On Zero Forcing and the Inverse Eigenvalue Problem for Graphs

Thomas R. Cameron

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Abstract

This note is intended to introduce the connection between the inverse eigenvalue problem of a graph and the zero forcing number of a graph.

1 Introduction

Let $\mathbb{G}$ denote the collection of all simple graphs (no loops nor multi-edges) of order n. With each graph $G \in \mathbb{G}$, there is a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and an edge set $E(G)$, where $\{v_i, v_j\} \in E(G)$ if and only if $i \neq j$ and v_i and v_j are adjacent vertices. For example, let G be the star graph of order 7 shown in Figure 1. Then, $V(G) = \{1, 2, 3, 4, 5, 6, 7\}$ and $E(G) = \{\{1, 7\}, \{2, 7\}, \{3, 7\}, \{4, 7\}, \{5, 7\}, \{6, 7\}\}$.

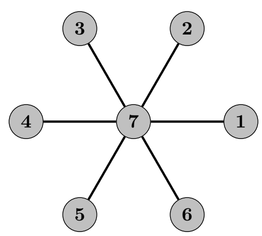

Figure 1: A star graph of order 7.

Every $n \times n$ symmetric matrix $A = [a_{ij}]_{i,j=1}^n$ is associated with a graph $G \in \mathbb{G}$, where $a_{ij} \neq 0$ if and only if $\{v_i, v_j\} \in E(G)$. Given a graph $G \in \mathbb{G}$, the set of all $n \times n$ symmetric matrices associated with G is denoted by

$$\mathcal{S}(G) = \left\{A = [a_{ij}]_{i,j=1}^n \colon a_{ij} = a_{ji}\ \forall i \neq j,\ a_{ij} \neq 0 \iff \{v_i, v_j\} \in E(G)\right\}.$$

For example, the star graph G shown in Figure 1 is associated with symmetric matrices of the form

$$A = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 & \beta_1 \\ 0 & \alpha_2 & \cdots & 0 & \beta_2 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \alpha_6 & \beta_6 \\ \beta_1 & \beta_2 & \cdots & \beta_6 & \alpha_7 \end{bmatrix}.$$

The inverse eigenvalue problem of a graph can be stated as follows: Given a graph $G \in \mathbb{G}$, what are all possible length n multi-sets σ such that there is a matrix $A \in \mathcal{S}(G)$ with spectrum σ. This important question is obviously difficult to answer in general; however, there are a few observations we can make.

- The empty graph of order n, denoted E_n, has no edges. Therefore, $\mathcal{S}(E_n)$ is the set of all $n \times n$ diagonal matrices. Since the eigenvalues of a diagonal matrix are equal to its diagonal entries, it follows that every length n multi-set σ corresponds to the eigenvalues of a matrix $A \in \mathcal{S}(E_n)$. It turns out, that the empty graph is the only graph with this property.
- Let K_n denote a complete graph of order n. Then, $\mathcal{S}(K_n)$ is made up of symmetric matrices where all off-diagonal entries are non-zero. Furthermore, for any $\lambda_1 \leq \lambda_2 \leq \cdots \lambda_n$, there is a matrix $A \in \mathcal{S}(K_n)$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ if and only if $\lambda_1 < \lambda_n$ [1].

- Let P_n denote a path graph of order n. Then, $\mathcal{S}(P_n)$ corresponds to tri-diagonal matrices with non-zero entries in the upper and lower diagonals. Furthermore, any length n multi-set σ corresponds to the eigenvalues of a matrix $A \in \mathcal{S}(P_n)$ if and only if the entries of σ are distinct [3].

Due to the difficulty of the inverse eigenvalue problem of a graph, there is a focus on solving related but seemingly easier problems. For instance, one may want to find the maximum multiplicity of a graph $G \in \mathbb{G}$, denoted $M(G)$, which is defined as the largest integer k such that there is a matrix $A \in \mathcal{S}(G)$ with an eigenvalue of multiplicity k. Note $M(E_n) = n$, $M(K_n) = (n-1)$, and $M(P_n) = 1$.

The multiplicity of an eigenvalue λ of the symmetric matrix A is equal to the nullity of $\lambda I - A$, i.e, the dimension of the null space of $\lambda I - A$. Note that for any $A \in \mathcal{S}(G)$, it follows that $\lambda I - A \in \mathcal{S}(G)$. Therefore,

$$M(G) = \max \{\text{nullity}\,(A) : A \in \mathcal{S}(G)\}.$$

For this reason, the maximum multiplicity of a graph is often refereed to as the max nullity of the graph.

In a similar manner, the minimum rank of G, denoted $mr(G)$, is defined by

$$mr(G) = \min \{\text{rank}\,(A) : A \in \mathcal{S}(G)\}.$$

Recall that the rank (A) is the dimension of the column space of A. By the rank-nullity theorem, we have

$$mr(G) + M(G) = n.$$

In 2006, at the AIMS Workshop on "Spectra of families of matrices described by graphs, digraphs, and sign pattern", a novel connection between the maximum nullity of a graph and its zero forcing number was presented. Zero forcing was introduced independently by Burgarth and Giovannetti in the control of quantum systems [2], where it was called graph infection. For our purposes, zero forcing is a coloring game on a graph, where an initial set of vertices are colored blue and all other vertices are colored white. Then, the blue vertices force neighboring white vertices blue using a color change rule. There are many color changing rules, but the standard rule is that a blue vertex b can force a white vertex w blue if w is the only white neighbor of b. We will reference this rule as the Z-color change rule to distinguish it from the other variants.

Given a graph $G \in \mathbb{G}$, let $B \subseteq V(G)$ denote the initial set of blue vertices; this is called the initial coloring of G. Then, denote by $B^{[t]}$ the blue vertices after t applications of the Z-color change rule, where $B^{[0]} = B$. Because the graph is finite, there exists a $t^* \geq 0$ for which $B^{[t^*]} = B^{[t^*+k]}$, for all $k \in \mathbb{N}$, i.e., no more changes are possible from applying the Z-color change rule. We reference $B^{[t^*]}$ as the final coloring of B. If $B^{[t^*]} = V(G)$, i.e., the final coloring is all of the vertices of G, then we say that B is a zero forcing set of G. The zero forcing number of G is defined as the cardinality of the smallest zero forcing set of G:

$$Z(G) = \min \left\{ |B| : B^{[t^*]} = V(G) \right\}.$$

For example, consider the application of the Z-color change rule to a path graph on 5 vertices shown in Figure 2. It is clear that $B^{[4]}$ is the final coloring of B since $B^{[5]} = B^{[4]}$. Furthermore, since $B^{[4]}$ encompasses all vertices in the graph, it follows that $B = \{1\}$ is a zero forcing set and, hence, the zero forcing number of this path graph is equal to 1.

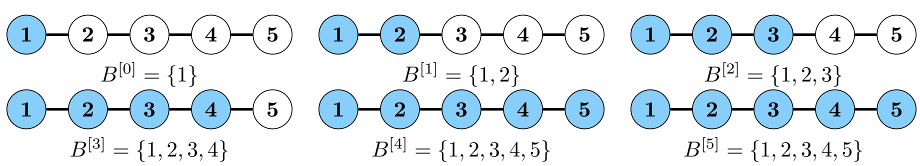

Figure 2: A zero forcing set for a path graph of order 5.

One can easily generalize the above example and conclude that the zero forcing number of all path graphs is equal to 1, i.e., $Z(P_n) = 1$. Note that not every initial coloring of 1 vertex is a zero forcing set of P_n. Indeed, consider the initial coloring of a path graph on 5 vertices shown in Figure 3.

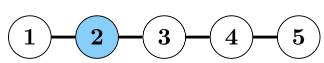

Figure 3: A non zero forcing set for a path graph of order 5.

We are now ready to present the connection between the zero forcing number and the maximum nullity of a graph. First we require the following lemma.

Lemma 1.1. *Suppose that $\alpha \subseteq \{1, 2, \ldots, n\}$ with $|\alpha| < k$ and U is a subspace of $\mathbb{R}^n$ of dimension k. Then, U contains a non-zero vector of $\boldsymbol{v}$ with $\boldsymbol{v}[\alpha] = 0$.*

Proof. Note that if U and V are subspaces of $\mathbb{R}^n$ whose dimensions satisfy

$$\dim V > n - \dim U,$$

then U and V must have a non-trivial intersection, i.e., there is a non-zero vector in $U \cap V$. Now, define

$$V = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v}[\alpha] = 0\},$$

and note that $\dim V = n - |\alpha| > n - k = \dim U$. Hence, there exists a non-zero vector $\mathbf{v} \in U \cap V$, and the result follows. □

Theorem 1.2. *Let $G \in \mathbb{G}$. Then, $M(G) \leq Z(G)$.*

Proof. Let $B \subseteq V(G)$ be a zero forcing set of G, and let $A \in \mathcal{S}(G)$. If $|B| < \text{nullity}\,(A)$, then Lemma 1.1 implies that there exists a non-zero vector $\mathbf{x} \in \text{null}\,(A)$ such that $\mathbf{x}_i = 0$ for all $i \in B$. However, if a vector $\mathbf{x} \in \text{null}\,(A)$ satisfies $\mathbf{x}_i = 0$ for all $i \in B$, then it follows that $\mathbf{x} = \mathbf{0}$.

Indeed, let $j \in V(G) \setminus B$ be any vertex that is forced by some $i \in B$; at least one such vertex must exist since B is a zero forcing set. Then, the equation $A\mathbf{x} = \mathbf{0}$ implies that $a_{ij}\mathbf{x}_j = 0$, which gives $\mathbf{x}_j = 0$ since $a_{ij} \neq 0$. If necessary, we can repeat the above argument on the zero forcing set $B \cup \{j\}$, and perhaps again, until all vertices in $V(G)$ are accounted for at which point it follows that $\mathbf{x} = \mathbf{0}$.

This contradiction with Lemma 1.1 implies that $|B| \geq \text{nullity}\,(A)$. Since B and A are arbitrary, it follows that $Z(G) \geq M(G)$. □

Theorem 1.2 can be used to determine the maximum nullity of a graph via its zero forcing number. For instance, we know that the path graph satisfies $Z(P_n) = 1$. Therefore, Theorem 1.2 implies that $M(P_n) \leq 1$. Since there is always a singular matrix in $\mathcal{S}(G)$, for any graph G, it follows that $M(P_n) = 1$.

As a more interesting example, consider the star graph in Figure 1. Note that $B = \{1, 2, 3, 4, 5\}$ is a zero-forcing set of minimal size. This observation can easily be generalized and we conclude that the zero forcing number of the star graph $K_{1,n-1}$ of order n satisfies $Z(K_{1,n-1}) = n - 2$. Therefore, the maximum nullity and minimum rank satisfy

$$M(K_{1,n-1}) \leq n - 2 \text{ and } mr(K_{1,n-1}) \geq 2,$$

respectively. Furthermore, the adjacency matrix of the star graph $K_{1,n-1}$:

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}.$$

is a $n \times n$ matrix in $\mathcal{S}(K_{1,n-1})$ that satisfies $\text{rank}\,(A) = 2$. Therefore, $M(K_{1,n-1}) = n - 2$ and $mr(K_{1,n-2}) = 2$.

2 Positive Semidefinite Zero Forcing

As mentioned in Section 1, there are many color change rules that can be used in the coloring game of zero forcing. The so-called standard zero forcing rule provides an upper bound on the maximum nullity of a graph via Theorem 1.2. The positive semidefinite zero forcing rule provides an upper bound on the maximum positive semidefinite nullity:

$$M_+(G) = \max \{\text{nullity}\,(A) : A \in S(G),\ \forall \lambda \in \sigma(A),\ \lambda \geq 0\}.$$

Note that the additional condition on the eigenvalues of A, i.e., $\lambda \geq 0$ for all $\lambda \in \sigma(A)$, implies that A must be positive semidefinite.

Given a graph $G \in \mathbb{G}$, let $B \subseteq V(G)$ denote the initial coloring of G. Let $G \setminus B$ denote the resulting graph after the blue vertices (and adjacent edges) have been removed, and let $W_1, \ldots, W_k$ denote the vertex sets of the connected components of $G - B$. If $b \in B$ and $w \in W_i$ is the only white neighbor of b in $G[W_i \cup B]$, for some $i \in \{1, \ldots, k\}$, then the positive semidefinite zero forcing rule says that b can force w blue.

As with the standard zero forcing rule, there exists a $t^* \geq 0$ such that $B^{[t^*]} = B^{[t^*+k]}$ for all $k \in \mathbb{N}$, and we say that B is a positive semidefinite zero forcing set if the final coloring satisfies $B^{[t^*]} = V(G)$. The positive semidefinite zero forcing number of G is defined as the cardinality of the smallest positive semidefinite zero forcing set of G:

$$Z_+(G) = \min \left\{ |B| : B^{[t^*]} = V(G) \right\}.$$

Theorem 2.1. *Let $G \in \mathbb{G}$. Then, $M_+(G) \leq Z_+(G)$.*

Proof. Let $A \in S(G)$ be positive semidefinite with nullity $(A) = M_+(G)$ and consider a non-zero vector $\mathbf{x} \in A$, which is guaranteed to exist since the max nullity is at least 1. Let $B \subset V(G)$ denote the set of vertices such that $\mathbf{x}_i = 0$, and let $W_1, \ldots, W_k$ denote the vertex sets of the connected components of $G \setminus B$. For every $i \in \{1, \ldots, k\}$, we claim that no $w \in W_i$ can be the only neighbor of a $b \in B$ in $G[W_i \cup B]$.

Indeed, note that we can write A, possibly after re-ordering the vertices of G, as follows

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 & C_1 \\ 0 & A_2 & \cdots & 0 & C_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_k & C_k \\ C_1 & C_2 & \cdots & C_k & D \end{bmatrix},$$

where $A_1, \ldots, A_k$ corresponds to the vertex sets $W_1, \ldots, W_k$, respectively, D corresponds to the vertex set B, and C_i corresponds to the edges between W_i and B, for all $i \in \{1, \ldots, k\}$.

Now, consider the partition $\mathbf{x} = [\mathbf{x}_1^T, \ldots, \mathbf{x}_k^T, \mathbf{0}^T]$, where $\mathbf{x}_i$ has size equal to $|W_i|$, for all $i \in \{1, \ldots, k\}$, and the zero vector $\mathbf{0}$ has size equal to $|B|$. Note that all entries of $\mathbf{x}_1, \ldots, \mathbf{x}_k$ must be non-zero. Moreover, the equation $A\mathbf{x} = \mathbf{0}$ implies that $A_i\mathbf{x}_i = \mathbf{0}$ for all $i \in \{1, \ldots, k\}$.

The column inclusion property for Hermitian positive semidefinite matrices [4], guarantees that each column in C_i is the span of the columns in A_i, for all $i \in \{1, \ldots, k\}$. Therefore, for each $i \in \{1, \ldots, k\}$, there exists a matrix Y_i such that $C_i = A_i Y_i$. Since A is symmetric, it follows that $C_i = Y_i^T A_i$; hence, $C_i\mathbf{x}_i = \mathbf{0}$ for all $i \in \{1, \ldots, k\}$. But if $w \in W_i$ were the only white neighbor of $b \in B$ in $G[W_i \cup B]$, then $C_i\mathbf{x}_i \neq \mathbf{0}$, which is a contradiction.

This contradiction implies that B is not a positive semidefinite zero forcing set of G. Furthermore, it follows that there is no non-trivial $\mathbf{x} \in \text{null}\,(A)$ with $\mathbf{x}_i = 0$ for all i in a positive semidefinite zero forcing set of G. Therefore, by Lemma 1.1, we have $M_+(G) \leq Z_+(G)$. □

As an example of how Theorem 2.1 could be applied, consider the star graph $K_{1,n-1}$. Note that the "center" vertex constitutes a positive semidefinite zero forcing set of $K_{1,n-1}$. Therefore,

$$Z_+(K_{1,n-1}) = 1$$

and it follows that $M_+(K_{1,n-1}) = 1$.

Finally, note that every standard zero forcing set of a graph $G \in \mathbb{G}$ constitutes a positive semidefinite zero forcing set; hence, $Z_+(G) \leq Z(G)$. Furthermore, it is also clear that $M_+(G) \leq M(G)$.

Question 2.2. *Is there a relationship between $Z_+(G)$ and $M(G)$, i.e., does the following hold*

$$Z_+(G) \leq M(G)$$

for all (or some) graphs $G \in \mathbb{G}$.

3 Computation of Max Nullity and Zero-Forcing

4 Colin de Verdiére Type Parameters

References

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