Bifurcations

week 4-5

1. Transcritical bifurcation

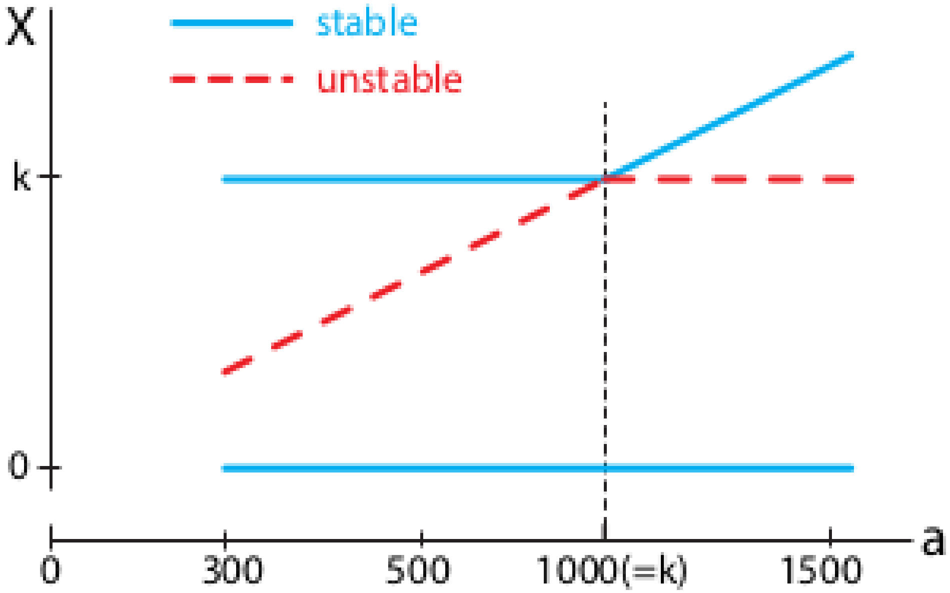
1. Transcritical bifurcation

A change in the parameters → can result in a qualitative change in the equilibrium points of a system

$$X' = 0.1X(1 - \frac{X}{k})(\frac{X}{a} - 1)$$

Exercise 4.1.1

1. Transcritical bifurcation



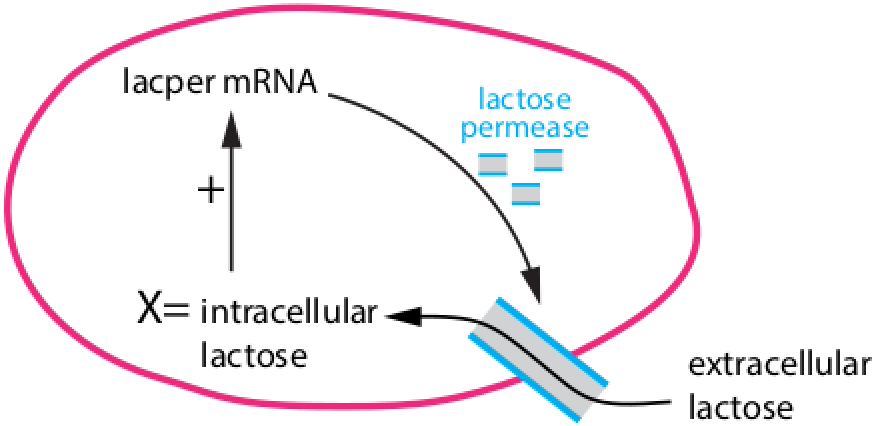
A pair of equilibrium points approach each other, collide, and exchange stability as a parameter smoothly varies

= **transcritical bifurcation**

2. Saddle-node bifurcation

2. Saddle-node bifurcation

The lac operon



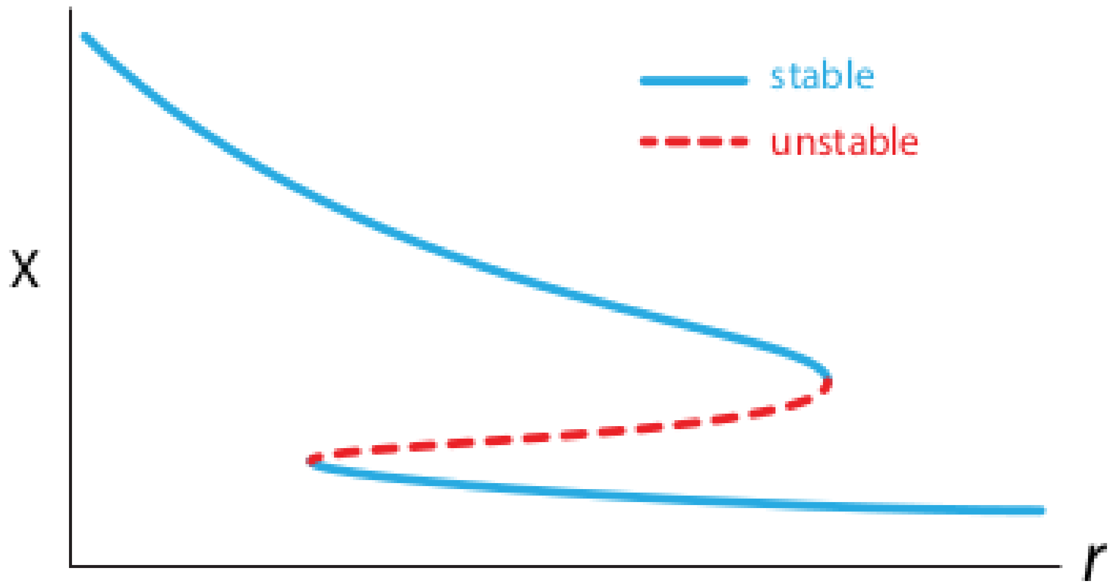
$$X' = \frac{a + X^2}{1 + X^2} - rX$$

a = 0.006

Exercise 4.2.1

2. Saddle-node bifurcation

The lac operon



One equilibrium. When r reaches a critical value, however, a new equilibrium point is born and immediately splits into two, one stable and one unstable = **saddle node bifurcation**

2. Saddle-node bifurcation

The spruce budworm

is a caterpillar that inhabits the forests of the northeastern United States

Typically: low numbers,

Sometimes: outbreak: dramatic increase population, to the point of defoliating large tracts of forest.

Why do these outbreaks happen?

2. Saddle-node bifurcation

X = budworm population

$$\text{growth of budworm} = rX(1 - \frac{X}{k})$$

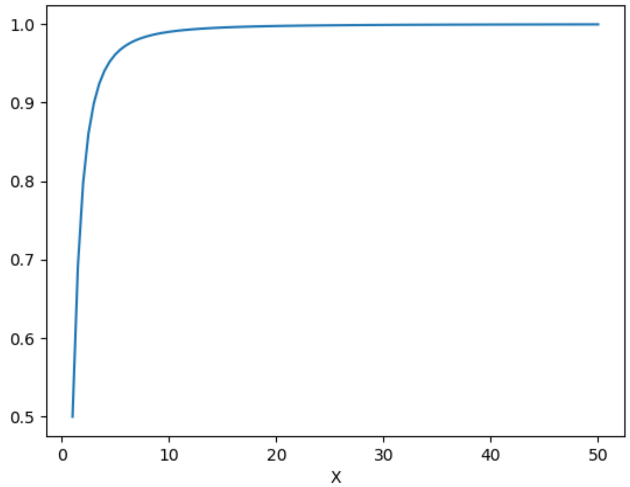
X is eaten by birds

→ X small: little hunting

→ as X rises: more hunting,

→ certain large X: birds have eaten their part, hunting saturates

→ larger X: no more extra hunting anymore



$$\boxed{\text{predation of budworm by birds} = \frac{X^2}{1 + X^2}}$$

2. Saddle-node bifurcation

$$X' = \underbrace{rX(1 - \frac{X}{k})}_{\text{growth of budworm}} - \underbrace{\frac{X^2}{1 + X^2}}_{\text{predation of budworm by birds}}$$

Equilibria? X' = 0

$$X' = 0$$

$$\Longrightarrow \quad rX(1 - \frac{X}{k}) = \frac{X^2}{1 + X^2}$$

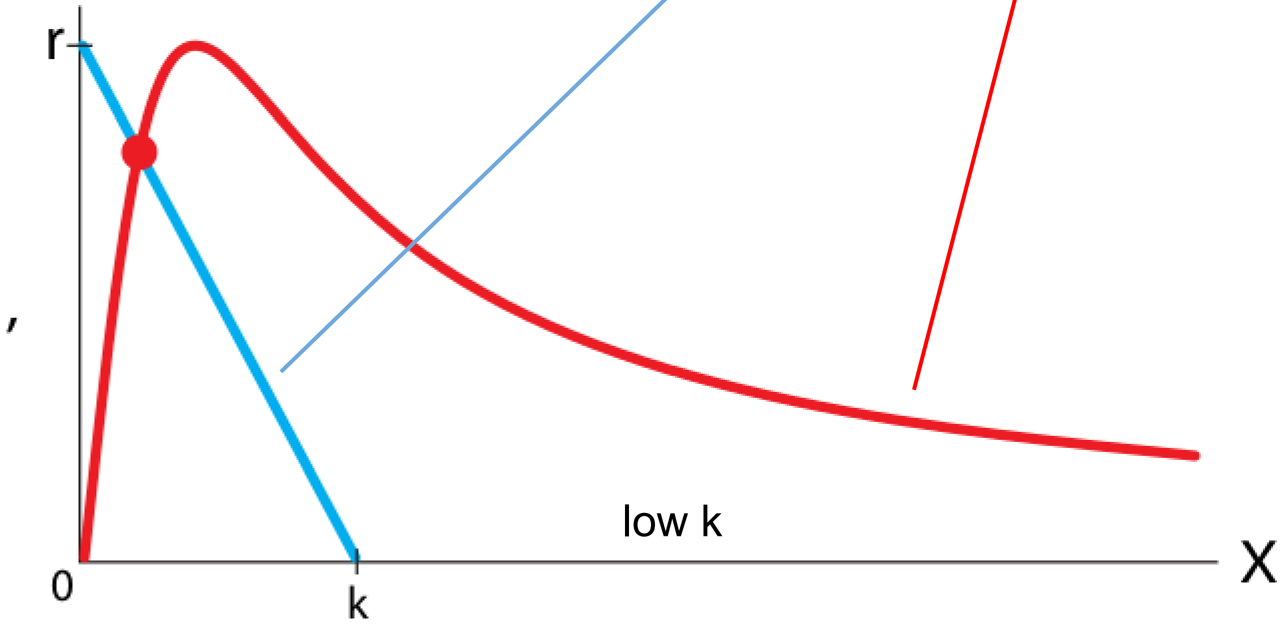
$$\text{dividing by } X \text{ gives} \quad r(1 - \frac{X}{k}) = \frac{X}{1 + X^2}$$

2. Saddle-node bifurcation

Equilibria $$X' = 0$$

$$\implies \quad rX(1 - \frac{X}{k}) = \frac{X^2}{1 + X^2}$$

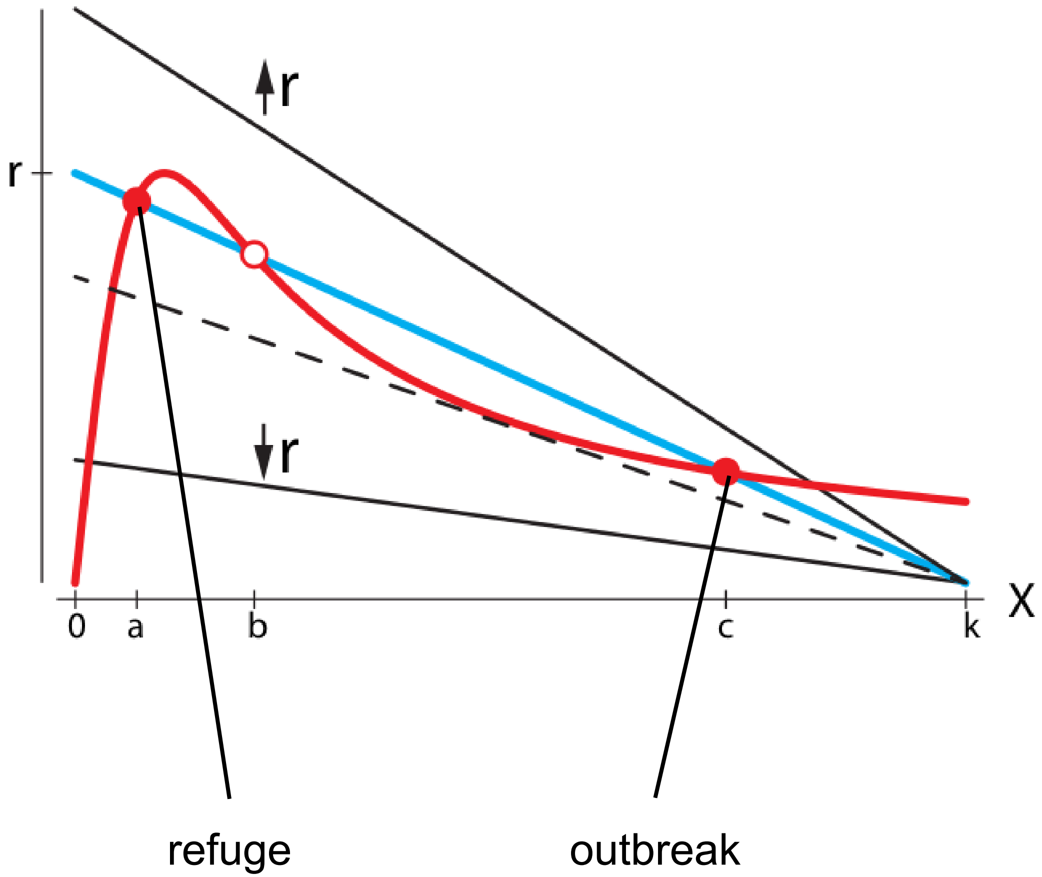
dividing by X gives $$r(1 - \frac{X}{k}) = \frac{X}{1 + X^2}$$



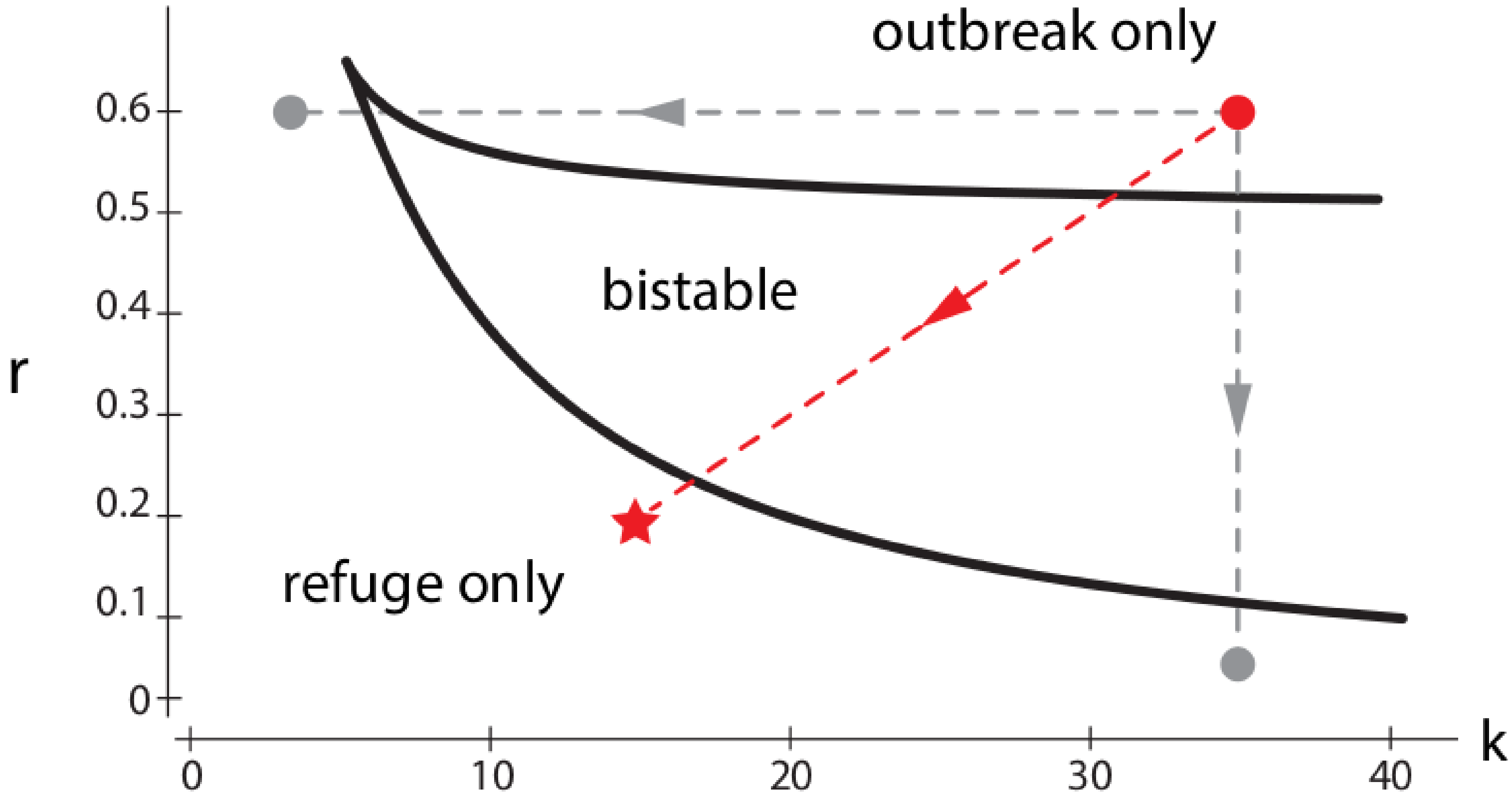
2. Saddle-node bifurcation

larger k: we increase r

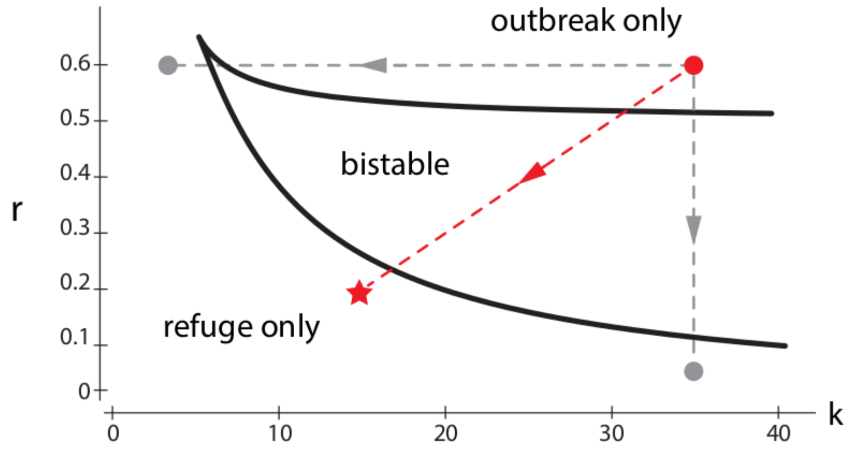
components of X′



2. Saddle-node bifurcation

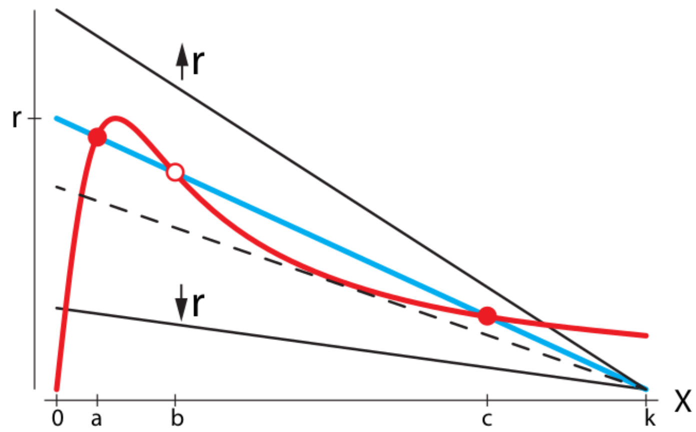


2. Saddle-node bifurcation

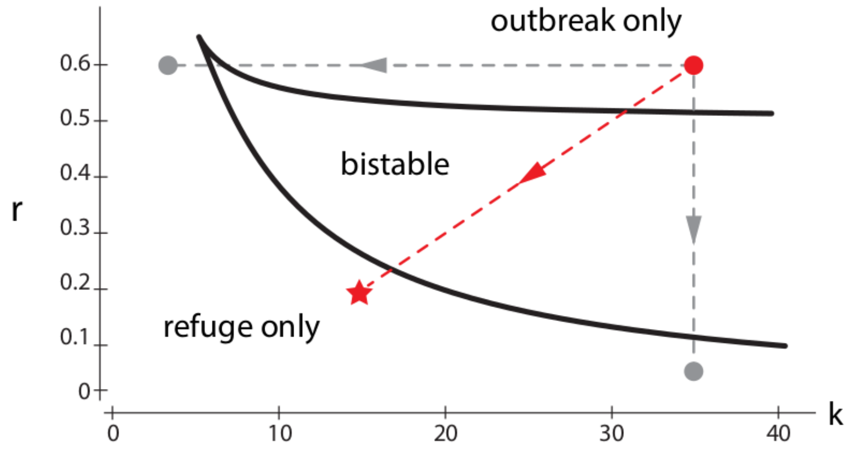


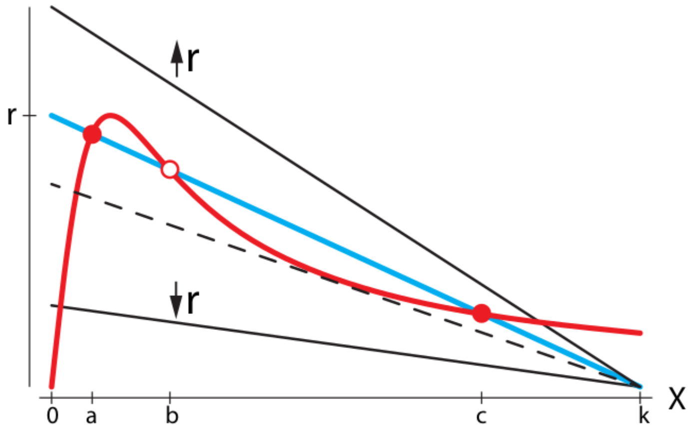
- lower k: defoliants
- lower r: the reproductive rate (as they try with mosquitos, by breeding them with infertile females)
- lower X: insecticide

- option 1: lowest cost: follow red line
- option 2: follow red line partially and then lower X to stable equilibrium



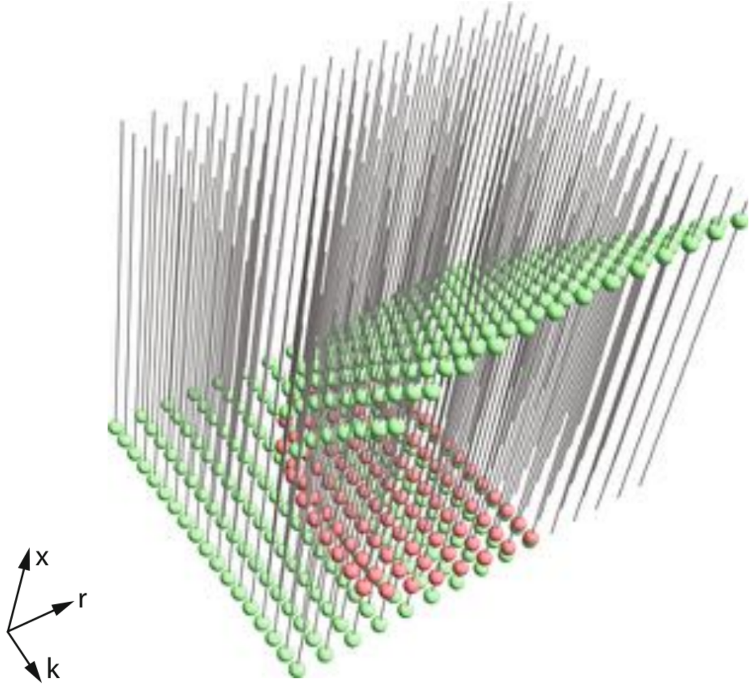
2. Saddle-node bifurcation





Exercise!

- option 1: lowest cost: follow red line
- option 2: follow red line partially and then lower X to stable equilibrium

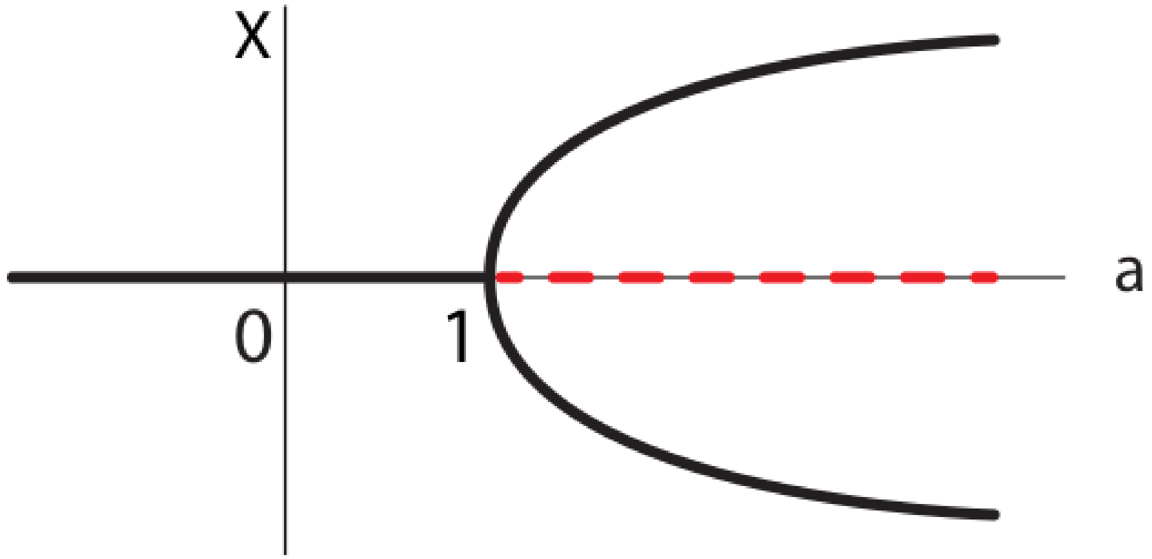


Assignment 1

3. Pitchfork bifurcation

3. Pitchfork bifurcation

less common in biology



3. Pitchfork bifurcation

Opinion model

$$\boxed{N} \underset{k_{PN}}{\overset{k_{NP}}{\rightleftarrows}} \boxed{P}$$

N = people with negative opinion

P = people with positive opinion

We set N + P = 2m (total population)

$$P' = k_{NP} \cdot N - k_{PN} \cdot P$$

$$N' = -k_{NP} \cdot N + k_{PN} \cdot P$$

the imbalance toward positive:

$$\boxed{X = \frac{P - N}{2m}}$$

- X = 0: P = N
- X = 1: P = 2m
- X −1: N = 2m

$$X' = \left(\frac{1}{2m}\right) \cdot (P' - N')$$

$$= \left(\frac{1}{2m}\right) \cdot (2k_{NP} \cdot N - 2k_{PN} \cdot P)$$

$$= \left(\frac{1}{m}\right) \cdot (k_{NP} \cdot N - k_{PN} \cdot P)$$

3. Pitchfork bifurcation

Opinion model

$$\boxed{N} \underset{k_{PN}}{\overset{k_{NP}}{\rightleftharpoons}} \boxed{P}$$

$$X = \frac{P - N}{2m}$$

N + P = 2m

$$X' = (\frac{1}{2m}) \cdot (P' - N')$$

$$= (\frac{1}{2m}) \cdot (2k_{NP} \cdot N - 2k_{PN} \cdot P)$$

$$= (\frac{1}{m}) \cdot (k_{NP} \cdot N - k_{PN} \cdot P)$$

$$X = \frac{P - (2m - P)}{2m} \Rightarrow P = m(1 + X)$$

$$X = \frac{(2m - N) - N}{2m} \Rightarrow N = m(1 - X)$$

$$= k_{NP} \cdot (1 - X) - k_{PN} \cdot (1 + X)$$

3. Pitchfork bifurcation

Opinion model

$$\boxed{N} \underset{k_{PN}}{\overset{k_{NP}}{\rightleftharpoons}} \boxed{P}$$

$$k_{NP} = v \cdot e^{ax}$$

$$X' = k_{NP} \cdot (1 - X) - k_{PN} \cdot (1 + X)$$

bandwagon effect: how sensitive is KNP to the degree of positive tilt

$$\frac{d(k_{NP})}{dX} \text{ is proportional to } k_{NP}$$

This says that the larger the per capita conversion rate, the more sensitive it is to the degree of positive tilt.

$$\frac{d(k_{NP})}{dx} = a \cdot k_{NP} \rightarrow k_{NP} = v \cdot e^{ax}$$

3. Pitchfork bifurcation

Opinion model

$$\boxed{N} \underset{k_{PN}}{\overset{k_{NP}}{\rightleftharpoons}} \boxed{P}$$

$$X' = k_{NP} \cdot (1 - X) - k_{PN} \cdot (1 + X)$$

$$\frac{d(k_{NP})}{dx} = a \cdot k_{NP} \quad \Rightarrow \quad k_{NP} = v \cdot e^{ax}$$

$$\frac{d(k_{PN})}{dx} = -a \cdot k_{PN} \quad \Rightarrow \quad k_{PN} = v \cdot e^{-ax}$$

$$X' = \underbrace{(1 - X) \cdot v \cdot e^{ax}}_{\text{increases } X} - \underbrace{(1 + X) \cdot v \cdot e^{-ax}}_{\text{decreases } X}$$

3. Pitchfork bifurcation

Opinion model

$$\boxed{\mathrm{N}} \underset{k_{PN}}{\overset{k_{NP}}{\rightleftharpoons}} \boxed{\mathrm{P}}$$

$$X' = \underbrace{(1 - X) \cdot v \cdot e^{ax}}_{\text{increases } X} - \underbrace{(1 + X) \cdot v \cdot e^{-ax}}_{\text{decreases } X}$$

Exercise 4.3.1

Bifurcations

Bifurcations: explaining system in a **qualitative** way $\leftrightarrow$ **quantitative** description of many other subjects in physics

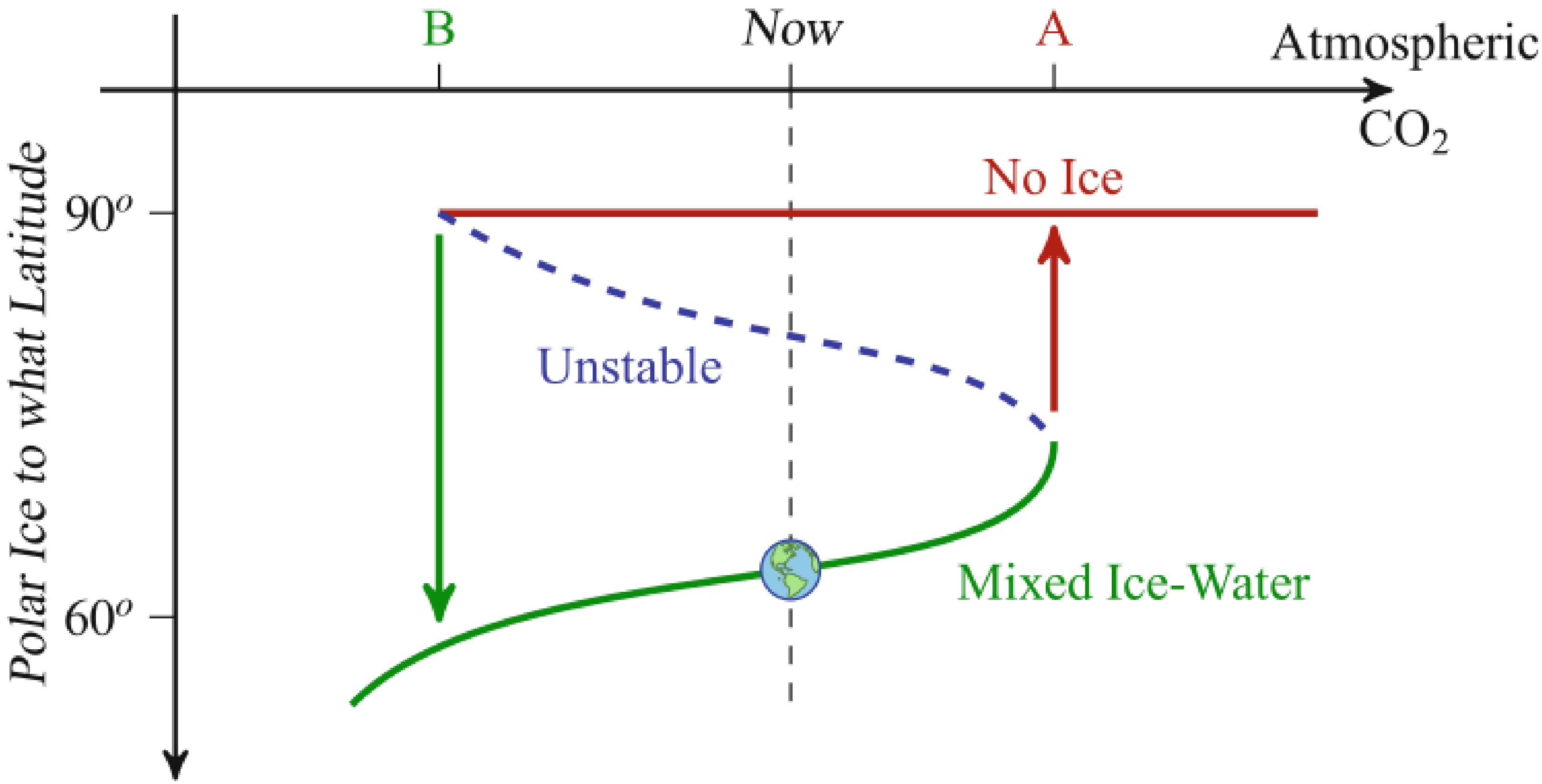
Very often in science: look for the qualitative behavior

- Can deer–moose coexist? Do they sometimes not coexist?
- Why does the lac operon have a bistable switch? What causes it to flip from mode A to mode B?
- In the model of public opinion, why did the middle “balanced opinions” equilibrium become unstable and the two extremes become stable?
- Why does the spruce budworm have outbreaks?

Exercise 4.4

Extra: climate change

Saddle-node bifurcation



An Introduction to Complex Systems Society, Ecology, and Nonlinear Dynamics - *Paul Fieguth*

Chapter 2 in the book