A Combinatorial Foundation for Analytic Geometry

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Abstract. A ruler measure of intervals allows a new class of combinatorial proofs providing insights into both measure theory and analytic geometry. Applying the ruler measure to the definition of a countable distance range converges to the taxicab distance equation as the upper boundary of the range, the Euclidean distance equation as the lower boundary of the range, and the triangle inequality over the full range, which provides an analytic motivation for the definition of metric space. A combinatorial definition of size (length/area/volume) converges to the product of the interval sizes (Euclidean volume) used in the Lebesgue measure and Euclidean integrals. Combinatorics limits a geometry having the properties of both symmetry and order to a cyclic set of at most 3 dimensions, which is the basis of the right-hand rule. Implications for non-Euclidean geometries and higher dimensional geometries are discussed. All the proofs are verified in Coq.

Contents

1. Introduction

The definitions of: a metric space, the n-volume equation of the Lebesgue measure, and the Euclidean distance equation are imported into analysis from elementary Euclidean geometry. Because the definitions are imported rather than derived from set and number theory, the definitions and equations do not provide insight into the counting principles that generate geometry.

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For example, fundamental to the notion of distance is "correspondence", where for each step (element) in a domain set there exists one element in a distance measure set, like one pebble for each step walked. The constraint that for every disjoint domain set there exists a distance set with the same number of elements, results in a defined range of distance measures, where each measure depends on the number of correspondences from each distance element to domain set elements.

Showing that this definition of a "countable distance range" applied to real-valued intervals converges to the real-valued triangle inequality, where the upper boundary of the range converges to the taxicab distance equation, and the lower boundary of the range converges to the Euclidean distance equation, provides both a motivation and deeper insight into the notion of a metric space independent of Euclidean geometry. And it provides new analytic (set and number theory-based) insights into the notions of shortest distance and Euclidean distance.

However, there have been no set and number theory-based derivations of the real-valued triangle inequality, taxicab and Euclidean distance equations. Further, there has been no derivation of the product of interval sizes, the Euclidean volume equation, used in the Lebesgue measure. Such derivations requires a using a different type of indefinite integral and a different type of interval measure.

The various traditional indefinite integrals (antiderivatives) are used to prove that a **real-valued, continuous function** relating the **size** of the subintervals of domain intervals to the **size** of the subintervals of an image interval converges to a real-valued function. In contrast, what is needed for counting-based (combinatorial) proofs is an indefinite integral that proves that a **combinatorial function** relating the **number** of same-sized subintervals of domain intervals to the **number** of same-sized subintervals in an image interval converges to a real-valued function.

Combinatorial integration requires a different method of dividing intervals into subintervals herein referred to as a ruler. A ruler measures each interval of a set of intervals to the nearest integer number of subintervals, each subinterval having the same size. The ruler is an approximate measure that ignores partial subintervals.

In the traditional method of dividing a set of intervals into subintervals, the number of subintervals is the same in both the domain and image intervals and the size of some subintervals can vary. In contrast, for the ruler measure, the number of subintervals in the domain and image intervals can vary and the size of the subintervals in each interval is always the exact same size.

Same-sized subintervals across both the set of domain intervals and image interval allows defining a countable relationship between the domain subintervals and image subintervals. For example, as the subinterval size goes to zero, the combinatorial relationships that define smallest countable distance and countable size (length/area/volume) converge to the n-dimensional Euclidean distance and volume equations.

Simple counting (combinatorial) principles generate geometry. And it will also shown that combinatorics also limits a geometry that has the properties of being both symmetric and ordered to a cyclic set of at most three dimensions, which is the basis of the right-hand rule.

The proofs in this article are verified formally using the Coq Proof Assistant [15] version 8.4p16. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at:
`https://github.com/treeck/CombinatorialGeometry`.

2. Ruler measure and convergence

DEFINITION 2.1. Ruler measure: A ruler measures the size of a closed, open, or semi-open interval as the nearest integer number of whole subintervals, p, times the subinterval size, c, where c is the independent variable. Notionally:

$$\forall\, c\, s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |b-a| \ \ \wedge\ \ c > 0\ \wedge \tag{2.1}$$
$$(p = floor(s/c)\ \ \vee\ \ p = ceiling(s/c)\ \ \wedge\ \ M = \lim_{c\to 0} \sum_{i=1}^{p} c = \lim_{c\to 0} pc.$$

The ruler measure has the three properties of measure in a σ-algebra:

(1) Non-negativity: $\forall\, E \in \Sigma,\ \mu(E) \geq 0: \quad s = |b-a|\ \wedge\ c > 0\ \Rightarrow$ $M = \lim_{c\to 0} pc \geq 0$.
(2) Zero-sized empty set: $\mu(\emptyset) = 0: \quad b = a \Rightarrow p = 0\ \Rightarrow M = \lim_{c\to 0} pc = 0$.
(3) Countable additivity: $\forall\, \{E_i\}_{i\in\mathbb{N}},\ |\cap_{i=1}^{\infty} E_i| = \emptyset \wedge \mu(\cup_{i=1}^{\infty} E_i) = \mu(\Sigma_{i=1}^{\infty} E_i)$. $(c \to 0 \Rightarrow p \to \infty)\ \wedge\ \mu(E_i) = c\ \Rightarrow\ \ \mu(\Sigma_{i=1}^{\infty} E_i) = \Sigma_{p=1}^{\infty} c = \lim_{c\to 0} pc$.

For example, showing convergence using the interval, $[0,\pi]$, $s = |\pi - 0|$, $c = 10^{-i}, i \in \mathbb{N}$, and $p = floor(s/c)$, then, $p \cdot c = 3.1,\ 3.14,\ 3.141, ..., \pi$.

THEOREM 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\ \ s = |b-a|\ \ \Rightarrow\ \ M = \lim_{c\to 0} pc = s.$

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0, \quad p = floor(s/c) \quad \Rightarrow \quad 0 \leq |p - s/c| < 1. \tag{2.2}$$

Multiply all sides by $|c|$:

$$\forall\, c > 0, \quad 0 \leq |p - s/c| < 1 \quad \Rightarrow \quad 0 \leq |pc - s| < |c|. \tag{2.3}$$

$$\forall\, c > 0, \exists\, \delta,\ \epsilon\ :\ 0 \leq |pc - s| < |c| = |c - 0| < \delta = \epsilon \tag{2.4}$$
$$\Rightarrow \quad 0 < |c - 0| < \delta \quad \wedge \quad 0 \leq |pc - s| < \epsilon = \delta \quad := \quad M = \lim_{c\to 0} pc = s. \quad \square$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure).

3. Distance

The primary principle of a countable distance measure is that the number of countable image (distance) set elements is equal to the number of domain set elements. Applying this principle to multiple disjoint (non-intersecting), domain sets, if there are p_i elements in the i^{th} domain set, then there exists p_i elements in the corresponding i^{th} distance set.

DEFINITION 3.1. Countable distance range, d_c:

$$\forall\, i\ n \in \mathbb{N},\ \ x_i \subseteq X,\ \ \bigcap_{i=1}^{n} x_i = \emptyset,\ \ \forall\, x_i\ \exists\, y_i \subseteq Y:\ \ |x_i| = |y_i| \quad \wedge \quad d_c = |Y|.$$

In the definition of countable distance range (3.1), the vertical bars around a set is the standard notation for indicating the cardinal (number of elements in the set). To prevent too much overloading on the vertical bar, the symbol for "such that" is the colon.

Note that the definition of a countable distance range (3.1) does **not** place a limitation on the distance sets being disjoint. Further, if a distance set element corresponds to more than one domain set element, then the size of the total (union) distance set, Y, will be smaller than the total (union) size of the disjoint domain sets, X. In other words, distance, $d_c = |Y| = |\cup_{i=1}^{n} y_i| \leq= \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} |x_i|$.

The property, $|x_i| = |y_i|$, of a countable distance range (3.1) implies a limitation on the number of correspondences of a distance set element to domain set elements. If each of the p_i number of elements of the i^{th} distance set has a one-to-one (bijective) correspondence to a domain set element, then the number of correspondences per distance set is: $1 \cdot |x_i| = 1 \cdot p_i = p_i = |y_i|$. A one-to-one correspondence of the distance set elements to the disjoint domain set elements implies that the distance sets are also disjoint. And therefore, the distance, $d_c = |Y| = |\cup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i$, is the largest possible distance and the upper bound of distance range.

If each of the p_i number of elements of the i^{th} distance set corresponds to all p_i number of domain set elements, then the largest number of correspondences per distance set is: $|y_i| \cdot |x_i| = p_i \cdot p_i = p_i^2$. The largest number of possible correspondences is related to the smallest possible distance and is the lower bound of the distance range, $\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |y_i|^2 = |\{(y, y) : y \in Y\}|$.

It is **not** possible for the sum of combinations, $|\{(y, y) : y \in Y\}|$, to always have an integer square root. But, the ruler (2.1) and ruler convergence theorem (2.2) is applied to real-valued intervals to show the shortest distance case converges to the real-valued Euclidean distance equation (always a real-valued square root).

The proof of equality for both real-valued taxicab and Euclidean distance equations requires the proof strategy of showing that the right and left sides of a proposed counting-based equation both converge to the same real value and therefore are equal. That is, the propositional logic, $A = B \wedge C = B \Rightarrow A = C$, is used.

THEOREM 3.2. *Taxicab (largest) distance, d, is the size of the distance interval,* $[y_0, y_m]$, *corresponding to a set of disjoint domain intervals,* $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, *where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|.$$

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals. Next, apply the definition of the countable distance range (3.1) and the rule of product:

$$\text{(3.1)} \quad \forall\, i\, n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\ |\{x_i : x_i \in \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\}| = |\{y_i : y_i \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_i}\}\}| = p_i.$$

$$\text{(3.2)} \qquad \forall\, i\, n \in \mathbb{N}, \quad i \in [1, n], \quad y \in y_i \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i = |\{y\}|.$$

Multiply both sides of 3.2 by c and apply the ruler convergence theorem (2.2):

$$(3.3)\quad s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n} (p_i \cdot c) = |\{y\}| \cdot c$$
$$\Rightarrow \quad \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) = \lim_{c\to 0} |\{y\}| \cdot c.$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$(3.4)\qquad \forall\, c > 0, \quad p_d = floor(d/c) = |\{y:\ y \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_d}\}\}|.$$

Multiply both sides of 3.4 by c and apply the ruler convergence theorem (2.2):

$$(3.5)\ d = \lim_{c\to 0} p_d \cdot c \ \wedge\ p_d \cdot c = |\{y\}| \cdot c \ \Rightarrow\ d = \lim_{c\to 0} p_d \cdot c = \lim_{c\to 0} |\{y\}| \cdot c.$$

Combine equations 3.5 and 3.3:

$$(3.6)\qquad d = \lim_{c\to 0} |\{y\}| \cdot c \quad \wedge \quad \sum_{i=1}^{n} s_i = \lim_{c\to 0} |\{y\}| \cdot c \quad \Rightarrow \quad d = \sum_{i=1}^{n} s_i. \qquad \square$$

THEOREM 3.3. *Euclidean (smallest) distance, d, is the size of the distance interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals, $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|.$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals. Next, apply the definition of the countable distance range (3.1) and the rule of product:

$$(3.7)\quad \forall\, i\ n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge$$
$$|\{x_i : x_i \in \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\}| = |\{y_i : y_i \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_i}\}\}| = p_i \quad \Rightarrow$$
$$\forall\, i \in [1, n], \quad |\{(x_i, y_i)\}| = |\{(y_i, y_i)\}| = p_i^2.$$

$$(3.8)\quad \forall\, i\ n \in \mathbb{N}, \quad i \in [1, n], \quad |\{(y_i, y_i)\}| = p_i^2 \quad \wedge \quad y = y_i \quad \Rightarrow$$
$$|\sum_{i=1}^{n} \{(y_i, y_i)\}| = \sum_{i=1}^{n} p_i^2 = |\{(y, y)\}|.$$

Multiply both sides of 3.8 by c^2 and apply the ruler convergence theorem (2.2):

$$(3.9)\quad s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n} (p_i \cdot c)^2 = |\{(y, y)\}| \cdot c^2$$
$$\Rightarrow \quad \sum_{i=1}^{n} s_i^2 = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c)^2 = \lim_{c\to 0} |\{(y, y)\}| \cdot c^2.$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$(3.10)\quad \forall\, c > 0, \quad p_d = floor(d/c) = |\{y:\ y \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_d}\}\}|$$
$$\Rightarrow \quad p_d^2 = |\{(y, y)\}|.$$

Multiply both sides of 3.10 by c^2 and apply the ruler convergence theorem (2.2):

$$(3.11)\quad d = \lim_{c\to 0} p_d \cdot c \quad \wedge \quad (p_d \cdot c)^2 = |\{(y,y)\}| \cdot c^2 \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(p_d \cdot c)^2 = \lim_{c\to 0} |\{(y,y)\}| \cdot c^2.$$

Combine equations 3.11 and 3.9:

$$(3.12)\quad d^2 = \lim_{c\to 0} |\{(y,y)\}| \cdot c^2 \quad \wedge \quad \sum_{i=1}^{n} s_i^2 = \lim_{c\to 0} |\{(y,y)\}| \cdot c^2 \quad \Rightarrow \quad d^2 = \sum_{i=1}^{n} s_i^2. \quad \square$$

3.1. Triangle inequality. The definition of a metric in real analysis is based on the triangle inequality, $\mathbf{d}(\mathbf{u},\mathbf{w}) \le \mathbf{d}(\mathbf{u},\mathbf{v}) + \mathbf{d}(\mathbf{v},\mathbf{w})$, that has been intuitively motivated by the triangle [**Gol76**]. Applying the inclusion-exclusion principle, ruler (2.1), and ruler convergence theorem (2.2) to the definition of a countable distance range (3.1) yields the real-valued triangle inequality:

$$(3.13)\quad d_c = |Y| = |\bigcup_{i=1}^{2} y_i| \le \sum_{i=1}^{2} |y_i| \quad \wedge$$
$$d_c = floor(\mathbf{d}(\mathbf{u},\mathbf{w})/c) \ \wedge \ |y_1| = floor(\mathbf{d}(\mathbf{u},\mathbf{v})/c) \ \wedge \ |y_2| = floor(\mathbf{d}(\mathbf{v},\mathbf{w})/c)$$
$$\Rightarrow \ \mathbf{d}(\mathbf{u},\mathbf{w}) = \lim_{c\to 0} d_c \cdot c \le \sum_{i=1}^{2} \lim_{c\to 0} |y_i| \cdot c = \mathbf{d}(\mathbf{u},\mathbf{v}) + \mathbf{d}(\mathbf{v},\mathbf{w}).$$

4. Size (length/area/volume)

The countable size measure is the number of combinations (correspondences) between members of disjoint domain sets, which is the Cartesian product of the domain set sizes.

Definition 4.1. countable size (length/area/volume) measure, S_c:

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1,n], \quad x_i \subseteq X, \quad |\bigcap_{i=1}^{n} x_i| = \emptyset \quad \wedge \quad \{(x_1,\ldots,x_n)\} = y \quad \wedge$$
$$S_c = |y| = |\{(x_1,\ldots,x_n)\}| = \prod_{i=1}^{n} |x_i|.$$

Theorem 4.2. *Euclidean size (length/area/volume), S, is the size of an image interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals:* $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, *where:*

$$S = \prod_{i=1}^{n} s_i, \quad S = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|, \quad i \in [1,n], \quad i,n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_size."

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$(4.1)\quad \forall\, i\; n \in \mathbb{N}, \quad i \in [1,n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \quad x_i = \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\} \quad \Rightarrow |x_i| = p_i.$$

Use the ruler (2.1) to divide the exact size, $S = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_S^n subintervals, where p_S^n satisfies the definition a countable size measure, S_c (4.1).

$$\forall\, c > 0 \quad \wedge \quad \exists\, r \in \mathbb{R},\ S = r^n \quad \wedge \quad p_S = floor(r/c) \quad \wedge \quad p_S^n = S_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i. \tag{4.2}$$

Multiply both sides of equation 4.2 by c^n to get the ruler measures:

$$p_S^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3:

$$S = r^n = \lim_{c\to 0} (p_S \cdot c)^n \quad \wedge \quad \lim_{c\to 0} (p_i \cdot c) = s_i \quad \wedge \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c) \tag{4.4}$$
$$\Rightarrow \quad S = \lim_{c\to 0} (p_S \cdot c)^n = \prod_{i=1}^{n} \lim\nolimits_{c\to 0} (p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square$$

5. Derived geometric definitions

5.1. Derived geometric primitives. There are no new mathematics in this section of derived geometric primitives. The purpose of this section is two fold: 1) Show a difference in perspective. In classical geometry, Euclidean distance is a product of lines and angles. Here, the perspective is that lines and angles are objects generated from Euclidean distance, which was generated from the counting principle of shortest distance. 2) The derivation of arc angle from Euclidean distance is used in combination with the section on symmetry and order to explain why the notion of perpendicular (right arc angle) is only valid up to three dimensions, while vector orthogonality (dot product equal to zero) is valid for any number of dimensions.

Definition 5.1. Straight line segment is the smallest (Euclidean) distance interval, $[y_0, y_m]$ (3.3).

Definition 5.2. Straight line segment orientation (slope): $\mathrm{d}b/\mathrm{d}a = b/a$, where $a = x_{m_1,1} - x_{0,1}$ and $b = x_{m_n,2} - x_{0,2}$ are the signed sizes of two domain intervals, $[x_{0,1}, x_{m_1,1}]$ and $[x_{0,2}, x_{m_n,2}]$.

The signed sizes, a and b, of the two domain intervals can be calculated from a single parametric distance, θ, and Euclidean distance, d.

Definition 5.3. Parametric distance (arc angle), θ:

$$b/a = \mathrm{d}b/\mathrm{d}a = \mathrm{d}b/\mathrm{d}\theta \cdot \mathrm{d}\theta/\mathrm{d}a = \sqrt{d^2 - a^2}/\sqrt{d^2 - b^2} \tag{5.1}$$

$$Case: \quad \mathrm{d}b/\mathrm{d}a = b/a = 1 \quad \Rightarrow \quad \mathrm{d}\theta/\mathrm{d}a = 1/\sqrt{d^2 - b^2} = 1/\sqrt{d^2 - a^2} \tag{5.2}$$

Applying Taylor's theorem [**Gol76**] and a table of integrals [**Wc11**]:

$$\int \mathrm{d}\theta = \int \mathrm{d}a/\sqrt{d^2 - a^2} \quad \Rightarrow \quad \theta = \sin^{-1}(a/d) = \cos^{-1}(b/d). \tag{5.3}$$

5.2. Vectors. Before discussing the implications of the proofs in this article on vector analysis for dimensions greater than three, the notions of vector, parallel, and orthogonal are defined here in terms of sets of intervals.

DEFINITION 5.4. Vector: A vector is the ordered set of the signed domain interval sizes, $\mathbf{s} = \{s_1, \ldots, s_n\}$, where $s_i = x_{m_i,i} - x_{0,i}$ for the domain interval, $[x_{0,i}, x_{m_i,i}]$.

DEFINITION 5.5. Parallel (congruent) vectors: Two vectors are parallel if each ratio of the signed sizes in one vector equals the ratio of the corresponding signed sizes in another vector (same rate of change in the same direction):

$$\frac{s_{1_i}}{s_{1_{i+1}}} = \frac{s_{2_i}}{s_{2_{i+1}}}, \quad i \in [1, n-1]. \tag{5.4}$$

DEFINITION 5.6. Orthogonal vectors: Two vectors are orthogonal if each ratio of the signed sizes in one vector is the inverse ratio and inverse sign of two corresponding signed sizes in another vector (inverse rate of change and inverse directions). Simplifying the equation yields the **dot (inner) product** equal to zero for any number of dimensions:

$$\frac{s_{1_i}}{s_{1_{i+1}}} = -\frac{s_{2_{i+1}}}{s_{2_i}}, \quad i \in [1, n-1] \quad \Leftrightarrow \quad \sum_{i=1}^{n} s_{1_i} \cdot s_{2_i} = 0. \tag{5.5}$$

6. Symmetric and ordered geometries

Euclidean size (area/volume) and distance are invariant for every order (permutation) of a set of intervals. A function (like size or distance) where every permutation of the arguments yields the same value(s) is called a symmetric function. If one can "jump" from any interval (element), x_i, of a set to another element, x_j to form the permutation, (x_i, x_j), then one can also jump back from element x_j to element x_i to form the permutation (x_j, x_i). In other words, every element x_i of a set having an immediate successor element x_j also requires that x_j has the immediate predecessor element x_i, such that traversing the elements in successor order and predecessor order yields both permutations.

DEFINITION 6.1. Symmetric geometry:

$$\forall\, i\; j\; n \in \mathbb{N},\ \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\ \textit{successor}\ x_i = x_j \ \wedge\ \textit{predecessor}\ x_j = x_i.$$

Two sets of intervals with the same volume and spanning distance (for example, $\{[0,2],\ [0,1],\ [0,5]\}$ and $\{[0,5],\ [0,2],\ [0,1]\}$) can only be distinguished by assigning a relative sequential order (orientation) to the elements of the interval (dimension) sets.

DEFINITION 6.2. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\ \forall\, x_i \in \{x_1, \ldots, x_n\},\ \textit{successor}\ x_i = x_{i+1} \ \wedge\ \textit{predecessor}\ x_{i+1} = x_i.$$

It will now be proved that any geometry, both Euclidean and non-Euclidean, that has both symmetry (every permutation of domain intervals yields the same distance and volume) and order (ability to discriminate distances and volumes by a relative sequential ordering), is a cyclic set limited to at most 3 domain intervals (dimensions), which is the basis for the right-hand rule. The implications with respect to vector operations and higher dimensioned geometries are discussed in the summary.

THEOREM 6.3. *A symmetric and ordered geometry is a cyclic set.*

$$\forall\, i\; j\; n \in \mathbb{N},\ \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\ i = n\ \wedge\ j = 1 \\ \Rightarrow\quad \mathit{successor}\ x_n = x_1\ \wedge\ \mathit{predecessor}\ x_1 = x_n.$$

The theorem and formal Coq-based proof is "ordered_symmetric_is_cyclic," which is located in the file threed.v.

PROOF. The property of order (6.2) defines unique successors and predecessors for all elements except for the successor of x_n and the predecessor of x_1. From the properties of a symmetric geometry (6.1):

$$i = n\ \wedge\ j = 1\ \wedge\ \mathit{successor}\ x_i = x_j\ \Rightarrow\ \mathit{successor}\ x_n = x_1. \tag{6.1}$$

$$i = n\ \wedge\ j = 1\ \wedge\ \mathit{predecessor}\ x_j = x_i\ \Rightarrow\ \mathit{predecessor}\ x_1 = x_n. \tag{6.2}$$

□

For example, using the cyclic set with elements labeled, $\{1,\ 2,\ 3\}$, starting with each element and counting through 3 cyclic successors and counting through 3 cyclic predecessors yields all possible permutations: $(1,2,3)$, $(2,3,1)$, $(3,1,2)$, $(1,3,2)$, $(3,2,1)$, and $(2,1,3)$. That is, a cyclically ordered set preserves sequential order while allowing a set of n-at-a-time permutations. If all possible n-at-a-time permutations are generated, then the cyclic set is also symmetric.

THEOREM 6.4. *An symmetric and ordered geometry is limited to at most 3 elements. That is, each element is sequentially adjacent (a successor or predecessor) to every other element in a set only where the number of elements (set sizes) are less than or equal to 3.*

The Coq-based lemmas and proofs in the file threed.v are:
`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn-like clauses (a subset of first-order logic) with unification and resolution. Horn clauses make it clear which facts satisfy a goal.

PROOF.
Because a symmetric and ordered set is a cyclic set (6.3), the successors and predecessors are cyclic:

DEFINITION 6.5. Successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (m + 1 \leq \mathit{setsize}). \tag{6.3}$$

DEFINITION 6.6. Predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (m - 1 \geq 1). \tag{6.4}$$

DEFINITION 6.7. Adjacent: element m is adjacent to element n (an allowed permutation), if the cyclic successor of m is n or the cyclic predccessor of m is n. Notionally:

$$\mathit{Adjacent}(m, n, \mathit{setsize}) \leftarrow \mathit{Successor}(m, n, \mathit{setsize}) \vee \mathit{Predecessor}(m, n, \mathit{setsize}). \tag{6.5}$$

Every element is adjacent to every other element, where $\mathit{setsize} \in \{1,\ 2,\ 3\}$:

$$\mathit{Adjacent}(1, 1, 1) \leftarrow \mathit{Successor}(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{6.6}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (1+1 \leq 2). \tag{6.7}$$
$$Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (2=2 \wedge 1=1). \tag{6.8}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (1+1 \leq 2). \tag{6.9}$$
$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (2-1 \geq 1). \tag{6.10}$$
$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (3=3 \wedge 1=1). \tag{6.11}$$
$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (1=1 \wedge 3=3). \tag{6.12}$$
$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (2+1 \leq 3). \tag{6.13}$$
$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (3-1 \geq 1). \tag{6.14}$$

For all $n = setsize > 3$, there exist non-adjacent elements (not every permutation allowed):

$$\forall\, n > 3,\ Successor(1,2,n) \Rightarrow \forall\, n > 3,\ \neg Successor(1,3,n). \tag{6.15}$$

That is, 2 is the only successor of 1 for all $n > 3$, which implies 3 is not a successor of 1 for all $n > 3$.

$$\forall\, n > 3,\ Predecessor(1,n,n) \Rightarrow \forall\, n > 3,\ \neg Predecessor(1,3,n). \tag{6.16}$$

That is, n is the only predecessor of 1 for all $n > 3$, which implies 3 is not a predecessor of n for all $n > 3$.

$$\forall\, n > 3,\ \neg Adjacent(1,3,n) \leftarrow \neg Successor(1,3,n) \wedge \neg Predecessor(1,3,n). \tag{6.17}$$

□

7. Summary

The ruler measure of intervals is an analytic tool allowing a new class of combinatorial proofs that provides insights into measure theory and analytic geometry:

(1) Combinatorial relations between the elements of sets converge to the Euclidean distance (3.3) and size (length/area/volume) (4.2) equations without notions of side, angle, and shape, and without motivation from diagrams. In particular, geometry is generated from the analytic properties of the real-value continuum.

(2) Taxicab distance (3.2), Euclidean distance (3.3), and the triangle inequality (3.1) are derived from the definition of the countable distance range (3.1), where taxicab distance is the largest possible monotonic distance and Euclidean distance is the smallest distance. This provides a counting-based motivation for the definition of the metric space without the need for Euclidean geometry.

(3) The Euclidean volume (product of interval sizes) of the Lebesgue measure is derived from use of the more fundamental ruler measure.

(4) Combinatorics limits a geometry having the properties of both order (6.2) and symmetry (6.1) to a cyclic set (6.3) of at most three elements (dimensions) (6.4), which is the basis of the right-hand rule. Empirical observation of physical space is that there are only three, cyclic dimensions (the basis for the right-hand rule in physics and engineering), which indicates physical space is both symmetric and ordered. Therefore, the properties

of symmetry and order explains why there can not be more than three dimensions of physical space.

(5) The countable distance range (3.1) underlying the triangle inequality and metric space defines that for every disjoint countable domain set there exists a corresponding image (distance) set with the same number of elements (where the amount of intersection of the distance sets defines the range of possible distances). Therefore, any elliptic or hyperbolic geometries that is a metric spaces will probably have the same counting-based constraint.

(6) Vector orthogonality (inner product equal to zero) was derived in this article for any number of dimensions (5.6) and only relies on the property of order. In contrast, arc angle was derived in this article from the symmetric function, Euclidean distance (5.3). Therefore, perpendicular (a right arc angle) is limited to at most three ordered dimensions (6.4). Further, the common assumption that vector orthogonality is the same as perpendicular for any number of dimensions is incorrect.

A cyclic set is a closed walk. An observer in a closed walk of three physical space dimensions would only be able to detect higher, non-physical space dimensions (other variables) indirectly via changes in the three closed walk dimensions (what physicists call "work"). In other words, distance and size in the three physical space dimensions are functions of variables in the higher dimensions.

Displaying higher dimensional manifolds in Euclidean coordinate diagrams (for example, three dimensional Cartesian coordinates and spherical coordinates) is probably only meaningful for the case where three of the modeled dimensions are both geometrically symmetric (6.1) and ordered (6.2).

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