The Real Analysis and Combinatorics of Geometry

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Abstract. A ruler-like measure of intervals allows real analysis proofs providing insights into the counting principles underlying geometry: A set-based definition of a countable distance range converges to the taxicab distance equation as the upper boundary, the Euclidean distance equation as the lower boundary, and the triangle inequality over the full range, which provides counting-based motivations for the definitions of metric space and Euclidean (smallest) distance. The Cartesian product of same-sized subintervals of intervals converges to the product of interval sizes used in the Lebesgue measure and Euclidean integrals. A cyclic set of at most 3 dimensions, which is the basis of the right-hand rule, emerges from the same countable set axioms from which distance and volume emerge. Implications for non-Euclidean geometries and higher dimensional geometries are discussed. All the proofs are verified in Coq.

Contents

1. Introduction

The triangle inequality of a metric space, Euclidean distance metric, and the volume equation (product of interval sizes) of the Lebesgue measure and Euclidean integrals are imported into mathematical analysis from Euclidean geometry [Gol76] as primitives rather than derived from set and number theory-based axioms. As a consequence, mathematical analysis has provided no insight into the counting principles that motivate and generate those geometric relations.

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Real analysis and measure theory have not provided any proofs that combinatorial relations between the same-sized subintervals of a set of domain intervals and the same-sized subintervals of an image interval converge to the triangle inequality, Euclidean distance, and volume equations, as the subinterval size goes to zero. For example, the differential proofs of the Pythagorean theorem [**Ber88**] [**Sta96**] [**Bog10**] derive Euclidean distance by integrating the differential equation, $\mathbf{d}y/\mathbf{d}x = x/y$, which specifies real-valued ratios rather than an integer-based combinatorial relationship derived from the same basic counting principle that also generates the real-valued triangle inequality.

Understanding the counting principle generating the triangle inequality, taxicab and Euclidean distance provides counting-based insights into the notions of a distance measure and smallest distance that importing as primitives from Euclidean geometry does not provide. Further, the Lebesgue measure and Euclidean integrals sum the product of interval sizes (Euclidean volumes) without proof that the Cartesian product of the same-sized subintervals of intervals converges to the product of intervals sizes.

The various traditional indefinite integrals (antiderivatives) derive a real-valued equation from a real-valued, continuous function relating the **sizes** of the subintervals. In contrast, what is needed for counting-based (combinatorial) proofs is an indefinite integration that derives a real-valued equation from a combinatorial function relating the integer **number** of same-sized subintervals of domain intervals to the integer **number** of same-sized subintervals in an image interval.

Combinatorial integration requires measuring the number of same-sized subintervals of intervals similar to using a ruler (measuring stick). Unlike traditional integration, the ruler approximates in **both** the domain and image intervals (rather than approximating only in the image or only in the domain intervals).

Using the ruler measure, the size of subintervals is the same in both the domain and image intervals and the number of subintervals in each domain and image interval can vary. In contrast, the traditional method of dividing a set of intervals into subintervals, the number of subintervals is the same in both the domain and image intervals and the size of some subintervals can vary.

The Euclidean volume and distance equations can be extended to any number of dimensions. So, why does classical Euclidean geometry appear to be limited to three dimensions? The counting principles generating distance and volume provides insight into the properties that can limit both Euclidean and non-Euclidean geometries to a cyclic set of at most three dimensions.

The proofs in this article are verified formally using the Coq Proof Assistant [**Coq15**] version 8.4p16. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at:
https://github.com/treeck/CombinatorialGeometry.

2. Ruler measure and convergence

Definition 2.1. Ruler measure: A ruler measures the size of a closed, open, or semi-open interval as the nearest integer number of whole subintervals, p, times the subinterval size, c, where c is the independent variable. Notionally:

$$\forall\, c\, s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |b-a| \ \wedge\ c > 0\ \wedge \\ (p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \Sigma_{p=1}^{\infty} c = \lim_{c\to 0} pc. \tag{2.1}$$

Theorem 2.2. *Ruler convergence:*
$\forall\ [a,b] \subset \mathbb{R},\ \ s = |b-a| \ \ \Rightarrow\ \ M = \lim_{c\to 0} pc = s.$

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\ c > 0, \quad p = floor(s/c) \quad \Rightarrow \quad 0 \leq |p - s/c| < 1. \tag{2.2}$$

Multiply all sides by $|c|$:

$$\forall\ c > 0, \quad 0 \leq |p - s/c| < 1 \quad \Rightarrow \quad 0 \leq |pc - s| < |c|. \tag{2.3}$$

$$\begin{aligned} &\forall\ c > 0,\ \exists\ \delta,\ \epsilon\ :\ 0 \leq |pc - s| < |c| = |c - 0| < \delta = \epsilon \\ &\Rightarrow \quad 0 < |c - 0| < \delta \quad \wedge \quad 0 \leq |pc - s| < \epsilon = \delta \quad := \quad M = \lim_{c\to 0} pc = s. \quad \square \end{aligned} \tag{2.4}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure).

For example, showing convergence using the interval, $[0, \pi]$, $s = |\pi - 0|$, $c = 10^{-i}, i \in \mathbb{N}$, and $p = floor(s/c)$, then, $p \cdot c = 3.1_{i=1},\ 3.14_{i=2},\ 3.141_{i=3}, ..., \pi$.

3. Distance

The most basic principle of distance is that an image (distance) set has the same number of elements as a corresponding domain set. For example, the number of steps in a distance set must equal the number pieces of land traversed. Therefore, for each i^{th} disjoint domain set containing p_i number of elements there exists a distance set with the same p_i number of elements.

The union size of the distance sets is less than or equal to the union size of the disjoint domain sets. In the case of intersecting distance sets, a single distance set element will correspond to multiple domain set elements. Notionally:

Definition 3.1. Countable distance range, d_c:

$$\begin{aligned} &\forall\ i\ n \in \mathbb{N}, \quad x_i \subseteq X, \quad \bigcap_{i=1}^{n} x_i = \emptyset, \quad \forall\ x_i\ \exists\ y_i \subseteq Y : \\ &\qquad |x_i| = |y_i| \quad \wedge \quad |\bigcap_{i=1}^{n} y_i| \geq 0, \quad \wedge \quad d_c = |\bigcup_{i=1}^{n} y_i| = |Y|. \end{aligned}$$

Notation conventions: The vertical bars around a set is the standard notation for indicating the cardinal (number of elements in the set). To prevent over use of the vertical bar, the symbol for "such that" is the colon.

The countable distance range principle (3.1), $|x_i| = |y_i| = p_i$, constrains each i^{th} distance set element to a range of correspondences from one domain set element to as many as p_i number of domain set elements. More than p_i number of surjective correspondences would be over counting like a step walked corresponding to the same piece of land multiple times.

Using the rule of product, there is a range of $|y_i| \cdot 1 = p_i$ to $|y_i| \cdot p_i = p_i^2$ number of distance-to-domain correspondences per distance set. Therefore, $d_c = f(\sum_{i=1}^{n} p_i)$ is the largest possible distance (a function of the smallest number of correspondences per distance set element, which is the case of disjoint distance sets). $d_c = f(\sum_{i=1}^{n} p_i^2)$ is the smallest possible distance (a function of the largest number of correspondences per distance set element, which is the case of the maximum allowed intersection of distance sets).

Using the ruler (2.1) to divide a set of real-valued domain intervals and distance interval into sets of same-sized subintervals, and applying the ruler convergence theorem (2.2) proves that the largest and shortest distance cases converges to the real-valued taxicab and Euclidean distance equations.

The convergence proofs of the taxicab and Euclidean distance equations use the strategy of showing that the right and left sides of a proposed counting-based equation both converge to the same real value and therefore are equal. In other words, the propositional logic, $A = C \wedge B = C \Rightarrow A = B$, is used.

THEOREM 3.2. *Taxicab (largest) distance, d, is the size of the distance interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals, $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|.$$

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\ |\{x_i : x_i \in \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\}| = |\{y_i : y_i \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_i}\}\}| = p_i. \tag{3.1}$$

Next, apply the definition of the countable distance range (3.1):

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad y \in y_i \subseteq Y \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i = |\{y : y \in Y\}|. \tag{3.2}$$

Multiply both sides of 3.2 by c and apply the ruler convergence theorem (2.2):

$$s_i = \lim_{c \to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n} (p_i \cdot c) = |\{y\}| \cdot c \\ \Rightarrow \quad \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) = \lim_{c \to 0} |\{y\}| \cdot c. \tag{3.3}$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$\forall\, c > 0, \quad p_d = floor(d/c) = |\{y : y \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_d}\} = Y\}|. \tag{3.4}$$

Multiply both sides of 3.4 by c and apply the ruler convergence theorem (2.2):

$$d = \lim_{c \to 0} p_d \cdot c \;\wedge\; p_d \cdot c = |\{y\}| \cdot c \;\Rightarrow\; d = \lim_{c \to 0} p_d \cdot c = \lim_{c \to 0} |\{y\}| \cdot c. \tag{3.5}$$

Combine equations 3.5 and 3.3:

$$d = \lim_{c \to 0} |\{y\}| \cdot c \quad \wedge \quad \sum_{i=1}^{n} s_i = \lim_{c \to 0} |\{y\}| \cdot c \quad \Rightarrow \quad d = \sum_{i=1}^{n} s_i. \tag{3.6}$$ □

THEOREM 3.3. *Euclidean (smallest) distance, d, is the size of the distance interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals, $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}].$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$(3.7)\quad \forall\, i\; n \in \mathbb{N},\quad i \in [1, n],\quad c > 0 \quad\wedge\quad p_i = floor(s_i/c) \quad\wedge \\ x_i = \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\} \quad\wedge\quad y_i = \{y_{1_i}, y_{2_i}, \ldots, y_{p_i}\}\}|.$$

Next, apply the definition of the countable distance range (3.1) and the rule of product:

$$(3.8)\qquad \textstyle\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} |\{(y_a, y_b) : y_a\, y_b \in y_i\}|.$$

where each pair, (y_a, y_b), represents a combination (correspondence) between two elements in the distance set, y_i. From the definition of countable distance range (3.1), the distance sets can intersect, which results in a range of possible distance set sizes. Applying the inclusion-exclusion principle:

$$(3.9)\qquad |\cap_{i=1}^{n} y_i| \geq 0 \quad\Rightarrow\quad \textstyle\sum_{i=1}^{n} |y_i| \geq |\cup_{i=1}^{n} y_i| = |Y|.$$

From combining equation 3.8 and the equality case of relation 3.9:

$$(3.10)\quad \textstyle\sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \geq |\cup_{i=1}^{n} y_i| = |Y| \\ \Rightarrow\ \exists\, y_i,\ Y\ :\ \textstyle\sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i = |\cup_{i=1}^{n} y_i| = |Y| \\ \Rightarrow\quad \textstyle\sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |\{(y_a, y_b) : y_a y_b \in y_i\}| = |\{(y_a, y_b) : y_a y_b \in Y\}|.$$

Multiply both sides of equation 3.10 by c^2 and apply the ruler convergence theorem.

$$(3.11)\quad s_i = \lim_{c\to 0} p_i \cdot c \quad\wedge\quad \sum_{i=1}^{n} (p_i \cdot c)^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \\ \Rightarrow\quad \sum_{i=1}^{n} s_i^2 = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c)^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2.$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$(3.12)\quad \forall\, c > 0,\quad p_d = floor(d/c) = |\{y_{1_i}, y_{2_i}, \ldots, y_{p_d}\}| = |Y| \\ \Rightarrow\quad p_d^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\}|,$$

where $\{(y_a, y_b)\}$ is the set of all combination pairs of elements of Y. Multiply both sides of 3.12 by c^2 and apply the ruler convergence theorem (2.2):

$$(3.13)\quad d = \lim_{c\to 0} p_d \cdot c \quad\wedge\quad (p_d \cdot c)^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \\ \Rightarrow\quad d^2 = \lim_{c\to 0} (p_d \cdot c)^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2.$$

Combine equations 3.12 and 3.13:

$$(3.14)\quad d^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \quad\wedge \\ \sum_{i=1}^{n} s_i^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \quad\Rightarrow\quad d^2 = \sum_{i=1}^{n} s_i^2. \quad \square$$

3.1. Triangle inequality. The definition of a metric in real analysis is based on the triangle inequality, $\mathbf{d}(\mathbf{u},\mathbf{w}) \leq \mathbf{d}(\mathbf{u},\mathbf{v}) + \mathbf{d}(\mathbf{v},\mathbf{w})$, that has been intuitively motivated by the triangle [**Gol76**]. Applying the inclusion-exclusion principle, ruler (2.1), and ruler convergence theorem (2.2) to the definition of a countable distance range (3.1) yields the real-valued triangle inequality:

$$
\begin{gathered}
d_c = |Y| = |\bigcup_{i=1}^{2} y_i| \leq \sum_{i=1}^{2} |y_i| \quad \wedge \\
d_c = floor(\mathbf{d}(\mathbf{u},\mathbf{w})/c) \ \wedge \ |y_1| = floor(\mathbf{d}(\mathbf{u},\mathbf{v})/c) \ \wedge \ |y_2| = floor(\mathbf{d}(\mathbf{v},\mathbf{w})/c) \\
\Rightarrow \ \mathbf{d}(\mathbf{u},\mathbf{w}) = \lim_{c\to 0} d_c \cdot c \leq \sum_{i=1}^{2} \lim_{c\to 0} |y_i| \cdot c = \mathbf{d}(\mathbf{u},\mathbf{v}) + \mathbf{d}(\mathbf{v},\mathbf{w}).
\end{gathered} \tag{3.15}
$$

4. Size (length/area/volume)

This section will use the ruler (2.1) and ruler convergence theorem (2.2) to prove that the Cartesian product of same-sized subintervals of intervals converges to the product of interval sizes. The first step is to define a set-based, countable size measure as the Cartesian product of disjoint domain set members.

DEFINITION 4.1. Countable size (length/area/volume) measure, S_c:

$$\forall\, i\ n \in \mathbb{N}, \quad i \in [1,n], \quad x_i \subseteq X, \quad |\bigcap_{i=1}^{n} x_i| = \emptyset \quad \wedge \quad S_c = |y| = \prod_{i=1}^{n} |x_i|.$$

THEOREM 4.2. *Euclidean size (length/area/volume), S, is the size of an image interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals:* $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, *where:*

$$S = \prod_{i=1}^{n} s_i, \quad S = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|, \quad i \in [1,n], \quad i, n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_size."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$
\begin{gathered}
\forall\, i\ n \in \mathbb{N}, \quad i \in [1,n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\
x_i = \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\} \quad \Rightarrow |x_i| = p_i.
\end{gathered} \tag{4.1}
$$

Use the ruler (2.1) to divide the exact size, $S = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_S^n subintervals, where p_S^n satisfies the definition a countable size measure, S_c (4.1).

$$
\begin{gathered}
\forall\, c > 0 \quad \wedge \quad \exists\, r \in \mathbb{R},\ S = r^n \quad \wedge \quad p_S = floor(r/c) \quad \wedge \\
p_S^n = S_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i.
\end{gathered} \tag{4.2}
$$

Multiply both sides of equation 4.2 by c^n to get the ruler measures:

$$p_S^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3:

$$S = r^n = \lim_{c\to 0}(p_S \cdot c)^n \quad \wedge \quad \lim_{c\to 0}(p_i \cdot c) = s_i \quad \wedge \quad (p_S \cdot c)^n = \prod_{i=1}^{n}(p_i \cdot c) \tag{4.4}$$

$$\Rightarrow \quad S = \lim_{c\to 0}(p_S \cdot c)^n = \prod_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square$$

5. Ordered and symmetric geometries

Neither classical nor modern analytic geometry has been able to provide any insight into why classical (physical) Euclidean geometry appears to be limited to at most three dimensions. The same counting principles that generate the triangle inequality, taxicab distance, Euclidean distance, and size (length/area/volume) also provide insight into what properties can limit a geometry (both Euclidean and non-Euclidean) to a cyclic set of at most three dimensions.

The previous derivations of taxicab distance (3.2), Euclidean distance (3.3), and Euclidean volume (4.2) show that the total number of combinations of subintervals of intervals converge to real-valued distance measures and Euclidean volume. By the commutative properties of addition and multiplication, all orderings (permutations) of the combinations of subintervals of intervals yield the same total distance and same total volume. Therefore, all orderings (permutations) of domain intervals corresponding to those subinterval combinations yield the same total distance and same total volume (a symmetric geometry).

All distance measures, size measures, and permutations emergent from the countable distance range principle and countable size exist. There is no axiom of choice about which distance measures, size measures, and permutations exist or does not exist (emerge or do not emerge).

For example, between any two distinct points, A and B, there is both a taxicab and Euclidean distance because both types of distance emerge from the same countable distance range definition (axiom). There is no choice about which type of distance exists and does not exist (emerge or does not emerge from the countable distance range axiom).

The same logic applies to "all permutations existing" (all possible permutations of intervals emerge from the countable distance range and countable size). Mathematics defines the ordering of a set in terms of a successor function and a predecessor (inverse order) function. For example, successor and predecessor functions can be defined that generate the left-to-right ordered set of elements, $\{A,\ B,\ C,\ D\}$. The successor function lists the permutation, $(A,\ B,\ C,\ D)$. And the predecessor function lists the permutation, $(D,\ C,\ B,\ A)$. In this case, only two permutations exist (emerge from) those successor and predecessor functions.

It will be proved that all permutations (a symmetric geometry) can only emerge from a successor function and predecessor function that defines a cyclic ordering on a set containing at most three elements (dimensions).

Definition 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i\; \in \{x_1, \ldots, x_n\},$$
$$\textit{successor}\ x_i = x_{i+1} \;\;\wedge\;\; \textit{predecessor}\ x_{i+1} = x_i.$$

where $\{x_1, \ldots, x_n\}$ are a set of real-valued intervals (dimensions).

Definition 5.2. Symmetric geometry (all permutations):

$$\forall\, i\; j\; n \in \mathbb{N},\ \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\ \textit{successor}\ x_i = x_j\ \ \wedge\ \ \textit{predecessor}\ x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric geometry is a cyclic set.*

$$\textit{successor}\ x_n = x_1\ \ \wedge\ \ \textit{predecessor}\ x_1 = x_n.$$

The theorem and formal Coq-based proof is "ordered_symmetric_is_cyclic," which is located in the file threed.v.

Proof. The property of order (5.1) defines unique successors and predecessors for all elements except for the successor of x_n and the predecessor of x_1. From the properties of a symmetric geometry (5.2):

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \textit{successor}\ x_i = x_j\ \ \Rightarrow\ \ \textit{successor}\ x_n = x_1. \tag{5.1}$$

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \textit{predecessor}\ x_j = x_i\ \ \Rightarrow\ \ \textit{predecessor}\ x_1 = x_n. \tag{5.2}$$

□

For example, using the cyclic set with elements labeled, $\{1,\ 2,\ 3\}$, starting with each element and counting through 3 cyclic successors and counting through 3 cyclic predecessors yields all possible permutations: $(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(1, 3, 2)$, $(3, 2, 1)$, and $(2, 1, 3)$. That is, a cyclically ordered set preserves sequential order while allowing some n-at-a-time permutations. If all possible n-at-a-time permutations are generated, then the cyclic set is also a symmetric geometry.

Theorem 5.4. *An ordered and symmetric geometry is limited to at most 3 elements. That is, each element is sequentially adjacent (a successor or predecessor) to every other element in a set only where the number of elements are less than or equal to 3.*

The Coq-based lemmas and proofs in the file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

Proof.
Because a symmetric and ordered set is a cyclic set (5.3), the successors and predecessors are cyclic:

Definition 5.5. Successor of m is n:

$$\textit{Successor}(m, n, \textit{setsize}) \leftarrow (m = \textit{setsize} \wedge n = 1) \vee (m + 1 \leq \textit{setsize}). \tag{5.3}$$

Definition 5.6. Predecessor of m is n:

$$\textit{Predecessor}(m, n, \textit{setsize}) \leftarrow (m = 1 \wedge n = \textit{setsize}) \vee (m - 1 \geq 1). \tag{5.4}$$

Definition 5.7. Adjacent: element m is adjacent to element n (an allowed permutation), if the cyclic successor of m is n or the cyclic predccessor of m is n. Notionally:

$$\textit{Adjacent}(m, n, \textit{setsize}) \leftarrow \textit{Successor}(m, n, \textit{setsize}) \vee \textit{Predecessor}(m, n, \textit{setsize}). \tag{5.5}$$

Every element is adjacent to every other element, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{5.6}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (1 + 1 \leq 2). \tag{5.7}$$

$$Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (2 = 2 \wedge 1 = 1). \tag{5.8}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (1 + 1 \leq 2). \tag{5.9}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (2 - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (3 = 3 \wedge 1 = 1). \tag{5.11}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (1 = 1 \wedge 3 = 3). \tag{5.12}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (2 + 1 \leq 3). \tag{5.13}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (3 - 1 \geq 1). \tag{5.14}$$

Must prove that for all $setsize > 3$, there exist non-adjacent elements (not every permutation allowed). For example, the first and third elements are not adjacent:

$$\forall\ setsize > 3: \quad \neg Successor(1,3,setsize) \\ \leftarrow Successor(1,2,setsize) \leftarrow (1 + 1 \leq setsize). \tag{5.15}$$

That is, 2 is the only successor of 1 for all $setsize > 3$, which implies 3 is not a successor of 1 for all $setsize > 3$.

$$\forall\ setsize > 3: \quad \neg Predecessor(1,3,setsize) \\ \leftarrow Predecessor(1,n,setsize) \leftarrow (1 = 1 \wedge n = setsize). \tag{5.16}$$

That is, $n = setsize$ is the only predecessor of 1 for all $setsize > 3$, which implies 3 is not a predecessor of 1 for all $setsize > 3$.

$$\forall\ setsize > 3: \quad \neg Adjacent(1,3,setsize) \\ \leftarrow \neg Successor(1,3,setsize) \wedge \neg Predecessor(1,3,setsize). \quad \square \tag{5.17}$$

6. Summary

Applying the ruler measure (2.1) and ruler convergence proof (2.2) to combinatorial relations between the same-sized subintervals of real-valued domain intervals and the same-sized subintervals an image interval yields some new insights into real analysis, measure theory, geometry, and physics.

(1) With respect to measure theory, the ruler measure both proves a real-valued function is derived from a counting relationship between the elements of more primitive sets and that the real-valued function is measurable. In contrast, the distance measures of a metric space and sigma algebra measures, like the Lebesgue measure, use a real-valued function as a primitive, which does not provide any insight into the principles that generate the real-valued function.

(2) With respect to distance measures, applying the ruler measure and ruler convergence proof to the countable distance range definition (3.1) provides the new insight that all real-valued measures of distance are based on the principle that for each disjoint domain set there exists a distance set containing the same number of elements:

(a) The countable distance range principle converges to the triangle inequality, which is the basis for the definition of metric space. Previous real analysis and measure theory have not provided this insight, because they imported the triangle inequality as a primitive.
(b) The upper bound of the countable distance range principle converging to taxicab distance provides the insight that largest (maximum) monotonic distance path is due to the smallest surjective mapping, which is the union of disjoint distance sets. Previous real analysis and measure theory have not provided this insight, because they imported taxicab distance as a primitive.
(c) The lower bound of the countable distance range principle converging to Euclidean distance provides the insight that smallest (minimum) monotonic distance path is due to the maximum surjective mapping, which is the union of intersecting distance sets, where each of the p_i number of elements in the i^{th} distance set corresponds to p_i number of elements in the i^{th} domain set. Previous real analysis and measure theory have not provided this insight, because they imported Euclidean distance as a primitive.
(d) All L^p norms where $p > 2$ generated from the countable distance range principle would require each distance set element to correspond to the same domain set element multiple times (over counts the number of surjective correspondences). It is not a useful measure of steps when walking from point A to B, where a step would correspond to the same piece of land multiple times. Previous real analysis and measure theory have not provided this over-counting insight.
(e) Euclidean distance (3.3) was derived without any notions of side, angle, or shape. Arc angle defined as a parametric variable relating the sizes of two domain intervals can be easily derived using using calculus (Taylor series) and the Euclidean distance equation to generate the arc sine and arc cosine functions of the parametric variable (arc angle). In other words, the notions of side and angle are derived from Euclidean distance, which is the reverse of what classical geometry [**Joy98**] [**Loo68**] [**Ber88**] and axiomatic foundations for geometry [**Bir32**] [**Hil**] [**TG99**] assume.

(3) Applying the ruler measure and ruler convergence proof to the countable size definition (4.1) provides the insight that the Cartesian product of same-sized subintervals of real-valued intervals converges to the product of the interval sizes (Euclidean volume):
(a) Euclidean size (length/area/volume) was derived without notions of sides, angles, and shape.
(b) The ruler measure proves that Euclidean volume (product of interval sizes) is derived from the Cartesian product of same-sized subintervals of real valued intervals. In contrast, the Lebesgue measure (a sigma algebra), uses Euclidean volume as a primitive.

(4) Combinatorics on same-sized subintervals generating the properties of distance and volume results in the same total distance and same total volume for all permutations (orderings) of domain intervals.

(a) Mathematics defines the ordering of a set in terms of a successor function and a predecessor (inverse order) function. When the successor and predecessor functions generate all permutations (a symmetric geometry), then the ordering must be cyclic (5.3) and the set size limited to at most three elements (dimensions) (5.4), which is the basis of the right-hand rule that permeates mathematics and physics.
(b) All permutations of dimensions (a symmetric geometry) emerge from the same axioms of countable distance range and countable size from which distance and volume measures emerge. Therefore, physical space is a cyclic set of three dimensions of the "type," space.
(c) Higher dimensional geometries beyond the set of three physical space dimensions must comprise sets having different types of dimensions (for example, time), where the result of arithmetic operations across sets must retain type information (for example, meter/second).

(5) Applying the ruler measure to countable relationships between same-sized subintervals of other types of intervals and the sets of same-sized subintervals of the three physical space intervals might converge to real-valued functions describing phenomena perceived as "particles", "waves", "mass", "forces", and "time." Our universe might emerge from a few simple counting principles in the real-valued continuum.

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