A Combinatorial Foundation for Analytic Geometry

George. M. Van Treeck

Abstract. Using a ruler-like measure of intervals with real analysis allows proofs that provide new insights into real analysis, measure theory, and geometry: A set-based definition of a countable distance range applied to sets of subintervals of intervals converges to the taxicab distance equation as the upper boundary, the Euclidean distance equation as the lower boundary, and the triangle inequality over the full range, which provides counting-based motivations for the definitions of metric space and Euclidean distance independent of elementary geometry. The Cartesian product of the subintervals of intervals converges to the product of interval sizes used in the Lebesgue measure and Euclidean integrals. Also combinatorics limits a geometry that is both ordered and symmetric to a cyclic set of at most 3 dimensions, which is the basis of the right-hand rule. Implications for non-Euclidean geometries and higher dimensional geometries are discussed. All the proofs are verified in Coq.

Contents

1. Introduction

The triangle inequality of a metric space, Euclidean distance metric, and the volume equation (product of interval sizes) of the Lebesgue measure and Euclidean integrals are imported into mathematical analysis from Euclidean geometry [**Gol76**] as primitives rather than derived from set and number theory-based definitions. As a consequence, mathematical analysis has provided no insight into the counting principles that motivate and generate those geometric relations.

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Real analysis and measure theory have not provided any proofs that combinatorial relations between the subintervals of a set of domain intervals and the subintervals of an image interval converge to the triangle inequality, Euclidean distance, and volume equations, as the subinterval size goes to zero. Understanding the combinatorial relations generating the triangle inequality and Euclidean distance provides counting-based insights into the notions of a distance measure and smallest distance that importing as primitives from Euclidean geometry does not provide. Further, the Lebesgue measure and Euclidean integrals sum the product of interval sizes (Euclidean volume) without proof that the Cartesian product of the subintervals of intervals converges to the product of intervals sizes.

The various traditional indefinite integrals (antiderivatives) derive a real-valued equation from a **real-valued, continuous function** relating the **sizes** of the subintervals. In contrast, what is needed for counting-based (combinatorial) proofs is an indefinite integration that derives a real-valued equation from a **combinatorial function** relating the integer **number** of same-sized subintervals of domain intervals to the integer **number** of same-sized subintervals in an image interval.

Combinatorial integration requires measuring the number of same-sized subintervals of intervals similar to using a ruler (measuring stick). Unlike traditional integration, the ruler is an approximate measure that ignores partial subintervals in **both** the domain and image intervals.

Using the ruler measure, the size of subintervals is the same in both the domain and image intervals and the number of subintervals in each domain and image interval can vary. In contrast, the traditional method of dividing a set of intervals into subintervals, the number of subintervals is the same in both the domain and image intervals and the size of some subintervals can vary.

The Euclidean volume and distance equations can be extended to any number of dimensions. So, why does classical Euclidean geometry appear to be limited to three dimensions? The counting principles generating distance and volume provides insight into the properties that can limit both Euclidean and non-Euclidean geometries to at most three, cyclic dimensions.

The proofs in this article are verified formally using the Coq Proof Assistant [**Coq15**] version 8.4p16. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at:
https://github.com/treeck/CombinatorialGeometry.

2. Ruler measure and convergence

Definition 2.1. Ruler measure: A ruler measures the size of a closed, open, or semi-open interval as the nearest integer number of whole subintervals, p, times the subinterval size, c, where c is the independent variable. Notionally:

$$\begin{aligned}(2.1)\quad &\forall\, c\, s \in \mathbb{R},\ \ [a, b] \subset \mathbb{R},\ \ s = |b - a| \ \ \wedge\ \ c > 0\ \wedge \\ &\qquad (p = floor(s/c)\ \ \vee\ \ p = ceiling(s/c))\ \ \wedge\ \ M = \Sigma_{p=1}^{\infty} c = \lim_{c\to 0} pc.\end{aligned}$$

Theorem 2.2. *Ruler convergence:*
$\forall\, [a, b] \subset \mathbb{R},\ \ s = |b - a| \ \ \Rightarrow\ \ M = \lim_{c\to 0} pc = s.$

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : \, y \leq x, \, y \in \mathbb{Z}, \, x \in \mathbb{R}\})$:

$$\forall\, c > 0, \quad p = floor(s/c) \quad \Rightarrow \quad 0 \leq |p - s/c| < 1. \tag{2.2}$$

Multiply all sides by $|c|$:

$$\forall\, c > 0, \quad 0 \leq |p - s/c| < 1 \quad \Rightarrow \quad 0 \leq |pc - s| < |c|. \tag{2.3}$$

$$\begin{aligned} &\forall\, c > 0, \, \exists\, \delta, \, \epsilon \, : \, 0 \leq |pc - s| < |c| = |c - 0| < \delta = \epsilon \\ &\Rightarrow \quad 0 < |c - 0| < \delta \quad \wedge \quad 0 \leq |pc - s| < \epsilon = \delta \quad := \quad M = \lim_{c \to 0} pc = s. \end{aligned} \tag{2.4}$$ □

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure).

For example, showing convergence using the interval, $[0, \pi]$, $s = |\pi - 0|$, $c = 10^{-i}, i \in \mathbb{N}$, and $p = floor(s/c)$, then, $p \cdot c = 3.1_{i=1}, \; 3.14_{i=2}, \; 3.141_{i=3}, ..., \pi$.

3. Distance

The most basic principle of distance is that an image (distance) set has the same number of elements as a corresponding domain set. For example, the number of steps in a distance set must equal the number pieces of land traversed. Therefore, for each i^{th} disjoint domain set containing p_i number of elements there exists a distance set with the same p_i number of elements.

The union size of the distance sets is less than or equal to the union size of the disjoint domain sets. In the case of intersecting distance sets, a single distance set element will correspond to multiple domain set elements. Notionally:

DEFINITION 3.1. Countable distance range, d_c:

$$\forall\, i\, n \in \mathbb{N}, \quad x_i \subseteq X, \quad \bigcap_{i=1}^{n} x_i = \emptyset, \quad \forall\, x_i \,\exists\, y_i \subseteq Y :$$

$$|x_i| = |y_i| \quad \wedge \quad |\bigcap_{i=1}^{n} y_i| \geq 0, \quad \wedge \quad d_c = |\bigcup_{i=1}^{n} y_i| = |Y|.$$

Notation conventions: The vertical bars around a set is the standard notation for indicating the cardinal (number of elements in the set). To prevent over use of the vertical bar, the symbol for "such that" is the colon.

The countable distance range principle (3.1), $|x_i| = |y_i| = p_i$, constrains each i^{th} distance set element to a range of correspondences from one domain set element to as many as p_i number of domain set elements. More than p_i number of correspondences would be over counting like a step walked corresponding to the same piece of land multiple times.

Using the rule of product, there is a range of $|y_i| \cdot 1 = p_i$ to $|y_i| \cdot p_i = p_i^2$ number of distance-to-domain correspondences per distance set. Therefore, $d_c = f(\sum_{i=1}^{n} p_i)$ is the largest possible distance (a function of the smallest number of correspondences per distance set element due to disjoint distance sets). $d_c = f(\sum_{i=1}^{n} p_i^2)$ is the smallest possible distance (a function of the largest number of correspondences per distance set element due to maximum allowed intersection of distance sets).

Using the ruler (2.1) to divide a set of real-valued domain intervals and distance interval into sets of same-sized subintervals, and applying the ruler convergence theorem (2.2) proves that the largest and shortest distance cases converges to the real-valued taxicab and Euclidean distance equations.

The convergence proofs of the taxicab and Euclidean distance equations requires the strategy of showing that the right and left sides of a proposed counting-based equation both converge to the same real value and therefore are equal. In other words, the propositional logic, $A = B \wedge C = B \Rightarrow A = C$, is used.

THEOREM 3.2. *Taxicab (largest) distance, d, is the size of the distance interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals, $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|.$$

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$\begin{aligned} \forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\ |\{x_i : x_i \in \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\}| = |\{y_i : y_i \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_i}\}\}| = p_i. \end{aligned} \tag{3.1}$$

Next, apply the definition of the countable distance range (3.1):

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad y \in y_i \subseteq Y \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i = |\{y : \; y \in Y\}|. \tag{3.2}$$

Multiply both sides of 3.2 by c and apply the ruler convergence theorem (2.2):

$$\begin{aligned} s_i = \lim_{c \to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n} (p_i \cdot c) = |\{y\}| \cdot c \\ \Rightarrow \quad \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) = \lim_{c \to 0} |\{y\}| \cdot c. \end{aligned} \tag{3.3}$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$\forall\, c > 0, \quad p_d = floor(d/c) = |\{y : \; y \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_d}\} = Y\}|. \tag{3.4}$$

Multiply both sides of 3.4 by c and apply the ruler convergence theorem (2.2):

$$d = \lim_{c \to 0} p_d \cdot c \; \wedge \; p_d \cdot c = |\{y\}| \cdot c \; \Rightarrow \; d = \lim_{c \to 0} p_d \cdot c = \lim_{c \to 0} |\{y\}| \cdot c. \tag{3.5}$$

Combine equations 3.5 and 3.3:

$$d = \lim_{c \to 0} |\{y\}| \cdot c \quad \wedge \quad \sum_{i=1}^{n} s_i = \lim_{c \to 0} |\{y\}| \cdot c \quad \Rightarrow \quad d = \sum_{i=1}^{n} s_i. \tag{3.6}$$ □

THEOREM 3.3. *Euclidean (smallest) distance, d, is the size of the distance interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals, $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}].$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1,n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\ x_i = \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\} \quad \wedge \quad y_i = \{y_{1_i}, y_{2_i}, \ldots, y_{p_i}\}\}|. \tag{3.7}$$

Next, apply the definition of the countable distance range (3.1) and the rule of product:

$$\textstyle\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} |\{(y_a, y_b) : y_a\, y_b \in y_i\}|, \tag{3.8}$$

where each pair, (y_a, y_b), represents a combination (correspondence) between two elements in the distance set, y_i. From the definition of countable distance range (3.1), the distance sets can intersect, which results in a range of possible distance set sizes. Applying the inclusion-exclusion principle:

$$|\cap_{i=1}^{n} y_i| \geq 0 \quad \Rightarrow \quad \textstyle\sum_{i=1}^{n} |y_i| \geq |\cup_{i=1}^{n} y_i| = |Y|. \tag{3.9}$$

From combining equation 3.8 and the equality case of relation 3.9:

$$\textstyle\sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \geq |\cup_{i=1}^{n} y_i| = |Y| \\ \Rightarrow\; \exists\, y_i,\; Y \;:\; \textstyle\sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i = |\cup_{i=1}^{n} y_i| = |Y| \\ \Rightarrow \quad \textstyle\sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |\{(y_a, y_b) : y_a y_b \in y_i\}| = |\{(y_a, y_b) : y_a y_b \in Y\}|. \tag{3.10}$$

Multiply both sides of equation 3.10 by c^2 and apply the ruler convergence theorem.

$$s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n} (p_i \cdot c)^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \\ \Rightarrow \quad \sum_{i=1}^{n} s_i^2 = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c)^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2. \tag{3.11}$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$\forall\, c > 0, \quad p_d = floor(d/c) = |\{y_{1_i}, y_{2_i}, \ldots, y_{p_d}\}| = |Y| \\ \Rightarrow \quad p_d^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\}|, \tag{3.12}$$

where $\{(y_a, y_b)\}$ is the set of all combination pairs of elements of Y. Multiply both sides of 3.12 by c^2 and apply the ruler convergence theorem (2.2):

$$d = \lim_{c\to 0} p_d \cdot c \quad \wedge \quad (p_d \cdot c)^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \\ \Rightarrow \quad d^2 = \lim_{c\to 0} (p_d \cdot c)^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2. \tag{3.13}$$

Combine equations 3.12 and 3.13:

$$d^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \quad \wedge \\ \sum_{i=1}^{n} s_i^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \quad \Rightarrow \quad d^2 = \sum_{i=1}^{n} s_i^2. \quad \square \tag{3.14}$$

3.1. Triangle inequality. The definition of a metric in real analysis is based on the triangle inequality, $\mathbf{d}(\mathbf{u},\mathbf{w}) \leq \mathbf{d}(\mathbf{u},\mathbf{v}) + \mathbf{d}(\mathbf{v},\mathbf{w})$, that has been intuitively motivated by the triangle [**Gol76**]. Applying the inclusion-exclusion principle, ruler (2.1), and ruler convergence theorem (2.2) to the definition of a countable distance range (3.1) yields the real-valued triangle inequality:

$$(3.15)\quad d_c = |Y| = |\bigcup_{i=1}^{2} y_i| \leq \sum_{i=1}^{2} |y_i| \quad \wedge$$
$$d_c = floor(\mathbf{d}(\mathbf{u},\mathbf{w})/c) \ \wedge\ |y_1| = floor(\mathbf{d}(\mathbf{u},\mathbf{v})/c) \ \wedge\ |y_2| = floor(\mathbf{d}(\mathbf{v},\mathbf{w})/c)$$
$$\Rightarrow\ \mathbf{d}(\mathbf{u},\mathbf{w}) = \lim_{c\to 0} d_c \cdot c \leq \sum_{i=1}^{2} \lim_{c\to 0} |y_i| \cdot c = \mathbf{d}(\mathbf{u},\mathbf{v}) + \mathbf{d}(\mathbf{v},\mathbf{w}).$$

4. Size (length/area/volume)

This section will use the ruler (2.1) and ruler convergence theorem (2.2) to prove that the Cartesian product of same-sized subintervals of intervals converges to the product of interval sizes. The first step is to define a set-based, countable size measure as the Cartesian product of disjoint domain set members.

DEFINITION 4.1. Countable size (length/area/volume) measure, S_c:

$$\forall\ i\ n \in \mathbb{N},\quad i \in [1,n],\quad x_i \subseteq X,\quad |\bigcap_{i=1}^{n} x_i| = \emptyset \quad \wedge \quad S_c = |y| = \prod_{i=1}^{n} |x_i|.$$

THEOREM 4.2. *Euclidean size (length/area/volume), S, is the size of an image interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals: $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, where:*

$$S = \prod_{i=1}^{n} s_i,\quad S = |y_m - y_0|,\quad s_i = |x_{m_i,i} - x_{0,i}|,\quad i \in [1,n],\quad i,n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_size."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$(4.1)\quad \forall\ i\ n \in \mathbb{N},\quad i \in [1,n],\quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge$$
$$x_i = \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\} \quad \Rightarrow |x_i| = p_i.$$

Use the ruler (2.1) to divide the exact size, $S = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_S^n subintervals, where p_S^n satisfies the definition a countable size measure, S_c (4.1).

$$(4.2)\quad \forall\ c > 0 \quad \wedge \quad \exists\ r \in \mathbb{R},\ S = r^n \quad \wedge \quad p_S = floor(r/c) \quad \wedge$$
$$p_S^n = S_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i.$$

Multiply both sides of equation 4.2 by c^n to get the ruler measures:

$$(4.3)\qquad p_S^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c).$$

Apply the ruler convergence theorem (2.2) to equation 4.3:

$$S = r^n = \lim_{c\to 0}(p_S \cdot c)^n \quad \wedge \quad \lim_{c\to 0}(p_i \cdot c) = s_i \quad \wedge \quad (p_S \cdot c)^n = \prod_{i=1}^{n}(p_i \cdot c) \tag{4.4}$$

$$\Rightarrow \quad S = \lim_{c\to 0}(p_S \cdot c)^n = \prod_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square$$

5. Ordered and symmetric geometries

Neither classical nor modern analytic geometry has been able to provide any insight into why classical Euclidean geometry appears to be limited to at most three dimensions. The same counting principles that generate the triangle inequality, taxicab distance, Euclidean distance, and size (length/area/volume) also provide insight into what properties can limit a geometry (both Euclidean and non-Euclidean) to a cyclic set of at most three dimensions.

The previous derivations of taxicab distance (3.2), Euclidean distance (3.3), and Euclidean volume (4.2) show that the total number of combinations of subintervals of intervals converge to real-valued distance measures and Euclidean volume. By the commutative property of addition and multiplication, all orderings (permutations) of the combinations of subintervals of intervals yield the same total distance and same total volume. Therefore, all orderings (permutations) of domain intervals corresponding to those subinterval combinations yield the same total distance and same total volume (a symmetric geometry).

All distance measures and permutations emergent from the countable distance range principle coexist. All permutations emergent from countable size coexist. There is no "choice" about which distance measures, size measures, and permutations exist or do not exist.

An order (permuation) is defined by successor and predecessor functions, where the predecessor function is defined as the inverse of the successor function. The existence of every permutation of intervals (dimensions) requires that each interval is sequentially adjacent (either a successor or predecessor) to every other interval.

For example, given the left-to-right ordered set of elements, $\{A,\ B,\ C,\ D\}$, the permutation, $(A,\ D,\ C,\ B)$, can **not** be generated by the element, D, being a successor of A, because the successor function defines B as the successor of A. Therefore, the permutation can only be generated if D is a predecessor of A (a cyclic set).

But, this implies that the permutation, $(A,\ C,\ D,\ B)$, does **not** exist, because C is neither a successor nor predecessor of A. It will be proved, that all permutations (a symmetric geometry) can only be generated if a set is a cyclic set limited to at most three elements (dimensions).

Definition 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i \in \{x_1, \ldots, x_n\},$$
$$\textit{successor}\ x_i = x_{i+1} \;\;\wedge\;\; \textit{predecessor}\ x_{i+1} = x_i.$$

where $\{x_1, \ldots, x_n\}$ are a set of real-valued intervals (dimensions).

Definition 5.2. Symmetric geometry:

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; \textit{successor}\ x_i = x_j \;\;\wedge\;\; \textit{predecessor}\ x_j = x_i.$$

THEOREM 5.3. *An ordered and symmetric geometry is a cyclic set.*

$$\textit{successor } x_n = x_1 \ \wedge \ \textit{predecessor } x_1 = x_n.$$

The theorem and formal Coq-based proof is "ordered_symmetric_is_cyclic," which is located in the file threed.v.

PROOF. The property of order (5.1) defines unique successors and predecessors for all elements except for the successor of x_n and the predecessor of x_1. From the properties of a symmetric geometry (5.2):

$$i = n \ \wedge \ j = 1 \ \wedge \ \textit{successor } x_i = x_j \ \Rightarrow \ \textit{successor } x_n = x_1. \tag{5.1}$$

$$i = n \ \wedge \ j = 1 \ \wedge \ \textit{predecessor } x_j = x_i \ \Rightarrow \ \textit{predecessor } x_1 = x_n. \tag{5.2}$$

□

For example, using the cyclic set with elements labeled, $\{1,\ 2,\ 3\}$, starting with each element and counting through 3 cyclic successors and counting through 3 cyclic predecessors yields all possible permutations: $(1,2,3)$, $(2,3,1)$, $(3,1,2)$, $(1,3,2)$, $(3,2,1)$, and $(2,1,3)$. That is, a cyclically ordered set preserves sequential order while allowing some n-at-a-time permutations. If all possible n-at-a-time permutations are generated, then the cyclic set is also a symmetric geometry.

THEOREM 5.4. *An ordered and symmetric geometry is limited to at most 3 elements. That is, each element is sequentially adjacent (a successor or predecessor) to every other element in a set only where the number of elements are less than or equal to 3.*

The Coq-based lemmas and proofs in the file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
Because a symmetric and ordered set is a cyclic set (5.3), the successors and predecessors are cyclic:

DEFINITION 5.5. Successor of m is n:

$$\textit{Successor}(m, n, \textit{setsize}) \leftarrow (m = \textit{setsize} \wedge n = 1) \vee (m + 1 \leq \textit{setsize}). \tag{5.3}$$

DEFINITION 5.6. Predecessor of m is n:

$$\textit{Predecessor}(m, n, \textit{setsize}) \leftarrow (m = 1 \wedge n = \textit{setsize}) \vee (m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: element m is adjacent to element n (an allowed permutation), if the cyclic successor of m is n or the cyclic predccessor of m is n. Notionally:

$$\textit{Adjacent}(m, n, \textit{setsize}) \leftarrow \textit{Successor}(m, n, \textit{setsize}) \vee \textit{Predecessor}(m, n, \textit{setsize}). \tag{5.5}$$

Every element is adjacent to every other element, where $\textit{setsize} \in \{1,\ 2,\ 3\}$:

$$\textit{Adjacent}(1, 1, 1) \leftarrow \textit{Successor}(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{5.6}$$

$$\textit{Adjacent}(1,2,2) \leftarrow \textit{Successor}(1,2,2) \leftarrow (1+1 \leq 2). \tag{5.7}$$
$$\textit{Adjacent}(2,1,2) \leftarrow \textit{Successor}(2,1,2) \leftarrow (2=2 \wedge 1=1). \tag{5.8}$$

$$\textit{Adjacent}(1,2,3) \leftarrow \textit{Successor}(1,2,3) \leftarrow (1+1 \leq 2). \tag{5.9}$$
$$\textit{Adjacent}(2,1,3) \leftarrow \textit{Predecessor}(2,1,3) \leftarrow (2-1 \geq 1). \tag{5.10}$$
$$\textit{Adjacent}(3,1,3) \leftarrow \textit{Successor}(3,1,3) \leftarrow (3=3 \wedge 1=1). \tag{5.11}$$
$$\textit{Adjacent}(1,3,3) \leftarrow \textit{Predecessor}(1,3,3) \leftarrow (1=1 \wedge 3=3). \tag{5.12}$$
$$\textit{Adjacent}(2,3,3) \leftarrow \textit{Successor}(2,3,3) \leftarrow (2+1 \leq 3). \tag{5.13}$$
$$\textit{Adjacent}(3,2,3) \leftarrow \textit{Predecessor}(3,2,3) \leftarrow (3-1 \geq 1). \tag{5.14}$$

Must prove that for all $setsize > 3$, there exist non-adjacent elements (not every permutation allowed). For example, the first and third elements are not adjacent:

$$\forall\, setsize > 3: \quad \neg \textit{Successor}(1,3,setsize) \leftarrow \textit{Successor}(1,2,setsize) \leftarrow (1+1 \leq setsize). \tag{5.15}$$

That is, 2 is the only successor of 1 for all $setsize > 3$, which implies 3 is not a successor of 1 for all $setsize > 3$.

$$\forall\, setsize > 3: \quad \neg \textit{Predecessor}(1,3,setsize) \leftarrow \textit{Predecessor}(1,n,setsize) \leftarrow (1=1 \wedge n = setsize). \tag{5.16}$$

That is, $n = setsize$ is the only predecessor of 1 for all $setsize > 3$, which implies 3 is not a predecessor of 1 for all $setsize > 3$.

$$\forall\, setsize > 3: \quad \neg \textit{Adjacent}(1,3,setsize) \leftarrow \neg \textit{Successor}(1,3,setsize) \wedge \neg \textit{Predecessor}(1,3,setsize). \quad \square \tag{5.17}$$

6. Summary

Applying the ruler measure (2.1) and ruler convergence proof (2.2) to combinatorial relations between the subintervals of real-valued domain intervals and the subintervals an image interval yields some new insights into real analysis, measure theory, geometry, and physics.

(1) Applying the ruler measure and ruler convergence proof (2.2) to the countable distance range provides the insight that all real-valued measures of distance are based on the principle that for each disjoint domain set there exists a distance set containing the same number of elements:
   (a) The countable distance range principle converges to the triangle inequality, which is the basis for the definition of metric space.
   (b) The upper bound of the countable distance range principle converging to taxicab distance provides the insight that largest (maximum) monotonic distance path is due to the union of disjoint distance sets.
   (c) The lower bound of the countable distance range principle converging to Euclidean distance provides the insight that smallest (minimum) monotonic distance path is due to the union of intersecting distance sets, where each of the p_i number of elements in the i^{th} distance set corresponds to p_i number of elements in the domain sets.

(d) All L^p norms where $p > 2$ generated from the countable distance range principle would require each distance set element to correspond to the same domain set element multiple times (over counts correspondences). It is not a useful measure of steps when walking from point A to B, where a step would correspond to the a piece of land multiple times. The triangle inequality and metric space does not provide that insight.

(e) Using the same constraints as the countable distance range with $d_c \leq |\bigcup_{i=1}^{n} y_i| = |Y|$ would define a closed countable ball. If each distance set is ordered, then the first element of each distance set would define the center point of the countable ball.

(f) The proof of Euclidean distance (3.3) was derived without any notions of side, angle, or shape. Arc angle defined as a parametric variable relating the sizes of two domain intervals can be easily derived using using calculus (Taylor series) and the Euclidean distance equation to generate the sine and cosine functions of the parametric variable. In other words, the notions of side and angle are derived from Euclidean distance, which is the reverse of what classical geometry [**Joy98**] [**Loo68**], axiomatic foundations for geometry [**Bir32**] [**Hil**] [**TG99**], and differential geometry [**Ber88**] [**Sta96**] [**Bog10**] assume. The ruler convergence applied to the countable distance range provides the set and number theory-based foundation that generates the common notions of geometry.

(2) Applying the ruler measure (2.1) and ruler convergence proof (2.2) to the countable size definition (4.1) provides the insight that the Cartesian product of same-sized subintervals of real-valued intervals converges to the product of the interval sizes (Euclidean volume):

(a) Euclidean size (length/area/volume) was derived without notions of sides, angles, and shape.

(b) The Euclidean volume (product of interval sizes) of the Lebesgue measure is derived from use of the more fundamental ruler measure.

(c) It is an open question whether the ruler measure can be restated as a σ-algebra.

(3) Combinatorics on infinitesimal subintervals generating the properties of distance and volume results in the same total distance and same total volume for all permutations (orderings) of domain intervals.

(a) Using successor and predecessor functions generates an ordering of the set of intervals (5.1). When the successor and predecessor functions generate every possible ordering (a symmetric geometry), then the set of intervals must be a cyclic set (5.3) limited to at most three intervals (5.4). which is the basis of the right-hand rule that permeates mathematics, physics, and engineering.

(b) A cyclic set is a closed walk. An observer in a closed walk of three dimensions would only be able to detect higher dimensions (other variables) indirectly via distance and size changes in the three closed walk dimensions, where a change in distance is what physicists call "work."

(4) The properties of distance, volume, three dimensions of space, and the right-hand rule (for example in electromagnetic fields and the vector cross product) are a consequence of simple combinatorial relations between the subintervals of real-valued intervals. Therefore, combinatorics on the subintervals of higher dimensions of real-valued intervals might also converge to real-valued functions describing phenomena perceived as "particles", "waves", "mass", and "forces." In other words, our universe might be a consequence of a few simple counting principles in the real-valued continuum. For example, current quantum qubit-based unified physics theories [**FGH+14**] [**VR16**] [**AHT13**] view gravity as "emergent" from the relationships between infinitesimal bits in space, where a qubit would be the equivalent of the infinitesimal subintervals of real-valued intervals explored in this article.

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George Van Treeck, 668 Westline Dr., Alameda, CA 94501