A Combinatorial Foundation for Geometry

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Abstract. A ruler-based measure of intervals is a new tool for combinatorial proofs and deeper insights into geometry and real analysis. Application of a ruler measure to cases of the inclusion-exclusion principle of set theory converge to the n-dimensional Euclidean distance equation and metric triangle inequality. The ruler measure is used to prove that the size of the Cartesian product of the subintervals of intervals converges to the product of the interval sizes, the n-dimensional volume equation. Combinatorics is also used to prove that order and symmetry are sufficient conditions for a geometry to be a cyclic set limited to at most three elements (root dimensions), which is the basis of the right-hand rule. Implications for higher dimensional geometries are discussed. All the proofs are verified in Coq.

Contents

1. Introduction

Real analysis an the axiomatic foundation for calculus based on set and number theory – except when developing the notions of distance and size (length/area/volume). The triangle inequality of a metric space, $d(u, w) \leq d(u, v) + d(v, w)$, and the volume equation of the Lebesgue measure are motivated by elementary geometry and defined rather than derived from the foundation of set and number theory. Further, there has been no derivations of Euclidean distance from set and number theory.

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As a consequence, real analysis and calculus have definitions that mimic the notions of side, angle, distance, and size (length/area/volume), but lacks a "self-contained" foundation to derive and motivate those notions. This article will show that applying combinatorics to real analysis provides the motivation and derivations of side, angle, distance, and size.

Combinatorics allows new proofs:

(1) A direct n-dimensional proof of the Euclidean distance equation.
(2) A direct n-dimensional proof of the Euclidean volume equation (there has not even been a proof of the 3-dimensional volume equation).
(3) A proof of the conditions that limit a geometry to a cyclic set having at most three elements (dimensions), which is the basis of the right-hand rule.

Proofs using combinatorics provides new insights into real analysis and geometry:

(1) Countable (combinatorial) relations between the elements of sets converge to the Euclidean distance and size (length/area/volume) equations without notions of side, angle, and without motivation from geometric diagrams.
(2) Taxicab distance, Euclidean distance, and the metric space triangle inequality are derived from cases of the inclusion-exclusion principle of set theory, where taxicab distance is the largest possible distance (all sets disjoint) and Euclidean distance is the smallest distance (largest possible intersections) satisfying the metric space triangle inequality.
(3) A geometry with the properties of geometric order and symmetry is a cyclic set and limited to at most three elements (dimensions), which is the basis of the right-hand rule.
(4) Vector orthogonality (inner product equal to zero) is valid for any number of dimensions and only requires the property of order. However, geometric orthogonality (perpendicular) requires the additional property of geometric symmetry, which is limited to at most three dimensions.

The Lebesgue measure of a volume, is the sum of a set of set volumes that converges to an outer measure of the volume, where the size of one interval of a sub-volume is often not countably related to the sizes of other intervals within that same sub-volume, which creates an obstacle to combinatorial proofs. Therefore, a new type of measure is required to implement combinatorial proofs.

A ruler measures each interval of a set of intervals to the nearest integer number of partitions (subintervals), p_i ($i \in [1, n]$), each partition of size, c. The ruler is an approximate measure that ignores partial partitions. Same-sized partitions, c, across a set of intervals allows defining a countable relationship between the number of partitions in one interval with the number of partitions in other intervals. As the partition size goes to zero, the combinatorial relationships that define countable size (length/area/volume) and smallest countable distance converge to the n-dimensional equations of Euclidean volume and distance.

The proofs in this article are verified formally using the Coq Proof Assistant [15] version 8.4p16. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at:
https://github.com/treeck/CombinatorialGeometry.

2. Ruler measure and convergence

DEFINITION 2.1. Ruler measure: A ruler measures the size of a closed, open, or semi-open interval as the nearest integer number of whole partitions (subintervals), p, times the partition size, c, where c is the independent variable. Notionally:

$$\forall\, c\; s \in \mathbb{R},\;\; [a,b] \subset \mathbb{R},\;\; s = |b-a| \;\wedge\; c > 0 \;\wedge \tag{2.1}$$
$$(p = floor(s/c) \;\vee\; p = ceiling(s/c) \;\wedge\; M = \lim_{c\to 0} \sum_{i=1}^{p} c = \lim_{c\to 0} pc.$$

The ruler has the three properties of measure in a σ-algebra:

(1) Non-negativity: $\forall\, E \in \Sigma,\;\; \mu(E) \geq 0 : \quad s = |b-a| \;\wedge\; c > 0 \;\Rightarrow$ $M = \lim_{c\to 0} pc \geq 0$.
(2) Zero-sized empty set: $\mu(\emptyset) = 0 : \quad b = a \;\Rightarrow M = \lim_{c\to 0} pc = 0$.
(3) Countable additivity: $\forall\, \{E_i\}_{i\in\mathbb{N}},\; |\cap_{i=1}^{\infty} E_i| = \emptyset \wedge \mu(\cup_{i=1}^{\infty} E_i) = \mu(\Sigma_{i=1}^{\infty} E_i)$. $(c \to 0 \Rightarrow p \to \infty) \wedge \mu(E_i) = c \Rightarrow \mu(\Sigma_{i=1}^{\infty} E_i) = \lim_{c\to 0} \Sigma_{i=1}^{p} c = \lim_{c\to 0} pc$.

For example showing convergence, using the interval, $[0,\pi]$, $s = |\pi - 0|$, $c = 10^{-i}, i \in \mathbb{N}$, and $p = floor(s/c)$, then, $p \cdot c = 3.1,\; 3.14,\; 3.141, ..., \pi$.

THEOREM 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\;\; s = |b-a| \;\;\Rightarrow\;\; M = \lim_{c\to 0} pc = s$.

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\; y \leq x,\; y \in \mathbb{Z},\; x \in \mathbb{R}\})$:

$$\forall\, c > 0, \quad p = floor(s/c) \quad \Rightarrow \quad 0 \leq |p - s/c| < 1. \tag{2.2}$$

Multiply all sides by $|c|$:

$$\forall\, c > 0, \quad 0 \leq |p - s/c| < 1 \quad \Rightarrow \quad 0 \leq |pc - s| < |c|. \tag{2.3}$$

$$\forall\, c > 0, \exists\, \delta,\, \epsilon \;:\; 0 \leq |pc - s| < |c| = |c - 0| < \delta = \epsilon \tag{2.4}$$
$$\Rightarrow \quad 0 < |c - 0| < \delta \quad \wedge \quad 0 \leq |pc - s| < \epsilon = \delta \quad := \quad M = \lim_{c\to 0} pc = s. \quad \square$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure).

3. Distance

Here, distance is the number of elements in an image (distance) set corresponding one-to-one with the elements of a domain set. This notion of distance is extended across disjoint (non-intersecting) domain sets, where an element in the image set may correspond to one element in each of the disjoint domain sets.

An example of a distance set is a set of equal-valued coins, C, corresponding to the members of sets of apples, A, and bananas, B. Using the correspondence distance measure, there exists $|C_A| = |A|$ number of coins and $|C_B| = |B|$ number of coins. By the inclusion-exclusion principle, the total number of coins, $|C| = |C_A \cup C_B| = |C_A| + |C_B| - |C_A \cap C_B|$, where the size of the intersection, $|C_A \cap C_B|$, represents the case of coins corresponding to both apples and bananas.

The Da Silva/Sylvester formula generalizes the inclusion-exclusion principle for n number of sets and is frequently used in combinatorics, measure theory, number theory (the sieve method), and probability theory [**Wc15**]:

Definition 3.1. Da Silva/Sylvester formula:

$$|\bigcup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| - \sum_{k=1}^{n} -1^{k+1}(\sum_{1 \leq i_1 < \ldots < i_k \leq n} |y_{i_1} \bigcap \ldots \bigcap y_{i_k}|). \tag{3.1}$$

The list of all set elements appended together is the set of "uniques" (union) plus the list of "duplicates." The intersection operations in the Da Silva/Sylvester formula is one way to calculate a list of duplicates. Therefore, a more general and simple formula for the inclusion-exclusion principle is used here that allows a more intuitive definition of distance:

Definition 3.2. General inclusion-exclusion formula:

$$|\bigcup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| - |duplicates(\{y_i\}_{i \in [1,n]})|. \tag{3.2}$$

Proofs of the inclusion-exclusion principle have already been published many times and won't be shown here. However, the more general duplicates-based, "inclusion_exclusion_principle" theorem and formal proof is located in the file, euclidrelations.v, which proves the list of all set elements appended together is the set of "uniques" (union) plus the list of "duplicates."

Definition 3.3. Countable Distance Measure, d_c:

$$\forall\, i\, j \in [1,n],\ x_i \subseteq X,\ |\bigcup_{i=1}^{n} x_i| = \sum_{i=1}^{n} |x_i| \ \wedge\ \forall\, x_i\, \exists\, y_i \subseteq Y :\ |x_i| = |y_i| \ \wedge$$
$$(\forall\ y_j \subseteq Y,\quad i \neq j \quad \wedge \quad y_i \not\subseteq y_j \quad \wedge \quad y_j \not\subseteq y_i) \ \wedge$$
$$0 \leq d_c = |\bigcup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| - |duplicates(\{y_i\}_{i \in [1,n]})|.$$

The condition, "$y_i \not\subseteq y_j \wedge y_j \not\subseteq y_i$," and the number of duplicates held constant guarantees any increase in the size of a domain set also increases the measure, d_c (countable additivity).

Definition 3.4. Countable taxicab (largest spanning) distance measure:

$$|duplicates(\{y_i\}_{i \in [1,n]})| = 0 \quad \Rightarrow \quad d_c = \sum_{i=1}^{n} |y_i|.$$

In the taxicab distance case, the number of coins is equal to the number of apples plus bananas, $|C| = |A| + |B|$, because $|C_A \cap C_B| = 0$.

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance." The proof for disjoint sets is common and won't be shown here.

Definition 3.5. Countable Euclidean (smallest spanning) distance measure:

$$|duplicates(\{y_i\}_{i \in [1,n]})| = max\_dups \quad \Rightarrow \quad d_c = \sum_{i=1}^{n} |y_i| - max\_dups.$$

The largest possible number of duplicate correspondences, "*max_dups*," is the case of the smallest (shortest) spanning distance.

"max_dups" and d_c can only be derived by indirect counting. For example, to determine the smallest, real-valued amount of coin, C_r, for any number of apples, $|A|$, and bananas, $|B|$, use the ruler measure (2.1) to partition sets of apples, bananas, and coins into pieces:

$$\begin{aligned}
(3.3)\quad & \forall\, c\ C_r \in \mathbb{R}, \quad c > 0 \quad \wedge \\
& p_1 = floor(|A|/c) \ \wedge\ p_2 = floor(|B|)/c) \ \wedge\ d_c = p_d = floor(C_r/c) \ \wedge \\
& |\{applePiece_1, applePiece_2, \ldots, applePiece_{p_1}\}| = p_1 \quad \wedge \\
& |\{bananaPiece_1, bananaPiece_2, \ldots, bananaPiece_{p_2}\}| = p_2 \quad \wedge \\
& |\{coinPiece_1, coinPiece_2, \ldots, coinPiece_{p_d}\}| = p_d = d_c.
\end{aligned}$$

The maximum possible number of duplicate correspondences is the set of the maximum possible number of $(applePiece, coinPiece)$ correspondences plus the maximum possible number of $(bananaPiece, coinPiece)$ correspondences. From the definition of a countable distance measure (3.3), each of p_1 number of coin pieces can correspond to a maximum of p_1 number of apple pieces, yielding a maximum of $p_1 \times p_1 = p_1^2$ number of possible $(applePiece, coinPiece)$ correspondences, which is also equal to p_1^2 number of $(coinPiece, coinPiece)$ combinations. Likewise, there are a maximum of p_2^2 number of possible $(banana\ Piece, coinPiece)$ correspondences, which is also equal to p_2^2 number of $(coinPiece, coinPiece)$ combinations. Therefore, there are a maximum of $p_1^2 + p_2^2 = |\{(fruitPiece, coinPiece)\}| = |\{(coinPiece, coinPiece)\}|$ combinations.

Multiply both sides by c^2 and apply the ruler convergence theorem (2.2):

$$\begin{aligned}
& (p_1 \cdot c)^2 + (p_2 \cdot c)^2 = |\{(coinPiece, coinPiece)\}| \cdot c^2 \quad \wedge \\
& \qquad |A| = \lim_{c \to 0} p_1 \cdot c \quad \wedge \quad |B| = \lim_{c \to 0} p_2 \cdot c \\
\Rightarrow \quad & |A|^2 + |B|^2 = \lim_{c \to 0} (p_1 \cdot c)^2 + \lim_{c \to 0} (p_2 \cdot c)^2 = \lim_{c \to 0} |\{(coinPiece, coinPiece)\}| \cdot c^2.
\end{aligned}$$

Equation 3.3 partitioned the C_r amount of coin into p_d number of pieces coin pieces. Therefore, there are a maximum possible $p_d^2 = |\{(coinPiece, coinPiece)\}|$ combinations.

Multiply both sides by c^2 and apply the ruler convergence theorem (2.2):

$$\begin{aligned}
& (p_d \cdot c)^2 = |\{(coinPiece, coinPiece)\}| \cdot c^2 \quad \wedge \quad C_r = \lim_{c \to 0} p_d \cdot c \\
& \qquad \Rightarrow \quad C_r^2 = \lim_{c \to 0} (p_d \cdot c)^2 = \lim_{c \to 0} |\{(coinPiece, coinPiece)\}| \cdot c^2.
\end{aligned}$$

$$\begin{aligned}
& |A|^2 + |B|^2 = \lim_{c \to 0} |\{(coinPiece, coinPiece)\}| \cdot c^2 \quad \wedge \\
& \qquad C_r^2 = \lim_{c \to 0} |\{(coinPiece, coinPiece)\}| \cdot c^2 \quad \Rightarrow \quad |A|^2 + |B|^2 = C_r^2.
\end{aligned}$$

THEOREM 3.6. *Euclidean (smallest) distance, d, is the size of the distance interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals, $\{[x_{0,1}, x_{m,1}], [x_{0,2}, x_{m,2}], \ldots, [x_{0,n}, x_{m,n}]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |y_m - y_0|, \quad s_i = |x_{m,i} - x_{0,i}|, \quad i \in [1, n], \quad i, n \in \mathbb{N}.$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

PROOF.
Use the ruler (2.1) to partition the exact size, $s_i = |x_{m,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m,i}]$, into p_i number of partitions. Next, apply the definition of the countable distance measure (3.3) and the rule of product:

$$\begin{aligned}(3.4)\quad & \forall\, i \in [1,n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\ & |\{x_i : x_i \in \{x_{1_i}, x_{2_i}, \dots, x_{p_i}\}\}| = |\{y_i : y_i \in \{y_{1_i}, y_{2_i}, \dots, y_{p_i}\}\}| = p_i \quad \Rightarrow \\ & \qquad \forall\, i \in [1,n], \quad |\{(x_i, y_i)\}| = |\{(y_i, y_i)\}| = p_i^2.\end{aligned}$$

$$\begin{aligned}(3.5)\quad & \forall\, i \in [1,n], \quad |\{(y_i, y_i)\}| = p_i^2 \quad \wedge \quad y \in \{y_i\} \quad \Rightarrow \\ & \qquad |\sum_{i=1}^{n} \{(y_i, y_i)\}| = \sum_{i=1}^{n} p_i^2 = |\{(y, y)\}|.\end{aligned}$$

Multiply both sides of 3.5 by c^2 and apply the ruler convergence theorem (2.2):

$$\begin{aligned}(3.6)\quad & s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n} (p_i \cdot c)^2 = |\{(y, y)\}| \cdot c^2 \\ & \Rightarrow \quad \sum_{i=1}^{n} s_i^2 = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c)^2 = \lim_{c\to 0} |\{(y, y)\}| \cdot c^2.\end{aligned}$$

Use the ruler to partition the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of partitions and apply the rule of product:

$$\begin{aligned}(3.7)\quad & \forall\, i \in [1,n],\ c > 0 \ \wedge\ p_d = floor(d/c) \ \wedge\ p_d = |\{y :\ y \in \{y_{1_i}, y_{2_i}, \dots, y_{p_d}\}\}| \\ & \qquad \Rightarrow \quad p_d^2 = |\{(y, y)\}|.\end{aligned}$$

Multiply both sides of 3.7 by c^2 and apply the ruler convergence theorem (2.2):

$$\begin{aligned}(3.8)\quad & d = \lim_{c\to 0} p_d \cdot c \quad \wedge \quad (p_d \cdot c)^2 = |\{(y, y)\}| \cdot c^2 \\ & \qquad \Rightarrow \quad d^2 = \lim_{c\to 0} (p_d \cdot c)^2 = \lim_{c\to 0} |\{(y, y)\}| \cdot c^2.\end{aligned}$$

Combine equations 3.6 and 3.8:

$$\begin{aligned}(3.9)\quad & d^2 = \lim_{c\to 0} |\{(y, y)\}| \cdot c^2 \quad \wedge \quad \sum_{i=1}^{n} s_i^2 = \lim_{c\to 0} |\{(y, y)\}| \cdot c^2 \\ & \qquad \Rightarrow \quad d^2 = \sum_{i=1}^{n} s_i^2. \quad \square\end{aligned}$$

3.1. Triangle inequality. The definition of a metric in real analysis is based on the triangle inequality, $\mathbf{d}(\mathbf{u}, \mathbf{w}) \le \mathbf{d}(\mathbf{u}, \mathbf{v}) + \mathbf{d}(\mathbf{v}, \mathbf{w})$, that has been intuitively motivated by the triangle [**Gol76**]. Applying the ruler (2.1) and convergence theorem (2.2) to the definition of a countable distance measure (3.3) (the inclusion-exclusion principle (3.2)) shows that the definition of the metric triangle inequality is derived from the inclusion-exclusion principle of set theory:

$$\begin{aligned}(3.10)\quad & d_c = |\bigcup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| - |duplicates(\{y_i\}_{i\in[1,n]})| \le \sum_{i=1}^{n} |y_i| \quad \wedge \\ & d_c = floor(\mathbf{d}(\mathbf{u}, \mathbf{w})/c) \ \wedge\ |y_1| = floor(\mathbf{d}(\mathbf{u}, \mathbf{v})/c) \ \wedge\ |y_2| = floor(\mathbf{d}(\mathbf{v}, \mathbf{w})/c) \\ & \qquad \Rightarrow \ \mathbf{d}(\mathbf{u}, \mathbf{w}) = \lim_{c\to 0} d_c \cdot c \le \sum_{i=1}^{2} \lim_{c\to 0} |y_i| \cdot c = \mathbf{d}(\mathbf{u}, \mathbf{v}) + \mathbf{d}(\mathbf{v}, \mathbf{w}).\end{aligned}$$

4. Size (length/area/volume)

The countable size measure is the number of combinations (correspondences) between members of disjoint domain sets, which is the Cartesian product of the domain set sizes. For example, given $|A|$ number of apples and $|B|$ number of bananas, the size measure is: $|\{(apple, banana)\}| = |A| \times |B|$ number of combinations.

DEFINITION 4.1. Countable size (length/area/volume) measure, S_c:

$$\forall\, i \in [1,n],\ \ x_i \subseteq X, \quad |\bigcup_{i=1}^{n} x_i| = \sum_{i=1}^{n} |x_i| \quad \wedge \quad \{(x_1,\ldots,x_n)\} = y \quad \wedge$$
$$S_c = |y| = |\{(x_1,\ldots,x_n)\}| = \prod_{i=1}^{n} |x_i|.$$

THEOREM 4.2. *Euclidean size (length/area/volume), S, is the size of an image interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals: $\{[x_{0,1}, x_{m,1}], [x_{0,2}, x_{m,2}], \ldots, [x_{0,n}, x_{m,n}]\}$, where:*

$$S = \prod_{i=1}^{n} s_i, \quad S = |y_m - y_0|, \quad s_i = |x_{m,i} - x_{0,i}|, \quad i \in [1,n], \quad i,n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_size."

PROOF.
Use the ruler (2.1) to partition the exact size, $s_i = |x_{m,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m,i}]$, into p_i number of partitions.

$$\text{(4.1)}\quad \forall\, i \in [1,n], c > 0 \wedge p_i = floor(s_i/c) \Rightarrow |\{x_i : x_i \in \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\}| = p_i.$$

Use the ruler (2.1) to partition the exact size, $S = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_S^n partitions, where p_S^n satisfies the definition a countable size measure, S_c.

$$\text{(4.2)}\quad \forall\, c > 0 \quad \wedge \quad \exists\, r \in \mathbb{R},\ S = r^n \quad \wedge \quad p_S = floor(r/c) \quad \wedge$$
$$p_S^n = S_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i.$$

Multiply both sides of equation 4.2 by c^n to get the ruler measures:

$$\text{(4.3)}\qquad p_S^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c).$$

Apply the ruler convergence theorem (2.2) to equation 4.3:

$$\text{(4.4)}\quad S = r^n = \lim_{c\to 0}(p_S \cdot c)^n \quad \wedge \quad \lim_{c\to 0}(p_i \cdot c) = s_i \quad \wedge \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c)$$
$$\Rightarrow \quad S = \lim_{c\to 0}(p_S \cdot c)^n = \prod_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square$$

5. Derived geometric definitions

5.1. Derived geometric primitives. There are no new mathematics in this section of derived geometric primitives. The purpose of this section is to show a difference in perspective. In classical geometry, Euclidean distance is a product of lines and angles. Here, the perspective is reversed to show that lines and angles are non-primitive relationships generated from the primitive relationship, Euclidean distance.

DEFINITION 5.1. Straight line segment is the smallest (Euclidean) distance interval, $[y_0, y_m]$ (3.6).

DEFINITION 5.2. Straight line segment orientation (slope): $\mathrm{d}b/\mathrm{d}a = b/a$, where $a = x_{m,1} - x_{0,1}$ and $b = x_{m,2} - x_{0,2}$ are the signed sizes of two domain intervals, $[x_{0,1}, x_{m,1}]$ and $[x_{0,2}, x_{m,2}]$.

The signed sizes, a and b, of the two domain intervals can be calculated from a single parametric distance, θ, and Euclidean distance, d.

DEFINITION 5.3. Parametric distance (arc angle), θ:

$$b/a = \mathrm{d}b/\mathrm{d}a = \mathrm{d}b/\mathrm{d}\theta \cdot \mathrm{d}\theta/\mathrm{d}a = \sqrt{d^2 - a^2}/\sqrt{d^2 - b^2} \tag{5.1}$$

$$Case: \quad \mathrm{d}b/\mathrm{d}a = b/a = 1 \quad \Rightarrow \quad \mathrm{d}\theta/\mathrm{d}a = 1/\sqrt{d^2 - b^2} = 1/\sqrt{d^2 - a^2} \tag{5.2}$$

Applying Taylor's theorem [Gol76] and a table of integrals [Wc11]:

$$\int \mathrm{d}\theta = \int \mathrm{d}a/\sqrt{d^2 - a^2} \quad \Rightarrow \quad \theta = \sin^{-1}(a/d) = \cos^{-1}(b/d). \tag{5.3}$$

5.2. Vectors. Before discussing the implications of the proofs in this article on vector analysis for dimensions greater than three, the notions of vector, parallel, and orthogonal are defined here in terms of sets of intervals.

DEFINITION 5.4. Vector: A vector is the ordered set of the signed domain interval sizes, $\mathbf{s} = \{s_1, \ldots, s_n\}$, where $s_i = x_{m,i} - x_{0,i}$ for the domain interval, $[x_{0,i}, x_{m,i}]$.

DEFINITION 5.5. Parallel (congruent) vectors: Two vectors are parallel if each ratio of the signed sizes in one vector equals the ratio of the corresponding signed sizes in another vector (same rate of change in the same direction):

$$\frac{s_{1_i}}{s_{1_{i+1}}} = \frac{s_{2_i}}{s_{2_{i+1}}}, \quad i \in [1, n-1]. \tag{5.4}$$

DEFINITION 5.6. Orthogonal vectors: Two vectors are orthogonal if each ratio of the signed sizes in one vector is the inverse ratio and inverse sign of two corresponding signed sizes in another vector (inverse rate of change and inverse directions). Simplifying the equation yields the **dot (inner) product** equal to zero for any number of dimensions:

$$\frac{s_{1_i}}{s_{1_{i+1}}} = -\frac{s_{2_{i+1}}}{s_{2_i}}, \quad i \in [1, n-1] \quad \Leftrightarrow \quad \sum_{i=1}^{n} s_{1_i} \cdot s_{2_i} = 0. \tag{5.5}$$

6. Ordered and symmetric geometries

Euclidean size (area/volume) and distance are invariant for every order (permutation) of a set of intervals. A function (like size or distance) where every permutation of the arguments yields the same value(s) is called a symmetric function. Two sets of intervals with the same volume and spanning distance (for example, $\{[0, 2],\ [0, 1],\ [0, 5]\}$ and $\{[0, 5],\ [0, 2],\ [0, 1]\}$) can be distinguished by assigning an order (relative position) to the elements of the sets.

DEFINITION 6.1. Ordered geometry:

$$\forall\, i\ n \in \mathbb{N},\ \forall\, x_i \in \{x_1, \ldots, x_n\},\ \ successor\ x_i = x_{i+1}\ \wedge\ predecessor\ x_{i+1} = x_i.$$

Order restricts counting (access) via successor and predecessor. Therefore, allowing every permutation of elements (symmetry) in an ordered and symmetric set requires every element to be a successor or predecessor of every other element.

DEFINITION 6.2. Symmetric geometry:

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; \mathit{successor}\; x_i = x_j \;\;\wedge\;\; \mathit{predecessor}\; x_j = x_i.$$

THEOREM 6.3. *An ordered and symmetric geometry is a cyclic set.*

$$\begin{aligned}&\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; i = n \;\;\wedge\;\; j = 1\\ &\qquad\qquad\Rightarrow \quad \mathit{successor}\; x_n = x_1 \;\;\wedge\;\; \mathit{predecessor}\; x_1 = x_n.\end{aligned}$$

The Coq theorem and proof in the file threed.v is "ordered_symmetric_is_cyclic."

PROOF. The property of order (6.1) defines unique successors and predecessors for all elements except for the successor of x_n and the predecessor of x_1. From the properties of a symmetric geometry (6.2):

$$i = n \;\;\wedge\;\; j = 1 \;\;\wedge\;\; \mathit{successor}\; x_i = x_j \;\;\Rightarrow\;\; \mathit{successor}\; x_n = x_1. \tag{6.1}$$

$$i = n \;\;\wedge\;\; j = 1 \;\;\wedge\;\; \mathit{predecessor}\; x_j = x_i \;\;\Rightarrow\;\; \mathit{predecessor}\; x_1 = x_n. \tag{6.2}$$

□

For example, using the cyclic set with elements labeled, $\{1,\ 2,\ 3\}$, starting with each element and counting through 3 cyclic successors and counting through 3 cyclic predecessors yields all possible permutations: $(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(1, 3, 2)$, $(3, 2, 1)$, and $(2, 1, 3)$. That is, a cyclically ordered set preserves sequential order while allowing a set of n-at-a-time permutations. If all possible n-at-a-time permutations are generated, then the cyclic ordered set is also symmetric.

THEOREM 6.4. *An ordered and symmetric geometry is limited to at most 3 elements. That is, each element is sequentially adjacent (a successor or predecessor) to every other element in a set only where the number of elements (set sizes) are less than or equal to 3.*

The Coq-based lemmas and proofs in the file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn-like clauses (a subset of first-order logic) with unification and resolution. Horn clauses make it clear which facts satisfy a goal.

PROOF.
Because an ordered and symmetric set is a cyclic set (6.3), the successors and predecessors are cyclic:

DEFINITION 6.5. Successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (m + 1 \leq \mathit{setsize}). \tag{6.3}$$

DEFINITION 6.6. Predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (m - 1 \geq 1). \tag{6.4}$$

DEFINITION 6.7. Adjacent: element m is adjacent to element n (an allowed permutation), if the cyclic successor of m is n or the cyclic predccessor of m is n. Notionally:

$$\mathit{Adjacent}(m, n, \mathit{setsize}) \leftarrow \mathit{Successor}(m, n, \mathit{setsize}) \vee \mathit{Predecessor}(m, n, \mathit{setsize}). \tag{6.5}$$

Every element is adjacent to every other element, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (1=1 \wedge 1=1). \tag{6.6}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (1+1 \leq 2). \tag{6.7}$$

$$Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (2=2 \wedge 1=1). \tag{6.8}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (1+1 \leq 2). \tag{6.9}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (2-1 \geq 1). \tag{6.10}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (3=3 \wedge 1=1). \tag{6.11}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (1=1 \wedge 3=3). \tag{6.12}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (2+1 \leq 3). \tag{6.13}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (3-1 \geq 1). \tag{6.14}$$

For all $n = setsize > 3$, there exist non-adjacent elements (not every permutation allowed):

$$\forall\, n > 3,\ Successor(1,2,n) \Rightarrow \forall\, n > 3,\ \neg Successor(1,3,n). \tag{6.15}$$

That is, 2 is the only successor of 1 for all $n > 3$, which implies 3 is not a successor of 1 for all $n > 3$.

$$\forall\, n > 3,\ Predecessor(1,n,n) \Rightarrow \forall\, n > 3,\ \neg Predecessor(1,3,n). \tag{6.16}$$

That is, n is the only predecessor of 1 for all $n > 3$, which implies 3 is not a predecessor of n for all $n > 3$.

$$\forall\, n > 3,\ \neg Adjacent(1,3,n) \leftarrow \neg Successor(1,3,n) \wedge \neg Predecessor(1,3,n). \tag{6.17}$$

□

7. Summary

It has been shown in this article that real analysis, a ruler measure of intervals (2.1), and combinatorics allow new proofs:

(1) A direct n-dimensional proof of the Euclidean distance equation (3.6).
(2) A direct n-dimensional proof of the Euclidean volume equation (4.2) (there has not even been a proof of the 3-dimensional volume equation). Such a proof is not possible with the Lebesgue measure.
(3) A proof of the conditions that limit a geometry to a cyclic set having at most three elements (dimensions), which is the basis of the right-hand rule (6.4)

. Proofs using combinatorics provides new insights into real analysis and geometry:

(1) Countable (combinatorial) relations between the elements of sets converge to the Euclidean distance and size (length/area/volume) equations without notions of side, angle, and without motivation from diagrams.
(2) Taxicab distance (3.4), Euclidean distance (3.5), and the metric space triangle inequality are derived from cases of the inclusion-exclusion principle of set theory, where taxicab distance is the largest possible distance (all sets disjoint) and Euclidean distance is the smallest distance (largest possible intersections) satisfying the metric space triangle inequality (3.1).

(3) The relationship, arc angle (5.3), is derived from the primitive relation, Euclidean distance rather than Euclidean distance being a derived from the notion of angle.
(4) A geometry with the properties of order and symmetry is a cyclic set (6.3) and limited to at most three elements (dimensions) (6.4), which is the basis of the right-hand rule.
(5) Vector orthogonality (inner product equal to zero) is valid for any number of dimensions (5.6) and only requires the property of order. However, geometric orthogonality (perpendicular) requires the additional property of geometric symmetry (6.2), which is limited to at most three dimensions (6.4).

The vector cross product and curl operations are based on the right-hand rule (a cyclic set of three dimensions) and can not be extended beyond three dimensions without losing either geometric order (orientation) or symmetry.

As was proved (6.4), it is not possible to have to have a vector with more than three dimensions that also has geometric symmetry. A means to preserve both geometric order and symmetry is to have a vector of three "root" dimensions of space, where size and distance in the three root dimensions are a function of other variables in a separate vector, forming a hierarchy of dimensions.

A cyclic set is a closed walk. An observer in the closed walk would only be able to detect higher dimensions indirectly via changes in the three closed walk dimensions (what physicists call "work").

Displaying higher dimensional manifolds in Euclidean coordinate diagrams (for example three dimensional Cartesian coordinates and spherical coordinates) is probably only meaningful for the case where three of the modeled dimensions are both geometrically ordered (6.2) and geometrically symmetric (6.2).

References

[15] *Coq proof assistant*, 2015. https://coq.inria.fr/documentation. ↑2

[Gol76] R. R. Goldberg, *Methods of real analysis*, John Wiley and Sons, 1976. ↑6, 8

[Wc11] Taylor theorem Wikipedia contributors, *Table of integrals*, Wikipedia, 2011. http://en.wikibooks.org/wiki/Calculus/Tables_of_Integrals. ↑8

[Wc15] Inclusion-exclusion Wikipedia contributors, *Inclusion exclusion principle*, Wikipedia, 2015. http://en.wikipedia.org/wiki/Inclusion-exclusion_principle. ↑4

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