NORTH-HOLLAND

Minimum-Rank Matrices With Prescribed Graph

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ABSTRACT

We study properties of real symmetric matrices with prescribed graph and lowest possible rank. We concentrate on the case where the graph is a tree and give an algorithm for computing the minimum possible rank in this case. A more general problem is also considered.

1. INTRODUCTION

Given an $n \times n$ real symmetric matrix $A = [a_{ij}]$, we may define an undirected graph $\Gamma(A)$ on n vertices $1, 2, \dots, n$ by including the unordered pair (i, j), i.e., the edge joining vertex i to vertex j in the edge set, if and only if $a_{ij} \neq 0$. [We always suppress the possible loops (i, i).] This graph, commonly called the (undirected) graph $\Gamma(A)$ of A, is useful for describing the zero-nonzero pattern of the off-diagonal portion of a matrix. Note that no information on the diagonal entries of A is contained in $\Gamma(A)$. See pp. 168 and 357 in [10] for further discussion of this notation.

Given a graph G on n vertices, we define $m(G)$ by

$$m(G) = \min\{\text{rank } A : A = A^T, \Gamma(A) = G\}.$$

The content of this paper is a study of the function m and matrices which attain the attendant minimum. Study of similar questions involving structured

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matrices (i.e., the graph of the matrix is specified) has become a common theme in matrix theory. Indeed, [8] and [7] are works on the eigenvalue possibilities of A and a principle submatrix, subject to $\Gamma(A)$ being a specified tree. The present work is related, in that $n - m(\cdot)$ may be thought of as the maximum multiplicity allowed for an eigenvalue. The groundbreaking work [6] deals directly with multiple eigenvalues of matrices A with $\Gamma(A)$ a tree, as do [1] and [5]. [2] gives some physical interpretation to [6]. There also may be some relation to the problem of minimizing the rank of a matrix by changing only the diagonal entries, since specifying the graph may be thought of as specifying some, typically many, of the off-diagonal entries to be zero, restricting the remaining off-diagonal entries to be nonzero, and leaving the diagonal entries entirely free. This last type of rank minimization problem is considered in [3] and [4]. Since we may view this problem as maximizing the algebraic multiplicity of the eigenvalue 0, the work of [9] needs to be mentioned. It follows directly from this work that if complex entries are allowed on the diagonal of A, then there is no restriction on the algebraic multiplicity, aside from the dimension of the matrix, regardless of the specified graph.

We are able to compute m via a recursive algorithm for the case of $G = T$, i.e., when the graph in question is a tree. This method of computation raises some graph-theoretic issues; in particular it is a corollary to this computation that a certain recursion relation defined on trees has the function m as a unique solution.

We also devote some effort to properties of the matrices A for which $\Gamma(A) = G$ and rank $A = m(G)$, in both the case of a general G and the case $G = T$. With respect to the latter case, there is much already known about matrices A with $\Gamma(A)$ a tree and rank $A \leqslant n - 2$, although the results are stated in terms of multiple eigenvalues. For instance, in [5], it is proven that if rank $A = k \leqslant n - 2$, then there exists an index p such that rank $A_p = k - 2$ [here, and henceforward, A_p denotes the $(n-1) \times (n-1)$ submatrix of A obtained by deleting row and column p from A]. This is quite a remarkable result in view of the fact that without tree structure, rank $A_p = k$ may hold for all p. Another result of this type is contained in [1], which states that under the same hypotheses on A, there again exists an index p, this time with the property that if $x \in R^n$ satisfies $Ax = 0$, then the pth entry of x is zero. We find that for matrices A satisfying $\Gamma(A) = G$, rank $A = m(G)$, the set of p having the first property is the same as the set of p having the second property. We are also able to provide more information on just which indices p may, or must, have these properties in the case that G is a tree, thus extending the results of [5] and [1] for minimum-rank matrices.

We shall close with a discussion of a more general problem, that of determining which sets $\{s_1, \dots, s_k\}$ of positive integers summing to n may occur as the algebraic multiplicities of the eigenvalues of an $n \times n$ real symmetric matrix A with $\Gamma(A)$ a specified tree T.

2. MINIMUM-RANK STRUCTURED MATRICES

Let A denote an $n \times n$ real symmetric matrix throughout. Let G be an undirected graph on n vertices. $\Gamma(A) = G$ means that $a_{ij} \neq 0$ if and only if (i, j) is an edge of the graph G. Note that the vertices of G are implicitly labeled in coordination with the rows (columns) of A by the statement $\Gamma(A) = G$. As defined above, $m(G)$ denotes the smallest value of rank A over all A with graph G. In this section we will give some elementary results concerning $m(\cdot)$ and matrices A that attain the defining minimum, which will be useful later. We also supply as an example the computation of $m(G)$ for G a simple cycle.

PROPOSITION 2.1.

(1) *If G is connected with k vertices, $1 \leqslant k \leqslant 2$, then $m(G) = k - 1$.*

(2) *If G is not connected, with connected components G^i, $i = 1, \dots, k$, then $m(G) = m(G^1) + \cdots + m(G^k)$.*

(3) *If G' is obtained from G by deleting a single vertex and each incident edge, then $m(G') + 2 \geqslant m(G) \geqslant m(G')$.*

(4) *If G' is obtained from G by deleting a single edge from G, then $m(G') + 1 \geqslant m(G) \geqslant m(G') - 1$.*

Proof. We provide a proof of the fourth statement. The first three statements are less involved and left to the reader. Let G be a graph on n vertices, labelled by the integers $1, \dots, n$. Select an edge (i, j) in G, and let G' denote the graph obtained from G by deleting this edge. Let A be an $n \times n$ symmetric matrix satisfying $\Gamma(A) = G$ and rank $A = m(G)$. Let $B = -a_{ij}(e_i + e_j)(e_i + e_j)^T$ (e_i denotes the ith standard basis vector). We then have $\Gamma(A + B) = G'$ and $m(G') \leqslant \operatorname{rank}(A + B) \leqslant \operatorname{rank} A + \operatorname{rank} B = \operatorname{rank} A + 1 = m(G) + 1$. This gives $m(G) \geqslant m(G') - 1$. To prove the second inequality contained in statement (4), let A satisfy $\Gamma(A) = G'$ and rank $A = m(G')$. Set $B = (e_i + e_j)(e_i + e_j)^T$. We then have $\Gamma(A + B) = G$ and $m(G) \leqslant \operatorname{rank}(A + B) \leqslant \operatorname{rank} A + \operatorname{rank} B = \operatorname{rank} A + 1 = m(G') + 1$. This completes the proof of statement (4). ∎

PROPOSITION 2.2. *Suppose that* $\Gamma(A) = G$ *and* $\operatorname{rank} A = m(G) = k$. *Let* A_p *denote the matrix obtained by deleting the* p*th row and column from* A. *Then* $\operatorname{rank} A_p = k$ *or* $\operatorname{rank} A_p = k - 2$ *for all* p, *i.e.*, $\operatorname{rank} A_p = k - 1$ *is not possible.*

Proof. Assume that $\operatorname{rank} A_p = k - 1$, and for notational expediency assume also that $p = 1$. Denote

$$A = \begin{bmatrix} a & b^T \\ b & A_1 \end{bmatrix}.$$

Since we have

$$k = \operatorname{rank} A \geqslant \operatorname{rank}\begin{bmatrix} b^T \\ A_1 \end{bmatrix} \geqslant \operatorname{rank} A_1 = k - 1, \tag{2.1}$$

one of the inequalities in (2.1) is an equality, while the other is not. The second inequality is an equality if and only if $A_1 x = b$ has a solution, while the first is an equality if and only if

$$\begin{bmatrix} b^T \\ A_1 \end{bmatrix} x = \begin{bmatrix} a \\ b \end{bmatrix}$$

has a solution. Thus, the second inequality must be an equality, while the first is a strict inequality. Now, let y solve $A_1 x = b$, and set $a' = y^T b$. Form A' by replacing a with a'. Clearly, A' has G as its graph, and inspection of (2.1) with A' in place of A reveals that $\operatorname{rank} A' = k - 1$, which is a contradiction. ∎

PROPOSITION 2.3. *Let* G *be a given graph on* n *vertices. Let* A *be an* $n \times n$ *real symmetric matrix satisfying* $\Gamma(A) = G$ *and* $\operatorname{rank} A = m(G)$. *Let* $p \in \{1, \dots, n\}$ *be given. The following statements are equivalent*:

(1) $Ax = e_p$ *has a solution* (e_p *denotes the* p*th standard basis vector*).
(2) $\operatorname{rank} A = \operatorname{rank} A_p + 2$
(3) *If* $Ax = 0$, *then the* p*th entry of* x *is* 0.

Furthermore, the number of such indices p *does not exceed* $m(G)$, *and does not exceed* $m(G) - 1$ *if* G *is connected.*

Proof. First, we note that for any real symmetric matrix, (1) and (3) are equivalent. Assuming the equivalences have been established, the "furthermore" statement holds, since if $p_1, \dots, p_j$ have the property described, then the column space of A contains $e_{p_1}, \dots, e_{p_j}$, a j-dimensional subspace. If $j = \operatorname{rank} A$, then all rows of A outside the index set $\{p_1, \dots, p_j\}$ must be zero, which implies that A is reducible; hence G is disconnected.

Now, we complete the proof of the equivalence of the three statements. First, we show that (2) implies (1). Assume for notational ease that $p = 1$, and partition A as

$$A = \begin{bmatrix} a & b^T \\ b & A_1 \end{bmatrix}.$$

Use of the argument in the proof of Proposition 2.2 shows that $A_1 x = b$ has no solution. Decompose $b = b_1 + b_2$, where $b_1 \in \text{RowSpace}(A_1)$ and $b_2 \in \text{NullSpace}(A_1)$. We have $b_2 \neq 0$, since $A_1 x = b$ has no solution. Now, let

$$y = \begin{bmatrix} 0 \\ \dfrac{1}{b_2^T b_2} b_2 \end{bmatrix}.$$

The claim then follows from this construction, since $Ay = e_1$.

Now, we demonstrate that (3) implies (2). Again, let $p = 1$, and use the partition of A previously described. We have that $x \in \text{NullSpace}(A)$ implies that

$$x = \begin{bmatrix} 0 \\ v \end{bmatrix},$$

where $v \in R^{n-1}$. Define

$$S(A) = \left\{ v \in R^{n-1} : \begin{bmatrix} 0 \\ v \end{bmatrix} \in \text{NullSpace(A)} \right\}.$$

We clearly have $S(A) \subseteq \text{NullSpace}(A_1)$. Hence, $n - \operatorname{rank} A = \dim S(A) \leqslant \dim \text{NullSpace}(A_1) = n - 1 - \operatorname{rank} A_1$; simplifying, this becomes $\operatorname{rank} A \geqslant \operatorname{rank} A_1 + 1$. Application of Proposition 2.2 finishes the proof. ∎

Now for an example:

EXAMPLE 2.4. If C is the simple cycle on n vertices, $n \geqslant 3$, then $m(C) = n - 2$.

We'll need the following lemma:

LEMMA 2.5. *Let J be a $k \times k$ real symmetric singular irreducible tridiagonal matrix. Then* $\operatorname{rank} J = k - 1$, *and $Jx = e_1$ has no solution.*

If A, real symmetric $n \times n$, satisfies $\Gamma(A) = C$, then we may assume, perhaps after application of a permutation similarity, that A is of the form

$$A = \begin{bmatrix} a & b^T \\ b & J \end{bmatrix}, \tag{2.2}$$

where J is an $(n-1) \times (n-1)$ irreducible tridiagonal matrix and $b = \alpha e_1 + \beta e_{n-1} \in R^{n-1}$, with both α and β nonzero. By Lemma 2.5, rank $J \geqslant n - 2$, so $m(C) \geqslant n - 2$. Now, let J be an arbitrary $(n-1) \times (n-1)$ irreducible tridiagonal symmetric singular matrix. Set $S = \operatorname{Span}\{e_2, \dots, e_{n-2}\} \subseteq R^{n-1}$. Since $\dim \operatorname{ColumnSpace}(J) = n - 2$ and $\dim S^{\perp} = 2$, there exists $b \neq 0$ in the intersection of these two subspaces. Since $Jx = e_1$ and $Jx = e_{n-1}$ are both not solvable (the former is part of Lemma 2.5; the latter is also easily verified), this vector b must be of the form $b = \alpha e_1 + \beta e_{n-1}$, with both α and β nonzero. If we then set $a = b^T y$, where y solves $Jx = b$, and construct A from a, b, J according to (2.2), we have $\Gamma(A) = C$ and rank $A = n - 2$.

In the following section on computing $m(G)$ for G a tree, the existence of a vertex p such that

$$m(G) = m(G \setminus \{p\}) + 2 \tag{2.3}$$

is pivotal. Consideration of Example 2.4 shows that for simple cycles, no such vertex exists. However, every vertex p satisfies

$$m(C) = m(C \setminus \{p\}). \tag{2.4}$$

This suggests that a possible approach to the computation of $m(G)$ for general G would be identification of vertices satisfying (2.3) and (2.4), provided that one or the other type of vertex could be shown to exist.

3. $G = T$

In this section, we specialize to the case where the graph G is a tree, and shall indicate this specialization by use of the symbol T. We first provide an algorithm for computing $m(T)$, and we then turn attention to the existence and location of the special indices described in the previous section.

Let T be a tree on n vertices ($n \geqslant 3$), labeled by the integers $1, \dots, n$, and let p be a vertex of T. Upon deletion of p and all edges incident to p from T, one obtains an acyclic, possibly disconnected graph $T_p = T \setminus \{p\}$. Denote the connected components of T_p by $T_p^1, \dots, T_p^k$. If $j \geqslant 2$ of the connected components are simple paths (one or more vertices) which were connected to p in T through an endpoint, then we shall call p an *appropriate vertex of* T. Equivalently, we could say that $j \geqslant 2$ of these connected components T_p^i have the property that every vertex of T_p^i has degree 2 or less, this being the degree in T prior to the deletion of p. We shall call the integer j the *index of appropriateness*.

The importance of appropriate vertices will become clear with Theorem 3.2 and its Corollary below. First, we need to be sure that such vertices exist.

LEMMA 3.1. *Every tree T with $n \geqslant 3$ vertices has an appropriate vertex.*

Proof. The truth of the assertion is clear when $n = 3$. Now, let $n \geqslant 3$ be arbitrary and let T be any tree on n vertices. Delete an end vertex u from T, letting x denote the vertex to which u was adjacent, to form a tree on $n - 1$ vertices T'. If we assume inductively that T' has an appropriate vertex, say v, then v is either appropriate in T or not. If not, then x is appropriate in T. ∎

THEOREM 3.2. *Let T be a tree on three or more vertices. Let p be an appropriate vertex of T of index j. If $j = 2$, then there exists A such that $\Gamma(A) = T$ and $m(T) = \operatorname{rank} A = \operatorname{rank} A_p + 2$. If $j > 2$, then every A such that $\Gamma(A) = T$ and $\operatorname{rank} A = m(T)$ has the property that $\operatorname{rank} A = \operatorname{rank} A_p + 2$.*

Proof. To simplify notation, we assume that $p = 1$ and that the remaining vertices of T are labelled so that with $\Gamma(B) = T$, we have the following block form of B:

$$B = \begin{bmatrix} a & b_1^T & \cdots & b_j^T & b_{j+1}^T & \cdots & b_k^T \\ b_1 & B^1 & & & & & \\ \vdots & & \ddots & & & & \\ b_j & & & B^j & & & \\ b_{j+1} & & & & B^{j+1} & & \\ \vdots & & & & & \ddots & \\ b_k & & & & & & B^k \end{bmatrix}, \tag{3.1}$$

where each b_i is a nonzero multiple of e_1 in the proper dimension, and $\Gamma(B^i) = T_1^i$ according to the previously described decomposition of T. Now, let us assume that rank $B = m(T)$, and further that $T_1^1, \dots, T_1^j$, $j \geqslant 2$, are the simple paths required by the appropriateness of vertex 1 in T. By application of Proposition 2.2, we need only consider the case of rank $B =$ rank B_1 to complete the proof. By the technique used to prove Proposition 2.2 and by Lemma 2.5, we see that each of $B^1, \dots, B^j$ is nonsingular. We then perturb the diagonal of B^i to form a singular matrix A^i of the same graph for each $i = 1, \dots, j$. Replace each of the B^i with the corresponding singular A^i to form the matrix A. Clearly, $\Gamma(A) = \Gamma(B)$. Computing ranks, we see that rank $A =$ rank $A_1 + 2 =$ rank $B_1 - j + 2 =$ rank $B - j + 2$. The conclusions of the theorem now follow. ∎

The following corollary gives a recursive formula for $m(T)$, making it quite easy to compute for a given tree T:

COROLLARY 3.3. *Let T be a tree and p an appropriate vertex of T, and let $T_p^1, \dots, T_p^k$ be the connected components of $T \setminus \{p\}$. Then*

$$m(T) = m\left(T_p^1\right) + \cdots + m\left(T_p^k\right) + 2. \tag{3.2}$$

Proof. By Proposition 2.1, we have $m(T) \leqslant m(T_p^1) + \cdots + m(T_p^k) + 2$ for every vertex p of T. Now, assume that p is appropriate and that A

satisfies $\Gamma(A) = T$ and rank $A = m(T) =$ rank $A_p + 2$ as provided for by Theorem 3.2. This gives, using the block form of (3.1),

$$m(T) = \text{rank } A_p^1 + \cdots + \text{rank } A_p^k + 2 \geqslant m\left(T_p^1\right) + \cdots + m\left(T_p^k\right) + 2.$$

■

This result is somewhat surprising, since typically trees have many different appropriate vertices, and hence the decomposition (3.2) may be performed in many different ways. However, $m(\cdot)$ is a well-defined function; hence the choice of appropriate vertex does not affect the value of the right side of (3.2). Complete induction on the number of vertices of T yields the following purely graph-theoretical result:

COROLLARY 3.4. *The recursion relation defined on trees by* $f(T) = f(T_p^1) + \cdots + f(T_p^k) + 2$, *for* p *appropriate, with "initial" condition* $f(T) = n - 1$ *when* T *has* n *vertices,* $n = 1, 2$, *has a unique solution,* $m(\cdot)$.

In [5] it is shown that if A is $n \times n$ real symmetric with graph $\Gamma(A) = T$ a tree and rank $A \leqslant n - 2$, then there exists p such that rank $A =$ rank $A_p + 2$. We now provide information on these indices p for the case that rank $A = m(T)$, beyond that appearing in Theorem 3.2, or in [1] or [5].

THEOREM 3.5. *Let* T *be a tree; let* A *be real symmetric satisfying* $\Gamma(A) = T$ *and* rank $A = m(T)$. *Let* q *be a vertex of* T *with degree* $k \geqslant 3$. *Then either* rank $A =$ rank $A_q + 2$ *or there exist* $k - 2$ *vertices* r *adjacent to* q *such that* rank $A =$ rank $A_r + 2$.

Proof. Let A satisfy $\Gamma(A) = T$, rank $A = m(T)$, and assume that the vertices of T are labeled so that vertex 1 has degree at least 3 (i.e., assume that $q = 1$). Of course, if rank $A =$ rank $A_1 + 2$, there is nothing to show. Thus, assume rank $A =$ rank A_1. Further assume that the labeling of T is such that

$$A = \begin{bmatrix} a & b_1^T & b_2^T & \cdots & b_k^T \\ b_1 & A_1^1 & & & \\ b_2 & & A_1^2 & & \\ \vdots & & & \ddots & \\ b_k & & & & A_1^k \end{bmatrix}, \tag{3.3}$$

where, as in (3.1), b_i is a nonzero multiple of e_1, and A_1^i, $i = 1, \dots, k$, correspond to the $k \geqslant 3$ connected components of $T \setminus \{1\}$.

If vertex 1 is appropriate of index 3 or larger, then application of Theorem 3.2 gives the desired result. Thus, assume that if vertex 1 is appropriate, then the index of appropriateness is 2. This assumption requires that at least $k - 2$ of the irreducible blocks A_1^j not be tridiagonal or 1×1. It also must be the case that rank $A_1^j = m(T_1^j)$ for all except at most two indices $j \in \{1, \dots, k\}$, due to the minimal rank of A, and furthermore, due to Lemma 2.5, A_1^j must be nonsingular if A_1^j is tridiagonal or a scalar. For convenience, reorder the blocks A_1^j and subgraphs T_1^j so that A_1^1 is such a nontridiagonal, nonscalar block satisfying rank $A_1^1 = m(T_1^1)$. There are at least $k - 2$ choices for the block A_1^1. Now, decompose A_1^1 in the style of (3.3), using the notation $A_1^1 = B$:

$$A_1^1 = B = \begin{bmatrix} b & c_1^T & c_2^T & \cdots & c_s^T \\ c_1 & B_1^1 & & & \\ c_2 & & B_1^2 & & \\ \vdots & & & \ddots & \\ c_s & & & & B_1^s \end{bmatrix}. \tag{3.4}$$

Application of the argument used to prove Proposition 2.2 reveals that $Bx = e_1$ is solvable; hence by this same proposition, rank B = rank $B_1 + 2$; hence for at least one $i \in \{1, \dots, s\}$, $B_1^i x = e_1$ is not solvable. Now, consider a decomposition of A (and T) of the type (3.3), except with vertex 2 of T in the key position formerly occupied by vertex 1. It is not hard to see then that each of the blocks $B_1^1, \dots, B_1^m$ will occur as a diagonal block of A_2, together with one additional block. (This additional block contains vertex 1 of T.) It then follows from Proposition 2.2 that rank A = rank $A_2 + 2$. ∎

4. MULTIPLICITY SETS

Let $\{s_1, \dots, s_k\}$ be a set of positive integers which sum to n. Let T be a tree on n vertices. Does there exist a real symmetric $n \times n$ matrix A such that $\Gamma(A) = T$ and the spectrum of A is $\{\lambda_1, \dots, \lambda_k\}$, where the algebraic multiplicity of λ_i is s_i, $i = 1, \dots, k$? The previous section characterizes $\max\{s_i : i = 1, \dots, k\}$. According to [5], at least two of the s_i are one. Using the techniques of the previous section, the possible *multiplicity sets* $\{s_1, \dots, s_k\}$ can be determined for trees T that have at most one vertex with degree greater that two. These graphs are of the form of a *star*, a central vertex with rays of various lengths emanating. When each ray is of length one, the matrix A can be taken to be a bordered diagonal matrix. In the general star case, the matrix A is permutationally similar to the form (3.3), with every diagonal block a scalar or an irreducible tridiagonal matrix.

We shall state the results for these special trees, omitting the details of proof which are not particularly illuminating. We shall also consider carefully a particular tree T on eight vertices, the one with two vertices of degree 4 and six of degree 1. This example is important, as it appears to rule out a majorization-type result suggested by the star case.

EXAMPLE 4.1. Let A be an $n \times n$ real symmetric bordered diagonal matrix:

$$A = \begin{bmatrix} a & & b^T & \\ & \lambda_1 I_{n_1} & & \\ b & & \ddots & \\ & & & \lambda_k I_{n_k} \end{bmatrix},$$

where b is an entrywise nonzero vector, $a \in R$ is arbitrary, and $n_1, \dots, n_k$ are positive integers summing to $n - 1$. The eigenvalues of A are the numbers λ_i, with multiplicity $n_i - 1$, $i = 1, \dots, k$ (suppress λ_i such that $n_i - 1 = 0$), together with $k + 1$ simple eigenvalues. Conclude that $\{s_1, \dots, s_k, 1, \dots, 1\}$, $s_i > 1$, can occur as a multiplicity set for T having only one vertex of degree greater than 1 if and only if $s_1 + \cdots + s_k \leqslant n - 1 - k$.

EXAMPLE 4.2. Let A be a real symmetric matrix of the form

$$A = \begin{bmatrix} a & b_1^T & b_2^T & \cdots & b_k^T \\ b_1 & J^1 & & & \\ b_2 & & J^2 & & \\ \vdots & & & \ddots & \\ b_k & & & & J^k \end{bmatrix},$$

where each b_i is a nonzero multiple of e_1, and each J^i is a scalar or an irreducible symmetric tridiagonal matrix. Let j_i denote the size of J^i. Since the eigenvalues of each J^i are simple, the eigenvalues of $A_1 = J^1 \oplus \cdots \oplus J^k$, repeated according to multiplicities, can be grouped into subsets $\sigma_1, \dots, \sigma_k$,

$|\sigma_i| = j_i$, where each σ_i contains only distinct eigenvalues. Let $s_1, \dots, s_r$ denote the multiplicities of the eigenvalues of A_1. Applying Lemma 2.5, we see then that the multiplicity set for A must be $\{s_1 - 1, \dots, s_r - 1, 1, \dots, 1\}$, where occurrences of $s_i - 1 = 0$ are suppressed and the ending tail of 1's has length chosen to make this set sum to n.

This characterization can be restated in terms of additive majorization of $\{s_1, \dots, s_r\}$ by the dual of $\{j_1, \dots, j_k\}$ as a partition of $n - 1$.

EXAMPLE 4.3. Let T be the tree on eight vertices with two vertices of degree 4 and six vertices of degree 1. Then $\{4, 1, 1, 1, 1\}$ and $\{3, 3, 1, 1\}$ are possible multiplicity sets, but $\{4, 2, 1, 1\}$ is not.

We first analyze the multiplicity sets containing 4. Assume that the multiplicity-4 eigenvalue is 0. We compute $m(T) = 4$, so 0 with multiplicity $8 - 4 = 4$ is possible. Let A be a real symmetric matrix with $\Gamma(A) = T$ and rank $A = 4$. By application of permutation similarity and application of Theorem 3.5, we may assume that

$$A = \begin{bmatrix} \bigstar & x & x & x & x & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & \bigstar & x & x & x \\ 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \end{bmatrix},$$

where x indicates a nonzero entry and $\bigstar$ indicates an arbitrary entry. Setting $Q = 1 \oplus H_1 \oplus 1 \oplus H_2$ for appropriately chosen 3×3 Householder reflections H_1, H_2, we obtain

$$Q^TAQ = \begin{bmatrix} \bigstar & x & 0 & 0 & x & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & \bigstar & x & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which is permutationally similar to

$$\begin{bmatrix} 0 & x & 0 & 0 \\ x & \bigstar & x & 0 \\ 0 & x & \bigstar & x \\ 0 & 0 & x & 0 \end{bmatrix} \oplus 0_{4\times 4}.$$

We conclude from this that the remaining nonzero eigenvalues of A are the eigenvalues of an irreducible tridiagonal matrix, and hence are distinct. This shows that $\{4, 1, 1, 1, 1\}$ is possible and that $\{4, 2, 1, 1\}$ is not.

Let

$$B = \begin{bmatrix} 1 & a & a & a & 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & a & a & a \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \end{bmatrix},$$

where $a = 1/\sqrt{3}$. The reader may verify that B has eigenvalues 0 and 1 with multiplicity 3 and two simple eigenvalues. Thus, $\{3, 3, 1, 1\}$ is possible.

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