A Combinatorial Foundation for Analytic Geometry

George. M. Van Treeck

Abstract. Using a ruler-like measure of intervals with real analysis provides insights into the combinatorial principles generating geometry: A set-based definition of a countable distance range converges to the taxicab distance equation as the upper boundary of the range, the Euclidean distance equation as the lower boundary of the range, and the triangle inequality over the full range, which provides an analytic motivation for the definition of metric space independent of elementary geometry. The Cartesian product of the subintervals of intervals converges to the product of interval sizes used in the Lebesgue measure and Euclidean integrals. Also combinatorics limits a geometry having the properties of both symmetry and order to a cyclic set of at most 3 dimensions, which is the basis of the right-hand rule. Implications for non-Euclidean geometries and higher dimensional geometries are discussed. All the proofs are verified in Coq.

Contents

1. Introduction

The triangle inequality of a metric space, Euclidean distance metric, and the volume equation of the Lebesgue measure and Euclidean integrals are imported into analysis from elementary Euclidean geometry. Because the definitions are imported rather than derived from set and number theory, the definitions do not provide insight into the counting principles that generate elementary geometry.

2010 *Mathematics Subject Classification.* Primary 28A75, 28A25, 51M25, 05C38; Secondary 05C90, 03E05, 51M05, 05A15.

For example, fundamental to the notion of distance is “correspondence”, where for each element in a domain set there exists one element in a distance measure set, like one pebble for each step walked. If one distance element can correspond to multiple domain set elements, then the distance set will contain fewer elements than the domain set. The constraint that for every disjoint domain subset there exists a distance subset with the same number of elements results in a defined range of possible correspondences (range of distance set sizes).

Applying real analysis, this definition of a “countable distance range” converges to the triangle inequality, which provides a motivation for the definition of a metric space independent of Euclidean geometry. The upper boundary of the range converges to the taxicab distance equation. And the lower boundary of the range converges to the Euclidean distance equation, which provides a new insight into the combinatorial principle generating the smallest distance spanning disjoint sets.

However, there have been no set and number theory-based derivations of the real-valued triangle inequality and Euclidean distance equations. Further, there has been no proof that the Cartesian product of the subintervals of intervals converges to the product of interval sizes, the Euclidean volume equation, used in the Lebesgue measure and Euclidean integrals. Such derivations requires a using a different type of indefinite integral and a different type of interval measure.

The various traditional indefinite integrals (antiderivatives) derive a real-valued equation from a **real-valued, continuous function** relating the **size** of the subintervals of domain intervals to the **size** of the subintervals of an image interval. In contrast, what is needed for counting-based (combinatorial) proofs is an indefinite integral that derives a real-valued equation from a discrete, **combinatorial function** relating the **number** of same-sized subintervals of domain intervals to the **number** of same-sized subintervals in an image interval.

Combinatorial integration requires a different method of dividing intervals into subintervals herein referred to as a ruler. In the traditional method of dividing a set of intervals into subintervals, the number of subintervals is the same in both the domain and image intervals and the size of some subintervals can vary. In contrast, for the ruler measure, the number of subintervals in the domain and image intervals can vary and the size of the subintervals in each interval is always the exact same size. The ruler is an approximate measure that ignores partial subintervals.

Same-sized subintervals across both the set of domain intervals and image interval allows defining a countable relationship between the number of domain subintervals and the number of image subintervals. For example, as the subinterval size goes to zero, the combinatorial relationships that define smallest countable distance and countable size (length/area/volume) converge to the n-dimensional Euclidean distance and volume equations.

It will be shown that simple counting (combinatorial) principles generate: the triangle inequality of a metric space, the Euclidean distance equation, and the product of interval sizes (volume equation) of the Lebesgue measure and Euclidean integrals. Combinatorics also limits a geometry that is both symmetric and ordered to a cyclic set of at most three dimensions, which is the basis of the right-hand rule.

The proofs in this article are verified formally using the Coq Proof Assistant [15] version 8.4p16. The Coq-based definitions, theorems, and proofs are in the files “euclidrelations.v” and “threed.v” located at:
https://github.com/treeck/CombinatorialGeometry.

2. Ruler measure and convergence

DEFINITION 2.1. Ruler measure: A ruler measures the size of a closed, open, or semi-open interval as the nearest integer number of whole subintervals, p, times the subinterval size, c, where c is the independent variable. Notionally:

$$\forall\, c\, s \in \mathbb{R},\ [a,b] \subset \mathbb{R},\ s = |b-a| \ \wedge\ c > 0\ \wedge \qquad (2.1)$$
$$(p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \Sigma_{p=1}^{\infty} c = \lim_{c\to 0} pc.$$

The ruler measure has the three properties of measure in a σ-algebra:

(1) Non-negativity: $\forall\, E \in \Sigma,\ \mu(E) \geq 0 :\quad s = |b-a| \geq 0\ \wedge\ c > 0\ \Rightarrow$ $pc = floor(s/c) \cdot c >= 0\ \Rightarrow\ M = \lim_{c\to 0} pc \geq 0$.
(2) Zero-sized empty set: $\mu(\emptyset) = 0 :\quad b = a\ \Rightarrow p = 0\ \Rightarrow M = \lim_{c\to 0} pc = 0$.
(3) Countable additivity: $\forall\, \{E_i\}_{i\in\mathbb{N}},\ |\cap_{i=1}^{\infty} E_i| = \emptyset \wedge \mu(\cup_{i=1}^{\infty} E_i) = \mu(\Sigma_{i=1}^{\infty} E_i)$. $(c \to 0 \Rightarrow p \to \infty)\ \wedge\ \mu(E_i) = c\ \Rightarrow\ \mu(\Sigma_{i=1}^{\infty} E_i) = \Sigma_{p=1}^{\infty} c = \lim_{c\to 0} pc$.

THEOREM 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\ s = |b-a|\ \Rightarrow\ M = \lim_{c\to 0} pc = s$.

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\quad p = floor(s/c)\quad \Rightarrow\quad 0 \leq |p - s/c| < 1. \qquad (2.2)$$

Multiply all sides by $|c|$:

$$\forall\, c > 0,\quad 0 \leq |p - s/c| < 1\quad \Rightarrow\quad 0 \leq |pc - s| < |c|. \qquad (2.3)$$

$$\forall\, c > 0,\ \exists\, \delta,\ \epsilon\ :\ 0 \leq |pc - s| < |c| = |c - 0| < \delta = \epsilon \qquad (2.4)$$
$$\Rightarrow\quad 0 < |c - 0| < \delta\quad \wedge\quad 0 \leq |pc - s| < \epsilon = \delta\quad :=\quad M = \lim_{c\to 0} pc = s. \quad \square$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure).

For example, showing convergence using the interval, $[0, \pi]$, $s = |\pi - 0|$, $c = 10^{-i}, i \in \mathbb{N}$, and $p = floor(s/c)$, then, $p \cdot c = 3.1,\ 3.14,\ 3.141, ..., \pi$.

3. Distance

Fundamental to the notion of distance is "correspondence", where for each element in a domain set there exists one element in a distance measure set, like one pebble for each step walked. If one distance element can correspond to multiple domain set elements, then the distance set will contain fewer elements than the domain set. The constraint that for each i^{th} disjoint domain subset containing p_i number of elements there exists a distance subset with the same p_i number of elements results in a defined range of possible correspondences (range of distance set sizes).

DEFINITION 3.1. Countable distance range, d_c:

$$\forall\, i\, n \in \mathbb{N},\ x_i \subseteq X,\ \bigcap_{i=1}^{n} x_i = \emptyset,\ \forall\, x_i\ \exists\, y_i \subseteq Y :\ |x_i| = |y_i|\quad \wedge\quad d_c = |Y|.$$

Notation conventions: In the definition of countable distance range (3.1), the vertical bars around a set is the standard notation for indicating the cardinal (number of elements in the set). To prevent too much overloading on the vertical bar, the symbol for "such that" is the colon.

From the definition of countable distance range (3.1), the amount of intersection of distance subsets is not defined, which results in a range of possible distance set sizes. Notionally:

$$|\cap_{i=1}^{n} y_i| \geq 0 \quad \Rightarrow \quad \textstyle\sum_{i=1}^{n} |y_i| \geq |\cup_{i=1}^{n} y_i| = |Y|. \tag{3.1}$$

The countable distance range property, $|x_i| = |y_i|$, implies a limitation on the number of possible correspondences of a distance subset element to domain subset elements. If each of the p_i number of elements of the i^{th} distance set has a one-to-one (bijective) correspondence to a domain subset element, then the number of correspondences per distance set is: $1 \cdot |x_i| = 1 \cdot p_i = p_i = |y_i|$. And therefore, the distance, $d_c = |Y| = |\cup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i$, is the largest possible distance and the upper bound of the distance range.

If each of the p_i number of elements of the i^{th} distance subset corresponds to all p_i number of domain subset elements, then the largest number of correspondences per distance subset is: $|y_i| \cdot |x_i| = p_i \cdot p_i = p_i^2$. The largest number of possible correspondences implies the smallest possible distance and is the lower bound of the distance range,

$$\textstyle\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} |\{(y_a, y_b) : y_a\, y_b \in y_i\}|, \tag{3.2}$$

where each pair, (y_a, y_b), represents a combination (correspondence) between two elements in the distance set, y_i. From combining equations 3.1 and 3.2:

$$\begin{aligned} &\textstyle\sum_{i=1}^{n} |y_i| \geq |\cup_{i=1}^{n} y_i| = |Y| \quad \Rightarrow \\ &\qquad \textstyle\sum_{i=1}^{n} |\{(y_a, y_b) : y_a\, y_b \in y_i\}| \geq |\{(y_a, y_b) : y_a\, y_b \in Y\}|. \end{aligned} \tag{3.3}$$

Choose the case of equality in equation 3.2:

$$\begin{aligned} \textstyle\sum_{i=1}^{n} |y_i| \cdot |x_i| &= \textstyle\sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |y_i|^2 \\ &= \textstyle\sum_{i=1}^{n} |\{(y_a, y_b) : y_a\, y_b \in y_i\}| = |\{(y_a, y_b) : y_a\, y_b \in Y\}|. \end{aligned} \tag{3.4}$$

It is **not** possible for the sum of combinations, $|\{(y_a, y_b) : y_a\, y_b \in Y\}|$, to always have an integer square root. But, the ruler (2.1) and ruler convergence theorem (2.2) is applied to real-valued intervals to show the shortest distance case converges to the real-valued Euclidean distance equation (always a real-valued square root).

The proof of the taxicab and Euclidean distance equations requires the strategy of showing that the right and left sides of a proposed counting-based equation both converge to the same real value and therefore are equal. That is, the propositional logic, $A = B \wedge C = B \Rightarrow A = C$, is used.

THEOREM 3.2. *Taxicab (largest) distance, d, is the size of the distance interval,* $[y_0, y_m]$*, corresponding to a set of disjoint domain intervals,* $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$*, where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|.$$

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals. Next, apply the definition of the countable distance range (3.1) and the rule of product:

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\ |\{x_i : x_i \in \{x_{1_i}, x_{2_i}, \dots, x_{p_i}\}\}| = |\{y_i : y_i \in \{y_{1_i}, y_{2_i}, \dots, y_{p_i}\}\}| = p_i. \tag{3.5}$$

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad y \in y_i \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i = |\{y\}|. \tag{3.6}$$

Multiply both sides of 3.6 by c and apply the ruler convergence theorem (2.2):

$$s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n} (p_i \cdot c) = |\{y\}| \cdot c \\ \Rightarrow \quad \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \lim_{c\to 0} |\{y\}| \cdot c. \tag{3.7}$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$\forall\, c > 0, \quad p_d = floor(d/c) = |\{y :\; y \in \{y_{1_i}, y_{2_i}, \dots, y_{p_d}\}\}|. \tag{3.8}$$

Multiply both sides of 3.8 by c and apply the ruler convergence theorem (2.2):

$$d = \lim_{c\to 0} p_d \cdot c \;\wedge\; p_d \cdot c = |\{y\}| \cdot c \;\Rightarrow\; d = \lim_{c\to 0} p_d \cdot c = \lim_{c\to 0} |\{y\}| \cdot c. \tag{3.9}$$

Combine equations 3.9 and 3.7:

$$d = \lim_{c\to 0} |\{y\}| \cdot c \quad \wedge \quad \sum_{i=1}^{n} s_i = \lim_{c\to 0} |\{y\}| \cdot c \quad \Rightarrow \quad d = \sum_{i=1}^{n} s_i. \tag{3.10}$$ □

THEOREM 3.3. *Euclidean (smallest) distance, d, is the size of the distance interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals, $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \dots, [x_{0,n}, x_{m_n,n}]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}].$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals. Next, apply the definition of the countable distance range (3.1) and the rule of product:

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\ x_i = \{x_{1_i}, x_{2_i}, \dots, x_{p_i}\}\} \quad \wedge \quad y_i = \{y_{1_i}, y_{2_i}, \dots, y_{p_i}\}\}|. \tag{3.11}$$

$$\textstyle\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} |\{(y_a, y_b) : y_a\; y_b \in y_i\}|, \tag{3.12}$$

where each pair, (y_a, y_b), represents a combination (correspondence) between two elements in the distance subset, $\{y_i\}$. From definition of countable distance range (3.1), the amount of intersection of distance subsets is not defined, which results in a range of possible distance set sizes. Notionally:

$$|\cap_{i=1}^{n} y_i| \geq 0 \quad \Rightarrow \quad \textstyle\sum_{i=1}^{n} |y_i| \geq |\cup_{i=1}^{n} y_i| = |Y|. \tag{3.13}$$

From combining equations 3.12 and 3.13:

$$\sum_{i=1}^{n}|y_i| \geq |\cup_{i=1}^{n} y_i| = |Y| \quad \Rightarrow \quad \sum_{i=1}^{n}|\{(y_a, y_b) : y_a\, y_b \in y_i\}| \geq |\{(y_a, y_b) : y_a\, y_b \in Y\}|. \tag{3.14}$$

Choose the case of equality in equation 3.14:

$$\sum_{i=1}^{n}|y_i|\cdot|x_i| = \sum_{i=1}^{n}p_i^2 = \sum_{i=1}^{n}|y_i|^2 = \sum_{i=1}^{n}|\{(y_a, y_b) : y_a\, y_b \in \{y_i\}\}| = |\{(y_a, y_b) : y_a\, y_b \in Y\}|. \tag{3.15}$$

$$s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n}(p_i \cdot c)^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\| \cdot c^2 \quad \Rightarrow \quad \sum_{i=1}^{n} s_i^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\| \cdot c^2. \tag{3.16}$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$\forall\, c > 0, \quad p_d = floor(d/c) = |\{y_{1_i}, y_{2_i}, \ldots, y_{p_d}\}| = Y \quad \Rightarrow \quad p_d^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\}|, \tag{3.17}$$

where $\{(y_a, y_b)\}$ is the set of all combination pairs of elements of Y. Multiply both sides of 3.16 by c^2 and apply the ruler convergence theorem (2.2):

$$d = \lim_{c\to 0} p_d \cdot c \quad \wedge \quad (p_d \cdot c)^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(p_d \cdot c)^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2. \tag{3.18}$$

Combine equations 3.16 and 3.18:

$$d^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \quad \wedge \quad \sum_{i=1}^{n} s_i^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \quad \Rightarrow \quad d^2 = \sum_{i=1}^{n} s_i^2. \quad \square \tag{3.19}$$

3.1. Triangle inequality. The definition of a metric in real analysis is based on the triangle inequality, $\mathbf{d}(\mathbf{u}, \mathbf{w}) \leq \mathbf{d}(\mathbf{u}, \mathbf{v}) + \mathbf{d}(\mathbf{v}, \mathbf{w})$, that has been intuitively motivated by the triangle [**Gol76**]. Applying the inclusion-exclusion principle, ruler (2.1), and ruler convergence theorem (2.2) to the definition of a countable distance range (3.1) yields the real-valued triangle inequality:

$$d_c = |Y| = |\bigcup_{i=1}^{2} y_i| \leq \sum_{i=1}^{2} |y_i| \quad \wedge \quad d_c = floor(\mathbf{d}(\mathbf{u}, \mathbf{w})/c) \ \wedge\ |y_1| = floor(\mathbf{d}(\mathbf{u}, \mathbf{v})/c) \ \wedge\ |y_2| = floor(\mathbf{d}(\mathbf{v}, \mathbf{w})/c) \quad \Rightarrow \quad \mathbf{d}(\mathbf{u}, \mathbf{w}) = \lim_{c\to 0} d_c \cdot c \leq \sum_{i=1}^{2} \lim_{c\to 0} |y_i| \cdot c = \mathbf{d}(\mathbf{u}, \mathbf{v}) + \mathbf{d}(\mathbf{v}, \mathbf{w}). \tag{3.20}$$

4. Size (length/area/volume)

Until now, there has not been a proof that the Cartesian product of the subintervals of intervals converges to the product of the interval sizes, the Euclidean volume equation, used by the Lebesgue measure and Euclidean integrals. This section will use the ruler (2.1) and ruler convergence theorem (2.2) to prove that the Cartesian product of the subintervals of intervals converges to the product of interval sizes.

The countable size measure is the number of combinations between members of disjoint domain sets, which is the Cartesian product of the domain set sizes.

DEFINITION 4.1. Countable size (length/area/volume) measure, S_c:

$$\forall\, i\, n \in \mathbb{N}, \quad i \in [1, n], \quad x_i \subseteq X, \quad |\bigcap_{i=1}^{n} x_i| = \emptyset \quad \wedge \quad \{(x_1, \dots, x_n)\} = y \quad \wedge$$
$$S_c = |y| = |\{(x_1, \dots, x_n)\}| = \prod_{i=1}^{n} |x_i|.$$

THEOREM 4.2. *Euclidean size (length/area/volume), S, is the size of an image interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals:* $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \dots, [x_{0,n}, x_{m_n,n}]\}$, *where:*

$$S = \prod_{i=1}^{n} s_i, \quad S = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|, \quad i \in [1, n], \quad i, n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_size."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$\forall\, i\, n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge$$
$$x_i = \{x_{1_i}, x_{2_i}, \dots, x_{p_i}\} \quad \Rightarrow |x_i| = p_i. \tag{4.1}$$

Use the ruler (2.1) to divide the exact size, $S = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_S^n subintervals, where p_S^n satisfies the definition a countable size measure, S_c (4.1).

$$\forall\, c > 0 \quad \wedge \quad \exists\, r \in \mathbb{R},\ S = r^n \quad \wedge \quad p_S = floor(r/c) \quad \wedge$$
$$p_S^n = S_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i. \tag{4.2}$$

Multiply both sides of equation 4.2 by c^n to get the ruler measures:

$$p_S^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3:

$$S = r^n = \lim_{c \to 0} (p_S \cdot c)^n \quad \wedge \quad \lim_{c \to 0} (p_i \cdot c) = s_i \quad \wedge \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c)$$
$$\Rightarrow \quad S = \lim_{c \to 0} (p_S \cdot c)^n = \prod_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square \tag{4.4}$$

5. Symmetric and ordered geometries

Euclidean size (area/volume) and distance are invariant for every order (permutation) of a set of intervals. A function (like size or distance) where every permutation of the arguments yields the same value(s) is called a symmetric function. If one can "jump" from any interval (element), x_i, of a set to another element, x_j to form the permutation, (x_i, x_j), then one can also jump back from element

x_j to element x_i to form the permutation (x_j, x_i). In other words, every element x_i of a set having an immediate successor element x_j also requires that x_j has the immediate predecessor element x_i, such that traversing the elements in successor order and predecessor order yields both permutations.

DEFINITION 5.1. Symmetric geometry:

$$\forall\, i\ j\ n \in \mathbb{N},\ \forall\, x_i\ x_j \in \{x_1, \ldots, x_n\},\ \mathit{successor}\ x_i = x_j\ \wedge\ \mathit{predecessor}\ x_j = x_i.$$

Two sets of intervals with the same volume and spanning distance (for example, $\{[0,2],\ [0,1],\ [0,5]\}$ and $\{[0,5],\ [0,2],\ [0,1]\}$) can only be distinguished by assigning a relative sequential order (orientation) to the elements of the interval (dimension) sets.

DEFINITION 5.2. Ordered geometry:

$$\forall\, i\ n \in \mathbb{N},\ \forall\, x_i \in \{x_1, \ldots, x_n\},\ \mathit{successor}\ x_i = x_{i+1}\ \wedge\ \mathit{predecessor}\ x_{i+1} = x_i.$$

It will now be proved that any geometry, both Euclidean and non-Euclidean, that has both symmetry (every permutation of domain intervals yields the same distance and volume) and order (ability to discriminate distances and volumes by a relative sequential ordering), is a cyclic set limited to at most 3 domain intervals (dimensions), which is the basis for the right-hand rule. The implications with respect to vector operations and higher dimensioned geometries are discussed in the summary.

THEOREM 5.3. *A symmetric and ordered geometry is a cyclic set.*

$$\begin{aligned}\forall\, i\ j\ n \in \mathbb{N},\ \forall\, x_i\ x_j \in \{x_1, \ldots, x_n\},\ i = n\ \wedge\ j = 1 \\ \Rightarrow\quad \mathit{successor}\ x_n = x_1\ \wedge\ \mathit{predecessor}\ x_1 = x_n.\end{aligned}$$

The theorem and formal Coq-based proof is "ordered_symmetric_is_cyclic," which is located in the file threed.v.

PROOF. The property of order (5.2) defines unique successors and predecessors for all elements except for the successor of x_n and the predecessor of x_1. From the properties of a symmetric geometry (5.1):

$$i = n\ \wedge\ j = 1\ \wedge\ \mathit{successor}\ x_i = x_j\ \Rightarrow\ \mathit{successor}\ x_n = x_1. \tag{5.1}$$

$$i = n\ \wedge\ j = 1\ \wedge\ \mathit{predecessor}\ x_j = x_i\ \Rightarrow\ \mathit{predecessor}\ x_1 = x_n. \tag{5.2}$$ □

For example, using the cyclic set with elements labeled, $\{1,\ 2,\ 3\}$, starting with each element and counting through 3 cyclic successors and counting through 3 cyclic predecessors yields all possible permutations: $(1,2,3)$, $(2,3,1)$, $(3,1,2)$, $(1,3,2)$, $(3,2,1)$, and $(2,1,3)$. That is, a cyclically ordered set preserves sequential order while allowing a set of n-at-a-time permutations. If all possible n-at-a-time permutations are generated, then the cyclic set is also symmetric.

THEOREM 5.4. *An symmetric and ordered geometry is limited to at most 3 elements. That is, each element is sequentially adjacent (a successor or predecessor) to every other element in a set only where the number of elements (set sizes) are less than or equal to 3.*

The Coq-based lemmas and proofs in the file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn-like clauses (a subset of first-order logic) with unification and resolution. Horn clauses make it clear which facts satisfy a goal.

PROOF.
Because a symmetric and ordered set is a cyclic set (5.3), the successors and predecessors are cyclic:

DEFINITION 5.5. Successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (m + 1 \leq setsize). \tag{5.3}$$

DEFINITION 5.6. Predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: element m is adjacent to element n (an allowed permutation), if the cyclic successor of m is n or the cyclic predccessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{5.5}$$

Every element is adjacent to every other element, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{5.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (1 + 1 \leq 2). \tag{5.7}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (2 = 2 \wedge 1 = 1). \tag{5.8}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (1 + 1 \leq 2). \tag{5.9}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (2 - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (3 = 3 \wedge 1 = 1). \tag{5.11}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (1 = 1 \wedge 3 = 3). \tag{5.12}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (2 + 1 \leq 3). \tag{5.13}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (3 - 1 \geq 1). \tag{5.14}$$

For all $n = setsize > 3$, there exist non-adjacent elements (not every permutation allowed):

$$\forall\, n > 3,\ Successor(1, 2, n) \Rightarrow \forall\, n > 3,\ \neg Successor(1, 3, n). \tag{5.15}$$

That is, 2 is the only successor of 1 for all $n > 3$, which implies 3 is not a successor of 1 for all $n > 3$.

$$\forall\, n > 3,\ Predecessor(1, n, n) \Rightarrow \forall\, n > 3,\ \neg Predecessor(1, 3, n). \tag{5.16}$$

That is, n is the only predecessor of 1 for all $n > 3$, which implies 3 is not a predecessor of n for all $n > 3$.

$$\forall\, n > 3,\ \neg Adjacent(1, 3, n) \leftarrow \neg Successor(1, 3, n) \wedge \neg Predecessor(1, 3, n). \tag{5.17}$$

□

6. Summary

In the past, the properties of the metric space, Euclidean distance metric, and volume equation of the Lebesgue measure and Euclidean integrals were imported from elementary Euclidean geometry. Importing from elementary geometry allows geometry to motivate real analysis, calculus, and measure theory. But, analysis has been unable to motivate geometry by deriving the properties and equations of elementary geometry from set and number theory. As a consequence, analysis has been unable to provide insights into the combinatorial properties that generate elementary geometry.

Using the ruler measure of intervals with real analysis is a tool allowing a new class of proofs that provides insights into combinatorial principles generating elementary geometry:

(1) The property that every disjoint domain subset has a corresponding distance subset with the same number of elements constrains the maximum number of possible correspondences from each distance subset element to domain subset elements, the smallest possible countable distance set size, converges to Euclidean distance provides a new insight into why Euclidean distance is smallest possible distance that elementary geometry, analytic geometry, real analysis, and measure theory have failed to provide.

(2) Combinatorial relations between the elements of sets converge to the Euclidean distance (3.3) and size (length/area/volume) (4.2) equations without notions of side, angle, and shape, and without motivation from diagrams.

(3) The triangle inequality (3.1) is derived from the definition of the countable distance range (3.1), which provides a counting-based motivation for the definition of the metric space without the need for Euclidean geometry.

(4) The Euclidean volume (product of interval sizes) of the Lebesgue measure is derived from use of the more fundamental ruler measure.

(5) Combinatorics limits a geometry (both Euclidean and non-Euclidean) having the properties of both order (5.2) and symmetry (5.1) to a cyclic set (5.3) of at most three elements (dimensions) (5.4), which is the basis of the right-hand rule. The properties of symmetry and order explains why there can not be more than three dimensions of physical space and why the right-hand rule permeates physics and engineering.

(6) A cyclic set is a closed walk. An observer in a closed walk of three dimensions would only be able to detect higher, non-closed walk dimensions (other variables) indirectly via changes in the three closed walk dimensions (what physicists call “work”). In other words, distance and size in the three closed walk dimensions are functions of variables in the higher dimensions.

References

[15] *Coq proof assistant*, 2015. https://coq.inria.fr/documentation. ↑2

[Gol76] R. R. Goldberg, *Methods of real analysis*, John Wiley and Sons, 1976. ↑6

[Wc11] Taylor theorem Wikipedia contributors, *Table of integrals*, Wikipedia, 2011. http://en.wikibooks.org/wiki/Calculus/Tables_of_Integrals. ↑

George Van Treeck, 668 Westline Dr., Alameda, CA 94501