# The Two Set Relations Generating Geometry

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ABSTRACT. Where each domain set has a corresponding range set, distance is the cardinal of the union of the range sets and volume is the cardinal of the Cartesian product of disjoint range sets. A ruler (measuring stick) partitions intervals approximately into sets of size c subintervals and sums the sizes. The distance and volume set operations on the sets of subintervals converge to the properties of metric space, Manhattan distance, Euclidean distance, and Euclidean volume as c goes to 0. The volume proof allows simpler derivations of Coulomb's charge force and Newton's gravity force equations without using other laws of physics or Gauss's divergence theorem. A type of symmetry limits physical space to at most 3 dimensions. All proofs are verified in Coq.

## CONTENTS

## 1. Introduction

Metric space, Euclidean distance as a metric/vector norm, and Euclidean area/ volume in measure and integration have been definitions [**Gol76**] [**Rud76**]. In this article, the properties of metric space, Euclidean distance and volume are theorems that are proved to be derived from operations on countable sets.

Most measure and integration divide both a domain interval, $[a, b]$, and range interval, $[f(a), f(b)]$, into the same number of subintervals, where the size of the subintervals can vary. In this article, a ruler (measuring stick) divides both domain and range intervals approximately into sets of same size, $c$, subintervals, where the number of subintervals in each interval can vary, which allows deriving geometric

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relations from the combinatorial mappings between sets of same-sized subintervals that would be obtuse using traditional measure and integration.

Where each disjoint domain set has a corresponding range set, distance is defined as the cardinal of the union of the range sets and volume as the cardinal of the Cartesian product of the disjoint range sets. The distance and volume set operations on the sets of size $c$ subintervals converges to: the properties of metric space, Manhattan distance, Euclidean distance, and Euclidean volume as $c \to 0$.

The derivations of metric space, distance equations, and volume from set operations provide some insights into geometry and physics, for example: the domain-to-range set mapping that makes Euclidean distance the smallest possible distance between two distinct points in $\mathbb{R}^n$ (flat space); the range set relationship common to Manhattan distance and Euclidean volume; how the Euclidean volume proof: 1) allows deriving Coulomb's charge force and Newton's gravity force equations without using other laws of physics or Gauss's divergence theorem, and 2) exposes the principle that generates the inverse square law and spacetime equations; how the order and symmetry properties of physical sets of intervals limit physical distance and volume to 3 dimensions.

All the proofs in this article have been formally verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

## 2. Ruler measure and convergence

The ruler measures the size, $M$, of an interval *approximately* as the sum of the nearest integer number (floor or ceiling) of whole subintervals, each subinterval having the same size, $c$. The ruler is both an inner and outer measure.

DEFINITION 2.1. Ruler measure: $\forall \, c, \, s \in \mathbb{R}, \, [a, b] \subset \mathbb{R}, \, s = |a - b| \, \wedge \, c > 0 \, \wedge$ $(p = floor(s/c) \, \vee \, p = ceiling(s/c)) \, \wedge \, M = \sum_{i=1}^{p} c = pc.$

THEOREM 2.2. *Ruler convergence:* $\forall \, [a, b] \subset \mathbb{R}, s = |a-b| : M = \lim_{c \to 0} pc = s.$

The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y \, : \, y \leq x, \, y \in \mathbb{Z}, \, x \in \mathbb{R}\})$:

(2.1) $\forall \, c > 0, \, p = floor(s/c) \, \wedge \, 0 \leq |floor(s/c) - s/c| < 1 \, \Rightarrow \, 0 \leq |p - s/c| < 1.$

Multiply all sides of inequality 2.1 by $|c|$:

(2.2) $\qquad\qquad \forall \, c > 0, \quad 0 \leq |p - s/c| < 1 \quad \Rightarrow \quad 0 \leq |pc - s| < |c|.$

(2.3) $\quad \forall \, \delta \, : \, |pc - s| < |c| = |c - 0| < \delta$
$\qquad\qquad \Rightarrow \quad \forall \, \epsilon = \delta : \, |c - 0| < \delta \, \wedge \, |pc - s| < \epsilon \, := \, M = \lim_{c \to 0} pc = s. \quad \square$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = |0 - \pi|$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}}, \, 3.14_{c=10^{-2}}, \, 3.141_{c=10^{-3}}, ..., \pi_{\lim_{c \to 0}}.$

## 3. Distance

**Notation convention:** Vertical bars around a set or list, $|\cdots|$, indicates the cardinal (number of members in the set or list).

**3.1. Countable distance.** A concrete example of a countable distance is the number of same-sized steps walked in a range (distance) set, $y$, which equals the number of same-sized pieces of land in a corresponding domain set, $x$: $|x| = |y|$. Generalizing, each disjoint domain set, $x_i$, has a corresponding range (distance) set, $y_i$. The countable distance spanning the disjoint domain sets is the number of members, $d_c$, in the union range set:

DEFINITION 3.1. Countable distance, $d_c$ :

$$d_c = |\bigcup_{i=1}^n y_i|, \quad \bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad |x_i| = |y_i|.$$

THEOREM 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^n y_i| \le \sum_{i=1}^n |y_i|$.

The inclusion-exclusion inequality follows from the inclusion-exclusion principle [**CG15**]. But, a more intuitive and simple proof follows from the associative law of addition where the sum of set sizes is equal to the size of all the set members appended into a list and the commutative law of addition that allows sorting that list into a list of unique members (the *union* set) and a list of duplicates. The duplicates being $\ge 0$ implies the union size is always $\le$ the sum of set sizes.

A formal proof, inclusion_exclusion_inequality, using sorting into a set of unique members (*union* set) and a list of duplicates, is in the file euclidrelations.v.

PROOF.

$$(3.1) \quad \sum_{i=1}^n |y_i| = |append_{i=1}^n y_i| = |sort(append_{i=1}^n y_i)|$$
$$= |\bigcup_{i=1}^n y_i| + |duplicates_{i=1}^n y_i|.$$

$$(3.2) \quad |\bigcup_{i=1}^n y_i| + |duplicates_{i=1}^n y_i| = \sum_{i=1}^n |y_i| \quad \wedge \quad |duplicates_{i=1}^n y_i| \ge 0$$
$$\Rightarrow \quad |\bigcup_{i=1}^n y_i| \le \sum_{i=1}^n |y_i|. \quad \square$$

**3.2. Countable distance range.** From the countable distance definition, $d_c = |\bigcup_{i=1}^n y_i|$, as the amount of intersection increases, more domain set members can map to a single range set member. Therefore, the countable distance, $d_c$, is a function of the number of domain-to-range set member mappings.

From the countable distance property (3.1), where $|x_i| = |y_i| = p_i = 1$, each domain set member maps to one unique range set member (a 1-1 correspondence), $1 \cdot |y_i| = p_i = 1$ mapping, and each domain set member maps to every range set member (a many-to-many/Cartesian product), $|x_i| \cdot |y_i| = p_i^2 = 1$ mapping. Therefore, the total number of domain-to-range set mappings varies from $\sum_{i=1}^n (1 \cdot |y_i|) = \sum_{i=1}^n p_i$ to $\sum_{i=1}^n (|y_i| \cdot |x_i|) = \sum_{i=1}^n p_i^2$ mappings.

The range of domain-to-range set mappings, $\sum_{i=1}^n p_i$ to $\sum_{i=1}^n p_i^2$, that are true for non-empty sets of one size are true for non-empty sets of all sizes. Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of domain-to-range set mapping cases converges to the real-valued Manhattan and Euclidean distance relations.

**3.3. Manhattan distance.**

THEOREM 3.3. *Manhattan (largest) distance, $d$, is the size of the range interval,* $[d_0, d_m]$, *corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$, *where:*

$$d = \sum_{i=1}^n s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "taxicab_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.

From the countable distance definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, $d_c$, is the equality case:

(3.3) $\quad d_c = |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i$
$$\Rightarrow \quad d_c \leq \sum_{i=1}^n |y_i| = \sum_{i=1}^n p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c : \ d_c = \sum_{i=1}^n p_i.$$

Multiply both sides of equation 3.5 by $c$ and take the limit:

(3.4) $\quad d_c = \sum_{i=1}^n p_i \Rightarrow d_c \cdot c = \sum_{i=1}^n (p_i \cdot c) \Rightarrow \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^n \lim_{c \to 0} (p_i \cdot c).$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of $d$:

(3.5) $\qquad d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d : \ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c.$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

(3.6) $\quad s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c \to 0} p_i \cdot c = s_i.$

Combine equations 3.5, 3.4, 3.6:

(3.7) $\quad d = \lim_{c \to 0} d_c \cdot c \quad \wedge \quad \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^n \lim_{c \to 0} (p_i \cdot c) \quad \wedge$
$$\lim_{c \to 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^n \lim_{c \to 0} (p_i \cdot c) = \sum_{i=1}^n s_i. \quad \square$$

### 3.4. Euclidean distance.

THEOREM 3.4. *Euclidean (smallest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,*
*$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*
$$d^2 = \sum_{i=1}^n s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "Euclidean_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.

Apply the rule of product to the largest number of domain-to-range set mappings, where all $p_i$ number of range set members, $y_i$, map to each of the $p_i$ number of members in the domain set, $x_i$, which is the Cartesian product, $|y_i| \cdot |x_i|$:

(3.8) $\qquad |x_i| = |y_i| = p_i \quad \Rightarrow \quad \sum_{i=1}^n |y_i| \cdot |x_i| = \sum_{i=1}^n p_i^2.$

From the countable distance definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

(3.9) $\quad d_c = |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i$
$$\Rightarrow \quad d_c \leq \sum_{i=1}^n |y_i| = \sum_{i=1}^n p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c : \ d_c = \sum_{i=1}^n p_i.$$

Square both sides of equation 3.9 ($x = y \Leftrightarrow f(x) = f(y)$):

(3.10) $\qquad \exists\, p_i,\, d_c : \ d_c = \sum_{i=1}^n p_i \quad \Leftrightarrow \quad \exists\, p_i,\, d_c : \ d_c^2 = (\sum_{i=1}^n p_i)^2.$

Apply the square of sum inequality, $(\sum_{i=1}^n p_i)^2 \geq \sum_{i=1}^n p_i^2$, to equation 3.10 and select the smallest area (the equality) case:

(3.11) $\quad d_c^2 = (\sum_{i=1}^n p_i)^2 = \sum_{i=1}^n p_i \sum_{j=1}^n p_j$
$$= \sum_{i=1}^n p_i^2 + \sum_{i=1}^n p_i \sum_{j=1, j \neq i}^n p_j \geq \sum_{i=1}^n p_i^2 \quad \Rightarrow \quad \exists\, p_i : \ d_c^2 = \sum_{i=1}^n p_i^2.$$

Multiply both sides of equation 3.11 by $c^2$, simplify, and take the limit.

(3.12) $\quad d_c^2 = \sum_{i=1}^n p_i^2 \;\Rightarrow\; d_c^2 \cdot c^2 = \sum_{i=1}^n p_i^2 \cdot c^2 \;\Leftrightarrow\; (d_c \cdot c)^2 = \sum_{i=1}^n (p_i \cdot c)^2$
$$\Rightarrow \quad \lim_{c\to0}(d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c\to0}(p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

(3.13) $\quad \exists\, c\, d \in \mathbb{R} : \; floor(d/c) = d_c \;\;\Rightarrow\;\; d = \lim_{c\to0} d_c \cdot c \;\;\Rightarrow\;\; d^2 = \lim_{c\to0}(d_c \cdot c)^2.$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

(3.14) $\quad s_i = |a_i - b_i| \;\;\wedge\;\; floor(s_i/c) = |x_i| = |y_i| = p_i \;\;\Rightarrow\;\; \lim_{c\to0} p_i \cdot c = s_i.$

Combine equations 3.13, 3.12, 3.14:

(3.15) $\quad d^2 = \lim_{c\to0}(d_c \cdot c)^2 \;\;\wedge\;\; \lim_{c\to0}(d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c\to0}(p_i \cdot c)^2 \;\;\wedge$

$\lim_{c\to0}(p_i \cdot c) = s_i \;\;\Rightarrow\;\; d^2 = \lim_{c\to0}(d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c\to0}(p_i \cdot c)^2 = \sum_{i=1}^n s_i^2. \quad \square$

**3.5. Metric Space.** All function range intervals, $d(u,w)$, satisfying the countable distance definition (3.1), where the ruler is applied, generates the properties of metric space: triangle inequality, non-negativity, identity of indiscernibles, and symmetry. The formal proofs: triangle_inequality, non_negativity, identity_of_indiscernibles and symmetry are in the Coq file, euclidrelations.v.

THEOREM 3.5. *Triangle Inequality:* $d_c = |y_1 \cup y_2| \Rightarrow d(u,w) \leq d(u,v) + d(v,w).$

PROOF. Apply the ruler measure (2.1), the countable distance condition (3.1), inclusion-exclusion inequality (3.2), and then ruler convergence (2.2).

(3.16) $\quad \forall\, c > 0, \; d(u,w), \; d(u,v), \; d(v,w) :$
$$|y_1| = floor(d(u,v)/c) \;\wedge\; |y_2| = floor(d(v,w)/c) \;\wedge$$
$$d_c = floor(d(u,w)/c) \;\wedge\; d_c = |y_1 \cup y_2| \leq |y_1| + |y_2|$$
$$\Rightarrow \; floor(d(u,w)/c) \leq floor(d(u,v)/c) + floor(d(v,w)/c)$$
$$\Rightarrow \; floor(d(u,w)/c) \cdot c \leq floor(d(u,v)/c) \cdot c + floor(d(v,w)/c) \cdot c$$
$$\Rightarrow \; \lim_{c\to0} floor(d(u,w)/c) \cdot c \leq \lim_{c\to0} floor(d(u,v)/c) \cdot c + \lim_{c\to0} floor(d(v,w)/c) \cdot c$$
$$\Rightarrow \; d(u,w) \leq d(u,v) + d(v,w). \quad \square$$

THEOREM 3.6. *Non-negativity:* $d_c = |y_1 \cup y_2| \Rightarrow d(u,w) \geq 0.$

PROOF. By definition, a set always has a size (cardinal) $\geq 0$:

(3.17) $\quad \forall\, c > 0, \; d(u,w) : \quad floor(d(u,w)/c) = d_c \;\;\wedge\;\; d_c = |y_1 \cup y_2| \geq 0$
$$\Rightarrow \quad floor(d(u,w)/c) = d_c \geq 0 \;\;\Rightarrow\;\; d(u,w) = \lim_{c\to0} d_c \cdot c \geq 0. \quad \square$$

THEOREM 3.7. *Identity of Indiscernibles:* $d(w,w) = 0.$

PROOF. Apply the triangle inequality property (3.5):

(3.18) $\quad \forall\, d(u,v) = d(v,w) = 0 \;\wedge\; d(u,w) \leq d(u,v) + d(v,w) \;\Rightarrow\; d(u,w) \leq 0.$

Combine the non-negativity property (3.6) and the previous inequality (3.18):

(3.19) $\quad d(u,w) \geq 0 \;\wedge\; d(u,w) \leq 0 \;\Leftrightarrow\; 0 \leq d(u,w) \leq 0 \;\Rightarrow\; d(u,w) = 0.$

Combine the result of step 3.19 and the condition, $d(u,v) = 0$, in step 3.18.

(3.20) $\qquad\qquad d(u,w) = 0 \;\wedge\; d(u,v) = 0 \;\Rightarrow\; w = v.$

Combine the condition, $d(v, w) = 0$, in step 3.18 and the result of step 3.20.

(3.21)                        $d(v, w) = 0 \ \wedge \ w = v \ \Rightarrow \ d(w, w) = 0.$                        □

THEOREM 3.8. *Symmetry: From the Euclidean distance* (3.4) *and Manhattan distance* (3.3) *proofs:* $(x^2 + y^2)^{1/2} \leq d(x, y) \leq x + y \Rightarrow d(u, v) = d(v, u)$ *in flat space, where* $|x_i| = |y_i|$.

PROOF.   The countable distance definition (3.1) constraint, $|x_i| = |y_i|$, which results in all countable distances being a function of the sum of domain-to-range set mappings. Where the type of mapping is the same for each domain-range set pair, distance is a function of sum of range set cardinals raised to the *same* power, which by the commutative law of addition results in the metric space property of symmetry.

(3.22)   $(x^2 + y^2)^{1/2} \leq d(x, y) \leq x + y$

$$\Rightarrow \quad \forall \, p : 1 \leq p \leq 2, \quad d(x, y) = (x^p + y^p)^{1/p}.$$

By the commutative law of addition:

(3.23)   $d(u, v) = (u^p + v^p)^{1/p} \quad \wedge \quad d(v, u) = (v^p + u^p)^{1/p}$

$\Rightarrow \quad d(u, v)^p = u^p + v^p = v^p + u^p = d(v, u)^p \quad \Rightarrow \quad d(u, v) = d(v, u).$   □

## 4. Euclidean Volume

Riemann and Lebesgue integration and the Lebesgue and Borel measures define Euclidean area/volume as the product of domain interval lengths. The goals, here, are: 1) to prove that Euclidean area/volume is derived from a set operation without assuming the product of interval lengths, and 2) show the close set-based relationship of Manhattan distance to Euclidean volume.

Manhattan and Euclidean distance were derived by applying the countable distance union operation, $d_c = \bigcup_{i=1}^{n} y_i$, to sets of size $c$ subintervals of range intervals. Therefore, there is a distance subinterval size corresponding to an ordered n-tuple of size $c$ subintervals, one size $c$ subinterval from each range interval. The largest number of ordered n-tuples is the Cartesian product of disjoint sets of range subintervals, $\bigcap_{i=1}^{n} y_i = \emptyset$. And Manhattan distance is also derived from the case of disjoint range sets.

DEFINITION 4.1. Countable Volume, $v_c$, is the largest number of all possible ordered n-tuples:

$$v_c = |\times_{i=1}^{n} y_i|, \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset.$$

THEOREM 4.2. *Euclidean volume,* $v$, *is length of the range interval,* $[v_0, v_m]$, *equal to product of domain interval lengths,* $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$:

$$v = \prod_{i=1}^{n} s_i, \ \ v = |v_0 - v_m|, \ \ s_i = |a_i - b_i|.$$

The theorem, "Euclidean_volume," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, $x_i$, containing $p_i$ number of subintervals.

(4.1)        $\forall \, i \, n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i| = |y_i|.$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$(4.2) \qquad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c \to 0}(p_i \cdot c) = s_i.$$

Apply the associative law of multiplication to derive the countable volume (4.1) in terms of $p_i$:

$$(4.3) \qquad v_c = |\times_{i=1}^n y_i| = \prod_{i=1}^n |y_i| \quad \wedge \quad |y_i| = p_i \quad \Rightarrow \quad v_c = \prod_{i=1}^n p_i.$$

Multiply both sides of equation 4.3 by $c^n$:

$$(4.4) \qquad v_c \cdot c^n = (\prod_{i=1}^n p_i) \cdot c^n = \prod_{i=1}^n (p_i \cdot c).$$

Use those cases, where $v_c$ has an integer $n^{th}$ root.

$$(4.5) \quad \forall \, n, \, p, \, v_c \in \mathbb{N} \; : \; p^n = v_c \; \Rightarrow \; v_c \cdot c^n = p^n \cdot c^n = (p \cdot c)^n = \prod_{i=1}^n (p_i \cdot c).$$

Apply the ruler (2.1) and ruler convergence (2.2) to the range interval, $[v_0, v_m]$ (where $v = |v_0 - v_m|$), and then combine with equations 4.5 and 4.2:

$$(4.6) \;\; floor(v/c^n) = p^n \Rightarrow v = \lim_{c \to 0}(p \cdot c)^n = \prod_{i=1}^n \lim_{c \to 0}(p_i \cdot c) = \prod_{i=1}^n s_i. \qquad \square$$

## 5. Applications to physics

**5.1. Coulomb's charge force.** The sizes, $q_1$ and $q_2$, of two charges are independent domain variables, where each infinitesimal size $c$ component of a charge exerts a force on each infinitesimal size $c$ component of the other charge. The total force, $F$, is proportionate to the total number of forces (the Cartesian product of the infinitesimal size $c$ components) multiplied times a quantum charge force, $m_C a_C$. From the volume proof (4.2), the Cartesian product converges to $q_1 q_2$:

$$(5.1) \qquad F \propto m_C a_C ((\lim_{c \to 0} floor(q_1/c) \cdot c)(\lim_{c \to 0} floor(q_2/c) \cdot c)) = m_C a_C (q_1 q_2).$$

From equation 5.1, a change in charge, $q$, causes a proportionate change in force, $F$. Solving for the quantum force, $F = m_C a_C$, requires a proportionate variable, $r$, to offset the effect of a change in $q$: $r \propto q \Rightarrow \exists \, q_C, \, r_C \in \mathbb{R} \; : \; r(q_C/r_C) = q$:

$$(5.2) \;\; \forall \, q_1, q_2 \geq 0 \; \exists \, q \in \mathbb{R} \; : \; q^2 = q_1 q_2 \; \wedge \; r(q_C/r_C) = q \; \Rightarrow \; (r(q_C/r_C))^2 = q_1 q_2.$$

$$(5.3) \quad (r(q_C/r_C))^2 = q_1 q_2 \; \wedge \; F \propto m_C a_C (q_1 q_2)$$
$$\Rightarrow \quad F \propto m_C a_C (r(q_C/r_C))^2 = m_C a_C (q_1 q_2)$$
$$\Rightarrow \quad F = m_C a_C = (m_C a_C r_C^2/q_C^2) q_1 q_2 / r^2 = k_c q_1 q_2 / r^2.$$

where $k_C = m_C a_C r_C^2/q_C^2$ corresponds to the SI units: $Nm^2 C^{-2}$.

**5.2. Newton's gravity force equation.** The sizes, $m_1$ and $m_2$, of two masses are independent domain variables, where each infinitesimal size $c$ component of a mass exerts a force on each infinitesimal size $c$ component of the other mass. The total force, $F$, is proportionate to the total number of forces (the Cartesian product of the size $c$ components) multiplied times a quantum gravity force, $m_G a_G$. From the volume proof (4.2), the Cartesian product converges to $m_1 m_2$:

$$(5.4) \quad F \propto m_G a_G ((\lim_{c \to 0} floor(m_1/c) \cdot c)(\lim_{c \to 0} floor(m_2/c) \cdot c)) = m_G a_G (m_1 m_2).$$

From equation 5.4, an change in mass, $m$, causes a proportionate change in force, $F$. Solving for the quantum force, $F = m_G a_G$, requires a proportionate variable, $r$, to offset the effect of a change in $m$: $r \propto m \Rightarrow \exists \, m_G, \, r_G \in \mathbb{R} \; : \; r(m_G/r_G) = m$:

(5.5)   $\forall\, m_1, m_2 \geq 0 \,\exists\, m \in \mathbb{R} : \quad m^2 = m_1 m_2 \quad \wedge \quad r(m_G/r_G) = m$

$$\Rightarrow \quad (r(m_G/r_G))^2 = m_1 m_2.$$

(5.6)   $(r(m_G/r_G))^2 = m_1 m_2 \;\wedge\; F \propto m_G a_G(m_1 m_2)$

$$\Rightarrow \quad F \propto m_G a_G (r(m_G/r_G))^2 = m_G a_G(m_1 m_2)$$

$$\Rightarrow \quad F = m_G a_G = (a_G r_G^2/m_G)m_1 m_2/r^2.$$

(5.7)   $\exists\, t_G \in \mathbb{R} : r_G/t_G^2 = a_G \quad \wedge \quad F = m_G a_G = (a_G r_G^2/m_G)m_1 m_2/r^2$

$$\Rightarrow \quad F = m_G a_G = (r_G^3/m_G t_G^2)m_1 m_2/r^2 = Gm_1 m_2/r^2,$$

where $G = r_G^3/m_G t_G^2$ corresponds to the SI units: $m^3 kg^{-1}s^{-2}$.

**5.3. Spacetime equations.** The charge (5.3) and gravity (5.7) force equations were derived from the principle that charge and mass are proportionate to distance: $r = (r_C/q_C)q = (r_G/m_G)m$. Suppose that time is also proportionate to distance, $r = (r_c/t_c)t = ct$.

Applying the ruler to two intervals, $[0, d_1]$ and $[0, d_2]$, in two inertial (independent, non-accelerating) frames of reference, the distance (and time) spanning the two domain intervals converges to a range of distances (and times) from Manhattan (3.3) to Euclidean distance (3.4).

(5.8)   $r^2 = d_1^2 + d_2^2 \quad \wedge \quad r = (r_c/t_c)t = ct$

$$\Rightarrow \quad (ct)^2 = d_1^2 + d_2^2 \quad \Rightarrow \quad d_2 = \sqrt{(ct)^2 - d_1^2}.$$

(5.9)   $d_2 = \sqrt{(ct)^2 - d_1^2} \quad \wedge \quad d = d_2 \quad \wedge \quad d_1 = vt$

$$\Rightarrow \quad d = \sqrt{(ct)^2 - (vt)^2} = ct\sqrt{1 - (v^2/c^2)},$$

which is the spacetime dilation equation. [**Bru17**].

(5.10)   $d_2^2 = (ct)^2 - d_1^2 \quad \wedge \quad s = d_2 \quad \wedge \quad d_1^2 = x^2 + y^2 + z^2$

$$\Rightarrow \quad s^2 = (ct)^2 - (x^2 + y^2 + z^2),$$

which is one form of the spacetime interval equation [**Bru17**].

**5.4. 3 dimensions of physical geometry.** Real-valued intervals are used to model a physical distance intervals. And the well-known, real-valued, physical distance and volume equations are generated by countable distance and volume operations, which require sequencing through a sequentially ordered set of intervals, where each interval has at most one successor and at most on predecessor. Further, the physical Manhattan distance intervals between two distinct points can be traversed sequentially in every possible order, which requires each physical interval/dimension to be sequentially adjacent (either a successor or predecessor) to every to every other interval/dimension, herein referred to as a symmetric geometry.

Sequentially ordered and symmetric sets with one and two members are the trivial cases. It will now be proved that the largest sequentially ordered and symmetric set is a cyclic set that contains at most 3 members, in this case, 3 dimensions of physical distance and volume.

DEFINITION 5.1. Ordered geometry:

$$\forall \, i \, n \in \mathbb{N}, \; i \in [1, n-1], \; \forall \, x_i \, \in \{x_1, \dots, x_n\},$$
$$successor \; x_i = x_{i+1} \; \wedge \; predecessor \; x_{i+1} = x_i.$$

DEFINITION 5.2. Symmetric geometry (every set member is sequentially adjacent to every other member):

$$\forall \, i \, j \, n \in \mathbb{N}, \; \forall \, x_i \, x_j \in \{x_1, \dots, x_n\}, \; successor \; x_i = x_j \; \Leftrightarrow \; predecessor \; x_j = x_i.$$

THEOREM 5.3. *An ordered and symmetric set is a cyclic set.*

$$i = n \; \wedge \; j = 1 \; \Rightarrow \; successor \; x_n = x_1 \; \wedge \; predecessor \; x_1 = x_n.$$

The theorem, "ordered_symmetric_is_cyclic," and formal proof is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of $x_n$ and the predecessor of $x_1$. Therefore, the only member that can be a successor of $x_n$, without creating a contradiction, is $x_1$. And the only member that can be a predecessor of $x_1$, without creating a contradiction, is $x_n$. From the properties of a symmetric geometry (5.2):

(5.11) $$i = n \; \wedge \; j = 1 \; \Rightarrow \; successor \; x_i = x_j = successor \; x_n = x_1.$$

Applying the definition of a symmetric geometry (5.2) to conclusion 5.11:

(5.12) $successor \; x_i = x_j \; \Rightarrow \; predecessor \; x_j = x_i \; \Rightarrow \; predecessor \; x_1 = x_n.$ □

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

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Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213,
adj313, adj323, and not_all_mutually_adjacent_gt_3.
```

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

DEFINITION 5.5. Cyclic successor of $m$ is $n$:

(5.13) $Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m+1 \leq setsize).$

DEFINITION 5.6. Cyclic predecessor of $m$ is $n$:

(5.14) $Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - q \geq 1).$

DEFINITION 5.7. Adjacent: member $m$ is sequentially adjacent to member $n$ if the cyclic successor of $m$ is $n$ or the cyclic predecessor of $m$ is $n$. Notionally:
(5.15)
$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize).$

Prove that every member is adjacent to every other member, where $setsize \in \{1, 2, 3\}$:

(5.16)        $Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1).$

(5.17)        $Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize).$

(5.18)        $Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1).$

(5.19)        $Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize).$

(5.20)        $Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - q \geq 1).$

(5.21)        $Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1).$

(5.22)        $Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize).$

(5.23)        $Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize).$

(5.24)        $Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - q \geq 1).$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

(5.25)   $\forall\, setsize > 3:$   $\neg Successor(1, 3, setsize > 3)$
$\leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize).$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

(5.26)   $\forall\, setsize > 3:$   $\neg Predecessor(1, 3, setsize > 3)$
$\leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

(5.27)   $\forall\, setsize > 3:$   $\neg Adjacent(1, 3, setsize > 3)$
$\leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3).$   $\square$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

## 6. Insights and implications

Applying the ruler measure (2.1) and ruler convergence (2.2) to the set relations, countable distance (3.1) and countable volume (4.1) yields the following insights and implications:

(1) The properties of metric space, Euclidean distance and area/volume are derived from two set operations without using the notions of Euclidean geometry like plane, side, angle, perpendicular, congruence, intersection, etc. [**Joy98**].

(2) Applying the countable set operations, countable distance (3.1) and countable volume (4.1), to sets of same-sized subintervals of intervals allows deriving geometric relations and insights into geometry that metric space, the Lebesgue, Borel, and Hausdorff measures are incapable of providing. For example:

(a) The constraint of disjoint range sets, $\bigcap_{i=1}^{n} y_i = \emptyset$, on the countable distance's union operation converges to Manhattan distance. And the same constraint on the countable volume's Cartesian product operation converges to Euclidean volume.

$d_c = |\bigcup_{i=1}^{n} y_i|, \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset \Rightarrow$
$v_c = |\times_{i=1}^{n} y_i|, \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset.$

(b) Countable distance depends on the amount range set intersection: $d_c = |\bigcup_{i=1}^{n} y_i|$. Manhattan distance is the case no intersection of the range sets: $d_c = \sum_{i=1}^{n} p_i$. The largest intersection (smallest distance), has the most domain set members mapping to each range set member. The equality constraint, $|x_i| = |y_i| = p_i$, on countable distance (3.1) limits the largest total number of domain-to-range set mappings to $\sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^2$. And the smallest distance case is the equality case, $d_c^2 = (\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$, which is the set-based reason Euclidean distance (3.4) is the smallest possible distance between two distinct points in $\mathbb{R}^n$ (in flat space, where $|x_i| = |y_i|$).

(c) The triangle inequality of metric space is derived from the countable distance, $d_c = |\bigcup_{i=1}^{n} y_i|$, (3.1) and the inclusion-exclusion inequality, $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$, (3.5).

(d) Generalizing the countable distance constraint, $|x_i| = |y_i|$, (3.1) to $|x_i| = |y_i|^q$, $q \geq 0$, generates all the $L^p$ norms, $||L||_p = (\sum_{i=1}^{n} s_i^p)^{1/p}$. For example, using the same proof pattern as for Euclidean distance (3.4): $p_i = |y_i| \Rightarrow |x_i| = p_i^q \Rightarrow \sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^{q+1} \leq d_c^{q+1} \ldots d = (\sum_{i=1}^{n} s_i^{q+1})^{1/(q+1)}.$

(e) The Euclidean volume proof was used to derive the Coulomb's charge force (5.1) and Newton's gravity force (5.4) without using other laws of physics or Gauss's divergence theorem.

(3) **The Proportionate Interval Principle:** The derivations of the charge force, gravity force, and spacetime equations using the Euclidean volume and distance derivations shows that all Euclidean distance intervals having a size, $r$, have proportionately sized intervals of other types: $r = (r_C/q_C)q = (r_G/m_G)m = (r_c/t_c)t.$

(a) The derivations of charge and gravity forces requiring the conversion ratios, $q = (q_C/r_C)r$ and $m = (m_G/r_G)r$, implies that if there are quantum values of charge, $q_C$, and mass, $m_G$, then there are quantum distances, $r_C$ and $r_G$, where the charge and gravity forces do not exist (are not defined) at smaller distances, which might have implications for the strong force, fine-structure constant, particle collisions, and the density of black holes.

(b) The proportionate interval principal generates the inverse square law. The derivations of charge and gravity forces requiring the conversion ratios, $q = (q_C/r_C)r$ and $m = (m_G/r_G)r$, show that rectangular geometric area $(r^2)$ maps to rectangular charge $(q_1 q_2)$ area and mass area $(m_1 m_2)$. But, some versions of the charge constant, vacuum magnetic permeability constant, fine structure constant, etc. contain the value $4\pi$ because the creators assumed geometric dilution (flux divergence on the surface of a sphere, $4\pi r^2$). It might be that those versions of the constants containing the value $4\pi$ are incorrect.

      (c) A countable set of values has measure 0 and thus no proportion rela-
tionship to distance. Therefore, a countable set of state value changes
with respect to time are independent of distance (for example, the
change in the spin values of two quantum coupled particles and the
change in polarization of two quantum coupled photons).

(4) Relativity theory assumes only 3 dimensions of physical space [**Bru17**].
The proof in this article shows that the properties order and symme-
try constrains physical distance and volume to at most three dimensions
(5.4). Higher dimensional Hausdorff (inner product vector) spaces are
valid if no more than 3 of the dimensions represent physical distance and
volume. More than 3 dimensions of physical distance and volume would
have sequential adjacency contradictions, where the dimensions of space
could not be deterministically sequenced.

      (a) The proof of at most 3 members in any ordered and symmetric set
(5.4), implies that each *physical* infinitesimal volume (ball) can have
at most 3 ordered and symmetric dimensions of discrete *physical*
states of the same type. And each dimension of discrete physical
states can have at most 3 ordered and symmetric discrete state val-
ues, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values
per infinitesimal ball.

      (b) If each of the three possible ordered and symmetric dimensions of
discrete physical states contained unordered sets of discrete state
values, for example, unordered binary values, then there would be
$2 \cdot 2 \cdot 2 = 8$ possible combinations of values. Unordered sets (states)
are non-deterministic. For example, for every sequential interval of
time an unordered state is physically measured, there is a 50 percent
chance of having one of the binary values.

      (c) Where infinitesimal balls intersect, an arithmetic of the interactions
of the discrete states with respect to time needs to be developed. The
interaction of the discrete states associated with overlapping infini-
tesimal balls with respect to time might result in what we perceive
as motion, waves, particles, spin, polarization, work, force, mass,
charge, etc.

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