
Linear Algebra

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Lecture Overview

- Elements in linear algebra
- Linear system
- Linear combination, vector equation,
Four views of matrix multiplication
- Linear independence, span, and subspace
- Linear transformation
- Least squares
- Eigendecomposition
- Singular value decomposition



Linear Combinations

- Given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in \mathbb{R}^n and given scalars c_1, c_2, \dots, c_p ,

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \dots, \mathbf{v}_p$ with **weights or coefficients** c_1, \dots, c_p .

- The weights in a linear combination can be any real numbers, including zero.

From Matrix Equation to Vector Equation

- Recall the matrix equation of a linear system:

Person ID	Weight	Height	Is_smoking	Life-span
1	60kg	5.5ft	Yes (=1)	66
2	65kg	5.0ft	No (=0)	74
3	55kg	6.0ft	Yes (=1)	78

$$\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$A \quad x = b$

- A matrix equation can be converted into a vector equation:

$$\rightarrow \begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$
$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$



Existence of Solution for $Ax = b$

- Consider its vector equation:

$$\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

- When does the solution exist for $Ax = b$?



Span

- **Definition:** Given a set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{R}^n$, $\text{Span } \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is defined as **the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$.**
- That is, $\text{Span } \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

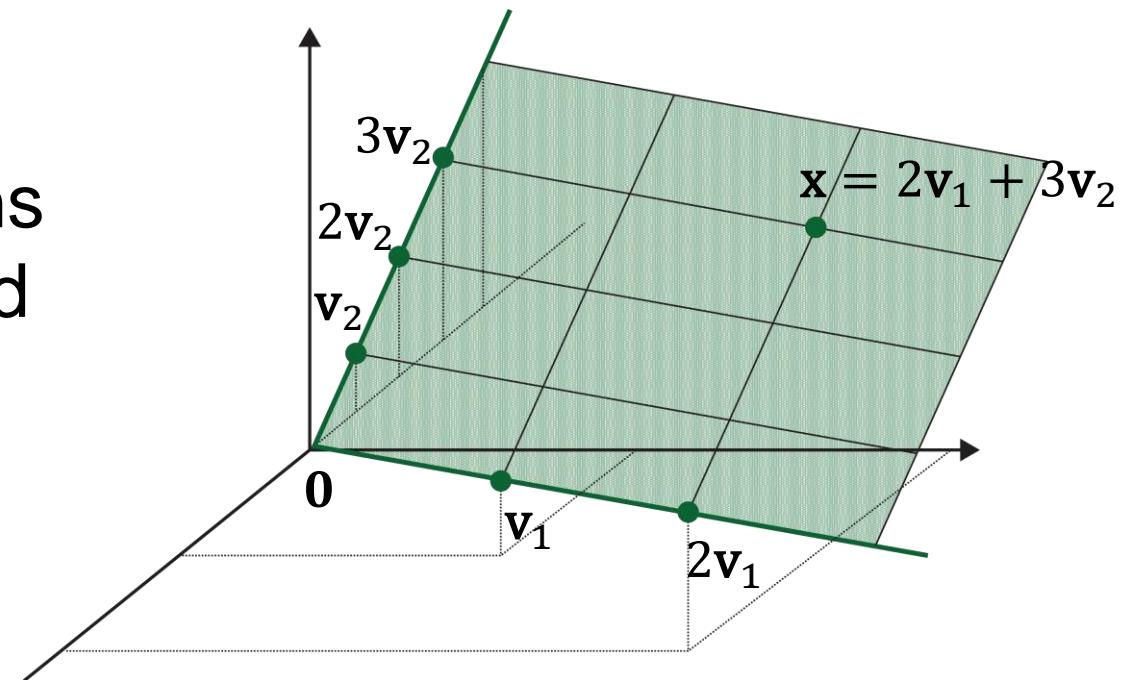
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p$$

with arbitrary scalars c_1, \dots, c_p .

- $\text{Span } \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is also called **the subset of \mathbb{R}^n spanned (or generated) by $\mathbf{v}_1, \dots, \mathbf{v}_p$.**

Geometric Description of Span

- If \mathbf{v}_1 and \mathbf{v}_2 are nonzero vectors in \mathbb{R}^3 , with \mathbf{v}_2 not a multiple of \mathbf{v}_1 , then $\text{Span } \{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane in \mathbb{R}^3 that contains $\mathbf{v}_1, \mathbf{v}_2$ and 0.
- In particular, $\text{Span } \{\mathbf{v}_1, \mathbf{v}_2\}$ contains the line in \mathbb{R}^3 through \mathbf{v}_1 and 0 and the line through \mathbf{v}_2 and 0.



Geometric Interpretation of Vector Equation

- Finding a linear combination of given vectors $\mathbf{a}_1, \mathbf{a}_2$, and \mathbf{a}_3 to be equal to \mathbf{b} :

x : 60
y : 65
z : 55

$$\begin{bmatrix} 60 \\ 65 \\ 55 \end{bmatrix} x_1 + \begin{bmatrix} 5.5 \\ 5.0 \\ 6.0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 66 \\ 74 \\ 78 \end{bmatrix}$$
$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 = \mathbf{b}$$

```
if      =      :
    return 가
if      >      :
    return usually
    cause
if      <      :
    return usually
    cause 2 3
```

- The solution exists only when $\mathbf{b} \in \text{Span } \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

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x1, x2, x3      a1, a2, a3  span
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Matrix Multiplications as Linear Combinations of Vectors

- **Recall:** we defined matrix-matrix multiplications as the inner product between the row on the left and the column on the right:

- e.g., $\begin{bmatrix} 1 & 6 \\ 3 & 4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 11 & 1 \\ 9 & -3 \end{bmatrix}$

- Inspired by the vector equation, we can view Ax as a linear combination of columns of the left matrix:

- $\begin{bmatrix} 60 & 5.5 & 1 \\ 65 & 5.0 & 0 \\ 55 & 6.0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = Ax = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3$



Matrix Multiplications as Column Combinations

- Linear combinations of columns
 - Left matrix: bases, right matrix: coefficients

One column on the right

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} 1 + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} 2 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} 3$$

Multi-columns on the right

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = [\mathbf{x} \ \mathbf{y}]$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} 1 + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} 2 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} 3$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (-1) + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} 0 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} 1$$



Matrix Multiplications as Row Combinations

- Linear combinations of rows of the right matrix
 - Right matrix: bases, left matrix: coefficients

One row on the left

$$[1 \ 2 \ 3] \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} = 1 \times [1 \ 1 \ 0] + 2 \times [1 \ 0 \ 1] + 3 \times [1 \ -1 \ 1]$$

Multiple rows on the left

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} \mathbf{x}^T \\ \mathbf{y}^T \end{bmatrix}$$

$$\mathbf{x}^T = [x_1 \ x_2 \ x_3] = 1[1 \ 1 \ 0] + 2[1 \ 0 \ 1] + 3[1 \ -1 \ 1]$$

$$\mathbf{y}^T = [y_1 \ y_2 \ y_3] = 1[1 \ 1 \ 0] + 0[1 \ 0 \ 1] + (-1)[1 \ -1 \ 1]$$



Matrix Multiplications as Sum of (Rank-1) Outer Products

- (Rank-1) outer product

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [1 \ 2 \ 3] = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

- Sum of (Rank-1) outer products

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [1 \ 2 \ 3] + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} [4 \ 5 \ 6]$$
$$= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 5 & 6 \\ -4 & -5 & -6 \\ 4 & 5 & 6 \end{bmatrix}$$



Matrix Multiplications as Sum of (Rank-1) Outer Products

- Sum of (Rank-1) outer products is widely used in machine learning
 - Covariance matrix in multivariate Gaussian
 - Gram matrix in style transfer