The Set Relations Generating Euclidean Geometry

George. M. Van Treeck

Abstract. A ruler-like measure divides sets of real-valued domain and range intervals into same-sized subintervals. The case, where each disjoint domain set of subintervals there exists a corresponding same-sized range set and the range sets in some cases intersect, converges to: the triangle inequality, Manhattan distance at the upper boundary and Euclidean distance at the lower boudary of the triangle inequality. The Cartesian product of domain set subintervals converges to the product of nterval interval sizes (Euclidean area/volume). The ordered and symmetric properties of the set-based distance and volume relations limit an Euclidean geometry to 3 dimensions. Proofs verified in Coq.

Contents

1. Introduction

Most of real analysis is derived from a foundation of number and set-based axioms. But, the triangle inequality, Manhattan distance, Euclidean distance, and Euclidean area/volume, are motivated by Euclidean geometry and used as definitions in measure (for example, metric space, Hausdorff, and Lebesgue) and integration (Lebesgue and Riemann) [**Gol76**] [**Rud76**]. This article will use some simple real analysis to motivate and derive the triangle inequality, the other properties of metric space, Manhattan and Euclidean distance metrics from a single set-based axiom and derive volume from another set-based axiom.

The relationships between countable sets generating distance and volume provides new insights into geometry and physics. For example, (*without any notions of*

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side, angle, and shapes): 1) the set-based reason Euclidean distance is the smallest distance between two distinct points in Euclidean space; 2) the single set relation generating all the properties of metric space; 3) the set-based reason metric space might not be a sufficient condition for a distance metric; 4) the set-based reason physical, Euclidean geometry is limited to 3 dimensions.

The proofs in this article are verified formally using the Coq Proof Assistant [**Coq15**] version 8.9.0. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at:
`https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

Deriving distance and volume from set and number axioms requires a measure that does not have Euclidean assumptions and also allows the full range of mappings from a one-to-one correspondence to a many-to-many mapping. A ruler (measuring stick) measures a real-valued interval as the nearest integer number of same-sized subintervals (units), where the partial subintervals are ignored.

The ruler measure allows defining relations, for example a many-to-many relation, between the set of same-sized subintervals in one interval and the set of same-sized subintervals in another interval. The countable relations converge to continuous, bijective functions as the subinterval size converges to zero.

Definition 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, p, each subinterval having the same size, c. Notionally:

$$\forall\, c\, s \in \mathbb{R},\;\; [a,b] \subset \mathbb{R},\;\; s = |a-b| \;\;\wedge\;\; c > 0 \;\;\wedge \\ (p = floor(s/c) \;\;\vee\;\; p = ceiling(s/c)) \;\;\wedge\;\; M = \textstyle\sum_{i=1}^{p} c = pc. \tag{2.1}$$

Theorem 2.2. *Ruler convergence:*
$\forall\; [a,b] \subset \mathbb{R},\;\; s = |a-b| \;\;\Rightarrow\;\; M = \lim_{c\to 0} pc = s.$

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y:\; y \le x,\, y \in \mathbb{Z},\, x \in \mathbb{R}\})$:

$$\forall\, c > 0,\, p = floor(s/c) \;\wedge\; 0 \le |floor(s/c) - s/c| < 1 \;\Rightarrow\; 0 \le |p - s/c| < 1. \tag{2.2}$$

Multiply all sides of inequality 2.2 by $|c|$:

$$\forall\, c > 0, \quad 0 \le |p - s/c| < 1 \quad\Rightarrow\quad 0 \le |pc - s| < |c|. \tag{2.3}$$

$$\forall\, \delta \,:\, |pc - s| < |c| = |c - 0| < \delta \\ \Rightarrow \quad \forall\, \epsilon = \delta :\; |c - 0| < \delta \;\wedge\; |pc - s| < \epsilon \;\; := \;\; M = \lim_{c\to 0} pc = s. \quad \square \tag{2.4}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence, where: $[0,\pi]$, $s = |0-\pi|$, $c = 10^{-i}$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{i=1},\; 3.14_{i=2},\; 3.141_{i=3}, ..., \pi$.

3. Distance

Notation conventions: The vertical bars around a set is the standard notation for indicating the cardinal (number of members in the set). To prevent over use of the vertical bar, the symbol for "such that" is the colon.

3.1. Countable distance space. The most fundamental notion of distance is that for each disjoint domain set, x_i, there exists a corresponding range set, y_i, containing the same number of members, p_i: $|x_i| = |y_i| = p_i$. For example, there should be as many steps walked in the range set, y_i, as there are pieces of traversed land in the corresponding domain set, x_i. And the distance, d_c, spanning one or more domain sets is the size of the union of range sets.

Definition 3.1. Countable distance space, d_c:

$$\bigcup_{i=1}^n x_i = \emptyset \quad \wedge \quad d_c = |\bigcup_{i=1}^n y_i| \quad \wedge \quad |x_i| = |y_i| = p_i.$$

Theorem 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$.

Proof. The following inequality is derived from the inclusion-exclusion principle [**CG15**] and the axiom, $u = v - w,\ w \geq 0 \Rightarrow u \leq v$. A different, formal proof, inclusion_exclusion_inequality, can be found in euclidrelations.v,

$$\begin{aligned}(3.1)\quad & |\bigcup_{i=1}^n y_i| = \sum_{i=1}^n |y_i| - \sum_{1\leq i<j\leq n} |y_i \cap y_j| + \cdots + (-1)^{n-1} |\bigcap_{i=1}^n y_i|) \quad \wedge \\ & \sum_{1\leq i<j\leq n} |y_i \cap y_j| + \cdots + (-1)^{n-1} |\bigcap_{i=1}^n y_i|) \geq 0 \\ & \Rightarrow \quad |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|. \quad \square\end{aligned}$$

3.2. Metric Space. Applying the ruler (2.1) and ruler convergence (2.2) to the set relation, $|\bigcup_{i=1}^2 y_i| \leq \sum_{i=1}^2 |y_i|$, generates all the properties of metric space. The formal proofs: triangle_inequality, non_negativity, identity_of_indiscernibles, and symmetry, are found in the Coq file, euclidrelations.v.

Theorem 3.3. *Triangle Inequality:* $d(u,w) \leq d(u,v) + d(v,w)$*:*

$$\begin{aligned}& \forall\, c > 0,\ |y_1| = floor(d(u,v)/c) \ \wedge\ |y_2| = floor(d(v,w)/c) \quad \wedge \\ & d_c = floor(d(u,w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \leq |y_1| + |y_2| \\ & \Rightarrow \quad d(u,w) \leq d(u,v) + d(v,w).\end{aligned}$$

Proof.
Divide the range intervals having the sizes, $d(u,w)$, $d(u,v)$, and $d(v,w)$, into sets of subintervals, each subinterval having the size, c.

$$\begin{aligned}(3.2)\quad & \forall\, c > 0,\ |y_1| = floor(d(u,v)/c) \ \wedge\ |y_2| = floor(d(v,w)/c) \quad \wedge \\ & d_c = floor(d(u,w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \leq |y_1| + |y_2| \\ & \Rightarrow\ floor(d(u,w)/c) \leq floor(d(u,v)/c) + floor(d(v,w)/c) \\ & \Rightarrow\ \lim_{c\to 0} floor(d(u,w)/c)\cdot c \leq \lim_{c\to 0} floor(d(u,v)/c)\cdot c + \lim_{c\to 0} floor(d(v,w)/c)\cdot c \\ & \Rightarrow\ d(u,w) \leq d(u,v) + d(v,w). \quad \square\end{aligned}$$

Theorem 3.4. *Non-negativity:* $d(u,w) \geq 0$.

Proof.
Divide the range intervals having the size, $d(u,w)$, into a set of $floor(d(u,w)/c)$ number of subintervals, each subinterval having the size, c. And apply the facts

that the number of members in every set is non-negative and the size, c, of every subinterval of an interval is non-negative.

$$\begin{aligned}(3.3)\quad \forall\, c > 0 : \quad & d_c = floor(d(u,w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \geq 0 \\ \Rightarrow \quad & floor(d(u,w)/c) = d_c \geq 0 \quad \Rightarrow \quad d(u,w) = \lim_{c \to 0} d_c \cdot c \geq 0. \quad \square\end{aligned}$$

THEOREM 3.5. *Identity of Indiscernibles:* $d(w,w) = 0$.

PROOF.

$$\begin{aligned}(3.4)\quad \forall\, d(u,v) = d(v,w) = 0 \;\wedge\; d(u,w) \leq d(u,v) + d(v,w) \;\wedge\; d(u,w) \geq 0 \\ \Rightarrow\; d(u,w) = 0.\end{aligned}$$

$$d(u,w) = 0 \;\wedge\; d(u,v) = 0 \;\Rightarrow\; w = v. \tag{3.5}$$

$$d(v,w) = 0 \;\wedge\; w = v \;\Rightarrow\; d(w,w) = 0. \tag{3.6}$$

$\square$

THEOREM 3.6. *Symmetry:* $d(v,w) = d(w,v)$.

PROOF.

$$w = v \;\Rightarrow\; d(w,w) = d(v,w) \;\wedge\; d(w,w) = d(w,v) \;\Rightarrow\; d(v,w) = d(w,v). \tag{3.7}$$

$\square$

3.3. Distance space range. Where the range sets intersect, multiple domain set members map to a single range set member. Therefore, the union set size, d_c, is function of the number of domain-to-range set member mappings.

The property, $|x_i| = |y_i| = p_i$, (3.1) constrains the range of domain-to-range set member mappings. Two facts are immediately obvious from the case, where $p_i = 1$: 1) Each of the p_i number of members of x_i map to *only one* member member of y_i, yielding $|x_i| \cdot 1 = p_i = 1$ number of domain-to-range mappings. 2) Each of the p_i number of members of x_i map to *each* member member of y_i, yielding $|x_i| \cdot |y_i| = p_i^2 = 1$ number of domain-to-range mappings.

Therefore, $\exists\, \mathbf{f} : d_c = \mathbf{f}(\sum_{i=1}^{n} p_i)$ is the largest possible distance because it is the case of the smallest number of domain-to-range mappings (no intersection of the range sets). And $\exists\, \mathbf{f} : d_c = \mathbf{f}(\sum_{i=1}^{n} p_i^2)$ is the smallest possible distance because it is the case of the largest number of domain-to-range mappings (largest allowed intersection of range sets). Applying the ruler (2.1) and ruler convergence theorem (2.2) to the largest and smallest countable distance cases yields the real-valued, Manhattan and Euclidean distance functions.

3.4. Manhattan distance.

THEOREM 3.7. *Manhattan (longest) distance, d, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals,* $\{[a_1,b_1],[a_2,b_2],\ldots,[a_n,b_n]\}$, *where:*

$$d = \textstyle\sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance."

PROOF.
From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$(3.8)\quad d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n}|y_i| \quad \wedge \quad |y_i| = p_i$$
$$\Rightarrow \quad d_c \leq \sum_{i=1}^{n}|y_i| = \sum_{i=1}^{n}p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c :\; d_c = \sum_{i=1}^{n}p_i.$$

Multiply both sides of equation 3.10 by c and take the limit:

$$(3.9)\quad d_c = \sum_{i=1}^{n}p_i \;\Rightarrow\; d_c \cdot c = \sum_{i=1}^{n}(p_i \cdot c) \;\Rightarrow\; \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c).$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$(3.10)\qquad d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d :\; floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of s_i:

$$(3.11)\quad \forall\, i \in [1,n],\; s_i = |a_i - b_i| \quad \Rightarrow \quad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i.$$

Combine equations 3.10, 3.9, 3.11:

$$(3.12)\quad d = \lim_{c\to 0} d_c \cdot c \quad \wedge \quad \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) \quad \wedge$$
$$\lim_{c\to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square$$

3.5. Euclidean distance.

THEOREM 3.8. *Euclidean (shortest) distance, d, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

PROOF.
Apply the rule of product to the largest number of domain-to-range set mappings, where all p_i number of domain set members, x_i, map to each of the p_i number of members in the range set, y_i:

$$(3.13)\qquad \sum_{i=1}^{n}|y_i| \cdot |x_i| = \sum_{i=1}^{n}p_i^2.$$

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$(3.14)\quad d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n}|y_i| \quad \wedge \quad |y_i| = p_i$$
$$\Rightarrow \quad d_c \leq \sum_{i=1}^{n}|y_i| = \sum_{i=1}^{n}p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c :\; d_c = \sum_{i=1}^{n}p_i.$$

Square both sides of equation 3.14 ($x = y \Leftrightarrow f(x) = f(y)$):

$$(3.15)\qquad \exists\, p_i,\, d_c :\; d_c = \sum_{i=1}^{n}p_i \quad \Leftrightarrow \quad \exists\, p_i,\, d_c :\; d_c^2 = (\sum_{i=1}^{n}p_i)^2.$$

Apply the Cauchy-Schwartz inequality to equation 3.15 and select the smallest distance (equality) case:

$$(3.16)\qquad d_c^2 = (\sum_{i=1}^{n}p_i)^2 \geq \sum_{i=1}^{n}p_i^2 \quad \Rightarrow \quad \exists\, p_i :\; d_c^2 = \sum_{i=1}^{n}p_i^2.$$

Multiply both sides of equation 3.16 by c^2, simplify, and take the limit.

$$(3.17)\quad d_c^2 = \sum_{i=1}^{n}p_i^2 \;\Rightarrow\; d_c^2 \cdot c^2 = \sum_{i=1}^{n}p_i^2 \cdot c^2 \;\Leftrightarrow\; (d_c \cdot c)^2 = \sum_{i=1}^{n}(p_i \cdot c)^2$$
$$\Rightarrow \quad \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$\exists\, c\, d: \; floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2. \tag{3.18}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$\forall\, i \in [1, n],\; floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i$$
$$\Rightarrow \quad \lim_{c\to 0}(p_i \cdot c)^2 = s_i^2 \quad \Rightarrow \quad \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \tag{3.19}$$

Combine equations 3.18, 3.17, 3.19:

$$d^2 = \lim_{c\to 0}(d_c \cdot c)^2 \quad \wedge \quad \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 \quad \wedge$$
$$\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2$$
$$\Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \quad \square \tag{3.20}$$

4. Euclidean Volume

The number of all possible combinations (all many-to-many mappings) between members in a countable set x_1 and a countable set x_2 is the Cartesian product, $|x_1| \cdot |x_2|$. This section will use the ruler (2.1) and ruler convergence theorem (2.2) to prove that the Cartesian product of the same-sized subintervals of intervals converges to the product of interval sizes as the subinterval size converges to zero. The first step is to define a countable set-based measure of area/volume as the Cartesian product (many-to-many mappings) of disjoint domain set members.

Definition 4.1. Countable volume measure, V_c:

$$\sum_{i=1}^{n} |x_i| = |\bigcup_{i=1}^{n} x_i|, \quad V_c = \prod_{i=1}^{n} |x_i|.$$

Theorem 4.2. *Euclidean volume, V, is the size of a range interval, $[v_0, v_m]$, corresponding to a set of disjoint intervals:*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:

$$V = \prod_{i=1}^{n} s_i, \quad V = |v_0 - v_m|, \quad s_i = |a_i - b_i|, \quad i \in [1, n], \quad i, n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_volume."

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, x_i of p_i number of subintervals.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i|. \tag{4.1}$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0}(p_i \cdot c) = s_i. \tag{4.2}$$

Use the ruler (2.1) to divide the exact size, $V = |v_0 - v_m|$, of the range interval, $[v_0, v_m]$, into p^n subintervals. Every integer number, V_c, does **not** have an integer n^{th} root. However, for those cases where V_c does have an integer n^{th} root, there is a p^n that satisfies the definition a countable volume measure, V_c (4.1). Notionally:

$$\forall\, p^n = V_c \in \mathbb{N},\, \exists\, V \in \mathbb{R},\, x_i \;: floor(V/c^n) = V_c = p^n = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i. \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3 and simplify:

$$floor(V/c^n) = p^n \quad \Rightarrow \quad V = \lim_{c\to 0} p^n \cdot c^n = \lim_{c\to 0} (p \cdot c)^n. \tag{4.4}$$

Multiply both sides of equation 4.3 by c^n and simplify:

$$p^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad p^n \cdot c^n = (\prod_{i=1}^{n} p_i) \cdot c^n \quad \Leftrightarrow \quad (p \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.5}$$

Combine equations 4.4, 4.5, and 4.2:

$$V = \lim_{c\to 0} (p \cdot c)^n \quad \wedge \quad (p \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c) \quad \wedge \quad \lim_{c\to 0}(p_i \cdot c) = s_i \tag{4.6}$$

$$\Rightarrow \quad V = \lim_{c\to 0} (p \cdot c)^n = \prod_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square$$

5. Ordered and symmetric geometries

Calculating the union and addition operations of distance and the union and multiplication operations of volume requires counting sequentially through each and every set. The commutative property of the union, addition, and multiplication also allows counting sequentially, where each set can be sequentially adjacent to any other set, herein referred to as a symmetric geometry.

But, a set can have only one sequential order *at a time* because each member can have at most one successor and at most one predecessor *at a time*. It will now be proved that a set (of sets) satisfying the constraints of a single sequencial order at a point in time and symmetry defines a cyclic set containing at most 3 members.

Definition 5.1. Ordered geometry:

$$\forall\, i\, n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i\ \in \{x_1, \ldots, x_n\},$$
$$successor\ x_i = x_{i+1}\ \ \wedge\ \ predecessor\ x_{i+1} = x_i.$$

Definition 5.2. Symmetric geometry (every set member is sequentially adjacent to any other member):

$$\forall\, i\, j\, n \in \mathbb{N},\ \ \forall\, x_i\, x_j \in \{x_1, \ldots, x_n\},\ \ successor\ x_i = x_j\ \ \wedge\ \ predecessor\ x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$successor\ x_n = x_1\ \ \wedge\ \ predecessor\ x_1 = x_n.$$

The theorem and formal Coq-based proof is "ordered_symmetric_is_cyclic," which is located in the file threed.v.

Proof. The property of order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ successor\ x_i = x_j\ \ \Rightarrow\ \ successor\ x_n = x_1. \tag{5.1}$$

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ predecessor\ x_j = x_i\ \ \Rightarrow\ \ predecessor\ x_1 = x_n. \quad \square \tag{5.2}$$

Theorem 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The Coq-based lemmas and proofs in the file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

DEFINITION 5.5. Cyclic successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (m + 1 \leq setsize). \tag{5.3}$$

DEFINITION 5.6. Cyclic predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{5.5}$$

Every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{5.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (1 + 1 \leq 2). \tag{5.7}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (2 = 2 \wedge 1 = 1). \tag{5.8}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (1 + 1 \leq 2). \tag{5.9}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (2 - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (3 = 3 \wedge 1 = 1). \tag{5.11}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (1 = 1 \wedge 3 = 3). \tag{5.12}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (2 + 1 \leq 3). \tag{5.13}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (3 - 1 \geq 1). \tag{5.14}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not adjacent:

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Successor(1, 3, setsize) \\ & \leftarrow Successor(1, 2, setsize) \leftarrow (1 + 1 \leq setsize). \end{aligned} \tag{5.15}$$

That is, 2 is the only successor of 1 for all $setsize > 3$, which implies 3 is not a successor of 1 for all $setsize > 3$.

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Predecessor(1, 3, setsize) \\ & \leftarrow Predecessor(1, n, setsize) \leftarrow (1 = 1 \wedge n = setsize). \end{aligned} \tag{5.16}$$

That is, $n = setsize$ is the only predecessor of 1 for all $setsize > 3$, which implies 3 is not a predecessor of 1 for all $setsize > 3$.

$$\forall\, setsize > 3 : \quad \neg Adjacent(1, 3, setsize) \leftarrow \neg Successor(1, 3, setsize) \wedge \neg Predecessor(1, 3, setsize). \quad \square \tag{5.17}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (violates the symmetry property).

6. Summary

Applying the ruler measure (2.1) and ruler convergence proof (2.2), to a set of real-valued domain intervals and a range interval yields the following new insights into geometry and physics.

(1) Distance is a function of the number of domain-to-*range* set member mappings. Area/volume is a function of the number of domain-to-*domain* set member mappings.
  (a) Other types of measure, like metric space, Borel, Hausdorff, and Lebesgue [**Gol76**] [**Rud76**], do not provide that insight.
  (b) Metric space allowing the non-domain-to-range set-based functions, area and volume, as metrics calls into question whether metric space is a sufficient condition for distance metrics.

(2) Applying the ruler measure to the countable distance space (3.1) provides the insight that all notions of distance are derived from the principle that for each domain set, x_i, there exists a corresponding range set, y_i, containing the same number of members, p_i: $|x_i| = |y_i| = p_i$ (3.1). For example, there should be as many steps walked in the range set, y_i, as there are pieces of traversed land in the corresponding domain set, x_i.

  And the union size depends on the amount of intersection of range sets: $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$.
  (a) Applying the ruler to the set relation, $|\bigcup_{i=1}^{2} y_i| \leq \sum_{i=1}^{2} |y_i|$, (3.1) generates all the metric space properties (3.2).
  (b) Where the range sets intersect, multiple domain set members can map to a single range set member. Therefore, distance is a function of domain-to-range set member mappings. The property, $|x_i| = |y_i| = p_i$, constrains the range of possible mappings from $\sum_{i=1}^{n} p_i$ to $\sum_{i=1}^{n} p_i^2$, which converge to Manhattan (largest possible) (3.7) distance and Euclidean (smallest possible) distance (3.8) respectively.
  (c) The type of constraint placed on the domain-to-range set member mappings yields three types of geometry: $|x_i| < |y_i|$, $|x_i| = |y_i|$, and $|x_i| > |y_i|$ (open, flat, and closed geometries).
  (d) A parametric variable relating the sizes of two domain intervals can be easily derived using using calculus (Taylor series) and the Euclidean distance equation to generate the arc sine and arc cosine functions of the parametric variable (arc angle). In other words, the notions of side and angle are derived from Euclidean distance, which is the reverse perspective of classical geometry [**Joy98**] [**Loo68**] [**Ber88**] and axiomatic foundations for geometry [**Bir32**] [**Hil80**] [**TG99**].

(3) Applying the ruler measure and ruler convergence proof to the countable volume definition (4.1) allows a proof that the Cartesian product of

same-sized subintervals of real-valued intervals converges to the product of the interval sizes (Euclidean length/area/volume)without notions of sides, angles, and shape:

(a) The Cartesian product is the largest possible number of domain-to-domain set member mappings (without over-counting). Therefore, all other types of volume-based measures must be functions of domain-to-domain set member mappings that are less than the Cartesian product number of mappings.

(b) Other types of measure, like Borel, Hausdorff, and Lebesgue [**Gol76**] [**Rud76**], do not provide those insights.

(4) A pure mathematician can extend the 2 and 3-dimensional Euclidean distance equations to as many dimensions as desired. But, the applied mathematian, engineer, and scientist must live within the constraints of the principles generating physical Euclidean geometry. Specifically, the constraints imposed by the ordered (5.1) and symmetric (5.2) properties of the set-based distance and volume relations limit an Euclidean geometry to 3 dimensions (5.4).

(5) The proof showing that more than 3 dimensions of geometric space would lead to contradictions (5.4) constrains all higher dimensional theories of physics to *hierarchical* 2 or 3-dimensional geometries. For example, the four-vectors common in physics [**Bru17**] are hierarchical, 2-dimensional geometries that have been "flattened."

The spacetime four-vector length, $d = \sqrt{(ct)^2 - (x^2 + y^2 + z^2)}$, where c is the speed of light and t is time, can be expressed in a form like, $(ct)^2 = d_1^2 + d_2^2$, where $d_1^2 = x^2 + y^2 + z^2$ and $d_2 = d$. Likewise, the energy-momentum four-vector has the 2-dimensional form: $E^2 = (mv^2)^2 + (pc)^2$, where E is energy, m is the resting mass, v is the *3-dimensional* velocity, c is the speed of light, and p is the relativistic momentum ($p = \gamma mv$, where $\gamma = (1/(1 - (v/c)^2))^{1/2}$ is the Lorentz factor).

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George Van Treeck, 668 Westline Dr., Alameda, CA 94501