Chaos

week 9

1. Chaos in a continuous model

1. Chaos in a continuous model

- We studied equilibrium behavior: modeled by stable equilibrium points = **point attractors**
- What about robust and stable oscillations in systems? **limit cycle attractors!**

Are point attractors and limit cycle attractors the only kinds of attractors that can occur in dynamical systems?

There are attractors of a third kind, called **chaotic attractors**

1. Chaos in a continuous model

So far, we saw prey predator models with only two species
→ real ecological systems: many more species

3 species!

X

Y

Z

1. Chaos in a continuous model

X

Y

Z

$$\boxed{X' = rX(1 - \frac{X}{K}) - \frac{a_1 X}{1 + b_1 X} Y}$$

logistic growth

The per herbivore consumption of plants saturates with increasing plant density

= holling tanner

1. Chaos in a continuous model


X


Y


Z

$$\boxed{X' = rX\left(1 - \frac{X}{K}\right) - \frac{a_1 X}{1 + b_1 X} Y}$$

$$\boxed{Y' = c_1 \frac{a_1 X}{1 + b_1 X} Y - d_1 Y - \frac{a_2 Y}{1 + b_2 Y} Z}$$

birth rate ~ food

death rate

holling tanner

1. Chaos in a continuous model

$$\boxed{X' = rX\left(1 - \frac{X}{K}\right) - \frac{a_1 X}{1 + b_1 X} Y}$$

$$\boxed{Y' = c_1 \frac{a_1 X}{1 + b_1 X} Y - d_1 Y - \frac{a_2 Y}{1 + b_2 Y} Z}$$

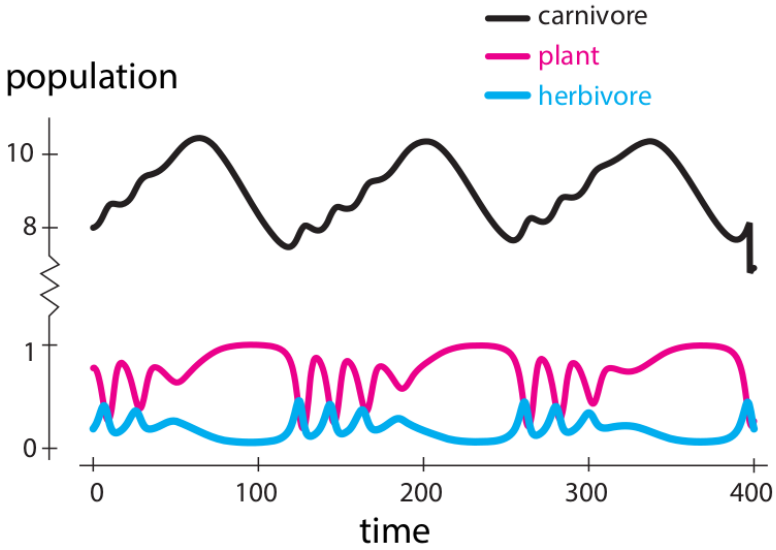
$$\boxed{Z' = c_2 \frac{a_2 Y}{1 + b_2 Y} Z - d_2 Z}$$

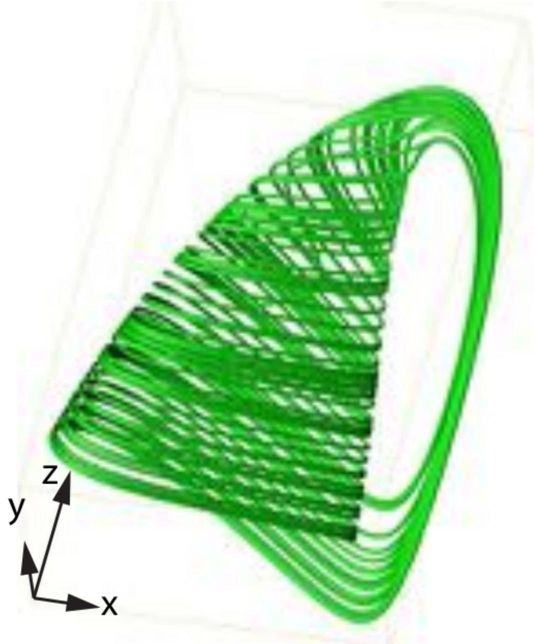
birth rate ~ food

death rate

Exercise 7.1.1

1. Chaos in a continuous model



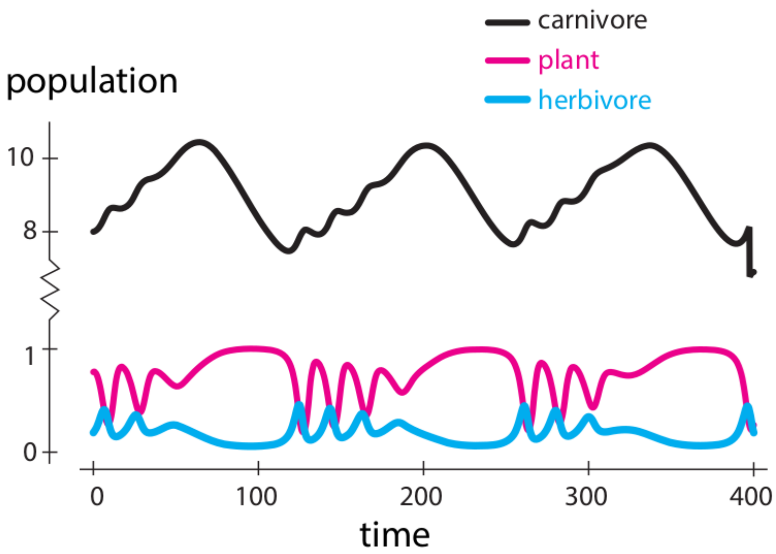


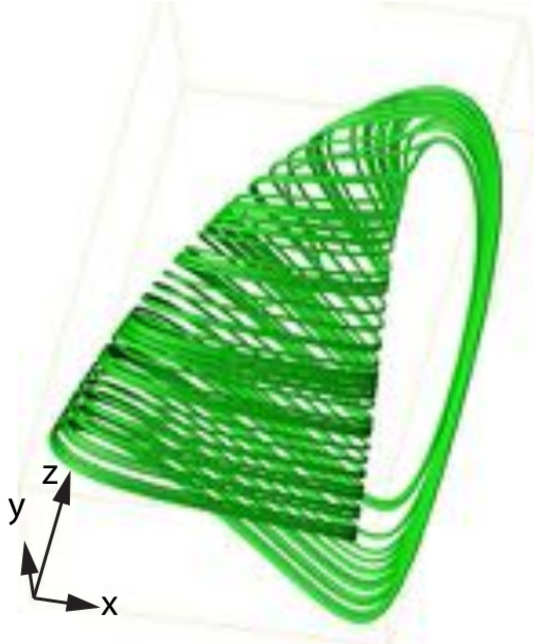
Does this system oscillate?

No, each cycle is different from the previous cycle!

It appears to be somewhat periodic, but also to have some kind of randomness.

1. Chaos in a continuous model





- 3D trajectory = complex shape = shape of an upside-down jug
- A typical state point: begins in the jug part → spirals inward mostly in the X-Y plane + slowly rising along the Z axis → the state point gets thrown into the handle of the jug, where it plummets down to begin another cycle
- Non-repeating time series + associated complex trajectories = chaos

1. Chaos in a continuous model

How did we get there?

$$\boxed{X' = rX(1 - \frac{X}{K}) - \frac{a_1 X}{1 + b_1 X} Y}$$

$$\boxed{Y' = c_1 \frac{a_1 X}{1 + b_1 X} Y - d_1 Y - \frac{a_2 Y}{1 + b_2 Y} Z}$$

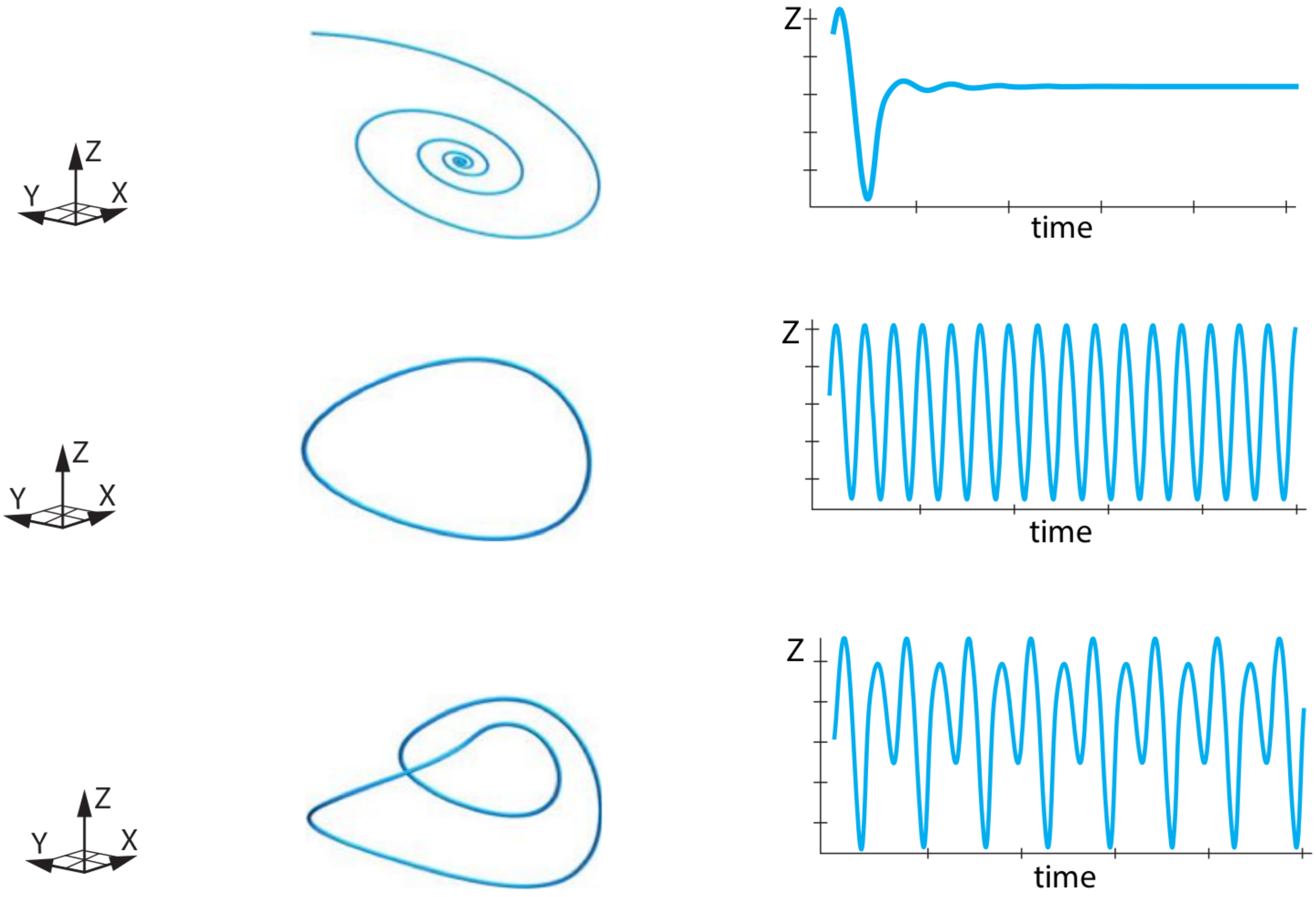
$$\boxed{Z' = c_2 \frac{a_2 Y}{1 + b_2 Y} Z - d_2 Z}$$

b1 = controls the level of plants that the herbivores can consume

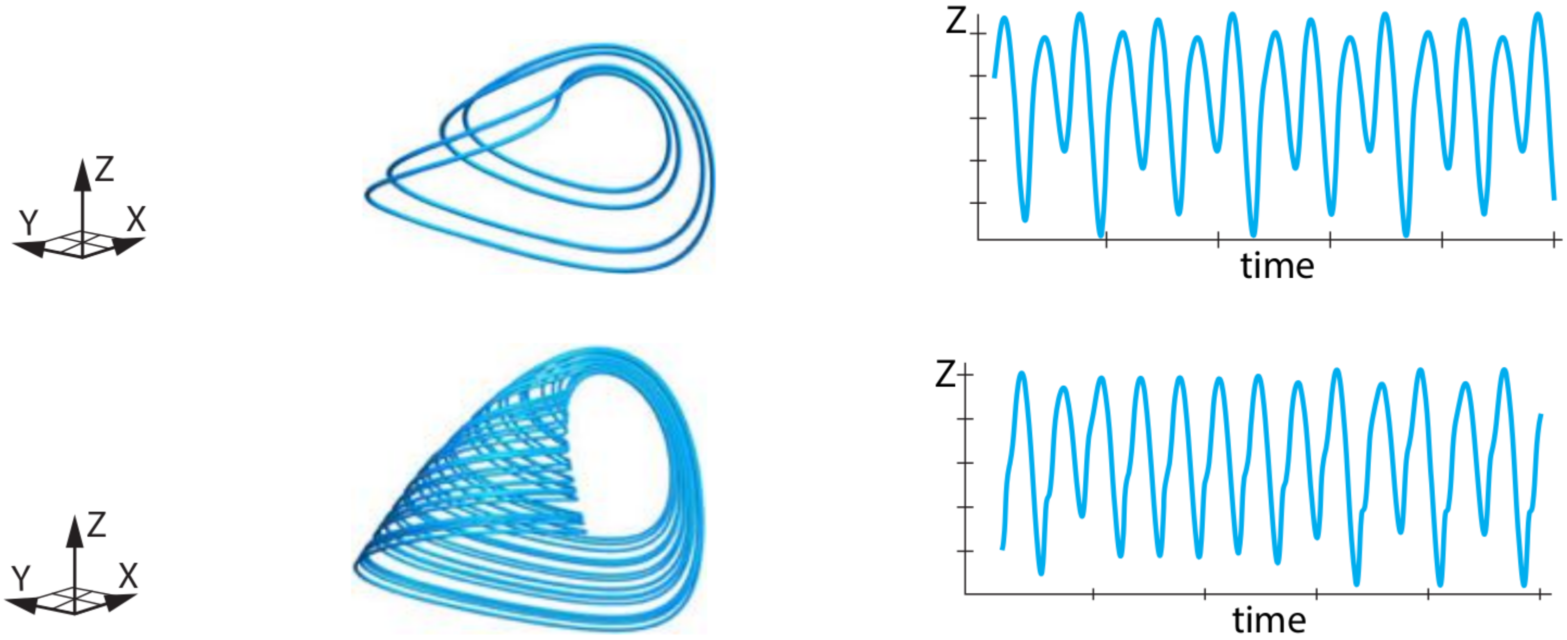
Exercise 7.1.2

1. Chaos in a continuous model

Path to chaos:



1. Chaos in a continuous model



equilibrium → oscillation → complex oscillation → chaos

1. Chaos in a continuous model

The picture tells a story...

- Starting in the jug itself: X (= plant) and Y (= herbivore) oscillate like a shark - tuna model in the X-Y plane, but with slowly diminishing amplitude (like the spring with friction), while Z (= carnivore) grows slowly
- Finally, Z grows so large → state point goes into the handle → plummets down rapidly (due to decrease in Y = food for Z) → Z crashes → takes the pressure off Y → ...
- Cycle starts over

1. Chaos in a continuous model

The picture tells a story...

- X-Y oscillation (like holling-tanner) is coupled to another oscillatory process, the Z-Y oscillation, in which the carnivore preys on the herbivore in a second cyclic process.
- Chaos as a result of the interaction of two coupled cyclic processes is a frequent scenario.

2. Characteristics of chaos

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1. Determinism

Is a chaotic system deterministic?

The system is deterministically producing its own irregular behavior without any randomness.

! Careful

Chaotic behaviour can look erratic, but it is a complex order.

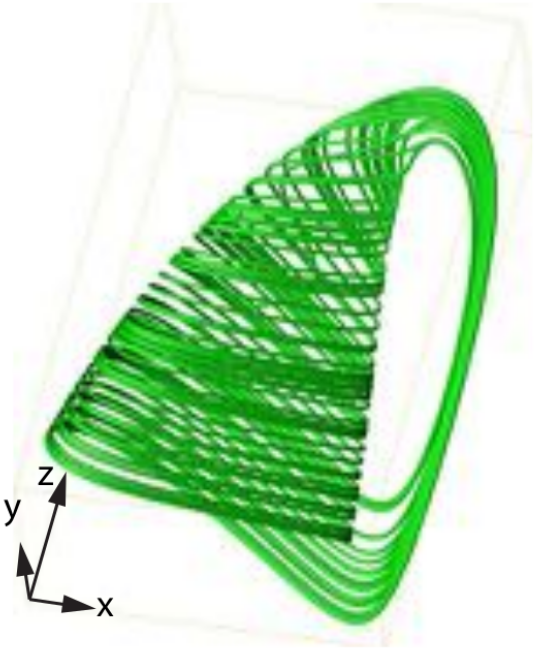
Chaos is a type of behaviour, not a type of system!

2. Characteristics of chaos

2. Boundedness

Is a chaotic system bounded or unbounded?

Boundedness means that the system does not go off to infinity. It stays within a certain region of state space = stays in a box.



2. Characteristics of chaos

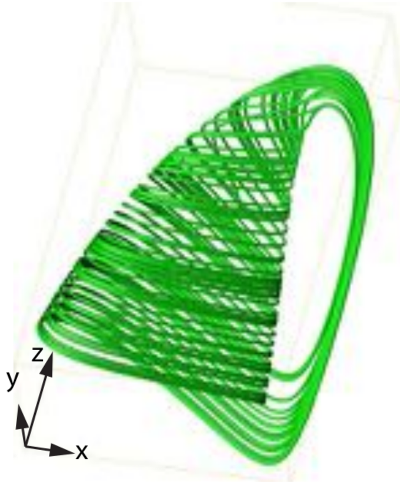
3. Irregularity

Is a chaotic system regular of irregular?

Chaotic system → Aperiodic behavior never exactly repeats. Not a closed orbit
Limit cycle attractors → periodic orbits = closed orbits

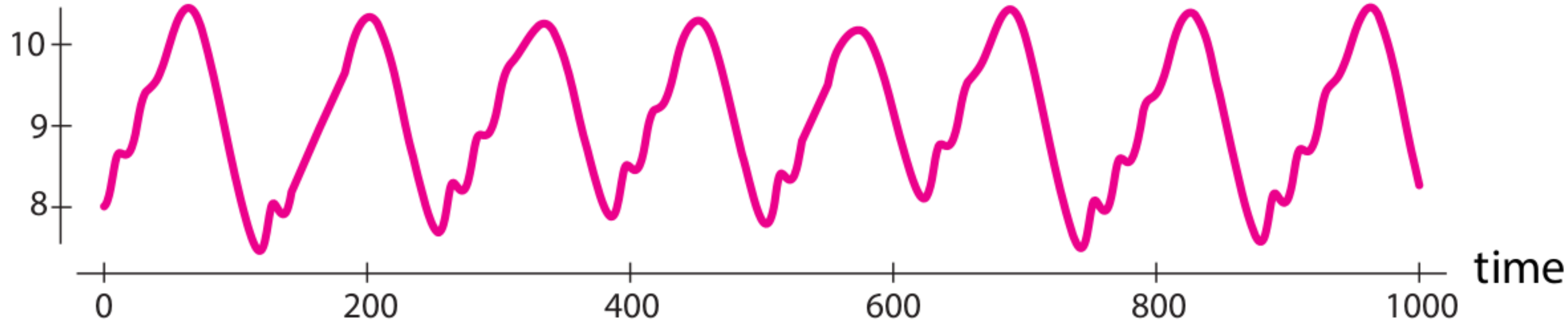
Can a 2D system be chaotic?

No! You can never draw an infinite curve in 2D which never intersects itself



2. Characteristics of chaos

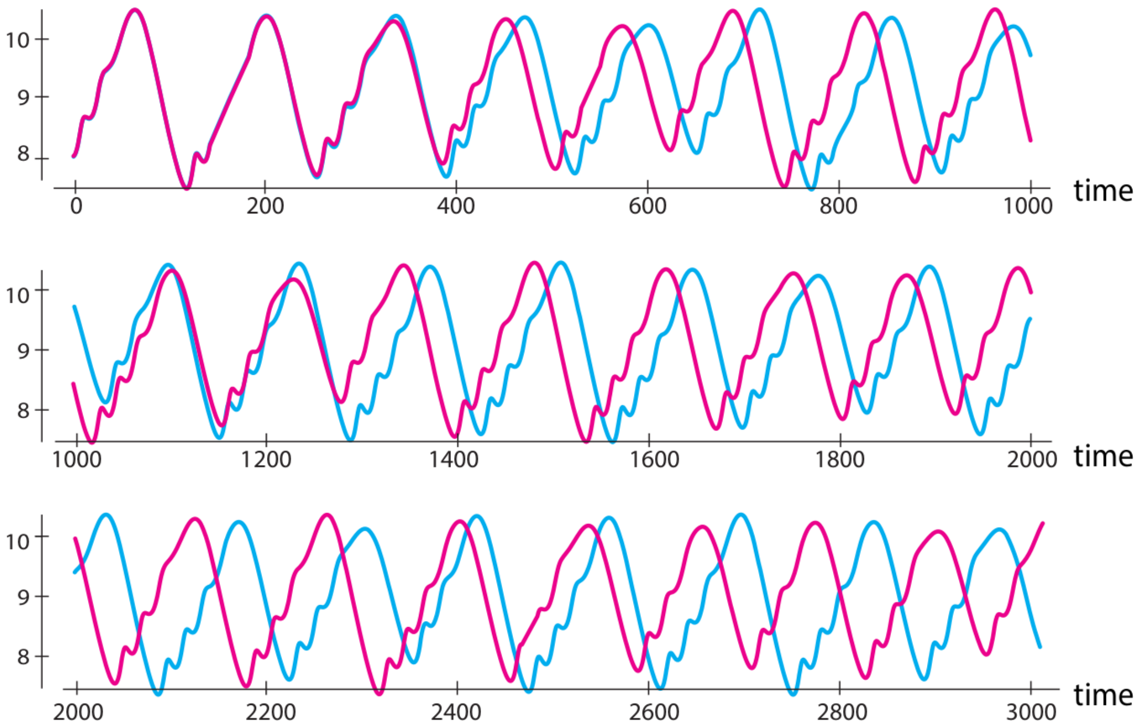
3. Irregularity



Despite a general qualitative similarity, the behaviour of the system never repeats and never approaches repetition! No two oscillations are the same

2. Characteristics of chaos

4. Sensitivity dependence on initial conditions

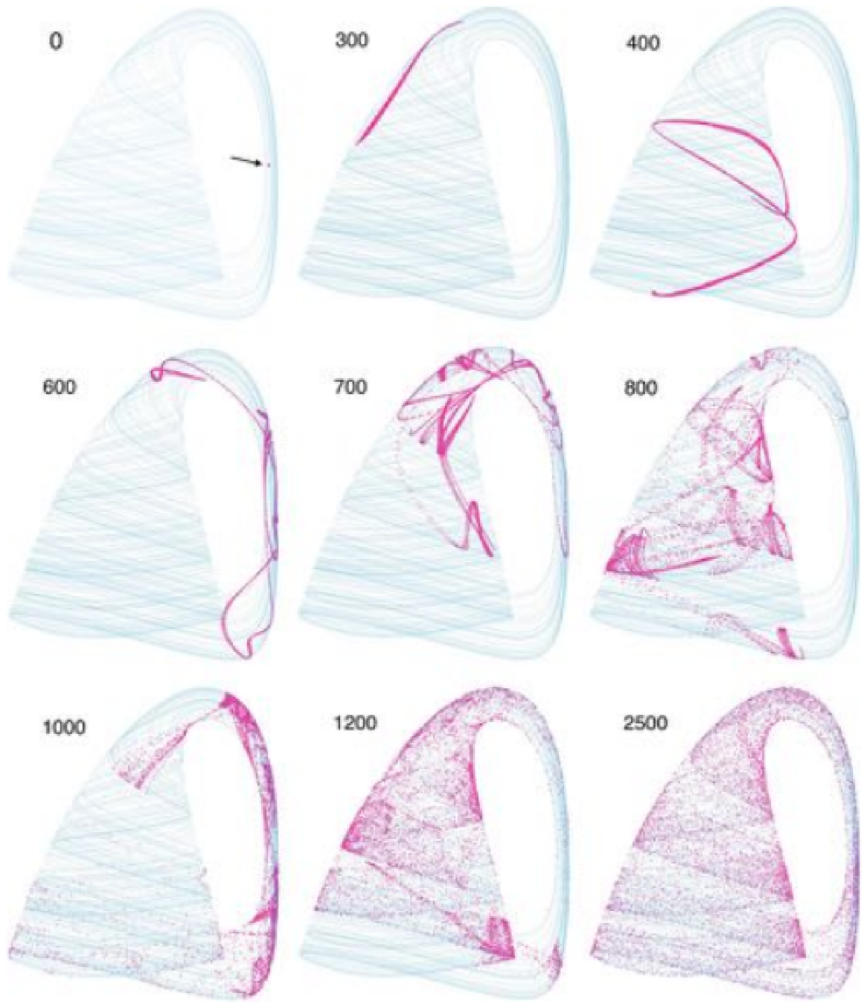


If you start with two values really close:

they start to diverge and become completely unrelated.

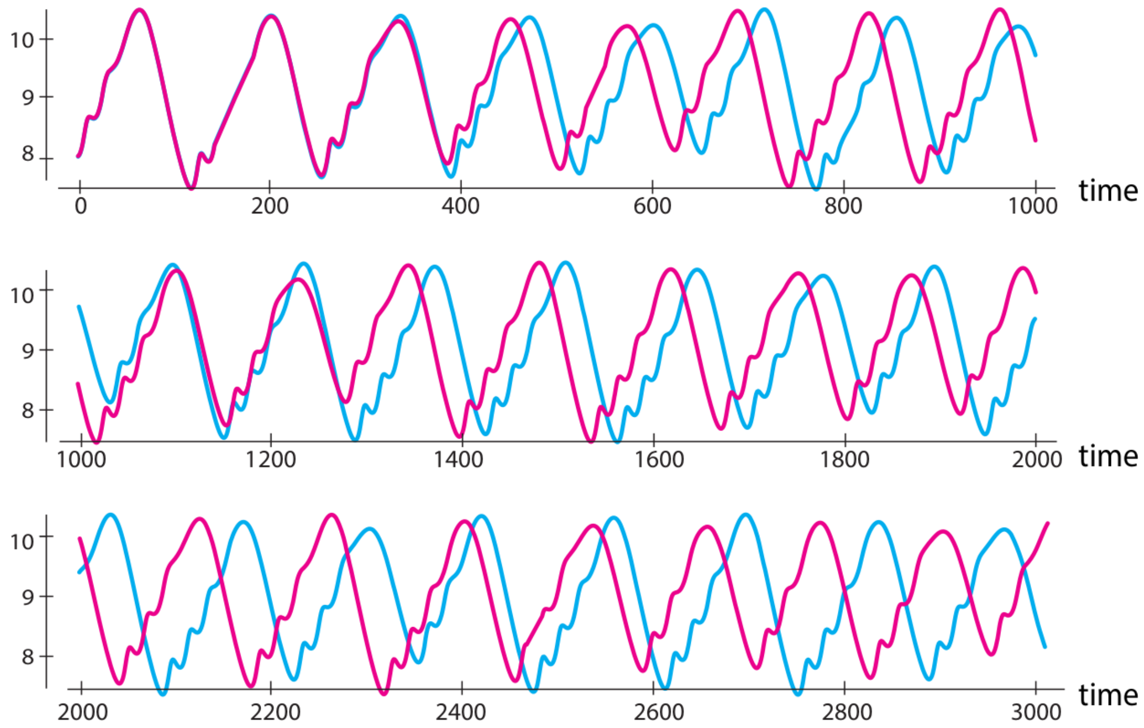
2. Characteristics of chaos

4. Sensitivity dependence on initial conditions



2. Characteristics of chaos

5. Unpredictability

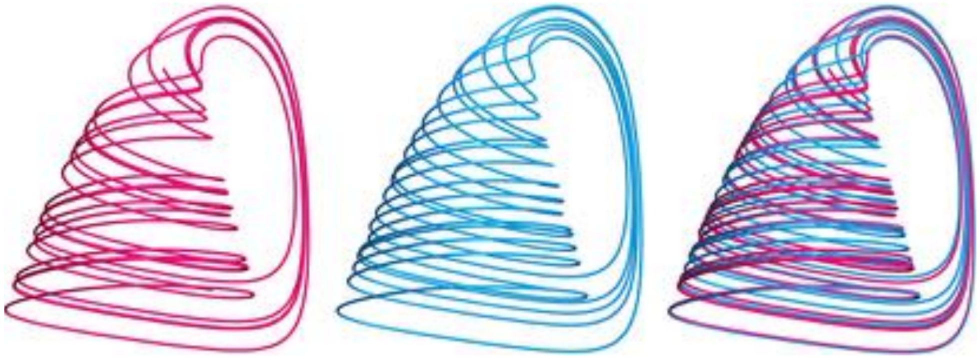


You can start with 2 very close values, but you can never predict how they will evolve in time → butterfly effect

Summary

Behavior	Mathematical model
equilibrium	stable equilibrium point ("point attractor")
oscillation	limit cycle attractor
chaos	chaotic attractor

- For a long time: mathematicians thought only two kinds of attractors existed
- Chaotic attractor = strange attractor: very complicated geometry
- Qualitative behavior versus quantitative behavior



Summary

Weather forecasting will not be accurate for more than a few days in advance
→ is a system that can display chaotic behavior