Numerical HW 4 Yu Sheng ys3171

Q2. a) the value of option by PDE method. with θ set to 0, 0.5 and 1 respectively (without k).

	spot	120
	USD zero rat	0.02
(domestic)	JPY zero rate	0.005
	strike	119
	vol	0.1
	time to expir	7
	m	20
	gamma max	24
	method	Explicit
	theta	0
on	x0	-0.05
	x10	0.0083682
	delta x	0.0058368
	x20	0.0667365
	tau max	9.589E-05
	delta tau	3.995E-06
	lambda	0.1172764
	S min	113.1963
	S max	127.21264
	qdelta	-3
	q	1
	pde price	1.2585934
	bs price	1.2520173
	discrepancy	0.0065762

	spot	120
	USD zero rat	0.02
(domestic)	JPY zero rate	0.005
	strike	119
	vol	0.1
	time to expir	7
	m	20
	gamma max	24
	method	Implicit
	theta	1
on	x0	-0.05
	x10	0.0083682
	delta x	0.0058368
	x20	0.0667365
	tau max	9.589E-05
	delta tau	3.995E-06
	lambda	0.1172764
	S min	113.1963
	S max	127.21264
	qdelta	-3
	q	1
	pde price	1.2695714
	bs price	1.2520173
	discrepancy	0.0175542

	spot	120
	USD zero rat	0.02
(domestic)	JPY zero rate	0.005
	strike	119
	vol	0.1
	time to expir	7
	m	20
	gamma max	24
	method	Crack Nicolson
	theta	0.5
on	x0	-0.05
	x10	0.0083682
	delta x	0.0058368
	x20	0.0667365
	tau max	9.589E-05
	delta tau	3.995E-06
	lambda	0.1172764
	S min	113.1963
	S max	127.21264
	qdelta	-3
	q	1
	pde price	1.2629209
	bs price	1.2520173
	discrepancy	0.0109037

b).

theta=0					
x0	-0.05	-0.04	-0.03	-0.02	-0.01
pde	1.25859342	1.29659004	1.74613487	5.14399049	-1.954E+12
lambda	0.11727639	0.17078243	0.27140632	0.49647668	1.18420909

We note when $X_0 = -0.01$. Pde value is meaningless. and $\lambda = 1.18 > 0.5$ in this case. When $\lambda \leq 0.5$, we see for other X_0. it produce some meaningful value. therefore this coherent with explicit method is stable when $\lambda \leq 0.5$.

(c)

theta=0.5					
x0	-0.05	-0.04	-0.03	-0.02	-0.01
pde	1.26292094	1.33384632	1.92110258	5.69517187	21.9115686
lambda	0.11727639	0.17078243	0.27140632	0.49647668	1.18420909

When $\theta = 0.5$. it employs crack nicolson method which is unconditionally stable. Therefore all X_0 and λ produce meaningful stable results.

$\sin 0 = \sin 2\pi = 0,$

Q3. a) $G\vec{v}^{(k)} = \begin{pmatrix} \alpha & \beta & & \\ \gamma & \alpha & \ddots & \\ & \ddots & \ddots & \beta \\ & & \gamma & \alpha \end{pmatrix} \times \begin{pmatrix} \sqrt{\frac{\gamma}{\beta}} \sin\frac{k\pi}{N+1} \\ \vdots \\ \sqrt{\frac{\gamma}{\beta}}^N \sin\frac{Nk\pi}{N+1} \end{pmatrix} = \begin{pmatrix} \alpha\sqrt{\frac{\gamma}{\beta}} \sin\frac{k\pi}{N+1} + \beta\sqrt{\frac{\gamma}{\beta}}^2 \sin\frac{2k\pi}{N+1} \\ \vdots \\ \gamma \times \sqrt{\frac{\gamma}{\beta}}^{N-1} \sin\frac{(N-1)k\pi}{N+1} + \alpha\sqrt{\frac{\gamma}{\beta}}^N \sin\frac{Nk\pi}{N+1} \end{pmatrix} = \begin{pmatrix} \gamma\sqrt{\frac{\gamma}{\beta}}^0 \sin\frac{0k\pi}{N+1} + \alpha\sqrt{\frac{\gamma}{\beta}} \sin\frac{k\pi}{N+1} + \beta\sqrt{\frac{\gamma}{\beta}}^2 \sin\frac{2k\pi}{N+1} \\ \vdots \\ \gamma\alpha\sqrt{\frac{\gamma}{\beta}}^{N-1} \sin\frac{(N-1)k\pi}{N+1} + \alpha\sqrt{\frac{\gamma}{\beta}}^N \sin\frac{Nk\pi}{N+1} + \beta\sqrt{\frac{\gamma}{\beta}}^{N+1} \sin\frac{(N+1)k\pi}{N+1} \end{pmatrix}$

Thus $G\vec{v}^{(k)}_{(i)} = \gamma \times \sqrt{\frac{\gamma}{\beta}}^{i-1} \sin\frac{(i-1)k\pi}{N+1} + \alpha\left(\sqrt{\frac{\gamma}{\beta}}\right)^i \sin\frac{ik\pi}{N+1} + \beta\left(\sqrt{\frac{\gamma}{\beta}}\right)^{i+1} \sin\frac{(i+1)k\pi}{N+1}$ for $i = 1 \ldots N$.

$\mu_{(k)}\vec{V}^{(\gamma)T}_{(i)} = \left(\alpha + 2\beta\sqrt{\frac{\gamma}{\beta}} \cos\frac{k\pi}{N+1}\right)\left(\sqrt{\frac{\gamma}{\beta}}^i \sin\frac{ik\pi}{N+1}\right)$

$= \alpha\sqrt{\frac{\gamma}{\beta}}^i \sin\frac{ik\pi}{N+1} + 2\beta\sqrt{\frac{\gamma}{\beta}}^{i+1} \cos\frac{k\pi}{N+1} \cdot \sin\frac{ik\pi}{N+1}$

$= \alpha\sqrt{\frac{\gamma}{\beta}}^i \sin\frac{ik\pi}{N+1} + \sqrt{\frac{\gamma}{\beta}}^{i+1} \cdot \beta \cdot \left(\sin\frac{(i-1)k\pi}{N+1} + \sin\frac{(i+1)k\pi}{N+1}\right)$

$= \gamma\sqrt{\frac{\gamma}{\beta}}^{i-1} \sin\frac{(i-1)k\pi}{N+1} + \alpha\sqrt{\frac{\gamma}{\beta}}^i \sin\frac{ik\pi}{N+1} + \beta\sqrt{\frac{\gamma}{\beta}}^{i+1} \sin\frac{(i+1)k\pi}{N+1}$

Therefore we have proven $\mu_k \vec{V}^{(\gamma)T}_{(i)} = G\vec{V}^{(\gamma)T}_{(i)} \Rightarrow \mu_k \vec{V}^{(\gamma)T} = G\vec{V}^{(\gamma)T}$

Thus $\vec{V}^{(\gamma)T}$ is an eigenvector of G for the eigenvalue μ_k.

b) $(G - \lambda I) = 0$. when $\beta = 0$. then $|G - \lambda I| = (\alpha - \lambda)^N = 0$

Thus $\lambda = \alpha$.

If $\gamma = 0$ then any non zero vector $X = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$ satisfy $GX = \lambda X$.

If $\gamma \neq 0$. then $(G - \alpha I)\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = 0$. thus $\begin{cases} \gamma x_1 = 0 \\ \gamma x_2 = 0 \\ \vdots \\ \gamma x_{N-1} = 0 \end{cases}$

then only vector $X = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_N \end{pmatrix}$ satisfy $GX = \lambda X$ for any $x_N \in \mathbb{R}$

Thus eigenvector will be $\begin{cases} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} & \text{if } \gamma = 0. \\ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_N \end{pmatrix} & \text{if } \gamma \neq 0. \end{cases}$

(c) To make G singular we want $|G| = 0$. that is G has an eigenvalue 0.

① If $\beta\gamma = 0$ ($\beta = 0 \vee \gamma = 0$) then $|G| = a^N = 0$ then $\alpha = 0$

② If $\beta\gamma > 0$. then $\mu_k = \alpha + 2\beta\sqrt{\frac{\gamma}{\beta}} \cos\frac{k\pi}{N+1} = 0$ for some $k = 1, \ldots N$.

③ If $\beta\gamma < 0$, $\mu_k = \alpha + 2\beta i\sqrt{\frac{\gamma}{\beta}} \cos\frac{k\pi}{N+1}$ then $\alpha = 0$. and $2\beta i \sqrt{\frac{\gamma}{\beta}} \cos\frac{k\pi}{N+1} = 0$. in order to have $\cos\frac{k\pi}{N+1} = 0$. N is odd.

thus for N is even, set is $\{\{\beta\gamma = 0, \alpha = 0\} \cup \{\beta\gamma > 0, \exists k \text{ in } \{1, \ldots, N\} \text{ s.t } \mu_k = \alpha + 2\beta\sqrt{\frac{\gamma}{\beta}} \cos\frac{k\pi}{N+1} = 0\}\}$

for N is odd, set is $\{\{\beta\gamma = 0, \alpha = 0\} \cup \{\beta\gamma > 0, \exists k \text{ in } \{1, \ldots, N\} \text{ s.t. } \mu_k = \alpha + 2\beta\sqrt{\frac{\gamma}{\beta}} \cos\frac{k\pi}{N+1} = 0\} \cup \{\beta\gamma < 0, \alpha = 0\}\}$

Q4 (a) $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial S} \cdot \frac{\partial S}{\partial x} = \frac{\partial V}{\partial S} \cdot ke^x = \frac{\partial V}{\partial S} \cdot S$

$\frac{\partial^2 V}{\partial x^2} = \left(\frac{\partial V}{\partial S} \cdot S\right)\frac{\partial}{\partial x} = \left(\frac{\partial V}{\partial x}\right)\frac{\partial V}{\partial S} S + \frac{\partial V}{\partial S} \cdot S = \frac{\partial^2 V}{\partial S^2} S^2 + \frac{\partial V}{\partial S} \cdot S$

(b) $\frac{\partial V(x,\tau)}{\partial \tau} = \beta k e^{\alpha x + \beta\tau} y(x,\tau) + ke^{\alpha x + \beta\tau} \frac{\partial y(x,\tau)}{\partial \tau}$

$\frac{\partial V(x,\tau)}{\partial x} = \alpha k e^{\alpha x + \beta\tau} y(x,\tau) + ke^{\alpha x + \beta\tau} \frac{\partial y(x,\tau)}{\partial \tau}$

$\frac{\partial^2 V(x,\tau)}{\partial x^2} = \alpha^2 k e^{\alpha x + \beta\tau} y(x,\tau) + 2\alpha k e^{\alpha x + \beta\tau} \frac{\partial y(x,\tau)}{\partial x} + ke^{\alpha x + \beta\tau} \frac{\partial^2 y(x,\tau)}{\partial x^2}$

Subsititute into equation and divided by $ke^{\alpha x + \beta\tau}$ on both side

$\beta y + \frac{\partial y}{\partial \tau} = \alpha^2 y + 2\alpha \frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2} + \alpha b y + b\frac{\partial y}{\partial x} + cy.$

Thus $\frac{\partial y}{\partial \tau} = (\alpha^2 + \alpha b + c - \beta)y + (2\alpha + b)\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}$

Q4 (c) $c(S,T) = \max\{S-K, 0\} = \max\{Ke^{x} - K, 0\}$

Also, $c(S,T) = V(x,0) = Ke^{-\frac{x}{2}(q\delta-1)} y(x,0)$ $\Leftarrow \alpha = -\frac{q\delta-1}{2}$

$y(x,0) = \frac{\max\{Ke^{x}-K, 0\}}{Ke^{-\frac{x}{2}(q\delta-1)}} = \max\{e^{\frac{x}{2}(q\delta+1)} - e^{\frac{x}{2}(q\delta-1)}, 0\}$

(d) $p(S,T) = \max\{K-S, 0\} = \max\{K - Ke^{x}, 0\}$

$p(S,T) = V(x,0) = Ke^{-\frac{x}{2}(q\delta-1)} y(x,0)$

$y(x,0) = \frac{\max\{K-Ke^{x}, 0\}}{Ke^{-\frac{x}{2}(q\delta-1)}} = \max\{e^{\frac{x}{2}(q\delta-1)} - e^{\frac{x}{2}(q\delta+1)}, 0\}$

forward = Call - Put $= \max\{e^{\frac{x}{2}(q\delta+1)} - e^{\frac{x}{2}(q\delta-1)}\} - \max\{e^{\frac{x}{2}(q\delta-1)} - e^{\frac{x}{2}(q\delta+1)}\}$

$= e^{\frac{x}{2}(q\delta+1)} - e^{\frac{x}{2}(q\delta-1)}$

Q5. $-\lambda\theta W_{i-1,\gamma+1} + (1+2\lambda\theta) W_{i,\gamma+1} - \lambda\theta W_{i+1,\gamma+1} = \lambda(1-\theta) W_{i-1,\gamma} + (1-2\lambda(1-\theta)) W_{i,\gamma} + \lambda(1-\theta) W_{i+1,\gamma}$

when $i=1$

$(1+2\lambda\theta) W_{1,\gamma+1} - \lambda\theta W_{2,\gamma+1} = (1-2\lambda(1-\theta)) W_{1,\gamma} + \lambda(1-\theta) W_{2,\gamma} + (\lambda(1-\theta) W_{0,\gamma} + \lambda\theta W_{0,\gamma+1})$

By $A\vec{W}^{(\gamma+1)} = B\vec{W}^{(\gamma)} + \vec{d}^{(\gamma)}$, thus $r_1(a, \tau_\gamma) = \lambda(1-\theta) W_{0,\gamma} + \lambda\theta W_{0,\gamma+1}$

when $i = N-1$.

$-\lambda\theta W_{N-1,\gamma+1} + (1+2\lambda\theta) W_{N,\gamma+1} = \lambda(1-\theta) W_{N-2,\gamma} + (1-2\lambda(1-\theta)) W_{N-1,\gamma} + (\lambda(1-\theta) W_{N,\gamma} + \lambda\theta W_{N,\gamma+1})$

Thus $r_2(b, \tau_\gamma) = \lambda(1-\theta) W_{N,\gamma} + \lambda\theta W_{N,\gamma+1}$

for $i = 2$ to $N-2$. $d_{i,\gamma} = 0$.

$$d^{(\gamma)} = \begin{pmatrix} r_1(a, \tau_\gamma) \\ 0 \\ \vdots \\ 0 \\ r_2(b, \tau_\gamma) \end{pmatrix} \quad \text{and} \quad \begin{array}{ccc} & \text{Call} & \text{Put.} \\ r_1(a, \tau_\gamma) & 0 & e^{\frac{x}{2}(q\delta-1) + \frac{\tau}{4}(q\delta-1)^2} \\ r_2(b, \tau_\gamma) & e^{\frac{x}{2}(q\delta+1) + \frac{\tau}{4}(q\delta+1)^2} & 0 \end{array}$$

$$\vec{d}^{(\gamma)} = \begin{cases} \begin{pmatrix} 0 & \cdots & 0 & \lambda\theta W_{N,\gamma_{max}} + \lambda(1-\theta) W_{N,\gamma_{max}-1} \\ 0 & \cdots & 0 & \lambda\theta W_{N,\gamma_{max}-1} + \lambda(1-\theta) W_{N,\gamma_{max}-2} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda\theta W_{N,1} + \lambda(1-\theta) W_{N,0} \end{pmatrix} & \text{for European call} \\ \begin{pmatrix} \lambda\theta W_{0,\gamma_{max}} + \lambda(1-\theta) W_{0,\gamma_{max}-1} & 0 & \cdots & 0 \\ \lambda\theta W_{0,\gamma_{max}-1} + \lambda(1-\theta) W_{0,\gamma_{max}-2} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \lambda\theta W_{0,1} + \lambda(1-\theta) W_{0,0} & 0 & \cdots & 0 \end{pmatrix} & \text{for European put} \end{cases}$$