

# On Zero Forcing and the Inverse Eigenvalue Problem for Graphs

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## Abstract

This note is intended to introduce the connection between the inverse eigenvalue problem of a graph and the zero forcing number of a graph.

## 1 Introduction

Let  $\mathbb{G}$  denote the collection of all simple graphs (no loops nor multi-edges) of order  $n$ . With each graph  $G \in \mathbb{G}$ , there is a vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and an edge set  $E(G)$ , where  $\{v_i, v_j\} \in E(G)$  if and only if  $i \neq j$  and  $v_i$  and  $v_j$  are adjacent vertices. For example, let  $G$  be the star graph of order 7 shown in Figure 1. Then,  $V(G) = \{1, 2, 3, 4, 5, 6, 7\}$  and  $E(G) = \{\{1, 7\}, \{2, 7\}, \{3, 7\}, \{4, 7\}, \{5, 7\}, \{6, 7\}\}$ .

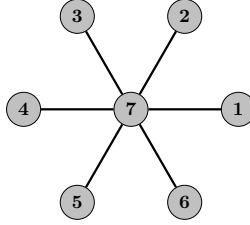


Figure 1: A star graph of order 7.

Every  $n \times n$  symmetric matrix  $A = [a_{ij}]_{i,j=1}^n$  is associated with a graph  $G \in \mathbb{G}$ , where  $a_{ij} \neq 0$  if and only if  $\{v_i, v_j\} \in E(G)$ . Given a graph  $G \in \mathbb{G}$ , the set of all  $n \times n$  symmetric matrices associated with  $G$  is denoted by

$$\mathcal{S}(G) = \{A = [a_{ij}]_{i,j=1}^n : a_{ij} = a_{ji} \forall i \neq j, a_{ij} \neq 0 \Leftrightarrow \{v_i, v_j\} \in E(G)\}.$$

For example, the star graph  $G$  shown in Figure 1 is associated with symmetric matrices of the form

$$A = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 & \beta_1 \\ 0 & \alpha_2 & \cdots & 0 & \beta_2 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \alpha_6 & \beta_6 \\ \beta_1 & \beta_2 & \cdots & \beta_6 & \alpha_7 \end{bmatrix}.$$

The inverse eigenvalue problem of a graph can be stated as follows: Given a graph  $G \in \mathbb{G}$ , what are all possible length  $n$  multi-sets  $\sigma$  such that there is a matrix  $A \in \mathcal{S}(G)$  with spectrum  $\sigma$ . This important question is obviously difficult to answer in general; however, there are a few observations we can make.

- The empty graph of order  $n$ , denoted  $E_n$ , has no edges. Therefore,  $\mathcal{S}(E_n)$  is the set of all  $n \times n$  diagonal matrices. Since the eigenvalues of a diagonal matrix are equal to its diagonal entries, it follows that every length  $n$  multi-set  $\sigma$  corresponds to the eigenvalues of a matrix  $A \in \mathcal{S}(E_n)$ . It turns out, that the empty graph is the only graph with this property.
- Let  $K_n$  denote a complete graph of order  $n$ . Then,  $\mathcal{S}(K_n)$  is made up of symmetric matrices where all off-diagonal entries are non-zero. Furthermore, for any  $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ , there is a matrix  $A \in \mathcal{S}(K_n)$  with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$  if and only if  $\lambda_1 < \lambda_n$  [1].

- Let  $P_n$  denote a path graph of order  $n$ . Then,  $\mathcal{S}(P_n)$  corresponds to tri-diagonal matrices with non-zero entries in the upper and lower diagonals. Furthermore, any length  $n$  multi-set  $\sigma$  corresponds to the eigenvalues of a matrix  $A \in \mathcal{S}(P_n)$  if and only if the entries of  $\sigma$  are distinct [5].

Due to the difficulty of the inverse eigenvalue problem of a graph, there is a focus on solving related but seemingly easier problems. For instance, one may want to find the maximum multiplicity of a graph  $G \in \mathbb{G}$ , denoted  $M(G)$ , which is defined as the largest integer  $k$  such that there is a matrix  $A \in \mathcal{S}(G)$  with an eigenvalue of multiplicity  $k$ . Note  $M(E_n) = n$ ,  $M(K_n) = (n - 1)$ , and  $M(P_n) = 1$ .

The multiplicity of an eigenvalue  $\lambda$  of the symmetric matrix  $A$  is equal to the nullity of  $\lambda I - A$ , i.e., the dimension of the null space of  $\lambda I - A$ . Note that for any  $A \in \mathcal{S}(G)$ , it follows that  $\lambda I - A \in \mathcal{S}(G)$ . Therefore,

$$M(G) = \max \{ \text{nullity}(A) : A \in \mathcal{S}(G) \}.$$

For this reason, the maximum multiplicity of a graph is often referred to as the max nullity of the graph.

In a similar manner, the minimum rank of  $G$ , denoted  $mr(G)$ , is defined by

$$mr(G) = \min \{ \text{rank}(A) : A \in \mathcal{S}(G) \}.$$

Recall that the  $\text{rank}(A)$  is the dimension of the column space of  $A$ . By the rank-nullity theorem, we have

$$mr(G) + M(G) = n.$$

In 2006, at the AIMS Workshop on “Spectra of families of matrices described by graphs, digraphs, and sign pattern”, a novel connection between the maximum nullity of a graph and its zero forcing number was presented. Zero forcing was introduced independently by Burgarth and Giovannetti in the control of quantum systems [3], where it was called graph infection. For our purposes, zero forcing is a coloring game on a graph, where an initial set of vertices are colored blue and all other vertices are colored white. Then, the blue vertices force neighboring white vertices blue using a color change rule. There are many color changing rules, but the standard rule is that a blue vertex  $b$  can force a white vertex  $w$  blue if  $w$  is the only white neighbor of  $b$ . We will reference this rule as the Z-color change rule to distinguish it from the other variants.

Given a graph  $G \in \mathbb{G}$ , let  $B \subseteq V(G)$  denote the initial set of blue vertices; this is called the initial coloring of  $G$ . Then, denote by  $B^{[t]}$  the blue vertices after  $t$  applications of the Z-color change rule, where  $B^{[0]} = B$ . Because the graph is finite, there exists a  $t^* \geq 0$  for which  $B^{[t^*]} = B^{[t^*+k]}$ , for all  $k \in \mathbb{N}$ , i.e., no more changes are possible from applying the Z-color change rule. We reference  $B^{[t^*]}$  as the final coloring of  $B$ . If  $B^{[t^*]} = V(G)$ , i.e., the final coloring is all of the vertices of  $G$ , then we say that  $B$  is a zero forcing set of  $G$ . The zero forcing number of  $G$  is defined as the cardinality of the smallest zero forcing set of  $G$ :

$$Z(G) = \min \{ |B| : B^{[t^*]} = V(G) \}.$$

For example, consider the application of the Z-color change rule to a path graph on 5 vertices shown in Figure 2. It is clear that  $B^{[4]}$  is the final coloring of  $B$  since  $B^{[5]} = B^{[4]}$ . Furthermore, since  $B^{[4]}$  encompasses all vertices in the graph, it follows that  $B = \{1\}$  is a zero forcing set and, hence, the zero forcing number of this path graph is equal to 1.

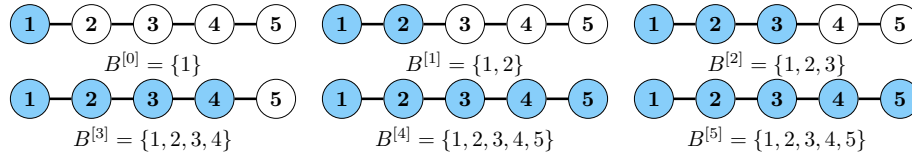


Figure 2: A zero forcing set for a path graph of order 5.

One can easily generalize the above example and conclude that the zero forcing number of all path graphs is equal to 1, i.e.,  $Z(P_n) = 1$ . Note that not every initial coloring of 1 vertex is a zero forcing set of  $P_n$ . Indeed, consider the initial coloring of a path graph on 5 vertices shown in Figure 3.

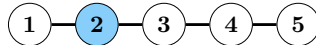


Figure 3: A non zero forcing set for a path graph of order 5.

We are now ready to present the connection between the zero forcing number and the maximum nullity of a graph. First we require the following lemma.

**Lemma 1.1.** *Suppose that  $\alpha \subseteq \{1, 2, \dots, n\}$  with  $|\alpha| < k$  and  $U$  is a subspace of  $\mathbb{R}^n$  of dimension  $k$ . Then,  $U$  contains a non-zero vector of  $\mathbf{v}$  with  $\mathbf{v}[\alpha] = 0$ .*

*Proof.* Note that if  $U$  and  $V$  are subspaces of  $\mathbb{R}^n$  whose dimensions satisfy

$$\dim V > n - \dim U,$$

then  $U$  and  $V$  must have a non-trivial intersection, i.e., there is a non-zero vector in  $U \cap V$ . Now, define

$$V = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v}[\alpha] = 0\},$$

and note that  $\dim V = n - |\alpha| > n - k = \dim U$ . Hence, there exists a non-zero vector  $\mathbf{v} \in U \cap V$ , and the result follows.  $\square$

**Theorem 1.2.** *Let  $G \in \mathbb{G}$ . Then,  $M(G) \leq Z(G)$ .*

*Proof.* Let  $B \subseteq V(G)$  be a zero forcing set of  $G$ , and let  $A \in \mathcal{S}(G)$ . If  $|B| < \text{nullity}(A)$ , then Lemma 1.1 implies that there exists a non-zero vector  $\mathbf{x} \in \text{null}(A)$  such that  $\mathbf{x}_i = 0$  for all  $i \in B$ . However, if a vector  $\mathbf{x} \in \text{null}(A)$  satisfies  $\mathbf{x}_i = 0$  for all  $i \in B$ , then it follows that  $\mathbf{x} = \mathbf{0}$ .

Indeed, let  $j \in V(G) \setminus B$  be any vertex that is forced by some  $i \in B$ ; at least one such vertex must exist since  $B$  is a zero forcing set. Then, the equation  $A\mathbf{x} = \mathbf{0}$  implies that  $a_{ij}\mathbf{x}_j = 0$ , which gives  $\mathbf{x}_j = 0$  since  $a_{ij} \neq 0$ . If necessary, we can repeat the above argument on the zero forcing set  $B \cup \{j\}$ , and perhaps again, until all vertices in  $V(G)$  are accounted for at which point it follows that  $\mathbf{x} = \mathbf{0}$ .

This contradiction with Lemma 1.1 implies that  $|B| \geq \text{nullity}(A)$ . Since  $B$  and  $A$  are arbitrary, it follows that  $Z(G) \geq M(G)$ .  $\square$

Theorem 1.2 can be used to determine the maximum nullity of a graph via its zero forcing number. For instance, we know that the path graph satisfies  $Z(P_n) = 1$ . Therefore, Theorem 1.2 implies that  $M(P_n) \leq 1$ . Since there is always a singular matrix in  $\mathcal{S}(G)$ , for any graph  $G$ , it follows that  $M(P_n) = 1$ .

As a more interesting example, consider the star graph in Figure 1. Note that  $B = \{1, 2, 3, 4, 5\}$  is a zero-forcing set of minimal size. This observation can easily be generalized and we conclude that the zero forcing number of the star graph  $K_{1,n-1}$  of order  $n$  satisfies  $Z(K_{1,n-1}) = n - 2$ . Therefore, the maximum nullity and minimum rank satisfy

$$M(K_{1,n-1}) \leq n - 2 \text{ and } mr(K_{1,n-1}) \geq 2,$$

respectively. Furthermore, the adjacency matrix of the star graph  $K_{1,n-1}$ :

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}.$$

is a  $n \times n$  matrix in  $\mathcal{S}(K_{1,n-1})$  that satisfies  $\text{rank}(A) = 2$ . Therefore,  $M(K_{1,n-1}) = n - 2$  and  $mr(K_{1,n-2}) = 2$ .

## 2 Positive Semidefinite Zero Forcing

As mentioned in Section 1, there are many color change rules that can be used in the coloring game of zero forcing. The so-called standard zero forcing rule provides an upper bound on the maximum nullity of a graph via Theorem 1.2. The positive semidefinite zero forcing rule provides an upper bound on the maximum positive semidefinite nullity:

$$M_+(G) = \max \{\text{nullity}(A) : A \in \mathcal{S}(G), \forall \lambda \in \sigma(A), \lambda \geq 0\}.$$

Note that the additional condition on the eigenvalues of  $A$ , i.e.,  $\lambda \geq 0$  for all  $\lambda \in \sigma(A)$ , implies that  $A$  must be positive semidefinite.

Given a graph  $G \in \mathbb{G}$ , let  $B \subseteq V(G)$  denote the initial coloring of  $G$ . Let  $G - B$  denote the resulting graph after the blue vertices (and adjacent edges) have been removed, and let  $W_1, \dots, W_k$  denote the vertex sets of the connected components of  $G - B$ . If  $b \in B$  and  $w \in W_i$  is the only white neighbor of  $b$  in  $G[W_i \cup B]$ , for some  $i \in \{1, \dots, k\}$ , then the positive semidefinite zero forcing rule says that  $b$  can force  $w$  blue.

As with the standard zero forcing rule, there exists a  $t^* \geq 0$  such that  $B^{[t^*]} = B^{[t^* + k]}$  for all  $k \in \mathbb{N}$ , and we say that  $B$  is a positive semidefinite zero forcing set if the final coloring satisfies  $B^{[t^*]} = V(G)$ . The positive semidefinite zero forcing number of  $G$  is defined as the cardinality of the smallest positive semidefinite zero forcing set of  $G$ :

$$Z_+(G) = \min \left\{ |B| : B^{[t^*]} = V(G) \right\}.$$

**Theorem 2.1.** *Let  $G \in \mathbb{G}$ . Then,  $M_+(G) \leq Z_+(G)$ .*

*Proof.* Let  $A \in S(G)$  be positive semidefinite with nullity  $(A) = M_+(G)$  and consider a non-zero vector  $\mathbf{x} \in A$ , which is guaranteed to exist since the max nullity is at least 1. Let  $B \subset V(G)$  denote the set of vertices such that  $\mathbf{x}_i = 0$ , and let  $W_1, \dots, W_k$  denote the vertex sets of the connected components of  $G \setminus B$ . For every  $i \in \{1, \dots, k\}$ , we claim that no  $w \in W_i$  can be the only neighbor of a  $b \in B$  in  $G[W_i \cup B]$ .

Indeed, note that we can write  $A$ , possibly after re-ordering the vertices of  $G$ , as follows

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 & C_1 \\ 0 & A_2 & \cdots & 0 & C_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_k & C_k \\ C_1 & C_2 & \cdots & C_k & D \end{bmatrix},$$

where  $A_1, \dots, A_k$  corresponds to the vertex sets  $W_1, \dots, W_k$ , respectively,  $D$  corresponds to the vertex set  $B$ , and  $C_i$  corresponds to the edges between  $W_i$  and  $B$ , for all  $i \in \{1, \dots, k\}$ .

Now, consider the partition  $\mathbf{x} = [\mathbf{x}_1^T, \dots, \mathbf{x}_k^T, \mathbf{0}^T]^T$ , where  $\mathbf{x}_i$  has size equal to  $|W_i|$ , for all  $i \in \{1, \dots, k\}$ , and the zero vector  $\mathbf{0}$  has size equal to  $|B|$ . Note that all entries of  $\mathbf{x}_1, \dots, \mathbf{x}_k$  must be non-zero. Moreover, the equation  $A\mathbf{x} = \mathbf{0}$  implies that  $A_i\mathbf{x}_i = \mathbf{0}$  for all  $i \in \{1, \dots, k\}$ .

The column inclusion property for Hermitian positive semidefinite matrices [7], guarantees that each column in  $C_i$  is the span of the columns in  $A_i$ , for all  $i \in \{1, \dots, k\}$ . Therefore, for each  $i \in \{1, \dots, k\}$ , there exists a matrix  $Y_i$  such that  $C_i = A_i Y_i$ . Since  $A$  is symmetric, it follows that  $C_i = Y_i^T A_i$ ; hence,  $C_i \mathbf{x}_i = \mathbf{0}$  for all  $i \in \{1, \dots, k\}$ . But if  $w \in W_i$  were the only white neighbor of  $b \in B$  in  $G[W_i \cup B]$ , then  $C_i \mathbf{x}_i \neq \mathbf{0}$ , which is a contradiction.

This contradiction implies that  $B$  is not a positive semidefinite zero forcing set of  $G$ . Furthermore, it follows that there is no non-trivial  $\mathbf{x} \in \text{null}(A)$  with  $\mathbf{x}_i = 0$  for all  $i$  in a positive semidefinite zero forcing set of  $G$ . Therefore, by Lemma 1.1, we have  $M_+(G) \leq Z_+(G)$ .  $\square$

As an example of how Theorem 2.1 could be applied, consider the star graph  $K_{1,n-1}$ . Note that the “center” vertex constitutes a positive semidefinite zero forcing set of  $K_{1,n-1}$ . Therefore,

$$Z_+(K_{1,n-1}) = 1$$

and it follows that  $M_+(K_{1,n-1}) = 1$ .

Finally, note that every standard zero forcing set of a graph  $G \in \mathbb{G}$  constitutes a positive semidefinite zero forcing set; hence,  $Z_+(G) \leq Z(G)$ . Furthermore, it is also clear that  $M_+(G) \leq M(G)$ .

**Question 2.2.** *Is there a relationship between  $Z_+(G)$  and  $M(G)$ , i.e., does the following hold*

$$Z_+(G) \leq M(G)$$

*for all (or some) graphs  $G \in \mathbb{G}$ .*

### 3 Historical Background of the Minimum Rank

Parter appears to be the first to study the spectral properties of matrices with a given graph; in 1960, he published his investigation of sign symmetric matrices with multiple eigenvalues whose underlying graph is a tree [9].

Recall that a *forest* is an acyclic graph, i.e., a graph with no cycles, and a *tree* is a connected forest. Moreover, given a tree  $T$  and a vertex  $v \in V(T)$ , a *branch* of  $T$  at  $v$  is a connected component of  $T - v$ . If  $A \in \mathcal{S}(T)$ , then each branch of  $T$  at  $v$  determines a principal submatrix of  $A$ , namely  $A[W]$ , where  $W$  is the vertex set of the branch. The following result is due to Parter [9].

**Theorem 3.1.** *Let  $T$  be a tree and let  $A \in \mathcal{S}(T)$ . Then,  $\text{nullity}(A) \geq 2$  if and only if there exists a vertex  $v \in V(T)$  and at least three branches  $T[W_i]$  of  $T$  at  $v$  such that the principal submatrix  $A[W_i]$  is singular.*

The following result is due to Fiedler [4]

**Theorem 3.2.** *Let  $T$  be a tree and let  $A \in \mathcal{S}(T)$  with  $\text{nullity}(A) \geq 2$ . Then, there exists an  $i \in \{1, 2, \dots, n\}$  such that every null vector  $\mathbf{x} = [x_i]$  has  $x_i = 0$ .*

An index  $i \in \{1, 2, \dots, n\}$  with the property that every eigenvector of  $A$  corresponding to an eigenvalue  $\lambda$  has its  $i$ th coordinate zero is known as a *Fiedler index* of  $A$  corresponding to  $\lambda$ .

In [4], Fiedler also showed that one can often determine the location of an eigenvalue  $\lambda$  of  $A \in \mathcal{S}(T)$  among the ordering of its eigenvalues from an eigenvector of  $A$  corresponding to  $\lambda$ .

**Theorem 3.3.** *Let  $T$  be a tree,  $A \in \mathcal{S}(A)$ , and  $\mathbf{y} = [y_i]$  be an eigenvector corresponding to an eigenvalue  $\lambda$ . Let  $p$  be the number of edges  $ij \in E(T)$  such that  $a_{ij}y_iy_j < 0$ ,  $q$  the number of edges  $ij \in E(T)$  such that  $a_{ij}y_iy_j > 0$ , and  $r$  the number of coordinates of  $\mathbf{y}$  that are zero. If there does not exist an index  $i$  such that  $y_i = 0$  and  $y_j = 0$  for all  $j$  adjacent to  $i$  in  $T$ , then there are  $p + r$  eigenvalues of  $A$  greater than  $\lambda$  and  $q + r$  eigenvalues of  $A$  less than  $\lambda$  (counting multiplicity).*

For example, consider the tree graph shown in Figure 4. The adjacency matrix of this graph is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and has spectrum  $\sigma(A) = \{-2, 2, -1, 1, 0, 0\}$ . To illustrate Theorem 3.1, note that there are three branches of the vertex 2, namely  $\{1\}$ ,  $\{3\}$ , and  $\{4, 5, 6\}$ , with corresponding principal submatrix of  $A$  that is singular. A similar argument can be made for the branches of the vertex 5. Moreover, the  $\text{null}(A)$  is spanned by the following basis vectors

$$\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}^T \\ \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T$$

Hence, every null vector of  $A$  has a zero in the index 2 and 5, which illustrates Theorem 3.2. Finally, note the following eigenvalue-eigenvector pairs for the matrix  $A$ :

$$\begin{aligned} \lambda = -2, \quad \mathbf{v} &= [-1 \quad 2 \quad -1 \quad 1 \quad -2 \quad 1]^T, \\ \lambda = 2, \quad \mathbf{v} &= [1 \quad 2 \quad 1 \quad 1 \quad 2 \quad 1]^T, \\ \lambda = -1, \quad \mathbf{v} &= [1 \quad -1 \quad 1 \quad 1 \quad -1 \quad 1]^T, \\ \lambda = 1, \quad \mathbf{v} &= [-1 \quad -1 \quad -1 \quad 1 \quad 1 \quad 1]^T. \end{aligned}$$

Note that Theorem 3.3 holds for each of the eigenvalue-eigenvector pairs since none of the eigenvectors do not have any zero elements. For instance, every entry of the eigenvector associated with  $\lambda = 2$  is positive. Hence, there are 5 (total number of edges) eigenvalues of  $A$  less than 2 (counting multiplicity). For the eigenvector associated with  $\lambda = -1$ , we have  $p = 4$  (corresponding to edges  $\{1, 2\}$ ,  $\{2, 3\}$ ,  $\{4, 5\}$ ,  $\{5, 6\}$ ) and  $q = 1$  (corresponding to the edge  $\{2, 5\}$ ). Hence, there are 4 eigenvalues of  $A$  greater than  $-1$  and 1 eigenvalue less than  $-1$ . A similar statement can be made for the eigenvectors associated with all other eigenvalues of  $A$ .

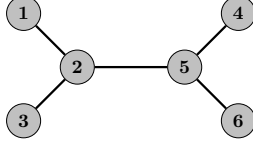


Figure 4: A tree graph of order 6.

In 1984, Wiener strengthened the result in Theorem 3.1 [10]. This stronger result, stated in Theorem 3.4, is now referred to as the Parter-Wiener theorem.

**Theorem 3.4.** *Let  $T$  be a tree and  $A \in \mathcal{S}(T)$  with  $\text{nullity}(A) \geq 2$ . Then, there is a vertex  $i \in V(T)$  such that  $\text{nullity}(A(i)) = \text{nullity}(A) + 1$ , and the principal submatrices of  $A$  corresponding to at least three branches of  $T$  at  $i$  are singular.*

Note that  $A(i)$  is the principal submatrix of  $A$  obtained by deleting row and column  $i$  from the matrix  $A$ . Moreover, the vertex  $i$  for which  $\text{nullity}((A - \lambda I)(i)) = \text{nullity}(A - \lambda I) + 1$  is known as a Parter-Wiener vertex for  $A$  corresponding to the eigenvalue  $\lambda$ . For example, the adjacency matrix of the tree graph in Figure 4 has Parter-Wiener vertices 2 and 5 corresponding to the eigenvalue 0. Over the subsequent years, the Parter-Wiener theorem has been further developed and produced many beautiful results, e.g., see Chapter 2 of [6].

Nylen appears to be the first to study the minimum rank of a graph, which appeared in his 1996 paper [8]. In that paper, Nylen made the following observations.

**Proposition 3.5.** *Let  $G \in \mathbb{G}$  have order  $n \geq 1$ . Then,*

- a. *If  $A \in \mathcal{S}(G)$  and  $v \in V(G)$ , then the principal submatrix  $A(v)$  corresponds to the induced subgraph  $G - v$ .*
- b. *If  $G$  is disconnected with connected components  $G_1, G_2, \dots, G_k$ , then*

$$\text{mr}(G) = \sum_{i=1}^k \text{mr}(G_i).$$

- c. *If  $G$  has  $k$  connected components, then*

$$\text{mr}(G) \leq n - k.$$

- d. *For each vertex  $v \in V(G)$ , we have*

$$\text{mr}(G) - 2 \leq \text{mr}(G - v) \leq \text{mr}(G).$$

- e. *For each edge  $e \in E(G)$ , we have*

$$\text{mr}(G) - 1 \leq \text{mr}(G - e) \leq \text{mr}(G) + 1.$$

In addition, Nylen proved the following more substantial result.

**Theorem 3.6.** *Let  $G \in \mathbb{G}$  and  $A \in \mathcal{S}(G)$  with  $\text{rank}(A) = \text{mr}(G)$ . Then, for any vertex  $i \in V(G)$ , either  $\text{rank}(A(i)) = \text{mr}(G) - 2$  or  $\text{rank}(A(i)) = \text{mr}(G)$ .*

The rest of Nylen's paper focuses on trees; in particular, he proved the following result, which provides an algorithm for computing the minimum rank of a tree. Note that a *high-degree* vertex of a graph is a vertex with degree at least 3; a *Nylen path* in a forest  $F$  is a path with at most one high-degree vertex, and whose initial and terminal vertices each have degree 1.

**Theorem 3.7.** *Let  $P$  be a Nylen path of the forest  $F$ . Then,*

- a. *If  $P$  has a high-degree vertex  $v$ , then  $\text{mr}(F) = \text{mr}(F - i) + 2$ .*
- b. *If  $P$  does not have a high-degree vertex, then  $\text{mr}(F) = |V(P)| - 1 + \text{mr}(F - V(P))$ .*

## 4 Colin de Verdière Type Parameters

## 5 Computational Approaches to Zero Forcing Number

### 5.1 Integer Program

In this section, we review the integer program model introduced by Brimkov et al. [2] to compute the zero forcing number of a graph  $G \in \mathbb{G}$ . This integer program is shown in (1a)–(1e), and we reference this program by ZIP( $G$ ). Note that every edge in  $E(G)$  is replaced by two directed edges with opposite direction. A binary variable  $s_v$  indicates whether vertex  $v$  is in the zero forcing set; an integer variable  $x_v$  ranging in  $\{0, \dots, T\}$  indicates at which timestep vertex  $v$  is forced, where  $T$  is the maximum difference between the forcing times of two vertices; finally, a binary variable  $y_e$  for each directed edge  $e = (u, v)$  indicates whether  $u$  forces  $v$ . The notation  $\delta^-(v)$  refers to the set of edges pointing towards the node  $v$ ,  $N(u)$  denotes the open neighborhood of  $u$ , i.e., all vertices  $v \in V(G)$  that are adjacent to  $u$ , and  $N[u]$  denotes the closed neighborhood of  $u$ , i.e.,  $N[u] = N(u) \cup \{u\}$ .

$$\text{minimize} \quad \sum_{v \in V} s_v \tag{1a}$$

$$\text{subject to} \quad s_v + \sum_{e \in \delta^-(v)} y_e = 1, \quad \forall v \in V, \tag{1b}$$

$$x_u - x_v + (T+1)y_e \leq T, \quad \forall e = (u, v) \in E, \tag{1c}$$

$$x_w - x_v + (T+1)y_e \leq T, \quad \forall e = (u, v) \in E, \forall w \in N(u) \setminus \{v\}, \tag{1d}$$

$$s \in \{0, 1\}^n, \quad x \in \{0, \dots, T\}^n, \quad y \in \{0, 1\}^m \tag{1e}$$

The triple  $(s, x, y)$ , where  $s \in \{0, 1\}^n$ ,  $x \in \{0, \dots, T\}^n$ ,  $y \in \{0, 1\}^m$ , is a *feasible solution* of ZIP( $G$ ) provided that  $s, x, y$  satisfy (1b)–(1e). If, in addition,  $s$  is minimal with respect to the objective function 1a, then we say that  $(s, x, y)$  is an *optimal solution* of ZIP( $G$ ). The corresponding minimal value of the objective function 1a is the optimal value of ZIP( $G$ ).

**Theorem 5.1.** *Let  $G \in \mathbb{G}$ . The optimal value of ZIP( $G$ ) is equal to the zero-forcing number of  $G$ .*

*Proof.* Let  $B \subseteq V$  denote a zero-forcing set of  $G$ . Also, fix some chronological list of forces that results in a total coloring of  $G$ , and let  $\mathcal{F}$  be the associated set of forcing chains. Since  $B$  is a zero forcing set, every vertex  $v \in V(G)$  is either in  $B$ , i.e.,  $s_v = 1$ , or is forced by some other vertex  $u \in V(G)$ , i.e.,  $y_e = 1$  where  $e = (u, v)$ . Thus, constraint (1b) must hold. Now, let  $x_v$  be the time step in which  $v \in V(G)$  is forced. Since a vertex cannot force until all but one of its neighbors are forced, it follows that for every edge  $e = (u, v)$  for which  $y_e = 1$ ,  $u$  must be forced before  $v$ , i.e.,  $x_u < x_v$ . Similarly,  $x_w < x_v$  for all neighbors  $w$  of  $u$ . Thus, constraints (1c) and (1d) are satisfied. If  $y_e = 0$ , then constraints (1c) and (1d) hold since  $T$  is the maximum difference between the forcing times of two vertices. Therefore, constraints (1b)–(1e) are valid for any zero-forcing set  $B$  and associated set of forcing chains  $\mathcal{F}$ .

Conversely, let  $(s, x, y)$  be a feasible solution of ZIP( $G$ ). Define  $B \subseteq V(G)$  to be the set of vertices for which  $s_v = 1$ . By constraint (1b), each vertex  $v \in V(G)$  is either in  $B$  or is the head of exactly one edge  $e = (u, v)$  such that  $y_e = 1$ . Consider any such edge; then, by constraints (1c) and (1d), there must be an integer  $x_v \in \{0, \dots, T\}$  such that  $x_w + 1 \leq x_v$  for all  $w \in N[u] \setminus \{v\}$ . Therefore, the edges  $e = (u, v)$  for which  $y_e = 1$  define a chronological list of forces. Since every vertex  $v \in V(G)$  is either in  $B$  or is the head of exactly one edge  $e = (u, v)$  such that  $y_e = 1$ , it follows that  $B$  is a zero-forcing set. If, in addition,  $s$  is minimal with respect to the objective function (1a), i.e.,  $(s, x, y)$  is an optimal solution of ZIP( $G$ ), then  $B$  is a minimal zero-forcing set and it follows that the objective value of ZIP( $G$ ) is the zero-forcing number of  $G$ .  $\square$

## 6 Propagation Time and Throttling Number

Given a color change rule and a graph  $G$ , the propagation time of  $B \subseteq V(G)$ , denoted  $pt(G, B)$ , is the smallest  $t$  such that  $B^{[t]} = V(G)$  or  $\infty$  if  $B$  is not a zero-forcing set. The propagation time of  $G$  is defined by

$$pt(G) = \min \{pt(G, B) : B \text{ is a minimal zero-forcing set}\}.$$

In Figure 5, we illustrate the standard propagation process and blue sets  $B^{[i]}$  for a tree  $T$  and initial set  $B = B^{[0]}$  such that  $pt(T) = pt(G, B)$ .

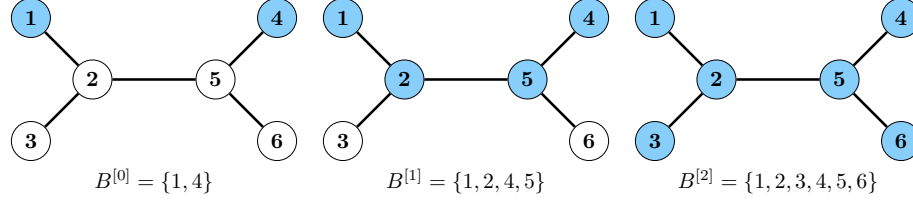


Figure 5: A zero forcing set for a tree graph of order 6.

Given a color change rule and a graph  $G$ , the throttling number of  $B \subseteq V(G)$  is defined by

$$th(G, B) = |B| + pt(G, B).$$

Furthermore, the throttling number of  $G$  is defined by

$$th(G) = \min_{B \subseteq V(G)} th(G, B).$$

Note that throttling number is not necessarily attained for minimal zero-forcing sets. Indeed, consider the path graph of order  $n$ , denoted by  $P_n$ . Note that  $Z(P_n) = 1$ , and any of the two leafs in  $P_n$  will serve as a zero forcing set, e.g., see Figure 2. Since every other node will be forced, one at a time, it follows that  $pt(P_n, B) = n - 1$ , for any minimal zero-forcing set. Therefore,

$$th(P_n, B) = n$$

for any minimal zero-forcing set  $B$ . However, if we let the initial coloring  $B$  be made up of both leafs in  $P_n$ , then  $pt(P_n, B) = \lceil (n - 2)/2 \rceil$ , and it follows that

$$th(P_n, B) = 2 + \lceil (n - 2)/2 \rceil.$$

Therefore,  $th(P_n) < n$  for all  $n > 3$ .

## References

- [1] W. BARRETT, A. LAZENBY, N. MALLOY, C. NELSON, W. SEXTON, R. SMITH, J. SINCKOVIC, AND T. YANG, *The combinatorial inverse eigenvalue problems: complete graphs and small graphs with strict inequality*, Electron. J. Linear Algebra, 26 (2013), pp. 656–672.
- [2] B. BRIMKOV, C. C. FAST, AND I. V. HICKS, *Computational approaches for zero forcing and related problems*, European Journal of Operational Research, 273 (2019), pp. 889–903.
- [3] D. BURGARTH AND V. GIOVANNETTI, *Full control by locally induced relaxation*, Phys. Rev. Lett., 99 (2007), p. 100501.
- [4] M. FIEDLER, *A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory*, Czech Math. J., 25 (1975), pp. 619–633.
- [5] H. HOCHSTADT, *On some inverse problems in matrix theory*, Archiv der Mathematik, 2 (1967), pp. 201–207.
- [6] C. JOHNSON AND C. SAIGO, *Eigenvalues, Multiplicities and Graphs*, Cambridge University Press, Cambridge, England, 2018.
- [7] C. R. JOHNSON AND THE ASSISTANCE OF SHAUN FALLAT, *Olga, matrix theory and the taussky unification problem*, Linear Algebra and its Applications, 280 (1998), pp. 39–49.
- [8] P. M. NYLEN, *Minimum-rank matrices with prescribed graph*, Linear Algebra and its Applications, 248 (1996), pp. 303–316.
- [9] S. PARTER, *On the eigenvalues and eigenvectors of a class of matrices*, J. Soc. Indust. Appl. Math., (1960), pp. 376–388.
- [10] G. WIENER, *Spectral multiplicity and splitting results for a class of qualitative matrices*, Linear Algebra and its Applications, 61 (1984), pp. 15–29.