

# The Two Set Relations Generating Geometry

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ABSTRACT. A ruler-like measure is used to prove that the properties of metric space, the Euclidean distance and volume equations are motivated and derived from two countable set relations. The ruler measure divides both domain and range intervals approximately into the nearest integer number of same-sized subintervals. As the subinterval size converges to zero: 1) Distance as the union size of range sets, where for each domain set there exists a corresponding same-sized range set, converges to: the triangle inequality with Manhattan distance at the upper boundary and Euclidean distance at the lower boundary. 2) The Cartesian product of the number of members in each domain set converges to the product of interval interval sizes (Euclidean area/volume). Time limits physical geometry to 3 dimensions. All proofs are verified in Coq.

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## 1. Introduction

The properties of metric space, Euclidean distance, and the product of interval sizes (Euclidean area/volume) are defined in real analysis [Gol76] [Rud76] rather than motivated and derived from set-based axioms. A “ruler” measure is introduced and used to prove that these geometric relations are motivated and derived from two countable set relations.

The derivation of geometric relations from set relations, *without notions of point, plane, line, angle, etc.*, identifies: 1) the single set relation generating all the properties of metric space; 2) the mapping between sets that makes Euclidean

distance the smallest possible distance between two distinct points in  $\mathbb{R}^n$ ; 3) the mapping between sets that makes distance different from area/volume; 4) how time places an additional constraint on physical sets, which limits physical geometry to 3 dimensions.

Proofs accepted by the Coq logic engine [Coq15] are internationally recognized to have a very high probability of being correct. All the proofs in this article have corresponding formal proofs in the Coq files, “euclidrelations.v” and “threed.v,” located at: <https://github.com/treeck/RASRGeometry>.

## 2. Ruler measure and convergence

A ruler (measuring stick) partitions both domain and range intervals approximately into the nearest integer number of same-sized subintervals. In contrast, Riemann and Lebesgue integrals partition the domain intervals *exactly* and partition the range intervals *approximately*, where each domain subinterval and corresponding range subinterval generally differ in size [Rud76]. The ruler measure allows counting the number of mappings, ranging from a one-to-one correspondence to a many-to-many mapping, between the set of same-sized subintervals in one interval and the set of same-sized subintervals in another interval. The mapping (combinatorial) relations converge to continuous, bijective relations as the subinterval size converges to zero.

**DEFINITION 2.1.** Ruler measure: A ruler measures the size,  $M$ , of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals,  $p$ , each subinterval having the same size,  $c$ . Notionally:

$$(2.1) \quad \forall c \ s \in \mathbb{R}, \ [a, b] \subset \mathbb{R}, \ s = |a - b| \wedge c > 0 \wedge \\ (p = \text{floor}(s/c) \vee p = \text{ceiling}(s/c)) \wedge M = \sum_{i=1}^p c = pc.$$

**THEOREM 2.2.** *Ruler convergence:*

$$\forall [a, b] \subset \mathbb{R}, \ s = |a - b| \Rightarrow M = \lim_{c \rightarrow 0} pc = s.$$

The theorem, “limit\_c\_0\_M\_eq\_exact\_size,” and formal proof is in the Coq file, euclidrelations.v.

**PROOF.** (epsilon-delta proof)

By definition of the floor function,  $\text{floor}(x) = \max(\{y : y \leq x, y \in \mathbb{Z}, x \in \mathbb{R}\})$ :

$$(2.2) \quad \forall c > 0, \ p = \text{floor}(s/c) \wedge 0 \leq |\text{floor}(s/c) - s/c| < 1 \Rightarrow 0 \leq |p - s/c| < 1.$$

Multiply all sides of inequality 2.2 by  $|c|$ :

$$(2.3) \quad \forall c > 0, \quad 0 \leq |p - s/c| < 1 \Rightarrow 0 \leq |pc - s| < |c|.$$

$$(2.4) \quad \forall \delta : |pc - s| < |c| = |c - 0| < \delta \\ \Rightarrow \forall \epsilon = \delta : |c - 0| < \delta \wedge |pc - s| < \epsilon := M = \lim_{c \rightarrow 0} pc = s. \quad \square$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence, where:  $[0, \pi]$ ,  $s = |0 - \pi|$ ,  $c = 10^{-i}$ , and  $p = \text{floor}(s/c) \Rightarrow p \cdot c = 3.1_{i=1}, 3.14_{i=2}, 3.141_{i=3}, \dots, \pi$ .

### 3. Distance

**Notation convention:** Curly brackets,  $\{\dots\}$ , delimit a set; square brackets,  $[\dots]$ , delimit a list; and vertical bars around a set or list,  $|\dots|$ , indicates the cardinal (number of members in the set or list).

**3.1. Countable distance space.** A simple measure of distance is the number of steps walked, which corresponds to an equal number of pieces of land. Abstracting, distance is the number of members in a range set,  $y_i$ , which equals the number of members in a corresponding domain set,  $x_i$ :  $|x_i| = |y_i|$ . And the distance spanning multiple, disjoint, domain sets,  $\bigcap_{i=1}^n x_i = \emptyset$ , is the number of members,  $d_c$ , in the union range set:  $d_c = |\bigcup_{i=1}^n y_i|$ .

DEFINITION 3.1. Countable distance space,  $d_c$ :

$$\bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad d_c = |\bigcup_{i=1}^n y_i| \quad \wedge \quad |x_i| = |y_i|.$$

THEOREM 3.2. *Inclusion-exclusion Inequality:*  $|\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$ .

This well-known inequality follows directly from the inclusion-exclusion principle [CG15]. But, a more intuitive and simple proof follows from the sum of the set sizes being equal to the number of unique members (the union set) plus the number of duplicate (intersection) members. For example,  $|\{a, b, c\}| + |\{c, d, e\}| = |\{a, b, c, c, d, e\}| = |\{a, b, c, d, e\}| + |[c]| = 6 \Rightarrow |\{a, b, c, d, e\}| = |\{a, b, c\}| + |\{c, d, e\}| - |[c]| = 5$ .

A formal proof, inclusion\_exclusion\_inequality, using sorting into unique members (union set) and duplicate members, is in the file euclidrelations.v.

PROOF. More generally:

$$\begin{aligned} (3.1) \quad |\bigcup_{i=1}^n y_i| + |\text{duplicates}_{i=1}^n y_i| &= \sum_{i=1}^n |y_i| \\ &\Rightarrow |\bigcup_{i=1}^n y_i| = \sum_{i=1}^n |y_i| - |\text{duplicates}_{i=1}^n y_i|. \end{aligned}$$

$$\begin{aligned} (3.2) \quad |\bigcup_{i=1}^n y_i| &= \sum_{i=1}^n |y_i| - |\text{duplicates}_{i=1}^n y_i| \quad \wedge \quad |\text{duplicates}_{i=1}^n y_i| \geq 0 \\ &\Rightarrow |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|. \quad \square \end{aligned}$$

**3.2. Metric Space.** Applying the ruler (2.1) and ruler convergence (2.2) to three range intervals having sizes:  $d(u, w)$ ,  $d(u, v)$ ,  $d(v, w)$ , and using the inequality,  $d_c = |\bigcup_{i=1}^2 y_i| \leq \sum_{i=1}^2 |y_i|$ , generates the properties of metric space. The formal proofs: triangle\_inequality, non\_negativity, identity\_of\_indiscernibles, and symmetry, are in the Coq file, euclidrelations.v.

THEOREM 3.3. *Triangle Inequality:*  $d(u, w) \leq d(u, v) + d(v, w)$ .

PROOF.

$$\begin{aligned} (3.3) \quad \forall c > 0, |y_1| &= \text{floor}(d(u, v)/c) \quad \wedge \quad |y_2| = \text{floor}(d(v, w)/c) \quad \wedge \\ d_c &= \text{floor}(d(u, w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \leq |y_1| + |y_2| \\ &\Rightarrow \text{floor}(d(u, w)/c) \leq \text{floor}(d(u, v)/c) + \text{floor}(d(v, w)/c) \\ &\Rightarrow \text{floor}(d(u, w)/c) \cdot c \leq \text{floor}(d(u, v)/c) \cdot c + \text{floor}(d(v, w)/c) \cdot c \\ &\Rightarrow \lim_{c \rightarrow 0} \text{floor}(d(u, w)/c) \cdot c \leq \lim_{c \rightarrow 0} \text{floor}(d(u, v)/c) \cdot c + \lim_{c \rightarrow 0} \text{floor}(d(v, w)/c) \cdot c \\ &\Rightarrow d(u, w) \leq d(u, v) + d(v, w). \quad \square \end{aligned}$$

**THEOREM 3.4.** *Non-negativity:*  $d(u, w) \geq 0$ .

**PROOF.**

$$(3.4) \quad \forall c > 0 : \quad d_c = \text{floor}(d(u, w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \geq 0 \\ \Rightarrow \quad \text{floor}(d(u, w)/c) = d_c \geq 0 \quad \Rightarrow \quad d(u, w) = \lim_{c \rightarrow 0} d_c \cdot c \geq 0. \quad \square$$

**THEOREM 3.5.** *Identity of Indiscernibles:*  $d(w, w) = 0$ .

**PROOF.**

$$(3.5) \quad \forall d(u, v) = d(v, w) = 0 \quad \wedge \quad d(u, w) \leq d(u, v) + d(v, w) \quad \wedge \quad d(u, w) \geq 0 \\ \Rightarrow \quad d(u, w) = 0.$$

$$(3.6) \quad d(u, w) = 0 \quad \wedge \quad d(u, v) = 0 \quad \Rightarrow \quad w = v.$$

$$(3.7) \quad d(v, w) = 0 \quad \wedge \quad w = v \quad \Rightarrow \quad d(w, w) = 0. \quad \square$$

**THEOREM 3.6.** *Symmetry:*  $d(v, w) = d(w, v)$ .

**PROOF.**

$$(3.8) \quad w = v \Rightarrow d(w, w) = d(v, w) \quad \wedge \quad d(w, w) = d(w, v) \Rightarrow d(v, w) = d(w, v). \quad \square$$

**3.3. Distance space range.** Distance,  $d_c = |\bigcup_{i=1}^n y_i|$ , implies that where the range sets intersect, multiple domain set members map to a single range set member. Therefore, distance is a function of domain-to-range set member mappings.

From the countable distance space definition (3.1),  $|x_i| = |y_i|$ . Where  $|x_i| = |y_i| = p_i = 1$ , each of the  $p_i$  number of domain set members in  $x_i$ : 1) maps 1-1 (bijective) to a *single*, unique range set member in  $y_i$ , yielding  $|x_i| \cdot 1 = p_i \cdot 1 = p_i = 1$  number of domain-to-range set mappings. 2) maps to *all* of the  $p_i$  number of range set members in  $y_i$ , yielding  $|x_i| \cdot |y_i| = p_i \cdot p_i = p_i^2 = 1$  number of domain-to-range set mappings.

Therefore, the total number of domain-to-range set mappings spanning  $n$  number of sets ranges from  $\sum_{i=1}^n p_i$  to  $\sum_{i=1}^n p_i^2$ . Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of domain-to-range set mapping cases converges to the real-valued, Manhattan and Euclidean distance functions.

### 3.4. Manhattan distance.

**THEOREM 3.7.** *Manhattan (longest non-increasing) distance,  $d$ , is the size of the distance interval,  $[d_0, d_m]$ , mapping to a set of disjoint domain intervals,  $\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$ , where:*

$$d = \sum_{i=1}^n s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, “taxicab.distance,” and formal proof is in the Coq file, euclidrelations.v.

PROOF.

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance,  $d_c$ , is the equality case:

$$(3.9) \quad d_c = |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \\ \Rightarrow \quad d_c \leq \sum_{i=1}^n |y_i| = \sum_{i=1}^n p_i \quad \Rightarrow \quad \exists p_i, d_c : d_c = \sum_{i=1}^n p_i.$$

Multiply both sides of equation 3.11 by  $c$  and take the limit:

$$(3.10) \quad d_c = \sum_{i=1}^n p_i \Rightarrow d_c \cdot c = \sum_{i=1}^n (p_i \cdot c) \Rightarrow \lim_{c \rightarrow 0} d_c \cdot c = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c).$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of  $d$ :

$$(3.11) \quad d = |d_0 - d_m| \Rightarrow \exists c d : \text{floor}(d/c) = d_c \Rightarrow d = \lim_{c \rightarrow 0} d_c \cdot c.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of  $s_i$ :

$$(3.12) \quad \forall i \in [1, n], s_i = |a_i - b_i| \wedge \text{floor}(s_i/c) = |x_i| = |y_i| = p_i \Rightarrow \lim_{c \rightarrow 0} p_i \cdot c = s_i.$$

Combine equations 3.11, 3.10, 3.12:

$$(3.13) \quad d = \lim_{c \rightarrow 0} d_c \cdot c \quad \wedge \quad \lim_{c \rightarrow 0} d_c \cdot c = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c) \quad \wedge \\ \lim_{c \rightarrow 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c \rightarrow 0} d_c \cdot c = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c) = \sum_{i=1}^n s_i. \quad \square$$

### 3.5. Euclidean distance.

**THEOREM 3.8.** *Euclidean (shortest) distance,  $d$ , is the size of the distance interval,  $[d_0, d_m]$ , mapping to a set of disjoint domain intervals,  $\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$ , where:*

$$d^2 = \sum_{i=1}^n s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, “Euclidean\_distance,” and formal proof is in the Coq file, euclidrelations.v.

PROOF.

Apply the rule of product to the largest number of domain-to-range set mappings, where all  $p_i$  number of domain set members,  $x_i$ , map to each of the  $p_i$  number of members in the range set,  $y_i$ :

$$(3.14) \quad \sum_{i=1}^n |y_i| \cdot |x_i| = \sum_{i=1}^n p_i^2.$$

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$(3.15) \quad d_c = |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \\ \Rightarrow \quad d_c \leq \sum_{i=1}^n |y_i| = \sum_{i=1}^n p_i \quad \Rightarrow \quad \exists p_i, d_c : d_c = \sum_{i=1}^n p_i.$$

Square both sides of equation 3.15 ( $x = y \Leftrightarrow f(x) = f(y)$ ):

$$(3.16) \quad \exists p_i, d_c : d_c = \sum_{i=1}^n p_i \Leftrightarrow \exists p_i, d_c : d_c^2 = (\sum_{i=1}^n p_i)^2.$$

Apply the Cauchy-Schwartz inequality to equation 3.16 and select the smallest distance (equality) case:

$$(3.17) \quad d_c^2 = (\sum_{i=1}^n p_i)^2 \geq \sum_{i=1}^n p_i^2 \Rightarrow \exists p_i : d_c^2 = \sum_{i=1}^n p_i^2.$$

Multiply both sides of equation 3.17 by  $c^2$ , simplify, and take the limit.

$$(3.18) \quad d_c^2 = \sum_{i=1}^n p_i^2 \Rightarrow d_c^2 \cdot c^2 = \sum_{i=1}^n p_i^2 \cdot c^2 \Leftrightarrow (d_c \cdot c)^2 = \sum_{i=1}^n (p_i \cdot c)^2 \\ \Rightarrow \lim_{c \rightarrow 0} (d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.19) \quad \exists c \, d : \text{floor}(d/c) = d_c \Rightarrow d = \lim_{c \rightarrow 0} d_c \cdot c \Rightarrow d^2 = \lim_{c \rightarrow 0} (d_c \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$(3.20) \quad \forall i \in [1, n], s_i = |a_i - b_i| \wedge \text{floor}(s_i/c) = |x_i| = |y_i| = p_i \Rightarrow \lim_{c \rightarrow 0} p_i \cdot c = s_i.$$

Combine equations 3.19, 3.18, 3.20:

$$(3.21) \quad d^2 = \lim_{c \rightarrow 0} (d_c \cdot c)^2 \quad \wedge \quad \lim_{c \rightarrow 0} (d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)^2 \quad \wedge \\ \lim_{c \rightarrow 0} p_i \cdot c = s_i \Rightarrow d^2 = \lim_{c \rightarrow 0} (d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)^2 = \sum_{i=1}^n s_i^2. \quad \square$$

#### 4. Euclidean Volume

The number of all possible combinations ( $n$ -tuples) taking one member from each disjoint set is the Cartesian product of the number of members in each set. Notionally:

DEFINITION 4.1. All Possible Combinations,  $V_c$ :

$$\bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad V_c = \prod_{i=1}^n |x_i|.$$

THEOREM 4.2. *Euclidean volume,  $V$ , is size of the range interval,  $[v_0, v_m]$ , corresponding to all the possible combinations of the members of disjoint domain intervals,  $\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$ . Notionally:*

$$V = \prod_{i=1}^n s_i, \quad V = |v_0 - v_m|, \quad s_i = |a_i - b_i|.$$

The theorem, “Euclidean\_volume,” and formal proof is in the Coq file, euclidrelations.v.

PROOF.

Use the ruler (2.1) to divide the exact size,  $s_i = |a_i - b_i|$ , of each of the domain intervals,  $[a_i, b_i]$ , into a set,  $x_i$  of  $p_i$  number of subintervals.

$$(4.1) \quad \forall i \, n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad \text{floor}(s_i/c) = p_i = |x_i|.$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$(4.2) \quad \text{floor}(s_i/c) = p_i \Rightarrow \lim_{c \rightarrow 0} (p_i \cdot c) = s_i.$$

Use the ruler (2.1) to divide the exact size,  $V = |v_0 - v_m|$ , of the range interval,  $[v_0, v_m]$ , into  $p^n$  subintervals. Use those cases, where  $V_c$  has an integer  $n^{\text{th}}$  root.

$$(4.3) \quad \forall p^n = V_c \in \mathbb{N}, \exists V \in \mathbb{R}, x_i : \text{floor}(V/c^n) = V_c = p^n = \prod_{i=1}^n |x_i| = \prod_{i=1}^n p_i.$$

Apply the ruler convergence theorem (2.2) to equation 4.3 and simplify:

$$(4.4) \quad \text{floor}(V/c^n) = p^n \Rightarrow V = \lim_{c \rightarrow 0} p^n \cdot c^n = \lim_{c \rightarrow 0} (p \cdot c)^n.$$

Multiply both sides of equation 4.3 by  $c^n$  and simplify:

$$(4.5) \quad p^n = \prod_{i=1}^n p_i \Rightarrow p^n \cdot c^n = (\prod_{i=1}^n p_i) \cdot c^n \Leftrightarrow (p \cdot c)^n = \prod_{i=1}^n (p_i \cdot c) \\ \Rightarrow \lim_{c \rightarrow 0} (p \cdot c)^n = \prod_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)$$

Combine equations 4.4, 4.5, and 4.2:

$$(4.6) \quad V = \lim_{c \rightarrow 0} (p \cdot c)^n \quad \wedge \quad \lim_{c \rightarrow 0} (p \cdot c)^n = \prod_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c) \quad \wedge \\ \lim_{c \rightarrow 0} (p_i \cdot c) = s_i \Rightarrow V = \lim_{c \rightarrow 0} (p \cdot c)^n = \prod_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c) = \prod_{i=1}^n s_i. \quad \square$$

## 5. Ordered and symmetric geometries

The set and arithmetic operations used to calculate distance and volume requires sequencing through a totally ordered set of dimensions. For example, from the countable distance space definition (3.1):  $d_c = |\bigcup_{i=1}^n y_i|$ . The commutative property of the set and arithmetic operations also allows sequencing through the dimensions in every possible order.

Asserting that a physical set has multiple sequential (total) orders at the same time to allow sequencing through every possible order creates successor and predecessor contradictions because each set member can have at most one successor and at most one predecessor at any point in time. Therefore, a physical set can have only one total order defined at any point in time.

If for each set member, every other member is adjacent (a successor or a predecessor), then for each member there exists successor and predecessor sequences in every possible order at the same time – without contradictions, herein referred to as a symmetric geometry. It will now be proved that a set satisfying the constraint of a single total order (at a point in time) and also symmetric defines a cyclic set containing at most 3 members, in this case, 3 dimensions of physical space.

DEFINITION 5.1. Ordered geometry:

$$\forall i \ n \in \mathbb{N}, \ i \in [1, n-1], \ \forall x_i \in \{x_1, \dots, x_n\}, \\ \text{successor } x_i = x_{i+1} \ \wedge \ \text{predecessor } x_{i+1} = x_i.$$

DEFINITION 5.2. Symmetric geometry (every set member is sequentially adjacent to any other member):

$$\forall i \ j \ n \in \mathbb{N}, \ \forall x_i \ x_j \in \{x_1, \dots, x_n\}, \ \text{successor } x_i = x_j \ \wedge \ \text{predecessor } x_j = x_i.$$

THEOREM 5.3. An ordered and symmetric set is a cyclic set.

$$\text{successor } x_n = x_1 \ \wedge \ \text{predecessor } x_1 = x_n.$$

The theorem, “ordered\_symmetric\_is\_cyclic,” and formal proof is in the Coq file, `threed.v`.

PROOF. The property of order (5.1) defines unique successors and predecessors for all set members except for the successor of  $x_n$  and the predecessor of  $x_1$ . Therefore, the only member that can be a successor of  $x_n$ , without creating a contradiction, is  $x_1$ . And the only member that can be a predecessor of  $x_1$ , without creating a contradiction, is  $x_n$ . From the properties of a symmetric geometry (5.2):

$$(5.1) \quad i = n \ \wedge \ j = 1 \ \wedge \ \text{successor } x_i = x_j \Rightarrow \text{successor } x_n = x_1.$$

$$(5.2) \quad i = n \ \wedge \ j = 1 \ \wedge \ \text{predecessor } x_j = x_i \Rightarrow \text{predecessor } x_1 = x_n. \quad \square$$

THEOREM 5.4. An ordered and symmetric set is limited to at most 3 members.

The lemmas and formal proofs in the Coq file `threed.v` are:

**Lemmas:** `adj111`, `adj122`, `adj212`, `adj123`, `adj133`, `adj233`, `adj213`, `adj313`, `adj323`, and `not_all_mutually_adjacent_gt_3`.

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.

It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

DEFINITION 5.5. Cyclic successor of  $m$  is  $n$ :

$$(5.3) \quad \text{Successor}(m, n, \text{setsize}) \leftarrow (m = \text{setsize} \wedge n = 1) \vee (m + 1 \leq \text{setsize}).$$

DEFINITION 5.6. Cyclic predecessor of  $m$  is  $n$ :

$$(5.4) \quad \text{Predecessor}(m, n, \text{setsize}) \leftarrow (m = 1 \wedge n = \text{setsize}) \vee (m - 1 \geq 1).$$

DEFINITION 5.7. Adjacent: member  $m$  is sequentially adjacent to member  $n$  if the cyclic successor of  $m$  is  $n$  or the cyclic predecessor of  $m$  is  $n$ . Notionally:

$$(5.5) \quad \text{Adjacent}(m, n, \text{setsize}) \leftarrow \text{Successor}(m, n, \text{setsize}) \vee \text{Predecessor}(m, n, \text{setsize}).$$

Every member is adjacent to every other member, where  $\text{setsize} \in \{1, 2, 3\}$ :

$$(5.6) \quad \text{Adjacent}(1, 1, 1) \leftarrow \text{Successor}(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1).$$

$$(5.7) \quad \text{Adjacent}(1, 2, 2) \leftarrow \text{Successor}(1, 2, 2) \leftarrow (1 + 1 \leq 2).$$

$$(5.8) \quad \text{Adjacent}(2, 1, 2) \leftarrow \text{Successor}(2, 1, 2) \leftarrow (2 = 2 \wedge 1 = 1).$$

$$(5.9) \quad \text{Adjacent}(1, 2, 3) \leftarrow \text{Successor}(1, 2, 3) \leftarrow (1 + 1 \leq 2).$$

$$(5.10) \quad \text{Adjacent}(2, 1, 3) \leftarrow \text{Predecessor}(2, 1, 3) \leftarrow (2 - 1 \geq 1).$$

$$(5.11) \quad \text{Adjacent}(3, 1, 3) \leftarrow \text{Successor}(3, 1, 3) \leftarrow (3 = 3 \wedge 1 = 1).$$

$$(5.12) \quad \text{Adjacent}(1, 3, 3) \leftarrow \text{Predecessor}(1, 3, 3) \leftarrow (1 = 1 \wedge 3 = 3).$$

$$(5.13) \quad \text{Adjacent}(2, 3, 3) \leftarrow \text{Successor}(2, 3, 3) \leftarrow (2 + 1 \leq 3).$$

$$(5.14) \quad \text{Adjacent}(3, 2, 3) \leftarrow \text{Predecessor}(3, 2, 3) \leftarrow (3 - 1 \geq 1).$$

Must prove that for all  $\text{setsize} > 3$ , there exist non-adjacent members. For example, the first and third members are not ( $\neg$ ) adjacent:

$$(5.15) \quad \forall \text{setsize} > 3: \quad \neg \text{Successor}(1, 3, \text{setsize} > 3) \\ \leftarrow \text{Successor}(1, 2, \text{setsize} > 3) \leftarrow (1 + 1 \leq \text{setsize} > 3).$$

That is, member 2 is the only successor of member 1 for all  $\text{setsize} > 3$ , which implies member 3 is not a successor of member 1 for all  $\text{setsize} > 3$ .

$$(5.16) \quad \forall \text{setsize} > 3: \quad \neg \text{Predecessor}(1, 3, \text{setsize} > 3) \\ \leftarrow \text{Predecessor}(1, n, \text{setsize} > 3) \leftarrow (1 = 1 \wedge n = \text{setsize} > 3).$$

That is, member  $n > 3$  is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all  $n > 3$ .

$$(5.17) \quad \forall \text{setsize} > 3: \quad \neg \text{Adjacent}(1, 3, \text{setsize} > 3) \\ \leftarrow \neg \text{Successor}(1, 3, \text{setsize} > 3) \wedge \neg \text{Predecessor}(1, 3, \text{setsize} > 3). \quad \square$$

That is, for all  $\text{setsize} > 3$ , some elements are not sequentially adjacent to every other element (not symmetric).



## 6. Insights and conjectures

Applying the ruler measure (2.1) and ruler convergence (2.2) to the set relations, countable distance space (3.1) and all possible combinations (4.1) yields the following insights and open questions:

- (1) Notions of point, plane, side, angle, perpendicular, congruence, intersection, etc. are not necessary to motivate and derive the properties of metric space, Euclidean distance and area/volume.
- (2) Distance is a function of the number of domain-to-range set member mappings. In contrast, area/volume is a function of the number of domain-to-domain set member mappings.
- (3) All notions of distance are derived from the principle that every domain set,  $x_i$ , has a corresponding range (distance) set,  $y_i$ , containing the same number of members:  $|x_i| = |y_i|$ . And the distance spanning multiple, disjoint, domain sets is the number of members,  $d_c$ , in the corresponding union range set:  $d_c = |\bigcup_{i=1}^n y_i|$  (3.1).
  - (a) A direct consequence of the inclusion-exclusion principle [CG15] is the set relation,  $d_c = |\bigcup_{i=1}^2 y_i| \leq \sum_{i=1}^2 |y_i|$  (3.2), which generates all the properties of metric space (3.2).
  - (b)  $|x_i| = |y_i| = p_i$  constrains the range of the total number of domain-to-range set member mappings from  $\sum_{i=1}^n p_i$  to  $\sum_{i=1}^n p_i^2$  mappings (3.3). The case of the largest possible number of domain-to-range set member mappings,  $\sum_{i=1}^n p_i^2$ , converges to the Euclidean distance equation (3.8) and is the set-based reason Euclidean distance is the smallest possible distance between two distinct points in  $\mathbb{R}^n$ .
  - (c) Using the Taylor series and the Euclidean distance equation with two domain intervals sizes yields the arc sine and arc cosine functions. In other words, the parametric variable equating arc sine and arc cosine maps to the notion of angle, where the two domain intervals map to the notion of two line segments (two sides).  
Euclidean geometry [Joy98] and axiomatic geometry (for example, Hilbert [Hil80] and Birkhoff [Bir32], Veblen [Veb04], and Tarski [TG99]) either use notions of line and angle as undefined primitives or as definitions in terms of other undefined primitives.
  - (d) In order to satisfy the triangle inequality of metric space, all hyperbolic geometry distance measures must be  $\leq$  Manhattan distance and  $>$  Euclidean distance.
  - (e) Conjecture: the constraints:  $|x_i| < |y_i|$ ,  $|x_i| = |y_i|$ , and  $|x_i| > |y_i|$  could yield three types of distance spaces: open, flat, and closed. But, a broader definition of a distance measure would be required because open space measures would not satisfy the triangle inequality of metric space. Closed distance space (elliptic geometry distances) would require changing the countable distance space relation from  $|x_i| = |y_i|$  to  $|x_i| \leq |y_i|$ .
- (4) Euclidean volume was derived, where a combination ( $n$ -tuple) of one member from each disjoint domain set corresponds 1-1 to a range set member and where the size of the range set is the Cartesian product of the number members in each domain set. Obviously, each  $n$ -tuple is a Cartesian coordinate.

- (a) Euclidean volume has as many range set elements,  $V_c$ , as  $n$ -tuples,  $T_c$ . Conjecture: the constraints:  $T_c < V_c$ ,  $T_c = V_c$ , and  $T_c > V_c$  yields three types of volume spaces: open, flat, and closed that correspond 1-1 to open, flat, and closed distance spaces.
- (5) Euclidean distance and volume were derived in this article for any number of dimensions. But, from the ordered and symmetric geometries theorem (5.4), time constrains a physical set of totally ordered members, where the commutative law applies to the set and arithmetic operations, is limited to at most three members. This applies to the dimensions of physical space, inertial frames of reference, and so on. Therefore, all higher-dimensional physics equations, where one side of the equations either sums or multiplies the same type of value from of a physical set, are hierarchical 2 or 3-dimensional equations. For example, the spacetime and energy-momentum four-vector lengths, are hierarchical 2-dimensional equations:
  - (a) The spacetime interval (relativistic change in 3-dimensional distance) has the four-vector length,  $\Delta d = \sqrt{(c\Delta t)^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2)}$ , where  $c$  is the speed of light and  $t$  is time [Bru17], can be expressed in the hierarchical 2-dimensional form,  $(c\Delta t)^2 = \Delta d_1^2 + \Delta d_2^2$ , where  $\Delta d_1^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$  and  $\Delta d_2 = \Delta d$ .
  - (b) The well known energy-momentum four-vector used in particle physics, has the hierarchical 2-dimensional length:  $E^2 = (mv^2)^2 + (pc)^2$ , where  $E$  is energy,  $m$  is the resting mass,  $v$  is the 3-dimensional velocity,  $c$  is the speed of light, and  $p$  is the relativistic momentum ( $p = \gamma mv$ , where  $\gamma = (1/(1 - (v/c)^2))^{1/2}$  is the Lorentz factor) [Bru17].

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