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LINEAR ALGEBRA AND ITS APPLICATIONS

Olga, matrix theory and the Taussky unification problem

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1. Introduction

Olga Taussky Todd had a profound role in the development of matrix theory as a subject, not only through her specific results but also through her style. Our purpose here is to briefly review the scope of her interests in matrix theory specifically and then discuss further two of her favorite topics (Gersgorin and Lyapunov), which are excellent examples of her influence. We then move on to our primary subject, which is another good example, the so-called "Taussky unification problem" by reviewing examples of commonalities *and differences* among properties of the positive semidefinite, totally nonnegative and M-matrices. We then complete this discussion with some new results that give further structural commonalilties generalizing an important known property of positive semidefinite matrices. For some earlier, personal thoughts on Olga's role in matrix theory alone, see [1].

2. Preliminaries

In addition to deep, bidirectional connections with most parts of mathematics and its applications, matrix theory has enjoyed extraordinary intellectual

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development as a distinct branch of mathematics in recent decades. In many ways, through her work, her students and her style, Olga Taussky Todd profoundly shaped the subject of "matrix theory" (as distinct from its intellectual forerunner and component, classical linear algebra). In addition to an abiding interest in some highly algebraic topics of matrix theory (commutators [T63, T111], commutativity and generalizations [T85,T137], integral matrices, the L-property [T56, T60, T69, T151], quadratic forms and matrix products [T133, T150, T147], etc., see [T225] for a lovely personal survey), she had two remarkable talents that continue to affect the subject. First, she had an excellent knack for recognizing potential for deep beauty and continuing development in facts previously of interest exclusively in an applied setting. The two best examples are Gersgorin's theorem [T43, T47, T51, T52, T181] that gives cheap inclusion regions for all the eigenvalues and Lyapunov's theorem [T101, T106, T110, T130] that relates matrix stability to positive definiteness. Second, she delighted in observing and understanding connections. An example is the "Taussky unification problem", stemming from [T90] in which explanations for commonalities among properties of positive definite, totally positive and M-matrices were sought.

3. Gersgorin's theorem

Gersgorin's theorem, in its basic form, states that all the eigenvalues of an n-by-n complex matrix lie among n discs whose centers are the diagonal entries and whose corresponding radii are the sums of the absolute values of the off-diagonal entries in the same row. For its simplicity and when extended by simple observations, this is perhaps the most powerful theorem in matrix theory. Olga knew of it through her work in the war effort with Jack (stability analysis of aircraft design) and recognized that it had much to offer, not only for specific applications but also as a topic of further research and use within matrix theory. She wrote several papers [T43, T47, T51, T52, T181], the most famous being her 1949 Monthly article "A Recurring Theorem on Determinants". She went much more deeply into such issues as how eigenvalues can occur on the boundary of the union of discs, the use of diagonal similarities to alter the radii to advantage and the role of irreducibility of the matrix. Subsequent to Olga's work many prominent researchers took an interest in Gersgorin's theorem, establishing further beautiful results. By now, the area has become a subject on its own in which attractive work is ongoing, and the perspective has expanded to Gersgorin-type estimation of other fundamental matricial objects, such as the field of values, singular values and permanental roots, etc. There is ample room for a book-length survey, which Olga had once contemplated with Jack and others, but to this day no one has actually completed such a project.

4. Lyapunov's theorem

The basic (matrix) version of Lyapunov's theorem states that the n-by-n complex matrix A is "positive stable" (all eigenvalues in the open right half plane) if and only if there is a positive definite matrix G such that GA = B satisfies: $H = B + B^*$ is positive definite. Moreover, if A is positive stable and *any* positive definite matrix H is chosen, the unique solution G to the linear matrix equation $GA + A^*G = H$ will be positive definite. Again, Olga knew of this theorem through applied work on stability analysis, but she also recognized its intrinsic, purely matrix theoretic, beauty. Somehow the location of the eigenvalues of a troublesome general matrix was related to the location of the eigenvalues of a much more friendly Hermitian matrix via a linear matrix equation! Surely, there was more to be understood in this connection; and there was. Olga found lovely generalizations [T101, T106], explored the matrix equation aspect ([T110, which was a forerunner of modern views of the subject) and made the connection between symmetric matrices and general matrices a theme of her work and her talks [T133, T150]. Again, others began to notice this attractive connection also, and Lyapunov theory (and related ideas such as special types of stability) became a major theme of matrix theoretic research in the second half of the century. There are now many views of Lyapunov's theorem (e.g. A has eigenvalues in the right half-plane if and only if a positive definite multiple of A has its field of values there) that lead to intriguing generalizations.

5. Taussky unification problem

The three highly important matrix classes: positive definite (PD), totally positive (TP, *all* minors positive) and M-matrices have manifest commonalities. They are all P-matrices (positive principal minors) with all eigenvalues in the open right half-plane. They also have less apparent similarities such as enjoying Hadamard's determinantal inequality, the more general inequality of Fischer (see [2], pp. 477–478), and the yet more general inequality attributable to Koteljanskii:

$$\det A[\alpha \cup \beta] \det A[\alpha \cap \beta] \leqslant \det A[\alpha] \det A[\beta]$$

for all index sets α, $\beta \subseteq \{1, 2, \ldots, n\}$, which implies many further inequalities (see [3,4]). They also all have LU factorizations, $A = LU$, with L lower triangular and U upper triangular, of special type: L and U are totally nonnegative (all minors nonnegative) if A is (see [5]), L and U are M-matrices if A is and $U^* = L$ if A is positive definite. With a natural interest in understanding the basis for commonalities, Olga posed a general problem [T90] which later became known as the rather more inclusive "Taussky unification problem". Recognizing and inquiring into commonalities of these three classes (not all mentioned explicitly by her) was not unique to Olga, but the problem and its broadening captured the attention of many researchers and, in some philosophical way, has been behind the thrust of much research. See [6] for an early discussion of some background details.

Though the three classes have much in common, their notable differences have received less attention in spite of also being important in understanding commonalities. Some examples include the following. The positive definite matrices are closed under Hadamard (entry-wise) multiplication (denoted $A \circ B$), while $A \circ B^{-1}$ is an M-matrix if A and B are (see [7]). But if A and B are TP and n-by-n with $n \geqslant 3$, then $A \circ B$ need *not* be TP and $A \circ B^{-1}$ need not be inverse TP (see [7]). (On the other hand, there are interesting large subclasses of totally nonnegative matrices that are closed under the Hadamard product, e.g. the Routh matrices of stable polynomials, the tridiagonals and the inverse tridagonals (see [8]).) The inverse M-matrices are closed under Hadamard product for $n \leqslant 3$ and not for $n \geqslant 6$, while $n = 4$ and 5 and the question of Hadamard squares of inverse M-matrices remain open (see [7]). There are also spectral differences. PD and TP matrices have real eigenvalues and are diagonalizable, while M-matrices can have complex eigenvalues and have nontrivial Jordan structure. The eigenvalues of a (single row and column deleted) principal submatrix of a PD matrix interlace the eigenvalues of the whole matrix, while this is true only for the deletion of the first or last row and column of a TP matrix, see [9] and there is little vestige of interlacing in M-matrices. Of course TP matrices are not closed under permutation similarity, while the other two classes are.

In the last section we add to the list of commonalities among the three classes.

6. Row and column inclusion results

In this section we generalize in a certain way a fact known for positive semidefinite (PSD) matrices to a much wider setting that includes most totally nonnegative matrices and most (possibly singular) M-matrices. In this way we provide a further direction of unification.

It is known that if $A = [a_{ij}]$ is PSD, then any column vector of the form

$$\begin{bmatrix} a_{i_1 j} \\ a_{i_2 j} \\ \vdots \\ a_{i_k j} \end{bmatrix}$$

must lie in the column space of the principal submatrix of A lying in rows and columns indexed by $i_1, i_2, \ldots, i_k$. (This classical fact may be seen in a variety of ways. For example, consider an Hermitian matrix of the form

$$\begin{bmatrix} B & c \\ c^* & d \end{bmatrix}.$$

If the column vector c is *not* in the range of B, null space/range orthogonality implies there is a vector x in the null space of B such that $x^*c < 0$. There is then an $\varepsilon > 0$ such that

$$[x^* \ \varepsilon]\begin{bmatrix} B & c \\ c^* & d \end{bmatrix}\begin{bmatrix} x \\ \varepsilon \end{bmatrix} < 0.)$$

Since A^{T} is also PSD, there is an analogous statement about rows. Moreover, for PSD matrices (though *both* inclusions are valid), it is equivalent to say that for each j either the row lies in the row space *or* the column lies in the column space of the indicated principal submatrix. It is precisely this statement that may be substantially generalized. The PSD fact has been known to several researchers for some time, but we do not know an initial reference. It is important in matrix completion theory [10], etc., was mentioned in [11] (where the term "principal submatrix rank property" to follow), and the row or column inclusion notion also arises in a more particular way in LU factorizations [12]. It should be noted that outside the context of PSD matrices, this notion of row *or* column inclusion seems to be novel and not to be compared with variants mentioned in other contexts.

We begin with some standard notation, preliminary definitions and results. For a given m-by-n matrix A, we let Row(A) (Col(A)) denote the row (column) space of A, and let rank(A) denote the rank of A. For $\alpha \subseteq \{1,2,\ldots,m\}$ and $\beta \subseteq \{1,2,\ldots,n\}$, the submatrix of A lying in rows indexed by α and columns indexed by β will be denoted $A[\alpha|\beta]$. If $\alpha = \beta$, then the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$. In the special case in which $\alpha = \{i\}$, we denote $A[\{i\}|\beta]$ ($A[\{i\}]$) by $A[i|\beta]$ ($A[i]$). Similarly, $A[\alpha|\{j\}]$ will be denoted $A[\alpha|j]$. As usual, $|\alpha|$ denotes the cardinallity of α. Let $\alpha \subseteq \{1,2,\ldots,n\}$ with $|\alpha| = k$. Suppose $i_1 < i_2 < \cdots < i_k$ are the elements of α arranged in increasing order. Then the *dispersion number* of α, denoted by $d(\alpha)$, is defined as

$$d(\alpha) = \sum_{j=1}^{k-1}(i_{j+1} - i_j - 1) = i_k - i_1 - (k-1),$$

with the convention that $d(\alpha) = 0$, when $k = 1$. Observe that $d(\alpha) = 0$ means that α consists of k consecutive integers.

We let P_0 (P) denote the class of P_0 (P)-matrices. Recall $A \in P_0$ (P) if $\det A[\alpha] \geqslant 0 (> 0)$, for all $\alpha \subseteq N \equiv \{1,2,\ldots,n\}$. For $A \in P_0$, we say that A satisfies *Fischer's inequality*, if for every α, $\beta \subseteq N$ with $\alpha \cap \beta = \phi$,

$$\det A[\alpha \cup \beta] \leqslant \det A[\alpha] \det A[\beta]$$

(see, for example [2]).

Let $\mathscr{F}_0$ ($\mathscr{F}$) denote the set of n-by-n P_0 (P) matrices which also satisfy Fischer's inequality. The class $\mathscr{F}_0$ contains many familiar classes of matrices, for example: positive semidefinite, M-matrices, totally nonnegative matrices (and their permutation similarities) and triangular matrices with nonnegative main diagonal.

Using the notation developed above, we can reformulate the column inclusion result for positive semidefinite matrices as follows: If A is an n-by-n positive semidefinite matrix, then for any $\alpha \subseteq N$ and for each j,

$$A[\alpha|j] \in \mathrm{Col}(A[\alpha]).$$

We now prove a similar result for the class $\mathscr{F}_0$.

Theorem 6.1. *If $A \in \mathscr{F}_0$ and $\alpha \subseteq N$, is such that* $\mathrm{rank}(A[\alpha]) \geqslant |\alpha| - 1$, *then for each j either $A[j|\alpha] \in \mathrm{Row}(A[\alpha])$ or $A[\alpha|j] \in Col(A[\alpha])$.*

Proof. In the case $\mathrm{rank}(A[\alpha]) = |\alpha|$, there is nothing to do. So suppose $\mathrm{rank}(A[\alpha]) = |\alpha| - 1$. We prove the claim by contradiction. Assume there exists j, $1 \leqslant j \leqslant n$ such that $A[j|\alpha] \notin \mathrm{Row}(A[\alpha])$ and $A[\alpha|j] \notin \mathrm{Col}(A|\alpha])$; necessarily $j \notin \alpha$. Consider the principal submatrix $A[\alpha \cup \{j\}]$. We then have $\mathrm{rank}(A[\alpha \cup \{j\}]) = \mathrm{rank}(A[\alpha]) + 2 = |\alpha| + 1$. Hence $A[\alpha \cup \{j\}]$ is nonsingular. However,

$$0 < \det A[\alpha \cup \{j\}] \leqslant \det A[\alpha] \cdot a_{jj} = 0.$$

This is a contradiction. Hence for each j, either $A[j|\alpha] \in \mathrm{Row}(A[\alpha])$ or $A[\alpha|j] \in \mathrm{Col}(A[\alpha])$. $\square$

Some condition imposed upon $\mathrm{rank}(A[\alpha])$ is necessary, as may be seen from the following example.

Example 6.2. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $A \in \mathscr{F}_0$. Let $\alpha = \{1, 3\}$. Then $A[\alpha] = 0$. However, $A[2|\alpha] \neq 0$ and $A[\alpha|2] \neq 0$.

To generalize further, we use the following concept.

Definition 6.3. An n-by-n matrix A satisfies the *principal rank property* (PRP), if every principal submatrix A' of A has in turn an invertible *principal* submatrix that is of the order rank(A').

The following is simply a recasting of the previous definition. A satisfies the PRP if for every α, there exists $\beta \subseteq \alpha$, with $|\beta| = \mathrm{rank}(A[\alpha])$ so that $A[\beta]$ is invertible.

Observe that the principal rank property is inherited by principal submatrices. The next lemma (perhaps of independent interest) gives a sufficient condition for a matrix to satisfy the PRP.

Lemma 6.4. *Let A be an n-by-n matrix. Suppose the algebraic multiplicity of 0 equals the geometric multiplicity of 0, for every principal submatrix of A. Then A satisfies the PRP.*

Proof. Let A' be any k-by-k principal submatrix of A. If A' is nonsingular, then there is nothing to do. Thus suppose $\mathrm{rank}(A') = r\ (r < k)$. Since the algebraic multiplicity of 0 equals the geometric multiplicity of 0 for A', it follows that the characteristic polynomial of A' is equal to

$$\det(xI - A') = x^{k-r}(x^r - s_1x^{r-1} + \cdots + (-1)^r s_r)$$

in which $s_r \neq 0$. Since s_r is equal to the sum of all the r-by-r principal minors of A', it follows that there exists at least one nonsingular r-by-r principal submatrix of A'. This completes the proof. □

Note that symmetric matrices are examples of matrices that satisfy the above lemma, and hence satisfy the PRP. The converse to the above general lemma is easily seen not to hold in general. The simplest example demonstrating this fact is

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

However, for P_0-matrices the condition given in the above lemma is clearlly also necessary, since for a P_0-matrix, the coefficient s_r will be nonzero if and only if there is a nonsingular r-by-r principal submatrix of A.

Theorem 6.5. *If $A \in \mathscr{F}_0$, satisfies the PRP, and $\alpha \subseteq N$, then for each j either $A[j|\alpha] \in \mathrm{Row}(A[\alpha])$ or $A[\alpha|j] \in \mathrm{Col}(A[\alpha])$.*

Proof. Fix $\alpha \subset N$. If $\mathrm{rank}(A[\alpha]) \geqslant |\alpha| - 1$, then the result holds by Theorem 5.1. Hence, suppose that $\mathrm{rank}(A[\alpha]) \leqslant |\alpha| - 2$. Assume, for the purpose of a contradiction that there exists a j (necessarily not in α) such that

$A[j|\alpha] \notin \mathrm{Row}(A[\alpha])$ and $A[\alpha|j] \notin \mathrm{Col}(A[\alpha])$. As before consider $A[\alpha \cup \{j\}]$. By construction, $\mathrm{rank}(A[\alpha \cup \{j\}]) = \mathrm{rank}(A[\alpha]) + 2 \leqslant |\alpha|$. Since A satisfies the PRP, there exists a principal submatrix B of $A[\alpha \cup \{j\}]$, which is nonsingular and has order $\mathrm{rank}(A[\alpha]) + 2$. It follows that $B = A[\beta \cup \{j\}]$ for some $\beta \subseteq \alpha$, and $|\beta| = \mathrm{rank}(A[\alpha]) + 1$. Furthermore, since $|\beta| > \mathrm{rank}(A[\alpha])$, $A[\beta|$ is singular. Thus

$$0 < \det A[\beta \cup \{j\}] \leqslant \det A[\beta] \cdot a_{jj} = 0.$$

This is a contradiction. Consequently for each j, either $A[j|\alpha] \in \mathrm{Row}(A|\alpha])$ or $A[\alpha|j] \in \mathrm{Col}(A[\alpha])$. □

Theorem 6.5 has two hypotheses: $\mathscr{F}_0$ and PRP. Neither of these can be omitted. For example,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

satisfies PRP but not the row or column inclusion conclusion of Theorem 6.5, and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is in $\mathscr{F}_0$ and does not satisfy the conclusion of Theorem 6.5.

Since any positive semidefinite matrix satisfies the PRP (by the previous lemma) and is in $\mathscr{F}_0$ (see [2]) we have the following corollary.

Corollary 6.6. *If A is an n-by-n positive semidefinite matrix, then for any $\alpha \subseteq N$ and for each j,*

$$A[\alpha|j] \in \mathrm{Col}(A[\alpha]).$$

We now specialize to the class of totally nonnegative matrices (*TN*), a subclass of $\mathscr{F}_0$. We may extend our row or column inclusion results somewhat to nonprincipal submatrices for this class. Before we state our next result we need the following notation. Let $\alpha, \beta \subseteq N$ and let $i_1 < i_2 < \cdots < i_k$ be the elements of α arranged in increasing order, and let $j_1 < j_2 < \cdots < j_k$ be the elements of β arranged in increasing order. Define $i_0 = 1$, $j_0 = 1$, $i_{k+1} = n$ and $j_{k+l} = n$. For $t = 0, 1, \ldots, k$, let

$$A[i_1, \ldots, i_t, (i_t, i_{t+i}), i_{t+1}, \ldots, i_k|\beta]$$

denote the submatrix of A obtained from $A[\alpha|\beta]$ by inserting any row s, for which $i_t < s < i_{t+1}$. Similarly, let

$$A[\alpha|j_1,\ldots,j_t,(j_t,j_{t+1}),j_{t+1},\ldots,j_k]$$

denote the submatrix obtained from $A[\alpha|\beta]$ by inserting any column q, for which $j_t < q < j_{t+1}$.

Theorem 6.7. *Let A be an n-by-n TN matrix and let $\alpha, \beta \subseteq N$, with $|\alpha| = |\beta|$. Suppose that* $\operatorname{rank}(A[\alpha|\beta]) = p$*, and either $p \geqslant |\alpha| - 1$ or at least one of $d(\alpha) = 0$ or $d(\beta) = 0$. Then either*

$$\operatorname{rank}(A[i_1,\ldots,i_t,(i_t,i_{t+1}),i_{t+1},\ldots,i_k|\beta]) = p$$

or

$$\operatorname{rank} A[\alpha|j_1,\ldots,j_t,(j_t,j_{t+1}),j_{t+1},\ldots,j_k] = p,$$

for $t = 0, 1, \ldots, k$.

Proof. Let $\alpha, \beta \subseteq N$, with $|\alpha| = |\beta|$. Firstly, suppose $d(\alpha) = 0$. The case when $d(\beta) = 0$ will then follow by transposition. Let $s, q \in N$, with $s < i_1$ and $q < j_1$ (if such an s,q exist). The case in which $s > i_k$ and $q > j_k$ is handled similarly. Assume for the purpose of contradiction that $\operatorname{rank}(A[\alpha \cup \{s\}|\beta]) > p$ and $\operatorname{rank}(A[\alpha|\beta \cup \{q\}) > p$. Consider the submatrix $A[\alpha \cup \{s\}|\beta \cup \{q\}]$, which is totally nonnegative and has rank equal to $p + 2$. Thus there exists a nonsingular submatrix A' of $A[\alpha \cup \{s\}|\beta \cup \{q\}]$ of order $p + 2$. Since the $\operatorname{rank}(A[\alpha|\beta]) = p$, it follows that $A' = A[\gamma \cup \{s\}|\delta \cup \{q\}]$, in which $\gamma \subseteq \alpha$, $|\gamma| = p + 1$ and $\delta \subseteq \beta$, $|\delta| = p + 1$. Further, $A[\gamma|\delta]$ is principal in A', since $s < i_1$ and $q < j_1$. Observe that since $\operatorname{rank}(A[\alpha|\beta]) = p$, $A[\gamma|\delta]$ is singular. Consequently,

$$0 < \det A' \leqslant \det A[\gamma|\delta]a_{sq} = 0.$$

This is a contradiction and hence the conclusion holds.

Finally, suppose $\operatorname{rank}(A[\alpha|\beta]) \geqslant |\alpha| - 1$, and let $|\alpha| = k$. Define $i_0 = j_0 = 1$ and $i_{k+1} = j_{k+1} = n$. Let t be any fixed integer, $0 \leqslant t \leqslant k$. Assume there exists a row indexed by $s, i_t < s < i_{t+1}$, and a column indexed by $q, j_t < q < j_{t+1}$, in which $\operatorname{rank}(A[i_1,\ldots,i_t,s,i_{t+1},\ldots i_k|\beta]) > p$ and $\operatorname{rank}(A[\alpha|j_1,\ldots,j_t,q,j_{t+1},\ldots,j_k]) > p$. In fact both must equal $p + 1$. Then as before, consider the principal submatrix (of A) $A[\alpha \cup \{s\}|\beta \cup \{q\}]$. Applying similar arguments as above we will arrive at a contradiction. This completes the proof. □

We wish to remark here that if for some $t\,(0 \leqslant t \leqslant k)\, i_t = i_{t+1} - 1$ and $j_t < j_{t+1} - 1$, then there may exist a q, with $j_t < q < j_{t+1}$ and

$$\operatorname{rank}(A[\alpha|\beta \cup \{q\}]) > \operatorname{rank}(A[\alpha|\beta]).$$

Thus, the conclusion of the above theorem holds only if for a fixed t $(0 \leqslant t \leqslant k)\, i_t < i_{t+1} - 1$ and $j_t < j_{t+1} - 1$.

We complete this discussion with an example, which illustrates three possibilities. The first two give insight into the necessity of the hypotheses in Theorem 6.7. The final one illustrates the previous remark.

Example 6.8. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is not difficult to determine that A is TN.

(i) Let $\alpha = \{2,4\}$ and $\beta = \{2,4\}$. Then $\operatorname{rank}(A[\alpha|\beta]) = 1$. However, if we let $s = 1$ and $q = 5$, then $\operatorname{rank}(A[\alpha \cup \{s\}|\beta]) = 2$ and $\operatorname{rank}(A[\alpha|\beta \cup \{q\}]) = 2$. Thus it is necessary to include a row and a column from the same "gap" in α and β.

(ii) Similar arguments can be applied in the case $\alpha = \{1,3\}$ and $\beta = \{4,5\}$, and with $s = q = 2$.

(iii) Let $\alpha = \{2,3\}$ and $\beta = \{1,3\}$. Suppose $q = 2$. Then

$$\operatorname{rank}(A[\alpha|\beta \cup \{q\}]) > \operatorname{rank}(A[\alpha|\beta]).$$

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