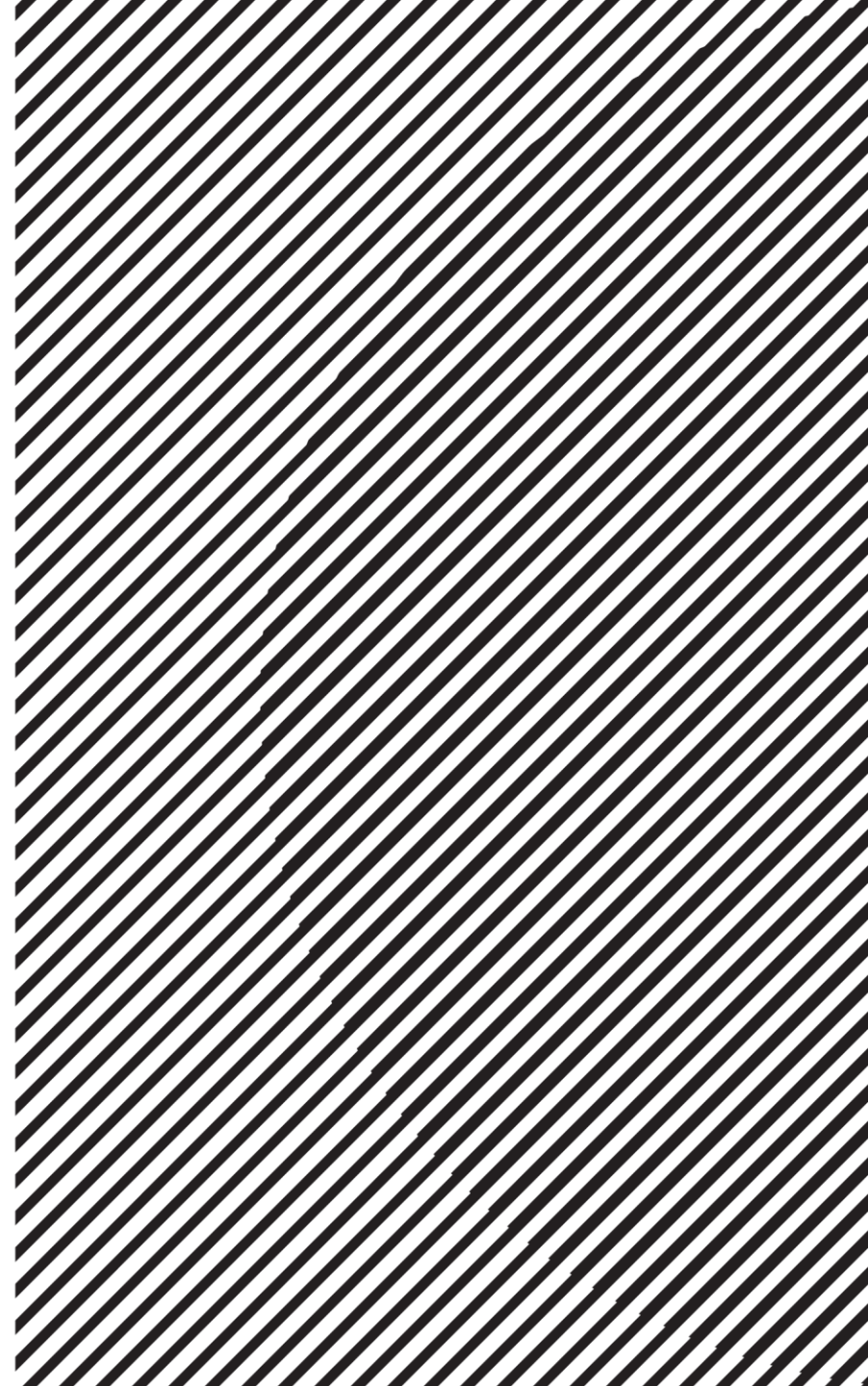

Linear Algebra

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Lecture Overview

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,
Four views of matrix multiplication
- Linear independence, span, and subspace
- Linear transformation
- Least squares
- Eigendecomposition
- Singular value decomposition



Span and Subspace

- **Definition:** A **subspace** H is defined as a subset of \mathbb{R}^n **closed under linear combination:**
- For any two vectors, $\mathbf{u}_1, \mathbf{u}_2 \in H$, and any two scalars c and d , $c\mathbf{u}_1 + d\mathbf{u}_2 \in H$.
- $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is always a subspace. Why?
 - $\mathbf{u}_1 = a_1\mathbf{v}_1 + \dots + a_p\mathbf{v}_p, \mathbf{u}_2 = b_1\mathbf{v}_1 + \dots + b_p\mathbf{v}_p$
 - $$\begin{aligned} c\mathbf{u}_1 + d\mathbf{u}_2 &= c(a_1\mathbf{v}_1 + \dots + a_p\mathbf{v}_p) + d(b_1\mathbf{v}_1 + \dots + b_p\mathbf{v}_p) \\ &= (ca_1 + db_1)\mathbf{v}_1 + \dots + (ca_p + db_p)\mathbf{v}_p \end{aligned}$$
- In fact, a subspace is always represented as $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.



Basis of a Subspace

- **Definition:** A **basis** of a subspace H is a set of vectors that satisfies both of the following:
 - Fully spans the given subspace H
 - Linearly independent (i.e., no redundancy)
- In the previous example, where $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ forms a plane, but $\mathbf{v}_3 = 2\mathbf{v}_1 + 3\mathbf{v}_2 \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of H , but not $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ nor $\{\mathbf{v}_1\}$ is a basis.

Non-Uniqueness of Basis

subset > span > subspace (closed under linear combination) < basis

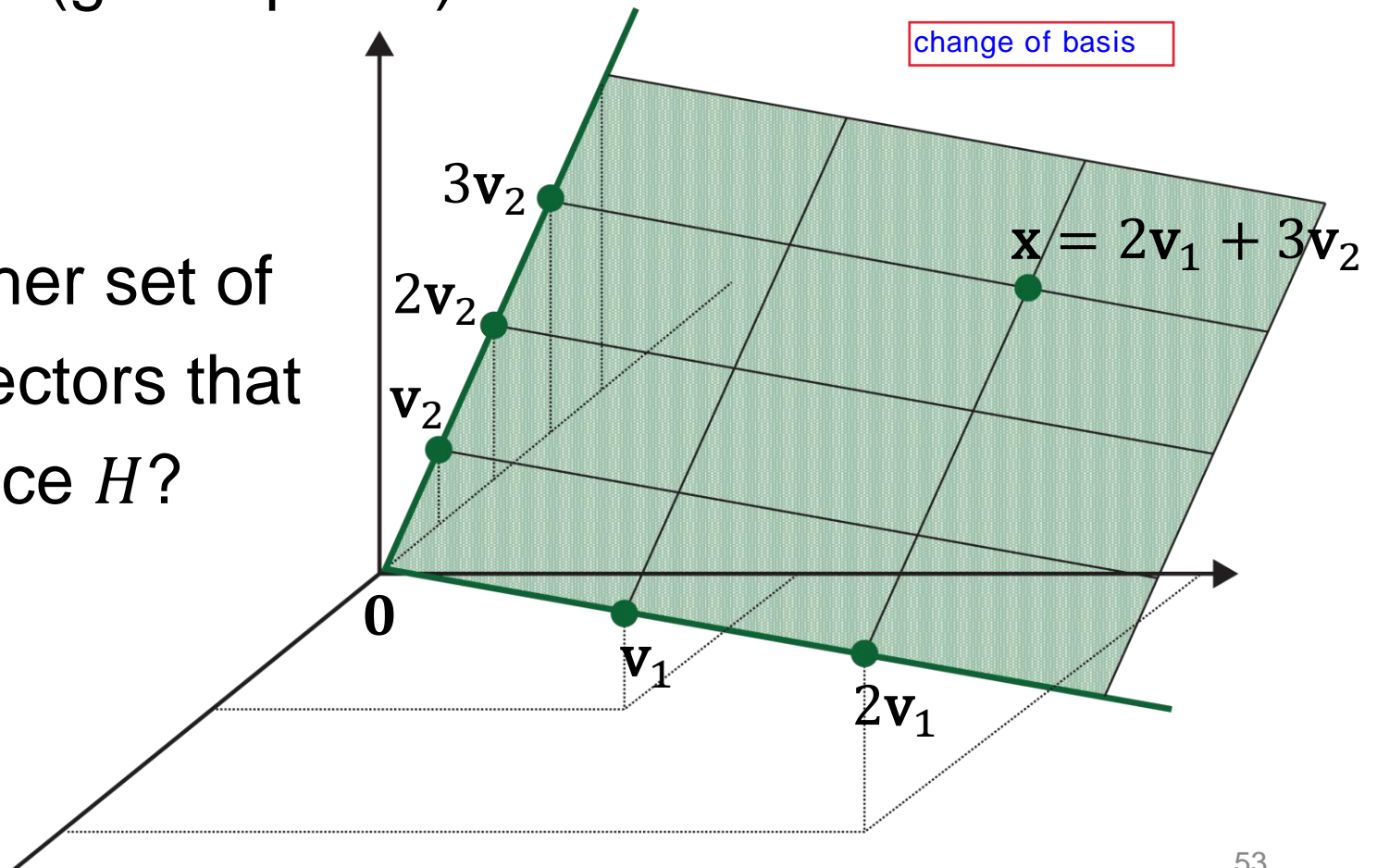
subspace > basis (fully span, independent &)

basis가
가 ()가 .

change of basis

- Consider a subspace H (green plane).
- Is a basis unique?
- That is, is there any other set of linearly independent vectors that span the same subspace H ?

standard basis / 3
(1,0,0)
(0,1,0)
(0,0,1)
->





Dimension of Subspace

basis

- What is then unique, given a particular subspace H ?
- Even though different bases exist for H , the number of vectors in any basis for H will be unique.
- We call this number as the dimension of H , denoted as $\dim H$.
- In the previous example, the dimension of the plane is 2, meaning any basis for this subspace contains exactly two vectors.

Column Space of Matrix

- **Definition:** The **column space** of a matrix A is the subspace spanned by the columns of A . We call the column space of A as **Col** A .

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \longrightarrow \quad \text{Col } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- What is $\dim \text{Col } A$?

Matrix with Linearly Dependent Columns

- Given $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, note that $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$,

i.e., the third column is a linear combination of the first two.

$$\text{Col } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\} \longrightarrow \text{Col } A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- What is $\dim \text{Col } A$?



Rank of Matrix

- **Definition:** The **rank** of a matrix A , denoted by **rank A** , is the dimension of the column space of A :

- $\text{rank } A = \dim \text{Col } A$

feature()가 100 rank가 5 , 가
Regularization



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Summary So Far

- Scalars, vectors, matrices, and their operations such as addition, scalar multiple, matrix multiplication, transpose
- Linear system: solving using inverse matrix
- Matrix equation and vector equation
- Linear combination and Span
 - When does the solution of a linear system exist?
- Four views of matrix multiplication: inner product, column combination, row combination, sum of rank-1 outer products
- Linear independence
 - If the solution of a linear system exists, when is it unique or many?
- Subspace
 - Subset of vectors in \mathbb{R}^n closed under linear combination
 - Basis and dimension
 - Column space and rank of a matrix