A Combinatorial Foundation for Analytic Geometry

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Abstract. Using a ruler-like measure of intervals with real analysis allows proofs that provide new insights into real analysis, measure theory, and geometry: A set-based definition of a countable distance range applied to sets of subintervals of intervals converges to the taxicab distance equation as the upper boundary, the Euclidean distance equation as the lower boundary, and the triangle inequality over the full range, which provides counting-based motivations for the definitions of metric space and Euclidean distance independent of elementary geometry. The Cartesian product of the subintervals of intervals converges to the product of interval sizes used in the Lebesgue measure and Euclidean integrals. Also combinatorics limits a geometry that is both ordered and symmetric to a cyclic set of at most 3 dimensions, which is the basis of the right-hand rule. Implications for non-Euclidean geometries and higher dimensional geometries are discussed. All the proofs are verified in Coq.

Contents

1. Introduction

The triangle inequality of a metric space, Euclidean distance metric, and the volume equation (product of interval sizes) of the Lebesgue measure and Euclidean integrals are imported into mathematical analysis from Euclidean geometry [**Gol76**] as primitives rather than derived from set and number theory-based definitions. As a consequence, mathematical analysis has provided no insight into the counting principles that motivate and generate those geometric relations.

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Real analysis and measure theory have not provided any proofs that combinatorial relations between the subintervals of a set of domain intervals and the subintervals of an image interval converge to the triangle inequality, Euclidean distance, and volume equations, as the subinterval size goes to zero. Understanding the combinatorial relations generating the triangle inequality and Euclidean distance provides counting-based insights into the notions of a distance measure and smallest distance that importing as primitives from Euclidean geometry does not provide. Further, the Lebesgue measure and Euclidean integrals sum the product of interval sizes (Euclidean volume) without proof that the Cartesian product of the subintervals of intervals converges to the product of intervals sizes.

The various traditional indefinite integrals (antiderivatives) derive a real-valued equation from a **real-valued, continuous function** relating the **sizes** of the subintervals. In contrast, what is needed for counting-based (combinatorial) proofs is an indefinite integration that derives a real-valued equation from a **combinatorial function** relating the integer **number** of same-sized subintervals of domain intervals to the integer **number** of same-sized subintervals in an image interval.

Combinatorial integration requires measuring the number of same-sized subintervals in both the domain and image intervals similar to using a ruler (measuring stick). Unlike traditional integration, the ruler is an approximate measure that ignores partial subintervals in **both** the domain and image intervals.

Using the ruler measure, the size of subintervals is the same in both the domain and image intervals and the number of subintervals in each domain and image interval can vary. In contrast, the traditional method of dividing a set of intervals into subintervals, the number of subintervals is the same in both the domain and image intervals and the size of some subintervals can vary.

The Euclidean volume and distance equations can be extended to any number of dimensions. So, why does classical Euclidean geometry appear to be limited to three dimensions? Lack of insight into the counting principles generating distance and volume has prevented identification of the properties that can limit both Euclidean and non-Euclidean geometries to at most three, cyclic dimensions.

The proofs in this article are verified formally using the Coq Proof Assistant [**Coq15**] version 8.4p16. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at:
https://github.com/treeck/CombinatorialGeometry.

2. Ruler measure and convergence

Definition 2.1. Ruler measure: A ruler measures the size of a closed, open, or semi-open interval as the nearest integer number of whole subintervals, p, times the subinterval size, c, where c is the independent variable. Notionally:

$$\begin{aligned}(2.1)\quad & \forall\, c\, s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |b-a| \ \ \wedge\ \ c > 0\ \wedge \\ & \qquad (p = floor(s/c)\ \ \vee\ \ p = ceiling(s/c))\ \ \wedge\ \ M = \Sigma_{p=1}^{\infty} c = \lim_{c\to 0} pc.\end{aligned}$$

Theorem 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\ \ s = |b-a|\ \ \Rightarrow\ \ M = \lim_{c\to 0} pc = s.$

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : \; y \leq x, \; y \in \mathbb{Z}, \; x \in \mathbb{R}\})$:

$$\forall \, c > 0, \quad p = floor(s/c) \quad \Rightarrow \quad 0 \leq |p - s/c| < 1. \tag{2.2}$$

Multiply all sides by $|c|$:

$$\forall \, c > 0, \quad 0 \leq |p - s/c| < 1 \quad \Rightarrow \quad 0 \leq |pc - s| < |c|. \tag{2.3}$$

$$\begin{gathered} \forall \, c > 0, \, \exists \, \delta, \, \epsilon \, : \, 0 \leq |pc - s| < |c| = |c - 0| < \delta = \epsilon \\ \Rightarrow \quad 0 < |c - 0| < \delta \quad \wedge \quad 0 \leq |pc - s| < \epsilon = \delta \quad := \quad M = \lim_{c \to 0} pc = s. \end{gathered} \tag{2.4}$$

□

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure).

For example, showing convergence using the interval, $[0, \pi]$, $s = |\pi - 0|$, $c = 10^{-i}, i \in \mathbb{N}$, and $p = floor(s/c)$, then, $p \cdot c = 3.1_{i=1}, \; 3.14_{i=2}, \; 3.141_{i=3}, ..., \pi$.

3. Distance

A discrete measure of distance is one image (distance) set element for each domain set element. If one distance element can correspond to multiple domain set elements, then the distance set will contain fewer elements than the domain set.

The constraint that for each i^{th} disjoint domain subset containing p_i number of elements there exists a distance subset with the same p_i number of elements results in a defined range of possible distance set sizes as a function of the number of distance-to-domain correspondences per distance subset. Notionally:

Definition 3.1. Countable distance range, d_c:

$$\forall \, i \; n \in \mathbb{N}, \;\; x_i \subseteq X, \;\; \bigcap_{i=1}^{n} x_i = \emptyset, \; \forall \, x_i \; \exists \, y_i \subseteq Y : \;\; |x_i| = |y_i| \; \wedge \; d_c = |Y|.$$

Notation conventions: In the definition of countable distance range (3.1), the vertical bars around a set is the standard notation for indicating the cardinal (number of elements in the set). To prevent too much over use of the vertical bar, the symbol for "such that" is the colon.

The countable distance range property, $|x_i| = |y_i|$, constrains the number of possible correspondences of a distance subset element to domain subset elements from one correspondence per distance element, up to as many as p_i number of correspondences per distance set element. Using the rule of product, there is a range of p_i to p_i^2 number of distance-to-domain correspondences per distance subset. Therefore, $d_c = f(\sum_{i=1}^{n} p_i)$ is the largest possible distance (a function of the smallest number of correspondences per distance subset element). $d_c = f(\sum_{i=1}^{n} p_i^2)$ is the smallest possible distance (a function of the largest number of correspondences per distance subset element).

Using the ruler (2.1) to divide a set of real-valued domain intervals and distance interval into sets of same-sized subintervals, and applying the ruler convergence theorem (2.2) proves that the largest and shortest distance cases converges to the real-valued taxicab and Euclidean distance equations. The convergence proofs of the taxicab and Euclidean distance equations requires the strategy of showing that the right and left sides of a proposed counting-based equation both converge to the same real value and therefore are equal. In other words, the propositional logic, $A = B \, \wedge \, C = B \; \Rightarrow \; A = C$, is used.

THEOREM 3.2. *Taxicab (largest) distance, d, is the size of the distance interval,* $[y_0, y_m]$*, corresponding to a set of disjoint domain intervals,* $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$*, where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|.$$

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\ |\{x_i : x_i \in \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\}| = |\{y_i : y_i \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_i}\}\}| = p_i. \tag{3.1}$$

Next, apply the definition of the countable distance range (3.1):

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad y \in y_i \subseteq Y \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i = |\{y : \; y \in Y\}|. \tag{3.2}$$

Multiply both sides of 3.2 by c and apply the ruler convergence theorem (2.2):

$$s_i = \lim_{c \to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n} (p_i \cdot c) = |\{y\}| \cdot c \\ \Rightarrow \quad \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) = \lim_{c \to 0} |\{y\}| \cdot c. \tag{3.3}$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$\forall\, c > 0, \quad p_d = floor(d/c) = |\{y : \; y \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_d}\} = Y\}|. \tag{3.4}$$

Multiply both sides of 3.4 by c and apply the ruler convergence theorem (2.2):

$$d = \lim_{c \to 0} p_d \cdot c \;\wedge\; p_d \cdot c = |\{y\}| \cdot c \;\Rightarrow\; d = \lim_{c \to 0} p_d \cdot c = \lim_{c \to 0} |\{y\}| \cdot c. \tag{3.5}$$

Combine equations 3.5 and 3.3:

$$d = \lim_{c \to 0} |\{y\}| \cdot c \quad \wedge \quad \sum_{i=1}^{n} s_i = \lim_{c \to 0} |\{y\}| \cdot c \quad \Rightarrow \quad d = \sum_{i=1}^{n} s_i. \tag{3.6}$$

□

THEOREM 3.3. *Euclidean (smallest) distance, d, is the size of the distance interval,* $[y_0, y_m]$*, corresponding to a set of disjoint domain intervals,* $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$*, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}].$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\ x_i = \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\} \quad \wedge \quad y_i = \{y_{1_i}, y_{2_i}, \ldots, y_{p_i}\}\}|. \tag{3.7}$$

Next, apply the definition of the countable distance range (3.1) and the rule of product:

$$\sum_{i=1}^{n}|y_i|\cdot|x_i| = \sum_{i=1}^{n}p_i^2 = \sum_{i=1}^{n}|y_i|^2 = \sum_{i=1}^{n}|\{(y_a, y_b) : y_a\, y_b \in y_i\}|, \tag{3.8}$$

where each pair, (y_a, y_b), represents a combination (correspondence) between two elements in the distance subset, y_i. From the definition of countable distance range (3.1), the distance subsets can intersect, which results in a range of possible distance set sizes. Applying the inclusion-exclusion principle:

$$|\cap_{i=1}^{n}y_i| \geq 0 \quad \Rightarrow \quad \sum_{i=1}^{n}|y_i| \geq |\cup_{i=1}^{n}y_i| = |Y|. \tag{3.9}$$

From combining equation 3.8 and the equality case of relation 3.9:

$$\begin{aligned}\sum_{i=1}^{n}|y_i| = \sum_{i=1}^{n}p_i \geq |\cup_{i=1}^{n}y_i| = |Y| \\ \Rightarrow\ \exists\, y_i,\ Y\ :\ \sum_{i=1}^{n}|y_i| = \sum_{i=1}^{n}p_i = |\cup_{i=1}^{n}y_i| = |Y| \\ \Rightarrow \quad \sum_{i=1}^{n}|y_i|^2 = \sum_{i=1}^{n}p_i^2 = \sum_{i=1}^{n}|\{(y_a, y_b) : y_a y_b \in y_i\}| = |\{(y_a, y_b) : y_a y_b \in Y\}|.\end{aligned} \tag{3.10}$$

Multiply both sides of equation 3.10 by c^2 and apply the ruler convergence theorem.

$$\begin{aligned}s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n}(p_i \cdot c)^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \\ \Rightarrow \quad \sum_{i=1}^{n} s_i^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2.\end{aligned} \tag{3.11}$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$\begin{aligned}\forall\, c > 0, \quad p_d = floor(d/c) = |\{y_{1_i}, y_{2_i}, \ldots, y_{p_d}\}| = |Y| \\ \Rightarrow \quad p_d^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\}|,\end{aligned} \tag{3.12}$$

where $\{(y_a, y_b)\}$ is the set of all combination pairs of elements of Y. Multiply both sides of 3.12 by c^2 and apply the ruler convergence theorem (2.2):

$$\begin{aligned}d = \lim_{c\to 0} p_d \cdot c \quad \wedge \quad (p_d \cdot c)^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \\ \Rightarrow \quad d^2 = \lim_{c\to 0}(p_d \cdot c)^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2.\end{aligned} \tag{3.13}$$

Combine equations 3.12 and 3.13:

$$\begin{aligned}d^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \quad \wedge \\ \sum_{i=1}^{n} s_i^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \quad \Rightarrow \quad d^2 = \sum_{i=1}^{n} s_i^2. \quad \square\end{aligned} \tag{3.14}$$

3.1. Triangle inequality. The definition of a metric in real analysis is based on the triangle inequality, $\mathbf{d}(\mathbf{u}, \mathbf{w}) \leq \mathbf{d}(\mathbf{u}, \mathbf{v}) + \mathbf{d}(\mathbf{v}, \mathbf{w})$, that has been intuitively motivated by the triangle [**Gol76**]. Applying the inclusion-exclusion principle, ruler (2.1), and ruler convergence theorem (2.2) to the definition of a countable distance range (3.1) yields the real-valued triangle inequality:

$$(3.15)\quad d_c = |Y| = |\bigcup_{i=1}^{2} y_i| \leq \sum_{i=1}^{2} |y_i| \quad \wedge$$
$$d_c = floor(\mathbf{d}(\mathbf{u},\mathbf{w})/c) \ \wedge\ |y_1| = floor(\mathbf{d}(\mathbf{u},\mathbf{v})/c) \ \wedge\ |y_2| = floor(\mathbf{d}(\mathbf{v},\mathbf{w})/c)$$
$$\Rightarrow\ \mathbf{d}(\mathbf{u},\mathbf{w}) = \lim_{c\to 0} d_c \cdot c \leq \sum_{i=1}^{2} \lim_{c\to 0} |y_i| \cdot c = \mathbf{d}(\mathbf{u},\mathbf{v}) + \mathbf{d}(\mathbf{v},\mathbf{w}).$$

4. Size (length/area/volume)

This section will use the ruler (2.1) and ruler convergence theorem (2.2) to prove that the Cartesian product of same-sized subintervals of intervals converges to the product of interval sizes. The first step is to define a set-based, countable size measure as the Cartesian product of disjoint domain subset members.

Definition 4.1. Countable size (length/area/volume) measure, S_c:

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1,n], \quad x_i \subseteq X, \quad |\bigcap_{i=1}^{n} x_i| = \emptyset \quad \wedge \quad \{(x_1,\ldots,x_n)\} = y \quad \wedge$$
$$S_c = |y| = |\{(x_1,\ldots,x_n)\}| = \prod_{i=1}^{n} |x_i|.$$

Theorem 4.2. *Euclidean size (length/area/volume), S, is the size of an image interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals:* $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, *where:*

$$S = \prod_{i=1}^{n} s_i, \quad S = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|, \quad i \in [1,n], \quad i,n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is “Euclidean_size.”

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$(4.1)\quad \forall\, i\; n \in \mathbb{N}, \quad i \in [1,n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge$$
$$x_i = \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\} \quad \Rightarrow |x_i| = p_i.$$

Use the ruler (2.1) to divide the exact size, $S = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_S^n subintervals, where p_S^n satisfies the definition a countable size measure, S_c (4.1).

$$(4.2)\quad \forall\, c > 0 \quad \wedge \quad \exists\, r \in \mathbb{R},\ S = r^n \quad \wedge \quad p_S = floor(r/c) \quad \wedge$$
$$p_S^n = S_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i.$$

Multiply both sides of equation 4.2 by c^n to get the ruler measures:

$$(4.3)\qquad p_S^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c).$$

Apply the ruler convergence theorem (2.2) to equation 4.3:

$$S = r^n = \lim_{c \to 0} (p_S \cdot c)^n \quad \wedge \quad \lim_{c \to 0} (p_i \cdot c) = s_i \quad \wedge \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c) \tag{4.4}$$
$$\Rightarrow \quad S = \lim_{c \to 0} (p_S \cdot c)^n = \prod_{i=1}^{n} \lim\nolimits_{c \to 0} (p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square$$

5. Ordered and symmetric geometries

Neither classical nor modern analytic geometry has been able to provide any insight into why classical Euclidean geometry appears to be limited to at most three dimensions. Again, the same counting principles that generate the triangle inequality, taxicab distance, Euclidean distance, and size (length/area/volume) also provide insight into what properties can limit a geometry (both Euclidean and non-Euclidean) to a cyclic set of at most three dimensions.

The previous derivations of taxicab distance (3.2), Euclidean distance (3.3), and Euclidean volume (4.2) show that the total number of combinations of subintervals of intervals converge to real-valued distance measures and Euclidean volume. By the commutative property of addition and multiplication, all orderings (permutations) of the combinations of subintervals of intervals yield the same total distance and same total volume. Therefore, all orderings (permutations) of domain intervals corresponding to those subinterval combinations yield the same total distance and same total volume (a symmetric geometry).

The axiom of choice can **not** be used to choose a subset of permutations and assume the unchosen permutations do not exist because a subset of things can not be chosen from a set of things that do not exist. Either all elements of a set exist or none exist. For example, between any two distinct points, A and B, there is both a taxicab and Euclidean distance. Therefore, there is no "choice" about which types of distance between points A and B exist and do not exist.

Likewise, there is no "choice" about which permutations of the subinterval combinations generating a distance or size measure exist and do not exist. And, there is no "choice" about which permutations of intervals (corresponding to subinterval combinations) exist and do not exist.

A permutation is an order. And order is defined by successor and predecessor functions, where traversing a set in successor order generates one permutation and traversing in predecessor order generates another permutation. Given the left-to-right ordered set of elements, $\{A, B, C, D\}$, the permutation, (A, D, C, B), can **not** be generated by the element, D, being a successor of A, because the successor function defines B as the successor of A. Therefore, the permutation can only be generated if D is a predecessor of A (a cyclic set).

But, this implies that the permutation, (A, C, D, B), does **not** exist, because C is neither a successor nor predecessor is A. It will be proved, that all permutations (a symmetric geometry) can only be generated by the same successor and predecessor functions if the set is a cyclic set of at most three dimensions.

Order is implemented via the usual successor and predecessor functions. Notionally:

Definition 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\; i \in [1, n-1],\; \forall\, x_i \;\in \{x_1, \ldots, x_n\},$$
$$\textit{successor}\ x_i = x_{i+1} \;\wedge\; \textit{predecessor}\ x_{i+1} = x_i.$$

where $\{x_1, \ldots, x_n\}$ are a set of real-valued intervals.

For elements, $x_i\ x_j \in \{x_1, \ldots, x_n\}$, such that traversing the ordered set in successor order yields the permutation, $(\ldots, x_i, x_j, \ldots)$, then traversing in predecessor order yields the permutation, $(\ldots, x_j, x_j, \ldots)$. A geometry is symmetric if permutations for all i and j exist.

DEFINITION 5.2. Symmetric geometry:

$$\forall\, i\ j\ n \in \mathbb{N},\ \forall\, x_i\ x_j \in \{x_1, \ldots, x_n\},\ \mathit{successor}\ x_i = x_j\ \ \wedge\ \ \mathit{predecessor}\ x_j = x_i.$$

THEOREM 5.3. *An ordered and symmetric geometry is a cyclic set.*

$$\mathit{successor}\ x_n = x_1\ \ \wedge\ \ \mathit{predecessor}\ x_1 = x_n.$$

The theorem and formal Coq-based proof is "ordered_symmetric_is_cyclic," which is located in the file threed.v.

PROOF. The property of order (5.1) defines unique successors and predecessors for all elements except for the successor of x_n and the predecessor of x_1. From the properties of a symmetric geometry (5.2):

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \mathit{successor}\ x_i = x_j\ \ \Rightarrow\ \ \mathit{successor}\ x_n = x_1. \tag{5.1}$$

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \mathit{predecessor}\ x_j = x_i\ \ \Rightarrow\ \ \mathit{predecessor}\ x_1 = x_n. \tag{5.2}$$

□

For example, using the cyclic set with elements labeled, $\{1,\ 2,\ 3\}$, starting with each element and counting through 3 cyclic successors and counting through 3 cyclic predecessors yields all possible permutations: $(1,2,3)$, $(2,3,1)$, $(3,1,2)$, $(1,3,2)$, $(3,2,1)$, and $(2,1,3)$. That is, a cyclically ordered set preserves sequential order while allowing some n-at-a-time permutations. If all possible n-at-a-time permutations are generated, then the cyclic set is also a symmetric geometry.

THEOREM 5.4. *An ordered and symmetric geometry is limited to at most 3 elements. That is, each element is sequentially adjacent (a successor or predecessor) to every other element in a set only where the number of elements are less than or equal to 3.*

The Coq-based lemmas and proofs in the file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
Because a symmetric and ordered set is a cyclic set (5.3), the successors and predecessors are cyclic:

DEFINITION 5.5. Successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (m + 1 \leq \mathit{setsize}). \tag{5.3}$$

DEFINITION 5.6. Predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: element m is adjacent to element n (an allowed permutation), if the cyclic successor of m is n or the cyclic predccessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{5.5}$$

Every element is adjacent to every other element, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{5.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (1 + 1 \leq 2). \tag{5.7}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (2 = 2 \wedge 1 = 1). \tag{5.8}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (1 + 1 \leq 2). \tag{5.9}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (2 - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (3 = 3 \wedge 1 = 1). \tag{5.11}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (1 = 1 \wedge 3 = 3). \tag{5.12}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (2 + 1 \leq 3). \tag{5.13}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (3 - 1 \geq 1). \tag{5.14}$$

Must prove that for all $setsize > 3$, there exist non-adjacent elements (not every permutation allowed). For example, the first and third elements are not adjacent:

$$\forall\ setsize > 3: \quad \neg Successor(1, 3, setsize) \\ \leftarrow Successor(1, 2, setsize) \leftarrow (1 + 1 \leq setsize). \tag{5.15}$$

That is, 2 is the only successor of 1 for all $setsize > 3$, which implies 3 is not a successor of 1 for all $setsize > 3$.

$$\forall\ setsize > 3: \quad \neg Predecessor(1, 3, setsize) \\ \leftarrow Predecessor(1, n, setsize) \leftarrow (1 = 1 \wedge n = setsize). \tag{5.16}$$

That is, $n = setsize$ is the only predecessor of 1 for all $setsize > 3$, which implies 3 is not a predecessor of 1 for all $setsize > 3$.

$$\forall\ setsize > 3: \quad \neg Adjacent(1, 3, setsize) \\ \leftarrow \neg Successor(1, 3, setsize) \wedge \neg Predecessor(1, 3, setsize). \quad \square \tag{5.17}$$

6. Summary

Applying ruler measure (2.1) and ruler convergence proof (2.2) to combinatorial relations between the subintervals of real-valued domain intervals and the subintervals an image interval yields some new insights into real analysis, geometry, physics, and the axiom of choice.

(1) Applying the ruler measure (2.1) and ruler convergence proof (2.2) to the countable distance range (3.1) provides the insight that for each disjoint domain subset there exists an image (distance) subset with the same number elements is the principle underlying the notion of distance:
   (a) The countable distance range principle converges to the triangle inequality, which is the basis for definition of metric space.

(b) All L^p norms where $p > 2$ satisfy the triangle inequality and definition of a metric space and are "smaller" distance measures between two points than Euclidean distance. However, no L^p norms where $p > 2$ are "logical" distance measures because they all over count, where each of the distance elements corresponds to each of the domain set elements multiple times. For example, it is not a useful measure of steps when walking from point A to B that each step corresponds to the same piece of land multiple times.

(c) A one-to-one correspondence (bijective map) of each distance subset element to one domain subset element converges to the largest (taxicab) distance equation (L^1 norm) and is the upper bound of the triangle inequality derived from the countable distance principle.

(d) The largest number of correspondences of each distance subset element to domain subset elements converges to the smallest (Euclidean) distance equation (L^2 norm) and is the lower bound of the triangle inequality derived from the countable distance principle.

(e) Any distance measure where for each of p_i number of distance subintervals there are **not** p_i number of correspondences to domain set subintervals is a non-Euclidean distance measure.

(f) The proof of Euclidean distance (3.3) was derived without any notions of side, angle, or shape. Arc angle defined as a parametric variable relating the sizes of two domain intervals can be easily derived using using calculus (Taylor series) and the Euclidean distance equation to generate the sine and cosine functions of the parametric variable. In other words, the notions of side and angle are derived from Euclidean distance.

(2) Applying the ruler measure (2.1) and ruler convergence proof (2.2) to the countable size definition (4.1) provides the insight that the Cartesian product of same-sized subintervals of real-valued intervals converges to the product of the interval sizes (Euclidean volume):

(a) Euclidean size (length/area/volume) was derived without notions of sides, angles, and shape.

(b) The Euclidean volume (product of interval sizes) of the Lebesgue measure is derived from use of the more fundamental ruler measure.

(3) Combinatorics on infinitesimal subintervals generating the properties of distance and volume results in the same total distance and same total volume for all permutations (orderings) of domain intervals.

(a) Using successor and predecessor functions generates an ordering of the set of intervals (5.1). When the successor and predecessor functions generate every possible ordering (a symmetric geometry), then the set of intervals must be a cyclic set (5.3) limited to at most three intervals (5.4). which is the basis of the right-hand rule that permeates mathematics, physics, and engineering.

(b) A cyclic set is a closed walk. An observer in a closed walk of three dimensions would only be able to detect higher dimensions (other variables) indirectly via distance and size changes in the three closed walk dimensions, where a change in distance is what physicists call "work."

(4) Just as the properties of distance, volume, three dimensions of space, and the right-hand rule are a consequence of simple combinatorial relations between the subintervals of real-valued intervals, combinatorics on the subintervals of higher dimensions of real-valued intervals might also converge to real-valued functions describing phenomena perceived as "particles", "waves", "mass", and "forces". In other words, our universe might be a consequence of a few simple counting principles in the real-valued continuum. These counting principles might provide a foundation for current quantum bit theories in physics.

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