2

Parter-Wiener, etc. Theory

2.1 Introduction

Because of the interlacing eigenvalue theorem (Section 0.1.2), $\left|\mathrm{m}_A(\lambda) - \mathrm{m}_{A(i)}(\lambda)\right| \leq 1$ for any Hermitian $A \in \mathcal{M}_n(\mathbb{C})$, any $\lambda \in \mathbb{R}$ and any index $i \in \{1, 2, \ldots, n\}$. For any given graph G, this means that when one vertex is removed from G, the multiplicity of an eigenvalue in any particular matrix $A \in \mathcal{H}(G)$ can change by at most 1. If the multiplicity goes down by 1, we call the removed vertex a ***downer*** in G (for the eigenvalue with respect to A and G). If it stays the same, we call the vertex ***neutral***, and if up by 1, we call it ***Parter***, in recognition of the first and surprisingly remarkable theorem on the subject [P]. This theorem was elaborated in a natural way by G. Wiener [Wie] in his PhD thesis work with John Maybee and finally to its fullest extent for trees by C. Johnson, A. Leal-Duarte and C. Saiago in [JL-DS03a]. This theory has dramatic implications for multiplicities, and our purpose is to present it in this chapter, along with some implications and closely related ideas. The theory is exactly only for trees and, in some sense, characterizes them, as we indicate. We note that in the case of "Parter," $\mathrm{m}_A(\lambda) = 0$ is allowed. In the case of "neutral," the possibility of $\mathrm{m}_A(\lambda) = 0$ does not arise, and it is convenient to assume that $\mathrm{m}_A(\lambda) \geq 1$. Any eigenvalue for which there is a Parter vertex is called a ***Parter eigenvalue*** (even if its initial multiplicity was 0). For a tree, any eigenvalue of multiplicity ≥ 2, and some of multiplicity 1, is a Parter eigenvalue, as we shall see.

As elsewhere, we benignly blur the distinction between graph and matrix and between subgraph and principal submatrix. In particular, for a given graph G, $A \in \mathcal{H}(G)$ and G' an induced subgraph of G, we often refer to the "eigenvalues of G'," meaning the "eigenvalues of the principal submatrix $A[G']$."

2.2 An Example

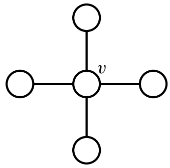
We begin with an example that motivates and illustrates the main result of this chapter.

Example 2.2.1

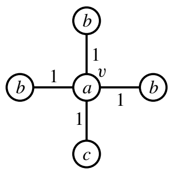
Consider the 5-by-5 matrix

$$A = \begin{bmatrix} a & 1 & 1 & 1 & 1 \\ 1 & b & 0 & 0 & 0 \\ 1 & 0 & b & 0 & 0 \\ 1 & 0 & 0 & b & 0 \\ 1 & 0 & 0 & 0 & c \end{bmatrix} \in \mathcal{S}(T)$$

with

$T =$ ,

which may be described as

.

Since $\operatorname{rank}(A - bI) = 3$ (when $c \neq b$), $\mathrm{m}_A(b) = 2$. Because

$$A(v) = \begin{bmatrix} b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{bmatrix},$$

we have $\mathrm{m}_{A(v)}(b) = 3 = \mathrm{m}_A(b) + 1$. This means that v is a Parter vertex (for the eigenvalue b with respect to A and T). Notice that this is independent of the value of a (the diagonal entry of A associated with the Parter vertex v) and of b (as long as three of the other diagonal entries are equal to b) and of the nonzero off-diagonal entries.

These occurrences will be seen to be common. Notice also that some branches adjacent to v have the property that when the vertex adjacent to v in them is deleted from them, the multiplicity of b in the branch goes down (these vertices are downers at v in their branches). This will be seen to be characteristic of a Parter vertex. Finally, notice also that v is an HDV in T (the only one) and that there are at least two eigenvalues of multiplicity 1 in A.

The three bs and c in Example 2.2.1 constitute an "assignment" of eigenvalues to the branches of T at v and entirely determine the occurrence of a multiple eigenvalue in A.

2.3 General Theory of the Existence of Parter Vertices for Trees

An important theorem about the existence of principal submatrices, of a real symmetric matrix whose graph is a tree, in which the multiplicity of an eigenvalue goes up was largely developed in separate papers by Parter [P] and Wiener [Wie]. The multiplicity of an eigenvalue of a real symmetric matrix can change by at most 1 if a principal submatrix of size one smaller is extracted. However, in [P], a very surprising observation was made: if T is a tree and $A \in \mathcal{S}(T)$ and $\mathrm{m}_A(\lambda) \geq 2$, then there is a vertex i such that $\mathrm{m}_{A(i)}(\lambda) \geq 3$ and λ is an eigenvalue of at least three components (branches) of $A(i)$. In particular, if $\mathrm{m}_A(\lambda) = 2$, then $\mathrm{m}_{A(i)}(\lambda) = \mathrm{m}_A(\lambda) + 1$! In [Wie], it was further shown that if $\mathrm{m}_A(\lambda) \geq 2$, then there is a vertex i such that $\mathrm{m}_{A(i)}(\lambda) = \mathrm{m}_A(\lambda) + 1$.

Curiously, Parter did not identify the multiplicity increase for all values of $\mathrm{m}_A(\lambda) \geq 2$ and Wiener did not explicitly mention the distribution of the eigenvalue among at least three branches, both of which are important, though it appears that each author might have, given the machinery they developed. When just one vertex is removed, we note that the "three branches" cannot generally be improved upon, as there are trees with maximum degree 3 and arbitrarily high possible multiplicities (see, e.g., [GeM], [JL-D99]).

These results have been important in recent work on possible multiplicities of eigenvalues among matrices in $\mathcal{S}(T)$ and, though not explicitly stated by either, we feel it appropriate to attribute the following theorem to Parter and Wiener.

Theorem 2.3.1 (Parter-Wiener Theorem) *Let T be a tree, $A \in \mathcal{S}(T)$ and suppose that $\lambda \in \mathbb{R}$ is such that $\mathrm{m}_A(\lambda) \geq 2$. Then there is a vertex i of T of degree at least 3 such that $\mathrm{m}_{A(i)}(\lambda) = \mathrm{m}_A(\lambda) + 1$ and λ occurs as an eigenvalue in direct summands of $A(i)$ that correspond to at least three branches of T at i.*

We call any vertex i meeting the requirements of Theorem 2.3.1's conclusion a ***strong Parter vertex*** of T for λ relative to A (a strong Parter vertex, for short).

We note that the referred results of neither [P] nor [Wie] apply when T is a path, as then $\mathrm{m}_A(\lambda) > 1$ cannot occur. We give a simple proof of this known fact in Section 2.7 (Theorem 2.7.1), and we present now a generalization of [P] and [Wie] that will apply to this case. Our approach also gives a clear identification of the elements necessary in a proof of the original observations.

This generalization of the Parter-Wiener theorem is as follows [JL-DS03a].

Theorem 2.3.2 *Let T be a tree, $A \in \mathcal{S}(T)$ and suppose that there is a vertex v of T and a real number λ such that $\lambda \in \sigma(A) \cap \sigma(A(v))$. Then*

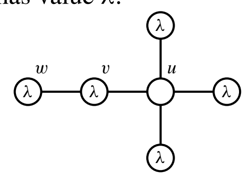
1. *there is a vertex u of T such that $\mathrm{m}_{A(u)}(\lambda) = \mathrm{m}_A(\lambda) + 1$, i.e., u is Parter for λ (with respect to A and T);*
2. *if $\mathrm{m}_A(\lambda) \geq 2$, then the prevailing hypothesis is automatically satisfied and u may be chosen so that $\deg_T(u) \geq 3$ and so that there are at least three components T_1, T_2 and T_3 of $T - u$ such that $\mathrm{m}_{A[T_i]}(\lambda) \geq 1$, $i = 1, 2, 3$; and*
3. *if $\mathrm{m}_A(\lambda) = 1$, then u may be chosen so that $\deg_T(u) \geq 2$ and so that there are two components T_1 and T_2 of $T - u$ such that $\mathrm{m}_{A[T_i]}(\lambda) = 1$, $i = 1, 2$.*

Before continuing, we should note that even when $\mathrm{m}_A(\lambda) \geq 2$ and $\mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_A(\lambda) + 1$, i.e., v is a Parter vertex for λ, it may happen that $\deg_T(v) = 1$ or $\deg_T(v) = 2$ or λ appears in only one or two components of $T - v$ even when $\deg_T(v) \geq 3$.

Example 2.3.3

Vertices u, v and w are Parter for λ in a real symmetric matrix A for which the graph is the following tree T. Assume that every diagonal entry corresponding to a labeled vertex has value λ.

$T =$

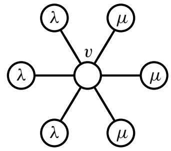


We have $\mathrm{m}_A(\lambda) = 2$ and $\mathrm{m}_{A(u)}(\lambda) = \mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_{A(w)}(\lambda) = 3$ with $\deg_T(u) = 4$, $\deg_T(v) = 2$ and $\deg_T(w) = 1$.

We also note that, depending on the tree T, several different vertices of T could be Parter for an eigenvalue of the same matrix in $\mathcal{S}(T)$. Also, depending on the tree T, the same vertex could be Parter for different eigenvalues of a matrix in $\mathcal{S}(T)$.

Example 2.3.4

The vertex v is Parter for λ and μ in a real symmetric matrix A for which the graph is the following tree. Assume that every diagonal entry corresponds to the label of the corresponding vertex.



It is clear that we have $\mathrm{m}_{A(v)}(\lambda) = 3 = \mathrm{m}_{A(v)}(\mu)$ and, as we shall see soon, $\mathrm{m}_A(\lambda) = 2 = \mathrm{m}_A(\mu)$, i.e., v is Parter for two different eigenvalues.

Of course, when $\lambda \in \sigma(A) \cap \sigma(A(v))$, it may also happen that v does not qualify as a Parter vertex (v need not increase the multiplicity of λ). However, as we shall see in Theorem 2.7.4, if A is a tridiagonal real symmetric matrix such that $\lambda \in \sigma(A) \cap \sigma(A(v))$, then v must be Parter for λ!

Example 2.3.5

In Section 2.7, we study in detail the case in which T is a path on n vertices, i.e., the case in which A is an n-by-n real symmetric matrix permutationally similar to an (irreducible) tridiagonal matrix. Here we discuss the simple trees (paths) T on $n = 2$ or $n = 3$ vertices.

Let $A \in \mathcal{S}(T)$ and λ be an eigenvalue of A. Suppose, *wlog*, $\lambda = 0$ so that $\operatorname{rank} A < n$.

If $n = 2$, then 0 cannot be an eigenvalue of $A(1)$ or $A(2)$ because any matrix $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \mathcal{S}(T)$ with $a = 0$ or $c = 0$ has full rank. Moreover, any $A \in \mathcal{S}(T)$ with 0 as an eigenvalue has rank 1 and must have the form $A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$, for some $a \neq 0$.

Let $n = 3$ and suppose, *wlog*, $A \in \mathcal{S}(T)$ is (irreducible) tridiagonal. Again, 0 cannot be an eigenvalue of $A(1)$ or $A(3)$. In fact, if $A(1)$ has the eigenvalue 0, then $A = \begin{bmatrix} c & b & 0 \\ b & a & a \\ 0 & a & a \end{bmatrix}$, with $a \neq 0$ and $b \neq 0$, which implies $\operatorname{rank} A = 3$.

Note that A and $A(i)$ may have both the eigenvalue 0 but if and only if i is the center vertex of T (degree 2 vertex) and $A(i)$ has 0 as an eigenvalue of multiplicity 2 (one more than in A). In fact, if $A = \begin{bmatrix} 0 & b & 0 \\ b & c & a \\ 0 & a & 0 \end{bmatrix}$, then $\operatorname{rank} A(i) = 0$ and $\operatorname{rank} A = 2$ (which gives 0 as eigenvalue of A with multiplicity 1). In the same way, we conclude that if i is the center vertex of T

and $A(i)$ has rank 1 ($A(i)$ has 0 as an eigenvalue of multiplicity 1), then 0 is not an eigenvalue of A because $\operatorname{rank} A = 3$.

Note that there is nothing special about considering $\lambda = 0$, as it could be any real number via translation (addition of a scalar multiple of I).

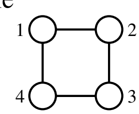
We should note that, if $G(A)$ is not a tree and $m_A(\lambda) \geq 2$, it may happen that $m_{A(v)}(\lambda) < m_A(\lambda)$ for any vertex v of $G(A)$ [JL-D06]. So the assumption that the graph is a tree is important in Theorems 2.3.1 and 2.3.2.

Example 2.3.6

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

whose graph, $G(A)$, is the cycle



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Then 1 is an eigenvalue of A such that $m_A(1) = 2$. However, $m_{A(v)}(1) = 1$, for each vertex v of $G(A)$, i.e., every vertex of $G(A)$ is a downer for the eigenvalue 1.

Note that, when T is a tree and $A \in \mathcal{S}(T)$, since $m_A(\lambda) \geq 2$ implies that $m_{A(v)}(\lambda) \geq 1$ for any vertex v, Theorem 2.3.1 is a special case of Theorem 2.3.2 because every vertex ensures the hypothesis of 2.3.2.

The proof of Theorem 2.3.2 rests, in part, on the following two key lemmas.

Lemma 2.3.7 *Let T be a tree and $A \in \mathcal{S}(T)$. If there is a vertex v of T and a real number λ such that $\lambda \in \sigma(A) \cap \sigma(A(v))$, then there are adjacent vertices u and u_0 of T such that the component T_0 of $T - u$ containing u_0 does not contain v and satisfies* $m_{A[T_0 - u_0]}(\lambda) = m_{A[T_0]}(\lambda) - 1$, *i.e., u_0 is a downer for λ in T_0.*

Proof. We argue by induction on the number n of vertices of T. If $n = 1$ or $n = 2$, the claimed implication is correct because it is not possible for the hypothesis to be satisfied, as may be easily checked (the case $n = 2$ is discussed in Example 2.3.5). If $n = 3$, then T is a path, and the hypothesis may be satisfied, but only by a matrix permutation similar to a tridiagonal matrix whose first and last diagonal entries are both λ and only for v the middle vertex (as we have discussed in Example 2.3.5). Then taking u to be the middle vertex v and u_0 to be either the first or the last vertex shows that the conclusion is satisfied (as the empty matrix cannot have λ as an eigenvalue).

Now suppose that the claim is valid for all trees on fewer than n vertices, $n > 3$, and consider a tree T on n vertices and $A \in \mathcal{S}(T)$ such that there is a vertex v such that $\lambda \in \sigma(A) \cap \sigma(A(v))$. First, try letting u be the vertex v. If there is a neighbor u_j of v such that $\mathrm{m}_{A[T_j]}(\lambda) \geq 1$ and $\mathrm{m}_{A[T_j-u_j]}(\lambda) = \mathrm{m}_{A[T_j]}(\lambda) - 1$, we are done. If not, there are, by the hypothesis, neighbors u_j such that $\mathrm{m}_{A[T_j-u_j]}(\lambda) \geq \mathrm{m}_{A[T_j]}(\lambda) \geq 1$ and, by replacing v with u_j and applying induction, the claim follows. □

The next lemma may be proven in two different ways, each giving different insights. Both proofs rely on an expansion of the characteristic polynomial for Hermitian matrices for which the graph is a tree, presented in Section 0.2.4. We give one proof here, and another may be found in [JL-DS03a]. See also [Wie, Theorem 12] for a variant of Lemma 2.3.8 and a different approach.

Lemma 2.3.8 *Let T be a tree and $A \in \mathcal{S}(T)$. If there is a vertex v of T and a real number λ for which*

$$\lambda \in \sigma(A) \cap \sigma(A(v)),$$

and there is a branch T_0 of T at v such that

$$\mathrm{m}_{A[T_0-u_0]}(\lambda) = \mathrm{m}_{A[T_0]}(\lambda) - 1,$$

in which u_0 is the neighbor of v in T_0, then

$$\mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_A(\lambda) + 1.$$

Proof. We employ the bridge formula for the characteristic polynomial of A (formula (2) in Section 0.2.4), with v and u_0 corresponding to the hypothesis of the lemma. Let $m = \mathrm{m}_A(\lambda)$ and $m_0 = \mathrm{m}_{A[T_0]}(\lambda)$. (We note that if it happens that $m_0 = m + 1$, the conclusion is immediate. Though the proof is technically correct in any event, it may be convenient to assume $m_0 \leq m$.) Since removal of u_0 from T leaves $A[T_v] \oplus A[T_0 - u_0]$, by the interlacing inequalities and the assumption that $\mathrm{m}_{A[T_0-u_0]}(\lambda) = m_0 - 1$, λ is a root of $p_{A[T_v]}$ at least $m - m_0$ times. Regarding formula (2) in Section 0.2.4, λ is a root of $p_A(t)$ m times and also a root of $p_{A[T_0]}(t)$ m_0 times. Since λ is a root of $p_{A[T_v]}$ at least $m - m_0$ times, λ is a root of the first term of the right-hand side of (2) at least m times. Thus, by a divisibility argument applied to (2), λ must be a root of the second term of its right-hand side at least m times. Since λ is a root of $p_{A[T_0-u_0]}(t)$ $m_0 - 1$ times, then λ is a root of $p_{A[T_v-v]}(t)$ at least $m - m_0 + 1$ times. Note that $A(v) = A[T_v - v] \oplus A[T_0]$. Since λ is a root of $p_{A[T_0]}(t)$ m_0 times and also a root of $p_{A[T_v-v]}(t)$ at least $m - m_0 + 1$ times, λ is a root of $p_{A(v)}(t)$ at least $m+1$ times and, by the interlacing inequalities, λ is a root of $p_{A(v)}(t)$ exactly $m+1$ times, i.e., $\mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_A(\lambda) + 1$. $\square$

Though we have made the statement in the form we wish to apply it, we note that the statement of Lemma 2.3.8 remains correct (trivially) if the hypothesis "$\lambda \in \sigma(A) \cap \sigma(A(v))$" is replaced by the weaker one, "$\lambda \in \sigma(A(v))$."

We next turn to a proof of Theorem 2.3.2.

Proof of Theorem 2.3.2. If $\lambda \in \sigma(A) \cap \sigma(A(v))$, by Lemma 2.3.7 there is a vertex u of T with a branch T_0 at u such that $\mathrm{m}_{A[T_0-u_0]}(\lambda) = \mathrm{m}_{A[T_0]}(\lambda) - 1$, in which u_0 is the neighbor of u in T_0. Since

$$\mathrm{m}_{A(u)}(\lambda) \geq \mathrm{m}_{A[T_0]}(\lambda) = \mathrm{m}_{A[T_0-u_0]}(\lambda) + 1 \geq 1,$$

we also have $\lambda \in \sigma(A) \cap \sigma(A(u))$. Thus, u in place of v satisfies the hypothesis of Lemma 2.3.8 and part **1** of the theorem follows.

For part **2**, we argue by induction on the number n of vertices of T. If $n \leq 3$, the claimed implication is correct because it is not possible that the hypothesis is satisfied, as may be easily checked (note that any tree on $n \leq 3$ vertices is a path, and we have discussed the cases $n = 2$ and $n = 3$ in Example 2.3.5).

If $n = 4$, the only tree on four vertices that is not a path is a star (one vertex of degree 3 and three pendent vertices). Since $\mathrm{m}_A(\lambda) = m \geq 2$, there is a vertex v in T such that $\mathrm{m}_{A(v)}(\lambda) = m + 1 \geq 3$. In that case, T must be a star and v its central vertex (the vertex of degree 3), since for any other vertex u, $T - u$ is a path. Thus, $T - v$ is a graph consisting of three isolated vertices with $\mathrm{m}_{A(v)}(\lambda) \leq 3$. Therefore, $m = 2$ and $\mathrm{m}_{A(v)}(\lambda) = 3$, i.e., λ is an eigenvalue of three components of $T - v$.

Now suppose that the claimed result is valid for all trees on fewer than n vertices, $n > 4$, and consider a tree T on n vertices and a matrix $A \in \mathcal{S}(T)$ such that λ is an eigenvalue of A with $\mathrm{m}_A(\lambda) = m \geq 2$. By part **1**, there is a vertex u in T such that $\mathrm{m}_{A(u)}(\lambda) = m + 1$. If λ is an eigenvalue of at least three components of $T - u$, we are done. If not, there are two possible situations: λ is an eigenvalue of two components of $T - u$ (Case (a)) or λ is an eigenvalue of one component of $T - u$ (Case (b)).

In Case (a), there is a component T' of $T - u$ with λ as an eigenvalue of $A[T']$ and $\mathrm{m}_{A[T']}(\lambda) \geq 2$. Applying induction to T', we have a vertex u' in T' such that $\mathrm{m}_{A[T'-u']}(\lambda) = \mathrm{m}_{A[T']}(\lambda) + 1$ and λ is an eigenvalue of at least three components of $T' - u'$. Observe that $\mathrm{m}_{A(\{u,u'\})}(\lambda) = m + 2$. Thus, by interlacing, we have $\mathrm{m}_{A(u')}(\lambda) = m + 1$. Consider the (unique) shortest path between u and u' in T, $P_{uu'}$, and let (u, u') denote the component of $T - \{u, u'\}$ containing vertices of $P_{uu'}$. Note that (u, u') is one of the components of $T' - u'$ (if not empty). If there are three components of $T' - u'$ having λ as an eigenvalue and none of these is (u, u'), then these three components are also components of $T - u'$, and we are done. If there are only three components of $T' - u'$ having λ as an eigenvalue and one of them is (u, u'), then by interlacing applied to the component of $T - u'$ containing u (since $T - u$ has another component with λ as an eigenvalue), we conclude that $T - u'$ still has three components with λ as an eigenvalue.

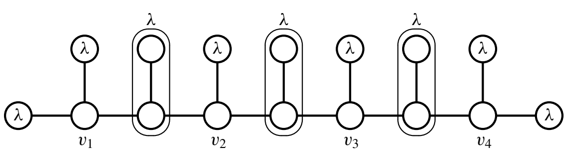
In Case (b), there is a component T' of $T - u$ with λ as an eigenvalue of $A[T']$ and $\mathrm{m}_{A[T']}(\lambda) = \mathrm{m}_A(\lambda) + 1$. Applying induction to T', we have a vertex u' in T' such that $\mathrm{m}_{A[T'-u']}(\lambda) = \mathrm{m}_{A[T']}(\lambda) + 1$ and λ is an eigenvalue of at least three components of $T' - u'$. By interlacing, $\mathrm{m}_{A(u')}(\lambda) = m + 1$. Thus, if there are three components of $T' - u'$ having λ as an eigenvalue and none of these is (u, u'), then these three components are also components of $T - u'$, and we are done. If there are only three components of $T' - u'$ having λ as an eigenvalue and one of these components is (u, u'), we may apply Case (a) to complete the consideration of Case (b).

For part **3**, the only contrary possibility is that λ is an eigenvalue of multiplicity 2 of one of the direct summands of $A(u)$. In that case, if T' is the component of $T - u$ with λ as an eigenvalue of multiplicity 2 of $A[T']$, then by part **2** applied to $A[T']$, we conclude the existence of a vertex u' of T such that λ is an eigenvalue of exactly two of the direct summands of $A(u')$. □

Clearly, the number two (components) cannot be improved in part **3**. But also, the number three (components) cannot be improved, in general, in part **2**. The original multiplicity may be high and the vertex deleted multiplicity one higher, without the vertex degrees in T being as high, as the following example shows.

Example 2.3.9

Let

$T =$ 

and $A \in \mathcal{S}(T)$. With the displayed assignment, we have $\mathrm{m}_A(\lambda) = 5$ and, by part **1** of Theorem 2.3.2, there is a vertex u of T such that $\mathrm{m}_{A(u)}(\lambda) = \mathrm{m}_A(\lambda) + 1 = 6$, a Parter vertex for λ. By part **2** of the same theorem, this Parter vertex u may be chosen so that $\deg_T(u) \geq 3$ and so that there are at least three components of $T - u$ having λ as an eigenvalue. This Parter vertex u can be any vertex in $\{v_1, v_2, v_3, v_4\}$, each one of degree 3, the maximum degree of a vertex in this tree.

Corollary 2.3.10 *Let T be a tree and $A \in \mathcal{S}(T)$. If for some vertex v of T, λ is an eigenvalue of $A(v)$, then there is a vertex u of T such that $\mathrm{m}_{A(u)}(\lambda) = \mathrm{m}_A(\lambda) + 1$.*

Proof. Suppose that λ is an eigenvalue of $A(v)$ for some vertex v of T. If λ is not an eigenvalue of A, then setting $u = v$, $\mathrm{m}_{A(u)}(\lambda) = \mathrm{m}_A(\lambda) + 1$. If λ is also an eigenvalue of A, by Theorem 2.3.2, there is a vertex u of T such that $\mathrm{m}_{A(u)}(\lambda) = \mathrm{m}_A(\lambda) + 1$. □

Lemma 2.3.8 indicates that a neighboring vertex u_0 in whose branch T_0 the multiplicity goes down (after removal of u_0) is important for the existence of a Parter vertex. According to Lemma 2.3.8, the existence of such a branch is sufficient for a vertex to be Parter. Importantly, it is also necessary, which provides a precise structural mechanism for recognition of Parter vertices. Notice that even when $\mathrm{m}_A(\lambda) = 0$, if there is a branch like this at v, then $\mathrm{m}_{A(v)}(\lambda) = 1$. It cannot be more by interlacing, nor less because $A[T_0]$ is a direct summand of $A(v)$.

2.4 Characterization of Parter Vertices

If a vertex v is removed from a tree T, a forest, composed of several branches at v, will result. Suppose $\deg_T(v) = k$. Then k branches $T_1, \ldots, T_k$ will result, and each neighbor u_i of v will be in exactly one of these branches T_i. If u_i is a downer vertex in T_i (for the eigenvalue λ of $A[T_i]$), then T_i is said to be a ***downer branch*** at v and u_i a ***downer vertex*** at v. We call such a neighbor u_i of the Parter vertex v, a ***downer neighbor*** of v. Note that u_i may or may not be a downer vertex in T in this event.

Almost as striking as the necessity of Parter vertices for a multiple eigenvalue, whether or not a vertex is Parter is characterized by the nature of its neighbors in their branches.

Theorem 2.4.1 *Let T be a tree, $A \in \mathcal{S}(T)$ and $\lambda \in \mathbb{R}$. Then a vertex v of T is Parter for λ if and only if there is at least one downer branch at v for λ.*

Proof. The sufficiency follows from the proof of part **1** of Theorem 2.3.2.

For necessity, consider the neighbors formula for the characteristic polynomial of A (formula (1) in Section 0.2.4). Suppose that none of the neighbors of v, vertices $u_1, \ldots, u_k$, is a downer vertex. Then the number of times λ is a root of the second term on the right is at least the number of times that λ is a root of $p_{A(v)}(t)$, i.e., $\prod_{j=1}^{k} p_{A[T_j]}(t)$. Thus, by a divisibility argument applied to (1), we conclude that $\mathrm{m}_A(\lambda) \geq \mathrm{m}_{A(v)}(\lambda)$, and v could not be Parter. □

Extending the divisibility argument of the proof of Theorem 2.4.1, if each neighbor of v is either Parter or neutral for λ in its branch, then v cannot be

Parter for λ. Moreover, if u_i is Parter in its branch, then it is still Parter in T, as its downer branch within its branch will be a downer branch in T.

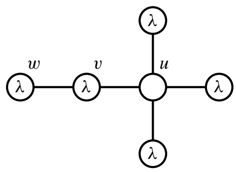
We refer to this characterization of Parter vertices as the ***downer branch mechanism***. Both the necessity and sufficiency of downer neighbors are quite powerful (and surprising) tools.

Note that whenever $\lambda \in \sigma(A(v))$ for any vertex v of T such that $A \in \mathcal{S}(T)$, there will be at least one Parter vertex for λ (not necessarily v), even if $\mathrm{m}_A(\lambda) = 0$ or $\mathrm{m}_A(\lambda) = 1$. In this event, the above downer branch mechanism is still in place. If $\mathrm{m}_A(\lambda) = 0$, v is Parter for λ and the branch at v in which λ appears will be the downer branch (by Theorem 2.4.1).

If there is only one downer branch at v, then v is called ***singly Parter*** (s-Parter, or 1P). If more than one, then ***multiply Parter*** (m-Parter, or mP). The ***downer index*** of a Parter vertex v is just the number of downer branches at v (the number of downer neighbors of v). The most important difference is between index 1 (s-Parter) and index greater than 1 (m-Parter).

Example 2.4.2

We return to the tree

$T =$ 

and matrix $A \in \mathcal{S}(T)$ of Example 2.3.3. It is now easy to conclude that $\mathrm{m}_A(\lambda) = 2$ and that u, v and w are all Parter vertices for λ. Note that λ is not an eigenvalue of $A[\{w, v\}]$, so $\mathrm{m}_{A(u)}(\lambda) = 3$. Since any of the pendent vertices at u in T is a downer neighbor of u (for λ), we conclude that $\mathrm{m}_A(\lambda) = 2$. By an argument similar to that for u, we conclude that v and w are also Parter vertices for λ in T.

Vertices v and w are each singly Parter, while vertex u, with three downer branches, is a multiply Parter vertex. Note that each downer neighbor of a singly Parter vertex is not a downer in the original graph! On the contrary, each downer neighbor of the multiply Parter vertex u is a downer in T! (See Theorem 2.5.3 below.)

The downer branch mechanism for identifying Parter vertices is subtle and very important. In general, there may be only one downer branch at a Parter vertex but, in Section 5.1, we show that if λ attains maximum multiplicity for a tree T (the maximum possible multiplicity for an eigenvalue among matrices in $\mathcal{S}(T)$), there must be at least two, a strong structural distinction in the maximum multiplicity case.

Nevertheless, each Parter eigenvalue has at least one m-Parter vertex.

Theorem 2.4.3 *Let T be a tree, $A \in \mathcal{S}(T)$ and λ be an eigenvalue of A. If λ is a Parter eigenvalue, then there is an* m-*Parter vertex in T for λ relative to A.*

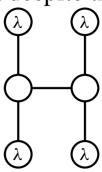
Proof. Let T be a tree on n vertices, $A \in \mathcal{S}(T)$, $\lambda \in \sigma(A)$, with $\mathrm{m}_A(\lambda) = k \geq 1$ and such that there is a Parter vertex v for λ in T. We argue by induction on the number n of vertices of T.

The smaller number of vertices for which the hypothesis may be satisfied is $n = 3$. T is a path on three vertices and the Parter vertex u for λ must be the degree 2 vertex, with λ in each pendent vertex at v, making each of the two pendent vertices a downer (vertex) branch at v.

Let $n > 3$ and suppose that the claimed result is valid for all trees on fewer than n vertices. By Theorem 2.3.2, there is a Parter vertex u for λ such that there are two branches (if $\mathrm{m}_A(\lambda) = 1$) or three branches (if $\mathrm{m}_A(\lambda) \geq 2$) at u having λ as an eigenvalue. If u has two downer branches, we are done. Suppose that u has only one downer branch. In that case, there is a branch T_0 at u, with λ as an eigenvalue, whose neighbor u_0 of u in T_0 is either Parter or neutral in T_0. Again by Theorem 2.3.2, there is a Parter vertex for λ in T_0 and, by the induction hypothesis, there is a Parter vertex v for λ in T_0 with at least two downer branches in T_0. We have two possibilities: none of these two downer branches is in the direction of u (Case 1; in this case, we are done), or one of these two downer branches at v in T_0 is in the direction of u (Case 2). In Case 2, let T_0' be the downer branch at v in T_0 that is in the direction of u and let v' be the neighbor of v in T_0'. Because v is Parter in $T - u$, we have $\mathrm{m}_{A(\{u,v\})}(\lambda) = k + 2$ and, since T_0' is a component of $T - \{u, v\}$ and v' is a downer in T_0', we have $\mathrm{m}_{A(\{u,v,v'\})}(\lambda) = k + 1$. Because u is Parter in $T - \{v, v'\}$ (by virtue of having a downer branch not in direction of v'), we obtain $\mathrm{m}_{A(\{v,v'\})}(\lambda) = k$ and, therefore, v' is still a downer vertex in $T - v$ and, thus, v has two downer branches at v for λ in T, completing the proof. $\square$

Example 2.4.4

The two downer neighbors in Theorem 2.4.3 is best possible, even when the eigenvalue is originally multiple, despite the necessity of three branches. In

$T =$ 

and $A \in \mathcal{S}(T)$ with the displayed assignment for λ, we have $\mathrm{m}_A(\lambda) = 2$ and both HDVs are m-Parter, while both have only two downer branches.

Recall Example 2.4.2, in which we noted that the downer neighbor of a singly Parter vertex is not a downer in the original tree. It is a good example of a change in "status" of a vertex after the removal of another vertex. This problem is investigated in the next section and, in the process, we give a characterization of an m-Parter vertex and an s-Parter vertex in a tree. We also investigate the possibilities of vertex adjacency in terms of the status of each vertex.

2.5 The Possible Changes in Status of One Vertex upon Removal of Another

By the ***status*** of a vertex in a graph G relative to an eigenvalue λ of $A \in \mathcal{S}(G)$, we mean one of the only three possibilities based upon its removal: Parter "P" (increase in multiplicity by 1), neutral "N" (no change in multiplicity) or downer "D" (a decrease by 1 in multiplicity). Which of these occurs for a given vertex is, of course, very important. In general, the status depends upon a number of features of its ingredients: the matrix and the combinatorial structure of G. In particular, the status may change upon the removal of other parts of G. Here, we are concerned with trees, and we give all the possible changes in status of one identified vertex when another identified vertex is removed. It can make a difference whether the two vertices are adjacent, so we include both possibilities. It may also matter, if one of the vertices is Parter, whether it is s-Parter or m-Parter.

This information was first presented in [Mc] and then also in [JL-DMc]. We start by presenting fundamental results and then a table giving all possibilities. Afterward, we summarize the proofs for the table, which are of just a few types, often involving counting arguments.

The following result states an important feature of the m-Parter vertices.

Theorem 2.5.1 *Let T be a tree. Upon removal of an* m*-Parter vertex u from T, no other vertex of T changes its status in $T - u$, including the downer index of a Parter vertex (relative to an identified eigenvalue of a matrix in $\mathcal{S}(T)$).*

Proof. Let u be an m-Parter vertex of T, for the eigenvalue λ, and v, $v \neq u$, be any other vertex of T. Since u is m-Parter in T, by Theorem 2.3.2 u is still Parter in $T - v$ (because the removal of v from T can "destroy," at most, one of the downer branches at u in T). If $\mathrm{m}_T(\lambda) = k$ and $\mathrm{m}_{T-v}(\lambda) = k + t_0$, in which $t_0 \in \{-1, 0, 1\}$ (depending on if v is a downer, neutral or Parter in T), then the multiplicity of λ in $T - \{u, v\}$ is

$$k \xrightarrow[\text{remove } v]{} k + t_0 \xrightarrow[\text{remove } u]{} k + t_0 + 1.$$

Removing v and u in the reverse order, we obtain

$$k \xrightarrow[\text{remove } u]{} k+1 \xrightarrow[\text{remove } v]{} k+1+t_1,$$

in which $t_1 \in \{-1, 0, 1\}$ (depending on if v is a downer, neutral or Parter in $T-u$). Since $k+t_0+1 = k+1+t_1$, we conclude that v must have in $T-u$ the same status as in T, completing the proof. □

The following result is used for the characterization of m-Parter and s-Parter vertices but it has independent interest.

Theorem 2.5.2 *Let T be a tree and u, v be two distinct vertices of T. Then u is Parter in T and v is a downer in T if and only if u is Parter in $T-v$ and v is a downer in $T-u$ (relative to an identified eigenvalue of a matrix in $\mathcal{S}(T)$).*

Proof. Let λ be an eigenvalue of a matrix in $\mathcal{S}(T)$ such that $\mathrm{m}_T(\lambda) = k$.

Suppose that u is Parter in T and v is a downer in T. Because of the status of each vertex in T and, by interlacing, we obtain

$$k \xrightarrow[\text{remove } u]{} k+1 \xrightarrow[\text{remove } v]{} k+2,\ k+1,\ \text{or } k$$

and

$$k \xrightarrow[\text{remove } v]{} k-1 \xrightarrow[\text{remove } u]{} k, k-1,\ \text{or } k-2,$$

so that we must have $m_{T-\{u,v\}}(\lambda) = k$ and, therefore, v is a downer in $T-u$ and u is Parter in $T-v$.

Conversely, suppose that u is Parter in $T-v$ and that v is a downer in $T-u$, i.e.,

$$k \xrightarrow[\text{remove } v]{} k+t_0 \xrightarrow[\text{remove } u]{} k+t_0+1$$

and

$$k \xrightarrow[\text{remove } u]{} k+t_1 \xrightarrow[\text{remove } v]{} k+t_1-1,$$

in which $t_0, t_1 \in \{-1, 0, 1\}$. Since $k+t_0+1 = k+t_1-1$, i.e., $t_0 = t_1 - 2$, and $t_0, t_1 \in \{-1, 0, 1\}$, we conclude that $t_0 = -1$ and $t_1 = 1$. Thus, v is a downer in T and u is Parter in T. □

In particular, from Theorem 2.5.2 we know that any downer vertex of T remains downer after the removal of a Parter vertex from T and, conversely, any Parter vertex of T remains Parter after the removal of a downer vertex from T.

We give now a characterization of an m-Parter vertex and of an s-Parter vertex.

Theorem 2.5.3 *Let T be a tree and u be a vertex of T. We have the following in T:*

1. *u is* m-*Parter if and only if u is Parter with a neighbor that is a downer in T. Moreover, the downer neighbors of an* m-*Parter vertex u of T are the neighbors of u in T that are downers in T.*
2. *u is* s-*Parter if and only if u is Parter with no neighbors that are downers in T.*

Proof. Let T be a tree and λ be the relevant eigenvalue of an $A \in \mathcal{S}(T)$, with $\mathrm{m}_A(\lambda) = k$.

1. Suppose that u is m-Parter, i.e., that there are at least two downer branches at u in T, and let v_1 and v_2 be two downer neighbors of u. We have

$$k \xrightarrow[\text{remove } u]{} k+1 \xrightarrow[\text{remove } v_1]{} k$$

and, because u is still Parter in $T - v_1$ (by virtue of having another downer branch not including v_1),

$$k \xrightarrow[\text{remove } v_1]{} k+t_0 \xrightarrow[\text{remove } u]{} k+t_0+1,$$

in which $t_0 \in \{-1, 0, 1\}$ (depending on if v_1 is a downer, neutral or Parter in T). From the equalities $\mathrm{m}_{T-\{u,v_1\}}(\lambda) = k = k + t_0 + 1$ we obtain $t_0 = -1$ and, therefore, v_1 must be a downer in T. Similarly, we conclude that v_2 is also a downer in T. In particular, we have shown that each downer neighbor of an m-Parter vertex is a downer in T.

For the converse, suppose that u is Parter and v is a neighbor of u that is a downer in T. By Theorem 2.5.2, v is still a downer in $T - u$ and u is still Parter in $T - v$. Since v is a downer in $T - u$, it means that v is a downer neighbor of u in T. Because u is still Parter in $T - v$, by Theorem 2.4.1 there must exist a downer neighbor w of u in $T - v$. Since u and v are adjacent, w is also a downer neighbor of u in T. Thus v and w are two downer neighbors of u in T making u m-Parter. In particular, we have shown that each downer vertex in T, that is a neighbor of an m-Parter vertex u of T, is a downer neighbor of u.

2. This follows from part **1**. □

Regarding the statement of Theorem 2.5.3, a Parter vertex may have two different kinds of downer neighbors. If v is a downer neighbor of u, i.e., v is a downer in a branch T' at u, but v is not a downer in T, then v is called a ***local downer neighbor*** of u. If v is still a downer in T, then a ***global downer neighbor***.

As a consequence of Theorem 2.5.3, we have

Corollary 2.5.4 *Let T be a tree.*

1. *The neighbors of an* s-*Parter vertex must be either Parter (*s-*Parter or* m-*Parter) or neutral in T.*
2. *No vertex has more than one local downer neighbor (relative to an identified eigenvalue).*
3. *The neighbors of a downer vertex must be either downer or* m-*Parter in T. (Or, equivalently, if u is a downer in T and v is either* s-*Parter or neutral in T, then u and v are not adjacent.)*
4. *In the unique path connecting a neutral vertex to a (necessarily nonadjacent) downer vertex in T, there is an* m-*Parter vertex.*

Proof. **1.** If u is s-Parter in T and a neighbor v of u was downer in T, then by part **1** of Theorem 2.5.3, vertex u would be m-Parter, a contradiction.

2. If a vertex u of T had more than one local downer neighbor, then u would be m-Parter in T. By part **1** of Theorem 2.5.3, each of the local downer neighbors of u would be also a downer in T, a contradiction.

3. Let u be a downer vertex of T and v be a neighbor of u. By part **1**, vertex v cannot be s-Parter so that we only need to show that v cannot also be neutral.

Suppose that v is neutral in T and that λ is the relevant eigenvalue, with multiplicity k in T. Then the multiplicity of λ in $T - \{u, v\}$ is

$$k \xrightarrow[\text{remove } u]{} k-1 \xrightarrow[\text{remove } v]{} k, k-1, \text{ or } k-2$$

and, removing u and v in the reverse order, we obtain

$$k \xrightarrow[\text{remove } v]{} k \xrightarrow[\text{remove } u]{} k+1, k, \text{ or } k-1$$

so that the multiplicity of λ in $T - \{u, v\}$ is k (Case 1) or $k - 1$ (Case 2).

In Case 1, v must be Parter in $T - u$, which means, by Theorem 2.4.1, that v has a downer neighbor v' in $T - u$. Because u and v are adjacent, v' is still a downer neighbor of v in T, making v Parter in T, a contradiction.

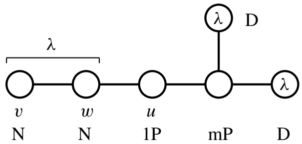
In Case 2, u must be a downer in $T - v$, i.e., u must be a downer neighbor of v in T. By Theorem 2.4.1, v is then Parter in T, a contradiction.

4. Let T' be the unique path in T connecting vertices u (neutral) and v (downer). By part **3**, vertices u and v cannot be adjacent. Again by part **3**, the neighbor of the downer vertex v in T' is either a downer or m-Parter. Since the interior vertices of T' cannot all be downers (again by part **3**), we conclude that at least one of them must be m-Parter. □

Example 2.5.5

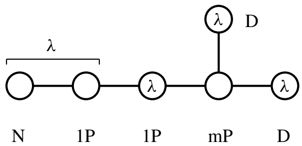
In each of the following trees T, consider an $A \in \mathcal{S}(T)$ with the displayed assignment for λ. If a vertex i is not labeled with λ, we mean that $A[i] \neq [\lambda]$. In each example, we identify the status of each vertex in T relative to the eigenvalue λ.

1.

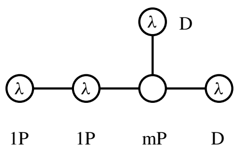
$T =$ 

Vertex w is a local downer neighbor of u so that u is s-Parter. Note that w is neutral in T.

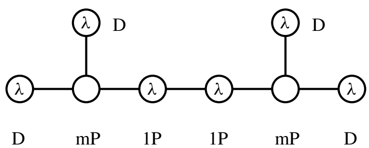
2.

$T =$ 

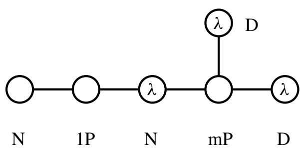
3.

$T =$ 

4.

$T =$ 

5.

$T =$ 

In the proof of part **3** of Corollary 2.5.4 (Case 1), we used one consequence of Theorem 2.4.1 (downer branch mechanism). For future reference, we record here this simple, but useful, fact.

Lemma 2.5.6 *Let u and v be adjacent vertices of a tree T. For $\lambda \in \mathbb{R}$ and a matrix in $\mathcal{S}(T)$, if u is Parter for λ in $T - v$, then u is Parter for λ in T.*

Proof. If u is Parter in $T - v$, then by Theorem 2.4.1, there is a downer branch T' at u in $T - v$. Because u and v are adjacent in T, the branch T' is still a downer branch at u in T, making u Parter in T. $\square$

We show now that the removal of a neutral vertex does not change the status of a downer vertex and conversely.

Theorem 2.5.7 *Let T be a tree and u, v be two nonadjacent vertices of T. If u is neutral in T and v is a downer in T, then u is neutral in $T - v$ and v is a downer in $T - u$ (relative to an identified eigenvalue of a matrix in $\mathcal{S}(T)$).*

Proof. Let λ be the relevant eigenvalue of a matrix in $\mathcal{S}(T)$, such that $\mathrm{m}_T(\lambda) = k$, and let u, v be two nonadjacent vertices of T. Suppose that u is neutral in T and v is a downer in T. Then the multiplicity of λ in $T - \{u, v\}$ is

$$k \xrightarrow[\text{remove } u]{} k \xrightarrow[\text{remove } v]{} k+1, k, \text{ or } k-1$$

and, removing u and v in the reverse order, we obtain

$$k \xrightarrow[\text{remove } v]{} k-1 \xrightarrow[\text{remove } u]{} k, k-1, \text{ or } k-2$$

so that the multiplicity of λ in $T - \{u, v\}$ is k (Case 1) or $k - 1$ (Case 2). In Case 1, u must be Parter in $T - v$ and v must be neutral in $T - u$; in Case 2, u must be neutral in $T - v$ and v must be a downer in $T - u$.

We show that Case 1 cannot occur. In order to obtain a contradiction, suppose that u is Parter in $T - v$ and v is neutral in $T - u$. First note that, considering the unique path T' in T connecting u and v, since u is neutral and v is a downer, there must exist on this path T' an m-Parter vertex (part **4** of Corollary 2.5.4). Let w be the m-Parter vertex on this path T' closest to u. Since w is m-Parter, by Theorem 2.5.1 u is still neutral in $T - w$ and v is still a downer in $T - w$, so that

$$k \xrightarrow[\text{remove } w]{} k+1 \xrightarrow[\text{remove } u]{} k+1 \xrightarrow[\text{remove } v]{} k,$$

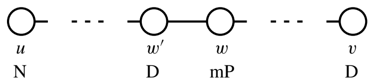
i.e., $\mathrm{m}_{T-\{u,v,w\}}(\lambda) = k$.

We consider now $T - v$ and, by our hypothesis, u is Parter in $T - v$. If w has a downer branch in $T - v$ that does not include u, then, after the removal of the Parter vertex u from $T - v$, w remains Parter in $T - \{u, v\}$ (by virtue of having a downer branch), so that

$$k \xrightarrow[\text{remove } v]{} k-1 \xrightarrow[\text{remove } u]{} k \xrightarrow[\text{remove } w]{} k+1,$$

i.e., $\mathrm{m}_{T-\{u,v,w\}}(\lambda) = k + 1$, which is a contradiction. We therefore only need to consider the case when w has exactly two downer branches in T, one including u and another including v.

We remove w from T. Because w is m-Parter, by Theorem 2.5.1 vertex u remains neutral in $T - w$ and, by part **1** of Theorem 2.5.3, the two downer neighbors of w in the path T' must be downers in $T - w$ and in T. Let w' be the downer vertex adjacent to w in the path connecting u and w.

$T' =$ 

Since w' is a downer and u is neutral, we have, by part **3** of Corollary 2.5.4, $w' \neq u$. By part **4** of Corollary 2.5.4, some other m-Parter vertex must separate w' from u in the path T', which is a contradiction because w is the m-Parter vertex closest to u in that path. Thus, Case 1 cannot occur, completing the proof. $\square$

Remark. The converse of Theorem 2.5.7 is not valid. In fact, if u is neutral in $T - v$ and v is a downer in $T - u$ (u and v nonadjacent) and λ is the relevant eigenvalue, with multiplicity k, then the multiplicity of λ in $T - \{u, v\}$ is

$$k \xrightarrow[\text{remove } v]{} k + t_0 \xrightarrow[\text{remove } u]{} k + t_0,$$

in which $t_0 \in \{-1, 0, 1\}$ (depending on whether v is a downer, neutral or Parter in T) and

$$k \xrightarrow[\text{remove } u]{} k + t_1 \xrightarrow[\text{remove } v]{} k + t_1 - 1,$$

in which $t_1 \in \{-1, 0, 1\}$ (depending on whether u is a downer, neutral or Parter in T). Since $k + t_0 = k + t_1 - 1$, i.e., $t_0 = t_1 - 1$, and $t_0, t_1 \in \{-1, 0, 1\}$, we conclude that we have two possibilities: $t_0 = -1$ and $t_1 = 0$, i.e., u is neutral in T and v is a downer in T (Case 1, which we know can occur), or $t_0 = 0$ and $t_1 = 1$, i.e., u is Parter in T and v is neutral in T (Case 2). Part **1** of Example 2.5.5 shows that Case 2 can occur. Note that, since u is Parter in T, but not Parter in $T - v$, it means that u has only one downer neighbor in T, so that u must be s-Parter in T.

We present now Table 2.1, in which we give all the possible changes in status of one identified vertex when another identified vertex is removed. It is now clear that it can make a difference whether the two vertices are adjacent, so we need to include both possibilities. When one of the vertices is Parter, it may also matter whether it is s-Parter or m-Parter.

Note that there are five differences between the Adjacent and Nonadjacent columns, all of the form “No” then “Yes.” Thus, differences occur only when a situation cannot happen adjacently, but can nonadjacently.

Table 2.1 *The Possible Changes in Status of a Vertex*

	i	j	Status of i when j is removed	Possible? (Adjacent)	Possible? (Nonadjacent)
1.	P	P	P	Yes	Yes
2.	P	P	N	No	No
3.	P	P	D	Yes*	Yes*
4.	P	N	P	Yes	Yes
5.	P	N	N	Yes*	Yes*
6.	P	N	D	No	No
7.	P	D	P	Yes**	Yes
8.	P	D	N	No	No
9.	P	D	D	No	No
10.	N	P	P	No	No
11.	N	P	N	Yes	Yes
12.	N	P	D	Yes*	Yes*
13.	N	N	P	No	Yes*
14.	N	N	N	Yes	Yes
15.	N	N	D	No	No
16.	N	D	P	No	No
17.	N	D	N	No	Yes
18.	N	D	D	No	No
19.	D	P	P	No	No
20.	D	P	N	No	No
21.	D	P	D	Yes**	Yes
22.	D	N	P	No	No
23.	D	N	N	No	No
24.	D	N	D	No	Yes
25.	D	D	P	No	Yes*
26.	D	D	N	Yes	Yes
27.	D	D	D	No	Yes

* Occurs only when the relevant Parter vertices are s-Parter.
** Only occurs if the Parter vertex is m-Parter in T.

We now present five arguments that justify some results stated in Table 2.1. In all arguments for a given tree T, there is an implicit matrix $A \in \mathcal{S}(T)$ and an identified eigenvalue λ of A. We will often refer to λ as an eigenvalue of T (resp. an induced subgraph T' of T), meaning an eigenvalue of A (resp. $A[T']$). We also write $\mathrm{m}_T(\lambda)$ (resp. $\mathrm{m}_{T'}(\lambda)$), meaning the multiplicity of λ as an eigenvalue of a given matrix $A \in \mathcal{S}(T)$ (resp. $A[T']$).

Argument 1 *The multiplicity of λ as an eigenvalue of $T - \{i, j\}$ has a unique value; thus, if we remove i and then j, the multiplicity of λ in the resulting subgraph should be the same as if we remove j and then i.*

This argument justifies some rows of the table. For example, consider row 6, in which i is Parter and j is neutral. Let $m_T(\lambda) = k$ and suppose that i is a downer in $T - j$. Then the possibilities for the multiplicity of λ in $T - \{i, j\}$ are

$$k \xrightarrow[\text{remove } j]{} k \xrightarrow[\text{remove } i]{} k - 1$$

and

$$k \xrightarrow[\text{remove } i]{} k + 1 \xrightarrow[\text{remove } j]{} k + 2,\ k + 1,\ \text{or } k.$$

There are different values for $m_{T-\{i,j\}}(\lambda)$ depending on the order in which vertices i and j are removed from T, a contradiction.

Argument 2 *If i an j are adjacent vertices of T and i is a downer in $T - j$, then by Theorem 2.4.1, j must be originally Parter in T.*

Argument 3 *If i an j are adjacent vertices of T and i is Parter in $T - j$, then by Lemma 2.5.6, i must be originally Parter in T.*

Argument 4 *Here we show that some cases are identical to each other. Consider row 17, in which i is neutral, j is a downer and i is neutral in $T - j$. If $m_T(\lambda) = k$, then the multiplicity of λ in $T - \{i, j\}$ is*

$$k \xrightarrow[\textit{remove } j]{} k - 1 \xrightarrow[\textit{remove } i]{} k - 1.$$

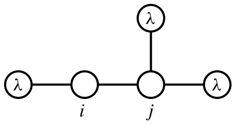
Since i is originally neutral, j must stay downer in $T - i$ to have $\mathrm{m}_{T-\{i,j\}} = k - 1$. This case is equivalent to the case in row 24, in which i is a downer, j is neutral and i is a downer in $T - j$ (simply switch the labels on i and j).

Argument 5 *By Theorem 2.4.1, if i is Parter in T and (1) j does not belong to a downer branch of T at i or (2) i is* m*-Parter, then i is still Parter in $T - j$. Moreover, if j is also Parter in T, then $\mathrm{m}_{T-\{i,j\}} = \mathrm{m}_T + 2$, and so j must also be Parter in $T - i$.*

We justify now the results stated in Table 2.1. The five arguments above are each used multiple times.

Proofs of Adjacent Cases. For each case, we either give an example of a tree with an assignment that justifies a "Yes" entry or prove that such a situation cannot occur. The classification of each vertex is with respect to the eigenvalue λ. In each displayed assignment, if a vertex i is not labeled with λ, we mean that $A[i] \neq [\lambda]$.

1.



2. By Theorem 2.4.1, i is Parter if and only if there is a downer branch at i. If the removal of j from T makes i neutral in $T - j$, then the downer branch at i in T must have been the branch including j, and thus, j becomes a downer vertex in $T - i$. Using Argument 1, we have a discrepancy in $\mathrm{m}_{T-\{i,j\}}(\lambda)$ because

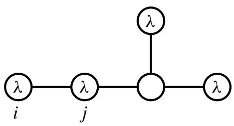
$$k \xrightarrow[\text{remove } j]{} k+1 \xrightarrow[\text{remove } i]{} k+1$$

and

$$k \xrightarrow[\text{remove } i]{} k+1 \xrightarrow[\text{remove } j]{} k,$$

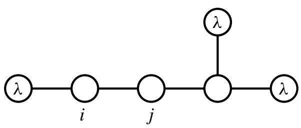
which is a contradiction.

3.

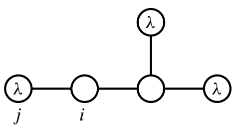


By Argument 5 and Theorem 2.5.1, both i and j must be s-Parter.

4.



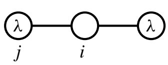
5.



By Argument 5, i must be s-Parter.

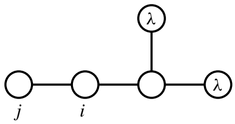
6. Use Argument 1 or Argument 2.

7.

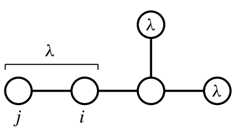


By Theorem 2.5.2, vertex j is still a downer in $T - i$ so that the branch of T at i including j is a downer branch T' at i. Since i is Parter in $T - j$, by Theorem 2.4.1 there must exist in $T - j$ a downer branch T'' at i. Since T' and T'' are distinct branches of T at i, we conclude that i must be m-Parter. (Alternatively, by part **1** of Theorem 2.5.3, i must be m-Parter.)

8. Use Argument 1. (Alternatively, use Theorem 2.5.2.)
9. Use Argument 1 or Argument 2. (Alternatively, use Theorem 2.5.2.)
10. Use Argument 1.
11. By Argument 4, **11.** is equivalent to **4.** so that we can switch the labels for i and j on the example graph for **4.** and obtain an example here.
12. By Argument 4, **12.** is equivalent to **5.** so that we can switch the labels for i and j on the example graph for **5.** and obtain an example here.
13. Use Argument 3.
14.



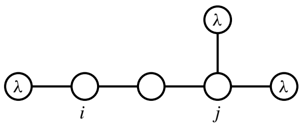
15. Use Argument 2.
16. Use Argument 3.
17. Use **24.** below. By Argument 4, **17.** is equivalent to **24.**. (Alternatively, use part **3** of Corollary 2.5.4.)
18. Use Argument 1 or Argument 2.
19. Use Argument 1 or Argument 3. (Alternatively, use Theorem 2.5.2.)
20. Use Argument 1. (Alternatively, use Theorem 2.5.2.)
21. By Argument 4, **21.** is equivalent to **7.** so that we can switch the labels for i and j on the example graph for **7.** and obtain an example here.
22. Use Argument 1 or Argument 3. (Alternatively, use part **3** of Corollary 2.5.4.)
23. By Argument 4, **23.** is equivalent to **16.**. (Alternatively, use part **3** of Corollary 2.5.4.)
24. Use Argument 2. By Argument 4, **24.** is equivalent to **17.**. (Alternatively, use part **3** of Corollary 2.5.4.)
25. Use Argument 3.
26.



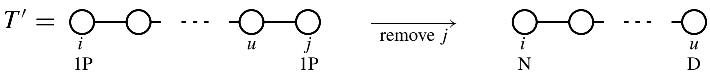
27. Use Argument 2. □

Proofs of Nonadjacent Cases. We use the same methods and conventions as in the proofs of the adjacent cases.

1.

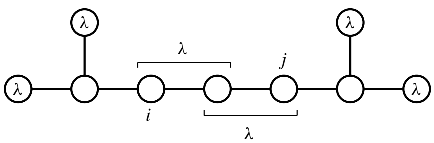


2. Assume that this case can occur: let there be a tree T with nonadjacent vertices i and j such that i is Parter in T, j is Parter in T and i is neutral in $T - j$. Since i is Parter in T, but not in $T - j$, by Argument 5 i must be s-Parter in T, and the unique downer branch at i in T is the branch including j. By Argument 1, we also conclude that j is neutral in $T - i$ so that j must be s-Parter in T and the unique downer branch at j in T is the branch including i. Let u be the downer neighbor of j in T, and we focus on the unique (because T is a tree) path T' connecting i and j.



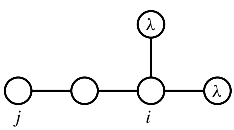
When we remove i from $T - j$, by Theorem 2.5.7 u is still a downer in $T - \{i, j\}$, so that the branch at j in $T - i$ including u is a downer branch at j in $T - i$, making j Parter in $T - i$, a contradiction.

3.

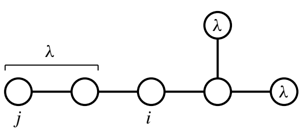


By Argument 5, both i and j must be s-Parter.

4.



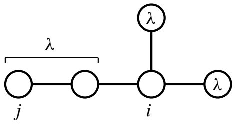
5.



By Argument 5, i must be s-Parter.

6. Use Argument 1.

7.



8. Use Argument 1. (Alternatively, use Theorem 2.5.2.)

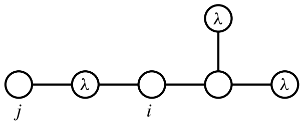
9. Use Argument 1. (Alternatively, use Theorem 2.5.2.)

10. Use Argument 1.

11. By Argument 4, **11.** is equivalent to **4.**. Thus, we can switch the labels for i and j on the example graph for **4.** and obtain an example here.

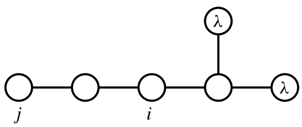
12. By Argument 4, **12.** is equivalent to **5.**. Thus, we can switch the labels for i and j on the example graph for **5.** and obtain an example here.

13.



Since i has no downer branch in T, it can have at most one downer branch in $T - j$ (the branch that initially contained j). So in this case, i is always s-Parter in $T - j$.

14.



15. Assume that this case can occur: let there be a tree T with nonadjacent vertices i and j, such that i and j are both neutral in T and i is a downer in $T - j$. Let λ be the relevant eigenvalue, with multiplicity k in T. Under our hypothesis, $\mathrm{m}_{T-\{i,j\}}(\lambda) = k - 1$, and, by Argument 1, we conclude that j is also a downer in $T - i$.

Since T is a tree, there is a unique path T' connecting i and j in T. We focus on this path T' and we start by showing that, under our hypothesis, each vertex on this path T' is neutral in T.

Assume that an m-Parter vertex w lies on the path T' connecting i and j. By Theorem 2.5.1, i and j remain neutral in $T - w$. Since i and j are in different disconnected components of $T - w$, we have $\mathrm{m}_{T-\{i,j,w\}}(\lambda) = k + 1$, which implies, by interlacing, that $\mathrm{m}_{T-\{i,j\}}(\lambda) \geq k$, a contradiction. Thus, there are no m-Parter vertices on the path T' connecting i and j; by part **3** of Corollary 2.5.4, we also know that there are no downer vertices on T'.

Assume now that an s-Parter vertex v lies on the path T'. If the (unique) downer branch at v (in T) does not contain i or j, it is easy to check that we may treat v as if it were m-Parter. So, *wlog*, we assume that j lies in the downer branch at v. (We can assume this because, under our hypothesis, i is a downer in $T - j$ and j is a downer in $T - i$.)

Since i is neutral in T and v is Parter in $T - i$ (by virtue of the unique downer branch at v to be the one including j), we conclude, by Argument 1, that (a) i is neutral in $T - v$.

The possibilities for $\mathrm{m}_{T-\{i,j,v\}}(\lambda)$ are

$$k \xrightarrow[\text{remove } i]{} k \xrightarrow[\text{remove } v]{} k+1 \xrightarrow[\text{remove } j]{} k+2,\ k+1,\ \text{or } k$$

and, under our hypothesis, i is a downer in $T - j$ so that

$$k \xrightarrow[\text{remove } j]{} k \xrightarrow[\text{remove } i]{} k-1 \xrightarrow[\text{remove } v]{} k,\ k-1,\ \text{or } k-2.$$

By Argument 1, we conclude that $\mathrm{m}_{T-\{i,j,v\}}(\lambda) = k$, which implies that (b) v must be necessarily Parter in $T - \{i, j\}$.

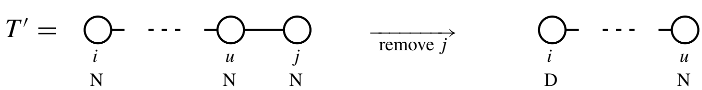
Since (a) i is neutral in $T - v$, and i and j are in different disconnected components of $T - v$, we have

$$k \xrightarrow[\text{remove } j]{} k \xrightarrow[\text{remove } v]{} k+t_0 \xrightarrow[\text{remove } i]{} k+t_0,$$

in which $t_0 \in \{-1, 0, 1\}$. Because $\mathrm{m}_{T-\{i,j,v\}}(\lambda) = k$, we get $t_0 = 0$ and, therefore, (c) v is neutral in $T - j$.

By (c), v is neutral in $T - j$, and, under our hypothesis, i is a downer in $T - j$. However, by (b), the removal of i (a downer) from $T - j$ changes the status of v (from neutral to Parter), a contradiction by Theorem 2.5.7. Thus, the path T' connecting i and j contains no Parter or downer vertices and, thus, consists only of neutral vertices in T.

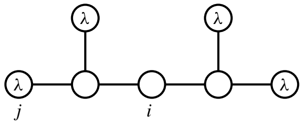
Let u be the vertex on T' adjacent to j. By our hypothesis, i is a downer in $T - j$, and, by rows 13–15 (adjacent cases) in the table, u is neutral in $T - j$.



If i and u are adjacent, we have a contradiction by part **3** of Corollary 2.5.4. Also by part **4** of Corollary 2.5.4, there must exist an m-Parter vertex w on the path connecting i and u. But such a vertex w would be at least s-Parter on the path T' connecting i and j, a contradiction. Therefore, this case is impossible.

16. By Theorem 2.5.7, this case is impossible.

17.



18. Use Argument 1. (Alternatively, use Theorem 2.5.7.)

19. Use Argument 1. (Alternatively, use Theorem 2.5.2.)

20. Use Argument 1. (Alternatively, use Theorem 2.5.2.)

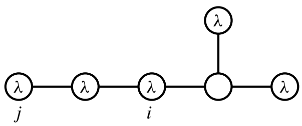
21. By Argument 4, **21.** is equivalent to **7.**. Thus, we can switch the labels for i and j on the example graph for **7.** and obtain an example here.

22. Use Argument 1. (Alternatively, use Theorem 2.5.7.)

23. By Theorem 2.5.7, this case is impossible.

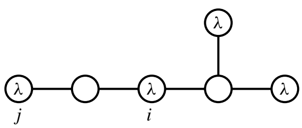
24. By Argument 4, **24.** is equivalent to **17.**. Thus, we can switch the labels for i and j on the example graph for **17.** and obtain an example here.

25.

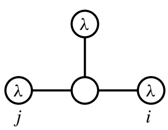


Since i has no downer branch in T, it can have at most one downer branch in $T - j$ (the branch in the direction of j). So, in this case, i is always s-Parter in $T - j$.

26.



27.



□

2.6 At Least Two Multiplicities Equal to 1

Another important fact that constrains multiplicity lists is that, for a tree, each list must contain two multiplicities equal to 1. Moreover, in an ordered list, the first and the last multiplicities must be 1. There are remarkably varied proofs of this fact, and we give two. One of them is based upon the Perron-Frobenius theory of nonnegative matrices (Section 0.1.5), in which it also makes clear that there can be no Parter vertices for the largest and smallest eigenvalues, so that every vertex is a downer for these two. The facts about two 1s in each list are also consequence of the relative position work in Section 6.8.

Earlier we mentioned that, for a tree T, $\mathcal{L}(T)$ and $\mathcal{L}_o(T)$ do not depend on whether $\mathcal{S}(T)$ or $\mathcal{H}(T)$ was in the background. This fact also becomes clear here. Note that if D is an invertible diagonal matrix and A is Hermitian, then $\mathrm{G}(D^*AD) = \mathrm{G}(A)$. The key is that if $\mathrm{G}(A)$ is a tree, then D may be chosen to be a diagonal unitary matrix such that D^*AD is real (and thus, symmetric), even with nonnegative off-diagonal entries.

Lemma 2.6.1 *Let T be a tree and $A \in \mathcal{H}(T)$. Then there is a diagonal unitary similarity of A, all of whose off-diagonal entries are nonnegative. The diagonal entries are the same as the necessarily real diagonal entries of A.*

Proof. The second claim is obvious for any diagonal similarity of any matrix.

For the first, we determine the matrix $D = \operatorname{diag}\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right)$, so that D^*AD realizes the claim, by induction on n. In case $n = 2$, let $\theta_1 = 0$ and $\theta_2 = \theta_1 - \arg a_{12}$ to verify the claim. Then for the induction step, label a pendent vertex of T 1 and its neighbor 2. Now begin choosing the θs as in the case $n = 2$ and realize that the removal of vertex 1 leaves a tree T_1 on $n - 1$ vertices covered by the induction hypothesis. Multiply the diagonal matrix D_1 for T_1 by $e^{i\theta}$ so that its first diagonal entry is $e^{i\theta_2}$ for the θ_2 just determined. Matrix multiplication then completes the proof. $\square$

We also present the following, as an alternative to Lemma 2.6.1, with a different proof.

Lemma 2.6.2 *Let T be a tree and $A \in \mathcal{H}(T)$. Then A is unitarily similar to a $B \in \mathcal{S}(T)$ whose diagonal entries are the same as the necessarily real diagonal entries of A. Moreover, A is also unitarily similar to a sign-symmetric matrix whose graph is T.*

Proof. Let $A = (a_{ij}) \in \mathcal{H}(T)$ and a_{uv} be any nonzero off-diagonal entry of A. By the bridge formula for the expansion of the characteristic polynomial of A (formula (2) in Section 0.2.4), we have

$$p_A(t) = p_{A[T_u]}(t)p_{A[T_v]}(t) - |a_{uv}|^2\, p_{A[T_u-u]}(t)p_{A[T_v-v]}(t),$$

in which $|a_{uv}|^2 = a_{uv} \times a_{vu} = a_{uv} \times \overline{a_{uv}}$ and T_u (resp. T_v) is the component of T resulting from deletion of v (resp. u) and containing u (resp. v). Replacing in A the entries a_{uv} and a_{vu} by real numbers b and c, respectively, such that $|a_{uv}|^2 = b \times c$, in particular replacing a_{uv} and a_{vu} by $|a_{uv}|$ (or both by $-|a_{uv}|$), we obtain a matrix $A' \in \mathcal{H}(T)$ with the same characteristic polynomial as A. Since a_{uv} is any nonzero off-diagonal entry of A, we conclude the claimed result. $\square$

From either Lemma 2.6.1 or Lema 2.6.2, we obtain

Corollary 2.6.3 *For a tree T, the spectra occurring among matrices in $\mathcal{S}(T)$ are the same as those occurring in $\mathcal{H}(T)$. Thus, to determine $\mathcal{L}(T)$ or $\mathcal{L}_o(T)$, it suffices to consider $\mathcal{S}(T)$.*

We may actually say more. Since the diagonal entries of $A \in \mathcal{S}(T)$ are real, there is a translation of A ($A \rightarrow A + \alpha I$, α real) so that its diagonal entries are positive. As translation adds α to every eigenvalue and thus does not change the ordered multiplicities of the eigenvalues, it follows from Lemma 2.6.1 that all ordered multiplicity lists for T are realizable by nonnegative (in fact, primitive) matrices in $\mathcal{S}(T)$. This means that Perron-Frobenius theory may be applied. We then have

Theorem 2.6.4 *If T is a tree, the largest and smallest eigenvalue of each $A \in \mathcal{S}(T)$ have multiplicity* 1. *Moreover, for any vertex v of T, the largest and smallest eigenvalues of the principal submatrix $A(v)$ lie strictly between those of A, i.e., the largest and smallest eigenvalue of a matrix $A \in \mathcal{S}(T)$ cannot occur as an eigenvalue of a submatrix $A(v)$, for any vertex v of T.*

We present two proofs of this important result, each giving different insights.

Proof 1. We verify the claims regarding the largest eigenvalue. The ones regarding the smallest follow by considering $-A \in \mathcal{S}(T)$.

Since there is a primitive nonnegative matrix $B \in \mathcal{S}(T)$ with the same ordered multiplicities as A, the largest eigenvalue of A has multiplicity 1 by **2** of Section 0.1.5. By **3** of Section 0.1.5, the largest eigenvalue of $A(v)$ is strictly smaller than that of A. $\square$

Proof 2. Let T be a tree and λ be the largest (resp. smallest) eigenvalue of a matrix $A \in \mathcal{S}(T)$. Suppose that there is a vertex v of T such that λ is an eigenvalue of $A(v)$. By Theorem 2.3.2, there is a vertex u of T such that $\mathrm{m}_{A(u)}(\lambda) = \mathrm{m}_A(\lambda) + 1$. But from the interlacing inequalities (Section 0.1.2), since λ is the largest (resp. smallest) eigenvalue of A, for any vertex i of T, $\mathrm{m}_{A(i)}(\lambda) \leq \mathrm{m}_A(\lambda)$, which is a contradiction. Thus, λ cannot occur as an eigenvalue of any submatrix $A(i)$ of A. Therefore, $\mathrm{m}_A(\lambda) = 1$. $\square$

It follows from Theorem 2.6.4 that for the largest or smallest eigenvalue of $A \in \mathcal{S}(T)$, every vertex of T is a downer. Thus, a branch of a tree at a vertex v, having λ as its largest or smallest eigenvalue, is automatically a downer branch for λ at v. It then follows from Theorem 2.4.1 that

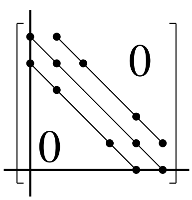
Corollary 2.6.5 *Let T be a tree and $A \in \mathcal{S}(T)$. If λ is the largest or smallest eigenvalue of at least one of the direct summands of $A(v)$, then* $\mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_A(\lambda) + 1$.

2.7 Eigenstructure of Tridiagonal Hermitian Matrices and Their Principal Submatrices

If T is a path on n vertices, then any $A \in \mathcal{S}(T)$ is permutationally similar to an (irreducible) tridiagonal matrix and conversely. Simply number the vertices consecutively along the path, beginning with "1" at one end. So the special case of our problem in which T is a path is just the study of the eigenstructure of tridiagonal matrices. This, historically, is a key problem in many areas and has long been studied. Some areas are recurrence relations, orthogonal polynomials, numerical analysis/computation, continued fractions, differential equations (PDE, ODE), etc. The two known, classical facts that are relevant here are the following.

Theorem 2.7.1 *If T is a path on n vertices, the only multiplicity list in $\mathcal{L}(T)$ is (1^n), i.e., all the eigenvalues of $A \in \mathcal{S}(T)$ are distinct.*

Proof. For Hermitian matrices, the algebraic and geometric multiplicities are the same. Up to permutation similarity, A, and thus the characteristic matrix $A - \lambda I$, is tridiagonal. Since A is irreducible, the result of deletion of the first column and last row of $A - \lambda I$ is an $(n-1)$-by-$(n-1)$ lower triangular matrix with nonzero diagonal, which is, therefore, nonsingular and has rank $n - 1$.



Since the rank of a submatrix cannot be greater than that of the matrix from which it comes, $\operatorname{rank}(A - \lambda I) \geq n - 1$, and, as A is Hermitian, the geometric (and, then, algebraic) multiplicity of no eigenvalue exceeds 1. □

Remark. The above proof also shows that for any irreducible tridiagonal matrix (not necessarily Hermitian), the geometric multiplicity of any eigenvalue is at most 1. The algebraic multiplicity may be as high as n. Theorem 2.7.1 also follows from the Parter-Wiener, etc. theorem (Theorem 2.3.2), as the existence of a multiple eigenvalue implies the existence of an HDV.

Theorem 2.7.2 *If A is an n-by-n tridiagonal irreducible Hermitian matrix, then $\sigma(A) \cap \sigma(A(1)) = \sigma(A) \cap \sigma(A(n)) = \emptyset$, i.e., all the interlacing inequalities are strict.*

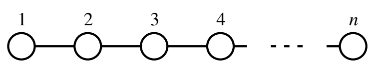
Proof. We induct on n. For $n \leq 3$, the validity of the claim is justified in Example 2.3.5. Let $n > 3$ and assume now the claim for tridiagonal matrices of size less than n. By symmetry, we need only verify the claim for $A(n)$. Suppose to the contrary that $\lambda \in \sigma(A) \cap \sigma(A(n))$. By the induction hypothesis, $\lambda \notin \sigma(A(\{n-1, n\}))$, so that $n-1$ is a downer vertex for λ at n. By Theorem 2.4.1, then, $\mathrm{m}_{A(n)}(\lambda) = 1 + 1$, a contradiction to Theorem 2.7.1, as the graph of $A(n)$ is again a path. $\square$

In terms of the graph of a tridiagonal matrix, the last result may be stated as follows: "Let T be a path whose pendent vertices are the vertices u_1 and u_n. If $A \in \mathcal{S}(T)$ then the eigenvalues of $A[T - u_1]$ (resp. $A[T - u_n]$) strictly interlace those of A."

If a branch of a tree T at v is a path and the neighbor of v in this branch is a degree 1 vertex of this path, we call such a branch a ***pendent path*** at v. By Theorem 2.7.2, a pendent path with λ as an eigenvalue must be a downer branch for λ, so that we may make the following observation using Theorem 2.4.1.

Corollary 2.7.3 *Let T be a tree and $A \in \mathcal{S}(T)$. If at least one of the direct summands of $A(v)$ has λ as an eigenvalue, and its graph is a pendent path at v, then* $\mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_A(\lambda) + 1$.

Suppose now that we consider a path, with vertices labeled $1, 2, \ldots, n$,

$T =$ 

and that we remove vertex i. Now, interlacing equalities are possible if $1 < i < n$. The general results, which are new because of the multiplicities literature, e.g., [JL-DS03a], are

Theorem 2.7.4 *If T is the above path, $i \in \{1, \ldots, n\}$ and $A \in \mathcal{S}(T)$, then $\lambda \in \sigma(A) \cap \sigma(A(i))$ if and only if $1 < i < n$ and i is Parter for λ, with $\lambda \in \sigma\left(A[\{1, \ldots, i-1\}]\right)$ and $\lambda \in \sigma\left(A[\{i+1, \ldots, n\}]\right)$.*

Proof. If $\lambda \in \sigma(A) \cap \sigma(A(i))$ then, by Theorem 2.7.2, $1 < i < n$ and, thus, λ is an eigenvalue of at least one of the two summands of $A(i)$. By Corollary 2.7.3, we have $\mathrm{m}_{A(i)}(\lambda) = \mathrm{m}_A(\lambda) + 1$. Since T is a path $\mathrm{m}_{A(i)}(\lambda) = 2$ and, because the two branches of T at i are paths, $\mathrm{m}_{A[\{1,\ldots,i-1\}]}(\lambda) = \mathrm{m}_{A[\{i+1,\ldots,n\}]}(\lambda) = 1$.

For the converse, since λ is an eigenvalue of the two summands of $A(i)$, $1 < i < n$, by interlacing we conclude that λ is still an eigenvalue of A. $\square$

From Theorems 2.7.4 and 2.7.1, we immediately have the following.

Corollary 2.7.5 *If T is the above path, then the maximum number of possible interlacing equalities resulting from the removal of vertex i is* $\min\{i-1, n-i\}$.

By Theorems 2.7.2 and 2.7.4 (or 2.3.2), when T is a path and $A \in \mathcal{S}(T)$, if $\lambda \in \sigma(A) \cap \sigma(A(v))$, then v must be a degree 2 vertex of T and v is necessarily Parter for λ.

> **Remark.** Note that this property characterizes the path. It is the only tree for which $\lambda \in \sigma(A) \cap \sigma(A(v))$ implies that v is Parter for λ in T.

Theorem 2.7.6 *Let T be a tree and v be a vertex of T such that, for any $A \in \mathcal{S}(T)$, if $\lambda \in \sigma(A) \cap \sigma(A(v))$ then $\mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_A(\lambda) + 1$. Then T is a path and v is a degree 2 vertex of T.*

Proof. Suppose that T is a tree but not a path, i.e., T is a tree with at least one HDV. We show that there exists a vertex v of T and a matrix $A \in \mathcal{S}(T)$ with an eigenvalue $\lambda \in \sigma(A) \cap \sigma(A(v))$ satisfying $\mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_A(\lambda) - 1$. In order to construct A, consider an HDV u of T whose removal leaves $k \geq 3$ components $T_1, \ldots, T_k$. For each of these components, construct $A_i \in \mathcal{S}(T_i)$ whose smallest eigenvalue is λ and let A be any matrix in $\mathcal{S}(T)$ with the submatrices A_i in appropriate positions. By Theorem 2.6.4, the smallest eigenvalue of a matrix whose graph is a tree does not occur as an eigenvalue of any principal submatrix of size one smaller. It means that any T_i is a downer branch at u for λ. Since $\mathrm{m}_{A(u)}(\lambda) = k$, by Theorem 2.4.1 it follows that $\mathrm{m}_A(\lambda) = k - 1 \geq 2$ and, by interlacing, $\lambda \in \sigma(A) \cap \sigma(A(v))$ for any vertex v of T.

Considering a vertex v of T, with $v \neq u$, let us show that $\mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_A(\lambda) - 1$. Observe that λ occurs as an eigenvalue of only one of the direct summands of $A(v)$, corresponding to the component T' of $T - v$ containing the vertex u. Since λ is now an eigenvalue of $k-1$ components of $A[T' - u]$ (in each one with multiplicity 1), again, by Theorem 2.4.1, it follows that $\mathrm{m}_{A[T']}(\lambda) = \mathrm{m}_{A[T'-u]}(\lambda) - 1 = k - 2$. Since $\mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_{A[T']}(\lambda)$, we have $\mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_A(\lambda) - 1$. □

2.8 Nontrees

In [JL-D06], it was shown that there is no analog for any nontrees of all the nice theory about Parter vertices in trees. This does not mean that there cannot be Parter vertices for some Hermitian matrices whose graphs are not trees, but there always exist matrices with multiple eigenvalues for which every vertex is actually a downer. Since M-matrices are used in the proof of the following result [JL-D06], see [HJ91, Chapter 2] as a general reference for this topic.

Theorem 2.8.1 *Let G be any connected graph that is not a tree. Then there is a matrix $A \in \mathcal{S}(G)$, with a multiple eigenvalue λ, such that every vertex of G is a downer for λ.*

Proof. We construct a positive semidefinite (PSD) matrix $A \in \mathcal{S}(G)$ such that $\operatorname{rank} A = k \leq n-2$, i.e., $\mathrm{m}_A(0) = n-k \geq 2$, and such that for any i, $1 \leq i \leq n$, $\operatorname{rank} A(i) = k$, i.e., $\mathrm{m}_{A(i)}(0) = (n-1)-k = \mathrm{m}_A(0) - 1$. Thus, in this case, the eigenvalue $\lambda = 0$ will satisfy the claims of the theorem.

First, note that for any G, there is a PSD matrix $A \in \mathcal{S}(G)$ that is not positive definite (PD): choose any $A' \in \mathcal{S}(G)$ and let $A = A' - \lambda_{\min A'} I$, in which $\lambda_{\min A'}$ denotes the smallest eigenvalue of A'.

We first suppose that G is connected (and not a tree). The case of any not connected G will be seen to be straightforward later.

Since G is connected and not a tree, there are vertices i and j such that (1) $\{i, j\}$ is an edge of G and (2) there is a path in G, not involving $\{i, j\}$, from i to j: $\{i, p_1\}, \{p_1, p_2\}, \ldots, \{p_k, j\}$. *Wlog*, suppose $i = 1$ and $j = 2$. We will construct the desired matrix A as follows:

$$A = \begin{bmatrix} A_1 & B \\ B^{\mathrm{T}} & A_2 \end{bmatrix}.$$

Let A_2 be a PD M-matrix with $A_2 \in \mathcal{S}(G')$ for $G' = G - \{1, 2\}$, the subgraph of G induced by vertices $\{3, 4, \ldots, n\}$. Such an A_2 may be easily found by choosing negative off-diagonal entries in the positions allowed by G' (0 off-diagonal elements otherwise) and then choosing positive diagonal entries to achieve strict diagonal dominance. Let B be nonnegative with positive entries corresponding to the edges of G and 0s elsewhere. Finally, let $A_1 = BA_2^{-1}B^{\mathrm{T}}$, a nonnegative PSD, 2-by-2 matrix. By Schur complements (see, e.g., [Ca]), $\operatorname{rank} A = \operatorname{rank} A_2 + \operatorname{rank}\left(A_1 - BA_2^{-1}B^{\mathrm{T}}\right) = \operatorname{rank} A_2 + \operatorname{rank} 0 = n-2+0 = n-2$, and A is PSD (in fact, the interlacing inequalities applied to the eigenvalues of A and A_2 show that A cannot have negative eigenvalues) of rank $n-2$.

Now it suffices to show that the two off-diagonal entries of A_1 are positive, so that $A \in \mathcal{S}(G)$. But B has a positive entry in the $(1, p_1)$ position and in the $(2, p_k)$ position. Moreover, because A_2 is an M-matrix, $A_2^{-1} \geq 0$, and as there is a path in G' from p_1 to p_k, the (p_1, p_k) (and (p_k, p_1)) entry of A_2^{-1} is positive. By matrix multiplication, the $(1, 2)$ entry of the symmetric matrix $BA_2^{-1}B^{\mathrm{T}}$ is then positive.

We turn now to the second claim in the connected case: $\operatorname{rank} A(i) = \operatorname{rank} A$, for $1 \leq i \leq n$ and the A just defined.

If $i \in \{1, 2\}$, then $A(i)$ contains A_2 as a principal submatrix and, as $\operatorname{rank} A = \operatorname{rank} A_2$, $\operatorname{rank} A(i) = \operatorname{rank} A$ as claimed. On the other hand, if $i \in \{3, 4, \ldots, n\}$, then $\operatorname{rank} A_2(i) = n-3$ and, as $A_2(i)^{-1} \leq A_2^{-1}(i)$ (entry-wise, because A_2 is an M-matrix; see, e.g., [JSm, Theorem 2.1]), we have (also entry-wise) $B(i)A_2(i)^{-1}B^{\mathrm{T}}(i) \neq (BA_2^{-1}B^{\mathrm{T}})(i) = A_1$. Here for i, we retain the numbering in A, and by $B(i)$ (resp. $B^{\mathrm{T}}(i)$), we mean B with only its ith column deleted (resp. B^{T} with only its ith row deleted). Thus, the Schur complement $A_1 - B(i)A_2(i)^{-1}B^{\mathrm{T}}(i) \neq 0$ and its rank must be 1. We conclude that $\operatorname{rank} A(i) = n-3+1 = n-2$ in this case as well, and the proof is complete in the case of connected graphs.

Finally, if G is not connected, A may be constructed for each connected component as follows. If the component is an isolated vertex, the corresponding submatrix is zero. If the component is a tree, let the corresponding principal submatrix be any PSD matrix of rank one less than the number of vertices in the graph that comprises that component. It follows from Theorem 2.6.4 that any proper principal submatrix of such a submatrix is then PD. If the component is neither a vertex nor a tree (a connected graph that is not a tree), construct the corresponding principal submatrix as in the earlier part of this proof. It is then easily checked that both parts of the conclusion of the theorem hold for such an A, completing the proof. □

2.9 Tree-Like Vertices

For a general (undirected) graph G and a vertex v of G, if there is a neighbor u_i of v such that the edge $\{v, u_i\}$ is a ***bridge*** or ***cut-edge*** (the removal of $\{v, u_i\}$ from G increases the number of components), we still denote by G_i the component of $G - v$ containing u_i and call G_i a ***pendent component*** of G at v. (If G_i is a tree, then G_i is the branch of G at v containing u_i.) We also denote by G_v the component of $G - u_i$ containing v. If $A \in \mathcal{H}(G)$ and u_i is a downer vertex in G_i (for the eigenvalue λ of $A[G_i]$), then G_i is said to be a ***downer component*** at v (for λ relative to A).

The bridge formula for the expansion of the characteristic polynomial of $A \in \mathcal{H}(G)$ (formula (2) in Section 0.2.4) does not depend on the graph of each of the summands $A[G_v]$ and $A[G_j]$; it only depends on the existence of the bridge $\{v, u_j\}$. So if λ is an eigenvalue of A, by the proof of Lemma 2.3.8, a sufficient condition to have $\mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_A(\lambda) + 1$ for a given vertex v is the existence of a downer component at v for λ. This may be stated as follows.

Theorem 2.9.1 *Let G be an undirected graph and $A \in \mathcal{H}(G)$. If $\{v, u_j\}$ is a bridge of G and λ is an eigenvalue of $A(v)$ such that*

$$\mathrm{m}_{A[G_j - u_j]}(\lambda) = \mathrm{m}_{A[G_j]}(\lambda) - 1,$$

in which G_j is the component of $G - v$ containing u_j, then

$$m_{A(v)}(\lambda) = m_A(\lambda) + 1,$$

i.e., v is Parter for λ.

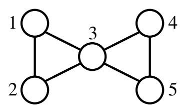
However, we should note that, when G is not a tree, we may have $m_{A(v)}(\lambda) = m_A(\lambda) + 1$, i.e., v is Parter, without a downer component at v for λ.

Example 2.9.2

Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

whose graph is



.

The matrix A has the eigenvalue 2 with multiplicity 1 and $m_{A(3)}(2) = 2$.

When T is a tree and v is a vertex of T, the removal of v from T leaves $\deg_T(v)$ components, i.e., any edge $\{v, u_i\}$ connecting v to a neighbor u_i is a bridge of T. Given a connected graph G, we call a vertex v in G a ***tree-like vertex*** when the removal of v from G leaves $\deg_G(v)$ components.

Observe that, if G is a graph with a tree-like vertex v whose neighbors are $u_1, \ldots, u_k$ and $A \in \mathcal{H}(G)$, the neighbors formula for the expansion of the characteristic polynomial of A (formula (1) in Section 0.2.4) does not depend on the graph of each of the summands $A[G_1], \ldots, A[G_k]$. So we can state the following result for Hermitian matrices whose graphs have tree-like vertices. The proof is similar to the one of Theorem 2.4.1.

Theorem 2.9.3 *Let G be a graph with a tree-like vertex v and $A \in \mathcal{H}(G)$. For an eigenvalue λ of A,* $m_{A(v)}(\lambda) = m_A(\lambda) + 1$ *if and only if there is a downer component at v for λ.*