The Real Analysis and Combinatorics of Geometry

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Abstract. The range of combinatorial relations between members of each disjoint domain set and members of a corresponding distance set containing the same number of members, where the distance sets sometimes intersect and the set members are the same-sized subintervals of intervals, converges to the triangle inequality, Manhattan distance at the upper boundary, and Euclidean distance at the lower boundary, which provides set-based definitions of: metric space, longest, and shortest distances spanning disjoint sets. The Cartesian product of the same-sized subintervals of intervals converges to the product of interval sizes used in the Lebesgue measure and Euclidean integrals. A cyclic set of 3 dimensions emerges from the same combinatorial relations generating distance and volume. Valid higher dimensional geometries, like the spacetime four-vector, collapse into hierarchical 2 or 3-dimensional geometries. Proofs are verified in Coq.

Contents

1. Introduction

The triangle inequality of a metric space, the Manhattan and Euclidean distance metrics, and the volume equation (product of interval sizes) of the Lebesgue measure and Euclidean integrals (for example, Riemann and Lebesgue integrals) are imported as definitions [Gol76] rather than derived from set-based relations. Using geometric relations as primitives can not expose the set-based relationships that generate those geometric relations and other fundamental, geometric properties.

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A ruler (measuring stick) measures an interval as the nearest integer number of same-sized subintervals (units), where the partial subintervals are ignored. The ruler measure allows defining combinatorial relations, for example a many-to-many relation, between the same-sized subintervals in one interval and the same-sized subintervals in another interval. The discrete, combinatorial relations converge to continuous, bijective functions as the subinterval size converges to zero.

The range of combinatorial relations, from a one-to-one correspondence to a many-to-many relation, between members of each disjoint domain set and members of a corresponding distance set containing the same number of members, where the distance sets sometimes intersect and the set members are the same-sized subintervals of intervals, converges to the triangle inequality, Manhattan distance at the upper boundary, and Euclidean distance at the lower boundary as the subinterval size converges to zero, which provides set-based definitions of: metric space, longest, and shortest distances spanning disjoint sets. The Cartesian product of the same-sized subintervals of intervals converges to the product of interval sizes used in the Lebesgue measure and Euclidean integrals.

A cyclic set of 3 dimensions emerges from the same combinatorial relations generating distance and volume. Valid higher dimensional geometries "collapse" into hierarchical 2 or 3-dimensional geometries. As shown in the summary, the four-vector lengths common in physics, like the spacetime four-vector length, are 2-dimensional Euclidean distance equations that have been "flattened."

The proofs in this article are verified formally using the Coq Proof Assistant [**Coq15**] version 8.7.0. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at:
`https://github.com/treeck/CombinatorialGeometry`.

2. Ruler measure and convergence

DEFINITION 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, p, each subinterval having the same size, c. Notionally:

$$\forall\, c\, s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |a-b| \ \wedge\ c > 0\ \wedge \\ (p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \textstyle\sum_{i=1}^{p} c = pc. \tag{2.1}$$

THEOREM 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\ \ s = |a-b|\ \Rightarrow\ M = \lim_{c\to 0} pc = s.$

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\quad p = floor(s/c)\quad \Rightarrow\quad 0 \le |p - s/c| < 1. \tag{2.2}$$

Multiply all sides of inequality 2.2 by $|c|$:

$$\forall\, c > 0,\quad 0 \le |p - s/c| < 1\quad \Rightarrow\quad 0 \le |pc - s| < |c|. \tag{2.3}$$

$$\forall\, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \\ \Rightarrow\quad \forall\, \epsilon = \delta :\ |c - 0| < \delta\ \wedge\ |pc - s| < \epsilon\ \ :=\ \ M = \lim_{c\to 0} pc = s. \quad \square \tag{2.4}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence, where: $[0, \pi]$, $s = |\pi - 0|$, $c = 10^{-i}$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{i=1},\ 3.14_{i=2},\ 3.141_{i=3}, ..., \pi$.

3. Distance

A simple countable distance measure is that an image (distance) set has the same number of members as a corresponding domain set. For example, the number of steps walked in a distance set must equal the number pieces of land traversed. Generalizing, for each distance set, y_i, containing p_i number of members there exists a corresponding domain set, x_i, with the same p_i number of members.

Notation conventions: The vertical bars around a set is the standard notation for indicating the cardinal (number of members in the set). To prevent over use of the vertical bar, the symbol for "such that" is the colon.

If the domain sets are disjoint ($\sum_{i=1}^{n} |x_i| = |\bigcup_{i=1}^{n} x_i|$) and the distance sets intersect ($\sum_{i=1}^{n} |y_i| > |\bigcup_{i=1}^{n} y_i|$), then a distance set member can map to (be combined with) multiple domain set members. Therefore, the size of the union of the distance sets, d_c, is a function of the number of combinations with domain set members. Notionally:

Definition 3.1. Countable distance range, d_c:

$$\forall\, i\; n \in \mathbb{N}, \quad x_i \subseteq X, \quad \sum_{i=1}^{n} |x_i| = |\bigcup_{i=1}^{n} x_i|, \quad \forall\, x_i\; \exists\, y_i \subseteq Y : \\ |x_i| = |y_i| = p_i \quad \wedge \quad d_c = |Y| = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|.$$

The countable distance range principle (3.1), $|x_i| = |y_i| = p_i$, constrains the range of combinatorial relations from each member of y_i combining with (mapping to) one member of x_i to each member of y_i combining with as many as p_i number of members of x_i. More than p_i number of combinations with a member of y_i would be over-counting combinations, because a member of y_i would map to the same member of x_i more than once.

Using the rule of product, there is a range from $|y_i| \cdot 1 = p_i$ to $|y_i| \cdot p_i = p_i^2$ number of distance-to-domain combinations per distance set. Therefore, $d_c = f(\sum_{i=1}^{n} p_i)$ is largest possible distance because it is the case of the smallest number of combinations (p_i) per distance set. And $d_c = f(\sum_{i=1}^{n} p_i^2)$ is the smallest possible distance because it is the case of the largest number of combinations (p_i^2) per distance set (largest allowed intersection of distance sets).

It will now be proved that using the ruler (2.1) to divide a set of real-valued domain intervals and a distance interval into sets of same-sized subintervals, and applying the ruler convergence theorem (2.2) to the longest and shortest distance cases converge to the real-valued, Manhattan and Euclidean distance equations.

Theorem 3.2. *Manhattan (longest) distance, d, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance."

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain

intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of subintervals and apply the definition of the countable distance range (3.1), where each domain set has a corresponding distance set containing the same p_i number of members.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i| = |y_i|. \tag{3.1}$$

Next, apply the rule of product to the case of one domain set member per distance set member:

$$|y_i| = p_i \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| \cdot 1 = \sum_{i=1}^{n} p_i. \tag{3.2}$$

Apply the countable distance range defintion (3.1) to equation 3.2:

$$\begin{aligned} \sum_{i=1}^{n} |y_i| \cdot 1 = \sum_{i=1}^{n} p_i \quad &\wedge \quad \sum_{i=1}^{n} |y_i| \geq d_c \\ &\Rightarrow \quad \sum_{i=1}^{n} p_i \geq d_c \quad \Rightarrow \quad \exists\, p_i,\, d_c : \sum_{i=1}^{n} p_i = d_c. \end{aligned} \tag{3.3}$$

Multiply both sides of 3.3 by c and apply the ruler convergence theorem (2.2):

$$\begin{aligned} s_i = \lim_{c\to 0} p_i \cdot c \quad &\wedge \quad \sum_{i=1}^{n} (p_i \cdot c) = d_c \cdot c \\ &\Rightarrow \quad \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) = \lim_{c\to 0} d_c \cdot c. \end{aligned} \tag{3.4}$$

Use the ruler to divide the exact size, $d = |d_0 - d_m|$, of the image interval, $[d_0, d_m]$, into a set, Y, containing d_c number of members:

$$\forall\, d_c \in \mathbb{N},\; c > 0\, \exists\, d \in \mathbb{R} : \;\; floor(d/c) = d_c. \tag{3.5}$$

Apply the ruler convergence theorem (2.2):

$$floor(d/c) = d_c \;\; \Rightarrow \;\; d = \lim_{c\to 0} d_c \cdot c. \tag{3.6}$$

Combine equations 3.6 and 3.4:

$$d = \lim_{c\to 0} d_c \cdot c \quad \wedge \quad \sum_{i=1}^{n} s_i = \lim_{c\to 0} d_c \cdot c \quad \Rightarrow \quad d = \sum_{i=1}^{n} s_i. \tag{3.7}$$ □

THEOREM 3.3. *Euclidean (shortest) distance, d, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of subintervals and apply the definition of the countable distance range (3.1), where each domain set has a corresponding distance set containing the same p_i number of members.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i| = |y_i|. \tag{3.8}$$

Apply the rule of product to the largest number of distance-to-domain set combinations, where all p_i number of domain set members, x_i, combine with (map to) each of the p_i number of members in the distance set, y_i:

$$\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2. \tag{3.9}$$

Choose the equality case of the Cauchy-Schwartz inequality:

$$\sum_{i=1}^{n} p_i^2 \leq \Big(\sum_{i=1}^{n} p_i\Big)^2 \;\; \Rightarrow \;\; \exists\, p_i : \sum_{i=1}^{n} p_i^2 = \Big(\sum_{i=1}^{n} p_i\Big)^2 \tag{3.10}$$

Choose the equality case of the countable distance range definition (3.1) and square both sides ($x = y \Rightarrow f(x) = f(y)$):

$$\begin{aligned}(3.11)\quad \textstyle\sum_{i=1}^{n}|y_i| = \sum_{i=1}^{n}p_i \geq d_c \;\; &\Rightarrow \;\; \exists\, p_i,\ d_c : \textstyle\sum_{i=1}^{n}p_i = d_c \\ &\Rightarrow \;\; \exists\, p_i,\ d_c : (\textstyle\sum_{i=1}^{n}p_i)^2 = d_c^2.\end{aligned}$$

Combine equations 3.10 and and 3.11:

$$\begin{aligned}(3.12)\quad \exists\, p_i : \textstyle\sum_{i=1}^{n}p_i^2 = (\sum_{i=1}^{n}p_i)^2 \;\wedge\; \exists\, p_i,\ d_c : (\sum_{i=1}^{n}p_i)^2 = d_c^2 \\ \Rightarrow \;\; \exists\, p_i,\ d_c : \textstyle\sum_{i=1}^{n}p_i^2 = d_c^2.\end{aligned}$$

Multiply both sides of equation 3.12 by c^2 and apply the ruler convergence theorem:

$$\begin{aligned}(3.13)\quad s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \textstyle\sum_{i=1}^{n}(p_i \cdot c)^2 = (d_c \cdot c)^2 \\ \Rightarrow \quad \textstyle\sum_{i=1}^{n}s_i^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \lim_{c\to 0}(d_c \cdot c)^2.\end{aligned}$$

Use the ruler to divide the exact size, $d = |d_0 - d_m|$, of the image interval, $[d_0, d_m]$ into a set, Y, containing d_c number of members:

$$(3.14)\qquad \forall\, d_c \in \mathbb{N},\ c > 0\, \exists\, d \in \mathbb{R} : \mathit{floor}(d/c) = d_c.$$

Apply the ruler convergence theorem (2.2) and then square both sides:

$$(3.15)\qquad \mathit{floor}(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2.$$

Combine equations 3.15 and 3.13:

$$(3.16)\quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 \;\wedge\; \textstyle\sum_{i=1}^{n}s_i^2 = \lim_{c\to 0}(d_c \cdot c)^2 \;\Rightarrow\; d^2 = \sum_{i=1}^{n}s_i^2. \qquad \square$$

3.1. Triangle inequality. The definition of a metric in real analysis is based on the triangle inequality, $\mathbf{d}(\mathbf{u}, \mathbf{w}) \leq \mathbf{d}(\mathbf{u}, \mathbf{v}) + \mathbf{d}(\mathbf{v}, \mathbf{w})$, that has been intuitively motivated by the triangle [**Gol76**]. Applying the ruler (2.1), and ruler convergence theorem (2.2) to the definition of a countable distance range (3.1) yields the real-valued triangle inequality:

$$\begin{aligned}(3.17)\quad & d_c = |Y| = |\textstyle\bigcup_{i=1}^{2}y_i| \leq \sum_{i=1}^{2}|y_i| \quad \wedge \\ & d_c = \mathit{floor}(\mathbf{d}(\mathbf{u}, \mathbf{w})/c) \;\wedge\; |y_1| = \mathit{floor}(\mathbf{d}(\mathbf{u}, \mathbf{v})/c) \;\wedge\; |y_2| = \mathit{floor}(\mathbf{d}(\mathbf{v}, \mathbf{w})/c) \\ & \Rightarrow\; \mathbf{d}(\mathbf{u}, \mathbf{w}) = \lim_{c\to 0} d_c \cdot c \leq \textstyle\sum_{i=1}^{2} \lim_{c\to 0} |y_i| \cdot c = \mathbf{d}(\mathbf{u}, \mathbf{v}) + \mathbf{d}(\mathbf{v}, \mathbf{w}).\end{aligned}$$

The other metric space properties: $\mathbf{d}(\mathbf{u}, \mathbf{w}) = 0 \Leftrightarrow u = w$, $\mathbf{d}(\mathbf{u}, \mathbf{w}) = \mathbf{d}(\mathbf{w}, \mathbf{u})$, and $\mathbf{d}(\mathbf{u}, \mathbf{w}) \geq 0$ also follow from the countable distance range definition.

4. Size (length/area/volume)

The combinatorial relations between all members in set x_1 to each member of set x_2 results in the Cartesian product of $|x_1|\cdot|x_2|$ number of combinations. This section will use the ruler (2.1) and ruler convergence theorem (2.2) to prove that the Cartesian product of the same-sized subintervals of intervals converges to the product of interval sizes as the subinterval converges to zero. The first step is to define a set-based, countable size measure as the Cartesian product of disjoint domain set members.

Definition 4.1. Countable size (length/area/volume) measure, S_c:

$$\forall\, i\ n \in \mathbb{N}, \quad i \in [1, n], \quad x_i \subseteq X, \quad \textstyle\sum_{i=1}^{n} |x_i| = |\bigcup_{i=1}^{n} x_i|, \quad \wedge \quad S_c = \displaystyle\prod_{i=1}^{n} |x_i|.$$

Theorem 4.2. *Euclidean size (length/area/volume), S, is the size of an image interval, $[v_0, v_m]$, corresponding to a set of disjoint intervals:* $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$S = \prod_{i=1}^{n} s_i, \quad S = |v_0 - v_m|, \quad s_i = |a_i - b_i|, \quad i \in [1, n], \quad i, n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_size."

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, x_i of p_i number of subintervals.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i|. \tag{4.1}$$

Use the ruler (2.1) to divide the exact size, $S = |v_0 - v_m|$, of the image interval, $[v_0, v_m]$, into p_S^n subintervals. Every integer number, S_c, does **not** have an integer n^{th} root. However, for those cases where S_c does have an integer n^{th} root, there is a p_S^n that satisfies the definition a countable size measure, S_c (4.1). Notionally:

$$\forall\, p_S^n = S_c \in \mathbb{N},\, \exists\, S \in \mathbb{R},\, x_i\; : floor(S/c) = p_S^n = S_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i. \tag{4.2}$$

Multiply both sides of equation 4.2 by c^n to get the ruler measures:

$$p_S^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3:

$$\begin{aligned} S = \lim_{c \to 0} (p_S \cdot c)^n \quad \wedge \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c) \quad \wedge \quad \lim_{c \to 0}(p_i \cdot c) = s_i \\ \Rightarrow \quad S = \lim_{c \to 0} (p_S \cdot c)^n = \prod_{i=1}^{n} \lim_{c \to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square \end{aligned} \tag{4.4}$$

5. Ordered and symmetric geometries

The commutative property of the union and addition operations in the countable distance range principle (3.1) that generates the triangle inequality, Manhattan and Euclidean distances and the commutative property of the union and multiplication operations in the countable size principle (4.1) that generates length/area/volume allow any sequential (total) ordering of the disjoint domain sets (dimensions) to exist. And the commutative property also allows every dimension to be sequentially adjacent to every other dimension (herein, referred to as a symmetric geometry).

Two equations can have ranges containing an infinite number of values, while having a very small, finite number of values satisfying both equations simultaneously (for example, only two values satisfying the equations of a circle and intersecting straight line). Likewise, each of the two properties, total order and symmetric geometry, individually allow any number of dimensions of distance and volume. But, it will now be proved that satisfying both properties simultaneously limits distance and volume to cyclic set of at most three dimensions.

DEFINITION 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i\; \in \{x_1, \dots, x_n\},$$
$$\textit{successor}\; x_i = x_{i+1} \;\;\wedge\;\; \textit{predecessor}\; x_{i+1} = x_i.$$

where each $x_i \in \{x_1, \dots, x_n\}$ is a set of subintervals of a real-valued domain interval (dimension).

DEFINITION 5.2. Symmetric geometry (every member is sequentally adjacent to every other member):

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \dots, x_n\},\;\; \textit{successor}\; x_i = x_j \;\;\wedge\;\; \textit{predecessor}\; x_j = x_i.$$

THEOREM 5.3. *An ordered and symmetric geometry is a cyclic set.*

$$\textit{successor}\; x_n = x_1 \;\;\wedge\;\; \textit{predecessor}\; x_1 = x_n.$$

The theorem and formal Coq-based proof is "ordered_symmetric_is_cyclic," which is located in the file threed.v.

PROOF. The property of order (5.1) defines unique successors and predecessors for all members except for the successor of x_n and the predecessor of x_1. From the properties of a symmetric geometry (5.2):

$$i = n \;\;\wedge\;\; j = 1 \;\;\wedge\;\; \textit{successor}\; x_i = x_j \;\;\Rightarrow\;\; \textit{successor}\; x_n = x_1. \tag{5.1}$$

$$i = n \;\;\wedge\;\; j = 1 \;\;\wedge\;\; \textit{predecessor}\; x_j = x_i \;\;\Rightarrow\;\; \textit{predecessor}\; x_1 = x_n. \tag{5.2}$$

□

THEOREM 5.4. *An ordered and symmetric geometry is limited to at most 3 members.*

The Coq-based lemmas and proofs in the file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
Because a symmetric and ordered set is a cyclic set (5.3), the successors and predecessors are cyclic:

DEFINITION 5.5. Successor of m is n:

$$\textit{Successor}(m, n, \textit{setsize}) \leftarrow (m = \textit{setsize} \wedge n = 1) \vee (m + 1 \leq \textit{setsize}). \tag{5.3}$$

DEFINITION 5.6. Predecessor of m is n:

$$\textit{Predecessor}(m, n, \textit{setsize}) \leftarrow (m = 1 \wedge n = \textit{setsize}) \vee (m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n (required for a "symmetric" set (5.2)), if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:

$$\textit{Adjacent}(m, n, \textit{setsize}) \leftarrow \textit{Successor}(m, n, \textit{setsize}) \vee \textit{Predecessor}(m, n, \textit{setsize}). \tag{5.5}$$

Every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{5.6}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (1 + 1 \leq 2). \tag{5.7}$$

$$Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (2 = 2 \wedge 1 = 1). \tag{5.8}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (1 + 1 \leq 2). \tag{5.9}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (2 - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (3 = 3 \wedge 1 = 1). \tag{5.11}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (1 = 1 \wedge 3 = 3). \tag{5.12}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (2 + 1 \leq 3). \tag{5.13}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (3 - 1 \geq 1). \tag{5.14}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not adjacent:

$$\begin{aligned} \forall\, setsize > 3: \quad & \neg Successor(1,3,setsize) \\ & \leftarrow Successor(1,2,setsize) \leftarrow (1 + 1 \leq setsize). \end{aligned} \tag{5.15}$$

That is, 2 is the only successor of 1 for all $setsize > 3$, which implies 3 is not a successor of 1 for all $setsize > 3$.

$$\begin{aligned} \forall\, setsize > 3: \quad & \neg Predecessor(1,3,setsize) \\ & \leftarrow Predecessor(1,n,setsize) \leftarrow (1 = 1 \wedge n = setsize). \end{aligned} \tag{5.16}$$

That is, $n = setsize$ is the only predecessor of 1 for all $setsize > 3$, which implies 3 is not a predecessor of 1 for all $setsize > 3$.

$$\begin{aligned} \forall\, setsize > 3: \quad & \neg Adjacent(1,3,setsize) \\ & \leftarrow \neg Successor(1,3,setsize) \wedge \neg Predecessor(1,3,setsize). \quad \square \end{aligned} \tag{5.17}$$

6. Summary

Applying some very simple real analysis, in the form of the ruler measure (2.1) and ruler convergence proof (2.2), to a set of real-valued domain intervals and an image interval yields some new insights into geometry and physics.

(1) Discrete, combinatorial relations converge to the continuous, bijective relations: triangle inequality, Manhattan distance, Euclidean distance and volume.
(2) The ruler measure is used to derive Euclidean distance and volume from many-to-many relationships between the elements of countable sets, where one domain element can map to many image elements. But, a function only allows a domain element to map to a single image element. Therefore, all calculus, functional analysis, analytic geometry, linear algebra, geometric measure theory, etc. that define real-valued geometric functions as primitives are **not** capable of providing insight into the many-to-many relationships generating Euclidean distance and volume.

(3) Ruler measure-based proofs expose the difference between distance and size (length/area/volume) measures: Distance is the combinatorial relation between the members of each disjoint domain set and members of a corresponding image (distance) set. In contrast, volume is a combinatorial relation between the members of disjoint domain sets.
(4) Applying the ruler measure to the countable distance range (3.1) provides the insight that all notions of distance are based on the principle that for each disjoint domain set there exists a corresponding distance set containing the same number of members, where the distance sets sometimes intersect:
  (a) The countable distance range principle converges to the real-valued triangle inequality (3.1), which is the basis for the definition of metric space. The other properties of metric space also come from the countable distance range principle. Therefore, a function is not a distance metric unless it satisfies the more fundamental countable distance range (3.1).
  (b) The upper bound of the countable distance range converging to Manhattan distance (3.2) provides the insight that the largest (longest) monotonic distance path is the case of the minimum number of combinations (mappings) per distance set, where each member in the i^{th} distance set combines with (maps to) only one member in the i^{th} domain set, which is the case of disjoint distance sets.
  (c) The lower bound of the countable distance range converging to Euclidean distance (3.3) provides the insight that the smallest (shortest) possible monotonic distance path is the case of the maximum number of combinations (mappings) per distance set, where each of the p_i number of members in the i^{th} distance set combine with (map to) all p_i number of members in the i^{th} domain set, which is the case of the maximum allowed intersection of the distance sets.
  (d) All $L^{p>2}$ norms generated from the countable distance range principle would require each member of the i^{th} distance set to map to (form a combination with) a member of the i^{th} domain set more than once, which would be over-counting the number of possible mappings (combinations). Therefore, $L^{p>2}$ norms are not valid distance measures. Measure theories have not provided this over-counting insight into $L^{p>2}$ norms.
  (e) Euclidean distance (3.3) was derived from a combinatorial relation without any notions of side, angle, or shape. A parametric variable relating the sizes of two domain intervals can be easily derived using using calculus (Taylor series) and the Euclidean distance equation to generate the arc sine and arc cosine functions of the parametric variable (arc angle). In other words, the notions of side and angle are derived from Euclidean distance, which is the reverse perspective of classical geometry [**Joy98**] [**Loo68**] [**Ber88**] and axiomatic foundations for geometry [**Bir32**] [**Hil80**] [**TG99**].
(5) Applying the ruler measure and ruler convergence proof to the countable size definition (4.1) allows a proof that the Cartesian product of same-sized subintervals of real-valued intervals converges to the product of the

interval sizes (Euclidean volume):
   (a) Euclidean size (length/area/volume) was derived from a combinatorial relation without notions of sides, angles, and shape.
(6) The set-based definitions of countable distance range converging to the triangle inequality, Manhattan distance, Euclidean distance and countable size converging to Euclidean volume provides a self-contained foundation under real analysis, calculus, and measure theory by not having to import those equations from Euclidean geometry as definitions and not having to rely on external geometry notions of side, angle, shape, side-angle-side relation, etc.
(7) The combinatorial relations of countable distance range (3.1) and countable size (4.1) that generate the real-valued triangle inequality, Manhattan distance, Euclidean distance, and volume equations also have a symmetry property (5.2) that limits distance and volume to a cyclic set (5.3) of three dimensions (5.4). This symmetry property explains why only three dimensions of physical space can be observed.
(8) Valid higher dimensional geometries "collapse" into hierarchical 2 or 3-dimensional geometries. The four-vector lengths common in physics are 2-dimensional Euclidean distance equations that have been "flattened." For example, the spacetime four-vector length, $d = \sqrt{(ct)^2 - (x^2 + y^2 + z^2)}$, is sometimes expressed in a form like, $d_2 = \sqrt{(ct)^2 - d_1^2}$, where $d_1 = \sqrt{x^2 + y^2 + z^2}$ and $d_2 = d$.

    Applying the Euclidean distance proof (3.3) to the 2-dimensional Poincaré form, $t^2 = d_1^2 + d_2^2$, $(c = 1)$, provides the perspective that d_1 and d_2 are lengths in two frames of reference (two domains) and the size of the time subintervals are the same size (same speed of light) in both frames of reference.

    A third frame of reference (dimension) allows spacetime to be a seven-vector that can be collapsed into a hierarchical 3-dimensional Euclidean distance.

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