Computational methods and heuristics for zero forcing

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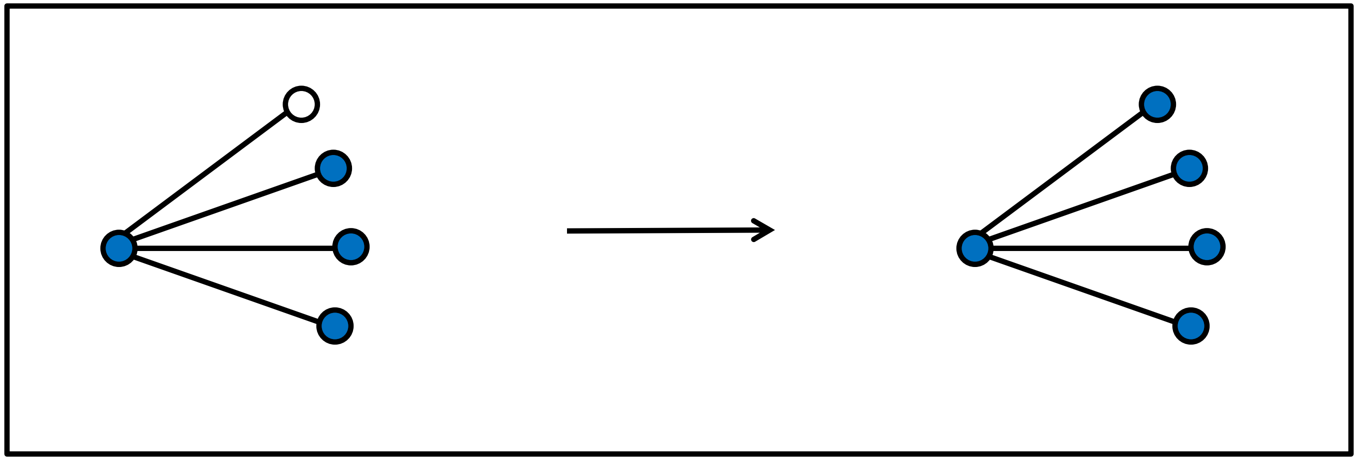
Mathematics and Statistics

Zero forcing



Blue vertex with exactly one white neighbor turns that neighbor blue

Related problems

- Power network monitoring (Haynes et al. 2002)
- Target set selection (Kempe et al. 2003)
- Zero forcing / minimum rank (AIM Group 2008)
- Quantum control (Burgarth et al. 2007)
- Fast mixed network searching (Yang 2013)
- k-forcing (Amos et al. 2015)
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All of these problems are computationally difficult

Computational methods

- Guess-and-check / brute force (Everyone)
- Dynamic programming (Butler et al. 2014)
- Integer programming and branch-and-bound (Aazami 2010, Ackerman et al. 2010, Ben-Zwi et al. 2011, Mahaei and Hagh 2012, Chiang et al. 2013, Agra et al. 2019)
- Boolean Satisfiability (Brimkov et al. 2020)
- Heuristics (Agra et al. 2019, Brimkov et al. 2020)

Dynamic programming

Wavefront Algorithm

Data: Graph $G = (V, E)$
Result: Zero forcing number of G
$\mathcal{C} \leftarrow \{(\emptyset, 0)\}$;
for $R \in [n]$ **do**
  for $(S, r) \in \mathcal{C}$ **do**
    for $v \in V$ **do**
      $S' \leftarrow cl(S \cup N[v])$;
      $r' \leftarrow r + |\{v\} \setminus S| + \max\{|N(v) \setminus S| - 1, 0\}$;
      if $r' \leq R$ **and** $(S', i) \notin \mathcal{C}$ **for** $i \leq R$ **then**
        $\mathcal{C} \leftarrow \mathcal{C} \cup \{(S', r')\}$;
        if $S' = V$ **then return** r';

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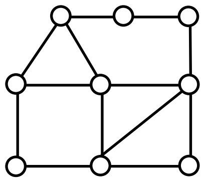

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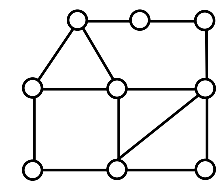

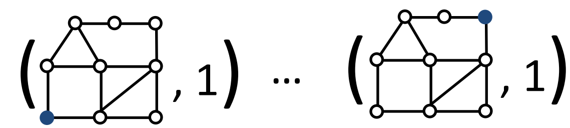

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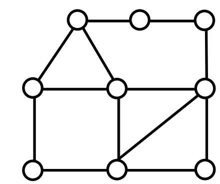

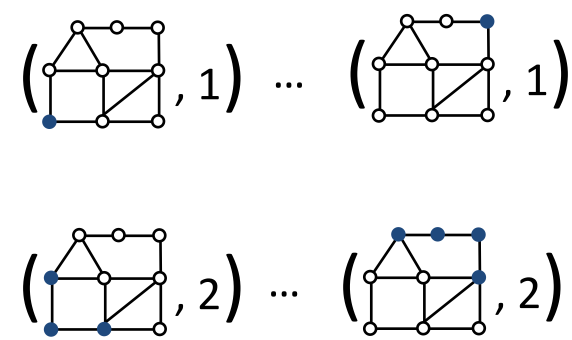

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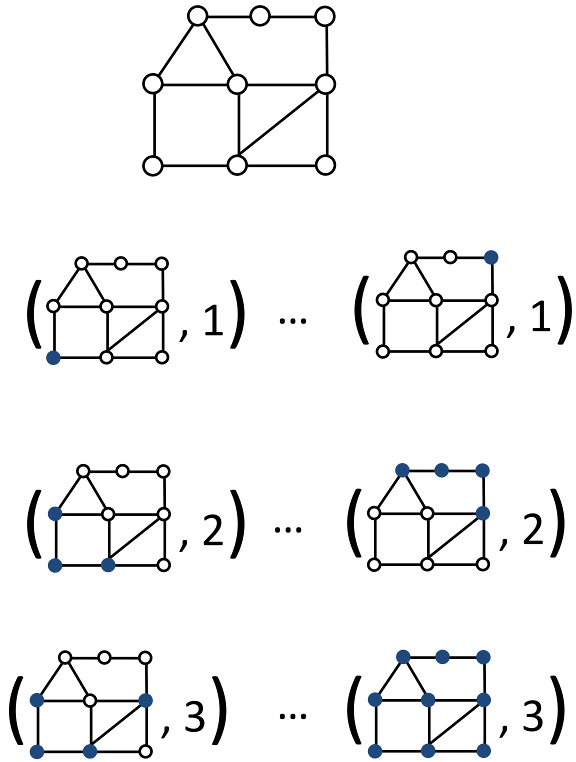

IP formulations

Compact formulation

$$
\begin{aligned}
\min \quad & \sum_{v \in V} s_v \\
\text{s.t.:} \quad & s_v + \sum_{e \in \delta^-(v)} y_e = 1 \qquad \forall v \in V \\
& x_u - x_v + (T+1) y_e \leq T \qquad \forall e = (u,v) \in E \\
& x_w - x_v + (T+1) y_e \leq T \qquad \forall e = (u,v) \in E, \forall w \in N(u) \backslash \{v\} \\
& x \in \{0, \ldots, T\}^n, s \in \{0,1\}^n, y \in \{0,1\}^m
\end{aligned}
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Noncompact formulation

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\min \quad & \sum_{v \in V} s_v \\
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\text{s.t.:} \quad & \sum_{v \in V} x_v \geq 1 \\
& x_w - x_v + \sum_{a \in N(w) \setminus \{v\}} x_a \geq 0 && \forall (v,w) \text{ with } v \in V, w \in N(v) \\
& x_v = 0 \qquad \forall v \in cl(S) && x \in \{0,1\}^n
\end{aligned}
$$

Boolean SAT formulation

$$\min \quad \sum_{v \in V} (\deg(v) z_v + s_v)$$

$$\text{s.t.:} \quad \bigwedge_{B \in \mathcal{B}} \left(\bigvee_{v \in B} s_v \vee \bigvee_{v \in T(B)} z_v \right) \wedge \left(\bigvee_{v \in V} z_v \right) \wedge \left(\bigwedge_{\substack{v \in V \\ w \in cl(N[v])}} (\neg\, s_w \vee \neg\, z_v) \right)$$

Boolean SAT formulation

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To generate violated constraints:

$$\min \quad \sum_{v\in V} x_v$$

$$\text{s.t.:} \quad \bigwedge_{\substack{v\in V\\ w\in N(v)}}\left(x_w \vee \neg\, x_v \vee \bigvee_{a\in N(w)\setminus\{v\}} x_a\right) \wedge \left(\bigvee_{v\in V} x_v\right) \wedge \left(\bigwedge_{v\in cl(S)} \neg\, x_v\right)$$

Alternate constraint generation

Algorithm 1: Greedy algorithm for finding minimal violated forts

1 **Input:** Graph $G = (V, E)$, set $S \subset V$ corresponding to solution of RMP;
2 **Output:** Minimal fort B which violates S;
3 $C \leftarrow cl(S)$;
4 **for** $v \in V \backslash C$ **do**
5   **if** $cl(C \cup \{v\}) \neq V$ **then**
6         $C \leftarrow C \cup \{v\}$;
7   **goto** line 4;
8 **return** $B \leftarrow V \backslash C$

Runtime comparisons

G	$\|V\|$	$Z(G)$	Wavefront	IP Models	Boolean
karate	34	13	329.10	0.16 (M)	**0.02**
chesapeake	39	14	10.91	35.43 (M)	**1.94**
dolphins	62	14	2405.56	246.24 (D)	**108.59**
lesmis	77	40	T	2708.95 (D)	**3.74**
polbooks	105	N/A	T	{14/27} (D)	{**18/27**}
adjnoun	112	N/A	T	{**9/30**} (M)	{9/32}
football	115	N/A	T	{**9/38**} (M)	{0/40}
jazz	198	N/A	T	{**27/96**} (M)	{10/102}
celegansneural	297	N/A	T	{**28/100**} (M)	{**15/89**}
IEEE 14	14	4	0.01	0.01 (M)	**0.001**
IEEE 24	24	6	0.07	0.01 (M)	**0.004**
IEEE 30	30	7	0.82	0.03 (M)	**0.010**
IEEE 39	39	7	6.69	0.04 (M)	**0.019**
IEEE 57	57	9	18.76	1.97 (M)	**0.282**
RTS 96	73	15	T	0.78 (M)	**0.641**
IEEE 118	118	26	T	734.96 (I)	**10.367**
IEEE 300	300	N/A	T	{**73/75**} (I)	{62/77}

Runtime comparisons

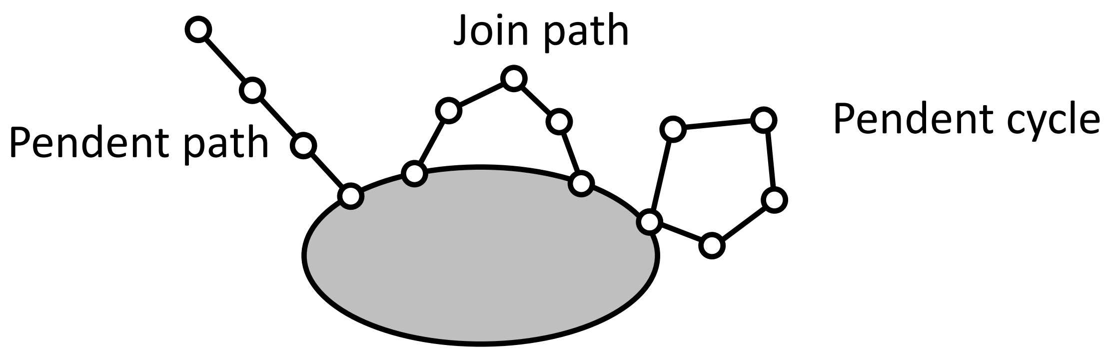
	$\lvert V\rvert$	$Z(G)$	Wavefront	IP Models	Boolean
Cubic graphs	10	3	0.01	0.01 (M)	**0.000**
	20	5	0.02	0.04 (M)	**0.009**
	30	6	**0.13**	0.42 (M)	0.142
	40	8	**1.95**	6.10 (E)	2.624
	50	9	**6.69**	30.90 (E)	17.527
	60	11	**156.83**	1363.05 (E)	3138.832
	70	12	**370.50**	2852.27 (E)	{10.8/12.4}
	80	N/A	**1615.85**	5341.15 (E)	{10.8/13.8}
	90	N/A	**3101.29**	{9.4/13.6} (F)	{9.6/14.4}
	100	N/A	T	{**9.6/15.4**} (F)	{9.6/15.8}
Avg-cubic	20	5.6	0.10	0.02 (M)	**0.003**
	40	10.0	796.89	0.63 (M)	**0.063**
	60	14.4	T	9.24 (I)	**0.867**
	80	19.4	T	15.45 (I)	**7.150**
	100	23.4	T	**37.23** (I)	345.232
	120	30.5	T	**109.13** (I)	382.478

Facets of zero forcing polytope

- If a polytope has a large number of facets, it is usually inherently difficult to solve with IP

Facets of zero forcing polytope

- If a polytope has a large number of facets, it is usually inherently difficult to solve with IP
- There can be $\Omega(1.3247^{n/3})$ facets defined by forts in pendant paths, join paths, and pendant cycles



Heuristics

- **Single vertex largest closure:** add a single white vertex to Z, so that the resulting closure is maximized
- **Neighborhood largest closure:** add a single white vertex and all-but-one of its white neighbors to Z, so that the resulting closure is maximized
- **Neighborhood scaled closure:** add a single white vertex and all-but-one of its white neighbors to Z, so that the *ratio* of the resulting closure to the number of vertices added is maximized

Closure based heuristics

Input: Graph $G = (V, E)$;
Output: Zero forcing set Z of G;
$Z \leftarrow \emptyset$;
$C \leftarrow \emptyset$;
while $C \neq V$ **do**
    $C^* \leftarrow \emptyset$;
    $A \leftarrow \emptyset$;
    for $v \in V \backslash C$ **do**
        $S \leftarrow C \cup A'(v, C)$;
        if $f(cl(S), v, C) > f(C^*, v, C)$ **then**
            $A \leftarrow A'(v, C)$;
            $C^* \leftarrow cl(S)$;
    $C \leftarrow C^*$;
    $Z \leftarrow Z \cup A$;
    for $v \in Z$ **do**
        if $cl(Z \setminus \{v\}) = V$ **then**
            $Z \leftarrow Z \setminus \{v\}$;
return Z

Single vertex largest closure

$$A'(v, C) = \{v\}$$
$$f(S, v, C) = |S|$$

Neighborhood largest closure

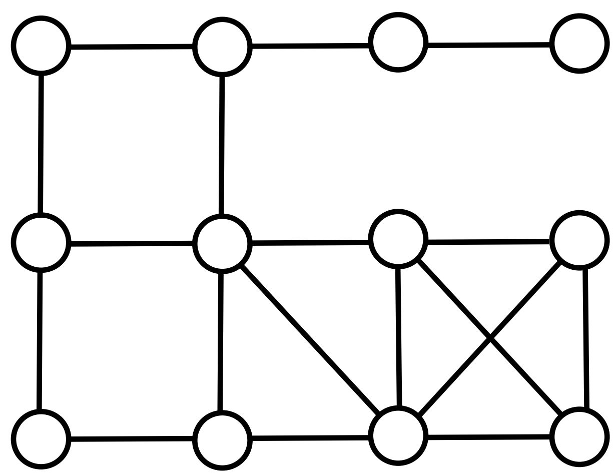
$$A'(v, C) = (N[v] \cap (V \backslash C)) \backslash \{u\}$$
for some $u \in N(v) \cap (V \backslash C)$
$$f(S, v, C) = |S|$$

Neighborhood scaled closure

$$A'(v, C) = (N[v] \cap (V \backslash C)) \backslash \{u\}$$
for some $u \in N(v) \cap (V \backslash C)$
$$f(S, v, C) = (|cl(S)| - |C|)/|A'(v, C)|$$

Heuristics

Single vertex largest closure: add a single white vertex to Z, so that the resulting closure is maximized



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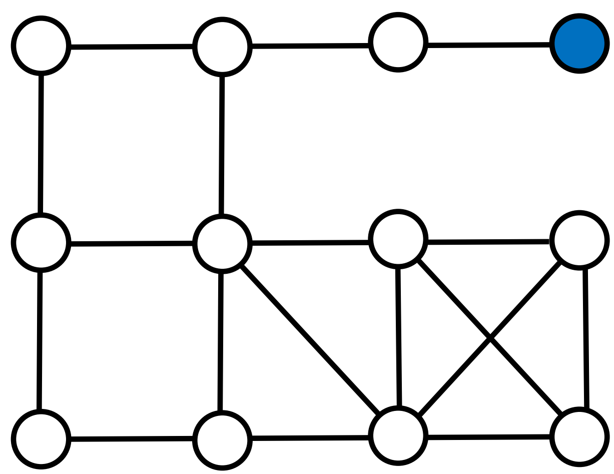

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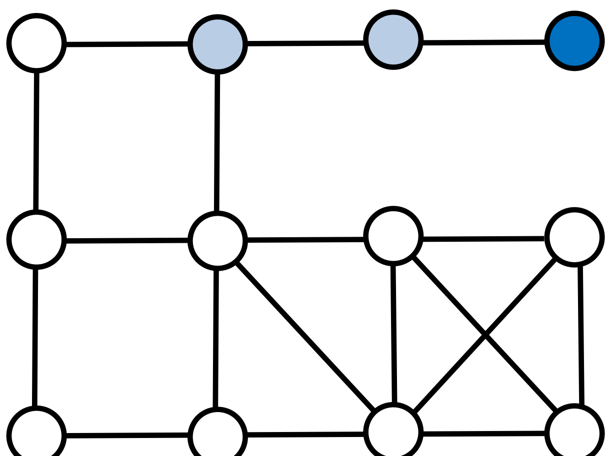

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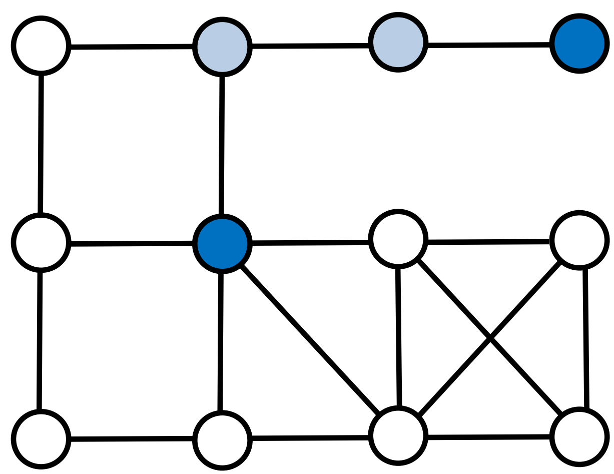

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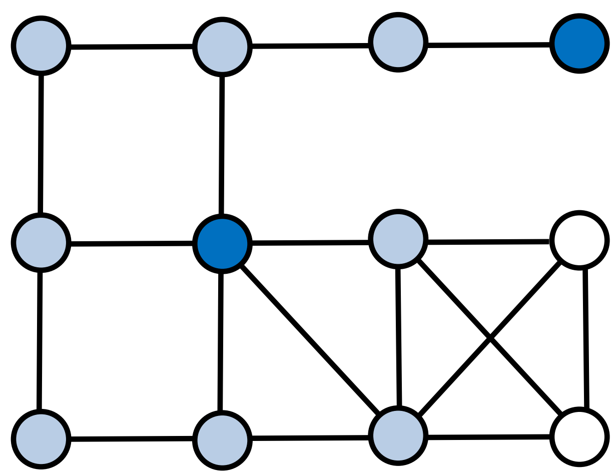

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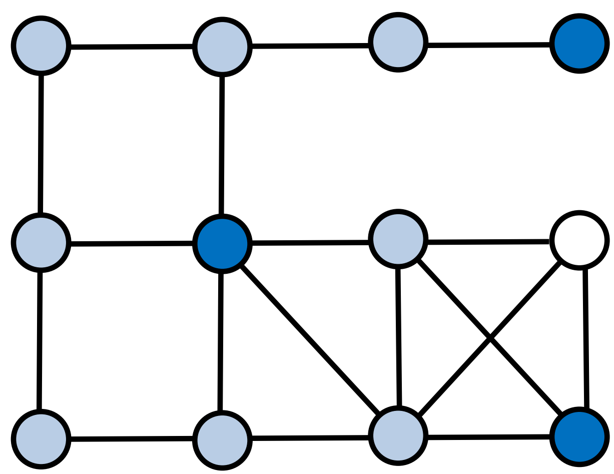

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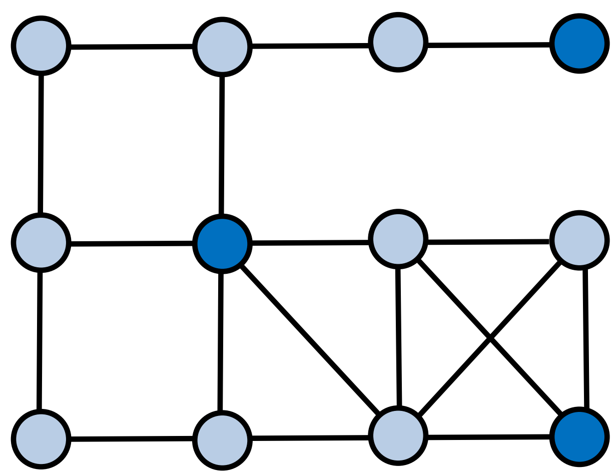

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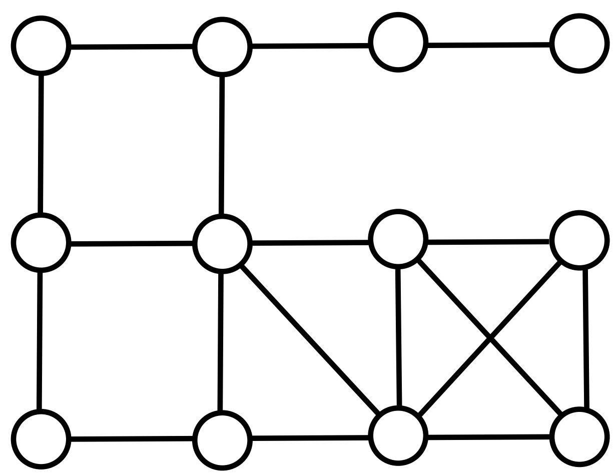

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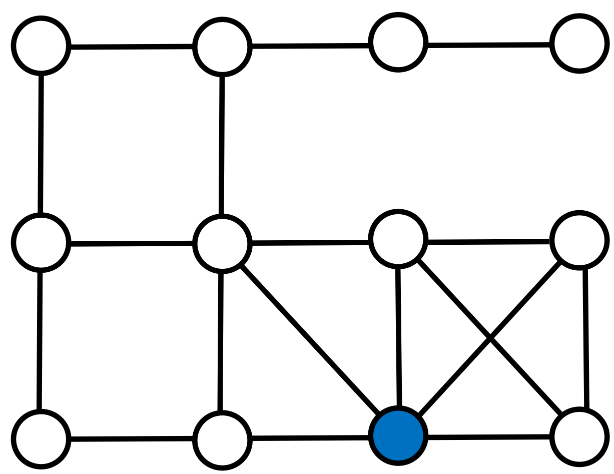

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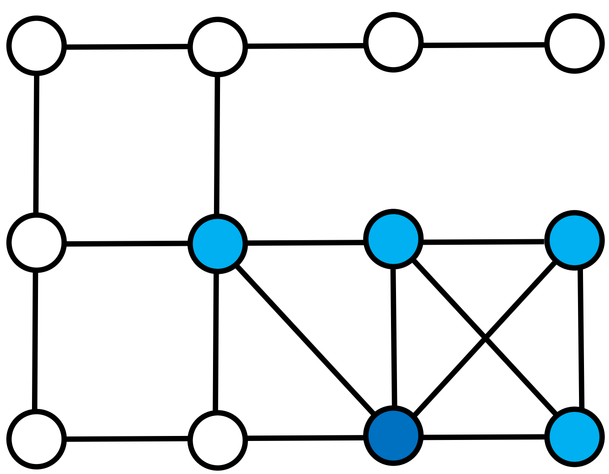

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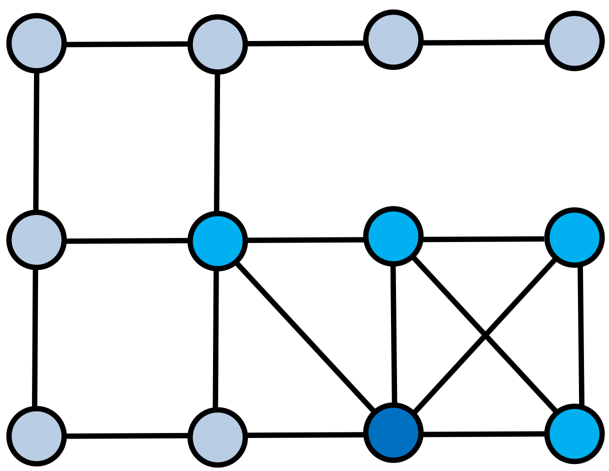

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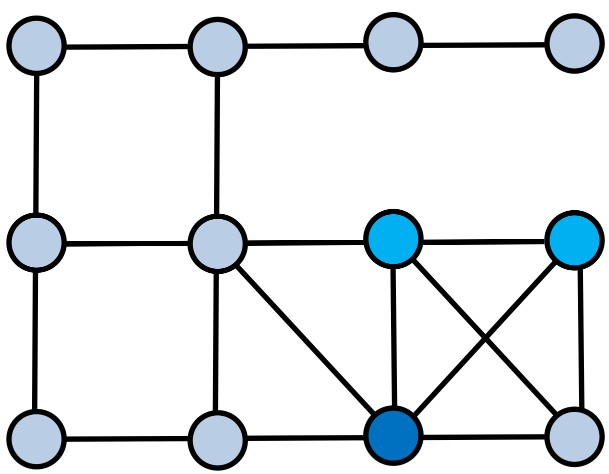

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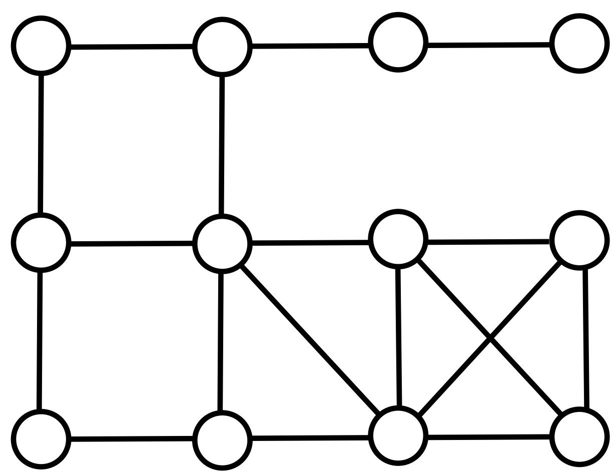

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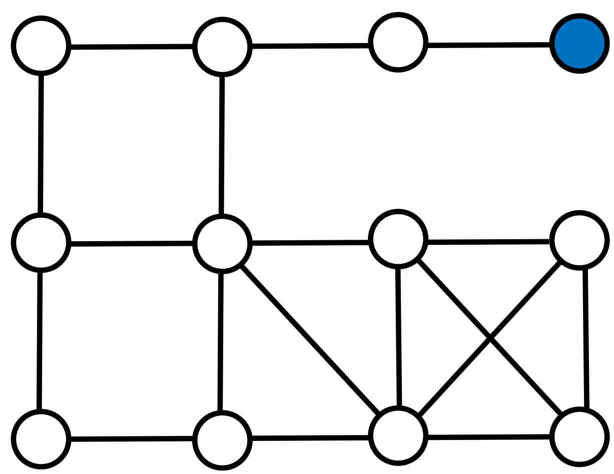

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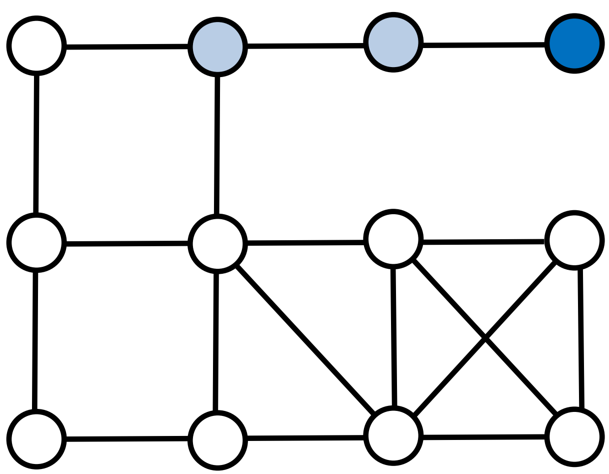

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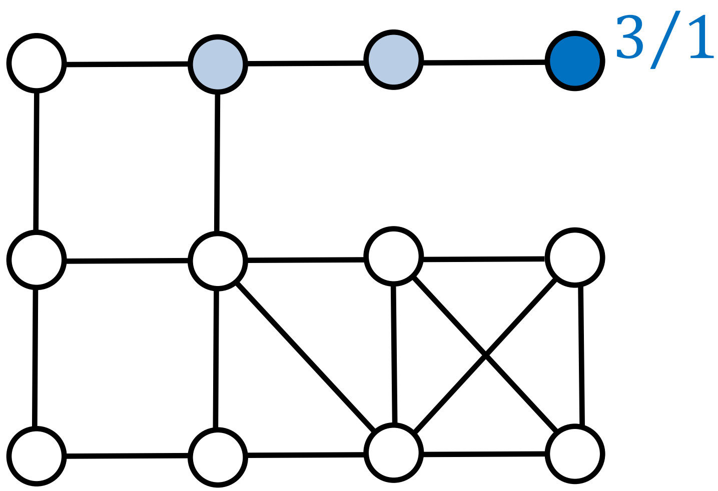

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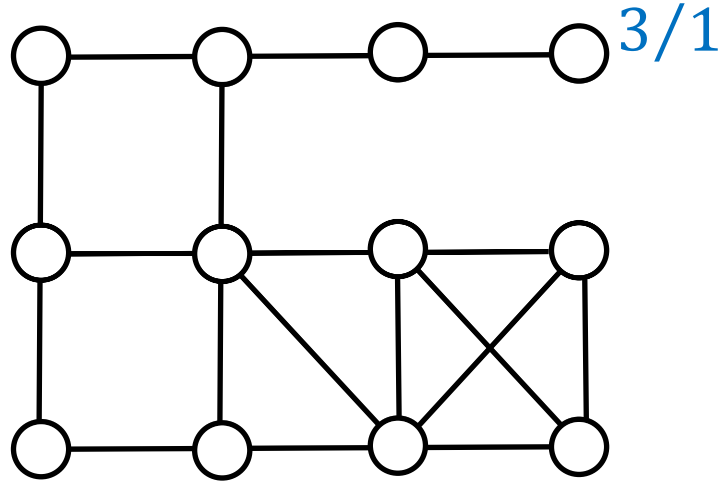

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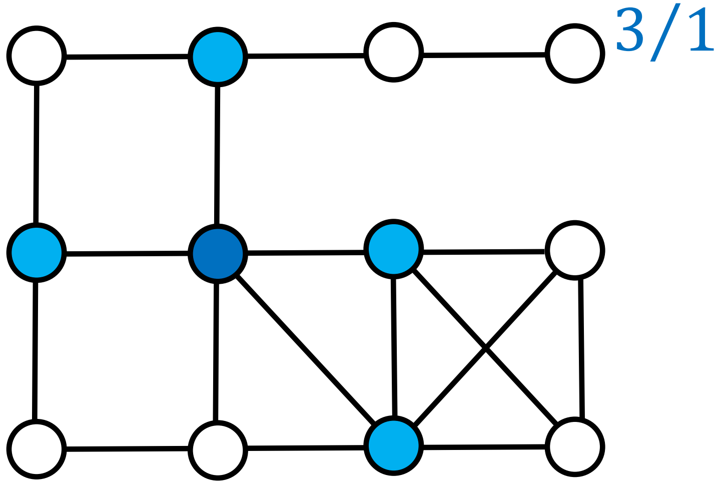

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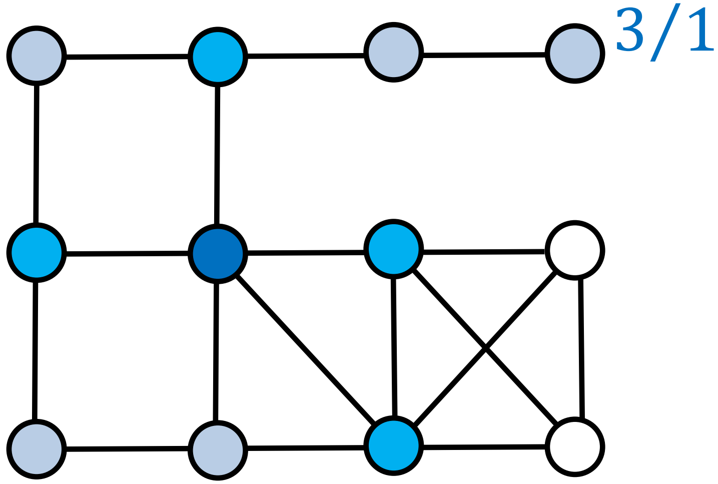

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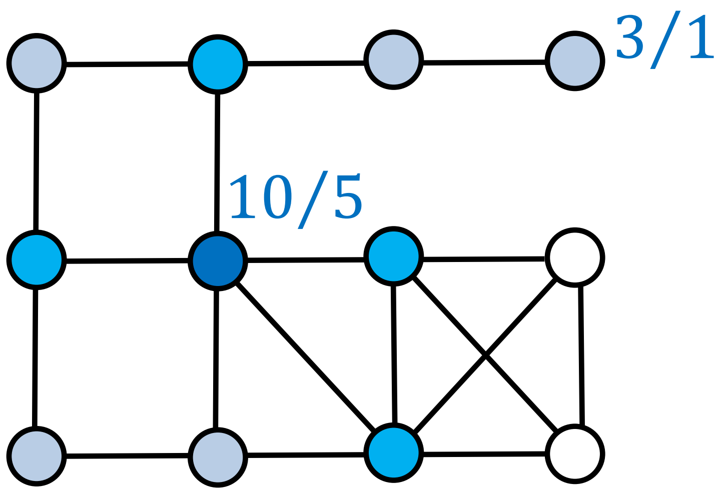

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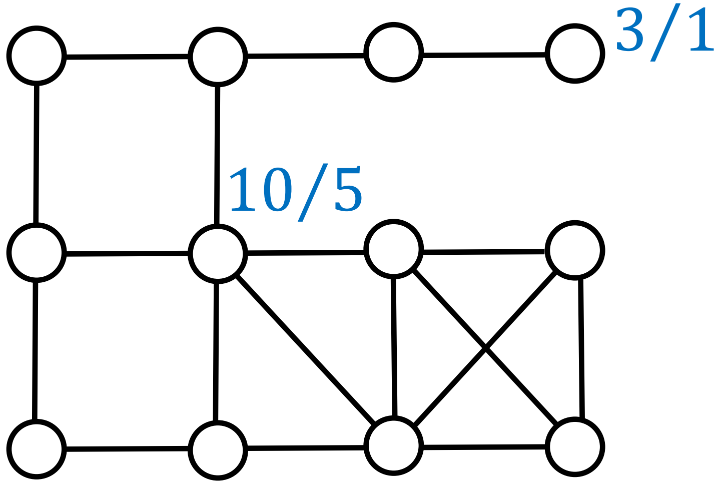

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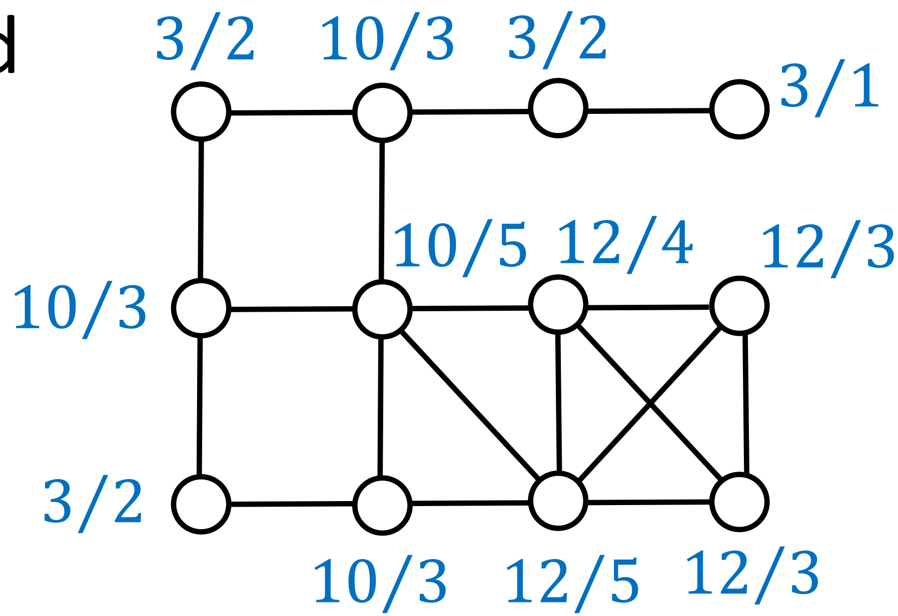

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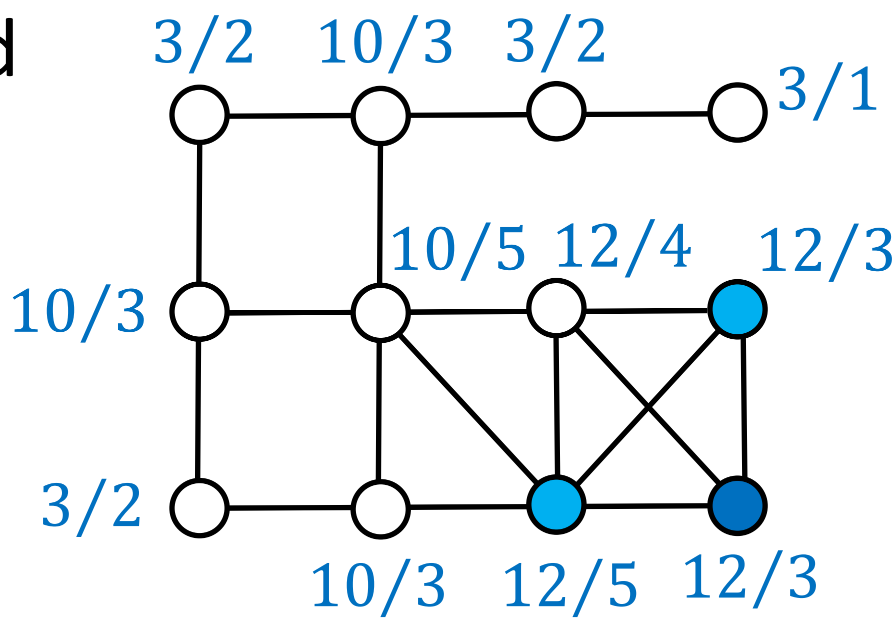

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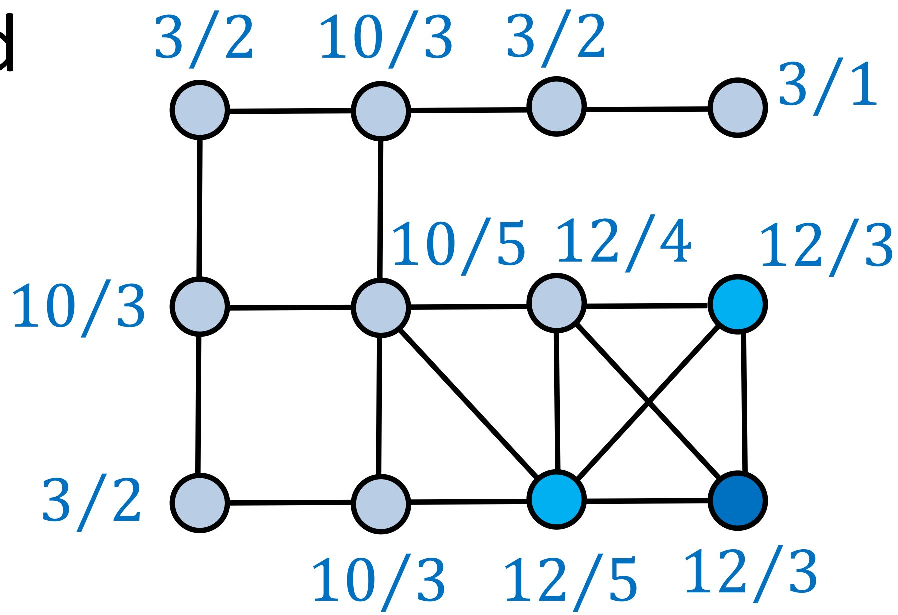

Heuristic results

			Single Node Largest Closure		Neighborhood Largest Closure		Neighborhood Scaled Closure		
G	$\|V\|$	$Z(G)$	Z_h	time	Z_h	time	Z_h	time	
IEEE 14	14	4	**4**	0.000	5	0.000	5	0.000	✓
IEEE 24	24	6	7	0.001	**6**	0.000	**6**	0.000	✓
IEEE 30	30	7	**7**	0.000	9	0.000	**7**	0.000	✓
IEEE 39	39	7	9	0.003	**8**	0.001	**8**	0.001	✓
IEEE 57	57	9	**10**	0.007	**10**	0.002	11	0.003	✓
RTS 96	73	15	17	0.019	17	0.008	**16**	0.006	✓
IEEE 118	118	26	**27**	0.048	32	0.017	28	0.029	✓
karate	34	13	**13**	0.003	14	0.001	14	0.002	✓
chesapeake	39	14	**15**	0.007	**15**	0.002	**15**	0.002	✓
dolphins	62	14	**17**	0.015	19	0.003	**17**	0.004	✗
lesmis	77	40	**41**	0.040	**41**	0.008	**41**	0.026	✓

Heuristic results

G	$\|V\|$	$Z(G)$	Single Node Largest Closure Z_h	Single Node Largest Closure time	Neighborhood Largest Closure Z_h	Neighborhood Largest Closure time	Neighborhood Scaled Closure Z_h	Neighborhood Scaled Closure time	
WS(10,.3)	20	12.0	**12.2**	0.000	**12.2**	0.000	**12.2**	0.000	✓
	30	15.4	16.2	0.002	16.8	0.000	**16.0**	0.000	✓
	40	18.0	18.4	0.004	18.4	0.001	**18.2**	0.001	✓
	50	21.8	**22.0**	0.009	24.0	0.002	23.6	0.002	✓
	60	24.6	**24.8**	0.015	25.6	0.003	25.8	0.003	✓
	70	27.4	28.8	0.025	**28.4**	0.005	**28.4**	0.004	✓
	80	31.2	**32.6**	0.036	33.8	0.008	33.2	0.007	✗
WS(5,.3)	10	4.4	4.8	0.000	**4.6**	0.000	**4.6**	0.000	✓
	20	6.2	**6.2**	0.000	6.4	0.000	6.4	0.000	✓
	30	7.0	8.0	0.001	**7.4**	0.000	7.6	0.000	✓
	40	9.4	10.2	0.002	10.6	0.000	**9.6**	0.000	✓
	50	10.8	**11.6**	0.005	12.2	0.002	**11.6**	0.002	✓
	60	11.6	13.6	0.008	14.0	0.002	**13.2**	0.002	✗
	70	14.0	15.0	0.016	16.2	0.003	**14.8**	0.004	✓
	80	14.8	**16.8**	0.015	18.2	0.005	17.2	0.006	✗

Heuristic results

			Single Node Largest Closure		Neighborhood Largest Closure		Neighborhood Scaled Closure		
G	$\lvert V \rvert$	$Z(G)$	Z_h	time	Z_h	time	Z_h	time	
Cubic	10	3.8	**3.8**	0.000	**3.8**	0.000	**3.8**	0.000	✓
	20	5.2	5.4	0.000	**5.2**	0.000	**5.2**	0.000	✓
	30	6.6	7.4	0.001	7.2	0.000	**7.0**	0.000	✓
	40	8.8	9.8	0.003	9.6	0.001	**9.2**	0.001	✓
	50	9.2	10.4	0.005	**9.4**	0.001	9.6	0.001	✓
	60	11.4	12.4	0.008	12.0	0.003	**11.8**	0.003	✓
	70	12.0	**12.6**	0.008	13.2	0.004	12.8	0.004	✓

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THANK YOU