A Combinatorial Foundation for Analytic Geometry

George. M. Van Treeck

Abstract. A ruler measure of intervals allows a new class of combinatorial proofs providing new insights into both measure theory and analytic geometry. Applying the ruler to the definition of a countable distance range converges to the taxicab distance equation as the upper boundary of the range, the Euclidean distance equation as the lower boundary of the range, and the triangle inequality over the full range. A combinatorial definition of size (length/area/volume) converges to the product of interval sizes (Euclidean volume) used in the Lebesgue measure. Combinatorics limits a geometry having the properties of both symmetry and order to a cyclic set of at most 3 dimensions, which is the basis of the right-hand rule. Implications for non-Euclidean geometries and higher dimensional geometries are discussed. All the proofs are verified in Coq.

Contents

1. Introduction

Definitions of the metric space, n-volume of the Lebesgue measure, and the Euclidean distance equation used in real analysis, calculus, and analytic geometry are imported from elementary geometry. Because the definitions are imported rather than derived from set and number theory, the definitions and equations do not provide insight into the counting principles that generate geometry.

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For example, there has been no proof that a counting-based definition of smallest distance converges to the real-valued Euclidean distance equation. Without a formal understanding of the counting-based relationships that converge to the real-valued triangle inequality, taxicab distance, and Euclidean distance, there is no formal motivation of the counting-based relationships that generate the properties of a metric space independent of elementary geometry. And there is no insight into the counting-based relationships that can generate some elliptic and hyperbolic distances

The traditional indefinite integral (antiderivative) of calculus is used to prove that a **real-valued, continuous function** relating the **size** of the subintervals of domain intervals to the **size** of the subintervals of an image interval converges to a real-valued function. Whereas, what is needed for counting-based (combinatorial) proofs is a indefinite integral that proves that a **counting-based function** relating the **number** of same-sized subintervals of domain intervals to the **number** of same-sized subintervals in an image interval converges to a real-valued function.

Without the combinatorial indefinite integral, there also has been no proof that the Cartesian product of the subintervals in a set of intervals converges to the product of the interval sizes (Euclidean volume). Therefore, Euclidean integrals and the Lebesgue measure assume the volume equation rather than deriving it.

Combinatorial proofs require a different method of dividing intervals into subintervals herein referred to as a ruler. A ruler measures each interval of a set of intervals to the nearest integer number of subintervals, each subinterval having the same size. The ruler is an approximate measure that ignores partial subintervals.

In the traditional method of dividing a set of intervals into subintervals, the number of subintervals is the same in both the domain and image intervals and the size of the subintervals in each interval varies. In contrast, for the ruler method, the number of subintervals in the domain and image intervals varies and the size of the subintervals in each interval is the same size.

Same-sized subintervals across both the set of domain intervals and image interval allows defining a countable relationship between the domain subintervals and image subintervals. For example, as the subinterval size goes to zero, the combinatorial relationships that define smallest countable distance and countable size (length/area/volume) converge to the n-dimensional Euclidean distance and volume equations.

The purpose of this article is to show that: 1) The principle that for every disjoint countable domain set there exists a distance set with the same number of elements results in a defined range of countable (combinatorial) relationships that converges to the real-valued triangle inequality, the upper boundary of the range converges to the taxicab distance equation, and the lower boundary of the range converges to the Euclidean distance equation. 2) The ruler measure can be used with combinatorics to derive the product of the interval sizes, the Euclidean n-volume, used in the Lebesgue measure. 3) Combinatorics limits a geometry that has the properties of being both ordered and symmetric to a cyclic set of at most three dimensions, which is the basis of the right-hand rule.

The proofs in this article are verified formally using the Coq Proof Assistant [15] version 8.4p16. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at:
https://github.com/treeck/CombinatorialGeometry.

2. Ruler measure and convergence

DEFINITION 2.1. Ruler measure: A ruler measures the size of a closed, open, or semi-open interval as the nearest integer number of whole subintervals, p, times the subinterval size, c, where c is the independent variable. Notionally:

$$\forall\, c\, s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |b-a| \ \ \wedge\ \ c > 0\ \wedge \tag{2.1}$$
$$(p = floor(s/c)\ \ \vee\ \ p = ceiling(s/c)\ \ \wedge\ \ M = \lim_{c\to 0} \sum_{i=1}^{p} c = \lim_{c\to 0} pc.$$

The ruler measure has the three properties of measure in a σ-algebra:

(1) Non-negativity: $\forall\, E \in \Sigma,\ \mu(E) \geq 0:\quad s = |b-a|\ \wedge\ c > 0\ \Rightarrow$ $M = \lim_{c\to 0} pc \geq 0$.
(2) Zero-sized empty set: $\mu(\emptyset) = 0:\quad b = a \Rightarrow p = 0 \Rightarrow M = \lim_{c\to 0} pc = 0$.
(3) Countable additivity: $\forall\, \{E_i\}_{i\in\mathbb{N}},\ |\cap_{i=1}^{\infty} E_i| = \emptyset \wedge \mu(\cup_{i=1}^{\infty} E_i) = \mu(\Sigma_{i=1}^{\infty} E_i)$. $(c \to 0 \Rightarrow p \to \infty)\ \wedge\ \mu(E_i) = c\ \Rightarrow\ \mu(\Sigma_{i=1}^{\infty} E_i) = \Sigma_{p=1}^{\infty} c = \lim_{c\to 0} pc$.

For example, showing convergence using the interval, $[0,\pi]$, $s = |\pi - 0|$, $c = 10^{-i}, i \in \mathbb{N}$, and $p = floor(s/c)$, then, $p \cdot c = 3.1,\ 3.14,\ 3.141, ..., \pi$.

THEOREM 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\ \ s = |b-a|\ \ \Rightarrow\ \ M = \lim_{c\to 0} pc = s$.

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\quad p = floor(s/c)\quad \Rightarrow\quad 0 \leq |p - s/c| < 1. \tag{2.2}$$

Multiply all sides by $|c|$:

$$\forall\, c > 0,\quad 0 \leq |p - s/c| < 1\quad \Rightarrow\quad 0 \leq |pc - s| < |c|. \tag{2.3}$$

$$\forall\, c > 0, \exists\, \delta,\ \epsilon\ :\ 0 \leq |pc - s| < |c| = |c - 0| < \delta = \epsilon \tag{2.4}$$
$$\Rightarrow\quad 0 < |c - 0| < \delta\quad \wedge\quad 0 \leq |pc - s| < \epsilon = \delta\quad :=\quad M = \lim_{c\to 0} pc = s.\quad \square$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure).

3. Distance

The primary characteristic of a countable distance measure is that the number of countable image (distance) set elements is equal to the number of domain set elements. Applying this characteristic to multiple domain sets, if there are p_i elements in the i^{th} disjoint (non-intersecting) domain set, then there exists p_i elements in the corresponding i^{th} distance set.

DEFINITION 3.1. Countable distance range, d_c:

$$\forall\, i\ n \in \mathbb{N},\ \ x_i \subseteq X,\ \ \bigcap_{i=1}^{n} x_i = \emptyset,\ \ \forall\, x_i\ \exists\, y_i \subseteq Y:\ \ |x_i| = |y_i|\quad \wedge\quad d_c = |Y|.$$

In the definition of countable distance range (3.1), the vertical bars around a set is the standard notation for indicating the cardinal (number of elements in the set). To prevent too much overloading on the vertical bar, the symbol for "such that" is the colon.

Note that the definition of a countable distance range (3.1) does **not** place a limitation on the distance sets being disjoint. Further, if a distance set element corresponds to more than one domain set element, then the size of the total (union) distance set, Y, will be smaller than the total (union) size of the disjoint domain sets, X. In other words, distance, $d_c = |Y| = |\cup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$.

The property, $|x_i| = |y_i|$, of a countable distance range (3.1) implies a limitation on the number of correspondences of a distance set element to domain set elements. If each of the p_i number of elements of the i^{th} distance set has one correspondence to a domain set element, then the smallest number of correspondences per distance set is: $1 \cdot p_i = p_i$. A one-to-one correspondence of the distance set elements to the disjoint domain set elements implies that the distance sets are also disjoint. And therefore, the distance, $d_c = |Y| = |\cup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i$, is the largest possible distance and the upper bound of distance range.

If each of the p_i number of elements of the i^{th} distance set corresponds to all p_i number of domain set elements, then the largest number of correspondences per distance set is: $p_i \cdot p_i = p_i^2$, which is the smallest possible distance and lower bound of the distance range that is related to the sum of all possible correspondences, $\sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} |\{(y, y) : y \in Y\}|^2$.

A **countable** (integer) equation for the smallest distance case is not possible for all sums, $\sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |\{(y, y) : y \in y_i\}|^2$ to equal $|\{(y, y) : y \in Y\}|^2$. But, the ruler (2.1) and ruler convergence theorem (2.2) is used to show the shortest distance case converges to the real-valued Euclidean distance equation.

The proof of equality for both real-valued taxicab and Euclidean distance equations requires the proof strategy of showing that the right and left sides of a proposed counting-based equation both converge to the same real value and therefore are equal. That is, the propositional logic, $A = B \wedge C = B \Rightarrow A = C$, is used.

THEOREM 3.2. *Taxicab (largest) distance, d, is the size of the distance interval,* $[y_0, y_m]$, *corresponding to a set of disjoint domain intervals,* $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, *where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|.$$

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals. Next, apply the definition of the countable distance range (3.1) and the rule of product:

$$\begin{aligned} (3.1) \quad & \forall\, i\, n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\ & |\{x_i : x_i \in \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\}| = |\{y_i : y_i \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_i}\}\}| = p_i. \end{aligned}$$

$$(3.2) \qquad \forall\, i\, n \in \mathbb{N}, \quad i \in [1, n], \quad y \in y_i \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i = |\{y\}|.$$

Multiply both sides of 3.2 by c and apply the ruler convergence theorem (2.2):

$$(3.3)\quad s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n}(p_i \cdot c) = |\{y\}| \cdot c \qquad \Rightarrow \quad \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \lim_{c\to 0} |\{y\}| \cdot c.$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$(3.4)\qquad \forall\, c > 0, \quad p_d = floor(d/c) = |\{y :\ y \in \{y_{1_i}, y_{2_i}, \dots, y_{p_d}\}\}|.$$

Multiply both sides of 3.4 by c and apply the ruler convergence theorem (2.2):

$$(3.5)\ d = \lim_{c\to 0} p_d \cdot c \ \wedge\ p_d \cdot c = |\{y\}| \cdot c \ \Rightarrow\ d = \lim_{c\to 0} p_d \cdot c = \lim_{c\to 0} |\{y\}| \cdot c.$$

Combine equations 3.5 and 3.3:

$$(3.6)\qquad d = \lim_{c\to 0} |\{y\}| \cdot c \quad \wedge \quad \sum_{i=1}^{n} s_i = \lim_{c\to 0} |\{y\}| \cdot c \quad \Rightarrow \quad d = \sum_{i=1}^{n} s_i. \qquad \square$$

THEOREM 3.3. *Euclidean (smallest) distance, d, is the size of the distance interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals, $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \dots, [x_{0,n}, x_{m_n,n}]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|.$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals. Next, apply the definition of the countable distance range (3.1) and the rule of product:

$$(3.7)\quad \forall\, i\ n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge$$
$$|\{x_i : x_i \in \{x_{1_i}, x_{2_i}, \dots, x_{p_i}\}\}| = |\{y_i : y_i \in \{y_{1_i}, y_{2_i}, \dots, y_{p_i}\}\}| = p_i \quad \Rightarrow$$
$$\forall\, i \in [1, n], \quad |\{(x_i, y_i)\}| = |\{(y_i, y_i)\}| = p_i^2.$$

$$(3.8)\quad \forall\, i\ n \in \mathbb{N}, \quad i \in [1, n], \quad |\{(y_i, y_i)\}| = p_i^2 \quad \wedge \quad y = y_i \quad \Rightarrow$$
$$|\sum_{i=1}^{n}\{(y_i, y_i)\}| = \sum_{i=1}^{n} p_i^2 = |\{(y, y)\}|.$$

Multiply both sides of 3.8 by c^2 and apply the ruler convergence theorem (2.2):

$$(3.9)\quad s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n}(p_i \cdot c)^2 = |\{(y, y)\}| \cdot c^2$$
$$\Rightarrow \quad \sum_{i=1}^{n} s_i^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \lim_{c\to 0} |\{(y, y)\}| \cdot c^2.$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$(3.10)\quad \forall\, c > 0, \quad p_d = floor(d/c) = |\{y :\ y \in \{y_{1_i}, y_{2_i}, \dots, y_{p_d}\}\}|$$
$$\Rightarrow \quad p_d^2 = |\{(y, y)\}|.$$

Multiply both sides of 3.10 by c^2 and apply the ruler convergence theorem (2.2):

$$d = \lim_{c \to 0} p_d \cdot c \quad \wedge \quad (p_d \cdot c)^2 = |\{(y,y)\}| \cdot c^2 \quad \Rightarrow \quad d^2 = \lim_{c \to 0} (p_d \cdot c)^2 = \lim_{c \to 0} |\{(y,y)\}| \cdot c^2. \tag{3.11}$$

Combine equations 3.11 and 3.9:

$$d^2 = \lim_{c \to 0} |\{(y,y)\}| \cdot c^2 \quad \wedge \quad \sum_{i=1}^{n} s_i^2 = \lim_{c \to 0} |\{(y,y)\}| \cdot c^2 \quad \Rightarrow \quad d^2 = \sum_{i=1}^{n} s_i^2. \quad \square \tag{3.12}$$

3.1. Triangle inequality. The definition of a metric in real analysis is based on the triangle inequality, $\mathbf{d}(\mathbf{u}, \mathbf{w}) \leq \mathbf{d}(\mathbf{u}, \mathbf{v}) + \mathbf{d}(\mathbf{v}, \mathbf{w})$, that has been intuitively motivated by the triangle [**Gol76**]. Applying the inclusion-exclusion principle, ruler (2.1), and ruler convergence theorem (2.2) to the definition of a countable distance range (3.1) yields the real-valued triangle inequality:

$$d_c = |Y| = |\bigcup_{i=1}^{2} y_i| \leq \sum_{i=1}^{2} |y_i| \quad \wedge \tag{3.13}$$

$$d_c = floor(\mathbf{d}(\mathbf{u}, \mathbf{w})/c) \ \wedge \ |y_1| = floor(\mathbf{d}(\mathbf{u}, \mathbf{v})/c) \ \wedge \ |y_2| = floor(\mathbf{d}(\mathbf{v}, \mathbf{w})/c)$$

$$\Rightarrow \ \mathbf{d}(\mathbf{u}, \mathbf{w}) = \lim_{c \to 0} d_c \cdot c \leq \sum_{i=1}^{2} \lim_{c \to 0} |y_i| \cdot c = \mathbf{d}(\mathbf{u}, \mathbf{v}) + \mathbf{d}(\mathbf{v}, \mathbf{w}).$$

4. Size (length/area/volume)

The countable size measure is the number of combinations (correspondences) between members of disjoint domain sets, which is the Cartesian product of the domain set sizes.

DEFINITION 4.1. countable size (length/area/volume) measure, S_c:

$$\forall\, i\ n \in \mathbb{N}, \quad i \in [1, n], \quad x_i \subseteq X, \quad |\bigcap_{i=1}^{n} x_i| = \emptyset \quad \wedge \quad \{(x_1, \ldots, x_n)\} = y \quad \wedge$$
$$S_c = |y| = |\{(x_1, \ldots, x_n)\}| = \prod_{i=1}^{n} |x_i|.$$

THEOREM 4.2. *Euclidean size (length/area/volume), S, is the size of an image interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals:* $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, *where:*

$$S = \prod_{i=1}^{n} s_i, \quad S = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|, \quad i \in [1, n], \quad i, n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_size."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$\forall\, i\ n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \quad x_i = \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\} \quad \Rightarrow |x_i| = p_i. \tag{4.1}$$

Use the ruler (2.1) to divide the exact size, $S = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_S^n subintervals, where p_S^n satisfies the definition a countable size measure, S_c (4.1).

$$\forall\, c > 0 \quad \wedge \quad \exists\, r \in \mathbb{R},\ S = r^n \quad \wedge \quad p_S = floor(r/c) \quad \wedge \quad p_S^n = S_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i. \tag{4.2}$$

Multiply both sides of equation 4.2 by c^n to get the ruler measures:

$$p_S^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3:

$$S = r^n = \lim_{c \to 0} (p_S \cdot c)^n \quad \wedge \quad \lim_{c \to 0} (p_i \cdot c) = s_i \quad \wedge \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c) \quad \Rightarrow \quad S = \lim_{c \to 0} (p_S \cdot c)^n = \prod_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) = \prod_{i=1}^{n} s_i. \tag{4.4}$$ □

5. Derived geometric definitions

5.1. Derived geometric primitives. There are no new mathematics in this section of derived geometric primitives. The purpose of this section is to show a difference in perspective. In classical geometry, Euclidean distance is a product of lines and angles. Here, the perspective is that lines and angles are objects generated from Euclidean distance, which was generated from the counting principle of shortest distance.

Definition 5.1. Straight line segment is the smallest (Euclidean) distance interval, $[y_0, y_m]$ (3.3).

Definition 5.2. Straight line segment orientation (slope): $\mathrm{d}b/\mathrm{d}a = b/a$, where $a = x_{m_1,1} - x_{0,1}$ and $b = x_{m_n,2} - x_{0,2}$ are the signed sizes of two domain intervals, $[x_{0,1}, x_{m_1,1}]$ and $[x_{0,2}, x_{m_n,2}]$.

The signed sizes, a and b, of the two domain intervals can be calculated from a single parametric distance, θ, and Euclidean distance, d.

Definition 5.3. Parametric distance (arc angle), θ:

$$b/a = \mathrm{d}b/\mathrm{d}a = \mathrm{d}b/\mathrm{d}\theta \cdot \mathrm{d}\theta/\mathrm{d}a = \sqrt{d^2 - a^2}/\sqrt{d^2 - b^2} \tag{5.1}$$

$$Case: \quad \mathrm{d}b/\mathrm{d}a = b/a = 1 \quad \Rightarrow \quad \mathrm{d}\theta/\mathrm{d}a = 1/\sqrt{d^2 - b^2} = 1/\sqrt{d^2 - a^2} \tag{5.2}$$

Applying Taylor's theorem [**Gol76**] and a table of integrals [**Wc11**]:

$$\int \mathrm{d}\theta = \int \mathrm{d}a/\sqrt{d^2 - a^2} \quad \Rightarrow \quad \theta = \sin^{-1}(a/d) = \cos^{-1}(b/d). \tag{5.3}$$

5.2. Vectors. Before discussing the implications of the proofs in this article on vector analysis for dimensions greater than three, the notions of vector, parallel, and orthogonal are defined here in terms of sets of intervals.

Definition 5.4. Vector: A vector is the ordered set of the signed domain interval sizes, $\mathbf{s} = \{s_1, \ldots, s_n\}$, where $s_i = x_{m_i,i} - x_{0,i}$ for the domain interval, $[x_{0,i}, x_{m_i,i}]$.

Definition 5.5. Parallel (congruent) vectors: Two vectors are parallel if each ratio of the signed sizes in one vector equals the ratio of the corresponding signed sizes in another vector (same rate of change in the same direction):

$$\frac{s1_i}{s1_{i+1}} = \frac{s2_i}{s2_{i+1}}, \quad i \in [1, n-1]. \tag{5.4}$$

Definition 5.6. Orthogonal vectors: Two vectors are orthogonal if each ratio of the signed sizes in one vector is the inverse ratio and inverse sign of two corresponding signed sizes in another vector (inverse rate of change and inverse directions). Simplifying the equation yields the **dot (inner) product** equal to zero for any number of dimensions:

$$\frac{s1_i}{s1_{i+1}} = -\frac{s2_{i+1}}{s2_i}, \quad i \in [1, n-1] \quad \Leftrightarrow \quad \sum_{i=1}^{n} s1_i \cdot s2_i = 0. \tag{5.5}$$

6. Ordered and symmetric geometries

Euclidean size (area/volume) and distance are invariant for every order (permutation) of a set of intervals. A function (like size or distance) where every permutation of the arguments yields the same value(s) is called a symmetric function. A permutation is a rearrangement of a sequence of set elements. Every element of a set can be placed sequentially adjacent to every other element, which implies that some element x_i having an immediate successor element x_j requires that x_j has the immediate predecessor element x_i.

Definition 6.1. Symmetric geometry:

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; \textit{successor}\; x_i = x_j \;\; \wedge \;\; \textit{predecessor}\; x_j = x_i.$$

Two sets of intervals with the same volume and spanning distance (for example, $\{[0,2],\ [0,1],\ [0,5]\}$ and $\{[0,5],\ [0,2],\ [0,1]\}$) can only be distinguished by assigning consistent sequential order (orientation) to the elements of the interval (dimension) sets.

Definition 6.2. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\;\; \forall\, x_i \in \{x_1, \ldots, x_n\},\;\; \textit{successor}\; x_i = x_{i+1} \;\; \wedge \;\; \textit{predecessor}\; x_{i+1} = x_i.$$

It will now be proved that any geometry, both Euclidean and non-Euclidean, that has both symmetry (every permutation of domain intervals yields the same distance and volume) and order (ability to discriminate distances and volumes by a consistent sequential ordering), is a cyclic set limited to at most 3 domain intervals (dimensions), which is the basis for the right-hand rule. The implications with respect to vector operations and higher dimensioned geometries are discussed in the summary.

If every element is sequentially adjacent to every other element, then traversing in both successor and predecessor order generates every possible permutation of elements (symmetry). This allows defining symmetry via successor and predecessor to be compatible with the notion of order.

Theorem 6.3. *An ordered and symmetric geometry is a cyclic set.*

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; i = n \;\; \wedge \;\; j = 1$$
$$\Rightarrow \quad \textit{successor}\; x_n = x_1 \;\; \wedge \;\; \textit{predecessor}\; x_1 = x_n.$$

The theorem and formal Coq-based proof is "ordered_symmetric_is_cyclic," which is located in the file threed.v.

PROOF. The property of order (6.2) defines unique successors and predecessors for all elements except for the successor of x_n and the predecessor of x_1. From the properties of a symmetric geometry (6.1):

$$i = n \ \wedge \ j = 1 \ \wedge \ \textit{successor}\ x_i = x_j \ \Rightarrow \ \textit{successor}\ x_n = x_1. \tag{6.1}$$

$$i = n \ \wedge \ j = 1 \ \wedge \ \textit{predecessor}\ x_j = x_i \ \Rightarrow \ \textit{predecessor}\ x_1 = x_n. \tag{6.2}$$

□

For example, using the cyclic set with elements labeled, $\{1,\ 2,\ 3\}$, starting with each element and counting through 3 cyclic successors and counting through 3 cyclic predecessors yields all possible permutations: $(1,2,3)$, $(2,3,1)$, $(3,1,2)$, $(1,3,2)$, $(3,2,1)$, and $(2,1,3)$. That is, a cyclically ordered set preserves sequential order while allowing a set of n-at-a-time permutations. If all possible n-at-a-time permutations are generated, then the cyclic set is also symmetric.

THEOREM 6.4. *An ordered and symmetric geometry is limited to at most 3 elements. That is, each element is sequentially adjacent (a successor or predecessor) to every other element in a set only where the number of elements (set sizes) are less than or equal to 3.*

The Coq-based lemmas and proofs in the file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn-like clauses (a subset of first-order logic) with unification and resolution. Horn clauses make it clear which facts satisfy a goal.

PROOF.
Because an ordered and symmetric set is a cyclic set (6.3), the successors and predecessors are cyclic:

DEFINITION 6.5. Successor of m is n:

$$\textit{Successor}(m, n, \textit{setsize}) \leftarrow (m = \textit{setsize} \wedge n = 1) \vee (m + 1 \leq \textit{setsize}). \tag{6.3}$$

DEFINITION 6.6. Predecessor of m is n:

$$\textit{Predecessor}(m, n, \textit{setsize}) \leftarrow (m = 1 \wedge n = \textit{setsize}) \vee (m - 1 \geq 1). \tag{6.4}$$

DEFINITION 6.7. Adjacent: element m is adjacent to element n (an allowed permutation), if the cyclic successor of m is n or the cyclic predccessor of m is n. Notionally:

$$\textit{Adjacent}(m, n, \textit{setsize}) \leftarrow \textit{Successor}(m, n, \textit{setsize}) \vee \textit{Predecessor}(m, n, \textit{setsize}). \tag{6.5}$$

Every element is adjacent to every other element, where $\textit{setsize} \in \{1,\ 2,\ 3\}$:

$$\textit{Adjacent}(1, 1, 1) \leftarrow \textit{Successor}(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{6.6}$$

$$\textit{Adjacent}(1, 2, 2) \leftarrow \textit{Successor}(1, 2, 2) \leftarrow (1 + 1 \leq 2). \tag{6.7}$$

$$\textit{Adjacent}(2, 1, 2) \leftarrow \textit{Successor}(2, 1, 2) \leftarrow (2 = 2 \wedge 1 = 1). \tag{6.8}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (1+1 \leq 2). \tag{6.9}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (2-1 \geq 1). \tag{6.10}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (3=3 \wedge 1=1). \tag{6.11}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (1=1 \wedge 3=3). \tag{6.12}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (2+1 \leq 3). \tag{6.13}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (3-1 \geq 1). \tag{6.14}$$

For all $n = setsize > 3$, there exist non-adjacent elements (not every permutation allowed):

$$\forall\, n > 3,\ Successor(1,2,n) \Rightarrow \forall\, n > 3,\ \neg Successor(1,3,n). \tag{6.15}$$

That is, 2 is the only successor of 1 for all $n > 3$, which implies 3 is not a successor of 1 for all $n > 3$.

$$\forall\, n > 3,\ Predecessor(1,n,n) \Rightarrow \forall\, n > 3,\ \neg Predecessor(1,3,n). \tag{6.16}$$

That is, n is the only predecessor of 1 for all $n > 3$, which implies 3 is not a predecessor of n for all $n > 3$.

$$\forall\, n > 3,\ \neg Adjacent(1,3,n) \leftarrow \neg Successor(1,3,n) \wedge \neg Predecessor(1,3,n). \tag{6.17}$$

□

7. Summary

A ruler-based measure of intervals is an analytic tool allowing combinatorial proofs that provides new insights into measure theory and analytic geometry:

1. Combinatorial relations between the elements of sets converge to the Euclidean distance (3.3) and size (length/area/volume) (4.2) equations without notions of side, angle, and shape, and without motivation from diagrams. In particular, geometry is generated from the analytic properties of real-value continuum.
2. Taxicab distance (3.2), Euclidean distance (3.3), and the triangle inequality are derived from the definition of the countable distance range (3.1), where taxicab distance is the largest possible monotonic distance and Euclidean distance is the smallest distance. This provides a more formal basis for the definition of the metric space.
3. Because Euclidean distance relies on the number of distance set elements being equal to the number of elements in a domain set, elliptic distance would require fewer distance set elements and hyperbolic distance would require more distance set elements than in a corresponding domain set.
4. The Euclidean volume (product of interval sizes) of the Lebesgue measure is derived from use of the more fundamental ruler measure.
5. Combinatorics limits a geometry having the properties of both order (6.2) and symmetry (6.1) is a cyclic set (6.3) of at most three elements (dimensions) (6.4), which is the basis of the right-hand rule.
6. Vector orthogonality (inner product equal to zero) was derived in this article for any number of dimensions (5.6). But, vector orthogonality is not a symmetric function. For example, re-arranging the order of the components of one orthogonal vector is likely to result in the dot product no longer being equal to zero. However, arc angle was derived in this article

from the symmetric function, Euclidean distance (5.3). Therefore, perpendicular (a right arc angle) is limited to at most three ordered dimensions (6.4).

A means to preserve both geometric order and symmetry in a higher dimensioned geometry (for example, Clifford geometry) is to have a vector of three "root" dimensions of space, where size and distance in the three root dimensions are a function of other variables in separate vectors, forming a hierarchy of dimensions.

A cyclic set is a closed walk. An observer in the closed walk would only be able to detect higher dimensions indirectly via changes in the three closed walk dimensions (what physicists call "work").

Displaying higher dimensional manifolds in Euclidean coordinate diagrams (for example, three dimensional Cartesian coordinates and spherical coordinates) is probably only meaningful for the case where three of the modeled dimensions are both geometrically symmetric (6.1) and ordered (6.2).

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George Van Treeck, 668 Westline Dr., Alameda, CA 94501