

# The Two Set Relations Generating Geometry

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ABSTRACT. A ruler-like measure is used to prove that the properties of metric space, Euclidean distance, and volume are motivated and derived from two countable set relations. The ruler measure divides both domain and range intervals approximately into the nearest integer number of same-sized subintervals. As the subinterval size converges to zero: 1) Distance as the union size of range sets, where for each domain set there exists a corresponding same-sized range set, converges to: the triangle inequality with Manhattan distance at the upper boundary and Euclidean distance at the lower boundary. 2) The Cartesian product of the number of members in each domain set converges to the product of interval interval sizes (Euclidean area/volume). Time limits physical geometry to 3 dimensions. The ruler measure is used to derive the charge force, gravity force, and spacetime interval equations. All proofs are verified in Coq.

## CONTENTS

1. Introduction	1
2. Ruler measure and convergence	2
3. Distance	3
4. Euclidean Volume	6
5. Ordered and symmetric geometries	7
6. Applying the ruler measure to physics	9
7. Insights and conjectures	10
References	11

## 1. Introduction

The properties of metric space, Euclidean distance, and the product of interval sizes (Euclidean area/volume) have been defined in real analysis [Gol76] [Rud76] rather than motivated and derived from set-based axioms. A “ruler” measure is used to prove that these geometric relations are motivated and derived from two countable set relations.

The derivation of geometric relations from set relations, *without notions of point, plane, line, angle, etc.*, identifies: 1) the single set relation generating the triangle inequality, non-negativity, and identity of indiscernibles properties of metric space; 2) the mapping between sets that makes Euclidean distance the smallest possible distance between two distinct points in  $\mathbb{R}^n$ ; 3) the mapping between sets that makes distance different from area/volume; 4) how time places an additional constraint on physical sets, which limits physical geometry to 3 dimensions.

Proofs accepted by the Coq logic engine [Coq15] are internationally recognized to have a very high probability of being correct. All the proofs in this article have corresponding formal proofs in the Coq files, “euclidrelations.v” and “threed.v,” located at: <https://github.com/treeck/RASRGeometry>.

## 2. Ruler measure and convergence

A ruler (measuring stick) partitions both domain and range intervals *approximately* into subintervals, where each subinterval has the *same size*,  $c$ , with the consequence that different sized intervals have a different number of subintervals. In contrast, the Riemann and Lebesgue integrals partition each domain interval and the range into the *same number* of subintervals, where different-sized intervals have *different-sized* subintervals [Gol76] [Rud76].

The ruler measure allows counting the number of mappings, ranging from a one-to-one correspondence to a many-to-many mapping, between the set of size  $c$  subintervals in one interval and the set of size  $c$  subintervals in another interval. The mapping (combinatorial) relations converge to continuous, bijective relations as the subinterval size,  $c$ , converges to zero.

**DEFINITION 2.1.** Ruler measure: A ruler measures the size,  $M$ , of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals,  $p$ , each subinterval having the same size,  $c$ . Notionally:

$$(2.1) \quad \forall c \, s \in \mathbb{R}, \, [a, b] \subset \mathbb{R}, \, s = |a - b| \wedge c > 0 \wedge \\ (p = \text{floor}(s/c) \vee p = \text{ceiling}(s/c)) \wedge M = \sum_{i=1}^p c = pc.$$

**THEOREM 2.2.** *Ruler convergence:*

$$\forall [a, b] \subset \mathbb{R}, \, s = |a - b| \Rightarrow M = \lim_{c \rightarrow 0} pc = s.$$

The theorem, “limit\_c\_0\_M.eq\_exact\_size,” and formal proof is in the Coq file, euclidrelations.v.

**PROOF.** (epsilon-delta proof)

By definition of the floor function,  $\text{floor}(x) = \max(\{y : y \leq x, y \in \mathbb{Z}, x \in \mathbb{R}\})$ :

$$(2.2) \quad \forall c > 0, \, p = \text{floor}(s/c) \wedge 0 \leq |\text{floor}(s/c) - s/c| < 1 \Rightarrow 0 \leq |p - s/c| < 1.$$

Multiply all sides of inequality 2.2 by  $|c|$ :

$$(2.3) \quad \forall c > 0, \quad 0 \leq |p - s/c| < 1 \Rightarrow 0 \leq |pc - s| < |c|.$$

$$(2.4) \quad \forall \delta : |pc - s| < |c| = |c - 0| < \delta \\ \Rightarrow \quad \forall \epsilon = \delta : |c - 0| < \delta \wedge |pc - s| < \epsilon := M = \lim_{c \rightarrow 0} pc = s. \quad \square$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence, where:  $[0, \pi]$ ,  $s = |0 - \pi|$ ,  $c = 10^{-i}$ , and  $p = \text{floor}(s/c) \Rightarrow p \cdot c = 3.1_{i=1}, 3.14_{i=2}, 3.141_{i=3}, \dots, \pi$ .

### 3. Distance

**Notation convention:** Curly brackets,  $\{\dots\}$ , delimit a set; square brackets,  $[\dots]$ , delimit a list; and vertical bars around a set or list,  $|\dots|$ , indicates the cardinal (number of members in the set or list).

**3.1. Countable distance space.** A simple measure of distance is the number of steps walked, which corresponds to an equal number of pieces of land. Abstracting, distance is the number of members in a range set,  $y_i$ , which equals the number of members in a corresponding domain set,  $x_i$ :  $|x_i| = |y_i|$ . And the distance spanning multiple, disjoint, domain sets,  $\bigcup_{i=1}^n x_i = \emptyset$ , is the number of members,  $d_c$ , in the union range set:  $d_c = |\bigcup_{i=1}^n y_i|$ .

DEFINITION 3.1. Countable distance space,  $d_c$ :

$$\bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad d_c = |\bigcup_{i=1}^n y_i| \quad \wedge \quad |x_i| = |y_i|.$$

THEOREM 3.2. *Inclusion-exclusion Inequality:*  $|\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$ .

This well-known inequality follows directly from the inclusion-exclusion principle [CG15]. But, a more intuitive and simple proof follows from the sum of the set sizes being equal to the size of all the set members appended into a list. And the list can be sorted into a list of unique members (the union set) and a list of duplicate members. For example,  $|\{a, b, c\}| + |\{c, d, e\}| = |[a, b, c, c, d, e]| = |\{a, b, c, d, e\}| + |[c]| = 6 \Rightarrow |\{a, b, c, d, e\}| = |\{a, b, c\}| + |\{c, d, e\}| - |[c]| = 5$ .

A formal proof, inclusion\_exclusion\_inequality, using sorting into unique members (union set) and duplicate members, is in the file euclidrelations.v.

PROOF. By the associative law of addition, append the sets into a list, sort into uniques and duplicates, and then subtract duplicates from both sides:

$$\begin{aligned} (3.1) \quad \sum_{i=1}^n |y_i| &= |\text{append}_{i=1}^n y_i| = |\bigcup_{i=1}^n y_i| + |\text{duplicates}_{i=1}^n y_i| \\ &\Rightarrow \sum_{i=1}^n |y_i| - |\text{duplicates}_{i=1}^n y_i| = |\bigcup_{i=1}^n y_i|. \end{aligned}$$

$$\begin{aligned} (3.2) \quad |\bigcup_{i=1}^n y_i| &= \sum_{i=1}^n |y_i| - |\text{duplicates}_{i=1}^n y_i| \quad \wedge \quad |\text{duplicates}_{i=1}^n y_i| \geq 0 \\ &\Rightarrow |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|. \quad \square \end{aligned}$$

**3.2. Metric Space.** The inequality,  $d_c = |\bigcup_{i=1}^2 y_i| \leq \sum_{i=1}^2 |y_i|$ , generates three of the metric space properties. The fourth property, symmetry  $[d(u, v) = d(v, u)]$ , is a consequence of the sum of set member sizes being the same for every ordering (commutative law of addition). The formal proofs: triangle\_inequality, non\_negativity, and identity\_of\_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 3.3. *Triangle Inequality:*  $d(u, w) \leq d(u, v) + d(v, w)$ .

PROOF.

$$\begin{aligned}
 (3.3) \quad & \forall c > 0, \quad |y_1| = \text{floor}(d(u, v)/c) \quad \wedge \quad |y_2| = \text{floor}(d(v, w)/c) \quad \wedge \\
 & d_c = \text{floor}(d(u, w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \leq |y_1| + |y_2| \\
 & \Rightarrow \text{floor}(d(u, w)/c) \leq \text{floor}(d(u, v)/c) + \text{floor}(d(v, w)/c) \\
 & \Rightarrow \text{floor}(d(u, w)/c) \cdot c \leq \text{floor}(d(u, v)/c) \cdot c + \text{floor}(d(v, w)/c) \cdot c \\
 & \Rightarrow \lim_{c \rightarrow 0} \text{floor}(d(u, w)/c) \cdot c \leq \lim_{c \rightarrow 0} \text{floor}(d(u, v)/c) \cdot c + \lim_{c \rightarrow 0} \text{floor}(d(v, w)/c) \cdot c \\
 & \Rightarrow d(u, w) \leq d(u, v) + d(v, w). \quad \square
 \end{aligned}$$

THEOREM 3.4. *Non-negativity:*  $d(u, w) \geq 0$ .

PROOF.

$$\begin{aligned}
 (3.4) \quad & \forall c > 0 : \quad \text{floor}(d(u, w)/c) = d_c \quad \wedge \quad d_c = |y_1 \cup y_2| \geq 0 \\
 & \Rightarrow \text{floor}(d(u, w)/c) = d_c \geq 0 \quad \Rightarrow \quad d(u, w) = \lim_{c \rightarrow 0} d_c \cdot c \geq 0. \quad \square
 \end{aligned}$$

THEOREM 3.5. *Identity of Indiscernibles:*  $d(w, w) = 0$ .

PROOF.

Apply the triangle inequality property:

$$(3.5) \quad \forall d(u, v) = d(v, w) = 0 \quad \wedge \quad d(u, w) \leq d(u, v) + d(v, w) \quad \Rightarrow \quad d(u, w) \leq 0.$$

Combine the non-negativity property (3.4) and the previous inequality (3.5):

$$(3.6) \quad d(u, w) \geq 0 \quad \wedge \quad d(u, w) \leq 0 \quad \Leftrightarrow \quad 0 \leq d(u, w) \leq 0 \quad \Rightarrow \quad d(u, w) = 0.$$

$$(3.7) \quad d(u, w) = 0 \quad \wedge \quad d(u, v) = 0 \quad \Rightarrow \quad w = v.$$

$$(3.8) \quad d(v, w) = 0 \quad \wedge \quad w = v \quad \Rightarrow \quad d(w, w) = 0. \quad \square$$

**3.3. Distance space range.** Distance,  $d_c = |\bigcup_{i=1}^n y_i|$ , implies that where the range sets intersect, multiple domain set members map to a single range set member. Therefore,  $d_c$  is a function of domain-to-range set member mappings.

From the countable distance space definition (3.1),  $|x_i| = |y_i|$ . Where  $|x_i| = |y_i| = p_i = 1$ , each of the  $p_i$  number of domain set members in  $x_i$ : 1) maps 1-1 (bijective) to a *single*, unique range set member in  $y_i$ , yielding  $|x_i| \cdot 1 = p_i \cdot 1 = p_i = 1$  number of domain-to-range set mappings. 2) maps to *all* of the  $p_i$  number of range set members in  $y_i$ , yielding  $|x_i| \cdot |y_i| = p_i \cdot p_i = p_i^2 = 1$  number of domain-to-range set mappings.

Therefore, the total number of domain-to-range set mappings ranges from  $\sum_{i=1}^n p_i$  to  $\sum_{i=1}^n p_i^2$ . Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of domain-to-range set mapping cases converges to the real-valued, Manhattan and Euclidean distance functions.

### 3.4. Manhattan distance.

THEOREM 3.6. *Manhattan (longest non-increasing) distance,  $d$ , is the size of the distance interval,  $[d_0, d_m]$ , mapping to a set of disjoint domain intervals,  $\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$ , where:*

$$d = \sum_{i=1}^n s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, “taxicab\_distance,” and formal proof is in the Coq file, euclidrelations.v.

PROOF.

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance,  $d_c$ , is the equality case:

$$(3.9) \quad d_c = |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \\ \Rightarrow \quad d_c \leq \sum_{i=1}^n |y_i| = \sum_{i=1}^n p_i \quad \Rightarrow \quad \exists p_i, d_c : d_c = \sum_{i=1}^n p_i.$$

Multiply both sides of equation 3.11 by  $c$  and take the limit:

$$(3.10) \quad d_c = \sum_{i=1}^n p_i \Rightarrow d_c \cdot c = \sum_{i=1}^n (p_i \cdot c) \Rightarrow \lim_{c \rightarrow 0} d_c \cdot c = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c).$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of  $d$ :

$$(3.11) \quad d = |d_0 - d_m| \Rightarrow \exists c d : \text{floor}(d/c) = d_c \Rightarrow d = \lim_{c \rightarrow 0} d_c \cdot c.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of  $s_i$ :

$$(3.12) \quad \forall i \in [1, n], s_i = |a_i - b_i| \wedge \text{floor}(s_i/c) = |x_i| = |y_i| = p_i \Rightarrow \lim_{c \rightarrow 0} p_i \cdot c = s_i.$$

Combine equations 3.11, 3.10, 3.12:

$$(3.13) \quad d = \lim_{c \rightarrow 0} d_c \cdot c \quad \wedge \quad \lim_{c \rightarrow 0} d_c \cdot c = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c) \quad \wedge \\ \lim_{c \rightarrow 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c \rightarrow 0} d_c \cdot c = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c) = \sum_{i=1}^n s_i. \quad \square$$

### 3.5. Euclidean distance.

**THEOREM 3.7.** *Euclidean (shortest) distance,  $d$ , is the size of the distance interval,  $[d_0, d_m]$ , mapping to a set of disjoint domain intervals,  $\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$ , where:*

$$d^2 = \sum_{i=1}^n s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, “Euclidean\_distance,” and formal proof is in the Coq file, euclidrelations.v.

PROOF.

Apply the rule of product to the largest number of domain-to-range set mappings, where all  $p_i$  number of domain set members,  $x_i$ , map to each of the  $p_i$  number of members in the range set,  $y_i$ :

$$(3.14) \quad \sum_{i=1}^n |y_i| \cdot |x_i| = \sum_{i=1}^n p_i^2.$$

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$(3.15) \quad d_c = |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \\ \Rightarrow \quad d_c \leq \sum_{i=1}^n |y_i| = \sum_{i=1}^n p_i \quad \Rightarrow \quad \exists p_i, d_c : d_c = \sum_{i=1}^n p_i.$$

Square both sides of equation 3.15 ( $x = y \Leftrightarrow f(x) = f(y)$ ):

$$(3.16) \quad \exists p_i, d_c : d_c = \sum_{i=1}^n p_i \Leftrightarrow \exists p_i, d_c : d_c^2 = (\sum_{i=1}^n p_i)^2.$$

Apply the Cauchy-Schwartz inequality to equation 3.16 and select the smallest distance (equality) case:

$$(3.17) \quad d_c^2 = (\sum_{i=1}^n p_i)^2 \geq \sum_{i=1}^n p_i^2 \Rightarrow \exists p_i : d_c^2 = \sum_{i=1}^n p_i^2.$$

Multiply both sides of equation 3.17 by  $c^2$ , simplify, and take the limit.

$$(3.18) \quad d_c^2 = \sum_{i=1}^n p_i^2 \Rightarrow d_c^2 \cdot c^2 = \sum_{i=1}^n p_i^2 \cdot c^2 \Leftrightarrow (d_c \cdot c)^2 = \sum_{i=1}^n (p_i \cdot c)^2 \\ \Rightarrow \lim_{c \rightarrow 0} (d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.19) \quad \exists c d : \text{floor}(d/c) = d_c \Rightarrow d = \lim_{c \rightarrow 0} d_c \cdot c \Rightarrow d^2 = \lim_{c \rightarrow 0} (d_c \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$(3.20) \quad \forall i \in [1, n], s_i = |a_i - b_i| \wedge \text{floor}(s_i/c) = |x_i| = |y_i| = p_i \Rightarrow \lim_{c \rightarrow 0} (p_i \cdot c) = s_i.$$

Combine equations 3.19, 3.18, 3.20:

$$(3.21) \quad d^2 = \lim_{c \rightarrow 0} (d_c \cdot c)^2 \wedge \lim_{c \rightarrow 0} (d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)^2 \wedge \\ \lim_{c \rightarrow 0} (p_i \cdot c) = s_i \Rightarrow d^2 = \lim_{c \rightarrow 0} (d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)^2 = \sum_{i=1}^n s_i^2. \quad \square$$

#### 4. Euclidean Volume

The number of all possible combinations ( $n$ -tuples) taking one member from each disjoint set is the Cartesian product of the number of members in each set. Notionally:

DEFINITION 4.1. All Possible Combinations,  $V_c$ :

$$\bigcap_{i=1}^n x_i = \emptyset \wedge V_c = \prod_{i=1}^n |x_i|.$$

THEOREM 4.2. *Euclidean volume,  $V$ , is size of the range interval,  $[v_0, v_m]$ , corresponding to all the possible combinations of the members of disjoint domain intervals,  $\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$ . Notionally:*

$$V = \prod_{i=1}^n s_i, \quad V = |v_0 - v_m|, \quad s_i = |a_i - b_i|.$$

The theorem, “Euclidean\_volume,” and formal proof is in the Coq file, euclidrelations.v.

PROOF.

Use the ruler (2.1) to divide the exact size,  $s_i = |a_i - b_i|$ , of each of the domain intervals,  $[a_i, b_i]$ , into a set,  $x_i$  of  $p_i$  number of subintervals.

$$(4.1) \quad \forall i \ n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \wedge \text{floor}(s_i/c) = p_i = |x_i|.$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$(4.2) \quad \text{floor}(s_i/c) = p_i \Rightarrow \lim_{c \rightarrow 0} (p_i \cdot c) = s_i.$$

Use the ruler (2.1) to divide the exact size,  $V = |v_0 - v_m|$ , of the range interval,  $[v_0, v_m]$ , into  $p^n$  subintervals. Use those cases, where  $V_c$  has an integer  $n^{th}$  root.

$$(4.3) \quad \forall p^n = V_c \in \mathbb{N}, \exists V \in \mathbb{R}, x_i : \text{floor}(V/c^n) = V_c = p^n = \prod_{i=1}^n |x_i| = \prod_{i=1}^n p_i.$$

Apply the ruler convergence theorem (2.2) to equation 4.3 and simplify:

$$(4.4) \quad \text{floor}(V/c^n) = p^n \Rightarrow V = \lim_{c \rightarrow 0} p^n \cdot c^n = \lim_{c \rightarrow 0} (p \cdot c)^n.$$

Multiply both sides of equation 4.3 by  $c^n$  and simplify:

$$(4.5) \quad p^n = \prod_{i=1}^n p_i \Rightarrow p^n \cdot c^n = (\prod_{i=1}^n p_i) \cdot c^n \Leftrightarrow (p \cdot c)^n = \prod_{i=1}^n (p_i \cdot c) \\ \Rightarrow \lim_{c \rightarrow 0} (p \cdot c)^n = \prod_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c)$$

Combine equations 4.4, 4.5, and 4.2:

$$(4.6) \quad V = \lim_{c \rightarrow 0} (p \cdot c)^n \quad \wedge \quad \lim_{c \rightarrow 0} (p \cdot c)^n = \prod_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c) \quad \wedge \\ \lim_{c \rightarrow 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad V = \lim_{c \rightarrow 0} (p \cdot c)^n = \prod_{i=1}^n \lim_{c \rightarrow 0} (p_i \cdot c) = \prod_{i=1}^n s_i. \quad \square$$

## 5. Ordered and symmetric geometries

The set and arithmetic operations used to calculate distance and volume requires sequencing through a totally ordered set of dimensions. For example, from the countable distance space definition (3.1):  $d_c = |\bigcup_{i=1}^n y_i|$ , where each range set,  $y_i$ , comes from dimension  $i$  of range sets. The commutative property of the set and arithmetic operations also allows sequencing through  $n$  number of dimensions in all  $n!$  number of possible orders.

But, a *physical* deterministic sequencer requires a *physical* set to have a single total order, at most one successor and at most one predecessor per set member, during the time of sequencing. If the set members are assigned a total order, 1, 2,  $\dots$ ,  $n$ , then the only way to determine that a sequencer traversed in the order, 2, 1,  $\dots$ , is if the same total order exists for both sequencers. Deterministic sequencing in every possible order via the same successor/predecessor relations requires each set member to be either a successor or predecessor to every other set member (mutually adjacent), herein referred to as a symmetric geometry.

It will now be proved that a set satisfying the constraints of a single total order and also symmetric defines a cyclic set containing at most 3 members, in this case, 3 dimensions of physical space.

DEFINITION 5.1. Ordered geometry:

$$\forall i \ n \in \mathbb{N}, \ i \in [1, n-1], \ \forall x_i \in \{x_1, \dots, x_n\}, \\ \text{successor } x_i = x_{i+1} \quad \wedge \quad \text{predecessor } x_{i+1} = x_i.$$

DEFINITION 5.2. Symmetric geometry (every set member is sequentially adjacent to any other member):

$$\forall i \ j \ n \in \mathbb{N}, \ \forall x_i \ x_j \in \{x_1, \dots, x_n\}, \ \text{successor } x_i = x_j \Leftrightarrow \text{predecessor } x_j = x_i.$$

THEOREM 5.3. An ordered and symmetric set is a cyclic set.

$$\text{successor } x_n = x_1 \quad \wedge \quad \text{predecessor } x_1 = x_n.$$

The theorem, “ordered\_symmetric\_is\_cyclic,” and formal proof is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of  $x_n$  and the predecessor of  $x_1$ . Therefore, the only member that can be a successor of  $x_n$ , without creating a contradiction, is  $x_1$ . And the only member that can be a predecessor of  $x_1$ , without creating a contradiction, is  $x_n$ . From the properties of a symmetric geometry (5.2):

$$(5.1) \quad i = n \quad \wedge \quad j = 1 \quad \wedge \quad \text{successor } x_i = x_j \quad \Rightarrow \quad \text{successor } x_n = x_1.$$

$$(5.2) \quad i = n \quad \wedge \quad j = 1 \quad \wedge \quad \text{predecessor } x_j = x_i \quad \Rightarrow \quad \text{predecessor } x_1 = x_n. \quad \square$$

THEOREM 5.4. An ordered and symmetric set is limited to at most 3 members.

The lemmas and formal proofs in the Coq file threed.v are:

**Lemmas:** `adj111`, `adj122`, `adj212`, `adj123`, `adj133`, `adj233`, `adj213`, `adj313`, `adj323`, and `not_all_mutually_adjacent_gt_3`.

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

**PROOF.**

It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

**DEFINITION 5.5.** Cyclic successor of  $m$  is  $n$ :

$$(5.3) \quad \text{Successor}(m, n, \text{setsize}) \leftarrow (m = \text{setsize} \wedge n = 1) \vee (n = m + 1 \leq \text{setsize}).$$

**DEFINITION 5.6.** Cyclic predecessor of  $m$  is  $n$ :

$$(5.4) \quad \text{Predecessor}(m, n, \text{setsize}) \leftarrow (m = 1 \wedge n = \text{setsize}) \vee (n = m - 1 \geq 1).$$

**DEFINITION 5.7.** Adjacent: member  $m$  is sequentially adjacent to member  $n$  if the cyclic successor of  $m$  is  $n$  or the cyclic predecessor of  $m$  is  $n$ . Notionally:

$$(5.5) \quad \text{Adjacent}(m, n, \text{setsize}) \leftarrow \text{Successor}(m, n, \text{setsize}) \vee \text{Predecessor}(m, n, \text{setsize}).$$

Every member is adjacent to every other member, where  $\text{setsize} \in \{1, 2, 3\}$ :

$$(5.6) \quad \text{Adjacent}(1, 1, 1) \leftarrow \text{Successor}(1, 1, 1) \leftarrow (m = \text{setsize} \wedge n = 1).$$

$$(5.7) \quad \text{Adjacent}(1, 2, 2) \leftarrow \text{Successor}(1, 2, 2) \leftarrow (n = m + 1 \leq \text{setsize}).$$

$$(5.8) \quad \text{Adjacent}(2, 1, 2) \leftarrow \text{Successor}(2, 1, 2) \leftarrow (n = \text{setsize} \wedge m = 1).$$

$$(5.9) \quad \text{Adjacent}(1, 2, 3) \leftarrow \text{Successor}(1, 2, 3) \leftarrow (n = m + 1 \leq \text{setsize}).$$

$$(5.10) \quad \text{Adjacent}(2, 1, 3) \leftarrow \text{Predecessor}(2, 1, 3) \leftarrow (n = m - 1 \geq 1).$$

$$(5.11) \quad \text{Adjacent}(3, 1, 3) \leftarrow \text{Successor}(3, 1, 3) \leftarrow (n = \text{setsize} \wedge m = 1).$$

$$(5.12) \quad \text{Adjacent}(1, 3, 3) \leftarrow \text{Predecessor}(1, 3, 3) \leftarrow (m = 1 \wedge n = \text{setsize}).$$

$$(5.13) \quad \text{Adjacent}(2, 3, 3) \leftarrow \text{Successor}(2, 3, 3) \leftarrow (n = m + 1 \leq \text{setsize}).$$

$$(5.14) \quad \text{Adjacent}(3, 2, 3) \leftarrow \text{Predecessor}(3, 2, 3) \leftarrow (n = m - 1 \geq 1).$$

Must prove that for all  $\text{setsize} > 3$ , there exist non-adjacent members. For example, the first and third members are not ( $\neg$ ) adjacent:

$$(5.15) \quad \forall \text{setsize} > 3 : \quad \neg \text{Successor}(1, 3, \text{setsize} > 3) \\ \leftarrow \text{Successor}(1, 2, \text{setsize} > 3) \leftarrow (n = m + 1 \leq \text{setsize}).$$

That is, member 2 is the only successor of member 1 for all  $\text{setsize} > 3$ , which implies member 3 is not a successor of member 1 for all  $\text{setsize} > 3$ .

$$(5.16) \quad \forall \text{setsize} > 3 : \quad \neg \text{Predecessor}(1, 3, \text{setsize} > 3) \\ \leftarrow \text{Predecessor}(1, \text{setsize}, \text{setsize} > 3) \leftarrow (m = 1 \wedge n = \text{setsize} > 3).$$



That is, member  $n > 3$  is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all  $n > 3$ .

$$(5.17) \quad \forall \text{setsize} > 3 : \neg \text{Adjacent}(1, 3, \text{setsize} > 3) \\ \leftarrow \neg \text{Successor}(1, 3, \text{setsize} > 3) \wedge \neg \text{Predecessor}(1, 3, \text{setsize} > 3). \quad \square$$

That is, for all  $\text{setsize} > 3$ , some elements are not sequentially adjacent to every other element (not symmetric).

## 6. Applying the ruler measure to physics

This section will show three examples of how the ruler measure can be used analyze and derive some of the fundamental relations of physics: the charge force, gravity force, and spacetime interval equations.

The principle of similitude in this context means the same dimension types and distance/area/volume types must be on both sides of the equation: type charge = charge (e.g.,  $q^2 = q_1 \cdot q_2$ ), distance = distance (e.g.,  $d = d_a + d_b$ ), area = area (e.g.,  $d^2 = d_a^2 + d_b^2$ ), and volume = volume (e.g.,  $V = d_a^3 + d_b^3$ ). And all arithmetic operations on a type must be with same type or multiplied/divided with a type-less number (like a ratio or scalar value). (e.g.,  $(t_a/t_b)(q_1 \cdot q_2)/4\pi$ ).

Consider the two independent domain intervals of type charge each having the sizes  $q_1$  and  $q_2$ . Applying the ruler, each countable member (subinterval) of charge in one interval interacts (corresponds to) each member in the other charge interval. The number of possible correspondences (interactions) is the product of the subintervals in each charge interval. And applying the volume proof (4.2), the product of the subinterval sizes converges to the area formula,  $q_1 \cdot q_2$ , as the subinterval sizes converge to zero.

Further, a characteristic of a charged object is that the strength of the charge is the same in all directions in 3-space. Therefore, we can pick a 3-space area that is equivalent to the charge area,  $q_1 \cdot q_2$ , for example, the area of a sphere:  $4\pi r^2 = q_1 q_2$ .

$$(6.1) \quad 4\pi r^2 = q_1 q_2 \quad \Rightarrow \quad 1 = q_1 q_2 / 4\pi r^2.$$

Use ratios that have no type:

$$(6.2) \quad \exists m_0, m_C, a, a_C \in \mathbb{R} : (m_0/m_C)(a/a_C) = 1 = q_1 q_2 / 4\pi r^2.$$

Multiple both sides by  $m_C a_C$ :

$$(6.3) \quad k_C = (m_C a_C / 4\pi) \Rightarrow F = m_0 a = k_C q_1 q_2 / r^2.$$

Applying the exact same logic and steps as for charge to gravity:

$$(6.4) \quad G = (m_G a_G / 4\pi) \Rightarrow F = m_0 a = G m_1 m_2 / r^2.$$

Note that gravity is proportionate to charge:

$$(6.5) \quad m^2 = m_1 m_2, \quad q^2 = q_1 q_2 \Rightarrow m^2 = (G/k_C) q^2 \Rightarrow m = (\sqrt{G/k_C}) q.$$

Consider the Euclidean distances,  $d_1$  and  $d_2$ , between the same two points, A and B, in two inertial frames of reference. We know from the Euclidean distance proof (3.7) that largest intersection of the range sets corresponding to the two distance intervals converges to the Euclidean distance equation:  $D^2 = d_1^2 + d_2^2$ .

The time ratio,  $t/t_c$  is a number without a type that can be multiplied times some distance value,  $d_c$ , and the result is still a distance:

$$(6.6) \quad D = (d_c/t_c)t = ct \Rightarrow (ct)^2 = d_1^2 + d_2^2.$$

$$(6.7) \quad (ct)^2 = d_1^2 + d_2^2 \Rightarrow d_2 = \sqrt{(ct)^2 - d_1^2}.$$

$$(6.8) \quad d = d_2 \wedge d_1^2 = x^2 + y^2 + z^2 \Rightarrow d = \sqrt{(ct)^2 - (x^2 + y^2 + z^2)},$$

which is the four-vector length of the spacetime interval (relativistic change in 3-dimensional distance) [Bru17].

## 7. Insights and conjectures

Applying the ruler measure (2.1) and ruler convergence (2.2) to the set relations, countable distance space (3.1) and all possible combinations (4.1) yields the following insights and open questions:

- (1) Notions of point, plane, side, angle, perpendicular, congruence, intersection, etc. are not necessary to motivate and derive the properties of metric space, Euclidean distance and area/volume.
- (2) Distance is a function of the number of domain-to-range set member mappings. In contrast, area/volume is a function of the number of domain-to-domain set member mappings.
- (3) All notions of distance are derived from the principle that every domain set,  $x_i$ , has a corresponding range (distance) set,  $y_i$ , containing the same number of members:  $|x_i| = |y_i|$ . And the distance spanning multiple, disjoint, domain sets is the number of members,  $d_c$ , in the corresponding union range set:  $d_c = |\bigcup_{i=1}^n y_i|$  (3.1).
  - (a) A direct consequence of the inclusion-exclusion principle [CG15] is the set relation,  $d_c = |\bigcup_{i=1}^2 y_i| \leq \sum_{i=1}^2 |y_i|$  (3.2), which generates the triangle inequality, non-negativity, and identity of indiscernibles properties of metric space (3.2). The fourth property, symmetry [ $d(u, v) = d(v, u)$ ], is a consequence of the sum of set member sizes being the same for every ordering (commutative law of addition).
  - (b)  $|x_i| = |y_i| = p_i$  constrains the range of the total number of domain-to-range set member mappings from  $\sum_{i=1}^n p_i$  to  $\sum_{i=1}^n p_i^2$  mappings (3.3). The case of the largest possible number of domain-to-range set member mappings,  $\sum_{i=1}^n p_i^2$ , converges to the Euclidean distance equation (3.7) and is the set-based reason Euclidean distance is the smallest possible distance between two distinct points in  $\mathbb{R}^n$ .
  - (c) Using the Taylor series and the Euclidean distance equation with two domain intervals sizes yields the arc sine and arc cosine functions. In other words, the parametric variable equating arc sine and arc cosine maps to the notion of angle, where the two domain intervals map to the notion of two line segments (two sides).

Euclidean geometry [Joy98] and axiomatic geometry (for example, Hilbert [Hil80] and Birkhoff [Bir32], Veblen [Veb04], and Tarski [TG99]) either use notions of line and angle as undefined primitives or as definitions in terms of other undefined primitives.

- (d) Conjecture: the constraints:  $|x_i| < |y_i|$ ,  $|x_i| = |y_i|$ , and  $|x_i| > |y_i|$  yield three types of distance spaces: open, flat, and closed. All hyperbolic distances are in flat space (satisfy the triangle inequality). All elliptic geometry distances are in closed space.
- (4) Euclidean volume has as many range set elements,  $|V_c|$ , as domain set  $n$ -tuples (coordinates),  $|T_c|$ . Conjecture: the constraints:  $|T_c| < |V_c|$ ,  $|T_c| = |V_c|$ , and  $|T_c| > |V_c|$  yields three types of volume spaces: open, flat, and closed that correspond 1-1 to open, flat, and closed distance spaces.
- (5) Euclidean distance and volume were derived in this article for any number of dimensions. But, from the ordered and symmetric geometries theorem (5.4), time constrains a physical set of totally ordered members, where the commutative law applies to the set and arithmetic operations, is limited to at most three members, for example, three dimensions of physical space. Of course, relativity theory assumes only 3 dimensions of space [Bru17].
- (6) The derivation of charge force, gravity force, and spacetime interval equations via the ruler measure demonstrates that the ruler measure is a useful tool to be added to the tool kit of PDEs and integrals.

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