2019. 3. 11

<u>Practical Greeks</u> (AAD output true theoretical derivatives)

Big "O" notation

$f = O(g)$. if there is a constant M, s.t $|f(x)| \le M|g(x)|$ for all x under consideration

Think of h to be very small (which is ε in calculus)

h^2 is smaller than h (e.g. $|h| < 1$)

$|h^3|$ is smaller than h^2 etc.

Ultimately, $\sum_{j=m}^{n} a_j h^j = a_m h^m + a_{m+1} h^{m+1} + \dots + a_n h^n = O(h^m)$

Nice property:

$$\alpha O(h^m) = O(h^m)$$
$$h^k O(h^m) = O(h^{k+m})$$
$$O(h^m) + O(h^n) = O(h^{\max\{m,n\}})$$

<u>Taylor's Expansion</u>

$$f(x+h) = \underbrace{f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \dots + \frac{h^n}{n!}f^{(n)}(x)}_{T_n(x)} + O(h^{n+1})$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + O(h^3)$$

$$\frac{f(x+h) - f(x)}{h} = f'(x) + \frac{h}{2}f''(x) + O(h^2)$$ * if f is twice differentiable ($f \in \mathcal{C}^2$)

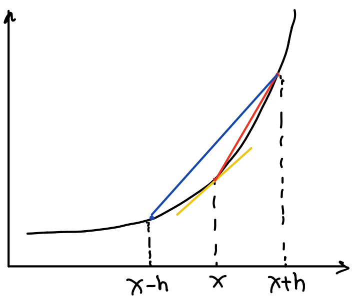
$h > 0$ $\boxed{\frac{f(x+h) - f(x)}{h} = f'(x) + O(h)}$ ~ Up Delta ①

$a_1 h + a_2 h^2 + \dots$

$h < 0$: $\frac{f(x) - f(x-h)}{h} = f'(x) + O(h)$ ~ down Delta ②

$\frac{① + ②}{2} = \frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h)$

You can prove $\frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h^2)$ * If $f \in \mathcal{C}^3$



<u>Proof:</u> $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + O(h^3)$

$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) + O(h^3)$

$f(x+h) - f(x-h) = 2hf'(x) + O(h^3)$

$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h^2)$

write f_k means $f(x+kh)$,

$$\frac{f_1 - f_0}{h} = f_0' + O(h)$$
$$\frac{f_0 - f_{-1}}{h} = f_0' + O(h)$$
$$\frac{f_1 - f_{-1}}{2h} = f_0' + O(h)$$

$$\frac{\partial}{\partial \sigma}\left[\frac{1}{\sigma\sqrt{T}x}\phi(-d_2(x))\right]$$

$$= \frac{\phi'(-d_2(x))\,\sigma\sqrt{T} - \sqrt{T}\cdot\phi(-d_2(x))}{\sigma^2 T}$$

$$= \phi(\mp d_2(x))' d_2(x) \sqrt{T}$$

<u>Theorem:</u> $f(x) - T_n(x) = \text{Remainder}$ ← M in definition

$= \boxed{\frac{f^{(n+1)}(\xi)}{(n+1)!}} h^{n+1}$

ξ lies between x, $x+h$

Let f = valuation function

$$\text{Delta} = \frac{\partial f}{\partial x} = \frac{f(x+h) - f(x)}{h}$$

<u>Example</u>: $\frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} = f'(x) + O(h^2)$

$\frac{-f_2 + 4f_1 - 3f_0}{2h} = f_0' + O(h^2)$

$-f(x+2h) = -f(x) - 2hf'(x) - \frac{(2h)^2}{2}f''(x) - O(h^3)$

$4f(x+h) = 4f(x) + 4hf'(x) + 4\cdot\frac{h^2}{2}f''(x) + O(h^3)$

$-3f(x) = -3f(x)$

$\sim \;= 0 + 2hf'(x) + 0 + O(h^3)$

$\text{LHS} = \frac{\sim}{2h} = f'(x) + O(h^2)$

If $f \in \mathcal{C}^4$, then $\frac{f_1 - 2f_0 + f_{-1}}{h^2} = f''(x) + O(h^2)$

Proof: $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3!} f^{(3)}(x) + O(h^4)$

$-2f(x) = -2f(x)$

$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{3!} f^{(3)}(x) + O(h^4)$

$\sim \quad = 0 + 0 + h^2 f''(x) + 0 + O(h^4)$

$\frac{\sim}{h^2} = f''(x) + O(h^2)$

Ladder Report

x	Value	Delta	Gamma
123	f_3		
122	f_2		
121	f_1	$f_2 - f_1$	
mkt x=120 → 120	f_0	$f_1 - f_0$	$f_1 - 2f_0 + f_{-1}$
119	f_{-1}		
118	f_{-2}		

At the boundary e.g. 123 & 118, use just up / down delta to get less precise results

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2019. 03. 13 Practical Greeks

if f is α - times differentiable,

$\sum_{j=-n_1}^{n_2} a_j f_j = f_m^k + O(h^\alpha)$

Task: Try solving a linear system with $\{a_{-n_1}, \dots, a_{n_2}\}$

$\frac{-f_2 + 4f_1 - 3f_0}{2h} = f_0' + O(h^2)$

Ex. Can this be done always ?

Remark: you can see some "pattern" in the formula

$\frac{+1f_1 - 1f_0}{h} = f_0' + O(h)$

$\frac{+1f_0 - 1f_{-1}}{h} = f_0' + O(h)$

$\frac{+1f_1 - 2f_0 + 1f_{-1}}{2h} = f_0'' + O(h^2)$

$a_1 \cdot f(x+h) = a_1 f(x) + a_1 h f'(x) + a_1 \dots$

$a_0 \cdot f(x) = a_0 f(x)$

$a_{-1} \cdot f(x-h) = a_{-1} \cdot f(x) - a_{-1} h f'(x) + a_{-1} \dots$

$\Rightarrow a_1 + a_0 + a_{-1} = 0$

Using a test function,

$f(x) = 1 \Rightarrow \sum$ coefficient $f = 0$

$\frac{f_1 - 2f_0 + f_{-1}}{h^2}$

Newton's Dividend Differences

Notation $f[x_0] \overset{def}{=} f(x_0)$

$f[x_0, x_1] \overset{def}{=} \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} \sim$ like Delta

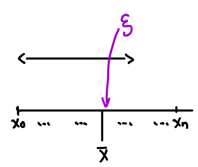
$f[x_0, x_1, x_2] \overset{def}{=} \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \sim$ like Gamma/2

In general,

$f[x_0, x_1, \dots, x_k] \overset{def}{=} \frac{f[x_1, \dots, x_k] - f[x_0, \dots, x_{k-1}]}{x_k - x_0} \sim$ like k^{th} derivative / $k!$

Theorem [S&B] if f has $(n+1)$ derivative, then for any $\bar{x}$, there is a ξ in the interval containing $x_0, x_1, \dots, x_n, \bar{x}$ such that,

$f[x_0, x_1, \dots, x_n, \bar{x}] = \frac{f^{(n+1)}(\xi)}{(n+1)!}$



Sell - side relatively wants greeks ≈ 0 (Don't always need to be 0 but delta close to 0)

Read textbook proof. (indirect)

Profit and Loss Explanation

$(P\&L) = f(x_1^*, \dots, x_n^*) - f(x_1, \dots, x_n)$

x_j^* - final market variable (i.e. today's close)

x_j - beginning market variable (i.e. yesterday's close)

$\Omega_i = f(x_1, \dots, x_i^*, \dots, x_n) - f(x_1, \dots, x_n) = \Delta_i \cdot (x_i^* - x_i)$

$\sim$ "P & L due to change in x_i"   Where Δ_i = Practical greek $= \frac{f(x_1, \dots, x_i^*, \dots, x_n) - f(x_1, \dots, x_n)}{(x_i^* - x_i)} \approx \frac{\partial f}{\partial x_i}\Big|_{x_i}$

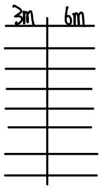
$\Omega - \underbrace{\sum_{i=1}^{n} \Omega_i}_{\text{all explanations}} =$ unexplained P&L ⇓ Higher order changes in Taylor expansion

$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \quad \Big| \quad f(\bar{x} + \Delta\bar{x}) = \sum \frac{\partial f}{\partial x_i} + \sum_{i>j} \frac{\partial^2 f}{\partial x_i \partial x_j} \Delta x_i \Delta x_j + \dots$

Implied Calculation $\longmapsto$ (Root Solving)

Imply the "spread" value = 0. $f(\ldots, H, \ldots) - \underset{30k}{V_0} = 0$

Call option
50k, only have 30 k



$f(x) = 0$ find "x"

4 main methods

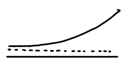
1. method of bisection
2. Regular Falsi (method of a false line)
3. Newton - Raphson method
4. Secant method

2019. 03. 25

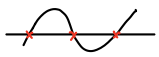
Implied Calculations / Root Solving

Solve $f(x) = 0$, Find α such that $f(\alpha) = 0$

→ no solution ($x \in \mathbb{R}$) → $e^x + 1 = 0$



many solutions



$y = 1.06 \times 10^{-203}$

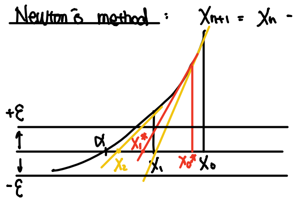
Let $\varepsilon = 10^{-9}$, Find x such that $|f(x)| \le \varepsilon$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

$$f(x_{new}^{(i+1)}) = f(x^{(i)})$$

$$m = \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n}.$$

Newton's method : $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$



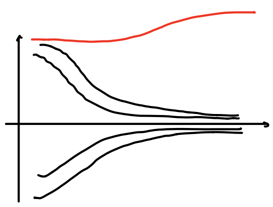
$|f(x_m)| \le \varepsilon$, stop

$x_0 \to x_1 \to x_2 \to \ldots \to x_m$
Ans

$x_0^* \to x_1^* \to x_2^* \to \ldots \to x_m^* \leftarrow$ Ans

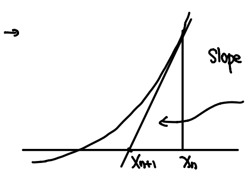
$|f(x_m^*)| \le \varepsilon$

Stability



Hope: x_m, x_m^* are close and do not affect things to come

→



Slope = $f'(x_n) = m$

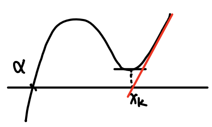
pass through $(x_n, f(x_n))$

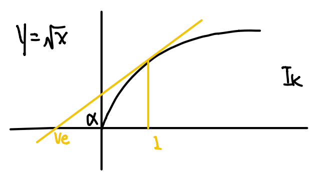
equation of line : $y - f(x_n) = m(x - x_n)$

$0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$

$\Rightarrow x_n - \frac{f(x_n)}{f'(x_n)} = x_{n+1}$

However, Newton's method may not always work → $f'(x_n) = 0$



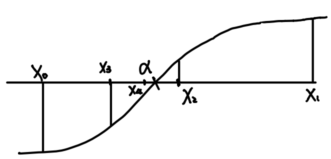


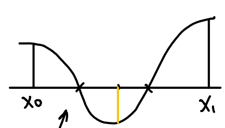
$I_k = \frac{1}{2^k}$

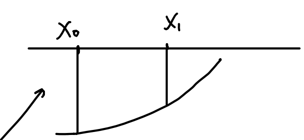
Initial Interval length

Newton's Method	Method of bisection [STBU]
No	Yes
(even if you can start it may not converge)	(if you can start, it never fails)
If it works, it converges very fast.	order = 1
order = 2	

↳ However, interval between initial points may contain several roots or may not contain any roots



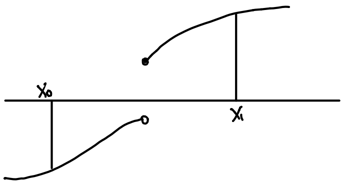




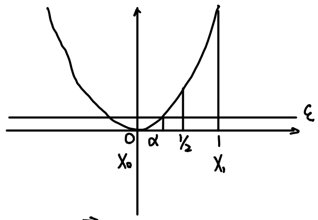
How to pick initial starting point(s)?

$\lim_{k\to\infty} f(x_k) = 0$

need f to be continuous on $[x_0, x_1]$



$x_0 = 0$
$x_1 = 1$
$x_2 = 0.5$
$x_3 = 0.25$
$x_4 = 0.125$
$x_5 = 0.0625$
$\cdots \quad 0.03125$
$\vdots$



$\rightarrow f(x) = x^2 - \varepsilon$

"linear convergence"

Let α be s.t. $f(\alpha) = 0$

when you get near α ($\exists$ a nbd $N(\alpha)$ s.t where $x^* \in N(\alpha)$), then

$|x_{n+1} - \alpha| < C|x_n - \alpha|$ $\Rightarrow x_{n+1} \in N(\alpha)$

$1 > C > 0$

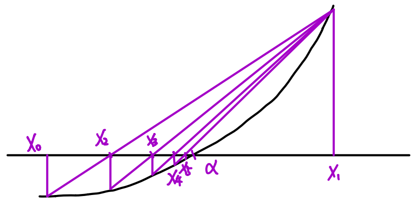
or order 1 convergence

Regular Falsi

Just drop a straight line between two points when it hits the x-axis, see if point is + or −.

Then repeat. Drop the point with same sign and make another line.

Like bisection method, if you can start, it never fails



Convergence Theorem for Newton

Let α be such that $f(\alpha) = 0$. Suppose on $N(\alpha)$, there exists $M > 0, k > 0$ s.t $\underbrace{|f''(x)| \le M}_{\text{2nd D bound above on } N(\alpha)}$ and $\underbrace{|f'(x)| \ge k}_{\text{1st D bound below on } N(\alpha)}$

then $|x_{n+1} - \alpha| \le \frac{M}{2k}|x_n - \alpha|^2$

2019.03.27

Theorem: (see last lecture)

to prove that, $|f''(x)| \le M$ $\quad f(\alpha) = 0$

$|f'(x)| \ge k > 0$ in $N(\alpha)$

$|e_{n+1}| \le \frac{M}{2k}|e_n|^2$ (write $e_n = x_n - \alpha$, "error")

Proof 1:

Taylor's expansion at x_n

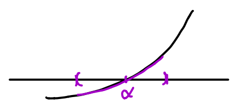
$\overset{0}{\overset{\parallel}{f(\alpha)}} = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{(\alpha - x_n)^2}{2!} f''(\xi)$, ξ is between α, x_n

$-\frac{f(x_n)}{f'(x_n)} = -e_n + \frac{e_n^2}{2!} \cdot \frac{f''(\xi)}{f'(x_n)}$

$x_{n+1} \Leftarrow x_n - \alpha - \frac{f(x_n)}{f'(x_n)} = \frac{e_n^2}{2!} \cdot \frac{f''(\xi)}{f'(x_n)}$

$x_{n+1} - \alpha = \frac{e_n^2}{2} \cdot \frac{f''(\xi)}{f'(x_n)}$

$|e_{n+1}| = \frac{e_n^2}{2}\left|\frac{f''(\xi)}{f'(x_n)}\right| \le \frac{e_n^2}{2} \cdot \frac{M}{k}$



" transverse intersect "

Proof 2:

Let $\phi(x) = x - \frac{f(x)}{f'(x)}$ Newton's Method

$x_{n+1} = \phi(x_n)$

If Newton's method converges to ξ, then $\xi = \phi(\xi)$

It converges to a fixed point of ϕ

$\vec{f}(x) = x^3$, $f'(x) = 3x^2$

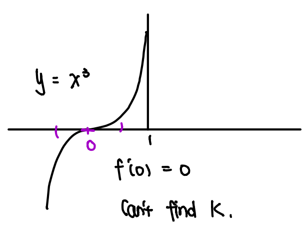
$x_{n+1} = x_n - \frac{x_n^3}{3x_n^2}$

$= x_n - \frac{1}{3}x_n$

$= \frac{2}{3}x_n$

$x_{n+1} - 0 = \frac{2}{3}(x_n - 0)$

* Newton is not always of order ≥ 2 convergence. $x_0 = 1$



$f'(0) = 0$

Can't find k.

Converges with order = 1

Definition for order convergence

if in $N(\alpha)$, $|e_{n+1}| \le C|e_n|$ with $C < 1$

$\longmapsto$ order of convergence = 1

If $p > 0$ & can find $C > 0$, $|e_{n+1}| \le C|e_n|^p \longmapsto$ order p convergence

Taylor's expansion at α.

$x_{n+1} = \phi(x_n)$

$= \phi(\alpha) + (x_n - \alpha)\phi'(\alpha) + \frac{(x_n - \alpha)^2}{2!}\phi''(\alpha) + \dots + \underbrace{\frac{(x_n - \alpha)^{p+1}}{(p+1)!}\phi^{(p+1)}(\xi)}$

$\phi'(\alpha) = 0$ $\quad \phi^{(p)}(\alpha) \ne 0$

$\phi''(\alpha) = 0$, $\phi^{(n)}(\alpha) = 0$

$\phi(x) = x - \frac{f(x)}{f'(x)}$, $\phi(\alpha) = \alpha - \frac{f(\alpha)}{f'(\alpha)} = \alpha$

$\phi'(x) = 1 - \frac{f'(x) \cdot f'(x) - f(x) f''(x)}{[f'(x)]^2} \longrightarrow \phi'(\alpha) = 0$

Write $f(x) = (x - \alpha)^m g(x)$ $\quad m \leftarrow$ multiplicity of α as a root for $f(x) = 0$

$g(\alpha) \ne 0$

If $m = 1$, α is a simple root

$\phi(x) = x - \frac{(x-\alpha)^m g(x)}{m(x-\alpha)^{m-1} g(x) + (x-\alpha)^m g'(x)}$

$= x - \frac{(x-\alpha) g(x)}{m g(x) + (x-\alpha) g'(x)}$

Claim: $\phi'(\alpha) = 1 - \frac{1}{m}$

$\phi'(x) = 1 - \frac{[\dots](g(x) + (x-\alpha)g'(x)) - (x-\alpha)g(x)\frac{d}{dx}[\dots]}{[m g(x) + (x-\alpha)g'(x)]^2}$

$\phi'(\alpha) = 1 - \frac{m g(\alpha)[g(\alpha)] - 0}{[m g(\alpha)]^2}$

$= 1 - \frac{1}{m}$

$x_{n+1} - \alpha = (x_n - \alpha)\phi'(\alpha) + \dots + \frac{(x_n - \alpha)^{p+1}}{(p+1)!}\phi^{(p+1)}(\xi)$

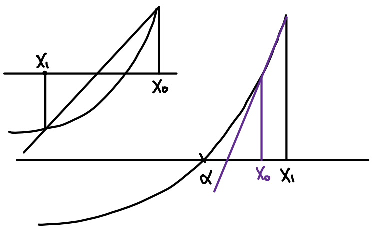
if α is simple, then $p \ge 2$.

$e_{n+1} = \frac{e_n^p}{p!}\phi^{(p)}(\alpha) + \frac{e_n^{p+1}}{(p+1)!}\phi^{(p+1)}(\xi)$ Quadratic Convergence

if α is not simple, α has multiplicity $m \ge 2$, $\phi'(\alpha) = 1 - \frac{1}{m} \ne 0$

$e_{n+1} = e_n(1 - \frac{1}{m})^1 + \dots$

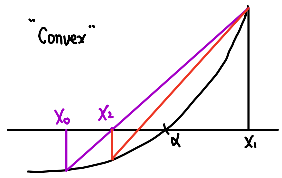
1. Bisection
2. Newton
3. Regular Fals
4. Secant Method



x_{n+1} = the x-intercept of the line $(x_n, f(x_n))$, $(x_{n-1}, f(x_{n-1}))$

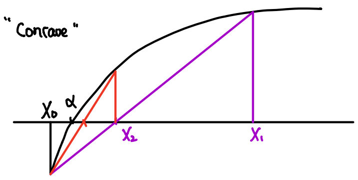
$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

"Convex"

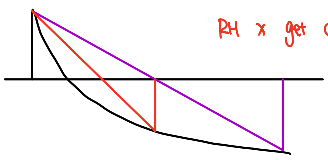


$f''(\alpha) > 0$. always LH x get dropped

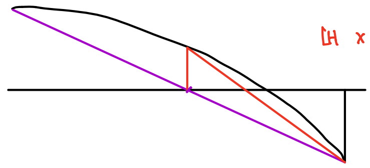
"Concave"



$f''(\alpha) < 0$. always RH x get dropped



RH x get dropped



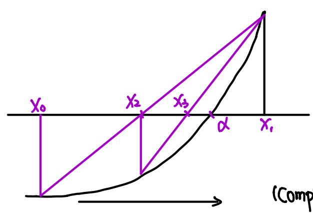
LH x get dropped

Good 1 It always converges

Good 2 It converges of order 1

Lemma One Sided "drop" property

$x_2 < x_3 < x_4 < \dots$



(Completeness of $\mathbb{R}$)

x_n = first n digits of π i.e. $x_4 = 3.1415$, $x_5 = 3.14159$

2018. 4. 1

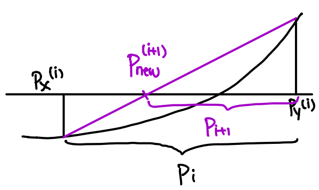
Regula Falsi [StBu] chap 5

Claim: Consider $P_i = (P_x^{(i)}, P_y^{(i)})$ with $P_x^{(i)} < P_y^{(i)}$, $f(P_x^{(i)}) < 0$ & $f(P_y^{(i)}) > 0$

If $f'' \geq 0$ on interval P_i, then either $f^{(i+1)}(P_{new}^{(i+1)}) = 0$ or $f^{(i+1)}(P_{new}^{(i+1)}) < 0$

Then left part $P_x^{(i)}$ is dropped

$\Rightarrow P_{i+1} = (P_{new}^{(i+1)}, P_y^{(i)})$



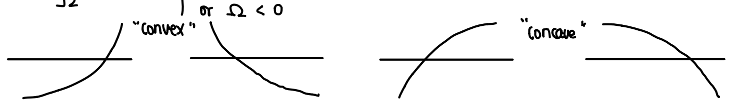
Proof: [StBu] (2.1.4.1) or from HW2

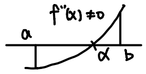
$\underbrace{f(P_{new}^{(i+1)})}_{\bar{x}} - \overset{0}{\overset{\shortparallel}{P_{01}}}\underbrace{(P_{new}^{(i+1)})}_{\bar{x}} = \underbrace{\frac{W(P_{new}^{(i+1)})}{2}}_{<0} \underbrace{f^{(2)}(\xi)}_{\geq 0}$, where $W(x) = (x - P_x^{(i)})(x - P_y^{(i)})$; ξ is in $I(P_x^{(i)}, P_y^{(i)}, P_{new}^{(i+1)})$

$\Rightarrow W(P_{new}^{(i+1)}) = (+ve)(-ve) < 0$

$\Rightarrow \underset{\Omega}{\underset{\shortparallel}{f(P_{new}^{(i+1)})}} \leq 0 \Rightarrow \begin{cases} \text{either } \Omega = 0. \text{ alg. stops} \\ \text{or } \Omega < 0 \end{cases}$

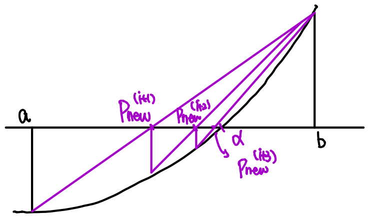
※ The proof works in all 4 different cases similarly





Theorem: If on $[a,b]$ f'' does not change sign and $f(a)\cdot f(b) < 0$

Then with Regula Falsi method, $\lim_{n\to\infty} P_{new}^{(n)} = \alpha$ with $f(\alpha) = 0$ and $\alpha \in (a,b)$



Let's consider $f(a) < 0$ & $f(b) > 0$, $f'' \geq 0$

From the claim, we know the left point gets dropped, since $P_{new}^{(i+1)} = P_x^{(i)} - f(P_x^{(i)}) \dfrac{P_y^{(i)} - P_x^{(i)}}{f(P_y^{(i)}) - f(P_x^{(i)})}$

$P_{new}^{(i+1)}, P_{new}^{(i+2)}, \ldots$ increasing, bounded above by b.

By completeness axiom, $\lim_{n\to\infty} P_{new}^{(n)} = \alpha$ exists, $\alpha \leq b$

(*) $P_{new}^{(i+1)} = P_x^{(i)} - f(P_x^{(i)}) \cdot \dfrac{b - P_x^{(i)}}{f(b) - f(P_x^{(i)})}$

take $\lim_{i\to\infty}$, $\alpha = \alpha - f(\alpha) \cdot \dfrac{b-\alpha}{f(b) - f(\alpha)}$

$0 = f(\alpha) \cdot \dfrac{b-\alpha}{f(b) - f(\alpha)}$

$\Rightarrow f(\alpha) = 0$

$(f(b) > 0,\ f(x_n) \leq 0 \Rightarrow \lim_{n\to\infty} f(x_n) \leq 0 \Leftrightarrow f(\lim_{n\to\infty} x_n) \leq 0 \Leftrightarrow f(\alpha) \leq 0 \Rightarrow \alpha \neq b)$

Claim: Same notation, $P_i = (P_x^{(i)}, P_y^{(i)})$, $P_x^{(i)} < P_y^{(i)}$, $f(P_x^{(i)}) < 0$ & $f(P_y^{(i)}) > 0$

If $f'' > 0$ on $[P_x^{(i)}, P_y^{(i)}]$, then Regula Falsi converges with order $= 1$

Proof: With (*), $\phi(x) = x - f(x) \cdot \dfrac{b-x}{f(b) - f(x)}$, where $b = P_y^{(i)}$

It suffices to show $\phi'(\alpha) \neq 0$. In fact, will show $\phi'(\alpha) > 0$

$\Rightarrow \phi'(x) = 1 - \dfrac{(f(b) - f(x))(f'(x)(b-x) + f(x)(-1)) - f(x)(b-x)\frac{d}{dx}(f(b) - f(x))}{(f(b) - f(x))^2}$

$\phi'(\alpha) = 1 - \dfrac{f(b) f'(\alpha)(b-\alpha)}{f(b)^2}$

$= 1 - \dfrac{f'(\alpha)(b-\alpha)}{f(b)}$

Suppose $\phi'(\alpha) \leq 0$, $1 \leq \dfrac{f'(\alpha)(b-\alpha)}{f(b)}$

because $f(b) > 0$, $b - \alpha > 0$, and $f(\alpha) = 0$, $\dfrac{f(b) - f(\alpha)}{b-\alpha} \leq f'(\alpha)$

$\quad\quad \| $

$f'(q)$ for some $\alpha < q < b$.

but $f'' > 0 \Rightarrow f'$ strictly increasing $\Rightarrow f'(\alpha) < f'(q)$

Now, since $\alpha < q$, $f'(\alpha) \geq f'(q)$, a contradiction.

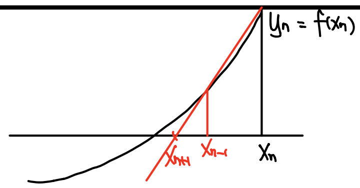
So $\phi'(\alpha) > 0$, convergence is order of 1.

Remark order of conve of Secant Method

Next Topic: PDE (textbook: "Tools for computational finance" Chap 4)

2019. 4. 3

Secant Method



Slope $m = \dfrac{y_n - y_{n-1}}{x_n - x_{n-1}}$

$y - y_n = m(x - x_n)$

$-y_n = m(x_{n+1} - x_n)$

(*) $\dfrac{-y_n(x_n - x_{n-1})}{y_n - y_{n-1}} + x_n = x_{n+1}$

$x_{n+1} = \dfrac{-x_n y_n + x_{n-1} y_n + x_n y_n - x_n y_{n-1}}{y_n - y_{n-1}}$

$x_{n+1} = \dfrac{y_n x_{n-1} - x_n y_{n-1}}{y_n - y_{n-1}}$ (Symmetric about "n-1", "n")

Start with (*):

$x_{n+1} - \alpha = x_n - \alpha - y_n \dfrac{x_n - x_{n-1}}{y_n - y_{n-1}}$, where α is true An i.e. $f(\alpha) = 0$

$x_i - \alpha = e_i$ $\quad$ $e_{n+1} = e_n - y_n \dfrac{1}{f[x_n, x_{n-1}]}$ $\quad \dfrac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$

What is q?

$e_{n+1} = e_n (M\ \ )$

$= e_n \left(1 - \dfrac{f(x_n)}{f[x_n, x_{n-1}]} \cdot \dfrac{1}{x_n - \alpha}\right)$

$= e_n \left(1 - \dfrac{f(x_n) - f(\alpha)}{f[x_n, x_{n-1}]} \cdot \dfrac{1}{x_n - \alpha}\right)$

$= e_n \left(1 - \dfrac{f[x_n, \alpha]}{f[x_n, x_{n-1}]}\right)$

$= e_n \left(\dfrac{f[x_n, x_{n-1}] - f[x_n, \alpha]}{f[x_n, x_{n-1}]}\right)$

$= e_n \left(\dfrac{f[x_{n-1}, x_n, \alpha]}{f[x_n, x_{n-1}]}\right) \cdot (x_{n-1} - \alpha)$

$e_{n+1} = e_n \cdot e_{n-1} \left(\dfrac{f[x_{n-1}, x_n, \alpha]}{f[x_n, x_{n-1}]}\right)$

$\dfrac{f''(\eta_2)}{2!}$ for some $\eta_2 \in I(x_{n-1}, x_n, \alpha)$

$f'(\eta_1)$ for some $\eta_1 \in I[x_{n-1}, x_n]$

$|e_{n+1}| \leq |e_n||e_{n-1}| W$

Recall Newton's method:

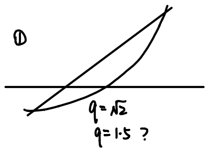
1^{st} proof if

$|f''| \leq M$

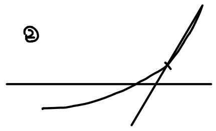
$|f'| > K$

$\Rightarrow$ there is $W > 0$ s.t. $\left|\dfrac{f''(\eta_2)}{f'(\eta_1)}\right| \leq W$

①



②



[StBu] $q = \frac{1+\sqrt{5}}{2} = 1.618\ldots$

Pretend $w = 1$

Experience $\quad E_{n+1} \le E_n E_{n-1}$

Suppose we can find q : $E_n \le E_{n-1}^{q}$

$E_1 \le E_0^{q} \qquad E_2 \le E_1^{q}$

$E_2 \le E_1 E_0 \qquad E_2 \le (E_0^{q})^{q}$

$E_2 \le E_0^{1+q} \qquad E_2 \le E_0^{q^2}$

$q^2 = 1 + q \Rightarrow q^2 - q - 1 = 0$

$q = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

Since $q > 0 \Rightarrow q = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$

In general, $E_{n+1} \le E_n \cdot E_{n-1}$

(By induction, $\le E_0^{q^n} E_0^{q^{n-1}} = E_0^{q^{n-1}(1+q)} = E_0^{q^{n-1} q^2} = E_0^{q^{n+1}}$

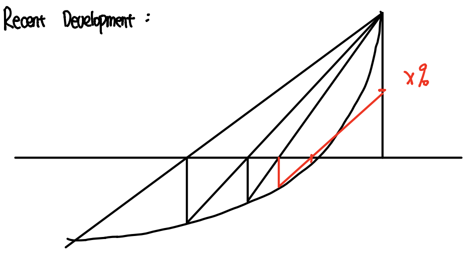
Order $= q^2 = 1+q \approx 2.618\ldots$

$E_0 \to E_2$ Secant method

$E_0 \to E_1$ Newton's method

A fair comparison: We can iterate twice with secant method for each Newton Iteration

Recent Development :



PDE 'Partial Differentiation Equation' Solver

Let V = value of a derivative security

B.S. PDE: $\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0$

dividend

Based on BS model : $\frac{dS}{S} = (r-\delta)dt + \sigma dW_t$

$V(t, S)$ underlying value $S = S_t$

Present time

Goal 1: You need only to solve (Heat equation : $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$)

Goal 2: Numerically solve $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$ using "PDE Grid".

2019. 04. 08

Last time :

dividend yield (Fx: foreign zero rate. Hull's "q")

B.S. PDE: $\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - r \cdot V = 0$

Goal: Turn this into Heat Equation : $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$ with the corr. boundary conditi

The PDE has terms $S^k \frac{\partial^k V}{\partial S^k}$ terms with $k = 0, 1, 2$ (Euler's Differential Equation)

$k > 0$

k = any constant

(Later on k is set to be strike)

Let $S = ke^{x} \Leftrightarrow$ let $x = \ln\left(\frac{S}{k}\right)$

$\frac{\partial^j S}{\partial x^j} = S$ for $j = 1, 2$,

$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial S} \cdot \frac{\partial S}{\partial x} = S \frac{\partial V}{\partial S} \sim$ ①

$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial S}\left(S \frac{\partial V}{\partial S}\right) \cdot \frac{\partial S}{\partial x}$

$= \left(\frac{\partial V}{\partial S} + S \frac{\partial^2 V}{\partial S^2}\right) \cdot S$

$= S^2 \frac{\partial^2 V}{\partial S^2} + S \frac{\partial V}{\partial S} \sim$ ②

// Boundary Condition in (S, t) - Space :

Payoff $\quad T$ = maturity of trade, k = Strike

(Like a tree-pricing) $\Rightarrow$ fwd contract : $S_T - k$

Std Call Option : $\max(S_T - k, 0)$

Std Put Option : $\max(k - S_T, 0)$

BS PDE domain $\begin{cases} 0 \le S < +\infty \\ 0 \le t \le T \end{cases}$

Into PDE : $\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x}\right) + (r-\delta)\frac{\partial V}{\partial x} - rV = 0$

$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \cdot \frac{\partial^2 V}{\partial x^2} + \left(r - \delta - \underbrace{\frac{\sigma^2}{2}}_{\text{Ito's term}}\right)\frac{\partial V}{\partial x} = rV$

q_δ

Ito : $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$

$d(\ln S_t) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$

$\frac{2}{\sigma^2} \cdot \frac{\partial V}{\partial t} + 1 \cdot \frac{\partial^2 V}{\partial x^2} + \left(\frac{2(r-\delta)}{\sigma^2} - 1\right)\frac{\partial V}{\partial x} = \underbrace{\frac{2r}{\sigma^2}}_{q_0 = q} V$

Let $t = T - \frac{2\tau}{\sigma^2} \Leftrightarrow \tau = \frac{\sigma^2}{2}(T - t) \to d\tau = \frac{\sigma^2}{2}(-dt)$, $\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = -\frac{2}{\sigma^2} \frac{\partial V}{\partial \tau}$

$$\frac{\partial^2 V}{\partial x^2} + \underbrace{(q_\delta - 1)}_{b}\frac{\partial V}{\partial x} \underbrace{- q}_{c} V = \frac{\partial V}{\partial \tau}$$

Suppose $\frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial x^2} + b\frac{\partial V}{\partial x} + cV$ Goal : get rid of b, c

Suppose $V(x,\tau) = Ke^{\alpha x + \beta\tau} y(x,\tau)$ what does y satisfy (PDE)

take the form

that α, β to be determined

(Ex in HW) $\longrightarrow \frac{\partial y}{\partial \tau} = \underbrace{(\alpha^2 + b\alpha + c - \beta)}_{A} y + \underbrace{(2\alpha + b)}_{B}\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}$

A, B = 0

$\Rightarrow 2\alpha + b = 0 \Leftrightarrow \alpha = -\frac{b}{2} \Leftrightarrow \alpha = -\frac{q_\delta - 1}{2}$

$\Rightarrow$ If I choose $\alpha = \frac{-(q_\delta - 1)}{2}$; $\beta = -\left(\frac{(q_\delta - 1)^2}{4} + q\right)$

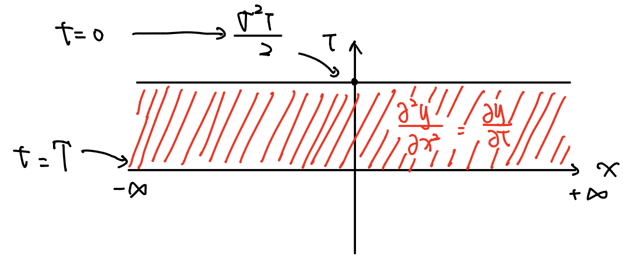
$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}$$

if $y(x,\tau) = \frac{1}{Ke^{\{-\frac{1}{2}(q_\delta - 1)x - \frac{1}{4}(q_\delta - 1)^2 + q\}\tau}} V(x,\tau)$

$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}$ Heat Equation

V	$S \to 0^+$	$S \to \infty$	$t = T$
fwd	$-Ke^{-r(T-t)}$	$Se^{-\delta(T-t)} - Ke^{-r(T-t)}$	$S_T - K$
Call	0	$Se^{-\delta(T-t)} - Ke^{-r(T-t)}$	$\max(S_T - K, 0)$
Put	$Ke^{-r(T-t)}$	0	$\max(K - S_T, 0)$
	①	②	③

y	$S \to 0^+$, $x \to -\infty$	$S \to \infty$, $x \to \infty$	$t = T$, $\tau = 0$
fwd	use put - call parity : fwd = buy call sell put		
Call	0	$e^{\frac{x}{2}(q_\delta + 1) + \frac{\tau}{4}(q_\delta + 1)^2 + \dots}$	$\max(e^{\frac{x}{2}(q_\delta + 1)} - e^{\frac{x}{2}(q_\delta - 1)}, 0)$
Put	$e^{\frac{x}{2}(q_\delta - 1) + \frac{\tau}{4}(q_\delta - 1)^2 + \dots}$	0	$\max(e^{\frac{x}{2}(q_\delta - 1)} - e^{\frac{x}{2}(q_\delta + 1)}, 0)$



$\tau = \frac{\sigma^2}{2}(T - t)$ t↑ from 0 to T

τ↓ from $\frac{\sigma^2 T}{2}$ to 0

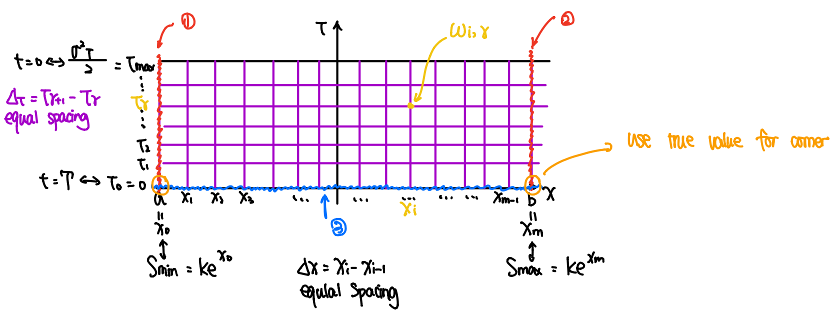
also dimensionless direction of "Time" is reversed

$x = \ln(\frac{S}{K})$ has no unit

is dimensionless

Task : Solve the PDE

Explicit Method : compute w as approximation for y



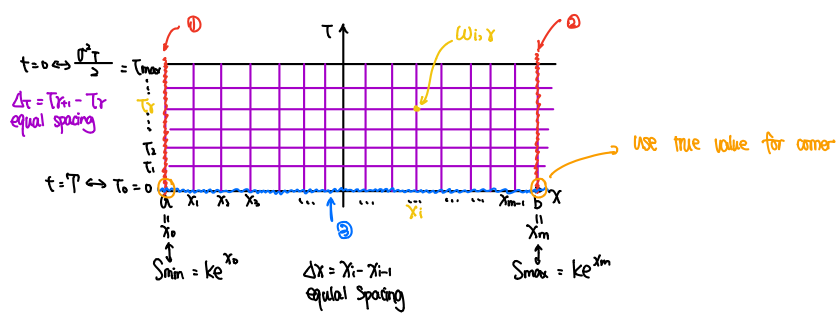
2019. 04. 10

$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}$

Hope : let $\Delta x \to 0^+$,

$\Delta\tau \to 0^+$,

the answer



Using practical Greeks :

$$\frac{\partial y_{i,\gamma}}{\partial \tau} \approx \frac{y_{i,\gamma+1} - y_{i,\gamma}}{\Delta\tau} + O(\Delta\tau)$$

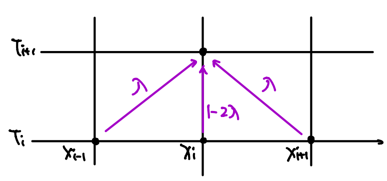
$$\frac{\partial^2 y_{i,\gamma}}{\partial x^2} \approx \frac{y_{i+1,\gamma} - 2y_{i,\gamma} + y_{i-1,\gamma}}{\Delta x^2} + O(\Delta x^2)$$

Plug into Heat Equations :

$$\frac{w_{i,\gamma+1} - w_{i,\gamma}}{\Delta\tau} = \frac{w_{i+1,\gamma} - 2w_{i,\gamma} + w_{i-1,\gamma}}{\Delta x^2}$$

$w_{i,\gamma+1} = \lambda w_{i+1,\gamma} + (1 - 2\lambda)w_{i,\gamma} + \lambda w_{i-1,\gamma}$, $\lambda = \frac{\Delta\tau}{\Delta x^2}$

for $i = 1, \dots, m-1$ and $\gamma = 0, \dots, \gamma_{max}$



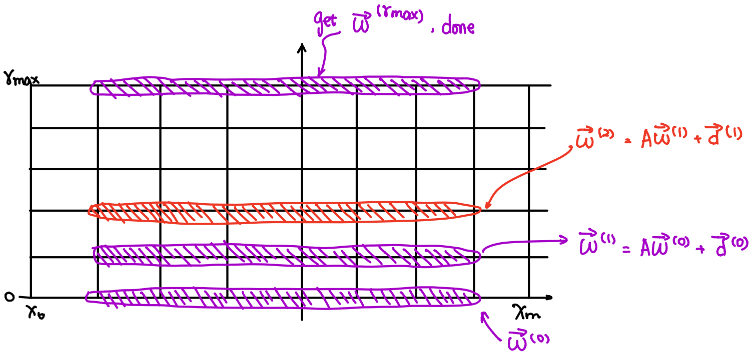
Write $\vec{\omega}^{(\tau)} = (\omega_1, \dots, \omega_{m-1})^T \sim$ the interior

$$\begin{pmatrix} \omega_1^{(\tau+1)} \\ \vdots \\ \vdots \\ \omega_{m-1}^{(\tau+1)} \end{pmatrix} = \begin{pmatrix} 1-2\lambda & \lambda & & & \\ \lambda & 1-2\lambda & \lambda & & \\ & \lambda & 1-2\lambda & \lambda & \\ & & \ddots & \ddots & \lambda \\ & & & \lambda & 1-2\lambda \end{pmatrix} \begin{pmatrix} \omega_1^{(\tau)} \\ \vdots \\ \vdots \\ \omega_{m-1}^{(\tau)} \end{pmatrix} + \begin{pmatrix} r_1(a, \tau_\gamma) \\ 0 \\ \vdots \\ 0 \\ r_2(b, \tau_\gamma) \end{pmatrix}$$

$$\vec{\omega}^{(\gamma+1)} = A\vec{\omega}^{(\gamma)} + \vec{d}^{(\gamma)}$$

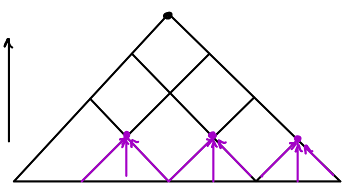
Ex. Using prior boundary condition

	$x \to -\infty$, $S \to 0^+$, $i = 0$	$x \to +\infty$, $S \to \infty$, $i = m$	$t = T$, $\tau = 0$
	$r_1(x,\tau) = 0$	$r_2(x,\tau) = e^{\frac{x}{2}(q_\delta+1)+\frac{\tau}{4}(q_\delta+1)^2}$	$\max(e^{\frac{x}{2}(q_\delta+1)} - e^{\frac{x}{2}(q_\delta-1)}, 0)$
	$r_1(x,\tau) = e^{\frac{x}{2}(q_\delta-1)+\frac{\tau}{4}(q_\delta-1)^2}$	0	$\max(e^{\frac{x}{2}(q_\delta-1)} - e^{\frac{x}{2}(q_\delta+1)}, 0)$



decide m, Δx, decide γ_{max}, $\Delta\tau$

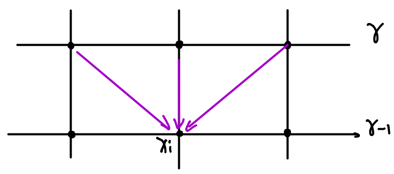
※ Tri-nomial Tree Pricing



<u>Implicit Method</u>

$$\frac{\partial y_{i,\gamma}}{\partial \tau} = \frac{y_{i,\gamma} - y_{i,\gamma-1}}{\Delta\tau} \quad \text{"Down Delta"}$$

$$\frac{\partial^2 y_{i,\gamma}}{\partial x^2} \approx \frac{y_{i+1,\gamma} - 2y_{i,\gamma} + y_{i-1,\gamma}}{\Delta x^2}$$



Use heat equation :

$$\Rightarrow \frac{\omega_{i,\gamma} - \omega_{i,\gamma-1}}{\Delta\tau} = \frac{\omega_{i+1,\gamma} - 2\omega_{i,\gamma} + \omega_{i-1,\gamma}}{\Delta x^2}, \quad \text{for } i = 1, \dots, m-1 \text{ and } \gamma = 0, \dots, \gamma_{max}$$

$$-\lambda\omega_{i+1,\gamma} + (2\lambda-1)\omega_{i,\gamma} - \lambda\omega_{i-1,\gamma} = \omega_{i,\gamma-1}$$

$$A^*\vec{\omega}^{(\gamma)} = \vec{\omega}^{(\gamma-1)} + \vec{d}^{(\gamma-1)} \qquad A^* = \begin{pmatrix} 2\lambda+1 & -\lambda & & \\ -\lambda & 2\lambda+1 & -\lambda & \\ & \ddots & \ddots & -\lambda \\ & & -\lambda & 2\lambda+1 \end{pmatrix}$$

$$\Updownarrow$$

$$A^*\vec{\omega}^{(\gamma+1)} = \vec{\omega}^{(\gamma)} + \vec{d}^{(\gamma)}$$

if A^* is invertible : $\vec{\omega}^{(\gamma+1)} = A^{*-1}\vec{\omega}^{(\gamma)} + A^{*-1}\vec{d}^{(\gamma)}$

What if we "linearly combine" the two

Explicit Method : $\dfrac{\omega_{i,\gamma+1} - \omega_{i,\gamma}}{\Delta\tau} = \dfrac{\omega_{i+1,\gamma} - 2\omega_{i,\gamma} + \omega_{i-1,\gamma}}{\Delta x^2}$ ①

Implicit Method : $\dfrac{\omega_{i,\gamma+1} - \omega_{i,\gamma}}{\Delta\tau} = \dfrac{\omega_{i+1,\gamma+1} - 2\omega_{i,\gamma+1} + \omega_{i-1,\gamma+1}}{\Delta x^2}$ ②

$$(1-\theta)\cdot ① + \theta \cdot ② \Leftrightarrow \frac{\omega_{i,\gamma+1} - \omega_{i,\gamma}}{\Delta\tau} = (1-\theta)(\quad) + \theta(\quad)$$

Collect terms :

$$-\lambda\theta\omega_{i-1,\gamma+1} + (1+2\lambda\theta)\omega_{i,\gamma+1} - \lambda\theta\omega_{i+1,\gamma+1} = \lambda(1-\theta)\omega_{i-1,\gamma} + (1-2\lambda(1-\theta))\omega_{i,\gamma} + \lambda(1-\theta)\omega_{i+1,\gamma}$$

Then,

$$A = \begin{pmatrix} 1+2\lambda\theta & -\lambda\theta & & \\ -\lambda\theta & 1+2\lambda\theta & -\lambda\theta & \\ & \ddots & \ddots & -\lambda\theta \\ & & -\lambda\theta & 1+2\lambda\theta \end{pmatrix}; \quad B = \begin{pmatrix} 1-2\lambda(1-\theta) & \lambda(1-\theta) & & \\ \lambda(1-\theta) & \ddots & \ddots & \\ & \ddots & \ddots & \lambda(1-\theta) \\ & & \lambda(1-\theta) & 1-2\lambda(1-\theta) \end{pmatrix}$$

$$A\vec{\omega}^{(\gamma+1)} = B\vec{\omega}^{(\gamma)} + \underline{\vec{d}}^{(\gamma)}$$ deriving this by tracing the boundary condition

2019. 04. 15

Stability of Method

Explicit Method form $\vec{w}^{(r+1)} = A\vec{w}^{(r)} + \vec{d}^{(r)}$

Start with $\vec{w}^{(0)}$

True answere there is $\vec{y}^{(0)}$

$\vec{e}^{(0)} = \vec{w}^{(0)} - \vec{y}^{(0)} \neq \vec{0}$ (due to rounding error etc.)

$\vec{w}^{(1)} = A\vec{w}^{(0)} + \vec{d}^{(0)}$

Compare with $A\vec{y}^{(0)} + \vec{d}^{(0)}$, there difference is $A\vec{e}^{(0)}$

$$\vec{W}^{(2)} = A\vec{w}^{(1)} + \vec{d}^{(1)}$$
$$= A(A\vec{w}^{(0)} + \vec{d}^{(0)}) + \vec{d}^{(1)}$$
$$= A^2\vec{w}^{(0)} + A\vec{d}^{(0)} + \vec{d}^{(1)}$$

$$\vec{W}^{(3)} = A^3\vec{w}^{(0)} + A^2\vec{d}^{(0)} + A\vec{d}^{(1)} + \vec{d}^{(2)}$$

$\vec{W}^{(n)} = A^n\vec{w}^{(0)} + A^{n-1}\vec{d}^{(0)} + \dots + A\vec{d}^{(n-2)} + \vec{d}^{(n-1)}$, n is large, $n \to r_{max}$

Use a slightly different $\vec{w}^{*(0)}$ (due to truncation, noise ...)

$$\vec{w}^{*(n)} = A^n\vec{w}^{*(0)} + A^{n-1}\vec{d}^{(0)} + \dots + A\vec{d}^{(n-2)} + \vec{d}^{(n-1)}$$

$$\underbrace{\vec{w}^{*(n)} - \vec{w}^{(n)}}_{\text{final error}} = A^n \underbrace{(\vec{w}^{*(0)} - \vec{w}^{(0)})}_{\text{initial error}} = A^n\vec{e}^{(0)}$$

Case : A has dim = 1

is $A^n\vec{e}^{(0)}$ small ? when e is small, is $a^n \cdot e$ small ?

$e \neq 0$,

$$a^n e = \begin{cases} \to +\infty\, \text{Sign}(e), & \text{if } a > 1 \\ e, & \text{if } a = 1 \\ \to 0, & \text{if } |a| < 1 \end{cases}$$

Case : A has dim = 2

e_1, e_2 small,
$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}^n \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} d_1^n & 0 \\ 0 & d_2^n \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} d_1^n e_1 \\ d_2^n e_2 \end{pmatrix} \longrightarrow 0^+ \text{ if } \begin{matrix} |d_1| < 1 \\ |d_2| < 1 \end{matrix}$$

$\lim_{n\to\infty} A^n = ?$ $A = P^{-1}DP$

$$A^n = (P^{-1}DP)\cdot(P^{-1}DP)\cdot \dots \cdot(P^{-1}DP)$$
$$= P^{-1}D^nP$$

d's are called eigenvalues

$$A^n = P^{-1}\begin{pmatrix} d_1^n & & \\ & \ddots & \\ & & d_m^n \end{pmatrix}P$$

Recall μ is an eigenvalue of a matrix A, if there is a non-zero vector $\vec{v}$ such that,

$A\vec{v} = \lambda\vec{v}$, $\vec{v}$ is called the corresponding eigenvector

$(A - \lambda I)\vec{v} = \vec{0} \Leftrightarrow \det(A - \lambda I) = 0$

For explicit method,

If I want method to be stable, I need all eigenvalues of A to satisfy $|\mu_k| < 1$.

$$A = \begin{pmatrix} 1-2\lambda & \lambda & & \\ \lambda & 1-2\lambda & \lambda & \\ & \ddots & \ddots & \lambda \\ & & \lambda & 1-2\lambda \end{pmatrix}$$

<u>Theorem:</u> Let $G = \begin{pmatrix} \alpha & \beta & & \\ \gamma & \ddots & \ddots & \\ & \ddots & \ddots & \beta \\ & & \gamma & \alpha \end{pmatrix}$ an $N \times N$ matrix with $\beta \neq 0$

For $k = 1, \dots, N$, $\mu_k = \alpha + 2\beta\sqrt{\frac{\gamma}{\beta}}\cos\left(\frac{k\pi}{N+1}\right)$

$$\vec{v}^{(k)} = \left(\sqrt{\frac{\gamma}{\beta}}\sin\left(\frac{k\pi}{N+1}\right), \left(\sqrt{\frac{\gamma}{\beta}}\right)^2\sin\left(\frac{2k\pi}{N+1}\right), \dots, \left(\sqrt{\frac{\gamma}{\beta}}\right)^N\sin\left(\frac{Nk\pi}{N+1}\right)\right)^T$$

Then $\vec{v}^{(k)}$ is an eigenvector of G with eigenvalue μ_k.

(Proof by equating both sides $G\vec{v}^{(k)} = \mu_k\vec{v}^{(k)}$

$\rightarrow$ With $\alpha = 1-2\lambda$, $\beta = \lambda$, $\gamma = \lambda$, $N = m-1$

$k = 1, 2, \dots, m-1$, by theorem,

$$\mu_k = (1-2\lambda) + 2\lambda\sqrt{\frac{\lambda}{\lambda}}\cos\left(\frac{k\pi}{m}\right)$$

is $|\mu_k| < 1$

$$= (1-2\lambda) + 2\lambda\cos\left(\frac{k\pi}{m}\right)$$

$\leftarrow \cos\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

$$= 1 - 2\lambda(1 - 1 + 2\sin^2\left(\frac{k\pi}{2m}\right))$$

$$\mu_k = 1 - 4\lambda\sin^2\frac{k\pi}{2m}, \quad k = 1, \dots, m-1$$

Clearly $\mu_k < 1$, want $-1 < \mu_k$

$$-1 < 1 - 4\lambda\sin^2\frac{k\pi}{2m}$$
$$-2 < -4\lambda\sin^2\frac{k\pi}{2m}$$
$$\circledast \quad \frac{1}{2} > \lambda\sin^2\frac{k\pi}{2m}, \quad k = 1, \dots, m-1$$

$\frac{k\pi}{2m} < \frac{\pi}{2}$

If $\frac{1}{2} > \lambda$, then $\circledast$ is satisfied and hence $|\mu_k| < 1$

Stability condition

2019. 04. 17

Stability Condition of Explicit Method $\quad \lambda = \frac{\Delta\tau}{\Delta x^2}$

$\lambda \le \frac{1}{2}$ (a sufficient condition)

Tri-nomial tree probabilities interpretation (all three $\mathbb{P} \ge 0$)

$\lambda = \frac{1}{2} \Rightarrow$ Binomial tree

Implicit Method

$A\vec{\omega}^{(\gamma)} = \vec{\omega}^{(\gamma-1)} + \vec{d}^{(\gamma-1)}$

$\vec{\omega}^{(\gamma)} = A^{-1}\vec{\omega}^{(\gamma-1)} + \dots$

$|\text{eigenvalues of } A^{-1}| < 1$?

$$A = \begin{pmatrix} 2\lambda+1 & -\lambda & & \\ -\lambda & 2\lambda+1 & -\lambda & \\ & \ddots & \ddots & -\lambda \\ & & -\lambda & 2\lambda+1 \end{pmatrix} \quad (m-1)\times(m-1) \text{ matrix}$$

$(N = m-1)$

Step 1 Show $\mu_k^A \ne 0$ Hence A^{-1} exists $\Leftarrow$

Using theorem (G): $\alpha = 2\lambda+1$; $\gamma = -\lambda$; $\beta = -\lambda$

$$\mu_k^A = (2\lambda+1) + 2(-\lambda)\cos\frac{k\pi}{m}, \quad k = 1, \dots, m-1$$
$$= 1 + 2\lambda - 2\lambda\cos\frac{k\pi}{m}$$
$$= 1 + 2\lambda\left(1 - \cos\frac{k\pi}{m}\right)$$
$$= 1 + 2\lambda\left(2\sin^2\frac{k\pi}{2m}\right)$$
$$= 1 + 4\lambda\sin^2\frac{k\pi}{2m} > 0, \quad k = 1, \dots, m-1$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\frac{\theta}{2} = 1 - \cos\theta$

$$\Rightarrow \mu_k^{(A^{-1})} = \frac{1}{1 + 4\lambda\sin^2\frac{k\pi}{2m}}, \quad k = 1, \dots, m-1$$

also, we see $|\mu_k^{(A^{-1})}| < 1$, unconditionally

Conclusion: Implicit method is stable unconditionally

Recall Theorem: Theorem: Let $G = \begin{pmatrix} \alpha & \beta & & \\ \gamma & \ddots & \ddots & \\ & \ddots & \ddots & \beta \\ & & \gamma & \alpha \end{pmatrix}$ an $N \times N$ matrix with $\beta \ne 0$

For $k = 1, \dots, N$, $\mu_k = \alpha + 2\beta\sqrt{\frac{\gamma}{\beta}}\cos\left(\frac{k\pi}{N+1}\right)$

$$\vec{v}^{(k)} = \left(\sqrt{\frac{\gamma}{\beta}}\sin\left(\frac{k\pi}{N+1}\right), \left(\sqrt{\frac{\gamma}{\beta}}\right)^2\sin\left(\frac{2k\pi}{N+1}\right), \dots, \left(\sqrt{\frac{\gamma}{\beta}}\right)^N\sin\left(\frac{Nk\pi}{N+1}\right)\right)^T$$

Then $\vec{v}^{(k)}$ is an eigenvector of G with eigenvalue μ_k.

$$\Rightarrow \mu_k^{(A^{-1})} = \frac{1}{\mu_k^A}$$

Proof: $\vec{v}_k \ne \vec{0}$ $\qquad$ $\vec{v}_k$ is eigenvector corresponding to μ_k^A (from theorem)

$A\vec{v}_k = \mu_k^A \cdot \vec{v}_k$

$\frac{1}{\mu_k^A}\vec{v}_k = A^{-1}\vec{v}_k$

$$A = P^{-1}\begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_N \end{pmatrix} P$$

$$\det(A) = \det(P^{-1})\det(D)\det(P) = \prod_{k=1}^{N}\mu_k^A$$

Crank Nicolson Method

$A\vec{\omega}^{(\gamma+1)} = B\vec{\omega}^{(\gamma)} + \vec{d}^{(\gamma)}$

$\vec{\omega}^{(\gamma+1)} = \underbrace{A^{-1}B}_{\text{eigenvalues ?}}\vec{\omega}^{(\gamma)} + \dots$

$(\theta = \frac{1}{2})$

$$A = \begin{pmatrix} 1+\lambda & -\lambda/2 & & \\ -\lambda/2 & 1+\lambda & -\lambda/2 & \\ & \ddots & \ddots & -\frac{\lambda}{2} \\ & & -\frac{\lambda}{2} & 1+\lambda \end{pmatrix} \qquad B = \begin{pmatrix} 1-\lambda & \lambda/2 & & \\ \lambda/2 & 1-\lambda & \lambda/2 & \\ & \ddots & \ddots & \frac{\lambda}{2} \\ & & \lambda/2 & 1-\lambda \end{pmatrix}$$

Proof: write $A = I + \frac{\lambda}{2}G$; $B = I - \frac{\lambda}{2}G$

$$A = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

G : $\alpha = 2$, $\beta = -1$, $\gamma = -1$ $\Rightarrow \mu_k^G = 2 - 2\cos\frac{k\pi}{m} = 4\sin^2\frac{k\pi}{2m}$

Let $C = 2A$, then $C = 2I + \lambda G$

$A\vec{\omega}^{(\gamma+1)} = B\vec{\omega}^{(\gamma)}$

$C\vec{\omega}^{(\gamma+1)} = 2B\vec{\omega}^{(\gamma)} = (2I - \lambda G)\vec{\omega}^{(\gamma)} = (2I - (C - 2I))\vec{\omega}^{(\gamma)} = (4I - C)\vec{\omega}^{(\gamma)}$

$\vec{\omega}^{(\gamma+1)} = (4C^{-1} - I)\vec{\omega}^{(\gamma)}$

※ In general, $\mu_k^{A+B} \ne \mu_k^A + \mu_k^B$

However, $\mu_k^{A+I} = \mu_k^A + 1$

$(A + I)\vec{v}_k^A = (\mu_k^A + 1)\vec{v}_k^A$

$$\Rightarrow \mu_k^{4C^{-1}-I} = \mu_k^{4C^{-1}} - 1 = 4\mu_k^{C^{-1}} - 1 = \frac{4}{\mu_k^C} - 1$$

$$\mu_k^C = \mu_k^{2I+\lambda G} = 2 + \mu_k^{\lambda G} = 2 + \lambda\mu_k^G = 2 + 4\lambda\sin^2\frac{k\pi}{2m}$$

$$\Rightarrow \mu_k^{4C^{-1}-I} = \frac{4}{2 + 4\lambda\sin^2\frac{k\pi}{2m}} - 1 = z$$
$$= \frac{2}{1 + 2\lambda\sin^2\frac{k\pi}{2m}} - 1$$

$\Rightarrow |z| < 1$

$\Rightarrow$ Crank Nicolson is unconditionally stable

2019. 04. 22

Monte Carlo

Fundamental : U = uniform distribution   0 ———— 1

→ Chap §2.1

Chap5 "Qusai - Random Number Generator" $\left\{ \text{Output } \frac{X_i}{N} \in [0,1) \right.$ X_0

$X_1 = aX_0 + b \pmod N$

$X_2 = aX_1 + b \pmod N$

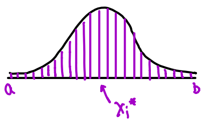
Space filling random generator is proven to be more superior

(Since there is a chance that 0 ×××××× ———— 1 would happen from time to time which would lead to inaccurate simulation)

$\int_0^1 f(x)dx = \mathbb{E}(f(U)) \; (= \int_x f(x) \mathbb{1}_{\{x \in [0,1]\}} dx)$

→ $V = \int_{y=0}^{\infty} e^{-rT} \underset{\text{payoff when } S_T = y}{f(y)} \mathbb{P}(S_T \in [y, dy])$

→ Riemann Sum



$\text{Sum} = \frac{b-a}{N} \cdot \sum_{i=1}^{n} f(x_i^*)$ ⇒ which is essentially a simulation

$\frac{1}{N} \cdot \underbrace{\sum_{i=1}^{n} f(x_i^*)}$ "U"

* A reflection of the space filling RN

Central Limit Theorem

Let f be a density function with finite mean μ, finite variance σ^2

Consider $X_1, X_2, X_3, \dots X_n$ to be independent sample of X and X has pdf f

Let $Z = \frac{\bar{X}_n - \mu}{(\sigma/\sqrt{n})}$, where $\bar{X}_n = \frac{X_1 + \dots + X_n}{n}$

Then $Z \sim N(0,1)$ when n is large.

Monte Carlo Pricing   Let S be the underlying (S_T - terminal stock price)

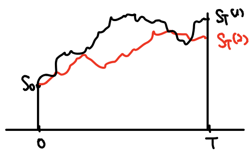
Step 1: Simulate $S_T^{(1)}, \dots, S_T^{(n)}$ (each one is a different "path")

Step 2: Compute the discounted payoff $Y^{(i)}(S_T^{(i)})$

Step 3: Output simulate value = $\frac{1}{n} \sum_{i=1}^{n} Y^{(i)}$

Step 4: Compute the standard deviation $\omega = \text{Std}(Y^{(1)}, Y^{(2)}, \dots, Y^{(n)})$

Step 5: 95% Confidence Interval that true value lying in $(V - \frac{1.96\omega}{\sqrt{n}}, V + \frac{1.96\omega}{\sqrt{n}})$



Recall a distribution function $F_X(x) = \mathbb{P}(X \le x)$ of a r.v. X

if X has a probability density function (pdf), then $F_X(x) = \mathbb{P}(X \le x) = \int \text{pdf}(x)dx$

→ $U \sim U[0,1]$: $F_U(\alpha) = \mathbb{P}(U \le \alpha) = \begin{cases} 1, & \alpha \ge 1 \\ \alpha, & 0 < \alpha < 1 \\ 0, & \alpha \le 0 \end{cases}$

Q: how to simulate a normal variate ?

→ B.S: $S_T = Se^{(r-q-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Z}$ ← Normal variate

$F_{N(\mu,\sigma^2)} = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(u-x)^2}{2\sigma^2}} du$

⇒ $F_{N(0,1)}(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$

Inverse Transform Method

Let F_X be the dist. function of X which you want to simulate.

Let U be uniform $[0,1]$ variate

Theorem: $F_X^{-1}(U)$ is random variable having distribution func. F_X.

Proof: Let $Y = F_X^{-1}(U)$, then dist. function of Y is

$F_Y(\alpha) = \mathbb{P}(Y \le \alpha) = \mathbb{P}(F_X^{-1}(U) \le \alpha) = \mathbb{P}(U \le F_X(\alpha)) = \mathbb{P}(U \le \alpha) = F_X(\alpha) \quad \forall \alpha$ (in $[0,1]$)

Exponential distribution

$F_X(x) = \mathbb{1}_{[0,\infty)}(1 - e^{-\lambda x})$ with $\lambda > 0$

$x = 1 - e^{-\lambda y}$

⇒ $y = \frac{-1}{\lambda} \ln(1-x)$

Hence, $F_X^{-1}(x) = -\frac{1}{\lambda} \ln(1-x)$

$F_X^{-1}(U) = -\frac{1}{\lambda} \ln(1-U)$

Since $U \in [0,1]$, when simulate can use $-\frac{1}{\lambda} \ln(U)$ ($U \sim U[0,1] \Rightarrow 1-U \sim U[0,1]$)

2019. 04. 24

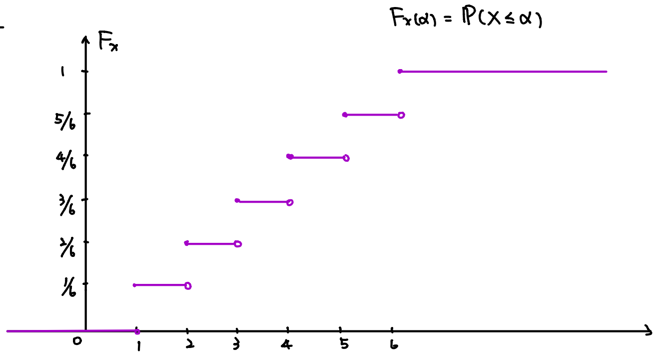
Inverse Transform Method

$F_X^{-1}(U)$ is random variable having distribution func. F_X.

→ Roll the dice

$F_X(\alpha) = \mathbb{P}(X \le \alpha)$

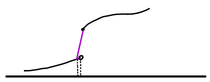
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6



Problem 1: F_X may not be injective, and can be many to one

Rule: Take the "first hit" → draw U between $(0, 1/6]$ ⇒ $X = 1$.

Problem 2: F_X may not cover the entire $(0,1)$



[G] "generalized inverse"

Conditional Sample

Claim: $Y = F_X^{-1}(F_X(a) + [F_X(b) - F_X(a)]U)$ has conditional distribution given $a < X \le b$

Check when $a = -\infty$, $b = +\infty$: since $F_X(a) = 0$, $F_X(b) = 1$, we get $Y = F_X^{-1}(U)$

Proof: Let $V = F_X(a) + (F_X(b) - F_X(a))U$

note $F_X(a) \le V \le F_X(b)$

$\Rightarrow \quad a \le F_X^{-1}(V) \le b$

For $a \le \alpha \le b$, $\mathbb{P}(Y \le \alpha) \overset{?}{=} \dfrac{F(\alpha) - F(a)}{F(b) - F(a)}$

$$\begin{aligned}
\mathbb{P}(Y \le \alpha) &= \mathbb{P}(V \le F_X(\alpha)) \\
&= \mathbb{P}(F_X(a) + (F_X(b) - F_X(a))U \le F_X(\alpha)) \\
&= \mathbb{P}\left(U \le \frac{F_X(\alpha) - F_X(a)}{F_X(b) - F_X(a)}\right) \\
&= \frac{F_X(\alpha) - F_X(a)}{F_X(b) - F_X(a)}
\end{aligned}$$

Box Muller Method

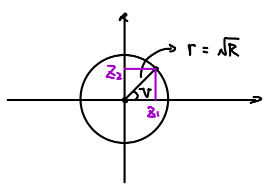
Step 1: Get U_1, U_2 independent uniform $U[0,1]$

Step 2: Let $R = -2\ln U_1$

Step 3: Let $Z_1 = \sqrt{R}\cos V$

Let $Z_2 = \sqrt{R}\sin V$

Step 4: Return Z_1, Z_2



⇒ Theorem: Z_1, Z_2 are independent $N(0,1)$ variables

Proof: Read [G], etc.

"Convincing Calculation": Let Z_1, Z_2 to be independent $N(0,1)$

⇒ Joint pdf of $(Z_1, Z_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Z_1^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{Z_2^2}{2}}$

$$\begin{aligned}
\mathbb{P}(r \le r_0) &= \int_0^{r_0}\int_0^{2\pi} \frac{1}{2\pi} e^{-\frac{r^2}{2}}\, r \, d\theta\, dr \\
&= 1 - e^{-\frac{r_0^2}{2}} \quad \sim \text{exponential}
\end{aligned}$$

$$\Rightarrow \mathbb{P}(\theta \le \theta_0 \mid r = r_0) = \frac{\mathbb{P}(\theta \le \theta_0 \text{ and } r = r_0)}{\mathbb{P}(r = r_0)} = \frac{\frac{1}{2\pi}\int_0^{\theta_0} e^{-\frac{r_0^2}{2}} \cdot r_0\, d\theta}{e^{-\frac{r_0^2}{2}} \cdot r_0} = \frac{\theta_0}{2\pi} \sim \text{Uniform}$$

$S_T = S_0 e^{(r-q-\frac{\sigma^2}{2})T + \sigma\sqrt{T} Z_1}$ ← $N(0,1)$

$S_T^{(1)}, S_T^{(2)}$

How to simulate Multivariate Normal Variable ?

$N = (\vec{\mu}, \Sigma)$
↑
d-dimentional normal

$$\vec{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_d \end{pmatrix} \qquad \Sigma = \begin{pmatrix} V_{11}^2 & \rho_{12}V_{12}V_{21} & \\ & \ddots & \\ & & V_{dd}^2 \end{pmatrix} = \begin{pmatrix} V_{11} & & 0 \\ & \ddots & \\ 0 & & V_{dd} \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1d} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{d1} & \rho_{d2} & \cdots & \rho_{dd} \end{pmatrix} \begin{pmatrix} V_{11} & & 0 \\ & \ddots & \\ 0 & & V_{dd} \end{pmatrix}$$

Recall : Σ is symmetric, semi-positive definite,

⇒ you can find lower triangular matrix A s.t $AA^T = \Sigma$ (cholesky factorization)

$d=2$. $\Sigma = \begin{pmatrix} V_1^2 & \rho V_1 V_2 \\ \rho V_1 V_2 & V_2^2 \end{pmatrix}$ ρ = correlation coefficient.

if $A = \begin{pmatrix} V_1 & 0 \\ \rho V_2 & \sqrt{1-\rho^2}\, V_2 \end{pmatrix}$, $AA^T = \begin{pmatrix} V_1 & 0 \\ \rho V_2 & \sqrt{1-\rho^2}\, V_2 \end{pmatrix} \begin{pmatrix} V_1 & \rho V_2 \\ 0 & \sqrt{1-\rho^2}\, V_2 \end{pmatrix} = \begin{pmatrix} V_1^2 & \rho V_1 V_2 \\ \rho V_1 V_2 & V_2^2 \end{pmatrix} = \Sigma$

Simulation. $X_1 = V_1 Z_1$

$X_2 = V_2 (\rho Z_1 + \sqrt{1-\rho^2}\, Z_2)$

$\langle X_1, X_2 \rangle = \rho V_1 V_2$ ← you see the lower triangular matrix effect