The Two Set Relations Generating Geometry

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Abstract. A ruler (measuring stick) divides both domain and range intervals approximately into the nearest integer number of same-sized subintervals. As the subinterval size converges to zero: 1) Distance as the union size of range sets converges to: the triangle inequality with Manhattan distance at the upper boundary and Euclidean distance at the lower boundary. 2) The Cartesian product of the number of members in each domain set converges to the product of interval interval sizes (Euclidean area/volume). The ruler measure-based proofs of Euclidean distance and area/volume are used to derive the spacetime, charge force, and Newtonian gravity force equations. Time limits physical geometry to 3 dimensions. All proofs are verified in Coq.

Contents

1. Introduction

Real analysis textbooks start with the step-by-step building of a foundation of continuity, convergence, limit, etc. from set definitions. But, metric space, Euclidean distance (metric/vector norm), and Euclidean area/volume (product of interval sizes) are defined in real analysis [**Gol76**] [**Rud76**] rather than built by applying that previously built foundation to yet more set definitions.

A "ruler" measure of intervals applies convergence to two types of set relations to derive metric space, Euclidean distance, and volume, which provides some new insights into geometry and physics, for example: 1) the single set relation generating the triangle inequality, non-negativity, and identity of indiscernibles properties

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of metric space; 2) the mapping between sets that makes Euclidean distance the smallest possible distance between two distinct points in $\mathbb{R}^n$; 3) the mapping between sets that makes distance different from area/volume; 4) the set-based reason the forces of charge and gravity vary inversely with the square of the distance between two infinitesimal objects; 5) how time places an additional constraint on those same two set relations, which limits physical geometry to 3 dimensions.

All the proofs in this article have been formally verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," located at:

`https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

A ruler (measuring stick) partitions both domain and range intervals *approximately* into the nearest integer number of subintervals, where each subinterval has the *same size*, c, with the consequence that different-sized intervals have a *different number* of subintervals. In contrast, the Riemann and Lebesgue integrals partition each domain interval and the range into the *same number* of subintervals, where different-sized intervals have *different-sized* subintervals [**Gol76**] [**Rud76**].

The ruler measure allows counting the number of mappings, ranging from a one-to-one correspondence to a many-to-many mapping, between the set of subintervals having size c in one interval and the set of subintervals having the same size c in another interval. The mapping (combinatorial) relations converge to continuous, bijective relations as the subinterval size, c, converges to zero.

Definition 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, p, each subinterval having the same size, c. Notionally:

$$\begin{aligned}(2.1)\quad &\forall\, c\, s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |a-b| \ \ \wedge\ \ c > 0\ \wedge \\ &\qquad (p = floor(s/c)\ \ \vee\ \ p = ceiling(s/c))\ \ \wedge\ \ M = \textstyle\sum_{i=1}^{p} c = pc.\end{aligned}$$

Theorem 2.2. *Ruler convergence:*
$\forall\ [a,b] \subset \mathbb{R},\ \ s = |a-b|\ \ \Rightarrow\ \ M = \lim_{c\to 0} pc = s.$

The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$(2.2)\ \forall\, c > 0,\ p = floor(s/c)\ \wedge\ 0 \le |floor(s/c) - s/c| < 1\ \Rightarrow\ 0 \le |p - s/c| < 1.$$

Multiply all sides of inequality 2.2 by $|c|$:

$$(2.3)\qquad \forall\, c > 0,\quad 0 \le |p - s/c| < 1 \quad\Rightarrow\quad 0 \le |pc - s| < |c|.$$

$$\begin{aligned}(2.4)\quad &\forall\, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \\ &\qquad \Rightarrow\quad \forall\, \epsilon = \delta :\ \ |c-0| < \delta\ \ \wedge\ \ |pc - s| < \epsilon\ \ :=\ \ M = \lim_{c\to 0} pc = s. \qquad \square\end{aligned}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence, where: $[0,\pi]$, $s = |0-\pi|$, $c = 10^{-i}$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{i=1},\ 3.14_{i=2},\ 3.141_{i=3}, ..., \pi$.

3. Distance

Notation convention: Curly brackets, $\{\cdots\}$, delimit a set; square brackets, $[\cdots]$, delimit a list; and vertical bars around a set or list, $|\cdots|$, indicates the cardinal (number of members in the set or list).

3.1. Countable distance space.

A simple measure of distance is the number of steps walked, which corresponds to an equal number of pieces of land. Abstracting, distance is proportionate to the number of members in a range set, y_i, which equals the number of members in a corresponding domain set, x_i: $|x_i| = |y_i|$. And the distance spanning multiple, disjoint, domain sets, $\bigcap_{i=1}^n x_i = \emptyset$, is proportionate to the number of members, d_c, in the union range set: $d_c = |\bigcup_{i=1}^n y_i|$.

Definition 3.1. Countable distance space, d_c:

$$\bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad d_c = |\bigcup_{i=1}^n y_i| \quad \wedge \quad |x_i| = |y_i|.$$

Theorem 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$.

This well-known inequality follows from the inclusion-exclusion principle [**CG15**]. But, a more intuitive and simple proof follows from the associative law of addition, which requires the sum of the set sizes to equal the size of all the set members appended into a list. And, by the commutative law of addition, the list can be sorted into a list of unique members (the union set) and a list of duplicate members. For example, $|\{a,b,c\}| + |\{c,d,e\}| = |[a,b,c,c,d,e]| = |\{a,b,c,d,e\}| + |[c]| \Rightarrow |\{a,b,c,d,e\}| = |\{a,b,c\}| + |\{c,d,e\}| - |[c]|$. The list of duplicates being ≥ 0 implies the union set size, $|\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$, the sum of the set sizes.

A formal proof, inclusion_exclusion_inequality, using sorting into a set of unique members (union set) and list of duplicate members, is in the file euclidrelations.v.

Proof. By the associative law of addition, append the sets into a list. Next, by the commutative law of addition, sort the list into uniques and duplicates, and then subtract duplicates from both sides:

$$\begin{aligned}(3.1)\quad & \sum_{i=1}^n |y_i| = |append_{i=1}^n y_i| = |sort(append_{i=1}^n y_i)| \\ &= |\bigcup_{i=1}^n y_i| + |duplicates_{i=1}^n y_i| \quad \Rightarrow \quad \sum_{i=1}^n |y_i| - |duplicates_{i=1}^n y_i| = |\bigcup_{i=1}^n y_i|.\end{aligned}$$

$$\begin{aligned}(3.2)\quad & |\bigcup_{i=1}^n y_i| = \sum_{i=1}^n |y_i| - |duplicates_{i=1}^n y_i| \quad \wedge \quad |duplicates_{i=1}^n y_i| \geq 0 \\ & \Rightarrow \quad |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|. \quad \square\end{aligned}$$

3.2. Metric Space.

All function range intervals, $d(u,w)$, satisfying the countable distance space definition, $d_c = |\bigcup_{i=1}^n y_i|$, where the ruler is applied, generates the three metric space properties: triangle inequality, non-negativity, and identity of indecernables. The fourth property of metric space, symmetry $[d(u,v) = d(v,u)]$, is motivated by Manhattan and Euclidean distance. The formal proofs: triangle_inequality, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

Theorem 3.3. *Triangle Inequality:* $d_c = |y_1 \cup y_2| \Rightarrow d(u,w) \leq d(u,v) + d(v,w)$.

PROOF. Apply the ruler measure (2.1), the countable distance space condition (3.1), inclusion-exclusion inequality (3.2), and then ruler convergence (2.2).

$$\begin{aligned}(3.3)\quad \forall\, c>0,\ d(u,w),\ d(u,v),\ d(v,w)\ :&\\ |y_1| = floor(d(u,v)/c)\ \wedge\ |y_2| = floor(d(v,w)/c)\ \wedge&\\ d_c = floor(d(u,w)/c)\quad \wedge\quad d_c = |y_1 \cup y_2| \le |y_1| + |y_2|&\\ \Rightarrow\ floor(d(u,w)/c) \le floor(d(u,v)/c) + floor(d(v,w)/c)&\\ \Rightarrow\ floor(d(u,w)/c)\cdot c \le floor(d(u,v)/c)\cdot c + floor(d(v,w)/c)\cdot c&\\ \Rightarrow\ \lim_{c\to 0} floor(d(u,w)/c)\cdot c \le \lim_{c\to 0} floor(d(u,v)/c)\cdot c + \lim_{c\to 0} floor(d(v,w)/c)\cdot c&\\ \Rightarrow\ d(u,w) \le d(u,v) + d(v,w).&\quad \square\end{aligned}$$

THEOREM 3.4. *Non-negativity:* $d_c = |y_1 \cup y_2| \Rightarrow d(u,w) \ge 0$.

PROOF.

$$\begin{aligned}(3.4)\quad \forall\, c>0,\ d(u,w)\ :\quad floor(d(u,w)/c) = d_c \quad \wedge \quad d_c = |y_1 \cup y_2| \ge 0&\\ \Rightarrow\quad floor(d(u,w)/c) = d_c \ge 0 \quad \Rightarrow \quad d(u,w) = \lim_{c\to 0} d_c \cdot c \ge 0.&\quad \square\end{aligned}$$

THEOREM 3.5. *Identity of Indiscernibles:* $d(w,w) = 0$.

PROOF. Apply the triangle inequality property (3.3):

$$(3.5)\qquad \forall\, d(u,v) = d(v,w) = 0\ \wedge\ d(u,w) \le d(u,v) + d(v,w)\ \Rightarrow\ d(u,w) \le 0.$$

Combine the non-negativity property (3.4) and the previous inequality (3.5):

$$(3.6)\qquad d(u,w) \ge 0\ \wedge\ d(u,w) \le 0\ \Leftrightarrow\ 0 \le d(u,w) \le 0\ \Rightarrow\ d(u,w) = 0.$$

$$(3.7)\qquad d(u,w) = 0\ \wedge\ d(u,v) = 0\ \Rightarrow\ w = v.$$

$$(3.8)\qquad d(v,w) = 0\ \wedge\ w = v\ \Rightarrow\ d(w,w) = 0. \qquad \square$$

3.3. Distance space range. $d_c = |\bigcup_{i=1}^n y_i|$ implies that distance varies inversely to the amount of intersection of the range sets. As the amount of intersection increases, a single range set member can map to more domain set members. Therefore, d_c is a function of range-to-domain set member mappings.

From the countable distance space property (3.1), where $|x_i| = |y_i| = p_i$, the total number of range-to-domain set member mappings vary from the sum of one-to-one correspondences (no intersection and largest distance), $\sum_{i=1}^n x_i \cdot 1 = \sum_{i=1}^n p_i$, to the sum of the most many-to-many correspondences (largest intersection and smallest possible distance), $\sum_{i=1}^n |x_i| \cdot |y_i| = \sum_{i=1}^n p_i^2$. Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of range-to-domain set mapping cases converges to the real-valued, Manhattan and Euclidean distance relations.

3.4. Manhattan distance.

THEOREM 3.6. *Manhattan (largest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$d = \sum_{i=1}^n s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "taxicab_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \\ \Rightarrow \quad d_c \leq \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c : \; d_c = \sum_{i=1}^{n} p_i. \tag{3.9}$$

Multiply both sides of equation 3.11 by c and take the limit:

$$d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \Rightarrow \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c). \tag{3.10}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d : \; floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c. \tag{3.11}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of s_i:

$$\forall\, i \in [1,n],\, s_i = |a_i - b_i| \wedge floor(s_i/c) = |x_i| = |y_i| = p_i \Rightarrow \lim_{c\to 0} p_i \cdot c = s_i. \tag{3.12}$$

Combine equations 3.11, 3.10, 3.12:

$$d = \lim_{c\to 0} d_c \cdot c \quad \wedge \quad \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) \quad \wedge \\ \lim_{c\to 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square \tag{3.13}$$

3.5. Euclidean distance.

THEOREM 3.7. *Euclidean (smallest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "Euclidean_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
Apply the rule of product to the largest number of range-to-domain set mappings, where all p_i number of range set members, y_i, map to each of the p_i number of members in the domain set, x_i:

$$\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2. \tag{3.14}$$

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \\ \Rightarrow \quad d_c \leq \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c : \; d_c = \sum_{i=1}^{n} p_i. \tag{3.15}$$

Square both sides of equation 3.15 ($x = y \Leftrightarrow f(x) = f(y)$):

$$\exists\, p_i,\, d_c : \; d_c = \sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\, d_c : \; d_c^2 = (\sum_{i=1}^{n} p_i)^2. \tag{3.16}$$

Apply the Cauchy-Schwartz inequality, $(\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$, to equation 3.16 and select the smallest area (the equality) case:

$$\begin{aligned}(3.17)\quad & \forall\, p_i,\ p_j \in \{p_1, \cdots, p_n\} \quad \wedge \quad p_i,\ p_j \geq 0 \quad \wedge \quad d_c = \textstyle\sum_{i=1}^{n} p_i \\ & \Rightarrow \quad d_c^2 = (\textstyle\sum_{i=1}^{n} p_i)^2 = (\sum_{j=1}^{n} p_j)\cdot(\sum_{i=1}^{n} p_i) = \sum_{j=1}^{n}(p_j \cdot \sum_{i=1}^{n} p_i) \\ & \qquad = \textstyle\sum_{j=1,\ i=j}^{n}(p_j \cdot p_i) + \sum_{j=1,\ i\neq j}^{n}(p_j \cdot \sum_{i=1}^{n} p_i) \\ & \qquad = \textstyle\sum_{i=1}^{n} p_i^2 + \sum_{j=1,\ i\neq j}^{n}(p_j \cdot \sum_{i=1}^{n} p_i) \geq \sum_{i=1}^{n} p_i^2 \\ & \qquad\qquad \Rightarrow \quad \exists\, p_i :\ d_c^2 = \textstyle\sum_{i=1}^{n} p_i^2.\end{aligned}$$

Multiply both sides of equation 3.17 by c^2, simplify, and take the limit.

$$\begin{aligned}(3.18)\quad & d_c^2 = \textstyle\sum_{i=1}^{n} p_i^2 \;\Rightarrow\; d_c^2 \cdot c^2 = \sum_{i=1}^{n} p_i^2 \cdot c^2 \;\Leftrightarrow\; (d_c \cdot c)^2 = \sum_{i=1}^{n}(p_i \cdot c)^2 \\ & \qquad \Rightarrow \quad \lim_{c\to 0}(d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2.\end{aligned}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.19)\qquad \exists\, c\, d :\ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$(3.20)\qquad \forall\, i \in [1, n],\ s_i = |a_i - b_i| \;\wedge\; floor(s_i/c) = |x_i| = |y_i| = p_i \;\Rightarrow\; \lim_{c\to 0}(p_i \cdot c) = s_i.$$

Combine equations 3.19, 3.18, 3.20:

$$\begin{aligned}(3.21)\quad & d^2 = \lim_{c\to 0}(d_c \cdot c)^2 \quad \wedge \quad \lim_{c\to 0}(d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 \quad \wedge \\ & \lim_{c\to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \quad \square\end{aligned}$$

4. Euclidean Volume

The number of all possible combinations (n-tuples) taking one member from each disjoint set is the Cartesian product of the number of members in each set. Notionally:

Definition 4.1. Countable Volume, V_c:

$$\textstyle\bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad V_c = \prod_{i=1}^{n} |x_i|.$$

Theorem 4.2. *Euclidean volume, V, is size of the range interval, $[v_0, v_m]$, corresponding to all the possible combinations of the members of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$. Notionally:*

$$V = \textstyle\prod_{i=1}^{n} s_i,\ \ V = |v_0 - v_m|,\ \ s_i = |a_i - b_i|.$$

The theorem, "Euclidean_volume," and formal proof is in the Coq file, euclidrelations.v.

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, x_i of p_i number of subintervals.

$$(4.1)\qquad \forall\, i\ n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i|.$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$(4.2)\qquad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0}(p_i \cdot c) = s_i.$$

Use the ruler (2.1) to divide the exact size, $V = |v_0 - v_m|$, of the range interval, $[v_0, v_m]$, into p^n subintervals. Use those cases, where V_c has an integer n^{th} root.

$$(4.3)\quad \forall\, p^n = V_c \in \mathbb{N}, \exists\, V \in \mathbb{R}, x_i : floor(V/c^n) = V_c = p^n = \prod_{i=1}^n |x_i| = \prod_{i=1}^n p_i.$$

Apply the ruler convergence theorem (2.2) to equation 4.3 and simplify:

$$(4.4)\qquad floor(V/c^n) = p^n \quad \Rightarrow \quad V = \lim_{c\to 0} p^n \cdot c^n = \lim_{c\to 0}(p \cdot c)^n.$$

Multiply both sides of equation 4.3 by c^n and simplify:

$$(4.5)\quad p^n = \prod_{i=1}^n p_i \quad \Rightarrow \quad p^n \cdot c^n = (\prod_{i=1}^n p_i) \cdot c^n \quad \Leftrightarrow \quad (p \cdot c)^n = \prod_{i=1}^n (p_i \cdot c)$$
$$\Rightarrow \quad \lim_{c\to 0}(p \cdot c)^n = \prod_{i=1}^n \lim_{c\to 0}(p_i \cdot c)$$

Combine equations 4.4, 4.5, and 4.2:

$$(4.6)\quad V = \lim_{c\to 0}(p \cdot c)^n \quad \wedge \quad \lim_{c\to 0}(p \cdot c)^n = \prod_{i=1}^n \lim_{c\to 0}(p_i \cdot c) \quad \wedge$$
$$\lim_{c\to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad V = \lim_{c\to 0}(p \cdot c)^n = \prod_{i=1}^n \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^n s_i. \quad \square$$

5. Applications to physics

5.1. Spacetime equations. Apply the ruler to two independent domain intervals, $[0, d_1]$ and $[0, d_2]$, and the range interval, $[0, D]$. For any interval, $[0, D]$, there is a proportionately sized interval, $[0, t]$, such that: $ct = D$, where the proportionality constant, c, is the ratio of a distance, d_c, and some value, t_c.

If each subinterval of $[0, t]$ corresponding to a proportionate subinterval of $[0, D]$, which corresponds to subintervals in $[0, d_1]$ and subintervals in $[0, d_2]$, then this is the case that converges to the Euclidean distance equation: $D^2 = d_1^2 + d_2^2$ (3.7).

$$(5.1)\quad D^2 = d_1^2 + d_2^2 \quad \wedge \quad D = (d_c/t_c)t = ct$$
$$\Rightarrow \quad D^2 = (ct)^2 = d_1^2 + d_2^2 \quad \Rightarrow \quad d_2 = \sqrt{(ct)^2 - d_1^2}.$$

$$(5.2)\quad d_2 = \sqrt{(ct)^2 - d_1^2} \quad \wedge \quad d = d_2 \quad \wedge \quad d_1 = vt$$
$$\Rightarrow \quad d = \sqrt{(ct)^2 - (vt)^2} = ct\sqrt{1 - (v^2/c^2)},$$

which is the spacetime dilation equation. [**Bru17**].

$$(5.3)\quad d_2 = \sqrt{(ct)^2 - d_1^2} \quad \wedge \quad d = d_2 \quad \wedge \quad d_1^2 = x^2 + y^2 + z^2$$
$$\Rightarrow \quad d = \sqrt{(ct)^2 - (x^2 + y^2 + z^2)},$$

which is the four-vector length of the spacetime interval (relativistic change in 3-dimensional Euclidean distance) [**Bru17**].

5.2. Charge and gravity force equations. Apply the ruler to the two independent domain intervals, $[0, q_1]$ and $[0, q_2]$. Each subinterval of $[0, q_1]$ is a force that interacts (corresponds to) all the subintervals in $[0, q_2]$. The number of possible correspondences (interactions) is the Cartesian product of the number of subintervals in each interval. And applying the volume proof (3.3), the Cartesian product of the subinterval sizes converges to the area formula, $q_1 \cdot q_2$, as the subinterval size converges to zero.

For any interval, $[0, q]$, there is a proportionately sized interval, $[0, r]$:

$$\text{(5.4)}\quad \forall\, q_1q_2 \in \mathbb{R}\, \exists\, q \in \mathbb{R} : \quad q^2 = q_1q_2 \quad \wedge \quad r(q_C/r_C) = q$$
$$\Rightarrow \quad (r(q_C/r_C))^2 = q_1q_2 \quad \Rightarrow \quad 1 = (r_C^2/q_C^2)q_1q_2/r^2.$$

Use force ($F = ma$) ratios equal to the scalar (unit-less) value one:

$$\text{(5.5)}\qquad \exists\, m_0,\ m_C,\ a,\ a_C \in \mathbb{R} \ : \ (m_0a/m_Ca_C) = 1 = (r_C^2/q_C^2)q_1q_2/r^2.$$

Multiplying both sides of equation 5.5 by m_Ca_C yields the charge force equation:

$$\text{(5.6)}\quad (m_0a/m_Ca_C)(m_Ca_C) = (r_C^2/q_C^2)(q_1q_2r^2)(m_Ca_C)$$
$$\Rightarrow \quad \exists\, F,\ k_C \in \mathbb{R} : \quad F = m_0a = (m_Ca_Cr_C^2/q_C^2)q_1q_2/r^2 = k_Cq_1q_2/r^2,$$

where $k_C = m_Ca_Cr_C^2/q_C^2$ agrees with the accepted standard units: Nm^2C^{-2}.

Using intervals of type, $[0, m]$, that are proportionate in size to $[0, r]$:

$$\text{(5.7)}\quad \forall\, m_1m_2 \in \mathbb{R}\, \exists\, m \in \mathbb{R} : \quad m^2 = m_1m_2 \quad \wedge \quad r(m_g/r_g) = m$$
$$\Rightarrow \quad (r(m_g/r_g))^2 = m_1m_2 \quad \Rightarrow \quad 1 = (r_g^2/m_g^2)m_1m_2/r^2.$$

Use force ($F = ma$) ratios equal to the scalar (unit-less) value one:

$$\text{(5.8)}\qquad \exists\, m_0,\ m_G,\ a,\ a_G \in \mathbb{R} \ : \ (m_0a/m_Ga_G) = 1 = (r_g^2/m_g^2)m_1m_2/r^2.$$

Multiplying both sides of equation 5.8 by m_Ga_G yields the charge force equation:

$$\text{(5.9)}\quad (m_0a/m_Ga_G)(m_Ga_G) = (r_g^2/m_g^2)(m_1m_2r^2)(m_Ga_G)$$
$$\Rightarrow \quad \exists\, F,\ k_C \in \mathbb{R} : \quad F = m_0a = (m_Ga_Gr_g^2/m_g^2)m_1m_2/r^2 = Gm_1m_2/r^2,$$

where $G = m_Ga_Gr_g^2/m_g^2$ agrees with the accepted standard units: $m^3kg^{-1}s^{-2}$.

5.3. 3 dimensions of physical geometry. The set and arithmetic operations used to calculate distance and volume requires sequencing through a totally ordered set of dimensions, for example, the countable distance space: $d_c = |\bigcup_{i=1}^{n} y_i|$, Euclidean distance: $d^2 = \sum_{i=1}^{n} s_i^2$, countable volume: $V_c = \prod_{i=1}^{n} |x_i|$, and Euclidean volume: $V = \prod_{i=1}^{n} s_i$. The commutative property of the set and arithmetic operations also allows sequencing through a set of n number of dimensions in all $n!$ number of possible orders.

But, a *physical* deterministic sequencer requires a *physical* set to have a single total order, at most one successor and at most one predecessor per set member, during the *time* of sequencing. A total order is the only way to "determine" that one sequencer traversed in the order, $[2,\ 1,\ \cdots]$, and another sequencer traversed in the order, $[1,\ 2,\ \cdots]$, during the time of sequencing. Deterministic sequencing in every possible order via the same successor/predecessor relations (same total order) requires each set member to be either a successor or predecessor to every other set member (sequentially adjacent), herein referred to as a symmetric geometry.

It will now be proved that a set satisfying the constraints of a single total order and also symmetric defines a cyclic set containing at most 3 members, in this case, 3 dimensions of physical space.

Definition 5.1. Ordered geometry:

$$\forall\, i\, n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i\ \in \{x_1, \ldots, x_n\},$$
$$\textit{successor}\ x_i = x_{i+1}\ \ \wedge\ \ \textit{predecessor}\ x_{i+1} = x_i.$$

Definition 5.2. Symmetric geometry (every set member is sequentially adjacent to any other member):

$$\forall\, i\, j\, n \in \mathbb{N},\ \ \forall\, x_i\, x_j \in \{x_1, \ldots, x_n\},\ \ \textit{successor}\ x_i = x_j\ \ \Leftrightarrow\ \ \textit{predecessor}\ x_j = x_i.$$

THEOREM 5.3. *An ordered and symmetric set is a cyclic set.*

$$\textit{successor } x_n = x_1 \ \wedge \ \textit{predecessor } x_1 = x_n.$$

The theorem, "ordered_symmetric_is_cyclic," and formal proof is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$i = n \ \wedge \ j = 1 \ \wedge \ \textit{successor } x_i = x_j \ \Rightarrow \ \textit{successor } x_n = x_1. \tag{5.10}$$

$$i = n \ \wedge \ j = 1 \ \wedge \ \textit{predecessor } x_j = x_i \ \Rightarrow \ \textit{predecessor } x_1 = x_n. \tag{5.11}$$ □

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

DEFINITION 5.5. Cyclic successor of m is n:

$$\textit{Successor}(m, n, \textit{setsize}) \leftarrow (m = \textit{setsize} \wedge n = 1) \vee (n = m + 1 \leq \textit{setsize}). \tag{5.12}$$

DEFINITION 5.6. Cyclic predecessor of m is n:

$$\textit{Predecessor}(m, n, \textit{setsize}) \leftarrow (m = 1 \wedge n = \textit{setsize}) \vee (n = m - 1 \geq 1). \tag{5.13}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:

$$\textit{Adjacent}(m, n, \textit{setsize}) \leftarrow \textit{Successor}(m, n, \textit{setsize}) \vee \textit{Predecessor}(m, n, \textit{setsize}). \tag{5.14}$$

Every member is adjacent to every other member, where $\textit{setsize} \in \{1,\ 2,\ 3\}$:

$$\textit{Adjacent}(1, 1, 1) \leftarrow \textit{Successor}(1, 1, 1) \leftarrow (m = \textit{setsize} \wedge n = 1). \tag{5.15}$$

$$\textit{Adjacent}(1, 2, 2) \leftarrow \textit{Successor}(1, 2, 2) \leftarrow (n = m + 1 \leq \textit{setsize}). \tag{5.16}$$

$$\textit{Adjacent}(2, 1, 2) \leftarrow \textit{Successor}(2, 1, 2) \leftarrow (n = \textit{setsize} \wedge m = 1). \tag{5.17}$$

$$\mathit{Adjacent}(1,2,3) \leftarrow \mathit{Successor}(1,2,3) \leftarrow (n = m+1 \leq \mathit{setsize}). \tag{5.18}$$

$$\mathit{Adjacent}(2,1,3) \leftarrow \mathit{Predecessor}(2,1,3) \leftarrow (n = m-1 \geq 1). \tag{5.19}$$

$$\mathit{Adjacent}(3,1,3) \leftarrow \mathit{Successor}(3,1,3) \leftarrow (n = \mathit{setsize} \wedge m = 1). \tag{5.20}$$

$$\mathit{Adjacent}(1,3,3) \leftarrow \mathit{Predecessor}(1,3,3) \leftarrow (m = 1 \wedge n = \mathit{setsize}). \tag{5.21}$$

$$\mathit{Adjacent}(2,3,3) \leftarrow \mathit{Successor}(2,3,3) \leftarrow (n = m+1 \leq \mathit{setsize}). \tag{5.22}$$

$$\mathit{Adjacent}(3,2,3) \leftarrow \mathit{Predecessor}(3,2,3) \leftarrow (n = m-1 \geq 1). \tag{5.23}$$

Must prove that for all $\mathit{setsize} > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\begin{aligned} \forall\, \mathit{setsize} > 3: \quad & \neg \mathit{Successor}(1,3,\mathit{setsize} > 3) \\ & \leftarrow \mathit{Successor}(1,2,\mathit{setsize} > 3) \leftarrow (n = m+1 \leq \mathit{setsize}). \end{aligned} \tag{5.24}$$

That is, member 2 is the only successor of member 1 for all $\mathit{setsize} > 3$, which implies member 3 is not a successor of member 1 for all $\mathit{setsize} > 3$.

$$\begin{aligned} \forall\, \mathit{setsize} > 3: \quad & \neg \mathit{Predecessor}(1,3,\mathit{setsize} > 3) \\ & \leftarrow \mathit{Predecessor}(1,\mathit{setsize},\mathit{setsize} > 3) \leftarrow (m = 1 \wedge n = \mathit{setsize} > 3). \end{aligned} \tag{5.25}$$

That is, member $n > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $n > 3$.

$$\begin{aligned} \forall\, \mathit{setsize} > 3: \quad & \neg \mathit{Adjacent}(1,3,\mathit{setsize} > 3) \\ & \leftarrow \neg \mathit{Successor}(1,3,\mathit{setsize} > 3) \wedge \neg \mathit{Predecessor}(1,3,\mathit{setsize} > 3). \quad \square \end{aligned} \tag{5.26}$$

That is, for all $\mathit{setsize} > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

Applying the ruler measure (2.1) and ruler convergence (2.2) to the set relations, countable distance space (3.1) and countable volume (4.1) yields the following insights and implications:

1. The properties of metric space, Euclidean distance and area/volume can be derived from two very simple set relations without using the notions of Euclidean geometry [**Joy98**] like plane, side, angle, perpendicular, congruence, intersection, etc.
2. The ruler measure-based proofs provide the insight that distance is a function of the combinatorial *range*-to-domain set member mappings. Whereas, area/volume is a function of the combinatorial *domain*-to-domain set member mappings.
3. The distance spanning multiple, disjoint, domain sets being proportionate to the number of members, d_c, in the union range set: $d_c = |\bigcup_{i=1}^{n} y_i|$ (3.1) generates the triangle inequality, non-negativity, and identity of indiscernibles properties of metric space (3.2). And combined with the constraint, $|x_i| = |y_i|$, also generates Manhattan and Euclidean distance, which motivates the fourth property of metric space, symmetry $[d(u,v) = d(v,u)]$. The reason Manhattan and Euclidean distance have the property of symmetry is that the type of combinatorial range-to-domain set mapping is the same for every set of mappings being summed.

(4) As the amount of intersection of the range sets increases, a single range set member can map to more domain set members, where the constraint, $|x_i| = |y_i| = p_i$, limits the largest total number of range-to-domain set mappings (largest intersection and smallest distance) to $\sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^2$, which is the set-based reason Euclidean distance (3.7) is the smallest possible distance between two distinct points in $\mathbb{R}^n$.

(5) Time, charge, and mass intervals can be defined that are directly proportionate in size to any distance interval. Applying the ruler to those intervals allows application of the Euclidean distance and area/volume proofs to derive the spacetime, charge force, and Newtonian gravity force equations.
   (a) The derivations of the spacetime equations did not require the notion of light and provides a purely geometric perspective on the relationship of time to the Euclidean distance in two independent frames of reference. Further, the derivation provides the insight that there is a proportionality constant that is a ratio of distance to time (a maximum velocity). However, a change in some non-distance variable with respect to time could have an effect at multiple locations at the same time, independent of the distance between the locations.
   (b) The charge and gravity force equations were derived from the more fundamental relations that the size, r, of a distance interval is proportionate to the size, q, of a charge interval: $r(q_C/r_C) = q$, and proportionate to the size, m, of a mass interval: $r(m_g/r_g) = m$.
      (i) Distance, r, proportionate to charge and mass combined with the ruler-based derivations of charge area, q_1q_2, and mass area, m_1m_2 implies there are proportionate geometric areas, r^2, which explains why the forces vary inversely with the square of the distance between two infinitesimal objects.
      (ii) If there are quantum values of charge, q_C, and mass, m_g, then there are quantum distances, r_C and r_g, where the forces do not exist at smaller distances. Quantum charge and mass would eliminate the need to invent stronger forces to override the charge and gravity forces at distances smaller than r_C and r_g.
   (c) Applying the ruler to sets of ordered intervals is critical to deriving geometric relations. But, applying the ruler to an unordered collection of intervals (a “bag” of intervals) might be a way to model some statistical mechanics and subatomic behavior, where the behavior would be described by probability equations.

(6) Relativity theory assumes only 3 dimensions of space [**Bru17**]. Time constrains the set relations generating the properties of distance and area/volume to at most 3 dimensions (5.4).

(7) Note that Euclidean distance (3.7) and volume (4.2) are range sets. All compressions, expansions, ripples, bends, bubbles, tunnels, etc. are in the range set space, where the range set is also a function of other variables.
   (a) Particles, waves, energy, force, etc. are range set phenomena that are projected onto (viewed from) our local, domain, Euclidean frame of reference.
   (b) An expanding universe is an expansion of the range set space.

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