The Real Analysis and Combinatorics of Geometry

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Abstract. A range from 1-to-1 to 1-to-many mappings between each disjoint domain set and each corresponding range set containing the same number of members, where the range sets in some cases intersect and the set members are the same-sized subintervals of intervals, converges to: the triangle inequality, Manhattan distance at the upper boundary, and Euclidean distance at the lower boundary, which provides set-based definitions of: metric space, longest, and shortest distances spanning disjoint sets. The Cartesian product of the same-sized subintervals of intervals converges to the product of interval sizes used in the Lebesgue measure and Euclidean integrals. A set of at most 3 dimensions emerges from the total ordering and symmetry properties of distance and volume. All ordered and symmetric, higher-dimensional geometries, like the spacetime four-vector, collapse into hierarchical 2 or 3-dimensional geometries. Proofs are verified in Coq.

Contents

1. Introduction

The triangle inequality of a metric space, the Manhattan and Euclidean distance metrics, and the volume equation (product of interval sizes) of the Lebesgue measure and Euclidean integrals (for example, Riemann and Lebesgue integrals) are defined [Gol76] rather than derived from set-based relations. Real analysis, topology, and measure theory have not exposed the set-based relationships that generate those geometric functions and other fundamental, geometric properties.

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A function only allows each domain set member to map to one range set member. Therefore, measures, like the Lebesgue and Hausdorff measures [Gol76] using distance, area, and volume functions as primitives are **not** capable of deriving the 1-to-many and many-to-many, set-based relationships that converge to the continuous, bijective Euclidean distance and volume functions.

A method of measurement that allows the full range of mappings from a one-to-one (bijective) mapping to a many-to-many mapping is required. A ruler (measuring stick) measures a real-valued interval as the nearest integer number of same-sized subintervals (units), where the partial subintervals are ignored. The ruler measure allows defining combinatorial relations, for example a many-to-many relation, between the same-sized subintervals in one interval and the same-sized subintervals in another interval. The discrete, combinatorial relations converge to continuous, bijective functions as the subinterval size converges to zero.

This article defines a set-based countable distance range having two properties: 1) For each disjoint domain set there exists a corresponding range (distance) set containing the same number of members; and 2) the distance sets in some cases intersect. When the distance sets intersect, multiple domain set members may map to the same distance set member. Therefore, the size of the union of distance sets is related to the number of domain-to-distance member mappings constrained to a range of no intersection (a one-to-one correspondence) to the maximum allowed intersection (a one-to-many mapping).

When the ruler is used to divide some disjoint, real-valued domain intervals and a real-valued distance interval into sets of same-sized subintervals: the one-to-one correspondence case converges to Manhattan distance; the one-to-many case converges to Euclidean distance; and the full-range converges to the triangle inequality, as the subinterval size converges to zero. The countable distance range provides set-based definitions of: metric space, longest, and shortest distances spanning disjoint domain sets.

Using the ruler measure, the Cartesian product of the same-sized subintervals of intervals converges to the product of interval sizes (length/area/volume functions) used in the Lebesgue measure and Euclidean integrals.

The commutative property of the set-based definitions from which distance and volume are derived: 1) Allow a sequential (total) ordering of the domain sets (dimensions), herein referred to as an ordered geometry; and 2) Allow every disjoint domain set (dimension) to be sequentially adjacent to any other domain set (dimension), herein referred to as a symmetric geometry. It is proved in this article that satisfying both a total ordering and symmetry results in a cyclic set of at most three dimensions, which explains why we live in a three dimensional world.

As shown in the summary, the four-vector lengths common in physics, like the spacetime four-vector length, are 2-dimensional Euclidean distance equations that have been "flattened." All ordered and symmetric, higher-dimensional geometries must collapse into hierarchical 2 or 3-dimensional geometries.

The proofs in this article are verified formally using the Coq Proof Assistant [Coq15] version 8.7.0. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at:
https://github.com/treeck/CombinatorialGeometry.

2. Ruler measure and convergence

DEFINITION 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, p, each subinterval having the same size, c. Notionally:

$$\begin{aligned}(2.1)\quad & \forall\, c\ s \in \mathbb{R},\ \ [a, b] \subset \mathbb{R},\ \ s = |a - b|\ \ \wedge\ \ c > 0\ \wedge \\ & \qquad (p = floor(s/c)\ \ \vee\ \ p = ceiling(s/c))\ \ \wedge\ \ M = \textstyle\sum_{i=1}^{p} c = pc.\end{aligned}$$

THEOREM 2.2. *Ruler convergence:*
$\forall\, [a, b] \subset \mathbb{R},\ \ s = |a - b|\ \ \Rightarrow\ \ M = \lim_{c \to 0} pc = s.$

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$(2.2)\qquad \forall\, c > 0,\quad p = floor(s/c)\quad \Rightarrow\quad 0 \le |p - s/c| < 1.$$

Multiply all sides of inequality 2.2 by $|c|$:

$$(2.3)\qquad \forall\, c > 0,\quad 0 \le |p - s/c| < 1\quad \Rightarrow\quad 0 \le |pc - s| < |c|.$$

$$\begin{aligned}(2.4)\quad & \forall\, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \\ & \qquad \Rightarrow\quad \forall\, \epsilon = \delta :\ \ |c - 0| < \delta\ \ \wedge\ \ |pc - s| < \epsilon\ \ :=\ \ M = \lim_{c \to 0} pc = s.\quad \square\end{aligned}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence, where: $[0, \pi]$, $s = |\pi - 0|$, $c = 10^{-i}$, and $p = floor(s/c) \Rightarrow\ p \cdot c = 3.1_{i=1},\ 3.14_{i=2},\ 3.141_{i=3}, ..., \pi$.

3. Distance

A simple countable distance measure is that an range (distance) set has the same number of members as a corresponding domain set. For example, the number of steps walked in a distance set must equal the number pieces of land traversed. Generalizing, for each distance set, y_i, containing p_i number of members there exists a corresponding domain set, x_i, with the same p_i number of members.

Notation conventions: The vertical bars around a set is the standard notation for indicating the cardinal (number of members in the set). To prevent over use of the vertical bar, the symbol for "such that" is the colon.

If the domain sets are disjoint ($\sum_{i=1}^{n} |x_i| = |\bigcup_{i=1}^{n} x_i|$) and the distance sets intersect ($\sum_{i=1}^{n} |y_i| > |\bigcup_{i=1}^{n} y_i|$), then multiple domain set member can map to a distance set member. Therefore, the size of the union of the distance sets, d_c, is related to the number of domain-to-distance member mappings. Notionally:

DEFINITION 3.1. Countable distance range, d_c:

$$\begin{aligned}& \forall\, i\ n \in \mathbb{N},\quad x_i \subseteq X,\quad \textstyle\sum_{i=1}^{n} |x_i| = |\bigcup_{i=1}^{n} x_i|,\quad \forall\, x_i\ \exists\, y_i \subseteq Y : \\ & \qquad |x_i| = |y_i| = p_i\quad \wedge\quad d_c = |Y| = |\textstyle\bigcup_{i=1}^{n} y_i| \le \sum_{i=1}^{n} |y_i|.\end{aligned}$$

For the countable distance range principle (3.1) case, $|x_i| = |y_i| = p_i = 1$: 1) Each member of x_i maps 1-to-1 (bijectively) to a member of y_i. 2) Each member of x_i maps 1-to-p_i number of members of y_i. And what is true for one member of a set must be true for any number of members of a set.

Using the rule of product, there is a range from $|y_i|\cdot 1 = p_i$ to $|y_i|\cdot p_i = p_i^2$ number of domain-to-distance mappings per distance set. Therefore, $d_c = f(\sum_{i=1}^{n} p_i)$ is largest possible distance because it is the case of the smallest number of mappings (p_i) per distance set (no intersection of the distance sets). And $d_c = f(\sum_{i=1}^{n} p_i^2)$ is the smallest possible distance because it is the case of the largest number of mappings (p_i^2) per distance set (largest allowed intersection of distance sets).

It will now be proved that using the ruler (2.1) to divide a set of real-valued domain intervals and a distance interval into sets of same-sized subintervals, and applying the ruler convergence theorem (2.2) to the longest and shortest distance cases converge to the real-valued, Manhattan and Euclidean distance equations.

Theorem 3.2. *Manhattan (longest) distance, d, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance."

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of subintervals and apply the definition of the countable distance range (3.1), where each domain set has a corresponding distance set containing the same p_i number of members.

$$\forall\, i\ n \in \mathbb{N}, \quad i \in [1, n], \quad s_i \in \mathbb{R}, \quad \exists\, c > 0 : \quad floor(s_i/c) = p_i = |x_i| = |y_i|. \tag{3.1}$$

Next, apply the rule of product to the case of one domain set member per distance set member:

$$|y_i| = p_i \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| \cdot 1 = \sum_{i=1}^{n} p_i. \tag{3.2}$$

Apply the countable distance range defintion (3.1) to equation 3.2:

$$\begin{aligned}\sum_{i=1}^{n} |y_i| \cdot 1 = \sum_{i=1}^{n} p_i \quad \wedge \quad \sum_{i=1}^{n} |y_i| \geq d_c \\ \Rightarrow \quad \sum_{i=1}^{n} p_i \geq d_c \quad \Rightarrow \quad \exists\, p_i,\ d_c : \ \sum_{i=1}^{n} p_i = d_c.\end{aligned} \tag{3.3}$$

Multiply both sides of 3.3 by c and apply the ruler convergence theorem (2.2):

$$\begin{aligned}s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n} (p_i \cdot c) = d_c \cdot c \\ \Rightarrow \quad \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) = \lim_{c\to 0} d_c \cdot c.\end{aligned} \tag{3.4}$$

Use the ruler to divide the exact size, $d = |d_0 - d_m|$, of the range interval, $[d_0, d_m]$, into a set, Y, containing d_c number of members:

$$\forall\, d_c \in \mathbb{N},\ c > 0\ \exists\, d \in \mathbb{R} : \ floor(d/c) = d_c. \tag{3.5}$$

Apply the ruler convergence theorem (2.2):

$$floor(d/c) = d_c \ \Rightarrow \ d = \lim_{c\to 0} d_c \cdot c. \tag{3.6}$$

Combine equations 3.6 and 3.4:

$$d = \lim_{c\to 0} d_c \cdot c \quad \wedge \quad \sum_{i=1}^{n} s_i = \lim_{c\to 0} d_c \cdot c \quad \Rightarrow \quad d = \sum_{i=1}^{n} s_i. \tag{3.7}$$

□

THEOREM 3.3. *Euclidean (shortest) distance, d, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of subintervals and apply the definition of the countable distance range (3.1), where each domain set has a corresponding distance set containing the same p_i number of members.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad s_i \in \mathbb{R}, \quad \exists\, c > 0 \;:\quad floor(s_i/c) = p_i = |x_i| = |y_i|. \tag{3.8}$$

Apply the rule of product to the largest number of domain-to-distance set mappings, where all p_i number of domain set members, x_i, map to each of the p_i number of members in the distance set, y_i:

$$\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2. \tag{3.9}$$

Choose the equality case of the Cauchy-Schwartz inequality:

$$\begin{aligned} \sum_{i=1}^{n} p_i^2 \;\le\; \sum_{i=1}^{n} p_i^2 + \sum_{i=1, j=1, i \neq j}^{n} (p_i \cdot p_j) \;=\; \left(\sum_{i=1}^{n} p_i\right)^2 \\ \Rightarrow\; \exists\, p_i \,:\, \sum_{i=1}^{n} p_i^2 = \left(\sum_{i=1}^{n} p_i\right)^2 \end{aligned} \tag{3.10}$$

Choose the equality case of the countable distance range definition (3.1) and square both sides ($x = y \Rightarrow f(x) = f(y)$):

$$\begin{aligned} \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \ge d_c \;\;\Rightarrow\;\; \exists\, p_i,\, d_c \,:\, \sum_{i=1}^{n} p_i = d_c \\ \Rightarrow\;\; \exists\, p_i,\, d_c \,:\, \left(\sum_{i=1}^{n} p_i\right)^2 = d_c^2. \end{aligned} \tag{3.11}$$

Combine equations 3.10 and and 3.11:

$$\begin{aligned} \exists\, p_i \,:\, \sum_{i=1}^{n} p_i^2 = \left(\sum_{i=1}^{n} p_i\right)^2 \;\wedge\; \exists\, p_i,\, d_c \,:\, \left(\sum_{i=1}^{n} p_i\right)^2 = d_c^2 \\ \Rightarrow\quad \exists\, p_i,\, d_c \,:\, \sum_{i=1}^{n} p_i^2 = d_c^2. \end{aligned} \tag{3.12}$$

Multiply both sides of equation 3.12 by c^2 and apply the ruler convergence theorem:

$$\begin{aligned} s_i = \lim_{c \to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n} (p_i \cdot c)^2 = (d_c \cdot c)^2 \\ \Rightarrow \quad \sum_{i=1}^{n} s_i^2 = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c)^2 = \lim_{c \to 0} (d_c \cdot c)^2. \end{aligned} \tag{3.13}$$

Use the ruler to divide the exact size, $d = |d_0 - d_m|$, of the range interval, $[d_0, d_m]$ into a set, Y, containing d_c number of members:

$$\forall\, d_c \in \mathbb{N},\; c > 0\, \exists\, d \in \mathbb{R} : floor(d/c) = d_c. \tag{3.14}$$

Apply the ruler convergence theorem (2.2) and then square both sides:

$$floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c \to 0} (d_c \cdot c)^2. \tag{3.15}$$

Combine equations 3.15 and 3.13:

$$d^2 = \lim_{c \to 0} (d_c \cdot c)^2 \;\wedge\; \sum_{i=1}^{n} s_i^2 = \lim_{c \to 0} (d_c \cdot c)^2 \;\Rightarrow\; d^2 = \sum_{i=1}^{n} s_i^2. \tag{3.16}$$

□

3.1. Triangle inequality. The definition of a metric in real analysis is based on the triangle inequality, $\mathbf{d}(\mathbf{u},\mathbf{w}) \leq \mathbf{d}(\mathbf{u},\mathbf{v}) + \mathbf{d}(\mathbf{v},\mathbf{w})$, that has been intuitively motivated by the triangle [**Gol76**]. Applying the ruler (2.1), and ruler convergence theorem (2.2) to the definition of a countable distance range (3.1) yields the real-valued triangle inequality:

$$\begin{gathered}
d_c = |Y| = |\textstyle\bigcup_{i=1}^{2} y_i| \leq \sum_{i=1}^{2} |y_i| \quad \wedge \\
d_c = floor(\mathbf{d}(\mathbf{u},\mathbf{w})/c) \;\wedge\; |y_1| = floor(\mathbf{d}(\mathbf{u},\mathbf{v})/c) \;\wedge\; |y_2| = floor(\mathbf{d}(\mathbf{v},\mathbf{w})/c) \\
\Rightarrow \mathbf{d}(\mathbf{u},\mathbf{w}) = \lim_{c\to 0} d_c \cdot c \leq \textstyle\sum_{i=1}^{2} \lim_{c\to 0} |y_i| \cdot c = \mathbf{d}(\mathbf{u},\mathbf{v}) + \mathbf{d}(\mathbf{v},\mathbf{w}).
\end{gathered} \tag{3.17}$$

The other metric space properties: $\mathbf{d}(\mathbf{u},\mathbf{w}) = 0 \Leftrightarrow u = w$, $\mathbf{d}(\mathbf{u},\mathbf{w}) = \mathbf{d}(\mathbf{w},\mathbf{u})$, and $\mathbf{d}(\mathbf{u},\mathbf{w}) \geq 0$ also follow from the countable distance range definition.

4. Size (length/area/volume)

The combinatorial relations between all members in set x_1 to each member of set x_2 results in the Cartesian product of $|x_1| \cdot |x_2|$ number of combinations. This section will use the ruler (2.1) and ruler convergence theorem (2.2) to prove that the Cartesian product of the same-sized subintervals of intervals converges to the product of interval sizes as the subinterval converges to zero. The first step is to define a set-based, countable size measure as the Cartesian product of disjoint domain set members.

Definition 4.1. Countable size (length/area/volume) measure, S_c:

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1,n], \quad x_i \subseteq X, \quad \textstyle\sum_{i=1}^{n} |x_i| = |\bigcup_{i=1}^{n} x_i|, \quad \wedge \quad S_c = \displaystyle\prod_{i=1}^{n} |x_i|.$$

Theorem 4.2. *Euclidean size (length/area/volume), S, is the size of an range interval, $[v_0, v_m]$, corresponding to a set of disjoint intervals:*
$\{[a_1,b_1],[a_2,b_2],\ldots,[a_n,b_n]\}$, *where:*

$$S = \prod_{i=1}^{n} s_i, \quad S = |v_0 - v_m|, \quad s_i = |a_i - b_i|, \quad i \in [1,n], \quad i,n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_size."

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, x_i of p_i number of subintervals.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1,n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i|. \tag{4.1}$$

Use the ruler (2.1) to divide the exact size, $S = |v_0 - v_m|$, of the range interval, $[v_0, v_m]$, into p_S^n subintervals. Every integer number, S_c, does **not** have an integer n^{th} root. However, for those cases where S_c does have an integer n^{th} root, there is a p_S^n that satisfies the definition a countable size measure, S_c (4.1). Notionally:

$$\forall\, p_S^n = S_c \in \mathbb{N},\, \exists\, S \in \mathbb{R},\, x_i \;: floor(S/c) = p_S^n = S_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i. \tag{4.2}$$

Multiply both sides of equation 4.2 by c^n to get the ruler measures:

$$p_S^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3:

$$S = \lim_{c\to 0}(p_S \cdot c)^n \quad \wedge \quad (p_S \cdot c)^n = \prod_{i=1}^{n}(p_i \cdot c) \quad \wedge \quad \lim_{c\to 0}(p_i \cdot c) = s_i \tag{4.4}$$

$$\Rightarrow \quad S = \lim_{c\to 0}(p_S \cdot c)^n = \prod_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i. \qquad \square$$

5. Ordered and symmetric geometries

The commutative property of the union and addition operations in the countable distance range principle (3.1) that generates the triangle inequality, Manhattan and Euclidean distances and the commutative property of the union and multiplication operations in the countable size principle (4.1) that generates length/area/volume allow a sequential (total) ordering of the disjoint domain sets (dimensions) to exist. And the commutative property also allows every dimension to be sequentially adjacent to any other dimension (herein, referred to as a symmetric geometry).

It will now be proved that satisfying both the total ordering and symmetry properties simultaneously limits distance and volume to a cyclic set of at most three dimensions.

Definition 5.1. Ordered geometry:

$$\forall\, i\ n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i \ \in \{x_1, \ldots, x_n\},$$
$$\textit{successor}\ x_i = x_{i+1} \ \wedge\ \textit{predecessor}\ x_{i+1} = x_i.$$

where each $x_i \in \{x_1, \ldots, x_n\}$ is a set of subintervals of a real-valued domain interval (dimension).

Definition 5.2. Symmetric geometry (every member is sequentally adjacent to every other member):

$$\forall\, i\ j\ n \in \mathbb{N},\ \ \forall\, x_i\ x_j \in \{x_1, \ldots, x_n\},\ \ \textit{successor}\ x_i = x_j \ \wedge\ \textit{predecessor}\ x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric geometry is a cyclic set.*

$$\textit{successor}\ x_n = x_1 \ \wedge\ \textit{predecessor}\ x_1 = x_n.$$

The theorem and formal Coq-based proof is "ordered_symmetric_is_cyclic," which is located in the file threed.v.

Proof. The property of order (5.1) defines unique successors and predecessors for all members except for the successor of x_n and the predecessor of x_1. From the properties of a symmetric geometry (5.2):

$$i = n \ \wedge\ j = 1 \ \wedge\ \textit{successor}\ x_i = x_j \ \Rightarrow\ \textit{successor}\ x_n = x_1. \tag{5.1}$$

$$i = n \ \wedge\ j = 1 \ \wedge\ \textit{predecessor}\ x_j = x_i \ \Rightarrow\ \textit{predecessor}\ x_1 = x_n. \tag{5.2}$$ $\square$

Theorem 5.4. *An ordered and symmetric geometry is limited to at most 3 members.*

The Coq-based lemmas and proofs in the file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
Because a symmetric and ordered set is a cyclic set (5.3), the successors and predecessors are cyclic:

DEFINITION 5.5. Successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (m + 1 \leq \mathit{setsize}). \tag{5.3}$$

DEFINITION 5.6. Predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n (required for a "symmetric" set (5.2)), if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:

$$\mathit{Adjacent}(m, n, \mathit{setsize}) \leftarrow \mathit{Successor}(m, n, \mathit{setsize}) \vee \mathit{Predecessor}(m, n, \mathit{setsize}). \tag{5.5}$$

Every member is adjacent to every other member, where $\mathit{setsize} \in \{1,\ 2,\ 3\}$:

$$\mathit{Adjacent}(1, 1, 1) \leftarrow \mathit{Successor}(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{5.6}$$

$$\mathit{Adjacent}(1, 2, 2) \leftarrow \mathit{Successor}(1, 2, 2) \leftarrow (1 + 1 \leq 2). \tag{5.7}$$

$$\mathit{Adjacent}(2, 1, 2) \leftarrow \mathit{Successor}(2, 1, 2) \leftarrow (2 = 2 \wedge 1 = 1). \tag{5.8}$$

$$\mathit{Adjacent}(1, 2, 3) \leftarrow \mathit{Successor}(1, 2, 3) \leftarrow (1 + 1 \leq 2). \tag{5.9}$$

$$\mathit{Adjacent}(2, 1, 3) \leftarrow \mathit{Predecessor}(2, 1, 3) \leftarrow (2 - 1 \geq 1). \tag{5.10}$$

$$\mathit{Adjacent}(3, 1, 3) \leftarrow \mathit{Successor}(3, 1, 3) \leftarrow (3 = 3 \wedge 1 = 1). \tag{5.11}$$

$$\mathit{Adjacent}(1, 3, 3) \leftarrow \mathit{Predecessor}(1, 3, 3) \leftarrow (1 = 1 \wedge 3 = 3). \tag{5.12}$$

$$\mathit{Adjacent}(2, 3, 3) \leftarrow \mathit{Successor}(2, 3, 3) \leftarrow (2 + 1 \leq 3). \tag{5.13}$$

$$\mathit{Adjacent}(3, 2, 3) \leftarrow \mathit{Predecessor}(3, 2, 3) \leftarrow (3 - 1 \geq 1). \tag{5.14}$$

Must prove that for all $\mathit{setsize} > 3$, there exist non-adjacent members. For example, the first and third members are not adjacent:

$$\forall\ \mathit{setsize} > 3: \quad \neg\mathit{Successor}(1, 3, \mathit{setsize}) \leftarrow \mathit{Successor}(1, 2, \mathit{setsize}) \leftarrow (1 + 1 \leq \mathit{setsize}). \tag{5.15}$$

That is, 2 is the only successor of 1 for all $\mathit{setsize} > 3$, which implies 3 is not a successor of 1 for all $\mathit{setsize} > 3$.

$$\forall\ \mathit{setsize} > 3: \quad \neg\mathit{Predecessor}(1, 3, \mathit{setsize}) \leftarrow \mathit{Predecessor}(1, n, \mathit{setsize}) \leftarrow (1 = 1 \wedge n = \mathit{setsize}). \tag{5.16}$$

That is, $n = \mathit{setsize}$ is the only predecessor of 1 for all $\mathit{setsize} > 3$, which implies 3 is not a predecessor of 1 for all $\mathit{setsize} > 3$.

$$\forall\ \mathit{setsize} > 3: \quad \neg\mathit{Adjacent}(1, 3, \mathit{setsize}) \leftarrow \neg\mathit{Successor}(1, 3, \mathit{setsize}) \wedge \neg\mathit{Predecessor}(1, 3, \mathit{setsize}). \tag{5.17}$$

□

6. Summary

Applying some very simple real analysis, in the form of the ruler measure (2.1) and ruler convergence proof (2.2), to a set of real-valued domain intervals and an range interval yields some new insights into geometry and physics.

(1) Discrete, combinatorial relations converge to the continuous, bijective relations: triangle inequality, Manhattan distance, Euclidean distance and volume. Other types of measures do not have that capability.

(2) Ruler measure-based proofs expose the difference between distance and size (length/area/volume) measures: Distance is a mapping relation between the members of each disjoint domain set and members of a corresponding range (distance) set. In contrast, volume is a combinatorial relation between the members of disjoint domain sets.

(3) Applying the ruler measure to the countable distance range (3.1) provides the insight that all notions of distance are based on the principle that for each disjoint domain set there exists a corresponding distance set containing the same number of members, where the distance sets in some cases intersect:

   (a) The countable distance range principle converges to the real-valued triangle inequality (3.1), which is the basis for the definition of metric space. The other properties of metric space also come from the countable distance range principle. Therefore, a function is not a distance metric unless it satisfies the more fundamental countable distance range (3.1).

   (b) All $L^{p>2}$ norms generated from the countable distance range principle would require each member of the i^{th} domain set to map to a member of the i^{th} distance set more than once, which would be over-counting the number of possible mappings. Therefore, $L^{p>2}$ norms are not valid distance measures. Other measure theories have not provided this over-counting insight into $L^{p>2}$ norms.

   (c) The upper bound of the countable distance range converging to Manhattan distance (3.2) provides the insight that the largest (longest) monotonic distance pathis is the case of disjoint distance sets, where each member in the i^{th} domain set maps to only one member in the i^{th} distance set.

   (d) The lower bound of the countable distance range converging to Euclidean distance (3.3) provides the insight that the smallest (shortest) possible monotonic distance path is the case of the maximum allowed intersection of the distance sets, where each of the p_i number of members in the i^{th} domain set maps to all p_i number of members of the i^{th} distance set.

   (e) Euclidean distance (3.3) was derived from a set-based, one-to-many relation without any notions of side, angle, or shape. A parametric variable relating the sizes of two domain intervals can be easily derived using using calculus (Taylor series) and the Euclidean distance equation to generate the arc sine and arc cosine functions of the parametric variable (arc angle). In other words, the notions of side and angle are derived from Euclidean distance, which is the reverse perspective of classical geometry [**Joy98**] [**Loo68**] [**Ber88**] and axiomatic foundations for geometry [**Bir32**] [**Hil80**] [**TG99**].

(4) Applying the ruler measure and ruler convergence proof to the countable size definition (4.1) allows a proof that the Cartesian product of same-sized subintervals of real-valued intervals converges to the product of the interval sizes (Euclidean length/area/volume):
   (a) Euclidean size (length/area/volume) was derived from a combinatorial relation without notions of sides, angles, and shape.

(5) This article provides a set and number theory-based foundation to derive the triangle inequality, Manhattan and Euclidean distance, and Eucldiean area/volume that has been lacking previously.

(6) The set-based relations of countable distance range (3.1) and countable size (4.1) that generate the real-valued triangle inequality, Manhattan distance, Euclidean distance, and volume equations have a symmetry property (5.2) that limits distance and volume to a cyclic set (5.3) of three dimensions (5.4). This symmetry property explains why only three dimensions of physical space can be observed.

(7) All valid ordered and symmetric, higher-dimensional geometries collapse into hierarchical 2 or 3-dimensional geometries. The four-vector lengths common in physics are 2-dimensional Euclidean lengths that have been "flattened." For example, the spacetime four-vector length,
$d = \sqrt{(ct)^2 - (x^2 + y^2 + z^2)}$, can be expressed in a form like,
$d_2 = \sqrt{(ct)^2 - d_1^2}$, where $d_1 = \sqrt{x^2 + y^2 + z^2}$ and $d_2 = d$.

   Applying the Euclidean distance proof (3.3) to the 2-dimensional Poincaré form (where $c = 1$), $t^2 = d_1^2 + d_2^2$, provides the perspective that d_1 and d_2 are lengths in two frames of reference (the lengths of two domain intervals) and the size of each time subinterval is the same size (same speed of light) in both frames of reference.

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