Numerical HW 1 YS3171

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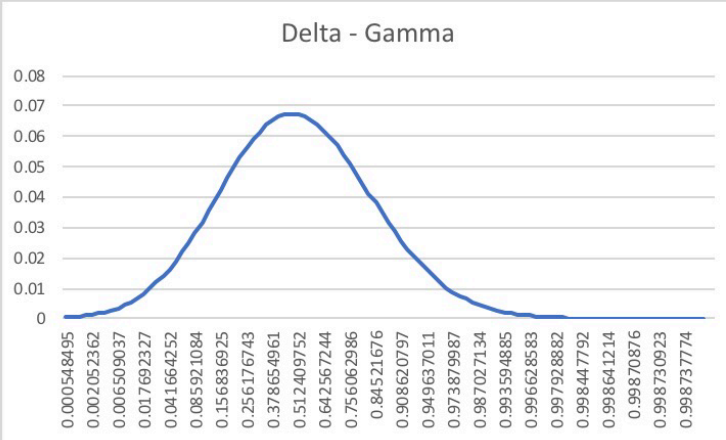
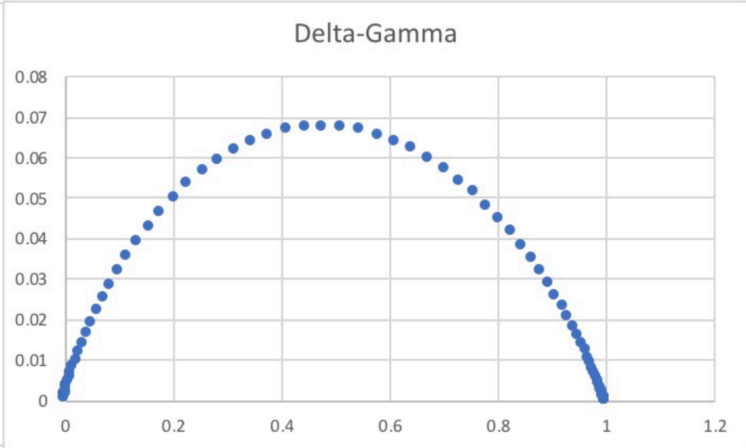
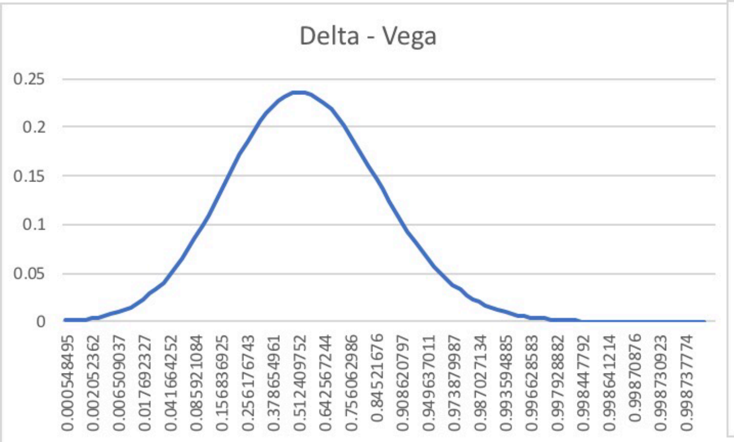
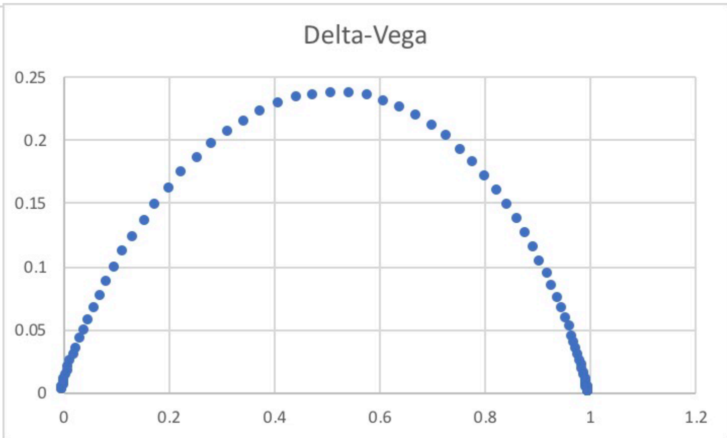
1. a)





b)



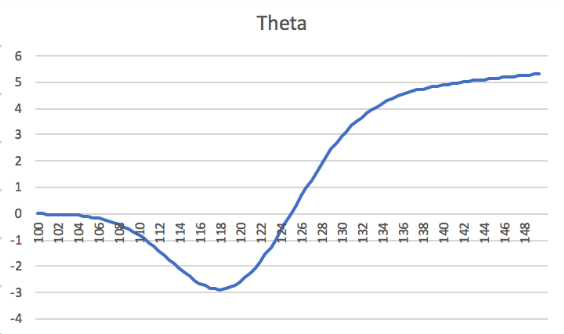


(c) $Gamma = S^2 \cdot \sigma t \cdot 100 \cdot Vega$

when spot price are very small/large, both Gamma and Vega approaches 0, therefore we only need to discuss when spot get very close to strike. As we can see from the scatter graph. those dots where spot gets very close to strike form the main shape of the curve. and for those points since $S \approx K$, we can regard S^2 as constant, since $\sigma, t, 100$ are also constant. Then $\frac{Gamma}{Vega}$ is constant. Therefore those points in Delta-gamma performs similary as in Delta-Vega. That is the reason why two graphs are similar.

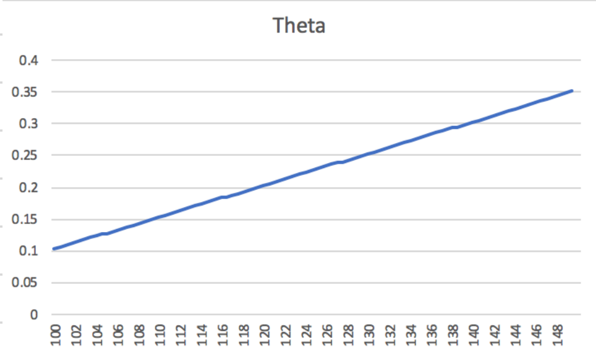
2. (a)

	FX OPTION TABLE		
Spot	149.5	Value	29.1502
Domestic Interest Rate	0.5%	Delta	0.9900
Foreign Interest Rate	4.0%	Gamma	0.0000
Volatility	10.0%	Vega	0.0000
Strike	119	Theta	5.3254
Today's date	2/12/19	Phi	-0.3730
Expiration date	5/15/19	Rho	0.2996
Time to maturity	92		



(b) Example

	FX OPTION TABLE		
Spot	149.5	Value	139.4120
Domestic Interest Rate	4.0%	Delta	0.9987
Foreign Interest Rate	0.5%	Gamma	0.0000
Volatility	10.0%	Vega	0.0000
Strike	10	Theta	0.3506
Today's date	2/12/19	Phi	-0.3763
Expiration date	5/15/19	Rho	0.0250
Time to maturity	92		



(b) Yes. θ can be positive

$$\theta = -\frac{S_0 N'(d_1)\sigma e^{-r_f T}}{2\sqrt{T}} + r_f S_0 N(d_1) e^{-r_f T} - rKe^{-rT}N(d_2)$$

When $S_0 >> K$, $N'(d_1) \to 0$, $N(d_1) \to 1$, $N(d_2) \to 1$

Above formula shows $\theta \approx r_f S_0 e^{-r_f T} - rKe^{-rT}N(d_2)$

Although $r_f < r$, but if S_0 is much bigger than K, theta is still positive in this case

for example 100

(c) The statement is false. Follow is counter example:

As in (b). when $S_0 >> K$, $N'(d_1) \to 0$, $N(d_1) \to 1$, $N(d_2) \to 1$

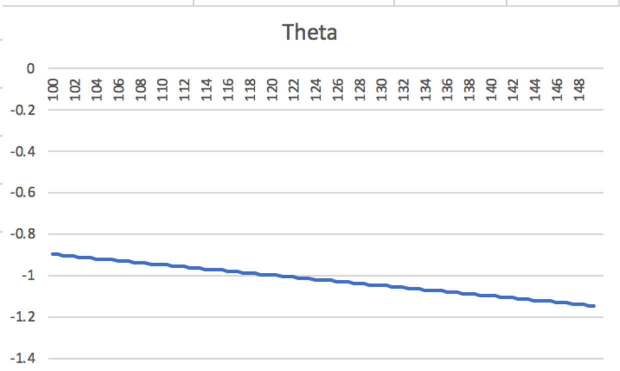
$\theta \approx r_f S_0 e^{-r_f T} - rKe^{-rT}N(d_2)$

Deep in the money means S_0 is much larger then K. however if $r_f < 0$, $r > 0$ then

$r_f S_0 e^{-r_f T} < 0$ and $-rKe^{-rT}N(d_2) < 0$

Therefore θ is negative in this case

	FX OPTION TABLE		
Spot	149.5	Value	139.7888
Domestic Interest Rate	4.0%	Delta	1.0013
Foreign Interest Rate	-0.5%	Gamma	0.0000
Volatility	10.0%	Vega	0.0000
Strike	10	Theta	-1.1444
Today's date	2/12/19	Phi	-0.3773
Expiration date	5/15/19	Rho	0.0250
Time to maturity	92		



3(a) Yes, the risk manager's statement is true.

For foreign exchange option, we have

$$V = S_f e^{-r_f T} N(d_1) - K e^{-r_d T} N(d_2)$$

where $d_1 = \frac{\ln(\frac{S}{K}) + (r_d - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$

$$\frac{\partial V}{\partial d_1} = S e^{-r_f T} N'(d_1) - K e^{-r_d T} N'(d_2)$$

$$= S e^{-r_f T} \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}} - K e^{-r_d T} \frac{e^{-\frac{1}{2}(d_1 - \sigma\sqrt{T})^2}}{\sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{2\pi}}\left(S e^{-r_f T - \frac{1}{2}d_1^2} - K e^{-r_d T - \frac{1}{2}d_1^2 - \frac{1}{2}\sigma^2 T + d_1\sigma\sqrt{T}}\right)$$

$$-r_d T - \frac{1}{2}\sigma^2 T + d_1\sigma\sqrt{T} = -\cancel{r_d T} - \cancel{\tfrac{1}{2}\sigma^2 T} + \ln(\tfrac{S}{K}) + \cancel{r_d T} - r_f T + \cancel{\tfrac{1}{2}\sigma^2 T}$$

$$= \ln(\tfrac{S}{K}) - r_f T$$

$$= \frac{1}{\sqrt{2\pi}}\left(S e^{-r_f T - \frac{1}{2}d_1^2} - K e^{-\frac{1}{2}d_1^2 + \ln(\frac{S}{K}) - r_f T}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(S e^{-r_f T - \frac{1}{2}d_1^2} - K e^{\ln(\frac{S}{K})} e^{-\frac{1}{2}d_1^2 - r_f T}\right)$$

$$\Longrightarrow S e^{-r_f T} N'(d_1) - K e^{-r_d T} N'(d_2) = 0 \quad (\star)$$

$$= \frac{1}{\sqrt{2\pi}}\left(S_f e^{-\frac{1}{2}d_1^2 - r_f T} - S e^{-\frac{1}{2}d_1^2 - r_f T}\right)$$

$$= 0$$

$$\frac{\partial V}{\partial \sigma} = S e^{-r_f T} N'(d_1) \cdot \frac{\partial d_1}{\partial \sigma} - K e^{-r_d T} N'(d_2) \cdot \frac{\partial d_2}{\partial \sigma} \qquad d_2 = d_1 - \sigma\sqrt{T}$$

$$= \left(S e^{-r_f T} N'(d_1) - K e^{-r_d T}\right) \frac{\partial d_1}{\partial \sigma} + K e^{-r_d T} N'(d_2) \frac{\partial \sigma\sqrt{T}}{\partial \sigma}$$

by $(\star)$

$$= K e^{-r_d T} N'(d_2) \sqrt{T}$$

$$\frac{\partial^2 V}{\partial \sigma^2} = K e^{-r_f T} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} (-d_2) \frac{\partial d_2}{\partial \sigma} \sqrt{T}$$

$$= K e^{-r_d T} N'(d_2) \sqrt{T} (-d_2) \frac{\partial d_2}{\partial \sigma}$$

$$= \frac{\partial V}{\partial \sigma} \times (-d_2) \times \frac{-d_1}{\sigma}$$

$$= \frac{\partial V}{\partial \sigma} \cdot \frac{d_1 d_2}{\sigma}$$

$$\frac{\partial d_2}{\partial \sigma} = \left(\frac{\ln(\frac{S}{K}) + (r_d - r_f - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)'$$

$$= -\ln(\tfrac{S}{K}) T^{-\frac{1}{2}} \sigma^{-2} - (r_d - r_f)\sqrt{T}\sigma^{-2} - \tfrac{1}{2}\sqrt{T}$$

$$= \frac{-\ln(\frac{S}{K}) + (r_d - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \times \frac{1}{\sigma}$$

$$= \frac{d_1}{\sigma}$$

(b) Yes, risk manager said truth

For both call & put options $\quad \Gamma = \frac{N'(d_1) e^{-r_f T}}{S\sigma\sqrt{T}}$

$$\frac{\partial^3 V}{\partial S^3} = \frac{\partial \Gamma}{\partial S} = \frac{N'(d_1) e^{-r_f T}}{S\sigma\sqrt{T}} \, dS$$

use quotient rule.

$$= \frac{1}{S^2\sigma^2 T} \times \left[-d_1 N'(d_1) \frac{\partial d_1}{\partial S} \cdot e^{-r_f T} \cdot S\sigma\sqrt{T} - N'(d_1) e^{-r_f T} \sigma\sqrt{T}\right)$$

$$= \frac{N'(d_1) e^{-r_d T}}{S^2 \sigma\sqrt{T}} \left(-d_1 \times \frac{\partial d_1}{\partial S} S - 1\right)$$

$$= \frac{\Gamma}{S} \times \left(-d_1 \times \frac{1}{S} \times \frac{1}{\sigma\sqrt{T}} \times S - 1\right)$$

$$= -\frac{\Gamma}{S}\left(\frac{d_1}{\sigma\sqrt{T}} + 1\right) \quad \text{as desired}$$

4. (1) We are proving that limiting portfolio $\lim_{k'\to k} \frac{1}{k'-k}(c(k)-c(k'))$ is equal to a digital call option with strike K

There are two case:

First case assume $k > k'$:

$$\text{payoff of } \frac{1}{k'-k}(c(k)-c(k')) = \begin{cases} \frac{1}{k'-k}(S-k) - \frac{1}{k'-k}(S-k') = \frac{k'-k}{k'-k} = 1 & \text{when } S \geq k \\ -\frac{1}{k'-k}(S-k') & k > S > k' \\ 0 & S \leq k' \end{cases}$$

When $k' \to k$, probability of $k > S > k'$ becomes 0. and $k' \approx k$, then

$$\text{Payoff of } \lim_{k'\to k} \frac{1}{k'-k}(c(k)-c(k')) = \begin{cases} 1 & \text{if } S \geq k \\ 0 & \text{if } S \leq k \end{cases}$$ which is the same as digital (k)

Second Case $k < k'$

$$\text{payoff of } \frac{1}{k'-k}(c(k)-c(k')) = \begin{cases} 1 & \text{if } S \geq k' \\ -\frac{1}{k'-k}(S-k) & \text{if } k' > S > k \\ 0 & \text{if } S \leq k \end{cases}$$

And similarly payoff of $\lim_{k\to k} \frac{1}{k'-k}(c(k)-c(k')) = \begin{cases} 1 & \text{if } S \geq k \\ 0 & \text{if } S < k \end{cases}$ also same as payoff of digital (k)

Thus a digital call option with strike K is equal limiting portfolio

Therefore digital $(k) = \lim_{k'\to k} \frac{c(k)-c(k')}{k'-k} = -\lim_{k'\to k} \frac{c(k)-c(k')}{k-k'} = -\frac{\partial c(k)}{\partial K}$

(2) In the presence of volatility smile,

$c = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2)$ where $d_1 = \frac{\ln(\frac{S}{K}) + (r-q+\frac{1}{2}\sigma^2(k))T}{\sigma(k)\sqrt{T}}$, $d_2 = d_1 - \sigma(k)\sqrt{T}$

$$\text{digital }(k) = -\frac{\partial C}{\partial K}$$

$$= -Se^{-qT}N'(d_1)\cdot\frac{\partial d_1}{\partial K} + Ke^{-rT}N'(d_2)\frac{\partial d_2}{\partial K} + e^{-rT}N(d_2) \quad \frac{\partial d_2}{\partial K} = \frac{\partial(d_1 - \sigma(k)\sqrt{T})}{\partial K} = \frac{\partial d_1}{\partial K} - \sigma'(k)\sqrt{T}$$

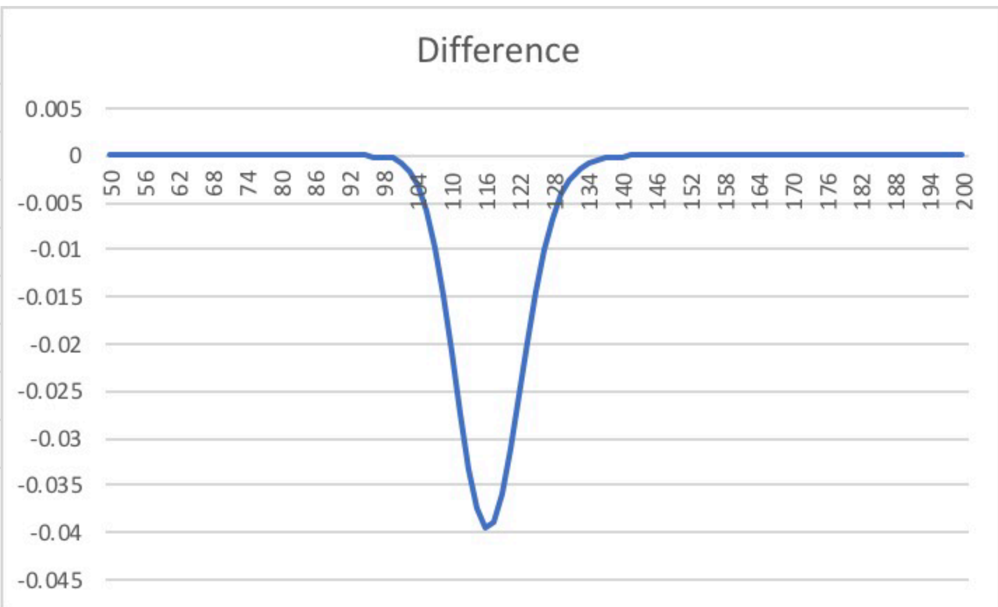
$$= -Se^{-qT}N'(d_1)\cdot\frac{\partial d_1}{\partial K} + Ke^{-rT}N'(d_2)\left(\frac{\partial d_1}{\partial K} - \sigma'(k)\sqrt{T}\right) + e^{-rT}N(d_2)$$

$=0$ by $(*)$

$$= \underbrace{[-Se^{-qT}N'(d_1) + Ke^{-rT}N'(d_2)]}_{=0 \text{ by } (*)}\frac{\partial d_1}{\partial K} - Ke^{-rT}N'(d_2)\sqrt{T}\sigma'(k) + e^{-rT}N(d_2)$$

$$= e^{-rT}N(d_2) - Ke^{-rT}N'(d_2)\sqrt{T}\sigma'(k) \quad \text{as desired}$$

5. (a)



(b) Yes, I agree with trader.

Out of the money call option means that stock price is much below the strike. Value of options are close to 0 and small change in stock price not change value (still 0). So delta goes 0, FRTBΔ(S) also approaches 0, thus difference goes to zero

As for deep in the money call option, where spot price is much higher than strike price, since value of option is already very high and show all value of option. thus even if stock price changes, it will not change rapidly, small or large change in spot rate will have same small effect on values of option. Thus difference of delta and small period delta approaches zero.

(c) $\boxed{S \to 0^+}$

$$\Delta(S) = e^{-r_f T} N(d_1), \quad d_1 = \frac{\ln(\frac{S}{K}) + (r_d - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad d_2 = d_1 - \sigma\sqrt{T}$$

when $S \to 0^+$, then $d_1, d_2 \to \infty$ since $\ln(\frac{S}{K}) \to -\infty$. then $N(d_1), N(d_2) \to 0$

$$\tilde{d}_1 = \frac{\ln(\frac{1.01S}{K}) + (r_d - r_f + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

therefore $\Delta(S) \to 0$ when $S \to 0^+$

$$\lim_{S\to 0^+} \frac{V(1.01S) - V(S)}{0.01S} = \lim_{S\to 0^+} \frac{1.01\,V'(1.01S) - V'(S)}{0.01} = \lim_{S\to 0^+} \frac{1.01 e^{-r_f T} N(\tilde{d}_1) - e^{-r_f T} N(d_1)}{0.01} = \frac{0}{0.01} = 0$$

Therefore $\lim_{S\to 0^+}\left(\Delta(S) - \frac{V(1.01) - V(S)}{0.01S}\right) = 0$

$\boxed{S \to \infty}$

$\Delta(S) = e^{-r_f T} N(d_1)$, when $S \to \infty$ then $d_1, d_2 \to \infty$ $N(d_1), N(d_2) \to 1$

$$\lim_{S\to\infty} \Delta(S) = e^{-r_f T} \lim_{S\to\infty} (N(d_1)) = e^{-r_f T}$$

$$\lim_{S\to\infty} \frac{V(1.01S) - V(S)}{0.01S} = \lim_{S\to\infty} \frac{1.01S e^{-r_f T} N(\tilde{d}_1) - K e^{-r_d T} N(\tilde{d}_2) - S e^{-r_f T} N(d_1) + K e^{-r_d T} N(d_2)}{0.01S}$$

$$\approx \lim_{S\to\infty} \frac{1.01S e^{-r_f T} - S e^{-r_f T}}{0.01S}$$

$$= e^{-r_f T}$$

Thus $\lim_{S\to\infty}\left(\Delta(S) - \frac{V(1.01) - V(S)}{0.01S}\right) = e^{-r_f T} - e^{-r_f T} = 0$