The Two Set Relations Generating Geometry

George. M. Van Treeck

Abstract. Where each domain set has a corresponding range set, distance is the cardinal of the union of the range sets and volume is the cardinal of the Cartesian product of disjoint range sets. A ruler (measuring stick) partitions intervals approximately into sets of size c subintervals and sums the sizes. The distance and volume set operations on sets of size c subintervals converge to the properties of metric space, the Manhattan distance, Euclidean distance, and volume equations as c goes to 0. The volume proof allows simpler derivations of Coulomb's charge force and Newton's gravity force equations without using other laws of physics or Gauss's divergence theorem. A type of symmetry limits physical space to 3 dimensions. All proofs are verified in Coq.

Contents

1. Introduction

Metric space, Euclidean distance as a metric/vector norm, and Euclidean area/volume in measure and integration have been definitions [Gol76] [Rud76]. In this article, the properties of metric space, Euclidean distance and volume are theorems that are proved to be derived from operations on countable sets.

Most measure and integration divide both a domain interval, $[a, b]$, and range interval, $[f(a), f(b)]$, into the same number of subintervals, where the size of the subintervals can vary. In this article, a ruler (measuring stick) divides both domain and range intervals approximately into sets of same size, c, subintervals, where the number of subintervals in each interval can vary, which allows deriving geometric

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relations from the combinatorial mappings between sets of same-sized subintervals that would be obtuse using traditional measure and integration.

Where each disjoint domain set has a corresponding range set, distance is defined as the cardinal of the union of the range sets and volume as the cardinal of the Cartesian product of the disjoint range sets. The distance and volume set operations on sets of size c subintervals converge to the properties of metric space, the Manhattan distance, Euclidean distance, and volume equations as $c \to 0$.

The derivations of metric space, distance equations, and volume from set operations provide some insights into geometry and physics, for example: the domain-to-range set mapping that makes Euclidean distance the smallest possible distance between two distinct points in $\mathbb{R}^n$; the range set relationship common to Manhattan distance and Euclidean volume; how the Euclidean volume proof: 1) allows deriving Coulomb's charge force and Newton's gravity force equations without using other laws of physics or Gauss's divergence theorem, and 2) exposes the principle that generates the inverse square law and spacetime equations; how a symmetry property of physical sets of intervals limits physical space to 3 dimensions.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

The ruler measures the size, M, of an interval *approximately* as the sum of the nearest integer number (floor or ceiling) of whole subintervals, each subinterval having the same size, c. The ruler is both an inner and outer measure. This article will not digress into other comparisons with the Lebesgue and Borel measures.

Definition 2.1. Ruler measure: $\forall\ c,\ s \in \mathbb{R},\ [a,b] \subset \mathbb{R},\ s = |a-b| \ \wedge\ c > 0\ \wedge$ $(p = floor(s/c)\ \ \vee\ \ p = ceiling(s/c))\ \ \wedge\ \ M = \sum_{i=1}^{p} c = pc.$

Theorem 2.2. *Ruler convergence:* $\forall\ [a,b] \subset \mathbb{R}, s = |a-b| : M = \lim_{c\to 0} pc = s.$

The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y:\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\ c > 0,\ p = floor(s/c)\ \wedge\ 0 \le |floor(s/c) - s/c| < 1\ \Rightarrow\ 0 \le |p - s/c| < 1. \tag{2.1}$$

Multiply all sides of inequality 2.1 by $|c|$:

$$\forall\ c > 0, \quad 0 \le |p - s/c| < 1 \quad \Rightarrow \quad 0 \le |pc - s| < |c|. \tag{2.2}$$

$$\begin{aligned} &\forall\ \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \\ &\qquad \Rightarrow \quad \forall\ \epsilon = \delta :\ \ |c - 0| < \delta\ \ \wedge\ \ |pc - s| < \epsilon\ \ :=\ \ M = \lim_{c\to 0} pc = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = |0-\pi|$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}},\ 3.14_{c=10^{-2}},\ 3.141_{c=10^{-3}}, ..., \pi_{\lim_{c\to 0}}.$

3. Distance

Notation convention: Vertical bars around a set or list, $|\cdots|$, indicates the cardinal (number of members in the set or list).

3.1. Countable distance. A concrete example of a countable distance is the number of same-sized steps walked in a range (distance) set, y, which equals the number of same-sized pieces of land in a corresponding domain set, x: $|x| = |y|$. Generalizing, each disjoint domain set, x_i, has a corresponding range (distance) set, y_i. The countable distance spanning the disjoint domain sets is the number of members, d_c, in the union range set:

DEFINITION 3.1. Countable distance, d_c :

$$d_c = |\bigcup_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i|.$$

Extending this definition beyond flat space, where $|x_i| = |y_i|$, is shown in the last section of this article.

THEOREM 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$.

The inclusion-exclusion inequality follows from the inclusion-exclusion principle [**CG15**]. But, a more intuitive and simple proof follows from the associative law of addition where the sum of set sizes is equal to the size of all the set members appended into a list and the commutative law of addition that allows sorting that list into a list of unique members (the *union* set) and a list of duplicates. The duplicates being ≥ 0 implies the union size is always $\leq$ the sum of set sizes.

A formal proof, inclusion_exclusion_inequality, using sorting into a set of unique members (*union* set) and a list of duplicates, is in the file euclidrelations.v.

PROOF.

$$\begin{aligned}(3.1)\quad \sum_{i=1}^{n} |y_i| &= |append_{i=1}^{n} y_i| = |sort(append_{i=1}^{n} y_i)| \\ &= |\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i|.\end{aligned}$$

$$\begin{aligned}(3.2)\quad |\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| \quad &\wedge \quad |duplicates_{i=1}^{n} y_i| \geq 0 \\ \Rightarrow \quad &|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|. \quad \square\end{aligned}$$

3.2. Countable distance range. From the countable distance definition, $d_c = |\bigcup_{i=1}^{n} y_i|$, as the amount of intersection increases, more domain set members can map to a single range set member. Therefore, the countable distance, d_c, is a function of the number of domain-to-range set member mappings.

From the countable distance property (3.1), where $|x_i| = |y_i| = p_i$: 1) Each domain set member can map to as few as one unique range set member (a 1-1 correspondence), $1 \cdot |y_i| = p_i$ number of mappings; 2) Each domain set member can map to as many as every range set member (the Cartesian product), $|x_i| \cdot |y_i| = p_i^2$ number of mappings. Therefore, the total number of domain-to-range set mappings varies from $\sum_{i=1}^{n}(1 \cdot |y_i|) = \sum_{i=1}^{n} p_i$ to $\sum_{i=1}^{n}(|y_i| \cdot |x_i|) = \sum_{i=1}^{n} p_i^2$ mappings.

Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of domain-to-range set mapping cases converges to the real-valued Manhattan and Euclidean distance relations.

3.3. Manhattan distance.

THEOREM 3.3. *Manhattan (largest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$*, where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "taxicab_distance," and formal proof is in the Coq file, euclidrelations.v.

Proof.
From the countable distance definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \\ \Rightarrow \quad d_c \leq \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i. \tag{3.3}$$

Multiply both sides of equation 3.5 by c and take the limit:

$$d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \Rightarrow \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c). \tag{3.4}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d :\ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c. \tag{3.5}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i. \tag{3.6}$$

Combine equations 3.5, 3.4, 3.6:

$$d = \lim_{c\to 0} d_c \cdot c \quad \wedge \quad \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) \quad \wedge \\ \lim_{c\to 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square \tag{3.7}$$

3.4. Euclidean distance.

Theorem 3.4. *Euclidean (smallest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "Euclidean_distance," and formal proof is in the Coq file, euclidrelations.v.

Proof.
Apply the rule of product to the largest number of domain-to-range set mappings, where all p_i number of range set members, y_i, map to each of the p_i number of members in the domain set, x_i, which is the Cartesian product, $|y_i| \cdot |x_i|$:

$$|x_i| = |y_i| = p_i \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2. \tag{3.8}$$

From the countable distance definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \\ \Rightarrow \quad d_c \leq \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i. \tag{3.9}$$

Square both sides of equation 3.9 ($x = y \Leftrightarrow f(x) = f(y)$):

$$\exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\ d_c :\ d_c^2 = (\sum_{i=1}^{n} p_i)^2. \tag{3.10}$$

Apply the square of sum inequality, $(\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$, to equation 3.10 and select the smallest area (the equality) case:

$$d_c^2 = (\sum_{i=1}^{n} p_i)^2 = \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j \\ = \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} p_i \sum_{j=1, j\neq i}^{n} p_j \geq \sum_{i=1}^{n} p_i^2 \quad \Rightarrow \quad \exists\, p_i :\ d_c^2 = \sum_{i=1}^{n} p_i^2. \tag{3.11}$$

Multiply both sides of equation 3.11 by c^2, simplify, and take the limit.

$$d_c^2 = \sum_{i=1}^{n} p_i^2 \;\Rightarrow\; d_c^2 \cdot c^2 = \sum_{i=1}^{n} p_i^2 \cdot c^2 \;\Leftrightarrow\; (d_c \cdot c)^2 = \sum_{i=1}^{n} (p_i \cdot c)^2 \;\Rightarrow\; \lim_{c \to 0} (d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c)^2. \tag{3.12}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$\exists\, c\, d \in \mathbb{R} : \; floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c \to 0} (d_c \cdot c)^2. \tag{3.13}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c \to 0} p_i \cdot c = s_i. \tag{3.14}$$

Combine equations 3.13, 3.12, 3.14:

$$d^2 = \lim_{c \to 0} (d_c \cdot c)^2 \quad \wedge \quad \lim_{c \to 0} (d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c)^2 \quad \wedge \quad \lim_{c \to 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d^2 = \lim_{c \to 0} (d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \tag{3.15}$$ □

3.5. Metric Space. All function range intervals, $d(u, w)$, satisfying the countable distance definition (3.1), where the ruler is applied, generates the properties of metric space: triangle inequality, non-negativity, identity of indiscernibles, and symmetry. The formal proofs: triangle_inequality, non_negativity, identity_of_ indiscernibles, and symmetry are in the Coq file, euclidrelations.v.

THEOREM 3.5. *Triangle Inequality:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \leq d(u, v) + d(v, w)$.

PROOF. Apply the ruler measure (2.1), the countable distance condition (3.1), inclusion-exclusion inequality (3.2), and then ruler convergence (2.2).

$$\begin{aligned} &\forall\, c > 0,\; d(u, w),\; d(u, v),\; d(v, w) \;: \\ &\quad |y_1| = floor(d(u, v)/c) \;\wedge\; |y_2| = floor(d(v, w)/c) \;\wedge \\ &\quad d_c = floor(d(u, w)/c) \;\wedge\; d_c = |y_1 \cup y_2| \leq |y_1| + |y_2| \\ &\quad \Rightarrow\; floor(d(u, w)/c) \leq floor(d(u, v)/c) + floor(d(v, w)/c) \\ &\quad \Rightarrow\; floor(d(u, w)/c) \cdot c \leq floor(d(u, v)/c) \cdot c + floor(d(v, w)/c) \cdot c \\ &\quad \Rightarrow\; \lim_{c \to 0} floor(d(u, w)/c) \cdot c \leq \lim_{c \to 0} floor(d(u, v)/c) \cdot c + \lim_{c \to 0} floor(d(v, w)/c) \cdot c \\ &\quad \Rightarrow\; d(u, w) \leq d(u, v) + d(v, w). \end{aligned} \tag{3.16}$$ □

THEOREM 3.6. *Non-negativity:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \geq 0$.

PROOF. By definition, a set always has a size (cardinal) ≥ 0:

$$\begin{aligned} &\forall\, c > 0,\; d(u, w) \;: \quad floor(d(u, w)/c) = d_c \quad \wedge \quad d_c = |y_1 \cup y_2| \geq 0 \\ &\quad \Rightarrow \quad floor(d(u, w)/c) = d_c \geq 0 \quad \Rightarrow \quad d(u, w) = \lim_{c \to 0} d_c \cdot c \geq 0. \end{aligned} \tag{3.17}$$ □

THEOREM 3.7. *Identity of Indiscernibles:* $d(w, w) = 0$.

PROOF. Apply the triangle inequality property (3.5):

$$\forall\, d(u, v) = d(v, w) = 0 \;\wedge\; d(u, w) \leq d(u, v) + d(v, w) \;\Rightarrow\; d(u, w) \leq 0. \tag{3.18}$$

Combine the non-negativity property (3.6) and the previous inequality (3.18):

$$d(u, w) \geq 0 \;\wedge\; d(u, w) \leq 0 \;\Leftrightarrow\; 0 \leq d(u, w) \leq 0 \;\Rightarrow\; d(u, w) = 0. \tag{3.19}$$

Combine the result of step 3.19 and the condition, $d(u, v) = 0$, in step 3.18.

$$d(u, w) = 0 \;\wedge\; d(u, v) = 0 \;\Rightarrow\; w = v. \tag{3.20}$$

Combine the condition, $d(v,w)=0$, in step 3.18 and the result of step 3.20.

$$d(v,w)=0 \ \wedge \ w=v \ \Rightarrow \ d(w,w)=0. \tag{3.21}$$
□

THEOREM 3.8. *Symmetry: From the Euclidean distance (3.4) and Manhattan distance (3.3) proofs:* $(x^2+y^2)^{1/2} \leq d(x,y) \leq x+y \Rightarrow d(u,v)=d(v,u)$ *in flat space, where* $|x_i|=|y_i|$.

PROOF.

$$(x^2+y^2)^{1/2} \leq d(x,y) \leq x+y \\ \Rightarrow \quad \forall\, p \,:\, 1 \leq p \leq 2, \quad d(x,y)=(x^p+y^p)^{1/p}. \tag{3.22}$$

By the commutative law of addition:

$$d(u,v)=(u^p+v^p)^{1/p} \quad \wedge \quad d(v,u)=(v^p+u^p)^{1/p} \\ \Rightarrow \quad d(u,v)^p=u^p+v^p=v^p+u^p=d(v,u)^p \quad \Rightarrow \quad d(u,v)=d(v,u). \tag{3.23}$$
□

4. Euclidean Volume

The Lebesgue and Borel area/volume measures sum range interval lengths, where each range interval length is defined (assumed) to be the product of an n-tuple domain interval lengths. The goal, here, is to *prove* that Euclidean area/volume is derived from the Cartesian product of countable sets without assuming the product of interval lengths.

The purpose of the structure of the following definition of countable volume is to show the close relationship to countable distance (3.1), which will be further developed in the last section of this article.

DEFINITION 4.1. Countable Volume, v_c, is the largest number of all possible ordered n-tuples:

$$v_c = |\times_{i=1}^n y_i| : \quad \bigcap_{i=1}^n x_i=\emptyset \quad \wedge \quad |x_i|=|y_i| \quad \wedge \quad \bigcap_{i=1}^n y_i=\emptyset.$$

THEOREM 4.2. *Euclidean volume, v, is length of the range interval,* $[v_0,v_m]$, *equal to product of domain interval lengths,* $\{[a_1,b_1],[a_2,b_2],\dots,[a_n,b_n]\}$*:*

$$v=\prod_{i=1}^n s_i, \ \ v=|v_0-v_m|, \ \ s_i=|a_i-b_i|.$$

The theorem, "Euclidean_volume," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i,b_i]$, into a set, x_i, containing p_i number of subintervals.

$$\forall\, i\ n \in \mathbb{N}, \quad i \in [1,n], \quad c>0 \quad \wedge \quad floor(s_i/c)=p_i=|x_i|=|y_i|. \tag{4.1}$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$floor(s_i/c)=p_i \quad \Rightarrow \quad \lim_{c\to 0}(p_i \cdot c)=s_i. \tag{4.2}$$

Apply the associative law of multiplication to derive the countable volume (4.1) in terms of p_i:

$$v_c=|\times_{i=1}^n y_i|=\prod_{i=1}^n|y_i| \quad \wedge \quad |y_i|=p_i \quad \Rightarrow \quad v_c=\prod_{i=1}^n p_i. \tag{4.3}$$

Multiply both sides of equation 4.3 by c^n:

$$v_c \cdot c^n=(\prod_{i=1}^n p_i)\cdot c^n=\prod_{i=1}^n(p_i \cdot c). \tag{4.4}$$

Use those cases, where v_c has an integer n^{th} root.

$$\forall\, n,\ p,\ v_c\ \in \mathbb{N}\ :\ p^n = v_c\ \Rightarrow\ v_c \cdot c^n = p^n \cdot c^n = (p \cdot c)^n = \textstyle\prod_{i=1}^{n}(p_i \cdot c). \tag{4.5}$$

Apply the ruler (2.1) and ruler convergence (2.2) to the range interval, $[v_0, v_m]$ (where $v = |v_0 - v_m|$), and then combine with equations 4.5 and 4.2:

$$floor(v/c^n) = p^n \Rightarrow v = \lim_{c\to 0}(p \cdot c)^n = \textstyle\prod_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i. \tag{4.6}$$ □

5. Applications to physics

5.1. Coulomb's charge force. The sizes, q_1 and q_2, of two charges are independent domain variables, where each infinitesimal size c component of a charge exerts a force on each infinitesimal size c component of the other charge. The total force, F, is proportionate to the total number of forces (the Cartesian product of the infinitesimal size c components) multiplied times a quantum charge force, $m_C a_C$. From the volume proof (4.2), the Cartesian product converges to $q_1 q_2$:

$$F \propto m_C a_C((\lim_{c\to 0} floor(q_1/c) \cdot c)(\lim_{c\to 0} floor(q_2/c) \cdot c)) = m_C a_C(q_1 q_2). \tag{5.1}$$

From equation 5.1, a change in charge, q, causes a proportionate change in force, F. Solving for the quantum force, $F = m_C a_C$, requires a proportionate variable, r, to offset the effect of a change in q. $r \propto q \Rightarrow \exists\, q_C,\ r_C \in \mathbb{R}\ :\ r(q_C/r_C) = q$, where q_C/r_C is a unit-factoring conversion ratio:

$$\forall\, q_1, q_2 \geq 0\ \exists\, q \in \mathbb{R}\ :\ q^2 = q_1 q_2\ \wedge\ r(q_C/r_C) = q\ \Rightarrow\ (r(q_C/r_C))^2 = q_1 q_2. \tag{5.2}$$

$$\begin{aligned}(r(q_C/r_C))^2 = q_1 q_2\ \wedge\ F \propto m_C a_C(q_1 q_2) \\ \Rightarrow\quad F \propto m_C a_C(r(q_C/r_C))^2 = m_C a_C(q_1 q_2) \\ \Rightarrow\quad F = m_C a_C = (m_C a_C r_C^2/q_C^2) q_1 q_2/r^2 = k_c q_1 q_2/r^2.\end{aligned} \tag{5.3}$$

where $k_C = m_C a_C r_C^2/q_C^2$ corresponds to the SI units: Nm^2C^{-2}.

5.2. Newton's gravity force equation. The sizes, m_1 and m_2, of two masses are independent domain variables, where each infinitesimal size c component of a mass exerts a force on each infinitesimal size c component of the other mass. The total force, F, is proportionate to the total number of forces (the Cartesian product of the size c components) multiplied times a quantum gravity force, $m_G a_G$. From the volume proof (4.2), the Cartesian product converges to $m_1 m_2$:

$$F \propto m_G a_G((\lim_{c\to 0} floor(m_1/c) \cdot c)(\lim_{c\to 0} floor(m_2/c) \cdot c)) = m_G a_G(m_1 m_2). \tag{5.4}$$

From equation 5.4, an change in mass, m, causes a proportionate change in force, F. Solving for the quantum force, $F = m_G a_G$, requires a proportionate variable, r, to offset the effect of a change in m. $r \propto m \Rightarrow \exists\, m_G,\ r_G \in \mathbb{R}\ :\ r(m_G/r_G) = m$, where m_G/r_G is a unit-factoring conversion ratio:

$$\begin{aligned}\forall\, m_1, m_2 \geq 0\, \exists\, m \in \mathbb{R}\ :\quad m^2 = m_1 m_2 \quad \wedge \quad r(m_G/r_G) = m \\ \Rightarrow\quad (r(m_G/r_G))^2 = m_1 m_2.\end{aligned} \tag{5.5}$$

$$\begin{aligned}(r(m_G/r_G))^2 = m_1 m_2\ \wedge\ F \propto m_G a_G(m_1 m_2) \\ \Rightarrow\quad F \propto m_G a_G(r(m_G/r_G))^2 = m_G a_G(m_1 m_2) \\ \Rightarrow\quad F = m_G a_G = (a_G r_G^2/m_G) m_1 m_2/r^2.\end{aligned} \tag{5.6}$$

$$(5.7)\quad \exists\, t_G \in \mathbb{R}:\; r_G/t_G^2 = a_G \quad\wedge\quad F = m_G a_G = (a_G r_G^2/m_G) m_1 m_2/r^2$$
$$\Rightarrow\quad F = m_G a_G = (r_G^3/m_G t_G^2) m_1 m_2/r^2 = G m_1 m_2/r^2,$$

where $G = r_G^3/m_G t_G^2$ corresponds to the SI units: $m^3 kg^{-1} s^{-2}$.

5.3. Spacetime equations. The charge (5.3) and gravity (5.7) force equations were derived from the principle that charge and mass are proportionate to distance: $r = (r_C/q_C)q = (r_G/m_G)m$. Suppose that time is also proportionate to distance, $r = (r_c/t_c)t = ct$, where $r_c/t_c = c$ is a unit-factoring conversion ratio.

Applying the ruler to two intervals, $[0, d_1]$ and $[0, d_2]$, in two inertial (independent, non-accelerating) frames of reference, the distance (and time) spanning the two domain intervals converges to a range of distances (and times) from Manhattan (3.3) to Euclidean distance (3.4).

$$(5.8)\quad r^2 = d_1^2 + d_2^2 \quad\wedge\quad r = (r_c/t_c)t = ct$$
$$\Rightarrow\quad (ct)^2 = d_1^2 + d_2^2 \quad\Rightarrow\quad d_2 = \sqrt{(ct)^2 - d_1^2}.$$

$$(5.9)\quad d_2 = \sqrt{(ct)^2 - d_1^2} \quad\wedge\quad d = d_2 \quad\wedge\quad d_1 = vt$$
$$\Rightarrow\quad d \;=\; \sqrt{(ct)^2 - (vt)^2} \;=\; ct\sqrt{1 - (v^2/c^2)},$$

which is the spacetime dilation equation. [**Bru17**].

$$(5.10)\quad d_2^2 = (ct)^2 - d_1^2 \quad\wedge\quad s = d_2 \quad\wedge\quad d_1^2 = x^2 + y^2 + z^2$$
$$\Rightarrow\quad s^2 = (ct)^2 - (x^2 + y^2 + z^2),$$

which is one form of the spacetime interval equation [**Bru17**].

5.4. 3 dimensions of physical geometry. Real-valued intervals are used to model physical distance intervals. And the countable distance, $d_c = |\bigcup_{i=1}^n y_i|$, (3.1) and volume, $d_c = |\times_{i=1}^n y_i|$, (4.1) generating the equations that model physical geometry require being able to assign a sequential order to a set of intervals/dimensions. Physical Manhattan distance intervals between two distinct points can be traversed in every possible order, which can be modeled by each physical interval/dimension being sequentially adjacent (either a successor or predecessor) to every other interval/dimension, herein referred to as a symmetric geometry.

Sets containing one and two members, that are both sequentially ordered and symmetric, are the trivial cases. It will now be proved that the largest sequentially ordered and symmetric set is a cyclic set that contains at most 3 members, in this case, 3 dimensions of physical distance and volume.

Definition 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i \;\in \{x_1, \ldots, x_n\},$$
$$\textit{successor } x_i = x_{i+1} \;\wedge\; \textit{predecessor } x_{i+1} = x_i.$$

Definition 5.2. Symmetric geometry (every set member is sequentially adjacent to every other member):

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; \textit{successor } x_i = x_j \;\Leftrightarrow\; \textit{predecessor } x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$i = n \;\wedge\; j = 1 \;\Rightarrow\; \textit{successor } x_n = x_1 \;\wedge\; \textit{predecessor } x_1 = x_n.$$

The theorem, "ordered_symmetric_is_cyclic," and formal proof is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$i = n \ \wedge \ j = 1 \ \wedge \ successor\ x_i = x_j \ \Rightarrow \ successor\ x_n = x_1. \tag{5.11}$$

Applying the definition of a symmetric geometry (5.2) to conclusion 5.11:

$$successor\ x_i = x_j \ \Rightarrow \ predecessor\ x_j = x_i \ \Rightarrow \ predecessor\ x_1 = x_n. \tag{5.12}$$ □

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

DEFINITION 5.5. Cyclic successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize). \tag{5.13}$$

DEFINITION 5.6. Cyclic predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - q \geq 1). \tag{5.14}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:
(5.15)
$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize).$$

Prove that every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \tag{5.16}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \tag{5.17}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1). \tag{5.18}$$

$$\text{(5.19)} \quad \mathit{Adjacent}(1,2,3) \leftarrow \mathit{Successor}(1,2,3) \leftarrow (n = m+1 \leq \mathit{setsize}).$$
$$\text{(5.20)} \quad \mathit{Adjacent}(2,1,3) \leftarrow \mathit{Predecessor}(2,1,3) \leftarrow (n = m-q \geq 1).$$
$$\text{(5.21)} \quad \mathit{Adjacent}(3,1,3) \leftarrow \mathit{Successor}(3,1,3) \leftarrow (n = \mathit{setsize} \wedge m = 1).$$
$$\text{(5.22)} \quad \mathit{Adjacent}(1,3,3) \leftarrow \mathit{Predecessor}(1,3,3) \leftarrow (m = 1 \wedge n = \mathit{setsize}).$$
$$\text{(5.23)} \quad \mathit{Adjacent}(2,3,3) \leftarrow \mathit{Successor}(2,3,3) \leftarrow (n = m+1 \leq \mathit{setsize}).$$
$$\text{(5.24)} \quad \mathit{Adjacent}(3,2,3) \leftarrow \mathit{Predecessor}(3,2,3) \leftarrow (n = m-q \geq 1).$$

Must prove that for all $\mathit{setsize} > 3$, there exist non-adjacent members. For example, the first and third members are not $(\neg)$ adjacent:

$$\text{(5.25)} \quad \forall\, \mathit{setsize} > 3: \quad \neg\mathit{Successor}(1,3,\mathit{setsize} > 3) \\ \leftarrow \mathit{Successor}(1,2,\mathit{setsize} > 3) \leftarrow (n = m+1 \leq \mathit{setsize}).$$

That is, member 2 is the only successor of member 1 for all $\mathit{setsize} > 3$, which implies member 3 is not a successor of member 1 for all $\mathit{setsize} > 3$.

$$\text{(5.26)} \quad \forall\, \mathit{setsize} > 3: \quad \neg\mathit{Predecessor}(1,3,\mathit{setsize} > 3) \\ \leftarrow \mathit{Predecessor}(1,\mathit{setsize},\mathit{setsize} > 3) \leftarrow (m = 1 \wedge n = \mathit{setsize} > 3).$$

That is, member $n = \mathit{setsize} > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $\mathit{setsize} > 3$.

$$\text{(5.27)} \quad \forall\, \mathit{setsize} > 3: \quad \neg\mathit{Adjacent}(1,3,\mathit{setsize} > 3) \\ \leftarrow \neg\mathit{Successor}(1,3,\mathit{setsize} > 3) \wedge \neg\mathit{Predecessor}(1,3,\mathit{setsize} > 3). \quad \square$$

That is, for all $\mathit{setsize} > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

Applying the ruler measure (2.1) and ruler convergence (2.2) to the set relations, countable distance (3.1) and countable volume (4.1) yields the following insights and implications:

(1) The properties of metric space, Euclidean distance and area/volume are derived from two set operations without using the notions of Euclidean geometry like plane, side, angle, perpendicular, congruence, intersection, etc. [**Joy98**].

(2) Applying the countable set operations, countable distance (3.1) and countable volume (4.1), to sets of same-sized subintervals of intervals allows deriving geometric relations and insights into geometry that metric space, the Lebesgue, Borel, and Hausdorff measures are incapable of providing. For example:

  (a) Countable distance depends on the amount range set intersection: $d_c = |\bigcup_{i=1}^{n} y_i|$. In flat space ($|x_i| = |y_i| = p_i$), Manhattan distance is the case of no intersection: $d_c = \sum_{i=1}^{n} 1 \cdot |y_i| = \sum_{i=1}^{n} p_i$. Euclidean distance is the case of the most domain set members mapping to each range set member: $\sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^2$. The case where both distances are true (coexist) is the inequality: $d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c^2 = (\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$, where the equality case is the smallest distance, which is the set-based reason Euclidean distance (3.4) is the smallest possible distance between two distinct points in $\mathbb{R}^n$.

(b) The triangle inequality of metric space is derived from the countable distance, $d_c = |\bigcup_{i=1}^{n} y_i|$, (3.1) and the inclusion-exclusion inequality, $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$, (3.5).

(c) Generalizing the countable distance constraint, $|x_i| = |y_i|$, (3.1) to $|x_i| = |y_i|^q$, $q \geq 0$, generates all the L^p norms, $||L||_p = (\sum_{i=1}^{n} s_i^p)^{1/p}$. For example, using the same proof pattern as for Euclidean distance (3.4): $p_i = |y_i| \Rightarrow |x_i| = p_i^q \Rightarrow \sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^{q+1} \leq d_c^{q+1} \ldots d = (\sum_{i=1}^{n} s_i^{q+1})^{1/(q+1)}$.

(d) Manhattan distance is largest distance between two distinct points, where each sub-Manhattan distance is the sum of an ordered n-tuple of size c subintervals, one size c subinterval from each disjoint range interval, $\bigcap_{i=1}^{n} y_i = \emptyset$. And, the "volume" containing all possible ordered n-tuples is also the Cartesian product of disjoint sets of range subintervals.

$$d_c = |\bigcup_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset \Rightarrow$$
$$v_c = |\times_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset.$$

(e) The Euclidean volume proof was used to derive the Coulomb's charge force (5.1) and Newton's gravity force (5.4) without using other laws of physics or Gauss's divergence theorem.

(3) **The Proportionate Interval Principle:** The derivations of the charge force, gravity force, and spacetime equations using the Euclidean volume and distance derivations shows that all Euclidean distance intervals having a size, r, have proportionately sized intervals of other types: $r = (r_C/q_C)q = (r_G/m_G)m = (r_c/t_c)t$, where the conversion ratios are for unit-factor analysis.

(a) The derivations of charge and gravity forces requiring the conversion ratios, $q = (q_C/r_C)r$ and $m = (m_G/r_G)r$, implies that if there are quantum values of charge, q_C, and mass, m_G, then there are quantum distances, r_C and r_G, where the charge and gravity forces do not exist (are not defined) at smaller distances, which agrees with the theory of the Planck length, l_P, and the Schwarzschild radius, r_s.

(b) The proportionate interval principal generates the inverse square law. The derivations of charge and gravity forces requiring the conversion ratios, $q = (q_C/r_C)r$ and $m = (m_G/r_G)r$, show that rectangular geometric area (r^2) maps to rectangular charge area ($q_1 q_2$) and mass area ($m_1 m_2$). But, some versions of the charge constant, vacuum magnetic permeability constant, fine structure constant, etc. contain the value 4π because the creators assumed geometric dilution (flux divergence on the surface of a sphere, $4\pi r^2$). It might be that those versions of the constants containing the value 4π are incorrect.

(c) A countable set of values has measure 0 and thus no proportion relationship to distance. Therefore, a countable set of state value changes with respect to time are independent of distance (for example, the change in the spin values of two quantum coupled particles and the change in polarization of two quantum coupled photons).

(4) Relativity theory assumes only 3 dimensions of physical space [**Bru17**]. The proof in this article shows that the properties, order and symmetry, constrains physical distance and volume to at most three dimensions

(5.4). Higher dimensional Hausdorff (inner product vector) spaces are valid if no more than 3 of the dimensions represent physical distance and volume. More than 3 dimensions of physical distance and volume would have sequential adjacency contradictions, where the dimensions of space could not be deterministically sequenced.

(a) The proof of at most 3 members in any ordered and symmetric set (5.4), implies that each *physical* infinitesimal volume (ball) can have at most 3 ordered and symmetric dimensions of discrete *physical* states of the same type. And each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values per infinitesimal ball.

(b) If each of the three possible ordered and symmetric dimensions of discrete physical states contained unordered sets of discrete state values, for example, unordered binary values, then there would be $2 \cdot 2 \cdot 2 = 8$ possible combinations of values. Unordered sets (states) are non-deterministic. For example, for every sequential interval of time an unordered state is physically measured, there is a 50 percent chance of having one of the binary values.

(c) Where infinitesimal balls intersect, an arithmetic of the interactions of the discrete states with respect to time needs to be developed. The interaction of the discrete states associated with overlapping infinitesimal balls with respect to time might result in what we perceive as motion, waves, particles, spin, polarization, work, force, mass, charge, etc.

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George Van Treeck, 668 Westline Dr., Alameda, CA 94501