27

Combinatorial Matrix Theory

Richard A. Brualdi
University of Wisconsin

27.1 Combinatorial Structure and Invariants **27**-1
27.2 Square Matrices and Strong Combinatorial Invariants **27**-3
27.3 Square Matrices and Weak Combinatorial Invariants **27**-5
27.4 The Class $\mathcal{A}(R, S)$ of $(0, 1)$-Matrices **27**-7
27.5 The Class $\mathcal{T}(R)$ of Tournament Matrices **27**-8
27.6 Convex Polytopes of Doubly Stochastic Matrices **27**-10
References **27**-12

27.1 Combinatorial Structure and Invariants

The **combinatorial structure of a matrix** generally refers to the locations of the nonzero entries of a matrix, or it might be used to refer to the locations of the zero entries. To study and take advantage of the combinatorial structure of a matrix, graphs are used as models. Associated with a matrix are several graphs that represent the combinatorial structure of a matrix in various ways. The type of graph (undirected graph, bipartite graph, digraph) used depends on the kind of matrices (symmetric, rectangular, square) being studied ([BR91], [Bru92], [BS04]). Conversely, associated with a graph, bipartite graph, or digraph are matrices that allow one to consider it as an algebraic object. These matrices — their algebraic properties — can often be used to obtain combinatorial information about a graph that is not otherwise obtainable. These are two of three general aspects of **combinatorial matrix theory**. A third aspect concerns intrinsic combinatorial properties of matrices viewed simply as an array of numbers.

Definitions:

Let $A = [a_{ij}]$ be an $m \times n$ matrix.

A **strong combinatorial invariant** of A is a quantity or property that does not change when the rows and columns of A are permuted, that is, which is shared by all matrices of the form PAQ, where P is a permutation matrix of order m and Q is a permutation matrix of order n.

A less restrictive definition can be considered when A is a square matrix of order n.

A **weak combinatorial invariant** is a quantity or property that does not change when the rows and columns are simultaneously permuted, that is, which is shared by all matrices of the form PAP^T where P is a permutation matrix of order n.

The $(0, 1)$-matrix obtained from A by replacing each nonzero entry with a 1 is the **pattern** of A. (In those situations where the actual value of the nonzero entries is unimportant, one may replace a matrix with its pattern, that is, one may assume that A itself is a $(0, 1)$-matrix.)

A **line** of a matrix is a row or column.

A **zero line** is a line of all zeros.

The **term rank** of a (0, 1)-matrix A is the largest size $\varrho(A)$ of a collection of 1s of A with no two 1s in the same line.

A **cover** of A is a collection of lines that contain all the 1s of A.

A **minimum cover** is a cover with the smallest number of lines. The number of lines in a minimum line cover of A is denoted by $c(A)$.

A **co-cover** of A is a collection of 1s of A such that each line of A contains at least one of the 1s.

A **minimum co-cover** is a co-cover with the smallest number of 1s. The number of 1s in a minimum co-cover is denoted by $c^*(A)$.

The quantity $\varrho^*(A)$ is the **largest size of a zero submatrix** of A, that is, the maximum of $r + s$ taken over all integers r and s with $0 \leq r \leq m$ and $0 \leq s \leq n$ such that A has an $r \times s$ zero (possibly vacuous) submatrix.

Facts:

The following facts are either elementary or can be found in Chapters 1 and 4 of [BR91].

1. These are strong combinatorial invariants:
   (a) The number of rows (respectively, columns) of a matrix.
   (b) The quantity $\max\{r, s\}$ taken over all $r \times s$ zero submatrices $(0 \leq r, s)$.
   (c) The maximum value of $r + s$ taken over all $r \times s$ zero submatrices $(0 \leq r, s)$.
   (d) The number of zeros (respectively, nonzeros) in a matrix.
   (e) The number of zero rows (respectively, zero columns) of a matrix.
   (f) The multiset of row sums (respectively, column sums) of a matrix.
   (g) The rank of a matrix.
   (h) The permanent (see Chapter 31) of a matrix.
   (i) The singular values of a matrix.
2. These are weak combinatorial invariants:
   (a) The largest order of a principal submatrix that is a zero matrix.
   (b) The number of A zeros on the main diagonal of a matrix.
   (c) The maximum value of $p + q$ taken over all $p \times q$ zero submatrices that do not meet the main diagonal.
   (d) Whether or not for some integer r with $1 \leq r \leq n$, the matrix A of order n has an $r \times n - r$ zero submatrix that does not meet the main diagonal of A.
   (e) Whether or not A is a symmetric matrix.
   (f) The trace $\mathrm{tr}(A)$ of a matrix A.
   (g) The determinant $\det A$ of a matrix A.
   (h) The eigenvalues of a matrix.
   (i) The multiset of elements on the main diagonal of a matrix.
3. $\varrho(A), c(A), \varrho^*(A)$, and $c^*(A)$ are all strong combinatorial invariants.
4. $\rho(A) = c(A)$.
5. A matrix A has a co-cover if and only if it does not have any zero lines. If A does not have any zero lines, then $\varrho^*(A) = c^*(A)$.
6. If A is an $m \times n$ matrix without zero lines, then $\varrho(A) + \varrho^*(A) = c(A) + c^*(A) = m + n$.

7. $\text{rank}(A) \le \varrho(A)$.
8. Let A be an $m \times n$ (0,1)-matrix. Then there are permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_1 & X & Y & Z \\ O & A_2 & O & O \\ O & S & A_3 & O \\ O & T & O & O \end{bmatrix},$$

where A_1, A_2, and A_3 are square, possibly vacuous, matrices with only 1s on their main diagonals, and $\rho(A)$ is the sum of the orders of A_1, A_2, and A_3. The rows, respectively columns, of A that are in every minimum cover of A are the rows, respectively columns, that meet A_1, respectively A_2. These rows and columns together with either the rows that meet A_3 or the columns that meet A_3 form minimum covers of A.

Examples:

1. Let

$$A = \left[\begin{array}{cc|cc|c|cc} \mathbf{1} & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \mathbf{1} & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \mathbf{1} & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & \mathbf{1} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right].$$

Then $\varrho(A) = c(A) = 5$ with the five **1**s in different lines, and rows 1, 2, and 5 and columns 3 and 4 forming a cover. The matrix is partitioned in the form given in Fact 8.

27.2 Square Matrices and Strong Combinatorial Invariants

In this section, we consider the strong combinatorial structure of square matrices.

Definitions:

Let A be a $(0, 1)$-matrix of order n.

A collection of n nonzero entries in A no two on the same line is a **diagonal** of A (this term is also applied to nonnegative matrices).

The next definitions are concerned with the existence of certain zero submatrices in A.

A is **partly decomposable** provided there exist positive integers p and q with $p + q = n$ such that A has a $p \times q$ zero submatrix. Equivalently, there are permutation matrices P and Q and an integer k with $1 \le k \le n - 1$ such that

$$PAQ = \begin{bmatrix} B & C \\ O_{k,n-k} & D \end{bmatrix}.$$

A is a **Hall matrix** provided there does not exist positive integers p and q with $p + q > n$ such that A has a $p \times q$ zero submatrix.

A has **total support** provided $A \ne O$ and each 1 of A is on a diagonal of A.

A is **fully indecomposable** provided it is not partly decomposable.

A is **nearly decomposable** provided it is fully indecomposable and each matrix obtained from A by replacing a 1 with a 0 is partly decomposable.

Facts:

Unless otherwise noted, the following facts can be found in Chapter 4 of [BR91].

1. [BS94] Each of the following properties is equivalent to the matrix A of order n being a Hall matrix:
   (a) $\rho(A) = n$, that is, A has a diagonal (*Frobenius–König theorem*).
   (b) For all nonempty subsets L of $\{1, 2, \ldots, n\}$, $A[\{1, 2, \ldots, n\}, L]$ has at least $|L|$ nonzero rows.
   (c) For all nonempty subsets K of $\{1, 2, \ldots, n\}$, $A[K, \{1, 2, \ldots, n\}]$ has at least $|K|$ nonzero columns.
2. Each of the following properties is equivalent to the matrix A of order n being a fully indecomposable matrix:
   (a) $\rho(A) = n$ and the only minimum line covers are the set of all rows and the set of all columns.
   (b) For all nonempty subsets L of $\{1, 2, \ldots, n\}$, $A[\{1, 2, \ldots, n\}, L]$ has at least $|L| + 1$ nonzero rows.
   (c) For all nonempty subsets K of $\{1, 2, \ldots, n\}$, $A[K, \{1, 2, \ldots, n\}]$ has at least $|K| + 1$ nonzero columns.
   (d) The term rank $\rho(A(i, j))$ of the matrix $A(i, j)$ obtained from A by deleting row i and column j equals $n - 1$ for all $i, j = 1, 2, \ldots, n$.
   (e) A^{n-1} is a positive matrix.
   (f) The determinant $\det A \circ X$ of the Hadamard product of A with a matrix $X = [x_{ij}]$ of distinct indeterminates over a field F is irreducible in the ring $F[\{x_{ij} : 1 \leq i, j \leq n\}]$.
3. Each of the following properties is equivalent to the matrix A of order n having total support:
   (a) $A \neq O$ and the term rank $\rho(A(i, j))$ equals $n - 1$ for all $i, j = 1, 2, \ldots, n$ with $a_{ij} \neq 0$.
   (b) There are permutation matrices P and Q such that PAQ is a direct sum of fully indecomposable matrices.
4. (*Dulmage–Mendelsohn Decomposition theorem*) If the matrix A of order n has term rank equal to n, then there exist permutation matrices P and Q and an integer $t \geq 1$ such that

$$PAQ = \begin{bmatrix} A_1 & A_{12} & \cdots & A_{1t} \\ O & A_2 & \cdots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_t \end{bmatrix},$$

   where $A_1, A_2, \ldots, A_t$ are square fully indecomposable matrices. The matrices $A_1, A_2, \ldots, A_t$ are called the **fully indecomposable components** of A and they are uniquely determined up to permutations of their rows and columns. The matrix A has total support if and only if $A_{ij} = O$ for all i and j with $i < j$; A is fully indecomposable if and only if $t = 1$.
5. (*Inductive structure of fully indecomposable matrices*) If A is a fully indecomposable matrix of order n, then there exist permutation matrices P and Q and an integer $k \geq 2$ such that

$$PAQ = \begin{bmatrix} B_1 & O & \cdots & O & E_1 \\ E_2 & B_2 & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & B_{k-1} & O \\ O & O & \cdots & E_k & B_k \end{bmatrix},$$

   where $B_1, B_2, \ldots, B_k$ are fully indecomposable and $E_1, E_2, \ldots, E_k$ each contain at least one nonzero entry. Conversely, a matrix of such a form is fully indecomposable.

6. (*Inductive structure of nearly decomposable matrices*) If A is a nearly decomposable $(0,1)$-matrix, then there exist permutation matrices P and Q and an integer p with $1 \leq p \leq n-1$ such that

$$PAQ = \left[\begin{array}{cccccc|c} 1 & 0 & 0 & \cdots & 0 & 0 & \\ 1 & 1 & 0 & \cdots & 0 & 0 & \\ 0 & 1 & 1 & \cdots & 0 & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & F_1 \\ 0 & 0 & 0 & \cdots & 1 & 0 & \\ 0 & 0 & 0 & \cdots & 1 & 1 & \\ \hline & & & F_2 & & & A' \end{array}\right],$$

where A' is a nearly decomposable matrix of order $n-p$, the matrix F_1 has exactly one 1 and this 1 occur in its first row, and the matrix F_2 has exactly one 1 and this 1 occurs in its last column. If $n-p \geq 2$, and the 1 in F_2 is in its column j and the 1 in F_2 is in its row i, then the (i,j) entry of A' is 0.
7. The number of nonzero entries in a nearly decomposable matrix A of order $n \geq 3$ is between $2n$ and $3(n-1)$.

Examples:

1. Let

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \ A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ A_4 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then A_1 is partly decomposable and not a Hall matrix. The matrix A_2 is a Hall matrix and is partly decomposable, but does not have total support. The matrix A_3 has total support. The matrix A_4 is nearly decomposable.

27.3 Square Matrices and Weak Combinatorial Invariants

In this section, we restrict our attention to the weak combinatorial structure of square matrices.

Definitions:

Let A be a matrix of order n.

B is **permutation similar** to A if there exists a permutation matrix P such that $B = P^T AP \ (= P^{-1}AP)$.

A is **reducible** provided $n \geq 2$ and for some integer r with $1 \leq r \leq n-1$, there exists an $r \times (n-r)$ zero submatrix which does not meet the main diagonal of A, that is, provided there is a permutation matrix P and an integer r with $1 \leq r \leq n-1$ such that

$$PAP^T = \begin{bmatrix} B & C \\ O_{r,n-r} & D \end{bmatrix}.$$

A is **irreducible** provided that A is not reducible.

A is **completely reducible** provided there exists an integer $k \geq 2$ and a permutation matrix P such that $PAP^T = A_1 \oplus A_2 \oplus \cdots \oplus A_k$ where $A_1, A_2, \ldots, A_k$ are irreducible.

A is **nearly reducible** provided A is irreducible and each matrix obtained from A by replacing a nonzero entry with a zero is reducible.

A **Frobenius normal form** of A is a block upper triangular matrix with irreducible diagonal blocks that is permutation similar to A; the diagonal blocks are called the **irreducible components** of A. (cf. Fact 27.3.)

The following facts can be found in Chapter 3 of [BR91].

Facts:

1. (*Frobenius normal form*) There is a permutation matrix P and an integer $r \geq 1$ such that

$$PAP^T = \begin{bmatrix} A_1 & A_{12} & \cdots & A_{1r} \\ O & A_2 & \cdots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_r \end{bmatrix},$$

where $A_1, A_2, \ldots, A_t$ are square irreducible matrices. The matrices $A_1, A_2, \ldots, A_r$ are the **irreducible components** of A and they are uniquely determined up to simultaneous permutations of their rows and columns.

2. There exists a permutation matrix Q such that AQ is irreducible if and only if A has at least one nonzero element in each line.
3. If A does not have any zeros on its main diagonal, then A is irreducible if and only if A is fully indecomposable. The matrix A is fully indecomposable if and only if there is a permutation matrix Q such that AQ has no zeros on its main diagonal and AQ is irreducible.
4. (*Inductive structure of irreducible matrices*) Let A be an irreducible matrix of order $n \geq 2$. Then there exists a permutation matrix P and an integer $m \geq 2$ such that

$$PAP^T = \begin{bmatrix} A_1 & O & \cdots & O & E_1 \\ E_2 & A_2 & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & A_{m-1} & O \\ O & O & \cdots & E_m & A_m \end{bmatrix},$$

where $A_1, A_2, \ldots, A_m$ are irreducible and $E_1, E_2, \ldots, E_m$ each have at least one nonzero entry.

5. (*Inductive structure of nearly reducible matrices*) If A is a nearly reducible $(0, 1)$-matrix, then there exist permutation matrix P and an integer m with $1 \leq m \leq n - 1$ such that

$$PAP^T = \left[\begin{array}{cccccc|c} 0 & 0 & 0 & \cdots & 0 & 0 & \\ 1 & 0 & 0 & \cdots & 0 & 0 & \\ 0 & 1 & 0 & \cdots & 0 & 0 & F_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 & \\ \hline & & & F_2 & & & A' \end{array}\right],$$

where A' is a nearly reducible matrix of order m, the matrix F_1 has exactly one 1 and it occurs in the first row and column j of F_1 with $1 \leq j \leq m$, and the matrix F_2 has exactly one 1 and it occurs in the last column and row i of F_2 where $1 \leq i \leq m$. The element in position (i, j) of A' is 0.

6. The number of nonzero entries in a nearly reducible matrix of order $n \geq 2$ is between n and $2(n - 1)$

Examples:

1. Let

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix},\ A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},\ A_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then A_1 is reducible but not completely reducible, and A_2 is irreducible. (Both A_1 and A_2 are partly decomposable.) The matrix A_3 is completely reducible. The matrix A_4 is nearly reducible.

27.4 The Class $\mathcal{A}(R,S)$ of (0,1)-Matrices

In the next definition, we introduce one of the most important and widely studied classes of $(0,1)$-matrices (see Chapter 6 of [Rys63] and [Bru80]).

Definitions:

Let $A = [a_{ij}]$ be an $m \times n$ matrix.

The **row sum vector** of A is $R = (r_1, r_2, \ldots, r_m)$, where $r_i = \sum_{j=1}^{n} a_{ij}$, $(i = 1, 2, \ldots, n)$.

The **column sum vector** of A is $S = (s_1, s_2, \ldots, s_n)$, where $s_j = \sum_{i=1}^{m} a_{ij}$, $(j = 1, 2, \ldots, n)$.

A real vector $(c_1, c_2, \ldots, c_n)$ is **monotone** provided $c_1 \geq c_2 \geq \cdots \geq c_n$.

The **class of all** $m \times n$ $(0,1)$**-matrices with row sum vector** R **and column sum vector** S is denoted by $\mathcal{A}(R,S)$.

The class $\mathcal{A}(R,S)$ is a **monotone class** provided R and S are both monotone vectors.

An **interchange** is a transformation on a $(0,1)$-matrix that replaces a submatrix equal to the identity matrix I_2 by the submatrix

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

or vice versa.

If $\theta(A)$ is any real numerical quantity associated with a matrix A, then the **extreme values** of θ are $\bar{\theta}(R,S)$ and $\tilde{\theta}(R,S)$, defined by

$$\bar{\theta}(R,S) = \max\{\theta(A) : A \in \mathcal{A}(R,S)\} \text{ and } \tilde{\theta}(R,S) = \min\{\theta(A) : A \in \mathcal{A}(R,S)\}.$$

Let $T = [t_{kl}]$ be the $(m+1) \times (n+1)$ matrix defined by

$$t_{kl} = kl - \sum_{j=1}^{l} s_j + \sum_{i=k+1}^{m} r_i, \quad (k = 0, 1, \ldots, m; l = 0, 1, \ldots, n).$$

The matrix T is the **structure matrix** of $\mathcal{A}(R,S)$.

Facts:

The following facts can be found in Chapter 6 of [Rys63], [Bru80], Chapter 6 of [BR91], and Chapters 3 and 4 of [Bru06].

1. A class $\mathcal{A}(R,S)$ can be transformed into a monotone class by row and column permutations.
2. Let $U = (u_1, u_2, \ldots, u_n)$ and $V = (v_1, v_2, \ldots, v_n)$ be monotone, nonnegative integral vectors. $U \preceq V$ if and only if $V^* \preceq U^*$, and $U^{**} = U$ or U extended with 0s.
3. (*Gale–Ryser theorem*) $\mathcal{A}(R,S)$ is nonempty if and only if $S \preceq R^*$.
4. Let the monotone class $\mathcal{A}(R,S)$ be nonempty, and let A be a matrix in $\mathcal{A}(R,S)$. Let $K = \{1, 2, \ldots, k\}$ and $L = \{1, 2, \ldots, l\}$. Then t_{kl} equals the number of 0s in the submatrix $A[K,L]$ plus the number of 1s in the submatrix $A(K,L)$; in particular, we have $t_{kl} \geq 0$.
5. (*Ford–Fulkerson theorem*) The monotone class $\mathcal{A}(R,S)$ is nonempty if and only if its structure matrix T is a nonnegative matrix.
6. If A is in $\mathcal{A}(R,S)$ and B results from A by an interchange, then B is in $\mathcal{A}(R,S)$. Each matrix in $\mathcal{A}(R,S)$ can be transformed to every other matrix in $\mathcal{A}(R,S)$ by a sequence of interchanges.
7. The maximum and minimum term rank of a nonempty monotone class $\mathcal{A}(R,S)$ satisfy:

$$\bar{\rho}(R,S) = \min\{t_{kl} + k + l; k = 0, 1, \ldots, m, l = 0, 1, \ldots, n\},$$
$$\tilde{\rho}(R,S) = \min\{k + l : \phi_{kl} \geq t_{kl}, k = 0, 1, \ldots, m, l = 0, 1, \ldots, n\},$$

where

$$\phi_{kl} = \min\{t_{i_1, l+j_2} + t_{k+i_2, j_1} + (k - i_1)(l - j_1)\},$$

the minimum being taken over all integers i_1, i_2, j_1, j_2 such that $0 \le i_1 \le k \le k + i_2 \le m$ and $0 \le j_1 \le l \le l + j_2 \le n$.

8. Let $\text{tr}(A)$ denote the trace of a matrix A. The maximum and minimum trace of a nonempty monotone class $\mathcal{A}(R, S)$ satisfy:

$$\overline{\text{tr}}(R, S) = \min\{t_{kl} + \max\{k, l\} : 0 \le k \le m, 0 \le l \le n\},$$
$$\tilde{\text{tr}}(R, S) = \max\{\min\{k, l\} - t_{kl} : 0 \le k \le m, 0 \le l \le n\}.$$

9. Let k and n be integers with $0 \le k \le n$, and let $\mathcal{A}(n, k)$ denote the class $\mathcal{A}(R, S)$, where $R = S = (k, k, \dots, k)$ (n k's). Let $\tilde{\nu}(n, k)$ and $\bar{\nu}(n, k)$ denote the minimum and maximum rank, respectively, of matrices in $\mathcal{A}(n, k)$.

   (a) $\bar{\nu}(n, k) = \begin{cases} 0, \text{ if } k = 0, \\ 1, \text{ if } k = n, \\ 3, \text{ if } k = 2 \text{ and } n = 4, \\ n, \text{ otherwise.} \end{cases}$

   (b) $\tilde{\nu}(n, k) = \tilde{\nu}(n, n - k)$ if $1 \le k \le n - 1$.

   (c) $\tilde{\nu}(n, k) \ge \lceil n/k \rceil$, $(1 \le k \le n - 1)$, with equality if and only if k divides n.

   (d) $\tilde{\nu}(n, k) \le \lfloor n/k \rfloor + k$, $(1 \le k \le n)$.

   (e) $\tilde{\nu}(n, 2) = n/2$ if n is even, and $(n + 3)/2$ if n is odd.

   (f) $\tilde{\nu}(n, 3) = n/3$ if 3 divides n and $\lfloor n/3 \rfloor + 3$ otherwise.

Additional properties of $\mathcal{A}(R, S)$ can be found in [Bru80] and in Chapters 3 and 4 of [Bru06].

Examples:

1. Let $R = (7, 3, 2, 2, 1, 1)$ and $S = (5, 5, 3, 1, 1, 1)$. Then $R^* = (6, 4, 2, 1, 1, 1, 1)$. Since $5 + 5 + 3 > 6 + 4 + 2$, $S \not\preceq R^*$ and, by Fact 3, $\mathcal{A}(R, S) = \emptyset$.
2. Let $R = S = (2, 2, 2, 2, 2)$. Then the matrices

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

   are in $\mathcal{A}(R, S)$. Then A can be transformed to B by two interchanges:

$$\begin{bmatrix} 1 & \mathbf{1} & 0 & 0 & \mathbf{0} \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & \mathbf{0} & 0 & 1 & \mathbf{1} \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{1} & 1 & 0 \\ 0 & \mathbf{1} & \mathbf{0} & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

27.5 The Class $\mathcal{T}(R)$ of Tournament Matrices

In the next definition, we introduce another important class of $(0, 1)$-matrices.

Definitions:

A $(0, 1)$-matrix $A = [a_{ij}]$ of order n is a **tournament matrix** provided $a_{ii} = 0$, $(1 \le i \le n)$ and $a_{ij} + a_{ji} = 1$, $(1 \le i < j \le n)$, that is, provided $A + A^T = J_n - I_n$.

The digraph of a tournament matrix is called a **tournament**.

Thinking of n teams $p_1, p_2, \ldots, p_n$ playing in a **round-robin tournament**, we have that $a_{ij} = 1$ signifies that team p_i beats team p_j.

The row sum vector R is also called the **score vector** of the tournament (matrix).

A **transitive tournament matrix** is one for which $a_{ij} = a_{jk} = 1$ implies $a_{ik} = 1$.

The **class of all tournament matrices with score vector** R is denoted by $\mathcal{T}(R)$.

A δ**-interchange** is a transformation on a tournament matrix that replaces a principal submatrix of order 3 equal to

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ with } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

or vice versa.

The following facts can be found in Chapters 2 and 5 of [Bru06].

Facts:

1. The row sum vector $R = (r_1, r_2, \ldots, r_n)$ and column sum vector $S = (s_1, s_2, \ldots, s_n)$ of a tournament matrix of order n satisfy $r_i + s_i = n - 1$, $(1 \leq i \leq n)$; in particular, the column sum vector is determined by the row sum vector.
2. If A is a tournament matrix and P is a permutation matrix, then PAP^T is a tournament matrix. Thus, one may assume without loss of generality that R is nondecreasing, that is, $r_1 \leq r_2 \leq \cdots \leq r_n$, so that the teams are ordered from worst to best.
3. (*Landau's theorem*) If $R = (r_1, r_2, \ldots, r_n)$ is a nondecreasing, nonnegative integral vector, then $\mathcal{T}(R)$ is nonempty if and only if

$$\sum_{i=1}^{k} r_i \geq \binom{k}{2}, \quad (1 \leq k \leq n)$$

   with equality when $k = n$. (A binomial coefficient $\binom{k}{s}$ is 0 if $k < s$.)
4. Let $R = (r_1, r_2, \ldots, r_n)$ be a nondecreasing nonnegative integral vector. The following are equivalent:

   (a) There exists an irreducible matrix in $\mathcal{T}(R)$.

   (b) $\mathcal{T}(R)$ is nonempty and every matrix in $\mathcal{T}(R)$ is irreducible.

   (c) $\sum_{i=1}^{k} r_i \geq \binom{k}{2}$, $(1 \leq k \leq n)$ with equality if and only if $k = n$.
5. If A is in $\mathcal{T}(R)$ and B results from A by a δ-interchange, then B is in $\mathcal{T}(R)$. Each matrix in $\mathcal{T}(R)$ can be transformed to every other matrix in $\mathcal{T}(R)$ by a sequence of δ-interchanges.
6. The rank, and so term rank, of a tournament matrix of order n is at least $n - 1$.
7. (*Strengthened Landau inequalities*) Let $R = (r_1, r_2, \ldots, r_n)$ be a nondecreasing nonnegative integral vector. Then $\mathcal{T}(R)$ is nonempty if and only if

$$\sum_{i \in I} r_i \geq \frac{1}{2} \sum_{i \in I} (i - 1) + \frac{1}{2} \binom{|I|}{2}, \quad (I \subseteq \{1, 2, \ldots, n\}),$$

   with equality if $I = \{1, 2, \ldots, n\}$.
8. Let $R = (r_1, r_2, \ldots, r_n)$ be a nondecreasing, nonnegative integral vector such that $\mathcal{T}(R)$ is nonempty. Then there exists a tournament matrix A such that the principal submatrices made out of the even-indexed and odd-indexed, respectively, rows and columns are transitive tournament matrices, that is, a matrix A in $\mathcal{T}(R)$ such that $A[\{1, 3, 5, \ldots\}]$ and $A[\{2, 4, 6, \ldots\}]$ are transitive tournament matrices.

Examples:

1. The following are tournament matrices:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

   The matrix A_2 is a transitive tournament matrix.
2. Let $R = (2, 2, 2, 2, 3, 4)$. A tournament matrix in $\mathcal{T}(R)$ satisfying Fact 7 is

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix},$$

   since the two submatrices

$$A[|[1, 3, 5\}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } A[\{2, 4, 6\}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

   are transitive tournament matrices.

27.6 Convex Polytopes of Doubly Stochastic Matrices

Doubly stochastic matrices (see Chapter 9.4) are widely studied because of their connection with probability theory, as every doubly stochastic matrix is the transition matrix of a Markov chain. The reader is referred to Chapter 28 for graph terminology.

Definitions:

Since a convex combination $cA + (1 - c)B$ of two doubly stochastic matrices A and B, where $0 \leq c \leq 1$, is doubly stochastic, the set Ω_n of doubly stochastic matrices of order n is a convex polytope in $\mathbb{R}^{n^2}$. Ω_n is also called the **assignment polytope** because of its appearance in the classical assignment problem.

If A is a $(0, 1)$-matrix of order n, then $\mathcal{F}(A)$ is the convex polytope of all doubly stochastic matrices whose patterns P satisfy $P \leq A$ (entrywise), that is, that have 0s at least wherever A has 0s.

The **dimension** of a convex polytope $\mathcal{P}$ is the smallest dimension of an affine space containing it and is denoted by $\dim \mathcal{P}$.

The **graph** $G(\mathcal{P})$ of a convex polytope $\mathcal{P}$ has the extreme points of $\mathcal{P}$ as its vertices and the pairs of extreme points of one-dimensional faces of $\mathcal{P}$ as its edges.

The **chromatic index** of a graph is the smallest integer t such that the edges can be partitioned into sets $E_1, E_2, \ldots, E_t$ such that no two edges in the same E_i meet.

A **scaling** of a matrix A is a matrix of the form $D_1 A D_2$, where D_1 and D_2 are diagonal matrices with positive diagonal entries.

If $D_1 = D_2$, then the scaling $D_1 A D_2$ is a **symmetric scaling**.

Let $A = [a_{ij}]$ have order n. A **diagonal product** of A is the product of the entries on a diagonal of A, that is, $a_{1j_1} a_{2j_2} \cdots a_{nj_n}$ where $j_1, j_2, \ldots, j_n$ is a permutation of $\{1, 2, \ldots, n\}$.

The following facts can be found in Chapter 9 of [Bru06].

Facts:

1. (*Birkhoff's theorem*) The extreme points of Ω_n are the permutation matrices of order n. Thus, each doubly stochastic matrix is a convex combination of permutation matrices.
2. The patterns of matrices in Ω_n are precisely the $(0, 1)$-matrices of order n with total support.
3. The faces of Ω_n are the sets $\mathcal{F}(A)$, where A is a $(0, 1)$-matrix of order n with total support. Ω_n is a face of itself with $\Omega_n = \mathcal{F}(J_n)$. The dimension of $\mathcal{F}(A)$ satisfies

$$\dim \mathcal{F}(A) = t - 2n + k,$$

where t is the number of 1s of A and k is the number of fully indecomposable components of A. The number of extreme points of $\mathcal{F}(A)$ (this number is the permanent per(A) of A), is at least $t - 2n + k + 1$. If A is fully indecomposable and $\dim \mathcal{F}(A) = d$, then $\mathcal{F}(A)$ has at most $2^{d-1} + 1$ extreme points. In general, $\mathcal{F}(A)$ has at most 2^d extreme points.
4. The graph $G(\Omega_n)$ has the following properties:
   (a) The number of vertices of $G(\Omega_n)$ is $n!$.
   (b) The degree of each vertex of $G(\Omega_n)$ is

$$d_n = \sum_{k=2}^{n} \binom{n}{k} (k-1)!.$$

   (c) $G(\Omega_n)$ is connected and its diameter equals 1 if $n = 1, 2,$ and 3, and equals 2 if $n \geq 4$.
   (d) $G(\Omega_n)$ has a Hamilton cycle.
   (e) The chromatic index of $G(\Omega_n)$ equals d_n.
5. (*Hardy, Littlewood, Pólya theorem*) Let $U = (u_1, u_2, \ldots, u_n)$ and $V = (v_1, v_2, \ldots, v_n)$ be monotone, nonnegative integral vectors. Then $U \preceq V$ if and only if there is a doubly stochastic matrix A such that $U = VA$.
6. The set Υ_n of symmetric doubly stochastic matrices of order n is a subpolytope of Ω_n whose extreme points are those matrices A such that there is a permutation matrix P for which PAP^T is a direct sum of matrices each of which is either the identity matrix I_1 of order 1, the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, or an odd order matrix of the type:

$$\begin{bmatrix} 0 & 1/2 & 0 & 0 & \cdots & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 & \cdots & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & \cdots & 0 & 0 \\ 0 & 0 & 1/2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & \cdots & 1/2 & 0 \end{bmatrix}.$$

7. Let A be a nonnegative matrix. Then there is a scaling $B = D_1 A D_2$ of A that is doubly stochastic if and only if A has total support. If A is fully indecomposable, then the doubly stochastic matrix B is unique and the diagonal matrices D_1 and D_2 are unique up to reciprocal scalar factors.
8. Let A be a nonnegative symmetric matrix with no zero lines. Then there is a symmetric scaling $B = DAD$ such that B is doubly stochastic if and only if A has total support. If A is fully indecomposable, then the doubly stochastic matrix B and the diagonal matrix D are unique.
9. Distinct doubly stochastic matrices of order n do not have proportional diagonal products; that is, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are doubly stochastic matrices of order n with $A \neq B$, there does not exist a constant c such that $a_{1j_1} a_{2j_2} \cdots a_{nj_n} = c b_{1j_1} b_{2j_2} \cdots b_{nj_n}$ for all permutations $j_1, j_2, \ldots, j_n$ of $\{1, 2, \ldots, n\}$.

The subpolytopes of Ω_n consisting of (i) the convex combinations of the $n! - 1$ nonidentity permutation matrices of order n and (ii) the permutation matrices corresponding to the even permutations of order n have been studied (see [Bru06]).

Polytopes of matrices more general than Ω_n have also been studied, for instance, the nonnegative generalizations of $\mathcal{A}(R, S)$ consisting of all nonnegative matrices with a given row sum vector R and a given column sum vector S (see Chapter 8 of [Bru06]).

Examples:

1. Ω_2 consists of all matrices of the form
$$\begin{bmatrix} a & 1-a \\ 1-a & a \end{bmatrix}, \quad (0 \leq a \leq 1).$$
All permutation matrices are doubly stochastic.
2. The matrix
$$\begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/6 & 1/3 & 1/2 \\ 1/3 & 5/12 & 1/4 \end{bmatrix}$$
is a doubly stochastic matrix of order 3.
3. If
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$
then
$$\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$
is in $\mathcal{F}(A)$.

References

[BR97] R.B. Bapat and T.E.S. Raghavan, *Nonnegative Matrices and Applications*, Encyclopedia of Mathematical Sciences, No. 64, Cambridge University Press, Cambridge, 1997.

[Bru80] R.A. Brualdi, Matrices of zeros and ones with fixed row and column sum vectors, *Lin. Alg. Appl.*, 33: 159–231, 1980.

[Bru92] R.A. Brualdi, The symbiotic relationship of combinatorics and matrix theory, *Lin. Alg. Appl.*, 162–164: 65–105, 1992.

[Bru06] R.A. Brualdi, *Combinatorial Matrix Classes*, Encyclopedia of Mathematics and Its Applications, Vol. 108, Cambridge Universty Press, Cambridge, 2006.

[BR91] R.A. Brualdi and H.J. Ryser, *Combinatorial Matrix Theory*, Encyclopedia of Mathematics and its Applications, Vol. 39, Cambridge University Press, Cambridge, 1991.

[BS94] R.A. Brualdi and B.L. Shader, Strong Hall matrices, *SIAM J. Matrix Anal. Appl.*, 15: 359–365, 1994.

[BS04] R.A. Brualdi and B.L. Shader, Graphs and matrices, in *Topics in Algebraic Graph Theory*, L. Beineke and R. Wilson, Eds., Cambridge University press, Cambridge, 2004, 56–87.

[Rys63] H.J. Ryser, *Combinatorial Mathematics*, Carus Mathematical Monograph No. 14, Mathematical Association of America, Washington, D.C., 1963.

28

Matrices and Graphs

Willem H. Haemers
Tilburg University

28.1 Graphs: Basic Notions **28**-1
28.2 Special Graphs **28**-3
28.3 The Adjacency Matrix and Its Eigenvalues **28**-5
28.4 Other Matrix Representations **28**-7
28.5 Graph Parameters **28**-9
28.6 Association Schemes **28**-11
References **28**-12

The first two sections of this chapter "Matrices and Graphs" give a short introduction to graph theory. Unfortunately much graph theoretic terminology is not standard, so we had to choose. We allow, for example, graphs to have multiple edges and loops, and call a graph simple if it has none of these. On the other hand, we assume that graphs are finite.

For all nontrivial facts, references are given, sometimes to the original source, but often to text books or survey papers. A recent global reference for this chapter is [BW04]. (This book was not available to the author when this chapter was written, so it is not referred to in the text below.)

28.1 Graphs: Basic Notions

Definitions:

A **graph** $G = (V, E)$ consists of a finite set $V = \{v_1, \ldots, v_n\}$ of **vertices** and a finite multiset E of **edges**, where each edge is a pair $\{v_i, v_j\}$ of vertices (not necessarily distinct). If $v_i = v_j$, the edge is called a **loop**. A vertex v_i of an edge is called an **endpoint** of the edge.

The **order** of graph G is the number of vertices of G.

A **simple graph** is a graph with no loops where each edge has multiplicity at most one.

Two graphs (V, E) and (V', E') are **isomorphic** whenever there exist bijections $\phi : V \to V'$ and $\psi : E \to E'$, such that $v \in V$ is an endpoint of $e \in E$ if and only if $\phi(v)$ is an endpoint of $\psi(e)$.

A **walk** of **length** ℓ in a graph is an alternating sequence $(v_{i_0}, e_{i_1}, v_{i_1}, e_{i_2}, \ldots, e_{i_\ell}, v_{i_\ell})$ of vertices and edges (not necessarily distinct), such that $v_{i_{j-1}}$ and v_{i_j} are endpoints of e_{i_j} for $j = 1, \ldots, \ell$.

A **path** of **length** ℓ in a graph is a walk of length ℓ with all vertices distinct.

A **cycle** of **length** ℓ in a graph is a walk $(v_{i_0}, e_{i_1}, v_{i_1}, e_{i_2}, \ldots, e_{i_\ell}, v_{i_\ell})$ with $v_{i_0} = v_{i_\ell}$, $\ell \neq 0$, and $v_{i_1}, \ldots, v_{i_\ell}$ all distinct.

A **Hamilton cycle** in a graph is a cycle that includes all vertices.

A graph (V, E) is **connected** if $V \neq \emptyset$ and there exists a walk between any two distinct vertices of V.

The **distance** between two vertices v_i and v_j of a graph G (denoted by $d_G(v_i, v_j)$ or $d(v_i, v_j)$) is the length of a shortest path between v_i and v_j. ($d(v_i, v_j) = 0$ if $i = j$, and $d(v_i, v_j)$ is infinite if there is no path between v_i and v_j.)

The **diameter** of a connected graph G is the largest distance that occurs between two vertices of G.

A **tree** is a connected graph with no cycles.

A **forest** is a graph with no cycles.

A graph (V', E') is a **subgraph** of a graph (V, E) if $V' \subseteq V$ and $E' \subseteq E$. If E' contains all edges from E with endpoints in V', (V', E') is an **induced** subgraph of (V, E).

A **spanning subgraph** of a connected graph (V, E) is a subgraph (V', E') with $V' = V$, which is connected.

A **spanning tree** of a connected graph (V, E) is a spanning subgraph, which is a tree.

A **connected component** of a graph (V, E) is an induced subgraph (V', E'), which is connected and such that there exists no edge in E with one endpoint in V' and one outside V'. A connected component with one vertex and no edge is called an **isolated vertex**.

Two graphs (V, E) and (V', E') are **disjoint** if V and V' are disjoint sets.

Two vertices u and v are **adjacent** if there exists an edge with endpoints u and v. A vertex adjacent to v is called a **neighbor** of v.

The **degree** or **valency** of a vertex v of a graph G (denoted by $\delta_G(v)$ or $\delta(v)$) is the number of times that v occurs as an endpoint of an edge (that is, the number of edges containing v, where loops count as 2).

A graph (V, E) is **bipartite** if the vertex set V admits a partition into two parts, such that no edge of E has both endpoints in one part (thus, there are no loops). More information on bipartite graphs is given in Chapter 30.

A simple graph (V, E) is **complete** if E consists of all unordered pairs from V. The (isomorphism class of the) complete graph on n vertices is denoted by K_n.

A graph (V, E) is **empty** if $E = \emptyset$. If also $V = \emptyset$, it is called the **null graph**.

A bipartite simple graph (V, E) with nonempty parts V_1 and V_2 is **complete bipartite** if E consists of all unordered pairs from V with one vertex in V_1 and one in V_2. The (isomorphism class of the) complete bipartite graph is denoted by K_{n_1,n_2}, where $n_1 = |V_1|$ and $n_2 = |V_2|$.

The (isomorphism class of the) simple graph that consists only of vertices and edges of a path of length ℓ is called **the path of length** ℓ, and denoted by $P_{\ell+1}$.

The (isomorphism class of the) simple graph that consists only of vertices and edges of a cycle of length ℓ is called **the cycle of length** ℓ, and denoted by C_ℓ.

The **complement** of a simple graph $G = (V, E)$ is the simple graph $\overline{G} = (V, \overline{E})$, where $\overline{E}$ consists of all unordered pairs from V that are not in E.

The **union** $G \cup G'$ of two graphs $G = (V, E)$ and $G' = (V', E')$ is the graph with vertex set $V \cup V'$, and edge (multi)set $E \cup E'$.

The **intersection** $G \cap G'$ of two graphs $G = (V, E)$ and $G' = (V', E')$ is the graph with vertex set $V \cap V'$, and edge (multi)set $E \cap E'$.

The **join** $G + G'$ of two disjoint graphs $G = (V, E)$ and $G' = (V', E')$ is the union of $G \cup G'$ and the complete bipartite graph with vertex set $V \cup V'$ and partition $\{V, V'\}$.

The **(strong) product** $G \cdot G'$ of two simple graphs $G = (V, E)$ and $G' = (V', E')$ is the simple graph with vertex set $V \times V'$, where two distinct vertices are adjacent whenever in both coordinate places the vertices are adjacent or equal in the corresponding graph. The strong product of ℓ copies of a graph G is denoted by G^ℓ.

Facts:

The facts below are elementary results that can be found in almost every introduction to graph theory, such as [Har69] or [Wes01].

1. For any graph, the sum of its degrees equals twice the number of edges; therefore, the number of vertices with odd degree is even.
2. For any simple graph, at least two vertices have the same degree.
3. A graph G is bipartite if and only if G has no cycles of odd length.
4. A tree with n vertices has $n - 1$ edges.

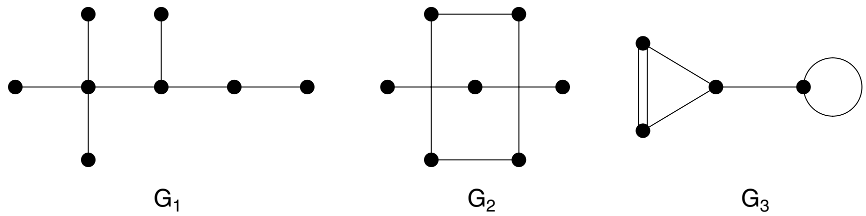


FIGURE 28.1 Three graphs. (Vertices are represented by points and an edge is represented by a line segment between the endpoints, or a loop.)

5. A graph is a tree if and only if there is a unique path between any two vertices.
6. A graph G is connected if and only if G cannot be expressed as the union of two or more mutually disjoint connected graphs.

Examples:

1. Consider the complete bipartite graph $K_{3,3} = (V_1 \cup V_2, E)$ with parts $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6\}$. Then

$$v_1, \{v_1, v_5\}, v_5, \{v_5, v_2\}, v_2, \{v_2, v_6\}, v_6, \{v_6, v_3\}, v_3$$

is a path of length 4 between v_1 and v_3,

$$v_1, \{v_1, v_5\}, v_5, \{v_5, v_2\}, v_2, \{v_2, v_5\}, v_5, \{v_5, v_3\}, v_3$$

is a walk of length 4, which is not a path, and

$$v_1, \{v_1, v_5\}, v_5, \{v_5, v_2\}, v_2, \{v_2, v_6\}, v_6, \{v_6, v_1\}, v_1$$

is a cycle of length 4.
2. Graphs G_1 and G_2 in Figure 28.1 are simple, but G_3 is not.
3. Graphs G_1 and G_2 are bipartite, but G_3 is not.
4. Graph G_1 is a tree. Its diameter equals 4.
5. Graph G_2 is not connected; it is the union of two disjoint graphs, a path P_3 and a cycle C_4. The complement $\overline{G_2}$ is connected and can be expressed as the join of $\overline{P_3}$ and $\overline{C_4}$.
6. Graph G_3 contains three kinds of cycles, cycles of length 3 (corresponding to the two triangles), cycles of length 2 (corresponding to the pair of multiple edges), and one of length 1 (corresponding to the loop).

28.2 Special Graphs

A graph G is **regular** (or k-**regular**) if every vertex of G has the same degree (equal to k).

A graph G is **walk-regular** if for every vertex v the number of walks from v to v of length ℓ depends only on ℓ (not on v).

A simple graph G is **strongly regular** with parameters (n, k, λ, μ) whenever G has n vertices and

- G is k-regular with $1 \leq k \leq n - 2$.
- Every two adjacent vertices of G have exactly λ common neighbors.
- Every two distinct nonadjacent vertices of G have exactly μ common neighbors.

An **embedding** of a graph in $\mathbb{R}^n$ consists of a representation of the vertices by distinct points in $\mathbb{R}^n$, and a representation of the edges by curve segments between the endpoints, such that a curve segment intersects another segment or itself only in an endpoint. (A curve segment between $\mathbf{x}$ and $\mathbf{y}$ is the range of a continuous map ϕ from $[0, 1]$ to $\mathbb{R}^n$ with $\phi(0) = \mathbf{x}$ and $\phi(1) = \mathbf{y}$.)

A graph is **planar** if it admits an embedding in $\mathbb{R}^2$.

A graph is **outerplanar** if it admits an embedding in $\mathbb{R}^2$, such that the vertices are represented by points on the unit circle, and the representations of the edges are contained in the unit disc.

A graph G is **linklessly embeddable** if it admits an embedding in $\mathbb{R}^3$, such that no two disjoint cycles of G are linked. (Two disjoint Jordan curves in $\mathbb{R}^3$ are linked if there is no topological 2-sphere in $\mathbb{R}^3$ separating them.)

Deletion of an edge e from a graph $G = (V, E)$ is the operation that deletes e from E and results in the subgraph $G - e = (V, E \setminus \{e\})$ of G.

Deletion of a vertex v from a graph $G = (V, E)$ is the operation that deletes v from V and all edges with endpoint v from E. The resulting subgraph of G is denoted by $G - v$.

Contraction of an edge e of a graph (V, E) is the operation that merges the endpoints of e in V, and deletes e from E.

A **minor** of a graph G is any graph that can be obtained from G by a sequence of edge deletions, vertex deletions, and contractions.

Let G be a simple graph. The **line graph** $L(G)$ of G has the edges of G as vertices, and vertices of $L(G)$ are adjacent if the corresponding edges of G have an endpoint in common.

The **cocktail party graph** $CP(a)$ is the graph obtained by deleting a disjoint edges from the complete graph K_{2a}. (Note that $CP(0)$ is the null graph.)

Let G be a simple graph with vertex set $\{v_1, \ldots, v_n\}$, and let $a_1, \ldots, a_n$ be nonnegative integers. The **generalized line graph** $L(G; a_1, \ldots, a_n)$ consists of the disjoint union of the line graph $L(G)$ and the cocktail party graphs $CP(a_1), \ldots, CP(a_n)$, together with all edges joining a vertex $\{v_i, v_j\}$ of $L(G)$ with each vertex of $CP(a_i)$ and $CP(a_j)$.

Facts:

If no reference is given, the fact is trivial or a classical result that can be found in almost every introduction to graph theory, such as [Har69] or [Wes01].

1. [God93, p. 81] A strongly regular graph is walk-regular.
2. A walk-regular graph is regular.
3. The complement of a strongly regular graph with parameters (n, k, λ, μ) is strongly regular with parameters $(n, n - k - 1, n - 2k + \mu - 2, n - 2k + \lambda)$.
4. Every graph can be embedded in $\mathbb{R}^3$.
5. [RS04] (Robertson, Seymour) For every graph property $\mathcal{P}$ that is closed under taking minors, there exists a finite list of graphs such that a graph G has property $\mathcal{P}$ if and only if no graph from the list is a minor of G.
6. The graph properties: planar, outerplanar, and linklessly embeddable are closed undertaking minors.
7. (Kuratowski, Wagner) A graph G is planar if and only if no minor of G is isomorphic to K_5 or $K_{3,3}$.
8. [CRS04, p. 8] A regular generalized line graph is a line graph or a cocktail party graph.
9. (Whitney) The line graphs of two connected nonisomorphic graphs G and G' are nonisomorphic, unless $\{G, G'\} = \{K_3, K_{1,3}\}$.

Examples:

1. Graph G_3 of Figure 28.1 is regular of degree 3.
2. The complete graph K_n is walk-regular and regular of degree $n - 1$.
3. The complete bipartite graph $K_{k,k}$ is regular of degree k, walk-regular and strongly regular with parameters $(2k, k, 0, k)$.
4. Examples of outerplanar graphs are all trees, C_n, and $\overline{P_5}$.
5. Examples of graphs that are planar, but not outerplanar are: K_4, $CP(3)$, $\overline{C_6}$, and $K_{2,n-2}$ for $n \geq 5$.

6. Examples of graphs that are not planar, but linklessly embeddable are: K_5, and $K_{3,n-3}$ for $n \geq 6$.
7. The Petersen graph (Figure 28.2) and K_n for $n \geq 6$ are not linklessly embeddable.
8. The complete graph K_5 can be obtained from the Petersen graph by contraction with respect to five mutually disjoint edges. Therefore, K_5 is a minor of the Petersen graph.
9. The cycle C_9 is a subgraph of the Petersen graph and, therefore, the Petersen graph has every cycle C_ℓ with $\ell \leq 9$ as a minor.
10. Figure 28.3 gives a simple graph G, the line graph $L(G)$, and the generalized line graph $L(G; 2, 1, 0, 0, 0)$ (the vertices of G are ordered from left to right).
11. For $n \geq 4$ and $k \geq 2$ the line graphs $L(K_n)$ and $L(K_{k,k})$ and their complements are strongly regular. The complement of $L(K_5)$ is the Petersen graph.

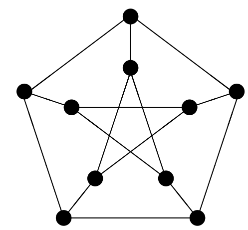


FIGURE 28.2 The Petersen graph.

28.3 The Adjacency Matrix and Its Eigenvalues

Definitions:

The **adjacency matrix** $\mathcal{A}_G$ of a graph G with vertex set $\{v_1, \ldots, v_n\}$ is the symmetric $n \times n$ matrix, whose (i, j)th entry is equal to the number of edges between v_i and v_j.

The **eigenvalues** of a graph G are the eigenvalues of its adjacency matrix.

The **spectrum** $\sigma(G)$ of a graph G is the multiset of eigenvalues (that is, the eigenvalues with their multiplicities).

Two graphs are **cospectral** whenever they have the same spectrum.

A graph G is **determined by its spectrum** if every graph cospectral with G is isomorphic to G.

The **characteristic polynomial** $p_G(x)$ of a graph G is the characteristic polynomial of its adjacency matrix $\mathcal{A}_G$, that is, $p_G(x) = \det(xI - \mathcal{A}_G)$.

A **Hoffman polynomial** of a graph G is a polynomial $h(x)$ of minimum degree such that $h(\mathcal{A}_G) = J$.

The **main angles** of a graph G are the cosines of the angles between the eigenspaces of $\mathcal{A}_G$ and the all-ones vector **1**.

Facts:

If no reference is given, the fact is trivial or a standard result in algebraic graph theory that can be found in the classical books [Big74] and [CDS80].

1. If $\mathcal{A}_G$ is the adjacency matrix of a simple graph G, then $J - I - \mathcal{A}_G$ is the adjacency matrix of the complement of G.
2. If $\mathcal{A}_G$ and $\mathcal{A}_{G'}$ are adjacency matrices of simple graphs G and G', respectively, then $((\mathcal{A}_G + I) \otimes (\mathcal{A}_{G'} + I)) - I$ is the adjacency matrix of the strong product $G \cdot G'$.
3. Isomorphic graphs are cospectral.

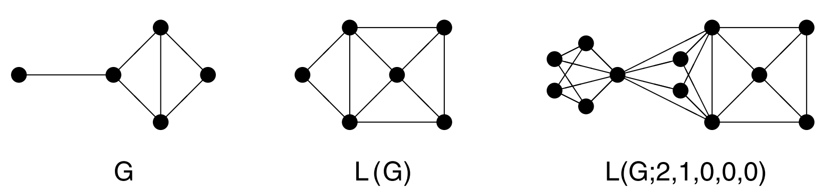


FIGURE 28.3 A graph with its line graph and a generalized line graph.

4. Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$ and adjacency matrix $\mathcal{A}_G$. The number of walks of length ℓ from v_i to v_j equals $(\mathcal{A}_G^{\ell})_{ij}$, i.e., the i, j-entry of $\mathcal{A}_G^{\ell}$.
5. The eigenvalues of a graph are real numbers.
6. The adjacency matrix of a graph is diagonalizable.
7. If $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of a graph G, then $|\lambda_i| \leq \lambda_1$. If $\lambda_1 = \lambda_2$, then G is disconnected. If $\lambda_1 = -\lambda_n$ and G is not empty, then at least one connected component of G is nonempty and bipartite.
8. [CDS80, p. 87] If $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of a graph G, then G is bipartite if and only if $\lambda_i = -\lambda_{n+1-i}$ for $i = 1, \ldots, n$. For more information on bipartite graphs see Chapter 30.
9. If G is a simple k-regular graph, then the largest eigenvalue of G equals k, and the multiplicity of k equals the number of connected components of G.
10. [CDS80, p. 94] If $\lambda_1 \geq \ldots \geq \lambda_n$ are the eigenvalues of a simple graph G with n vertices and m edges, then $\sum_i \lambda_i^2 = 2m \leq n\lambda_1$. Equality holds if and only if G is regular.
11. [CDS80, p. 95] A simple graph G has a Hoffman polynomial if and only if G is regular and connected.
12. [CRS97, p. 99] Suppose G is a simple graph with n vertices, r distinct eigenvalues $\nu_1, \ldots, \nu_r$, and main angles $\beta_1, \ldots, \beta_r$. Then the complement $\overline{G}$ of G has characteristic polynomial
$$p_{\overline{G}}(x) = (-1)^n p_G(-x-1) \left(1 - n \sum_{i=1}^{r} \beta_i^2/(x+1+\nu_i)\right).$$
13. [CDS80, p. 103], [God93, p. 179] A connected simple regular graph is strongly regular if and only if it has exactly three distinct eigenvalues. The eigenvalues ($\nu_1 > \nu_2 > \nu_3$) and parameters (n, k, λ, μ) are related by $\nu_1 = k$ and
$$\nu_2, \nu_3 = \frac{1}{2}\left(\lambda - \mu \pm \sqrt{(\mu-\lambda)^2 + 4(k-\mu)}\right).$$
14. [BR91, p. 150], [God93, p. 180] The multiplicities of the eigenvalues ν_1, ν_2, and ν_3 of a connected strongly regular graph with parameters (n, k, λ, μ) are 1 and
$$\frac{1}{2}\left(n - 1 \pm \frac{(n-1)(\mu-\lambda) - 2k}{\sqrt{(\mu-\lambda)^2 + 4(k-\mu)}}\right) \quad \text{(respectively)}.$$
15. [GR01, p. 190] A regular simple graph with at most four distinct eigenvalues is walk-regular.
16. [CRS97, p. 79] Cospectral walk-regular simple graphs have the same main angles.
17. [Sch73] Almost all trees are cospectral with another tree.
18. [DH03] The number of nonisomorphic simple graphs on n vertices, not determined by the spectrum, is asymptotically bounded from below by $n^3 g_{n-1}(\frac{1}{24} - o(1))$, where g_{n-1} denotes the number of nonisomorphic simple graphs on $n-1$ vertices.
19. [DH03] The complete graph, the cycle, the path, the regular complete bipartite graph, and their complements are determined by their spectrum.
20. [DH03] Suppose G is a regular connected simple graph on n vertices, which is determined by its spectrum. Then also the complement $\overline{G}$ of G is determined by its spectrum, and if $n+1$ is not a square, also the line graph $L(G)$ of G is determined by its spectrum.
21. [CRS04, p. 7] A simple graph G is a generalized line graph if and only if the adjacency matrix $\mathcal{A}_G$ can be expressed as $\mathcal{A}_G = C^T C - 2I$, where C is an integral matrix with exactly two nonzero entries in each column. (It follows that the nonzero entries are ± 1.)
22. [CRS04, p. 7] A generalized line graph has smallest eigenvalue at least -2.
23. [CRS04, p. 85] A connected simple graph with more than 36 vertices and smallest eigenvalue at least -2 is a generalized line graph.
24. [CRS04, p. 90] There are precisely 187 connected regular simple graphs with smallest eigenvalue at least -2 that are not a line graph or a cocktail party graph. Each of these graphs has smallest eigenvalue equal to -2, at most 28 vertices, and degree at most 16.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

FIGURE 28.4 Two cospectral graphs with their adjacency matrices.

Examples:

1. Figure 28.4 gives a pair of nonisomorphic bipartite graphs with their adjacency matrices. Both matrices have spectrum $\{2,\ 0^3,\ -2\}$ (exponents indicate multiplicities), so the graphs are cospectral.
2. The main angles of the two graphs of Figure 28.4 (with the given ordering of the eigenvalues) are $2/\sqrt{5},\ 1/\sqrt{5},\ 0$ and $3/\sqrt{10},\ 0, 1/\sqrt{10}$, respectively.
3. The spectrum of K_{n_1,n_2} is $\{\sqrt{n_1 n_2},\ 0^{n-2},\ -\sqrt{n_1 n_2}\}$.
4. By Fact 14, the multiplicities of the eigenvalues of any strongly regular graph with parameters $(n, k, 1, 1)$ would be nonintegral, so no such graph can exist (this result is known as the Friendship theorem).
5. The Petersen graph has spectrum $\{3,\ 1^5,\ -2^4\}$ and Hoffman polynomial $(x-1)(x+2)$. It is one of the 187 connected regular graphs with least eigenvalue -2, which is neither a line graph nor a cocktail party graph.
6. The eigenvalues of the path P_n are $2\cos\frac{i\pi}{n+1}$ $(i = 1, \ldots, n)$.
7. The eigenvalues of the cycle C_n are $2\cos\frac{2i\pi}{n}$ $(i = 1, \ldots, n)$.

28.4 Other Matrix Representations

Definitions:

Let G be a simple graph with adjacency matrix $\mathcal{A}_G$. Suppose D is the diagonal matrix with the degrees of G on the diagonal (with the same vertex ordering as in $\mathcal{A}_G$). Then $L_G = D - \mathcal{A}_G$ is the **Laplacian matrix** of G (often abbreviated to the **Laplacian**, and also known as **admittance matrix**), and the matrix $|L_G| = D + \mathcal{A}_G$ is (sometimes) called the **signless Laplacian matrix**.

The **Laplacian eigenvalues** of a simple graph G are the eigenvalues of the Laplacian matrix L_G.

If $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$ are the Laplacian eigenvalues of G, then μ_2 is called the **algebraic connectivity** of G. (See section 28.6 below.)

Let G be simple graph with vertex set $\{v_1, \ldots, v_n\}$. A symmetric real matrix $M = [m_{ij}]$ is called a **generalized Laplacian** of G, whenever $m_{ij} < 0$ if v_i and v_j are adjacent, and $m_{ij} = 0$ if v_i and v_j are nonadjacent and distinct (nothing is required for the diagonal entries of M).

Let G be a graph without loops with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{e_1, \ldots, e_m\}$. The **(vertex-edge) incidence matrix** of G is the $n \times m$ matrix N_G defined by $(N_G)_{ij} = 1$ if vertex v_i is an endpoint of edge e_j and $(N_G)_{ij} = 0$ otherwise.

An **oriented (vertex-edge) incidence matrix** of G is a matrix N'_G obtained from N_G by replacing a 1 in each column by a -1, and thereby orienting each edge of G.

If $\mathcal{A}_G$ is the adjacency matrix of a simple graph G, then $S_G = J - I - 2\mathcal{A}_G$ is the **Seidel matrix** of G.

Let G be a simple graph with Seidel matrix S_G, and let I' be a diagonal matrix with ± 1 on the diagonal. Then the simple graph G' with Seidel matrix $S_{G'} = I' S_G I'$ is **switching equivalent** to G. The graph operation that changes G into G' is called **Seidel switching**.

Facts:

In all facts below, G is a simple graph. If no reference is given, the fact is trivial or a classical result that can be found in [BR91].

1. Let G be a simple graph. The Laplacian matrix L_G and the signless Laplacian $|L_G|$ are positive semidefinite.
2. The nullity of L_G is equal to the number of connected components of G.
3. The nullity of $|L_G|$ is equal to the number of connected components of G that are bipartite.
4. [DH03] The Laplacian and the signless Laplacian of a graph G have the same spectrum if and only if G is bipartite.
5. (Matrix-tree theorem) Let G be a graph with Laplacian matrix L_G, and let c_G denote the number of spanning trees of G. Then $\operatorname{adj}(L_G) = c_G J$.
6. Suppose N_G is the incidence matrix of G. Then $N_G N_G^T = |L_G|$ and $N_G^T N_G - 2I = \mathcal{A}_{L(G)}$.
7. Suppose N'_G is an oriented incidence matrix of G. Then $N'_G N'^T_G = L_G$.
8. If $\mu_1 \leq \ldots \leq \mu_n$ are the Laplacian eigenvalues of G, and $\overline{\mu}_1 \leq \ldots \leq \overline{\mu}_n$ are the Laplacian eigenvalues of $\overline{G}$, then $\mu_1 = \overline{\mu}_1 = 0$ and $\overline{\mu}_i = n - \mu_{n+2-i}$ for $i = 2, \ldots, n$.
9. [DH03] If $\mu_1 \leq \ldots \leq \mu_n$ are the Laplacian eigenvalues of a graph G with n vertices and m edges, then $\sum_i \mu_i = 2m \leq \sqrt{n \sum_i \mu_i(\mu_i - 1)}$ with equality if and only if G is regular.
10. [DH98] A connected graph G has at most three distinct Laplacian eigenvalues if and only if there exist integers μ and $\overline{\mu}$, such that any two distinct nonadjacent vertices have exactly μ common neighbors, and any two adjacent vertices have exactly $\overline{\mu}$ common nonneighbors.
11. If G is k-regular and $\mathbf{v} \notin \operatorname{span}\{\mathbf{1}\}$, then the following are equivalent:
    - λ is an eigenvalue of $\mathcal{A}_G$ with eigenvector $\mathbf{v}$.
    - $k - \lambda$ is an eigenvalue of L_G with eigenvector $\mathbf{v}$.
    - $k + \lambda$ is an eigenvalue of $|L_G|$ with eigenvector $\mathbf{v}$.
    - $-1 - 2\lambda$ is an eigenvalue of S_G with eigenvector $\mathbf{v}$.
12. [DH03] Consider a simple graph G with n vertices and m edges. Let $\nu_1 \leq \ldots \leq \nu_n$ be the eigenvalues of $|L_G|$, the signless Laplacian of G. Let $\lambda_1 \geq \ldots \geq \lambda_m$ be the eigenvalues of $L(G)$, the line graph of G. Then $\lambda_i = \nu_{n-i+1} - 2$ if $1 \leq i \leq \min\{m, n\}$, and $\lambda_i = -2$ if $\min\{m, n\} < i \leq m$.
13. [GR01, p. 298] Let G be a connected graph, let M be a generalized Laplacian of G, and let $\mathbf{v}$ be an eigenvector for M corresponding to the second smallest eigenvalue of M. Then the subgraph of G induced by the vertices corresponding to the positive entries of $\mathbf{v}$ is connected.
14. The Seidel matrices of switching equivalent graphs have the same spectrum.

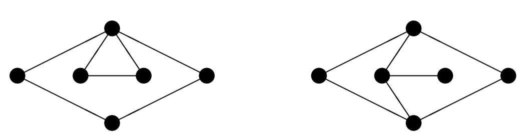


FIGURE 28.5 Graphs with cospectral Laplacian matrices.

FIGURE 28.6 Graphs with cospectral signless Laplacian matrices.

Examples:

1. The Laplacian eigenvalues of the Petersen graph are $\{0,\ 2^5,\ 5^4\}$.
2. The two graphs of Figure 28.5 are nonisomorphic, but the Laplacian matrices have the same spectrum. Both Laplacian matrices have $12J$ as adjugate, so both have 12 spanning trees. They are not cospectral with respect to the adjacency matrix because one is bipartite and the other one is not.
3. Figure 28.6 gives two graphs with cospectral signless Laplacian matrices. They are not cospectral with respect to the adjacency matrix because one is bipartite and the other one is not. They also do not have cospectral Laplacian matrices because the numbers of components differ.
4. The eigenvalues of the Laplacian and the signless Laplacian matrix of the path P_n are $2 + 2\cos\frac{i\pi}{n}$ $(i = 1, \ldots, n)$.
5. The complete bipartite graph K_{n_1,n_2} is Seidel switching equivalent to the empty graph on $n = n_1+n_2$ vertices. The Seidel matrices have the same spectrum, being $\{n-1, -1^{n-1}\}$.

28.5 Graph Parameters

Definitions:

A subgraph G' on n' vertices of a simple graph G is a **clique** if G' is isomorphic to the complete graph $K_{n'}$. The largest value of n' for which a clique with n' vertices exists is called the **clique number** of G and is denoted by $\omega(G)$.

An induced subgraph G' on n' vertices of a graph G is a **coclique** or **independent set of vertices** if G' has no edges. The largest value of n' for which a coclique with n' vertices exists is called the **vertex independence number** of G and is denoted by $\iota(G)$. Note that the standard notation for the vertex independence number of G is $\alpha(G)$, but $\iota(G)$ is used here due to conflict with the use of $\alpha(G)$ to denote the algebraic connectivity of G in Chapter 36.

The **Shannon capacity** $\Theta(G)$ of a simple graph G is defined by $\Theta(G) = \sup_\ell \sqrt[\ell]{\iota(G^\ell)}$

A **vertex coloring** of a graph is a partition of the vertex set into cocliques. A coclique in such a partition is called a **color class**.

The **chromatic number** $\chi(G)$ of a graph G is the smallest number of color classes of any vertex coloring of G. (The chromatic number is not defined if G has loops.)

For a simple graph $G = (V, E)$, the **conductance** or **isoperimetric number** $\Phi(G)$ is defined to be the minimum value of $\partial(V')/|V'|$ over any subset $V' \subset V$ with $|V'| \leq |V|/2$, where $\partial(V')$ equals the number of edges in E with one endpoint in V' and one endpoint outside V'.

An infinite family of graphs with constant degree and isoperimetric number bounded from below is called a family of **expanders**.

A symmetric real matrix M is said to satisfy the **Strong Arnold Hypothesis** provided there does not exist a symmetric nonzero matrix X with zero diagonal, such that $MX = \mathbf{0}$, $M \circ X = \mathbf{0}$.

The **Colin de Verdière parameter** $\mu(G)$ of a simple graph G is the largest nullity of any generalized Laplacian M of G satisfying the following:

- M has exactly one negative eigenvalue of multiplicity 1.
- The Strong Arnold Hypothesis.

Consider a simple graph G with vertex set $\{v_1, \ldots, v_n\}$. The **Lovász parameter** $\vartheta(G)$ is the minimum value of the largest eigenvalue $\lambda_1(M)$ of any real symmetric $n \times n$ matrix $M = [m_{ij}]$, which satisfies $m_{ij} = 1$ if v_i and v_j are nonadjacent (including the diagonal).

Consider a simple graph G with vertex set $\{v_1, \ldots, v_n\}$. The integer $\eta(G)$ is defined to be the smallest rank of any $n \times n$ matrix M (over any field), which satisfies $m_{ii} \neq 0$ for $i = 1, \ldots, n$ and $m_{ij} = 0$, if v_i and v_j are distinct nonadjacent vertices.

Facts:

In the facts below, all graphs are simple.

1. [Big74, p. 13] A connected graph with r distinct eigenvalues (for the adjacency, the Laplacian or the signless Laplacian matrix) has diameter at most $r - 1$.
2. [CDS80, pp. 90–91], [God93, p. 83] The chromatic number $\chi(G)$ of a graph G with adjacency eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ satisfies: $1 - \lambda_1/\lambda_n \leq \chi(G) \leq 1 + \lambda_1$.
3. [CDS80, p. 88] For a graph G, let m_+ and m_- denote the number of nonnegative and nonpositive adjacency eigenvalues, respectively. Then $\iota(G) \leq \min\{m_+, m_-\}$.
4. [GR01, p. 204] If G is a k-regular graph with adjacency eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$, then $\omega(G) \leq \frac{n(\lambda_2+1)}{n-k+\lambda_2}$ and $\iota(G) \leq \frac{-n\lambda_n}{k-\lambda_n}$.
5. [Moh97] Suppose G is a graph with maximum degree Δ and algebraic connectivity μ_2. Then the isoperimetric number $\Phi(G)$ satisfies $\mu_2/2 \leq \Phi(G) \leq \sqrt{\mu_2(2\Delta - \mu_2)}$.
6. [HLS99] The Colin de Verdière parameter $\mu(G)$ is minor monotonic, that is, if H is a minor of G, then $\mu(H) \leq \mu(G)$.
7. [HLS99] If G has at least one edge, then $\mu(G) = \max\{\mu(H) \mid H \text{ is a component of } G\}$.
8. [HLS99] The Colin de Verdière parameter $\mu(G)$ satisfies the following:
   - $\mu(G) \leq 1$ if and only if G is the disjoint union of paths.
   - $\mu(G) \leq 2$ if and only if G is outerplanar.
   - $\mu(G) \leq 3$ if and only if G is planar.
   - $\mu(G) \leq 4$ if and only if G is linklessly embeddable.
9. (Sandwich theorems)[Lov79], [Hae81] The parameters $\vartheta(G)$ and $\eta(G)$ satisfy: $\iota(G) \leq \vartheta(G) \leq \chi(\overline{G})$ and $\iota(G) \leq \eta(G) \leq \chi(\overline{G})$.
10. [Lov79], [Hae81] The parameters $\vartheta(G)$ and $\eta(G)$ satisfy: $\vartheta(G \cdot H) = \vartheta(G)\vartheta(H)$ and $\eta(G \cdot H) \leq \eta(G)\eta(H)$.
11. [Lov79], [Hae81] The Shannon capacity $\Theta(G)$ of a graph G satisfies: $\iota(G) \leq \Theta(G)$, $\Theta(G) \leq \vartheta(G)$, and $\Theta(G) \leq \eta(G)$.
12. [Lov79], [Hae81] If G is a k-regular graph with eigenvalues $k = \lambda_1 \geq \ldots \geq \lambda_n$, then $\vartheta(G) \leq -n\lambda_n/(k - \lambda_n)$. Equality holds if G is strongly regular.
13. [Lov79] The Lovász parameter $\vartheta(G)$ can also be defined as the maximum value of $\mathrm{tr}(MJ_n)$, where M is any positive semidefinite $n \times n$ matrix, satisfying $\mathrm{tr}(M) = 1$ and $m_{ij} = 0$ if v_i and v_j are adjacent vertices in G.

Examples:

1. Suppose G is the Petersen graph. Then $\iota(G) = 4$, $\vartheta(G) = 4$ (by Facts 9 and 12). Thus, $\Theta(G) = 4$ (by Fact 11). Moreover, $\chi(G) = 3$, $\chi(\overline{G}) = 5$, $\mu(G) = 5$ (take $M = L_G - 2I$), and $\eta(G) = 4$ (take $M = \mathcal{A}_G + I$ over the field with two elements).
2. The isoperimetric number $\Phi(G)$ of the Petersen graph equals 1. Indeed, $\Phi(G) \geq 1$, by Fact 5, and any pentagon gives $\Phi(G) \leq 1$.
3. $\mu(K_n) = n - 1$ (take $M = -J$).
4. If G is the empty graph with at least two vertices, then $\mu(G) = 1$. (M must be a diagonal matrix with exactly one negative entry, and the Strong Arnold Hypothesis forbids two or more diagonal entries to be 0.)

5. By Fact 12, $\vartheta(C_5) = \sqrt{5}$. If $(v_1, \ldots, v_5)$ are the vertices of C_5, cyclically ordered, then (v_1, v_1), $(v_2, v_3), (v_3, v_5), (v_4, v_2), (v_5, v_4)$ is a coclique of size 5 in $C_5 \cdot C_5$. Thus, $\iota(C_5 \cdot C_5) \geq 5$ and, therefore, $\Theta(C_5) = \sqrt{5}$.

28.6 Association Schemes

Definitions:

A set of graphs $G_0, \ldots, G_d$ on a common vertex set $V = \{v_1, \ldots, v_n\}$ is an **association scheme** if the adjacency matrices $A_0, \ldots, A_d$ satisfy:

- $A_0 = I$.
- $\sum_{i=0}^{d} A_i = J$.
- $\text{span}\{A_0, \ldots, A_d\}$ is closed under matrix multiplication.

The numbers $p_{i,j}^k$ defined by $A_i A_j = \sum_{i=0}^{d} p_{i,j}^k A_k$ are called the **intersection numbers** of the association scheme.

The (associative) algebra spanned by $A_0, \ldots, A_d$ is the **Bose–Mesner algebra** of the association scheme.

Consider a connected graph $G_1 = (V, E_1)$ with diameter d. Define $G_i = (V, E_i)$ to be the graph wherein two vertices are adjacent if their distance in G_1 equals i. If $G_0, \ldots, G_d$ is an association scheme, then G_1 is a **distance-regular graph**.

Let V' be a subset of the vertex set V of an association scheme. The **inner distribution a** $= [a_0, \ldots, a_d]^T$ of V' is defined by $a_i|V'| = \mathbf{c}^T A_i \mathbf{c}$, where $\mathbf{c}$ is the characteristic vector of V' (that is, $c_i = 1$ if $v_i \in V$ and $c_i = 0$ otherwise).

Facts:

Facts 1 to 7 below are standard results on association schemes that can be found in any of the following references: [BI84], [BCN89], [God93].

1. Suppose $G_0, \ldots, G_d$ is an association scheme. For any three integers $i, j, k \in \{0, \ldots, d\}$ and for any two vertices x and y adjacent in G_k, the number of vertices z adjacent to x in G_i and to y in G_j equals the intersection number p_{ij}^k. In particular, G_i is regular of degree $k_i = p_{ii}^0$ $(i \neq 0)$.
2. The matrices of a Bose–Mesner algebra $\mathcal{A}$ can be diagonalized simultaneously. In other words, there exists a nonsingular matrix S such that SAS^{-1} is a diagonal matrix for every $A \in \mathcal{A}$.
3. A Bose–Mesner algebra has a basis $\{E_0 = \frac{1}{n}J, E_1, \ldots, E_d\}$ of idempotents, that is, $E_i E_j = \delta_{i,j} E_i$ ($\delta_{i,j}$ is the Kronecker symbol).
4. The change-of-coordinates matrix $P = [p_{ij}]$ defined by $A_j = \sum_i p_{ij} E_i$ satisfies:
   - p_{ij} is an eigenvalue of A_j with eigenspace $\text{range}(E_i)$.
   - $p_{i0} = 1$, $p_{0i} = k_i$ (the degree of G_i $(i \neq 0)$).
   - $nk_j(P^{-1})_{ji} = m_i p_{ij}$, where $m_i = \text{rank}(E_i)$ (the multiplicity of eigenvalue p_{ij}).
5. (Krein condition) The Bose–Mesner algebra of an association scheme is closed under Hadamard multiplication. The numbers $q_{i,j}^k$, defined by $E_i \circ E_j = \sum_k q_{i,j}^k E_k$, are nonnegative.
6. (Absolute bound) The multiplicities $m_0 = 1, m_1, \ldots, m_d$ of an association scheme satisfy
$$\sum_{k:q_{i,j}^k > 0} m_k \leq m_i m_j \quad \text{and} \quad \sum_{k:q_{i,i}^k > 0} m_k \leq m_i(m_i + 1)/2.$$
7. A connected strongly regular graph is distance-regular with diameter two.
8. [BCN89, p. 55] Let V' be a subset of the vertex set V of an association scheme with change-of-coordinates matrix P. The inner distribution $\mathbf{a}$ of V' satisfies $\mathbf{a}^T P^{-1} \geq \mathbf{0}$.

Examples:

1. The change-of-coordinates matrix P of a strongly regular graph with eigenvalues k, ν_2, and ν_3 is equal to

$$\begin{bmatrix} 1 & k & n-k-1 \\ 1 & \nu_2 & -\nu_2 - 1 \\ 1 & \nu_3 & -\nu_3 - 1 \end{bmatrix}.$$

2. A strongly regular graph with parameters $(28, 9, 0, 4)$ cannot exist, because it violates Facts 5 and 6.
3. The Hamming association scheme $H(d, q)$ has vertex set $V = Q^d$, the set of all vectors with d entries from a finite set Q of size q. Two such vectors are adjacent in G_i if they differ in exactly i coordinate places. The graph G_1 is distance-regular. The matrix P of a Hamming association scheme can be expressed in terms of Kravčuk polynomials, which gives

$$p_{ij} = \sum_{k=0}^{j} (-\mathbf{1})^k (q-1)^{j-k} \binom{i}{k} \binom{d-i}{j-k}.$$

4. An error correcting code with minimum distance δ is a subset V' of the vertex set V of a Hamming association scheme, such that V' induces a coclique in $G_1, \ldots, G_{\delta-1}$. If $\mathbf{a}$ is the inner distribution of V', then $a_0 = 1$, $a_1 = \cdots = a_{\delta-1} = 0$, and $|V'| = \sum_i a_i$. Therefore, by Fact 8, the linear programming problem "Maximize $\sum_{i \geq \delta} a_i$, subject to $\mathbf{a}^T P^{-1} \geq 0$ (with $a_0 = 1$ and $a_1 = \cdots = a_{\delta-1} = 0$)" leads to an upper bound for the size of an error correcting code with given minimum distance. This bound is known as Delsarte's Linear Programming Bound.
5. The Johnson association scheme $J(d, \ell)$ has as vertex set V all subsets of size d of a set of size ℓ ($\ell \geq 2d$). Two vertices are adjacent in G_i if the intersection of the corresponding subsets has size $d - i$. The graph G_1 is distance-regular. The matrix P of a Johnson association scheme can be expressed in terms of Eberlein polynomials, which gives:

$$p_{ij} = \sum_{k=0}^{j} (-1)^k \binom{i}{k} \binom{d-i}{j-k} \binom{\ell-d-i}{j-k}.$$

References

[BW04] L.W. Beineke and R.J. Wilson (Eds.). *Topics in Algebraic Graph Theory.* Cambridge University Press, Cambridge, 2005.

[BI84] Eiichi Bannai and Tatsuro Ito. *Algebraic Combinatorics I: Association Schemes.* The Benjamin/Cummings Publishing Company, London, 1984.

[Big74] N.L. Biggs, *Algebraic Graph Theory.* Cambridge University Press, Cambridge, 1974. (2nd ed., 1993.)

[BCN89] A.E. Brouwer, A.M. Cohen, and A. Neumaier. *Distance-Regular Graphs.* Springer, Heidelberg, 1989.

[BR91] Richard A. Brualdi and Herbert J. Ryser. *Combinatorial Matrix Theory.* Cambridge University Press, Cambridge, 1991.

[CDS80] Dragŏs M. Cvetković, Michael Doob, and Horst Sachs. *Spectra of Graphs: Theory and Applications.* Deutscher Verlag der Wissenschaften, Berlin, 1980; Academic Press, New York, 1980. (3rd ed., Johann Abrosius Barth Verlag, Heidelberg-Leipzig, 1995.)

[CRS97] Dragŏs Cvetković, Peter Rowlinson, and Slobodan Simić. *Eigenspaces of Graphs.* Cambridge University Press, Cambridge, 1997.

[CRS04] Dragŏs Cvetković, Peter Rowlinson, and Slobodan Simić. *Spectral Generalizations of Line Graphs: On graphs with Least Eigenvalue* -2. Cambridge University Press, Cambridge, 2004.

[DH98] Edwin R. van Dam and Willem H. Haemers. Graphs with constant μ and $\overline{\mu}$. *Discrete Math.* 182: 293–307, 1998.

[DH03] Edwin R. van Dam and Willem H. Haemers. Which graphs are determined by their spectrum? *Lin. Alg. Appl.,* 373: 241–272, 2003.

[God93] C.D. Godsil. *Algebraic Combinatorics.* Chapman and Hall, New York, 1993.

[GR01] Chris Godsil and Gordon Royle. *Algebraic Graph Theory.* Springer-Verlag, New York, 2001.

[Hae81] Willem H. Haemers. An upper bound for the Shannon capacity of a graph. *Colloqua Mathematica Societatis János Bolyai 25* (proceedings "Algebraic Methods in Graph Theory," Szeged, 1978). North-Holland, Amsterdam, 1981, pp. 267–272.

[Har69] Frank Harary. *Graph Theory.* Addison-Wesley, Reading, MA, 1969.

[HLS99] Hein van der Holst, László Lovász, and Alexander Schrijver. The Colin de Verdière graph parameter. *Graph Theory and Combinatorial Biology* (L. Lovász, A. Gyárfás, G. Katona, A. Recski, L. Székely, Eds.). János Bolyai Mathematical Society, Budapest, 1999, pp. 29–85.

[Lov79] László Lovász. On the Shannon Capacity of a graph. *IEEE Trans. Inform. Theory,* 25: 1–7, 1979.

[Moh97] Bojan Mohar. Some applications of Laplace eigenvalues of Graphs. *Graph Symmetry: Algebraic Methods and Applications* (G. Hahn, G. Sabidussi, Eds.). Kluwer Academic Publishers, Dordrecht, 1997, pp. 225–275.

[RS04] Neil Robertson and P.D. Seymour. Graph Minors XX: Wagner's conjecture. *J. Combinatorial Theory, Ser. B.* 92: 325–357, 2004.

[Sch73] A.J. Schwenk. Almost all trees are cospectral, in *New directions in the theory of graphs* (F. Harary, Ed.). Academic Press, New York 1973, pp. 275–307.

[Wes01] Douglas West. *Introduction to Graph Theory.* 2nd ed., Prentice Hall, Upper Saddle River, NJ, 2001.

29

Digraphs and Matrices

29.1 Digraphs **29**-1
29.2 The Adjacency Matrix of a Directed Graph and the Digraph of a Matrix **29**-3
29.3 Walk Products and Cycle Products **29**-4
29.4 Generalized Cycle Products **29**-5
29.5 Strongly Connected Digraphs and Irreducible Matrices **29**-6
29.6 Primitive Digraphs and Primitive Matrices **29**-8
29.7 Irreducible, Imprimitive Matrices and Cyclic Normal Form **29**-9
29.8 Minimally Connected Digraphs and Nearly Reducible Matrices **29**-12
References **29**-13

Jeffrey L. Stuart

Pacific Lutheran University

Directed graphs, often called digraphs, have much in common with graphs, which were the subject of the previous chapter. While digraphs are of interest in their own right, and have been the subject of much research, this chapter focuses on those aspects of digraphs that are most useful to matrix theory. In particular, it will be seen that digraphs can be used to understand how the zero–nonzero structure of square matrices affects matrix products, determinants, inverses, and eigenstructure. Basic material on digraphs and their adjacency matrices can be found in many texts on graph theory, nonnegative matrix theory, or combinatorial matrix theory. For all aspects of digraphs, except their spectra, see [BG00]. Perhaps the most comprehensive single source for results, proofs, and references to original papers on the interplay between digraphs and matrices is [BR91, Chapters 3 and 9]. Readers preferring a matrix analytic rather than combinatorial approach to irreducibility, primitivity, and their consequences, should consult [BP94, Chapter 2].

29.1 Digraphs

Definitions:

A **directed graph** $\Gamma = \Gamma(V, E)$ consists of a finite, nonempty set V of **vertices** (sometimes called **nodes**), together with a multiset E of elements of $V \times V$, whose elements are called **arcs** (sometimes called **edges**, **directed edges**, or **directed arcs**).

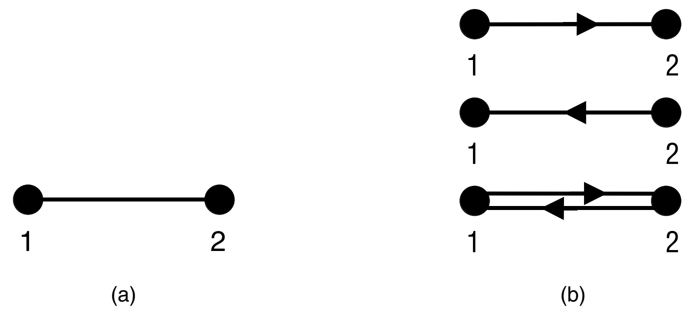


FIGURE 29.1

A **loop** is an arc of the form (v, v) for some vertex v.

If there is more than one arc (u, v) for some u and v in V, then Γ is called a **directed multigraph**.

If there is at most one arc (u, v) for each u and v in V, then Γ is called a **digraph**.

If the digraph Γ contains no loops, then Γ is called a **simple** digraph.

A **weighted digraph** is a digraph Γ with a **weight function** $w : E \to \mathbf{F}$, where the set $\mathbf{F}$ is often the real or complex numbers.

A **subdigraph** Γ' of a digraph Γ is a digraph $\Gamma' = \Gamma'(V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$.

A **proper** subdigraph Γ' of a digraph Γ is a subdigraph of Γ such that $V' \subset V$ or $E' \subset E$.

If V' is a nonempty subset of V, then the **induced subdigraph of** Γ **induced by** V' is the digraph with vertex set V' whose arcs are those arcs in E that lie in $V' \times V'$.

A **walk** is a sequence of arcs $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, where one or more vertices may be repeated.

The **length** of a walk is the number of arcs in the walk. (Note that some authors define the length to be the number of vertices rather than the number of arcs.)

A **simple walk** is a walk in which all vertices, except possibly v_0 and v_k, are distinct. (Note that some authors use **path** to mean what we call a simple walk.)

A **cycle** is a simple walk for which $v_0 = v_k$. A cycle of length k is called a k-**cycle**.

A **generalized cycle** is either a cycle passing through all vertices in V or else a union of cycles such that every vertex in V lies on exactly one cycle.

Let $\Gamma = \Gamma(V, E)$ be a digraph. The **undirected graph** G **associated with the digraph** Γ is the undirected graph with vertex set V, whose edge set is determined as follows: There is an edge between vertices u and v in G if and only if at least one of the arcs (u, v) and (v, u) is present in Γ.

The digraph Γ is **connected** if the associated undirected graph G is connected.

The digraph Γ is a **tree** if the associated undirected graph G is a tree.

The digraph Γ is a **doubly directed tree** if the associated undirected graph is a tree and if whenever (i, j) is an arc in Γ, (j, i) is also an arc in Γ.

Examples:

1. The key distinction between a graph on a vertex set V and a digraph on the same set is that for a graph we refer to the *edge between vertices u and v*, whereas for a digraph, we have two arcs, the *arc from u to v* and the *arc from v to u*. Thus, there is one connected, simple graph on two vertices, K_2 (see Figure 29.1a), but there are three possible connected, simple digraphs on two vertices (see Figure 29.1b). Note that all graphs in Figure 29.1 are trees, and that the graph in Figure 29.1a is the undirected graph associated with each of the digraphs in Figure 29.1b. The third graph in Figure 29.1b is a doubly directed tree.

2. Let Γ be the digraph in Figure 29.2. Then (1,1), (1,1), (1,3), (3,1), (1,2) is a walk of length 5 from vertex 1 to vertex 2. $(1, 2), (2, 3)$ is a simple walk of length 2. $(1, 1)$ is a 1-cycle; $(1, 3), (3, 1)$ is a 2-cycle; and $(1, 2), (2, 3), (3, 1)$ is a 3-cycle. $(1, 1), (2, 3), (3, 2)$ and $(1, 2), (2, 3), (3, 1)$ are two generalized cycles. Not all digraphs contain generalized cycles; consider the digraph obtained by deleting the arc $(2, 3)$ from Γ, for example. Unless we are emphasizing a particular vertex on a cycle, such as all cycles starting (and ending) at vertex v, we view cyclic permutations of a cycle as equivalent. That is, in Figure 29.2, we would speak of *the* 3-cycle, although technically $(1, 2), (2, 3), (3, 1)$; $(2, 3), (3, 1), (1, 2)$; and $(3, 1), (1, 2), (2, 3)$ are distinct cycles.

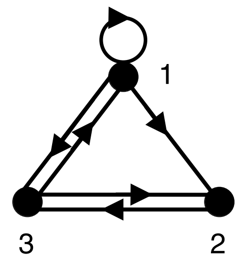


FIGURE 29.2

29.2 The Adjacency Matrix of a Directed Graph and the Digraph of a Matrix

If Γ is a directed graph on n vertices, then there is a natural way that we can record the arc information for Γ in an $n \times n$ matrix. Conversely, if A is an $n \times n$ matrix, we can naturally associate a digraph Γ on n vertices with A.

Definitions:

Let Γ be a digraph with vertex set V. Label the vertices in V as $v_1, v_2, \ldots, v_n$. Once the vertices have been ordered, the **adjacency matrix** for Γ, denoted $\mathcal{A}_\Gamma$, is the 0, 1-matrix whose entries a_{ij} satisfy: $a_{ij} = 1$ if (v_i, v_j) is an arc in Γ, and $a_{ij} = 0$ otherwise. When the set of vertex labels is $\{1, 2, \ldots, n\}$, the default labeling of the vertices is $v_i = i$ for $1 \le i \le n$.

Let A be an $n \times n$ matrix. Let V be the set $\{1, 2, \ldots, n\}$. Construct a digraph denoted $\Gamma(A)$ on V as follows. For each i and j in V, let (i, j) be an arc in Γ exactly when $a_{ij} \neq 0$. $\Gamma(A)$ is called the **digraph of the matrix** A. Commonly Γ is viewed as a weighted digraph with weight function $w((i, j)) = a_{ij}$ for all (i, j) with $a_{ij} \neq 0$.

Facts: [BR91, Chap. 3]

1. If a digraph H is obtained from a digraph Γ by adding or removing an arc, then $\mathcal{A}_H$ is obtained by changing the corresponding entry of $\mathcal{A}_\Gamma$ to a 1 or a 0, respectively. If a digraph H is obtained from a digraph Γ by deleting the i^{th} vertex in the ordered set V and by deleting all arcs in E containing the i^{th} vertex, then $\mathcal{A}_H = \mathcal{A}_\Gamma(i)$. That is, $\mathcal{A}_H$ is obtained by deleting row i and column i of $\mathcal{A}_\Gamma$.
2. Given one ordering of the vertices in V, any other ordering of those vertices is simply a permutation of the original ordering. Since the rows and columns of $A = \mathcal{A}_\Gamma$ are labeled by the ordered vertices, reordering the vertices in Γ corresponds to simultaneously permuting the rows and columns of A. That is, if P is the permutation matrix corresponding to a permutation of the vertices of V, then the new adjacency matrix is PAP^T. Since $P^T = P^{-1}$ for a permutation matrix, all algebraic properties preserved by similarity transformations are invariant under changes of the ordering of the vertices.
3. Let Γ be a digraph with vertex set V. The Jordan canonical form of an adjacency matrix for Γ is independent of the ordering applied to the vertices in V. Consequently, all adjacency matrices for Γ have the same rank, trace, determinant, minimum polynomial, characteristic polynomial, and spectrum.
4. If A is an $n \times n$ matrix and if $v_i = i$ for $1 \le i \le n$, then A and $\mathcal{A}_{\Gamma(A)}$ have the same zero–nonzero pattern and, hence, $\Gamma(A) = \Gamma(\mathcal{A}_{\Gamma(A)})$.

Examples:

1. For the digraph Γ given in Figure 29.3, if we order the vertices as $v_i = i$ for $i = 1, 2, \ldots, 7$, then

$$A = \mathcal{A}_\Gamma = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

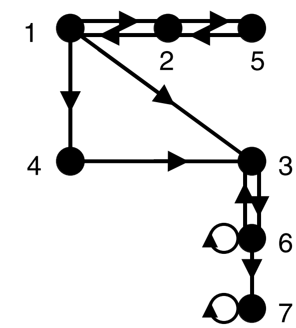


FIGURE 29.3

If we reorder the vertices in the digraph Γ given in Figure 29.3 so that $v_1, v_2, \cdots, v_7$ is the sequence $1, 2, 5, 4, 3, 6, 7$, then the new adjacency matrix is

$$B = \mathcal{A}_\Gamma = \left[\begin{array}{ccc|c|cc|c} 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

2. If A is the 3×3 matrix

$$A = \begin{bmatrix} 3 & -2 & 5 \\ 0 & 0 & -11 \\ 9 & -6 & 0 \end{bmatrix},$$

then $\Gamma(A)$ is the digraph given in Figure 29.2. Up to permutation similarity,

$$\mathcal{A}_\Gamma = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

29.3 Walk Products and Cycle Products

For a square matrix A, $a_{12}a_{23}$ is nonzero exactly when both a_{12} and a_{23} are nonzero. That is, exactly when both $(1, 2)$ and $(2, 3)$ are arcs in $\Gamma(A)$. Note also that $a_{12}a_{23}$ is one summand in $(A^2)_{13}$. Consequently, there is a close connection between powers of a matrix A and walks in its digraph $\Gamma(A)$. In fact, the signs (complex arguments) of walk products play a fundamental role in the study of the matrix sign patterns for real matrices (matrix ray patterns for complex matrices). See Chapter 33 [LHE94] or [Stu03].

Definitions:

Let A be an $n \times n$ matrix. Let W given by $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$ be a walk in $\Gamma(A)$. The **walk product** for the walk W is

$$\prod_{j=1}^{k} a_{v_{j-1}, v_j},$$

and is often denoted by $\prod_W a_{ij}$. This product is a generic summand of the (v_0, v_k)-entry of A^k. If $s_1, s_2 \ldots, s_n$ are scalars, $\prod_W s_i$ denotes the ordinary product of the s_i over the index set $v_0, v_1, v_2, \ldots, v_{k-1}, v_k$.

If W is a cycle in the directed graph $\Gamma(A)$, then the walk product for W is called a **cycle product**.

Let A be an $n \times n$ real or complex matrix. Define $|A|$ to be the matrix obtained from A by replacing a_{ij} with $|a_{ij}|$ for all i and j.

Facts:

1. Let A be a square matrix. The walk W given by $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$ occurs in $\Gamma(A)$ exactly when $a_{v_0 v_1} a_{v_1 v_2} \cdots a_{v_{k-1} v_k}$ is nonzero.
2. [BR91, Sec. 3.4] [LHE94] Let A be a square matrix. For each positive integer k, and for all i and j, the (i, j)-entry of A^k is the sum of the walk products for all length k walks in $\Gamma(A)$ from i to j. Further, there is a walk of length k from i to j in $\Gamma(A)$ exactly when the (i, j)-entry of $|A|^k$ is nonzero.
3. [BR91, Sec. 3.4] [LHE94] Let A be a square, real or complex matrix. For all positive integers k, $\Gamma(A^k)$ is a subdigraph of $\Gamma(|A|^k)$. Further, $\Gamma(A^k)$ is a proper subgraph of $\Gamma(|A|^k)$ exactly when additive cancellation occurs in summing products for length k walks from some i to some j.
4. [LHE94] Let A be a square, real matrix. The sign pattern of the k^{th} power of A is determined solely by the sign pattern of A when the signs of the entries in A are assigned so that for each ordered pair of vertices, all products of length k walks from the first vertex to the second have the same sign.
5. [FP69] Let A and B be irreducible, real matrices with $\Gamma(A) = \Gamma(B)$. There exists a nonsingular, real diagonal matrix D such that $B = DAD^{-1}$ if and only if the cycle product for every cycle in $\Gamma(A)$ equals the cycle product for the corresponding cycle in $\Gamma(B)$.
6. Let A be an irreducible, real matrix. There exists a nonsingular, real diagonal matrix D such that DAD^{-1} is nonnegative if and only if the cycle product for every cycle in $\Gamma(A)$ is positive.

Examples:

1. If $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, then $(A^2)_{12} = a_{11}a_{12} + a_{12}a_{22} = (1)(1) + (1)(-1) = 0$, whereas $(|A|^2)_{12} = 2$.
2. If A is the matrix in Example 2 of the previous section, then $\Gamma(A)$ contains four cycles: the loop $(1, 1)$; the two 2-cycles $(1, 3), (3, 1)$ and $(2, 3), (3, 2)$; and the 3-cycle $(1, 2), (2, 3), (3, 1)$. Each of these cycles has a positive cycle product and using $D = diag(1, -1, 1)$, DAD^{-1} is nonnegative.

29.4 Generalized Cycle Products

If the matrix A is 2×2, then $\det(A) = a_{11}a_{22} - a_{12}a_{22}$. Assuming that the entries of A are nonzero, the two summands $a_{11}a_{22}$ and $a_{12}a_{21}$ are exactly the walk products for the two generalized cycles of $\Gamma(A)$. From Chapter 4.1, the determinant of an $n \times n$ matrix A is the sum of all terms of the form $(-1)^{sign(\sigma)} a_{1j_1} a_{2j_2} \cdots a_{nj_n}$, where $\sigma = (j_1, j_2, \ldots, j_n)$ is a permutation of the ordered set $\{1, 2, \ldots, n\}$. Such a summand is nonzero precisely when $(1, j_1), (2, j_2), \cdots, (n, j_n)$ are all arcs in $\Gamma(A)$. For this set of arcs, there is exactly one arc originating at each of the n vertices and exactly one arc terminating at each of the n vertices. Hence, the arcs correspond to a generalized cycle in $\Gamma(A)$. See [BR91, Sect. 9.1]. Since the eigenvalues of A are the roots of $\det(\lambda I - A)$, it follows that results connecting the cycle structure of a matrix to its determinant should play a key role in determining the spectrum of the matrix. For further results connecting determinants and generalized cycles, see [MOD89] or [BJ86]. Generalized cycles play a crucial role in the study of the nonsingularity of sign patterns. (See Chapter 33 or [Stu91]). In general, there are fewer results for the spectra of digraphs than for the spectra of graphs. There are generalizations of Geršgorin's Theorem for spectral inclusion regions for complex matrices that depend on directed cycles. (See Chapter 14.2 [Bru82] or [BR91, Sect. 3.6], or especially, [Var04]).

Definitions:

Let A be an $n \times n$ matrix. Let $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$ be a permutation of the ordered set $\{1, 2, \ldots, n\}$. When (i, j_i) is an arc in $\Gamma(A)$ for each i, the cycle or vertex disjoint union of cycles with arcs $(1, j_1)$, $(2, j_2), \ldots, (n, j_n)$ is called the **generalized cycle induced by** σ.

The product of the cycle products for the cycle(s) comprising the generalized cycle induced by σ is called the **generalized cycle product** corresponding to σ.

Facts:

1. The entries in A that correspond to a generalized cycle are a diagonal of A and vice versa.
2. If σ is a permutation of the ordered set $\{1, 2, \ldots, n\}$, then the nonzero entries of the $n \times n$ permutation matrix P corresponding to σ are precisely the diagonal of P corresponding to the generalized cycle induced by σ.
3. [BR91, Sec. 9.1] Let A be an $n \times n$ real or complex matrix. Then $\det(A)$ is the sum over all permutations of the ordered set $\{1, 2, \ldots, n\}$ of all of the signed generalized cycle products for $\Gamma(A)$ where the sign of a generalized cycle is determined as $(-1)^{n-k}$, where k is the number of disjoint cycles in the generalized cycle. If $\Gamma(A)$ contains no generalized cycle, then A is singular. If $\Gamma(A)$ contains at least one generalized cycle, then A is nonsingular unless additive cancellation occurs in the sum of the signed generalized cycle products.
4. [Cve75] [Har62] Let A be an $n \times n$ real or complex matrix. The coefficient of x^{n-k} in $\det(xI - A)$ is the sum over all induced subdigraphs H of $\Gamma(A)$ on k vertices of the signed generalized cycle products for H.
5. [BP94, Chap. 3], [BR97, Sec. 1.8] Let A be an $n \times n$ real or complex matrix. Let g be the greatest common divisor of the lengths of all cycles in $\Gamma(A)$. Then $\det(xI - A) = x^k p(x^g)$ for some nonnegative integer k and some polynomial $p(z)$ with $p(0) \neq 0$. Consequently, the spectrum of A is invariant under $\frac{2\pi}{g}$ rotations of the complex plane. Further, if A is nonsingular, then g divides n, and $\det(xI - A) = p(x^g)$ for some polynomial $p(z)$ with $p(0) = (-1)^n \det(A)$.

Examples:

1. If A is the matrix in Example 2 of the previous section, then $\Gamma(A)$ contains two generalized cycles — the loop $(1, 1)$ together with the 2-cycle $(2, 3), (3, 2)$; and the 3-cycle $(1, 2), (2, 3), (3, 1)$. The corresponding generalized cycle products are $(3)(-11)(-6) = 198$ and $(-2)(-11)(9) = 198$, with corresponding signs -1 and 1, respectively. Thus, $\det(A) = 0$ is a consequence of additive cancellation.
2. If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then the only cycles in $\Gamma(A)$ are loops, so $g = 1$. The spectrum of A is clearly invariant under $\frac{2\pi}{g}$ rotations, but it is also invariant under rotations through the smaller angle of π.

29.5 Strongly Connected Digraphs and Irreducible Matrices

Irreducibility of a matrix, which can be defined in terms of permutation similarity (see Section 27.3), and which Frobenius defined as an algebraic property in his extension of Perron's work on the spectra of positive matrices, is equivalent to the digraph property of being strongly connected, defined in this section. Today, most discussions of the celebrated Perron–Frobenius Theorem (see Chapter 9) use digraph theoretic terminology.

Definitions:

Vertex u **has access to** vertex v in a digraph Γ if there exists a walk in Γ from u to v. By convention, every vertex has access to itself even if there is no walk from that vertex to itself.

If u and v are vertices in a digraph Γ such that u has access to v, and such that v has access to u, then u and v are **access equivalent** (or u and v **communicate**).

Access equivalence is an equivalence relation on the vertex set V of Γ that partitions V into **access equivalence classes**.

If V_1 and V_2 are nonempty, disjoint subsets of V, then V_1 **has access to** V_2 if some vertex in v_1 in V_1 has access in Γ to some vertex v_2 in V_2.

For a digraph Γ, the subdigraphs induced by each of the access equivalence classes of V are the **strongly connected components** *of* Γ.

When all of the vertices of Γ lie in a single access equivalence class, Γ is **strongly connected**.

Let $V_1, V_2, \ldots, V_k$ be the access equivalence classes for some digraph Γ. Define a new digraph, $R(\Gamma)$, called the **reduced digraph** (also called the **condensation digraph**) for Γ as follows. Let $W = \{1, 2, \ldots, k\}$ be the vertex set for $R(\Gamma)$. If $i, j \in W$ with $i \neq j$, then (i, j) is an arc in $R(\Gamma)$ precisely when V_i has access to V_j.

Facts: [BR91, Chap. 3]

1. [BR91, Sec. 3.1] A digraph Γ is strongly connected if and only if there is a walk from each vertex in Γ to every other vertex in Γ.
2. The square matrix A is irreducible if and only if $\Gamma(A)$ is strongly connected. A reducible matrix A is completely reducible if $\Gamma(A)$ is a disjoint union of two or more strongly connected digraphs.
3. [BR91, Sec. 3.1] Suppose that V_1 and V_2 are distinct access equivalence classes for some digraph Γ. If any vertex in V_1 has access to any vertex in V_2, then every vertex in V_1 has access to every vertex in V_2. Further, exactly one of the following holds: V_1 has access to V_2, V_2 has access to V_1, or neither has access to the other. Consequently, access induces a partial order on the access equivalence classes of vertices.
4. [BR91, Lemma 3.2.3] The access equivalence classes for a digraph Γ can be labelled as $V_1, V_2, \ldots, V_k$ so that whenever there is an arc from a vertex in V_i to a vertex in V_j, $i \leq j$.
5. [Sch86] If Γ is a digraph, then $R(\Gamma)$ is a simple digraph that contains no cycles. Further, the vertices in $R(\Gamma)$ can always be labelled so that if (i, j) is an arc in $R(\Gamma)$, then $i < j$.
6. [BR91, Theorem 3.2.4] Suppose that Γ is not strongly connected. Then there exists at least one ordering of the vertices in V so that $\mathcal{A}_\Gamma$ is block upper triangular, where the diagonal blocks of $\mathcal{A}_\Gamma$ are the adjacency matrices of the strongly connected components of Γ.
7. [BR91, Theorem 3.2.4] Let A be a square matrix. Then A has a Frobenius normal form. (See Chapter 27.3.)
8. The Frobenius normal form of a square matrix A is not necessarily unique. The set of Frobenius normal forms for A is preserved by permutation similarities that correspond to permutations that reorder the vertices within the access equivalence classes of $\Gamma(A)$. If B is a Frobenius normal form for A, then all the arcs in $R(\Gamma(B))$ satisfy (i, j) is an edge implies $i < j$. Let σ be any permutation of the vertices of $R(\Gamma(B))$ such that $(\sigma(i), \sigma(j))$ is an edge in $R(\Gamma(B))$ implies $\sigma(i) < \sigma(j)$. Let B be block partitioned by the access equivalence classes of $\Gamma(B)$. Applying the permutation similarity corresponding to σ to the blocks of B produces a Frobenius normal form for A. All Frobenius normal forms for A are produced using combinations of the above two types of permutations.
9. [BR91, Sect. 3.1] [BP94, Chap. 3] Let A be an $n \times n$ matrix for $n \geq 2$. The following are equivalent:

   (a) A is irreducible.

   (b) $\Gamma(A)$ is strongly connected.

   (c) For each i and j, there is a positive integer k such that $(|A|^k)_{ij} > 0$.

   (d) There does not exist a permutation matrix P such that

   $$PAP^T = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline \mathbf{0} & A_{22} \end{array}\right],$$

   where A_{11} and A_{22} are square matrices.

10. Let A be a square matrix. A is completely reducible if and only if there exists a permutation matrix P such that PAP^T is a direct sum of at least two irreducible matrices.
11. All combinations of the following transformations preserve irreducibility, reducibility, and complete reducibility: Scalar multiplication by a nonzero scalar; transposition; permutation similarity; left or right multiplication by a nonsingular, diagonal matrix.
12. Complex conjugation preserves irreducibility, reducibility, and complete reducibility for square, complex matrices.

Examples:

1. In the digraph Γ in Figure 29.3, vertex 4 has access to itself and to vertices 3, 6, and 7, but not to any of vertices 1, 2, or 5. For Γ, the access equivalence classes are: $V_1 = \{1, 2, 5\}$, $V_2 = \{3, 6\}$, $V_3 = \{4\}$, and $V_4 = \{7\}$. The strongly connected components of Γ are given in Figure 29.4, and $R(\Gamma)$ is given in Figure 29.5. If the access equivalence classes for Γ are relabeled so that $V_2 = \{4\}$ and $V_3 = \{3, 6\}$, then the labels on vertices 2 and 3 switch in the reduced digraph given in Figure 29.5. With this labeling, if (i, j) is an arc in the reduced digraph, then $i \leq j$.
2. If A and B are the adjacency matrices of Example 1 of Section 29.2, then $A(3, 4, 6, 7) = A[1, 2, 5]$ is the adjacency matrix for the largest strongly connected component of the digraph Γ in Figure 29.3 using the first ordering of V; and using the second ordering of V, B is block-triangular and the irreducible, diagonal blocks of B are the adjacency matrices for each of the four strongly connected components of Γ. B is a Frobenius normal form for A.

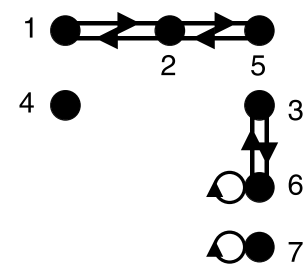


FIGURE 29.4

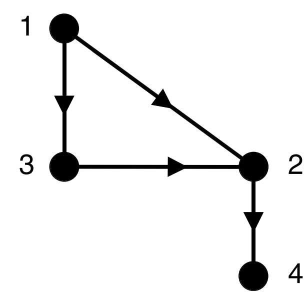


FIGURE 29.5

3. The matrix A in Example 2 of Section 29.2 is irreducible, hence, it is its own Frobenius normal form.

29.6 Primitive Digraphs and Primitive Matrices

Primitive matrices, defined in this section, are necessarily irreducible. Unlike irreducibility, primitivity depends not just on the matrix A, but also its powers, and, hence, on the signs of the entries of the original matrix. Consequently, most authors restrict discussions of primitivity to nonnegative matrices. Much work has been done on bounding the exponent of a primitive matrix; see [BR91, Sec. 3.5] or [BP94, Sec. 2.4]. One consequence of the fifth fact stated below is that powers of sparse matrices and inverses of sparse matrices can experience substantial fill-in.

Definitions:

A digraph Γ with at least two vertices is **primitive** if there is a positive integer k such that for every pair of vertices u and v (not necessarily distinct), there exists at least one walk of length k from u to v. A digraph on a single vertex is **primitive** if there is a loop on that vertex.

The **exponent** of a digraph Γ (sometimes called **the index of primitivity**) is the smallest value of k that works in the definition of primitivity.

A digraph Γ is **imprimitive** if it is not primitive. This includes the simple digraph on one vertex.

If A is a square, nonnegative matrix such that $\Gamma(A)$ is primitive with exponent k, then A is called a **primitive matrix** *with* **exponent** k.

Facts:

1. A primitive digraph must be strongly connected, but not conversely.
2. A strongly connected digraph with at least one loop is primitive.
3. [BP94, Chap. 3] [BR91, Sections 3.2 and 3.4] Let Γ be a strongly connected digraph with at least two vertices. The following are equivalent:

   (a) Γ is primitive.

   (b) The greatest common divisor of the cycle lengths for Γ is 1 (i.e., Γ is aperiodic, cf. Chapter 9.2).

   (c) There is a smallest positive integer k such that for each $t \geq k$ and each pair of vertices u and v in Γ, there is a walk of length t from u to v.

4. [BR91, Sect. 3.5] Let Γ be a primitive digraph with $n \geq 2$ vertices and exponent k. Then

   (a) $k \leq (n-1)^2 + 1$.

   (b) If s is the length of the shortest cycle in Γ, then $k \leq n + s(n-2)$.

   (c) If Γ has $p \geq 1$ loops, then $k \leq 2n - p - 1$.

5. [BR91, Theorem 3.4.4] Let A be an $n \times n$ nonnegative matrix with $n \geq 2$. The matrix A is primitive if and only if there exists a positive integer k such that A^k is positive. When such a positive integer k exists, the smallest such k is the exponent of A. Further, if A^k is positive, then A^h is positive for all integers $h \geq k$. A nonnegative matrix A with the property that some power of A is positive is also called **regular**. See Chapter 9 and Chapter 54 for more information about primitive matrices and their uses.
6. [BP94, Chap. 3] If A is an irreducible, nonnegative matrix with positive trace, then A is primitive.
7. [BP94, Chap. 6] Let A be a nonnegative, tridiagonal matrix with all entries on the first superdiagonal and on the first subdiagonal positive, and at least one entry on the main diagonal positive. Then A is primitive and, hence, some power of A is positive. Further, if $s > \rho(A)$ where $\rho(A)$ is the spectral radius of A, then the tridiagonal matrix $sI - A$ is a nonsingular M-matrix with a positive inverse.

Examples:

1. The digraph Γ in Figure 29.2 is primitive with exponent 3; the strongly connected digraph in Figure 29.1*b* is not primitive.
2. Let $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and let $A_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. Note that $A_1 = |A_2|$. Clearly, $\Gamma(A_1) = \Gamma(A_2)$ is an irreducible, primitive digraph with exponent 1. For all positive integers k, A_1^k is positive, so it makes sense to call A_1 a primitive matrix. In contrast, $A_2^k = \mathbf{0}$ for all integers $k \geq 2$.

29.7 Irreducible, Imprimitive Matrices and Cyclic Normal Form

While most authors restrict discussions of matrices with primitive digraphs to nonnegative matrices, many authors have exploited results for imprimitive digraphs to understand the structure of real and complex matrices with imprimitive digraphs.

Definitions:

Let A be an irreducible $n \times n$ matrix with $n \geq 2$ such that $\Gamma(A)$ is imprimitive. The greatest common divisor $g > 1$ of the lengths of all cycles in $\Gamma(A)$ is called the **index of imprimitivity of** A (or **period** of A).

If there is a permutation matrix P such that

$$PAP^T = \begin{bmatrix} \mathbf{0} & A_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A_{g-1} \\ A_g & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix},$$

where each of the diagonal blocks is a square zero matrix, then the matrix PAP^T is called a **cyclic normal form** for A.

By convention, when A is primitive, A is said to be its own cyclic normal form.

Facts:

1. [BP94, Sec. 2.2] [BR91, Sections 3.4] [Min88, Sec. 3.3–3.4] Let A be an irreducible matrix with index of imprimitivity $g > 1$. Then there exists a permutation matrix P such that PAP^T is a cyclic normal form for A. Further, the cyclic normal form is unique up to cyclic permutation of the blocks A_j and permutations within the partition sets of V of $\Gamma(A)$ induced by the partitioning of PAP^T. Finally, if A is real or complex, then $|A_1|\,|A_2| \cdots |A_g|$ is irreducible and nonzero.
2. [BP94, Sec. 3.3] [BR91, Sec. 3.4] If A is an irreducible matrix with index of imprimitivity $g > 1$, and if there exists a permutation matrix P and a positive integer k such that

$$PAP^T = \begin{bmatrix} \mathbf{0} & A_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A_{k-1} \\ A_k & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix},$$

   where each diagonal block is square zero matrix, then k divides g. Conversely, if A is real or complex, if PAP^T has the specified form for some positive integer k, if PAP^T has no zero rows and no zero columns, and if $|A_1|\,|A_2| \cdots |A_k|$ is irreducible, then A is irreducible, and k divides g.
3. [BR91, Sec. 3.4] Let A be an irreducible, nonnegative matrix with index of imprimitivity $g > 1$. Let m be a positive integer. Then A^m is irreducible if and only if m and g are relatively prime. If A^m is reducible, then it is completely reducible, and it is permutation similar to a direct sum of r irreducible matrices for some positive integer r. Further, either each of these summands is primitive (when $g/r = 1$), or each of these summands has index of imprimitivity $g/r > 1$.
4. [Min88, Sec. 3.4] Let A be an irreducible, nonnegative matrix in cyclic normal form with index of imprimitivity $g > 1$. Suppose that for $1 \leq i \leq k-1$, A_i is $n_i \times n_{i+1}$ and that A_g is $n_g \times n_1$. Let $k = \min\left(n_1, n_2, \ldots, n_g\right)$. Then 0 is an eigenvalue for A with multiplicity at least $n - gk$; and if A is nonsingular, then each $n_i = n/g$.
5. [Min88, Sec. 3.4] If A is an irreducible, nonnegative matrix in cyclic normal form with index of imprimitivity g, then for $j = 1, 2, \ldots, g$, reading the indices modulo g, $B_j = \prod_{i=j}^{j+g-1} A_i$ is irreducible. Further, all of the matrices B_j have the same nonzero eigenvalues. If the nonzero eigenvalues (not necessarily distinct) of B_1 are $\omega_1, \omega_2, \ldots, \omega_m$ for some positive integer m, then the spectrum of A consists of 0 with multiplicity $n - gm$ together with the complete set of g^{th} roots of each of the ω_i.

6. Let A be a square matrix. Then A has a Frobenius normal form for which each irreducible, diagonal block is in cyclic normal form.
7. Let A be a square matrix. If A is reducible, then the spectrum of A (which is a multiset) is the union of the spectra of the irreducible, diagonal blocks of any Frobenius normal form for A.
8. Explicit, efficient algorithms for computing the index of imprimitivity and the cyclic normal form for an imprimitive matrix and for computing the Frobenius normal form for a matrix can be found in [BR91, Sec. 3.7].
9. All results stated here for block upper triangular forms have analogs for block lower triangular forms.

Examples:

1. If A is the 4×4 matrix $A = \begin{bmatrix} \mathbf{0} & A_1 \\ A_2 & \mathbf{0} \end{bmatrix}$, where $A_1 = I_2$ and $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then A and $A_1 A_2 = A_2$ are irreducible but $g \neq 2$. In fact, $g = 4$ since A is actually a permutation matrix corresponding to the permutation (1324). Also note that when $A_1 = A_2 = I_2$, A is completely reducible since $\Gamma(A)$ consists of two disjoint cycles.
2. If M is the irreducible matrix $M = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 6 & 0 & 3 & 0 \\ 0 & 2 & 0 & 2 \\ 6 & 0 & 3 & 0 \end{bmatrix}$, then $g = 2$, and using the permutation matrix $Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $N = QMQ^T = \left[\begin{array}{cc|cc} 0 & 0 & 3 & 6 \\ 0 & 0 & 3 & 6 \\ \hline 2 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right]$ is a cyclic normal form for M. Note that $|N_1||N_2| = \begin{bmatrix} 3 & 6 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 12 \\ 12 & 12 \end{bmatrix}$ is irreducible even though $N_1 N_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not irreducible.
3. The matrix B in Example 1 of Section 29.2 is a Frobenius normal form for the matrix A in that example. Observe that B_{11} is imprimitive with $g = 2$, but B_{11} is not in cyclic normal form. The remaining diagonal blocks, B_{22} and B_{33}, have primitive digraphs, hence, they are in cyclic normal form. Let $Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $QB_{11}Q^T = \left[\begin{array}{c|cc} 0 & 3 & 6 \\ \hline 2 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right]$ is a cyclic normal form for B_{11}. Using the permutation matrix $P = Q \bigoplus I_1 \bigoplus I_2 \bigoplus I_1$, PBP^T is a Frobenius normal form for A with each irreducible, diagonal block in cyclic normal form.
4. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and let $B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Observe that A and B are each in Frobenius normal form, each with two irreducible, diagonal blocks, and that $A_{11} = B_{11}$ and $A_{22} = B_{22}$. Consequently, $\sigma(A) = \sigma(B) = \sigma(A_{11}) \cup \sigma(A_{22}) = \{1, 3, 3\}$. However, B has only one independent eigenvector for eigenvalue 3, whereas A has two independent eigenvectors for eigenvalue 3. The underlying cause for this difference is the difference in the access relations as captured in the reduced digraphs $R(\Gamma(A))$ (two isolated vertices) and $R(\Gamma(B))$ (two vertices joined by a single arc). The role that the reduced digraph of a matrix plays in connections between the eigenspaces for each of the irreducible, diagonal blocks of a matrix and those of the entire matrix is discussed in [Sch86] and in [BP94, Theorem 2.3.20]. For connections between $R(\Gamma(A))$ and the structure of the generalized eigenspaces for A, see also [Rot75].

29.8 Minimally Connected Digraphs and Nearly Reducible Matrices

Replacing zero entries in an irreducible matrix with nonzero entries preserves irreducibility, and equivalently, adding arcs to a strongly connected digraph preserves strong connectedness. Consequently, it is of interest to understand how few nonzero entries are needed in a matrix and in what locations to guarantee irreducibility; or equivalently, how few arcs are needed in a digraph and between which vertices to guarantee strong connectedness. Except as noted, all of the results in this section can be found in both [BR91, Sec. 3.3] and [Min88, Sec. 4.5]. Further results on nearly irreducible matrices and on their connections to nearly decomposable matrices can be found in [BH79].

Definitions:

A digraph is **minimally connected** if it is strongly connected and if the deletion of any arc in the digraph produces a subdigraph that is not strongly connected.

Facts:

1. A digraph Γ is minimally connected if and only if $\mathcal{A}_\Gamma$ is nearly reducible.
2. A matrix A is nearly reducible if and only if $\Gamma(A)$ is minimally connected.
3. The only minimally connected digraph on one vertex is the simple digraph on one vertex.
4. The only nearly reducible 1×1 matrix is the zero matrix.
5. [BR91, Theorem 3.3.5] Let Γ be a minimally connected digraph on n vertices with $n \geq 2$. Then Γ has no loops, at least n arcs, and at most $2(n-1)$ arcs. When Γ has exactly n arcs, Γ is an n-cycle. When Γ has exactly $2(n-1)$ arcs, Γ is a doubly directed tree.
6. Let A be an $n \times n$ nearly reducible matrix with $n \geq 2$. Then A has no nonzeros on its main diagonal, A has at least n nonzero entries, and A has at most $2(n-1)$ nonzero entries. When A has exactly n nonzero entries, $\Gamma(A)$ is an n-cycle. When A has exactly $2(n-1)$ nonzero entries, $\Gamma(A)$ is a doubly directed tree.
7. [Min88, Theorem 4.5.1] Let A be an $n \times n$ nearly reducible matrix with $n \geq 2$. Then there exists a positive integer m and a permutation matrix P such that

$$PAP^T = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & \cdots & 0 & \mathbf{0}^T \\ 0 & 0 & 1 & \cdots & 0 & \mathbf{0}^T \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \mathbf{0}^T \\ 0 & 0 & \cdots & 0 & 0 & \mathbf{v}^T \\ \hline \mathbf{u} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & B \end{array}\right],$$

where the upper left matrix is $m \times m$, B is nearly reducible, $\mathbf{0}$ is a $(n-m) \times 1$ vector, both of the vectors $\mathbf{u}$ and $\mathbf{v}$ are $(n-m) \times 1$, and each of $\mathbf{u}$ and $\mathbf{v}$ contains a single nonzero entry.

Examples:

1. Let Γ be the third digraph in Figure 29.1b. Then Γ is minimally connected. The subdigraph Γ' obtained by deleting arc $(1, 2)$ from Γ is no longer strongly connected, however, it is still connected since its associated undirected graph is the graph in Figure 29.1a.

2. For $n \geq 1$, let $A^{(n)}$ be the $(n+1) \times (n+1)$ matrix be given by

$$A^{(n)} = \left[\begin{array}{c|cccc} 0 & 1 & 1 & \cdots & 1 \\ \hline 1 & & & & \\ 1 & & & & \\ \vdots & & & \mathbf{0}_{n\times n} & \\ 1 & & & & \end{array}\right].$$

Then $A^{(n)}$ is nearly reducible. The digraph $\Gamma(A^{(n)})$ is called a **rosette**, and has the most arcs possible for a minimally connected digraph on $n+1$ vertices. Suppose that $n \geq 2$, and let $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bigoplus I_{n-2}$. Then

$$PA^{(n)}P^T = \left[\begin{array}{c|cccc} 0 & 1 & 0 & \cdots & 0 \\ \hline 1 & & & & \\ 0 & & & & \\ \vdots & & & A^{(n-1)} & \\ 0 & & & & \end{array}\right]$$

is the decomposition for $A^{(n)}$ given in Fact 7 with $m = 1$.

References

[BG00] J. Bang-Jensen and G. Gutin. *Digraphs: Theory, Algorithms and Applications*. Springer-Verlag, London, 2000.

[BH79] R.A. Brualdi and M.B. Hendrick. A unified treatment of nearly reducible and nearly decomposable matrices. *Lin. Alg. Appl.*, 24 (1979) 51–73.

[BJ86] W.W. Barrett and C.R. Johnson. Determinantal Formulae for Matrices with Sparse Inverses, II: Asymmetric Zero Patterns. *Lin. Alg. Appl.*, 81 (1986) 237–261.

[BP94] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. SIAM, Philadelphia, 1994.

[Bru82] R.A. Brualdi. Matrices, eigenvalues and directed graphs. *Lin. Multilin. Alg. Applics.*, 8 (1982) 143–165.

[BR91] R.A. Brualdi and H.J. Ryser. *Combinatorial Matrix Theory*. Cambridge University Press, Cambridge, 1991.

[BR97] R.B. Bapat and T.E.S. Raghavan. *Nonnegative Matrices and Applications*. Cambridge University Press, Cambridge, 1997.

[Cve75] D.M. Cvetković. The determinant concept defined by means of graph theory. *Mat. Vesnik,* 12 (1975) 333–336.

[FP69] M. Fiedler and V. Ptak. Cyclic products and an inequality for determinants. *Czech Math J.,* 19 (1969) 428–450.

[Har62] F. Harary. The determinant of the adjacency matrix of a graph. *SIAM Rev.*, 4 (1962) 202–210.

[LHE94] Z. Li, F. Hall, and C. Eschenbach. On the period and base of a sign pattern matrix. *Lin. Alg. Appl.,* 212/213 (1994) 101–120.

[Min88] H. Minc. *Nonnegative Matrices*. John Wiley & Sons, New York, 1988.

[MOD89] J. Maybee, D. Olesky, and P. van den Driessche. Matrices, digraphs and determinants. *SIAM J. Matrix Anal. Appl.,* 4, (1989) 500–519.

[Rot75] U.G. Rothblum. Algebraic eigenspaces of nonnegative matrices. *Lin. Alg. Appl.*, 12 (1975) 281–292.

[Sch86] H. Schneider. The influence of the marked reduced graph of a nonnegative matrix on the Jordan form and related properties: a survey. *Lin. Alg. Appl.*, 84 (1986) 161–189.

[Stu91] J.L. Stuart. The determinant of a Hessenberg L-matrix, *SIAM J. Matrix Anal.*, 12 (1991) 7–15.

[Stu03] J.L. Stuart. Powers of ray pattern matrices. *Conference proceedings of the SIAM Conference on Applied Linear Algebra*, July 2003, at http://www.siam.org/meetings/la03/proceedings/stuartjl.pdf .

[Var04] R.S. Varga. *Gershgorin and His Circles*. Springer, New York, 2004.

30

Bipartite Graphs and Matrices

Bryan L. Shader
University of Wyoming

30.1 Basics of Bipartite Graphs **30**-1
30.2 Bipartite Graphs Associated with Matrices **30**-4
30.3 Factorizations and Bipartite Graphs **30**-8
References **30**-11

An $m \times n$ matrix is naturally associated with a bipartite graph, and the structure of the matrix is reflected by the combinatorial properties of the associated bipartite graph. This section discusses the fundamental structural theorems for matrices that arise from this association, and describes their implications for linear algebra.

30.1 Basics of Bipartite Graphs

This section introduces the various properties and families of bipartite graphs that have special significance for linear algebra.

Definitions:

A graph G is **bipartite** provided its vertices can be partitioned into disjoint subsets U and V such that each edge of G has the form $\{u, v\}$, where $u \in U$ and $v \in V$. The set $\{U, V\}$ is a **bipartition** of G.

A **complete bipartite graph** is a simple bipartite graph with bipartition $\{U, V\}$ such that each $\{u, v\}$ ($u \in U$, $v \in V$) is an edge. The complete bipartite graph with $|U| = m$ and $|V| = n$ is denoted by $K_{m,n}$.

A **chordal graph** is one in which every cycle of length 4 or more has a chord, that is, an edge joining two nonconsecutive vertices on the cycle.

A **chordal bipartite graph** is a bipartite graph in which every cycle of length 6 or more has a chord.

A bipartite graph is **quadrangular** provided it is simple and each pair of vertices with a common neighbor lies on a cycle of length 4.

A **weighted bipartite graph** consists of a simple bipartite graph G and a function $w : E \to X$, where E is the edge set of G and X is a set (usually $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, $\{-1, 1\}$, or a set of indeterminates). A **signed bipartite graph** is a weighted bipartite graph with $X = \{-1, 1\}$. In a signed bipartite graph, the **sign** of a set α of edges, denoted $\text{sgn}(\alpha)$, is the product of the weights of the edges in α. The set α is **positive** or **negative** depending on whether $\text{sgn}(\alpha)$ is $+1$ or -1.

Let G be a bipartite graph with bipartition $\{U, V\}$ and let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ be orderings of the distinct elements of U and V, respectively. The **biadjacency matrix** of G is the $m \times n$ matrix $\mathcal{B}_G = [b_{ij}]$, where b_{ij} is the multiplicity of the edge $\{u_i, v_j\}$. Note that if U, respectively, V, is empty, then $\mathcal{B}_G$ is a matrix with no rows, respectively, no columns. For a weighted bipartite graph, b_{ij} is defined to be the weight of the edge $\{u_i, v_j\}$ if present, and 0 otherwise. For a signed bipartite graph, b_{ij} is the sign of the edge $\{u_i, v_j\}$ if present, and 0 otherwise.

Let N'_G be an oriented incidence matrix of simple graph G. The **cut space** of G is the column space of ${N'_G}^T$, and the **cut lattice** of G is the set of integer vectors in the cut space of G. The **flow space** of G is $\{x \in \mathbb{R}^m : N'_G x = 0\}$, and the **flow lattice** of G is $\{x \in \mathbb{Z}^m : N'_G x = 0\}$.

A **matching** of G is a set M of mutually disjoint edges. If M has k edges, then M is a k-**matching**, and if each vertex of G is in some (and hence exactly one) edge of M, then M is a **perfect matching**.

Facts:

Unless otherwise noted, the following can be found in [BR91, Chap. 3] or [Big93]. In the references, the results are stated and proven for simple graphs, but still hold true for graphs.

1. A bipartite graph has no loops. It has more than one bipartition if and only if the graph is disconnected. Each forest (and, hence, each tree and each path) is bipartite. The cycle C_n is bipartite if and only if n is even.
2. The following statements are equivalent for a graph G:
   (a) G is bipartite.
   (b) The vertices of G can be labeled with the colors red and blue so that each edge of G has a red vertex and a blue vertex.
   (c) G has no cycles of odd length.
   (d) There exists a permutation matrix P such that $P^T \mathcal{A}_G P$ has the form
   $$\begin{bmatrix} O & B \\ B^T & O \end{bmatrix}.$$
   (e) G is loopless and every minor of the vertex-edge incidence matrix N_G of G is 0, 1, or -1.
   (f) The characteristic polynomial $p_{\mathcal{A}_G}(x) = \sum_{i=0}^{n} c_i x^{n-i}$ of $\mathcal{A}_G$ satisfies $c_k = 0$ for each odd integer k.
   (g) $\sigma(G) = -\sigma(G)$ (as multisets), where $\sigma(G)$ is the spectrum of $\mathcal{A}_G$.
3. The connected graph G is bipartite if and only if $-\rho(\mathcal{A}_G)$ is an eigenvalue of $\mathcal{A}_G$.
4. The bipartite graph G disconnected if and only if there exist permutation matrices P and Q such that $P\mathcal{B}_G Q$ has the form
   $$\begin{bmatrix} B_1 & O \\ O & B_2 \end{bmatrix},$$
   where both B_1 and B_2 have at least one column or at least one row. More generally, if G is bipartite and has k connected components, then there exist permutation matrices P and Q such that
   $$P\mathcal{B}_G Q = \begin{bmatrix} B_1 & O & \cdots & O \\ O & B_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & B_k \end{bmatrix},$$
   where the B_i are the biadjacency matrices of the connected components of G.
5. [GR01] If G is a simple graph with n vertices, m edges, and c components, then its cut space has dimension $n - c$, and its flow space has dimension $m - n + c$. If G is a plane graph, then the edges of G can be oriented and ordered so that the flow space of G equals the cut space of its dual graph. The norm, $\mathbf{x}^T\mathbf{x}$, is even for each vector $\mathbf{x}$ in the cut lattice of G if and only if each vertex has even degree. The norm of each vector in the flow lattice of G is even if and only if G is bipartite.

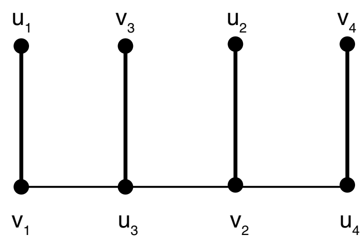


FIGURE 30.1

6. [Bru66], [God85], [Sim89] Let G be a bipartite graph with a unique perfect matching. Then there exist permutation matrices P and Q such that $P\mathcal{B}_G Q$ is a square, lower triangular matrix with all 1s on its main diagonal. If G is a tree, then the inverse of $P\mathcal{B}_G Q$ is a $(0, 1, -1)$-matrix. Let n be the order of $\mathcal{B}_G$, and let H be the simple graph with vertices $1, 2, \ldots, n$ and $\{i, j\}$ an edge if and only if $i \neq j$ and either the (i, j)- or (j, i)-entry of $P\mathcal{B}_G Q$ is nonzero. If H is bipartite, then $(P\mathcal{B}_G Q)^{-1}$ is diagonally similar to a nonnegative matrix, which equals $P\mathcal{B}_G Q$ if and only if G can be obtained by appending a pendant edge to each vertex of a bipartite graph.

Examples:

1. Up to matrix transposition and permutations of rows and columns, the biadjacency matrix of the path P_{2n}, the path P_{2n+1}, and the cycle C_{2n} are

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{n\times n}, \quad \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}_{n\times (n+1)}, \quad \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix}_{n\times n.}$$

2. The biadjacency matrix of the complete bipartite graph $K_{m,n}$ is $J_{m,n}$, the $m \times n$ matrix of all ones.
3. Up to row and column permutations, the biadjacency matrix of the graph obtained from $K_{n,n}$ by removing the edges of a perfect matching is $J_n - I_n$.
4. Let G be the bipartite graph (Figure 30.1).
   Then

$$\mathcal{B}_G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

G has a unique perfect matching, and the graph H defined in Fact 6 is the path 1–3–2–4. Hence, $\mathcal{B}_G$ is diagonally similar to a nonnegative matrix. Also, since G is obtained from the bipartite graph v_1—u_3—v_2—u_4 by appending pendant vertices to each vertex, $\mathcal{B}_G^{-1}$ is diagonally similar to $\mathcal{B}_G$. Indeed,

$$S\mathcal{B}_G^{-1}S = S \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} S^{-1} = \mathcal{B}_G,$$

where S is the diagonal matrix with main diagonal $(1, 1, -1, -1)$.

30.2 Bipartite Graphs Associated with Matrices

This section presents some of the ways that matrices have been associated to bipartite graphs and surveys resulting consequences.

Definitions:

The **bigraph** of the $m \times n$ matrix $A = [a_{ij}]$ is the simple graph with vertex set $U \cup V$, where $U = \{1, 2, \ldots, m\}$ and $V = \{1', 2', \ldots, n'\}$, and edge set $\{\{i, j'\} : a_{ij} \neq 0\}$. If A is a nonnegative integer matrix, then the **multi-bigraph** of A has vertex set $U \cup V$ and edge $\{i, j'\}$ of multiplicity a_{ij}. If A is a general matrix, then the **weighted bigraph** of A has vertex set $U \cup V$ and edge $\{i, j'\}$ of weight a_{ij}. If A is a real matrix, then the **signed bigraph** of A is obtained by weighting the edge $\{i, j'\}$ of the bigraph by $+1$ if $a_{ij} > 0$, and by -1 if $a_{ij} < 0$.

The **(zero) pattern** of the $m \times n$ matrix $A = [a_{ij}]$ is the $m \times n$ $(0, 1)$-matrix whose (i, j)-entry is 1 if and only if $a_{ij} \neq 0$.

The **sign pattern** of the real $m \times n$ matrix $A = [a_{ij}]$ is the $m \times n$ matrix whose (i, j)-entry is $+$, 0, or $-$, depending on whether a_{ij} is positive, zero, or negative. (See Chapter 33 for more information on sign patterns.)

A $(0, 1)$-matrix is a **Petrie matrix** provided the 1s in each of its columns occur in consecutive rows. A $(0, 1)$-matrix A has the **consecutive ones property** if there exists a permutation P such that PA is a Petrie matrix.

The **directed bigraph** of the real $m \times n$ matrix $A = [a_{ij}]$ is the directed graph with vertices $1, 2, \ldots, m$, $1', 2', \ldots, n'$, the arc (i, j') if and only if $a_{ij} > 0$, and the arc (j', i) if and only if $a_{ij} < 0$.

An $m \times n$ matrix A is a **generic matrix** with respect to the field F provided its nonzero elements are independent indeterminates over the field F. The matrix A can be viewed as a matrix whose elements are in the ring of polynomials in these indeterminates with coefficients in F.

Let A be an $n \times n$ matrix with each diagonal entry nonzero. The **bipartite fill-graph** of A, denoted $G^+(A)$, is the simple bipartite graph with vertex set $\{1, 2, \ldots, n\} \cup \{1', 2', \ldots, n'\}$ with an edge joining i and j' if and only if there exists a path from i to j in the digraph, $\Gamma(A)$, of A each of whose intermediate vertices has label less than $\min\{i, j\}$. If A is symmetric, then (by identifying vertices i and i' for $i = 1, 2, \ldots, n$ and deleting loops), $G^+(A)$ can be viewed as a simple graph, and is called the **fill-graph** of A.

The square matrix B has a **perfect elimination ordering** provided there exist permutation matrices P and Q such that the bipartite fill-graph, $G^+(PBQ)$, and the bigraph of PBQ are the same.

Associated with the $n \times n$ matrix $A = [a_{ij}]$ is the sequence $H_0, H_1, \ldots, H_{n-1}$ of bipartite graphs as defined by:

1. H_0 consists of vertices $1, 2, \ldots, n$, and $1', 2', \ldots, n'$, and edges of the form $\{i, j'\}$, where $a_{ij} \neq 0$.
2. For $k = 1, \ldots, n-1$, H_k is the graph obtained from H_{k-1} by deleting vertices k and k' and inserting each edge of the form $\{r, c'\}$, where $r > k$, $c > k$, and both $\{r, k'\}$ and $\{k, c'\}$ are edges of H_{k-1}.

The **4-cockades** are the bipartite graphs recursively defined by: A 4-cycle is a 4-cockade, and if G is a 4-cockade and e is an edge of G, then the graph obtained from G by identifying e with an edge of a 4-cycle disjoint from G is a 4-cockade. A **signed 4-cockade** is a 4-cockade whose edges are weighted by ± 1 in such a way that every 4-cycle is negative.

Facts:

General references for bipartite graphs associated with matrices are [BR91, Chap. 3] and [BS04].

1. [Rys69] (See also [BR91, p. 18].) If A is an $m \times n$ $(0, 1)$-matrix such that each entry of AA^T is positive, then either A has a column with no zeros or the bigraph of A has a chordless cycle of length 6. The converse is not true.
2. [RT76], [GZ98] If $G = (V, E)$ is a connected quadrangular graph, then $|E| \leq 2|V| - 4$. The connected quadrangular graphs with $|E| = 2|V| - 4$ are characterized in the first reference.

3. [RT76], If A is an $m \times n$ $(0,1)$-matrix such that no entry of AA^T or A^TA is 1, and the bigraph of A is connected, then A has at most $2(m+n)-4$ nonzero entries.
4. [Tuc70] The $(0,1)$-matrix A has the consecutive ones property if and only if it does not have a submatrix whose rows and columns can be permuted to have one of the following forms for $k \geq 1$.

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & 1 & 1 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(k+2)\times(k+2)}, \quad \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & & 0 & 1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(k+3)\times(k+2)},$$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & & 0 & 1 & 1 \\ 0 & 1 & \ddots & \vdots & \vdots & \vdots \\ \vdots & & \ddots & 1 & 1 & 1 \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}_{(k+3)\times(k+3)}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}_{4\times 6}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}_{4\times 5}.$$

5. [ABH99] Let A be a $(0,1)$-matrix and let $L = D - AA^T$, where D is the diagonal matrix whose ith diagonal entry is the ith row sum of AA^T. Then L is a symmetric, singular matrix each of whose eigenvalues is nonnegative. Let $\mathbf{v}$ be a eigenvector of L corresponding to the second smallest eigenvalue of L. If A has the consecutive ones property and the entries of $\mathbf{v}$ are distinct, then PA is a Petrie matrix, where P is the permutation matrix such that the entries of $P\mathbf{v}$ are in increasing order. In addition, the reference gives a recursive method for finding a P such that PA is a Petrie matrix when the elements of $\mathbf{v}$ are not distinct.
6. The directed bigraph of the real matrix A contains at most one of the arcs (i, j') or (j', i).
7. [FG81] The directed bigraph of the real matrix A is strongly connected if and only if there do not exist subsets α and β such that $A[\alpha, \beta] \geq 0$ and $A(\alpha, \beta) \leq 0$. Here, either α or β may be the empty set, and a vacuous matrix M satisfies both $M \geq 0$ and $M \leq 0$.
8. [FG81] If $A = [a_{ij}]$ is a fully indecomposable, $n \times n$ sign pattern, then the following are equivalent:

   (a) There is a matrix $\widehat{A}$ with sign pattern A such that $\widehat{A}$ is invertible and its inverse is a positive matrix.

   (b) There do not exist subsets α and β such that $A[\alpha, \beta] \geq O$ and $A(\alpha, \beta) \leq O$.

   (c) The bipartite directed graph of A is strongly connected.

   (d) There exists a matrix with sign pattern A each of whose line sums is 0.

   (e) There exists a rank $n-1$ matrix with sign pattern A each of whose line sums is 0.

9. [Gol80] Up to relabeling of vertices, G is the fill-graph of some $n \times n$ symmetric matrix if and only if G is chordal.
10. [GN93] Let A be an $n \times n$ $(0,1)$-matrix with each diagonal entry equal to 1. Suppose that B is a matrix with zero pattern A, and that B can be factored as $B = LU$, where $L = [\ell_{ij}]$ is a lower triangular matrix and $U = [u_{ij}]$ is an upper triangular matrix. If $i \neq j$ and either $\ell_{ij} \neq 0$ or $u_{ij} \neq 0$, then $\{i, j'\}$ is an edge of $G^+(A)$. Moreover, if B is a generic matrix with zero pattern A, then such a factorization $B = LU$ exists, and for each edge $\{i, j'\}$ of $G^+(A)$ either $\ell_{ij} \neq 0$ or $u_{ij} \neq 0$.

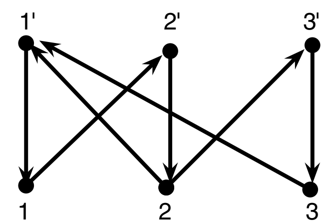


FIGURE 30.2

11. [GG78] If the bigraph of the generic, square matrix A is chordal bipartite, then A has a perfect elimination ordering and, hence, there exist permutation matrices P and Q such that performing Gaussian elimination on PAQ has no fill-in. The converse is not true; see Example 3.
12. [GN93] If A is a generic $n \times n$ matrix with each diagonal entry nonzero, and $\alpha = \{1, 2, \ldots, r\}$, then the bigraph of the Schur complement of $A[\alpha]$ in A is the bigraph H_r defined above.
13. [DG93] For each matrix with a given pattern, small relative perturbations in the nonzero entries cause only small relative perturbations in the singular values (independent of the values of the matrix entries) if and only if the bigraph of the pattern is a forest. The singular values of such a matrix can be computed to high relative accuracy.
14. [DES99] If the signed bipartite graph of the real matrix is a signed 4-cockade, then small relative perturbations in the nonzero entries cause only small relative perturbations in the singular values (independent of the values of the matrix entries). The singular values of such a matrix can be computed to high relative accuracy.

Examples:

1. Let

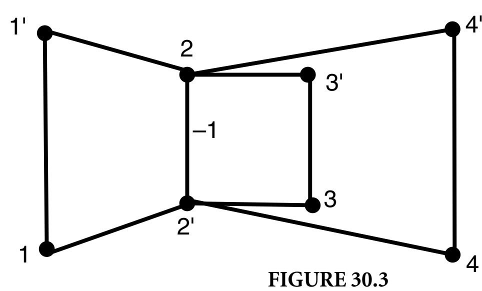
$$A = \begin{bmatrix} - & + & 0 \\ + & - & + \\ + & 0 & - \end{bmatrix}.$$

The directed bigraph of A is (Figure 30.2).

Since this is strongly connected, Fact 7 implies that there do not exist subsets α and β such that $A[\alpha, \beta] \geq O$ and $A(\alpha, \beta) \leq O$. Also, there is a matrix with sign pattern A whose inverse is positive. One such matrix is

$$\begin{bmatrix} -3/2 & 2 & 0 \\ 1 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

2. A signed 4-cockade on 8 vertices (unlabeled edges have sign $+1$) and its biadjacency matrix are (Figure 30.3)



$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

FIGURE 30.3

3. Both the bipartite fill-graph and the bigraph of the matrix (Figure 30.4) below

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

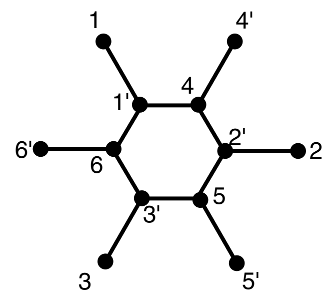


FIGURE 30.4

are the graph illustrated. Since its bigraph has a chordless 6-cycle, this example shows that the converse to Fact 11 is false.

4. Let

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & 0 & 0 \\ x_7 & 0 & x_8 & 0 \\ x_9 & 0 & 0 & x_{10} \end{bmatrix},$$

where $x_1, \ldots, x_{10}$ are independent indeterminates. The bigraph of A is chordal bipartite. The biadjacency matrix of H_1 is J_3. Thus, by Fact 12, the pattern of the Schur complement of $A[\{1\}]$ in A is J_3. The bipartite fill-graph of A has biadjacency matrix J_4. If

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

then the bipartite fill-graph of PAP^T and the bigraph of PAP^T are the same. Hence, it is possible to perform Gaussian elimination (without pivoting) on PAP^T without any fill-in.

Applications:

1. [Ken69], [ABH99] Petrie matrices are named after the archaeologist Flinders Petrie and were first introduced in the study of seriation, that is, the chronological ordering of archaeological sites. If the rows of the matrix A represent archaeological sites ordered by their historical time period, the columns of A represent artifacts, and $a_{ij} = 1$ if and only if artifact j is present at site i, then one would expect A to be a Petrie matrix. More recently, matrices with the consecutive ones property have arisen in genome sequencing (see [ABH99]).
2. [BBS91], [Sha97] If U is a unitary matrix and A is the pattern of U, then the bigraph of A is quadrangular. If U is fully indecomposable, then U has at most $4n-4$ nonzero entries. The matrices achieving equality are characterized in the first reference. (See Fact 2 for more on quadrangular graphs.)

30.3 Factorizations and Bipartite Graphs

This section discusses the combinatorial interpretations and applications of certain matrix factorizations.

Definitions:

A **biclique** of a graph G is a subgraph that is a complete bipartite graph. For disjoint subsets X and Y of vertices, $B(X, Y)$ denotes the biclique consisting of all edges of the form $\{x, y\}$ such that $x \in X$ and $y \in Y$ (each of multiplicity 1). If G is bipartite with bipartition $\{U, V\}$, then it is customary to take $X \subseteq U$ and $Y \subseteq V$.

A **biclique partition** of $G = (V, E)$ is a collection $B(X_1, Y_1), \ldots, B(X_k, Y_k)$ of bicliques of G whose edges partition E.

A **biclique cover** of $G = (V, E)$ is a collection of bicliques such that each edge of E is in at least one biclique.

The **biclique partition number of** G, denoted $\mathrm{bp}(G)$, is the smallest k such that there is a partition of G into k bicliques. The **biclique cover number of** G, denoted $\mathrm{bc}(G)$, is the smallest k such that there is a cover of G by k bicliques. If G does not have a biclique partition, respectively, cover, then $\mathrm{bp}(G)$, respectively, $\mathrm{bc}(G)$, is defined to be infinite.

If G is a graph, then $n_+(G)$, respectively, $n_-(G)$, denotes the number of positive, respectively, negative, eigenvalues of $\mathcal{A}_G$ (including multiplicity).

If $X \subseteq \{1, 2, \ldots, n\}$, then the **characteristic vector** of X is the $n \times 1$ vector $\vec{\mathbf{X}} = [\mathbf{x_i}]$, where $x_i = 1$ if $i \in X$, and $x_i = 0$ otherwise.

The **nonnegative integer rank** of the nonnegative integer matrix A is the minimum k such that there exist an $m \times k$ nonnegative integer matrix B and a $k \times n$ nonnegative integer matrix C with $A = BC$.

The **(0,1)-Boolean algebra** consists of the elements 0 and 1, endowed with the operations defined by $0 + 0 = 0, 0 + 1 = 1 = 1 + 0, 1 + 1 = 1, 0 * 1 = 0 = 1 * 0, 0 * 0 = 0$, and $1 * 1 = 1$. A **Boolean matrix** is a matrix whose entries belong to the (0,1)-Boolean algebra. Addition and multiplication of Boolean matrices is defined as usual, except Boolean arithmetic is used.

The **Boolean rank** of the $m \times n$ Boolean matrix A is the minimum k such that there exists an $m \times k$ Boolean matrix B and a $k \times n$ Boolean matrix C such that $A = BC$.

Let G be a bipartite graph with bipartition $\{\{1, 2, \ldots, m\}, \{1', 2', \ldots, n'\}\}$. Then $\mathcal{M}(G)$ denotes the set of all $m \times n$ matrices $A = [a_{ij}]$ such that if $a_{ij} \neq 0$, then $\{i, j'\}$ is an edge of G, that is, the bigraph of A is a subgraph of G. The graph G **supports rank decompositions** provided each matrix $A \in \mathcal{M}(G)$ is the sum of $\mathrm{rank}(A)$ elements of $\mathcal{M}(G)$ each having rank 1.

If G is a signed bipartite graph, then $\mathcal{M}(G)$ denotes the set of all matrices $A = [a_{ij}]$ such that if $a_{ij} > 0$, then $\{i, j'\}$ is a positive edge of G, and if $a_{ij} < 0$, then $\{i, j'\}$ is a negative edge of G. The signed bigraph G **supports rank decompositions** provided each matrix $A \in \mathcal{M}(G)$ is the sum of $\mathrm{rank}(A)$ elements of $\mathcal{M}(G)$ each having rank 1.

Facts:

1. [GP71]
   - A graph has a biclique partition (and, hence, cover) if and only if it has no loops.
   - For every graph G, $\mathrm{bc}(G) \leq \mathrm{bp}(G)$.
   - Every simple graph G with n vertices has a biclique partition with at most $n - 1$ bicliques, namely, $B(\{i\}, \{j : \{i, j\}$ is an edge of G and $j > i\})$ $(i = 1, 2, \ldots, n - 1)$.
2. [CG87] Let G be a bipartite graph with bipartition (U, V), where $|U| = m$ and $|V| = n$. Let $B(X_1, Y_1), B(X_2, Y_2), \ldots, B(X_k, Y_k)$ be bicliques with $X_i \subseteq U$ and $Y_i \subseteq V$ for all i. The following are equivalent:

   (a) $B(X_1, Y_1), B(X_2, Y_2), \ldots, B(X_k, Y_k)$ is a biclique partition of G.

(b) $\sum_{i=1}^{k} \vec{\mathbf{X_i}}\vec{\mathbf{Y_i}}^T = \mathcal{B}_G$.

(c) $XY^T = \mathcal{B}_G$, where X is the $n \times k$ matrix whose ith column is $\vec{\mathbf{X_i}}$, and Y is the $n \times k$ matrix whose ith column is $\vec{\mathbf{Y_i}}$.

3. [CG87] For a simple bipartite graph G, $\text{bp}(G)$ equals the nonnegative integer rank of $\mathcal{B}_G$.
4. [CG87]

- Let G be the bipartite graph obtained from $K_{n,n}$ by removing a perfect matching. Then $\text{bp}(G) = n$. Furthermore, if $B(X_i, Y_i)$ $(i = 1, 2, \ldots, n)$ is a biclique partition of G, then there exist positive integers r and s such that $rs = n - 1$, $|X_i| = r$ and $|Y_i| = s$ $(i = 1, 2, \ldots, n)$, k is in exactly r of the X_i's and exactly s of the Y_i's $(k = 1, 2, \ldots, n)$, and $X_i \cap Y_j = 1$ for $i \neq j$.
- In matrix terminology, if X and Y are $n \times n$ $(0, 1)$-matrices such that $XY^T = J_n - I_n$, then there exist integers r and s such that $rs = n - 1$, X has constant line sums r, Y has constant line sums s, and $Y^T X = J_n - I_n$.
- In particular, if $n - 1$ is prime, then either X is a permutation matrix and $Y = (J_n - I_n)X$, or Y is a permutation matrix and $X = (J_n - I_n)Y$.

5. [BS04, see p. 67] Let G be a graph on n vertices with adjacency matrix $\mathcal{A}_G$, and let $B(X_1, Y_1)$, $B(X_2, Y_2), \ldots, B(X_k, Y_k)$ be bicliques of G. Then the following are equivalent:

(a) $B(X_1, Y_1), B(X_2, Y_2), \ldots, B(X_k, Y_k)$ is a biclique partition of G.

(b) $\sum_{i=1}^{k} \vec{\mathbf{X_i}}\vec{\mathbf{Y_i}}^T + \sum_{i=1}^{k} \vec{\mathbf{Y_i}}\vec{\mathbf{X_i}}^{\mathbf{T}} = \mathcal{A}_G$.

(c) $XY^T + YX^T = \mathcal{A}_G$, where X is the $n \times k$ matrix whose ith column is $\vec{\mathbf{X_i}}$, and Y is the $n \times k$ matrix whose ith column is $\vec{\mathbf{Y_i}}$.

(d) $\mathcal{A}_G = M \begin{bmatrix} O & I_m \\ I_m & O \end{bmatrix} M^T$, where M is the $n \times 2k$ matrix $\begin{bmatrix} X & Y \end{bmatrix}$ formed from the matrices X and Y defined in (c).

6. [CH89] The bicliques $B(X_1, Y_1), B(X_2, Y_2), \ldots, B(X_k, Y_k)$ partition K_n if and only if XY^T is an $n \times n$ tournament matrix, where X is the $n \times k$ matrix whose ith column is $\vec{\mathbf{X_i}}$, and Y is the $n \times k$ matrix whose ith column is $\vec{\mathbf{Y_i}}$. Thus, $\text{bp}(K_n)$ is the minimum nonnegative integer rank among all the $n \times n$ tournament matrices.
7. [CH89] The rank of an $n \times n$ tournament matrix is at least $n - 1$.
8. (Attributed to Witsenhausen in [GP71])

$$\text{bp}(K_n) = n - 1,$$

that is, it is impossible to partition the complete graph into $n - 2$ or fewer bicliques.

9. [GP71] *The Graham–Pollak Theorem*: If G is a loopless graph, then

$$\text{bp}(G) \geq \max\{n_+(G), n_-(G)\}. \tag{30.1}$$

The graph G is **eigensharp** if equality holds in (30.1). It is conjectured in [CGP86] that for all λ, and n sufficiently large, the complete graph λK_n with each edge of multiplicity k is eigensharp.

10. [ABS91] If $B(X_1, Y_1), B(X_2, Y_2), \ldots, B(X_k, Y_k)$ is a biclique partition of G, then there exists an acyclic subgraph of G with $\max\{(n_+(G), n_-(G)\}$ edges no two in the same $B(X_i, Y_i)$.
In particular, for each biclique partition of K_n there exists a spanning tree no two of whose edges belong to the same biclique of the partition.
11. [CH89] For all positive integers r and s with $2 \leq r < s$, the edges of the complete graph K_{2rs} cannot be partitioned into copies of the complete bipartite graph $K_{r,s}$.

12. [Hof01] If m and n are positive integers with $2m \leq n$, and $G_{m,n}$ is the graph obtained from the complete graph K_n by duplicating the edges of an m-matching, then $n_+(G) = n - m - 1$, $n_-(G) = m + 1$, and $\text{bp}(G) \geq n - m + \lfloor\sqrt{2m}\rfloor - 1$.
13. [CGP86] Let A be an $m \times n$ $(0,1)$-matrix with bigraph G and let $B(X_1, Y_1)$, $B(X_2, Y_2)$, ..., $B(X_k, Y_k)$ be bicliques. The following are equivalent:
    (a) $B(X_1, Y_1), B(X_2, Y_2), \ldots, B(X_k, Y_k)$ is a biclique cover of G.
    (b) $\sum_{i=1}^{k} \vec{\mathbf{X_i}}\vec{\mathbf{Y_i}}^{\mathbf{T}} = A$ (using Boolean arithmetic).
    (c) $XY^T = B$ (using Boolean arithmetic), where X is the $m \times k$ matrix whose jth column is $\vec{\mathbf{X_j}}$, and Y is the $m \times k$ matrix whose j column is $\vec{\mathbf{Y_j}}$.
14. [CGP86] The Boolean rank of a $(0, 1)$-matrix A equals the biclique cover number of its bigraph.
15. [CSS87] Let k be a positive integer and let $t(k)$ be the largest integer n such that there exists an $n \times n$ tournament matrix with Boolean rank k. Then for $k \geq 2$, $t(k) < k^{\log_2(2k)}$, and $n(n^2+n+1)+2 \leq t(n^2 + n + 1)$.
    It is still an open problem to determine the minimum Boolean rank among $n \times n$ tournament matrices.
16. [DHM95, JM97] The bipartite graph G supports rank decompositions if and only if G is chordal bipartite.
17. [GMS96] The signed bipartite graph G support rank decompositions if and only if

$$\text{sgn}(\gamma) = (-1)^{(\ell(\gamma)/2)-1} \tag{30.2}$$

for every cycle γ of G of length $\ell(\gamma) \geq 6$. Additionally, every matrix in $\mathcal{M}(G)$ has its rank equal to its term rank if and only if (30.2) holds for every cycle of G.

Examples:

1. Below, the edges of different textures form the bicliques (Figure 30.5) in a biclique partition of the graph $G_{2,4}$ obtained from K_4 by duplicating two disjoint edges.
2. Let n be an integer and r and s positive integers with $n - 1 = rs$. Then $XY^T = J_n - I_n$, where $X = I + C^s + C^{2s} + \cdots + C^{s(r-1)}$, $Y = C + C^2 + C^3 + \cdots + C^s$, and C is the $n \times n$ permutation matrix with 1s in positions $(1, 2), (2, 3), \ldots, (n-1, n)$, and $(n, 1)$.
   This shows that for each pair of positive integers r and s with $rs = n-1$, there is a biclique partition of $J_n - I_n$ with X_i and Y_i satisfying the conditions in Fact 4.
3. For n odd, $B(\{i\}, \{i+1, i+2, \ldots, i+\frac{n-1}{2}\})$ $(i = 1, 2, \ldots, n)$ is a partition of K_n into bicliques each isomorphic to $K_{1,\frac{n-1}{2}}$, where the indices are read mod n (see Fact 11).

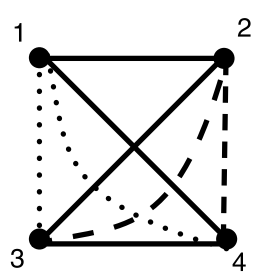


FIGURE 30.5

References

[ABS91] N. Alon, R.A. Brualdi, and B.L. Shader. Multicolored forests in bipartite decompositions of graphs. *J. Combin. Theory Ser. B*, 53:143–148, 1991.

[ABH99] J. Atkins, E. Boman, and B. Hendrickson. A spectral algorithm for seriation and the consecutive ones problem. *SIAM J. Comput.*, 28:297–310, 1999.

[BBS91] L.B. Beasley, R.A. Brualdi, and B.L. Shader. Combinatorial orthogonality. *Combinatorial and Graph-Theoretical Problems in Linear Algebra*, The IMA Volumes in Mathematics and Its Applications, vol. 50, Springer-Verlag, New York, 207–218, 1991.

[Big93] N. Biggs. *Algebraic Graph Theory*. Cambridge University Press, Cambridge, 2nd ed., 1993.

[BR91] R.A. Brualdi and H.J. Ryser. *Combinatorial Matrix Theory*. Cambridge University Press, Cambridge, 1991.

[Bru66] R.A. Brualdi. Permanent of the direct product of matrices. *Pac. J. Math.*, 16:471–482, 1966.

[BS04] R.A. Brualdi and B.L. Shader. Graphs and matrices. *Topics in Algebraic Graph Theory* (L. Beineke and R. J. Wilson, Eds.), Cambridge University Press, Cambridge, 56–87, 2004.

[CG87] D. de Caen and D.A. Gregory. On the decomposition of a directed graph into complete bipartite subgraphs. *Ars Combin.*, 23:139–146, 1987.

[CGP86] D. de Caen, D.A. Gregory, and N.J. Pullman. The Boolean rank of zero-one matrices. *Proceedings of the Third Caribbean Conference on Combinatorics and Computing*, 169–173, 1981.

[CH89] D. de Caen and D.G. Hoffman. Impossibility of decomposing the complete graph on n points into $n-1$ isomorphic complete bipartite graphs. *SIAM J. Discrete Math.*, 2:48–50, 1989.

[CSS87] K.L. Collins, P.W. Shor, and J.R. Stembridge. A lower bound for $(0, 1, *)$ tournament codes. *Discrete Math.*, 63:15–19, 1987.

[DES99] J. Demmel, M. Gu, S. Eisenstat, I. Slapnicar, K. Veselic, and Z. Drmac. Computing the singular value decompostion with high relative accuracy. *Lin. Alg. Appls.*, 119:21–80, 1999.

[DG93] J. Demmel and W. Gragg. On computing accurate singular values and eigenvalues of matrices with acyclic graphs. *Lin. Alg. Appls.*, 185:203–217, 1993.

[DHM95] K.R. Davidson, K.J. Harrison, and U.A. Mueller. Rank decomposability in incident spaces. *Lin. Alg. Appls.*, 230:3–19, 1995.

[FG81] M. Fiedler and R. Grone. Characterizations of sign patterns of inverse-positive matrices. *Lin. Alg. Appls.*, 40:237–45, 1983.

[Gol80] M.C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs*. Academic Press, New York, 1980.

[GG78] M.C. Golumbic and C.F. Goss. Perfect elimination and chordal bipartite graphs. *J. Graph Theory*, 2:155–263, 1978.

[GMS96] D.A. Gregory, K.N. vander Meulen, and B.L. Shader. Rank decompositions and signed bigraphs. *Lin. Multilin. Alg.*, 40:283–301, 1996.

[GN93] J.R. Gilbert and E.G. Ng. Predicting structure in nonsymmetric sparse matrix factorizations. *Graph Theory and Sparse Matrix Computations*, The IMA Volumes in Mathematics and Its Applications, v. 56, 1993, 107–140.

[God85] C.D. Godsil. Inverses of trees. *Combinatorica*, 5:33–39, 1985.

[GP71] R.L. Graham and H.O. Pollak. On the addressing problem for loop switching. *Bell Sys. Tech. J.*, 50:2495–2519, 1971.

[GR01] C.D. Godsil and G. Royle. *Algebraic Graph Theory,* Graduate Texts in Mathematics, 207, Springer-Verlag, Heidelberg 2001.

[GZ98] P.M. Gibson and G.-H. Zhang. Combinatorially orthogonal matrices and related graphs. *Lin. Alg. Appls.*, 282:83–95, 1998.

[Hof01] A.J. Hoffman. On a problem of Zaks. *J. Combin. Theory Ser. A*, 93:371–377, 2001.

[JM97] C.R. Johnson and J. Miller. Rank decomposition under combinatorial constraints. *Lin. Alg. Appls.*, 251:97–104, 1997.

[Ken69] D.G. Kendall. Incidence matrices, interval graphs and seriation in archaeology. *Pac. J. Math.*, 28:565–570, 1969.