

# The Set Relations Generating Euclidean Geometry

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ABSTRACT. Applying a ruler-like measure to domain-to-range set mappings, where for each disjoint domain set there exists a corresponding same-sized range set and the range sets in some cases intersect, converges to: the triangle inequality and non-negativity properties of a distance metric, and converges to Manhattan distance at the upper boundary and Euclidean distance at the lower boundary of the triangle inequality. Applying the ruler-like measure to a domain-to-domain set mapping converges to the product of interval interval sizes (Euclidean area/volume). The total order and symmetry properties of the set-based distance and volume relations limit an Euclidean geometry to 3 dimensions. Proofs verified in Coq.

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## 1. Introduction

Much of mathematical analysis is derived from a foundation of set-based axioms, except for: triangle inequality, Manhattan distance, Euclidean distance, and Euclidean area/volume, which are motivated by Euclidean geometry and used as definitions in measure (for example, metric space, Hausdorff, and Lebesgue) and integration (Lebesgue and Riemann) [Gol76]. This article will use some very simple real analysis to motivate and derive the triangle inequality, Manhattan and Euclidean distance metrics from a single set-based axiom and derive volume from another set-based axiom.

The relationships between countable sets generating distance and volume provides new insights into Euclidean geometry, metric space, and physics. For example, (*without any notions of side, angle, and shapes*): 1) the set-based reason Euclidean distance is the smallest distance between two distinct points in Euclidean space; 2) the set-based reason for the triangle inequality and non-negativity properties of a distance metric; 3) the set-based reason metric space might not be a sufficient condition for a distance metric; 4) the set-based reason physical, Euclidean geometry is limited to 3 dimensions.

The proofs in this article are verified formally using the Coq Proof Assistant [Coq15] version 8.7.0. The Coq-based definitions, theorems, and proofs are in the files “euclidrelations.v” and “threed.v” located at:

<https://github.com/treeck/RASRGeometry>.

## 2. Ruler measure and convergence

Deriving distance and volume from set and number theory requires a measure that does not have Euclidean assumptions and also allows the full range of mappings from a one-to-one correspondence to a many-to-many mapping. A ruler (measuring stick) measures a real-valued interval as the nearest integer number of same-sized subintervals (units), where the partial subintervals are ignored.

The ruler measure allows defining relations, for example a many-to-many relation, between the set of same-sized subintervals in one interval and the set of same-sized subintervals in another interval. The countable relations converge to continuous, bijective functions as the subinterval size converges to zero.

**DEFINITION 2.1.** Ruler measure: A ruler measures the size,  $M$ , of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals,  $p$ , each subinterval having the same size,  $c$ . Notionally:

$$(2.1) \quad \forall c \, s \in \mathbb{R}, \, [a, b] \subset \mathbb{R}, \, s = |a - b| \wedge c > 0 \wedge \\ (p = \text{floor}(s/c) \vee p = \text{ceiling}(s/c)) \wedge M = \sum_{i=1}^p c = pc.$$

**THEOREM 2.2.** *Ruler convergence:*

$$\forall [a, b] \subset \mathbb{R}, \, s = |a - b| \Rightarrow M = \lim_{c \rightarrow 0} pc = s.$$

The Coq-based theorem and proof in the file euclidrelations.v is “limit\_c\_0\_M\_eq\_exact\_size.”

**PROOF.** (epsilon-delta proof)

By definition of the floor function,  $\text{floor}(x) = \max(\{y : y \leq x, y \in \mathbb{Z}, x \in \mathbb{R}\})$ :

$$(2.2) \quad \forall c > 0, p = \text{floor}(s/c) \wedge 0 \leq |\text{floor}(s/c) - s/c| < 1 \Rightarrow 0 \leq |p - s/c| < 1.$$

Multiply all sides of inequality 2.2 by  $|c|$ :

$$(2.3) \quad \forall c > 0, \quad 0 \leq |p - s/c| < 1 \Rightarrow 0 \leq |pc - s| < |c|.$$

$$(2.4) \quad \forall \delta : |pc - s| < |c| = |c - 0| < \delta \\ \Rightarrow \forall \epsilon = \delta : |c - 0| < \delta \wedge |pc - s| < \epsilon := M = \lim_{c \rightarrow 0} pc = s. \quad \square$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence, where:  $[0, \pi]$ ,  $s = |0 - \pi|$ ,  $c = 10^{-i}$ , and  $p = \text{floor}(s/c) \Rightarrow p \cdot c = 3.1_{i=1}, 3.14_{i=2}, 3.141_{i=3}, \dots, \pi$ .

### 3. Distance

**Notation conventions:** The vertical bars around a set is the standard notation for indicating the cardinal (number of members in the set). To prevent over use of the vertical bar, the symbol for “such that” is the colon.

**3.1. Countable distance space.** The most fundamental notion of distance is that for each disjoint domain set,  $x_i$ , there exists a corresponding range set,  $y_i$  containing the same number of members,  $p_i$ :  $|x_i| = |y_i| = p_i$ . For example, there should be as many steps walked in the range set,  $y_i$ , as there are pieces of traversed land in the corresponding domain set,  $x_i$ . The size of the union set depends on the amount of range set intersection, where  $|\bigcup_{i=1}^n y_i| = \sum_{i=1}^n |y_i| \Leftrightarrow \bigcap_{i=1}^n y_i = \emptyset$  and  $|\bigcup_{i=1}^n y_i| < \sum_{i=1}^n |y_i| \Leftrightarrow \bigcap_{i=1}^n y_i \neq \emptyset$ .

DEFINITION 3.1. Countable distance space,  $d_c$ :

$$|\bigcup_{i=1}^n x_i| = \sum_{i=1}^n |x_i| \quad \wedge \quad d_c = |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i.$$

**3.2. Metric Space.** Applying the ruler (2.1) and ruler convergence (2.2) to the countable distance space property,  $d_c = |\bigcup_{i=1}^2 y_i| \leq \sum_{i=1}^2 |y_i|$ , (3.1) generates the real-valued triangle inequality and non-negativity properties of metric space:

$$\begin{aligned} (3.1) \quad d_c &= |\bigcup_{i=1}^2 y_i| \leq \sum_{i=1}^2 |y_i| \quad \wedge \quad \forall c > 0 : \\ d_c &= \text{floor}(d(u, w)/c) \quad \wedge \quad |y_1| = \text{floor}(d(u, v)/c) \quad \wedge \quad |y_2| = \text{floor}(d(v, w)/c) \\ &\Rightarrow d(u, w) = \lim_{c \rightarrow 0} d_c \cdot c \leq \sum_{i=1}^2 \lim_{c \rightarrow 0} |y_i| \cdot c = d(u, v) + d(v, w). \end{aligned}$$

The number of members in any countable set is always non-negative. And the product of two non-negative numbers,  $d_c \cdot c$ , is always a non-negative number:

$$\begin{aligned} (3.2) \quad \forall c > 0 : \quad d_c &= \text{floor}(d(u, w)/c) \quad \wedge \quad d_c = |\bigcup_{i=1}^n y_i| \geq 0 \\ &\Rightarrow \text{floor}(d(u, w)/c) = d_c \geq 0 \quad \Rightarrow \quad d(u, w) = \lim_{c \rightarrow 0} d_c \cdot c \geq 0. \end{aligned}$$

**3.3. Distance space range.** Where the range sets intersect, multiple domain set members map to a single range set member. Therefore, the union set size,  $d_c$ , is function of the number of domain-to-range set member mappings.

The property,  $|x_i| = |y_i| = p_i$ , (3.1) constrains the range of possible domain-to-range set member mapping from  $|x_i| \cdot 1 = p_i$  (where members of  $x_i$  corresponds 1-1 to members of  $y_i$ ) to  $|x_i| \cdot |y_i| = p_i^2$  (where all  $p_i$  number of members of  $x_i$  map to each member in  $y_i$ , which is a many-to-many relation).

Therefore,  $\exists \mathbf{f} : d_c = \mathbf{f}(\sum_{i=1}^n p_i)$  is the largest possible distance because it is the case of the smallest number of domain-to-range mappings (no intersection of the range sets). And  $\exists \mathbf{f} : d_c = \mathbf{f}(\sum_{i=1}^n p_i^2)$  is the smallest possible distance because it is the case of the largest number of domain-to-range mappings (largest allowed intersection of range sets). Applying the ruler (2.1) and ruler convergence theorem (2.2) to the largest and smallest countable distance cases yields the real-valued, Manhattan and Euclidean distance functions.

### 3.4. Manhattan distance.

THEOREM 3.2. *Manhattan (longest) distance,  $d$ , is the size of the distance interval,  $[d_0, d_m]$ , mapping to a set of disjoint domain intervals,  $\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$ , where:*

$$d = \sum_{i=1}^n s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The formal Coq-based theorem and proof in file euclidrelations.v is “taxicab.distance.”

PROOF.

From the countable distance space definition (3.1), the largest possible countable distance,  $d_c$ , is the equality case:

$$(3.3) \quad d_c \leq \sum_{i=1}^n |y_i| \quad \wedge \quad |y_i| = p_i \\ \Rightarrow \quad d_c \leq \sum_{i=1}^n |y_i| = \sum_{i=1}^n p_i \quad \Rightarrow \quad \exists p_i, d_c : d_c = \sum_{i=1}^n p_i.$$

Multiply both sides of equation 3.5 by  $c$  and take the limit:

$$(3.4) \quad d_c = \sum_{i=1}^n p_i \Rightarrow d_c \cdot c = \sum_{i=1}^n (p_i \cdot c) \Rightarrow \lim_{c \rightarrow 0} d_c \cdot c = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c).$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of  $d$ :

$$(3.5) \quad d = |d_0 - d_m| \Rightarrow \exists c d : \text{floor}(d/c) = d_c \Rightarrow d = \lim_{c \rightarrow 0} d_c \cdot c.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of  $s_i$ :

$$(3.6) \quad \forall i \in [1, n], s_i = |a_i - b_i| \Rightarrow \text{floor}(s_i/c) = p_i \Rightarrow \lim_{c \rightarrow 0} p_i \cdot c = s_i.$$

Combine equations 3.5, 3.4, 3.6:

$$(3.7) \quad d = \lim_{c \rightarrow 0} d_c \cdot c \quad \wedge \quad \lim_{c \rightarrow 0} d_c \cdot c = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c) \quad \wedge \\ \lim_{c \rightarrow 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c \rightarrow 0} d_c \cdot c = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c) = \sum_{i=1}^n s_i. \quad \square$$

### 3.5. Euclidean distance.

**THEOREM 3.3.** *Euclidean (shortest) distance,  $d$ , is the size of the distance interval,  $[d_0, d_m]$ , mapping to a set of disjoint domain intervals,  $\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$ , where:*

$$d^2 = \sum_{i=1}^n s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The formal Coq-based theorem and proof in the file euclidrelations.v is “Euclidean.distance.”

PROOF.

Apply the rule of product to the largest number of domain-to-range set mappings, where all  $p_i$  number of domain set members,  $x_i$ , map to each of the  $p_i$  number of members in the range set,  $y_i$ :

$$(3.8) \quad \sum_{i=1}^n |y_i| \cdot |x_i| = \sum_{i=1}^n p_i^2.$$

From the countable distance space definition (3.1), choose the equality case:

$$(3.9) \quad d_c \leq \sum_{i=1}^n |y_i| \quad \wedge \quad |y_i| = p_i \\ \Rightarrow \quad d_c \leq \sum_{i=1}^n |y_i| = \sum_{i=1}^n p_i \quad \Rightarrow \quad \exists p_i, d_c : d_c = \sum_{i=1}^n p_i.$$

Square both sides of equation 3.9 ( $x = y \Leftrightarrow f(x) = f(y)$ ):

$$(3.10) \quad \exists p_i, d_c : d_c = \sum_{i=1}^n p_i \Leftrightarrow \exists p_i, d_c : d_c^2 = (\sum_{i=1}^n p_i)^2.$$

Apply the Cauchy-Schwartz inequality to equation 3.10 and select the smallest distance (equality) case:

$$(3.11) \quad d_c^2 = (\sum_{i=1}^n p_i)^2 \geq \sum_{i=1}^n p_i^2 \Rightarrow \exists p_i : d_c^2 = \sum_{i=1}^n p_i^2.$$

Multiply both sides of equation 3.11 by  $c^2$ , simplify, and take the limit.

$$(3.12) \quad d_c^2 = \sum_{i=1}^n p_i^2 \Rightarrow d_c^2 \cdot c^2 = \sum_{i=1}^n p_i^2 \cdot c^2 \Leftrightarrow (d_c \cdot c)^2 = \sum_{i=1}^n (p_i \cdot c)^2 \\ \Rightarrow \lim_{c \rightarrow 0} (d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.13) \quad \exists c d : \text{floor}(d/c) = d_c \Rightarrow d = \lim_{c \rightarrow 0} d_c \cdot c \Rightarrow d^2 = \lim_{c \rightarrow 0} (d_c \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.14) \quad \forall i \in [1, n], \text{floor}(s_i/c) = p_i \Rightarrow \lim_{c \rightarrow 0} p_i \cdot c = s_i \\ \Rightarrow \lim_{c \rightarrow 0} (p_i \cdot c)^2 = s_i^2 \Rightarrow \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)^2 = \sum_{i=1}^n s_i^2.$$

Combine equations 3.13, 3.12, 3.14:

$$(3.15) \quad d^2 = \lim_{c \rightarrow 0} (d_c \cdot c)^2 \wedge \lim_{c \rightarrow 0} (d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)^2 \wedge \\ \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)^2 = \sum_{i=1}^n s_i^2 \\ \Rightarrow d^2 = \lim_{c \rightarrow 0} (d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)^2 = \sum_{i=1}^n s_i^2. \quad \square$$

#### 4. Euclidean Volume

The number of all possible combinations (all many-to-many mappings) between members in a countable set  $x_1$  and a countable set  $x_2$  is the Cartesian product,  $|x_1| \cdot |x_2|$ . This section will use the ruler (2.1) and ruler convergence theorem (2.2) to prove that the Cartesian product of the same-sized subintervals of intervals converges to the product of interval sizes as the subinterval size converges to zero. The first step is to define a countable set-based measure of area/volume as the Cartesian product (many-to-many mappings) of disjoint domain set members.

DEFINITION 4.1. Countable volume measure,  $V_c$ :

$$\sum_{i=1}^n |x_i| = |\bigcup_{i=1}^n x_i|, \quad V_c = \prod_{i=1}^n |x_i|.$$

THEOREM 4.2. *Euclidean volume,  $V$ , is the size of a range interval,  $[v_0, v_m]$ , corresponding to a set of disjoint intervals:  $\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$ , where:*

$$V = \prod_{i=1}^n s_i, \quad V = |v_0 - v_m|, \quad s_i = |a_i - b_i|, \quad i \in [1, n], \quad i, n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is “Euclidean\_volume.”

PROOF.

Use the ruler (2.1) to divide the exact size,  $s_i = |a_i - b_i|$ , of each of the domain intervals,  $[a_i, b_i]$ , into a set,  $x_i$  of  $p_i$  number of subintervals.

$$(4.1) \quad \forall i n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \wedge \text{floor}(s_i/c) = p_i = |x_i|.$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$(4.2) \quad \text{floor}(s_i/c) = p_i \Rightarrow \lim_{c \rightarrow 0} (p_i \cdot c) = s_i.$$

Use the ruler (2.1) to divide the exact size,  $V = |v_0 - v_m|$ , of the range interval,  $[v_0, v_m]$ , into  $p^n$  subintervals. Every integer number,  $V_c$ , does **not** have an integer  $n^{th}$  root. However, for those cases where  $V_c$  does have an integer  $n^{th}$  root, there is a  $p^n$  that satisfies the definition a countable volume measure,  $V_c$  (4.1). Notionally:

$$(4.3) \quad \forall p^n = V_c \in \mathbb{N}, \exists V \in \mathbb{R}, x_i : \text{floor}(V/c^n) = V_c = p^n = \prod_{i=1}^n |x_i| = \prod_{i=1}^n p_i.$$

Apply the ruler convergence theorem (2.2) to equation 4.3 and simplify:

$$(4.4) \quad \text{floor}(V/c^n) = p^n \Rightarrow V = \lim_{c \rightarrow 0} p^n \cdot c^n = \lim_{c \rightarrow 0} (p \cdot c)^n.$$

Multiply both sides of equation 4.3 by  $c^n$  and simplify:

$$(4.5) \quad p^n = \prod_{i=1}^n p_i \Rightarrow p^n \cdot c^n = \left( \prod_{i=1}^n p_i \right) \cdot c^n \Leftrightarrow (p \cdot c)^n = \prod_{i=1}^n (p_i \cdot c).$$

Combine equations 4.4, 4.5, and 4.2:

$$(4.6) \quad V = \lim_{c \rightarrow 0} (p \cdot c)^n \quad \wedge \quad (p \cdot c)^n = \prod_{i=1}^n (p_i \cdot c) \quad \wedge \quad \lim_{c \rightarrow 0} (p_i \cdot c) = s_i \\ \Rightarrow V = \lim_{c \rightarrow 0} (p \cdot c)^n = \prod_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c) = \prod_{i=1}^n s_i. \quad \square$$

## 5. Ordered and symmetric geometries

Calculating the union and addition operations of distance and the union and multiplication operations of volume requires iterating sequentially through each and every set, which implies a total order of the set of sets. The commutative property of the union, addition, and multiplication also allows each set to be sequentially adjacent to any other set, herein referred to as a symmetric geometry.

But, a sequentially ordered set can have only one order *at a point in time* because each member can have at most one successor and at most one predecessor. It will now be proved that a set satisfying the *simultaneous* constraints of a single total order and symmetry defines a cyclic set containing at most 3 members.

DEFINITION 5.1. Ordered geometry:

$$\forall i \ n \in \mathbb{N}, \ i \in [1, n-1], \ \forall x_i \in \{x_1, \dots, x_n\}, \\ \text{successor } x_i = x_{i+1} \ \wedge \ \text{predecessor } x_{i+1} = x_i,$$

where each  $x_i \in \{x_1, \dots, x_n\}$  is a set.

DEFINITION 5.2. Symmetric geometry (every member of a set is sequentially adjacent to every other member):

$$\forall i \ j \ n \in \mathbb{N}, \ \forall x_i \ x_j \in \{x_1, \dots, x_n\}, \ \text{successor } x_i = x_j \ \wedge \ \text{predecessor } x_j = x_i.$$

THEOREM 5.3. An ordered and symmetric set is a cyclic set.

$$\text{successor } x_n = x_1 \ \wedge \ \text{predecessor } x_1 = x_n.$$

The theorem and formal Coq-based proof is “ordered\_symmetric\_is\_cyclic,” which is located in the file `threed.v`.

PROOF. The property of order (5.1) defines unique successors and predecessors for all set members except for the successor of  $x_n$  and the predecessor of  $x_1$ . Therefore, the only member that can be a successor of  $x_n$ , without creating a contradiction, is  $x_1$ . And the only member that can be a predecessor of  $x_1$ , without creating a contradiction, is  $x_n$ . From the properties of a symmetric geometry (5.2):

$$(5.1) \quad i = n \ \wedge \ j = 1 \ \wedge \ \text{successor } x_i = x_j \Rightarrow \text{successor } x_n = x_1.$$

$$(5.2) \quad i = n \ \wedge \ j = 1 \ \wedge \ \text{predecessor } x_j = x_i \Rightarrow \text{predecessor } x_1 = x_n. \quad \square$$

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The Coq-based lemmas and proofs in the file `threed.v` are:

**Lemmas:** `adj111`, `adj122`, `adj212`, `adj123`, `adj133`, `adj233`, `adj213`,  
`adj313`, `adj323`, and `not_all_mutually_adjacent_gt_3`.

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.

It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

DEFINITION 5.5. Cyclic successor of  $m$  is  $n$ :

$$(5.3) \quad \text{Successor}(m, n, \text{setsize}) \leftarrow (m = \text{setsize} \wedge n = 1) \vee (m + 1 \leq \text{setsize}).$$

DEFINITION 5.6. Cyclic predecessor of  $m$  is  $n$ :

$$(5.4) \quad \text{Predecessor}(m, n, \text{setsize}) \leftarrow (m = 1 \wedge n = \text{setsize}) \vee (m - 1 \geq 1).$$

DEFINITION 5.7. Adjacent: member  $m$  is sequentially adjacent to member  $n$  (required for a "symmetric" set (5.2)), if the cyclic successor of  $m$  is  $n$  or the cyclic predecessor of  $m$  is  $n$ . Notionally:

$$(5.5) \quad \text{Adjacent}(m, n, \text{setsize}) \leftarrow \text{Successor}(m, n, \text{setsize}) \vee \text{Predecessor}(m, n, \text{setsize}).$$

Every member is adjacent to every other member, where  $\text{setsize} \in \{1, 2, 3\}$ :

$$(5.6) \quad \text{Adjacent}(1, 1, 1) \leftarrow \text{Successor}(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1).$$

$$(5.7) \quad \text{Adjacent}(1, 2, 2) \leftarrow \text{Successor}(1, 2, 2) \leftarrow (1 + 1 \leq 2).$$

$$(5.8) \quad \text{Adjacent}(2, 1, 2) \leftarrow \text{Successor}(2, 1, 2) \leftarrow (2 = 2 \wedge 1 = 1).$$

$$(5.9) \quad \text{Adjacent}(1, 2, 3) \leftarrow \text{Successor}(1, 2, 3) \leftarrow (1 + 1 \leq 2).$$

$$(5.10) \quad \text{Adjacent}(2, 1, 3) \leftarrow \text{Predecessor}(2, 1, 3) \leftarrow (2 - 1 \geq 1).$$

$$(5.11) \quad \text{Adjacent}(3, 1, 3) \leftarrow \text{Successor}(3, 1, 3) \leftarrow (3 = 3 \wedge 1 = 1).$$

$$(5.12) \quad \text{Adjacent}(1, 3, 3) \leftarrow \text{Predecessor}(1, 3, 3) \leftarrow (1 = 1 \wedge 3 = 3).$$

$$(5.13) \quad \text{Adjacent}(2, 3, 3) \leftarrow \text{Successor}(2, 3, 3) \leftarrow (2 + 1 \leq 3).$$

$$(5.14) \quad \text{Adjacent}(3, 2, 3) \leftarrow \text{Predecessor}(3, 2, 3) \leftarrow (3 - 1 \geq 1).$$

Must prove that for all  $\text{setsize} > 3$ , there exist non-adjacent members. For example, the first and third members are not adjacent:

$$(5.15) \quad \forall \text{setsize} > 3 : \quad \neg \text{Successor}(1, 3, \text{setsize}) \\ \leftarrow \text{Successor}(1, 2, \text{setsize}) \leftarrow (1 + 1 \leq \text{setsize}).$$

That is, 2 is the only successor of 1 for all  $\text{setsize} > 3$ , which implies 3 is not a successor of 1 for all  $\text{setsize} > 3$ .

$$(5.16) \quad \forall \text{setsize} > 3 : \quad \neg \text{Predecessor}(1, 3, \text{setsize}) \\ \leftarrow \text{Predecessor}(1, n, \text{setsize}) \leftarrow (1 = 1 \wedge n = \text{setsize}).$$

That is,  $n = \text{setsize}$  is the only predecessor of 1 for all  $\text{setsize} > 3$ , which implies 3 is not a predecessor of 1 for all  $\text{setsize} > 3$ .

$$(5.17) \quad \forall \text{setsize} > 3: \quad \neg \text{Adjacent}(1, 3, \text{setsize}) \\ \leftarrow \neg \text{Successor}(1, 3, \text{setsize}) \wedge \neg \text{Predecessor}(1, 3, \text{setsize}). \quad \square$$

That is, for all  $\text{setsize} > 3$ , some elements are not sequentially adjacent to every other element (violates the symmetry property).

## 6. Summary

Applying the ruler measure (2.1) and ruler convergence proof (2.2), to a set of real-valued domain intervals and a range interval yields some new insights into geometry and physics.

- (1) Distance is a function of the number of domain-to-range set member mappings. Area/volume is a function of the number of domain-to-domain set member mappings. Other types of measure, like metric space, Borel, Hausdorff, and Lebesgue, do not provide that insight. Metric space allowing area/volume as a metric calls into question whether metric space is a sufficient condition for distance metrics.
- (2) Applying the ruler measure to the countable distance space (3.1) provides the insight that all notions of distance are derived from the principle that for each domain set,  $x_i$ , there exists a corresponding range set,  $y_i$ , containing the same number of members,  $p_i$ :  $|x_i| = |y_i| = p_i$  (3.1). For example, there should be as many steps walked in the range set,  $y_i$ , as there are pieces of traversed land in the corresponding domain set,  $x_i$ .

And the union size depends on the amount of intersection of range sets:  $|\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$ .

- (a) Applying the ruler to the set relation,  $|\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$ , (3.1) generates the real-valued triangle inequality and non-negativity properties of metric space (3.2).
- (b) Where the range sets intersect, multiple domain set members can map to a single range set member. Therefore, distance is a function of domain-to-range set member mappings. The property,  $|x_i| = |y_i| = p_i$ , constrains the range of possible domain-to-range set member mapping from  $\sum_{i=1}^n p_i$  to  $\sum_{i=1}^n p_i^2$ , which converge to Manhattan (3.2) and Euclidean distance (3.3) respectively.
- (c) Euclidean distance was derived from a many-to-many relation between each disjoint domain set and a corresponding same-sized range set, where the range sets intersect *without any notions of side, angle, or shape*. A parametric variable relating the sizes of two domain intervals can be easily derived using calculus (Taylor series) and the Euclidean distance equation to generate the arc sine and arc cosine functions of the parametric variable (arc angle). In other words, the notions of side and angle are derived from Euclidean distance, which is the reverse perspective of classical geometry [Joy98] [Loo68] [Ber88] and axiomatic foundations for geometry [Bir32] [Hil80] [TG99].



- (3) Applying the ruler measure and ruler convergence proof to the countable volume definition (4.1) allows a proof that the Cartesian product of same-sized subintervals of real-valued intervals converges to the product of the interval sizes (Euclidean length/area/volume):
  - (a) Euclidean volume was derived from a many-to-many (combinatorial) relation without notions of sides, angles, and shape.
  - (b) The Lebesgue measure, Riemann and Lebesgue integration use Euclidean volume/space ( $\mathbb{R}^n$ ) as a definition rather than deriving Euclidean volume from more fundamental set-based relations.
- (4) The set-based relations generating distance and volume have the *simultaneous* properties of a single total order of dimensions (5.1) and symmetry (5.2), which limits distance and volume to a cyclic set (5.3) of three dimensions (5.4), which explains why there are only three dimensions of physical space (width, height, and depth).
- (5) The *simultaneously* ordered and symmetric properties that generate the three dimensions of geometric space constrain all higher dimensional theories of physics to *hierarchical* 2 or 3-dimensional geometries to prevent contradictions. For example, the four-vectors common in physics [Bru17] are hierarchical, 2-dimensional geometries that have been "flattened."

The spacetime four-vector length,  $d = \sqrt{(ct)^2 - (x^2 + y^2 + z^2)}$ , where  $c$  is the speed of light and  $t$  is time, can be expressed in a form like,  $(ct)^2 = d_1^2 + d_2^2$ , where  $d_1^2 = x^2 + y^2 + z^2$  and  $d_2 = d$ . Likewise, the energy-momentum four-vector has the 2-dimensional length:  $E^2 = (mv^2)^2 + (pc)^2$ , where  $E$  is energy,  $m$  is the resting mass,  $v$  is the 3-dimensional velocity,  $c$  is the speed of light, and  $p$  is the relativistic momentum ( $p = \gamma mv$ , where  $\gamma = (1/(1 - (v/c)^2))^{1/2}$  is the Lorentz factor).

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