# Throttling as a Forbidden Subgraph Problem

Joshua Carlson [1] Jürgen Kritschgau [2]

jkritschgau.com

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[1] Dept. of Mathematics and Statistics, Williams College, Williamstown, MA, USA (jc31@williams.edu)

[2] Dept. of Mathematics, Iowa State University, Ames, IA, USA (jkritsch@iastate.edu) Research is supported by NSF grant DMS-1839918

# Outline

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1) Definitions
2) General Motivation and Background
3) Results

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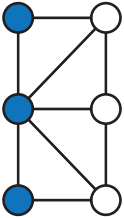

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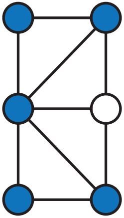

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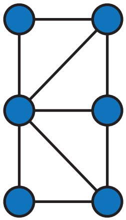

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**We can always find a standard witness!**

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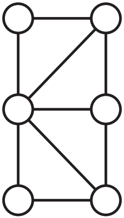
- If $G$ is triangle-free, then any subgraph of $G$ is triangle-free.
- If $G$ is not triangle-free, then any supergraph of $G$ is not triangle-free.
- $F = \{K_3\}$ characterizes the set of triangle-free graphs.



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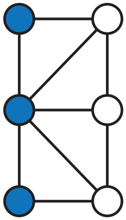
# Making Throttling a Forbidden Subgraph Problem



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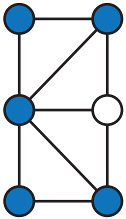

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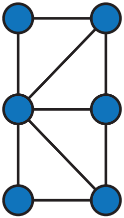

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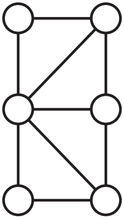

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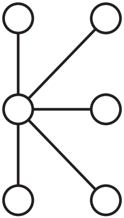

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# Making Throttling a Forbidden Subgraph Problem

**Theorem (Carlson 2019 [7])**

Given a graph $G$ and a positive integer $t$, $\text{th}(G) \leq t$ if and only if there exists integers $a \geq 0$, $b \geq 1$ such that $a + b = t$ and $G$ can be obtained from $K_a \square P_{b+1}$ by contracting path edges and deleting complete edges.

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- This is nice because $\text{th}(G) \leq |V(G)|$ and $|V(G)|$ is always attainable.
- Now the questions is, given $G$, how much can I improve on the naive strategy of coloring everything blue.

# Some PSD Throttling Characterizations

**Proposition (Carlson, Hogben, K., Lorenzen, Ross, Selken, Valle Martinez 2019 [8])**

Let $G$ be a graph. Then $\text{th}_+(G) = |V(G)|$ if and only if $G$ does not contain an induced $\bar{K}_2$ (that is, $G$ is complete).

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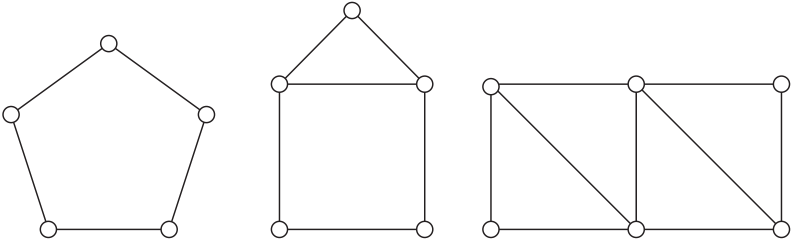

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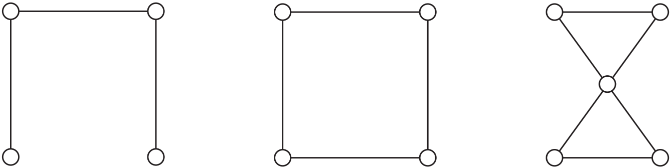

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### Corollary (Carlson, K. (2021))

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### Question 1

Is there a stronger connection between the spectral radius and throttling number?

# Throttling is a Forbidden Induced Subgraph Problem

**Proposition (Carlson, K. (2021))**

Let $k$ be a constant. The set of graphs $G$ such that $\mathrm{th}(G) \geq |V(G)| - k$ and $|V(G)| \geq k$ is characterized by a family of forbidden induced subgraphs.

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# Forbidden Families can be Finite

**Theorem (Carlson, K. (2021))**

The set of graphs $G$ such that $\mathrm{th}_R(G) \geq |V(G)| - k$ and $|V(G)| \geq k$ is characterized by a finite family of forbidden induced subgraphs.

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## Ideas!

Forbid graphs $H$ with $|V(H)| \leq 4k + 4$ and $\mathrm{th}(H) < |V(H)| - k$, and suppose $\mathrm{th}(G) < |V(G)| - k$.

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2) Bound how large this "fast" part of the graph is.
3) Since the subgraph is small and throttles quickly, it was forbidden.

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## Lemma

There exists an $R$ forcing set $B \subseteq V(G)$ such that

$$\sum_{i=1}^{\mathrm{pt}_R(G;B)} |B^{(i)}| - 1 \geq k + 1$$

if and only if $\mathrm{th}_R(G) < |V(G)| - k$.

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## Let's use a **standard witness**!

Suppose $B$ is a standard witness for $\mathrm{th}(G) < |V(G)| - k$. Let $r$ be the earliest time such that

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- By $R$ forcing rules, $|U^{(i)}| \leq |B^{(i)}|$.

$$|X| \leq \sum_{i=1}^{r} \left( |B^{(i)}| + |U^{(i)}| \right) \leq 2 \sum_{i=1}^{r} |B^{(i)}| = 2(k + r + 1) \leq 4k + 4$$

# Some Take-Aways

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Notice that the forbidden family we used to prove this Theorem contains only graphs with

$$\mathrm{th}_R(G) \leq 3k + 3.$$

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The set of graphs $G$ such that $\mathrm{th}_R(G) \geq |V(G)| - k$ and $|V(G)| \geq k$ is characterized by a finite family of forbidden induced subgraphs.

Notice that the forbidden family we used to prove this Theorem contains only graphs with

$$\mathrm{th}_R(G) \leq 3k + 3.$$

Therefore, understanding graphs with

$$\mathrm{th}_R(G) \geq |V(G)| - k$$

is **in some sense dual** to understanding graphs with

$$\mathrm{th}_R(G) \leq 3k + 3.$$

# Future Work

## Question 2

What conditions must be imposed on an abstract color change rule $R$ so that $\mathrm{th}_R(G) \geq |V(G)| - k$ and $|V(G)| \geq k$ is a forbidden induced subgraph problem?

## Question 2+

What further conditions are necessary to conclude that the set of graphs with $\mathrm{th}_R(G) \geq |V(G)| - k$ can be characterized by a finite set of forbidden subgraphs?

## Question 3

Does there exists a local abstract color change rule $R$ such that the set of graphs with $\mathrm{th}_R(G) \geq |V(G)| - k$ and $|V(G)| \geq k$ can be characterized as a forbidden subgraph problem, but no finite set of graphs is a corresponding set of forbidden subgraphs?

# The End

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