The Two Set Relations Generating Geometry

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Abstract. Each domain set has a corresponding range set. Distance is the cardinal of the union of the range sets and volume is the cardinal of the Cartesian product of disjoint range sets. A ruler (measuring stick) partitions intervals approximately into sets of size c subintervals and sums the sizes. The distance and volume set operations on the sets of subintervals converge to the properties of metric space, the Lp norms (like Manhattan and Euclidean distance), and Euclidean volume as c goes to 0. The volume proof allow simpler derivations of Coulomb's charge force and Newton's gravity force equations without using other laws of physics or Gauss's divergence theorem. Spacetime limits physical distance and volume to 3 dimensions. All proofs are verified in Coq.

Contents

1. Introduction

Metric space, the Euclidean distance metric/vector norm, and Euclidean area/volume in the Lebesgue, Borel, and Hausdorff measures have been definitions [**Gol76**] [**Rud76**] motivated by Euclidean geometry. These measures do not separate the measure of interval length from the measures of distance and area/volume.

In this article, each domain set has a corresponding range set. Distance is the cardinal of the union of the range sets and volume is the cardinal of the Cartesian product of disjoint range sets. A ruler (measuring stick) measures interval lengths by partitioning all domain and range intervals approximately into sets of size c

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subintervals and summing the sizes. The distance and volume set operations on the sets of subintervals converge to the properties of metric space, the L^p norms (in particular, Manhattan and Euclidean distance), and Euclidean volume as $c \to 0$.

The derivations of metric space, distance equations, and volume from the set operations provide some insights into geometry and physics, for example: the constraint on the union operation generating the symmetry property of metric space; the domain-to-range set mapping that makes Euclidean distance the smallest possible distance between two distinct points in $\mathbb{R}^n$; the constraints on domain and range sets that generate both Manhattan distance and Euclidean volume; the Euclidean distance and volume proofs allow deriving Coulomb's charge force, Newton's gravity force, and spacetime equations without using other laws of physics or Gauss's divergence theorem; the proportionate interval principle that generates the inverse square law; how spacetime places an additional constraint on physical sets, which limits physical distance and volume to 3 dimensions.

All the proofs in this article have been formally verified using using the Coq proof verification system [**Coq15**]. The formal proofs are located in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

DEFINITION 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, p, each subinterval having the same size, c. Notionally:

$$\forall\, c,\ s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |a-b| \ \wedge\ c > 0\ \wedge \\ (p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \textstyle\sum_{i=1}^{p} c = pc. \tag{2.1}$$

THEOREM 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\ \ s = |a-b| \ \Rightarrow\ M = \lim_{c\to 0} pc = s.$

The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\ p = floor(s/c)\ \wedge\ 0 \le |floor(s/c) - s/c| < 1\ \Rightarrow\ 0 \le |p - s/c| < 1. \tag{2.2}$$

Multiply all sides of inequality 2.2 by $|c|$:

$$\forall\, c > 0,\quad 0 \le |p - s/c| < 1 \quad\Rightarrow\quad 0 \le |pc - s| < |c|. \tag{2.3}$$

$$\forall\, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \\ \Rightarrow\quad \forall\, \epsilon = \delta :\ \ |c - 0| < \delta\ \wedge\ |pc - s| < \epsilon\ \ := \ \ M = \lim_{c\to 0} pc = s. \quad \square \tag{2.4}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence for the interval, $[0,\pi]$: $s = |0 - \pi|$, and $p = floor(s/c) \Rightarrow p\ \cdot c = 3.1_{c=10^{-1}},\ 3.14_{c=10^{-2}},\ 3.141_{c=10^{-3}}, \ldots, \pi_{\lim_{c\to 0}}$.

3. Distance

Notation convention: Vertical bars around a set or list, $|\cdots|$, indicates the cardinal (number of members in the set or list).

3.1. Countable distance. A concrete example of a countable distance is the number of same-sized steps walked in a range (distance) set, y, which equals the number of same-sized pieces of land in a corresponding domain set, x: $|x| = |y|$. Generalizing, each disjoint domain set, x_i, has a corresponding range (distance) set, y_i. The countable distance spanning the disjoint domain sets is the number of members, d_c, in the union range set:

DEFINITION 3.1. Countable distance, d_c :

$$d_c = |\bigcup_{i=1}^{n} y_i|, \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i|^q,\ q \geq 0.$$

THEOREM 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$.

The inclusion-exclusion inequality follows from the inclusion-exclusion principle [**CG15**]. But, a more intuitive and simple proof follows from the associative law of addition where the sum of set sizes is equal to the size of all the set members appended into a list and the commutative law of addition that allows sorting that list into a list of unique members (the *union* set) and a list of duplicates. The duplicates being ≥ 0 implies the union size is always $\leq$ the sum of set sizes.

A formal proof, inclusion_exclusion_inequality, using sorting into a set of unique members (*union* set) and a list of duplicates, is in the file euclidrelations.v.

PROOF.

$$\sum_{i=1}^{n} |y_i| = |append_{i=1}^{n} y_i| = |sort(append_{i=1}^{n} y_i)| = |\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i|. \tag{3.1}$$

$$|\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| \quad \wedge \quad |duplicates_{i=1}^{n} y_i| \geq 0 \quad \Rightarrow \quad |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|. \quad \square \tag{3.2}$$

3.2. Metric Space. All function range intervals, $d(u, w)$, satisfying the countable distance space definition, $d_c = |\bigcup_{i=1}^{n} y_i|$, where the ruler is applied, generates three of the four metric space properties: triangle inequality, non-negativity, and identity of indiscernibles. The set-based reason for the fourth property of metric space, symmetry $[d(u, v) = d(v, u)]$, will be identified in the last section of this article. The formal proofs: triangle_inequality, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 3.3. *Triangle Inequality:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \leq d(u, v) + d(v, w)$.

PROOF. Apply the ruler measure (2.1), the countable distance condition (3.1), inclusion-exclusion inequality (3.2), and then ruler convergence (2.2).

$$\begin{aligned} &\forall\, c > 0,\ d(u, w),\ d(u, v),\ d(v, w)\ : \\ &\quad |y_1| = floor(d(u, v)/c) \ \wedge\ |y_2| = floor(d(v, w)/c) \quad \wedge \\ &\quad d_c = floor(d(u, w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \leq |y_1| + |y_2| \\ &\quad \Rightarrow\ floor(d(u, w)/c) \leq floor(d(u, v)/c) + floor(d(v, w)/c). \\ &\quad \Rightarrow\ floor(d(u, w)/c) \cdot c \leq floor(d(u, v)/c) \cdot c + floor(d(v, w)/c) \cdot c \\ &\Rightarrow \lim_{c \to 0} floor(d(u, w)/c) \cdot c \leq \lim_{c \to 0} floor(d(u, v)/c) \cdot c + \lim_{c \to 0} floor(d(v, w)/c) \cdot c \\ &\quad \Rightarrow\ d(u, w) \leq d(u, v) + d(v, w). \quad \square \end{aligned} \tag{3.3}$$

THEOREM 3.4. *Non-negativity:* $d_c = |y_1 \cup y_2| \Rightarrow d(u,w) \geq 0$.

PROOF. By definition, a set always has a size (cardinal) ≥ 0:

$$\forall\, c > 0,\ d(u,w)\ :\quad floor(d(u,w)/c) = d_c \quad \wedge \quad d_c = |y_1 \cup y_2| \geq 0 \tag{3.4}$$
$$\Rightarrow \quad floor(d(u,w)/c) = d_c \geq 0 \quad \Rightarrow \quad d(u,w) = \lim_{c \to 0} d_c \cdot c \geq 0. \quad \square$$

THEOREM 3.5. *Identity of Indiscernibles:* $d(w,w) = 0$.

PROOF. Apply the triangle inequality property (3.3):

$$\forall\, d(u,v) = d(v,w) = 0 \ \wedge\ d(u,w) \leq d(u,v) + d(v,w) \ \Rightarrow\ d(u,w) \leq 0. \tag{3.5}$$

Combine the non-negativity property (3.4) and the previous inequality (3.5):

$$d(u,w) \geq 0 \ \wedge\ d(u,w) \leq 0 \ \Leftrightarrow\ 0 \leq d(u,w) \leq 0 \ \Rightarrow\ d(u,w) = 0. \tag{3.6}$$

Combine the result of step 3.6 and the condition, $d(u,v) = 0$, in step 3.5.

$$d(u,w) = 0 \ \wedge\ d(u,v) = 0 \ \Rightarrow\ w = v. \tag{3.7}$$

Combine the condition, $d(v,w) = 0$, in step 3.5 and the result of step 3.7.

$$d(v,w) = 0 \ \wedge\ w = v \ \Rightarrow\ d(w,w) = 0. \tag{3.8}$$
$\square$

3.3. Distance space range. From the countable distance definition, $d_c = |\bigcup_{i=1}^{n} y_i|$, as the amount of intersection increases, more domain set members can map to a single range set member. Therefore, the number of domain-to-range set member mappings is a function of the amount of range set intersection.

From the countable distance property (3.1), where $|x_i| = |y_i| = p_i$, the range of possible domain-to-range set member mappings is the number of one-to-one (bijective), $1 \cdot |y_i| = p_i$, mappings, to the number of many-to-many, $|x_i| \cdot |y_i| = p_i^2$, mappings. Therefore, the total number of domain-to-range set mappings varies from $\sum_{i=1}^{n}(1 \cdot |y_i|) = \sum_{i=1}^{n} p_i$ to $\sum_{i=1}^{n}(|y_i| \cdot |x_i|) = \sum_{i=1}^{n} p_i^2$ mappings. Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of domain-to-range set mapping cases converges to the real-valued Manhattan and Euclidean distance relations.

3.4. Manhattan distance.

THEOREM 3.6. *Manhattan (largest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots,\ [a_n, b_n]\}$*, where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "taxicab_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
From the countable distance definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \tag{3.9}$$
$$\Rightarrow \quad d_c \leq \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i.$$

Multiply both sides of equation 3.11 by c and take the limit:

$$d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \Rightarrow \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c). \tag{3.10}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d:\ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c. \tag{3.11}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i. \tag{3.12}$$

Combine equations 3.11, 3.10, 3.12:

$$d = \lim_{c\to 0} d_c \cdot c \quad \wedge \quad \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) \quad \wedge \tag{3.13}$$
$$\lim_{c\to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square$$

3.5. Euclidean distance.

THEOREM 3.7. *Euclidean (smallest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,* $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "Euclidean_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
Apply the rule of product to the largest number of domain-to-range set mappings, where all p_i number of range set members, y_i, map to each of the p_i number of members in the domain set, x_i, which is the Cartesian product, $|y_i| \cdot |x_i|$:

$$|x_i| = |y_i| = p_i \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2. \tag{3.14}$$

From the countable distance definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \tag{3.15}$$
$$\Rightarrow \quad d_c \leq \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\ d_c:\ d_c = \sum_{i=1}^{n} p_i.$$

Square both sides of equation 3.15 ($x = y \Leftrightarrow f(x) = f(y)$):

$$\exists\, p_i,\ d_c:\ d_c = \sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\ d_c:\ d_c^2 = (\sum_{i=1}^{n} p_i)^2. \tag{3.16}$$

Apply the square of sum inequality, $(\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$, to equation 3.16 and select the smallest area (the equality) case:

$$d_c^2 = (\sum_{i=1}^{n} p_i)^2 = \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j \tag{3.17}$$
$$= \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} p_i \sum_{j=1, j\neq i}^{n} p_j \geq \sum_{i=1}^{n} p_i^2 \quad \Rightarrow \quad \exists\, p_i:\ d_c^2 = \sum_{i=1}^{n} p_i^2.$$

Multiply both sides of equation 3.17 by c^2, simplify, and take the limit.

$$d_c^2 = \sum_{i=1}^{n} p_i^2 \ \Rightarrow\ d_c^2 \cdot c^2 = \sum_{i=1}^{n} p_i^2 \cdot c^2 \ \Leftrightarrow\ (d_c \cdot c)^2 = \sum_{i=1}^{n} (p_i \cdot c)^2 \tag{3.18}$$
$$\Rightarrow \quad \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$\exists\, c\, d \in \mathbb{R}:\ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2. \tag{3.19}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i. \tag{3.20}$$

Combine equations 3.19, 3.18, 3.20:

$$(3.21) \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 \quad \wedge \quad \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 \quad \wedge$$
$$\lim_{c\to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \quad \square$$

4. Euclidean Volume

The case of a 1-1 correspondence of domain-to-range set mappings and disjoint range sets, $d_c = |\bigcup_{i=1}^n y_i|, \quad \bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^n y_i = \emptyset.$ converges to Manhattan distance (3.6). The sum of same-sized sub-Manhattan distances is the sum of ordered n-tuples of subintervals, where each n-tuple is an ordered set containing one same-sized subinterval from each range set, y_i. The number of all possible ordered sub-Manhattan distance n-tuples, v_c, is the number of ordered n-tuples generated by Cartesian product of the range sets:

Definition 4.1. Countable Volume, v_c:

$$v_c = |\times_{i=1}^n y_i|, \quad \bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^n y_i = \emptyset.$$

Theorem 4.2. *Euclidean volume, v, is size of the range interval, $[v_0, v_m]$, corresponding to the Cartesian product of all the members of the domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$. Notionally:*

$$v = \prod_{i=1}^n s_i, \;\; v = |v_0 - v_m|, \;\; s_i = |a_i - b_i|.$$

The theorem, "Euclidean_volume," and formal proof is in the Coq file, euclidrelations.v.

Proof.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of subintervals.

$$(4.1) \qquad \forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i| = |y_i|.$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$(4.2) \qquad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0}(p_i \cdot c) = s_i.$$

Apply the associative law of multiplication to derive the countable volume (4.1) in terms of p_i:

$$(4.3) \quad v_c = |\times_{i=1}^n y_i| \quad \wedge \quad |y_i| = p_i \quad \Rightarrow \quad v_c = |\times_{i=1}^n y_i| = \prod_{i=1}^n |y_i| = \prod_{i=1}^n p_i.$$

Multiply both sides of equation 4.3 by c^n:

$$(4.4) \qquad v_c \cdot c^n = (\prod_{i=1}^n p_i) \cdot c^n = \prod_{i=1}^n (p_i \cdot c).$$

Use those cases, where v_c has an integer n^{th} root.

$$(4.5) \quad \forall\, n,\; p,\; v_c \;\in \mathbb{N} \;:\; p^n = v_c \;\Rightarrow\; v_c \cdot c^n = p^n \cdot c^n = (p \cdot c)^n = \prod_{i=1}^n (p_i \cdot c).$$

Apply the ruler (2.1) and ruler convergence (2.2) to the range interval, $[v_0, v_m]$ (where $v = |v_0 - v_m|$), and then combine with equations 4.5 and 4.2:

$$(4.6) \;\; floor(v/c^n) = p^n \Rightarrow v = \lim_{c\to 0}(p \cdot c)^n = \prod_{i=1}^n \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^n s_i. \qquad \square$$

5. Applications to physics

5.1. Coulomb's charge force. The sizes, q_1 and q_2, of two charges are independent domain variables, where each infinitesimal size, c, component of a charge exerts a force on each infinitesimal size, c, component of the other charge. The total force, F, is proportionate to the total number of forces (the Cartesian product of the infinitesimal size, c, components) multiplied times a quantum charge force, $m_C a_C$. From the volume proof (4.2), the Cartesian product converges to $q_1 q_2$:

$$F \propto (m_C a_C)(\lim_{c \to 0} floor(q_1/c) \cdot c)(\lim_{c \to 0} floor(q_2/c) \cdot c) = (m_C a_C) \iint \mathrm{d}q_1 \mathrm{d}q_2 = (m_C a_C)(q_1 q_2). \tag{5.1}$$

From equation 5.1, a change in charge, q, causes a proportionate change in force, F. Solving for the constant force, $F = m_C a_C$, requires a proportionate variable, r, to offset the effect of a change in q: $r \propto q \Rightarrow \exists\, q_C/r_C \in \mathbb{R} \,:\, r(q_C/r_C) = q$:

$$\forall\, q_1, q_2 \geq 0 \,\exists\, q \in \mathbb{R} \,:\, q^2 = q_1 q_2 \;\wedge\; r(q_C/r_C) = q \;\Rightarrow\; (r(q_C/r_C))^2 = q_1 q_2. \tag{5.2}$$

$$\begin{aligned} &(r(q_C/r_C))^2 = q_1 q_2 \;\wedge\; F \propto (m_C a_C)(q_1 q_2) \\ &\quad\Rightarrow\quad F \propto (m_C a_C)(r(q_C/r_C))^2 = (m_C a_C)(q_1 q_2) \\ &\quad\quad\Rightarrow\quad F = m_C a_C = (m_C a_C r_C^2/q_C^2) q_1 q_2/r^2 = k_c q_1 q_2/r^2. \end{aligned} \tag{5.3}$$

where $k_C = m_C a_C r_C^2/q_C^2$ corresponds to the SI units: Nm^2C^{-2}.

5.2. Newton's gravity force equation. The sizes, m_1 and m_2, of two masses are independent domain variables, where each infinitesimal size, c, component of a mass exerts a force on each infinitesimal size, c, component of the other mass. The total force, F, is proportionate to the total number of forces (the Cartesian product of the size, c, components) multiplied times a quantum gravity force, $m_G a_G$. From the volume proof (4.2), the Cartesian product converges to $m_1 m_2$:

$$F \propto (m_G a_G)(\lim_{c \to 0} floor(m_1/c) \cdot c)(\lim_{c \to 0} floor(m_2/c) \cdot c) = (m_G a_G) \iint \mathrm{d}m_1 \mathrm{d}m_2 = (m_G a_G)(m_1 m_2). \tag{5.4}$$

From equation 5.4, an change in mass, m, causes a proportionate change in force, F. Solving for the constant force, $F = m_G a_G$, requires a proportionate variable, r, to offset the effect of a change in m: $r \propto m \Rightarrow \exists\, m_G/r_G \in \mathbb{R} \,:\, r(m_G/r_G) = m$:

$$\forall\, m_1, m_2 \geq 0 \,\exists\, m \in \mathbb{R} \,:\; m^2 = m_1 m_2 \;\wedge\; r(m_G/r_G) = m \;\Rightarrow\; (r(m_G/r_G))^2 = m_1 m_2. \tag{5.5}$$

$$\begin{aligned} &(r(m_G/r_G))^2 = m_1 m_2 \;\wedge\; F \propto (m_G a_G)(m_1 m_2) \\ &\quad\Rightarrow\quad F \propto (m_G a_G)(r(m_G/r_G))^2 = (m_G a_G)(m_1 m_2) \\ &\quad\quad\Rightarrow\quad F = m_G a_G = (m_G a_G r_G^2/q_G^2) m_1 m_2/r^2. \end{aligned} \tag{5.6}$$

$$\begin{aligned} &\exists\, t_G \in \mathbb{R} :\, r_G/t_G^2 = a_G \;\wedge\; F = m_G a_G = (m_G a_G r_G^2/m_G^2) m_1 m_2/r^2 \\ &\quad\Rightarrow\quad F = m_G a_G = (r_G^3/m_G t_G^2) m_1 m_2/r^2 = G m_1 m_2/r^2, \end{aligned} \tag{5.7}$$

where $G = r_G^3/m_G t_G^2$ corresponds to the SI units: $m^3 kg^{-1} s^{-2}$.

5.3. Spacetime equations. The charge (5.3) and gravity (5.7) force equations were derived from the principle that charge and mass are proportionate to distance: $r = (r_C/q_C)q = (r_G/m_G)m$. Suppose that time is also proportionate to distance, $r = (r_c/t_c)t = ct$.

Applying the ruler to two intervals, $[0, d_1]$ and $[0, d_2]$, in two inertial (independent, non-accelerating) frames of reference, the distance and time spanning the two domain intervals converges to a range of distances (and times) from Manhattan (3.6) to Euclidean distance (3.7).

$$r^2 = d_1^2 + d_2^2 \quad \wedge \quad r = (r_c/t_c)t = ct \qquad \Rightarrow \quad (ct)^2 = d_1^2 + d_2^2 \quad \Rightarrow \quad d_2 = \sqrt{(ct)^2 - d_1^2}. \tag{5.8}$$

$$d_2 = \sqrt{(ct)^2 - d_1^2} \quad \wedge \quad d = d_2 \quad \wedge \quad d_1 = vt \qquad \Rightarrow \quad d \; = \; \sqrt{(ct)^2 - (vt)^2} \; = \; ct\sqrt{1 - (v^2/c^2)}, \tag{5.9}$$

which is the spacetime dilation equation. [**Bru17**].

$$d_2^2 = (ct)^2 - d_1^2 \quad \wedge \quad s = d_2 \quad \wedge \quad d_1^2 = x^2 + y^2 + z^2 \qquad \Rightarrow \quad s^2 = (ct)^2 - (x^2 + y^2 + z^2), \tag{5.10}$$

which is one form of the spacetime interval equation [**Bru17**].

5.4. 3 dimensions of physical geometry. From the derivation of the spacetime equations (5.3), a non-zero amount of time is required to sequence through non-zero length distance intervals. Each physical interval can be traversed sequentially (the Manhattan distance), which requires that physical sets of distance intervals have a static, total order during the *time* of sequencing.

And the Manhattan distance intervals can be sequenced in any order. Sequencing in all possible orders without changing the total order requires every interval to be sequentially adjacent (either a successor or predecessor) to every other interval, herein referred to as a symmetric geometry. Ordered and symmetric sets with one and two members are the trivial cases. It will now be proved that the largest ordered and symmetric set is a cyclic set that contains at most 3 members, in this case, 3 dimensions of physical distance and volume.

Definition 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i \;\in \{x_1, \ldots, x_n\}, \qquad \mathit{successor}\; x_i = x_{i+1} \;\;\wedge\;\; \mathit{predecessor}\; x_{i+1} = x_i.$$

Definition 5.2. Symmetric geometry (every set member is sequentially adjacent to every other member):

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; \mathit{successor}\; x_i = x_j \;\;\Leftrightarrow\;\; \mathit{predecessor}\; x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$i = n \;\;\wedge\;\; j = 1 \;\;\Rightarrow\;\; \mathit{successor}\; x_n = x_1 \;\;\wedge\;\; \mathit{predecessor}\; x_1 = x_n.$$

The theorem, “ordered_symmetric_is_cyclic,” and formal proof is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$i = n \ \wedge \ j = 1 \ \Rightarrow \ \textit{successor}\ x_i = x_j = \textit{successor}\ x_n = x_1. \tag{5.11}$$

Applying the definition of a symmetric geometry (5.2) to conclusion 5.11:

$$\textit{successor}\ x_i = x_j \ \Rightarrow \ \textit{predecessor}\ x_j = x_i \ \Rightarrow \ \textit{predecessor}\ x_1 = x_n. \tag{5.12}$$ □

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

DEFINITION 5.5. Cyclic successor of m is n:

$$\textit{Successor}(m, n, \textit{setsize}) \leftarrow (m = \textit{setsize} \wedge n = 1) \vee (n = m + 1 \leq \textit{setsize}). \tag{5.13}$$

DEFINITION 5.6. Cyclic predecessor of m is n:

$$\textit{Predecessor}(m, n, \textit{setsize}) \leftarrow (m = 1 \wedge n = \textit{setsize}) \vee (n = m - q \geq 1). \tag{5.14}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:

$$\textit{Adjacent}(m, n, \textit{setsize}) \leftarrow \textit{Successor}(m, n, \textit{setsize}) \vee \textit{Predecessor}(m, n, \textit{setsize}). \tag{5.15}$$

Prove that every member is adjacent to every other member, where $\textit{setsize} \in \{1,\ 2,\ 3\}$:

$$\textit{Adjacent}(1,1,1) \leftarrow \textit{Successor}(1,1,1) \leftarrow (m = \textit{setsize} \wedge n = 1). \tag{5.16}$$

$$\textit{Adjacent}(1,2,2) \leftarrow \textit{Successor}(1,2,2) \leftarrow (n = m + 1 \leq \textit{setsize}). \tag{5.17}$$

$$\textit{Adjacent}(2,1,2) \leftarrow \textit{Successor}(2,1,2) \leftarrow (n = \textit{setsize} \wedge m = 1). \tag{5.18}$$

$$\textit{Adjacent}(1,2,3) \leftarrow \textit{Successor}(1,2,3) \leftarrow (n = m + 1 \leq \textit{setsize}). \tag{5.19}$$

$$\textit{Adjacent}(2,1,3) \leftarrow \textit{Predecessor}(2,1,3) \leftarrow (n = m - q \geq 1). \tag{5.20}$$

$$\textit{Adjacent}(3,1,3) \leftarrow \textit{Successor}(3,1,3) \leftarrow (n = \textit{setsize} \wedge m = 1). \tag{5.21}$$

$$\textit{Adjacent}(1,3,3) \leftarrow \textit{Predecessor}(1,3,3) \leftarrow (m = 1 \wedge n = \textit{setsize}). \tag{5.22}$$

$$\textit{Adjacent}(2,3,3) \leftarrow \textit{Successor}(2,3,3) \leftarrow (n = m + 1 \leq \textit{setsize}). \tag{5.23}$$

$$\textit{Adjacent}(3,2,3) \leftarrow \textit{Predecessor}(3,2,3) \leftarrow (n = m - q \geq 1). \tag{5.24}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\forall\, setsize > 3: \quad \neg Successor(1, 3, setsize > 3) \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.25}$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$\forall\, setsize > 3: \quad \neg Predecessor(1, 3, setsize > 3) \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \tag{5.26}$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$\forall\, setsize > 3: \quad \neg Adjacent(1, 3, setsize > 3) \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \tag{5.27}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

Applying the ruler measure (2.1) and ruler convergence (2.2) to the set relations, countable distance (3.1) and countable volume (4.1) yields the following insights and implications:

(1) The properties of metric space, Euclidean distance and area/volume are derived from two set operations without using the notions of Euclidean geometry like plane, side, angle, perpendicular, congruence, intersection, etc. [**Joy98**].

(2) Making the measures, countable distance (3.1) and volume (4.1), independent of the measure of intervals (2.1) allows deriving geometric relations and insights into geometry that metric space, the Lebesgue, Borel, and Hausdorff measures are incapable of providing.

  (a) The same constraints on countable distance (3.1) that converges to Manhattan distance also cause countable volume (4.1) to converge to Euclidean volume. The number of all possible ordered sub-Manhattan distance n-tuples, v_c, is the number of ordered n-tuples generated by Cartesian product of the range sets, which is a set-based pattern relating volume to distance:
  $d_c = |\bigcup_{i=1}^{n} y_i|, \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset \Rightarrow$
  $v_c = |\times_{i=1}^{n} y_i|, \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset$
  that metric space, the Lebesgue, Borel, and Hausdorff measures have not shown.

  (b) The largest intersection (smallest distance), $d_c = |\bigcup_{i=1}^{n} y_i|$, has the most domain set members mapping to each range set member. The equality case, $|x_i| = |y_i| = p_i$, of the countable distance space constraint, $|x_i| = |y_i|^q$, $q \geq 1$, (3.1) limits the largest total number of domain-to-range set mappings to $\sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^2$, which is the set-based reason Euclidean distance (3.7) is the smallest possible distance between two distinct points in $\mathbb{R}^n$ (flat space, where $|x_i| = |y_i|$).

(c) $|x_i| = |y_i|^q$, $q \geq 1$, (3.1) generates all the L^p norms, $||L||_p = (\sum_{i=1}^{n} s_i^p)^{1/p}$. For example, using the same proof pattern as for Euclidean distance (3.7): $p_i = |y_i| \Rightarrow |x_i| = p_i^q \Rightarrow \sum_{i=1}^{n}|x_i|\cdot|y_i| = \sum_{i=1}^{n} p_i^{q+1} \leq d_c^{q+1} \ \ldots\ d = (\sum_{i=1}^{n} s_i^{q+1})^{1/(q+1)}$.

(d) Note that $q \geq 1$ because $0 \leq q < 1$ would violate the triangle inequality property, $d(u, w) \leq d(u, v) + d(v, w)$, of metric space.

(e) It was shown that the countable distance set operation, $d_c = |\bigcup_{i=1}^{n} y_i|$, (3.1) generates three of the metric space properties (3.2). The countable distance constraint, $|x_i| = |y_i|^q$, $q \geq 1$, (3.1) is the reason for the fourth property of metric space, symmetry, $d(u, v) = d(v, u)$, where the value, q, causes the same combinatorial domain-to-range set mapping for every domain-range set pair that manifests as the same value, p, in each term of the L^p norm.

(f) The Euclidean volume proof was used to derive the Coulomb's charge force (5.1) and Newton's gravity force (5.4) without using other laws of physics or Gauss's divergence theorem. The Euclidean distance proof was used to derive the spacetime equations (5.8) without a constant speed of light assumption or even the notion of light.

(3) **The Proportionate Interval Principle:** The derivations of the charge force, gravity force, and spacetime equations using the Euclidean volume and distance proofs shows that all Euclidean distance range intervals having a size, r, have proportionately sized intervals of other types, for example: $r = (r_C/q_C)q = (r_G/m_G)m = (r_c/t_c)t$.

(a) Using the proportionate interval principal with the Euclidean volume proof to derive the charge and gravity force equations proves that the inverse square law is a mapping of the Cartesian product (*rectangular*) geometric area (r^2) to *rectangular* charge (q_1q_2) and gravity (m_1m_2) areas.

(i) "Geometric dilution" (flux divergence) on the surface area of a sphere ($4\pi r^2$) applies to finite sized particles in a fluid but probably does **not** apply to field forces.

(ii) The charge constant, $k_c = 1/(4\pi\epsilon_0)$, assumes flux divergence and is probably invalid. The correct constant should align with the rectangular area, inverse square law, $k_c = 1/\epsilon_0$, where the vacuum permittivity constant, ϵ_0, is 4π times larger. The same applies to the vacuum magnetic permeability constant and the fine-structure constant.

(b) The ratios, $r = (r_C/q_C)q = (r_G/m_G)m$, imply that if there are quantum values of charge, q_C, and mass, m_G, then there are quantum distances, r_C and r_G, where the charge and gravity forces do not exist (are not defined) at smaller distances, which might have implications for the strong force, fine-structure constant, particle collisions, and the density of black holes.

(c) A countable set of values has measure 0 and thus no proportion relationship to distance. Therefore, a countable set of state value changes with respect to time are independent of distance (for example, the change in the spin values of two quantum coupled particles).

(4) Relativity theory assumes that only 3 dimensions of physical space [**Bru17**]. The proof in this article shows that spacetime constrains physical distance and volume to at most three dimensions (5.4). Higher dimensional Hilbert spaces (inner product vector spaces) are valid if no more than three of the dimensions represent physical space because higher dimensions of physical distance and volume would have order contradictions.
   (a) The proof of at most 3 members in any ordered and symmetric set (5.4), implies that each *physical* infinitesimal volume (ball) can have at most 3 ordered and symmetric dimensions of discrete *physical* states of the same type. And each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values per infinitesimal ball.
   (b) If each of the three possible ordered and symmetric dimensions of discrete physical states contained unordered sets of discrete state values, for example, unordered binary values, then there would be $2 \cdot 2 \cdot 2 = 8$ possible combinations of values. Unordered sets (states) are non-deterministic. For example, every time an unordered state is physically measured, there is a 50 percent chance of having one of the binary values.
   (c) Where infinitesimal balls intersect, an algebra of the interactions of the discrete states with respect to time needs to be developed. The interaction of the discrete states associated with overlapping infinitesimal balls with respect to time might result in what we perceive as motion, particles, waves, mass, charge, etc.

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