $\omega' \simeq \omega(1 \mp \beta)$.

- For $\theta = \frac{\pi}{2}$, we have the transversal Doppler effect

$$\omega' = \gamma\omega, \tag{6.63}$$

which for $\beta \ll 1$ becomes

$$\omega' \simeq \omega\left(1 + \frac{v^2}{2c^2}\right). \tag{6.64}$$

In contrast with the classical expectation, we see that there is also a nonzero frequency shift for waves traveling in the transverse direction.

Back to the general case and denoting θ' the angle between $\vec{k}'$ and $\vec{\beta}$, we can write

$$\sin\theta = \frac{|\vec{k}_\perp|}{|\vec{k}|}, \qquad \text{and} \qquad \sin\theta' = \frac{|\vec{k}'_\perp|}{|\vec{k}'|}, \tag{6.65}$$

so that

$$\sin\theta' = \frac{\omega}{\omega'}\sin\theta. \tag{6.66}$$

Using the expression (6.61) for the Doppler frequency shift, we arrive at

$$\sin\theta' = \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta}\sin\theta. \tag{6.67}$$

This modification of the direction of propagation of the wave in different inertial frames is known as *aberration* and is well-known in astronomy. For $\theta = \pi/2$ and $\beta \ll 1$, we find

$$\sin\theta' = \cos\left(\frac{\pi}{2} - \theta'\right) \simeq \sqrt{1-\beta^2} \qquad \rightarrow \qquad \frac{\pi}{2} - \theta' \simeq \frac{v}{c}. \tag{6.68}$$

This means that when observing a star through a telescope that is moving with respect to the star, it must be slightly inclined with respect to the actual position of the star as shown in figure 6.5. The inclination angle is approximately

$$\alpha \simeq \frac{v}{c}. \tag{6.69}$$

A typical value found experimentally is $\alpha \sim 10^{-4}$, from which we can learn that the speed of the earth in its rotation around the sun is $v \sim 30\,\mathrm{km/s}$.

6.4 The Lorentz and Poincaré groups

The Lorentz group. In the same way that three-dimensional rotations corresponds to the group of transformations that leave the norm of a vector $\vec{x}$ invariant, we want to find the

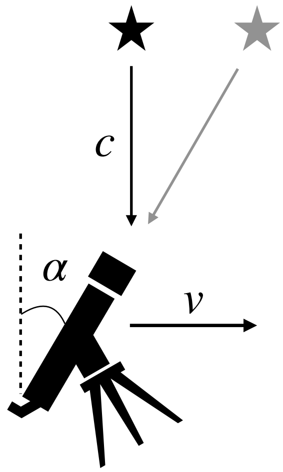

Figure 6.5

group of transformations that leave the interval $s^2 = c^2t^2 - |\vec{x}|^2$ invariant in 4-dimensional spacetime. Clearly, the Lorentz boosts previously presented will be part of this group.

The condition for s^2 to remain invariant under a linear transformation $x'^\mu = \Lambda^\mu{}_\nu$ is

$$g_{\mu\nu}x^\mu x^\nu = x^2 = x'^2 = g_{\rho\sigma}x'^\rho x'^\sigma = g_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu x^\mu x^\nu. \tag{6.70}$$

Since this must hold for arbitrary x, we can read out the following condition on Λ:

$$g_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu = g_{\mu\nu}, \tag{6.71}$$

which can be interpreted as the invariance of the Minkowski metric under the transformation. Any transformation Λ satisfying this condition is known as a *Lorentz transformation.* It is easy to see that this set of transformation defines a group with respect to the composition, which we call the *Lorentz group* SO(3,1).

The left side of (6.71) can also be written as $\Lambda^\rho{}_\mu g_{\rho\sigma}\Lambda^\sigma{}_\nu$. Then, (6.71) can be rewritten in matrix notation as $\Lambda^T g\Lambda = g$, after noting that $\Lambda^\rho{}_\mu = (\Lambda^T)_\mu{}^\rho$. This shows that one has to be careful with the placing of the indices in non-symmetric tensors like $\Lambda^\mu{}_\nu$. With this, we see that the inverse Lorentz transformation matrix is given by $\Lambda^{-1} = g\Lambda^T g$, or, using index notation,

$$\Lambda_\rho{}^\mu \equiv (\Lambda^{-1})^\mu{}_\rho = g^{\mu\nu}\Lambda^\sigma{}_\nu g_{\sigma\rho}. \tag{6.72}$$

We can generalize the transformation properties of 4-vectors (or vector fields) in the expected way. Contravariant vector fields transform with Λ,

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x), \tag{6.73}$$

while covariant vector fields transform with the inverse Λ^{-1},

$$A'_\mu(x') = \Lambda_\mu{}^\nu A_\nu(x). \tag{6.74}$$

Note that the Lorentz transformations can also be given in terms of the relation between original and transformed coordinates as

$$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} \qquad \text{and} \qquad \Lambda_\mu{}^\nu \frac{\partial x^\nu}{\partial x'^\mu}. \tag{6.75}$$

With this, you can easily show that the scalar product between two vector fields is of course a Lorentz scalar, $A'^\mu B'_\mu = A^\mu B_\mu$. It is also clear that ∂_μ remains a covariant 4-vector,

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu} = \Lambda_\mu{}^\nu \partial_\nu. \tag{6.76}$$

The dual derivative $\partial^\mu \equiv \partial/\partial x_\mu$ is naturally a contravariant vector,

$$\partial'^\mu = g^{\mu\sigma}\partial'_\sigma = g^{\mu\sigma}\Lambda_\sigma{}^\rho \partial_\rho = g^{\mu\sigma}\Lambda_\sigma{}^\rho g_{\rho\nu}\partial^\nu = \Lambda^\mu{}_\nu \partial^\nu, \tag{6.77}$$

and thus the d'Alembert operator

$$\Box = \partial^\mu \partial_\mu = \frac{\partial^2}{\partial (x^0)^2} - \nabla^2 \tag{6.78}$$

is a Lorentz invariant (or scalar) quantity.

Topology of the Lorentz group. We can learn some properties of the Lorentz group from the relation $\Lambda^T g \Lambda = g$. First, taking determinants at both sides of the equation, we have

$$(\det \Lambda^T)(\det g)(\det \Lambda) = \det g. \tag{6.79}$$

Using $\det g = -1$ and $\det \Lambda^T = \det \Lambda$, this implies that

$$\det \Lambda = \pm 1. \tag{6.80}$$

On the other hand, from the $\mu = \nu = 0$ component of $g_{\rho\sigma} = g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma$, we have

$$1 = g_{\mu\nu}\Lambda^\mu{}_0 \Lambda^\nu{}_0 = (\Lambda^0{}_0)^2 - \sum_k (\Lambda^k{}_0)^2, \tag{6.81}$$

meaning that

$$\Lambda^0{}_0 \geq 1 \qquad \text{or} \qquad \Lambda^0{}_0 \leq -1. \tag{6.82}$$

The two conditions (6.80) and (6.82) show that the Lorentz groups has four disconnected components, that cannot be reached by continuously varying the entries of Λ. They are often denoted as

$$L^\uparrow_+, \quad L^\downarrow_+, \quad L^\uparrow_-, \quad L^\downarrow_-, \tag{6.83}$$

where $+(-)$ signals $\det\Lambda = +1(-1)$ and $\uparrow(\downarrow)$ signals ${\Lambda^0}_0 \geq +1(\leq -1)$. Lorentz transformations with $\det\Lambda = +1$ are called *proper*, while those with $\det\Lambda = -1$ are called *improper*. If ${\Lambda^0}_0 > +1$, we say that the Lorentz transformation is *orthochronous*, while those with ${\Lambda^0}_0 > -1$ are said to be *antichronous*.

Some special examples of Lorentz transformations of each kind are

$$\mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in L_+^\uparrow, \tag{6.84}$$

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in L_-^\uparrow, \tag{6.85}$$

$$\mathcal{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in L_-^\downarrow, \tag{6.86}$$

$$\mathcal{PT} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in L_+^\downarrow. \tag{6.87}$$

Among the four disconnected components, only $L_+^\uparrow$ forms a subgroup, as it is the only one that contains the identity transformation. $L_+^\uparrow$ is known as the proper orthochronous Lorentz group.

Generators of the Lorentz group. A Lorentz transformation in $L_+^\uparrow$ is completely specified by 6 parameters. They can conveniently be thought of as 3 parameters (e.g. Euler angles) that specify the relative orientation of the coordinate axes, and 3 parameters to specify the relative velocity between the two inertial frames (the three components of $\vec{w}$). This means that we can write

$$\Lambda = \Lambda_B(\vec{w})\,\mathcal{R}(\vec{\theta}), \tag{6.88}$$

where $\Lambda_B(\vec{w})$ is a Lorentz boost with rapidity $\vec{w}$, and $\mathcal{R}(\vec{\theta})$ is a spatial rotation

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}, \tag{6.89}$$

with $R \in \mathrm{SO}(3)$. Any finite rotation can be constructed infinitesimally as

$$\mathcal{R} = \mathrm{e}^{-\vec{\theta}\cdot\vec{S}}, \tag{6.90}$$

where $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ is a constant 3-vector containing the parameters of the rotation, and $\vec{S} = (S_1, S_2, S_3)$ contains the three *infinitesimal generators*

$$S_1 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right), \qquad S_2 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right), \qquad S_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{6.91}$$

You can check explicitly that, for instance, for $\vec{\theta} = (0, 0, \theta)$, we recover

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.92}$$

that is, a rotation of angle θ around the 3rd axis. Similarly, a Lorentz boost can be written as

$$\Lambda_B = \mathrm{e}^{-\vec{w}\cdot\vec{K}}, \tag{6.93}$$

and in this case the generators arranged in the vector $\vec{K} = (K_1, K_2, K_3)$ are

$$K_1 = \left(\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad K_2 = \left(\begin{array}{c|ccc} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad K_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right). \tag{6.94}$$

As an example, you can check that for $\vec{w} = (w, 0, 0)$, the exponentiation gives

$$\Lambda_B = \begin{pmatrix} \cosh w & -\sinh w & 0 & 0 \\ -\sinh w & \cosh w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.95}$$

which is a Lorentz boost in the x^1 direction with rapidity w.

The six matrices $S_1, S_2, S_3, K_1, K_2, K_3$ constitute the infinitesimal generators of the (proper ortochronous) Lorentz group. It is straightforward to show that these generators satisfy the commutation relations

$$[S_i, S_j] = \epsilon_{ijk} S_k, \tag{6.96}$$
$$[S_i, K_j] = \epsilon_{ijk} K_k, \tag{6.97}$$
$$[K_i, K_j] = -\epsilon_{ijk} S_k, \tag{6.98}$$

where, as usual, the commutator is $[A, B] = AB - BA$. These are the characteristic commutation relations of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ or $\mathfrak{o}(3, 1)$, which is the algebraic structure that underlies the Lorentz group. Physically, the first commutation relation is just the well-known one for angular momentum, the second one shows that K transforms as a vector under rotations, and the third one implies that Lorentz boosts do not commute.

The Poincaré group. In addition to rotations and boosts, we also need to consider time and spatial translations

$$x^\mu \to x'^\mu = x^\mu + a^\mu, \tag{6.99}$$

with a^μ an arbitrary constant 4-vector. Thus, the most general transformation under which the Minkowski metric remains invariant is a combination of a Lorentz transformation and a translation,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{6.100}$$

and is known as a *Poincaré transformation.* These transformations also form a group with respect to composition

$$(\Lambda_2, a_2) \odot (\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2), \tag{6.101}$$

constituting the 10-parameter *Poincaré group*, which is the fundamental symmetry group of Minkowski spacetime.

Chapter 7

The covariant formulation of electrodynamics

7.1 Units and dimensions.

In our discussion so far, we have been using SI units for electromagnetic quantities. However, for the discussion of relativistic electrodynamics it will be more convenient to use a different system of units, the Heavyside-Lorentz system (Gaussian units are another popular choice). To understand where the freedom in the choice of units stems from, let us proceed step by step.

We start with the commonly accepted definition of the current being the rate of change of charge per unit time,

$$I = \frac{\mathrm{d}q}{\mathrm{d}t}, \tag{7.1}$$

from which the form of the continuity equation follows as

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0. \tag{7.2}$$

We continue with electrostatics. In Coulomb's law a free constant k_1 can be chosen,

$$F = k_1 \frac{qq'}{r^2}. \tag{7.3}$$

The choice of the proportionality constant k_1 is tied to the choice of the unit of charge. The choice of k_1 also fixes the dimensionality of the electric field via

$$E = k_1 \frac{q}{r^2}. \tag{7.4}$$

For magnetostatics, we can fix an additional proportionality constant k_2 in Ampère's law,

$$\frac{\mathrm{d}F}{\mathrm{d}l} = 2k_2 \frac{II'}{d}, \tag{7.5}$$

which fixes the relative units of the force per unit length between to infinitely long, parallel wires separated by a distance d and carrying currents I and I'. As we have already defined the current as being charge per unit time, the dimensions of k_1 relative to k_2 are in fact already fixed. The ratio k_1/k_2 has dimensions of velocity squared, and experiments show that the magnitude of the ratio is

$$\frac{k_1}{k_2} = c^2. \tag{7.6}$$

The definition of the magnetic induction introduces a further free constant. For a long straight wire carrying current I, the magnetic induction at a distance d is

$$B = 2k_2\alpha\frac{I}{d}, \tag{7.7}$$

where we have denoted the proportionality constant as α. Finally, one may thing that another free constant k_3 representing the relative dimensions of the electric and magnetic field appears in Faraday's law of induction,

$$\vec{\nabla}\times\vec{E} + k_3\frac{\partial\vec{B}}{\partial t} = 0. \tag{7.8}$$

However, given that the dimensions of $\vec{E}$ and $\vec{B}$ are already fixed, we see that k_3 must have the same dimensions as $1/\alpha$. In fact, experimentally one finds that $k_3 = 1/\alpha$. With these, we can write the following most general form of the Maxwell equations:

$$\vec{\nabla}\cdot\vec{E} = 4\pi k_1\rho, \tag{7.9}$$

$$\vec{\nabla}\times\vec{B} = 4\pi k_2\alpha\vec{J} + \frac{k_2\alpha}{k_1}\frac{\partial\vec{E}}{\partial t}, \tag{7.10}$$

$$\vec{\nabla}\times\vec{E} + k_3\frac{\partial\vec{B}}{\partial t} = 0, \tag{7.11}$$

$$\vec{\nabla}\times\vec{B} = 0. \tag{7.12}$$

As we already know, from these one can derive the wave equation

$$\nabla^2\vec{B} - k_3\frac{k_2\alpha}{k_1}\frac{\partial^2\vec{B}}{\partial t^2} = 0. \tag{7.13}$$

Since we know that electromagnetic waves in vacuum propagate at the speed of light, it must be that

$$\frac{k_1}{k_3k_2\alpha} = c^2, \tag{7.14}$$

in addition to the already mentioned relation

$$k_3 = \frac{1}{\alpha}. \tag{7.15}$$

In light of this, there are really only two constants to be freely fixed. There are three main systems of units that are commonly employed, and whose choices for k_1, k_2, k_3, and α are summarized in the following table:

System	k_1	k_2	α	k_3
Gaussian	1	$\frac{1}{c^2}$	c	$\frac{1}{c}$
Heavyside-Lorentz	$\frac{1}{4\pi}$	$\frac{1}{4\pi c^2}$	c	$\frac{1}{c}$
SI	$\frac{1}{4\pi\epsilon_0} = 10^{-7}\, c^2\, \frac{\mathrm{A\,s}}{\mathrm{V\,m}}$	$\frac{\mu_0}{4\pi} = 10^{-7}\, \frac{\mathrm{V\,s}}{\mathrm{A\,m}}$	1	1

The symbol c stands here for the velocity of light in vacuum, which is fixed by definition to be $c = 299\,792\,458\,\mathrm{m/s}$, exactly. Usually, we just approximate $c = 3\times 10^8\,\mathrm{m/s}$.

The SI is the most common system employed in nonrelativistic setups. For relativistic electrodynamics, the Gaussian system has traditionally been the most widely adopted one in textbooks. However, the Heavyside-Lorentz system has the advantage of making the factor of 4π explicit in equations where one would expect it to be there due to, e.g., spherical symmetry. The latter system is thus the one preferred in theoretical and high-energy physics, and is the one that we will be employing from this point on.

It is handy to have a dictionary that we can use to quickly translate expression between the three main systems of units. The following table lists the substitutions to be made in order to transform from one system to another:

SI units	Heaviside-Lorentz units	Gaussian units
$\sqrt{\epsilon_0}(\vec{E}^{\mathrm{SI}}, \Phi^{\mathrm{SI}})$	$(\vec{E}^{\mathrm{HL}}, \Phi^{\mathrm{HL}})$	$\frac{1}{\sqrt{4\pi}}(\vec{E}^{\mathrm{G}}, \Phi^{\mathrm{G}})$
$\frac{1}{\sqrt{\epsilon_0}}\vec{D}^{\mathrm{SI}}$	$\vec{D}^{\mathrm{HL}}$	$\frac{1}{\sqrt{4\pi}}\vec{D}^{\mathrm{G}}$
$\frac{1}{\sqrt{\epsilon_0}}(q^{\mathrm{SI}}, \rho^{\mathrm{SI}}, I^{\mathrm{SI}}, \vec{J}^{\mathrm{SI}}, \vec{P}^{\mathrm{SI}}, \vec{p}^{\mathrm{SI}})$	$(q^{\mathrm{HL}}, \rho^{\mathrm{HL}}, I^{\mathrm{HL}}, \vec{J}^{\mathrm{HL}}, \vec{P}^{\mathrm{HL}}, \vec{p}^{\mathrm{HL}})$	$\sqrt{4\pi}(q^{\mathrm{G}}, \rho^{\mathrm{G}}, I^{\mathrm{G}}, \vec{J}^{\mathrm{G}}, \vec{P}^{\mathrm{G}}, \vec{p}^{\mathrm{G}})$
$\frac{1}{\sqrt{\mu_0}}(\vec{B}^{\mathrm{SI}}, \Phi_m^{\mathrm{SI}}, \vec{A}_m^{\mathrm{SI}})$	$(\vec{B}^{\mathrm{HL}}, \Phi_m^{\mathrm{HL}}, \vec{A}_m^{\mathrm{HL}})$	$\frac{1}{\sqrt{4\pi}}(\vec{B}^{\mathrm{G}}, \Phi_m^{\mathrm{G}}, \vec{A}_m^{\mathrm{G}})$
$\sqrt{\mu_0}\vec{H}^{\mathrm{SI}}$	$\vec{H}^{\mathrm{HL}}$	$\frac{1}{\sqrt{4\pi}}\vec{H}^{\mathrm{G}}$
$\sqrt{\mu_0}(\vec{m}^{\mathrm{SI}}, \vec{M}^{\mathrm{SI}})$	$(\vec{m}^{\mathrm{HL}}, \vec{M}^{\mathrm{HL}})$	$\sqrt{4\pi}(\vec{m}^{\mathrm{G}}, \vec{M}^{\mathrm{G}})$
$\left(\frac{\epsilon^{\mathrm{SI}}}{\epsilon_0}, \frac{\mu^{\mathrm{SI}}}{\mu_0}\right)$	$\left(\epsilon^{\mathrm{HL}}, \mu^{\mathrm{HL}}\right)$	$\left(\epsilon^{\mathrm{G}}, \mu^{\mathrm{G}}\right)$
$\left(\chi_e^{\mathrm{SI}}, \chi_m^{\mathrm{SI}}\right)$	$\left(\chi_e^{\mathrm{HL}}, \chi_m^{\mathrm{HL}}\right)$	$4\pi\left(\chi_e^{\mathrm{G}}, \chi_m^{\mathrm{G}}\right)$
$\frac{1}{\epsilon_0}\sigma^{\mathrm{SI}}$	σ^{HL}	$4\pi\sigma^{\mathrm{G}}$

Heavyside-Lorentz units. Explicitly, in Heavyside-Lorentz (HL) units, the microscopic Maxwell equations read

$$\vec{\nabla} \cdot \vec{E} = \rho, \tag{7.16}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}, \tag{7.17}$$

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0, \tag{7.18}$$

$$\vec{\nabla} \times \vec{B} = 0, \tag{7.19}$$

and the Lorentz force equation becomes

$$\vec{F} = q\left(\vec{E} + \frac{1}{c}\,\vec{v} \times \vec{B}\right). \tag{7.20}$$

As you can see, ϵ_0 or μ_0 do not explicitly appear in the equations: these are absorbed in the definitions of $\vec{E}$ and $\vec{B}$. Only c appears along with each time derivative. Furthermore, the electric and magnetic fields have the same units in this system. In a homogeneous, linear, isotropic, and nondispersive medium, the macroscopic fields are given by

$$\vec{D} = \vec{E} + \vec{P}, \tag{7.21}$$

$$\vec{B} = \vec{H} + \vec{M}, \tag{7.22}$$

with

$$\vec{D} = \epsilon\vec{E}, \tag{7.23}$$

$$\vec{P} = \chi_e\vec{E}, \tag{7.24}$$

$$\vec{B} = \mu\vec{H}, \tag{7.25}$$

$$\vec{M} = \chi_m\vec{H}. \tag{7.26}$$

The electric and magnetic susceptibilities in the HL system are equal to those in the SI one, while the permittivity and permeability are related by $\epsilon^{\mathrm{HL}} = \epsilon^{\mathrm{SI}}/\epsilon_0$ and $\mu^{\mathrm{HL}} = \mu^{\mathrm{SI}}/\mu_0$.

7.2 Covariant form of the Maxwell equations

In the previous chapter we showed that the Poincaré group is the symmetry group describing the invariance properties of the Maxwell equations. We will now work to rewrite these equations in a form that is explicitly Lorentz invariant and is the most convenient way of doing calculations in relativistic electrodynamics.

Four-potential and field strength tensor. In the Lorentz family of gauges, we obtained wave equations for the vector and scalar potentials, which in HL units correspond to

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \frac{1}{c}\vec{J}, \tag{7.27}$$

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = \rho. \tag{7.28}$$

The Lorentz gauge condition is

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0, \tag{7.29}$$

which we can rewrite in a more suggestive form as

$$\frac{\partial}{\partial x^0}\Phi + \frac{\partial}{\partial x^k}A^k = 0. \tag{7.30}$$

These leads us to hypothesize that the scalar and vector potentials form a contravariant 4-vector

$$(A^\mu) = \begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix}. \tag{7.31}$$

This is indeed required for the Lorentz gauge condition to be gauge invariant, and we will explicitly confirm it later. The Lorentz condition can with this be compactly written as

$$\partial_\mu A^\mu = 0. \tag{7.32}$$

This expression is explicitly invariant under Lorentz transformations. Similarly, the gauge transformation law for the four potential is given by

$$A^\mu \to A'^\mu = A^\mu - \partial^\mu \Lambda. \tag{7.33}$$

Using

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi, \tag{7.34}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}, \tag{7.35}$$

the components of the electric and magnetic fields can be expressed as

$$E_1 = \partial_1 \Phi - \partial_0 A^1 = \partial^1 A^0 - \partial^0 A^1, \tag{7.36}$$

$$E_2 = \partial_2 \Phi - \partial_0 A^2 = \partial^2 A^0 - \partial^0 A^2, \tag{7.37}$$

$$E_3 = \partial_3 \Phi - \partial_0 A^3 = \partial^3 A^0 - \partial^0 A^3, \tag{7.38}$$

and

$$B_1 = \partial_2 A^3 - \partial_3 A^2 = \partial^3 A^2 - \partial^2 A^3, \tag{7.39}$$
$$B_2 = \partial_3 A^1 - \partial_1 A^3 = \partial^1 A^3 - \partial^3 A^1, \tag{7.40}$$
$$B_3 = \partial_1 A^2 - \partial_2 A^1 = \partial^2 A^1 - \partial^1 A^2. \tag{7.41}$$

These equations imply that the six components of the electric and magnetic fields are the elements of a rank-2, antisymmetric tensor $F^{\mu\nu}$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{7.42}$$

which is known as the *field strength tensor* and is *manifestly covariant* under Lorentz transformations. The antisymmetry $F^{\mu\nu} = -F^{\nu\mu}$ is explicit in the above definition. In matrix form, the field strength tensor is

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}. \tag{7.43}$$

Or, written with two covariant indices,

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}. \tag{7.44}$$

Using index notation, we can write

$$E_i = F_{0i}, \tag{7.45}$$
$$B_i = -\frac{1}{2}\epsilon_{ijk} F^{jk}, \tag{7.46}$$

or the inverse relations

$$F^{0i} = -E^i, \tag{7.47}$$
$$F^{ij} = -\epsilon^{ijk} B_k. \tag{7.48}$$

It is worth noting that $F^{\mu\nu}$ is a gauge-invariant tensor. This can be easily see by applying the transformation law (7.33) or by noting that $F^{\mu\nu}$ only contains the gauge-invariant fields $\vec{E}$ and $\vec{B}$.

Transformation of the electromagnetic fields. Under a Lorentz transformation, the field strength tensor transforms as

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}F^{\rho\sigma} = \Lambda^{\mu}{}_{\rho}F^{\rho\sigma}(\Lambda^T)_{\sigma}{}^{\nu} = \begin{pmatrix} 0 & -E'_1 & -E'_2 & -E'_3 \\ E'_1 & 0 & -B'_3 & B'_2 \\ E'_2 & B'_3 & 0 & -B'_1 \\ E'_3 & -B'_2 & B'_1 & 0 \end{pmatrix}. \tag{7.49}$$

For a Lorentz boost along the first axis,

$$(\Lambda^{\mu}{}_{\rho}) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7.50}$$

we find the transformations

$$E'_1 = E_1, \qquad B'_1 = B_1 \tag{7.51}$$
$$E'_2 = \gamma(E_2 - \beta B_3), \qquad B'_2 = \gamma(B_2 + \beta E_3), \tag{7.52}$$
$$E'_3 = \gamma(E_3 + \beta B_2), \qquad B'_3 = \gamma(B_3 - \beta E_2). \tag{7.53}$$

For a general boost in an arbitrary $\vec{\beta}$ direction, one has

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}, \tag{7.54}$$
$$\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{\beta} \times \vec{B}), \tag{7.55}$$
$$\vec{B}'_{\parallel} = \vec{B}_{\parallel}, \tag{7.56}$$
$$\vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - \vec{\beta} \times \vec{E}). \tag{7.57}$$

We can also explicitly write the transformation of the 4-potential,

$$A'0 = \gamma(A^0 - \beta A_{\parallel}), \tag{7.58}$$
$$\vec{A}'_{\parallel} = \gamma(\vec{A}_{\parallel} - \vec{\beta} A^0), \tag{7.59}$$
$$\vec{A}'_{\perp} = \vec{A}_{\perp}, \tag{7.60}$$

in terms of the spatial components parallel and perpendicular to $\vec{\beta}$.

As an example, let us compute the 4-potential generated by an electron moving at a constant speed v in the x^1 direction. In the rest frame of the electron, we have simply

$$A^0_r = \frac{e}{r}, \qquad \vec{A}_r = 0, \tag{7.61}$$

where the r subindex denotes quantities in the rest frame. To move to the system in which the electron is moving with velocity $\vec{v} = v\vec{e}_1$, we perform a boost of parameter $-\vec{\beta}$, with $\vec{\beta} = \vec{v}/c$, to find

$$A^0(x) = \gamma A_r^0(x_r), \tag{7.62}$$
$$A^1(x) = \gamma\beta A_r^0(x_r), \tag{7.63}$$

with

$$x_r^0 = \gamma(x^0 + \beta x^1), \tag{7.64}$$
$$x_r^1 = \gamma(x^1 - \beta x^0), \tag{7.65}$$
$$x_r^2 = x^2, \tag{7.66}$$
$$x_r^3 = x^3. \tag{7.67}$$

Using

$$r = \sqrt{\gamma^2(x^1)^2 + (x^2)^2 + (x^3)^2}, \tag{7.68}$$

the potential at any point $(x^1, x^2, x^3) \equiv (x, y, z)$ and time t, assuming that the point charge is located at position $(vt, 0, 0)$ at that time, can be obtained as

$$\Phi = A^0 = \frac{e}{\sqrt{1-\beta^2}} \frac{1}{\sqrt{\frac{(x-vt)^2}{1-\beta^2} + y^2 + z^2}}, \tag{7.69}$$
$$A_x = \frac{e\beta}{\sqrt{1-\beta^2}} \frac{1}{\sqrt{\frac{(x-vt)^2}{1-\beta^2} + y^2 + z^2}}, \tag{7.70}$$
$$A_y = A_z = 0. \tag{7.71}$$

These equations are written in terms of the coordinates measured from the current position $x = vt$ of the electron (see figure 7.1). Of course, given that the action of the field travels at the speed of light c, the field is actually generated by the charge back at the retarded position $x = vt'$, with $t' = t - r'/c$. Even if it is less physical, we see that it is just more convenient to express the potentials with respect to the current position of the particle, so we will stick to that.

The next step is to calculate the fields, which we obtain by differentiation of the 4-potential as in (7.34) and (7.35), for instance,

$$E_x = -\frac{\partial \Phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t} \tag{7.72}$$

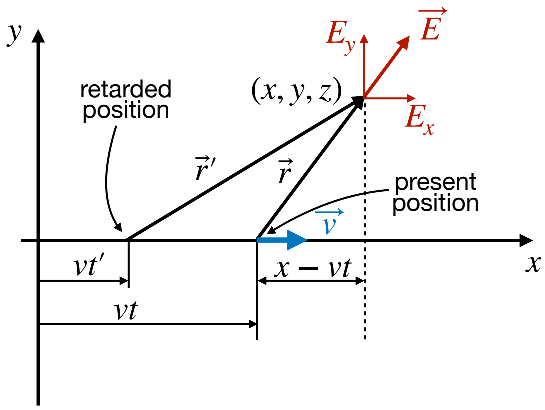

Figure 7.1

and similarly for E_y and E_z. Carrying out the differentiations, we arrive at

$$E_x = \frac{e}{\sqrt{1-\beta^2}} \frac{x - vt}{\left[\frac{(x-vt)^2}{1-\beta^2} + y^2 + z^2\right]^{\frac{3}{2}}}, \tag{7.73}$$

$$E_y = \frac{e}{\sqrt{1-\beta^2}} \frac{y}{\left[\frac{(x-vt)^2}{1-\beta^2} + y^2 + z^2\right]^{\frac{3}{2}}}, \tag{7.74}$$

$$E_z = \frac{e}{\sqrt{1-\beta^2}} \frac{z}{\left[\frac{(x-vt)^2}{1-\beta^2} + y^2 + z^2\right]^{\frac{3}{2}}}. \tag{7.75}$$

Let us interpret this result. The first thing to note is that ratio of the components of $\vec{E}$ is equal to the ratio of the components of the vector $\vec{r} = (x - vt, y, z)$. This means that $\vec{E}$ is parallel to $\vec{r}$, and thus the electric field is radial from the current position of the electron (not the retarded one!), as illustrated in figure 7.1.

The second is that the field is different to the one by a stationary charge due to the factors of $(1 - \beta^2)$. For points directly orthogonal to the trajectory of the particle, that is, for $(x - vt) = 0$, the distance from the charge is $\sqrt{y^2 + z^2}$, and the total strength of the electric field is

$$E_\perp = \sqrt{E_y^2 + E_z^2} = \frac{e/\sqrt{1-\beta^2}}{y^2 + z^2}. \tag{7.76}$$

This is the same as for the static case except for the extra factor of $\gamma = 1/\sqrt{1-\beta^2} > 1$: at the sides of a moving charge, the electric field is stronger than for a particle at rest.

For points ahead or behind the electron, we have $y = z = 0$ and thus

$$E_\parallel = E_x = \frac{e(1-\beta^2)}{(x - vt)^2}. \tag{7.77}$$

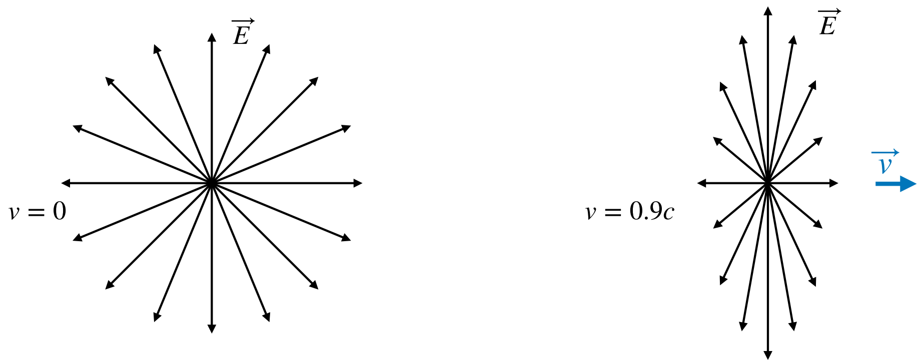

Figure 7.2

Now, the electric field is reduced by a factor of $1/\gamma^2 = 1 - \beta^2 < 1$ compared to the static case. We conclude that for particles moving at velocities close to the speed of light, the field in the forward and backward directions is enormously reduced, while the one in the sidewise directions is greatly enhanced. In terms of electric field lines, the lines are spread out ahead and behind the charge and are squeezed together around the sides, as shown in figure 7.2.

For the magnetic field, we could proceed analogously by differentiation, but there is an easier way since $A_x = \beta\Phi$. First,

$$B_x = \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y} = 0 \tag{7.78}$$

since $A_y = A_z = 0$. Second,

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = \beta\frac{\partial \Phi}{\partial z} = -\beta E_z, \tag{7.79}$$

and

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \beta\frac{\partial \Phi}{\partial y} = -\beta E_y. \tag{7.80}$$

Putting everything together, we have that

$$\vec{B} = \vec{\beta} \times \vec{E}. \tag{7.81}$$

As the electric field is radial, we find that $\vec{B}$ circles around the line of motion of the particle, as illustrated in figure 7.3. This is the same shape that we had for a nonrelativistic particle, but similarly to what happens with the electric field, for relativistic particles the magnitude of $\vec{B}$ is greatly enhanced at the sides of the particle and reduced in the area behind and ahead of it.

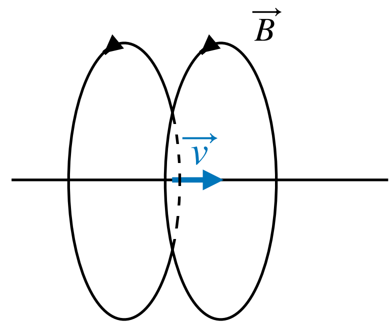

Figure 7.3

4-current density and Maxwell equations. We now want to rewrite the Maxwell equations in covariant notation. Let us start with the homogeneous ones,

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0, \tag{7.82}$$

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{7.83}$$

The easiest way to rewrite these equations is to introduce the *dual field strength tensor* $\tilde{F}^{\mu\nu}$, defined by

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}. \tag{7.84}$$

It is easy to see that $\tilde{F}^{\mu\nu} = -\tilde{F}^{\nu\mu}$, so it is also an antisymmetric tensor. Explicitly in terms of electric and magnetic fields, one finds by direct calculation that

$$(\tilde{F}^{\mu\nu}) = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix}. \tag{7.85}$$

This can also be summarized in

$$\tilde{F}^{kl} = \epsilon^{klm} E_m, \tag{7.86}$$

$$\tilde{F}^{k0} = B_k. \tag{7.87}$$

Now, it is easy to show that

$$\partial_\mu \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu(\partial_\rho A_\sigma - \partial_\sigma A_\rho) \tag{7.88}$$

$$= \epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\rho A_\sigma \tag{7.89}$$

$$= 0. \tag{7.90}$$

The last equality follows from the fact that we are contracting a tensor that is antisymmetric in μ and ρ with one that is symmetric in those two indices. We thus arrive the equation

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{7.91}$$

which, as you will show in the exercises, is equivalent to the homogeneous Maxwell equations. An analogous way to write this equation is

$$\epsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} = 0. \tag{7.92}$$

The inhomogeneous Maxwell equations involve the 4-divergence of $F^{\mu\nu}$. In the Lorentz gauge ($\partial_\mu A^\mu = 0$), we have

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= \partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu && (7.93)\\ &= \partial_\mu\partial^\mu A^\nu - \partial^\nu\partial_\mu A^\mu && (7.94)\\ &= \Box A^\nu. && (7.95) \end{aligned}$$

The right-hand side constitutes a 4-vector, since $F^{\mu\nu}$ is a rank-2 tensor and $\Box = \partial_\mu\partial^\mu$ is a scalar under Lorentz transformations. In components, we have

$$\begin{aligned} \Box A^0 = \Box\Phi = \rho &= \frac{1}{c}(c\rho), && (7.96)\\ (\Box A^k) = \Box\vec{A} &= \frac{1}{c}\vec{J}, && (7.97) \end{aligned}$$

so that

$$\partial_\mu F^{\mu\nu} = \Box A^\nu = \frac{1}{c}\begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix}. \tag{7.98}$$

This result motivates us to define the *4-current density vector*

$$(j^\nu) = \begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix}, \tag{7.99}$$

such that the inhomogeneous Maxwell equations can be written in covariant form as

$$\partial_\mu F^{\mu\nu} = \frac{1}{c} j^\nu. \tag{7.100}$$

This expression is explicitly independent of the choice of gauge, since the 4-potential doesn't appear in the equation anymore.

A consequence of the antisymmetry of $F^{\mu\nu}$ is that

$$\partial_\nu j^\nu = c\partial_\nu\partial\mu F^{\mu\nu} = 0, \tag{7.101}$$

and thus

$$\partial_\nu j^\nu = \partial_0 j^0 + \partial_l j^l = \frac{\partial}{\partial t}\rho + \vec{\nabla} \cdot \vec{J} = 0. \tag{7.102}$$

This is thus the covariant form of the continuity equation,

$$\partial_\mu j^\mu = 0. \tag{7.103}$$

Any 4-current satisfying this condition is said to be a *conserved current*.

Let us investigate some more properties of the 4-current density. For that, consider a charge and current distribution for which $\vec{J} = 0$ in some inertial system I. This means that, in that system, we have

$$j^0(x) = c\rho, \tag{7.104}$$

$$j^k(x) = 0. \tag{7.105}$$

In another inertial system I' moving at a speed $\vec{v}$ in the x^1 direction with respect to I, we can write

$$\rho' = \gamma\rho = \frac{\rho}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{7.106}$$

$$j'^1 = -\gamma\beta j^0 = -\frac{v\rho}{\sqrt{1 - \frac{v^2}{c^2}}} = v\rho', \tag{7.107}$$

$$j'^2 = 0, \tag{7.108}$$

$$j'^3 = 0. \tag{7.109}$$

The charge density is therefore a factor of γ larger in system I'. However, the total charge in a given volume V,

$$Q = \int_V \mathrm{d}^3x\, \rho(x) = \frac{1}{c}\int_V \mathrm{d}^3x\, j^0(x), \tag{7.110}$$

must be invariant. How can this be? The reason is that the volume element is itself not invariant under Lorentz transformations! Indeed, due to length contraction along the x^1 direction, we have that

$$\mathrm{d}^3x' = \frac{1}{\gamma}\,\mathrm{d}^3x, \tag{7.111}$$

and thus the integral (7.110) is invariant under Lorentz transformations. This is in fact a general result: the spatial integral of the zeroth component of a conserved current is a Lorentz-invariant conserved quantity.

Finally, from the relation

$$\Box A^\nu = \frac{1}{c} j^\nu, \tag{7.112}$$

and given that j^ν is a 4-vector and $\Box$ a Lorentz scalar, we confirm that A^ν is indeed a 4-vector, as we had anticipated.

Chapter 8

Relativistic electrodynamics

8.1 Units in relativistic electrodynamics

We have seen that the HL system of units is a convenient choice when dealing with electromagnetic fields in relativistic situations. However, there is still something that is fairly cumbersome, and it is that various powers of the speed of light c appear in the formulas. To simplify this, it is customary to suppress all factors of c by measuring all velocities in units of the speed of light. This amounts to setting $c = 1$ everywhere. To recover the factors of c when necessary, it is enough to replace

$$p \to cp, \quad E \to E, \quad m \to mc^2, \quad v \to \frac{v}{c} \tag{8.1}$$

in our formulas. As an example, the relation between energy, momentum, and mass with $c = 1$ becomes simply

$$E^2 = p^2 + m^2. \tag{8.2}$$

Thus, we are measuring all momenta, energies, and masses in units of energy. A particularly convenient energy unit is the electronvolt (eV), which is the energy gained by a particle with charge e after traversing a potential difference of one volt, $1\,\mathrm{eV} = 1.602 \times 10^{-19}$ joule.

8.2 Relativistic kinematics

Worldline of a particle. When parametrizing the *worldline* of a particle (i.e. the trajectory of a particle in the Minkowski spacetime), it is most convenient to do so using proper time as defined in (6.38),

$$x^\mu(\tau). \tag{8.3}$$

With this, we can directly calculate the 4-velocity and 4-acceleration of the particle as

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \quad a^\mu = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}. \tag{8.4}$$

Note that because $u^2 = 1$, after differentiating with respect to τ we find that

$$u^\mu a_\mu = 0, \tag{8.5}$$

which means that the 4-velocity and the 4-acceleration are always orthogonal to each other.

Relativistic version of the Lorentz force. Recall that the motion of a particle in an electromagnetic field is given by the Lorentz force

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{8.6}$$

(remember that we are setting $c = 1$). We can rewrite this in terms of the proper time and the 4-velocity by multiplying by γ,

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}\tau} = \gamma\vec{F} = q(\gamma\vec{E} + \gamma\vec{v} \times \vec{B}) = q(u^0\vec{E} + \vec{u} \times \vec{B}). \tag{8.7}$$

The left-hand side of this equation is the spatial part of a 4-vector. If we want to extend it to a full 4-vector, we expect that the temporal component would be

$$\frac{\mathrm{d}p^0}{\mathrm{d}\tau} = \frac{\mathrm{d}E}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}W}{\mathrm{d}t} = \gamma q\vec{v} \cdot \vec{E} = q\vec{u} \cdot \vec{E}. \tag{8.8}$$

Here, we have used that the rate at which the energy of the particle changes must equal the rate at which the electromagnetic field does work on the particle (recall that the magnetic field does no work). Combining both equations in 4-vector notation, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} p^0 \\ \vec{p} \end{pmatrix} = q\begin{pmatrix} \vec{u} \cdot \vec{E} \\ u^0\vec{E} + \vec{u} \times \vec{B} \end{pmatrix}. \tag{8.9}$$

The left-hand side of the equation is a 4-vector ($\mathrm{d}p^\mu/\mathrm{d}\tau$), so the right-hand side one must also be a 4-vector, which we will denote by f^μ and refer to as the *4-force*. In terms of the 4-force, the relativistic version of Newton's second law reads

$$f^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = m\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = m\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} = ma^\mu. \tag{8.10}$$

Note that because $u_\mu a^\mu = 0$, the 4-force is orthogonal to the 4-velocity,

$$u_\mu f^\mu = mu_\mu a^\mu = 0. \tag{8.11}$$

As you will show explicitly in the exercises, we can write f^μ in a manifestly Lorentz-covariant form as

$$f^\mu = qF^{\mu\nu}u_\nu, \tag{8.12}$$

so that the relativistic generalization of the Lorentz force equation becomes

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = m\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = qF^{\mu\nu}u_\nu. \tag{8.13}$$

8.3 Lagrangian and Hamiltonian formalisms

Although the equations of motion (8.13) are enough to completely describe the motion of particle in an external electromagnetic field, it is instructive to study the problem using the Lagrangian and Hamiltonian formalisms. These will prove useful when we consider the electromagnetic field as a dynamical quantity in the next section.

The basic entity in Lagrangian mechanics is the action

$$S = \int_{t_1}^{t_2} \mathrm{d}t\, L[q_i(t), \dot{q}_i(t), t], \tag{8.14}$$

where the Lagrangian L is a functional that depends on the generalized coordinates q_i and velocities $\dot{q}_i$, and perhaps explicitly on time. The equations of motion are obtained evoking the principle of stationary action, which amounts to the vanishing of the variation of the action for small variations of the coordinates with fixed endpoints,

$$\delta S = 0 \quad \text{with} \quad \delta q(t_1) = \delta q(t_2) = 0. \tag{8.15}$$

This can be shown to lead to the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \tag{8.16}$$

which can also be written in terms of the canonical (or conjugate) momentum

$$P_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{8.17}$$

The Lagrangian formulation is clearly not manifestly invariant under Lorentz transformations, as we would like it to be in relativistic situations. A physically consistent description should lead to Poincaré-invariant equations of motion, which is only possible if the action is a scalar under Lorentz transformations.

Free particle. Let us construct such a Lorentz-invariant action for a free particle. As a starting point, a reasonable ansatz is

$$S[x(\tau)] = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, f(x, u), \tag{8.18}$$

in terms of the proper time, the coordinate 4-vector x^μ and the 4-velocity $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$. The action is Lorentz invariant if and only if $f(x,u)$ is. The requirement of translational invariance means that f cannot depend on x. Furthermore, in order to be invariant under Lorentz boosts, f can only contain terms quadratic in the 4-velocity. But as $u^2 = 1$, the only possibility is that f is a constant:

$$S[x(\tau)] = \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, \kappa. \tag{8.19}$$

The constant κ can be fixed by dimensional considerations. Because we want the action to have units of energy×time (or momentum×length), κ must have units of energy. The only constant at hand that can play that role is m, the mass of the particle under consideration. Reinstating the factors of c for a moment, this means that the action must be

$$S = -mc^2 \int_{\tau_1}^{\tau_2} \mathrm{d}\tau = -mc \int_{\mathcal{W}} \mathrm{d}s, \tag{8.20}$$

where $\mathcal{W}$ refers to the worldline of the particle and in the last equality we have expressed it in terms of the line element $\mathrm{d}s$. The overall minus sign (as well as the overall normalization) is conventional and is chosen so that the extremum of the action corresponds to a minimum.

It is interesting to see that written in terms of $\mathrm{d}s$, the action can be interpreted as the 4-dimensional length of the worldline of the particle. The minimum of the action is achieved when the 4-dimensional length is maximal (worldlines must lie within the light cone and thus have $\mathrm{d}s \geq 0$ everywhere). We can understand this by thinking a bit about the geometry of the Minkowski spacetime. If the particle where to move from point 1 to 2 in a zigzag made up of light-like segments with $\mathrm{d}s = 0$, then the 4-dimensional length of its worldline would be zero. Thus, a trajectory with maximal 4-dimensional length corresponds to one with minimal 3-dimensional length, as we expect for a free particle.

In terms of the time measured by an arbitrary inertial observer, $\mathrm{d}t = \gamma \mathrm{d}\tau$, we can write the action as (now back to $c = 1$)

$$S = -m \int_{t_1}^{t_2} \mathrm{d}t \sqrt{1 - \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}. \tag{8.21}$$

From this, we can read out what the relativistic Lagrangian of a free particle is:

$$L = -m\sqrt{1 - v^2}. \tag{8.22}$$

In the nonrelativistic limit $v \ll 1$, we have

$$\sqrt{1 - v^2} \simeq 1 - \frac{1}{2}v^2, \tag{8.23}$$

which leads to the action

$$S \simeq \text{const} + \int \mathrm{d}t \, \frac{mv^2}{2}, \tag{8.24}$$

and thus the expected non-relativistic Lagrangian $L \simeq mv^2/2$.

Back to the relativistic case, we can calculate the canonical momentum as

$$P_i = \frac{\partial L}{\partial v_i} = \frac{mv_i}{\sqrt{1 - v^2}} = \gamma m v_i, \tag{8.25}$$

which is precisely how we defined the relativistic momentum $\vec{p} = \gamma m \vec{v}$. Analogously, we can compute the Hamiltonian

$$H = \vec{p} \cdot \vec{v} - L = \gamma m = \sqrt{m^2 + |\vec{p}|^2} = E, \tag{8.26}$$

corresponding to the relativistic expression for the energy.

Finally, we compute the Euler-Lagrange equations for the Lagrangian (8.22). As L does not depend on x, we simply have

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma m \vec{v}) = 0, \tag{8.27}$$

which amounts to the conservation of relativistic momentum, as expected. Given that $\mathrm{d}\tau = \mathrm{d}t/\gamma$ and the constancy of $\vec{p}$ implies the constancy of E, we can alternatively write the equations of motion as

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = 0. \tag{8.28}$$

Particle in an electromagnetic field. Let us now construct the Lagrangian for a particle of mass m and charge q in an external electromagnetic field. For that, we split the action in two parts,

$$S = S_{\text{free}} + S_{\text{int}}, \tag{8.29}$$

with S_{free} the action for a free particle as in (8.20) and S_{int} describes the interaction between the particle and the electromagnetic field.

Of course, S_{int} must also be Poincaré invariant. Furthermore, if we want to reproduce the term o the right-hand side of the equations of motion (8.13), we need to involve the field along the particle's worldline and its 4-velocity. To construct a Lorentz scalar, we need some 4-vector to contract with u^μ. The natural (in fact, the only) candidate for that is the 4-potential A^μ. Thus, we conjecture the interaction part action to be

$$S_{\text{int}} = q \int_{\mathcal{W}} \mathrm{d}\tau\, u^\mu\, A_\mu = q \int_{\mathcal{W}} \mathrm{d}\tau\, \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\, A_\mu = q \int_{\mathcal{W}} \mathrm{d}x^\mu\, A_\mu. \tag{8.30}$$

Thus, the total action becomes

$$S = -m \int_{\mathcal{W}} \mathrm{d}\tau + q \int_{\mathcal{W}} \mathrm{d}x^\mu\, A_\mu(x). \tag{8.31}$$

The term in the action involving the 4-potential is not gauge invariant, but transforms under a gauge transformation $A_\mu \to A_\mu - \partial_\mu \chi$ as

$$S \to S - q \int_{\mathcal{W}} \mathrm{d}x^\mu\, \partial_\mu \chi \tag{8.32}$$

$$= S - q \int_{\mathcal{W}} \mathrm{d}x^\mu\, \frac{\mathrm{d}\chi}{\mathrm{d}x^\mu} \tag{8.33}$$

$$= S - q \int_{\mathcal{W}} \mathrm{d}\chi \tag{8.34}$$

$$= q(\chi_2 - \chi_1), \tag{8.35}$$

which only depends on the values of the function χ at the endpoints of the worldline. For a variation of the action with fixed endpoints, the term above doesn't change and thus it has no impact on the equations of motion.

We can rewrite the action (8.31) in terms of the vector and scalar potentials using physical time as

$$S = \int_{\mathcal{W}} \mathrm{d}t \left(-m\sqrt{1-v^2} + q\vec{A}\cdot\vec{v} - q\Phi\right), \tag{8.36}$$

from which we can read out the Lagrangian

$$L = -m\sqrt{1-v^2} + q\vec{A}\cdot\vec{v} - q\Phi. \tag{8.37}$$

As you will show explicitly in the exercises, the Euler-Lagrange equations lead to the equations of motion

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = q(\partial^\mu A^\nu - \partial^\nu A^\mu)u_\nu = qF^{\mu\nu}u_\nu, \tag{8.38}$$

as expected. These equations of motion only involve the $\vec{E}$ and $\vec{B}$ fields and are thus manifestly gauge invariant.

As for the free particle, we can calculate the canonical momentum $P_i = \partial L/\partial v_i$,

$$\vec{P} = \frac{m\vec{v}}{\sqrt{1-v^2}} + q\vec{A} = \vec{p} + q\vec{A}, \tag{8.39}$$

where $\vec{p}$ is the relativistic momentum. The Hamiltonian therefore reads

$$H = \vec{P}\cdot\vec{v} - L = v_i\frac{\partial L}{\partial v_i} - L = \gamma m + q\Phi. \tag{8.40}$$

The Hamiltonian is a constant of motion if the Lagrangian does not explicitly depend on time, as is the case here. We can write H as a function of only $\vec{x}$ and $\vec{P}$ (and not $\vec{v}$) by noting that

$$(H - q\Phi)^2 = m^2 + (\vec{P} - q\vec{A})^2, \tag{8.41}$$

and thus

$$H = \sqrt{m^2 + (\vec{P} - q\vec{A})^2} + q\Phi, \tag{8.42}$$

which corresponds to the total energy of the particle. From this form of the Hamiltonian we can also derive the equations of motion by means of Hamilton's equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial H}{\partial P_i}, \quad \frac{\mathrm{d}P_i}{\mathrm{d}t} = -\frac{\partial H}{\partial x_i}. \tag{8.43}$$

In the nonrelativistic limit and after subtracting the constant term corresponding to the rest energy $E_0 = m$, we recover the Lagrangian

$$L \simeq \frac{1}{2}mv^2 + q\vec{A}\cdot\vec{v} - q\Phi. \tag{8.44}$$

Taking into account that

$$\vec{p} \simeq m\vec{v} = \vec{P} - q\vec{A}, \tag{8.45}$$

the Hamiltonian becomes

$$H \simeq \frac{1}{2m}\left(\vec{P} - q\vec{A}\right)^2 + q\Phi. \tag{8.46}$$

You may be familiar with this Hamiltonian from quantum mechanics.

Motion in a uniform, static magnetic field. As an example, let us consider the dynamics of a charged particle in a constant magnetic field. In this case, the equations of motion (8.9) reduce to

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = q\vec{v} \times \vec{B}. \tag{8.47}$$

Since the energy is a constant, so must $\gamma = E/m$ and the magnitude of the velocity be. We can therefore rewrite the second equation in (8.47) as

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \vec{v} \times \vec{\omega}_g, \tag{8.48}$$

where

$$\vec{\omega}_g = \frac{q\vec{B}}{\gamma m} = \frac{q\vec{B}}{E} \tag{8.49}$$

is the gyrofrequency or *cyclotron frequency*. Because the acceleration is always orthogonal to the velocity of the particle, the motion is circular in the plane perpendicular to $\vec{B}$, on top of a constant velocity parallel to $\vec{B}$. Considering for definiteness that $\vec{B}$ is oriented in the x^3 direction, we find

$$\vec{v}(t) = \mathrm{Re}\Big[v_\parallel \vec{e}_3 + \omega_g r_g(\vec{e}_1 - i\vec{e}_2)\mathrm{e}^{-i\omega_g t}\Big], \tag{8.50}$$

where $v_\parallel$ is the component of the velocity parallel to $\vec{B}$, and

$$r_g = \frac{v_\perp}{\omega_g} = \frac{\gamma m v_\perp}{qB} = \frac{p_\perp}{qB} \tag{8.51}$$

is the *cyclotron radius* (also known as radius of gyration or Larmor radius). For a positive charge, the motion viewed in the direction of $\vec{B}$ is that of a conterclockwise rotation. The radius of curvature of the patch of a charged particle in a known magnetic field therefore allows to determine the momentum of the particle. The trajectory can be found by direct integration,

$$\vec{x}(t) = \mathrm{Re}\Big[\vec{x}_0 + v_\parallel t\vec{e}_3 + ir_g(\vec{e}_1 - i\vec{e}_2)\mathrm{e}^{-i\omega_g t}\Big]. \tag{8.52}$$

This represents a helix of radius r_g and pitch angle $\tan\alpha = v_\parallel/\omega_g r_g$.

8.4 Lagragian for the electromagnetic field

In the previous section we have constructed the Lagrangian for a charged particle in an external (nondynamical) electromagnetic field. Now, we want to extend the Lagrangian description to dynamical fields interacting with charge and current sources. This means that we have to extend the Lagrangian approach from discrete particles to continuous fields.

The first step is to replace the finite number of coordinates $q_i(t)$ and $\dot{q}_i(t)$, with $i = 1, \ldots, n$ to an infinite numbers of degrees of freedom, one for each spacetime point x^μ. The generalized coordinate q_i is replaced by the field $\phi(x)$, where we can see x^μ as a continuous index. Similarly, the generalized velocities $\dot{q}_i$ are replaced by the gradient 4-vector $\partial^\mu\phi$. The Euler-Lagrange equations are obtained by demanding the action be stationary with respect to functional variations $\delta\phi$ and $\delta(\partial^\mu\phi)$.

All in all and allowing for the existence of multiple fields labelled by the finite index k, we have the correspondences:

$$i \longrightarrow x^\mu, k \tag{8.53}$$

$$q_i \longrightarrow \phi_k(x) \tag{8.54}$$

$$\dot{q}_i \longrightarrow \partial^\mu\phi_k(x) \tag{8.55}$$

$$L = \sum_i L_i(q_i, \dot{q}_i) \longrightarrow L = \int \mathrm{d}^3x\, \mathcal{L}(\phi_k, \partial^\mu\phi_k) \tag{8.56}$$

$$S = \int \mathrm{d}t\, L \longrightarrow S = \int\int \mathrm{d}t\, \mathrm{d}^3x\, \mathcal{L} = \int \mathrm{d}^4x\, \mathcal{L} \tag{8.57}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \longrightarrow \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu\phi_k)} = \frac{\partial \mathcal{L}}{\partial \phi_k}. \tag{8.58}$$

Here, we have introduced the *Lagrangian density* $\mathcal{L}$, which is in some sense equivalent to the individual terms L_i in the Lagrangian of a collection of discrete particles. The action thus can be written as the spacetime integral of the Lagrangian density, and the Euler-Lagrange equations can be directly written in terms of $\mathcal{L}$ rather than L.

For the electromagnetic field, we expect that the relevant field entering the Lagrangian is the 4-potential A^μ, as this is what ended appearing in the Lagrangian (8.37). The action will be Lorentz invariant if the Lagrangian density $\mathcal{L}$ is Lorentz invariant.

In order to have a dynamical (time evolving) electromagnetic field, we need derivatives of the field to appear, $\partial^\mu A^\nu$ or, equivalently $F^{\mu\nu}$. At the quadratic level, there are only two possible Lorentz-invariant forms: $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}\tilde{F}^{\mu\nu}$. The latter is invariant under proper Lorentz transformations, but it is odd under time reversal transformations, so we will drop it.

The interaction between the electromagnetic field and the sources must involve A^μ and the 4-current j^μ. In order to agree with (8.37), the term must in fact be $-j_\mu A^\mu$. Putting everything together, we have the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu, \tag{8.59}$$

where we have already fixed the normalization to reproduce the Maxwell equations, as we will now confirm. To use the Euler-Lagrange equations, it is useful to write $\mathcal{L}$ in terms of A^μ only,

$$\mathcal{L} = -\frac{1}{4} g_{\mu\nu} g_{\rho\sigma} (\partial^\mu A^\rho - \partial^\rho A^\mu)(\partial^\nu A^\sigma - \partial^\sigma A^\nu) - j_\mu A^\mu. \tag{8.60}$$

When calculating $\partial\mathcal{L}/\partial(\partial^\alpha A^\beta)$, we have to be careful with collecting all the terms:

$$\frac{\partial \mathcal{L}}{\partial(\partial^\alpha A^\beta)} = -\frac{1}{4} g_{\mu\nu} g_{\rho\sigma} \left[\delta^\mu_\alpha \delta^\rho_\beta F^{\nu\sigma} - \delta^\rho_\alpha \delta^\mu_\beta F^{\nu\sigma} + \delta^\nu_\alpha \delta^\sigma_\beta F^{\mu\rho} - \delta^\sigma_\alpha \delta^\nu_\beta F^{\mu\rho}\right] \tag{8.61}$$

$$= -F_{\alpha\beta} = F_{\beta\alpha}. \tag{8.62}$$

In the second line, we have used the antisymmetry of $F^{\mu\nu}$ together with the symmetry of $g_{\mu\nu}$. The other functional derivative in the Euler-Lagrange equations is

$$\frac{\partial \mathcal{L}}{\partial A^\mu} = -j_\mu. \tag{8.63}$$

Therefore the equations of motion for the electromagnetic field are

$$\partial^\mu F_{\mu\nu} = j_\nu, \tag{8.64}$$

which is the covariant form of the inhomogeneous Maxwell equations (7.100). As we saw in the previous chapter, the homogeneous ones are automatically satisfied by construction, given that $\partial_\mu \tilde{F}^{\mu\nu} = 0$.

Hamiltonian for the electromagnetic field. As was the case for the Lagrangian, we expect that the Hamiltonian of a field can be written as the spatial integral of a *Hamiltonian density* $\mathcal{H}$,

$$H = \int \mathrm{d}^3x\, \mathcal{H}. \tag{8.65}$$

The Hamiltonian density should be a function of the fields ϕ_k and the canonical momenta $\Pi_k = \partial\phi_k/\partial t$. The procedure for the construction of the Hamiltonian for fields is analogous to the one for a finite number of generalized coordinates q_i with the correspondences:

$$P_i = \frac{\partial L}{\partial \dot{q}_i} \longrightarrow \Pi_k = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_k)}, \tag{8.66}$$

$$H = \sum_i P_i \dot{q}_i - L \longrightarrow \mathcal{H} = \sum_k \Pi_k \partial_t \phi_k - \mathcal{L}, \tag{8.67}$$

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial P_i}, \quad \frac{\mathrm{d}P_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i} \longrightarrow \frac{\mathrm{d}\phi_k}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial \Pi_k}, \quad \frac{\mathrm{d}\Pi_k}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \phi_k}. \tag{8.68}$$

Let us apply this procedure to the Lagrangian for the electromagnetic field,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \tag{8.69}$$

$$= \frac{1}{2}\left(\partial_t \vec{A} + \vec{\nabla}\Phi\right)^2 - \frac{1}{2}\left(\vec{\nabla} \times \vec{A}\right)^2 - \Phi\rho + \vec{A} \cdot \vec{J}, \tag{8.70}$$

which we have written explicitly in terms of the 4-potential. When calculating the canonical momenta Π^μ conjugate to A^μ, we find that

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial(\partial_t A_i)} = \partial_t A_i + \partial_i \Phi, \tag{8.71}$$

$$\Pi_0 = \frac{\partial \mathcal{L}}{\partial(\partial_t \Phi)} = 0. \tag{8.72}$$

The fact that Π_0 vanishes reflects the fact that the Lagrangian contains no time derivative for Φ, which would indicate that Φ is not a dynamical field. This is a direct consequence of the gauge invariance of the action: not all the components in A^μ are actually independent because they are related by gauge transformations. This fact slightly complicates the derivation of the Hamiltonian for the electromagnetic field.

The condition $\Pi_0 = 0$ is a constraint that must be imposed in the Hamiltonian. Constrained systems can be studied by introducing Lagrange multipliers, but here we will follow a less rigorous approach that will lead to the same result. Before applying the constraint, the Hamiltonian density for the electromagnetic field is

$$\mathcal{H} = \Pi_0\, \partial_t \Phi + \vec{\Pi} \cdot \partial_t \vec{A} - \mathcal{L} \tag{8.73}$$

$$= \Pi_0\, \partial_t \Phi + \left(\partial_t \vec{A} + \vec{\nabla}\Phi\right) \partial_t \vec{A} - \frac{1}{2}\left(\partial_t \vec{A} + \vec{\nabla}\Phi\right)^2 + \frac{1}{2}\left(\vec{\nabla} \times \vec{A}\right)^2 + \Phi\rho - \vec{A} \cdot \vec{J} \tag{8.74}$$

$$= \Pi_0\, \partial_t \Phi + \frac{1}{2}\vec{\Pi}^2 + \Phi\left(\rho + \vec{\nabla} \cdot \vec{\Pi}\right) + \frac{1}{2}\left(\vec{\nabla} \times \vec{A}\right)^2 - \vec{A} \cdot \vec{J}. \tag{8.75}$$

In the last line, we have rewritten everything in terms of the canonical momentum and integrated by parts (recall that $\mathcal{H}$ is to be integrated over space) to replace $\vec{\Pi} \cdot \vec{\nabla}\Phi \to -\Phi \vec{\nabla} \cdot \vec{\Pi}$.

Now we have to be careful to impose the constraint. Clearly, we have to impose $\Pi_0 = 0$, but that is not sufficient: the Hamiltonian equation of motion for Π_0 is

$$\frac{\mathrm{d}\Pi_0}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \Phi} = -(\rho + \vec{\nabla} \cdot \vec{\Pi}). \tag{8.76}$$

For consistency, this must also be set to vanish. Indeed, note that

$$0 = \rho + \vec{\nabla} \cdot \vec{\Pi} = \rho + \vec{\nabla} \cdot \vec{E} \tag{8.77}$$

is nothing but Gauss' law. After imposing the two constraints, the Hamiltonian simplifies to

$$\mathcal{H} = \frac{1}{2}\vec{\Pi}^2 + \frac{1}{2}\left(\vec{\nabla} \times \vec{A}\right)^2 - \vec{A} \cdot \vec{J} \tag{8.78}$$

$$= \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right) - \vec{A} \cdot \vec{J}. \tag{8.79}$$

In the case where the sources vanish, $\vec{J} = 0$, we recover the expected expression for the energy density of the electromagnetic field.

8.5 Stress-energy tensor, conservation laws

We have interpreted the Hamiltonian above as the energy density of the electromagnetic field. However, the energy is not a relativistic covariant quantity, so we would like to embed it into an object with nice transformation properties under Lorentz transformations. For that, let us start with the expression for the 4-force f^μ exerted by the electromagnetic field. Using the Maxwell equations in covariant form, we can obtain an expression for it that only depends on the electromagnetic field:

$$\begin{aligned} f^\mu &= F^{\mu\nu} q u_\nu = F^{\mu\nu} j_\nu && (8.80)\\ &= F^{\mu\nu}\partial^\rho F_{\rho\nu} && (8.81)\\ &= \partial^\rho (F^{\mu\nu}F_{\rho\nu}) - F_{\rho\nu}\partial^\rho F^{\mu\nu}. && (8.82) \end{aligned}$$

The last term can be transformed in the following way:

$$\begin{aligned} F_{\rho\nu}\partial^\rho F^{\mu\nu} &= \frac{1}{2}\left(F_{\rho\nu}\partial^\rho F^{\mu\nu} + F_{\nu\rho}\partial^\nu F^{\mu\rho}\right) && (8.83)\\ &= \frac{1}{2}\left(F_{\rho\nu}\partial^\rho F^{\mu\nu} + F_{\rho\nu}\partial^\nu F^{\rho\mu}\right) && (8.84)\\ &= -\frac{1}{2}F_{\rho\nu}\partial^\mu F^{\nu\rho} && (8.85)\\ &= \frac{1}{4}\partial^\mu\left(F_{\rho\nu}F^{\rho\nu}\right). && (8.86) \end{aligned}$$

In the first line, we have only relabelled indices, while in the second one we have used the antisymmetry of $F^{\mu\nu}$. In the third one, we have used the homogeneous Maxwell equations in the form

$$\partial^\rho F^{\mu\nu} + \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} = 0. \tag{8.87}$$

Finally, to get to the last line we have simply used the chain rule. With this, we have

$$\begin{aligned} f^\mu &= \partial^\rho (F^{\mu\nu}F_{\rho\nu}) - \frac{1}{4}\partial^\mu\left(F_{\sigma\nu}F^{\sigma\nu}\right) && (8.88)\\ &= \partial_\rho\left(g^{\rho\sigma}F^{\mu\nu}F_{\sigma\nu} - \frac{1}{4}g^{\rho\mu}F_{\sigma\nu}F^{\sigma\nu}\right) && (8.89)\\ &= -\partial_\rho T^{\rho\mu}, && (8.90) \end{aligned}$$

where we have defined the *stress-energy tensor* of the electromagnetic field

$$T^{\mu\nu} = -g_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma} + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \tag{8.91}$$

In this form, it is clear that the stress-energy tensor is symmetric,

$$T^{\mu\nu} = T^{\nu\mu}. \tag{8.92}$$

By explicit calculation, we can rewrite its components in terms of the electric and magnetic fields,

$$T^{00} = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right), \tag{8.93}$$

$$T^{0i} = \left(\vec{E} \times \vec{B}\right)_i, \tag{8.94}$$

$$T^{ij} = -\left[E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}(E^2 + B^2)\right]. \tag{8.95}$$

In matrix form, we can write this as

$$(T^{\mu\nu}) = \left(\begin{array}{c|c} u & \vec{g}^T \\ \hline \vec{g} & -T^{\mathrm{M}}_{ij} \end{array}\right). \tag{8.96}$$

In the time-time component, we recognize the energy density (5.131), while the time-space components correspond to the momentum density (5.151), or, equivalently with $c = 1$, the Poynting vector (5.136). The space-space components are just minus the Maxwell stress tensor that we introduced in (5.153).

From the relation $\partial_\mu T^{\mu\nu} = f^\nu$, we can obtain the known conservation equations in differential form:

- for $\nu = 0$ we recover the conservation of energy or Poynting's theorem (5.135)

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E}, \tag{8.97}$$

- for $\nu = i$ we reconstruct the conservation of momentum (5.156)

$$\frac{\partial g_i}{\partial t} - \sum_j \frac{\partial}{\partial x_j} T^{\mathrm{M}}_{ij} = -\left[\rho E_i + (\vec{J} \times \vec{B})_i\right] \tag{8.98}$$

In the absence of sources $j_\nu = 0$ so $f^\mu = F^{\mu\nu} j_\nu = 0$, and the conservation law reduces to

$$\partial_\mu T^{\mu\nu} = 0, \tag{8.99}$$

which represents the conservation of energy and momentum for free electromagnetic fields.

Chapter 9

Radiation by moving charges

In this chapter we will put into practice the framework of relativistic electrodynamics. The goal is to describe the radiation generated by accelerated charged particles, including those moving at velocities close to the speed of light. As a first approximation, we will consider that the radiation emitted by the particle has no effect on its trajectory. In reality, a fully consistent treatment would take into account the loss of energy of the particle through the radiation.

9.1 Liénard-Wiechert potentials and fields for a point charge

To describe the electromagnetic field generated by a moving point charge, we make use of the retarded potentials that we derived in chapter 5, but reformulated in a covariant way. The first step is to write the charge and current densities associated with a point particle of charge q and trajectory $\vec{\xi}(t)$ in a given inertial frame I,

$$\rho(\vec{x},t) = q\delta[\vec{x} - \vec{\xi}(t)], \tag{9.1}$$

$$\vec{J}(\vec{x},t) = q\vec{v}(t)\delta[\vec{x} - \vec{\xi}(t)], \tag{9.2}$$

where $\vec{v}(t) = \mathrm{d}\vec{\xi}(t)/\mathrm{d}t$ is the particle's velocity in the frame I. To rewrite these in a covariant way, we make use of the worldline of the particle $\xi^\mu(\tau) = (t,\, \vec{x})$ parametrized by its proper time τ, and its 4-velocity $u^\mu = \gamma(1,\, \vec{v})^T$. We can construct the components of the 4-current density j^μ as

$$\begin{aligned} j^0 &= \rho(\vec{x},t) && (9.3)\\ &= q\,\delta[\vec{x} - \vec{\xi}(t)] && (9.4)\\ &= q\int \frac{\mathrm{d}t'}{\gamma}\,\gamma\,\delta(t-t')\delta[\vec{x} - \vec{\xi}(t')] && (9.5)\\ &= q\int \mathrm{d}\tau\, u^0\delta^{(4)}[x - \xi(\tau)] && (9.6) \end{aligned}$$

for the temporal component, and

$$\begin{aligned} j^i &= J^i(\vec{x}, t) &(9.7)\\ &= qv^i(t)\,\delta[\vec{x} - \vec{\xi}(t)] &(9.8)\\ &= q\int \frac{\mathrm{d}t'}{\gamma}\,\gamma v^i(t')\,\delta(t-t')\delta[\vec{x} - \vec{\xi}(t')] &(9.9)\\ &= q\int \mathrm{d}\tau\, u^i \delta^{(4)}[x - \xi(\tau)] &(9.10) \end{aligned}$$

for the spatial ones. Putting both together, j^μ can be expressed as

$$j^\mu(x) = q\int \mathrm{d}\tau\, u^\mu(\tau)\,\delta^{(4)}[x - \xi(\tau)]. \tag{9.11}$$

This expression for the 4-current in terms of the proper time and the 4-dimensional delta function $\delta^{(4)}(x - x') = \delta(x_0 - x'_0)\,\delta(\vec{x} - \vec{x}')$ is explicitly covariant.

Our task is now to solve the inhomogeneous Maxwell equations with the source j^μ,

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{9.12}$$

or, in terms of the 4-potential,

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu. \tag{9.13}$$

In Lorenz gauge, $\partial_\mu A^\mu = 0$ and we have to solve the 4-dimensional wave equation

$$\Box A^\mu = j^\mu. \tag{9.14}$$

As we know, we can solve wave equations by making use of the retarded (or advanced) Green function

$$A^\mu(x) = \int \mathrm{d}^4x'\, D_{\mathrm{ret}}(x - x')\, j^\mu(x'). \tag{9.15}$$

Recall that the Green function is defined to satisfy

$$\Box D_{\mathrm{ret}}(x) = \delta^{(4)}(x), \tag{9.16}$$

and in chapter 5 we found it to be (5.100)

$$D_{\mathrm{ret}}(x) = \frac{1}{4\pi} G^{(+)}(\vec{x}, t) = \frac{1}{2\pi}\delta(x^2)\,\theta(x^0). \tag{9.17}$$

A similar expression holds for the advanced Green function, but it is the retarded one that will be useful for the physical situations of interest to us. With this and the expression (9.11) for the 4-current, the 4-potential (9.15) becomes

$$\begin{aligned} A^\mu(x) &= \frac{q}{2\pi}\int \mathrm{d}\tau \int \mathrm{d}^4x'\,\theta[x^0 - x'^0]\,\delta\left([x - x']^2\right)\, u^\mu(\tau)\,\delta^{(4)}[x' - \xi(\tau)] &(9.18)\\ &= \frac{q}{2\pi}\int \mathrm{d}\tau\,\theta[x^0 - \xi^0(\tau)]\,\delta\left([x - \xi(\tau)]^2\right)\, u^\mu(\tau). &(9.19) \end{aligned}$$

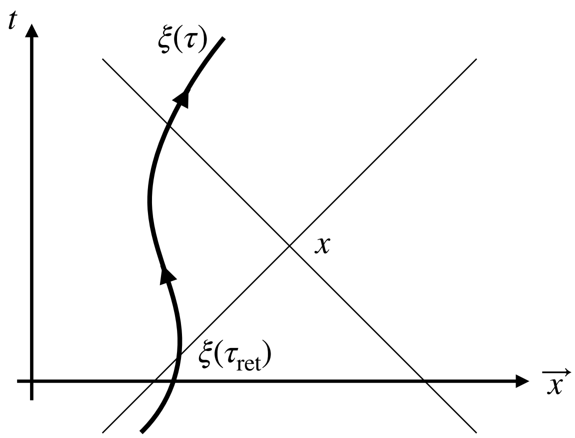

Figure 9.1

The theta function restricts the points contributing to the integral to only those lying in the past of the observation point x. Furthermore, the delta function requires the points to lie in the lightcone of x. Thus, as is exemplified in figure 9.1, the only points contributing to the integral are those where the past light cone of x intersects the worldline of the particle. Because the particle moves at a speed smaller than the speed of light, there is only one such point, corresponding to

$$\tau = \tau_{\rm ret} \quad \text{with} \quad x^0 - \xi^0(\tau_{\rm ret}) = |\vec{x} - \vec{\xi}(\tau_{\rm ret})|. \tag{9.20}$$

To evaluate (9.19), we use the rule

$$\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_i}}, \tag{9.21}$$

where x_i denote the zeroes of $f(x)$. We thus only need to compute

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[(x - \xi(\tau)\right]^2 = -2\left[(x - \xi(\tau)\right]_\mu u^\mu \tag{9.22}$$

to obtain the covariant form of the *Liénard-Wiechert potentials*

$$A^\mu(x) = \frac{q}{4\pi} \left.\frac{u^\mu(\tau)}{u(\tau)\cdot[x - \xi(\tau)]}\right|_{\tau=\tau_{\rm ret}}. \tag{9.23}$$

It is common to write these potentials in a non-covariant form for a given reference system in which

$$u \cdot (x - \xi) = u^0(x^0 - \xi^0) - \vec{u} \cdot (\vec{x} - \vec{\xi}) \tag{9.24}$$

$$= \gamma(t_x - t_\xi) - \gamma v \cdot (\vec{x} - \vec{\xi}) \tag{9.25}$$

$$= \gamma|\vec{x} - \vec{\xi}|(1 - \vec{v} \cdot \vec{n}), \tag{9.26}$$

where $\vec{v}$ is the velocity of the particle in that frame, and we have defined

$$\vec{n} = \frac{\vec{x} - \vec{\xi}(\tau)}{|\vec{x} - \vec{\xi}(\tau)|}, \tag{9.27}$$

that is, a unit vector in the direction of $\vec{x} - \vec{\xi}(\tau)$. With this, we recover the usual form of the Liénard-Wiechert potentials,

$$\Phi(\vec{x}, t) = \frac{q}{4\pi} \left. \frac{1}{(1 - \vec{v} \cdot \vec{n})\, |\vec{x} - \vec{\xi}|} \right|_{\tau = \tau_{\rm ret}}, \tag{9.28}$$

$$\vec{A}(\vec{x}, t) = \frac{q}{4\pi} \left. \frac{\vec{v}}{(1 - \vec{v} \cdot \vec{n})\, |\vec{x} - \vec{\xi}|} \right|_{\tau = \tau_{\rm ret}}. \tag{9.29}$$

The subscript indicates that the quantities are to be evaluated at the retarded time, given by (9.20). Clearly, these potentials reduce to the familiar expressions in the case of nonrelativistic motion.

The electromagnetic fields can be calculated directly by differentiating the potentials above. However, it is a bit simpler to return to the integral expression (9.19). In that expression, the derivatives in $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ will act only on the theta and delta functions as follows:

- Differentiation of the theta function results in a delta function,

$$\partial^\mu[\theta(x^0 - \xi^0(\tau)] = \delta^\mu_0\, \delta(x^0 - \xi^0(\tau)), \tag{9.30}$$

  and so this term only gives a nonzero contribution for $x^0 - \xi^0 = |\vec{x} - \vec{\xi}| = 0$, that is, at points along the worldline of the particle. We can exclude those points from consideration and drop this term.

- To differentiate the delta function, we can use the identity

$$\partial^\mu \delta[f] = \partial^\mu f \cdot \frac{\mathrm{d}}{\mathrm{d}f}\delta[f] = \partial^\mu f \cdot \frac{\mathrm{d}\tau}{\mathrm{d}f} \cdot \frac{\mathrm{d}}{\mathrm{d}\tau}\delta[f] \tag{9.31}$$

  and apply it to $f = [x - \xi(\tau)]^2$. This leads to

$$\partial^\mu \delta\left([x - \xi(\tau)]^2\right) = -\frac{(x - \xi)^\mu}{u \cdot (x - \xi)} \frac{\mathrm{d}}{\mathrm{d}\tau} \delta\left([x - \xi(\tau)]^2\right). \tag{9.32}$$

Putting everything together, we find

$$\partial^\mu A^\nu = -\frac{q}{2\pi}\int \mathrm{d}\tau\, \theta[x^0 - \xi^0(\tau)]\, \frac{(x-\xi)^\mu u^\nu}{u\cdot(x-\xi)}\, \frac{\mathrm{d}}{\mathrm{d}\tau}\delta\left([x-\xi(\tau)]^2\right) \tag{9.33}$$

$$= \frac{q}{2\pi}\int \mathrm{d}\tau\, \theta[x^0 - \xi^0(\tau)]\, \delta\left([x-\xi(\tau)]^2\right)\, \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{(x-\xi)^\mu u^\nu}{u\cdot(x-\xi)}\right], \tag{9.34}$$

where in the second line we have integrated by parts (the differentiation of the theta function gives no contribution as discussed above). We notice that this expression is in fact of the same form as (9.19), meaning that we can read off the result of the integral from (9.23) by making the appropriate substitutions. Doing this, we arrive at

$$F^{\mu\nu} = \frac{q}{4\pi}\frac{1}{u\cdot(x-\xi)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{(x-\xi)^\mu u^\nu - (x-\xi)^\nu u^\mu}{u\cdot(x-\xi)}\right]_{\tau=\tau_{\mathrm{ret}}} \tag{9.35}$$

It is interesting to rewrite this expression in terms of a^μ, the 4-acceleration of the particle,

$$F^{\mu\nu} = \frac{q}{4\pi}\frac{1}{[u\cdot(x-\xi)]^2}\Bigg\{(x-\xi)^\mu a^\nu - (x-\xi)^\nu a^\mu \tag{9.36}$$

$$- \frac{(x-\xi)^\mu u^\nu - (x-\xi)^\nu u^\mu}{u\cdot(x-\xi)}\,\frac{\mathrm{d}}{\mathrm{d}\tau}[u\cdot(x-\xi)]\Bigg\}_{\tau=\tau_{\mathrm{ret}}}. \tag{9.37}$$

Further using

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[u\cdot(x-\xi)] = -u^2 + a\cdot(x-\xi) = -1 + a\cdot(x-\xi), \tag{9.38}$$

we can separate the terms in $F^{\mu\nu}$ between those that only depend on u and those that contain both u and a. Thus, we can write

$$F^{\mu\nu} = F^{\mu\nu}_{(u)} + F^{\mu\nu}_{(a)}, \tag{9.39}$$

with

$$F^{\mu\nu}_{(u)} = \frac{q}{4\pi}\frac{1}{[u\cdot(x-\xi)]^3}\left[(x-\xi)^\mu u^\nu - (x-\xi)^\nu u^\mu\right]_{\tau=\tau_{\mathrm{ret}}}, \tag{9.40}$$

$$F^{\mu\nu}_{(a)} = \frac{q}{4\pi}\frac{1}{[u\cdot(x-\xi)]^2}\Bigg\{(x-\xi)^\mu a^\nu - (x-\xi)^\nu a^\mu \tag{9.41}$$

$$- \frac{a\cdot(x-\xi)}{u\cdot(x-\xi)}\left[(x-\xi)^\mu u^\nu - (x-\xi)^\nu u^\mu\right]\Bigg\}_{\tau=\tau_{\mathrm{ret}}}. \tag{9.42}$$

From this, one can easily read off the expressions for the electric and magnetic fields. For the electric field, one finds

$$\vec{E}_{(u)} = \frac{q}{4\pi} \frac{(1-v^2)(\vec{n}-\vec{v})}{|\vec{x}-\vec{\xi}|^2(1-\vec{v}\cdot\vec{n})^3}\bigg|_{\tau=\tau_{\rm ret}}, \tag{9.43}$$

$$\vec{E}_{(a)} = \frac{q}{4\pi} \frac{\vec{n}\times\left[(\vec{n}-\vec{v})\times\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right]}{|\vec{x}-\vec{\xi}|(1-\vec{v}\cdot\vec{n})^3}\bigg|_{\tau=\tau_{\rm ret}}, \tag{9.44}$$

while the magnetic field can be written as

$$\vec{B} = \vec{n}\times\vec{E}\Big|_{\tau=\tau_{\rm ret}}. \tag{9.45}$$

These expressions allow to calculate the 4-force $f^\mu = F^{\mu\nu} j_\nu$ that the particle following the trajectory $\xi(\tau)$ exerts on the 4-current j^ν located at the point x. In the special case of a particle following a uniform motion, the terms involving the acceleration are obviously absent.

It is interesting to note that the $F^{\mu\nu}_{(u)}$ and $F^{\mu\nu}_{(a)}$ terms have different asymptotic scaling at large distances $R = |\vec{x}-\vec{\xi}|$:

- $F^{\mu\nu}_{(u)}$ falls off as R^{-2}, as static fields do, and
- $F^{\mu\nu}_{(a)}$ falls off as R^{-1}, as is typical for radiation fields ($\vec{E}$ and $\vec{E}$ are transverse to the radial vector $\vec{n}$).

We thus expect the terms involving the acceleration to describe the electromagnetic radiation being emitted by the particle. In the next few sections, we will study this phenomenon in more detail.

9.2 Radiation in nonrelativistic scenarios

Let us consider a setup where there is a time-dependent distribution of charges and currents that is localized within a volume V around the origin, and we are looking at the fields far away from this region. As we will be interested in the nonrelativistic limit, we will recover the factors of c in this section.

We start from the basic formula for the retarded potential for an arbitrary 4-current j^μ,

$$A^\mu(\vec{x},t) = \frac{1}{4\pi c}\int_V \mathrm{d}^3x' \, \frac{j^\mu(\vec{x}', t_{\rm ret})}{|\vec{x}-\vec{x}'|}, \tag{9.46}$$

which is the straightforward covariant generalization of (5.108) and (5.109) (it can also be easily derived from (9.15)). The region V can be enclosed in a sphere of radius d so that the current j^μ is only nonzero for $|\vec{x}'| \le d$. Denoting $r = |\vec{x}|$, for $r \gg d$ we can expand

$$|\vec{x}-\vec{x}'| = r - \frac{\vec{x}\cdot\vec{x}'}{r} + \dots \qquad \Rightarrow \qquad \frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r} + \frac{\vec{x}\cdot\vec{x}'}{r^3} + \dots \tag{9.47}$$

Compared to the static case, there is an extra dependence in (9.46) on $|\vec{x}-\vec{x}'|$ coming from the retarded time

$$t_{\text{ret}} = t - \frac{\vec{x}-\vec{x}'}{c}, \tag{9.48}$$

for which we can apply the above expansion to find

$$j^\mu(\vec{x}', t_{\text{ret}}) = j^\mu\left(\vec{x}', t - \frac{r}{c} + \frac{\vec{x}\cdot\vec{x}'}{rc} + \dots\right). \tag{9.49}$$

To further simplify this expression, we take the motion of the source charges and currents to be nonrelativistic, meaning that j^μ does not change much over the time $T \sim d/c$ that it takes light to travel through the region V. If the current varies with characteristic frequency ω, this requirement amounts to demanding that $d/c \ll 1/\omega$. This allows to Taylor expand the 4-current as

$$j^\mu(\vec{x}', t_{\text{ret}}) = j^\mu(\vec{x}', t - r/c) + \partial_t j^\mu(\vec{x}', t - r/c)\frac{\vec{x}\cdot\vec{x}'}{rc} + \dots. \tag{9.50}$$

Electric dipole radiation. At zeroth order in the d/r expansion, the retarded potential becomes

$$A^\mu(\vec{x}, t) = \frac{1}{4\pi c r}\int_V \mathrm{d}^3x'\, j^\mu(\vec{x}', t - r/c). \tag{9.51}$$

We now want to derive the associated electric and magnetic fields. It is easier to start with the magnetic field, for which we only need the vector potential

$$\vec{A}(\vec{x}, t) \simeq \frac{1}{4\pi c r}\int_V \mathrm{d}^3x'\, \vec{J}(\vec{x}', t - r/c). \tag{9.52}$$

We can rewrite this using the relation (4.20) that we proved for magnetostatics, with $f = 1$ and $g = x_i'$. The only difference that now we need to use the continuity equation for time-dependent charge distributions $\partial_t\rho + \vec{\nabla}\cdot\vec{J} = 0$, to arrive at

$$\int_V \mathrm{d}^3x'\, \vec{J}(\vec{x}') = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}^3x'\, \vec{x}'\,\rho(\vec{x}') = \partial_t\vec{p}. \tag{9.53}$$

Here, we have identified the expression for the electric dipole moment $\vec{p}$ of the charge distribution. With this, the vector potential can be rewritten as

$$\vec{A}(\vec{x}, t) \simeq \frac{1}{4\pi c r}\partial_t\vec{p}(t - r/c), \tag{9.54}$$

which is known as the *electric dipole approximation.*

The magnetic field can be computed as $\vec{B} = \vec{\nabla}\times\vec{A}$. We get two terms as the curl acts on both the $1/r$ factor and the argument of $\partial_t\vec{p}$ (recall that $\vec{\nabla}r = \hat{x}$),

$$\vec{B} \simeq -\frac{1}{4\pi c r^3}\vec{x}\times\partial_t\vec{p}(t - r/c) - \frac{1}{4\pi c^2 r^2}\vec{x}\times\partial_t^2\vec{p}(t - r/c). \tag{9.55}$$

Far away from the source, the second term scales with $1/r$ while the first one scales with $1/r^2$, so we expect the former to dominate, but we have to be careful with the difference between the first and second time derivatives of $\vec{p}$. If the characteristic frequency at which the charge distribution varies is ω so that $\partial_t^2\vec{p} \sim \omega\partial_t\vec{p}$, the second term will dominate as long as $r \gg \lambda$, where $\lambda = c/\omega$ is the characteristic wavelength of the emitted radiation. The region satisfying $r \gg \lambda$ is known as the *far-field zone* or *radiation zone*. Thus, as long as $r \gg \lambda \gg d$, we can approximate the magnetic field as

$$\vec{B}(\vec{x}, t) \simeq -\frac{1}{4\pi c^2 r^2}\vec{x} \times \partial_t^2\vec{p}(t - r/c). \tag{9.56}$$

The electric field can be obtained from the Maxwell equation $\partial_t\vec{E} = c\vec{\nabla} \times \vec{B}$. In the radiation zone, the curl of $\vec{B}$ is dominated by $\vec{\nabla}$ acting on the argument of $\partial_t^2\vec{p}$,

$$\vec{\nabla} \times \vec{B} \simeq \frac{1}{4\pi r^3 c^3}\vec{x} \times \left[\vec{x} \times \partial_t^3\vec{p}(t - r/c)\right], \tag{9.57}$$

from which we can read off the electric field,

$$\vec{E} \simeq \frac{1}{4\pi c^2 r^3}\vec{x} \times \left[\vec{x} \times \partial_t^2\vec{p}(t - r/c)\right]. \tag{9.58}$$

This form is suggestively reminiscent of (9.44) in the nonrelativistic limit.

The electric and magnetic fields fall off as $1/r$, as expected for radiation fields. In the radiation zone, the electric and magnetic field are related by

$$\vec{E} = -\hat{x} \times \vec{B} \tag{9.59}$$

as for a plane wave propagating in the $\vec{x}$ direction. Of course, the oscillating dipole actually emits spherical waves, but at large distances $r \gg \lambda$ the waves look approximately planar.

Larmor formula for the radiated power. The instantaneous energy flux radiated away from the source is given by the Poynting vector

$$\vec{S} = c\,\vec{E} \times \vec{B} = c|\vec{B}|^2\hat{x} = \frac{1}{16\pi^2 c^3 r^5}|\vec{x} \times \partial_t^2\vec{p}|^2\,\vec{x}, \tag{9.60}$$

which points in the direction of $\vec{x}$, signaling that the power is emitted radially from the dipole. The overall scaling is $1/r^2$ as required by conservation of energy: the total energy flux integrated over a surface enclosing the source is constant and independent of r. The total power can be found by integrating over a sufficiently large sphere,

$$P = \int_{S^2} \mathrm{d}^2\vec{x} \cdot \vec{S} = \int_{S^2} \mathrm{d}\Omega\,\frac{\mathrm{d}P}{\mathrm{d}\Omega}, \tag{9.61}$$

where the power radiated per unit solid angle at a distance r is

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = r^2 \hat{x} \cdot \vec{S} = \frac{|\partial_t^2 \vec{p}|^2}{16\pi^2 c^3} \sin^2\Theta, \tag{9.62}$$

where Θ is the angle between $\partial_t^2 \vec{p}$ and the direction of observation $\vec{x}$. The power is maximal in the plane orthogonal to $\partial_t^2 \vec{p}$ and vanishes along its direction.

The total instantaneous radiated power is thus found to be

$$P = \frac{1}{16\pi^2 c^3} |\partial_t^2 \vec{p}|^2 \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \sin^3\theta \tag{9.63}$$

$$= \frac{1}{6\pi c^3} |\partial_t^2 \vec{p}|^2, \tag{9.64}$$

which is known as the *Larmor formula.*

Often, the time dependence of the dipole is oscillatory with frequency ω, so that $|\partial_t^2 \vec{p}|^2 = \omega^4 p_0^2 \cos^2(\omega t)$. In those situations, rather than in the instantaneous power, one is often interested in the time averaged one. Averaging over a period $T = 2\pi/\omega$ then results in the time-averaged radiated power

$$\bar{P} = \frac{p_0^2 \omega^4}{12\pi c^3}. \tag{9.65}$$

Higher-order corrections. Recall that to obtain the previous result we only kept zeroth-order terms in the r/d expansion of the retarded 4-potential. Let us now continue expanding to next order. We find

$$A^\mu(\vec{x}, t) = \frac{1}{4\pi c} \int_V \mathrm{d}^3 x' \, \frac{j^\mu(\vec{x}', t_{\mathrm{ret}})}{|\vec{x} - \vec{x}'|} \tag{9.66}$$

$$= \frac{1}{4\pi c r} \int_V \mathrm{d}^3 x' \left[j^\mu(\vec{x}', t - r/c) + \partial_t j^\mu(\vec{x}', t - r/c) \frac{\vec{x} \cdot \vec{x}'}{rc} \right] \left[1 + \frac{\vec{x} \cdot \vec{x}'}{r^2} \right] + \ldots \tag{9.67}$$

The first term, which we will denote A^μ_{ED} gives the electric dipole approximation. To next order we only need to consider the second term in the first bracket as the second term in the second bracket is suppressed by an extra power of $1/r$. Therefore, we can write

$$A^\mu(\vec{x}, t) \simeq A^\mu_{\mathrm{ED}}(\vec{x}, t) + \frac{1}{4\pi c^2 r^2} \int_V \mathrm{d}^3 x' (\vec{x} \cdot \vec{x}') \partial_t j^\mu(\vec{x}', t - r/c), \tag{9.68}$$

so that the vector potential becomes

$$\vec{A}(\vec{x}, t) \simeq \vec{A}_{\mathrm{ED}}(\vec{x}, t) + \frac{1}{4\pi c^2 r^2} \int_V \mathrm{d}^3 x' (\vec{x} \cdot \vec{x}') \partial_t \vec{J}(\vec{x}', t - r/c). \tag{9.69}$$

Similar to (9.53), we can use the relation (4.20), this time with $f = x_i$ and $g = x'_i x'_j$ and $\partial_t \vec{J}$ instead of $\vec{J}$, together with the time-dependent continuity equation, to prove that

$$\left[\int \mathrm{d}^3x' (\vec{x}\cdot\vec{x}')\partial_t \vec{J}\right]_i = \int \mathrm{d}^3x'\, x_j x'_j \partial_t J_i \tag{9.70}$$

$$= \frac{1}{2}\int \mathrm{d}^3x' \left[x_j x'_j \partial_t J_i - x_j x'_i \partial_t J_j + x_j x'_i x'_j \partial_t^2 \rho\right] \tag{9.71}$$

$$= \frac{1}{2}\int \mathrm{d}^3x' \left[-x_j(\delta_{mj}\delta_{li} - \delta_{mi}\delta_{lj})x'_l \partial_t J_m + x_j x'_i x'_j \partial_t^2 \rho\right] \tag{9.72}$$

$$= \frac{1}{2}\int \mathrm{d}^3x' \left[-\epsilon_{ijk}\epsilon_{klm} x_j x'_l \partial_t J_m + x_j x'_i x'_j \partial_t^2 \rho\right] \tag{9.73}$$

$$= -\frac{1}{2}\left[\vec{x}\times\int \mathrm{d}^3x'\, \vec{x}'\times\partial_t \vec{J}\right]_i + \frac{1}{2}\left[\int \mathrm{d}^3x' (\vec{x}\cdot\vec{x}')\vec{x}'\partial_t^2\rho\right]_i . \tag{9.74}$$

In the third line, we have just rewritten the first two terms using Kronecker deltas; in the fourth one, we have used the identity $\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$; and in the fifth one the definition of the cross product.

Using the result above, we can split the first-order expansion of the retarded vector potential into three terms,

$$\vec{A}(\vec{x},t) = \vec{A}_{\mathrm{ED}}(\vec{x},t) + \vec{A}_{\mathrm{MD}}(\vec{x},t) + \vec{A}_{\mathrm{EQ}}(\vec{x},t), \tag{9.75}$$

including the *magnetic dipole* contribution

$$\vec{A}_{\mathrm{MD}}(\vec{x},t) = -\frac{1}{8\pi c^2 r^2}\,\vec{x}\times\int \mathrm{d}^3x'\,\vec{x}'\times\partial_t\vec{J}(\vec{x},t-r/c) \tag{9.76}$$

and the *electric quadrupole* contribution

$$\vec{A}_{\mathrm{EQ}}(\vec{x},t) = -\frac{1}{8\pi c^2 r^2}\int \mathrm{d}^3x'(\vec{x}\cdot\vec{x}')\vec{x}'\partial_t^2\rho(\vec{x},t-r/c). \tag{9.77}$$

The origin of the names will become clear shortly.

Magnetic dipole radiation. Recall that in chapter 4 we defined the magnetic dipole moment $\vec{m}$ of an arbitrary current distribution as

$$\vec{m} = \frac{1}{2c}\int \mathrm{d}^3x'\,\vec{x}'\times\vec{J}(\vec{x}'). \tag{9.78}$$

With this, we can rewrite (9.76) as

$$\vec{A}_{\mathrm{MD}}(\vec{x},t) = -\frac{1}{4\pi c r^2}\,\vec{x}\times\partial_t\vec{m}(\vec{x},t-r/c), \tag{9.79}$$

justifying the name of magnetic dipole radiation. We can calculate the associated magnetic field by taking the curl of this expression. As before, the leading-order contribution corresponds to the derivative acting on the argument of $\partial_t\vec{m}$, resulting in

$$\vec{B}_{\mathrm{MD}}(\vec{x},t) \simeq \frac{1}{4\pi c^2 r^3}\,\vec{x}\times\left(\vec{x}\times\partial_t^2\vec{m}(\vec{x},t-r/c)\right). \tag{9.80}$$

The electric field can once again be obtained from $\partial_t \vec{E} = c\vec{\nabla} \times \vec{B}$, which yields

$$\vec{E}_{\mathrm{MD}}(\vec{x}, t) \simeq \frac{1}{4\pi c^2 r^2}\, \vec{x} \times \partial_t^2 \vec{m}(\vec{x}, t - r/c). \tag{9.81}$$

This expression is completely analogous to the one for the electric dipole radiation after exchange of $\vec{p} \to \vec{m}$. In particular, in the far field regime we still have $\vec{B}_{\mathrm{MD}} = \hat{x} \times \vec{E}_{\mathrm{MD}}$ and the Poynting vector has the same radial profile and angular dependence as (9.60). Integrating it over an sphere of radius r gives the total instantaneous emitted power,

$$P_{\mathrm{MD}} = \frac{1}{6\pi c^3} |\partial_t^2 \vec{m}|^2. \tag{9.82}$$

Although this expression has the same form as the Larmor formula with the magnetic dipole replacing the electric one, the power emitted by magnetic dipole radiation is substantially smaller in nonrelativistic situations. To illustrate this, consider a particle of charge q oscillating with frequency ω and amplitude d, for which $p \sim qd$ and $m \sim qd^2\omega/c$. The ratio of the emitted powers is in this case

$$\frac{P_{\mathrm{MD}}}{P_{\mathrm{ED}}} \sim \frac{d^2\omega^2}{c^2} \sim \frac{v^2}{c^2}, \tag{9.83}$$

where v is the velocity of the particle.

In an analogous (but significantly more tedious) way, one can show that $\vec{A}_{\mathrm{EQ}}$ represents the radiating field of an electric quadrupole that is varying in time. We will skip the quantitative discussion of that case.

9.3 Scattering of light.

The formulas derived above can be used to understand the phenomenon of scattering, which occurs when an incoming electromagnetic wave accelerates a particle, which in turn emits electromagnetic radiation itself. As we will see, it is convenient to think of the incoming radiation as inducing oscillating electric and magnetic multipoles which radiate energy in directions that can be different to that of the incoming wave.

Thomson scattering. The simplest case corresponds to the scatterer being a free charged particle. For an incident plane wave with $\vec{E} = \vec{E}_0 \mathrm{e}^{i(\vec{k}\cdot\vec{x} - \omega t)}$, the particle feels an acceleration

$$m\partial_t^2 \vec{x} = q\vec{E}. \tag{9.84}$$

If the displacement of the particle, which we take to be at $\vec{x} = 0$, is small compared to the wavelength of the radiation, we have $\vec{E} \simeq \vec{E}_0 \mathrm{e}^{-i\omega t}$, and (9.84) describes a simple harmonic oscillation

$$\vec{x}(t) = -\frac{q\vec{E}_0}{m\omega^2} \sin(\omega t). \tag{9.85}$$

The amplitude of the oscillation is $x_{\text{max}} = qE_0/m\omega^2$, so our assumption of small displacement is satisfied if

$$x_{\text{max}} \ll \frac{c}{\omega} \quad \Leftrightarrow \quad \frac{qE_0}{m\omega c} \ll 1, \tag{9.86}$$

which implies that the maximum velocity of the particle $v_{\text{max}} = qE_0/m\omega \ll c$ is nonrelativistic. Thus, we can employ our formula (9.65) for the dipole radiation to calculate the time-averaged power that is radiated by the particle,

$$\bar{P}_{\text{rad}} = \frac{q^4 E_0^2}{12\pi m^2 c^3}. \tag{9.87}$$

We can compare the energy in the emitted radiation with that of the incoming wave, which is given by its time-averaged Poynting vector (5.137), whose magnitude is

$$\bar{S}_{\text{inc}} = \frac{c}{2} E_0^2. \tag{9.88}$$

The ratio of outgoing to incoming power powers is known as the *cross-section* of the scattering process. This example of the scattering of light by free charges gives the Thomson scattering cross-section

$$\sigma = \frac{\bar{P}_{\text{rad}}}{\bar{S}_{\text{inc}}} = \frac{q^4}{6\pi m^2 c^4}. \tag{9.89}$$

We see that the Thomson cross-section does not depend on the frequency of the incoming radiation, which means that free charges scatter all wavelengths of light equally.

The cross-sections has units of area, which can be made explicit by writing it as

$$\sigma = \frac{8\pi}{3} r_q^2 \tag{9.90}$$

with the *classical radius of the particle* being

$$r_q = \frac{q^2}{4\pi m c^2}. \tag{9.91}$$

This corresponds to the radius of the charged sphere whose Coulomb energy equals the relativistic rest mass of the particle. Note that written this way, the Thomson cross-section is slightly smaller than the geometric cross section of the particle, which would be $4\pi r_q^2$.

Rayleigh scattering. Free charges are not so common around us: most matter is electrically neutral, with charges bound in neutral atoms or molecules. When an electromagnetic wave hits a neutral atom, its center of mass does not accelerate but, rather, a polarization of the atom is generated.

Let us consider a simple harmonic model in which the electron is bound in the atom by a restoring force

$$\vec{F} = -m\omega_0^2 \vec{x}, \tag{9.92}$$

where m is the mass of the electron and ω_0 is the natural oscillation frequency of the atom. If the frequency of the incoming radiation is much smaller than this natural frequency, $\omega \ll \omega_0$, as is for many cases of interest, the electron responds quasi statically to the action of the electric field, producing a displacement from its equilibrium position

$$m\omega_0^2 \, \Delta\vec{x}(t) = q\vec{E}(t). \tag{9.93}$$

This displacement induces an oscillating dipole moment

$$\vec{p}(t) = q \, \Delta\vec{x}(t) = \frac{q^2}{m\omega_0^2}\vec{E}(t). \tag{9.94}$$

As before, we can use (9.65) to calculate the time-averaged radiated power in this case,

$$\bar{P}_{\rm rad} = \frac{q^4 E_0^2}{12\pi m^2 c^3}\left(\frac{\omega}{\omega_0}\right)^4. \tag{9.95}$$

The associated cross-section for Rayleigh scattering is thus

$$\sigma = \frac{\bar{P}_{\rm rad}}{\bar{S}_{\rm inc}} = \frac{q^4}{6\pi m^2 c^4}\left(\frac{\omega}{\omega_0}\right)^4 = \frac{8\pi r_q^2}{3}\left(\frac{\omega}{\omega_0}\right)^4, \tag{9.96}$$

which as opposed to the Thomson one has a sharp frequency dependence: high frequencies (or short wavelengths) are scattered more often. This is the reason why the sky is blue: nitrogen and oxygen in the atmosphere scatter short-wavelength blue light more than long-wavelength red light. Thus, the blue light from the Sun undergoes many scatterings and we see it as coming from all regions in the sky. In contrast, the yellow and red components get scattered less and the Sun appears yellow in midday or red at sunrise and sunset, when its light traverses a much longer path through the atmosphere.

9.4 Radiation by relativistic charges

The goal of this section is to study the radiation emitted by a point charge moving at relativistic speed. We already know that the fields generated by the particle can be calculated from the Liénard-Wiechert potentials and are given by (9.39). In particular, we will care about the piece that is proportional to the acceleration of the particle, as we know that that is the one that corresponds to radiation fields. In this section, we will once again set $c = 1$.

First and to compare with the nonrelativistic results derived in the previous section, we can reduce (9.44) in the nonrelativistic limit to

$$\vec{E}_{(a)} = \frac{q}{4\pi}\frac{\vec{n}\times(\vec{n}\times\vec{a})}{|\vec{x}-\vec{\xi}|}\bigg|_{t=t_{\rm ret}}, \tag{9.97}$$

where $\vec{a} = \mathrm{d}\vec{v}/\mathrm{d}t$ is the acceleration of the particle. This, together with $\vec{B} = \vec{n} \times \vec{E}$, leads to the Poynting vector

$$\vec{S} = |\vec{E}_{(a)}|^2 \, \vec{n}, \tag{9.98}$$

and to the power radiated per solid angle

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{q^2}{16\pi^2} |\vec{a}|^2 \sin^2 \Theta, \tag{9.99}$$

where Θ is the angle between the acceleration $\vec{a}$ and $\vec{n}$. Note that the radiation is polarized in the plane containing those two vectors. The total instantaneous power is thus

$$P = \frac{1}{6\pi} q^2 |\vec{a}|^2, \tag{9.100}$$

which matches (9.64) for the particular case at hand where $\partial_t \vec{p} = q\vec{v}$.

Relativistic Larmor formula. Working with the full relativistic expression (9.44) now, we can calculate the Poynting vector to be

$$\begin{aligned}
\vec{S} &= \vec{E}_{(a)} \times \vec{B}_{(a)} && (9.101)\\
&= \vec{E}_{(a)} \times \left(\vec{n} \times \vec{E}_{(a)} \right) && (9.102)\\
&= |\vec{E}_{(a)}|^2 \vec{n} && (9.103)\\
&= \frac{q^2}{16\pi^2} \frac{|\vec{n} \times [(\vec{n} - \vec{v}) \times \vec{a}]|^2}{|\vec{x} - \vec{\xi}|^2 (1 - \vec{v} \cdot \vec{n})^6} \, \vec{n}, && (9.104)
\end{aligned}$$

where in the second line we have used (9.45), in the third one we have taken into account the fact that $\vec{E}_{(a)}$ is perpendicular to $\vec{n}$, and the last one implements (9.44). Everything in the formula above is assumed to be evaluated at the retarded time t_{ret}, satisfying

$$t_{\mathrm{ret}} + |\vec{x} - \vec{\xi}(t_{\mathrm{ret}})| = t. \tag{9.105}$$

When calculating the power emitted by the particle, there is a subtlety related to the Doppler shifting of the radiation. The energy emitted per unit time t does not equal the energy emitted per unit time t_{ret}, both differ by a factor of

$$\frac{\mathrm{d}t}{\mathrm{d}t_{\mathrm{ret}}} = 1 - \vec{v} \cdot \vec{n}. \tag{9.106}$$

If we want to integrate $\vec{S}$ over a sphere of radius $|\vec{x} - \vec{\xi}|$ around the particle, we have to use the retarded time as a time coordinate. Thus, the power emitted per unit time t_{ret}, per solid angle $\mathrm{d}\Omega$, is

$$\begin{aligned}
\frac{\mathrm{d}P(t_{\mathrm{ret}})}{\mathrm{d}\Omega} &= \frac{\mathrm{d}t}{\mathrm{d}t_{\mathrm{ret}}} \, |(\vec{x} - \vec{\xi})|^2 \, \vec{n} \cdot \vec{S} && (9.107)\\
&= \frac{q^2}{16\pi^2} \frac{|\vec{n} \times [(\vec{n} - \vec{v}) \times \vec{a}]|^2}{(1 - \vec{v} \cdot \vec{n})^5} && (9.108)
\end{aligned}$$

The total instantaneous power emitted can be calculated by integrating over the sphere. This calculation is a bit tedious so we will just quote the result here,

$$P = \frac{q^2}{6\pi}\gamma^6\left[a^2 - (\vec{v}\times\vec{a})^2\right] \tag{9.109}$$

$$= \frac{q^2}{6\pi}\gamma^4\left[a^2 + \gamma^2(\vec{v}\cdot\vec{a})^2\right] \tag{9.110}$$

which is the relativistic generalization of the Larmor formula (9.100). Let us apply this formula to two scenarios of particular interest.

Bremsstrahlung. Assume that the particle is traveling in a straight line, so that $\vec{v}$ and $\vec{a}$ are parallel to each other. The most common situation of this kind occurs when a particle decelerates, and the emitted radiation is known as *bremsstrahlung* (German for stopping radiation).

At the observation point $\vec{x}$, the radiation reaches us from the retarded point in the particle's trajectory, $\vec{\xi}(t_{\rm ret})$. We denote θ the angle between the observation point and the common direction of $\vec{v}$ and $\vec{a}$, so that

$$\vec{n}\cdot\vec{v} = v\cos\theta. \tag{9.111}$$

The angular dependence (9.108) simplifies in this case as $\vec{v}\times\vec{a} = 0$, reducing to

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{q^2a^2}{16\pi}\,\frac{\sin^2\theta}{(1-v\cos\theta)^5}. \tag{9.112}$$

For $v \ll 1$, this is Larmor's nonrelativistic result, for which we know that the power emitted is largest in the plane perpendicular to the direction of travel of the particle. In contrast, at relativistic speeds $v \to 1$, the radiation has a larger magnitude and becomes increasingly beamed in the forward direction of the particle's trajectory. As shown in Fig. 9.2, the radiation becomes confined to a narrow cone in the direction of motion of the particle.

The total power emitted (9.110) simplifies in this case to

$$P = \frac{q^2\gamma^6a^2}{6\pi}, \tag{9.113}$$

which agains showcases the growth of power when $v \to 1$.

Cyclotron and synchrotron radiation. Another example that comes up frequently is that of a relativistic particle traveling in a circle, with $\vec{v}\cdot\vec{a} = 0$. We choose a coordinate system in which $\vec{v}$ is instantaneously aligned with the z axis and $\vec{a}$ is in the x direction. In terms of the polar angles, we have

$$\vec{n} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}. \tag{9.114}$$

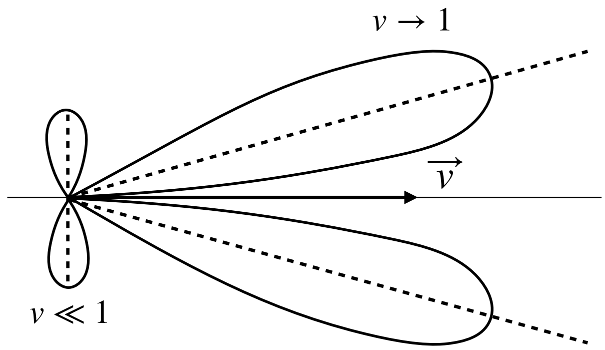

Figure 9.2

A bit of algebra leads to the following expression for the angular dependence of the emitted radiation,

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{q^2}{16\pi}\ \frac{a^2}{(1-v\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-v\cos\theta)^2}\right]. \tag{9.115}$$

At nonrelativistic speeds, the angular dependence simplifies and is proportional to $(1 - \sin^2\theta\cos^2\phi)$, which is known as *cyclotron radiation*.

In contrast, in the relativistic limit $v \to 1$ the radiation is once again beamed mostly in the forward direction. This radiation is referred to as *synchrotron radiation*. The total emitted power (9.110) is in this case

$$P = \frac{q^2\gamma^4 a^2}{6\pi}. \tag{9.116}$$

It is interesting to compare this result with the one for a linear trajectory in (9.113) for the same magnitude of applied force. For circular motion, the magnitude of the velocity stays constant and thus

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \gamma m\vec{a}, \tag{9.117}$$

so we have

$$P_{\mathrm{circular}} = \frac{q^2}{6\pi m^2}\gamma^2\left(\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\right)^2. \tag{9.118}$$

For linear motion, the magnitude of the velocity changes, so

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = m\frac{\mathrm{d}(\gamma\vec{v})}{\mathrm{d}t} = m\gamma\vec{a}(1+\gamma^2 v) \simeq m\gamma^3\vec{a}, \tag{9.119}$$

where in the last step we have taken the relativistic limit $v \to 1$ ($\Rightarrow \gamma \gg 1$). Thus, from (9.113) we find

$$P_{\mathrm{linear}} = \frac{q^2}{6\pi m^2}\left(\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\right)^2. \tag{9.120}$$

Comparing (9.118) with (9.120), we see that the radiation emitted by a transversal acceleration is a factor of γ^2 larger than that of a longitudinal acceleration (for the same magnitude of applied force). This means that the radiation emitted by a particle in an arbitrary relativistic motion is approximately equal to that of a particle moving instantaneously along a circular path with radius of curvature $\rho = v^2/a_\perp \simeq 1/a_\perp$, where $a_\perp$ is the component of the acceleration perpendicular to the trajectory. This radiation is highly beamed along the instantaneous velocity vector of the particle.