The Two Set Relations Generating Geometry

George. M. Van Treeck

Abstract. A ruler (measuring stick) divides both domain and range intervals approximately into the nearest integer number of same-sized subintervals. As the subinterval size converges to zero: 1) Where each domain interval has a corresponding same-sized range interval, distance as the union size of range sets converges to: the triangle inequality with Manhattan distance at the upper boundary and Euclidean distance at the lower boundary. 2) The Cartesian product of the number of members in each domain set converges to the product of interval interval sizes (Euclidean area/volume). The ruler measure-based proofs of Euclidean distance and area/volume are used to derive the charge force, Newtonian gravity force, and spacetime equations. Time limits physical geometry to 3 dimensions. All proofs are verified in Coq.

Contents

1. Introduction

Introduction to real analysis textbooks start with the step-by-step building of a foundation of continuity, convergence, limit, etc. from set definitions. But, metric space, Euclidean distance (metric/vector norm), and Euclidean area/volume (the product of interval sizes used by the Riemann and Lebesgue integrals and used by the Lebesgue measure) are defined in real analysis [**Gol76**] [**Rud76**] rather than derived by applying the previously constructed foundation to more set definitions.

A "ruler" measure of intervals applies convergence to two types of set relations to derive the properties of distance and volume, which provides some new insights

2010 *Mathematics Subject Classification.* Primary 28A75, 28E15. Secondary 03E75, 51M99.

into geometry and physics, for example: 1) the single set relation generating the triangle inequality, non-negativity, and identity of indiscernibles properties of metric space; 2) the mapping between sets that makes Euclidean distance the smallest possible distance between two distinct points in $\mathbb{R}^n$; 3) the mapping between sets that makes distance different from area/volume; 4) the set-based reason the forces of charge and gravity vary inversely with the square of the distance between two objects; 5) how time places an additional constraint on those same two set relations, which limits physical geometry to 3 dimensions.

All the proofs in this article have been formally verified using using the Coq proof verification system [**Coq15**]. The formal proofs are located in the Coq files, "euclidrelations.v" and "threed.v," at:

`https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

A ruler (measuring stick) partitions both domain and range intervals *approximately* into the nearest integer number of subintervals, where each subinterval has the *same size*, c, with the consequence that different-sized intervals have a *different number* of subintervals. In contrast, the Riemann and Lebesgue integrals partition each domain interval and the range into the *same number* of subintervals, where different-sized intervals have *different-sized* subintervals [**Gol76**] [**Rud76**].

The ruler measure allows counting the number of mappings, ranging from a one-to-one correspondence to a many-to-many mapping, between the set of subintervals having size c in one interval and the set of subintervals having the same size, c, in another interval. The mapping (combinatorial) relations converge to continuous, bijective relations as the subinterval size, c, converges to zero.

Definition 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, p, each subinterval having the same size, c. Notionally:

$$\forall\, c\, s \in \mathbb{R},\;\; [a,b] \subset \mathbb{R},\;\; s = |a-b| \;\wedge\; c > 0 \;\wedge \\ (p = floor(s/c) \;\vee\; p = ceiling(s/c)) \;\wedge\; M = \textstyle\sum_{i=1}^{p} c = pc. \tag{2.1}$$

Theorem 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\;\; s = |a-b| \;\Rightarrow\; M = \lim_{c\to 0} pc = s.$

The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\; y \le x,\, y \in \mathbb{Z},\, x \in \mathbb{R}\})$:

$$\forall\, c > 0,\, p = floor(s/c) \;\wedge\; 0 \le |floor(s/c) - s/c| < 1 \;\Rightarrow\; 0 \le |p - s/c| < 1. \tag{2.2}$$

Multiply all sides of inequality 2.2 by $|c|$:

$$\forall\, c > 0, \quad 0 \le |p - s/c| < 1 \quad\Rightarrow\quad 0 \le |pc - s| < |c|. \tag{2.3}$$

$$\forall\, \delta \;:\; |pc - s| < |c| = |c - 0| < \delta \\ \Rightarrow \quad \forall\, \epsilon = \delta :\; |c-0| < \delta \;\wedge\; |pc - s| < \epsilon \;:=\; M = \lim_{c\to 0} pc = s. \quad \square \tag{2.4}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence, where: $[0, \pi]$, $s = |0 - \pi|$, $c = 10^{-i}$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{i=1},\ 3.14_{i=2},\ 3.141_{i=3}, ..., \pi$.

3. Distance

Notation convention: Curly brackets, $\{\cdots\}$, delimit a set; square brackets, $[\cdots]$, delimit a list; and vertical bars around a set or list, $|\cdots|$, indicates the cardinal (number of members in the set or list).

3.1. Countable distance space. A simple measure of distance is the number of steps walked, which corresponds to an equal number of pieces of land. Abstracting, distance is proportionate to the number of members in a range set, y_i, which equals the number of members in a corresponding domain set, x_i: $|x_i| = |y_i|$. And the distance spanning multiple, disjoint, domain sets, $\bigcap_{i=1}^{n} x_i = \emptyset$, is proportionate to the number of members, d_c, in the union range set: $d_c = |\bigcup_{i=1}^{n} y_i|$.

DEFINITION 3.1. Countable distance space, d_c:

$$\bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad d_c = |\bigcup_{i=1}^{n} y_i| \quad \wedge \quad |x_i| \geq |y_i|.$$

Note: $|x_i| < |y_i|$ would violate the triangle inequality property of metric space.

THEOREM 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$.

This well-known inequality follows from the inclusion-exclusion principle [**CG15**]. But, a more intuitive and simple proof follows from the associative law of addition, which requires the sum of the set sizes to equal the size of all the set members appended into a list. And, by the commutative law of addition, the list can be sorted into a list of unique members (the union set) and a list of duplicate members. For example, $|\{a, b, c\}| + |\{c, d, e\}| = |[a, b, c, c, d, e]| = |\{a, b, c, d, e\}| + |[c]| \Rightarrow |\{a, b, c, d, e\}| = |\{a, b, c\}| + |\{c, d, e\}| - |[c]|$. The list of duplicates being ≥ 0 implies the union set size is always $\leq$ the sum of the set sizes.

A formal proof, inclusion_exclusion_inequality, using sorting into a set of unique members (union set) and a list of duplicate members, is in the file euclidrelations.v.

PROOF. By the associative law of addition, append the sets into a list. Next, by the commutative law of addition, sort the list into uniques and duplicates, and then subtract duplicates from both sides:

$$\sum_{i=1}^{n} |y_i| = |append_{i=1}^{n} y_i| = |sort(append_{i=1}^{n} y_i)| \\ = |\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i| \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| - |duplicates_{i=1}^{n} y_i| = |\bigcup_{i=1}^{n} y_i|. \tag{3.1}$$

$$|\bigcup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| - |duplicates_{i=1}^{n} y_i| \quad \wedge \quad |duplicates_{i=1}^{n} y_i| \geq 0 \\ \Rightarrow \quad |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|. \tag{3.2}$$

□

3.2. Metric Space. All function range intervals, $d(u, w)$, satisfying the countable distance space definition, $d_c = |\bigcup_{i=1}^{n} y_i|$, where the ruler is applied, generates the three metric space properties: triangle inequality, non-negativity, and identity of indecernables. The fourth property of metric space, symmetry $[d(u, v) = d(v, u)]$, is motivated by Manhattan and Euclidean distance. The formal proofs: triangle_inequality, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 3.3. *Triangle Inequality:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \leq d(u, v) + d(v, w)$.

PROOF. Apply the ruler measure (2.1), the countable distance space condition (3.1), inclusion-exclusion inequality (3.2), and then ruler convergence (2.2).

$$\begin{aligned}
(3.3)\quad & \forall\, c > 0,\ d(u, w),\ d(u, v),\ d(v, w)\ : \\
& |y_1| = floor(d(u, v)/c) \ \wedge\ |y_2| = floor(d(v, w)/c) \ \wedge \\
& d_c = floor(d(u, w)/c) \ \wedge\ d_c = |y_1 \cup y_2| \leq |y_1| + |y_2| \\
& \Rightarrow\ floor(d(u, w)/c) \leq floor(d(u, v)/c) + floor(d(v, w)/c) \\
& \Rightarrow\ floor(d(u, w)/c) \cdot c \leq floor(d(u, v)/c) \cdot c + floor(d(v, w)/c) \cdot c \\
& \Rightarrow\ \lim_{c \to 0} floor(d(u, w)/c) \cdot c \leq \lim_{c \to 0} floor(d(u, v)/c) \cdot c + \lim_{c \to 0} floor(d(v, w)/c) \cdot c \\
& \Rightarrow\ d(u, w) \leq d(u, v) + d(v, w). \qquad \square
\end{aligned}$$

THEOREM 3.4. *Non-negativity:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \geq 0$.

PROOF.
By definition, a set always has a size (cardinal) ≥ 0:

$$\begin{aligned}
(3.4)\quad & \forall\, c > 0,\ d(u, w)\ :\quad floor(d(u, w)/c) = d_c \quad \wedge \quad d_c = |y_1 \cup y_2| \geq 0 \\
& \Rightarrow\quad floor(d(u, w)/c) = d_c \geq 0 \quad \Rightarrow \quad d(u, w) = \lim_{c \to 0} d_c \cdot c \geq 0. \qquad \square
\end{aligned}$$

THEOREM 3.5. *Identity of Indiscernibles:* $d(w, w) = 0$.

PROOF. Apply the triangle inequality property (3.3):

$$\forall\, d(u, v) = d(v, w) = 0 \ \wedge\ d(u, w) \leq d(u, v) + d(v, w) \ \Rightarrow\ d(u, w) \leq 0. \tag{3.5}$$

Combine the non-negativity property (3.4) and the previous inequality (3.5):

$$d(u, w) \geq 0 \ \wedge\ d(u, w) \leq 0 \ \Leftrightarrow\ 0 \leq d(u, w) \leq 0 \ \Rightarrow\ d(u, w) = 0. \tag{3.6}$$

$$d(u, w) = 0 \ \wedge\ d(u, v) = 0 \ \Rightarrow\ w = v. \tag{3.7}$$

$$d(v, w) = 0 \ \wedge\ w = v \ \Rightarrow\ d(w, w) = 0. \tag{3.8}$$

$\square$

3.3. Distance space range. From the countable distance space definition, $d_c = |\bigcup_{i=1}^{n} y_i|$. As the amount of intersection increases, a single range set member can map to more domain set members. Therefore, d_c is a function of range-to-domain set member mappings.

From the countable distance space property (3.1), where $|x_i| = |y_i| = p_i$, the total number of range-to-domain set member mappings vary from the sum of one-to-one correspondences (no intersection and largest distance), $\sum_{i=1}^{n} x_i \cdot 1 = \sum_{i=1}^{n} p_i$, to the sum of the most many-to-many correspondences (largest intersection and smallest possible distance), $\sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^2$. Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of range-to-domain set mapping cases converges to the real-valued Manhattan and Euclidean distance relations. The cases where $|x_i| > |y_i|$ are elliptic geometry distances.

3.4. Manhattan distance.

THEOREM 3.6. *Manhattan (largest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "taxicab_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$\begin{aligned} d_c = |\textstyle\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad &\wedge \quad |x_i| = |y_i| = p_i \\ &\Rightarrow \quad d_c \leq \textstyle\sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c : \; d_c = \sum_{i=1}^{n} p_i. \end{aligned} \tag{3.9}$$

Multiply both sides of equation 3.11 by c and take the limit:

$$d_c = \textstyle\sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \Rightarrow \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c). \tag{3.10}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d : \; floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c. \tag{3.11}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of s_i:

$$\forall\, i \in [1, n],\, s_i = |a_i - b_i| \wedge floor(s_i/c) = |x_i| = |y_i| = p_i \Rightarrow \lim_{c\to 0} p_i \cdot c = s_i. \tag{3.12}$$

Combine equations 3.11, 3.10, 3.12:

$$\begin{aligned} &d = \lim_{c\to 0} d_c \cdot c \quad \wedge \quad \lim_{c\to 0} d_c \cdot c = \textstyle\sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) \quad \wedge \\ &\lim_{c\to 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c = \textstyle\sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square \end{aligned} \tag{3.13}$$

3.5. Euclidean distance.

THEOREM 3.7. *The smallest Euclidean distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "Euclidean_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
Apply the rule of product to the largest number of range-to-domain set mappings, where all p_i number of range set members, y_i, map to each of the p_i number of members in the domain set, x_i, and where $|x_i| = |y_i| = p_i$:

$$|x_i| = |y_i| = p_i \quad \Rightarrow \quad \textstyle\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2. \tag{3.14}$$

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$\begin{aligned} d_c = |\textstyle\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad &\wedge \quad |x_i| = |y_i| = p_i \\ &\Rightarrow \quad d_c \leq \textstyle\sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c : \; d_c = \sum_{i=1}^{n} p_i. \end{aligned} \tag{3.15}$$

Square both sides of equation 3.15 ($x = y \Leftrightarrow f(x) = f(y)$):

$$\exists\, p_i,\, d_c : \; d_c = \textstyle\sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\, d_c : \; d_c^2 = (\sum_{i=1}^{n} p_i)^2. \tag{3.16}$$

Apply the square of sum inequality, $(\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$, to equation 3.16 and select the smallest area (the equality) case:

$$\begin{aligned}(3.17)\quad d_c = \textstyle\sum_{i=1}^{n} p_i,\ p_i \geq 0 \quad &\Rightarrow \quad d_c^2 = (\textstyle\sum_{i=1}^{n} p_i)^2 = \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \\ &= \textstyle\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} p_i p_j \geq \sum_{i=1}^{n} p_i^2 \\ &\Rightarrow \quad \exists\, p_i : d_c^2 = \textstyle\sum_{i=1}^{n} p_i^2.\end{aligned}$$

Multiply both sides of equation 3.17 by c^2, simplify, and take the limit.

$$\begin{aligned}(3.18)\quad d_c^2 = \textstyle\sum_{i=1}^{n} p_i^2 \Rightarrow d_c^2 \cdot c^2 = \sum_{i=1}^{n} p_i^2 \cdot c^2 \Leftrightarrow (d_c \cdot c)^2 = \sum_{i=1}^{n} (p_i \cdot c)^2 \\ \Rightarrow \quad \lim_{c\to 0}(d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2.\end{aligned}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.19)\quad \exists\, c\, d : \mathit{floor}(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

(3.20)

$$\forall\, i \in [1, n],\ s_i = |a_i - b_i| \ \wedge\ \mathit{floor}(s_i/c) = |x_i| = |y_i| = p_i \ \Rightarrow\ \lim_{c\to 0}(p_i \cdot c) = s_i.$$

Combine equations 3.19, 3.18, 3.20:

$$\begin{aligned}(3.21)\quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 \quad \wedge \quad \lim_{c\to 0}(d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 \quad \wedge \\ \lim_{c\to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \quad \square\end{aligned}$$

4. Euclidean Volume

The number of all possible combinations (n-tuples) taking one member from each disjoint set is the Cartesian product of the number of members in each set. Notionally:

Definition 4.1. Countable Volume, V_c:

$$\textstyle\bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad V_c = \prod_{i=1}^{n} |x_i|.$$

Theorem 4.2. *Euclidean volume, V, is size of the range interval, $[v_0, v_m]$, corresponding to all the possible combinations of the members of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$. Notionally:*

$$V = \textstyle\prod_{i=1}^{n} s_i,\ \ V = |v_0 - v_m|,\ \ s_i = |a_i - b_i|.$$

The theorem, "Euclidean_volume," and formal proof is in the Coq file, euclidrelations.v.

Proof.

Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, x_i of p_i number of subintervals.

$$(4.1)\qquad \forall\, i\, n \in \mathbb{N},\quad i \in [1, n],\quad c > 0 \quad \wedge \quad \mathit{floor}(s_i/c) = p_i = |x_i|.$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$(4.2)\qquad \mathit{floor}(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0}(p_i \cdot c) = s_i.$$

Use the ruler (2.1) to divide the exact size, $V = |v_0 - v_m|$, of the range interval, $[v_0, v_m]$, into p^n subintervals. Use those cases, where V_c has an integer n^{th} root.

$$(4.3)\ \forall\, p^n = V_c \in \mathbb{N}, \exists\, V \in \mathbb{R}, x_i : \mathit{floor}(V/c^n) = V_c = p^n = \textstyle\prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i.$$

Apply the ruler convergence theorem (2.2) to equation 4.3 and simplify:

$$(4.4) \qquad floor(V/c^n) = p^n \quad \Rightarrow \quad V = \lim_{c\to 0} p^n \cdot c^n = \lim_{c\to 0}(p\cdot c)^n.$$

Multiply both sides of equation 4.3 by c^n and simplify:

$$(4.5) \quad p^n = \prod_{i=1}^n p_i \quad \Rightarrow \quad p^n \cdot c^n = (\prod_{i=1}^n p_i)\cdot c^n \quad \Leftrightarrow \quad (p\cdot c)^n = \prod_{i=1}^n (p_i \cdot c)$$
$$\Rightarrow \quad \lim_{c\to 0}(p\cdot c)^n = \prod_{i=1}^n \lim_{c\to 0}(p_i\cdot c)$$

Combine equations 4.4, 4.5, and 4.2:

$$(4.6) \quad V = \lim_{c\to 0}(p\cdot c)^n \quad \wedge \quad \lim_{c\to 0}(p\cdot c)^n = \prod_{i=1}^n \lim_{c\to 0}(p_i\cdot c) \quad \wedge$$
$$\lim_{c\to 0}(p_i\cdot c) = s_i \quad \Rightarrow \quad V = \lim_{c\to 0}(p\cdot c)^n = \prod_{i=1}^n \lim_{c\to 0}(p_i\cdot c) = \prod_{i=1}^n s_i. \quad \square$$

5. Applications to physics

5.1. Charge force equation. Apply the ruler to the two independent domain intervals, $[0, q_1]$ and $[0, q_2]$, where each size c component of $[0, q_1]$ acts on (corresponds to) each of the size c component in $[0, q_2]$. with a quantum charge force, $m_C a_C$. The total force, F, is proportionate to the total number of infinitesimal correspondences (the Cartesian product of components) multiplied times the quantum charge force, $m_C a_C$:

$$(5.1) \qquad F \propto (m_C a_C)(\lim_{c\to 0} floor(q_1/c)\cdot c)(\lim_{c\to 0} floor(q_2/c)\cdot c) = (m_C a_C)(q_1 q_2).$$

For any interval, $[0, q]$, a proportionately sized interval, $[0, r]$, can be defined with a proportion ratio, $q_e/r_e \in \mathbb{R}$, such that $r(q_e/r_e) = q$:

$$(5.2) \qquad \forall\, q_1 q_2 \in \mathbb{R}\, \exists\, q \in \mathbb{C} \;:\; q^2 = q_1 q_2 \;\wedge\; r(q_e/r_e) = q \;\Rightarrow\; (r(q_e/r_e))^2 = q_1 q_2.$$

$$(5.3) \quad (r(q_e/r_e))^2 = q_1 q_2 \quad \wedge \quad F \propto (m_C a_C)(q_1 q_2) \quad \wedge \quad \exists\, m_0 a \in \mathbb{R} \;:\; m_0 a = m_C a_C$$
$$\Rightarrow \quad F \propto (m_0 a)(r(q_e/r_e))^2 = (m_C a_C)(q_1 q_2)$$
$$\Rightarrow \quad \exists\, k_C \in \mathbb{R} : \quad F = m_0 a = (m_C a_C r_e^2/q_e^2) q_1 q_2/r^2 = k_C q_1 q_2/r^2,$$

where $k_C = m_C a_C r_e^2/q_e^2$ corresponds to the accepted standard units: Nm^2C^{-2}.

5.2. Newtonian gravity force equation. Apply the ruler to the two independent domain intervals, $[0, m_1]$ and $[0, m_2]$, where each size c component of $[0, m_1]$ acts on (corresponds to) each of the size c component in $[0, m_2]$ with a quantum gravity force, $m_G a_G$. The total force, F, is proportionate to the total number of infinitesimal correspondences (the Cartesian product of components) multiplied times the quantum gravity force, $m_G a_G$:

$$(5.4) \quad F \propto (m_G a_G)(\lim_{c\to 0} floor(m_1/c)\cdot c)(\lim_{c\to 0} floor(m_2/c)\cdot c) = (m_G a_G)(m_1 m_2).$$

For any interval, $[0, m]$, a proportionately sized interval, $[0, r]$, can be defined with a proportion ratio, $m_G/r_G \in \mathbb{R}$, such that $r(m_G/r_G) = m$:

$$(5.5) \quad \forall\, m_1 m_2 \in \mathbb{R}\, \exists\, m \in \mathbb{R} \;:\quad m^2 = m_1 m_2 \quad \wedge \quad r(m_G/r_G) = m$$
$$\Rightarrow \quad (r(m_G/r_G))^2 = m_1 m_2.$$

$$(5.6) \quad (r(m_G/r_G))^2 = m_1 m_2 \wedge F \propto (m_G a_G)(m_1 m_2) \wedge \exists\, m_0 a \in \mathbb{R} :\; m_0 a = m_G a_G$$
$$\Rightarrow \quad F \propto (m_0 a)(r(m_G/r_G))^2 = (m_G a_G)(m_1 m_2)$$

$$F \propto (m_0 a)(r(m_G/r_G))^2 = (m_G a_G)(m_1 m_2) \ \wedge \ \exists\, r_G, t_G \in \mathbb{R}: \ a_G = r_G/t_G^2 \\ \Rightarrow \quad \exists\, G \in \mathbb{R}: \quad F = m_0 a = (r_G^3/m_G t_G^2) m_1 m_2/r^2 = G m_1 m_2/r^2, \tag{5.7}$$

where $G = r_G^3/m_G t_G^2$ corresponds to the accepted standard units: $m^3 kg^{-1} s^{-2}$.

5.3. Spacetime equations. The purpose of the derivations of the spacetime dilation and interval equations is to show the same pattern of an Euclidean distance interval having a proportionate interval of any other type, in this case, time. Any interval, $[0, t]$, proportionate to the Euclidean range interval, $[0, r]$, is measured in terms of a conversion constant, c, that is the ratio of some value, r_e to some value, t_c, such that $r = (r_c/t_c)t = ct$.

$$r^2 = d_1^2 + d_2^2 \quad \wedge \quad r = (r_c/t_c)t = ct \\ \Rightarrow \quad (ct)^2 = d_1^2 + d_2^2 \quad \Rightarrow \quad d_2 = \sqrt{(ct)^2 - d_1^2}. \tag{5.8}$$

$$d_2 = \sqrt{(ct)^2 - d_1^2} \quad \wedge \quad d' = d_2 \quad \wedge \quad d_1 = vt \\ \Rightarrow \quad d' \ = \ \sqrt{(ct)^2 - (vt)^2} \ = \ ct\sqrt{1 - (v^2/c^2)}, \tag{5.9}$$

which is the spacetime dilation equation. [**Bru17**].

$$d_2^2 = (ct)^2 - d_1^2 \quad \wedge \quad s = d_2 \quad \wedge \quad d_1^2 = x^2 + y^2 + z^2 \\ \Rightarrow \quad s^2 = (ct)^2 - (x^2 + y^2 + z^2), \tag{5.10}$$

which is the spacetime interval equation [**Bru17**].

5.4. 3 dimensions of physical geometry. The set and arithmetic operations used to calculate distance and volume requires sequencing through a totally ordered set of dimensions, for example, the countable distance space: $d_c = |\bigcup_{i=1}^{n} y_i|$, Euclidean distance: $d^2 = \sum_{i=1}^{n} s_i^2$, countable volume: $V_c = \prod_{i=1}^{n} |x_i|$, and Euclidean volume: $V = \prod_{i=1}^{n} s_i$. The commutative property of the union, addition, and multiplication operations also allows sequencing through a set of n number of dimensions in all $n!$ number of possible orders.

But, a *physical*, deterministic sequencer requires a *physical* set to have a single total order, at most one successor and at most one predecessor per set member, during the *time* of sequencing. A total order is the only way to "determine" that a sequencer traversed in the order, $[x_2,\ x_1,\ \cdots]$, and next sequenced in the order, $[x_1,\ x_2,\ \cdots]$, during some time period. Deterministic sequencing in every possible order via the same successor/predecessor relations (same total order) requires each set member to be either a successor or predecessor to every other set member (sequentially adjacent), herein referred to as a symmetric geometry.

It will now be proved that a set satisfying the constraints of a single total order and also symmetric defines a cyclic set containing at most 3 members, in this case, 3 dimensions of physical space.

Definition 5.1. Ordered geometry:

$$\forall\, i\, n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i \ \in \{x_1, \ldots, x_n\}, \\ \mathit{successor}\ x_i = x_{i+1} \ \wedge \ \mathit{predecessor}\ x_{i+1} = x_i.$$

Definition 5.2. Symmetric geometry (every set member is sequentially adjacent to any other member):

$$\forall\, i\ j\ n \in \mathbb{N},\ \ \forall\, x_i\ x_j \in \{x_1, \ldots, x_n\},\ \ \mathit{successor}\ x_i = x_j \ \ \Leftrightarrow\ \ \mathit{predecessor}\ x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$\mathit{successor}\ x_n = x_1 \ \ \wedge\ \ \mathit{predecessor}\ x_1 = x_n.$$

The theorem, "ordered_symmetric_is_cyclic," and formal proof is in the Coq file, threed.v.

Proof. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$i = n \ \ \wedge\ \ j = 1 \ \ \wedge\ \ \mathit{successor}\ x_i = x_j \ \ \Rightarrow\ \ \mathit{successor}\ x_n = x_1. \tag{5.11}$$

$$i = n \ \ \wedge\ \ j = 1 \ \ \wedge\ \ \mathit{predecessor}\ x_j = x_i \ \ \Rightarrow\ \ \mathit{predecessor}\ x_1 = x_n. \tag{5.12}$$

□

Theorem 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

Proof.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

Definition 5.5. Cyclic successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (n = m + 1 \leq \mathit{setsize}). \tag{5.13}$$

Definition 5.6. Cyclic predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (n = m - 1 \geq 1). \tag{5.14}$$

Definition 5.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:

$$\mathit{Adjacent}(m, n, \mathit{setsize}) \leftarrow \mathit{Successor}(m, n, \mathit{setsize}) \vee \mathit{Predecessor}(m, n, \mathit{setsize}). \tag{5.15}$$

Prove that every member is adjacent to every other member, where $\mathit{setsize} \in \{1,\ 2,\ 3\}$:

$$\mathit{Adjacent}(1, 1, 1) \leftarrow \mathit{Successor}(1, 1, 1) \leftarrow (m = \mathit{setsize} \wedge n = 1). \tag{5.16}$$

$$\mathit{Adjacent}(1, 2, 2) \leftarrow \mathit{Successor}(1, 2, 2) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{5.17}$$

$$\mathit{Adjacent}(2, 1, 2) \leftarrow \mathit{Successor}(2, 1, 2) \leftarrow (n = \mathit{setsize} \wedge m = 1). \tag{5.18}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m+1 \leq setsize). \tag{5.19}$$
$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m-1 \geq 1). \tag{5.20}$$
$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \wedge m = 1). \tag{5.21}$$
$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \wedge n = setsize). \tag{5.22}$$
$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m+1 \leq setsize). \tag{5.23}$$
$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m-1 \geq 1). \tag{5.24}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\begin{aligned} \forall\, setsize > 3 : \quad & \neg Successor(1,3,setsize > 3) \\ & \leftarrow Successor(1,2,setsize > 3) \leftarrow (n = m+1 \leq setsize). \end{aligned} \tag{5.25}$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\, setsize > 3 : \quad & \neg Predecessor(1,3,setsize > 3) \\ & \leftarrow Predecessor(1,setsize,setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \end{aligned} \tag{5.26}$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\, setsize > 3 : \quad & \neg Adjacent(1,3,setsize > 3) \\ & \leftarrow \neg Successor(1,3,setsize > 3) \wedge \neg Predecessor(1,3,setsize > 3). \quad \square \end{aligned} \tag{5.27}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

Applying the ruler measure (2.1) and ruler convergence (2.2) to the set relations, countable distance space (3.1) and countable volume (4.1) yields the following insights and implications:

(1) The properties of metric space, Euclidean distance and area/volume can be derived from two set relations without using the notions of Euclidean geometry [**Joy98**] like plane, side, angle, perpendicular, congruence, intersection, etc.
(2) The ruler measure-based proofs provide the insight that distance is a function of the combinatorial *range*-to-domain set member mappings. Whereas, area/volume is a function of the combinatorial *domain*-to-domain set member mappings.
(3) The distance spanning multiple, disjoint, domain sets being proportionate to the number of members, d_c, in the union range set: $d_c = |\bigcup_{i=1}^{n} y_i|$ (3.1) generates the triangle inequality, non-negativity, and identity of indiscernibles properties of metric space (3.2). And combined with the constraint, $|x_i| = |y_i|$, also generates Manhattan and Euclidean distance, which motivates the fourth property of metric space, symmetry $[d(u,v) = d(v,u)]$. The reason Manhattan and Euclidean distance have the property of symmetry is that the type of combinatorial range-to-domain set mapping is the same for every domain-range set pair. For example, all continuous, symmetric distances, including elliptic geometry distances, generated

from the constraint, $|x_i| \geq |y_i|$, have the form: $d = (\sum_{i=1}^{n} s_i^{2/k})^{k/2}$, where $s_i = |a_i - b_i|$ and $0 < k \leq 2$.

(4) The case, $|x_i| = |y_i| = p_i$, limits the largest total number of range-to-domain set mappings (largest intersection and smallest distance) to $\sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^2$, which is the set-based reason Euclidean distance (3.7) is the smallest possible distance between two distinct points in $\mathbb{R}^n$.

(5) The charge, Newtonian gravity, and spacetime equations were all derived from the law that all Euclidean distance range intervals having a size, r, have proportionately sized intervals of other types, for example: $r = (r_c/q_e)q = (r_G/m_G)m = (r_c/t_c)/t$. Applying the ruler to those intervals allows application of the Euclidean area/volume and distance proofs to derive the charge force, Newtonian gravity force, and spacetime equations.

   (a) Time is a range variable proportionate to Euclidean distance, $(ct)^2 = r^2 = d_1^2 + d_2^2$ (5.8), where d_1^2 and d_2^2 are distances in two inertial frames of reference. But, there are area relationships between two domain variable charges and two domain variable masses, $q_1 q_2 = q^2 = (q_e/r_e)^2 r^2$ and $m_1 m_2 = m^2 = (m_G/r_G)^2 r^2$, which explains why the forces vary inversely with the square of the distance.

   (b) $ct^2 = r^2$ and $F = G m_1 m_2 / r^2 = k_C q_1 q_2 / r^2$ implies that as the distance, r, between two infinitesimal masses and between two infinitesimal charges goes gets smaller, the time interval also gets smaller (time slows). Time slowing as two infinitesimal masses or charges approach each other, implies they are decelerating, acting like particles with boundaries.

   (c) If there are quantum values of charge, q_e, and mass, m_G, then there are quantum distances, r_e and r_G, where the forces do not exist (not defined) at smaller distances. Quantum charge and mass might eliminate the need to invent stronger forces to override the charge and gravity forces at distances smaller than r_e and r_G.

   (d) Discrete valued variables (discrete states like spin or quark color) would not have proportionate continuous distance intervals. Therefore, discrete value changes with respect to time could appear at multiple locations, at the same time, independent of the distance between locations (for example, the change in spin of two quantum coupled particles).

   (e) Time, charge, and mass being proportionate to the Euclidean distance implies that time, charge, and mass (energy) would all increase if the Euclidean distance spanning the universe also increased, which would violate the conservation of energy law unless as new space (and energy in some forms) between galaxies is being created an equivalent amount of energy in some other form is disappearing (for example, disappearing in black holes).

(6) Relativity theory assumes that only 3 dimensions of space exist [**Bru17**]. The proof in this article (5.4) explains why time constrains physical distance and volume to at most three dimensions. There may be more dimensions of "space," but those dimensions have different "types" than the geometric dimensions (for example, time).

(7) The proof of at most 3 dimensions of any set of ordered and symmetric members (5.4), implies that each infinitesimal volume (ball) can have at most 3 ordered and symmetric dimensions of discrete values. And each dimension of discrete values can have at most 3 ordered and symmetric discrete values, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values corresponding to 27 possible "types" of infinitesimal balls.

(8) Where infinitesimal balls intersect, an algebra of the interactions of the discrete values needs to be developed. The interaction of the discrete values associated with the intersecting infinitesimal balls might result in what we perceive as particles, waves, and motion.

References

[Bru17] P. Bruskiewich, *A very simple introduction to special relativity: Part two - four vectors, the lorentz transformation and group velocity (the new mathematics for the millions book 38)*, Pythagoras Publishing, 2017. ↑8, 11

[CG15] W. Conradie and V. Goranko, *Logic and discrete mathematics*, Wiley, 2015. ↑3

[Coq15] Coq, *Coq proof assistant*, 2015. https://coq.inria.fr/documentation. ↑2

[Gol76] R. R. Goldberg, *Methods of real analysis*, John Wiley and Sons, 1976. ↑1, 2

[Joy98] D. E. Joyce, *Euclid's elements*, 1998. http://aleph0.clarku.edu/~djoyce/java/elements/elements.html. ↑10

[Rud76] W. Rudin, *Principles of mathematical analysis*, McGraw Hill Education, 1976. ↑1, 2

George Van Treeck, 668 Westline Dr., Alameda, CA 94501