Linear Programming, Zero-Forcing, and Maximally Diverse Optima
(Progress Documentation)

1 Integer Program for Zero Forcing Diameter

Denote the zero forcing diameter model as $\text{ZFD}\,(G, s^*)$, where $G \in \mathbb{G}$ and s^* is an optimal solution to $\text{ZIP}\,(G)$. That is, s^* is the zero forcing number of G. The integer program $\text{ZFD}\,(G, s^*)$ presented in (1a) -(1k) computes the minimum intersection of two distinct zero forcing sets of G. We reference these sets by B and B'. Note that every edge in $E(G)$ is represented by two directed edges with opposite direction. The binary variables s_v and s'_v indicate whether v is in B or B' respectively. The variables $x_v, x'_v \in \mathbb{Z}$ ranging in {0,1,...,T} specify at which time steps vertex v is forced given its corresponding initial zero forcing set. Note that T is the maximum difference in the forcing times of two vertices. The binary variables y_e, y'_e, for each directed edge $e = (u, v)$, indicate whether u forces v given B or B' initially. Lastly, the notation $\delta^-(v)$ references the set of edges pointing toward vertex v and $N(u)$ denotes the set of all vertices $v \in V(G)$ that are adjacent to u. Note that the vertex and edge sets of G are denoted by V and E respectively.

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\begin{aligned}
\text{minimize} \quad & \sum_{v \in V} z_v && \text{(1a)}\\
\text{subject to} \quad & s_v + \sum_{e \in \delta^-(v)} y_e = 1, \quad \forall\, v \in V, && \text{(1b)}\\
& s'_v + \sum_{e \in \delta^-(v)} y'_e = 1, \quad \forall\, v \in V, && \text{(1c)}\\
& x_u - x_v + (T+1)y_e \le T, \quad \forall\, e = (u,v) \in E, && \text{(1d)}\\
& x'_u - x'_v + (T+1)y'_e \le T, \quad \forall\, e = (u,v) \in E, && \text{(1e)}\\
& x_w - x_v + (T+1)y_e \le T, \quad \forall\, e = (u,v) \in E,\ \forall\, w \in N(u) \setminus \{v\}, && \text{(1f)}\\
& x'_w - x'_v + (T+1)y'_e \le T, \quad \forall\, e = (u,v) \in E,\ \forall\, w \in N(u) \setminus \{v\}, && \text{(1g)}\\
& \sum_{v \in V} s_v = s^*, && \text{(1h)}\\
& \sum_{v \in V} s'_v = s^*, && \text{(1i)}\\
& s_v + s'_v - z_v \le 1 \quad \forall\, v \in V, && \text{(1j)}\\
& s, s', z \in \{0,1\}^n,\ x, x' \in \{0, \ldots, T\}^n,\ y, y' \in \{0,1\}^m && \text{(1k)}
\end{aligned}
$$

The triples (s, x, y) and (s', x', y'), where $s, s' \in \{0,1\}^n$; $x, x' \in \{0, ..., T\}^n$, and $y, y' \in \{0,1\}^m$, make a *feasible solution* of $\text{ZFD}\,(G, s^*)$ provided that (s, x, y) and (s', x', y') satisfy (1b)-(1k). Additionally, if s and s' are minimal with respect to the objective function (1a), then we say that (s, x, y) and (s', x', y') is an *optimal solution* of $\text{ZFD}\,(G, s^*)$. The corresponding minimal value of the objective function (1a) is the *optimal value* of $\text{ZFD}\,(G, s^*)$.

Proposition 1.1. *Given an optimal solution to* $\text{ZFD}\,(G, s^*)$*,* $z_v = 1$ *for some* $v \in V$ *if and only if* $s_v = s'_v = 1$.

Proof. Suppose $z_v = 1$ for some $v \in V$. By way of contradiction, also suppose either $s_v = 0$ or $s'_v = 0$. Then constraint (1j) will be true if $z_v = 0$. This contradicts the fact that we have an optimal solution to $\text{ZFD}\,(G, s^*)$, i.e., the objective function (1a) would not be minimized. Thus, our assumption that $s_v = 0$ or $s'_v = 0$ must be false. Hence, $s_v = s'_v = 1$.

Conversely, let $s_v = s'_v = 1$ for some $v \in V$. Then constraint (1j) holds only when $z_v = 1$. Therefore, given an optimal solution to $\text{ZFD}\,(G, s^*)$, $z_v = 1$ for some $v \in V$ if and only if $s_v = s'_v = 1$. □

2 Zero Forcing Diameter/Graphical Interpretation of $\mathrm{ZFD}\,(G, s^*)$

Given a *feasible solution* of $\mathrm{ZFD}\,(G, s^*)$, the value of the objective function (1a) can be defined as $|B \cap B^{'}|$, where B and $B^{'}$ are zero forcing sets of G. Furthermore, the *optimal value* of $\mathrm{ZFD}\,(G, s^*)$ is equal to $\min|B \cap B^{'}|$, where $B, B^{'} \in S$ and S is the set of all minimal zero forcing sets of G. We define the *zero forcing diameter* of a graph $G \in \mathbb{G}$ as

$$d(G) = zf(G) - \min|B \cap B^{'}| \tag{2}$$

where $B, B^{'} \in S$ and $zf(G)$ is the zero forcing number of G. Note that $zf(G) = s^*$ in terms of the inter program $\mathrm{ZFD}\,(G, s^*)$ given in equations (1a) -(1k).

Proposition 2.1. *Given a graph $G \in \mathbb{G}$, $d(G) = 0$ if and only if G is an empty graph.*

Proof. Suppose $G \in \mathbb{G}$ and $d(G) = 0$, i.e., $zf(G) = \min|B \cap B^{'}|$. This implies there is only one unique zero-forcing set of G, which is only possible if no forcings occur. Hence, our zero-forcing set must include all vertices in G. This implies that G has no edges. Thus, G is an empty graph by definition.

Conversely, suppose G is an empty graph. Since G has no edges by definition, then the only way to force all of its vertices blue is to include them in our zero-forcing set. Hence, $\min|B \cap B^{'}| = zf(G)$ where B, $B^{'}$ are minimal zero-forcing sets of G. Thus, $d(G) = 0$. Therefore, $d(G) = 0$ if and only if G is an empty graph. □

Proposition 2.2. *Given a path graph P_n, $d(P_n) = 1\ \forall\ n \in \mathbb{N}$.*

Proof. Suppose P_n is a path graph such that $n \in \mathbb{N}$. By definition, P_n contains at least 2 vertices and $zf(P_n) = 1$ [citation]. Thus, $\min|B \cap B^{'}| = 0$, where B and $B^{'}$ are minimum zero-forcing sets containing the leftmost and rightmost vertices of P_n respectively. Therefore, $d(G) = 1$ by definition. □

Proposition 2.3. *Given a complete graph K_n, $d(K_n) = 1\ \forall\ n \in \mathbb{N}$.*

Proof. Suppose K_n is a complete graph such that $n \in \mathbb{N}$. Note that $zf(K_n) = n-1$ [citation]. To calculate $d(K_n)$, let B be a minimum zero-forcing set containing $n-1$ arbitrary vertices in $V(K_n)$. Let $B^{'}$ be another minimum zero-forcing set containing the vertex in the set $V(K_n) - B$ in addition to $n-2$ unique, arbitrary vertices in $V(K_n)$. Since $|B| = n-1$ and $V(K_n) - B \subseteq B^{'}$, then $\min|B \cap B^{'}| = n-2$. Therefore, $d(K_n) = (n-1) - (n-2) = 1$. □

Proposition 2.4. *Given a star graph S_n, $d(S_n) = 1\ \forall\ n \in \mathbb{N}$.*

Proof. Suppose S_n is a star graph such that $n \in \mathbb{N}$ and let $L(S_n) \subseteq V(S_n)$ be the set of leaves in the graph S_n. Note that $zf(S_n) = n-2$ [citation]. To calculate $d(S_n)$, let B be a minimum zero-forcing set of S_n containing $|L(S_n)| - 1$ arbitrary leaves in $L(S_n)$, i.e., $n-2$ vertices in $V(S_n)$. Similarly, let $B^{'}$ be another minimum zero-forcing set of S_n containing the vertex in the set $L(S_n) - B$ and $|L(S_n)| - 2$ unique, arbitrary leaves in $L(S_n)$. Since $|B| = n-2$, $L(S_n) - B \subseteq B^{'}$, and the root of S_n is not in B or $B^{'}$, then $\min|B \cap B^{'}| = n-3$. Therefore, $d(S_n) = (n-2) - (n-3) = 1$. □

Proposition 2.5. *Given a cycle graph C_n, $d(C_n) = 2\ \forall\ n \geq 4 \in \mathbb{N}$.*

Theorem 2.6. *For all graphs $G \in \mathbb{G}$, we have that $0 \leq d(G) \leq \lfloor n/2 \rfloor$ where n is the order of G.*

Proof. Suppose $G \in \mathbb{G}$. By definition, $0 \leq d(G) \leq zf(G)$. Hence, $d(G) \geq 0$. By way of contradiction, suppose $d(G) > \lfloor n/2 \rfloor$, i.e., $d(G) = \lfloor n/2 \rfloor + c$ where $0 < c < \lfloor n/2 \rfloor$. Then $zf(G) - \min|B \cap B^{'}| = \lfloor n/2 \rfloor + c$. This implies $zf(G) \geq \lfloor n/2 \rfloor + c$, and it follows that $\min|B \cap B^{'}| \geq c$ for all zero-forcing sets B and B'. Hence, $d(G) \leq \lfloor n/2 \rfloor$, which contradicts our assumption that $d(G) > \lfloor n/2 \rfloor$. Therefore, $0 \leq d(G) \leq \lfloor n/2 \rfloor$ for all $G \in \mathbb{G}$. □