The Two Set Relations Generating Euclidean Geometry

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Abstract. A ruler-like measure divides both domain and range intervals into countable sets of same-sized subintervals, ignoring partial subintervals. As the subinterval size converges to zero: 1) Distance as the union size of range sets, where for each domain set there exists a corresponding same-sized range set, converges to: the triangle inequality with Manhattan distance at the upper boundary and Euclidean distance at the lower boundary. 2) The Cartesian product of the number of members in each domain set converges to the product of interval interval sizes (Euclidean area/volume). All proofs are verified in Coq.

Contents

1. Introduction

Triangle inequality, distance non-negativity, Euclidean distance, and Euclidean area/volume are motivated by Euclidean geometry and used as primitives in real analysis (metric space, Hilbert spaces, Hausdorff and Lebesgue measures, and Riemann and Lebesgue integration) [Gol76] [Rud76]. A "ruler" measure is introduced and used to prove that these geometric relations are derived from two countable set relations.

For example, historically, the size of the Cartesian product of all values from a set of disjoint intervals is defined as the product of interval sizes (Euclidean area/volume). The ruler measure allows a short and simple proof.

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The derivation of geometric relations from set relations (*without notions of plane, line, angle, etc.*) provides some new insights. For example: 1) the single set relation generating all the properties of metric space. 2) the set relation that makes Euclidean distance the smallest possible distance between two distinct points in $\mathbb{R}^n$; 3) the mapping between sets that makes distance different from area/volume; 4) how time places an additional constraint on physical sets, which constrains physical Euclidean geometry to 3 dimensions.

To give the reader confidence that the proofs in this article are correct, all the proofs have corresponding formal proofs in the Coq files, "euclidrelations.v" and "threed.v," located at: `https://github.com/treeck/RASRGeometry`. Mathematicians all over the world use Coq [**Coq15**] to verify their proofs because proofs accepted by the Coq logic engine have a high probability of being correct.

2. Ruler measure and convergence

A ruler (measuring stick) partitions both domain and range intervals to the nearest integer number of same-sized subintervals, where the partial subintervals are ignored. In contrast, Riemann and Lebesgue integrals partition the domain intervals *exactly*, where each domain subinterval and corresponding range subinterval generally differ in size [**Rud76**]. The ruler measure allows counting the number of mappings, ranging from a one-to-one correspondence to a many-to-many mapping, between the set of same-sized subintervals in one interval and the set of same-sized subintervals in another interval. The mapping (combinatorial) relations converge to continuous, bijective relations as the subinterval size converges to zero.

Definition 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, p, each subinterval having the same size, c. Notionally:

$$\forall\, c\, s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |a-b|\ \ \wedge\ \ c > 0\ \ \wedge \\ (p = floor(s/c)\ \ \vee\ \ p = ceiling(s/c))\ \ \wedge\ \ M = \textstyle\sum_{i=1}^{p} c = pc. \tag{2.1}$$

Theorem 2.2. *Ruler convergence:*
$\forall\ [a,b] \subset \mathbb{R},\ \ s = |a-b|\ \ \Rightarrow\ \ M = \lim_{c\to 0} pc = s.$

The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\, p = floor(s/c)\ \wedge\ 0 \le |floor(s/c) - s/c| < 1\ \Rightarrow\ 0 \le |p - s/c| < 1. \tag{2.2}$$

Multiply all sides of inequality 2.2 by $|c|$:

$$\forall\, c > 0,\quad 0 \le |p - s/c| < 1\quad \Rightarrow\quad 0 \le |pc - s| < |c|. \tag{2.3}$$

$$\forall\, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \\ \Rightarrow\quad \forall\, \epsilon = \delta :\ \ |c-0| < \delta\ \ \wedge\ \ |pc - s| < \epsilon\ \ :=\ \ M = \lim_{c\to 0} pc = s. \quad \square \tag{2.4}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence, where: $[0,\pi]$, $s = |0-\pi|$, $c = 10^{-i}$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{i=1},\ 3.14_{i=2},\ 3.141_{i=3}, ..., \pi$.

3. Distance

Notation convention: Curly brackets, $\{\cdots\}$, delimit a set; square brackets, $[\cdots]$, delimit a list; and vertical bars around a set or list indicates, $|\{\cdots\}|$, the cardinal (number of members in the set or list).

3.1. Countable distance space. The most primitive notion of distance is the number of steps walked to some destination. Abstracting, distance is the cardinal of the range set, y_i, (for example, the number of steps walked) which equals the cardinal of a corresponding domain set, x_i, (for example, the number of corresponding pieces of land): $|x_i| = |y_i|$. If the distance spanning one domain set is the cardinal of the range set, then the distance spanning disjoint domain sets, $\bigcap_{i=1}^n x_i = \emptyset$, is the cardinal of the union range set, $d_c = |\bigcup_{i=1}^n y_i|$.

Definition 3.1. Countable distance space, d_c:

$$\bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad d_c = |\bigcup_{i=1}^n y_i| \quad \wedge \quad |x_i| = |y_i|.$$

Theorem 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$.

This well-known inequality follows directly from the inclusion-exclusion principle [CG15]. But, a more intuitive and simple proof follows from the sum of the set sizes being equal to number of unique members (the union set) plus the number of duplicate (intersection) members. For example, $|\{a,b,c\}| + |\{c,d,e\}| = |[a,b,c,c,d,e]| = |\{a,b,c,d,e\}| + |[c]| = 6 \quad \Rightarrow \quad |\{a,b,c,d,e\}| = |\{a,b,c\}| + |\{c,d,e\}| - |[c]| = 5$.

Proof. More generally:

$$|\bigcup_{i=1}^n y_i| + |duplicates_{i=1}^n y_i| = \sum_{i=1}^n |y_i| \quad \wedge \quad |duplicates_{i=1}^n y_i| \geq 0 \tag{3.1}$$
$$\Rightarrow \quad |\bigcup_{i=1}^n y_i| = \sum_{i=1}^n |y_i| - |duplicates_{i=1}^n y_i| \quad \wedge \quad |duplicates_{i=1}^n y_i| \geq 0$$
$$\Rightarrow \quad |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|. \quad \square$$

A formal proof, inclusion_exclusion_inequality, using sorting into unique members (union set) and duplicate members, is in the file euclidrelations.v.

3.2. Metric Space. Applying the ruler (2.1) and ruler convergence (2.2) to three range intervals having sizes: $d(u,w)$, $d(u,v)$, $d(v,w)$, and using the inequality, $d_c = |\bigcup_{i=1}^2 y_i| \leq \sum_{i=1}^2 |y_i|$, generates the properties of metric space. The formal proofs: triangle_inequality, non_negativity, identity_of_indiscernibles, and symmetry, are in the Coq file, euclidrelations.v.

Theorem 3.3. *Triangle Inequality:* $d(u,w) \leq d(u,v) + d(v,w)$.

Proof.

$$\forall\, c > 0, \;\; |y_1| = floor(d(u,v)/c) \;\; \wedge \;\; |y_2| = floor(d(v,w)/c) \;\; \wedge \tag{3.2}$$
$$d_c = floor(d(u,w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \leq |y_1| + |y_2|$$
$$\Rightarrow \;\; floor(d(u,w)/c) \leq floor(d(u,v)/c) + floor(d(v,w)/c)$$
$$\Rightarrow \;\; floor(d(u,w)/c)\cdot c \leq floor(d(u,v)/c)\cdot c + floor(d(v,w)/c)\cdot c$$
$$\Rightarrow \;\; \lim_{c\to 0} floor(d(u,w)/c)\cdot c \leq \lim_{c\to 0} floor(d(u,v)/c)\cdot c + \lim_{c\to 0} floor(d(v,w)/c)\cdot c$$
$$\Rightarrow \;\; d(u,w) \leq d(u,v) + d(v,w). \quad \square$$

Theorem 3.4. *Non-negativity:* $d(u,w) \geq 0$.

Proof.

$$\begin{aligned} \forall\, c > 0 : \quad & d_c = floor(d(u,w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \geq 0 \\ \Rightarrow \quad & floor(d(u,w)/c) = d_c \geq 0 \quad \Rightarrow \quad d(u,w) = \lim_{c \to 0} d_c \cdot c \geq 0. \quad \square \end{aligned} \tag{3.3}$$

Theorem 3.5. *Identity of Indiscernibles:* $d(w,w) = 0$.

Proof.

$$\begin{aligned} \forall\, d(u,v) = d(v,w) = 0 \ \wedge \ d(u,w) \leq d(u,v) + d(v,w) \ \wedge \ d(u,w) \geq 0 \\ \Rightarrow \ d(u,w) = 0. \end{aligned} \tag{3.4}$$

$$d(u,w) = 0 \ \wedge \ d(u,v) = 0 \ \Rightarrow \ w = v. \tag{3.5}$$

$$d(v,w) = 0 \ \wedge \ w = v \ \Rightarrow \ d(w,w) = 0. \quad \square \tag{3.6}$$

Theorem 3.6. *Symmetry:* $d(v,w) = d(w,v)$.

Proof.

$$w = v \ \Rightarrow \ d(w,w) = d(v,w) \ \wedge \ d(w,w) = d(w,v) \ \Rightarrow \ d(v,w) = d(w,v). \quad \square \tag{3.7}$$

3.3. Distance space range. Where the range sets intersect, multiple domain set members map to a single range set member. Therefore, the union set size, d_c, is function of the number of domain-to-range set member mappings.

The property, $|x_i| = |y_i| = p_i$, (3.1) constrains the range of domain-to-range set member mappings. Two facts are immediately obvious from the case, where $p_i = 1$: 1) Each of the p_i number of members in x_i corresponds 1-1 (bijective) to a single, unique member in y_i, yielding $|x_i| \cdot 1 = p_i = 1$ number of domain-to-range mappings. 2) Each of the p_i number of members in x_i map to *each* of the p_i number of members in y_i, yielding $|x_i| \cdot |y_i| = p_i^2 = 1$ number of domain-to-range mappings.

Therefore, $d_c = \sum_{i=1}^{n} p_i$ is the largest possible distance because it is the case of the smallest number of domain-to-range mappings (no intersection of the range sets). And $\exists\ \mathbf{f} : d_c = \mathbf{f}(\sum_{i=1}^{n} p_i^2)$ is the smallest possible distance because it is the case of the largest number of domain-to-range mappings (largest allowed intersection of range sets). Applying the ruler (2.1) and ruler convergence theorem (2.2) to the largest and smallest countable distance cases yields the real-valued, Manhattan and Euclidean distance functions.

3.4. Manhattan distance.

Theorem 3.7. *Manhattan (longest) distance, d, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "taxicab_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$(3.8)\quad d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |y_i| = p_i \quad \Rightarrow \quad d_c \leq \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i.$$

Multiply both sides of equation 3.10 by c and take the limit:

$$(3.9)\quad d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \Rightarrow \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c).$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$(3.10)\qquad d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d :\ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of s_i:

$$(3.11)\qquad \forall\, i \in [1, n],\ s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i.$$

Combine equations 3.10, 3.9, 3.11:

$$(3.12)\quad d = \lim_{c\to 0} d_c \cdot c \quad \wedge \quad \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) \quad \wedge \quad \lim_{c\to 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square$$

3.5. Euclidean distance.

THEOREM 3.8. *Euclidean (shortest) distance, d, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "Euclidean_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
Apply the rule of product to the largest number of domain-to-range set mappings, where all p_i number of domain set members, x_i, map to each of the p_i number of members in the range set, y_i:

$$(3.13)\qquad \sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2.$$

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$(3.14)\quad d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |y_i| = p_i \quad \Rightarrow \quad d_c \leq \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i.$$

Square both sides of equation 3.14 ($x = y \Leftrightarrow f(x) = f(y)$):

$$(3.15)\qquad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\ d_c :\ d_c^2 = (\sum_{i=1}^{n} p_i)^2.$$

Apply the Cauchy-Schwartz inequality to equation 3.15 and select the smallest distance (equality) case:

$$(3.16)\qquad d_c^2 = (\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2 \quad \Rightarrow \quad \exists\, p_i :\ d_c^2 = \sum_{i=1}^{n} p_i^2.$$

Multiply both sides of equation 3.16 by c^2, simplify, and take the limit.

$$(3.17)\quad d_c^2 = \sum_{i=1}^{n} p_i^2 \Rightarrow d_c^2 \cdot c^2 = \sum_{i=1}^{n} p_i^2 \cdot c^2 \Leftrightarrow (d_c \cdot c)^2 = \sum_{i=1}^{n} (p_i \cdot c)^2 \quad \Rightarrow \quad \lim_{c\to 0} (d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$\exists\, c\, d:\ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2. \tag{3.18}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$\forall\, i \in [1,n],\ s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i. \tag{3.19}$$

Combine equations 3.18, 3.17, 3.19:

$$d^2 = \lim_{c\to 0}(d_c \cdot c)^2 \quad \wedge \quad \lim_{c\to 0}(d_c \cdot c)^2 = \textstyle\sum_{i=1}^n \lim_{c\to 0}(p_i \cdot c)^2 \quad \wedge \tag{3.20}$$

$$\lim_{c\to 0} p_i{\cdot}c = s_i \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c{\cdot}c)^2 = \textstyle\sum_{i=1}^n \lim_{c\to 0}(p_i{\cdot}c)^2 = \sum_{i=1}^n s_i^2. \quad \square$$

4. Euclidean Volume

Where a combination (n-tuple) of one member from each disjoint domain set corresponds 1-1 to a range set member, the size of the range set is the Cartesian product of the number members in each domain set. Notionally:

Definition 4.1. All Possible Combinations, V_c:

$$\textstyle\bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad V_c = \prod_{i=1}^n |x_i|.$$

Theorem 4.2. *Euclidean volume, V, is size of the range interval, $[v_0, v_m]$, corresponding to all the possible combinations of the members of disjoint domain intervals, $\{[a_1,b_1],[a_2,b_2],\ldots,[a_n,b_n]\}$. Notionally:*

$$V = \textstyle\prod_{i=1}^n s_i,\ \ V = |v_0 - v_m|,\ \ s_i = |a_i - b_i|.$$

The theorem, "Euclidean_volume," and formal proof is in the Coq file, euclidrelations.v.

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, x_i of p_i number of subintervals.

$$\forall\, i\, n \in \mathbb{N}, \quad i \in [1,n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i|. \tag{4.1}$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0}(p_i \cdot c) = s_i. \tag{4.2}$$

Use the ruler (2.1) to divide the exact size, $V = |v_0 - v_m|$, of the range interval, $[v_0, v_m]$, into p^n subintervals. Use those cases, where V_c has an integer n^{th} root.

$$\forall\, p^n = V_c \in \mathbb{N}, \exists\, V \in \mathbb{R},\, x_i\ :floor(V/c^n) = V_c = p^n = \textstyle\prod_{i=1}^n |x_i| = \prod_{i=1}^n p_i. \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3 and simplify:

$$floor(V/c^n) = p^n \quad \Rightarrow \quad V = \lim_{c\to 0} p^n{\cdot}\, c^n = \lim_{c\to 0}(p \cdot c)^n. \tag{4.4}$$

Multiply both sides of equation 4.3 by c^n and simplify:

$$\begin{aligned} p^n = \textstyle\prod_{i=1}^n p_i \quad \Rightarrow \quad p^n \cdot c^n = (\prod_{i=1}^n p_i) \cdot c^n \quad \Leftrightarrow \quad (p \cdot c)^n = \prod_{i=1}^n (p_i \cdot c) \\ \Rightarrow \quad \lim_{c\to 0}(p \cdot c)^n = \textstyle\prod_{i=1}^n \lim_{c\to 0}(p_i \cdot c) \end{aligned} \tag{4.5}$$

Combine equations 4.4, 4.5, and 4.2:

$$V = \lim_{c\to 0}(p \cdot c)^n \quad \wedge \quad \lim_{c\to 0}(p \cdot c)^n = \textstyle\prod_{i=1}^n \lim_{c\to 0}(p_i \cdot c) \quad \wedge \tag{4.6}$$

$$\lim_{c\to 0}(p_i{\cdot}c) = s_i \quad \Rightarrow \quad V = \lim_{c\to 0}(p{\cdot}c)^n = \textstyle\prod_{i=1}^n \lim_{c\to 0}(p_i{\cdot}c) = \prod_{i=1}^n s_i. \quad \square$$

5. Ordered and symmetric geometries

The set operations of countable distance range (3.1) and all possible combinations (4.1) requires sequencing through each set. The commutative property of the set operations also allows sequencing, where each set can be sequentially adjacent to any other set, herein referred to as a symmetric geometry.

From a combinatoric perspective, there are $n!$ number of sequential arrangements of any n number of sets, where there are two arrangements having a set, x_i, that is sequentially adjacent (once as a predecessor and once as a successor) to any set, x_j. Where all arrangements exist, the properties of sequential order and symmetry are satisfied for any n number of sets (dimensions).

But, time places an additional constraint on physical sets. A physical set can have only one sequential order *at a time* because each set member can have at most one successor and at most one predecessor *at a time*. It will now be proved that a physical set satisfying the constraints of a single sequential (total) order and symmetric *at the same time* defines a cyclic set containing at most 3 members (in this case, 3 dimensions of physical space).

Definition 5.1. Ordered geometry:

$$\forall\, i\, n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i\, \in \{x_1, \ldots, x_n\},$$
$$\textit{successor}\ x_i = x_{i+1}\ \ \wedge\ \ \textit{predecessor}\ x_{i+1} = x_i.$$

Definition 5.2. Symmetric geometry (every set member is sequentially adjacent to any other member):

$$\forall\, i\, j\, n \in \mathbb{N},\;\; \forall\, x_i\, x_j \in \{x_1, \ldots, x_n\},\;\; \textit{successor}\ x_i = x_j\ \ \wedge\ \ \textit{predecessor}\ x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$\textit{successor}\ x_n = x_1\ \ \wedge\ \ \textit{predecessor}\ x_1 = x_n.$$

The theorem, "ordered_symmetric_is_cyclic," and formal proof is in the Coq file, threed.v.

Proof. The property of order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \textit{successor}\ x_i = x_j\ \ \Rightarrow\ \ \textit{successor}\ x_n = x_1. \tag{5.1}$$

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \textit{predecessor}\ x_j = x_i\ \ \Rightarrow\ \ \textit{predecessor}\ x_1 = x_n. \tag{5.2}$$

□

Theorem 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

DEFINITION 5.5. Cyclic successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (m + 1 \leq setsize). \tag{5.3}$$

DEFINITION 5.6. Cyclic predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{5.5}$$

Every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{5.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (1 + 1 \leq 2). \tag{5.7}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (2 = 2 \wedge 1 = 1). \tag{5.8}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (1 + 1 \leq 2). \tag{5.9}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (2 - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (3 = 3 \wedge 1 = 1). \tag{5.11}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (1 = 1 \wedge 3 = 3). \tag{5.12}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (2 + 1 \leq 3). \tag{5.13}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (3 - 1 \geq 1). \tag{5.14}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\forall\ setsize > 3: \quad \neg Successor(1, 3, setsize) \\ \leftarrow Successor(1, 2, setsize) \leftarrow (1 + 1 \leq setsize). \tag{5.15}$$

That is, 2 is the only successor of 1 for all $setsize > 3$, which implies 3 is not a successor of 1 for all $setsize > 3$.

$$\forall\ setsize > 3: \quad \neg Predecessor(1, 3, setsize) \\ \leftarrow Predecessor(1, n, setsize) \leftarrow (1 = 1 \wedge n = setsize). \tag{5.16}$$

That is, $n = setsize$ is the only predecessor of 1 for all $setsize > 3$, which implies 3 is not a predecessor of 1 for all $setsize > 3$.

$$\forall\ setsize > 3: \quad \neg Adjacent(1, 3, setsize) \\ \leftarrow \neg Successor(1, 3, setsize) \wedge \neg Predecessor(1, 3, setsize). \quad \square \tag{5.17}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (violates the symmetry property).

6. New Insights and open questions

Applying the ruler measure (2.1) and ruler convergence (2.2) to the set relations, countable distance space (3.1) and all possible combinations (4.1) yields the following insights and open questions:

(1) Notions of point, plane, side, angle, perpendicular, congruence, intersection, etc. are all completely unnecessary to derive the properties of metric space, the Euclidean distance equation, and the area/volume equations.

(2) All notions of distance are derived from the principle that every domain set, x_i, has corresponding range (distance) set, y_i, containing the same number of elements: $|x_i| = |y_i|$. And the distance spanning domain sets is the cardinal, d_c, of the corresponding union range set: $d_c = |\bigcup_{i=1}^{n} y_i|$ (3.1).

   (a) A direct consequence of the inclusion-exclusion principle [**CG15**] is the set relation, $d_c = |\bigcup_{i=1}^{2} y_i| \leq \sum_{i=1}^{2} |y_i|$ (3.2), which generates all the properties of metric space (3.2).

   (b) Where range sets intersect, multiple domain set members may map to a single range set member. $|x_i| = |y_i| = p_i \Rightarrow \sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^2$ is the largest possible number of domain-to-range set member mappings (largest allowed intersection) and converging to Euclidean distance (3.8) is the set-based reason that Euclidean distance is the smallest possible distance between two distinct points in $\mathbb{R}^n$.

   (c) Using the Taylor series and the Euclidean distance equation with two domain intervals sizes yields the arc sine and arc cosine functions. In other words, the parametric variable equating arc sine and arc cosine maps to the notion of angle, where the two domain intervals map to the notion of two line segments (two sides).
   Euclidean geometry [**Joy98**] and axiomatic geometry (for example, Hilbert [**Hil80**] and Birkhoff [**Bir32**], Veblen [**Veb04**], and Tarski [**TG99**]) either use notions of line and angle as undefined primitives or as definitions in terms of other undefined primitives.

   (d) Conjecture: the constraints: $|x_i| < |y_i|$, $|x_i| = |y_i|$, and $|x_i| > |y_i|$ yields three types of distance spaces: open, flat, and closed.

(3) Euclidean volume was derived, where a combination (n-tuple) of one member from each countable disjoint domain set corresponds 1-1 to a range set member and where the size of the range set is the Cartesian product of the number members in each domain set. Obviously, each domain set member and each range set member correspond to the geometric notion point. And each n-tuple is a Cartesian coordinate.

   (a) Euclidean volume has as many range set elements, V_c, as n-tuples, T_c. Conjecture: The constraints: $T_c < V_c$, $T_c = V_c$, and $T_c > V_c$ yields three types of volume spaces: open, flat, and closed.

   (b) Conjecture: Open, flat, and closed volume spaces correspond 1-1 to open, flat, and closed distance spaces.

(4) Distance is a function of the number of domain-to-*range* set member mappings. In contrast, area/volume is a function of the number of domain-to-*domain* set member mappings.

(5) Time constrains physical, ordered, and symmetric sets, where more than 3 members (dimensions) would lead to contradictions (5.4).

(a) Conjecture: All higher dimensional physics theories must be *hierarchical* 2 or 3-dimensional geometries with at most 3 dimensions of physical space. As shown below, the four-vectors common in physics [**Bru17**] are hierarchical, 2-dimensional geometries.

(b) The spacetime four-vector length, $d = \sqrt{(ct)^2 - (x^2 + y^2 + z^2)}$, where c is the speed of light and t is time, can be expressed in the 2-dimensional form, $(ct)^2 = d_1^2 + d_2^2$, where $d_1^2 = x^2 + y^2 + z^2$ and $d_2 = d$. Likewise, the energy-momentum four-vector has the 2-dimensional form: $E^2 = (mv^2)^2 + (pc)^2$, where E is energy, m is the resting mass, v is the *3-dimensional* velocity, c is the speed of light, and p is the relativistic momentum ($p = \gamma mv$, where $\gamma = (1/(1-(v/c)^2))^{1/2}$ is the Lorentz factor).

References

[Bir32] G. D. Birkhoff, *A set of postulates for plane geometry (based on scale and protractors)*, Annals of Mathematics **33** (1932). ↑9

[Bru17] P. Bruskiewich, *A very simple introduction to special relativity: Part two - four vectors, the lorentz transformation and group velocity (the new mathematics for the millions book 38)*, Pythagoras Publishing, 2017. ↑10

[CG15] W. Conradie and V. Goranko, *Logic and discrete mathematics*, Wiley, 2015. ↑3, 9

[Coq15] Coq, *Coq proof assistant*, 2015. https://coq.inria.fr/documentation. ↑2

[Gol76] R. R. Goldberg, *Methods of real analysis*, John Wiley and Sons, 1976. ↑1

[Hil80] D. Hilbert, *The foundations of geometry (2cd ed)*, Chicago: Open Court, 1980. http://www.gutenberg.org/ebooks/17384. ↑9

[Joy98] D. E. Joyce, *Euclid's elements*, 1998. http://aleph0.clarku.edu/~djoyce/java/elements/elements.html. ↑9

[Rud76] W. Rudin, *Principles of mathematical analysis*, McGraw Hill Education, 1976. ↑1, 2

[TG99] A. Tarski and S. Givant, *Tarski's system of geometry*, The Bulletin of Symbolic Logic **5** (1999), no. 2, 175–214. ↑9

[Veb04] O. Veblen, *A system of axioms for geometry*, Trans. Amer. Math. Soc **5** (1904), 343–384. https://doi.org/10.1090/S0002-9947-1904-1500678-X. ↑9

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