Data Structures and Algorithms

CS245-2013S-18

Spanning Trees

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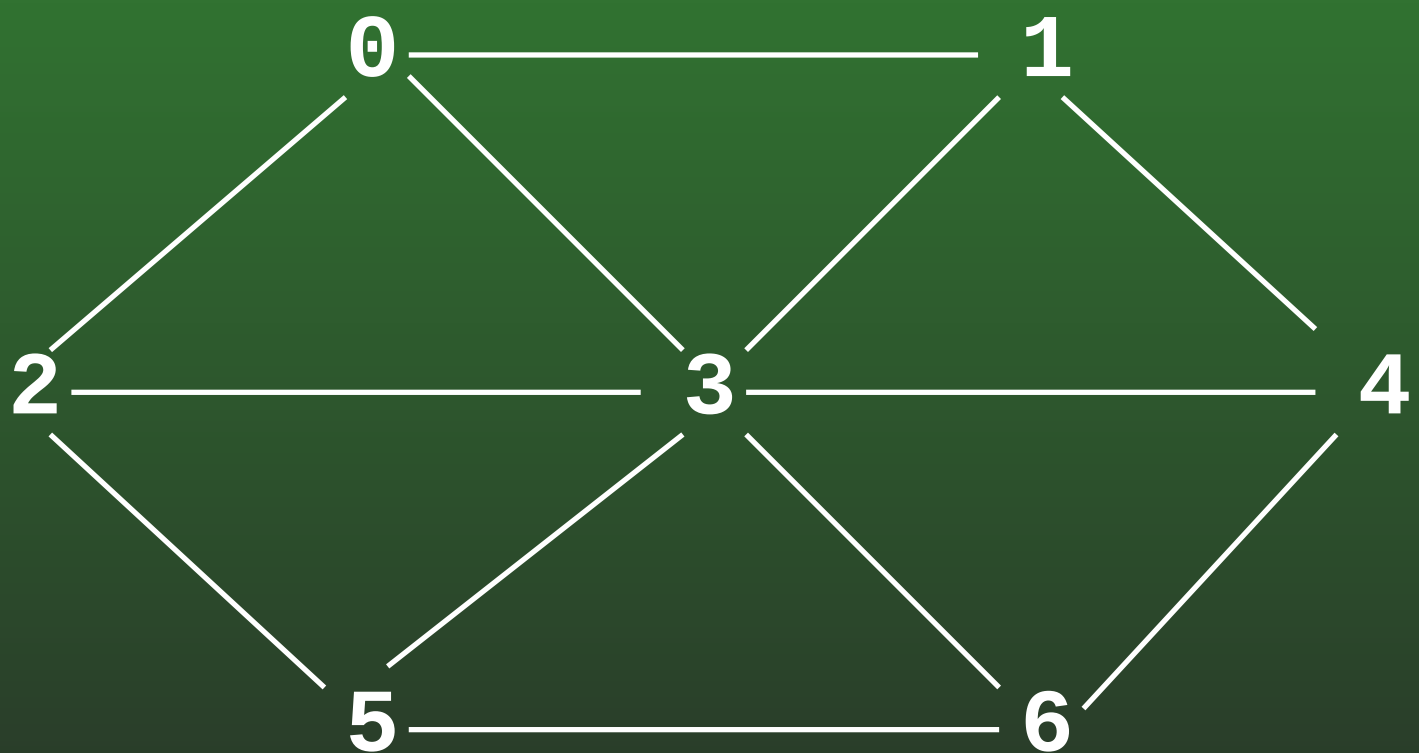
18-0: Spanning Trees

- Given a connected, undirected graph G
  - A *subgraph* of G contains a subset of the vertices and edges in G
  - A *Spanning Tree* T of G is:
    - subgraph of G
    - contains all vertices in G
    - connected
    - acyclic

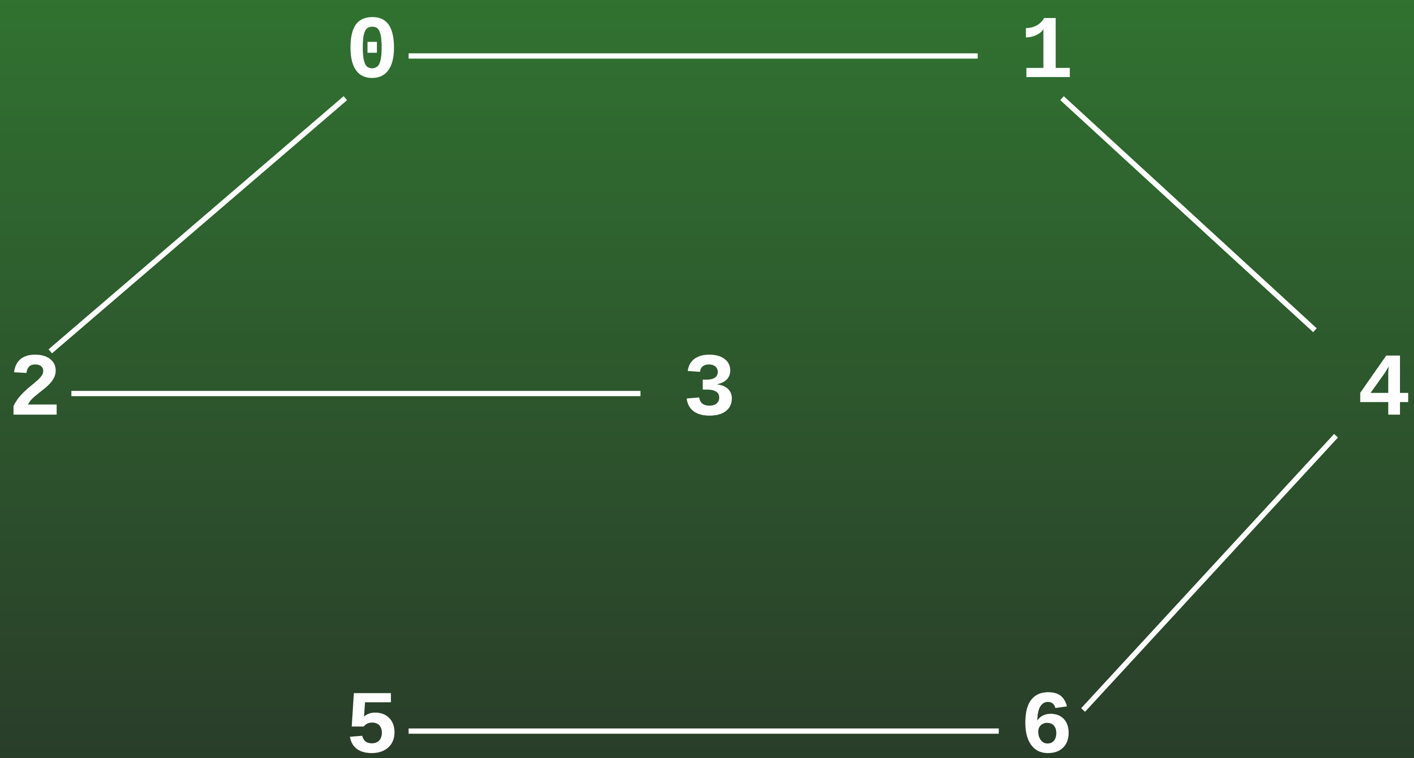
18-1: Spanning Tree Examples

- Graph



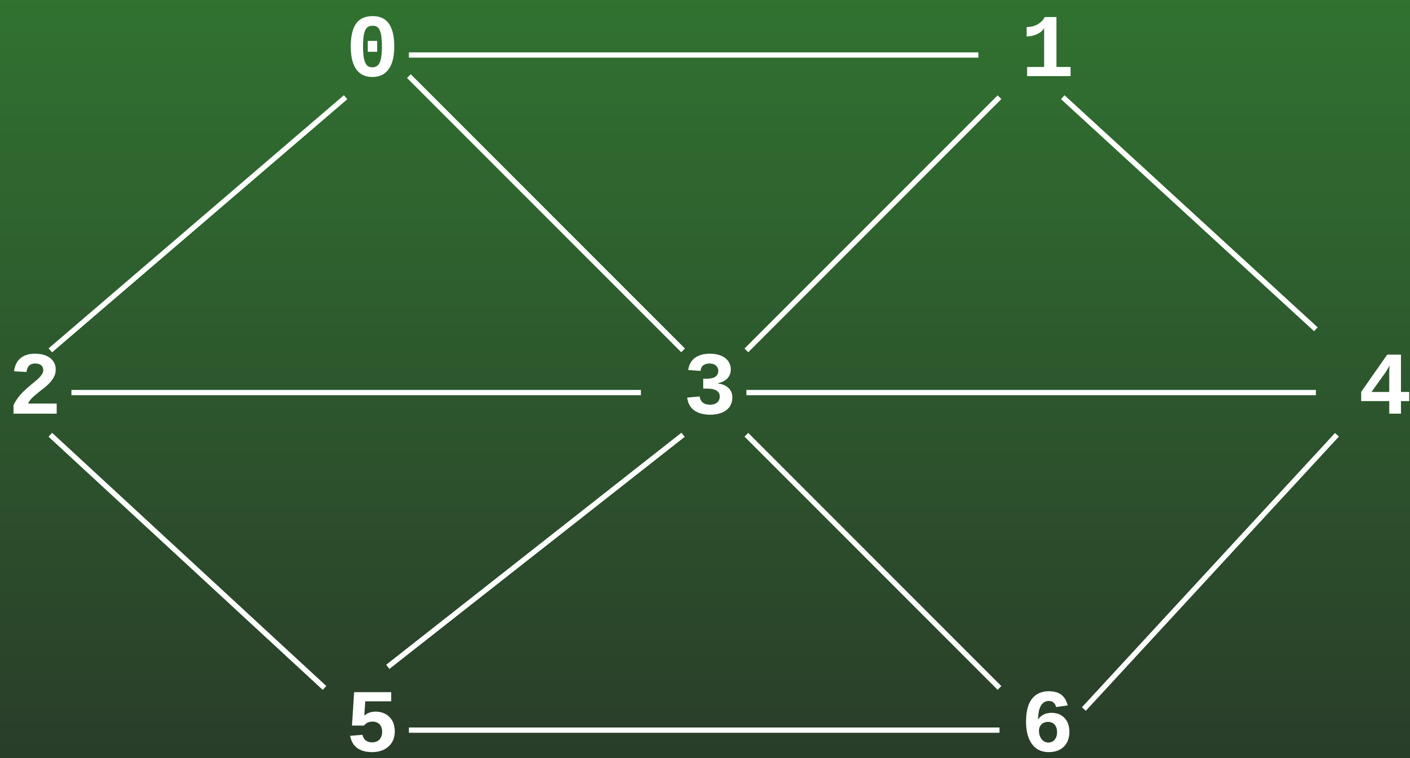
18-2: Spanning Tree Examples

- Spanning Tree



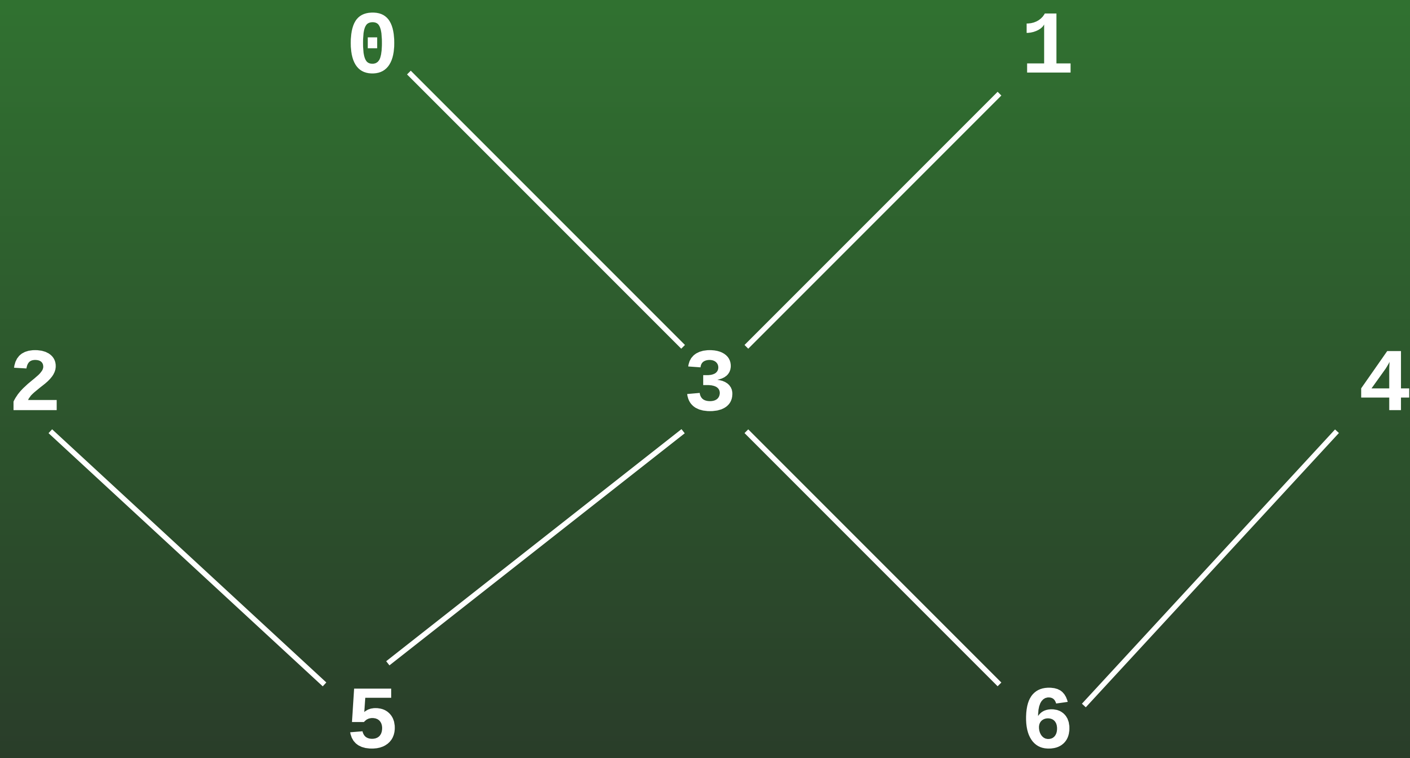
18-3: Spanning Tree Examples

- Graph



18-4: Spanning Tree Examples

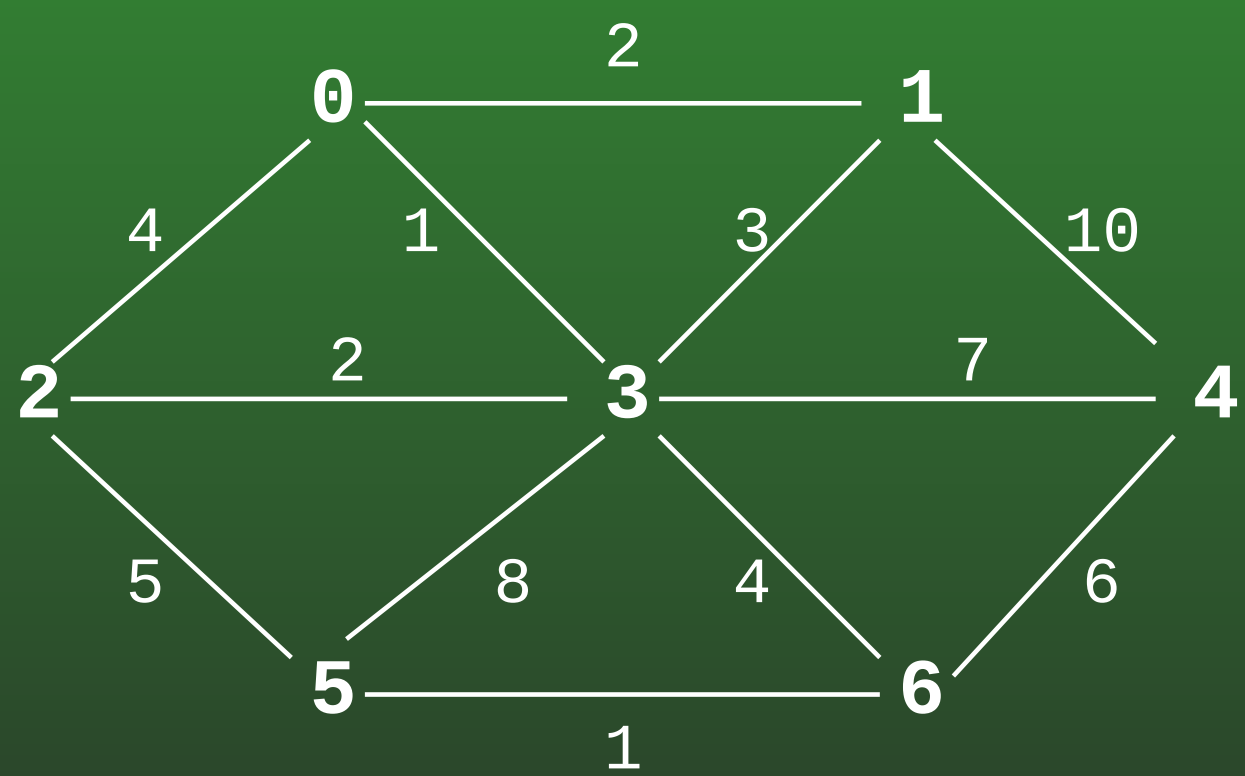
- Spanning Tree



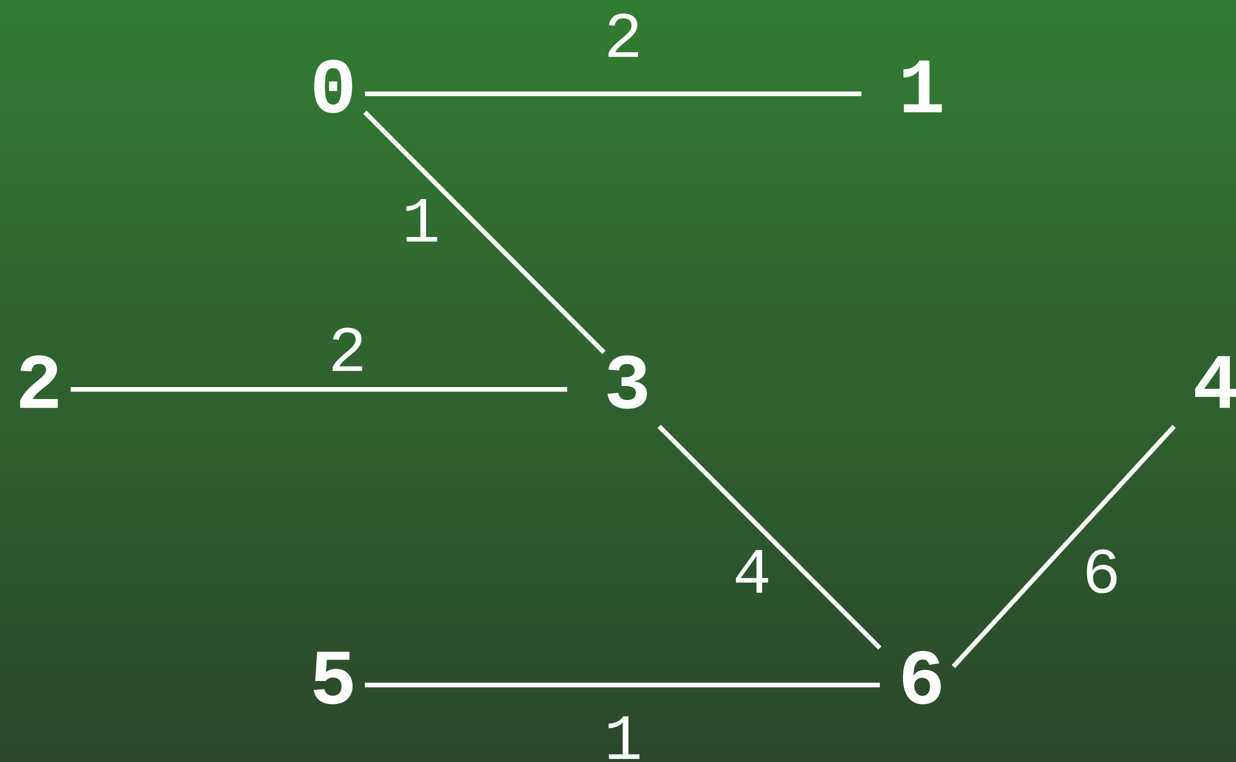
Minimal Cost Spanning Tree

- Minimal Cost Spanning Tree
  - Given a weighted, undirected graph G
  - Spanning tree of G which minimizes the sum of all weights on edges of spanning tree

MST Example



18-7: MST Example



18-8: Minimal Cost Spanning Trees

- Can there be more than one minimal cost spanning tree for a particular graph?

18-9: Minimal Cost Spanning Trees

- Can there be more than one minimal cost spanning tree for a particular graph?
- YES!
  - What happens when all edges have unit cost?

18-10: Minimal Cost Spanning Trees

- Can there be more than one minimal cost spanning tree for a particular graph?
- YES!
  - What happens when all edges have unit cost?
  - All spanning trees are MSTs

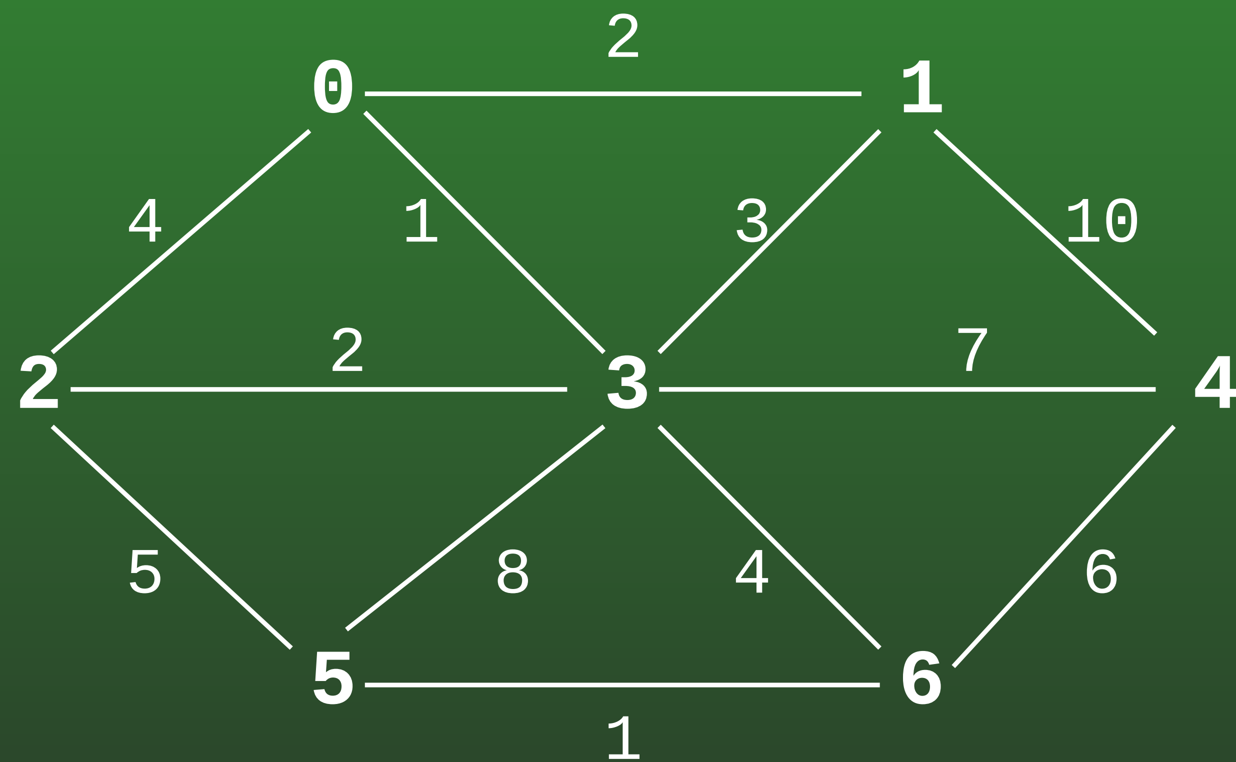
18-11: Calculating MST

- Two algorithms to calculate MST:
  - Kruskal's Algorithm
    - Build a "forest" of spanning trees
    - Combine into one tree
  - Prims Algorithm
    - Grow a single tree out from a start vertex

Kruskal's Algorithm

- Start with an empty graph (no edges)
- Sort the edges by cost
- For each edge e (in increasing order of cost)
  - Add e to G if it would not cause a cycle

18-13: Kruskal's Algorithm Examples



18-14: Kruskal's Algorithm

- Proof (by contradiction)
- Assume that *no* optimal MST T contains the minimum cost edge e
- Add e to T, which causes a cycle
- Remove an edge other than e to break the cycle
- cost $T' \leq T$, a contradiction

18-15: Kruskal's Algorithm

- Coding Kruskal's Algorithm:
  - Place all edges into a list
  - Sort list of edges by cost
  - For each edge in the list
    - Select the edge if it does not form a cycle with previously selected edges
    - How can we do this?

18-16: Kruskal's Algorithm

- Determining of adding an edge will cause a cycle
  - Start with a forest of V trees (each containing one node)
  - Each added edge merges two trees into one tree
  - An edge causes a cycle if both vertices are in the same tree
    - (examples)

18-17: Kruskal's Algorithm

- We need to:
  - Put each vertex in its own tree
  - Given any two vertices v_1 and v_2, determine if they are in the same tree
  - Given any two vertices v_1 and v_2, merge the tree containing v_1 and the tree containing v_2
- ... sound familiar?

Kruskal's Algorithm

- Disjoint sets!
- Create a list of all edges
- Sort list of edges
- For each edge $e = (v_1, v_2)$ in the list
  - if FIND(v_1) != FIND(v_2)
    - Add e to spanning tree
    - UNION(v_1, v_2)

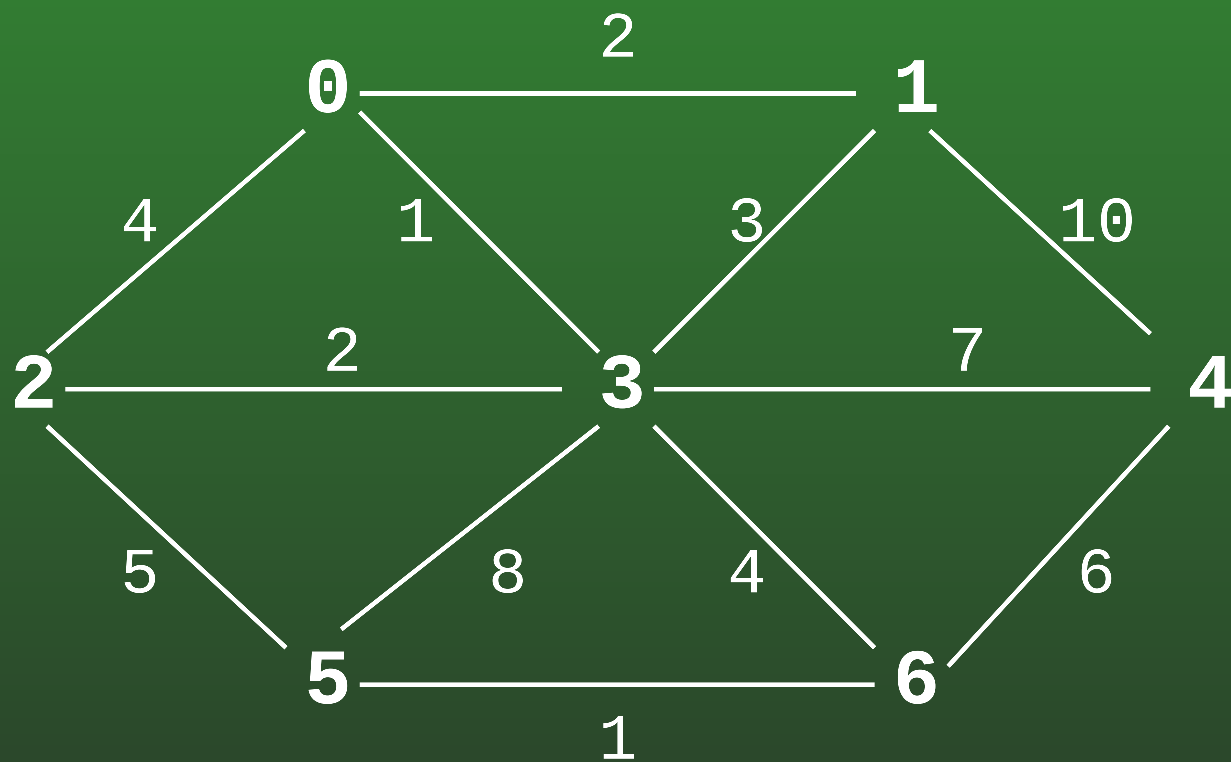
18-19: Prim's Algorithm

- Grow that spanning tree out from an initial vertex
- Divide the graph into two sets of vertices
  - vertices in the spanning tree
  - vertices *not* in the spanning tree
- Initially, Start vertex is in the spanning tree, all other vertices are not in the tree
  - Pick the initial vertex arbitrarily

18-20: Prim's Algorithm

- While there are vertices not in the spanning tree
  - Add the cheapest vertex to the spanning tree

18-21: Prims's Algorithm Examples



Prim's Algorithm

- Use a table – much like Dijkstra table
- Path has the same meaning
- Cost is for vertex v_k
  - cost to add v_k to the tree
  - (instead of length of path to v_k)

18-23: Prim's Algorithm

- Code for Prim's algorithm is very similar to the code for Dijkstra's algorithm
- Make *one small change* to Dijkstra's algorithm to get Prim's algorithm

18-24: Dijkstra Code

```
void Dijkstra(Edge G[], int s, tableEntry T[]) {
  int i, v;
  Edge e;
  for(i=0; i<G.length; i++) {
    T[i].distance = Integer.MAX_VALUE;
    T[i].path = -1;
    T[i].known = false;
  }
  T[s].distance = 0;
  for (i=0; i < G.length; i++) {
    v = minUnknownVertex(T);
    T[v].known = true;
    for (e = G[v]; e != null; e = e.next) {
      if (T[e.neighbor].distance >
            T[v].distance + e.cost) {
        T[e.neighbor].distance = T[v].distance + e.cost;
        T[e.neighbor].path = v;
      }
    }
  }
}
```

18-25: Prim Code

```
void Dijkstra(Edge G[], int s, tableEntry T[]) {
  int i, v;
  Edge e;
  for(i=0; i<G.length; i++) {
    T[i].distance = Integer.MAX_VALUE;
    T[i].path = -1;
    T[i].known = false;
  }
  T[s].distance = 0;
  for (i=0; i < G.length; i++) {
    v = minUnknownVertex(T);
    T[v].known = true;
    for (e = G[v]; e != null; e = e.next) {
      if (T[e.neighbor].distance >
            e.cost) {
        T[e.neighbor].distance = e.cost;
        T[e.neighbor].path = v;
      }
    }
  }
}
```