A Combinatorial Foundation for Geometry

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Abstract. A ruler-like measure of intervals provides a new tool for combinatorial proofs and deeper insights into geometry and analysis. Application of a ruler measure to cases of the inclusion-exclusion principle of set theory converge to the n-dimensional Euclidean distance equation and metric triangle inequality. The ruler measure is used to prove that the size of the Cartesian product of the subintervals of intervals converges to the product of the interval sizes, the n-dimensional volume equation. Combinatorics is also used to prove that all geometries that have the property of being both ordered and symmetric are cyclic sets and limited to at most three dimensions, which is the basis of the right-hand rule. Implications for higher dimensional geometries are discussed. All the proofs are verified in Coq.

Contents

1. Introduction

It will be shown that counting (combinatorial) relationships are the foundation generating the primitive geometric relationships: Euclidean distance, triangle inequality, area, volume, and the conditions for a three dimensional geometry. The non-primitive relationship, arc angle, is derived from Euclidean distance.

A ruler-like measure of intervals provides a new tool for combinatorial proofs and deeper insights into geometry and analysis. For example, a ruler measure is

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used to create a combinatorial proof of the n-dimensional Euclidean distance equation, which provides these insights: 1) a case of the inclusion-exclusion principle defining the smallest countable distance spanning one or more sets converges to Euclidean distance; 2) the sum of squares relationship is the result of summing Cartesian products of same-sized image and domain sets. 3) counting (combinatorial) relationships generate Euclidean distance; 4) Euclidean distance is independent of any notions of side, angle, and shape.

In contrast, the Pythagorean theorem (Euclidean distance in two dimensions) has hundreds of proofs, where the proofs fall into one of five categories: construction, algebraic, [**Loo68**], trigonometric [**Zim09**], differential [**Ber88**] [**Sta96**] [**Bog10**], and axiomatic [**Bir32**], [**Hil**], [**SST83**]. However, none of these five categories of proof have provided the aforementioned insights of the combinatorial proof.

The definition of a metric space depends on the triangle inequality, which has been intuitively motivated by the triangle [**Gol76**] rather than being derived from set theory and limits like most of real analysis. The ruler measure adds rigor by deriving the metric triangle inequality from a property of the inclusion-exclusion principle of set theory.

The Lebesgue measure and Riemann integral define volume as the product of a set of interval sizes, rather than proving the volume equation. There is an algebraic proof of rectangular area [**Spi68**]. But, that proof fails to expose the counting relationship that generates area. Further, the ancient definition of volume from Euclid's Elements, Book 7, Definition 17 [**Joy98**]) as the product of the lengths of a solid has still been assumed to this day without proof. In contrast, the ruler measure is used to prove that the size of the Cartesian product of the subintervals of a set of intervals converges to the n-dimensional Euclidean distance equation.

A ruler measure of an interval is the nearest integer number of partitions (subintervals), p, each partition of size, c, where the partition size, c, is the only independent variable, and p is the dependent variable (an approximate measure ignoring partial partitions). Same-sized partitions, c, across a set of intervals allows defining a countable relationship between the number of partitions in one interval with the number of partitions in other intervals. As the partition size goes to zero, the combinatorial relationships that define countable size and smallest countable distance converge to the n-dimensional equations of Euclidean volume and distance.

For example, the Lebesgue measure, measures an area by creating p number of sub-areas, each sub-area contains two intervals with an assumed size of $c_{1,i}$ and $c_{2,i}$, $1 \leq i < \infty$. Whereas, the ruler measure would partition a pair of intervals into p_1 and p_2 number of partitions, each partition of size c. With the ruler measure, there are no sub-areas.

Every permutation of the arguments to area, volume, and distance functions return the same value (symmetric functions). However, being able to distinguish sets with the same area/volume and spanning distance, requires applying an order to the dimensions. Again, combinatorics is used to prove that a geometry (Euclidean and non-Euclidean) that is both ordered and symmetric is a cyclic set and limited to at most three elements (dimensions), which is the basis of the right-hand rule. Issues with respect to three ordered and symmetric dimensions as a subset of a higher dimensional geometry are discussed.

The proofs in this article are verified using the Coq Proof Assistant [15] version 8.4p16. The Coq-based definitions, theorems, and proofs are in the files “euclidrelations.v” and “threed.v” located at:
https://github.com/treeck/CombinatorialGeometry.

2. Ruler measure and convergence

A measure, μ, in a σ-algebra has the three properties:

(1) Non-negativity: $\forall\, E \in \Sigma,\ \ \mu(E) \geq 0$.
(2) Zero-sized empty set: $\mu(\emptyset) = 0$.
(3) Countable additivity: $\forall\, \{E_i\}_{i \in \mathbb{N}},\, E_i \cap E_{i+1} = \emptyset \wedge \mu(\cup_{i=1}^{\infty} E_i) = \mu(\Sigma_{i=1}^{\infty} E_i)$.

Definition 2.1. Ruler measure: A ruler measures the size of a closed, open, or semi-open interval as the nearest integer number of whole partitions (subintervals), p, times the partition size, c, where c is the independent variable. Notionally:

$$\forall\, c\, s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |b-a| \ \ \wedge\ \ c > 0\ \wedge \tag{2.1}$$
$$(p = floor(s/c)\ \ \vee\ \ p = ceiling(s/c)\ \ \wedge\ \ M = \lim_{c \to 0} \sum_{i=1}^{p} c = \lim_{c \to 0} pc.$$

The ruler measure has countable additivity:
$(c \to 0 \Rightarrow p \to \infty)\ \ \wedge\ \ \mu(E_i) = c\ \ \Rightarrow\ \ \mu(\Sigma_{i=1}^{\infty} E_i) = \lim_{c \to 0} \Sigma_{i=1}^{p} c = \lim_{c \to 0} pc.$

Theorem 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\ \ s = |b-a|\ \ \Rightarrow\ \ M = \lim_{c \to 0} pc = s.$

The Coq-based theorem and proof in the file euclidrelations.v is “limit_c_0_M_eq_exact_size.”

Proof.
By definition of the floor function, $floor(x) = max(\{y :\ y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0, \quad p = floor(s/c) \quad \Rightarrow \quad 0 \leq |p - s/c| < 1. \tag{2.2}$$

Multiply all sides by $|c|$:

$$\forall\, c > 0, \quad 0 \leq |p - s/c| < 1 \quad \Rightarrow \quad 0 \leq |pc - s| < |c|. \tag{2.3}$$

$$\forall\, c > 0,\, \exists\, \delta,\, \epsilon\ :\ 0 \leq |pc - s| < |c| = |c - 0| < \delta = \epsilon \tag{2.4}$$
$$\Rightarrow \quad 0 < |c - 0| < \delta \quad \wedge \quad 0 \leq |pc - s| < \epsilon = \delta \quad := \quad M = \lim_{c \to 0} pc = s. \quad \square$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure).

3. Distance

In cultures that did not have a number system for counting, some people counted by correspondence, where for example the size of a herd or army was “tallied” with corresponding cuts on a ”tally” stick or adding pebbles to a pile. Judgments were made from the relative lengths of equally spaced cuts on two tally sticks or the relative size of two piles of pebbles. The Latin word, talea, meaning cutting and the Latin word, calculus, meaning pebble. [Dan54]

Here, distance is the number of elements in an image (distance) set corresponding one-to-one with the elements of a domain set, like a set of cuts or pebbles. This notion of distance is extended across disjoint (non-intersecting) domain sets, where some elements in the image may correspond to one element in each domain set.

An example of a distance set is a set of equal-valued coins, C, corresponding with the members of sets of apples, A, and bananas, B. Using the correspondence distance measure, there exists $|C_A| = |A|$ number of coins and $|C_B| = |B|$ number of coins. By the inclusion-exclusion principle, the total number of coins, $|C| = |C_A \cup C_B| = |C_A| + |C_B| - |C_A \cap C_B|$, where the size of the intersection, $|C_A \cap C_B|$, represents the case of coins corresponding to both apples and bananas.

The Da Silva/Sylvester formula generalizes the inclusion-exclusion principle for n number of sets and is frequently used in combinatorics, measure theory, number theory (the sieve method), and probability theory [**Wc15**]:

Definition 3.1. Da Silva/Sylvester formula:

$$|\bigcup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| - \sum_{k=1}^{n} -1^{k+1}(\sum_{1 \le i_1 < \ldots < i_k \le n} |y_{i_1} \bigcap \cdots \bigcap y_{i_k}|). \tag{3.1}$$

The basis of the inclusion-exclusion principle is that the list of all set elements appended together is the set of "uniques" (union) plus the list of "duplicates." The intersection operations in the Da Silva/Sylvester formula calculate the list of duplicates.

While the Da Silva/Sylvester formula is an exponential (expensive) time algorithm, it is very useful in cases where direct counting is not possible. However, when direct counting is possible, a single-pass through the list of all elements can partition the elements into a "uniques" (union) set and duplicates list in linear time, as shown in the Coq-based "inclusion_exclusion_principle" theorem and proof that is located in the file, euclidrelations.v.

Therefore, a more general and simple formula for the inclusion-exclusion principle is used here that allows a more intuitive definition of distance:

Definition 3.2. General inclusion-exclusion formula:

$$|\bigcup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| - |duplicates(\{y_i\}_{i \in [1,n]})|. \tag{3.2}$$

Proofs of the inclusion-exclusion principle have already been published many times and won't be shown here.

Definition 3.3. Countable Distance Measure, d_c:

$$\forall\, i\, j \in [1,n],\ x_i \subseteq X,\ |\bigcup_{i=1}^{n} x_i| = \sum_{i=1}^{n} |x_i| \ \wedge\ \forall\, x_i\, \exists\, y_i \subseteq Y:\ |x_i| = |y_i| \ \wedge$$
$$(\forall\ y_j \subseteq Y,\quad i \neq j \quad \wedge \quad y_i \not\subseteq y_j \quad \wedge \quad y_j \not\subseteq y_i) \ \wedge$$
$$0 \le d_c = |\bigcup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| - |duplicates(\{y_i\}_{i \in [1,n]})|.$$

The condition, "$y_i \not\subseteq y_j \wedge y_j \not\subseteq y_i$," and the number of duplicates held constant guarantees any increase in the size of a domain set also increases the measure, d_c (countable additivity).

Definition 3.4. Countable taxicab (largest spanning) distance measure:

$$|duplicates(\{y_i\}_{i \in [1,n]})| = 0 \quad \Rightarrow \quad d_c = \sum_{i=1}^{n} |y_i|.$$

In the taxicab distance case, the number of coins is equal to the number of apples plus bananas, $|C| = |A| + |B|$, because $|C_A \cap C_B| = 0$.

The Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance." The proof for disjoint sets is common and won't be shown here.

Definition 3.5. Countable Euclidean (smallest spanning) distance measure:

$$|duplicates(\{y_i\}_{i\in[1,n]})| = max\_dups \quad \Rightarrow \quad d_c = \sum_{i=1}^{n} |y_i| - max\_dups.$$

The largest possible number of duplicate correspondences, "max_dups," is the case of the smallest (shortest) spanning distance.

For example, to determine the smallest, real-valued amount of coin, C_r, for a specified number of apples, $|A|$, and bananas, $|B|$, use the ruler measure (2.1) to partition sets of apples, bananas, and coins into pieces:

$$\begin{aligned}(3.3)\quad & \forall\, c\ C_r \in \mathbb{R}, \quad c > 0 \quad \wedge \\ & p_1 = floor(|A|/c) \ \wedge\ p_2 = floor(|B|)/c) \ \wedge\ p_d = floor(C_r/c) \ \wedge \\ & |\{applePiece_1, applePiece_2, \ldots, applePiece_{p_1}\}| = p_1 \quad \wedge \\ & |\{bananaPiece_1, bananaPiece_2, \ldots, bananaPiece_{p_2}\}| = p_2 \quad \wedge \\ & |\{coinPiece_1, coinPiece_2, \ldots, coinPiece_{p_d}\}| = p_d.\end{aligned}$$

The maximum number of duplicate correspondences creating the smallest possible distance is the set of the maximum number of possible $(applePiece, coinPiece)$ correspondences plus the maximum number of $(bananaPiece, coinPiece)$ correspondences. From the definition of a countable distance measure (3.3), each of p_1 number of coin pieces can correspond to a maximum of p_1 number of apple pieces, yielding a maximum of $p_1 \times p_1 = p_1^2$ number of possible $(applePiece, coinPiece)$ correspondences. Likewise, there are a maximum of p_2^2 number of possible $(banana\ Piece, coinPiece)$ correspondences. Therefore, there are a maximum of $p_1^2 + p_2^2 = |\{(fruitPiece, coinPiece)\}|$ correspondences.

Multiply both sides by c^2 and apply the ruler convergence theorem (2.2):

$$\begin{aligned}& (p_1 \cdot c)^2 + (p_2 \cdot c)^2 = |\{(fruitPiece, coinPiece)\}| \cdot c^2 \quad \wedge \\ & |A| = \lim_{c\to 0} p_1 \cdot c \quad \wedge \quad |B| = \lim_{c\to 0} p_2 \cdot c \\ \Rightarrow\quad & |A|^2 + |B|^2 = \lim_{c\to 0}(p_1 \cdot c)^2 + \lim_{c\to 0}(p_2 \cdot c)^2 = \lim_{c\to 0} |\{(fruitPiece, coinPiece)\}| \cdot c^2.\end{aligned}$$

Equation 3.3 partitioned the C_r amount of coin into p_d number of pieces coin pieces. And by the correspondence definition of distance (3.3), there are an equal, p_d, number of fruit pieces, yielding $p_d^2 = |\{(fruitPiece, coinPiece)\}|$ number of possible correspondences.

Multiply both sides by c^2 and apply the ruler convergence theorem (2.2):

$$\begin{aligned}& (p_d \cdot c)^2 = |\{(fruitPiece, coinPiece)\}| \cdot c^2 \quad \wedge \quad C_r = \lim_{c\to 0} p_d \cdot c \\ & \Rightarrow\quad C_r^2 = \lim_{c\to 0}(p_d \cdot c)^2 = \lim_{c\to 0} |\{(fruitPiece, coinPiece)\}| \cdot c^2.\end{aligned}$$

$$\begin{aligned}& |A|^2 + |B|^2 = \lim_{c\to 0} |\{(fruitPiece, coinPiece)\}| \cdot c^2 \quad \wedge \\ & C_r^2 = \lim_{c\to 0} |\{(fruitPiece, coinPiece)\}| \cdot c^2 \quad \Rightarrow \quad |A|^2 + |B|^2 = C_r^2.\end{aligned}$$

THEOREM 3.6. *Euclidean (smallest) distance, d, is the size of the distance interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals, $\{[x_{0,1}, x_{m,1}], [x_{0,2}, x_{m,2}], \ldots, [x_{0,n}, x_{m,n}]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |y_m - y_0|, \quad s_i = |x_{m,i} - x_{0,i}|, \quad i \in [1, n], \quad i, n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

PROOF.
Use the ruler (2.1) to partition the exact size, $s_i = |x_{m,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m,i}]$, into p_i number of partitions. Next, apply the definition of the countable distance measure (3.3) and the rule of product:

$$\begin{aligned}(3.4)\quad & \forall\, i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\ & |\{x_i : x_i \in \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\}| = |\{y_i : y_i \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_i}\}\}| = p_i \quad \Rightarrow \\ & \qquad \forall\, i \in [1, n], \quad |\{(x_i, y_i)\}| = p_i^2.\end{aligned}$$

$$\begin{aligned}(3.5)\quad & \forall\, i \in [1, n], \quad |\{(x_i, y_i)\}| = p_i^2 \quad \wedge \quad x \in \{x_i\} \quad \wedge \quad y \in \{y_i\} \quad \Rightarrow \\ & \qquad |\sum_{i=1}^{n} \{(x_i, y_i)\}| = \sum_{i=1}^{n} p_i^2 = |\{(x, y)\}|.\end{aligned}$$

Multiply both sides of 3.5 by c^2 and apply the ruler convergence theorem (2.2):

$$\begin{aligned}(3.6)\quad & s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n} (p_i \cdot c)^2 = |\{(x, y)\}| \cdot c^2 \\ & \Rightarrow \quad \sum_{i=1}^{n} s_i^2 = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c)^2 = \lim_{c\to 0} |\{(x, y)\}| \cdot c^2.\end{aligned}$$

Use the ruler to partition the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of partitions and apply the rule of product:

$$\begin{aligned}(3.7)\quad & \forall\, i \in [1, n], \quad c > 0 \quad \wedge \quad p_d = floor(d/c) \quad \wedge \quad |\{x\}| = |\{y\}| \quad \wedge \\ & p_d = |\{x : x \in \{x_{1_i}, x_{2_i}, \ldots, x_{p_d}\}\}| = |\{y : y \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_d}\}\}| \\ & \qquad \Rightarrow \quad p_d^2 = |\{(x, y)\}|.\end{aligned}$$

Multiply both sides of 3.7 by c^2 and apply the ruler convergence theorem (2.2):

$$\begin{aligned}(3.8)\quad & d = \lim_{c\to 0} p_d \cdot c \quad \wedge \quad (p_d \cdot c)^2 = |\{(x, y)\}| \cdot c^2 \\ & \qquad \Rightarrow \quad d^2 = \lim_{c\to 0} (p_d \cdot c)^2 = \lim_{c\to 0} |\{(x, y)\}| \cdot c^2.\end{aligned}$$

Combine equations 3.6 and 3.8:

$$\begin{aligned}(3.9)\quad & d^2 = \lim_{c\to 0} |\{(x, y)\}| \cdot c^2 \quad \wedge \quad \sum_{i=1}^{n} s_i^2 = \lim_{c\to 0} |\{(x, y)\}| \cdot c^2 \\ & \qquad \Rightarrow \quad d^2 = \sum_{i=1}^{n} s_i^2. \quad \square\end{aligned}$$

3.1. Triangle inequality. The definition of a metric in real analysis is based on the triangle inequality, $\mathbf{d}(\mathbf{u}, \mathbf{w}) \leq \mathbf{d}(\mathbf{u}, \mathbf{v}) + \mathbf{d}(\mathbf{v}, \mathbf{w})$, that has been intuitively motivated by the triangle [**Gol76**]. Applying the ruler (2.1) and convergence theorem (2.2) to the definition of a countable distance measure (3.3) (the inclusion-exclusion principle (3.2)) shows that the definition of the metric triangle inequality is derived from the inclusion-exclusion principle of set theory:

$$(3.10)\quad d_c = |\bigcup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| - |duplicates(\{y_i\}_{i\in[1,n]})| \leq \sum_{i=1}^{n} |y_i| \quad \wedge$$
$$d_c = floor(\mathbf{d}(\mathbf{u},\mathbf{w})/c) \ \wedge\ |y_1| = floor(\mathbf{d}(\mathbf{u},\mathbf{v})/c) \ \wedge\ |y_2| = floor(\mathbf{d}(\mathbf{v},\mathbf{w})/c)$$
$$\Rightarrow\ \mathbf{d}(\mathbf{u},\mathbf{w}) = \lim_{c\to 0} d_c \cdot c \leq \sum_{i=1}^{2} \lim_{c\to 0} |y_i| \cdot c = \mathbf{d}(\mathbf{u},\mathbf{v}) + \mathbf{d}(\mathbf{v},\mathbf{w}).$$

4. Size (length/area/volume)

The countable size measure is the number of combinations (correspondences) between members of disjoint domain sets, which is the Cartesian product of the domain set sizes. For example, given $|A|$ number of apples and $|B|$ number of bananas, the size measure is: $|\{(apple, banana)\}| = |A| \times |B|$ number of combinations.

DEFINITION 4.1. Countable size (length/area/volume) measure, S_c:

$$\forall\, i \in [1,n],\ x_i \subseteq X, \quad |\bigcup_{i=1}^{n} x_i| = \sum_{i=1}^{n} |x_i| \quad \wedge \quad \{(x_1,\ldots,x_n)\} = y \quad \wedge$$
$$S_c = |y| = |\{(x_1,\ldots,x_n)\}| = \prod_{i=1}^{n} |x_i|.$$

Distance (3.3) and size (4.1) are equivalent for $n = 1$.

THEOREM 4.2. *Euclidean size (length/area/volume), S, is the size of an image interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals: $\{[x_{0,1}, x_{m,1}], [x_{0,2}, x_{m,2}], \ldots, [x_{0,n}, x_{m,n}]\}$, where:*

$$S = \prod_{i=1}^{n} s_i, \quad S = |y_m - y_0|, \quad s_i = |x_{m,i} - x_{0,i}|, \quad i \in [1,n], \quad i, n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_size."

PROOF.
Use the ruler (2.1) to partition the exact size, $s_i = |x_{m,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m,i}]$, into p_i number of partitions.

$$(4.1)\ \forall\, i \in [1,n],\ c > 0 \wedge p_i = floor(s_i/c) \Rightarrow |\{x_i : x_i \in \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\}| = p_i.$$

Use the ruler (2.1) to partition the exact size, $S = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_S^n partitions, where p_S^n satisfies the definition a countable size measure, S_c.

$$(4.2)\quad \forall\, c > 0 \quad \wedge \quad \exists\, r \in \mathbb{R},\ S = r^n \quad \wedge \quad p_S = floor(r/c) \quad \wedge$$
$$p_S^n = S_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i.$$

Multiply both sides of equation 4.2 by c^n to get the ruler measures:

$$(4.3)\qquad p_S^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c).$$

Apply the ruler convergence theorem (2.2) to equation 4.3:

$$S = r^n = \lim_{c\to 0}(p_S \cdot c)^n \quad \wedge \quad \lim_{c\to 0}(p_i \cdot c) = s_i \quad \wedge \quad (p_S \cdot c)^n = \prod_{i=1}^{n}(p_i \cdot c) \tag{4.4}$$

$$\Rightarrow \quad S = \lim_{c\to 0}(p_S \cdot c)^n = \prod_{i=1}^{n} \lim\nolimits_{c\to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square$$

5. Derived geometric definitions

5.1. Derived geometric primitives. There are no new mathematics in this section of derived geometric primitives. The purpose of this section is to show a difference in perspective. In classical geometry, Euclidean distance is a product of lines and angles. Here, the perspective is reversed to show that lines and angles are non-primitive relationships generated from the primitive relationship, Euclidean distance.

Definition 5.1. Straight line segment is the smallest (Euclidean) distance interval, $[y_0, y_m]$ (3.6).

Definition 5.2. Straight line segment orientation (slope): $\mathrm{d}b/\mathrm{d}a = b/a$, where $a = x_{m,1} - x_{0,1}$ and $b = x_{m,2} - x_{0,2}$ are the signed sizes of two domain intervals, $[x_{0,1}, x_{m,1}]$ and $[x_{0,2}, x_{m,2}]$.

The signed sizes, a and b, of the two domain intervals can be calculated from a single parametric size, θ, and Euclidean distance, d.

Definition 5.3. Parametric size (arc angle), θ:

$$b/a = \mathrm{d}b/\mathrm{d}a = \mathrm{d}b/\mathrm{d}\theta \cdot \mathrm{d}\theta/\mathrm{d}a = \sqrt{d^2 - a^2}/\sqrt{d^2 - b^2} \tag{5.1}$$

$$Case: \quad \mathrm{d}b/\mathrm{d}a = b/a = 1 \quad \Rightarrow \quad \mathrm{d}\theta/\mathrm{d}a = 1/\sqrt{d^2 - b^2} = 1/\sqrt{d^2 - a^2} \tag{5.2}$$

Applying Taylor's theorem [**Gol76**] and a table of integrals [**Wc11**]:

$$\int \mathrm{d}\theta = \int \mathrm{d}a/\sqrt{d^2 - a^2} \quad \Rightarrow \quad \theta = \sin^{-1}(a/d) = \cos^{-1}(b/d). \tag{5.3}$$

5.2. Vectors. Before discussing the implications of the proofs in this article on vector analysis for dimensions greater than three, the notions of vector, parallel, and orthogonal are defined here in terms of sets of intervals.

Definition 5.4. Vector: A vector is the ordered set of the signed domain interval sizes, $\mathbf{s} = \{s_1, \ldots, s_n\}$, where $s_i = x_{m,i} - x_{0,i}$ for the domain interval, $[x_{0,i}, x_{m,i}]$.

Definition 5.5. Parallel (congruent) vectors: Two vectors are parallel if each ratio of the signed sizes in one vector equals the ratio of the corresponding signed sizes in another vector (same rate of change in the same direction):

$$\frac{s_{1_i}}{s_{1_{i+1}}} = \frac{s_{2_i}}{s_{2_{i+1}}}, \quad i \in [1, n-1]. \tag{5.4}$$

Definition 5.6. Orthogonal vectors: Two vectors are orthogonal if each ratio of the signed sizes in one vector is the inverse ratio and inverse sign of two corresponding signed sizes in another vector (inverse rate of change and inverse directions). Simplifying the equation yields the **dot (inner) product** equal to zero for any number of dimensions:

$$\frac{s_{1_i}}{s_{1_{i+1}}} = -\frac{s_{2_{i+1}}}{s_{2_i}}, \quad i \in [1, n-1] \quad \Leftrightarrow \quad \sum_{i=1}^{n} s_{1_i} \cdot s_{2_i} = 0. \tag{5.5}$$

6. Ordered and symmetric geometries

Euclidean size (area/volume) and distance are invariant for every order (permutation) of a set of intervals. A function (like size or distance) where every permutation of the arguments yields the same value(s) is called a symmetric function. Two sets of intervals with the same volume and spanning distance (for example, $\{[0,2],\ [0,1],\ [0,5]\}$ and $\{[0,5],\ [0,2],\ [0,1]\}$) can be distinguished by assigning an order (relative position) to the elements of the sets.

Definition 6.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\ \forall\, x_i \in \{x_1, \ldots, x_n\},\ \mathit{successor}\ x_i = x_{i+1}\ \wedge\ \mathit{predecessor}\ x_{i+1} = x_i.$$

Order restricts counting (access) via successor and predecessor. Therefore, allowing every permutation of elements (symmetry) in an ordered and symmetric set requires every element to be a successor or predecessor of every other element.

Definition 6.2. Symmetric geometry:

$$\forall\, i\; j\; n \in \mathbb{N},\ \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\ \mathit{successor}\ x_i = x_j\ \wedge\ \mathit{predecessor}\ x_j = x_i.$$

Theorem 6.3. *An ordered and symmetric geometry is a cyclic set.*

$$\forall\, i\; j\; n \in \mathbb{N},\ \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\ i = n\ \wedge\ j = 1 \\ \Rightarrow\quad \mathit{successor}\ x_n = x_1\ \wedge\ \mathit{predecessor}\ x_1 = x_n.$$

The Coq theorem and proof in the file threed.v is "ordered_symmetric_is_cyclic."

Proof. The property of order (6.1) defines unique successors and predecessors for all elements except for the successor of x_n and the predecessor of x_1. From the properties of a symmetric geometry (6.2):

$$i = n\ \wedge\ j = 1\ \wedge\ \mathit{successor}\ x_i = x_j\ \Rightarrow\ \mathit{successor}\ x_n = x_1. \tag{6.1}$$

$$i = n\ \wedge\ j = 1\ \wedge\ \mathit{predecessor}\ x_j = x_i\ \Rightarrow\ \mathit{predecessor}\ x_1 = x_n. \tag{6.2}$$

□

For example, using the cyclic set with elements labeled, $\{1,\ 2,\ 3\}$, starting with each element and counting through 3 cyclic successors and counting through 3 cyclic predecessors yields all possible permutations: $(1,2,3)$, $(2,3,1)$, $(3,1,2)$, $(1,3,2)$, $(3,2,1)$, and $(2,1,3)$. That is, a cyclically ordered set preserves sequential order while allowing a set of n-at-a-time permutations. If all possible n-at-a-time permutations are generated, then the cyclic ordered set is also symmetric.

Theorem 6.4. *An ordered and symmetric geometry is limited to at most 3 elements. That is, each element is sequentially adjacent (a successor or predecessor) to every other element in a set only where the number of elements (set sizes) are less than or equal to 3.*

The Coq-based lemmas and proofs in the file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn-like clauses (a subset of first-order logic) with unification and resolution. Horn clauses make it clear which facts satisfy a goal.

Proof.
Because an ordered and symmetric set is a cyclic set (6.3), the successors and predecessors are cyclic:

DEFINITION 6.5. Successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (m + 1 \leq n). \tag{6.3}$$

DEFINITION 6.6. Predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (m - 1 \geq 1). \tag{6.4}$$

DEFINITION 6.7. Adjacent: element m is adjacent to element n (an allowed permutation), if the cyclic successor of m is n or the cyclic predccessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{6.5}$$

Every element is adjacent to every other element, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{6.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (1 + 1 \leq 2). \tag{6.7}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (2 = 2 \wedge 1 = 1). \tag{6.8}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (1 + 1 \leq 2). \tag{6.9}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (2 - 1 \geq 1). \tag{6.10}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (3 = 3 \wedge 1 = 1). \tag{6.11}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (1 = 1 \wedge 3 = 3). \tag{6.12}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (2 + 1 \leq 3). \tag{6.13}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (3 - 1 \geq 1). \tag{6.14}$$

For all $n = setsize > 3$, there exist non-adjacent elements (not every permutation allowed):

$$\forall\, n > 3,\ Successor(1, 2, n) \Rightarrow \forall\, n > 3,\ \neg Successor(1, 3, n). \tag{6.15}$$

That is, 2 is the only successor of 1 for all $n > 3$, which implies 3 is not a successor of 1 for all $n > 3$.

$$\forall\, n > 3,\ Predecessor(1, n, n) \Rightarrow \forall\, n > 3,\ \neg Predecessor(1, 3, n). \tag{6.16}$$

That is, n is the only predecessor of 1 for all $n > 3$, which implies 3 is not a predecessor of n for all $n > 3$.

$$\forall\, n > 3,\ \neg Adjacent(1, 3, n) \leftarrow \neg Successor(1, 3, n) \wedge \neg Predecessor(1, 3, n). \tag{6.17}$$

□

7. Summary

A ruler measure of intervals (2.1) provides a new tool for combinatorial proofs and deeper insights into geometry and analysis.

A ruler-based combinatorial proof of an n-dimensional Euclidean distance equation (3.6) provides these insights: 1) a case of the inclusion-exclusion principle defining the smallest countable distance spanning one or more sets converges to Euclidean distance (3.3) (3.5); 2) the sum of squares relationship is the result of summing Cartesian products of same-sized image and domain sets. 3) counting (combinatorial) relationships generate Euclidean distance; 4) the derivation of Euclidean distance from sets of apples, bananas, and coins shows that Euclidean distance is independent of any notions of side, angle, and shape (3.5). The other five categories of proof of Euclidean distance (Pythagorean theorem) have not provided those insights.

The definition of a metric space depends on the triangle inequality, which has been intuitively motivated by the triangle [**Gol76**] rather than being derived from set theory and limits like most of real analysis. The ruler measure adds rigor by deriving the metric triangle inequality from a property of the inclusion-exclusion principle of set theory (3.1).

The ruler measure is also used to prove that the size of the Cartesian product of the subintervals of a set of intervals converges to the n-dimensional Euclidean volume equation (4.1). This is also the first proof of Euclidean volume.

Distance (3.3) and size (4.1) are equivalent for $n = 1$ intervals.

In both Euclidean and non-Euclidean geometries, the number of ruler partitions, p, depends solely on partition size, c, where c is the same in all intervals (dimensions). In an Euclidean geometry, the sizes of the domain intervals are independent. Whereas, in a non-Euclidean geometry, the sizes of the domain intervals are related by some function.

Because area/volume and distance in non-Euclidean geometries is calculated by summing infinitesimal Euclidean area/volumes and distances, area/volume and distance in non-Euclidean geometries is also symmetric. Therefore, the proofs of order and symmetry also apply to non-Euclidean geometries.

Proofs were presented that an ordered and symmetric set is a cyclic set (6.3) with at most three elements (6.4), which is the basis of the right-hand rule. Because the definitions of vector cross product and curl operations are based on the right-hand rule, these proofs provide the insight that cross product and curl can not be extended beyond three dimensions without losing either orientation (order) or symmetry.

The inner product was derived in this article for any number of dimensions (5.6). However, the proofs of an ordered and symmetric geometry imply that there are at most three mutually (symmetrically) perpendicular lines. Therefore, the proofs provide the insight that vector orthogonality is only equivalent up to three dimensions of symmetrically perpendicular lines. More study is needed to determine if a higher dimensional vector that contains a subset of three ordered and symmetric dimensions can use the inner and outer products of geometric algebra (Clifford geometry) and still preserve the symmetry of the subset. This is an important question because current unified theories in physics rely on Clifford geometry.

The proofs about ordered and symmetric sets may explain why we appear to live in a three dimensional world. An ordered and symmetric set is a cyclic set, which is a closed walk. An observer in the closed walk might only be able to detect higher dimensions indirectly via changes in the three closed walk dimensions (what physicists call "work"). A closed walk is a more parsimonious explanation for the higher dimensions of string theory not being directly observable than the currently popular hypothesis that the higher dimensions are not observable because the infinitely large dimensions are rolled into infinitesimally small balls in such a manner that they are still orthogonal at all points to the Euclidean dimensions.

Displaying higher dimensional manifolds in Euclidean coordinate diagrams (for example three dimensional Cartesian coordinates and spherical coordinates) is probably only meaningful for the case where three of the modeled dimensions are ordered and symmetric.

The order and symmetry proofs might also have relevance to symmetric groups and symmetry groups.

It has been shown that counting (combinatorial) relationships are the foundation generating the primitive geometric relationships: Euclidean distance (3.6), triangle inequality (3.1), size (length/area/volume) (4.2), and the conditions for a three dimensional geometry (6.4). The non-primitive relationship, arc angle (5.3), is derived from the primitive relation, Euclidean distance.

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