Linear Programming, Zero-Forcing, and Maximally Diverse Optima
(Progress Documentation)

1 Zero Forcing Diameter Model/Integer Program

Denote the zero forcing diameter model as $\mathrm{ZFD}\,(G, s^*)$, where $G \in \mathbb{G}$ and s^* is an optimal solution to $\mathrm{ZIP}\,(G)$.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{v \in V} z_v && \text{(1a)}\\
\text{subject to} \quad & s_v + \sum_{e \in \delta^-(v)} y_e = 1, \quad \forall\, v \in V, && \text{(1b)}\\
& s_v^{'} + \sum_{e \in \delta^-(v)} y_e^{'} = 1, \quad \forall\, v \in V, && \text{(1c)}\\
& x_u - x_v + (T+1)y_e \le T, \quad \forall\, e = (u,v) \in E, && \text{(1d)}\\
& x_u^{'} - x_v^{'} + (T+1)y_e^{'} \le T, \quad \forall\, e = (u,v) \in E, && \text{(1e)}\\
& x_w - x_v + (T+1)y_e \le T, \quad \forall\, e = (u,v) \in E,\ \forall\, w \in N(u) \setminus \{v\}, && \text{(1f)}\\
& x_w^{'} - x_v^{'} + (T+1)y_e^{'} \le T, \quad \forall\, e = (u,v) \in E,\ \forall\, w \in N(u) \setminus \{v\}, && \text{(1g)}\\
& \sum_{v \in V} s_v = s^*, && \text{(1h)}\\
& \sum_{v \in V} s_v^{'} = s^*, && \text{(1i)}\\
& s_v + s_v^{'} - z_v \le 1 \quad \forall\, v \in V, && \text{(1j)}\\
& s, s^{'}, z \in \{0,1\}^n,\ x, x^{'} \in \{0, \ldots, T\}^n,\ y, y^{'} \in \{0,1\}^m && \text{(1k)}
\end{aligned}
$$

Proposition 1.1. *Given an optimal solution to* $\mathrm{ZFD}\,(G, s^*)$, $z_v = 1$ *for some* $v \in V$ *if and only if* $s_v = s_v^{'} = 1$.

Proof. Suppose $z_v = 1$ for some $v \in V$. By way of contradiction, also suppose either $s_v = 0$ or $s_v^{'} = 0$. Then constraint (1j) will be true if $z_v = 0$. This contradicts the fact that we have an optimal solution to $\mathrm{ZFD}\,(G, s^*)$, i.e., the objective function (1a) would not be minimized. Thus, our assumption that $s_v = 0$ or $s_v^{'} = 0$ must be false. Hence, $s_v = s_v^{'} = 1$.

Conversely, let $s_v = s_v^{'} = 1$ for some $v \in V$. Then constraint (1j) holds only when $z_v = 1$. Therefore, given an optimal solution to $\mathrm{ZFD}\,(G, s^*)$, $z_v = 1$ for some $v \in V$ if and only if $s_v = s_v^{'} = 1$. □