Linear Programming, Zero-Forcing, and Maximally Diverse Optima
(Progress Documentation)

1 Integer Program for Zero Forcing Diameter

Denote the zero forcing diameter model as $\mathrm{ZFD}\,(G, s^*)$, where $G \in \mathbb{G}$ and s^* is an optimal solution to $\mathrm{ZIP}\,(G)$. That is, s^* is the zero forcing number of G. The integer program $\mathrm{ZFD}\,(G, s^*)$ presented in (1a) -(1k) computes the minimum intersection of two distinct zero forcing sets of G. We reference these sets by B and $B^{'}$. Note that every edge in $E(G)$ is represented by two directed edges with opposite direction. The binary variables s_v and $s_v^{'}$ indicate whether v is in B or $B^{'}$ respectively. The variables $x_v, x_v^{'} \in \mathbb{Z}$ ranging in {0,1,...,T} specify at which time steps vertex v is forced given its corresponding initial zero forcing set. Note that T is the maximum difference in the forcing times of two vertices. The binary variables $y_e, y_e^{'}$, for each directed edge $e = (u, v)$, indicate whether u forces v given B or $B^{'}$ initially. Lastly, the notation $\delta^-(v)$ references the set of edges pointing toward vertex v and $N(u)$ denotes the set of all vertices $v \in V(G)$ that are adjacent to u. Note that the vertex and edge sets of G are denoted by V and E respectively.

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\begin{aligned}
\text{minimize} \quad & \sum_{v\in V} z_v && \text{(1a)}\\
\text{subject to} \quad & s_v + \sum_{e\in\delta^-(v)} y_e = 1, \quad \forall\, v \in V, && \text{(1b)}\\
& s_v^{'} + \sum_{e\in\delta^-(v)} y_e^{'} = 1, \quad \forall\, v \in V, && \text{(1c)}\\
& x_u - x_v + (T+1)y_e \le T, \quad \forall\, e = (u,v) \in E, && \text{(1d)}\\
& x_u^{'} - x_v^{'} + (T+1)y_e^{'} \le T, \quad \forall\, e = (u,v) \in E, && \text{(1e)}\\
& x_w - x_v + (T+1)y_e \le T, \quad \forall\, e = (u,v) \in E,\ \forall\, w \in N(u)\setminus\{v\}, && \text{(1f)}\\
& x_w^{'} - x_v^{'} + (T+1)y_e^{'} \le T, \quad \forall\, e = (u,v) \in E,\ \forall\, w \in N(u)\setminus\{v\}, && \text{(1g)}\\
& \sum_{v\in V} s_v = s^*, && \text{(1h)}\\
& \sum_{v\in V} s_v^{'} = s^*, && \text{(1i)}\\
& s_v + s_v^{'} - z_v \le 1 \quad \forall\, v \in V, && \text{(1j)}\\
& s, s^{'}, z \in \{0,1\}^n,\ x, x^{'} \in \{0,\dots,T\}^n,\ y, y^{'} \in \{0,1\}^m && \text{(1k)}
\end{aligned}
$$

The triples (s, x, y) and $(s^{'}, x^{'}, y^{'})$, where $s, s^{'} \in \{0,1\}^n$; $x, x^{'} \in \{0,...,T\}^n$, and $y, y^{'} \in \{0,1\}^m$, make a *feasible solution* of $\mathrm{ZFD}\,(G, s^*)$ provided that (s, x, y) and $(s^{'}, x^{'}, y^{'})$ satisfy (1b)-(1k). Additionally, if s and $s^{'}$ are minimal with respect to the objective function (1a), then we say that (s, x, y) and $(s^{'}, x^{'}, y^{'})$ is an *optimal solution* of $\mathrm{ZFD}\,(G, s^*)$. The corresponding minimal value of the objective function (1a) is the *optimal value* of $\mathrm{ZFD}\,(G, s^*)$.

Proposition 1.1. *Given an optimal solution to* $\mathrm{ZFD}\,(G, s^*)$*,* $z_v = 1$ *for some* $v \in V$ *if and only if* $s_v = s_v^{'} = 1$.

Proof. Suppose $z_v = 1$ for some $v \in V$. By way of contradiction, also suppose either $s_v = 0$ or $s_v^{'} = 0$. Then constraint (1j) will be true if $z_v = 0$. This contradicts the fact that we have an optimal solution to $\mathrm{ZFD}\,(G, s^*)$, i.e., the objective function (1a) would not be minimized. Thus, our assumption that $s_v = 0$ or $s_v^{'} = 0$ must be false. Hence, $s_v = s_v^{'} = 1$.

Conversely, let $s_v = s_v^{'} = 1$ for some $v \in V$. Then constraint (1j) holds only when $z_v = 1$. Therefore, given an optimal solution to $\mathrm{ZFD}\,(G, s^*)$, $z_v = 1$ for some $v \in V$ if and only if $s_v = s_v^{'} = 1$. □

2 Zero Forcing Diameter/Graphical Interpretation of $\mathrm{ZFD}\,(G, s^*)$

Given a *feasible solution* of $\mathrm{ZFD}\,(G, s^*)$, the value of the objective function (1a) can be defined as $|B \cap B^{'}|$, where B and $B^{'}$ are zero forcing sets of G. Furthermore, the *optimal value* of $\mathrm{ZFD}\,(G, s^*)$ is equal to $\min|B \cap B^{'}|$, where $B, B^{'} \in S$ and S is the set of all minimal zero forcing sets of G. We define the *zero forcing diameter* of a graph $G \in \mathbb{G}$ as

$$d(G) = zf(G) - \min|B \cap B^{'}| \tag{2}$$

where $B, B^{'} \in S$ and $zf(G)$ is the zero forcing number of G. Note that $zf(G) = s^*$ in terms of the inter program $\mathrm{ZFD}\,(G, s^*)$ given in equations (1a) -(1k).

Proposition 2.1. *Given a graph $G \in \mathbb{G}$, $d(G) = 0$ if and only if G is an empty graph.*

Proof. Suppose $G \in \mathbb{G}$ and $d(G) = 0$. This implies that $zf(G) = \min|B \cap B^{'}|$, which means there is only one unique zero forcing set for G. □

Proposition 2.2. *Given a path graph P_n, $d(P_n) = 1\ \forall\ n \in \mathbb{N}$.*

Proposition 2.3. *Given a complete graph K_n, $d(K_n) = 1\ \forall\ n \in \mathbb{N}$.*

Proposition 2.4. *Given a star graph S_n, $d(S_n) = 1\ \forall\ n \in \mathbb{N}$.*

Proposition 2.5. *Given a cycle graph C_n , $d(C_n) = 2\ \forall\ n \geq 4 \in \mathbb{N}$.*

Theorem 2.6. *For all graphs $G \in \mathbb{G}$, we have that $0 \leq d(G) \leq \lfloor n/2 \rfloor$ where n is the order of G.*