The Boustrophedon* Conjecture

An approach to the Graph Complement Conjecture

H. Tracy Hall

Hall Labs, LLC

7 January 2021

Haystacks MRC Special Session,
JMM Virtual

**boustrophedon* (from Greek "ox-turning-wise", as in ploughing):
text in which alternating lines are read in opposite directions.

Graph parameters from matrix nullity

Graph $G \longleftrightarrow$ real symmetric matrix A:

$$ij \text{ is an edge} \longleftrightarrow a_{ij} \neq 0 \text{ (off-diagonal)}$$

(requires vertex ordering—usually ignored)

Graph parameters: $x(G) \in \mathbb{N}$

$M(G)$: maximum nullity

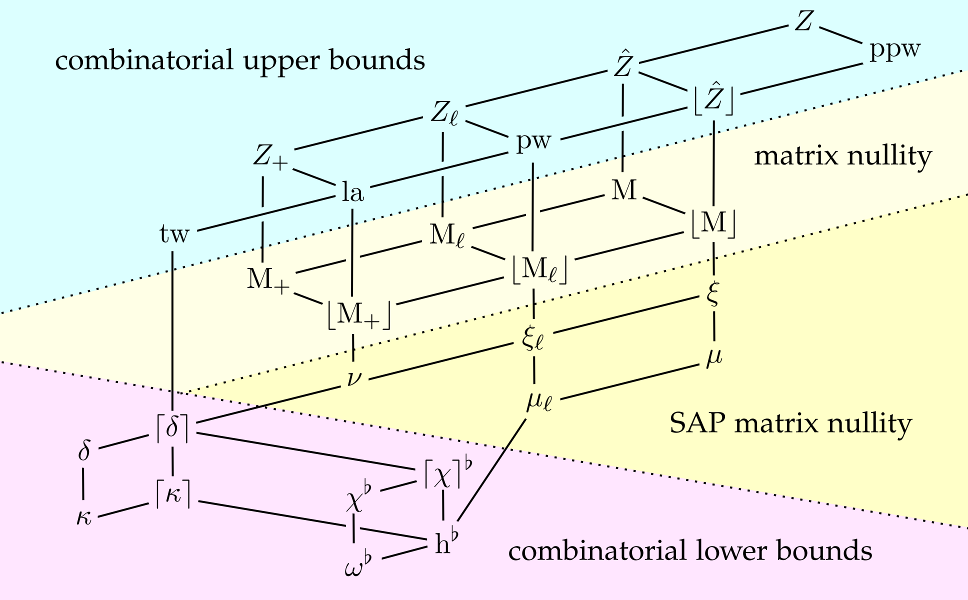
$M_+(G)$: also positive semidefinite

$\nu(G)$: also SAP

More requirements $\Rightarrow$ smaller maximum nullity

$$\nu(G) \leq \mathrm{M}_+(G) \leq \mathrm{M}(G)$$

The graph parameter zoo



The Strong Arnold Property (SAP)

Real symmetric A has *Strong Arnold Property* (SAP) when only $X = 0$ satisfies:

- X real symmetric.
- $X \circ I = 0$.
- $X \circ A = 0$.
- $XA = 0$.

A kind of "generic". Can make nullity minor-monotone.

Given A, easy to check. ($\cap$ spaces trivial?)

Not easy: Does A exist?

Upper-zero generic

Easier: Are certain row sets of A independent?

Real symmetric A is *upper-zero generic* ("upzegen") if:

- No zeros on diagonal.
- In each column, zeros above diagonal select independent rows.

$$\begin{bmatrix}2&1&0&0\\1&2&0&0\\0&0&1&1\\0&0&1&1\end{bmatrix} \qquad \begin{bmatrix}1&1&0&0\\1&1&0&0\\0&0&2&1\\0&0&1&2\end{bmatrix}$$

YES NO

Proposition: Upper-zero generic $\Rightarrow$ Strong Arnold Property.

upzegen $\Rightarrow$ SAP

Proof of Proposition. By contraposition. (Goal: not SAP $\Rightarrow$ not upzegen)

Suppose A not SAP, so nonzero X exists.

Let $\mathbf{r}$ be the last nonzero row of X.

- By X symmetry, $\mathbf{r}$ zero past diagonal.
- By $X \circ I = 0$, $\mathbf{r}$ zero on diagonal.
- By $X \circ A = 0$, $\mathbf{r}$ & $\mathbf{r}^T$ zero wherever A nonzero.
  $\Rightarrow \mathbf{r}^T$ nonzeros $\subseteq$ an upper-zero set of A.
- By $XA = 0$, $\mathbf{r}$ gives dependency on upper-zero set of rows.
  $\Rightarrow A$ is not upper-zero generic. $\square$

Bonus claim: If A upzegen of target rank exists, easy to find.

$\mathrm{M_s}(G, \sigma)$ (sequential maximum nullity)

Upper-zero generic A depends on vertex ordering σ for G.

Sequential maximum nullity $\mathrm{M_s}(G, \sigma)$:

Maximum nullity of A such that

- A is real symmetric PSD,
- A has pattern G, ordered by σ, and
- A is upper-zero generic.

(Bonus claim says: Fix G and σ, then $\mathrm{M_s}(G, \sigma)$ easy.)

Max over all σ: $\mathrm{M_s}(G)$

Upzegen more restrictive than SAP, so

$$\mathrm{M_s}(G, \sigma) \leq \mathrm{M_s}(G) \leq \nu(G).$$

Graph Complement Conjecture(s), GCC_x

Nordhaus-Gaddum type problems: combine $x(G)$ and $x(\overline{G})$.
(additive or multiplicative)

Graph Complement Conjecture: For all simple G on n vertices,

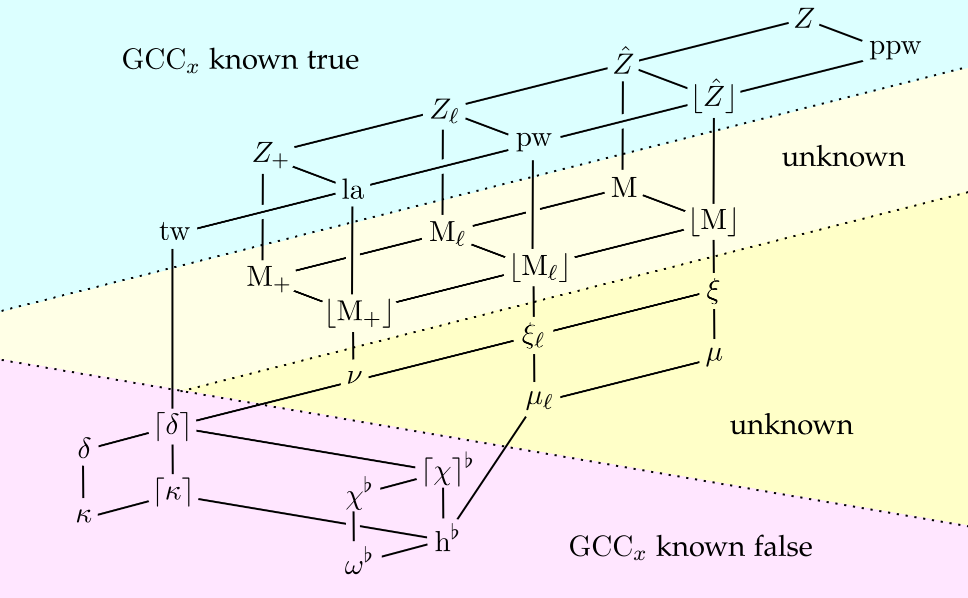
$$\mathrm{M}(G) + \mathrm{M}(\overline{G}) \geq n - 2.$$

Apply to any graph parameter $x(G)$.

GCC_x: For all simple G on n vertices,

$$x(G) + x(\overline{G}) \geq n - 2.$$

What is known about GCC_x?



Computer verification of $\mathrm{GCC}_{\mathrm{M_s}}$ for low n

$\mathrm{GCC}_{\mathrm{M_s}}$ implies GCC_x for many other nullity parameters ...

... and it's not hard to check.

(Not quite combinatorial, but randomized process usually succeeds.)

Results so far:

True for all G: $n \leq 10$.

True for $G = \overline{G}$: $n \leq 17$.

The Boustrophedon Conjecture

Which σ for $\mathrm{M_s}(G)$ vs. for $\mathrm{M_s}(\overline{G})$?

$\mathrm{GCC}_{\mathrm{M_s}}$ doesn't know. Stronger conjecture:

Boustrophedon Conjecture. *Let G be a graph with $|V(G)| = n$. Then there exists an ordering σ of $V(G)$ such that*

$$\mathrm{M_s}(G, \sigma) \; + \; \mathrm{M_s}(\overline{G}, \overleftarrow{\sigma}) \; \geq \; n - 2,$$

where $\overline{G}$ denotes the complement of G and $\overleftarrow{\sigma}$ denotes the reversal of σ.

(Same order σ fails for $n = 8$.)

Results so far: Also true for all G, $n = 10$.

Bonus: proof idea for bonus claim

Geometric perspective:

A is PSD, rank $\leq d$
$\iff A$ is Gram matrix, vectors in $\mathbb{R}^d$.

Lovàsz, Saks, and Schrijver (LSS) process:

- choose vectors at random,
- in order σ,
- with orthogonality constraints from upper zeros.

Upzegen LSS:

- and require independent constraints.

$\implies$ polynomials instead of random
$\implies$ succeeds never or with probability 1 (easy to tell).