The Two Set Relations Generating Geometry

George. M. Van Treeck

Abstract. A ruler (measuring stick) partitions both domain and range intervals approximately into sets of same-sized subintervals. As the subinterval size converges to zero: 1) The union of range sets of subintervals, where each domain set corresponds to a less or equal-sized range set, generates the properties of metric space and all Lp norms, in particular, Manhattan and Euclidean distance. 2) The domain and range set constraints that generates Manhattan distance also generates Euclidean volume. The proofs allow simpler derivations of Coulomb's charge force, Newton's gravity force, and spacetime equations without using other laws of physics or Gauss's divergence theorem. Time limits physical distance and volume to 3 dimensions. All proofs are verified in Coq.

Contents

1. Introduction

Metric space, the Euclidean distance metric/vector norm, and volume have been definitions in mathematical analysis [**Gol76**] [**Rud76**] rather than derived from more fundamental set definitions. Further, measure theory has lacked coherent set-based definitions of distance and volume, and has lacked a measure that applies to both distance and volume. For example, the function-based definition of the distance-specific metric space is very different and independent of the set and σ-algebra-based definition of the length/area/volume-specific Lebesgue measure.

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In this article, both distance and volume are defined as the cardinals of operations on abstract, countable sets, where measurable sets conforming to the definitions are distance and volume measures. Using a ruler (measuring stick) to partition domain and range intervals approximately into sets of subintervals, each subinterval having the same size, c, that conform to the distance and volume definitions converge to the properties of metric space, the L^p norms, and Euclidean volume as $c \to 0$.

The derivations of metric space, distance equations, and volume from set definitions provide some insights into geometry and physics, for example: the set operations and constraint generating the properties of metric space; the mapping between sets that makes Euclidean distance the smallest possible distance between two distinct points in $\mathbb{R}^n$; the domain and range set constraints that generates Manhattan distance also generates Euclidean volume; how the Euclidean distance and volume proofs allow deriving Coulomb's charge force, Newton's gravity force, and space time equations without using other laws of physics or Gauss's divergence theorem; the proportionate interval principle generating the inverse square law and flux divergence; how time places an additional constraint on physical sets, which limits physical distance and volume to 3 dimensions.

All the proofs in this article have been formally verified using using the Coq proof verification system [**Coq15**]. The formal proofs are located in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

Definition 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, p, each subinterval having the same size, c. Notionally:

$$\forall\, c,\ s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |a-b| \ \wedge\ c > 0\ \wedge \\ (p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \textstyle\sum_{i=1}^{p} c = pc. \tag{2.1}$$

Theorem 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\ \ s = |a-b| \ \Rightarrow\ M = \lim_{c\to 0} pc = s.$

The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\ p = floor(s/c)\ \wedge\ 0 \le |floor(s/c) - s/c| < 1\ \Rightarrow\ 0 \le |p - s/c| < 1. \tag{2.2}$$

Multiply all sides of inequality 2.2 by $|c|$:

$$\forall\, c > 0,\quad 0 \le |p - s/c| < 1 \quad \Rightarrow \quad 0 \le |pc - s| < |c|. \tag{2.3}$$

$$\forall\, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \\ \Rightarrow\quad \forall\, \epsilon = \delta :\ |c-0| < \delta\ \wedge\ |pc - s| < \epsilon\ \ :=\ \ M = \lim_{c\to 0} pc = s. \quad \square \tag{2.4}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = |0 - \pi|$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}},\ 3.14_{c=10^{-2}},\ 3.141_{c=10^{-3}}, \ldots, \pi_{\lim_{c\to 0}}$.

3. Distance

Notation convention: Vertical bars around a set or list, $|\cdots|$, indicates the cardinal (number of members in the set or list).

3.1. Countable distance space.

Countable distance is a set operation-based abstraction of distance, that allows derivation of the properties of metric space, and the L^p norms, in particular, Manhattan and Euclidean distance. An example of a countable distance is the number of members in a range set, y, which equals the number of members in a corresponding domain set, x: $|x| = |y|$. If every disjoint domain set has a corresponding range set, then the countable distance spanning multiple, disjoint, domain sets, $\bigcap_{i=1}^{n} x_i = \emptyset$, is the number of members, d_c, in the union range set: $d_c = |\bigcup_{i=1}^{n} y_i|$. Generalizing:

Definition 3.1. Countable distance space, d_c :

$$\bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad d_c = |\bigcup_{i=1}^{n} y_i| \quad \wedge \quad |x_i| = |y_i|^q,\ q \geq 1.$$

Theorem 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$.

The inclusion-exclusion inequality follows from the inclusion-exclusion principle [CG15]. But, a more intuitive and simple proof follows from the associative law of addition where the sum of set sizes is equal to the size of all the set members appended into a list and the commutative law of addition that allows sorting that list into a list of unique members (the *union* set) and a list of duplicates. The duplicates being ≥ 0 implies the union size is always $\leq$ the sum of set sizes.

A formal proof, inclusion_exclusion_inequality, using sorting into a set of unique members (*union* set) and a list of duplicates, is in the file euclidrelations.v.

Proof.

$$\sum_{i=1}^{n} |y_i| = |append_{i=1}^{n} y_i| = |sort(append_{i=1}^{n} y_i)| = |\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i|. \tag{3.1}$$

$$|\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| \quad \wedge \quad |duplicates_{i=1}^{n} y_i| \geq 0 \Rightarrow |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|. \quad \square \tag{3.2}$$

3.2. Metric Space.

All function range intervals, $d(u, w)$, satisfying the countable distance space definition, $d_c = |\bigcup_{i=1}^{n} y_i|$, where the ruler is applied, generates three of the four metric space properties: triangle inequality, non-negativity, and identity of indiscernibles. The set-based reason for the fourth property of metric space, symmetry $[d(u, v) = d(v, u)]$, will be identified in the last section of this article. The formal proofs: triangle_inequality, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

Theorem 3.3. *Triangle Inequality:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \leq d(u, v) + d(v, w)$.

Proof. Apply the ruler measure (2.1), the countable distance space condition (3.1), inclusion-exclusion inequality (3.2), and then ruler convergence (2.2).

$$\begin{aligned}(3.3)\quad &\forall\, c>0,\ d(u,w),\ d(u,v),\ d(v,w)\ :\\ &|y_1| = floor(d(u,v)/c) \ \wedge\ |y_2| = floor(d(v,w)/c) \ \wedge\\ &d_c = floor(d(u,w)/c) \ \wedge\ d_c = |y_1 \cup y_2| \le |y_1| + |y_2|\\ &\Rightarrow\ floor(d(u,w)/c) \le floor(d(u,v)/c) + floor(d(v,w)/c)\\ &\Rightarrow\ floor(d(u,w)/c)\cdot c \le floor(d(u,v)/c)\cdot c + floor(d(v,w)/c)\cdot c\\ &\Rightarrow\ \lim_{c\to 0} floor(d(u,w)/c)\cdot c \le \lim_{c\to 0} floor(d(u,v)/c)\cdot c + \lim_{c\to 0} floor(d(v,w)/c)\cdot c\\ &\Rightarrow\ d(u,w) \le d(u,v) + d(v,w). \quad \square\end{aligned}$$

Theorem 3.4. *Non-negativity:* $d_c = |y_1 \cup y_2| \Rightarrow d(u,w) \ge 0$.

Proof. By definition, a set always has a size (cardinal) ≥ 0:

$$\begin{aligned}(3.4)\quad &\forall\, c>0,\ d(u,w)\ :\quad floor(d(u,w)/c) = d_c \quad \wedge \quad d_c = |y_1 \cup y_2| \ge 0\\ &\Rightarrow\quad floor(d(u,w)/c) = d_c \ge 0 \quad\Rightarrow\quad d(u,w) = \lim_{c\to 0} d_c \cdot c \ge 0. \quad \square\end{aligned}$$

Theorem 3.5. *Identity of Indiscernibles:* $d(w,w) = 0$.

Proof. Apply the triangle inequality property (3.3):

$$(3.5)\qquad \forall\, d(u,v) = d(v,w) = 0 \ \wedge\ d(u,w) \le d(u,v) + d(v,w) \ \Rightarrow\ d(u,w) \le 0.$$

Combine the non-negativity property (3.4) and the previous inequality (3.5):

$$(3.6)\qquad d(u,w) \ge 0 \ \wedge\ d(u,w) \le 0 \ \Leftrightarrow\ 0 \le d(u,w) \le 0 \ \Rightarrow\ d(u,w) = 0.$$

Combine the result of step 3.6 and the condition, $d(u,v) = 0$, in step 3.5.

$$(3.7)\qquad d(u,w) = 0 \ \wedge\ d(u,v) = 0 \ \Rightarrow\ w = v.$$

Combine the condition, $d(v,w) = 0$, in step 3.5 and the result of step 3.7.

$$(3.8)\qquad d(v,w) = 0 \ \wedge\ w = v \ \Rightarrow\ d(w,w) = 0. \qquad \square$$

3.3. Distance space range. From the countable distance space definition, $d_c = |\bigcup_{i=1}^{n} y_i|$, as the amount of intersection increases, more domain set members can map to a single range set member. Therefore, the number of domain-to-range set member mappings is a function of the amount of range set intersection.

From the countable distance space property (3.1), where $|x_1| = |y_1| = p_1$, the range of possible domain-to-range set member mappings is the number of one-to-one (bijective), $|y_1|\cdot 1 = p_1$, mappings, to the number of many-to-many, $|x_1|\cdot|y_1| = p_1^2$, mappings. The range of domain-to-range set mappings that are true in one dimension must be true in every dimension. Therefore, the total number of domain-to-range set mappings varies from $\sum_{i=1}^{n}|y_i|\cdot 1 = \sum_{i=1}^{n} p_i$ to $\sum_{i=1}^{n}|y_i|\cdot|x_i| = \sum_{i=1}^{n} p_i^2$ mappings. Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of domain-to-range set mapping cases converges to the real-valued Manhattan and Euclidean distance relations.

3.4. Manhattan distance.

THEOREM 3.6. *Manhattan (largest) distance, d, is the size of the range interval,* $[d_0, d_m]$*, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$*, where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "taxicab_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$(3.9) \quad d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i$$
$$\Rightarrow \quad d_c \leq \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i.$$

Multiply both sides of equation 3.11 by c and take the limit:

$$(3.10) \quad d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \Rightarrow \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c).$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$(3.11) \qquad d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d :\ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$(3.12) \quad s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c \to 0} p_i \cdot c = s_i.$$

Combine equations 3.11, 3.10, 3.12:

$$(3.13) \quad d = \lim_{c \to 0} d_c \cdot c \quad \wedge \quad \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) \quad \wedge$$
$$\lim_{c \to 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square$$

3.5. Euclidean distance.

THEOREM 3.7. *Euclidean (smallest) distance, d, is the size of the range interval,* $[d_0, d_m]$*, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$*, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "Euclidean_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
Apply the rule of product to the largest number of domain-to-range set mappings, where all p_i number of range set members, y_i, map to each of the p_i number of members in the domain set, x_i, which is the Cartesian product, $|y_i| \cdot |x_i|$:

$$(3.14) \qquad |x_i| = |y_i| = p_i \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2.$$

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$(3.15) \quad d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i$$
$$\Rightarrow \quad d_c \leq \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i.$$

Square both sides of equation 3.15 ($x = y \Leftrightarrow f(x) = f(y)$):

$$(3.16) \qquad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\ d_c :\ d_c^2 = (\sum_{i=1}^{n} p_i)^2.$$

Apply the square of sum inequality, $(\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$, to equation 3.16 and select the smallest area (the equality) case:

$$\begin{aligned}(3.17)\quad d_c^2 &= (\textstyle\sum_{i=1}^{n} p_i)^2 = \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j \\ &= \textstyle\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} p_i \sum_{j=1, j\neq i}^{n} p_j \geq \sum_{i=1}^{n} p_i^2 \quad \Rightarrow \quad \exists\, p_i : d_c^2 = \sum_{i=1}^{n} p_i^2.\end{aligned}$$

Multiply both sides of equation 3.17 by c^2, simplify, and take the limit.

$$\begin{aligned}(3.18)\quad d_c^2 = \textstyle\sum_{i=1}^{n} p_i^2 \;\Rightarrow\; d_c^2 \cdot c^2 = \sum_{i=1}^{n} p_i^2 \cdot c^2 \;\Leftrightarrow\; (d_c \cdot c)^2 = \sum_{i=1}^{n} (p_i \cdot c)^2 \\ \Rightarrow \quad \lim_{c\to 0} (d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c)^2.\end{aligned}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.19)\quad \exists\, c\, d \in \mathbb{R} : floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c\to 0} (d_c \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$(3.20)\quad s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i.$$

Combine equations 3.19, 3.18, 3.20:

$$\begin{aligned}(3.21)\quad d^2 = \lim_{c\to 0} (d_c \cdot c)^2 \quad \wedge \quad \lim_{c\to 0} (d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c)^2 \quad \wedge \\ \lim_{c\to 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d^2 = \lim_{c\to 0} (d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \quad \square\end{aligned}$$

4. Euclidean Volume

The countable distance (3.1) case, $\bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset \quad \wedge \quad d_c = |\bigcup_{i=1}^{n} y_i| \quad \wedge \quad |x_i| = |y_i|$, generates Manhattan distance (3.6), where each sub-Manhattan distance is the sum of a unique n-tuple (coordinate) of one same-sized subinterval from each range set, y_i. The number of possible unique sub-Manhattan distance n-tuples, v_c, is the Cartesian product of the range sets:

Definition 4.1. Countable Volume, v_c:

$$\textstyle\bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset \quad \wedge \quad v_c = |\times_{i=1}^{n} y_i| \quad \wedge \quad |x_i| = |y_i|.$$

Theorem 4.2. *Euclidean volume, v, is size of the range interval, $[v_0, v_m]$, corresponding to the Cartesian product of all the members of the domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$. Notionally:*

$$v = \textstyle\prod_{i=1}^{n} s_i, \;\; v = |v_0 - v_m|, \;\; s_i = |a_i - b_i|.$$

The theorem, "Euclidean_volume," and formal proof is in the Coq file, euclidrelations.v.

Proof.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of subintervals.

$$(4.1)\qquad \forall\, i\, n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i| = |y_i|.$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$(4.2)\qquad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0} (p_i \cdot c) = s_i.$$

State the countable volume (4.1) in terms of p_i:

$$(4.3)\quad v_c = |\times_{i=1}^{n} y_i| \quad \wedge \quad |y_i| = p_i \quad \Rightarrow \quad v_c = |\times_{i=1}^{n} y_i| = \textstyle\prod_{i=1}^{n} |y_i| = \prod_{i=1}^{n} p_i.$$

Multiply both sides of equation (4.3) by c^n:

$$(4.4)\qquad v_c \cdot c^n = (\textstyle\prod_{i=1}^{n} p_i) \cdot c^n = \prod_{i=1}^{n} (p_i \cdot c).$$

Use those cases, where v_c has an integer n^{th} root.

$$\forall\, p^n = v_c \in \mathbb{N} \,:\, v_c \cdot c^n = p^n \cdot c^n = (p \cdot c)^n = \textstyle\prod_{i=1}^{n} (p_i \cdot c). \tag{4.5}$$

Use the ruler (2.1) and ruler convergence (2.2) to the range interval, $[v_0, v_m]$ and apply equations 4.5 and 4.2:

$$floor(v/c^n) = p^n \Rightarrow v = \lim_{c\to 0}(p \cdot c)^n = \textstyle\prod_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i. \tag{4.6}$$ □

5. Applications to physics

5.1. Coulomb's charge force. The sizes, q_1 and q_2, of two charges are independent domain variables, where each size, c, component of a charge exerts a force on each same size, c, component of the other charge. The total force, F, is proportionate to the total number of forces (the Cartesian product of the number of *same-sized*, infinitesimal components) multiplied times a quantum charge force, $m_C a_C$. From the volume proof (4.2), the Cartesian product converges to $q_1 q_2$:

$$F \propto (m_C a_C)(\lim_{c\to 0} floor(q_1/c) \cdot c)(\lim_{c\to 0} floor(q_2/c) \cdot c) = (m_C a_C)(q_1 q_2). \tag{5.1}$$

From equation 5.1, an increase in charge, q, causes a proportionate increase in force, F. Solving for the constant force, $F = m_C a_C$, requires a proportionate variable, r, to offset the effect of a change in q: $r \propto q \Rightarrow \exists\, q_C/r_C \in \mathbb{R} \,:\, r(q_C/r_C) = q$:

$$\forall\, q_1, q_2 \geq 0 \,\exists\, q \in \mathbb{R} \,:\, q^2 = q_1 q_2 \;\wedge\; r(q_C/r_C) = q \;\Rightarrow\; (r(q_C/r_C))^2 = q_1 q_2. \tag{5.2}$$

$$\begin{aligned} &(r(q_C/r_C))^2 = q_1 q_2 \;\wedge\; F \propto (m_C a_C)(q_1 q_2) \\ &\quad\Rightarrow\quad F \propto (m_C a_C)(r(q_C/r_C))^2 = (m_C a_C)(q_1 q_2) \\ &\quad\quad\Rightarrow\quad F = m_C a_C = (m_C a_C r_C^2/q_C^2) q_1 q_2/r^2 = k_c q_1 q_2/r^2. \end{aligned} \tag{5.3}$$

where $k_C = m_C a_C r_C^2/q_C^2$ corresponds to the SI units: Nm^2C^{-2}.

5.2. Newton's gravity force equation. The sizes, m_1 and m_2, of two masses are independent domain variables, where each size, c, component of a mass exerts a force on each same size, c, component of the other mass. The total force, F, is proportionate to the total number of forces (the Cartesian product of the number of *same-sized*, infinitesimal components) multiplied times a quantum gravity force, $m_G a_G$. From the volume proof (4.2), the Cartesian product converges to $m_1 m_2$:

$$F \propto (m_G a_G)(\lim_{c\to 0} floor(m_1/c) \cdot c)(\lim_{c\to 0} floor(m_2/c) \cdot c) = (m_G a_G)(m_1 m_2). \tag{5.4}$$

From equation 5.4, an increase in mass, m, causes a proportionate increase in force, F. Solving for the constant force, $F = m_G a_G$, requires a proportionate variable, r, to offset the effect of a change in m: $r \propto m \Rightarrow \exists\, m_G/r_G \in \mathbb{R} \,:\, r(m_G/r_G) = m$:

$$\begin{aligned} &\forall\, m_1, m_2 \geq 0 \,\exists\, m \in \mathbb{R} \,:\quad m^2 = m_1 m_2 \quad\wedge\quad r(m_G/r_G) = m \\ &\qquad\qquad\Rightarrow\quad (r(m_G/r_G))^2 = m_1 m_2. \end{aligned} \tag{5.5}$$

$$\begin{aligned} &(r(m_G/r_G))^2 = m_1 m_2 \;\wedge\; F \propto (m_G a_G)(m_1 m_2) \\ &\quad\Rightarrow\quad F \propto (m_G a_G)(r(m_G/r_G))^2 = (m_G a_G)(m_1 m_2) \\ &\quad\quad\Rightarrow\quad F = m_G a_G = (m_G a_G r_G^2/q_G^2) m_1 m_2/r^2. \end{aligned} \tag{5.6}$$

$$\exists\, t_G \in \mathbb{R} : \, r_G/t_G^2 = a_G \quad \wedge \quad F = m_G a_G = (m_G a_G r_G^2/m_G^2) m_1 m_2/r^2$$
$$\Rightarrow \quad F = m_G a_G = (r_G^3/m_G t_G^2) m_1 m_2/r^2 = G m_1 m_2/r^2, \tag{5.7}$$

where $G = r_G^3/m_G t_G^2$ corresponds to the SI units: $m^3 kg^{-1} s^{-2}$.

5.3. Spacetime equations. Applying the ruler measure, if sequencing across each same-sized subinterval of a *physical*, Euclidean distance (range) interval, $[0, r]$, corresponds to a proportionate number of same-sized subintervals of a time interval, $[0, t]$, then, as the subinterval size converges to zero, the interval, $[0, t]$, is proportionate to the range interval, $[0, r]$, where there is a conversion constant, c, that is the ratio of some value, r_c to some value, t_c, such that $r = (r_c/t_c)t = ct$.

Applying the ruler, to two intervals, $[0, d_1]$ and $[0, d_2]$, in two inertial (independent, non-accelerating) frames of reference, the distance and time to sequence over the subintervals the two intervals converges to a range of distances (and times) from Manhattan (3.6) to Euclidean distance (3.7).

$$r^2 = d_1^2 + d_2^2 \quad \wedge \quad r = (r_c/t_c)t = ct$$
$$\Rightarrow \quad (ct)^2 = d_1^2 + d_2^2 \quad \Rightarrow \quad d_2 = \sqrt{(ct)^2 - d_1^2}. \tag{5.8}$$

$$d_2 = \sqrt{(ct)^2 - d_1^2} \quad \wedge \quad d = d_2 \quad \wedge \quad d_1 = vt$$
$$\Rightarrow \quad d = \sqrt{(ct)^2 - (vt)^2} = ct\sqrt{1 - (v^2/c^2)}, \tag{5.9}$$

which is the spacetime dilation equation. [**Bru17**].

$$d_2^2 = (ct)^2 - d_1^2 \quad \wedge \quad s = d_2 \quad \wedge \quad d_1^2 = x^2 + y^2 + z^2$$
$$\Rightarrow \quad s^2 = (ct)^2 - (x^2 + y^2 + z^2), \tag{5.10}$$

which is one form of the spacetime interval equation [**Bru17**].

5.4. 3 dimensions of physical geometry. The set and arithmetic operations used to calculate distance and volume requires sequencing through a totally ordered set of dimensions, for example, the countable distance space: $d_c = |\bigcup_{i=1}^{n} y_i|$, Euclidean distance: $d^2 = \sum_{i=1}^{n} s_i^2$, countable volume: $v_c = |\times_{i=1}^{n} x_i|$, and Euclidean volume: $v = \prod_{i=1}^{n} s_i$. The commutative property of the union, addition, and multiplication operations also allows sequencing through a set of n number of dimensions in all $n!$ number of possible orders.

Physical sets have the additional constraint that sequencing across the members of a non-empty set takes some greater than zero amount of *time*. *Determining* that a *physical* sequencer sequenced a physical set in the order, $[x_5, x_4, \cdots, x_1]$, and next sequenced in the order, $[x_1, x_2, \cdots, x_5]$, requires the total order, at most one successor and at most one predecessor per set member, to not change during the *time* of the sequencing. Deterministic sequencing of a totally ordered set via successor/predecessor links in each possible order requires each set member to be either a successor or predecessor to every other set member (sequentially adjacent) during the *time* of sequencing, herein referred to as a symmetric geometry. **Note** that "indexing" to jump around an ordered set is implicitly calculating the number of successor/predecessor links (implicitly traversing the links) to reach the next member to be sequenced.

It will now be proved that a set satisfying the constraints of a single total order and also symmetric defines a cyclic set containing at most 3 members, in this case, 3 dimensions of physical space.

Definition 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i\;\; \in \{x_1, \ldots, x_n\},$$
$$\mathit{successor}\ x_i = x_{i+1}\;\; \wedge\;\; \mathit{predecessor}\ x_{i+1} = x_i.$$

Definition 5.2. Symmetric geometry (every set member is sequentially adjacent to every other member):

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; \mathit{successor}\ x_i = x_j\;\; \Leftrightarrow\;\; \mathit{predecessor}\ x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$\mathit{successor}\ x_n = x_1\;\; \wedge\;\; \mathit{predecessor}\ x_1 = x_n.$$

The theorem, "ordered_symmetric_is_cyclic," and formal proof is in the Coq file, threed.v.

Proof. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$i = n\;\; \wedge\;\; j = 1\;\; \wedge\;\; \mathit{successor}\ x_i = x_j\;\; \Rightarrow\;\; \mathit{successor}\ x_n = x_1. \tag{5.11}$$

Applying the definition of a symmetric geometry (5.2) to conclusion 5.11:

$$i = n\;\; \wedge\;\; j = 1\;\; \wedge\;\; \mathit{predecessor}\ x_j = x_i\;\; \Rightarrow\;\; \mathit{predecessor}\ x_1 = x_n. \tag{5.12}$$

□

Theorem 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

Proof.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

Definition 5.5. Cyclic successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (n = m + 1 \leq \mathit{setsize}). \tag{5.13}$$

Definition 5.6. Cyclic predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (n = m - q \geq 1). \tag{5.14}$$

Definition 5.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:
(5.15)

$$\mathit{Adjacent}(m, n, \mathit{setsize}) \leftarrow \mathit{Successor}(m, n, \mathit{setsize}) \vee \mathit{Predecessor}(m, n, \mathit{setsize}).$$

Prove that every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (m = setsize \wedge n = 1). \tag{5.16}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (n = m+1 \leq setsize). \tag{5.17}$$

$$Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (n = setsize \wedge m = 1). \tag{5.18}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m+1 \leq setsize). \tag{5.19}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m-q \geq 1). \tag{5.20}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \wedge m = 1). \tag{5.21}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \wedge n = setsize). \tag{5.22}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m+1 \leq setsize). \tag{5.23}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m-q \geq 1). \tag{5.24}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\begin{aligned} \forall\, setsize > 3: \quad & \neg Successor(1,3,setsize > 3) \\ & \leftarrow Successor(1,2,setsize > 3) \leftarrow (n = m+1 \leq setsize). \end{aligned} \tag{5.25}$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\, setsize > 3: \quad & \neg Predecessor(1,3,setsize > 3) \\ & \leftarrow Predecessor(1,setsize,setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \end{aligned} \tag{5.26}$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\, setsize > 3: \quad & \neg Adjacent(1,3,setsize > 3) \\ & \leftarrow \neg Successor(1,3,setsize > 3) \wedge \neg Predecessor(1,3,setsize > 3). \quad \square \end{aligned} \tag{5.27}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

Applying the ruler measure (2.1) and ruler convergence (2.2) to the set relations, countable distance space (3.1) and countable volume (4.1) yields the following insights and implications:

(1) The properties of metric space, Euclidean distance and area/volume are derived from two set relations without using the notions of Euclidean geometry [**Joy98**] like plane, side, angle, perpendicular, congruence, intersection, etc., which makes analysis a more self-contained discipline and also provides a formal (set-based) understanding of geometry.

(2) The number of possible unique sub-Manhattan distance n-tuples (one size, c, subinterval from each range set, y_i), v_c, is the Cartesian product of the range sets:

$$\bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset \quad \wedge \quad d_c = |\bigcup_{i=1}^{n} y_i| \quad \wedge \quad |x_i| = |y_i| \rightarrow$$

$\bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset \quad \wedge \quad v_c = |\times_{i=1}^{n} y_i| \quad \wedge \quad |x_i| = |y_i|$,
which converges to Euclidean volume as $c \to 0$. Volume as the set of all possible unique sub-Manhattan distance n-tuples is a coherent, set-based theory relating volume to distance that metric space, the Lebesgue, Borel, and Hausdorff measures have failed to show.

(3) Distance is a function of the combinatorial *domain*-to-range set mappings. Whereas, area/volume is a function of the combinatorial *range*-to-range set mappings.
(4) The largest intersection and smallest distance, $d_c = |\bigcup_{i=1}^{n} y_i|$, has the most domain set members mapping to each range set member. The equality case, $|x_i| = |y_i| = p_i$, of the countable distance space constraint, $|x_i| = |y_i|^q$, $q \geq 1$, (3.1) limits the largest total number of domain-to-range set mappings to $\sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^2$, which is the set-based reason Euclidean distance (3.7) is the smallest possible distance between two distinct points in $\mathbb{R}^n$.
(5) $|x_i| = |y_i|^q$, $q \geq 1$, (3.1) generates all the L^p norms, $||L||_p = (\sum_{i=1}^{n} s_i^p)^{1/p}$. For example, using the same proof pattern as for Euclidean distance (3.7): $p_i = |y_i| \;\Rightarrow\; |x_i| = p_i^q \;\Rightarrow\; \sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^{q+1} \leq d_c^{q+1} \;\ldots\;.$
(6) It was shown that the countable distance set operation, $d_c = |\bigcup_{i=1}^{n} y_i|$, (3.1) generates three of the metric space properties. The countable distance constraint, $|x_i| = |y_i|^q$, $q \geq 1$, (3.1) is the reason for the fourth property of metric space, symmetry, $d(u, v) = d(v, u)$, where the value, q, causes the same combinatorial domain-to-range set mapping for every domain-range set pair that manifests as the same value, p, in each term of the L^p norm.
(7) The Euclidean volume proof was used to derive the Coulomb's charge force (5.1) and Newton's gravity force (5.4) without other laws of physics or Gauss's divergence theorem. The Euclidean distance proof was used to derive the spacetime equations (5.8) without a constant speed of light assumption or even the notion of light.
(8) **The Proportionate Interval Principle:** The derivations of the charge force, gravity force, and spacetime equations shows that all Euclidean distance range intervals having a size, r, have proportionately sized intervals of other types, for example: $r = (r_C/q_C)q = (r_G/m_G)m = (r_c/t_c)t$.
    (a) The derivations of the charge and gravity force equations show that the proportionate interval principle is the cause of the inverse square law and flux divergence.
    (b) If there are quantum values of charge, q_C, and mass, m_G, then there are quantum distances, r_C and r_G, where the forces do not exist (not defined) at smaller distances, which might have implications for particle collisions and the density of black holes.
    (c) Discrete states (like spin) do not have proportionate continuous distance intervals. Therefore, discrete state changes with respect to time are independent of distance (for example, the change in spin of two quantum coupled particles).
(9) Relativity theory assumes that only 3 dimensions of physical space exist [**Bru17**]. The proof in this article (5.4) shows that time constrains

physical distance and volume to at most three dimensions. Higher dimensional Hilbert spaces (vector spaces) are valid if no more than three of the dimensions represent physical space. Theories of higher dimensions of *physical* distance and volume would cause logical contradictions.

(10) The proof of at most 3 members in any set of ordered and symmetric members (5.4), implies that each *physical* infinitesimal volume (ball) can have at most 3 ordered and symmetric dimensions of discrete *physical* states of the same type. And each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values corresponding to 27 possible "types" of infinitesimal balls.

(11) If each of the three possible ordered and symmetric dimensions of discrete physical states contained unordered sets of discrete state values, for example, unordered binary values, then there would be $2 \cdot 2 \cdot 2 = 8$ possible combinations of values. Unordered sets (states) are non-deterministic with respect to time. For example, every time an unordered state is physically measured, there is a 50-50 chance of having one of the binary values.

(12) Where infinitesimal balls intersect, an algebra of the interactions of the discrete states with respect to time needs to be developed. The interaction of the discrete states associated with overlapping infinitesimal balls with respect to time might result in what we perceive as waves, motion, mass, charge, etc.

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George Van Treeck, 668 Westline Dr., Alameda, CA 94501