**LINEAR PROGRAMMING, ZERO-FORCING, AND MAXIMALLY DIVERSE OPTIMA**
(PROGRESS DOCUMENTATION)

# 1 Integer Program for Zero Forcing Diameter

Denote the zero forcing diameter model as $\text{ZFD}(G, s^*)$, where $G \in \mathbb{G}$ and $s^*$ is an optimal solution to $\text{ZIP}(G)$.

$$\text{minimize} \quad \sum_{v \in V} z_v \tag{1a}$$

$$\text{subject to} \quad s_v + \sum_{e \in \delta^-(v)} y_e = 1, \quad \forall\, v \in V, \tag{1b}$$

$$s_v^{'} + \sum_{e \in \delta^-(v)} y_e^{'} = 1, \quad \forall\, v \in V, \tag{1c}$$

$$x_u - x_v + (T+1)y_e \leq T, \quad \forall\, e = (u,v) \in E, \tag{1d}$$

$$x_u^{'} - x_v^{'} + (T+1)y_e^{'} \leq T, \quad \forall\, e = (u,v) \in E, \tag{1e}$$

$$x_w - x_v + (T+1)y_e \leq T, \quad \forall\, e = (u,v) \in E,\ \forall\, w \in N(u) \setminus \{v\}, \tag{1f}$$

$$x_w^{'} - x_v^{'} + (T+1)y_e^{'} \leq T, \quad \forall\, e = (u,v) \in E,\ \forall\, w \in N(u) \setminus \{v\}, \tag{1g}$$

$$\sum_{v \in V} s_v = s^*, \tag{1h}$$

$$\sum_{v \in V} s_v^{'} = s^*, \tag{1i}$$

$$s_v + s_v^{'} - z_v \leq 1 \quad \forall\, v \in V, \tag{1j}$$

$$s, s^{'}, z \in \{0,1\}^n,\ x, x^{'} \in \{0, \ldots, T\}^n,\ y, y^{'} \in \{0,1\}^m \tag{1k}$$

The triples $(s, x, y)$ and $(s^{'}, x^{'}, y^{'})$, where $s, s^{'} \in \{0,1\}^n$; $x, x^{'} \in \{0, ..., T\}^n$, and $y, y^{'} \in \{0,1\}^m$, make a *feasible solution* of $\text{ZFD}(G, s^*)$ provided that $(s, x, y)$ and $(s^{'}, x^{'}, y^{'})$ satisfy (1b)-(1k). Additionally, if $s$ and $s^{'}$ are minimal with respect to the objective function (1a), then we say that $(s, x, y)$ and $(s^{'}, x^{'}, y^{'})$ is an *optimal solution* of $\text{ZFD}(G, s^*)$. The corresponding minimal value of the objective function (1a) is the *optimal value* of $\text{ZFD}(G, s^*)$.

**Proposition 1.1.** *Given an optimal solution to* $\text{ZFD}(G, s^*)$, $z_v = 1$ *for some* $v \in V$ *if and only if* $s_v = s_v^{'} = 1$.

*Proof.* Suppose $z_v = 1$ for some $v \in V$. By way of contradiction, also suppose either $s_v = 0$ or $s_v^{'} = 0$. Then constraint (1j) will be true if $z_v = 0$. This contradicts the fact that we have an optimal solution to $\text{ZFD}(G, s^*)$, i.e., the objective function (1a) would not be minimized. Thus, our assumption that $s_v = 0$ or $s_v^{'} = 0$ must be false. Hence, $s_v = s_v^{'} = 1$.

Conversely, let $s_v = s_v^{'} = 1$ for some $v \in V$. Then constraint (1j) holds only when $z_v = 1$. Therefore, given an optimal solution to $\text{ZFD}(G, s^*)$, $z_v = 1$ for some $v \in V$ if and only if $s_v = s_v^{'} = 1$. □

# 2 Graphical Interpretation of Integer Program

Given a *feasible solution* of $\text{ZFD}(G, s^*)$, the value of the objective function (1a) can be defined as $|B \cap B^{'}|$, where $B$ and $B^{'}$ are zero forcing sets of $G$. Furthermore, the *optimal value* of $\text{ZFD}(G, s^*)$ is equal to $\min|B \cap B^{'}|$, where $B, B^{'} \in S$ and $S$ is the set of all minimal zero forcing sets of $G$. Considering the model $\text{ZFD}(G, s^*)$, denote the *zero forcing diameter* of $G$ as $d(G) = s^* - min|B \cap B^{'}|$ where $B, B^{'} \in S$. Note that $s^*$ is an optimal solution to $\text{ZIP}(G)$, which is equivalent to the zero forcing number of $G$.