# The Real Analysis and Combinatorics of Geometry

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Abstract. A range from 1-to-1 to many-to-many mappings between each disjoint domain set and each corresponding range set containing the same number of members, where the range sets in some cases intersect and the set members are the same-sized subintervals of intervals, converges to: the triangle inequality, Manhattan distance at the upper boundary, and Euclidean distance at the lower boundary, which provides set-based definitions of: metric space, longest, and shortest distances spanning disjoint sets. The Cartesian product of the same-sized subintervals of intervals converges to the product of interval sizes (Euclidean length/area/volume). The total ordering and symmetry properties of the set-based relations limit the number of dimensions of the same type to 3 dimensions. All ordered and symmetric, higher-dimensional geometries, like the spacetime four-vector, collapse into hierarchical 2 or 3-dimensional geometries. Proofs are verified in Coq.

Contents



## 1. Introduction

All of real analysis, measure, and integration is based on axioms from set and number theory – except for the notions of distance and volume. Metric space, the Manhattan and Euclidean distance metrics, and the product of interval sizes (Euclidean area/volume) are all defined [Gol76] rather than motivated and derived from more fundamental relations between countable sets.

The purpose of this article is motivate and derive real-valued distance and Euclidean volume from the relationships between countable sets. It will also be

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shown that these set-based relations have properties that constrain the number of dimensions of the same type and also constrain how one type of dimension, like time, relates to dimensions of another type, like space (length, width, and height).

The proofs in this article are verified formally using the Coq Proof Assistant [**Coq15**] version 8.7.0. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at:
`https://github.com/treeck/CombinatorialGeometry`.

## 2. Ruler measure and convergence

Euclidean distance and volume are derived from many-to-many relations. But, a function only allows each domain set member to map to one range set member. Therefore, a measure is needed that does not have Euclidean assumptions and also allows the full range of mappings from a one-to-one correspondence to a many-to-many mapping.

A ruler (measuring stick) measures a real-valued interval as the nearest integer number of same-sized subintervals (units), where the partial subintervals are ignored. The ruler measure allows defining combinatorial relations, for example a many-to-many relation, between the same-sized subintervals in one interval and the same-sized subintervals in another interval. The discrete, combinatorial relations converge to continuous, bijective functions as the subinterval size converges to zero.

Definition 2.1. Ruler measure: A ruler measures the size, $M$, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, $p$, each subinterval having the same size, $c$. Notionally:

$$\begin{aligned}(2.1)\quad &\forall\, c\ s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |a-b| \ \wedge\ c > 0\ \wedge \\ &\qquad (p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \textstyle\sum_{i=1}^{p} c = pc.\end{aligned}$$

Theorem 2.2. *Ruler convergence:*
$\forall\ [a,b] \subset \mathbb{R},\ \ s = |a-b| \ \Rightarrow\ M = \lim_{c\to 0} pc = s.$

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y:\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$(2.2)\qquad \forall\, c > 0,\quad p = floor(s/c) \quad\Rightarrow\quad 0 \le |p - s/c| < 1.$$

Multiply all sides of inequality 2.2 by $|c|$:

$$(2.3)\qquad \forall\, c > 0,\quad 0 \le |p - s/c| < 1 \quad\Rightarrow\quad 0 \le |pc - s| < |c|.$$

$$\begin{aligned}(2.4)\quad &\forall\, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \\ &\qquad \Rightarrow\quad \forall\, \epsilon = \delta:\ |c-0| < \delta\ \wedge\ |pc - s| < \epsilon\ \ := \ \ M = \lim_{c\to 0} pc = s. \quad \square\end{aligned}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence, where: $[0,\pi]$, $s = |0-\pi|$, $c = 10^{-i}$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{i=1},\ 3.14_{i=2},\ 3.141_{i=3}, ..., \pi$.

## 3. Distance

**3.1. Countable metric space.** A simple countable distance measure is that a range (distance) set has the same number of members as a corresponding domain set. For example, the number of steps walked in a distance set must equal the number pieces of land traversed. Generalizing, for each distance set, $y_i$, containing $p_i$ number of members there exists a corresponding domain set, $x_i$, with the same $p_i$ number of members.

**Notation conventions:** The vertical bars around a set is the standard notation for indicating the cardinal (number of members in the set). To prevent over use of the vertical bar, the symbol for "such that" is the colon.

If the domain sets are disjoint ($\sum_{i=1}^{n} |x_i| = |\bigcup_{i=1}^{n} x_i| \Leftrightarrow \bigcap_{i=1}^{n} x_i = \emptyset$) and the distance sets intersect ($\sum_{i=1}^{n} |y_i| > |\bigcup_{i=1}^{n} y_i| \Leftrightarrow \bigcap_{i=1}^{n} y_i \neq \emptyset$), then multiple domain set members can map to a distance set member. Therefore, the size of the union of the distance sets, $d_c$, is related to the number of domain-to-distance member mappings. Notionally:

Definition 3.1. Countable metric space, $d_c$:

$$\forall\, i\; n \in \mathbb{N}, \quad x_i \subseteq X, \quad \textstyle\sum_{i=1}^{n} |x_i| = |\bigcup_{i=1}^{n} x_i|, \quad \forall\, x_i\; \exists\, y_i \subseteq Y :$$
$$|x_i| = |y_i| = p_i \quad \wedge \quad d_c = |Y| = |\textstyle\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|.$$

**3.2. Metric Space.** Applying the ruler (2.1), and ruler convergence theorem (2.2) to the definition of a countable metric space (3.1) yields the real-valued triangle inequality and non-negativity properties of metric space:

$$\begin{aligned}(3.1)\quad & d_c = |Y| = |\textstyle\bigcup_{i=1}^{2} y_i| \leq \sum_{i=1}^{2} |y_i| \quad \wedge \\ & d_c = floor(\mathbf{d}(\mathbf{u},\mathbf{w})/c) \;\wedge\; |y_1| = floor(\mathbf{d}(\mathbf{u},\mathbf{v})/c) \;\wedge\; |y_2| = floor(\mathbf{d}(\mathbf{v},\mathbf{w})/c) \\ & \Rightarrow\; \mathbf{d}(\mathbf{u},\mathbf{w}) = \lim_{c\to 0} d_c \cdot c \leq \textstyle\sum_{i=1}^{2} \lim_{c\to 0} |y_i| \cdot c = \mathbf{d}(\mathbf{u},\mathbf{v}) + \mathbf{d}(\mathbf{v},\mathbf{w}).\end{aligned}$$

$$\begin{aligned}(3.2)\quad & d(u,w) \geq 0 : \quad \forall\, c > 0, \quad d_c = floor(d(u,w)/c) \quad \wedge \quad d_c = |Y| \geq 0 \\ & \Rightarrow \quad \forall\, c > 0, \quad d(u,w) = \lim_{c\to 0} d_c \cdot c \geq 0.\end{aligned}$$

**3.3. Metric space range.** Consider the trivial case of the countable metric space principle (3.1), where a domain set has only one member: $|x_i| = |y_i| = p_i = 1$: 1) Each member of $x_i$ maps to the each member of $y_i$, bijectively, yielding $p_i$ number of domain-to-distance member mappings. 2) Each member of $x_i$ maps to the all $p_i$ number of members in $y_i$, yielding $p_i^2$ number of domain-to-distance member mappings. And every member of a set having the same properties implies the same range of domain-to-distance mappings, $p_i$ to $p_i^2$, for all sizes of domain sets.

Therefore, $d_c = f(\sum_{i=1}^{n} p_i)$ is the largest possible distance because it is the case the smallest number of domain-to-distance mappings (no intersection of the distance sets). And $d_c = f(\sum_{i=1}^{n} p_i^2)$ is the smallest possible distance because it is the case the largest number of domain-to-distance mappings (largest allowed intersection of distance sets).

It will now be proved that using the ruler (2.1) to divide a set of real-valued domain intervals and a distance interval into sets of same-sized subintervals, and applying the ruler convergence theorem (2.2) to the longest and shortest distance cases converge to the real-valued, Manhattan and Euclidean distance functions.

### 3.4. Manhattan distance.

Theorem 3.2. *Manhattan (longest) distance, $d$, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$d = \textstyle\sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance."

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, $x_i$, containing $p_i$ number of subintervals and apply the definition of the countable metric space (3.1), where each domain set has a corresponding distance set containing the same $p_i$ number of members.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad s_i \in \mathbb{R}, \quad \exists\, c > 0 \;:\; \mathit{floor}(s_i/c) = p_i = |x_i| = |y_i|. \tag{3.3}$$

Next, apply the rule of product to the case of one domain set member per distance set member:

$$|y_i| = p_i \quad \Rightarrow \quad \textstyle\sum_{i=1}^{n} |y_i| \cdot 1 = \sum_{i=1}^{n} p_i. \tag{3.4}$$

Apply the countable metric space defintion (3.1) to equation 3.4:

$$\begin{aligned} &\textstyle\sum_{i=1}^{n} |y_i| \cdot 1 = \sum_{i=1}^{n} p_i \quad \wedge \quad d_c \le \sum_{i=1}^{n} |y_i| \\ &\qquad \Rightarrow \quad d_c \le \textstyle\sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c :\; d_c = \sum_{i=1}^{n} p_i. \end{aligned} \tag{3.5}$$

Multiply both sides of equation 3.5 by $c$:

$$d_c = \textstyle\sum_{i=1}^{n} p_i \quad \Rightarrow \quad d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c). \tag{3.6}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definiton of $d$:

$$d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d :\; \mathit{floor}(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c. \tag{3.7}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definiton of $s_i$:

$$s_i = |a_i - b_i| \quad \Rightarrow \quad \mathit{floor}(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c \to 0} p_i \cdot c = s_i. \tag{3.8}$$

Combine equations 3.7, 3.6, 3.8:

$$\begin{aligned} &d = \lim\nolimits_{c \to 0} d_c \cdot c \quad \wedge \quad d_c \cdot c = \textstyle\sum_{i=1}^{n} (p_i \cdot c) \quad \wedge \quad \lim\nolimits_{c \to 0} p_i \cdot c = s_i \\ &\qquad \Rightarrow \quad d = \lim\nolimits_{c \to 0} d_c \cdot c = \textstyle\sum_{i=1}^{n} \lim\nolimits_{c \to 0} (p_i \cdot c) = \sum_{i=1}^{n} s_i. \end{aligned} \tag{3.9}$$

□

### 3.5. Euclidean distance.

Theorem 3.3. *Euclidean (shortest) distance, $d$, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$d^2 = \textstyle\sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, $x_i$, containing $p_i$ number of subintervals and apply

the definition of the countable metric space (3.1), where each domain set has a corresponding distance set containing the same $p_i$ number of members.

(3.10)
$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1,n], \quad s_i \in \mathbb{R}, \quad \exists\, c > 0 \;: \quad floor(s_i/c) = p_i = |x_i| = |y_i|.$$

Apply the rule of product to the largest number of domain-to-distance set mappings, where all $p_i$ number of domain set members, $x_i$, map to each of the $p_i$ number of members in the distance set, $y_i$:

$$\textstyle\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2. \tag{3.11}$$

Choose the equality case of the Cauchy-Schwartz inequality:

$$\begin{aligned}\textstyle\sum_{i=1}^{n} p_i^2 \;\le\; \sum_{i=1}^{n} p_i^2 + \sum_{i=1,j=1,i\neq j}^{n} (p_i \cdot p_j) \;&=\; (\textstyle\sum_{i=1}^{n} p_i)^2 \\ &\Rightarrow\; \exists\, p_i : (\textstyle\sum_{i=1}^{n} p_i)^2 = \sum_{i=1}^{n} p_i^2.\end{aligned} \tag{3.12}$$

Choose the equality case of the countable metric space definition (3.1) and square both sides ($x = y \Rightarrow f(x) = f(y)$):

$$\begin{aligned} d_c \le \textstyle\sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \;\Rightarrow\; \exists\, p_i,\, d_c : d_c = \sum_{i=1}^{n} p_i \\ \Rightarrow\; \exists\, p_i,\, d_c : d_c^2 = (\textstyle\sum_{i=1}^{n} p_i)^2.\end{aligned} \tag{3.13}$$

Combine equations 3.13 and and 3.12:

$$\begin{aligned}\exists\, p_i,\, d_c : d_c^2 = (\textstyle\sum_{i=1}^{n} p_i)^2 \quad \wedge \quad \exists\, p_i : (\sum_{i=1}^{n} p_i)^2 = \sum_{i=1}^{n} p_i^2 \\ \Rightarrow \quad \exists\, p_i,\, d_c : d_c^2 = \textstyle\sum_{i=1}^{n} p_i^2.\end{aligned} \tag{3.14}$$

Multiply both sides of equation 3.14 by $c^2$.

$$d_c^2 = \textstyle\sum_{i=1}^{n} p_i^2 \quad \Rightarrow \quad (d_c \cdot c)^2 = \sum_{i=1}^{n} (p_i \cdot c)^2. \tag{3.15}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definiton of $d$:

$$d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d : floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c. \tag{3.16}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definiton of $s_i$:

$$s_i = |a_i - b_i| \quad \Rightarrow \quad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i. \tag{3.17}$$

Combine equations 3.16, 3.15, 3.17:

$$\begin{aligned} d = \lim\nolimits_{c\to 0} d_c \cdot c \quad \wedge \quad (d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} (p_i \cdot c)^2 \quad \wedge \quad \lim\nolimits_{c\to 0} p_i \cdot c = s_i \\ \Rightarrow \quad d^2 = \lim\nolimits_{c\to 0} (d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim\nolimits_{c\to 0} (p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \quad \square\end{aligned} \tag{3.18}$$

## 4. Euclidean Volume

The number of all possible combinations between members in a countable set $x_1$ and a countable set $x_2$ is the Cartesian product, $|x_1| \cdot |x_2|$. This section will use the ruler (2.1) and ruler convergence theorem (2.2) to prove that the Cartesian product of the same-sized subintervals of intervals converges to the product of interval sizes as the subinterval converges to zero. The first step is to define a countable set-based measure of area/volume as the Cartesian product of disjoint domain set members.

Definition 4.1. Countable volume measure, $V_c$:

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1,n], \quad x_i \subseteq X, \quad \textstyle\sum_{i=1}^{n} |x_i| = |\bigcup_{i=1}^{n} x_i|, \quad V_c = \displaystyle\prod_{i=1}^{n} |x_i|.$$

THEOREM 4.2. *Euclidean volume, $V$, is the size of a range interval, $[v_0, v_m]$, corresponding to a set of disjoint intervals:*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$V = \prod_{i=1}^{n} s_i, \quad V = |v_0 - v_m|, \quad s_i = |a_i - b_i|, \quad i \in [1, n], \quad i, n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_volume."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, $x_i$ of $p_i$ number of subintervals.

$$\forall\, i\ n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i|. \tag{4.1}$$

Use the ruler (2.1) to divide the exact size, $V = |v_0 - v_m|$, of the range interval, $[v_0, v_m]$, into $p^n$ subintervals. Every integer number, $V_c$, does **not** have an integer $n^{th}$ root. However, for those cases where $V_c$ does have an integer $n^{th}$ root, there is a $p^n$ that satisfies the definition a countable volume measure, $V_c$ (4.1). Notionally:

$$\forall\, p^n = V_c \in \mathbb{N},\ \exists\, V \in \mathbb{R},\ x_i\ : floor(V/c) = V_c = p^n = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i. \tag{4.2}$$

Multiply both sides of equation 4.2 by $c^n$ to get the ruler measures:

$$p^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad (p \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3:

$$V = \lim_{c \to 0} (p \cdot c)^n \quad \wedge \quad (p \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c) \quad \wedge \quad \lim_{c \to 0}(p_i \cdot c) = s_i \tag{4.4}$$

$$\Rightarrow \quad V = \lim_{c \to 0} (p \cdot c)^n = \prod_{i=1}^{n} \lim_{c \to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i. \qquad \square$$

## 5. Ordered and symmetric geometries

Inspecting the equations of distance and volume, there is no reason to assume any limit to the number dimensions. If there are any limitations to the number dimensions, then those limitations probably come from the underlying principles that geneate distance and volume.

The union operations in the countable metric space principle (3.1) generating all real-valued distance equations and the countable volume principle (4.1) generating Euclidean volume requires being able to iterate sequentially through each set (dimension), which implies a total ordering exists. The commutative property of union also allows each domain set (dimension) to be sequentially adjacent to every other dimension (herein, referred to as a symmetric geometry).

The set of every possible order of $n$ number of dimensions allows an order where some dimension is sequentially adjacent to some other dimension. Using that logic, then any number of dimensions are possible. But, that logic is **flawed**!

Asserting that a specific order exists (is true), for example, $\{x_1, x_2, x_3, x_4\}$, contradicts the assertion that $x_1$ is sequentially adjacent to $x_3$. It will now be proved that satisfying both the total ordering and symmetry properties limits the number of dimensions of distance and volume.

DEFINITION 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i\; \in \{x_1, \ldots, x_n\},$$
$$\textit{successor } x_i = x_{i+1} \;\;\wedge\;\; \textit{predecessor } x_{i+1} = x_i,$$

where each $x_i \in \{x_1, \ldots, x_n\}$ is a set of subintervals of a real-valued domain interval (dimension).

DEFINITION 5.2. Symmetric geometry (every member is sequentially adjacent to every other member):

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; \textit{successor } x_i = x_j \;\;\wedge\;\; \textit{predecessor } x_j = x_i.$$

THEOREM 5.3. *An ordered and symmetric geometry is a cyclic set.*

$$\textit{successor } x_n = x_1 \;\;\wedge\;\; \textit{predecessor } x_1 = x_n.$$

The theorem and formal Coq-based proof is "ordered_symmetric_is_cyclic," which is located in the file threed.v.

PROOF. The property of order (5.1) defines unique successors and predecessors for all members except for the successor of $x_n$ and the predecessor of $x_1$. Therefore, the only member that can be a successor of $x_n$, without creating a contradiction, is $x_1$. And the only member that can be a predecssor of $x_1$, without creating a constradiction, is $x_n$. From the properties of a symmetric geometry (5.2):

$$i = n \;\;\wedge\;\; j = 1 \;\;\wedge\;\; \textit{successor } x_i = x_j \;\;\Rightarrow\;\; \textit{successor } x_n = x_1. \tag{5.1}$$

$$i = n \;\;\wedge\;\; j = 1 \;\;\wedge\;\; \textit{predecessor } x_j = x_i \;\;\Rightarrow\;\; \textit{predecessor } x_1 = x_n. \tag{5.2}$$

□

THEOREM 5.4. *An ordered and symmetric geometry is limited to at most 3 members.*

The Coq-based lemmas and proofs in the file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
Because a symmetric and ordered set is a cyclic set (5.3), the successors and predecessors are cyclic:

DEFINITION 5.5. Successor of $m$ is $n$:

$$\textit{Successor}(m, n, \textit{setsize}) \leftarrow (m = \textit{setsize} \wedge n = 1) \vee (m + 1 \leq \textit{setsize}). \tag{5.3}$$

DEFINITION 5.6. Predecessor of $m$ is $n$:

$$\textit{Predecessor}(m, n, \textit{setsize}) \leftarrow (m = 1 \wedge n = \textit{setsize}) \vee (m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: member $m$ is sequentially adjacent to member $n$ (required for a "symmetric" set (5.2)), if the cyclic successor of $m$ is $n$ or the cyclic predecessor of $m$ is $n$. Notionally:

$$\textit{Adjacent}(m, n, \textit{setsize}) \leftarrow \textit{Successor}(m, n, \textit{setsize}) \vee \textit{Predecessor}(m, n, \textit{setsize}). \tag{5.5}$$

Every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{5.6}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (1 + 1 \leq 2). \tag{5.7}$$

$$Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (2 = 2 \wedge 1 = 1). \tag{5.8}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (1 + 1 \leq 2). \tag{5.9}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (2 - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (3 = 3 \wedge 1 = 1). \tag{5.11}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (1 = 1 \wedge 3 = 3). \tag{5.12}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (2 + 1 \leq 3). \tag{5.13}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (3 - 1 \geq 1). \tag{5.14}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not adjacent:

$$\forall\ setsize > 3: \quad \neg Successor(1,3,setsize) \\ \leftarrow Successor(1,2,setsize) \leftarrow (1 + 1 \leq setsize). \tag{5.15}$$

That is, 2 is the only successor of 1 for all $setsize > 3$, which implies 3 is not a successor of 1 for all $setsize > 3$.

$$\forall\ setsize > 3: \quad \neg Predecessor(1,3,setsize) \\ \leftarrow Predecessor(1,n,setsize) \leftarrow (1 = 1 \wedge n = setsize). \tag{5.16}$$

That is, $n = setsize$ is the only predecessor of 1 for all $setsize > 3$, which implies 3 is not a predecessor of 1 for all $setsize > 3$.

$$\forall\ setsize > 3: \quad \neg Adjacent(1,3,setsize) \\ \leftarrow \neg Successor(1,3,setsize) \wedge \neg Predecessor(1,3,setsize). \quad \square \tag{5.17}$$

## 6. Summary

Applying some very simple real analysis, in the form of the ruler measure (2.1) and ruler convergence proof (2.2), to a set of real-valued domain intervals and a range interval yields some new insights into geometry and physics.

(1) Discrete, combinatorial relations converge to the continuous, bijective relations: triangle inequality, Manhattan distance, Euclidean distance and volume. Other types of measures do not have that capability.

(2) Ruler measure-based proofs expose the difference between distance and volume measures: Distance is a mapping relation between the members of each disjoint domain set and members of a corresponding range (distance) set. In contrast, volume is a combinatorial relation between the members of disjoint domain sets. Other types of measures can not provide that insight.

(3) Applying the ruler measure to the countable metric space (3.1) provides the insight that all notions of distance are based on the principle that for

each disjoint domain set there exists a corresponding distance set containing the same number of members, where the distance sets in some cases intersect:

(a) Applying the ruler and ruler convergence to the countable metric space principle (3.1) generates the real-valued triangle inequality and non-negativity properties of metric space (3.2). Therefore, a function is not a distance metric unless it satisfies the more fundamental countable metric space.

(b) All $L^{p>2}$ norms generated from the countable metric space principle would require each member of the $i^{th}$ domain set to map to a member of the $i^{th}$ distance set more than once, which would be over-counting the number of possible mappings. Therefore, $L^{p>2}$ norms are not valid distance measures. Other measure theories have not provided this over-counting insight into $L^{p>2}$ norms.

(c) The upper bound of the countable metric space converging to Manhattan distance (3.2) provides the insight that the largest (longest) monotonic distance path is the case of disjoint distance sets, where there is a 1-1 correspondence between the domain and distance set members.

(d) The lower bound of the countable metric space converging to Euclidean distance (3.3) provides the insight that the smallest (shortest) possible monotonic distance path is the case of the maximum allowed intersection of the distance sets, where there is a many-to-many mapping from domain to distance set members.

(e) Euclidean distance (3.3) was derived from a set-based, many-to-many relation without any notions of side, angle, or shape. A parametric variable relating the sizes of two domain intervals can be easily derived using using calculus (Taylor series) and the Euclidean distance equation to generate the arc sine and arc cosine functions of the parametric variable (arc angle). In other words, the notions of side and angle are derived from Euclidean distance, which is the reverse perspective of classical geometry [**Joy98**] [**Loo68**] [**Ber88**] and axiomatic foundations for geometry [**Bir32**] [**Hil80**] [**TG99**].

(4) Applying the ruler measure and ruler convergence proof to the countable volume definition (4.1) allows a proof that the Cartesian product of same-sized subintervals of real-valued intervals converges to the product of the interval sizes (Euclidean length/area/volume):

(a) Euclidean volume was derived from a combinatorial relation without notions of sides, angles, and shape.

(b) Other types of measures, like the Lebesgue and Hausdorff measures, Riemann and Lebesgue integration, and vector analysis can only assume Euclidean space.

(5) The set-based relations of countable metric space (3.1) and countable volume (4.1) that generate metric space, Manhattan distance, Euclidean distance, and volume equations have the properties of total ordering (5.1) and symmetry (5.2). A geometry that is simultaneously both ordered and symmetric limits distance and volume to a cyclic set (5.3) of three dimensions (5.4), which explains why there are only three dimensions of

physical space.

(6) All valid higher dimensional theories of physics must collapse into hierarchical 2 or 3-dimensional geometries, where all domain dimensions at each level in the hierarchy are the same type. The four-vectors common in physics are 2-dimensional geometries that have been "flattened." For example, the spacetime four-vector length, $d = \sqrt{(ct)^2 - (x^2 + y^2 + z^2)}$, can be expressed in a form like, $d_2 = \sqrt{(ct)^2 - d_1^2}$, where $d_1 = \sqrt{x^2 + y^2 + z^2}$ and $d_2 = d$.

Applying the Euclidean distance proof (3.3) to the 2-dimensional spacetime equation, $(ct)^2 = d_1^2 + d_2^2$, provides the perspective that $d_1$ and $d_2$ are lengths in two frames of reference (the sizes of two domain intervals of the same distance type) and the size of each range subinterval is the same size (same speed of light) in both frames of reference.

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