The Two Set Relations Generating Geometry

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Abstract. A ruler (measuring stick) partitions both domain and range intervals approximately into the nearest integer number of same-sized subintervals. As the subinterval size converges to zero: 1) Countable distance as the cardinal of the intersection of range sets generates the properties of metric space and all Lp norms, in particular, Manhattan and Euclidean distances. 2) Countable volume as the cardinal of the Cartesian product of domain sets converges to the product of interval interval sizes (Euclidean area/volume). The ruler measure-based proofs of Euclidean distance and area/volume are used to derive the charge force, Newtonian gravity force, and spacetime equations. Time limits physical geometry to 3 dimensions. All proofs are verified in Coq.

Contents

1. Introduction

Metric space, the Euclidean distance metric/vector norm, and Euclidean volume as product of interval sizes have been opaque definitions in mathematical analysis [**Gol76**] [**Rud76**] rather than derived from set operations. A "ruler" (measuring stick) partitions both domain and range intervals approximately into the nearest integer number of *same-sized* subintervals. Where the sets of subintervals conform to the set operation-based abstractions of distance and volume, the limit as the subinterval size converges to zero are the properties of metric space, all L^p norms, particularly, Manhattan and Euclidean distance, and Euclidean volume.

2010 *Mathematics Subject Classification.* Primary 28A75, 28E15. Secondary 03E75, 51M99.

The derivations of distance and volume provides some new insights into geometry and physics, for example: the single set operation generating the triangle inequality, non-negativity, and identity of indiscernibles properties of metric space; the mapping between sets that makes Euclidean distance the smallest possible distance between two distinct points in $\mathbb{R}^n$; the mapping between sets that makes distance different from area/volume; why the ruler-based derivation of Euclidean volume is a necessary condition to derive the charge force and Newtonian gravity force equations; how time places an additional constraint on the set operation-based definitions of distance and volume, which limits physical geometry to 3 dimensions.

All the proofs in this article have been formally verified using using the Coq proof verification system [**Coq15**]. The formal proofs are located in the Coq files, "euclidrelations.v" and "threed.v," at:
`https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

The ruler measure differs from σ-algebra measures like Lebesgue, Borel, and Hausdorff in three ways: 1) Abstract measures where specific examples are measurable versus a specific measure of continuous, real-valued intervals; 2) Volume defined as the product of interval sizes versus no notion of volume; 3) Outer measures versus both an inner and outer measure.

A ruler partitions both domain and range intervals approximately into the nearest integer number of *same-sized subintervals*, where different-sized intervals have a *different number of subintervals*. In contrast, the Riemann and Lebesgue integrals partition each interval, $[a_i, b_i]$ $(i \in [1, n])$, into the *same number of subintervals*, $m : I_{i_{k\in[1,m]}} \in [a_i, b_i]$, where different-sized intervals have *different-sized subintervals* [**Gol76**] [**Rud76**] $[(b_1 - a_1) \neq (b_2 - a_2) \Rightarrow I_{1_k} \neq I_{2_k}]$.

The ruler measure allows counting the number of mappings, ranging from a one-to-one correspondence to a many-to-many mapping, between the set of subintervals having size c in one interval and the set of subintervals having the same size, c, in another interval. The mapping (combinatorial) relations converge to continuous relations as the subinterval size, c, converges to zero.

Definition 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, p, each subinterval having the same size, c. Notionally:

$$\forall\, c\, s \in \mathbb{R},\ [a, b] \subset \mathbb{R},\ s = |a - b| \ \wedge\ c > 0\ \wedge \\ (p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \textstyle\sum_{i=1}^{p} c = pc. \tag{2.1}$$

Theorem 2.2. *Ruler convergence:*
$\forall\, [a, b] \subset \mathbb{R},\ s = |a - b|\ \Rightarrow\ M = \lim_{c\to 0} pc = s.$

The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\ p = floor(s/c)\ \wedge\ 0 \leq |floor(s/c) - s/c| < 1\ \Rightarrow\ 0 \leq |p - s/c| < 1. \tag{2.2}$$

Multiply all sides of inequality 2.2 by $|c|$:

$$\forall\, c > 0,\quad 0 \leq |p - s/c| < 1 \quad \Rightarrow \quad 0 \leq |pc - s| < |c|. \tag{2.3}$$

$$\text{(2.4)} \quad \forall\, \delta \,:\, |pc - s| < |c| = |c - 0| < \delta$$
$$\Rightarrow \quad \forall\, \epsilon = \delta:\ |c - 0| < \delta \ \wedge\ |pc - s| < \epsilon \ := \ M = \lim_{c \to 0} pc = s. \quad \square$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = |0 - \pi|$, $c = 10^{-i}$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{i=1},\ 3.14_{i=2},\ 3.141_{i=3}, ..., \pi_{\lim_{i\to\infty \Rightarrow c \to 0}}$.

3. Distance

Notation convention: Curly brackets, $\{\cdots\}$, delimit a set; square brackets, $[\cdots]$, delimit a list; and vertical bars around a set or list, $|\cdots|$, indicates the cardinal (number of members in the set or list).

3.1. Countable distance space.

Metric space is an abstraction of a distance measure used to classify which functions are distance measures. In this article, countable distance is a more fundamental, set operation-based abstraction of distance, that allows derivation of the properties of metric space, and the L^p norms, like Manhattan and Euclidean distance.

An example of a countable distance is the number of members in a range set, y_i, which equals the number of members in a corresponding domain set, x_i: $|x_i| = |y_i|$. And the countable distance spanning multiple, disjoint, domain sets, $\bigcap_{i=1}^{n} x_i = \emptyset$, is the number of members, d_c, in the union range set: $d_c = |\bigcup_{i=1}^{n} y_i|$.

Definition 3.1. Countable distance space, d_c:

$$\bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad d_c = |\bigcup_{i=1}^{n} y_i| \quad \wedge \quad |x_i| \geq |y_i|.$$

Theorem 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$.

The inclusion-exclusion inequality follows from the inclusion-exclusion principle [CG15]. But, a more intuitive and simple proof follows from the sum equal to size of all the set members appended into a list and sorting that list into a list of unique members (the union set) and a list of duplicates. For example, $|\{a, b, c\}| + |\{c, d, e\}| = |[a, b, c, c, d, e]| = |\{a, b, c, d, e\}| + |[c]| > |\{a, b, c, d, e\}|$. The list of duplicates being ≥ 0 implies the union set size is always $\leq$ the sum of set sizes.

A formal proof, inclusion_exclusion_inequality, using sorting into a set of unique members (union set) and a list of duplicate members, is in the file euclidrelations.v.

Proof. By the associative law of addition, append the sets into a list. Next, by the commutative law of addition, sort the list into a set of unique members and a list of duplicate members, and then subtract the duplicates from both sides:

$$\text{(3.1)} \quad \sum_{i=1}^{n} |y_i| = |append_{i=1}^{n} y_i| = |sort(append_{i=1}^{n} y_i)|$$
$$= |\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i| \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| - |duplicates_{i=1}^{n} y_i| = |\bigcup_{i=1}^{n} y_i|.$$

$$\text{(3.2)} \quad |\bigcup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| - |duplicates_{i=1}^{n} y_i| \quad \wedge \quad |duplicates_{i=1}^{n} y_i| \geq 0$$
$$\Rightarrow \quad |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|. \quad \square$$

3.2. Metric Space. All function range intervals, $d(u,w)$, satisfying the countable distance space definition, $d_c = |\bigcup_{i=1}^{n} y_i|$, where the ruler is applied, generates the three metric space properties: triangle inequality, non-negativity, and identity of indiscernibles. The set-based reason for the fourth property of metric space, symmetry $[d(u,v) = d(v,u)]$, will be identified in the last section of this article. The formal proofs: triangle_inequality, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 3.3. *Triangle Inequality:* $d_c = |y_1 \cup y_2| \Rightarrow d(u,w) \leq d(u,v) + d(v,w)$.

PROOF. Apply the ruler measure (2.1), the countable distance space condition (3.1), inclusion-exclusion inequality (3.2), and then ruler convergence (2.2).

$$\begin{aligned}
(3.3)\quad & \forall\, c > 0,\ d(u,w),\ d(u,v),\ d(v,w)\ :\\
& |y_1| = floor(d(u,v)/c)\ \wedge\ |y_2| = floor(d(v,w)/c)\ \wedge\\
& d_c = floor(d(u,w)/c)\ \wedge\ d_c = |y_1 \cup y_2| \leq |y_1| + |y_2|\\
& \Rightarrow\ floor(d(u,w)/c) \leq floor(d(u,v)/c) + floor(d(v,w)/c)\\
& \Rightarrow\ floor(d(u,w)/c)\cdot c \leq floor(d(u,v)/c)\cdot c + floor(d(v,w)/c)\cdot c\\
& \Rightarrow\ \lim_{c\to 0} floor(d(u,w)/c)\cdot c \leq \lim_{c\to 0} floor(d(u,v)/c)\cdot c + \lim_{c\to 0} floor(d(v,w)/c)\cdot c\\
& \Rightarrow\ d(u,w) \leq d(u,v) + d(v,w). \quad \square
\end{aligned}$$

THEOREM 3.4. *Non-negativity:* $d_c = |y_1 \cup y_2| \Rightarrow d(u,w) \geq 0$.

PROOF.
By definition, a set always has a size (cardinal) ≥ 0:

$$\begin{aligned}
(3.4)\quad & \forall\, c > 0,\ d(u,w)\ :\ floor(d(u,w)/c) = d_c\ \wedge\ d_c = |y_1 \cup y_2| \geq 0\\
& \Rightarrow\ floor(d(u,w)/c) = d_c \geq 0\ \Rightarrow\ d(u,w) = \lim_{c\to 0} d_c \cdot c \geq 0. \quad \square
\end{aligned}$$

THEOREM 3.5. *Identity of Indiscernibles:* $d(w,w) = 0$.

PROOF. Apply the triangle inequality property (3.3):

$$(3.5)\quad \forall\, d(u,v) = d(v,w) = 0\ \wedge\ d(u,w) \leq d(u,v) + d(v,w)\ \Rightarrow\ d(u,w) \leq 0.$$

Combine the non-negativity property (3.4) and the previous inequality (3.5):

$$(3.6)\quad d(u,w) \geq 0\ \wedge\ d(u,w) \leq 0\ \Leftrightarrow\ 0 \leq d(u,w) \leq 0\ \Rightarrow\ d(u,w) = 0.$$

$$(3.7)\quad d(u,w) = 0\ \wedge\ d(u,v) = 0\ \Rightarrow\ w = v.$$

$$(3.8)\quad d(v,w) = 0\ \wedge\ w = v\ \Rightarrow\ d(w,w) = 0. \quad \square$$

3.3. Distance space range. From the countable distance space definition, $d_c = |\bigcup_{i=1}^{n} y_i|$, as the amount of intersection increases, a single range set member can map to more domain set members. Therefore, the number of range-to-domain set member mappings is a function of the amount of range set intersection.

From the countable distance space property (3.1), where $|x_i| = |y_i| = p_i = 1$, each range member maps one-to-one to a unique domain set member (no intersection and largest distance), $|y_i| \cdot 1 = p_i = 1$ mapping, and also each range member maps to every domain set member (largest intersection and smallest possible distance), $|y_i| \cdot |x_i| = p_i^2 = 1$ mapping. The types of range-to-domain set mappings that are true for one range set member are true for all members in a range set. Therefore, the total number of range-to-domain set mappings varies from $\sum_{i=1}^{n} |y_i| \cdot 1 = \sum_{i=1}^{n} p_i$ to $\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2$ mappings. Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of range-to-domain set mapping cases converges to the real-valued Manhattan and Euclidean distance relations.

3.4. Manhattan distance.

THEOREM 3.6. *Manhattan (largest) distance, d, is the size of the range interval,* $[d_0, d_m]$, *corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "taxicab_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \tag{3.9}$$
$$\Rightarrow \quad d_c \leq \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\ d_c : \ d_c = \sum_{i=1}^{n} p_i.$$

Multiply both sides of equation 3.11 by c and take the limit:

$$d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \Rightarrow \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c). \tag{3.10}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d : \ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c. \tag{3.11}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c \to 0} p_i \cdot c = s_i. \tag{3.12}$$

Combine equations 3.11, 3.10, 3.12:

$$d = \lim_{c \to 0} d_c \cdot c \quad \wedge \quad \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) \quad \wedge \tag{3.13}$$
$$\lim_{c \to 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) = \sum_{i=1}^{n} s_i. \qquad \square$$

3.5. Euclidean distance.

THEOREM 3.7. *Euclidean (smallest) distance, d, is the size of the range interval,* $[d_0, d_m]$, *corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "Euclidean_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
Apply the rule of product to the largest number of range-to-domain set mappings, where all p_i number of range set members, y_i, map to each of the p_i number of members in the domain set, x_i, and where $|x_i| = |y_i| = p_i$:

$$|x_i| = |y_i| = p_i \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2. \tag{3.14}$$

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$d_c = |\bigcup_{i=1}^n y_i| \le \sum_{i=1}^n |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i$$
$$\Rightarrow \quad d_c \le \sum_{i=1}^n |y_i| = \sum_{i=1}^n p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c :\; d_c = \sum_{i=1}^n p_i. \tag{3.15}$$

Square both sides of equation 3.15 ($x = y \Leftrightarrow f(x) = f(y)$):

$$\exists\, p_i,\, d_c :\; d_c = \sum_{i=1}^n p_i \quad \Leftrightarrow \quad \exists\, p_i,\, d_c :\; d_c^2 = (\sum_{i=1}^n p_i)^2. \tag{3.16}$$

Apply the square of sum inequality, $(\sum_{i=1}^n p_i)^2 \ge \sum_{i=1}^n p_i^2$, to equation 3.16 and select the smallest area (the equality) case:

$$d_c = \sum_{i=1}^n p_i,\; p_i \ge 0 \quad \Rightarrow \quad d_c^2 = (\sum_{i=1}^n p_i)^2 = \sum_{i=1}^n p_i \sum_{j=1}^n p_j$$
$$= \sum_{i=1}^n p_i^2 + \sum_{i=1}^n p_i \sum_{j=1, j\ne i}^n p_j \ge \sum_{i=1}^n p_i^2$$
$$\Rightarrow \quad \exists\, p_i :\; d_c^2 = \sum_{i=1}^n p_i^2. \tag{3.17}$$

Multiply both sides of equation 3.17 by c^2, simplify, and take the limit.

$$d_c^2 = \sum_{i=1}^n p_i^2 \;\Rightarrow\; d_c^2 \cdot c^2 = \sum_{i=1}^n p_i^2 \cdot c^2 \;\Leftrightarrow\; (d_c \cdot c)^2 = \sum_{i=1}^n (p_i \cdot c)^2$$
$$\Rightarrow \quad \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c\to 0}(p_i \cdot c)^2. \tag{3.18}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$\exists\, c\, d \in \mathbb{R} :\; floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2. \tag{3.19}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i. \tag{3.20}$$

Combine equations 3.19, 3.18, 3.20:

$$d^2 = \lim_{c\to 0}(d_c \cdot c)^2 \quad \wedge \quad \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c\to 0}(p_i \cdot c)^2 \quad \wedge$$
$$\lim_{c\to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^n s_i^2. \tag{3.21}$$

□

4. Euclidean Volume

Cartesian geometry, the Lebesgue n-volume, the Borel σ-algebra on $\mathbb{R}^n$, Riemann and Lebesgue integrals, etc. all assume (define) the size of the Cartesian product of the members in the intervals ($[a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$) as the product of interval sizes: $(b_1 - a_1)(b_2 - a_2)\ldots(b_n - a_n)$. The ruler measure makes a proof of that assumption trivial, where if each member of a countable, Cartesian set is a same-sized, c, interval, then the sum of the interval sizes converges to the product of the interval sizes. Notionally:

Definition 4.1. Countable Volume, V_c:

$$\bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad V_c = \prod_{i=1}^n |x_i|.$$

Theorem 4.2. *Euclidean volume, V, is size of the range interval, $[v_0, v_m]$, corresponding to the Cartesian product of all the members of the domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$. Notionally:*

$$V = \prod_{i=1}^n s_i,\;\; V = |v_0 - v_m|,\;\; s_i = |a_i - b_i|.$$

The theorem, "Euclidean_volume," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of subintervals.

$$\forall\, i\, n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i|. \tag{4.1}$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0}(p_i \cdot c) = s_i. \tag{4.2}$$

Multiply both sides of the countable volume definition 4.1 by c^n:

$$V_c \cdot c^n = (\textstyle\prod_{i=1}^n |x_i|) \cdot c^n = (\prod_{i=1}^n p_i) \cdot c^n = \prod_{i=1}^n (p_i \cdot c). \tag{4.3}$$

Use those cases, where V_c has an integer n^{th} root.

$$\forall\, p^n = V_c \in \mathbb{N} \;:\; V_c \cdot c^n = p^n \cdot c^n = (p \cdot c)^n = \textstyle\prod_{i=1}^n (p_i \cdot c). \tag{4.4}$$

Use the ruler (2.1) to partition the range interval, $[v_0, v_m]$, into $floor(V/c^n) = p^n$ subintervals and then apply the ruler convergence theorem (2.2) to equation 4.4:

$$V = \lim_{c\to 0} p^n \cdot c^n = \lim_{c\to 0}(p \cdot c)^n = \textstyle\prod_{i=1}^n \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^n s_i. \tag{4.5}$$ □

5. Applications to physics

5.1. Charge force equation. The sizes, q_1 and q_2, of two charges are independent domain variables, where each size, c, component of a charge exerts a force on each same size, c, component of the other charge. The total force, F, is proportionate to the total number of forces (the Cartesian product of the number of infinitesimal components) multiplied times a quantum charge force, $m_C a_C$. From the volume proof (4.2), the Cartesian product converges to $q_1 q_2$. Therefore:

$$F \propto (m_C a_C)(\lim_{c\to 0} floor(q_1/c) \cdot c)(\lim_{c\to 0} floor(q_2/c) \cdot c) = (m_C a_C)(q_1 q_2). \tag{5.1}$$

For any interval, $[0, q]$, a proportionately sized interval, $[0, r]$, can be defined with a proportion ratio, $q_C/r_C \in \mathbb{R}$, such that $r(q_C/r_C) = q$:

$$\forall\, q_1, q_2 \geq 0\, \exists\, q \in \mathbb{R} \;: q^2 = q_1 q_2 \;\wedge\; r(q_C/r_C) = q \;\Rightarrow\; (r(q_C/r_C))^2 = q_1 q_2. \tag{5.2}$$

$$\begin{aligned} &(r(q_C/r_C))^2 = q_1 q_2 \;\wedge\; F \propto (m_C a_C)(q_1 q_2) \\ &\qquad \Rightarrow \quad F \propto (m_C a_C)(r(q_C/r_C))^2 = (m_C a_C)(q_1 q_2). \end{aligned} \tag{5.3}$$

$$\begin{aligned} &m_C a_C = (x/x)(m_C a_C) = (m_C x)(a_C/x) \\ &\wedge \quad \exists\, m_0,\; a \in \mathbb{R} :\; m_0 = m_C x,\; a = a_C/x \quad \Rightarrow \quad \exists\, m_0,\; a \in \mathbb{R} :\; m_0 a = m_C a_C. \end{aligned} \tag{5.4}$$

Combine equations 5.4 and 5.3:

$$\begin{aligned} &m_0 a = m_C a_C \quad \wedge \quad F \propto (m_C a_C)(r(q_C/r_C))^2 = (m_C a_C)(q_1 q_2) \\ &\qquad \Rightarrow \quad F \propto (m_0 a)(r(q_C/r_C))^2 = (m_C a_C)(q_1 q_2) \\ &\qquad \Rightarrow \quad \exists\, k_C \in \mathbb{R} : \quad F = m_0 a = (m_C a_C r_C^2/q_C^2) q_1 q_2/r^2 = k_C q_1 q_2/r^2, \end{aligned} \tag{5.5}$$

where $k_C = m_C a_C r_C^2/q_C^2$ corresponds to the accepted standard units: Nm^2C^{-2}.

5.2. Newtonian gravity force equation. The sizes, m_1 and m_2, of two masses are independent domain variables, where each size, c, component of a mass exerts a force on each same size, c, component of the other mass. The total force, F, is proportionate to the total number of forces (the Cartesian product of the number of infinitesimal components) multiplied times a quantum gravity force, $m_G a_G$. From the volume proof (4.2), the Cartesian product converges to $m_1 m_2$.

$$F \propto (m_G a_G)(\lim_{c \to 0} floor(m_1/c) \cdot c)(\lim_{c \to 0} floor(m_2/c) \cdot c) = (m_G a_G)(m_1 m_2). \tag{5.6}$$

For any interval, $[0, m]$, a proportionately sized interval, $[0, r]$, can be defined with a proportion ratio, $m_G/r_G \in \mathbb{R}$, such that $r(m_G/r_G) = m$:

$$\begin{aligned} &\forall\, m_1, m_2 \geq 0\, \exists\, m \in \mathbb{R} : \quad m^2 = m_1 m_2 \quad \wedge \quad r(m_G/r_G) = m \\ &\qquad \Rightarrow \quad (r(m_G/r_G))^2 = m_1 m_2. \end{aligned} \tag{5.7}$$

$$\begin{aligned} &(r(m_G/r_G))^2 = m_1 m_2 \ \wedge \ F \propto (m_G a_G)(m_1 m_2) \\ &\qquad \Rightarrow \quad F \propto (m_G a_G)(r(m_G/r_G))^2 = (m_G a_G)(m_1 m_2). \end{aligned} \tag{5.8}$$

$$\begin{aligned} &m_G a_G = (x/x)(m_G a_G) = (m_G x)(a_G/x) \\ &\wedge \quad \exists\, m_0,\ a \in \mathbb{R} :\ m_0 = m_G x,\ a = a_G/x \quad \Rightarrow \quad \exists\, m_0,\ a \in \mathbb{R} :\ m_0 a = m_G a_G. \end{aligned} \tag{5.9}$$

Combine equations 5.9 and 5.8:

$$\begin{aligned} &m_0 a = m_G a_G \quad \wedge \quad F \propto (m_G a_G)(r(m_G/r_G))^2 = (m_G a_G)(m_1 m_2) \\ &\qquad \Rightarrow \quad F \propto (m_0 a)(r(m_G/r_G))^2 = (m_G a_G)(m_1 m_2) \\ &\Rightarrow \exists\, t_G, G \in \mathbb{R} : r_G/t_G^2 = a_G \quad \wedge \quad F = m_0 a = (r_G^3/m_G t_G^2) m_1 m_2/r^2 = G m_1 m_2/r^2, \end{aligned} \tag{5.10}$$

where $G = r_G^3/m_G t_G^2$ corresponds to the accepted standard units: $m^3 kg^{-1} s^{-2}$.

5.3. Spacetime equations. The purpose of the derivations of the spacetime dilation and interval equations is to show the same pattern of an Euclidean distance interval having a proportionate interval of any other type, in this case, time. Any interval, $[0, t]$, proportionate to the Euclidean range interval, $[0, r]$, is measured in terms of a conversion constant, c, that is the ratio of some value, r_c to some value, t_c, such that $r = (r_c/t_c)t = ct$.

Two inertial (moving) frames of reference have the respective interval sizes d_1 and d_2. In flat space (the countable distance space (3.1), where $|x_i| = |y_i|$), the spanning distance where each spanning distance subinterval member maps to members of two domain (two independent frames of reference) subintervals converges to the range of distances from Manhattan (3.6) to Euclidean distance (3.7). And combined with t proportionate to the Euclidean distance, r, we have:

$$\begin{aligned} &r^2 = d_1^2 + d_2^2 \quad \wedge \quad r = (r_c/t_c)t = ct \\ &\qquad \Rightarrow \quad (ct)^2 = d_1^2 + d_2^2 \quad \Rightarrow \quad d_2 = \sqrt{(ct)^2 - d_1^2}. \end{aligned} \tag{5.11}$$

$$\begin{aligned} &d_2 = \sqrt{(ct)^2 - d_1^2} \quad \wedge \quad d = d_2 \quad \wedge \quad d_1 = vt \\ &\qquad \Rightarrow \quad d = \sqrt{(ct)^2 - (vt)^2} = ct\sqrt{1 - (v^2/c^2)}, \end{aligned} \tag{5.12}$$

which is the spacetime dilation equation. [Bru17].

$$d_2^2 = (ct)^2 - d_1^2 \quad \wedge \quad s = d_2 \quad \wedge \quad d_1^2 = x^2 + y^2 + z^2 \\ \Rightarrow \quad s^2 = (ct)^2 - (x^2 + y^2 + z^2), \tag{5.13}$$

which is the spacetime interval equation [Bru17].

5.4. 3 dimensions of physical geometry. The set and arithmetic operations used to calculate distance and volume requires sequencing through a totally ordered set of dimensions, for example, the countable distance space: $d_c = |\bigcup_{i=1}^{n} y_i|$, Euclidean distance: $d^2 = \sum_{i=1}^{n} s_i^2$, countable volume: $V_c = \prod_{i=1}^{n} |x_i|$, and Euclidean volume: $V = \prod_{i=1}^{n} s_i$. The commutative property of the union, addition, and multiplication operations also allows sequencing through a set of n number of dimensions in all $n!$ number of possible orders.

Consider a set of 5 identical items arranged in a horizontal row on a table, enumerated left-to-right as: $[x_1,\ x_2,\ \cdots\ ,\ x_5]$. *Determining* that a *physical* sequencer sequenced the set in the order, $[x_5,\ x_4,\ \cdots\ ,\ x_1]$, and next sequenced in the order, $[x_1,\ x_2,\ \cdots\ ,\ x_5]$, requires the total order, at most one successor and at most one predecessor per set member, to not change during the *time* of the sequencing. Deterministic sequencing in every possible order requires each set member to be either a successor or predecessor to every other set member (sequentially adjacent) during the *time* of sequencing, herein referred to as a symmetric geometry.

It will now be proved that a set satisfying the constraints of a single total order and also symmetric defines a cyclic set containing at most 3 members, in this case, 3 dimensions of physical space.

Definition 5.1. Ordered geometry:

$$\forall\, i\, n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i\ \in \{x_1, \ldots, x_n\}, \\ \textit{successor}\ x_i = x_{i+1}\ \ \wedge\ \ \textit{predecessor}\ x_{i+1} = x_i.$$

Definition 5.2. Symmetric geometry (every set member is sequentially adjacent to any other member):

$$\forall\, i\, j\, n \in \mathbb{N},\ \ \forall\, x_i\, x_j \in \{x_1, \ldots, x_n\},\ \ \textit{successor}\ x_i = x_j \ \ \Leftrightarrow\ \ \textit{predecessor}\ x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$\textit{successor}\ x_n = x_1 \ \ \wedge\ \ \textit{predecessor}\ x_1 = x_n.$$

The theorem, "ordered_symmetric_is_cyclic," and formal proof is in the Coq file, threed.v.

Proof. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \textit{successor}\ x_i = x_j\ \ \Rightarrow\ \ \textit{successor}\ x_n = x_1. \tag{5.14}$$

Applying the definition of a symmetric geometry (5.2) to conclusion 5.14:

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \textit{predecessor}\ x_j = x_i\ \ \Rightarrow\ \ \textit{predecessor}\ x_1 = x_n. \tag{5.15}$$

□

Theorem 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

DEFINITION 5.5. Cyclic successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m+1 \leq setsize). \tag{5.16}$$

DEFINITION 5.6. Cyclic predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1). \tag{5.17}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{5.18}$$

Prove that every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \tag{5.19}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \tag{5.20}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1). \tag{5.21}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.22}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \tag{5.23}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1). \tag{5.24}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize). \tag{5.25}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.26}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \tag{5.27}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\begin{aligned} \forall\, setsize > 3: \quad & \neg Successor(1, 3, setsize > 3) \\ & \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \end{aligned} \tag{5.28}$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\, setsize > 3: \quad & \neg Predecessor(1, 3, setsize > 3) \\ & \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \end{aligned} \tag{5.29}$$

That is, member $n = \mathit{setsize} > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $\mathit{setsize} > 3$.

$$\begin{aligned} (5.30)\quad \forall\, \mathit{setsize} > 3: &\quad \neg \mathit{Adjacent}(1, 3, \mathit{setsize} > 3) \\ &\leftarrow \neg \mathit{Successor}(1, 3, \mathit{setsize} > 3) \wedge \neg \mathit{Predecessor}(1, 3, \mathit{setsize} > 3). \quad \square \end{aligned}$$

That is, for all $\mathit{setsize} > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

Applying the ruler measure (2.1) and ruler convergence (2.2) to the set relations, countable distance space (3.1) and countable volume (4.1) yields the following insights and implications:

(1) The properties of metric space, Euclidean distance and area/volume can be derived from two set relations without using the notions of Euclidean geometry [**Joy98**] like plane, side, angle, perpendicular, congruence, intersection, etc.

(2) The ruler measure-based proofs provide the insight that distance is a function of the combinatorial *range*-to-domain set member mappings. Whereas, area/volume is a function of the combinatorial *domain*-to-domain set member mappings.

(3) The equality case, $|x_i| = |y_i| = p_i$, of the countable distance space (3.1) constraint, $|x_i| \geq |y_i|$, limits the largest total number of range-to-domain set mappings (largest intersection, $d_c = |\bigcup_{i=1}^{n} y_i|$, and smallest distance) to $\sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^2$, which is the set-based reason Euclidean distance (3.7) is the smallest possible distance between two distinct points in $\mathbb{R}^n$.

(4) $|x_i|^{1/q} = |y_i|$, $q \geq 1$, generate all the L^p norms, $||L||_p = (\sum_{i=1}^{n} s_i^p)^{1/p}$. L^p norms, where $p > 2$, have shorter distances than Euclidean distance, L^2. But, the shorter distances, where $p > 2$, do not exist in "flat" ($|x_i| = |y_i|$, $\mathbb{R}^n$) space.

(5) Manhattan and Euclidean distance are the intuitive motivations for the fourth property of metric space, symmetry: $d(u, v) = d(v, u)$. But, the formal reason is that the type of combinatorial range-to-domain set mapping is the same for every domain-range set pair.

(6) All Euclidean distance range intervals having a size, r, can have proportionately sized intervals of other types, for example: $r = (r_C/q_C)q = (r_G/m_G)m = (r_c/t_c)/t$. Applying the ruler to those intervals allows application of the Euclidean area/volume and distance proofs to derive the charge force, Newtonian gravity force, and spacetime equations.
   (a) The $q_1 q_2$ (5.1) and $m_1 m_2$ (5.6) parts of the charge and gravity force equations is trivial to derive using the ruler measure and cannot be derived using methods like Riemann or Lebesgue integration. Without the ruler measure proof of volume, the charge and gravity force equations would remain purely empirical equations.
   (b) Euclidean distance proportionate to charge and mass, as shown in equations 5.2 and 5.7, explains why the charge and gravity forces vary inversely with the square of the distance.

   (c) If there are quantum values of charge, q_C, and mass, m_G, then there are quantum distances, r_C and r_G, where the forces do not exist (not defined) at smaller distances. This eliminates the need to invent stronger counteracting forces at smaller distances.
   (d) Discrete valued variables (discrete states like spin) do not have proportionate continuous distance intervals. Therefore, discrete value changes with respect to time are independent of distance (for example, the change in spin of two quantum coupled particles).

(7) Relativity theory assumes that only 3 dimensions of geometric space exist [**Bru17**]. The proof in this article (5.4) explains why time constrains physical, geometric space to at most three dimensions. If any higher dimensions of "space" exist, those higher dimensions must have types that differ from the 3 dimensions of geometric space.

(8) The proof of at most 3 dimensions of any set of ordered and symmetric members (5.4), implies that each infinitesimal volume (ball) can have at most 3 ordered and symmetric dimensions of discrete values of the same type. And each dimension of discrete values can have at most 3 ordered and symmetric discrete values, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values corresponding to 27 possible "types" of infinitesimal balls.

(9) If each of the three possible ordered and symmetric dimensions of discrete values contained unordered sets of discrete values, for example, unordered binary values, then there would be $2 \cdot 2 \cdot 2 = 8$ possible combinations of values. These unordered values would be non-deterministic. For example, every time a value is physically measured, there would be a 50-50 chance of having one of the binary values.

(10) Where infinitesimal balls intersect, an algebra of the interactions of the discrete values needs to be developed. The interaction of the discrete values associated with overlapping infinitesimal balls might result in what we perceive as particles, waves, motion, mass, charge, etc.

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