The Two Set Relations Generating Geometry

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ABSTRACT. Where each domain set has a corresponding range set, distance is the cardinal of the union of the range sets and volume is the cardinal of the Cartesian product of disjoint range sets. A ruler (measuring stick) partitions intervals approximately into sets of size c subintervals and sums the sizes. The distance and volume set operations on sets of size c subintervals converge to the properties of metric space, the Manhattan distance, Euclidean distance, and volume equations as c goes to 0. The volume proof allows simpler derivations of Coulomb's charge force and Newton's gravity force equations without using other laws of physics or Gauss's divergence theorem. A symmetry constraint can limit physical space to 3 dimensions. All proofs are verified in Coq.

CONTENTS

1. Introduction

What constraint between domain and range sets generates flat space? What is the domain-to-range set mapping that generates the Euclidean distance equation and makes Euclidean distance the smallest possible distance between two distinct points in flat space? Mathematical analysis has not answered these and other basic questions because metric space, Euclidean distance, and Euclidean area/volume have been definitions [**Gol76**] [**Rud76**] motivated by the notions of Euclidean geometry, like side, angle, plane, congruent, etc. [**Joy98**] rather than deriving the geometric relations from an abstract set and limit-based foundation.

Riemann and Lebesgue integration divide both a domain interval, $[a, b]$, and range interval, $[f(a), f(b)]$, into the same, n, number of subintervals, where the size of the subintervals can vary. In contrast, a ruler (measuring stick) divides both domain and range intervals approximately into sets of same size, c, subintervals, where the number of subintervals in each interval can vary. A ruler allows deriving geometric relations from the combinatorial mappings between sets of same-sized subintervals as $c \to 0$ that are not possible using traditional integration methods.

Where each disjoint domain set has a corresponding range set, distance is defined, here, as the cardinal of the union of the range sets and volume as the cardinal of the Cartesian product of the disjoint range sets. The distance and volume set operations applied to sets of size c subintervals converge to the properties of metric space, the Manhattan distance, Euclidean distance, and volume equations as $c \to 0$. The proofs provide a deeper understanding of geometry and physics.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

Like integrals, the ruler is a "concrete" measure of intervals, which is more useful for deriving equations and inequalities than abstractions like metric space, and the Lebesgue, Borel, Hausdorff, etc. measures. The ruler measures the size, M, of an interval *approximately* as the sum of the nearest integer number (floor or ceiling) of whole subintervals, each subinterval having the same size, c. And like integrals, the ruler is both an inner and outer measure.

DEFINITION 2.1. Ruler measure: $\forall\, c,\ s \in \mathbb{R},\ [a, b] \subset \mathbb{R},\ s = |a - b|\ \wedge\ c > 0\ \wedge$ $(p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \lim_{c \to 0} \sum_{i=1}^{p} c = \lim_{c \to 0} pc.$

THEOREM 2.2. *Ruler convergence:* $\forall\, [a, b] \subset \mathbb{R}, s = |a - b| : M = \lim_{c \to 0} pc = s.$

The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

(2.1) $\forall\, c > 0,\ p = floor(s/c)\ \wedge\ 0 \le |floor(s/c) - s/c| < 1\ \Rightarrow\ 0 \le |p - s/c| < 1.$

Multiply all sides of inequality 2.1 by $|c|$:

(2.2) $\qquad \forall\, c > 0, \quad 0 \le |p - s/c| < 1 \quad \Rightarrow \quad 0 \le |pc - s| < |c|.$

(2.3) $\quad \forall\, \delta :\ |pc - s| < |c| = |c - 0| < \delta$
$\qquad\qquad \Rightarrow \quad \forall\, \epsilon = \delta :\ |c - 0| < \delta\ \wedge\ |pc - s| < \epsilon\ :=\ M = \lim_{c \to 0} pc = s. \quad \square$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = |0 - \pi|$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}},\ 3.14_{c=10^{-2}},\ 3.141_{c=10^{-3}}, ..., \pi_{\lim_{c \to 0}}.$ $M = \lim_{c \to 0} pc = \int_{a}^{b} dc = s.$

3. Distance

Notation convention: Vertical bars around a set or list, $|\cdots|$, indicates the cardinal (number of members in the set or list).

3.1. Countable distance. A concrete example of a countable distance is the number of same-sized steps walked in a range (distance) set, y, which equals the number of same-sized pieces of land in a corresponding domain set, x: $|x| = |y|$. Generalizing, each disjoint domain set, x_i, has a corresponding range (distance) set, y_i. The countable distance spanning the disjoint domain sets is the number of members, d_c, in the union range set:

DEFINITION 3.1. Countable distance, d_c :
$$d_c = |\bigcup_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i|.$$

Flat space is where $|x_i| = |y_i|$. Extending distance beyond flat space is shown in the last section of this article.

The following trivial proof of the well-known inequality, $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$, is nothing new and is included here for completeness. The inequality follows directly from the inclusion-exclusion principle [**CG15**]. But, a more intuitive and simple proof follows from the associative law of addition where the sum of set sizes is equal to the size of all the set members appended into a list and the commutative law of addition that allows sorting that list into a list of unique members (the *union* set) and a list of duplicates. The duplicates being ≥ 0 implies the union size is always $\leq$ the sum of set sizes.

THEOREM 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|.$

PROOF. A formal proof, inclusion_exclusion_inequality, using sorting into a set of unique members (*union* set) and a list of duplicates, is in the file euclidrelations.v.

$$(3.1) \quad \sum_{i=1}^{n} |y_i| = |append_{i=1}^{n} y_i| = |sort(append_{i=1}^{n} y_i)|$$
$$= |\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i|.$$

$$(3.2) \quad |\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| \quad \wedge \quad |duplicates_{i=1}^{n} y_i| \geq 0$$
$$\Rightarrow \quad |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|. \quad \square$$

3.2. Countable distance range. From the countable distance definition (3.1), $d_c = |\bigcup_{i=1}^{n} y_i|$, as the amount of intersection increases, more domain set members can map to a single range set member. Therefore, the countable distance, d_c, is a function of the number of domain-to-range set member mappings.

From the countable distance constraint (3.1), where $|x_i| = |y_i| = p_i$: 1) The inclusion-exclusion inequality (3.2), $d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$, puts an upper bound on distance, where there is a 1-1 domain-to-range set correspondence, $d_c = \sum_{i=1}^{n} (1 \cdot |y_i|) = \sum_{i=1}^{n} p_i$ number of mappings; 2) Each domain set member can map to as many as every range set member (the Cartesian product), $\sum_{i=1}^{n} (|y_i| \cdot |x_i|) = \sum_{i=1}^{n} p_i^2$ number of mappings and $\exists\, f : d_c = f(\sum_{i=1}^{n} p_i^2)$.

Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of domain-to-range set mapping cases converges to the real-valued Manhattan and Euclidean distance relations.

3.3. Manhattan distance.

THEOREM 3.3. *Manhattan (largest) distance, d, is the size of the range interval,* $[d_0, d_m]$, *corresponding to a set of disjoint domain intervals,* $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*
$$d = \sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "taxicab_distance," and formal proof are in the Coq file, euclidrelations.v.

PROOF.

From the countable distance definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$(3.3) \quad d_c = |\textstyle\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n}|y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \quad \Rightarrow \quad d_c \leq \sum_{i=1}^{n} p_i$$
$$\Rightarrow \quad \exists\, p_i,\, d_c : \; d_c = \sum_{i=1}^{n} p_i.$$

Multiply both sides of equation 3.3 by c and take the limit:

$$(3.4) \quad d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n}(p_i \cdot c) \Rightarrow \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0}(p_i \cdot c).$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$(3.5) \quad d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d : \; floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$(3.6) \quad s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c \to 0} p_i \cdot c = s_i.$$

Combine equations 3.5, 3.4, 3.6:

$$(3.7) \quad d = \lim_{c \to 0} d_c \cdot c \quad \wedge \quad \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0}(p_i \cdot c) \quad \wedge$$
$$\lim_{c \to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0}(p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square$$

3.4. Euclidean distance.

THEOREM 3.4. *Euclidean (smallest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*
$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "Euclidean_distance," and formal proof are in the Coq file, euclidrelations.v.

PROOF.

Apply the rule of product to the largest number of domain-to-range set mappings, where all p_i number of range set members, y_i, map to each of the p_i number of members in the domain set, x_i, which is the Cartesian product, $|y_i| \cdot |x_i|$:

$$(3.8) \quad |x_i| = |y_i| = p_i \quad \Rightarrow \quad \sum_{i=1}^{n}|y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2.$$

From the countable distance definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$(3.9) \quad d_c = |\textstyle\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n}|y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \quad \Rightarrow \quad d_c \leq \sum_{i=1}^{n} p_i$$
$$\Rightarrow \quad \exists\, p_i,\, d_c : \; d_c = \sum_{i=1}^{n} p_i.$$

Square both sides of equation 3.9 ($x = y \Leftrightarrow f(x) = f(y)$):

$$(3.10) \quad \exists\, p_i,\, d_c : \; d_c = \sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\, d_c : \; d_c^2 = (\sum_{i=1}^{n} p_i)^2.$$

Apply the square of sum inequality, $(\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$, to equation 3.10 and select the smallest area (the equality) case:

$$(3.11) \quad d_c^2 = (\sum_{i=1}^{n} p_i)^2 = \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j$$
$$= \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} p_i \sum_{j=1, j \neq i}^{n} p_j \geq \sum_{i=1}^{n} p_i^2 \quad \Rightarrow \quad \exists\, p_i : \; d_c^2 = \sum_{i=1}^{n} p_i^2.$$

Multiply both sides of equation 3.11 by c^2, simplify, and take the limit.

(3.12) $\quad d_c^2 = \sum_{i=1}^n p_i^2 \;\Rightarrow\; d_c^2 \cdot c^2 = \sum_{i=1}^n p_i^2 \cdot c^2 \;\Leftrightarrow\; (d_c \cdot c)^2 = \sum_{i=1}^n (p_i \cdot c)^2$

$$\Rightarrow \quad \lim_{c \to 0}(d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \to 0}(p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

(3.13) $\quad \exists\, c\, d \in \mathbb{R} : floor(d/c) = d_c \;\Rightarrow\; d = \lim_{c \to 0} d_c \cdot c \;\Rightarrow\; d^2 = \lim_{c \to 0}(d_c \cdot c)^2.$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

(3.14) $\quad s_i = |a_i - b_i| \;\;\wedge\;\; floor(s_i/c) = |x_i| = |y_i| = p_i \;\Rightarrow\; \lim_{c \to 0} p_i \cdot c = s_i.$

Combine equations 3.13, 3.12, 3.14:

(3.15) $\quad d^2 = \lim_{c \to 0}(d_c \cdot c)^2 \;\;\wedge\;\; \lim_{c \to 0}(d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \to 0}(p_i \cdot c)^2 \;\;\wedge$

$\lim_{c \to 0}(p_i \cdot c) = s_i \;\Rightarrow\; d^2 = \lim_{c \to 0}(d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \to 0}(p_i \cdot c)^2 = \sum_{i=1}^n s_i^2. \quad \square$

3.5. Metric Space. All function range intervals, $d(u, w)$, satisfying the countable distance definition (3.1), where the ruler is applied, generates the properties of metric space: triangle inequality, non-negativity, identity of indiscernibles, and symmetry. The formal proofs: triangle_inequality, non_negativity, identity_of_ indiscernibles, and symmetry are in the Coq file, euclidrelations.v.

THEOREM 3.5. *Triangle Inequality:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \leq d(u, v) + d(v, w).$

PROOF. Apply the ruler measure (2.1), the countable distance condition (3.1), inclusion-exclusion inequality (3.2), and then ruler convergence (2.2).

(3.16) $\quad \forall\, c > 0, \; d(u, w), \; d(u, v), \; d(v, w) :$

$$|y_1| = floor(d(u, v)/c) \;\wedge\; |y_2| = floor(d(v, w)/c) \;\wedge$$
$$d_c = floor(d(u, w)/c) \;\wedge\; d_c = |y_1 \cup y_2| \leq |y_1| + |y_2|$$
$$\Rightarrow\; floor(d(u, w)/c) \leq floor(d(u, v)/c) + floor(d(v, w)/c)$$
$$\Rightarrow\; floor(d(u, w)/c) \cdot c \leq floor(d(u, v)/c) \cdot c + floor(d(v, w)/c) \cdot c$$
$$\Rightarrow\; \lim_{c \to 0} floor(d(u, w)/c) \cdot c \leq \lim_{c \to 0} floor(d(u, v)/c) \cdot c + \lim_{c \to 0} floor(d(v, w)/c) \cdot c$$
$$\Rightarrow\; d(u, w) \leq d(u, v) + d(v, w). \quad \square$$

THEOREM 3.6. *Non-negativity:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \geq 0.$

PROOF. By definition, a set always has a size (cardinal) ≥ 0:

(3.17) $\quad \forall\, c > 0, \; d(u, w) : \quad floor(d(u, w)/c) = d_c \;\;\wedge\;\; d_c = |y_1 \cup y_2| \geq 0$

$$\Rightarrow\; floor(d(u, w)/c) = d_c \geq 0 \;\Rightarrow\; d(u, w) = \lim_{c \to 0} d_c \cdot c \geq 0. \quad \square$$

THEOREM 3.7. *Identity of Indiscernibles:* $d(w, w) = 0.$

PROOF. Apply the triangle inequality property (3.5):

(3.18) $\quad \forall\, d(u, v) = d(v, w) = 0 \;\wedge\; d(u, w) \leq d(u, v) + d(v, w) \;\Rightarrow\; d(u, w) \leq 0.$

Combine the non-negativity property (3.6) and the previous inequality (3.18):

(3.19) $\quad d(u, w) \geq 0 \;\wedge\; d(u, w) \leq 0 \;\Leftrightarrow\; 0 \leq d(u, w) \leq 0 \;\Rightarrow\; d(u, w) = 0.$

Combine the result of step 3.19 and the condition, $d(u, v) = 0$, in step 3.18.

(3.20) $\qquad\qquad d(u, w) = 0 \;\wedge\; d(u, v) = 0 \;\Rightarrow\; w = v.$

Combine the condition, $d(v, w) = 0$, in step 3.18 and the result of step 3.20.

(3.21) $d(v, w) = 0 \;\wedge\; w = v \;\Rightarrow\; d(w, w) = 0.$ □

THEOREM 3.8. *Symmetry: From the Euclidean distance* (3.4) *and Manhattan distance* (3.3) *proofs:* $(x^2 + y^2)^{1/2} \leq d(x, y) \leq x + y \Rightarrow d(u, v) = d(v, u)$ *in flat space, where* $|x_i| = |y_i|$.

PROOF.

(3.22) $(x^2 + y^2)^{1/2} \leq d(x, y) \leq x + y$

$$\Rightarrow \quad \forall\, p \,:\, 1 \leq p \leq 2, \quad d(x, y) = (x^p + y^p)^{1/p}.$$

By the commutative law of addition:

(3.23) $d(u, v) = (u^p + v^p)^{1/p} \quad \wedge \quad d(v, u) = (v^p + u^p)^{1/p}$

$\qquad \Rightarrow \quad d(u, v)^p = u^p + v^p = v^p + u^p = d(v, u)^p \quad \Rightarrow \quad d(u, v) = d(v, u).$ □

4. Euclidean Volume

The Lebesgue and Borel area/volume measures and Riemann and Lebesgue integrals sum range interval lengths, where each range interval length is defined (assumed) to be the product of an n-tuple of domain interval lengths. The goal, here, is to *prove* that Euclidean area/volume is derived from the Cartesian product of countable sets without assuming the product of an n-tuple of interval lengths.

Both Manhattan distance and Euclidean volume are cases of a 1-1 correspondence of a domain-to-range set mapping and the case of disjoint range sets.

$$d_c = |\textstyle\bigcup_{i=1}^n y_i| : \quad \bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^n y_i = \emptyset.$$

DEFINITION 4.1. *Countable Volume,* v_c, *is the largest number of all possible ordered n-tuples:*

$$v_c = |\textstyle\times_{i=1}^n y_i| : \quad \bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^n y_i = \emptyset.$$

THEOREM 4.2. *Euclidean volume,* v, *is length of the range interval,* $[v_0, v_m]$, *equal to product of domain interval lengths,* $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$:

$$v = \textstyle\prod_{i=1}^n s_i, \;\; v = |v_0 - v_m|, \;\; s_i = |a_i - b_i|.$$

The theorem, "Euclidean_volume," and formal proof are in the Coq file, euclidrelations.v.

PROOF.

Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of subintervals.

(4.1) $\forall\, i \, n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i| = |y_i|.$

Apply the ruler convergence theorem (2.2) to equation 4.1:

(4.2) $floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c \to 0}(p_i \cdot c) = s_i.$

Apply the associative law of multiplication to derive the countable volume (4.1) in terms of p_i:

(4.3) $v_c = |\textstyle\times_{i=1}^n y_i| = \prod_{i=1}^n |y_i| \quad \wedge \quad |y_i| = p_i \quad \Rightarrow \quad v_c = \prod_{i=1}^n p_i.$

Multiply both sides of equation 4.3 by c^n:

(4.4) $v_c \cdot c^n = (\textstyle\prod_{i=1}^n p_i) \cdot c^n = \prod_{i=1}^n (p_i \cdot c).$

Use those cases, where v_c has an integer n^{th} root.

(4.5) $\quad \forall\, n,\, p,\, v_c \in \mathbb{N}\, :\, p^n = v_c \;\Rightarrow\; v_c \cdot c^n = p^n \cdot c^n = (p \cdot c)^n = \prod_{i=1}^{n}(p_i \cdot c).$

Apply the ruler (2.1) and ruler convergence (2.2) to the range interval, $[v_0, v_m]$ (where $v = |v_0 - v_m|$), and then combine with equations 4.5 and 4.2:

(4.6) $\; floor(v/c^n) = p^n \Rightarrow v = \lim_{c \to 0}(p \cdot c)^n = \prod_{i=1}^{n} \lim_{c \to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i.$ $\quad\square$

5. Applications to physics

5.1. Coulomb's charge force. The sizes, q_1 and q_2, of two charges are independent domain variables, where each infinitesimal size c component of a charge exerts a force on each infinitesimal size c component of the other charge. The total force, F, is proportionate to the total number of forces (the Cartesian product of the infinitesimal size c components) multiplied times a quantum charge force, $m_C a_C$. From the volume proof (4.2), the Cartesian product converges to $q_1 q_2$:

(5.1) $\qquad F \propto m_C a_C((\lim_{c \to 0} floor(q_1/c) \cdot c)(\lim_{c \to 0} floor(q_2/c) \cdot c)) = m_C a_C(q_1 q_2).$

From equation 5.1, a change in charge, q, causes a proportionate change in force, F. Solving for the quantum force, $F = m_C a_C$, requires a proportionate variable, r, to offset the effect of a change in q. $r \propto q \Rightarrow \exists\, q_C,\, r_C \in \mathbb{R}\, :\, r(q_C/r_C) = q$, where q_C/r_C is a unit-factoring conversion ratio:

(5.2) $\;\forall\, q_1, q_2 \geq 0 \;\exists\, q \in \mathbb{R}\, :\, q^2 = q_1 q_2\;\wedge\; r(q_C/r_C) = q\;\Rightarrow\;(r(q_C/r_C))^2 = q_1 q_2.$

(5.3) $\;(r(q_C/r_C))^2 = q_1 q_2\;\wedge\; F \propto m_C a_C(q_1 q_2)$

$\qquad\Rightarrow\quad F \propto m_C a_C(r(q_C/r_C))^2 = m_C a_C(q_1 q_2)$

$\qquad\qquad\Rightarrow\quad F = m_C a_C = (m_C a_C r_C^2/q_C^2)q_1 q_2/r^2 = k_C q_1 q_2/r^2.$

where $k_C = m_C a_C r_C^2/q_C^2$ corresponds to the SI units: $Nm^2 C^{-2}$.

5.2. Newton's gravity force equation. The sizes, m_1 and m_2, of two masses are independent domain variables, where each infinitesimal size c component of a mass exerts a force on each infinitesimal size c component of the other mass. The total force, F, is proportionate to the total number of forces (the Cartesian product of the size c components) multiplied times a quantum gravity force, $m_G a_G$. From the volume proof (4.2), the Cartesian product converges to $m_1 m_2$:

(5.4) $\quad F \propto m_G a_G((\lim_{c \to 0} floor(m_1/c) \cdot c)(\lim_{c \to 0} floor(m_2/c) \cdot c)) = m_G a_G(m_1 m_2).$

From equation 5.4, an change in mass, m, causes a proportionate change in force, F. Solving for the quantum force, $F = m_G a_G$, requires a proportionate variable, r, to offset the effect of a change in m. $r \propto m \Rightarrow \exists\, m_G,\, r_G \in \mathbb{R}\, :\, r(m_G/r_G) = m$, where m_G/r_G is a unit-factoring conversion ratio:

(5.5) $\;\forall\, m_1, m_2 \geq 0 \;\exists\, m \in \mathbb{R}\, :\quad m^2 = m_1 m_2\quad\wedge\quad r(m_G/r_G) = m$

$\qquad\qquad\qquad\qquad\qquad\Rightarrow\quad(r(m_G/r_G))^2 = m_1 m_2.$

(5.6) $\;(r(m_G/r_G))^2 = m_1 m_2\;\wedge\; F \propto m_G a_G(m_1 m_2)$

$\qquad\qquad\Rightarrow\quad F \propto m_G a_G(r(m_G/r_G))^2 = m_G a_G(m_1 m_2)$

$\qquad\qquad\qquad\Rightarrow\quad F = m_G a_G = (a_G r_G^2/m_G)m_1 m_2/r^2.$

(5.7) $\exists\, t_G \in \mathbb{R}: r_G/t_G^2 = a_G \quad \wedge \quad F = (a_G r_G^2/m_G)m_1 m_2/r^2$
$$\Rightarrow \quad F = (r_G^3/m_G t_G^2)m_1 m_2/r^2 = G m_1 m_2/r^2,$$

where $G = r_G^3/m_G t_G^2$ corresponds to the SI units: $m^3 kg^{-1} s^{-2}$.

5.3. Spacetime equations. The charge (5.3) and gravity (5.7) force equations were derived from the principle that charge and mass are proportionate to distance: $r = (r_C/q_C)q = (r_G/m_G)m$. Suppose that time is also proportionate to distance, $r = (r_c/t_c)t = ct$, where the unit-factoring conversion ratio, $r_c/t_c = c$, is some constant speed.

Applying the ruler to two intervals, $[0, d_1]$ and $[0, d_2]$, in two inertial (independent, non-accelerating) frames of reference, the distance (and time at a constant speed) spanning the two domain intervals converges to a range of distances (and times) from Manhattan (3.3) to Euclidean distance (3.4).

(5.8) $r^2 = d_1^2 + d_2^2 \quad \wedge \quad r = (r_c/t_c)t = ct$
$$\Rightarrow \quad (ct)^2 = d_1^2 + d_2^2 \quad \Rightarrow \quad d_2 = \sqrt{(ct)^2 - d_1^2}.$$

(5.9) $d_2 = \sqrt{(ct)^2 - d_1^2} \quad \wedge \quad d = d_2 \quad \wedge \quad d_1 = vt$
$$\Rightarrow \quad d = \sqrt{(ct)^2 - (vt)^2} = ct\sqrt{1 - (v^2/c^2)},$$

which is the spacetime dilation equation. [**Bru17**].

(5.10) $d_2^2 = (ct)^2 - d_1^2 \quad \wedge \quad s = d_2 \quad \wedge \quad d_1^2 = x^2 + y^2 + z^2$
$$\Rightarrow \quad s^2 = (ct)^2 - (x^2 + y^2 + z^2),$$

which is one form of the spacetime interval equation [**Bru17**].

5.4. 3 dimensions of physical geometry. The countable distance, $d_c = |\bigcup_{i=1}^{n} y_i|$, (3.1) and volume, $d_c = |\times_{i=1}^{n} y_i|$, (4.1) generating geometric relations require being able to assign a sequential order (a total order, 1 through n) to a set of intervals (dimensions). The physical intervals (dimensions): width, height, and depth have fixed relative positions that allow being assigned a sequential order.

The commutative properties of union, addition, and product allow sequencing through each interval (dimension) in every possible order. But, *deterministic* sequencing in every possible order through a sequentially ordered set requires each set member to be sequentially adjacent (either a successor or predecessor) to every other member, herein referred to as a symmetric geometry.

It will now be proved that the constraint (coexistence) of symmetry on a sequentially ordered set defines a cyclic set that contains at most 3 members, in this case, 3 dimensions of physical distance and volume.

DEFINITION 5.1. Ordered geometry:

$$\forall\, i\, n \in \mathbb{N}, \ i \in [1, n-1], \ \forall\, x_i \in \{x_1, \ldots, x_n\},$$
$$successor\ x_i = x_{i+1} \ \wedge \ predecessor\ x_{i+1} = x_i.$$

DEFINITION 5.2. Symmetric geometry (every set member is sequentially adjacent to every other member):

$$\forall\, i\, j\, n \in \mathbb{N}, \ \forall\, x_i\, x_j \in \{x_1, \ldots, x_n\}, \ successor\ x_i = x_j \ \Leftrightarrow \ predecessor\ x_j = x_i.$$

THEOREM 5.3. *An ordered and symmetric set is a cyclic set.*

$$i = n \; \wedge \; j = 1 \; \Rightarrow \; successor \; x_n = x_1 \; \wedge \; predecessor \; x_1 = x_n.$$

The theorem, "ordered_symmetric_is_cyclic," and formal proof is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

(5.11) $i = n \; \wedge \; j = 1 \; \wedge \; successor \; x_i = x_j \; \Rightarrow \; successor \; x_n = x_1.$

Applying the definition of a symmetric geometry (5.2) to conclusion 5.11:

(5.12) $successor \; x_i = x_j \; \Rightarrow \; predecessor \; x_j = x_i \; \Rightarrow \; predecessor \; x_1 = x_n.$ □

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

DEFINITION 5.5. Cyclic successor of m is n:

(5.13) $Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \le setsize).$

DEFINITION 5.6. Cyclic predecessor of m is n:

(5.14) $Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - q \ge 1).$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:
(5.15)
$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize).$

Prove that every member is adjacent to every other member, where $setsize \in \{1, 2, 3\}$:

(5.16) $Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1).$

(5.17) $Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \le setsize).$

(5.18) $Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1).$

(5.19) $Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize)$.

(5.20) $Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - q \geq 1)$.

(5.21) $Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1)$.

(5.22) $Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize)$.

(5.23) $Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize)$.

(5.24) $Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - q \geq 1)$.

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

(5.25) $\forall\, setsize > 3:$ $\neg Successor(1, 3, setsize > 3)$
$$\leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize).$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

(5.26) $\forall\, setsize > 3:$ $\neg Predecessor(1, 3, setsize > 3)$
$$\leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

(5.27) $\forall\, setsize > 3:$ $\neg Adjacent(1, 3, setsize > 3)$
$$\leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

(1) The ruler measure (2.1) of an interval, $[a, b]$, and an integral converge to the same value: $c \geq 0, s = |a - b|, p = floor(s/c) \Rightarrow \lim_{c \to 0} pc = \int_a^b dc = s$. The integer, p, as the cardinal of a countable set of size c subintervals, makes it easy to derive the properties of metric space and equations like Euclidean distance and volume as $c \to 0$. But, it probably is not possible to use integrals to make the same simple derivations from the same countable distance set (3.1) and countable volume set (4.1) defintions.

(2) Generalizing the countable distance constraint, $|x_i| = |y_i|$, (3.1) to $|x_i| = |y_i|^q$, $q \in \mathbb{R}$, generates all the L^p norms, $||L||_p = (\sum_{i=1}^n s_i^p)^{1/p}$. For example, using the same proof pattern as for Euclidean distance (3.4): $p_i = |y_i| \Rightarrow |x_i| = p_i^q \Rightarrow \sum_{i=1}^n |x_i| \cdot |y_i| = \sum_{i=1}^n p_i^q \cdot p_i = \sum_{i=1}^n p_i^{q+1} \leq d_c^{q+1} \dots d = (\sum_{i=1}^n s_i^{q+1})^{1/(q+1)}$.

(3) The countable distance (3.1) combined with the inclusion-exclusion inequality (3.5), $d_c = |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$, generates all the properties defining metric space (3.5) and distances in the form, $d(x, y) = (x^p + y^p)^{1/p}$ (because the domain-to-range set mapping is the same for each domain-range set pair, which causes all terms to have the same exponent, p, where the same exponent, p, on each term causes the symmetry property, $d(u, v) = d(v, u)$ (3.8)). But, there are functions that satisfy the definition of metric space (for example, $d(x, y) = |x - y|$ and $d(x, y) = |log(y/x)|$)

that are not in the form, $d(x, y) = (x^p + y^p)^{1/p}$. Therefore, the definition of metric space is not sufficiently restrictive to model the distance in geometry.

(4) The Manhattan (largest) distance is the case of a 1-1 domain-to-range set correspondence: $d_c = \sum_{i=1}^{n} 1 \cdot |y_i| = \sum_{i=1}^{n} p_i$. The Euclidean distance is the case of the largest many-to-man domain-to-range set mappings: $\sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^2$. The case where both domain-to-range set mappings, $\sum_{i=1}^{n} p_i$ and $\sum_{i=1}^{n} p_i^2$, coexist is: $d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c^2 = (\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$. The equality case is where the smallest distance, $d_c^2 = \sum_{i=1}^{n} p_i^2$, coexists with the largest (Manhattan) distance, $d_c = \sum_{i=1}^{n} p_i$, in flat space ($|x_i| = |y_i|$), which is the set-based reason Euclidean distance (3.4) is the smallest possible distance between two distinct points in $\mathbb{R}^n$.

(5) Coulomb's charge force (5.1) and Newton's gravity force (5.4) equations were each derived in a few steps using the Euclidean volume proof (4.2), without using other laws of physics or Gauss's divergence theorem. The Cartesian product of infinitesimal charge forces using integrals would be: $(\lim_{c \to 0} floor(q_1/c) \cdot c)(\lim_{c \to 0} floor(q_2/c) \cdot c) = \int_0^{q_2} \int_0^{q_1} \mathrm{d}c \mathrm{d}c = q_1 q_2$.

(6) **The Proportionate Interval Principle:** The derivations of the charge force, gravity force, and spacetime equations shows that all Euclidean distance intervals having a size, r, have proportionately sized intervals of other types: $r = (r_C/q_C)q = (r_G/m_G)m = (r_c/t_c)t$, where the conversion ratios are for unit-factor analysis.

   (a) The derivations of charge and gravity forces requiring the conversion ratios, $q = (q_C/r_C)r$ and $m = (m_G/r_G)r$, implies that if there are quantum values of charge, q_C, and mass, m_G, then there are quantum distances, r_C and r_G, where the charge and gravity forces do not exist (are not defined) at smaller distances, which agrees with the theories of the Planck length, l_P, and the Schwarzschild radius, r_s.

   (b) The charge and gravity force derivations show that the proportionate interval principal generates the inverse square law.

   (c) The derivations of charge and gravity forces requiring the conversion ratios, $q = (q_C/r_C)r$ and $m = (m_G/r_G)r$, show that rectangular geometric area (r^2) maps to rectangular charge area ($q_1 q_2$) and mass area ($m_1 m_2$). But, some versions of the charge constant, vacuum magnetic permeability constant, fine structure constant, etc. contain the value 4π because the creators assumed geometric dilution (flux divergence on the surface of a sphere, $4\pi r^2$). The proportionate interval principle is a more parsimonious explanation of the inverse square law than flux divergence, which indicates that those versions of the constants containing the value 4π might be incorrect.

   (d) Time proportionate to Euclidean distance, $r = ct \Rightarrow r^2 = (ct)^2 = d_1^2 + d_2^2$, (5.8) shows an intuitive relationship between time and distance and also generates the spacetime dilation (5.9) and spacetime interval equations (5.10).

   (e) A countable set of values has measure 0. And 0 times and any distance is 0. In other words, there is no proportion relationship of a countable set of values to distance. Therefore, a countable set of

state value changes with respect to time are independent of distance (for example, the change in the spin values of two quantum coupled particles and the change in polarization of two quantum coupled photons).

(7) The proof of at most 3 members in any ordered and symmetric set (5.4) has implications beyond limiting physical space to 3 dimensions.

    (a) Each *physical* infinitesimal volume (ball) can have at most 3 ordered and symmetric dimensions of discrete *physical* states of the same type. And each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values per ball.

    (b) If each of the three possible ordered and symmetric dimensions of discrete physical states contained unordered sets of discrete state values, for example, unordered binary values, then there would be $2 \cdot 2 \cdot 2 = 8$ possible combinations of values. Unordered sets (states) are non-deterministic. For example, every time that an unordered state is physically measured, there is a 50 percent chance of having one of the binary values.

    (c) Where infinitesimal balls intersect, an arithmetic of the interactions of the discrete states with respect to time needs to be developed. The interaction of the discrete states associated with overlapping infinitesimal balls with respect to time might result in what we perceive as motion, waves, particles, spin, polarization, work, force, mass, charge, etc.

References

[Bru17] P. Bruskiewich, *A very simple introduction to special relativity: Part two - four vectors, the lorentz transformation and group velocity (the new mathematics for the millions book 38)*, Pythagoras Publishing, 2017. ↑8

[CG15] W. Conradie and V. Goranko, *Logic and discrete mathematics*, Wiley, 2015. ↑3

[Coq15] Coq, *Coq proof assistant*, 2015. https://coq.inria.fr/documentation. ↑2

[Gol76] R. R. Goldberg, *Methods of real analysis*, John Wiley and Sons, 1976. ↑1

[Joy98] D. E. Joyce, *Euclid's elements*, 1998. http://aleph0.clarku.edu/~djoyce/java/elements/elements.html. ↑1

[Rud76] W. Rudin, *Principles of mathematical analysis*, McGraw Hill Education, 1976. ↑1

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