The Two Set Relations Generating Geometry

George. M. Van Treeck

Abstract. Where each countable domain set has a corresponding range set, countable distance is defined as the cardinal of the union of the range sets and countable volume as the cardinal of the set of n-tuples of members from the disjoint range sets. The countable distance and volume set operations applied to sets of same-sized subintervals of domain and range intervals generate the properties of metric space, all Lp norms (for example, Manhattan and Euclidean distance), and the volume equation as the subinterval size goes to 0. The volume proof is used to derive Coulomb's charge force and Newton's gravity force equations without using other laws of physics or Gauss's divergence theorem. A symmetry constraint on a totally ordered set limits the set to at most 3 members, for example, 3 dimensions. All proofs are verified in Coq.

Contents

1. Introduction

Metric space, Euclidean distance, and Euclidean area/volume have been definitions in mathematical analysis [Gol76] [Rud76] motivated by Euclidean geometry [Joy98], rather than derived from an abstract set and limit-based foundation. A consequence of no set and limit-based foundation is that analysis has not been able to identify: the constraint between countable domain and range sets that generates flat space; the countable domain-to-range set mapping that makes Euclidean distance the smallest possible distance between two distinct points in flat space; the set operation and constraint generating the properties of metric space, etc.

2010 *Mathematics Subject Classification.* Primary 28A75, 28E15. Secondary 03E75, 51M99.

Where each disjoint, countable domain set has a corresponding countable range set, countable distance is defined as the cardinal of the union of the range sets and countable volume as the cardinal of the set of n-tuples of members from disjoint range sets. The countable distance and volume set operations applied to the sets of same-sized, size c, subintervals of domain and range intervals generate the properties of metric space, all L_p norms (Minkowski distances, for example, Manhattan and Euclidean distance), and the volume equation as $c \to 0$. And some applications to physics are shown.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

In this article, geometric relations are determined by the number of mappings between the size c subintervals in each interval, as $c \to 0$, which requires knowing the relative number of size c subintervals in each interval. But, anti-derivative integrals (for example, the Riemann and Lebesgue integrals) divide all the domain intervals and the range interval into the *same* number of subintervals, where the *size* of the subintervals may *differ*, which makes the integrals an inappropriate tool.

A ruler (measuring stick) measures the size, M, of each interval, i, *approximately* as the sum of the nearest integer number, p_i, of whole subintervals, where each subinterval has the *same* size, c. The ruler format, $\lim_{c\to 0} p_i c$, makes it easy to derive geometric relations from the number of mappings between the size c subintervals in each interval.

Definition 2.1. Ruler measure, M: $\forall\, c,\ s \in \mathbb{R},\ [a, b] \subset \mathbb{R},\ s = b-a \wedge c > 0 \wedge (p = floor(s/c) \ \vee\ p = ceiling(s/c)) \ \wedge\ M = \sum_{i=1}^{p} c = pc.$

Theorem 2.2. *Ruler convergence:* $M = \lim_{c\to 0} pc = s$.

The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : \ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\ p = floor(s/c) \ \wedge\ 0 \le |floor(s/c) - s/c| < 1 \ \Rightarrow\ 0 \le |p - s/c| < 1. \tag{2.1}$$

Multiply all sides of inequality 2.1 by c:

$$\forall\, c > 0, \quad 0 \le |p - s/c| < 1 \quad \Rightarrow \quad 0 \le |pc - s| < |c|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \delta \ :\ |pc - s| < |c| = |c - 0| < \delta \\ &\qquad \Rightarrow \quad \forall\, \epsilon = \delta :\ \ |c - 0| < \delta \ \wedge\ |pc - s| < \epsilon \ \ := \ \ M = \lim_{c\to 0} pc = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}},\ 3.14_{c=10^{-2}},\ 3.141_{c=10^{-3}}, ..., \pi_{\lim_{c\to 0}}$.

3. Distance

Notation convention: Vertical bars around a set or list, $|\cdots|$, indicates the cardinal (number of members in the set or list).

3.1. Countable distance. Distance in one direction/dimension is independent of distance in every other other direction/dimension. Therefore, each disjoint domain set, x_i, has its own independent range set, y_i, with the same number of members, $|x_i| = |y_i|$. The countable distance spanning the disjoint domain sets is the number of members, d_c, in the union range (distance) set:

DEFINITION 3.1. Countable distance, d_c :

$$d_c = |\bigcup_{i=1}^n y_i| : \quad \bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad |x_i| = |y_i|.$$

It will be shown in the next subsections that the constraint, $|x_i| = |y_i|$, generates Manhattan and Euclidean distance at the boundaries (generates flat space). Generalizing distance beyond flat space is shown in the last section of this article.

3.2. Inclusion-Exclusion Inequality. The inequality, $|\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$, follows directly from the inclusion-exclusion principle [**CG15**]. Because this inequality is used often in this article, a trivial proof is included here. The proof follows from the associative law of addition where the sum of set sizes is equal to the size of all the set members appended into a list and the commutative law of addition that allows sorting that list into a list of unique members (the *union* set) and a list of duplicates. For example, for $y_1 = \{a, b, c\}$ and $y_2 = \{c, d, e\}$: $\sum_{i=1}^2 |y_i| = 6 = |[a, b, c, c, d, e]| = |[a, b, c, d, e, c]| = |\{a, b, c, d, e\}| + |[c]|$. The duplicates being ≥ 0 implies the union size is always $\leq$ the sum of set sizes.

LEMMA 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$.

PROOF. A formal proof, inclusion_exclusion_inequality, using sorting into a set of unique members (*union* set) and a list of duplicates, is in the file euclidrelations.v.

$$\begin{aligned} \sum_{i=1}^n |y_i| = |append_{i=1}^n y_i| &= |sort(append_{i=1}^n y_i)| \\ &= |\bigcup_{i=1}^n y_i| + |duplicates_{i=1}^n y_i|. \end{aligned} \tag{3.1}$$

$$\begin{aligned} |\bigcup_{i=1}^n y_i| + |duplicates_{i=1}^n y_i| = \sum_{i=1}^n |y_i| \quad &\wedge \quad |duplicates_{i=1}^n y_i| \geq 0 \\ &\Rightarrow \quad |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|. \quad \square \end{aligned} \tag{3.2}$$

3.3. Countable distance range. From the countable distance definition (3.1), $d_c = |\bigcup_{i=1}^n y_i|$, as the amount of intersection increases, more domain set members can map to a single range set member. Therefore, the countable distance, d_c, is a function of the total number of domain-to-range set member mappings.

Each domain set, x_i has its own independent range set, y_i. From the countable distance constraint (3.1), where $|x_i| = |y_i| = p_i$, the countable distance, d_c, ranges from a function of the sum of 1-1 mappings, $d_c = \mathrm{f}(\sum_{i=1}^n (1 \cdot |y_i|)) = \mathrm{f}(\sum_{i=1}^n p_i)$, to a function of the sum of many-to-many (Cartesian product) mappings, $d_c = \mathrm{f}(\sum_{i=1}^n (|x_i| \cdot |y_i|)) = \mathrm{f}(\sum_{i=1}^n p_i^2)$.

Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of domain-to-range set mapping cases converges to the real-valued Manhattan and Euclidean distance relations.

3.4. Manhattan distance.

THEOREM 3.3. *Manhattan (largest) distance, d, is the size of the range interval,* $[d_0, d_m]$, *corresponding to a set of disjoint domain intervals,*

$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = d_m - d_0, \quad s_i = b_i - a_i.$$

The formal proof, "taxicab_distance," is in the Coq file, euclidrelations.v.

PROOF.

From the countable distance definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \quad \Rightarrow \quad d_c \leq \sum_{i=1}^{n} p_i \\ \Rightarrow \quad \exists\, p_i,\, d_c : \; d_c = \sum_{i=1}^{n} p_i. \tag{3.3}$$

Multiply both sides of equation 3.3 by c and take the limit:

$$d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \Rightarrow \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c). \tag{3.4}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$d = d_m - d_0 \quad \Rightarrow \quad \exists\, c\, d : \; floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c. \tag{3.5}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$s_i = b_i - a_i \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c \to 0} p_i \cdot c = s_i. \tag{3.6}$$

Combine equations 3.5, 3.4, 3.6:

$$d = \lim_{c \to 0} d_c \cdot c \quad \wedge \quad \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) \quad \wedge \\ \lim_{c \to 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) = \sum_{i=1}^{n} s_i. \tag{3.7}$$

□

3.5. Euclidean distance.

THEOREM 3.4. *Euclidean (smallest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,* $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = d_m - d_0, \quad s_i = b_i - a_i.$$

The formal proof, "Euclidean_distance," is in the Coq file, euclidrelations.v.

PROOF.

Apply the rule of product to the largest number of domain-to-range set mappings, where all p_i number of range set members, y_i, map to each of the p_i number of members in the domain set, x_i, which is the Cartesian product, $|y_i| \cdot |x_i|$:

$$|x_i| = |y_i| = p_i \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2. \tag{3.8}$$

From the countable distance definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \quad \Rightarrow \quad d_c \leq \sum_{i=1}^{n} p_i \\ \Rightarrow \quad \exists\, p_i,\, d_c : \; d_c = \sum_{i=1}^{n} p_i. \tag{3.9}$$

Square both sides of equation 3.9 ($x = y \Leftrightarrow f(x) = f(y)$):

$$\exists\, p_i,\, d_c : \; d_c = \sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\, d_c : \; d_c^2 = (\sum_{i=1}^{n} p_i)^2. \tag{3.10}$$

Apply the square of sum inequality, $(\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$, to equation 3.10 and select the smallest area (the equality) case:

$$d_c^2 = (\sum_{i=1}^{n} p_i)^2 = \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j \\ = \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} p_i \sum_{j=1, j \neq i}^{n} p_j \geq \sum_{i=1}^{n} p_i^2 \quad \Rightarrow \quad \exists\, p_i : \; d_c^2 = \sum_{i=1}^{n} p_i^2. \tag{3.11}$$

Multiply both sides of equation 3.11 by c^2, simplify, and take the limit.

$$d_c^2 = \sum_{i=1}^{n} p_i^2 \;\Rightarrow\; d_c^2 \cdot c^2 = \sum_{i=1}^{n} p_i^2 \cdot c^2 \;\Leftrightarrow\; (d_c \cdot c)^2 = \sum_{i=1}^{n} (p_i \cdot c)^2 \\ \Rightarrow\; \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2. \tag{3.12}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$\exists\, c\, d \in \mathbb{R} : \; floor(d/c) = d_c \quad\Rightarrow\quad d = \lim_{c\to 0} d_c \cdot c \quad\Rightarrow\quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2. \tag{3.13}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$s_i = b_i - a_i \quad\wedge\quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad\Rightarrow\quad \lim_{c\to 0} p_i \cdot c = s_i. \tag{3.14}$$

Combine equations 3.13, 3.12, 3.14:

$$d^2 = \lim_{c\to 0}(d_c \cdot c)^2 \quad\wedge\quad \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 \quad\wedge \tag{3.15}$$
$$\lim_{c\to 0}(p_i \cdot c) = s_i \quad\Rightarrow\quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \quad \square$$

3.6. Metric Space. All function range intervals, $d(u, w)$, satisfying the countable distance definition (3.1), where the ruler is applied, generates the properties of metric space. The formal proofs: triangle_inequality, non_negativity, identity_of_indiscernibles, and symmetry are in the Coq file, euclidrelations.v.

THEOREM 3.5. *Triangle Inequality:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \leq d(u, v) + d(v, w)$.

PROOF. Apply the ruler measure (2.1), the countable distance condition (3.1), inclusion-exclusion inequality (3.2), and then ruler convergence (2.2).

$$\begin{aligned} &\forall\, c > 0,\; d(u, w),\; d(u, v),\; d(v, w) \;: \\ &\quad |y_1| = floor(d(u, v)/c) \;\wedge\; |y_2| = floor(d(v, w)/c) \;\wedge \\ &\quad d_c = floor(d(u, w)/c) \;\wedge\; d_c = |y_1 \cup y_2| \leq |y_1| + |y_2| \\ &\quad \Rightarrow\; floor(d(u, w)/c) \leq floor(d(u, v)/c) + floor(d(v, w)/c) \\ &\quad \Rightarrow\; floor(d(u, w)/c) \cdot c \leq floor(d(u, v)/c) \cdot c + floor(d(v, w)/c) \cdot c \\ &\quad \Rightarrow\; \lim_{c\to 0} floor(d(u, w)/c) \cdot c \leq \lim_{c\to 0} floor(d(u, v)/c) \cdot c + \lim_{c\to 0} floor(d(v, w)/c) \cdot c \\ &\quad \Rightarrow\; d(u, w) \leq d(u, v) + d(v, w). \quad \square \end{aligned} \tag{3.16}$$

THEOREM 3.6. *Non-negativity:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \geq 0$.

PROOF. By definition, a set always has a size (cardinal) ≥ 0:

$$\begin{aligned} &\forall\, c > 0,\; d(u, w) \;:\quad floor(d(u, w)/c) = d_c \quad\wedge\quad d_c = |y_1 \cup y_2| \geq 0 \\ &\quad \Rightarrow\quad floor(d(u, w)/c) = d_c \geq 0 \quad\Rightarrow\quad d(u, w) = \lim_{c\to 0} d_c \cdot c \geq 0. \quad \square \end{aligned} \tag{3.17}$$

THEOREM 3.7. *Identity of Indiscernibles:* $d(w, w) = 0$.

PROOF. Apply the triangle inequality property (3.5):

$$\forall\, d(u, v) = d(v, w) = 0 \;\wedge\; d(u, w) \leq d(u, v) + d(v, w) \;\Rightarrow\; d(u, w) \leq 0. \tag{3.18}$$

Combine the non-negativity property (3.6) and the previous inequality (3.18):

$$d(u, w) \geq 0 \;\wedge\; d(u, w) \leq 0 \;\Leftrightarrow\; 0 \leq d(u, w) \leq 0 \;\Rightarrow\; d(u, w) = 0. \tag{3.19}$$

Combine the result of step 3.19 and the condition, $d(u, v) = 0$, in step 3.18.

$$d(u, w) = 0 \;\wedge\; d(u, v) = 0 \;\Rightarrow\; w = v. \tag{3.20}$$

Combine the condition, $d(v, w) = 0$, in step 3.18 and the result of step 3.20.

$$d(v, w) = 0 \;\wedge\; w = v \;\Rightarrow\; d(w, w) = 0. \tag{3.21}$$

$\square$

THEOREM 3.8. *Symmetry:* $|x_i| = |y_i| \Rightarrow d(u,v) = d(v,u)$.

PROOF.
The range of countable distances (3.3) is a function of domain-to-range set members, under the constraint, $|x_i| = |y_i|$:

$$(3.22)\quad |x_i| = |y_i| = p_i \quad \Rightarrow \quad d_c = \mathrm{f}(\textstyle\sum_{i=1}^{n} |x_i| \cdot |y_i|^q) = \mathrm{f}(\sum_{i=1}^{n} p_i^{1+q}),\ 0 \le q \le 1.$$

Applying the ruler and ruler convergence to real-valued domain and range intervals, generates the range of distances from Manhattan distance (3.3), $d(x,y) = \mathrm{f}(\sum_{i=1}^{n} s_i^1)$, to Euclidean distance (3.4), $d(x,y) = \mathrm{f}(\sum_{i=1}^{n} s_i^2)$. Therefore:

$$(3.23)\qquad \forall\, p\ :\ 1 \le p \le 2, \quad d(x,y) = \mathrm{f}(\textstyle\sum_{i=1}^{n} s_i^p).$$

There are two cases:

Case #1: 1-dimensional distance: This is the absolute value metric, where x and y are the bounds of a domain interval, with interval length, $s = |x - y|$. And applying equation (3.23) to the case of $n = 1$ dimension:

$$(3.24)\quad d(x,y) = \mathrm{f}(s^p) = \mathrm{f}(|x-y|^p),\ 1 \le p \le 2$$
$$\Rightarrow \quad d(u,v) = \mathrm{f}(|u-v|^p) = \mathrm{f}(|v-u|^p) = d(v,u).$$

Case #2: 2-dimensional distance:
By the commutative law of addition:

$$(3.25)\qquad d(u,v) = \mathrm{f}(u^p + v^p) = \mathrm{f}(v^p + u^p) = d(v,u). \qquad \square$$

4. Euclidean Volume

$\mathbb{R}^n$, the Lebesgue measure, Riemann integral, and Lebesgue integral define (assume) area/volume to be the product of domain interval lengths. The goal here is to derive the area/volume equation from an abstract, set-based definition of volume.

DEFINITION 4.1. Countable Volume, v_c, is the cardinal of the set of n-tuples of members from countable, disjoint, range sets:

$$v_c = |\times_{i=1}^{n} y_i| : \quad \textstyle\bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset.$$

THEOREM 4.2. *Euclidean volume, v, is length of the range interval, $[v_0, v_m]$, equal to product of domain interval lengths, $\{[a_1,b_1],[a_2,b_2],\ldots,[a_n,b_n]\}$:*

$$v = \textstyle\prod_{i=1}^{n} s_i, \ \ v = v_m - v_0, \ \ s_i = b_i - a_i.$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of subintervals.

$$(4.1)\qquad \forall\, i\ n \in \mathbb{N}, \quad i \in [1,n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i| = |y_i|.$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$(4.2)\qquad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c \to 0}(p_i \cdot c) = s_i.$$

Apply the associative law of multiplication to derive the countable volume (4.1) in terms of p_i:

$$(4.3)\qquad v_c = |\times_{i=1}^{n} y_i| = \textstyle\prod_{i=1}^{n} |y_i| \quad \wedge \quad |y_i| = p_i \quad \Rightarrow \quad v_c = \prod_{i=1}^{n} p_i.$$

Multiply both sides of equation 4.3 by c^n:

$$v_c \cdot c^n = (\textstyle\prod_{i=1}^n p_i) \cdot c^n = \prod_{i=1}^n (p_i \cdot c). \tag{4.4}$$

Use those cases, where v_c has an integer n^{th} root.

$$\forall\, n,\ p,\ v_c\ \in \mathbb{N}\ :\ p^n = v_c\ \Rightarrow\ v_c \cdot c^n = p^n \cdot c^n = (p \cdot c)^n = \textstyle\prod_{i=1}^n (p_i \cdot c). \tag{4.5}$$

Apply the ruler (2.1) and ruler convergence (2.2) to the range interval, $[v_0, v_m]$ (where $v = v_m - v_0$), and then combine with equations 4.5 and 4.2:

$$floor(v/c^n) = p^n \Rightarrow v = \lim_{c\to 0}(p \cdot c)^n = \textstyle\prod_{i=1}^n \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^n s_i. \tag{4.6}$$ □

5. Applications to physics

5.1. Coulomb's charge force. The sizes, q_1 and q_2, of two charges are independent domain intervals, where each infinitesimal size c subinterval of a charge exerts a quantum force, $m_C a_C$, on each infinitesimal size c subinterval of the other charge. The total force, F, is proportionate to the total number of forces (the Cartesian product of the infinitesimal size c components) multiplied times the quantum charge force, $m_C a_C$. Applying the ruler, $p_1 = floor(q_1/c)$ and $p_2 = floor(q_2/c)$, and the Cartesian product, $p_1 \times p_2$, of size c components yields:

$$F \propto m_C a_C(\lim_{c\to 0} p_1 c \cdot \lim_{c\to 0} p_2 c) = m_C a_C \int_0^{q_2}\int_0^{q_1} \mathrm{d}^2 c = m_C a_C(q_1 q_2). \tag{5.1}$$

If each quantum charge size, q_C has a corresponding distance size, r_C, then the total charge size, q, is related to the total distance size, $r:\ q = (q_C/r_C)r$, where q_C/r_C, where is a unit-factoring conversion ratio:

$$\forall\, q_1, q_2 \geq 0\ \exists\, q \in \mathbb{R}\ :\ q^2 = q_1 q_2\ \wedge\ (q_C/r_C)r = q\ \Rightarrow\ ((q_C/r_C)r)^2 = q_1 q_2. \tag{5.2}$$

$$\begin{aligned} &((q_C/r_C)r)^2 = q_1 q_2\ \wedge\ F \propto m_C a_C(q_1 q_2) \\ &\quad\Rightarrow\quad F \propto m_C a_C((q_C/r_C)r)^2 = m_C a_C(q_1 q_2) \\ &\quad\quad\Rightarrow\quad F = m_C a_C = (m_C a_C r_C^2/q_C^2) q_1 q_2/r^2 = k_c q_1 q_2/r^2. \end{aligned} \tag{5.3}$$

where $k_C = m_C a_C r_C^2/q_C^2$ corresponds to the SI units: Nm^2C^{-2}. Where r and q can be varied independently, $F = m_0 a$ instead of $F = m_C a_C$.

5.2. Newton's gravity force equation. The sizes, m_1 and m_2, of two masses are independent domain intervals, where each infinitesimal size c subinterval of a mass exerts a quantum force, $m_G a_G$, on each infinitesimal size c subinterval of the other mass. The total force, F, is proportionate to the total number of forces (the Cartesian product of the size c components) multiplied times the quantum gravity force, $m_G a_G$. Applying the ruler, $p_1 = floor(m_1/c)$ and $p_2 = floor(m_2/c)$, and the Cartesian product, $p_1 \times p_2$, of size c components yields:

$$F \propto m_G a_G(\lim_{c\to 0} p_1 c \cdot \lim_{c\to 0} p_2 c) = m_G a_G \int_0^{m_2}\int_0^{m_1} \mathrm{d}^2 c = m_G a_G(m_1 m_2). \tag{5.4}$$

If each quantum mass size, m_G has a corresponding distance size, r_G, then the total charge size, m, is related to the total distance size, $r:\ m = (m_G/r_G)r$, where m_G/r_G is a unit-factoring conversion ratio:

$$\begin{aligned} &\forall\, m_1, m_2 \geq 0\, \exists\, m \in \mathbb{R}\ :\quad m^2 = m_1 m_2\quad \wedge\quad (m_G/r_G)r = m \\ &\qquad\qquad\qquad\Rightarrow\quad ((m_G/r_G)r)^2 = m_1 m_2. \end{aligned} \tag{5.5}$$

$$((m_G/r_G)r)^2 = m_1m_2 \;\wedge\; F \propto m_G a_G(m_1m_2) \tag{5.6}$$
$$\Rightarrow \quad F \propto m_G a_G((m_G/r_G)r)^2 = m_G a_G(m_1m_2)$$
$$\Rightarrow \quad F = m_G a_G = (a_G r_G^2/m_G)m_1m_2/r^2.$$

$$\exists\, t_G \in \mathbb{R}:\; r_G/t_G^2 = a_G \quad \wedge \quad F = (a_G r_G^2/m_G)m_1m_2/r^2 \tag{5.7}$$
$$\Rightarrow \quad F = (r_G^3/m_G t_G^2)m_1m_2/r^2 = Gm_1m_2/r^2,$$

where $G = r_G^3/m_G t_G^2$ corresponds to the SI units: $m^3kg^{-1}s^{-2}$. Where r and m can be varied independently, $F = m_0a$ instead of $F = m_Ga_G$.

5.3. Spacetime equations. The charge (5.3) and gravity (5.7) force equations were derived from the principle that charge and mass are proportionate to distance: $r = (r_C/q_C)q = (r_G/m_G)m$. If time is also proportionate to distance, then $r = (r_c/t_c)t = ct$, where $r_c/t_c = c$ is a unit-factoring conversion ratio.

Applying the ruler to two intervals, $[0, d_1]$ and $[0, d_2]$, in two inertial (independent, non-accelerating) frames of reference, the distance (and time) spanning the two domain intervals converges to a range of distances (and times) from Manhattan (3.3) to Euclidean distance (3.4).

$$r^2 = d_1^2 + d_2^2 \quad \wedge \quad r = (r_c/t_c)t = ct \quad \Rightarrow \quad (ct)^2 = d_1^2 + d_2^2 \tag{5.8}$$
$$\Leftrightarrow \quad d_1^2 = (ct)^2 - (x^2 + y^2 + z^2),$$

where $d_2^2 = x^2+y^2+z^2$, which is a form of Minkowski's well-known spacetime interval equation (in flat space) [**Bru17**]. And, the time dilation and length contraction equations also follow directly from $(ct)^2 = d_1^2 + d_2^2$.

5.4. 3 dimensional balls. Countable distance, $d_c = |\bigcup_{i=1}^n y_i|$, (3.1), countable volume, $d_c = |\times_{i=1}^n y_i|$, (4.1), Manhattan distance (3.3), Euclidean distance (3.4), and volume (4.2) requires a *strict total order* ($i = 1$ to n) of a set of intervals/dimensions. And the commutative properties of union, addition, and product allow sequencing through each interval (dimension) in every possible order. But, sequencing via the successor and predecessor relations of a strict totally ordered set in every possible order requires each set member to be sequentially adjacent (either a successor or predecessor) to every other member, herein referred to as a symmetric geometry.

It will now be proved that the constraint (coexistence) of symmetry on a sequentially ordered set defines a cyclic set that contains at most 3 members, in this case, 3 dimensions of ordered and symmetric distance and volume. If there are higher dimension of space, then the cyclic property prevents sequencing from the 3 lower, cyclic set of dimensions to any higher dimensions.

Definition 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i \;\in \{x_1, \ldots, x_n\},$$
$$\textit{successor } x_i = x_{i+1} \;\wedge\; \textit{predecessor } x_{i+1} = x_i.$$

Definition 5.2. Symmetric geometry (every set member is sequentially adjacent to every other member):

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; \textit{successor } x_i = x_j \;\Leftrightarrow\; \textit{predecessor } x_j = x_i.$$

THEOREM 5.3. *An ordered and symmetric set is a cyclic set.*

$$i = n \ \wedge\ j = 1 \ \Rightarrow\ \textit{successor } x_n = x_1 \ \wedge\ \textit{predecessor } x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$i = n \ \wedge\ j = 1 \ \wedge\ \textit{successor } x_i = x_j \ \Rightarrow\ \textit{successor } x_n = x_1. \tag{5.9}$$

Applying the definition of a symmetric geometry (5.2) to conclusion 5.9:

$$\textit{successor } x_i = x_j \ \Rightarrow\ \textit{predecessor } x_j = x_i \ \Rightarrow\ \textit{predecessor } x_1 = x_n. \tag{5.10}$$ □

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

DEFINITION 5.5. Cyclic successor of m is n:

$$\textit{Successor}(m, n, \textit{setsize}) \leftarrow (m = \textit{setsize} \wedge n = 1) \vee (n = m + 1 \le \textit{setsize}). \tag{5.11}$$

DEFINITION 5.6. Cyclic predecessor of m is n:

$$\textit{Predecessor}(m, n, \textit{setsize}) \leftarrow (m = 1 \wedge n = \textit{setsize}) \vee (n = m - q \ge 1). \tag{5.12}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:
(5.13)
$$\textit{Adjacent}(m, n, \textit{setsize}) \leftarrow \textit{Successor}(m, n, \textit{setsize}) \vee \textit{Predecessor}(m, n, \textit{setsize}).$$

Prove that every member is adjacent to every other member, where $\textit{setsize} \in \{1,\ 2,\ 3\}$:

$$\textit{Adjacent}(1, 1, 1) \leftarrow \textit{Successor}(1, 1, 1) \leftarrow (m = \textit{setsize} \wedge n = 1). \tag{5.14}$$

$$\textit{Adjacent}(1, 2, 2) \leftarrow \textit{Successor}(1, 2, 2) \leftarrow (n = m + 1 \le \textit{setsize}). \tag{5.15}$$

$$\textit{Adjacent}(2, 1, 2) \leftarrow \textit{Successor}(2, 1, 2) \leftarrow (n = \textit{setsize} \wedge m = 1). \tag{5.16}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m+1 \leq setsize). \tag{5.17}$$
$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m-q \geq 1). \tag{5.18}$$
$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \wedge m = 1). \tag{5.19}$$
$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \wedge n = setsize). \tag{5.20}$$
$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m+1 \leq setsize). \tag{5.21}$$
$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m-q \geq 1). \tag{5.22}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not $(\neg)$ adjacent:

$$\forall\, setsize > 3: \quad \neg Successor(1,3,setsize > 3) \\ \leftarrow Successor(1,2,setsize > 3) \leftarrow (n = m+1 \leq setsize). \tag{5.23}$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$\forall\, setsize > 3: \quad \neg Predecessor(1,3,setsize > 3) \\ \leftarrow Predecessor(1,setsize,setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \tag{5.24}$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$\forall\, setsize > 3: \quad \neg Adjacent(1,3,setsize > 3) \\ \leftarrow \neg Successor(1,3,setsize > 3) \wedge \neg Predecessor(1,3,setsize > 3). \quad \square \tag{5.25}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

(1) The countable distance (3.1), d_c, is a function of the domain-to-range set mappings, where the constraint, $|x_i| = |y_i| = p_i$, allows a range of domain-to-range set mappings from Manhattan distance, $d_c = \mathrm{f}(\sum_{i=1}^{n} 1 \cdot |y_i|) = \mathrm{f}(\sum_{i=1}^{n} p_i)$ (3.3) to Euclidean distance, $d_c = \mathrm{f}(\sum_{i=1}^{n} |x_i| \cdot |y_i|) = \mathrm{f}(\sum_{i=1}^{n} p_i^2)$ (3.4). The case where both domain-to-range set mappings, $\sum_{i=1}^{n} p_i$ and $\sum_{i=1}^{n} p_i^2$, coexist is: $d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c^2 = (\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$. The equality case is where the smallest distance, $d_c^2 = \sum_{i=1}^{n} p_i^2$, coexists with the largest (Manhattan) distance, $d_c = \sum_{i=1}^{n} p_i$, in flat space ($|x_i| = |y_i|$), which is the set-based reason Euclidean distance (3.4) is the smallest possible distance between two distinct points in $\mathbb{R}^n$.

(2) Generalizing the countable distance and volume constraint, $|x_i| = |y_i|$, to $|x_i| = |y_i|^q$, $q > -1$, generates all the L^p norms (Minkowski distances), $||L||_p = (\sum_{i=1}^{n} s_i^p)^{1/p}$. For example, using the same proof pattern as for Euclidean distance (3.4): $|y_i| = p_i \Rightarrow |x_i| = p_i^q \Rightarrow \sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^q \cdot p_i = \sum_{i=1}^{n} p_i^{q+1} \leq d_c^{q+1} \ \ldots\ d = (\sum_{i=1}^{n} s_i^{q+1})^{1/(q+1)}$.

(3) Manhattan distance is the largest distance and Euclidean volume is the largest volume in flat space (where $|x_i| = |y_i|$) because both are the case of disjoint range sets, $\bigcap_{i=1}^{n} y_i = \emptyset$. This is why each n-tuple of size c subintervals corresponding 1-1 to a unique, sub-Manhattan distance also

corresponds 1-1 to a unique sub-area/volume.

$$d_c = |\bigcup_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset.$$

$$v_c = | \times_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset.$$

(4) There are combinatorial relationships between countable sets of subintervals of intervals in statistics, probability, physics, etc., where the ruler is an applicable tool. For example, applying the ruler (2.1) and ruler convergence (2.2) to the Cartesian product of same-sized, infinitesimal charge forces and mass forces allowed deriving Coulomb's charge force (5.1) and Newton's gravity force (5.4) equations in a few steps each, without using other laws of physics or Gauss's divergence theorem.

(5) **The Proportionate Interval Principle:** The derivations of the charge force, gravity force, and spacetime equations shows that all Euclidean distance intervals having a size, r, have proportionately sized intervals of other types: $r = (r_C/q_C)q = (r_G/m_G)m = (r_c/t_c)t$, where the conversion ratios are for unit-factor analysis.
   (a) The derivations of charge and gravity forces requiring the conversion ratios, $q = (q_C/r_C)r$ and $m = (m_G/r_G)r$, implies that if there are quantum values of charge, q_C, and mass, m_G, then there are quantum distances, r_C and r_G, where the charge and gravity forces do not exist (are not defined) at smaller distances, which agrees with the theories of the Planck length, l_{P}, and the Schwarzschild radius, r_s.
   (b) The charge and gravity force derivations show that the proportionate interval principal generates the inverse square law, where rectangular geometric area (r^2) maps to rectangular charge area ($q_1 q_2$) and mass area ($m_1 m_2$). But, some versions of the charge constant, vacuum magnetic permeability constant, fine structure constant, Einstein's gravity constant, etc. contain the value 4π because the creators assumed geometric dilution (flux divergence on the surface of a sphere, $4\pi r^2$). Using Occam's razor, the proportionate interval principle is a more parsimonious derivation of the inverse square law than flux divergence. Therefore, those versions of the constants containing the value 4π might be incorrect.
   (c) Time proportionate to Euclidean distance, $r = ct \Rightarrow r^2 = (ct)^2 = d_1^2 + d_2^2$, (5.8) allows short and simple derivations of Minkowski's spacetime interval, time dilation, and length contraction equations.
   (d) $r^2 = (ct)^2 \quad \wedge \quad F = m_0 a = G m_1 m_2 / r^2 \quad \Rightarrow$
   $F = m_0 a = G m_1 m_2 / c^2 t^2 \quad \Rightarrow$
   $E = m_0 c^2 = G m_1 m_2 / t^2 a = G m_1 m_2 / t^2 (d/t^2) = G m_1 m_2 / d$. And $E = k_C q_1 q_2 / d$.
   (e) A countable set of values has measure 0. And because 0 times any distance is 0, there is no proportion relationship of a countable set of values to distance. Therefore, a countable set of state value changes with respect to time are independent of distance. For example, the change in the spin values of two quantum coupled particles and the change in polarization of two quantum coupled photons are independent of the distance between the coupled particles.

(6) Any higher dimensions of space not being sequentially reachable from the lower 3 dimensions because the lower 3 dimensions are a cyclic set is a more parsimonious explanation than the higher dimensions being rolled into infinitesimal balls, which requires an additional explanation of what causes the dimensions to be rolled up and additional equations describing the balls.

(7) If there are higher dimensions of space, then there is most likely an ordered and symmetric set of three members, each member being an ordered and symmetric set of dimensions (three boxes), yielding a total of 9 dimensions.

(8) The proof of at most 3 members in any ordered and symmetric set (5.4) has implications beyond only 3 ordered and symmetric dimensions of space.
   (a) Each *physical* infinitesimal volume (ball) can have at most 3 ordered and symmetric dimensions of discrete *physical* states of the same type, for example, a set of 3 binary values, 1 and -1, indicating vector orientation.
   (b) And each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values of the same type per ball.
   (c) Each of the three possible ordered and symmetric dimensions of discrete physical states could contain unordered sets (bags) of discrete state values, for example, unordered binary values. Bags (of states) are non-deterministic. For example, every time that an unordered binary state is physically measured, there is a 50 percent chance of having one of the binary values. Bags of discrete values might be a way to model some quantum physics.
   (d) Where infinitesimal (or perhaps Schwarzschild radius) balls intersect, an arithmetic of the interactions of the discrete states with respect to time needs to be developed. The interaction of the discrete states associated with intersecting balls with respect to time might result in what we perceive as motion, waves, particles, spin, polarization, work, force, mass, charge, etc.

References

[Bru17] P. Bruskiewich, *A very simple introduction to special relativity: Part two - four vectors, the lorentz transformation and group velocity (the new mathematics for the millions book 38)*, Pythagoras Publishing, 2017. ↑8

[CG15] W. Conradie and V. Goranko, *Logic and discrete mathematics*, Wiley, 2015. ↑3

[Coq15] Coq, *Coq proof assistant*, 2015. https://coq.inria.fr/documentation. ↑2

[Gol76] R. R. Goldberg, *Methods of real analysis*, John Wiley and Sons, 1976. ↑1

[Joy98] D. E. Joyce, *Euclid's elements*, 1998. http://aleph0.clarku.edu/~djoyce/java/elements/elements.html. ↑1

[Rud76] W. Rudin, *Principles of mathematical analysis*, McGraw Hill Education, 1976. ↑1

George Van Treeck, 668 Westline Dr., Alameda, CA 94501