Lab 1 | Collaborated w/ Juan, Phillip, Yousuf

Q1) $OP_1 = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$ $OP_2 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$ $OP_3 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$ $OP_4 = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$ $T_{so}(x) = A_{so}x + b_{so}$

$sP_1 = \begin{bmatrix} -1.3840 \\ 4.5620 \\ -0.1280 \end{bmatrix}$ $sP_2 = \begin{bmatrix} -0.9608 \\ 1.3110 \\ -1.6280 \end{bmatrix}$ $sP_3 = \begin{bmatrix} 1.3250 \\ -2.3890 \\ 1.7020 \end{bmatrix}$ $sP_4 = \begin{bmatrix} -1.3140 \\ 0.2501 \\ -0.7620 \end{bmatrix}$

$$M = \begin{bmatrix} R_{so} & t_{so} \\ 0 & 1 \end{bmatrix} \qquad T_{so} = R_{so}x + t_{so}$$

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \cdot \begin{bmatrix} OP_x \\ OP_y \\ OP_z \end{bmatrix} + \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} SP_x \\ SP_y \\ SP_z \end{bmatrix}$$

$$OP_{1x}(R_{11}) + OP_{1y}(R_{12}) + OP_{1z}(R_{13}) + T_1 = SP_{1x}$$

$$OP_{2x}(R_{21}) + OP_{2y}(R_{22}) + OP_{2z}(R_{23}) + T_2 = SP_{2x}$$

$$OP_{3x}(R_{31}) + OP_{3y}(R_{32}) + OP_{3z}(R_{33}) + T_3 = SP_{3x}$$

let

$$\begin{bmatrix} OP_{1x} & OP_{1y} & OP_{1z} & 1 \\ OP_{2x} & OP_{2y} & OP_{2z} & 1 \\ OP_{3x} & OP_{3y} & OP_{3z} & 1 \\ OP_{4x} & OP_{4y} & OP_{4z} & 1 \end{bmatrix} = A \Rightarrow$$

Plug in Calculator

$$A^{-1} \cdot \begin{bmatrix} SP_{1x} \\ SP_{2x} \\ SP_{3x} \\ SP_{4x} \end{bmatrix} = \begin{bmatrix} R_{11} \\ R_{12} \\ R_{13} \\ T_1 \end{bmatrix} \rightarrow \begin{bmatrix} 0.7068 \\ -0.6123 \\ 0.3536 \\ 0.1000 \end{bmatrix}$$

$$A^{-1} \cdot \begin{bmatrix} SP_{1y} \\ SP_{2y} \\ SP_{3y} \\ SP_{4y} \end{bmatrix} = \begin{bmatrix} R_{21} \\ R_{22} \\ R_{23} \\ T_2 \end{bmatrix} \rightarrow \begin{bmatrix} 0.7072 \\ 0.6122 \\ -0.3537 \\ 0.2500 \end{bmatrix}$$

$$A^{-1} \cdot \begin{bmatrix} SP_{1z} \\ SP_{2z} \\ SP_{3z} \\ SP_{4z} \end{bmatrix} = \begin{bmatrix} R_{31} \\ R_{32} \\ R_{33} \\ T_3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0.5000 \\ 0.8660 \\ 0.9700 \end{bmatrix}$$

$$R = \begin{bmatrix} 0.7068 & -0.6123 & 0.3536 \\ 0.7072 & 0.6122 & -0.3537 \\ 0 & 0.5000 & 0.8660 \end{bmatrix}$$

$$T = \begin{bmatrix} 0.100 \\ 0.250 \\ 0.970 \end{bmatrix}$$

$$\boxed{T_{so} = \begin{bmatrix} 0.7068 & -0.6123 & 0.3536 \\ 0.7072 & 0.6122 & -0.3537 \\ 0 & 0.5000 & 0.8660 \end{bmatrix} x + \begin{bmatrix} 0.100 \\ 0.250 \\ 0.970 \end{bmatrix}}$$

(Q2)

a) $\boxed{L_v(x) = v + x}$ → from vector group operation defined in problem

b) $dL_x : T_e(G) \to T_x(G) \to T_{xL}$

$\to dL_v(x) = T_{v+x}(G) = I$

c) from notes: $dL_g V(x) = V(gx)$, given vector ξ

$\Rightarrow dL_v V(x) = V(vx)$

$V(vx) = I \cdot \xi$

$\boxed{V(vx) = \xi}$ → geometrically, the field is the set of vectors all pointing in the same direction

d) $\boxed{L_A(x) = A \cdot x}$ → $GL(n)$ is set of $n \times n$ invertible matrices w/ matrix multiplication

e) from d using simple derivative with A as the constant

$\rightarrow \boxed{dL_A(x) = A}$

f) $dL_g V(x) = V(gx)$

from e,

$dL_A(x) = A \to dL_\Omega(x) = \Omega$

$\boxed{V(\Omega x) = \Omega V_\Omega}$

Q3) a) $\Omega = \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix}$

$\Omega^2 = \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix}\begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix} = \begin{bmatrix} -w^2 & 0 \\ 0 & -w^2 \end{bmatrix}$

$\Omega^3 = \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix}\begin{bmatrix} -w^2 & 0 \\ 0 & -w^2 \end{bmatrix} = \begin{bmatrix} 0 & -w^3 \\ w^3 & 0 \end{bmatrix}$

$\Omega^4 = \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix}\begin{bmatrix} 0 & -w^3 \\ w^3 & 0 \end{bmatrix} = \begin{bmatrix} w^4 & 0 \\ 0 & w^4 \end{bmatrix}$

$\Omega^5 = \begin{bmatrix} w^4 & 0 \\ 0 & w^4 \end{bmatrix}\begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix} = \begin{bmatrix} 0 & w^5 \\ -w^5 & 0 \end{bmatrix}$

$\Omega^6 = \begin{bmatrix} 0 & w^5 \\ -w^5 & 0 \end{bmatrix}\begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix} = \begin{bmatrix} -w^6 & 0 \\ 0 & -w^6 \end{bmatrix}$

k-even $\quad \Omega^{2k} = \begin{bmatrix} (-1)^k w^{2k} & 0 \\ 0 & (-1)^k w^{2k} \end{bmatrix}$

k-odd$_1$ $\quad \Omega^{4k+1} = \begin{bmatrix} 0 & w^{4k+1} \\ -(w)^{4k+1} & 0 \end{bmatrix}$

k-odd$_2$ $\quad \Omega^{4k+3} = \begin{bmatrix} 0 & -(w)^{4k+3} \\ w^{4k+3} & 0 \end{bmatrix}$

b)

$$\exp(X) \triangleq \sum_{k=0}^{\infty} \frac{X^k}{k!} \Rightarrow \exp(\Omega) = \sum_{k=0}^{\infty} \frac{\Omega^k}{k!} = \sum_{k=0}^{\infty}\left(\frac{\Omega^{2k}}{(2k)!} + \frac{\Omega^{4k+1}}{(4k+1)!} + \frac{\Omega^{4k+3}}{(4k+3)!}\right)$$

$$= \sum_{k=0}^{\infty} \frac{\Omega^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{\Omega^{4k+1}}{(4k+1)!} + \sum_{k=0}^{\infty} \frac{\Omega^{4k+3}}{(4k+3)!}$$

$$= \begin{bmatrix} \sum_{x=0}^{\infty} \frac{(-1)^x w^{2x}}{(2x)!} & 0 \\ 0 & \sum_{x=0}^{\infty} \frac{(-1)^x w^{2x}}{(2x)!} \end{bmatrix} + \begin{bmatrix} 0 & \sum_{x=0}^{\infty} \frac{w^{4x+1}}{(4x+1)!} \\ -\sum_{x=0}^{\infty} \frac{w^{4x+1}}{(4x+1)!} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\sum_{x=0}^{\infty} \frac{w^{4x+3}}{(4x+3)!} \\ \sum_{x=0}^{\infty} \frac{w^{4x+3}}{(4x+3)!} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{x=0}^{\infty} \frac{(-1)^x w^{2x}}{(2x)!} & 0 \\ & \sum_{x=0}^{\infty} \frac{(-1)^x w^{2x}}{(2x)!} \end{bmatrix} + \begin{bmatrix} 0 & \sum_{x=0}^{\infty} \frac{(-1)^x w^{2x+1}}{(2x+1)!} \\ -\sum_{x=0}^{\infty} \frac{(-1)^x w^{2x+1}}{(2x+1)!} & 0 \end{bmatrix}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Thus,

$$= \begin{bmatrix} \cos(w) & 0 \\ 0 & \cos(w) \end{bmatrix} + \begin{bmatrix} 0 & \sin(w) \\ -\sin(w) & \end{bmatrix} = \boxed{\begin{bmatrix} \cos(w) & \sin(w) \\ -\sin(w) & \cos(w) \end{bmatrix}}$$

$\Rightarrow$ Geometrically, represent rotation of angle w

Q4) a) $\gamma(t) \stackrel{?}{=} x * \exp(tw)$ $\quad x = \exp(\log(x))$ $\quad \gamma(0) = x$ $\quad \gamma(1) = y$

$\boxed{\gamma(t) = x \exp\left(t \cdot \log(x^{-1}y)\right)}$ $\quad y = x \cdot \exp(1 \cdot w)$

$y \cdot x^{-1} = \exp(w)$

$\log(x^{-1}y) = w$

b) $\exp(\xi) = \xi$

↳ because log is inverse of exp → $\log(\xi) = \xi$

from a,

$r(t) = x \exp(t \cdot \log(x^{-1}y))$

$\Downarrow$

$\boxed{\gamma(t) = x + \boxed{t \cdot (y - x)}}$

c) $X_0 = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.4330 & 0.1768 & 0.8839 \\ 0.2500 & 0.9186 & -0.3062 \\ -0.8660 & 0.3536 & 0.3536 \end{pmatrix} \right) \Rightarrow \begin{bmatrix} 0.433 & 0.1768 & 0.8839 & 1 \\ 0.25 & 0.9186 & -0.3062 & 1 \\ -0.866 & 0.3536 & 0.3536 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$X_1 = \left(\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 0.7500 & 0.1768 & 0.8839 \\ 0.4330 & 0.9186 & -0.3062 \\ -0.5000 & 0.3536 & 0.3536 \end{pmatrix} \right) \Rightarrow \begin{bmatrix} 0.75 & 0.1768 & 0.8839 & 2 \\ 0.433 & 0.9186 & -0.3062 & 4 \\ -0.5 & 0.3536 & 0.3536 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Given X_0, X_1 → $\gamma(t) = X_0 \exp(t \cdot \log(x^{-1}y))$

$\gamma(1/2) = X_0 \exp(t \cdot \log(x^{-1}y))$

↓ calculator ⟶ python: x@expm(0.5·logm(inv(x)@y))

$$\boxed{\gamma(1/2) = \begin{bmatrix} 0.5972 & 0.1768 & 0.8839 & 1.5602 \\ 0.3448 & 0.9186 & -0.3062 & 2.5347 \\ -0.6764 & 0.3536 & 0.3536 & 1.5695 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

(Q5) $P(\check{Y}_i|\theta) = \dfrac{P(\theta|\check{Y}) \cdot P(\check{Y})}{P(\theta)}$ $\check{Y}_i = A_i\theta + b_i + \epsilon_i$

$\epsilon_i \sim N(\mu_i, \Sigma_i)$

a) posterior = $p(\theta|\check{Y}_1, \dots, \check{Y}_m)$ likelihood function ⇒ $L(\theta) = \prod_{i=1}^{n} f_i(y_i|\theta)$

prior = $p(\theta)$ ↳ $L(\theta) = \prod_{i=1}^{n} p(\check{Y}_i|\theta)$

likelihoods = $p(\check{Y}_i|\theta)$ $L(\theta) = p(\check{Y}_1, \dots, \check{Y}_m|\theta)$

Bayes

↳ $P(\theta|\check{Y}_1, \dots, \check{Y}_m) = \dfrac{P(\check{Y}_1, \dots, \check{Y}_m|\theta)\, P(\theta)}{P(\check{Y}_1, \dots, \check{Y}_m)}$ ← drop normalizing constant since problem states we can leave it unnormalized

$\Rightarrow P(\check{Y}_1, \dots, \check{Y}_m|\theta) P\theta$

$= L(\theta)\, p(\theta)$

$$\boxed{p(\theta|\check{Y}_m) = \prod_{i=1}^{n} p(\check{Y}_i|\theta)\, p(\theta)}$$