Collaborated w/ Ibrahim and Juan

Problem 1

a) $\Psi(t,\theta) = \begin{pmatrix} R(\theta) & t \\ 0 & 1 \end{pmatrix}$ $\quad R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ $\quad t = \begin{pmatrix} X \\ Y \end{pmatrix}$

$$\Psi(t,\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x \\ \sin(\theta) & \cos(\theta) & Y \\ 0 & 0 & 1 \end{pmatrix} \quad \frac{d}{dt}R(\theta) = \begin{pmatrix} -\dot\theta\sin(\theta) & -\dot\theta\cos(\theta) \\ \dot\theta\cos(\theta) & -\dot\theta\sin(\theta) \end{pmatrix}$$

$$\frac{d}{dt} = \begin{pmatrix} \dot X \\ \dot Y \end{pmatrix}$$

Plug in:

$$d\,\Psi(t,\theta) = \begin{pmatrix} -\dot\theta\sin(\theta) & -\dot\theta\cos(\theta) & \dot x \\ \dot\theta\cos(\theta) & -\dot\theta\sin(\theta) & \dot Y \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \boxed{d\,\Psi(0,0) = \begin{pmatrix} 0 & -\dot\theta & \dot x \\ \dot\theta & 0 & \dot Y \\ 0 & 0 & 0 \end{pmatrix}}$$

b) $V_r = \dot\theta_r(R + \frac{w}{2})$, $V_L = \dot\theta_r(R - \frac{w}{2})$ $\Rightarrow$ $\dot\theta_r = \frac{V_r - V_L}{w}$ $\quad V_L = r\dot\varphi_L \quad V_r = r\dot\varphi_r \quad$ robot @ origin $(0,0)$

$= \frac{r(\dot\varphi_r - \dot\varphi_L)}{w}$ $\quad \dot X = \frac{r}{2}(\dot\varphi_r + \dot\varphi_L)$ $\quad \dot Y = 0$ at Identity

plugging into prt a:

$$\dot\Omega(\dot\varphi_L,\dot\varphi_r) = \begin{pmatrix} -\frac{r(\dot\varphi_r-\dot\varphi_L)}{w}\sin(0) & -\frac{r(\dot\varphi_r-\dot\varphi_L)}{w}\cos(0) & \frac{r(\dot\varphi_r+\dot\varphi_L)}{2} \\ \frac{r(\dot\varphi_r-\dot\varphi_L)}{w}\cos(0) & -\frac{r(\dot\varphi_r-\dot\varphi_L)}{w}\sin(0) & 0 \\ 0 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & -\frac{r(\dot\varphi_r-\dot\varphi_L)}{w} & \frac{r(\dot\varphi_r+\dot\varphi_L)}{2} \\ \frac{r(\dot\varphi_r-\dot\varphi_L)}{w} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

c) $\begin{pmatrix} R(\theta_{WR}) & 0 \\ 0 & 1 \end{pmatrix} \cdot \dot\Omega(\dot\varphi_L,\dot\varphi_r) \Leftarrow X \cdot \dot\Omega$

$$= \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{r(\dot\varphi_r-\dot\varphi_L)}{w} & \frac{r(\dot\varphi_r+\dot\varphi_L)}{2} \\ \frac{r(\dot\varphi_r-\dot\varphi_L)}{w} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} -\sin(\theta)\frac{r(\dot\varphi_r-\dot\varphi_L)}{w} & -\cos(\theta)\frac{r(\dot\varphi_r-\dot\varphi_L)}{w} & \cos(\theta)\frac{r(\dot\varphi_r-\dot\varphi_L)}{w} \\ \cos(\theta)\frac{r(\dot\varphi_r-\dot\varphi_L)}{w} & -\sin(\theta)\frac{r(\dot\varphi_r-\dot\varphi_L)}{w} & \sin(\theta)\frac{r(\dot\varphi_r-\dot\varphi_L)}{w} \\ 0 & 0 & 0 \end{pmatrix}}$$

$X \Rightarrow \begin{pmatrix} R(\theta_{WR}) & 0 \\ 0 & 1 \end{pmatrix}$ given from perspective of robot frame

d) from prev HW $V_\Omega(X) = d(L_x)_I(\Omega) = X \cdot \Omega$

$$\Rightarrow V_{(\dot{e}_c, \dot{e}_r)} X = X \dot{\Omega}(\dot{e}_c, \dot{e}_r) = d(L_x), \dot{\Omega}(\dot{e}_c, \dot{e}_r)$$

Because $\dot{e}_r$ and $\dot{e}_c$ are constant, we can use $V(x)$ as the vector field of X and take the derivative of the left-multiplication map in order to map the tangent space $T_x(SE(2))$

↳ thus, left-invariant

e) Because from (d) we showed that $V_{(\dot{e}_r, \dot{e}_c)}(X)$ is left-invariant with Tangent mapping of $SE(2)$ with $d(L_x)$ and $\dot{e}_r$ and $\dot{e}_c$ are constant. We can integrate using the exponential map in order to find $SE(2)$

$c(t) = x \exp(t\,\omega)$,

substituting for $x = X_0$, $\omega = \Omega(\dot{e}_c, \dot{e}_r)$

$$\boxed{\gamma(t) = X_0 \exp(t\,\dot{\Omega}(\dot{e}_c, \dot{e}_r))}$$

Created With Tiny Scanner

Problem 2

a) $\ddot{h} = \frac{4K_T u}{m} - g$ $\quad h(0)=0 \quad \dot{h}=0 \quad u = PID + \frac{mg}{4K_T}$ $\quad r = 1m$

$m = 65g = 0.065kg$ $\quad K_T = 5.276\times10^{-4}$

$g = 9.81 m/s^2$

$m = 65$

Design P controller, thus $\Rightarrow u = P + \frac{mg}{4K_T}$

$\Rightarrow u = K_p(r-h) + \frac{mg}{4K_T}$

$\ddot{h} = \frac{4K_T(K_p(r-h) + \frac{mg}{4K_T})}{m} - g$

$\ddot{h} = \frac{4K_T K_p(r-h)}{m} + \frac{\cancel{4K_T} mg}{4mK_T} - g$

$\ddot{h} = \frac{4K_T K_p r}{m} - \frac{4K_T K_p h}{m}$

$\ddot{h} + \frac{4K_T K_p}{m} h - \frac{4K_T K_p}{m} = 0$

$\ddot{h} + \frac{4K_T K_p}{m}(h-1) = 0$

for $K_p = 5$

$h(t) = 1 - \cos(0.403t)$

for $K_p = 15$

$h(t) = 1 - \cos(0.698t)$

for $K_p = 50$

$h(t) = 1 - \cos(1.274t)$

See Images for graph + discussion

b) PD controller, thus $u = PD + \frac{mg}{4K_T}$ $\quad u = K_p(r-h) - K_d\dot{h} + \frac{mg}{4K_T}$

underdamped $\Rightarrow \zeta^2 - 1 < 0$

let $\zeta = 0.5$

Want settling time = 3s

$\ddot{h} = \frac{4K_T}{m}\left(K_p(r-h) - K_d\dot{h} + \frac{mg}{4K_T}\right) - g$

$\ddot{h} = \frac{4K_T K_p(r-h)}{m} - \frac{4K_T K_d}{m}\dot{h}$

$\ddot{h} + \underbrace{\frac{4K_T K_d}{m}}_{2\zeta\omega_n}\dot{h} + \underbrace{\frac{4K_T K_p}{m}}_{\omega_n^2} h - \frac{4K_T K_p}{m} = 0$

$\ddot{h} + \frac{4K_T}{m}\left(K_d\dot{h} + (h-1)K_p\right) = 0$

$\Rightarrow \frac{3}{\zeta\omega_n} = 3s$

$\zeta\omega_n = 1$

$2\zeta\omega_n = 2$

$2 = 4\frac{K_T}{m}K_d$

$K_d = \frac{1}{2}\cdot\frac{0.065}{5.276\times10^{-4}}$

$\boxed{K_d = 61.60}$

from underdamped,

$0.5(\omega_n) = 1$

$\omega_n = 2$

$\omega_n^2 = 4$

$\frac{4K_T K_p}{m} = \omega_n^2$

$\frac{4K_T K_p}{m} = 4$

$K_p = \frac{m}{K_T} = \frac{0.065}{5.276\times10^{-4}}$

$\boxed{K_p = 123.20}$

$h(t) = -0.577e^{-1.t}\sin(1.732t) - e^{-1.t}\cos(1.732t) + 1$

See Image for graph + discussion

c) Same as part b. except want over damped, thus $\xi > 1$

$\ddot{h} + \frac{4K_T}{m}(K_d\dot{h} + (h-1)K_p) = 0$ $\boxed{K_d = 61.60}$ let $\xi = 2$ from settling time = 3s

$\hookrightarrow \xi\omega_n = 1$

$\omega_n = 1/2$

$\omega_n = 1/4$

$h(t) = 0.077e^{-1.866t} - 1.077e^{-0.134t} + 1$

$\omega_n^2 = 4\frac{K_T K_p}{m}$

$K_p = \frac{0.065}{16 \cdot 5.276 \cdot 10^{-4}}$

$\boxed{K_p = 7.70}$

See attatched Images for graph + discussion

d) $u' = 0.95u$ $u = K_p(1-h) - K_d\dot{h} + \frac{mg}{4K_T}$ $K_p = 61.6$ $K_d = 123.2$

$\ddot{h} = \frac{4K_T u}{m} - g$ $u' = 0.95(K_p(1-h) - K_d\dot{h} + \frac{mg}{4K_T})$

$\ddot{h} = \frac{4K_T}{m}\left(0.95(K_p(1-h) - K_d\dot{h} + \frac{mg}{4K_T}\right) - g$

$\ddot{h} = \frac{3.8K_T K_p}{m}(1-h) - \frac{3.8K_T K_d}{m}\dot{h} - 0.05g$

$\ddot{h} + \frac{3.8K_T K_d}{m}\dot{h} + \frac{3.8K_T K_p}{m}h - \frac{3.8K_T K_p}{m} + 0.05g = 0$

$\hookrightarrow h(t) = -0.0879e^{-1.765t} + 1.153e^{-0.135t} - 1.065$

with Integral

$u' = 0.95(K_p(1-h) - K_d\dot{h} + K_i\int_0^t e(\tau)d\tau + \frac{mg}{4K_T})$

$\ddot{h} = \frac{4K_T}{m}\left(0.95(K_p(1-h) - K_d\dot{h} + K_i\int_0^t e(\tau)d\tau + \frac{mg}{4K_T}\right) - g$

$\ddot{h} = \frac{3.8K_T K_p(1-h)}{m} - \frac{3.8K_T K_d}{m}\dot{h} + \frac{3.8K_T K_i}{m}\int_0^t (1-h)d_t - 0.05g$

take derivative $\rightarrow$ $\dddot{h} = -\frac{3.8K_T K_p}{m}\dot{h} - \frac{3.8K_T K_d}{m}\ddot{h} + \frac{3.8K_T K_i}{m}(1-h)$

$\dddot{h} + \frac{3.8K_T K_p}{m}\dot{h} + \frac{3.8K_T K_d}{m}\ddot{h} - \frac{3.8K_T K_i}{m}(1-h) = 0$

$\dddot{h} + \frac{3.8K_T}{m}\left(K_p\dot{h} + K_d\ddot{h} - K_i(1-h)\right) = 0$, $h(0) = 0$, $h'(0) = 0$

$K_p = 123.2$ $K_d = 61.6$

let $K_i = 5$, $h''(0) = 1$

$h(t) = -0.727e^{0.0398t} - 0.137e^{-0.9689t}\sin(1.714t - 0.273e^{-0.9689t}\cos(1.714t) + 1$

See Images

Problem 3

a) $m l^2 \ddot{\theta} = -mgl\sin(\theta) - \mu\dot{\theta} + \tau$ $\qquad x = (\theta, \dot{\theta})$

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) - \frac{\mu}{ml^2}\dot{\theta} + \frac{\tau}{ml^2} \qquad \dot{x} = \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix}$$

$$\dot{x} = \begin{pmatrix} \dot{\theta} \\ -\frac{g}{l}\sin(\theta) - \frac{\mu}{ml^2}\dot{\theta} + \frac{\tau}{ml^2} \end{pmatrix}$$

let, $x_1 = \theta \quad x_2 = \dot{\theta}$

$$\boxed{f(x,\tau) = \dot{x} = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{\mu}{ml^2}x_2 + \frac{\tau}{ml^2} \end{pmatrix}}$$

b) linearizing f at $x^* = (\pi, 0)$ and $\tau = 0$

$$\frac{df}{dx} = \begin{pmatrix} & 1 \\ -\frac{g}{l}\cos(x_1) & -\frac{\mu}{ml^2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos(x_1) & -\frac{\mu}{ml^2} \end{pmatrix}$$

$$\frac{df}{dx}(\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos(\pi) & -\frac{\mu}{ml^2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{\mu}{ml^2} \end{pmatrix}$$

get eigen values,

$$\begin{pmatrix} 0-\lambda & 1 \\ \frac{g}{l} & -\frac{\mu}{ml^2} - \lambda \end{pmatrix} \Rightarrow (-\lambda)\left(-\frac{\mu}{ml^2} - \lambda\right) - \frac{g}{l} \Rightarrow \lambda^2 + \frac{\mu\lambda}{ml^2} - \frac{g}{l} = 0$$

$$\lambda = \frac{-\frac{\mu}{ml^2} \pm \sqrt{\frac{\mu^2}{m^2l^4} + 4\frac{g}{l}}}{2}$$

because the right side has $4\,g/l$ added under the square root, we can see that if μ were positive and the root was added to $-\frac{\mu}{ml^2}$, the eigen value would be positive!

$\boxed{\text{thus, } \lambda \text{ is positive}}$

$\boxed{\rightarrow \text{unstable}}$

c) $\tau(x) = K_p\sin(x_1) + K_d x_2$

$$f(x, \tau(x)) = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{\mu}{ml^2}x_2 + \frac{K_p\sin(x_1) + K_d x_2}{ml^2} \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{-mgl\sin(x_1) - \mu x_2 + K_p\sin(x_1) + K_d x_2}{ml^2} \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} x_2 \\ \frac{(K_p - mgl)\sin(x_1) + (K_d - \mu)x_2}{ml^2} \end{pmatrix}}$$

d) linearizing $f(x, \pi(x))$ at $x^* = (\pi, 0)$

$$\frac{df}{dx} = \begin{pmatrix} \frac{df_1}{dx_1} & \frac{df_2}{dx_2} \\ \frac{df_1}{dx_1} & \frac{df_2}{dx_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{(K_p - mgl)\cos(x_1)}{ml^2} & \frac{(K_d - \mu)}{ml^2} \end{pmatrix} \Rightarrow \frac{df(\pi, 0)}{d(x)} = \begin{pmatrix} 0 & 1 \\ \frac{mgl - K_p}{ml^2} & \frac{K_d - \mu}{ml^2} \end{pmatrix}$$

eigen:
$$\begin{pmatrix} 0-\lambda & 1 \\ \frac{mgl - K_p}{ml^2} & \frac{K_d - \mu}{ml^2} - \lambda \end{pmatrix} \Rightarrow (-\lambda)\left(\frac{K_d - \mu}{ml^2} - \lambda\right) - \frac{mgl - K_p}{ml^2}$$

$$\lambda^2 - \frac{(K_d - \mu)}{ml^2}\lambda - \frac{mgl - K_p}{ml^2}$$

$$\lambda = \frac{\frac{K_d - \mu}{ml^2} \pm \sqrt{\left(\frac{K_d - \mu}{ml^2}\right)^2 + \frac{4g - K_p}{l}}}{2}$$

to be asymptomatically stable want λ to be negative, thus given $\mu > 0$ K_d must be less then μ to allow the left side of $\pm$ to be negative. In order to keep the right side of the $\pm$ always less than the left side we want the $\frac{4g - K_p}{l}$ term to take value away from $\left(\frac{K_d - \mu}{ml^2}\right)^2$ term thus $K_p > 4g$

$$\boxed{K_d > \mu;\ K_p > 4g;\ \mu > 0}$$

e) $V(x) = -mgl(1 + \cos(x_1)) + \alpha\, mgl(1 - \cos^2(x_1)) + \frac{1}{2} ml^2 x_2^2$

gradient $\Rightarrow \nabla V(x) = \left(mgl\sin(x_1) + 2\alpha\, mgl\sin(x_1)\cos(x_1),\ ml^2 x_2\right)$

Hessian $\Rightarrow$ $Hf(x_1, x_2) = \begin{pmatrix} \frac{d^2f}{dx_1^2} & \frac{d^2f}{dx_1 dx_2} \\ \frac{d^2f}{dx_1 dx_2} & \frac{d^2f}{dx_2^2} \end{pmatrix} = \begin{pmatrix} mgl\cos(x_1) + 2\alpha\, mgl\cos(2x_1) & 0 \\ 0 & ml^2 \end{pmatrix}$

Want Hessian eigen values to be positive @ $V(x^*) = 0 \rightarrow$ concave up

eigen: $\begin{pmatrix} mgl(\cos(x_1) + 2\alpha\, mgl\cos(2x_1) - \lambda & 0 \\ 0 & ml^2 - \lambda \end{pmatrix} \Rightarrow (ml^2 - \lambda)(mgl\cos(x_1) + 2\alpha\, mgl\cos(2x_1) - \lambda) = 0$

Continue →

3c) $V(x^*) = 0 \Rightarrow V(\pi, 0)$

$$(mL^2 - \lambda)(mgl\cos(x_1) + 2\alpha mgl\cos(2x_1) - \lambda) = 0$$

for $V(\pi,0) \Rightarrow (mL^2 - \lambda)(mgl\cos(\pi) + 2\alpha mgl\cos(2\pi) - \lambda) = 0$

$$\Rightarrow (mL^2 - \lambda)(mgl + 2\alpha mgl - \lambda) = 0$$

$$\Rightarrow m^2gl^3 + 2\alpha m^2gl^3 - \lambda mL^2 - \lambda mgl - 2\lambda\alpha mgl + \lambda^2 = 0$$

$$= \lambda^2 - \lambda(mL^2 + mgl + 2\alpha mgL) + 2\alpha m^2gl^3 + m^2gl^3 = 0$$

want $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} > 0$ because 'b' portion is already negative (cancel negatives in quadratic). Just need $4ac > 0$

$$\Rightarrow 4ac > 0$$

$$4(2\alpha m^2gl^3 + m^2gl^3) > 0$$

$$2\alpha m^2gl^3 + m^2gl^3 > 0 \rightarrow 2\alpha m^2gl > m^2gl^3$$

$$\boxed{\alpha > -\frac{1}{2}}$$

3f) $\dot{V} = \nabla V(x) \cdot f(x, \tau)$ from a: $f(x,\tau) = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{\mu}{mL^2}x_2 + \frac{\tau}{mL^2} \end{pmatrix}$

from e: $\nabla V(x) = \begin{pmatrix} mgl\sin(x_1) + 2\alpha mgl\sin(x_1)\cos(x_1) \\ mL^2 x_2 \end{pmatrix}$

$$\begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{\mu}{mL^2}x_2 + \frac{\tau}{mL^2} \end{pmatrix} \cdot \begin{pmatrix} mgl\sin(x_1) + 2\alpha mgl\sin(x_1)\cos(x_1) \\ mL^2 x_2 \end{pmatrix}$$

$$= \cancel{x_2 mgl\sin(x_1)} + 2x_2\alpha mgl\sin(x_1)\cos(x_1) - \cancel{x_2 mgl\sin(x_1)} - x_2^2\mu + x_2\tau$$

$$\boxed{\dot{V} = -\mu x_2^2 + (2\alpha mgl\sin(x_1)\cos(x_1) + \tau)x_2}$$

3g) $\tau(x) = -2\alpha mgl\sin(x_1)\cos(x_1)$ $\quad f(x,\tau) = \dot{x} = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{\mu}{ml^2}x_2 + \frac{\tau}{ml^2} \end{pmatrix}$

$$\Rightarrow f(x, \tau(x)) = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{\mu}{ml^2}x_2 - \frac{2\alpha g\sin(x_1)\cos(x_1)}{l} \end{pmatrix}$$

$$\boxed{f(x,\tau(x)) = \begin{pmatrix} x_2 \\ -\frac{g\sin(x_1)}{l}\left(1+2\alpha\cos(x_1)\right) - \frac{\mu}{ml^2}x_2 \end{pmatrix}}$$

3h) to find set of stationary points for the closed loop system

$\rightarrow f(x,\tau(x)) = 0$

thus, $x_2 = 0$, $\quad -\frac{g\sin(x_1)(1+2\alpha\cos(x_1))}{l} - \frac{\mu}{ml^2}x_2 = 0$

$$\Rightarrow -\frac{g\sin(x_1)(1+2\alpha\cos(x_1))}{l} = 0$$

$$\Rightarrow \sin(x_1)(1+2\alpha\cos(x_1)) = 0$$

$\sin(x_1) = 0$ or $1 + 2\alpha\cos(x_1) = 0$

$\Rightarrow x_1 = 0, \pi$ $\qquad \cos(x_1) = -\frac{1}{2\alpha}$

$\pi/2 \leq x_1 \leq \pi$

$\boxed{x_1 = n\pi}$ $\boxed{x_2 = 0}$

3i) $\dot{V} = -\mu x_2^2 + (2\alpha mgl\sin(x_1)\cos(x_1) + \tau)x_2$ $\quad \tau = -2\alpha mgl\sin(x_1)\cos(x_1)$

$= -\mu x_2^2 + (2\alpha mgl\sin(x_1)\cos(x_1) - 2\alpha mgl\sin(x_1)\cos(x_1))x_2$

$\dot{V} = -\mu x_2^2$

$\boxed{\text{Clearly } \dot{V} \leq 0 \text{ for all } x_2 \text{ and } \mu}$

3j) Given $\dot{V}(x) = 0$ as the set constraint. We can see from (f) $\dot{V} = -\mu x_2^2$ that $x_2 = 0$ satisfies the condition. Thus for points to be contained entirely in the set $\mathcal{I}$ they must retain $x_2 = 0$ where x_2 is the velocity. Thus, this implies only stationary points can remain in $\tilde{\mathcal{I}}$, otherwise, for the case they aren't stationary they have some velocity which directly contradicts the constraint on the set $\mathcal{I}$

3k) See images