Euler's method

week 2

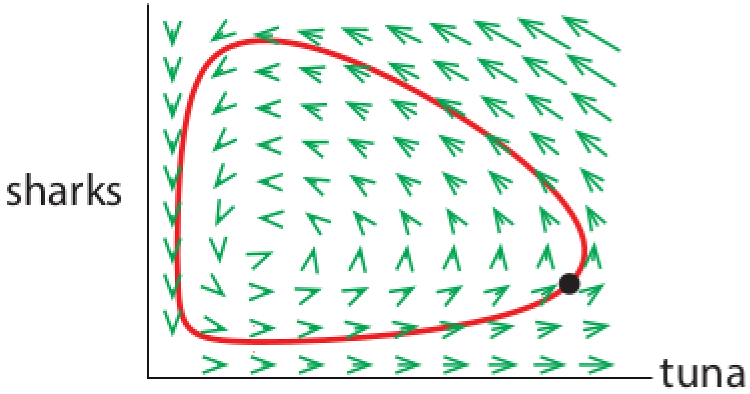
5. Trajectories

5. Trajectories

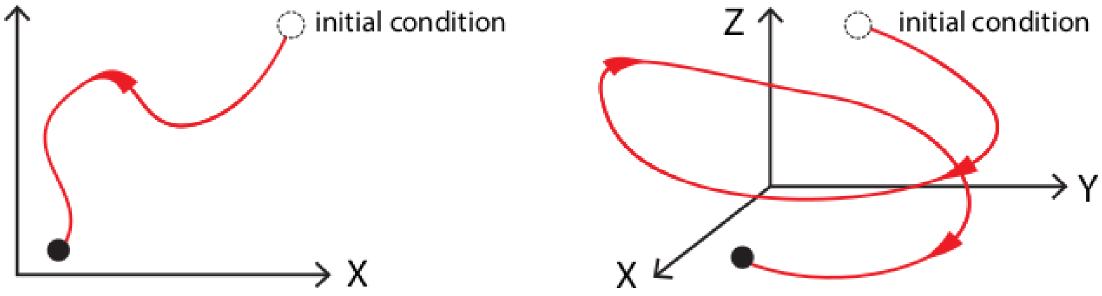
The shark-tuna model

$$\boxed{\begin{aligned} S' &= ST - S \\ T' &= T - ST \end{aligned}}$$

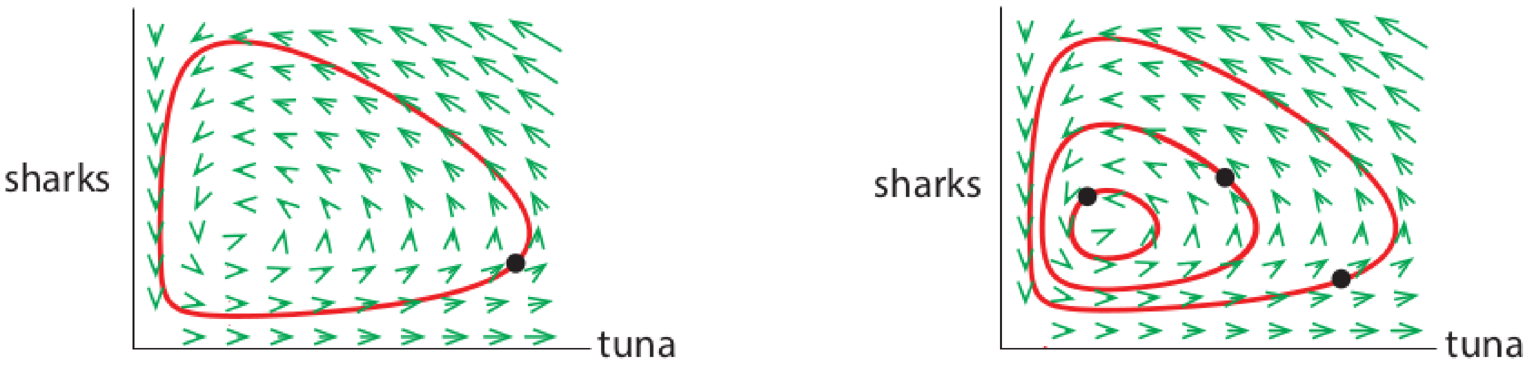
Given an initial condition, what is the exact solution of the model?



5. Trajectories



- You start with an initial condition
- Point moves trough state space = solution curve or trajectory

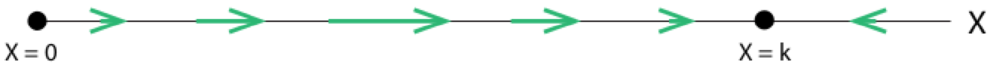


- Trajectory follows the change vectors
- Every initial condition gives a different trajectory
- We do not see how fast it travels

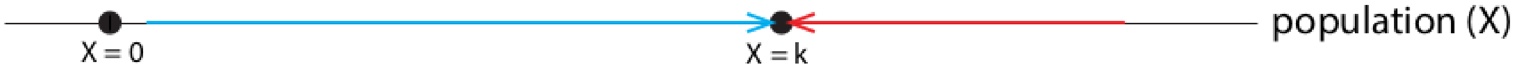
5. Trajectories

see exercises!

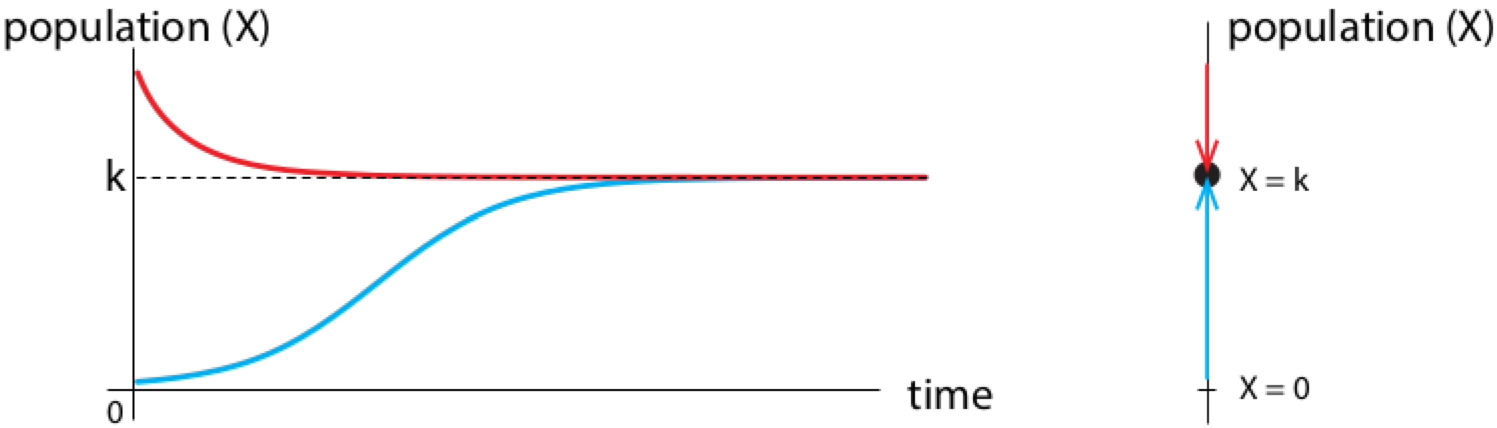
population model with crowding: $X' = r X (1 - X/k)$



two possible trajectories:



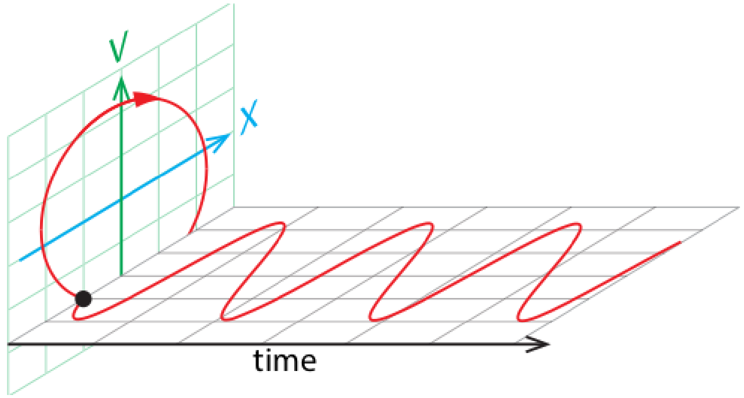
time series:



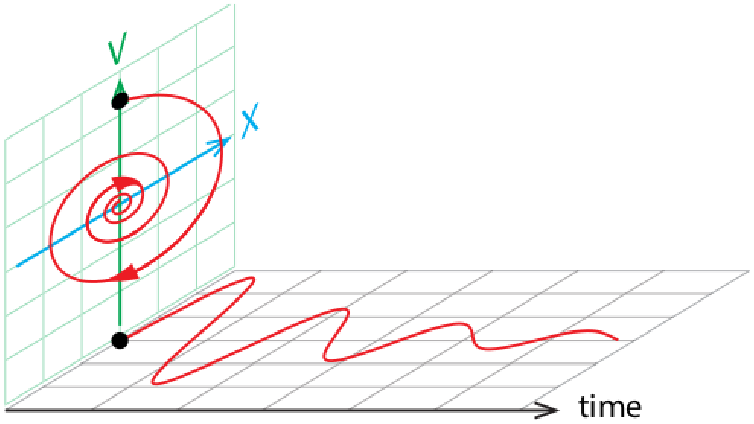
5. Trajectories

see exercises!

circular: like the spring



spring with friction: what will happen?

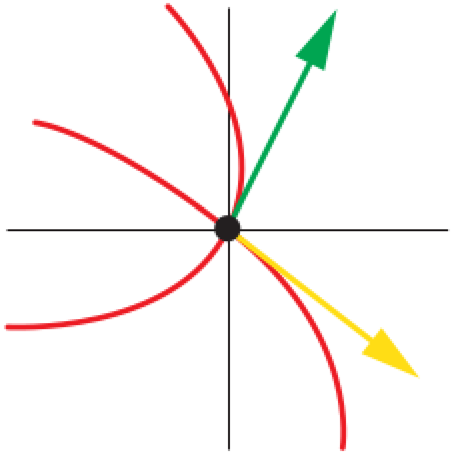


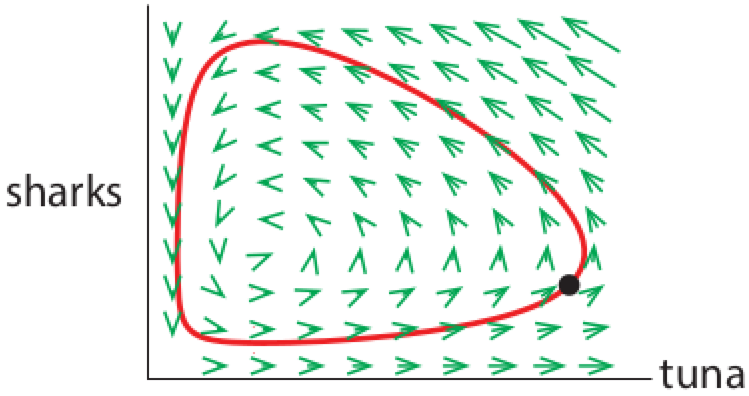
5. Trajectories

uniqueness of a trajectory

$$\boxed{\begin{aligned} S' &= ST - S \\ T' &= T - ST \end{aligned}}$$

change vectors: exactly one for each point in state space

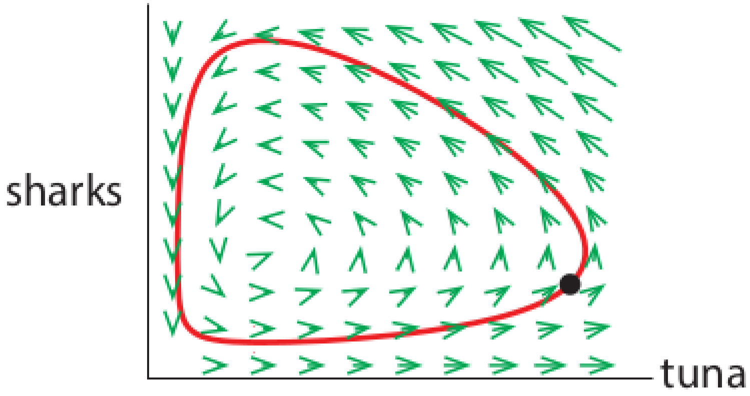




what does this say about a trajectory?

They cannot cross! (and also do not touch: theorem of uniqueness)

5. Trajectories



Q1: Does the red curve really exist? Is there really a single trajectory through a given point that everywhere follows the change arrows?

Yes, almost always

Q2: Can we figure out the equation for the red curve from the equation for the vector field?

No, almost never

6. Euler integration

6. Euler integration

- Start from initial condition X0
- Follow the change vector out of X0: X0′
- But for how long do you follow the change vector?

infinitesimal

Does this mean you can never find the exact solution?

6. Euler integration

make Δt very small but not zero

one dimensional example: X' = 0.2 X

$$\text{new } X = \text{old } X + \Delta t \cdot X'$$

$$\begin{aligned} \text{new } X &= X_0 + 0.01 \cdot f(X_0) \\ &= 100 + 0.01 \cdot 20 \\ &= 100.2 \end{aligned}$$

= X1

$$\begin{aligned} \text{new new } X &= X_1 + 0.01 \cdot f(X_1) \\ &= 100.2 + 0.01 \cdot 20.04 \\ &= 100.4004 \\ &= X_2 \end{aligned}$$

6. Euler integration

see exercises!

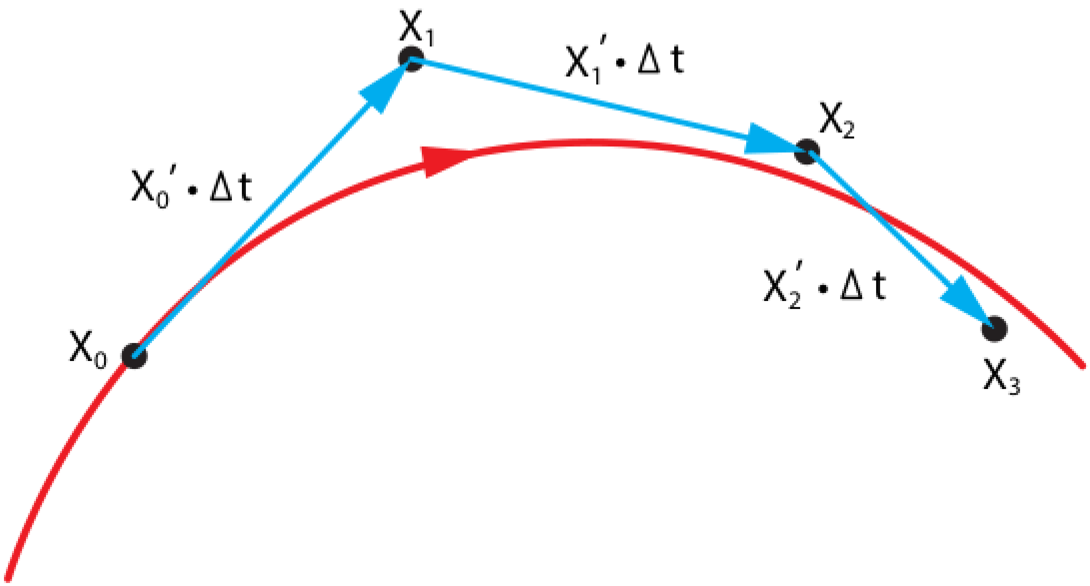
two dimensional example

$$\text{new } X = \text{old } X + \Delta t \cdot X'(\text{old } X,\ \text{old } Y)$$
$$\text{new } Y = \text{old } Y + \Delta t \cdot Y'(\text{old } X,\ \text{old } Y)$$

6. Euler integration

Shadowing lemma

as Δt gets smaller and smaller, the blue jagged line gets closer and closer to a true red curve, possibly from a slightly perturbed initial condition.



6. Euler integration

https://www.youtube.com/embed/gdxYsVniOYo?enablejsapi=1

Exercise 2.1
2.2
2.3

7. Epidemiology

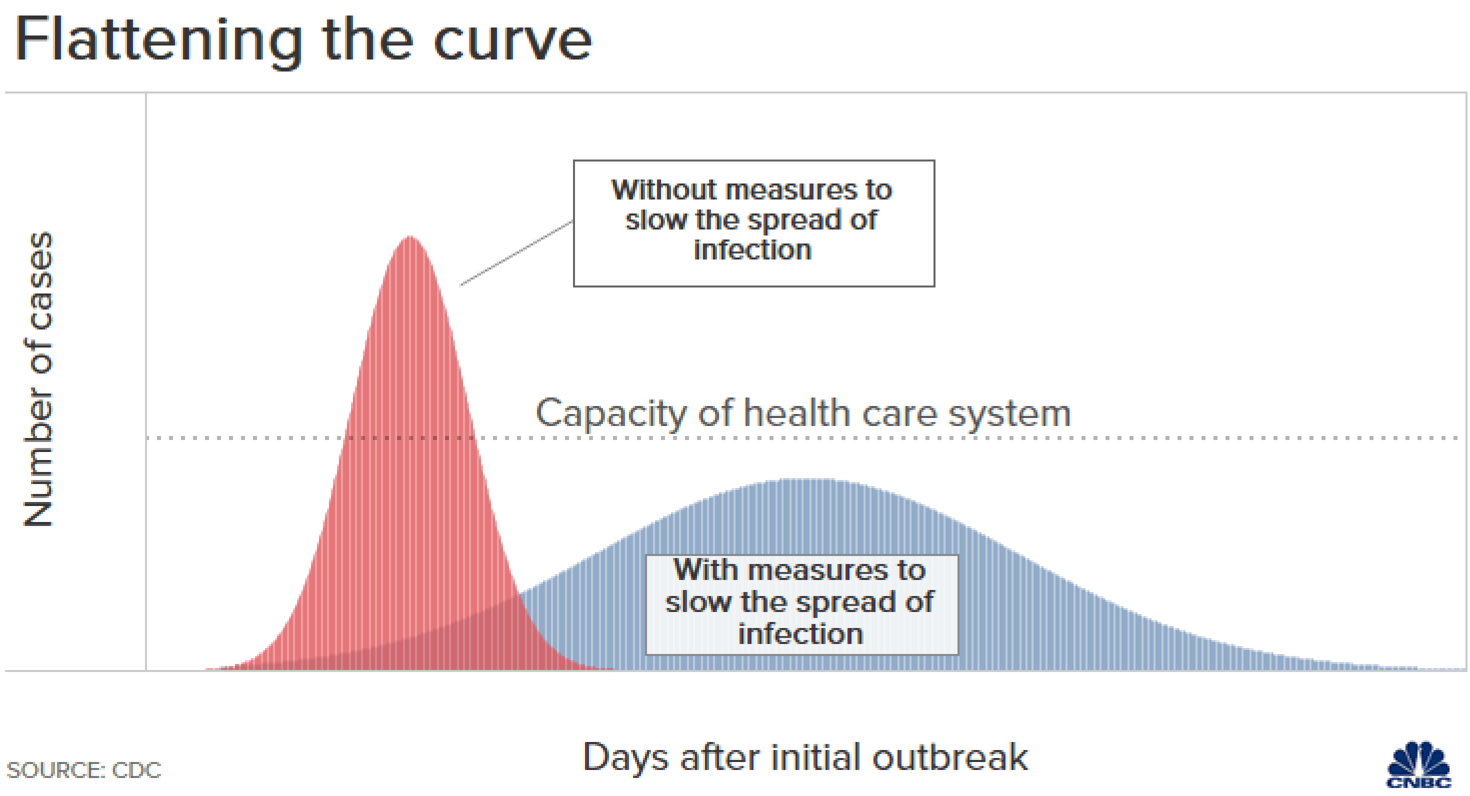
7. Epidemiology

Epidemics

- Black death 1347- 1350: Europe lost 1/3 of population
- Great Plague of London 1664–66: Killed more than 75,000 of total population of 460,000 in London
- Spanish flue 1918-1919: 25 million killed in Europe
- SARS-CoV-1 2002: 774 death, but was only stopped by quarantine of sick people (disease was only transmissible when person had symtoms)
- Covid-19 2020: put the whole world in lockdown

7. Epidemiology

Mathematical models predict rate of spread, peak, and effect of taking different measures



7. Epidemiology

Most simple model - the SIR model

- **S** - Susceptible class: those who may catch the disease but currently are not infected
- **I** - Infected class: those who are infected and currently contagious
- **R** - Removed class: those who cannot get the disease, because they either have recovered permanently, are naturally immune, or have died

7. Epidemiology

$$\frac{dS}{dt} = -\frac{\beta IS}{N},$$

$$\frac{dI}{dt} = \frac{\beta IS}{N} - \gamma I,$$

$$\frac{dR}{dt} = \gamma I,$$

- β = infection rate
- γ = removal rate

When does an epidemic occur?

Epidemics occur when dI/dt > 0

7. Epidemiology - SIR model

$$\frac{dS}{dt} = -\frac{\beta IS}{N},$$

$$\frac{dI}{dt} = \frac{\beta IS}{N} - \gamma I,$$

$$\frac{dR}{dt} = \gamma I,$$

- Assume I > 0
- If $\beta/\gamma > N/S$ than $dI/dt > 0$
- R0 = β/γ is called the basic reproduction number
- N/S is initially about 1
- R0 > 1 gives rise to an epidemic

7. Epidemiology - SIR model

$$\frac{dS}{dt} = -\frac{\beta I S}{N},$$

$$\frac{dI}{dt} = \frac{\beta I S}{N} - \gamma I,$$

$$\frac{dR}{dt} = \gamma I,$$

Meaning of γ:
If the period of contagion lasts 7 days, then each day we expect roughly or approximately 14% of the total number of infected to move from the infected class I t into the removed class R.
$\gamma = 1/7$ (days) => $1/\gamma$ = total duration illness (days)

7. Epidemiology - SIR model

$$\frac{dS}{dt} = -\frac{\beta IS}{N},$$

$$\frac{dI}{dt} = \frac{\beta IS}{N} - \gamma I,$$

$$\frac{dR}{dt} = \gamma I,$$

Meaning of β:
β describes the infection rate: an infected subject will infect βS/N people per unit of time. If N is close to S, this means that per unit of time (a day) about β people will be infected

7. Epidemiology - SIR model

$$\frac{dS}{dt} = -\frac{\beta IS}{N},$$

$$\frac{dI}{dt} = \frac{\beta IS}{N} - \gamma I,$$

$$\frac{dR}{dt} = \gamma I,$$

Meaning of R0 = β/γ:

- Ratio between new infections per unit of time (a day) and removal of infections per unit of time (a day)
- As $1/\gamma$ is the total duration of the illness R0 = total number of infection of a single person in a fully susceptible population

If this ratio is larger then N/S(0), we get an epidemic

7. Epidemiology - SIR model

$$\frac{dS}{dt} = -\frac{\beta IS}{N},$$

$$\frac{dI}{dt} = \frac{\beta IS}{N} - \gamma I,$$

$$\frac{dR}{dt} = \gamma I,$$

In the SIR model, after falling ill, a subject can heal or die; moreover, if healed, the subject cannot be infected again (he becomes immune).

7. Epidemiology - SIR model

$$\frac{dS}{dt} = -\frac{\beta IS}{N},$$

$$\frac{dI}{dt} = \frac{\beta IS}{N} - \gamma I,$$

$$\frac{dR}{dt} = \gamma I,$$

Exercise 2.4.1

7. Epidemiology - SEIR model

One extra step - the SEIR model

- **S** - Susceptible class: those who may catch the disease but currently are not infected
- **E** - Exposed class: subjects exposed to the infection (subjects who come into contact with infected individuals, but are not necessarily infected).
  THIS ADDS A LATENCY TO THE MODEL
- **I** - Infected class: those who are infected and currently contagious
- **R** - Removed class: those who cannot get the disease, because they either have recovered permanently, are naturally immune, or have died

7. Epidemiology - SEIR model

One extra step - the SEIR model

1. $dS/dt = -\beta SI / N$
2. $dE/dt = \beta SI / N - \alpha E$
3. $dI/dt = \alpha E - \gamma I$
4. $dR/dt = \gamma I$

α (alpha) is the inverse of the average time of incubation, that is, of the period that elapses between when an individual has been infected and when it becomes contagious in turn.

7. Epidemiology - SEIR model

The effect of social distancing:

1. $dS/dt = -\beta SI / N$
2. $dE/dt = \beta SI / N - \alpha E$
3. $dI/dt = \alpha E - \gamma I$
4. $dR/dt = \gamma I$

ρ = index of social distancing

- will modify the evolution of susceptible and exposed
- can have a value from 0 (in this case the whole population is in quarantine, complete lock down) to 1 (no social distancing = SEIR model)

7. Epidemiology - SEIR model

The effect of social distancing:

1. $dS/dt = -\beta SI / N$
2. $dE/dt = \beta SI / N - \alpha E$
3. $dI/dt = \alpha E - \gamma I$
4. $dR/dt = \gamma I$

Simulate the effect of trying to flatten the curve!

Exercise 2.4.2