The Real Analysis and Combinatorics of Geometry

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ABSTRACT. A ruler measures an interval in terms of the number of same-sized subintervals (units), ignoring partial subintervals. The range of possible surjective (many-to-one/combinatorial) mappings constrained by every disjoint domain set (of same-sized subintervals) having a corresponding distance set containing the same number of elements (same-sized subintervals) converges to the triangle inequality with taxicab (Manhattan) distance at the upper boundary and Euclidean distance at the lower boundary, which provides a set-based foundation for the definitions of metric space, longest, and shortest distance measures. A surjective definition of countable size converges to the product of interval sizes used in the Lebesgue measure and Euclidean integrals. A cyclic set of at most 3 dimensions emerges from the same surjective relations generating distance and volume. Implications for higher dimensional geometries are discussed. Proofs are verified in Coq.

CONTENTS

1. Introduction

The triangle inequality of a metric space, Euclidean distance metric, and the volume equation (product of interval sizes) of the Lebesgue measure and Euclidean integrals (for example, Riemann and Lebesgue integrals) are imported from Euclidean geometry as definitions [**Gol76**] rather than derived from set-based axioms. As a consequence, mathematical analysis has provided no insight into the relationships between countable sets that motivate and generate those geometric relations.

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A ruler measures a real-valued interval in terms of the nearest number of same-sized subintervals (units), where the partial subintervals are ignored. The ruler measure allows defining surjective (many-to-one/combinatorial) mappings from one or more of the subintervals in one interval to a same-sized subinterval in another interval. The discrete, surjective mappings converge to continuous, bijective functions as the subinterval size converges to zero.

The range of possible surjective mappings constrained by every disjoint domain set (of same-sized subintervals) having a corresponding distance set containing the same number of elements (same-sized subintervals) converges to the triangle inequality with taxicab (Manhattan) distance at the upper boundary and Euclidean distance at the lower boundary, which provides a set-based foundation for the definitions of metric space, longest, and shortest distance measures. A surjective definition of countable size converges to the product of interval sizes used in the Lebesgue measure and Euclidean integrals.

A cyclic set of at most 3 dimensions emerges from the same surjective relations generating distance and volume. Implications for higher dimensional geometries are discussed in the summary.

The proofs in this article are verified formally using the Coq Proof Assistant [**Coq15**] version 8.4p16. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at:
`https://github.com/treeck/CombinatorialGeometry`.

2. Ruler measure and convergence

Definition 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the nearest integer number of whole subintervals, p, times the subinterval size, c, where c is the independent variable. Notionally:

$$\forall\, c\, s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |b-a| \ \wedge\ c > 0\ \wedge \\ (p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \textstyle\sum_{i=1}^{p} c = pc. \tag{2.1}$$

Theorem 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\ \ s = |b-a|\ \ \Rightarrow\ \ M = \lim_{c\to 0} pc = s.$

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y:\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\quad p = floor(s/c)\quad \Rightarrow\quad 0 \le |p - s/c| < 1. \tag{2.2}$$

Multiply all sides of inequality 2.2 by $|c|$:

$$\forall\, c > 0,\quad 0 \le |p - s/c| < 1\quad \Rightarrow\quad 0 \le |pc - s| < |c|. \tag{2.3}$$

$$\forall\, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \\ \Rightarrow\quad \forall\, \epsilon = \delta,\ |c-0| < \delta\ \wedge\ |pc - s| < \epsilon\ \ :=\ \ M = \lim_{c\to 0} pc = s. \quad \square \tag{2.4}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure).

The following is an example of ruler convergence, where: $[0,\pi]$, $s = |\pi - 0|$, $c = 10^{-i}$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{i=1},\ 3.14_{i=2},\ 3.141_{i=3}, ..., \pi$.

3. Distance

A simple countable distance measure is that an image (distance) set has the same number of elements as a corresponding domain set. For example, the number of steps walked in a distance set must equal the number pieces of land traversed. Generalizing, for each disjoint domain set, x_i, containing p_i number of elements there exists a corresponding distance set, y_i, with the same p_i number of elements.

Notation conventions: The vertical bars around a set is the standard notation for indicating the cardinal (number of elements in the set). To prevent over use of the vertical bar, the symbol for "such that" is the colon.

If the distance sets intersect ($\sum_{i=1}^{n} |y_i| > |\bigcup_{i=1}^{n} y_i|$), then multiple domain set elements can correspond to a single distance element. Therefore, the size of the union of distance sets, d_c, is a function of the number of surjective (many-to-one) correspondences to each distance set element. Notionally:

Definition 3.1. Countable distance range, d_c:

$$\forall\, i\, n \in \mathbb{N}, \quad x_i \subseteq X, \quad \sum_{i=1}^{n} |x_i| = |\bigcup_{i=1}^{n} x_i|, \quad \forall\, x_i\ \exists\, y_i \subseteq Y :$$
$$|x_i| = |y_i| = p_i \quad \wedge \quad d_c = |Y| = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|.$$

The countable distance range principle (3.1), $|x_i| = |y_i| = p_i$, constrains the range of surjective correspondences from only one element of x_i corresponding to an element of y_i to as many as p_i number of elements of x_i corresponding to an element of y_i. More than p_i number of surjective correspondences to an element of y_i would be over-counting correspondences.

Using the rule of product, there is a range from $|y_i|\cdot 1 = p_i$ to $|y_i|\cdot p_i = p_i^2$ number of domain-to-distance surjective correspondences per distance set. The case of no intersecting distance sets yields the largest union size of the distance sets (largest countable distance, d_c). Therefore, $d_c = f(\sum_{i=1}^{n} p_i)$, is the longest possible distance because d_c is directly proportionate to the number of the number of correspondences. $d_c = f(\sum_{i=1}^{n} p_i^2)$ is the shortest possible distance because it is the case of the largest number of surjective correspondences per distance set (largest intersection of distance sets).

Using the ruler (2.1) to divide a set of real-valued domain intervals and a distance interval into sets of same-sized subintervals, and applying the ruler convergence theorem (2.2) proves that the longest and shortest distance cases converge to the real-valued taxicab (Manhattan) and Euclidean distance equations.

The following convergence proofs of the taxicab and Euclidean distance equations use the strategy of showing that the right and left sides of a proposed counting-based equation both converge to the same real value and therefore are equal. In other words, the propositional logic, $A = C \wedge B = C \Rightarrow A = B$, is used.

Theorem 3.2. *Taxicab (longest) distance, d, is the size of the distance interval,* $[y_0, y_m]$, *corresponding to a set of disjoint domain intervals,* $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_2,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, *where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|.$$

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance."

Proof.

Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into a set, x_i, containing p_i number of subintervals and apply the definition of the countable distance range (3.1), where each domain set has a corresponding distance set containing the same p_i number of elements.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i| = |y_i|. \tag{3.1}$$

Next, apply the rule of product to the case of one domain set element per distance set element:

$$\forall\, y_i \in Y,\; \textstyle\sum_{i=1}^{n} |y_i| \cdot 1 = \sum_{i=1}^{n} p_i. \tag{3.2}$$

Apply the countable distance range defintion (3.1) to 3.2:

$$\begin{aligned} &\forall\, y_i \in Y,\; \textstyle\sum_{i=1}^{n} |y_i| \cdot 1 = \sum_{i=1}^{n} p_i \quad \wedge \quad \sum_{i=1}^{n} |y_i| \le |Y| \\ &\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow \quad \exists\, y_i \in Y :\; \textstyle\sum_{i=1}^{n} p_i = |Y|. \end{aligned} \tag{3.3}$$

Multiply both sides of 3.3 by c and apply the ruler convergence theorem (2.2):

$$\begin{aligned} &s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \textstyle\sum_{i=1}^{n} (p_i \cdot c) = |Y| \cdot c \\ &\qquad\qquad \Rightarrow \quad \textstyle\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \lim_{c\to 0} |Y| \cdot c. \end{aligned} \tag{3.4}$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into a set, Y, containing p_d, number of subintervals:

$$|Y| \in \mathbb{N},\; c > 0 \quad \Rightarrow \quad \exists\, d \in \mathbb{R} :\; floor(d/c) = p_d = |Y|. \tag{3.5}$$

Multiply both sides of 3.5 by c and apply the ruler convergence theorem (2.2):

$$d = \lim_{c\to 0} p_d \cdot c \;\wedge\; p_d \cdot c = |Y| \cdot c \;\Rightarrow\; d = \lim_{c\to 0} p_d \cdot c = \lim_{c\to 0} |Y| \cdot c. \tag{3.6}$$

Combine equations 3.6 and 3.4:

$$d = \lim_{c\to 0} |Y| \cdot c \quad \wedge \quad \textstyle\sum_{i=1}^{n} s_i = \lim_{c\to 0} |Y| \cdot c \quad \Rightarrow \quad d = \sum_{i=1}^{n} s_i. \tag{3.7}$$ $\square$

THEOREM 3.3. *Euclidean (shortest) distance, d, is the size of the distance interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals, $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_2,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, where:*

$$d^2 = \textstyle\sum_{i=1}^{n} s_i^2, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}].$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into a set, x_i, containing p_i number of subintervals and apply the definition of the countable distance range (3.1), where each domain set has a corresponding distance set containing the same p_i number of elements.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i| = |y_i|. \tag{3.8}$$

Apply the rule of product to largest number of domain-to-distance surjective correspondences, where each of the p_i number of distance set elements in y_i corresponds to all p_i number of elements in the domain set x_i:

$$\textstyle\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} |\{(y_a, y_b) : y_a\, y_b \in y_i\}|, \tag{3.9}$$

where each pair, (y_a, y_b), represents a combination between two elements in the distance set, y_i. From the countable distance range definition (3.1):

$$|\textstyle\bigcup_{i=1}^{n} y_i| = |Y| \;\Rightarrow\; |\bigcup_{i=1}^{n} \{(y_a, y_b) : y_a\, y_b \in y_i\}| = |\{(y_a, y_b) : y_a\, y_b \in Y\}|. \tag{3.10}$$

$$\begin{aligned}(3.11)\quad & |\bigcup_{i=1}^{n}\{(y_a,y_b) : y_a\, y_b \in y_i\}| = |\{(y_a,y_b) : y_a\, y_b \in Y\}| \\ & \wedge \quad \sum_{i=1}^{n}|\{(y_a,y_b) : y_a\, y_b \in y_i\}| \geq |\bigcup_{i=1}^{n}\{(y_a,y_b) : y_a\, y_b \in y_i\}| \\ & \Rightarrow \quad \sum_{i=1}^{n}|\{(y_a,y_b) : y_a\, y_b \in y_i\}| \geq |\{(y_a,y_b) : y_a\, y_b \in Y\}|.\end{aligned}$$

From combining equation 3.9 and relation 3.11:

$$\begin{aligned}(3.12)\quad & \sum_{i=1}^{n}p_i^2 = \sum_{i=1}^{n}|\{(y_a,y_b) : y_a\, y_b \in y_i\}| \geq |\{(y_a,y_b) : y_a\, y_b \in Y\}| \\ & \Rightarrow \quad \exists\, y_i, Y : \sum_{i=1}^{n}p_i^2 = |\{(y_a,y_b) : y_a\, y_b \in Y\}|.\end{aligned}$$

Multiply both sides of equation 3.12 by c^2 and apply the ruler convergence theorem.

$$\begin{aligned}(3.13)\quad & s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n}(p_i \cdot c)^2 = |\{(y_a,y_b) : y_a\, y_b \in Y\}| \cdot c^2 \\ & \Rightarrow \quad \sum_{i=1}^{n}s_i^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \lim_{c\to 0} |\{(y_a,y_b) : y_a\, y_b \in Y\}| \cdot c^2.\end{aligned}$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$\begin{aligned}(3.14)\quad & |Y| \in \mathbb{N},\ c > 0 \quad \Rightarrow \\ & \exists\, d \in \mathbb{R} :\ \mathit{floor}(d/c) = p_d = |Y| \ \Rightarrow\ p_d^2 = |Y|^2 = |\{(y_a,y_b) : y_a\, y_b \in Y\}|,\end{aligned}$$

where $\{(y_a,y_b)\}$ is the set of all combination pairs of elements of Y. Multiply both sides of 3.14 by c^2 and apply the ruler convergence theorem (2.2):

$$\begin{aligned}(3.15)\quad & d = \lim_{c\to 0} p_d \cdot c \quad \wedge \quad (p_d \cdot c)^2 = |\{(y_a,y_b) : y_a\, y_b \in Y\}| \cdot c^2 \\ & \Rightarrow \quad d^2 = \lim_{c\to 0}(p_d \cdot c)^2 = \lim_{c\to 0} |\{(y_a,y_b) : y_a\, y_b \in Y\}| \cdot c^2.\end{aligned}$$

Combine equations 3.15 and 3.13:

$$\begin{aligned}(3.16)\quad & d^2 = \lim_{c\to 0} |\{(y_a,y_b) : y_a\, y_b \in Y\}| \cdot c^2 \quad \wedge \\ & \sum_{i=1}^{n}s_i^2 = \lim_{c\to 0} |\{(y_a,y_b) : y_a\, y_b \in Y\}| \cdot c^2 \quad \Rightarrow \quad d^2 = \sum_{i=1}^{n}s_i^2. \quad \square\end{aligned}$$

3.1. Triangle inequality. The definition of a metric in real analysis is based on the triangle inequality, $\mathbf{d}(\mathbf{u},\mathbf{w}) \leq \mathbf{d}(\mathbf{u},\mathbf{v}) + \mathbf{d}(\mathbf{v},\mathbf{w})$, that has been intuitively motivated by the triangle [**Gol76**]. Applying the ruler (2.1), and ruler convergence theorem (2.2) to the definition of a countable distance range (3.1) yields the real-valued triangle inequality:

$$\begin{aligned}(3.17)\quad & d_c = |Y| = |\bigcup_{i=1}^{2}y_i| \leq \sum_{i=1}^{2}|y_i| \quad \wedge \\ & d_c = \mathit{floor}(\mathbf{d}(\mathbf{u},\mathbf{w})/c) \ \wedge\ |y_1| = \mathit{floor}(\mathbf{d}(\mathbf{u},\mathbf{v})/c) \ \wedge\ |y_2| = \mathit{floor}(\mathbf{d}(\mathbf{v},\mathbf{w})/c) \\ & \Rightarrow \ \mathbf{d}(\mathbf{u},\mathbf{w}) = \lim_{c\to 0} d_c \cdot c \leq \sum_{i=1}^{2} \lim_{c\to 0} |y_i| \cdot c = \mathbf{d}(\mathbf{u},\mathbf{v}) + \mathbf{d}(\mathbf{v},\mathbf{w}).\end{aligned}$$

The other metric space properties: $\mathbf{d}(\mathbf{u},\mathbf{w}) = 0 \Leftrightarrow u = w$, $\mathbf{d}(\mathbf{u},\mathbf{w}) = \mathbf{d}(\mathbf{w},\mathbf{u})$, and $\mathbf{d}(\mathbf{u},\mathbf{w}) \geq 0$ also follow from the countable distance range definition.

4. Size (length/area/volume)

The surjective (many-to-one) relationship between all elements in domain set x_1 to each element of domain set x_2 results in the Cartesian product of $|x_1|\cdot|x_2|$ number of correspondences. This section will use the ruler (2.1) and ruler convergence theorem (2.2) to prove that the Cartesian product of the number of same-sized subintervals of intervals converges to the product of interval sizes. The first step is to define a set-based, countable size measure as the Cartesian product of disjoint domain set members.

Definition 4.1. Countable size (length/area/volume) measure, S_c:

$$\forall\, i\, n \in \mathbb{N}, \quad i \in [1, n], \quad x_i \subseteq X, \quad \textstyle\sum_{i=1}^{n} |x_i| = |\bigcup_{i=1}^{n} x_i|, \quad \wedge \quad S_c = \prod_{i=1}^{n} |x_i|.$$

Theorem 4.2. *Euclidean size (length/area/volume), S, is the size of an image interval, $[y_0, y_m]$, corresponding to a set of disjoint intervals:*
$\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_2,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, *where:*

$$S = \prod_{i=1}^{n} s_i, \quad S = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|, \quad i \in [1, n], \quad i, n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_size."

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into a set, x_i of p_i number of subintervals.

$$\forall\, i\, n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i|. \tag{4.1}$$

Use the ruler (2.1) to divide the exact size, $S = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_S^n subintervals. Every integer number, S_c, does **not** have an integer n^{th} root. However, for those cases where S_c does have an integer n^{th} root, there is a p_S^n that satisfies the definition a countable size measure, S_c (4.1). Notionally:

$$\forall\, p_S^n = S_c \in \mathbb{N}, \exists\, S \in \mathbb{R}, x_i \ : floor(S/c) = p_S^n = S_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i. \tag{4.2}$$

Multiply both sides of equation 4.2 by c^n to get the ruler measures:

$$p_S^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3:

$$S = \lim_{c \to 0} (p_S \cdot c)^n \quad \wedge \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c) \quad \wedge \quad \lim_{c \to 0} (p_i \cdot c) = s_i \tag{4.4}$$

$$\Rightarrow \quad S = \lim_{c \to 0} (p_S \cdot c)^n = \prod_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square$$

5. Ordered and symmetric geometries

Neither classical nor modern analytic geometry has been able to provide any insight into why physical Euclidean geometry appears to be limited to at most three dimensions. The same combinatorial relationships that generate the triangle inequality, taxicab distance, Euclidean distance, and volume also limit distance and volume to a cyclic set of at most three dimensions.

All distance measures, size measures, and orderings of domain intervals allowed by the countable distance range (3.1) and countable size (4.1) definitions exist. For example, there is no axiom of choice about which type of distance, taxicab or Euclidean, exists and does not exist between two distinct points because both types of distance are allowed by the countable distance range axiom.

Likewise, all orderings (permutations) of the domain intervals are allowed by the countable distance range and countable size axioms because the commutative properties of addition and multiplication allow orderings to yield the same distance and volume. Therefore, all orderings exist.

Mathematics defines the ordering (permutation) of a set in terms of a successor function and a predecessor (inverse order) function. A successor function and predecessor function can only generate every possible permutation if every element of a set is: 1) either a successor or predecessor of every other element of the set (herein refereed to as a symmetric geometry), 2) the first element of an ordering, and 3) the last element of an ordering.

It will now be proved that all permutations (a symmetric geometry) can only emerge from a successor function and predecessor function that defines a cyclic ordering on a set containing at most three elements (dimensions).

Definition 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i \;\in \{x_1, \ldots, x_n\},$$
$$\mathit{successor}\; x_i = x_{i+1} \;\;\wedge\;\; \mathit{predecessor}\; x_{i+1} = x_i.$$

where $\{x_1, \ldots, x_n\}$ are a set of real-valued intervals (dimensions).

Definition 5.2. Symmetric geometry (all permutations):

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; \mathit{successor}\; x_i = x_j \;\;\wedge\;\; \mathit{predecessor}\; x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric geometry is a cyclic set.*

$$\mathit{successor}\; x_n = x_1 \;\;\wedge\;\; \mathit{predecessor}\; x_1 = x_n.$$

The theorem and formal Coq-based proof is "ordered_symmetric_is_cyclic," which is located in the file threed.v.

Proof. The property of order (5.1) defines unique successors and predecessors for all elements except for the successor of x_n and the predecessor of x_1. From the properties of a symmetric geometry (5.2):

$$i = n \;\;\wedge\;\; j = 1 \;\;\wedge\;\; \mathit{successor}\; x_i = x_j \;\;\Rightarrow\;\; \mathit{successor}\; x_n = x_1. \tag{5.1}$$

$$i = n \;\;\wedge\;\; j = 1 \;\;\wedge\;\; \mathit{predecessor}\; x_j = x_i \;\;\Rightarrow\;\; \mathit{predecessor}\; x_1 = x_n. \tag{5.2}$$ □

In a cyclic set, every element is the first element of the set and every element the the last element of the set. Therefore, a cyclic set has a successor function and a predecessor function that starts at each element.

For example, using the cyclic set with elements labeled, $\{1,\ 2,\ 3\}$, starting with each element and counting through 3 cyclic successors and counting through 3 cyclic predecessors yields the permutations: $(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(1, 3, 2)$, $(3, 2, 1)$, and $(2, 1, 3)$. That is, a cyclically ordered set preserves sequential order while allowing some n-at-a-time permutations. If all possible n-at-a-time permutations are generated, then the cyclic set is also a symmetric geometry.

Theorem 5.4. *An ordered and symmetric geometry is limited to at most 3 elements.*

The Coq-based lemmas and proofs in the file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

Proof.
Because a symmetric and ordered set is a cyclic set (5.3), the successors and predecessors are cyclic:

Definition 5.5. Successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (m + 1 \leq setsize). \tag{5.3}$$

Definition 5.6. Predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (m - 1 \geq 1). \tag{5.4}$$

Definition 5.7. Adjacent: element m is adjacent to element n (an allowed permutation), if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{5.5}$$

Every element is adjacent to every other element, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{5.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (1 + 1 \leq 2). \tag{5.7}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (2 = 2 \wedge 1 = 1). \tag{5.8}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (1 + 1 \leq 2). \tag{5.9}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (2 - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (3 = 3 \wedge 1 = 1). \tag{5.11}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (1 = 1 \wedge 3 = 3). \tag{5.12}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (2 + 1 \leq 3). \tag{5.13}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (3 - 1 \geq 1). \tag{5.14}$$

Must prove that for all $setsize > 3$, there exist non-adjacent elements (not every permutation allowed). For example, the first and third elements are not adjacent:

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Successor(1, 3, setsize) \\ & \leftarrow Successor(1, 2, setsize) \leftarrow (1 + 1 \leq setsize). \end{aligned} \tag{5.15}$$

That is, 2 is the only successor of 1 for all $setsize > 3$, which implies 3 is not a successor of 1 for all $setsize > 3$.

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Predecessor(1, 3, setsize) \\ & \leftarrow Predecessor(1, n, setsize) \leftarrow (1 = 1 \wedge n = setsize). \end{aligned} \tag{5.16}$$

That is, $n = setsize$ is the only predecessor of 1 for all $setsize > 3$, which implies 3 is not a predecessor of 1 for all $setsize > 3$.

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Adjacent(1, 3, setsize) \\ & \leftarrow \neg Successor(1, 3, setsize) \wedge \neg Predecessor(1, 3, setsize). \end{aligned} \tag{5.17}$$

□

6. Summary

Applying the ruler measure (2.1) and ruler convergence proof (2.2) to a set of real-valued domain intervals and an image interval yields some new insights into geometry and physics.

(1) Countable, surjective functions (many-to-one/combinatorial relationships) converge to the continuous, bijective functions: triangle inequality, taxicab (Manhattan) distance, Euclidean distance and volume.
(2) Ruler-based proofs expose the difference between distance and size (length/area/volume) measures. The number of elements in a distance set is a function of the surjective correspondences from the elements of each disjoint domain set to the elements of an image (distance) set. In contrast, the number of elements in a size (length/area/volume) set is a function solely of the surjective correspondences between the elements of disjoint domain set elements.
(3) Applying the ruler measure to the surjective, countable distance range (3.1) provides the insight that all notions of distance are based on the principle that for each disjoint domain set there exists a corresponding distance set containing the same number of elements:
   (a) The countable distance range principle converges to the real-valued triangle inequality, which is the basis for the definition of metric space. The other properties of metric space also come from the countable distance range principle. Therefore, a function is not a distance metric unless it satisfies the more fundamental countable distance range principle (3.1).
   (b) The upper bound of the countable distance range converging to taxicab (Manhattan) distance shows that the longest possible path starting at point A and going to point B where each step decreases the distance to point B, is due to the union of disjoint distance sets, where there is a one-to-one correspondence (bijective mapping) of each domain set element to each distance set element.
   (c) The lower bound of the countable distance range converging to Euclidean distance provides the insight that the shortest possible distance path is due to the maximum intersection of distance sets within the constraint of the maximum number of surjective correspondences, where all of the p_i number of elements in the i^{th} domain set correspond to each of the p_i number of elements in the i^{th} distance set.
   (d) All $L^{p>2}$ norms generated from the countable distance range principle would require more than all the p_i number of elements in the i^{th} domain set corresponding to an element of the i^{th} distance set, which would be over-counting the number of possible surjective (many-to-one) correspondences. The definition of metric space and number theory have not provide this over-counting insight into $L^{p>2}$ norms.
   (e) Euclidean distance (3.3) was derived without any notions of side, angle, or shape. A parametric variable relating the sizes of two domain intervals can be easily derived using using calculus (Taylor series) and the Euclidean distance equation to generate the arc sine and arc cosine functions of the parametric variable (arc angle).

In other words, the notions of side and angle are derived from Euclidean distance, which is the reverse perspective of classical geometry [**Joy98**] [**Loo68**] [**Ber88**] and axiomatic foundations for geometry [**Bir32**] [**Hil80**] [**TG99**], which derive Euclidean distance from the Pythagorean Theorem stating that the sum of the squares on the sides of a right triangle are equal to square on the hypotenuse.

(4) Applying the ruler measure and ruler convergence proof to the countable size definition (4.1) allows a proof that the Cartesian product of same-sized subintervals of real-valued intervals converges to the product of the interval sizes (Euclidean volume):
   (a) Euclidean size (length/area/volume) was derived from a countable set-based notion of size without notions of sides, angles, and shape.
   (b) The countable set-based definition of size converging to Euclidean volume provides a more self-contained foundation under real analysis and calculus by not having to import volume from Euclidean geometry as a definition.
   (c) In real (functional) analysis and measure theory, the Lebesgue measure of a volume is defined as the sum of subset volumes [**Gol76**]. In this article, it is proved that the volume definition in the Lebesgue measure and Euclidean integrals, like the Riemann and Lebesgue integrals, is derived from the more fundamental ruler measure (2.1) and more fundamental notion of countable size (4.1) .

(5) Extending the notions of "distance" and "volume" beyond three dimensions violates the surjective (many-to-one/combinatorial) relations that generates the real-valued triangle inequality, taxicab (Manhattan) distance, Euclidean distance, and volume equations. This explains why we can only observe three physical dimensions of distance and volume.
   (a) Just as all distances and volumes allowed by the countable distance range (3.1) and size (volume) (4.1) exist, all orderings of dimensions allowed by the countable distance range and countable size also exist.
   (b) Mathematics defines the ordering of a set in terms of a successor function and a predecessor (inverse order) function. When the successor and predecessor functions generate all permutations, then the ordering must be cyclic (5.3) and the set size limited to at most three elements (dimensions) (5.4).
   (c) A higher dimensional geometry can be defined where distance and volume in three dimensions is a function of other (non-distance, non-volume) variables, like time. For example, applying the ruler measure to surjective relations between the same-sized subintervals of the set of three "distance" intervals and other sets of intervals might converge to real-valued functions describing phenomena in the three distance dimensions that are perceived as “particles”, “waves”, “mass”, “forces”, and “time.” Our universe might emerge from a few simple combinatorial relationships between the infinitesimal subintervals (quantum qubits) in multiple dimensions of real-valued intervals.

References

[Ber88] B. C. Berndt, *Ramanujan-100 years old (fashioned) or 100 years new (fangled)?*, The Mathematical Intelligencer **10** (1988), no. 3. ↑10

[Bir32] G. D. Birkhoff, *A set of postulates for plane geometry (based on scale and protractors)*, Annals of Mathematics **33** (1932). ↑10

[Coq15] Coq, *Coq proof assistant*, 2015. https://coq.inria.fr/documentation. ↑2

[Gol76] R. R. Goldberg, *Methods of real analysis*, John Wiley and Sons, 1976. ↑1, 5, 10

[Hil80] D. Hilbert, *The foundations of geometry (2cd ed)*, Chicago: Open Court, 1980. http://www.gutenberg.org/ebooks/17384. ↑10

[Joy98] D. E. Joyce, *Euclid's elements*, 1998. http://aleph0.clarku.edu/~djoyce/java/elements/elements.html. ↑10

[Loo68] E. S. Loomis, *The pythagorean proposition*, NCTM, 1968. ↑10

[TG99] A. Tarski and S. Givant, *Tarski's system of geometry*, The Bulletin of Symbolic Logic **5** (1999), no. 2. ↑10

[Wil95] A. Wiles, *Modular elliptic curves and fermat's last theorem*, Annals of Mathematics **141** (1995), 443–551. https://dx.doi.org/10.2307%2F2118559. ↑

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