The Two Set Relations Generating Geometry

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Abstract. A ruler (measuring stick) divides both domain and range intervals approximately into the nearest integer number of same-sized subintervals. As the subinterval size converges to zero: 1) Distance as the union size of countable range sets converges to: the triangle inequality with Manhattan distance at the upper boundary and Euclidean distance at the lower boundary. 2) The Cartesian product of the number of members in each domain set converges to the product of interval interval sizes (Euclidean area/volume). The ruler measure-based proofs of Euclidean distance and area/volume are used to derive the spacetime interval, charge force, and Newtonian gravity force equations. Time limits physical geometry to 3 dimensions. All proofs are verified in Coq.

Contents

1. Introduction

The properties of metric space, Euclidean distance, and the product of interval sizes (Euclidean area/volume) have been defined in real analysis [**Gol76**] [**Rud76**] rather than motivated and derived from set-based axioms. A "ruler" measure of intervals is used to prove that these geometric relations are motivated and derived from two set relations.

The derivation of geometric relations from set relations, *without notions of point, plane, side, angle, etc.*, identifies: 1) the single set relation generating the triangle inequality, non-negativity, and identity of indiscernibles properties of metric

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space; 2) the mapping between sets that makes Euclidean distance the smallest possible distance between two distinct points in $\mathbb{R}^n$; 3) the mapping between sets that makes distance different from area/volume; 4) the set-based reason the forces of charge and gravity vary inversely with the square of the distance between two objects; 5) how time places an additional constraint on physical sets, which limits physical geometry to 3 dimensions.

Proofs accepted by the Coq logic engine [**Coq15**] are internationally recognized to have a very high probability of being correct. All the proofs in this article have corresponding formal proofs in the Coq files, "euclidrelations.v" and "threed.v," located at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

A ruler (measuring stick) partitions both domain and range intervals *approximately* into the nearest integer number of subintervals, where each subinterval has the *same size*, c, with the consequence that different-sized intervals have a *different number* of subintervals. In contrast, the Riemann and Lebesgue integrals partition each domain interval and the range into the *same number* of subintervals, where different-sized intervals have *different-sized* subintervals [**Gol76**] [**Rud76**].

The ruler measure allows counting the number of mappings, ranging from a one-to-one correspondence to a many-to-many mapping, between the set of subintervals having size c in one interval and the set of subintervals having the same size c in another interval. The mapping (combinatorial) relations converge to continuous, bijective relations as the subinterval size, c, converges to zero.

Definition 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, p, each subinterval having the same size, c. Notionally:

$$\begin{aligned}(2.1)\quad & \forall\, c\, s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |a-b| \ \wedge\ c > 0\ \wedge \\ & \qquad (p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \textstyle\sum_{i=1}^{p} c = pc.\end{aligned}$$

Theorem 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\ \ s = |a-b|\ \Rightarrow\ M = \lim_{c\to 0} pc = s.$

The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y:\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$(2.2)\ \forall\, c > 0,\ p = floor(s/c)\ \wedge\ 0 \le |floor(s/c) - s/c| < 1\ \Rightarrow\ 0 \le |p - s/c| < 1.$$

Multiply all sides of inequality 2.2 by $|c|$:

$$(2.3)\qquad \forall\, c > 0,\quad 0 \le |p - s/c| < 1 \quad\Rightarrow\quad 0 \le |pc - s| < |c|.$$

$$\begin{aligned}(2.4)\quad & \forall\, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \\ & \qquad \Rightarrow\quad \forall\, \epsilon = \delta:\ \ |c - 0| < \delta\ \wedge\ |pc - s| < \epsilon\ \ :=\ \ M = \lim_{c\to 0} pc = s. \quad \square\end{aligned}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence, where: $[0,\pi]$, $s = |0 - \pi|$, $c = 10^{-i}$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{i=1},\ 3.14_{i=2},\ 3.141_{i=3}, ..., \pi$.

3. Distance

Notation convention: Curly brackets, $\{\cdots\}$, delimit a set; square brackets, $[\cdots]$, delimit a list; and vertical bars around a set or list, $|\cdots|$, indicates the cardinal (number of members in the set or list).

3.1. Countable distance space.

A simple measure of distance is the number of steps walked, which corresponds to an equal number of pieces of land. Abstracting, distance is prortionate to the number of members in a range set, y_i, which equals the number of members in a corresponding domain set, x_i: $|x_i| = |y_i|$. And the distance spanning multiple, disjoint, domain sets, $\bigcap_{i=1}^n x_i = \emptyset$, is proportionate to the number of members, d_c, in the union range set: $d_c = |\bigcup_{i=1}^n y_i|$.

Definition 3.1. Countable distance space, d_c:

$$\bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad d_c = |\bigcup_{i=1}^n y_i| \quad \wedge \quad |x_i| = |y_i|.$$

Theorem 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$.

This well-known inequality follows from the inclusion-exclusion principle [**CG15**]. But, a more intuitive and simple proof follows from the associative law of addition, which requires the sum of the set sizes to equal the size of all the set members appended into a list. And, by the commutative law of addition, the list can be sorted into a list of unique members (the union set) and a list of duplicate members. For example, $|\{a,b,c\}| + |\{c,d,e\}| = |[a,b,c,c,d,e]| = |\{a,b,c,d,e\}| + |[c]| \Rightarrow |\{a,b,c,d,e\}| = |\{a,b,c\}| + |\{c,d,e\}| - |[c]|$. The list of duplicates being ≥ 0 implies the union set size, $|\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$, the sum of the set sizes.

A formal proof, inclusion_exclusion_inequality, using sorting into a set of unique members (union set) and list of duplicate members, is in the file euclidrelations.v.

Proof. By the associative law of addition, append the sets into a list. Next, by the commutative law of addition, sort the list into uniques and duplicates, and then subtract duplicates from both sides:

$$\sum_{i=1}^n |y_i| = |append_{i=1}^n y_i| = |sort(append_{i=1}^n y_i)| \\ = |\bigcup_{i=1}^n y_i| + |duplicates_{i=1}^n y_i| \quad \Rightarrow \quad \sum_{i=1}^n |y_i| - |duplicates_{i=1}^n y_i| = |\bigcup_{i=1}^n y_i|. \tag{3.1}$$

$$|\bigcup_{i=1}^n y_i| = \sum_{i=1}^n |y_i| - |duplicates_{i=1}^n y_i| \quad \wedge \quad |duplicates_{i=1}^n y_i| \geq 0 \\ \Rightarrow \quad |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|. \quad \square \tag{3.2}$$

3.2. Metric Space.

All function range intervals, $d(u,w)$, satisfying the countable distance space definition, $d_c = |\bigcup_{i=1}^n y_i|$, where the ruler is applied, generates the three metric space properties: triangle inequality, non-negativity, and identity of indecernables. The fourth property of metric space, symmetry $[d(u,v) = d(v,u)]$, is motivated by Manhattan and Euclidean distance. The formal proofs: triangle_inequality, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

Theorem 3.3. *Triangle Inequality:* $d_c = |y_1 \cup y_2| \Rightarrow d(u,w) \leq d(u,v) + d(v,w)$.

Proof. Apply the ruler measure (2.1), the countable distance space condition (3.1), inclusion-exclusion inequality (3.2), and then ruler convergence (2.2).

$$\begin{aligned}
(3.3)\quad & \forall\, c>0,\ d(u,w),\ d(u,v),\ d(v,w)\ :\\
& |y_1| = floor(d(u,v)/c)\ \wedge\ |y_2| = floor(d(v,w)/c)\ \wedge\\
& d_c = floor(d(u,w)/c)\ \wedge\ d_c = |y_1 \cup y_2| \le |y_1| + |y_2|\\
& \Rightarrow\ floor(d(u,w)/c) \le floor(d(u,v)/c) + floor(d(v,w)/c)\\
& \Rightarrow\ floor(d(u,w)/c)\cdot c \le floor(d(u,v)/c)\cdot c + floor(d(v,w)/c)\cdot c\\
& \Rightarrow\ \lim_{c\to 0} floor(d(u,w)/c)\cdot c \le \lim_{c\to 0} floor(d(u,v)/c)\cdot c + \lim_{c\to 0} floor(d(v,w)/c)\cdot c\\
& \Rightarrow\ d(u,w) \le d(u,v) + d(v,w). \quad \square
\end{aligned}$$

Theorem 3.4. *Non-negativity:* $d_c = |y_1 \cup y_2| \Rightarrow d(u,w) \ge 0$.

Proof.

$$\begin{aligned}
(3.4)\quad & \forall\, c>0,\ d(u,w)\ :\quad floor(d(u,w)/c) = d_c \quad \wedge \quad d_c = |y_1 \cup y_2| \ge 0\\
& \Rightarrow \quad floor(d(u,w)/c) = d_c \ge 0 \quad \Rightarrow \quad d(u,w) = \lim_{c\to 0} d_c \cdot c \ge 0. \quad \square
\end{aligned}$$

Theorem 3.5. *Identity of Indiscernibles:* $d(w,w) = 0$.

Proof. Apply the triangle inequality property (3.3):

$$(3.5)\qquad \forall\, d(u,v) = d(v,w) = 0\ \wedge\ d(u,w) \le d(u,v) + d(v,w)\ \Rightarrow\ d(u,w) \le 0.$$

Combine the non-negativity property (3.4) and the previous inequality (3.5):

$$(3.6)\qquad d(u,w) \ge 0\ \wedge\ d(u,w) \le 0\ \Leftrightarrow\ 0 \le d(u,w) \le 0\ \Rightarrow\ d(u,w) = 0.$$

$$(3.7)\qquad d(u,w) = 0\ \wedge\ d(u,v) = 0\ \Rightarrow\ w = v.$$

$$(3.8)\qquad d(v,w) = 0\ \wedge\ w = v\ \Rightarrow\ d(w,w) = 0. \qquad \square$$

3.3. Distance space range. $d_c = |\bigcup_{i=1}^{n} y_i|$ implies that where the range sets intersect, multiple domain set members map to a single range set member. Therefore, d_c is a function of domain-to-range set member mappings.

From the countable distance space definition (3.1), $|x_i| = |y_i|$. Where $|x_i| = |y_i| = p_i = 1$, each of the p_i number of domain set members in x_i: 1) maps 1-1 (bijective) to a *single*, unique range set member in y_i, yielding $|x_i|\cdot 1 = p_i \cdot 1 = p_i = 1$ number of domain-to-range set mappings. 2) maps to *all* of the p_i number of range set members in y_i, yielding $|x_i|\cdot|y_i| = p_i \cdot p_i = p_i^2 = 1$ number of domain-to-range set mappings.

Therefore, the total number of domain-to-range set mappings ranges from $\sum_{i=1}^{n} p_i$ to $\sum_{i=1}^{n} p_i^2$. Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of domain-to-range set mapping cases converges to the real-valued, Manhattan and Euclidean distance relations.

3.4. Manhattan distance.

Theorem 3.6. *Manhattan (largest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "taxicab_distance," and formal proof is in the Coq file, euclidrelations.v.

Proof.
From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$d_c = |\bigcup_{i=1}^{n} y_i| \le \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i$$
$$\Rightarrow \quad d_c \le \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i. \tag{3.9}$$

Multiply both sides of equation 3.11 by c and take the limit:

$$d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \Rightarrow \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c). \tag{3.10}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d :\ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c. \tag{3.11}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of s_i:

$$\forall\, i \in [1,n],\, s_i = |a_i - b_i| \wedge floor(s_i/c) = |x_i| = |y_i| = p_i \Rightarrow \lim_{c\to 0} p_i \cdot c = s_i. \tag{3.12}$$

Combine equations 3.11, 3.10, 3.12:

$$d = \lim_{c\to 0} d_c \cdot c \quad \wedge \quad \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) \quad \wedge$$
$$\lim_{c\to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square \tag{3.13}$$

3.5. Euclidean distance.

Theorem 3.7. *Euclidean (smallest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "Euclidean_distance," and formal proof is in the Coq file, euclidrelations.v.

Proof.
Apply the rule of product to the largest number of domain-to-range set mappings, where all p_i number of domain set members, x_i, map to each of the p_i number of members in the range set, y_i:

$$\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2. \tag{3.14}$$

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$d_c = |\bigcup_{i=1}^{n} y_i| \le \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i$$
$$\Rightarrow \quad d_c \le \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i. \tag{3.15}$$

Square both sides of equation 3.15 ($x = y \Leftrightarrow f(x) = f(y)$):

$$\exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\ d_c :\ d_c^2 = (\sum_{i=1}^{n} p_i)^2. \tag{3.16}$$

Apply the Cauchy-Schwartz inequality to equation 3.16 and select the smallest distance (equality) case:

$$d_c^2 = (\sum_{i=1}^{n} p_i)^2 \ge \sum_{i=1}^{n} p_i^2 \quad \Rightarrow \quad \exists\, p_i :\ d_c^2 = \sum_{i=1}^{n} p_i^2. \tag{3.17}$$

Multiply both sides of equation 3.17 by c^2, simplify, and take the limit.

$$(3.18)\quad d_c^2 = \sum_{i=1}^{n} p_i^2 \;\Rightarrow\; d_c^2 \cdot c^2 = \sum_{i=1}^{n} p_i^2 \cdot c^2 \;\Leftrightarrow\; (d_c \cdot c)^2 = \sum_{i=1}^{n} (p_i \cdot c)^2$$
$$\Rightarrow\; \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.19)\quad \exists\, c\, d:\; floor(d/c) = d_c \;\Rightarrow\; d = \lim_{c\to 0} d_c \cdot c \;\Rightarrow\; d^2 = \lim_{c\to 0}(d_c \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:
(3.20)
$$\forall\, i \in [1,n],\; s_i = |a_i - b_i| \;\wedge\; floor(s_i/c) = |x_i| = |y_i| = p_i \;\Rightarrow\; \lim_{c\to 0}(p_i \cdot c) = s_i.$$

Combine equations 3.19, 3.18, 3.20:

$$(3.21)\quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 \;\wedge\; \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 \;\wedge$$
$$\lim_{c\to 0}(p_i \cdot c) = s_i \;\Rightarrow\; d^2 = \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \quad \square$$

4. Euclidean Volume

The number of all possible combinations (n-tuples) taking one member from each disjoint set is the Cartesian product of the number of members in each set. Notionally:

DEFINITION 4.1. Countable Volume, V_c:

$$\bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad V_c = \prod_{i=1}^{n} |x_i|.$$

THEOREM 4.2. *Euclidean volume, V, is size of the range interval, $[v_0, v_m]$, corresponding to all the possible combinations of the members of disjoint domain intervals, $\{[a_1,b_1],[a_2,b_2],\ldots,[a_n,b_n]\}$. Notionally:*

$$V = \prod_{i=1}^{n} s_i,\;\; V = |v_0 - v_m|,\;\; s_i = |a_i - b_i|.$$

The theorem, "Euclidean_volume," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, x_i of p_i number of subintervals.

$$(4.1)\quad \forall\, i\; n \in \mathbb{N},\quad i \in [1,n],\quad c > 0 \;\wedge\; floor(s_i/c) = p_i = |x_i|.$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$(4.2)\quad floor(s_i/c) = p_i \;\Rightarrow\; \lim_{c\to 0}(p_i \cdot c) = s_i.$$

Use the ruler (2.1) to divide the exact size, $V = |v_0 - v_m|$, of the range interval, $[v_0, v_m]$, into p^n subintervals. Use those cases, where V_c has an integer n^{th} root.

$$(4.3)\quad \forall\, p^n = V_c \in \mathbb{N},\, \exists\, V \in \mathbb{R},\, x_i\,:\, floor(V/c^n) = V_c = p^n = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i.$$

Apply the ruler convergence theorem (2.2) to equation 4.3 and simplify:

$$(4.4)\quad floor(V/c^n) = p^n \;\Rightarrow\; V = \lim_{c\to 0} p^n \cdot c^n = \lim_{c\to 0}(p \cdot c)^n.$$

Multiply both sides of equation 4.3 by c^n and simplify:

$$(4.5)\quad p^n = \prod_{i=1}^{n} p_i \;\Rightarrow\; p^n \cdot c^n = \Big(\prod_{i=1}^{n} p_i\Big) \cdot c^n \;\Leftrightarrow\; (p \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c)$$
$$\Rightarrow\; \lim_{c\to 0}(p \cdot c)^n = \prod_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)$$

Combine equations 4.4, 4.5, and 4.2:

$$V = \lim_{c\to 0}(p \cdot c)^n \quad \wedge \quad \lim_{c\to 0}(p \cdot c)^n = \prod_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) \quad \wedge \quad \lim_{c\to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad V = \lim_{c\to 0}(p \cdot c)^n = \prod_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square \tag{4.6}$$

5. Applying the ruler measure to physics

Apply the ruler to two independent domain intervals, $[0, d_1]$ and $[0, d_2]$, and the range interval, $[0, D]$, where each subinterval of $[0, D]$ corresponds to both a nonzero number of subintervals in $[0, d_1]$ and a nonzero number of subintervals of $[0, d_2]$. This is the case that converges to the Euclidean distance equation: $D^2 = d_1^2 + d_2^2$ (3.7).

For any interval, $[0, D]$, there is a proportionately sized interval, $[0, t]$, such that: $ct = D$, where the proportionality constant, c, is the ratio of a distance, d_c, and some value, t_c:

$$\forall\, D,\ d_1,\ d_2,\ t\ \in \mathbb{R} : \quad D^2 = d_1^2 + d_2^2 \\ \exists\, d_c,\ t_c\ c,\ \in \mathbb{R} :\ D = (d_c/t_c)t = ct\ \Rightarrow\ D^2 = (ct)^2 = d_1^2 + d_2^2. \tag{5.1}$$

$$(ct)^2 = d_1^2 + d_2^2.\ \Rightarrow\ d_2 = \sqrt{(ct)^2 - d_1^2}. \tag{5.2}$$

$$d = d_2\ \wedge\ d_1^2 = x^2 + y^2 + z^2\ \Rightarrow\ d = \sqrt{(ct)^2 - (x^2 + y^2 + z^2)}, \tag{5.3}$$

which is the four-vector length of the spacetime interval (relativistic change in 3-dimensional Euclidean distance) [**Bru17**].

Apply the ruler to two independent domain intervals having the sizes q_1 and q_2. Each subinterval of $[0, q_1]$ is a force that affects (corresponds to) all the subintervals in $[0, q_2]$. The number of possible correspondences (affects) is the Cartesian product of the number of subintervals in each interval. And applying the volume proof (3.3), the Cartesian product of the subinterval sizes converges to the area formula, $q_1 \cdot q_2$, as the subinterval size converges to zero.

For any interval, $[0, q]$, there is a proportionately sized interval, $[0, r]$, such that: $r(q_C/r_C) = q$.

$$\forall\, q_C,\ r_C,\ r,\ q,\ q_1,\ q_2 \in \mathbb{R} : \quad r(q_C/r_C)) = q \quad \wedge \quad q^2 = q_1 q_2 \\ \Rightarrow \quad (r(q_C/r_C))^2 = q^2 = q_1 q_2\ \Rightarrow\ 1 = (r_C^2/q_C^2)q_1 q_2/r^2. \tag{5.4}$$

Use force ($F = ma$) ratios equal to the scalar (unit-less) value one:

$$\exists\, m_0,\ m_C,\ a,\ a_C \in \mathbb{R}\ :\ (m_0 a/m_C a_C) = 1 = (r_C^2/q_C^2)q_1 q_2/r^2. \tag{5.5}$$

Multiplying both sides of equation 5.5 by $m_C a_C$ yields the charge force equation:

$$(m_0 a/m_C a_C)(m_C a_C) = (r_C^2/q_C^2)(q_1 q_2 r^2)(m_C a_C) \\ \Rightarrow \quad \exists\, F,\ k_C \in \mathbb{R} : \quad F = m_0 a = (m_C a_C r_C^2/q_C^2)q_1 q_2/r^2 = k_C q_1 q_2/r^2. \tag{5.6}$$

Using intervals of type, $[0, m]$, that are proportionate in size to $[0, r]$, yields the Newtonian gravity equation:

$$\exists\, m_G,\ a_G.\ F,\ G \in \mathbb{R} : \\ F = m_0 a = (m_G a_G r_g^2/m_g^2)m_1 m_2/r^2 = G m_1 m_2/r^2. \tag{5.7}$$

6. Ordered and symmetric geometries

The set and arithmetic operations used to calculate distance and volume requires sequencing through a totally ordered set of dimensions. For example, from Euclidean distance, $d^2 = \sum_{i=1}^{n} s_i^2$, where s_i is the size of a domain interval from a dimension i of intervals. The commutative property of the set and arithmetic operations also allows sequencing through n number of dimensions in all $n!$ number of possible orders.

But, a *physical* deterministic sequencer requires a *physical* set to have a single total order, at most one successor and at most one predecessor per set member, during the *time* of sequencing. Assigning a total order, $[1,\ 2,\ \cdots,\ n]$, is the only way to determine that one sequencer traversed in the order, $[2,\ 1,\ \cdots]$, and another sequencer traversed in the order, $[1,\ 2,\ \cdots]$. Deterministic sequencing in every possible order via the same successor/predecessor relations (same total order) requires each set member to be either a successor or predecessor to every other set member (mutually adjacent), herein referred to as a symmetric geometry.

It will now be proved that a set satisfying the constraints of a single total order and also symmetric defines a cyclic set containing at most 3 members, in this case, 3 dimensions of physical space.

Definition 6.1. Ordered geometry:

$$\forall\, i\ n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i\ \in \{x_1, \ldots, x_n\},$$
$$\textit{successor}\ x_i = x_{i+1}\ \ \wedge\ \ \textit{predecessor}\ x_{i+1} = x_i.$$

Definition 6.2. Symmetric geometry (every set member is sequentially adjacent to any other member):

$$\forall\, i\ j\ n \in \mathbb{N},\ \ \forall\, x_i\ x_j \in \{x_1, \ldots, x_n\},\ \ \textit{successor}\ x_i = x_j\ \ \Leftrightarrow\ \ \textit{predecessor}\ x_j = x_i.$$

Theorem 6.3. *An ordered and symmetric set is a cyclic set.*

$$\textit{successor}\ x_n = x_1\ \ \wedge\ \ \textit{predecessor}\ x_1 = x_n.$$

The theorem, "ordered_symmetric_is_cyclic," and formal proof is in the Coq file, threed.v.

Proof. A total order (6.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (6.2):

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \textit{successor}\ x_i = x_j\ \ \Rightarrow\ \ \textit{successor}\ x_n = x_1. \tag{6.1}$$

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \textit{predecessor}\ x_j = x_i\ \ \Rightarrow\ \ \textit{predecessor}\ x_1 = x_n. \tag{6.2}$$

□

Theorem 6.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

Proof.
It was proved that an ordered and symmetric set is a cyclic set (6.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

Definition 6.5. Cyclic successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize). \tag{6.3}$$

Definition 6.6. Cyclic predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1). \tag{6.4}$$

Definition 6.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{6.5}$$

Every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \tag{6.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \tag{6.7}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1). \tag{6.8}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize). \tag{6.9}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \tag{6.10}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1). \tag{6.11}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize). \tag{6.12}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize). \tag{6.13}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \tag{6.14}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Successor(1, 3, setsize > 3) \\ & \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \end{aligned} \tag{6.15}$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Predecessor(1, 3, setsize > 3) \\ & \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \end{aligned} \tag{6.16}$$

That is, member $n > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $n > 3$.

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Adjacent(1, 3, setsize > 3) \\ & \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \end{aligned} \tag{6.17}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

7. Insights and implications

Applying the ruler measure (2.1) and ruler convergence (2.2) to the set relations, countable distance space (3.1) and countable volume (4.1) yields the following insights and implications:

(1) Notions of point, plane, side, angle, perpendicular, congruence, intersection, etc. are not necessary to motivate and derive the properties of metric space, Euclidean distance and area/volume.

(2) The Riemann and Lebesgue integrals sum infinitesimal Euclidean areas and volumes, where area/volume is defined rather than derived from a set and number theory. The line integral sums infinitesimal Euclidean lengths, where Euclidean distance is defined rather than derived from set and number theory. The ruler measure-based proofs of Euclidean distance (3.7) and area/volume (4.2) put a more complete set and number theory-based foundation under integral calculus.

(3) The ruler measure-based proofs provide the insight that distance is a function of the number of domain-to-*range* set member mappings. Whereas, area/volume is a function of the number of domain-to-*domain* set member mappings.

    Classical geometry[**Joy98**], axiomatic geometry (for example, Hilbert [**Hil80**], Birkhoff [**Bir32**], Veblen [**Veb04**], and Tarski [**TG99**]), calculus [**Rud76**], and previous measure theories (like Lebesgue) have not provided those set mapping insights about distance and area/volume.

(4) The distance spanning multiple, disjoint, domain sets is proportionate to the number of members, d_c, in the corresponding union range set: $d_c = |\bigcup_{i=1}^{n} y_i|$ (3.1), which generates the triangle inequality, non-negativity, and identity of indiscernibles properties of metric space (3.2) and also generates Manhattan and Euclidean distance. The fourth property of metric space, symmetry $[d(u,v) = d(v,u)]$, is motivated by Manhattan and Euclidean distance.

(5) The range of domain-to-range set mappings is from $\sum_{i=1}^{n} p_i$ to $\sum_{i=1}^{n} p_i^2$, which is the range of sums of 1-1 correspondences to sums of maximum many-to-many mappings between each domain set, x_i, and a corresponding same-sized range set, y_i: $|x_i| = |y_i|$. Every symmetric $[d(u,v) = d(v,u)]$, hyperbolic space distance, $d(u,v) : \sqrt{u^2+v^2} < d(u,v) \leq |u|+|v|$, are of the form, $d^{2/k} = \sum_{i=1}^{n} s_i^{2/k}$, where $1 < k \leq 2$. In order to be symmetric, the type of domain-to-range set mapping must be the same for each domain set and corresponding range set, which is satisfied by the constant, k.

(6) The case of the largest possible number of domain-to-range set member mappings, $\sum_{i=1}^{n} p_i^2$, is the set-based reason that Euclidean distance (3.7) is the smallest possible distance between two distinct points in $\mathbb{R}^n$.

(7) The spacetime interval, charge force, and Newtonian gravity force equations were derived using the same ruler measure and combinatorics that was also used in the derivation of Euclidean area/volume and distance.
    (a) The derivations of the charge and gravity equations both have ratio (proportion) constants (for charge: $r(q_C/r_C) = q$, and mass: $r(m_g/r_g) = m$). If there are quantum values of charge, q_C, and mass, m_g, then there are lower bound, quantum distances, r_C and

r_g. In other words, there could be lower bound distances at which the charge and gravity forces between two objects reach finite maximums (the forces at smaller distances are not defined). The notion of particle might be related to such quantum distances. The empirical charge and gravity equations do not provide that lower bound/quantum distance insight.

   (b) Using the ruler measure with counting methods (combinatorics, compounding, series, probability, etc.) might also be a useful tool to derive set-based models and equations at the subatomic level.
   (c) The notion of sets of ordered members is critical to deriving geometric relations. But, sets having no order of members (bags) might be a way to model some subatomic behavior, where the behavior might be described by probability equations.

(8) Time constrains a physical set of totally ordered members, where the commutative law applies to the set and arithmetic operations, to at most three members (6.4), for example, three dimensions of physical space. Of course, relativity theory assumes only 3 dimensions of space [**Bru17**].
(9) Note that Euclidean distance (3.7) and volume (4.2) are range sets. All compressions, expansions, ripples, bends, bubbles, tunnels, etc. are in the range set space, where the range set is also a function of other variables.
   (a) Particles, waves, energy, force, etc. are range set phenomena that are projected onto (viewed from) our local, domain, Euclidean frame of reference.
   (b) An expanding universe would probably be an expansion of the range set space.

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