Linear Algebra

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Eigenvectors and Eigenvalues

- **Definition**: An **eigenvector** of a **square** matrix $A \in \mathbb{R}^{n\times n}$ is a **nonzero** vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ
  In this case, λ is called an **eigenvalue** of A, and
  such an $\mathbf{x}$ is called an ***eigenvector corresponding to λ***.

Transformation Perspective

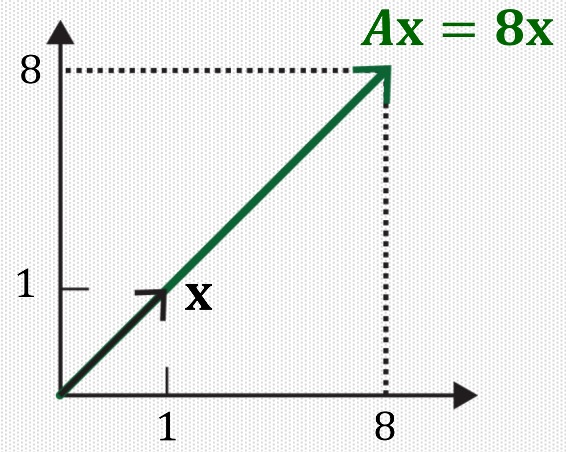
일반적으로 선형변환을 하면 방향과 크기가 변하게 되는데,
eigenvector는 크기만 변하고 방향은 변하지 않는다

- Consider a linear transformation $T(\mathbf{x}) = A\mathbf{x}$.
- If $\mathbf{x}$ is an eigenvector, then $T(\mathbf{x}) = A\mathbf{x} = \lambda\mathbf{x}$, which means the output vector has **the same direction** as $\mathbf{x}$, but the length is scaled by a factor of λ.

방향은 그대로이고 크기만 변하는 벡터

- **Example:** For $A = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$, an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ since

$$T(\mathbf{x}) = A\mathbf{x} = \underbrace{\begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\mathbf{x}} \underset{=}{=} \begin{bmatrix} 8 \\ 8 \end{bmatrix} = \underbrace{8}_{8} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\mathbf{x}}$$



Computational Advantage

- Which computation is faster between $\begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $8\begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

eignvector와 eigenvalue를 이용한 계산이 더 빠르다

Eigenvectors and Eigenvalues

- The equation $A\mathbf{x} = \lambda\mathbf{x}$ can be re-written as

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

독립이면 x는 반드시 zero이어야 한다

A행렬이 독립일지라도 여기에 를 빼줌
으로써 의존이 될 수 있다

그런 남다 와 X를 찾는 것이다

- λ is an eigenvalue of an $n \times n$ matrix A if and only if this equation has a **nontrivial** solution (since $\mathbf{x}$ should be a nonzero vector).

역행렬이 존재한다는 것은 서로 독립이고 해가 존재한다
역행렬이 존재한다면 x는 0이 될수밖에 없다 (p.160참조)
0이 아닌 x값을 얻기 위해서는 역행렬이 없어야 한다

그래서, 역행렬이 0가 되는 남다(eigenvalue)를 구한다

Eigenvectors and Eigenvalues

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

- The set of *all* solutions of the above equation is the **null space** of the matrix $(A - \lambda I)$, which we call the **eigenspace** of A **corresponding to λ**.
- The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ, satisfying the above equation.

Null Space

- **Definition**: The **null space** of a matrix $A \in \mathbb{R}^{m \times n}$ is the set of all solutions of a homogeneous linear system, $A\mathbf{x} = \mathbf{0}$.

  We denote the null space of A as Nul A.

- For $A = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}$, $\mathbf{x}$ should satisfy the following:

$$\mathbf{a}_1^T\mathbf{x} = 0, \mathbf{a}_2^T\mathbf{x} = 0, \dots, \mathbf{a}_m^T\mathbf{x} = 0$$

- That is, $\mathbf{x}$ should be orthogonal to every row vector in A.

Null Space is a Subspace

- **Theorem**: The null space of a matrix $A \in \mathbb{R}^{m \times n}$, denoted as Nul A is a subspace of $\mathbb{R}^n$. In other words, the set of all the solutions of a system $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.

- **Note**: An eigenspace thus have a set of basis vectors with a particular dimension.

Orthogonal Complement

- If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then $\mathbf{z}$ is said to be **orthogonal to** W.
- The set of all vectors $\mathbf{z}$ that are orthogonal to W is called the **orthogonal complement** of W and is denoted by $W^\perp$ (and read as "W perpendicular" or simply "*W* perp").
- A vector $\mathbf{x} \in \mathbb{R}^n$ is in $W^\perp$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans W.
- $W^\perp$ is a subspace of $\mathbb{R}^n$.
- Nul $A = (\mathrm{Row}\, A)^\perp$.
- Likewise, Nul $A^T = (\mathrm{Col}\, A)^\perp$.

Fundamental Subspaces Given by A

- Nul $A = (\text{Row } A)^{\perp}$.
- Nul $A^T = (\text{Col } A)^{\perp}$.

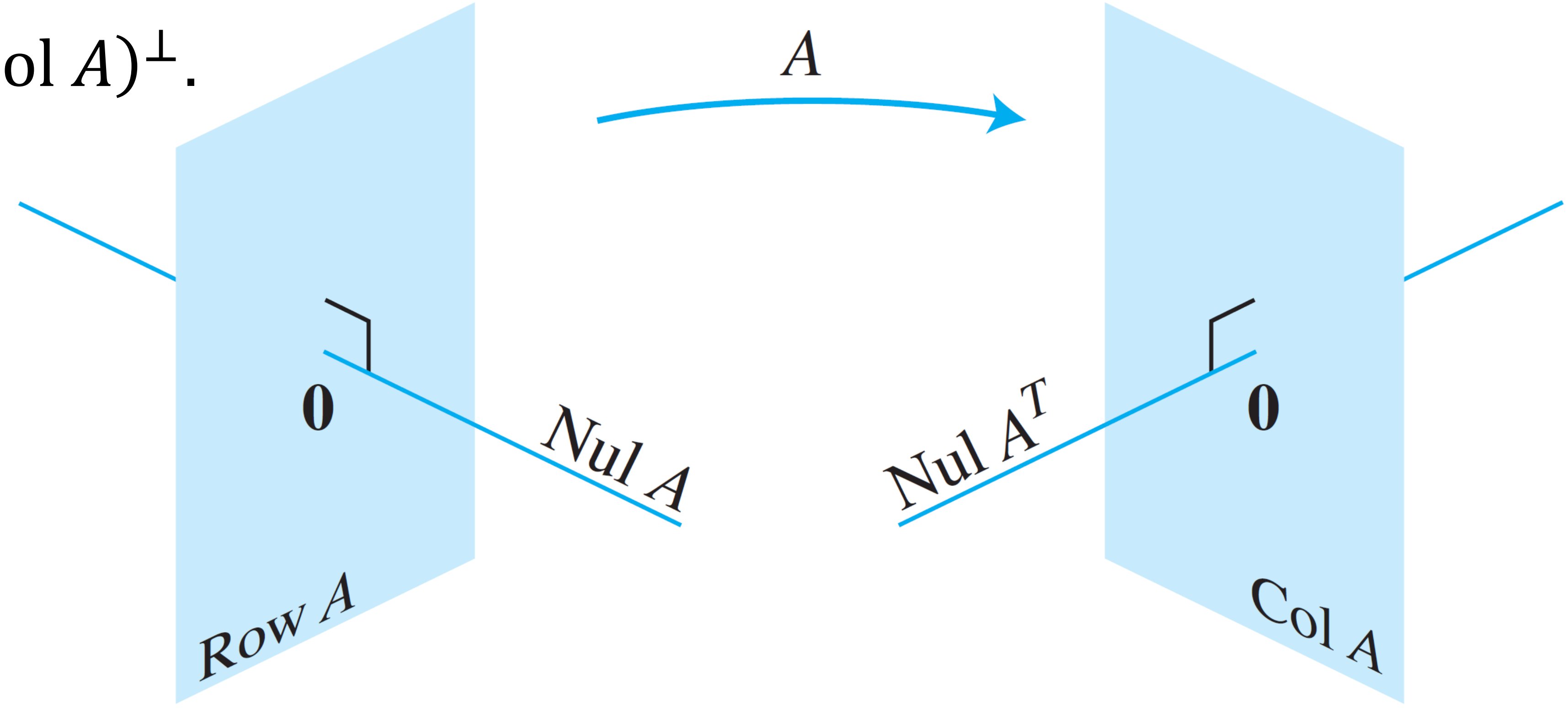


FIGURE 8 The fundamental subspaces determined by an $m \times n$ matrix A.

Example: Eigenvalues and Eigenvectors

- **Example**: Show that 8 is an eigenvalue of a matrix

  $A = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$ and find the corresponding eigenvectors.

- **Solution**: The scalar 8 is an eigenvalue of A if and only if the equation $(A - 8I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution:

$$(A - 8I)\mathbf{x} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

- The solution is $\mathbf{x} = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for any nonzero scalar c,

  which is Span $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

Example: Eigenvalues and Eigenvectors

- In the previous example, -3 is also an eigenvalue:

$$(A+3I)\mathbf{x} = \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

- The solution is $\mathbf{x} = c\begin{bmatrix} 1 \\ -5/6 \end{bmatrix}$ for any nonzero scalar c, which is Span $\left\{\begin{bmatrix} 1 \\ -5/6 \end{bmatrix}\right\}$.

Characteristic Equation

- How can we find the eigenvalues such as 8 and -3?
- If $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then the columns of $(A - \lambda I)$ should be noninvertible.
- If it is invertible, $\mathbf{x}$ cannot be a nonzero vector since
$$(A - \lambda I)^{-1}(A - \lambda I)\mathbf{x} = (A - \lambda I)^{-1}\mathbf{0} \Longrightarrow \mathbf{x} = \mathbf{0}$$
- Thus, we can obtain eigenvalues by solving
$$\det(A - \lambda I) = 0$$
called a **characteristic equation**.
- Also, the solution is not unique, and thus $A - \lambda I$ has linearly dependent columns.

Example: Characteristic Equation

- In the previous example, $A = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$ is originally invertible since

$$\det(A) = \det \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix} = 6 - 30 = -24 \neq 0.$$

- By solving the characteristic equation, we want to find λ that makes $A - \lambda I$ non-invertible:

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 6 \\ 5 & 3-\lambda \end{bmatrix}$$

$$= (2-\lambda)(3-\lambda) - 30$$

$$= -\lambda^2 - 5\lambda - 25 = (8-\lambda)(-3-\lambda) = 0$$

$$\lambda = -3 \text{ or } 8$$

Example: Characteristic Equation

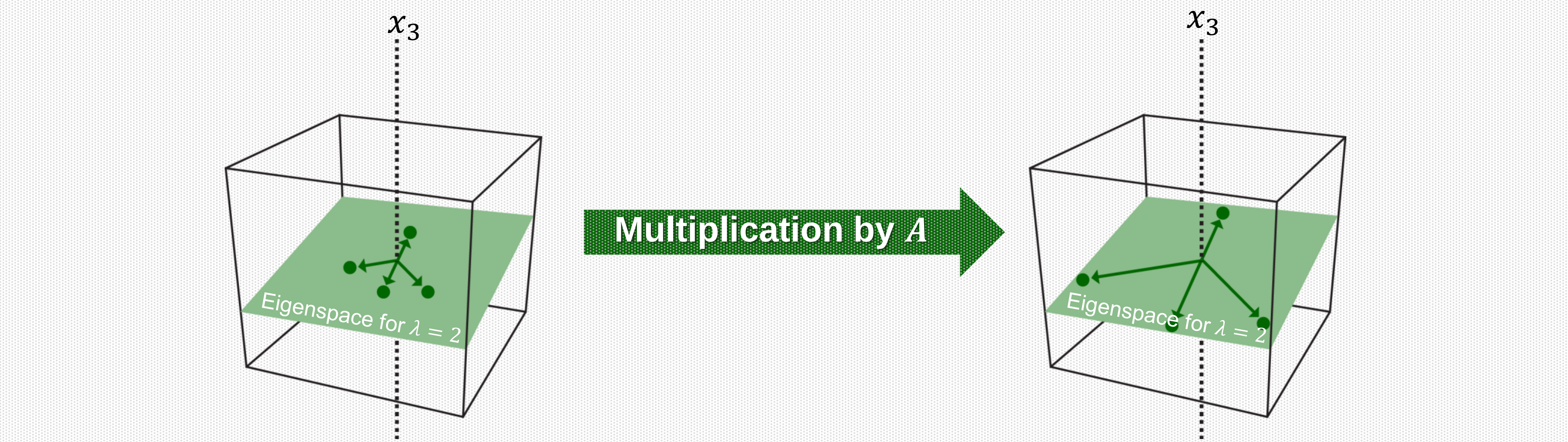
- Once obtaining eigenvalues, we compute the eigenvectors for each λ by solving

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

Eigenspace

- Note that the dimension of the eigenspace (corresponding to a particular λ) can be more than one. In this case, any vector in the eigenspace satisfies

$$T(\mathbf{x}) = A\mathbf{x} = \lambda\mathbf{x}$$



A acts as a dilation on the eigenspace.

Finding all eigenvalues and eigenvectors

- In summary, we can find all the possible eigenvalues and eigenvectors, as follows.
- First, find all the eigenvalue by solving the **characteristic equation**:

$$\det(A - \lambda I) = 0$$

- Second, for each eigenvalue λ, solve for $(A - \lambda I)\mathbf{x} = \mathbf{0}$ and obtain the set of basis vectors of the corresponding eigenspace.