

# The Two Set Relations Generating Euclidean Geometry

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**ABSTRACT.** A ruler-like measure divides sets of real-valued domain and range intervals into same-sized subintervals. The case, where for each disjoint domain set of subintervals there exists a corresponding same-sized range set and the range sets in some cases intersect, converges to: the triangle inequality, Manhattan distance at the upper boundary and Euclidean distance at the lower boundary of the triangle inequality. The Cartesian product of the number of subintervals in each domain interval converges to the product of interval interval sizes (Euclidean area/volume). Time places a constraint on physical sets, which limits physical Euclidean geometry to at most 3 dimensions. All proofs are verified in Coq.

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## 1. Introduction

Most of real analysis is derived from a foundation of number and set-based axioms. But: the triangle inequality, the other properties of metric space, Manhattan distance, Euclidean distance, and Euclidean area/volume, are motivated by Euclidean geometry and used as primitives in measure (metric space, Hausdorff, and Lebesgue) and integration (Lebesgue and Riemann) [Gol76] [Rud76]. This article will use some simple real analysis to motivate and derive the triangle inequality, the other properties of metric space, Manhattan and Euclidean distance from a single set-based axiom and derive volume from another set-based axiom.

The relationships between countable sets generating distance and volume provides new insights into geometry and physics. For example, (*without any notions of side, angle, and shapes*): 1) the set-based reason Euclidean distance is the smallest distance between two distinct points in  $\mathbb{R}^n$ ; 2) the single set relation generating all the properties of metric space; 3) the set-based difference between distance and volume; 4) the set-based reason metric space might not be a sufficient condition for a distance measure; 5) how time constrains the set relations generating *physical* Euclidean geometry to at most 3 dimensions.

The proofs in this article are verified formally using the Coq Proof Assistant [Coq15] version 8.9.0. The Coq-based definitions, theorems, and proofs are in the files “euclidrelations.v” and “threed.v” located at:  
<https://github.com/treeck/RASRGeometry>.

## 2. Ruler measure and convergence

Deriving distance and volume from set and number axioms requires a measure that does not have Euclidean assumptions and also allows the full range of mappings from a one-to-one correspondence to a many-to-many mapping. A ruler (measuring stick) measures a real-valued interval as the nearest integer number of same-sized subintervals (units), where the partial subintervals are ignored.

The ruler measure allows defining relations, for example a many-to-many relation, between the set of same-sized subintervals in one interval and the set of same-sized subintervals in another interval. The countable relations converge to continuous, bijective functions as the subinterval size converges to zero.

**DEFINITION 2.1.** Ruler measure: A ruler measures the size,  $M$ , of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals,  $p$ , each subinterval having the same size,  $c$ . Notionally:

$$(2.1) \quad \forall c \, s \in \mathbb{R}, \, [a, b] \subset \mathbb{R}, \, s = |a - b| \wedge c > 0 \wedge \\ (p = \text{floor}(s/c) \vee p = \text{ceiling}(s/c)) \wedge M = \sum_{i=1}^p c = pc.$$

**THEOREM 2.2.** *Ruler convergence:*

$$\forall [a, b] \subset \mathbb{R}, \, s = |a - b| \Rightarrow M = \lim_{c \rightarrow 0} pc = s.$$

The theorem, “limit\_c\_0\_M.eq\_exact\_size,” and formal proof is in the Coq file, euclidrelations.v.

**PROOF.** (epsilon-delta proof)

By definition of the floor function,  $\text{floor}(x) = \max(\{y : y \leq x, y \in \mathbb{Z}, x \in \mathbb{R}\})$ :

$$(2.2) \quad \forall c > 0, p = \text{floor}(s/c) \wedge 0 \leq |\text{floor}(s/c) - s/c| < 1 \Rightarrow 0 \leq |p - s/c| < 1.$$

Multiply all sides of inequality 2.2 by  $|c|$ :

$$(2.3) \quad \forall c > 0, \quad 0 \leq |p - s/c| < 1 \Rightarrow 0 \leq |pc - s| < |c|.$$

$$(2.4) \quad \forall \delta : |pc - s| < |c| = |c - 0| < \delta \\ \Rightarrow \quad \forall \epsilon = \delta : |c - 0| < \delta \wedge |pc - s| < \epsilon := M = \lim_{c \rightarrow 0} pc = s. \quad \square$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence, where:  $[0, \pi]$ ,  $s = |0 - \pi|$ ,  $c = 10^{-i}$ , and  $p = \text{floor}(s/c) \Rightarrow p \cdot c = 3.1_{i=1}, 3.14_{i=2}, 3.141_{i=3}, \dots, \pi$ .

### 3. Distance

**Notation conventions:** The vertical bars around a set is the standard notation for indicating the cardinal (number of members in the set). To prevent over use of the vertical bar, the symbol for “such that” is the colon.

**3.1. Countable distance space.** The most fundamental notion of distance is that for each disjoint domain set,  $x_i$ , there exists a corresponding range set,  $y_i$ , containing the same number of members,  $p_i$ :  $|x_i| = |y_i| = p_i$ . For example, there should be as many steps walked in the range set,  $y_i$ , as there are pieces of traversed land in the corresponding domain set,  $x_i$ . And the distance,  $d_c$ , spanning one or more domain sets is the size of the union of range sets.

DEFINITION 3.1. Countable distance space,  $d_c$ :

$$\bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad d_c = |\bigcup_{i=1}^n y_i| \quad \wedge \quad |x_i| = |y_i| = p_i.$$

THEOREM 3.2. *Inclusion-exclusion Inequality:*  $|\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$ .

PROOF. This well-known inequality and proof is derived from the inclusion-exclusion principle [CG15] and the axiom,  $u = v - w$ ,  $w \geq 0 \Rightarrow u \leq v$  and shown at a high-level here for completeness. A formal proof using partitioning rather than the da Silva formula, inclusion\_exclusion\_inequality, is in the file euclidrelations.v.

$$(3.1) \quad |\bigcup_{i=1}^n y_i| = \sum_{i=1}^n |y_i| - \sum_{1 \leq i < j \leq n} |y_i \cap y_j| + \cdots + (-1)^{n-1} |\bigcap_{i=1}^n y_i| \quad \wedge \\ \sum_{1 \leq i < j \leq n} |y_i \cap y_j| + \cdots + (-1)^{n-1} |\bigcap_{i=1}^n y_i| \geq 0 \\ \Rightarrow \quad |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|. \quad \square$$

**3.2. Metric Space.** Applying the ruler (2.1) and ruler convergence (2.2) to three range intervals having sizes,  $d(u, w)$ ,  $d(u, v)$ , and  $d(v, w)$ , and using the inequality,  $d_c = |\bigcup_{i=1}^2 y_i| \leq \sum_{i=1}^2 |y_i|$  generates all the properties of metric space. The formal proofs: triangle\_inequality, non\_negativity, identity\_of\_indiscernibles, and symmetry, are in the Coq file, euclidrelations.v.

THEOREM 3.3. *Triangle Inequality:*  $d(u, w) \leq d(u, v) + d(v, w)$ :

PROOF.

$$(3.2) \quad \forall c > 0, |y_1| = \text{floor}(d(u, v)/c) \quad \wedge \quad |y_2| = \text{floor}(d(v, w)/c) \quad \wedge \\ d_c = \text{floor}(d(u, w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \leq |y_1| + |y_2| \\ \Rightarrow \quad \text{floor}(d(u, w)/c) \leq \text{floor}(d(u, v)/c) + \text{floor}(d(v, w)/c) \\ \Rightarrow \quad \text{floor}(d(u, w)/c) \cdot c \leq \text{floor}(d(u, v)/c) \cdot c + \text{floor}(d(v, w)/c) \cdot c \\ \Rightarrow \quad \lim_{c \rightarrow 0} \text{floor}(d(u, w)/c) \cdot c \leq \lim_{c \rightarrow 0} \text{floor}(d(u, v)/c) \cdot c + \lim_{c \rightarrow 0} \text{floor}(d(v, w)/c) \cdot c \\ \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w). \quad \square$$

THEOREM 3.4. *Non-negativity:*  $d(u, w) \geq 0$ .

PROOF.

$$(3.3) \quad \forall c > 0 : \quad d_c = \text{floor}(d(u, w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \geq 0 \\ \Rightarrow \quad \text{floor}(d(u, w)/c) = d_c \geq 0 \quad \Rightarrow \quad d(u, w) = \lim_{c \rightarrow 0} d_c \cdot c \geq 0. \quad \square$$

THEOREM 3.5. *Identity of Indiscernibles:*  $d(w, w) = 0$ .

PROOF.

$$(3.4) \quad \forall d(u, v) = d(v, w) = 0 \wedge d(u, w) \leq d(u, v) + d(v, w) \wedge d(u, w) \geq 0 \\ \Rightarrow d(u, w) = 0.$$

$$(3.5) \quad d(u, w) = 0 \wedge d(u, v) = 0 \Rightarrow w = v.$$

$$(3.6) \quad d(v, w) = 0 \wedge w = v \Rightarrow d(w, w) = 0. \quad \square$$

THEOREM 3.6. *Symmetry*:  $d(v, w) = d(w, v)$ .

PROOF.

$$(3.7) \quad w = v \Rightarrow d(w, w) = d(v, w) \wedge d(w, w) = d(w, v) \Rightarrow d(v, w) = d(w, v). \quad \square$$

**3.3. Distance space range.** Where the range sets intersect, multiple domain set members map to a single range set member. Therefore, the union set size,  $d_c$ , is function of the number of domain-to-range set member mappings.

The property,  $|x_i| = |y_i| = p_i$ , (3.1) constrains the range of domain-to-range set member mappings. Two facts are immediately obvious from the case, where  $p_i = 1$ : 1) Each of the  $p_i$  number of members of  $x_i$  corresponds 1-1 to each member of  $y_i$ , yielding  $|x_i| \cdot 1 = p_i = 1$  number of domain-to-range mappings. 2) Each of the  $p_i$  number of members of  $x_i$  map to *each* member of  $y_i$ , yielding  $|x_i| \cdot |y_i| = p_i^2 = 1$  number of domain-to-range mappings.

Therefore,  $\exists \mathbf{f} : d_c = \mathbf{f}(\sum_{i=1}^n p_i)$  is the largest possible distance because it is the case of the smallest number of domain-to-range mappings (no intersection of the range sets). And  $\exists \mathbf{f} : d_c = \mathbf{f}(\sum_{i=1}^n p_i^2)$  is the smallest possible distance because it is the case of the largest number of domain-to-range mappings (largest allowed intersection of range sets). Applying the ruler (2.1) and ruler convergence theorem (2.2) to the largest and smallest countable distance cases yields the real-valued, Manhattan and Euclidean distance functions.

### 3.4. Manhattan distance.

THEOREM 3.7. *Manhattan (longest) distance,  $d$ , is the size of the distance interval,  $[d_0, d_m]$ , mapping to a set of disjoint domain intervals,  $\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$ , where:*

$$d = \sum_{i=1}^n s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, “taxicab\_distance,” and formal proof is in the Coq file, euclidrelations.v.

PROOF.

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance,  $d_c$ , is the equality case:

$$(3.8) \quad d_c = |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i| \wedge |y_i| = p_i \\ \Rightarrow d_c \leq \sum_{i=1}^n |y_i| = \sum_{i=1}^n p_i \Rightarrow \exists p_i, d_c : d_c = \sum_{i=1}^n p_i.$$

Multiply both sides of equation 3.10 by  $c$  and take the limit:

$$(3.9) \quad d_c = \sum_{i=1}^n p_i \Rightarrow d_c \cdot c = \sum_{i=1}^n (p_i \cdot c) \Rightarrow \lim_{c \rightarrow 0} d_c \cdot c = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c).$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of  $d$ :

$$(3.10) \quad d = |d_0 - d_m| \Rightarrow \exists c d : \text{floor}(d/c) = d_c \Rightarrow d = \lim_{c \rightarrow 0} d_c \cdot c.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of  $s_i$ :

$$(3.11) \quad \forall i \in [1, n], s_i = |a_i - b_i| \Rightarrow \text{floor}(s_i/c) = p_i \Rightarrow \lim_{c \rightarrow 0} p_i \cdot c = s_i.$$

Combine equations 3.10, 3.9, 3.11:

$$(3.12) \quad d = \lim_{c \rightarrow 0} d_c \cdot c \quad \wedge \quad \lim_{c \rightarrow 0} d_c \cdot c = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c) \quad \wedge \\ \lim_{c \rightarrow 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c \rightarrow 0} d_c \cdot c = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c) = \sum_{i=1}^n s_i. \quad \square$$

### 3.5. Euclidean distance.

**THEOREM 3.8.** *Euclidean (shortest) distance,  $d$ , is the size of the distance interval,  $[d_0, d_m]$ , mapping to a set of disjoint domain intervals,  $\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$ , where:*

$$d^2 = \sum_{i=1}^n s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, “Euclidean\_distance,” and formal proof is in the Coq file, euclidrelations.v.

**PROOF.**

Apply the rule of product to the largest number of domain-to-range set mappings, where all  $p_i$  number of domain set members,  $x_i$ , map to each of the  $p_i$  number of members in the range set,  $y_i$ :

$$(3.13) \quad \sum_{i=1}^n |y_i| \cdot |x_i| = \sum_{i=1}^n p_i^2.$$

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$(3.14) \quad d_c = |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i| \quad \wedge \quad |y_i| = p_i \\ \Rightarrow \quad d_c \leq \sum_{i=1}^n |y_i| = \sum_{i=1}^n p_i \quad \Rightarrow \quad \exists p_i, d_c : d_c = \sum_{i=1}^n p_i.$$

Square both sides of equation 3.14 ( $x = y \Leftrightarrow f(x) = f(y)$ ):

$$(3.15) \quad \exists p_i, d_c : d_c = \sum_{i=1}^n p_i \Leftrightarrow \exists p_i, d_c : d_c^2 = (\sum_{i=1}^n p_i)^2.$$

Apply the Cauchy-Schwartz inequality to equation 3.15 and select the smallest distance (equality) case:

$$(3.16) \quad d_c^2 = (\sum_{i=1}^n p_i)^2 \geq \sum_{i=1}^n p_i^2 \quad \Rightarrow \quad \exists p_i : d_c^2 = \sum_{i=1}^n p_i^2.$$

Multiply both sides of equation 3.16 by  $c^2$ , simplify, and take the limit.

$$(3.17) \quad d_c^2 = \sum_{i=1}^n p_i^2 \Rightarrow d_c^2 \cdot c^2 = \sum_{i=1}^n p_i^2 \cdot c^2 \Leftrightarrow (d_c \cdot c)^2 = \sum_{i=1}^n (p_i \cdot c)^2 \\ \Rightarrow \quad \lim_{c \rightarrow 0} (d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.18) \quad \exists c d : \text{floor}(d/c) = d_c \Rightarrow d = \lim_{c \rightarrow 0} d_c \cdot c \Rightarrow d^2 = \lim_{c \rightarrow 0} (d_c \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.19) \quad \forall i \in [1, n], \text{floor}(s_i/c) = p_i \Rightarrow \lim_{c \rightarrow 0} p_i \cdot c = s_i \\ \Rightarrow \quad \lim_{c \rightarrow 0} (p_i \cdot c)^2 = s_i^2 \Rightarrow \quad \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)^2 = \sum_{i=1}^n s_i^2.$$

Combine equations 3.18, 3.17, 3.19:

$$(3.20) \quad d^2 = \lim_{c \rightarrow 0} (d_c \cdot c)^2 \quad \wedge \quad \lim_{c \rightarrow 0} (d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)^2 \quad \wedge \\ \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)^2 = \sum_{i=1}^n s_i^2 \\ \Rightarrow \quad d^2 = \lim_{c \rightarrow 0} (d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)^2 = \sum_{i=1}^n s_i^2. \quad \square$$

#### 4. Euclidean Volume

A location is a range set member that corresponds 1-1 to a combination (tuple) of one member from each domain set. The number of all possible locations is the Cartesian product of the number members in each domain set. Notionally:

DEFINITION 4.1. All Possible Locations,  $V_c$ :

$$\bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad V_c = \prod_{i=1}^n |x_i|.$$

THEOREM 4.2. *Euclidean volume is the largest possible set of all real-valued locations,  $V$ , corresponding to a disjoint set of real-valued domain intervals:  $\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$ , where:*

$$V = \prod_{i=1}^n s_i, \quad V = |v_0 - v_m|, \quad s_i = |a_i - b_i|.$$

The theorem, “Euclidean\_volume,” and formal proof is in the Coq file, euclidrelations.v.

PROOF.

Use the ruler (2.1) to divide the exact size,  $s_i = |a_i - b_i|$ , of each of the domain intervals,  $[a_i, b_i]$ , into a set,  $x_i$  of  $p_i$  number of subintervals.

$$(4.1) \quad \forall i \ n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad \text{floor}(s_i/c) = p_i = |x_i|.$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$(4.2) \quad \text{floor}(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c \rightarrow 0} (p_i \cdot c) = s_i.$$

Use the ruler (2.1) to divide the exact size,  $V = |v_0 - v_m|$ , of the range interval,  $[v_0, v_m]$ , into  $p^n$  subintervals. Use those cases, where  $V_c$  has an integer  $n^{\text{th}}$  root.

$$(4.3) \quad \forall p^n = V_c \in \mathbb{N}, \exists V \in \mathbb{R}, x_i : \text{floor}(V/c^n) = V_c = p^n = \prod_{i=1}^n |x_i| = \prod_{i=1}^n p_i.$$

Apply the ruler convergence theorem (2.2) to equation 4.3 and simplify:

$$(4.4) \quad \text{floor}(V/c^n) = p^n \quad \Rightarrow \quad V = \lim_{c \rightarrow 0} p^n \cdot c^n = \lim_{c \rightarrow 0} (p \cdot c)^n.$$

Multiply both sides of equation 4.3 by  $c^n$  and simplify:

$$(4.5) \quad p^n = \prod_{i=1}^n p_i \quad \Rightarrow \quad p^n \cdot c^n = \left( \prod_{i=1}^n p_i \right) \cdot c^n \quad \Leftrightarrow \quad (p \cdot c)^n = \prod_{i=1}^n (p_i \cdot c).$$

Combine equations 4.4, 4.5, and 4.2:

$$(4.6) \quad V = \lim_{c \rightarrow 0} (p \cdot c)^n \quad \wedge \quad (p \cdot c)^n = \prod_{i=1}^n (p_i \cdot c) \quad \wedge \quad \lim_{c \rightarrow 0} (p_i \cdot c) = s_i$$

$$\Rightarrow \quad V = \lim_{c \rightarrow 0} (p \cdot c)^n = \prod_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c) = \prod_{i=1}^n s_i. \quad \square$$

#### 5. Ordered and symmetric geometries

The set operations of countable distance range (3.1) and all possible locations (4.1) requires sequencing through each set. The commutative property of the set operations also allows sequencing, where each set can be sequentially adjacent to any other set, herein referred to as a symmetric geometry.

From a combinatoric perspective, there are  $n!$  number of sequential arrangements of any  $n$  number of the sets, where there are two arrangements having a set,

$x_i$ , that is sequentially adjacent (once as a predecessor and once as a successor) to a set,  $x_j$ . But, physical sets have the additional constraint of time.

Arrange a set of beans at time  $t_0$  into a sequential order and label them,  $\{b_1, b_2, b_3, b_4\}$ . Rearranging into a new sequential order,  $\{b_1, b_2, b_4, b_3\}$ , requires changing some of the bean successors and predecessors at time  $t_1$ . And the previous order is no longer true (no longer exists).

A physical set can have only one sequential order *at a time* because each member can have at most one successor and at most one predecessor *at a time*. It will now be proved that a set (of sets or dimensions) satisfying the constraints of a single sequential (total) order and symmetric *at the same time* defines a cyclic set containing at most 3 members (in this case, 3 dimensions of physical space).

DEFINITION 5.1. Ordered geometry:

$$\forall i \ n \in \mathbb{N}, \ i \in [1, n-1], \ \forall x_i \in \{x_1, \dots, x_n\}, \\ \text{successor } x_i = x_{i+1} \ \wedge \ \text{predecessor } x_{i+1} = x_i.$$

DEFINITION 5.2. Symmetric geometry (every set member is sequentially adjacent to any other member):

$$\forall i \ j \ n \in \mathbb{N}, \ \forall x_i \ x_j \in \{x_1, \dots, x_n\}, \ \text{successor } x_i = x_j \ \wedge \ \text{predecessor } x_j = x_i.$$

THEOREM 5.3. *An ordered and symmetric set is a cyclic set.*

$$\text{successor } x_n = x_1 \ \wedge \ \text{predecessor } x_1 = x_n.$$

The theorem, “ordered\_symmetric\_is\_cyclic,” and formal proof is in the Coq file, `threed.v`.

PROOF. The property of order (5.1) defines unique successors and predecessors for all set members except for the successor of  $x_n$  and the predecessor of  $x_1$ . Therefore, the only member that can be a successor of  $x_n$ , without creating a contradiction, is  $x_1$ . And the only member that can be a predecessor of  $x_1$ , without creating a contradiction, is  $x_n$ . From the properties of a symmetric geometry (5.2):

$$(5.1) \quad i = n \ \wedge \ j = 1 \ \wedge \ \text{successor } x_i = x_j \ \Rightarrow \ \text{successor } x_n = x_1.$$

$$(5.2) \quad i = n \ \wedge \ j = 1 \ \wedge \ \text{predecessor } x_j = x_i \ \Rightarrow \ \text{predecessor } x_1 = x_n. \quad \square$$

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file `threed.v` are:

**Lemmas:** `adj111`, `adj122`, `adj212`, `adj123`, `adj133`, `adj233`, `adj213`, `adj313`, `adj323`, and `not_all_mutually_adjacent_gt_3`.

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.

It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

DEFINITION 5.5. Cyclic successor of  $m$  is  $n$ :

$$(5.3) \quad \text{Successor}(m, n, \text{setsize}) \leftarrow (m = \text{setsize} \wedge n = 1) \vee (m + 1 \leq \text{setsize}).$$

DEFINITION 5.6. Cyclic predecessor of  $m$  is  $n$ :

$$(5.4) \quad \text{Predecessor}(m, n, \text{setsize}) \leftarrow (m = 1 \wedge n = \text{setsize}) \vee (m - 1 \geq 1).$$

DEFINITION 5.7. Adjacent: member  $m$  is sequentially adjacent to member  $n$  if the cyclic successor of  $m$  is  $n$  or the cyclic predecessor of  $m$  is  $n$ . Notionally:

$$(5.5) \quad \text{Adjacent}(m, n, \text{setsize}) \leftarrow \text{Successor}(m, n, \text{setsize}) \vee \text{Predecessor}(m, n, \text{setsize}).$$

Every member is adjacent to every other member, where  $\text{setsize} \in \{1, 2, 3\}$ :

$$(5.6) \quad \text{Adjacent}(1, 1, 1) \leftarrow \text{Successor}(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1).$$

$$(5.7) \quad \text{Adjacent}(1, 2, 2) \leftarrow \text{Successor}(1, 2, 2) \leftarrow (1 + 1 \leq 2).$$

$$(5.8) \quad \text{Adjacent}(2, 1, 2) \leftarrow \text{Successor}(2, 1, 2) \leftarrow (2 = 2 \wedge 1 = 1).$$

$$(5.9) \quad \text{Adjacent}(1, 2, 3) \leftarrow \text{Successor}(1, 2, 3) \leftarrow (1 + 1 \leq 2).$$

$$(5.10) \quad \text{Adjacent}(2, 1, 3) \leftarrow \text{Predecessor}(2, 1, 3) \leftarrow (2 - 1 \geq 1).$$

$$(5.11) \quad \text{Adjacent}(3, 1, 3) \leftarrow \text{Successor}(3, 1, 3) \leftarrow (3 = 3 \wedge 1 = 1).$$

$$(5.12) \quad \text{Adjacent}(1, 3, 3) \leftarrow \text{Predecessor}(1, 3, 3) \leftarrow (1 = 1 \wedge 3 = 3).$$

$$(5.13) \quad \text{Adjacent}(2, 3, 3) \leftarrow \text{Successor}(2, 3, 3) \leftarrow (2 + 1 \leq 3).$$

$$(5.14) \quad \text{Adjacent}(3, 2, 3) \leftarrow \text{Predecessor}(3, 2, 3) \leftarrow (3 - 1 \geq 1).$$

Must prove that for all  $\text{setsize} > 3$ , there exist non-adjacent members. For example, the first and third members are not adjacent:

$$(5.15) \quad \forall \text{setsize} > 3 : \neg \text{Successor}(1, 3, \text{setsize}) \\ \leftarrow \text{Successor}(1, 2, \text{setsize}) \leftarrow (1 + 1 \leq \text{setsize}).$$

That is, 2 is the only successor of 1 for all  $\text{setsize} > 3$ , which implies 3 is not a successor of 1 for all  $\text{setsize} > 3$ .

$$(5.16) \quad \forall \text{setsize} > 3 : \neg \text{Predecessor}(1, 3, \text{setsize}) \\ \leftarrow \text{Predecessor}(1, n, \text{setsize}) \leftarrow (1 = 1 \wedge n = \text{setsize}).$$

That is,  $n = \text{setsize}$  is the only predecessor of 1 for all  $\text{setsize} > 3$ , which implies 3 is not a predecessor of 1 for all  $\text{setsize} > 3$ .

$$(5.17) \quad \forall \text{setsize} > 3 : \neg \text{Adjacent}(1, 3, \text{setsize}) \\ \leftarrow \neg \text{Successor}(1, 3, \text{setsize}) \wedge \neg \text{Predecessor}(1, 3, \text{setsize}). \quad \square$$

That is, for all  $\text{setsize} > 3$ , some elements are not sequentially adjacent to every other element (violates the symmetry property).

## 6. Summary

Applying the ruler measure (2.1) and ruler convergence proof (2.2), to a set of real-valued domain intervals and a range interval yields the following new insights into geometry and physics.

- (1) Distance is a function of the number of domain-to-range set member mappings. Area/volume is a function of the number of domain-to-domain set member mappings.



- (a) Area and volume satisfying the criteria of metric space, while not being a function of domain-to-range set mappings, calls into question whether the definition of metric space is a sufficient criteria for a distance measure.
  - (b) Other types of measure, like metric space, Hausdorff, and Lebesgue [Gol76] [Rud76], can not provide these mapping insights.
- (2) All notions of distance are derived from the principle that there should be as many steps walked,  $p_i$ , in the range set,  $y_i$ , as there are pieces of traversed land in the corresponding domain set,  $x_i$ :  $|x_i| = |y_i| = p_i$ . And the countable distance,  $d_c$ , spanning disjoint domain sets is the size of the union of the range sets:  $d_c = |\bigcup_{i=1}^n y_i|$  (3.1).
- (a) A direct consequence of the inclusion-exclusion principle [CG15] is the set relation,  $d_c = |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$  (3.2). The ruler measure makes possible proofs that this single set relation generates all the properties of metric space (3.2).
  - (b) The countable distance space property,  $|x_i| = |y_i|$  (3.1), constrains the range of domain-to-range set member mappings from  $\sum_{i=1}^n p_i$  to  $\sum_{i=1}^n p_i^2$ .  $d_c = \sum_{i=1}^n p_i \Leftrightarrow d_c^2 = (\sum_{i=1}^n p_i)^2 \geq \sum_{i=1}^n p_i^2$ . The smallest possible distance case is the equality case:  $d_c^2 = \sum_{i=1}^n p_i^2$ , where applying the ruler measure converges to the smallest possible real-valued distance, Euclidean distance (3.8). Other types of measures can only assume Euclidean distance is the smallest possible distance between two points in  $\mathbb{R}^n$ .
  - (c) The constraints:  $|x_i| < |y_i|$ ,  $|x_i| = |y_i|$ , and  $|x_i| > |y_i|$  yields three types of distance spaces: open, flat, and closed.
- (3) The notion of a location (a point) as a 1-1 correspondence to a tuple of domain set members is the notion of a coordinate in geometry. The ruler measure allows proof that the set of all possible locations generates the real-valued area and volume equations.
- (a) Euclidean volume has as many range set elements,  $V_r$ , as domain set combinations,  $V_c$ . The constraints:  $V_c < V_r$ ,  $V_c = V_r$ , and  $V_c > V_r$  yields three types of volume spaces: open, flat, and closed.
  - (b) The open, flat, and closed volume spaces correspond to the open, flat, and closed distance spaces.
- (4) The two axioms, countable distance space (3.1) and all possible locations (4.1) in this article, comprise an axiomatic foundation for Euclidean geometry that is purely set-based and does not rely on primitives like line, angle, shape, between, congruence, vector, etc. When each set member has the property of size, then the two axioms generate all the properties of metric space, Manhattan distance, Euclidean distance, and Euclidean length/area/volume.
- (a) Using the Taylor series and the Euclidean distance equation with two domain intervals sizes yields the arc sine and arc cosine functions. In other words, the parametric variable equating arc sine and arc cosine maps to the notion of angle, where the two domain intervals map to the notion of two line segments or two sides.
  - (b) In contrast, other axiomatic foundations for geometry either use notions of line and angle as undefined primitives (for example, Birkhoff

- [Bir32] and Hilbert [Hil80]) or as definitions in terms of other undefined primitives (for example, Veblen [Veb04] and Tarski [TG99]).
- (c) Variations of the domain-to-range set mappings easily extend this foundation to open and closed spaces.
  - (d) This foundation makes real analysis and measure theory more self-contained by deriving the properties of metric space, Euclidean distance, and Euclidean volume from set-based axioms rather than importing them as primitives from Euclidean geometry.
- (5) The set and arithmetic operations of the set-based axioms of distance and volume require sequencing through sets (dimensions of sets), where any dimension can be sequentially adjacent to any other dimension (symmetric). But, a physical set can have only one sequential order *at a time* because each member can have at most one successor and at most one predecessor *at a time*. The properties of a single, total order (5.1) and symmetric (5.2) defines a cyclic set (5.3) of at most 3 members (5.4) (3 dimensions of physical space).
  - (6) The proof showing that more than 3 dimensions of physical space would lead to contradictions (5.4) constrains all higher dimensional physics theories to *hierarchical* 2 or 3-dimensional geometries. For example, the four-vectors common in physics [Bru17] are hierarchical, 2-dimensional geometries that have been “flattened.”

The spacetime four-vector length,  $d = \sqrt{(ct)^2 - (x^2 + y^2 + z^2)}$ , where  $c$  is the speed of light and  $t$  is time, can be expressed in a form like,  $(ct)^2 = d_1^2 + d_2^2$ , where  $d_1^2 = x^2 + y^2 + z^2$  and  $d_2 = d$ . Likewise, the energy-momentum four-vector has the 2-dimensional form:  $E^2 = (mv^2)^2 + (pc)^2$ , where  $E$  is energy,  $m$  is the resting mass,  $v$  is the *3-dimensional* velocity,  $c$  is the speed of light, and  $p$  is the relativistic momentum ( $p = \gamma mv$ , where  $\gamma = (1/(1 - (v/c)^2))^{1/2}$  is the Lorentz factor).

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