The Two Set Relations Generating Euclidean Geometry

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Abstract. A ruler-like measure divides sets of real-valued domain and range intervals into same-sized subintervals. The case, where for each disjoint domain set of subintervals there exists a corresponding same-sized range set and the range sets in some cases intersect, converges to: the triangle inequality, Manhattan distance at the upper boundary and Euclidean distance at the lower boundary of the triangle inequality. The Cartesian product of the number of subintervals in each domain interval converges to the product of interval interval sizes (Euclidean area/volume). Time places a constraint on physical sets, which limits physical distance and volume to at most 3 physical dimensions. All proofs are verified in Coq.

Contents

1. Introduction

Most of real analysis is derived from a foundation of number and set-based axioms. But, the triangle inequality, Manhattan distance, Euclidean distance, and Euclidean area/volume, are motivated by Euclidean geometry and used as definitions (primitives) in measure (metric space, Hausdorff, and Lebesgue) and integration (Lebesgue and Riemann) [**Gol76**] [**Rud76**]. This article will use some simple real analysis to motivate and derive the triangle inequality, the other properties of metric space, Manhattan and Euclidean distance from a single set-based axiom and derive volume from another set-based axiom.

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The relationships between countable sets generating distance and volume provides new insights into geometry and physics. For example, (*without any notions of side, angle, and shapes*): 1) the set-based reason Euclidean distance is the smallest distance between two distinct points in $\mathbb{R}^n$; 2) the single set relation generating all the properties of metric space; 3) the set-based difference between distance and volume; 4) the set-based reason metric space might not be a sufficient condition for a distance measure; 5) how time constrains the set relations generating *physical* Euclidean geometry to at most 3 dimensions.

The proofs in this article are verified formally using the Coq Proof Assistant [**Coq15**] version 8.9.0. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at:
https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

Deriving distance and volume from set and number axioms requires a measure that does not have Euclidean assumptions and also allows the full range of mappings from a one-to-one correspondence to a many-to-many mapping. A ruler (measuring stick) measures a real-valued interval as the nearest integer number of same-sized subintervals (units), where the partial subintervals are ignored.

The ruler measure allows defining relations, for example a many-to-many relation, between the set of same-sized subintervals in one interval and the set of same-sized subintervals in another interval. The countable relations converge to continuous, bijective functions as the subinterval size converges to zero.

Definition 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, p, each subinterval having the same size, c. Notionally:

$$\forall\, c\, s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |a-b| \ \wedge\ c > 0\ \wedge \\ (p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \textstyle\sum_{i=1}^{p} c = pc. \tag{2.1}$$

Theorem 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\ \ s = |a-b|\ \Rightarrow\ M = \lim_{c\to 0} pc = s.$

The theorem, "limit_c_0_M_eq_exact_size," and formal proof is located in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\ p = floor(s/c)\ \wedge\ 0 \le |floor(s/c) - s/c| < 1\ \Rightarrow\ 0 \le |p - s/c| < 1. \tag{2.2}$$

Multiply all sides of inequality 2.2 by $|c|$:

$$\forall\, c > 0, \quad 0 \le |p - s/c| < 1 \quad \Rightarrow \quad 0 \le |pc - s| < |c|. \tag{2.3}$$

$$\forall\, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \\ \Rightarrow\quad \forall\, \epsilon = \delta :\ \ |c-0| < \delta\ \wedge\ |pc - s| < \epsilon\ \ :=\ \ M = \lim_{c\to 0} pc = s. \quad \square \tag{2.4}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence, where: $[0,\pi]$, $s = |0-\pi|$, $c = 10^{-i}$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{i=1},\ 3.14_{i=2},\ 3.141_{i=3}, ..., \pi$.

3. Distance

Notation conventions: The vertical bars around a set is the standard notation for indicating the cardinal (number of members in the set). To prevent over use of the vertical bar, the symbol for "such that" is the colon.

3.1. Countable distance space. The most fundamental notion of distance is that for each disjoint domain set, x_i, there exists a corresponding range set, y_i, containing the same number of members, p_i: $|x_i| = |y_i| = p_i$. For example, there should be as many steps walked in the range set, y_i, as there are pieces of traversed land in the corresponding domain set, x_i. And the distance, d_c, spanning one or more domain sets is the size of the union of range sets.

Definition 3.1. Countable distance space, d_c:

$$\bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad d_c = |\bigcup_{i=1}^n y_i| \quad \wedge \quad |x_i| = |y_i| = p_i.$$

Theorem 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$.

Proof. This inequality is derived from the inclusion-exclusion principle [**CG15**] and the axiom, $u = v - w,\ w \geq 0 \Rightarrow u \leq v$. A formal proof using partitioning rather than the da Silva formula, inclusion_exclusion_inequality, can be found in the file euclidrelations.v.

$$\begin{aligned} |\bigcup_{i=1}^n y_i| = \textstyle\sum_{i=1}^n |y_i| - \sum_{1\leq i<j\leq n} |y_i \cap y_j| + \cdots + (-1)^{n-1} |\bigcap_{i=1}^n y_i|) \quad \wedge \\ \textstyle\sum_{1\leq i<j\leq n} |y_i \cap y_j| + \cdots + (-1)^{n-1} |\bigcap_{i=1}^n y_i|) \geq 0 \\ \Rightarrow \quad |\bigcup_{i=1}^n y_i| \leq \textstyle\sum_{i=1}^n |y_i|. \end{aligned} \tag{3.1}$$ □

3.2. Metric Space. Applying the ruler (2.1) and ruler convergence (2.2) to three range intervals having sizes, $d(u,w)$, $d(u,v)$, and $d(v,w)$, and using the inequality, $d_c = |\bigcup_{i=1}^2 y_i| \leq \sum_{i=1}^2 |y_i|$ generates all the properties of metric space. The formal proofs: triangle_inequality, non_negativity, identity_of_indiscernibles, and symmetry, are found in the Coq file, euclidrelations.v.

Theorem 3.3. *Triangle Inequality:* $d(u,w) \leq d(u,v) + d(v,w)$*:*

Proof.

$$\begin{aligned} \forall\, c > 0,\ |y_1| = floor(d(u,v)/c) \ \wedge\ |y_2| = floor(d(v,w)/c) \quad \wedge \\ d_c = floor(d(u,w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \leq |y_1| + |y_2| \\ \Rightarrow\ floor(d(u,w)/c) \leq floor(d(u,v)/c) + floor(d(v,w)/c) \\ \Rightarrow\ floor(d(u,w)/c)\cdot c \leq floor(d(u,v)/c)\cdot c + floor(d(v,w)/c)\cdot c \\ \Rightarrow\ \lim_{c\to 0} floor(d(u,w)/c)\cdot c \leq \lim_{c\to 0} floor(d(u,v)/c)\cdot c + \lim_{c\to 0} floor(d(v,w)/c)\cdot c \\ \Rightarrow\ d(u,w) \leq d(u,v) + d(v,w). \end{aligned} \tag{3.2}$$ □

Theorem 3.4. *Non-negativity:* $d(u,w) \geq 0$.

Proof.

$$\begin{aligned} \forall\, c > 0 : \quad d_c = floor(d(u,w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \geq 0 \\ \Rightarrow \quad floor(d(u,w)/c) = d_c \geq 0 \quad \Rightarrow \quad d(u,w) = \lim_{c\to 0} d_c \cdot c \geq 0. \end{aligned} \tag{3.3}$$ □

Theorem 3.5. *Identity of Indiscernibles:* $d(w,w) = 0$.

PROOF.

$$\forall\, d(u,v) = d(v,w) = 0 \;\wedge\; d(u,w) \leq d(u,v) + d(v,w) \;\wedge\; d(u,w) \geq 0 \Rightarrow d(u,w) = 0. \tag{3.4}$$

$$d(u,w) = 0 \;\wedge\; d(u,v) = 0 \;\Rightarrow\; w = v. \tag{3.5}$$

$$d(v,w) = 0 \;\wedge\; w = v \;\Rightarrow\; d(w,w) = 0. \tag{3.6}$$

□

THEOREM 3.6. *Symmetry:* $d(v,w) = d(w,v)$.

PROOF.

$$w = v \;\Rightarrow\; d(w,w) = d(v,w) \wedge d(w,w) = d(w,v) \;\Rightarrow\; d(v,w) = d(w,v). \tag{3.7}$$

□

3.3. Distance space range. Where the range sets intersect, multiple domain set members map to a single range set member. Therefore, the union set size, d_c, is function of the number of domain-to-range set member mappings.

The property, $|x_i| = |y_i| = p_i$, (3.1) constrains the range of domain-to-range set member mappings. Two facts are immediately obvious from the case, where $p_i = 1$: 1) Each of the p_i number of members of x_i map to *only one* member member of y_i, yielding $|x_i| \cdot 1 = p_i = 1$ number of domain-to-range mappings. 2) Each of the p_i number of members of x_i map to *each* member member of y_i, yielding $|x_i| \cdot |y_i| = p_i^2 = 1$ number of domain-to-range mappings.

Therefore, $\exists\, \mathbf{f} : d_c = \mathbf{f}(\sum_{i=1}^{n} p_i)$ is the largest possible distance because it is the case of the smallest number of domain-to-range mappings (no intersection of the range sets). And $\exists\, \mathbf{f} : d_c = \mathbf{f}(\sum_{i=1}^{n} p_i^2)$ is the smallest possible distance because it is the case of the largest number of domain-to-range mappings (largest allowed intersection of range sets). Applying the ruler (2.1) and ruler convergence theorem (2.2) to the largest and smallest countable distance cases yields the real-valued, Manhattan and Euclidean distance functions.

3.4. Manhattan distance.

THEOREM 3.7. *Manhattan (longest) distance, d, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals,* $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "taxicab_distance," and formal proof is located in the Coq file, euclidrelations.v.

PROOF.

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$d_c = \left|\bigcup_{i=1}^{n} y_i\right| \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |y_i| = p_i \Rightarrow \quad d_c \leq \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c :\; d_c = \sum_{i=1}^{n} p_i. \tag{3.8}$$

Multiply both sides of equation 3.10 by c and take the limit:

$$d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \Rightarrow \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c). \tag{3.9}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d :\; \mathit{floor}(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c. \tag{3.10}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of s_i:

$$\forall\, i \in [1, n],\ s_i = |a_i - b_i| \quad \Rightarrow \quad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i. \tag{3.11}$$

Combine equations 3.10, 3.9, 3.11:

$$d = \lim_{c\to 0} d_c \cdot c \quad \wedge \quad \lim_{c\to 0} d_c \cdot c = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) \quad \wedge \tag{3.12}$$
$$\lim_{c\to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square$$

3.5. Euclidean distance.

THEOREM 3.8. *Euclidean (shortest) distance, d, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$d^2 = \textstyle\sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "Euclidean_distance," and formal proof is located in the Coq file, euclidrelations.v.

PROOF.
Apply the rule of product to the largest number of domain-to-range set mappings, where all p_i number of domain set members, x_i, map to each of the p_i number of members in the range set, y_i:

$$\textstyle\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2. \tag{3.13}$$

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$d_c = |\textstyle\bigcup_{i=1}^{n} y_i| \le \sum_{i=1}^{n} |y_i| \quad \wedge \quad |y_i| = p_i \tag{3.14}$$
$$\Rightarrow \quad d_c \le \textstyle\sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i.$$

Square both sides of equation 3.14 ($x = y \Leftrightarrow f(x) = f(y)$):

$$\exists\, p_i,\ d_c :\ d_c = \textstyle\sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\ d_c :\ d_c^2 = (\sum_{i=1}^{n} p_i)^2. \tag{3.15}$$

Apply the Cauchy-Schwartz inequality to equation 3.15 and select the smallest distance (equality) case:

$$d_c^2 = (\textstyle\sum_{i=1}^{n} p_i)^2 \ge \sum_{i=1}^{n} p_i^2 \quad \Rightarrow \quad \exists\, p_i :\ d_c^2 = \sum_{i=1}^{n} p_i^2. \tag{3.16}$$

Multiply both sides of equation 3.16 by c^2, simplify, and take the limit.

$$d_c^2 = \textstyle\sum_{i=1}^{n} p_i^2 \ \Rightarrow\ d_c^2 \cdot c^2 = \sum_{i=1}^{n} p_i^2 \cdot c^2 \ \Leftrightarrow\ (d_c \cdot c)^2 = \sum_{i=1}^{n} (p_i \cdot c)^2 \tag{3.17}$$
$$\Rightarrow \quad \lim_{c\to 0}(d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$\exists\, c\, d :\ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2. \tag{3.18}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$\forall\, i \in [1, n],\ floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i \tag{3.19}$$
$$\Rightarrow \quad \lim_{c\to 0}(p_i \cdot c)^2 = s_i^2 \quad \Rightarrow \quad \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2.$$

Combine equations 3.18, 3.17, 3.19:

$$d^2 = \lim_{c\to 0}(d_c \cdot c)^2 \quad \wedge \quad \lim_{c\to 0}(d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 \quad \wedge \tag{3.20}$$
$$\textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2$$
$$\Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \quad \square$$

4. Euclidean Volume

A location is a range set member that corresponds 1-1 to a combination (tuple) of one member from each domain set. The number of all possible locations is the Cartesian product of the number members in each domain set. Notionally:

Definition 4.1. All Possible Locations, V_c:

$$\bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad V_c = \prod_{i=1}^{n} |x_i|.$$

Theorem 4.2. *Euclidean volume is the largest possible set of all real-valued locations, V, corresponding to a disjoint set of real-valued domain intervals: $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$V = \prod_{i=1}^{n} s_i, \quad V = |v_0 - v_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "Euclidean_volume," and formal proof is located in the Coq file, euclidrelations.v.

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, x_i of p_i number of subintervals.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i|. \tag{4.1}$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c \to 0}(p_i \cdot c) = s_i. \tag{4.2}$$

Use the ruler (2.1) to divide the exact size, $V = |v_0 - v_m|$, of the range interval, $[v_0, v_m]$, into p^n subintervals. Use those cases, where V_c has an integer n^{th} root.

$$\forall\, p^n = V_c \in \mathbb{N},\ \exists\, V \in \mathbb{R},\ x_i\ : floor(V/c^n) = V_c = p^n = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i. \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3 and simplify:

$$floor(V/c^n) = p^n \quad \Rightarrow \quad V = \lim_{c \to 0} p^n \cdot c^n = \lim_{c \to 0}(p \cdot c)^n. \tag{4.4}$$

Multiply both sides of equation 4.3 by c^n and simplify:

$$p^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad p^n \cdot c^n = (\prod_{i=1}^{n} p_i) \cdot c^n \quad \Leftrightarrow \quad (p \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.5}$$

Combine equations 4.4, 4.5, and 4.2:

$$V = \lim_{c \to 0}(p \cdot c)^n \quad \wedge \quad (p \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c) \quad \wedge \quad \lim_{c \to 0}(p_i \cdot c) = s_i \tag{4.6}$$

$$\Rightarrow \quad V = \lim_{c \to 0}(p \cdot c)^n = \prod_{i=1}^{n} \lim_{c \to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square$$

5. Ordered and symmetric geometries

The union operations of countable distance range (3.1) and all possible (countable) locations (4.1) requires sequencing through each set. The commutative property of the union also allows sequencing, where each set can be sequentially adjacent to any other set, herein referred to as a symmetric geometry.

From a combinatoric perspective, there are $n!$ number of sequential arrangements of any n number of the set members, where there are two arrangements having a set member, x_i, that is sequentially adjacent to a set member, x_j. But, physical sets have the additional constraint of time.

Arrange a set of beans at time t_0 into a sequential order and label them, $\{b_1, b_2, b_3, b_4\}$. Rearranging into a new sequential order, $\{b_1, b_2, b_4, b_3\}$, requires changing some of the bean successors and predecessors at time t_1. And the previous order is no longer true (no longer exists).

A physical set can have only one sequential order *at a time* because each member can have at most one successor and at most one predecessor *at a time*. It will now be proved that a set satisfying the constraints of a single sequential (total) order at a point in time and symmetry defines a cyclic set containing at most 3 members (in this case, 3 dimensions of physical space).

Definition 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i\; \in \{x_1, \dots, x_n\},$$
$$\textit{successor}\; x_i = x_{i+1} \;\;\wedge\;\; \textit{predecessor}\; x_{i+1} = x_i.$$

Definition 5.2. Symmetric geometry (every set member is sequentially adjacent to any other member):

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \dots, x_n\},\;\; \textit{successor}\; x_i = x_j \;\;\wedge\;\; \textit{predecessor}\; x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$\textit{successor}\; x_n = x_1 \;\;\wedge\;\; \textit{predecessor}\; x_1 = x_n.$$

The theorem, "ordered_symmetric_is_cyclic," and formal proof is located in the Coq file, threed.v.

Proof. The property of order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$i = n \;\wedge\; j = 1 \;\wedge\; \textit{successor}\; x_i = x_j \;\Rightarrow\; \textit{successor}\; x_n = x_1. \tag{5.1}$$

$$i = n \;\wedge\; j = 1 \;\wedge\; \textit{predecessor}\; x_j = x_i \;\Rightarrow\; \textit{predecessor}\; x_1 = x_n. \tag{5.2}$$

□

Theorem 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

Proof.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

Definition 5.5. Cyclic successor of m is n:

$$\textit{Successor}(m, n, \textit{setsize}) \leftarrow (m = \textit{setsize} \wedge n = 1) \vee (m + 1 \leq \textit{setsize}). \tag{5.3}$$

Definition 5.6. Cyclic predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (m - 1 \geq 1). \tag{5.4}$$

Definition 5.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{5.5}$$

Every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{5.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (1 + 1 \leq 2). \tag{5.7}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (2 = 2 \wedge 1 = 1). \tag{5.8}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (1 + 1 \leq 2). \tag{5.9}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (2 - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (3 = 3 \wedge 1 = 1). \tag{5.11}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (1 = 1 \wedge 3 = 3). \tag{5.12}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (2 + 1 \leq 3). \tag{5.13}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (3 - 1 \geq 1). \tag{5.14}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not adjacent:

$$\forall\ setsize > 3: \quad \neg Successor(1, 3, setsize) \\ \leftarrow Successor(1, 2, setsize) \leftarrow (1 + 1 \leq setsize). \tag{5.15}$$

That is, 2 is the only successor of 1 for all $setsize > 3$, which implies 3 is not a successor of 1 for all $setsize > 3$.

$$\forall\ setsize > 3: \quad \neg Predecessor(1, 3, setsize) \\ \leftarrow Predecessor(1, n, setsize) \leftarrow (1 = 1 \wedge n = setsize). \tag{5.16}$$

That is, $n = setsize$ is the only predecessor of 1 for all $setsize > 3$, which implies 3 is not a predecessor of 1 for all $setsize > 3$.

$$\forall\ setsize > 3: \quad \neg Adjacent(1, 3, setsize) \\ \leftarrow \neg Successor(1, 3, setsize) \wedge \neg Predecessor(1, 3, setsize). \quad \square \tag{5.17}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (violates the symmetry property).

6. Summary

Applying the ruler measure (2.1) and ruler convergence proof (2.2), to a set of real-valued domain intervals and a range interval yields the following new insights into geometry and physics.

(1) Distance is a function of the number of domain-to-*range* set member mappings. Area/volume is a function of the number of domain-to-*domain* set member mappings.

   (a) Area and volume satisfying the criteria of metric space, while not being a function of domain-to-range set mappings, calls into question whether the definition of metric space is a sufficient criteria for a distance measure.
   (b) Other types of measure, like metric space, Hausdorff, and Lebesgue [**Gol76**] [**Rud76**], can not provide these mapping insights.
(2) All notions of distance are derived from the principle that there should be as many steps walked, p_i, in the range set, y_i, as there are pieces of traversed land in the corresponding domain set, x_i: $|x_i| = |y_i| = p_i$. And the countable distance, d_c, spanning disjoint domain sets is the size of the union of the range sets: $d_c = |\bigcup_{i=1}^{n} y_i|$ (3.1).
   (a) A direct consequence of the inclusion-exclusion principle [**CG15**] is the set relation, $d_c = |\bigcup_{i=1}^{2} y_i| \leq \sum_{i=1}^{2} |y_i|$ (3.2). The ruler measure makes possible proofs that this single set relation generates all the properties of metric space (3.2).
   (b) The countable distance space property, $|x_i| = |y_i|$ (3.1), constrains the range of domain-to-range set member mappings from $\sum_{i=1}^{n} p_i$ to $\sum_{i=1}^{n} p_i^2$. $d_c = \sum_{i=1}^{n} p_i \Leftrightarrow d_c^2 = (\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$. The smallest possible distance case is the equality case: $d_c^2 = \sum_{i=1}^{n} p_i^2$, where applying the ruler measure converges to the smallest possible real-valued distance, Euclidean distance (3.8). Other types of measures can only assume Euclidean distance is the smallest possible distance between two points in $\mathbb{R}^n$.
   (c) The constraints: $|x_i| < |y_i|$, $|x_i| = |y_i|$, and $|x_i| > |y_i|$ yields three types of distance spaces: open, flat, and closed.
(3) The notion of a location (a point) 1-1 correspondence to a tuple of domain set members corresponds to the notion coordinate in geometry. The ruler measure allows proof that the set of all possible locations generates the real-valued volume equation.
   (a) The Hausdorff, and Lebesgue measures [**Gol76**] [**Rud76**], cannot be used to derive the volume equation because they use volume as a primitive.
   (b) Euclidean volume has as many range set elements, V_r, as locations, V_c. The constraints: $V_c < V_r$, $V_c = V_r$, and $V_c > V_r$ yields three types of volume spaces: open, flat, and closed.
   (c) The open, flat, and closed volume spaces correspond to the open, flat, and closed distance spaces.
(4) The two axioms, countable distance space (3.1) and all possible locations (4.1) in this article, comprise an axiomatic foundation for Euclidean geometry that is purely set-based and does not rely on primitives like line, angle, shape, between, congruence, vector, etc.
   (a) In contrast, other axiomatic foundations for geometry [**Bir32**] [**Hil80**] use of primitives like line, angle, plane, between, congruence, etc. that are not defined purely in terms of sets and set operations. Tarski's first-order logic foundation does not use set theory at all [**TG99**].
   (b) Variations of the domain-to-range set mappings provides set-based axiomatic foundations for open and closed spaces.

(c) This foundation makes real analysis and measure theory more self-contained by deriving the properties of metric space, Euclidean distance, and Euclidean volume from set-based axioms rather than importing them as primitives from Euclidean geometry.

(5) The union operations of the set-based axioms of distance and volume require sequencing through sets (dimensions of sets), where any dimension can be sequentially adjacent to any other dimension (symmetric). But, a physical set can have only one sequential order *at a time* because each member can have at most one successor and at most one predecessor *at a time*. The properties of a single, total order (5.1) and symmetry (5.2) defines a cyclic set (5.3) of at most 3 members (5.4) (3 dimensions of physical space).

(6) The proof showing that more than 3 dimensions of physical space would lead to contradictions (5.4) constrains all higher dimensional physics theories to *hierarchical* 2 or 3-dimensional geometries. For example, the four-vectors common in physics [**Bru17**] are hierarchical, 2-dimensional geometries that have been "flattened."

The spacetime four-vector length, $d = \sqrt{(ct)^2 - (x^2 + y^2 + z^2)}$, where c is the speed of light and t is time, can be expressed in a form like, $(ct)^2 = d_1^2 + d_2^2$, where $d_1^2 = x^2 + y^2 + z^2$ and $d_2 = d$. Likewise, the energy-momentum four-vector has the 2-dimensional form: $E^2 = (mv^2)^2 + (pc)^2$, where E is energy, m is the resting mass, v is the *3-dimensional* velocity, c is the speed of light, and p is the relativistic momentum ($p = \gamma mv$, where $\gamma = (1/(1 - (v/c)^2))^{1/2}$ is the Lorentz factor).

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