A Combinatorial Foundation for Analytic Geometry

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Abstract. Using a ruler-like measure of intervals with real analysis allows proofs that provide new insights into geometry: A set-based definition of a countable distance range applied to sets of subintervals of intervals converges to the taxicab distance equation as the upper boundary, the Euclidean distance equation as the lower boundary, and the triangle inequality over the full range, which provides counting-based motivations for the definitions of metric space and Euclidean distance independent of elementary geometry. The Cartesian product of the subintervals of intervals converges to the product of interval sizes used in the Lebesgue measure and Euclidean integrals. Also combinatorics limits a geometry that is both ordered and symmetric to a cyclic set of at most 3 dimensions, which is the basis of the right-hand rule. Implications for non-Euclidean geometries and higher dimensional geometries are discussed. All the proofs are verified in Coq.

Contents

1. Introduction

The triangle inequality of a metric space, Euclidean distance metric, and the volume equation (product of interval sizes) of the Lebesgue measure and Euclidean integrals are imported into mathematical analysis from Euclidean geometry [Gol76]. Because the geometric relations are imported as primitives rather than derived from set and number theory, mathematical analysis has provided no insight into the counting principles that motivate and generate those geometric relations.

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Real analysis and measure theory have not provided any proofs that combinatorial relations between the subintervals of a set of domain intervals and the subintervals of an image interval converge to the triangle inequality, Euclidean distance, and volume equations, as the subinterval size goes to zero. Understanding the combinatorial relations generating the triangle inequality and Euclidean distance provides counting-based insights into the notions of a distance measure and smallest distance that importing as primitives from Euclidean geometry does not provide. Further, the product of interval sizes (Euclidean volume) used in the Lebesgue measure and Euclidean integrals is imported as a primitive because there has not been a proof that the Cartesian product of the subintervals of intervals converges to the product of intervals sizes.

The various traditional indefinite integrals (antiderivatives) derive a real-valued equation from a **real-valued, continuous function** relating the **sizes** of the subintervals. In contrast, what is needed for counting-based (combinatorial) proofs is an indefinite integration that derives a real-valued equation from a **combinatorial function** relating the integer **number** of same-sized subintervals of domain intervals to the integer **number** of same-sized subintervals in an image interval.

Combinatorial integration requires measuring the number of same-sized subintervals in both the domain and image intervals similar to using a ruler (measuring stick). The ruler is an approximate measure that ignores partial subintervals. Using the ruler measure, the size of subintervals is the same in both the domain and image intervals and the number of subintervals in each domain and image interval can vary. In contrast, the traditional method of dividing a set of intervals into subintervals, the number of subintervals is the same in both the domain and image intervals and the size of some subintervals can vary.

The Euclidean volume and distance equations can be extended to any number of dimensions. So, why does "physical" Euclidean geometry appear to be limited to three dimensions? Lack of insight into the counting principles generating distance and volume has prevented identification of the properties that can limit both Euclidean and non-Euclidean geometries to at most three dimensions.

The total number of distance and volume subinterval combinations is independent of the order of the combinations. Therefore, volume and all distance metrics are the same for any ordering of the set of domain intervals corresponding to those combinations (a symmetric geometry). However, differentiating between two sets of domain intervals (for example, $\{[0,2],\ [0,1],\ [0,5]\}$ and $\{[0,5],\ [0,2],\ [0,1]\}$) having the same spanning distance magnitude and volume magnitude, both Euclidean and non-Euclidean, requires each set of domain intervals to be ordered. It will be shown that a geometry that is both ordered and symmetric is limited to a cyclic set of at most three dimensions, which is the basis of the right-hand rule.

The proofs in this article are verified formally using the Coq Proof Assistant [**Coq15**] version 8.4p16. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at:
`https://github.com/treeck/CombinatorialGeometry`.

2. Ruler measure and convergence

Definition 2.1. Ruler measure: A ruler measures the size of a closed, open, or semi-open interval as the nearest integer number of whole subintervals, p, times the subinterval size, c, where c is the independent variable. Notionally:

$$\forall\, c\, s \in \mathbb{R},\;\; [a,b] \subset \mathbb{R},\;\; s = |b-a| \;\;\wedge\;\; c > 0 \;\wedge \\ (p = floor(s/c) \;\;\vee\;\; p = ceiling(s/c)) \;\;\wedge\;\; M = \lim_{c\to 0} \Sigma_{p=1}^{\infty} c = \lim_{c\to 0} pc. \tag{2.1}$$

THEOREM 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\;\; s = |b-a| \;\;\Rightarrow\;\; M = \lim_{c\to 0} pc = s.$

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\; y \leq x,\, y \in \mathbb{Z},\, x \in \mathbb{R}\})$:

$$\forall\, c > 0, \quad p = floor(s/c) \quad \Rightarrow \quad 0 \leq |p - s/c| < 1. \tag{2.2}$$

Multiply all sides by $|c|$:

$$\forall\, c > 0, \quad 0 \leq |p - s/c| < 1 \quad \Rightarrow \quad 0 \leq |pc - s| < |c|. \tag{2.3}$$

$$\forall\, c > 0,\, \exists\, \delta,\, \epsilon \,:\, 0 \leq |pc - s| < |c| = |c - 0| < \delta = \epsilon \\ \Rightarrow \quad 0 < |c - 0| < \delta \quad \wedge \quad 0 \leq |pc - s| < \epsilon = \delta \quad := \quad M = \lim_{c\to 0} pc = s. \quad \square \tag{2.4}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure).
For example, showing convergence using the interval, $[0,\pi]$, $s = |\pi - 0|$, $c = 10^{-i}, i \in \mathbb{N}$, and $p = floor(s/c)$, then, $p \cdot c = 3.1,\; 3.14,\; 3.141, ..., \pi$.

3. Distance

Consider the distance from one corner of a rectangular park to the cater-corner. Distance can be measured both by the number of steps around the circumference of the park and by the number of steps diagonally from one corner to the cater-corner. There is an implicit notion of a one-to-one correspondence (bijective map) of each step walked (distance set of steps) to each piece of land (domain set of land pieces).
If one distance element can correspond to multiple domain set elements, then the distance set will contain fewer elements than the domain set. Here, the distance set size (cardinal) is defined as a function of the number of correspondences between each distance set element and the domain set elements. The constraint that for each i^{th} disjoint domain subset containing p_i number of elements there exists a distance subset with the same p_i number of elements results in a defined range of possible distance set sizes as a function of the number of correspondences.

DEFINITION 3.1. Countable distance range, d_c:

$$\forall\, i\, n \in \mathbb{N},\;\; x_i \subseteq X,\;\; \bigcap_{i=1}^{n} x_i = \emptyset,\; \forall\, x_i\; \exists\, y_i \subseteq Y :\;\; |x_i| = |y_i| \;\wedge\; d_c = |Y|.$$

Notation conventions: In the definition of countable distance range (3.1), the vertical bars around a set is the standard notation for indicating the cardinal (number of elements in the set). To prevent too much over use of the vertical bar, the symbol for "such that" is the colon.
From the definition of countable distance range (3.1), the amount of intersection of the distance subsets is not defined. Applying the inclusion-exclusion principle:

$$|\cap_{i=1}^{n} y_i| \geq 0 \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| \geq |\cup_{i=1}^{n} y_i| = |Y|. \tag{3.1}$$

The countable distance range property, $|x_i| = |y_i|$, implies a limitation on the number of possible correspondences of a distance subset element to domain subset elements. If each of the p_i number of elements of the i^{th} distance set has a one-to-one correspondence (bijective map) to a domain subset element, then the number of correspondences per distance set is (using the rule of product): $1 \cdot |x_i| = 1 \cdot p_i = p_i = |y_i|$. And therefore, the distance, $d_c = |Y| = |\cup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i$, is the largest possible distance and the upper bound of the distance range.

If each of the p_i number of elements of the i^{th} distance subset corresponds to all p_i number of domain subset elements, then the largest number of correspondences per distance subset is (using the rule of product): $|y_i| \cdot |x_i| = p_i \cdot p_i = p_i^2$. The largest number of possible correspondences implies the smallest possible distance and is the lower bound of the distance range,

$$\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} |\{(y_a, y_b) : y_a\ y_b \in y_i\}|, \tag{3.2}$$

where each pair, (y_a, y_b), represents a combination (correspondence) between two elements in the distance set, y_i. From combining equations 3.1 and 3.2:

$$\begin{aligned} &\sum_{i=1}^{n} |y_i| \geq |\cup_{i=1}^{n} y_i| = |Y| \quad \Rightarrow \\ &\qquad \sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} |\{(y_a, y_b) : y_a\ y_b \in y_i\}| \geq |\{(y_a, y_b) : y_a\ y_b \in Y\}|. \end{aligned} \tag{3.3}$$

The ruler (2.1) and ruler convergence theorem (2.2) can be applied to real-valued intervals to show the largest and shortest distance cases converges to the real-valued taxicab and Euclidean distance equations. The convergence proofs of the taxicab and Euclidean distance equations requires the strategy of showing that the right and left sides of a proposed counting-based equation both converge to the same real value and therefore are equal. That is, the propositional logic, $A = B \wedge C = B \Rightarrow A = C$, is used.

Theorem 3.2. *Taxicab (largest) distance, d, is the size of the distance interval,* $[y_0, y_m]$, *corresponding to a set of disjoint domain intervals,* $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, *where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|.$$

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance."

Proof.

Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$\begin{aligned} &\forall\, i\ n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\ &\qquad |\{x_i : x_i \in \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\}| = |\{y_i : y_i \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_i}\}\}| = p_i. \end{aligned} \tag{3.4}$$

Next, apply the definition of the countable distance range (3.1):

$$\forall\, i\ n \in \mathbb{N}, \quad i \in [1, n], \quad y \in y_i \subseteq Y \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i = |\{y : y \in Y\}|. \tag{3.5}$$

Multiply both sides of 3.5 by c and apply the ruler convergence theorem (2.2):

$$(3.6)\quad s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n} (p_i \cdot c) = |\{y\}| \cdot c \\ \Rightarrow \quad \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) = \lim_{c\to 0} |\{y\}| \cdot c.$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$(3.7)\qquad \forall\, c > 0, \quad p_d = floor(d/c) = |\{y :\ y \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_d}\} = Y\}|.$$

Multiply both sides of 3.7 by c and apply the ruler convergence theorem (2.2):

$$(3.8)\ \ d = \lim_{c\to 0} p_d \cdot c \ \wedge\ p_d \cdot c = |\{y\}| \cdot c \ \Rightarrow\ d = \lim_{c\to 0} p_d \cdot c = \lim_{c\to 0} |\{y\}| \cdot c.$$

Combine equations 3.8 and 3.6:

$$(3.9)\qquad d = \lim_{c\to 0} |\{y\}| \cdot c \quad \wedge \quad \sum_{i=1}^{n} s_i = \lim_{c\to 0} |\{y\}| \cdot c \quad \Rightarrow \quad d = \sum_{i=1}^{n} s_i. \qquad \square$$

THEOREM 3.3. *Euclidean (smallest) distance, d, is the size of the distance interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals, $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|.$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$(3.10)\quad \forall\, i\ n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\ x_i = \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\} \quad \wedge \quad y_i = \{y_{1_i}, y_{2_i}, \ldots, y_{p_i}\}\}|.$$

Next, apply the definition of the countable distance range (3.1) and the rule of product:

$$(3.11)\qquad \textstyle\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} |\{(y_a, y_b) : y_a\ y_b \in y_i\}|,$$

where each pair, (y_a, y_b), represents a combination (correspondence) between two elements in the distance subset, y_i. From the definition of countable distance range (3.1), the distance subsets can intersect, which results in a range of possible distance set sizes. Applying the inclusion-exclusion principle:

$$(3.12)\qquad |\cap_{i=1}^{n} y_i| \geq 0 \quad \Rightarrow \quad \textstyle\sum_{i=1}^{n} |y_i| \geq |\cup_{i=1}^{n} y_i| = |Y|.$$

From combining equation 3.11 and the equality case of relation 3.12:

$$(3.13)\quad \textstyle\sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \geq |\cup_{i=1}^{n} y_i| = |Y| \\ \Rightarrow\ \exists\, y_i,\ Y\ :\ \textstyle\sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i = |\cup_{i=1}^{n} y_i| = |Y| \\ \Rightarrow\quad \textstyle\sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |\{(y_a, y_b) : y_a y_b \in y_i\}| = |\{(y_a, y_b) : y_a y_b \in Y\}|.$$

Multiply both sides of equation 3.13 by c^2 and apply the ruler convergence theorem.

$$(3.14)\quad s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n} (p_i \cdot c)^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2$$
$$\Rightarrow \quad \sum_{i=1}^{n} s_i^2 = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c)^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2.$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$(3.15)\quad \forall\, c > 0, \quad p_d = floor(d/c) = |\{y_{1_i}, y_{2_i}, \ldots, y_{p_d}\}| = |Y|$$
$$\Rightarrow \quad p_d^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\}|,$$

where $\{(y_a, y_b)\}$ is the set of all combination pairs of elements of Y. Multiply both sides of 3.15 by c^2 and apply the ruler convergence theorem (2.2):

$$(3.16)\quad d = \lim_{c\to 0} p_d \cdot c \quad \wedge \quad (p_d \cdot c)^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2$$
$$\Rightarrow \quad d^2 = \lim_{c\to 0} (p_d \cdot c)^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2.$$

Combine equations 3.15 and 3.16:

$$(3.17)\quad d^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \quad \wedge$$
$$\sum_{i=1}^{n} s_i^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \quad \Rightarrow \quad d^2 = \sum_{i=1}^{n} s_i^2. \quad \square$$

3.1. Triangle inequality. The definition of a metric in real analysis is based on the triangle inequality, $\mathbf{d}(\mathbf{u}, \mathbf{w}) \leq \mathbf{d}(\mathbf{u}, \mathbf{v}) + \mathbf{d}(\mathbf{v}, \mathbf{w})$, that has been intuitively motivated by the triangle [**Gol76**]. Applying the inclusion-exclusion principle, ruler (2.1), and ruler convergence theorem (2.2) to the definition of a countable distance range (3.1) yields the real-valued triangle inequality:

$$(3.18)\quad d_c = |Y| = |\bigcup_{i=1}^{2} y_i| \leq \sum_{i=1}^{2} |y_i| \quad \wedge$$
$$d_c = floor(\mathbf{d}(\mathbf{u}, \mathbf{w})/c) \;\; \wedge \;\; |y_1| = floor(\mathbf{d}(\mathbf{u}, \mathbf{v})/c) \;\; \wedge \;\; |y_2| = floor(\mathbf{d}(\mathbf{v}, \mathbf{w})/c)$$
$$\Rightarrow \; \mathbf{d}(\mathbf{u}, \mathbf{w}) = \lim_{c\to 0} d_c \cdot c \leq \sum_{i=1}^{2} \lim_{c\to 0} |y_i| \cdot c = \mathbf{d}(\mathbf{u}, \mathbf{v}) + \mathbf{d}(\mathbf{v}, \mathbf{w}).$$

4. Size (length/area/volume)

This section will use the ruler (2.1) and ruler convergence theorem (2.2) to prove that the Cartesian product of the subintervals of intervals converges to the product of interval sizes. The first step is to define of a countable size measure. The countable size measure is the number of combinations between members of disjoint domain sets, which is the Cartesian product of the domain set sizes.

Definition 4.1. Countable size (length/area/volume) measure, S_c:

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad x_i \subseteq X, \quad |\bigcap_{i=1}^{n} x_i| = \emptyset \quad \wedge \quad \{(x_1, \ldots, x_n)\} = y \quad \wedge$$
$$S_c = |y| = |\{(x_1, \ldots, x_n)\}| = \prod_{i=1}^{n} |x_i|.$$

THEOREM 4.2. *Euclidean size (length/area/volume), S, is the size of an image interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals:* $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, *where:*

$$S = \prod_{i=1}^{n} s_i, \quad S = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|, \quad i \in [1, n], \quad i, n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_size."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \quad x_i = \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\} \quad \Rightarrow |x_i| = p_i. \tag{4.1}$$

Use the ruler (2.1) to divide the exact size, $S = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_S^n subintervals, where p_S^n satisfies the definition a countable size measure, S_c (4.1).

$$\forall\, c > 0 \quad \wedge \quad \exists\, r \in \mathbb{R},\; S = r^n \quad \wedge \quad p_S = floor(r/c) \quad \wedge \quad p_S^n = S_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i. \tag{4.2}$$

Multiply both sides of equation 4.2 by c^n to get the ruler measures:

$$p_S^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3:

$$S = r^n = \lim_{c \to 0} (p_S \cdot c)^n \quad \wedge \quad \lim_{c \to 0} (p_i \cdot c) = s_i \quad \wedge \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c) \quad \Rightarrow \quad S = \lim_{c \to 0} (p_S \cdot c)^n = \prod_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square \tag{4.4}$$

5. Ordered and symmetric geometries

The previous derivations of the triangle inequality, (3.1), taxicab distance (3.2), Euclidean distance (3.3), and Euclidean volume (4.2) show that the total number of combinations of subintervals of intervals converges to real-valued distance measures and Euclidean volume. Therefore, all orderings (permutations) of a set of intervals corresponding to those combinations of subintervals (by the commutative property of addition) yield the same total distance and same total volume.

A set of successor and predecessor functions define each ordering (permutation) of a set of domain intervals. Notionally:

DEFINITION 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N}, \;\; \forall\, x_i \in \{x_1, \ldots, x_{n-1}\}, \;\; successor\ x_i = x_{i+1} \;\; \wedge \;\; predecessor\ x_{i+1} = x_i,$$

where x_i represents a real-valued domain interval.

For all elements, $x_i\, x_j \in \{x_1, \ldots, x_n\}$: if traversing the ordered set in successor order yields the permutation, $(\ldots, x_i, x_j, \ldots)$, then traversing in predecessor order yields the permutation, $(\ldots, x_j, x_j, \ldots)$. If permutations for all i and j are valid (a symmetric geometry), then every element has a sequentially adjacent successor and every element has a sequentially adjacent predecessor. Notionally:

DEFINITION 5.2. Symmetric geometry:

$$\forall\, i\; j\; n \in \mathbb{N},\ \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\ \textit{successor}\ x_i = x_j\ \wedge\ \textit{predecessor}\ x_j = x_i.$$

THEOREM 5.3. *An ordered and symmetric geometry is a cyclic set.*

$$\textit{successor}\ x_n = x_1\ \wedge\ \textit{predecessor}\ x_1 = x_n.$$

The theorem and formal Coq-based proof is "ordered_symmetric_is_cyclic," which is located in the file threed.v.

PROOF. The property of order (5.1) defines unique successors and predecessors for all elements except for the successor of x_n and the predecessor of x_1. From the properties of a symmetric geometry (5.2):

$$i = n\ \wedge\ j = 1\ \wedge\ \textit{successor}\ x_i = x_j\ \Rightarrow\ \textit{successor}\ x_n = x_1. \tag{5.1}$$

$$i = n\ \wedge\ j = 1\ \wedge\ \textit{predecessor}\ x_j = x_i\ \Rightarrow\ \textit{predecessor}\ x_1 = x_n. \tag{5.2}$$

□

For example, using the cyclic set with elements labeled, $\{1,\ 2,\ 3\}$, starting with each element and counting through 3 cyclic successors and counting through 3 cyclic predecessors yields all possible permutations: $(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(1, 3, 2)$, $(3, 2, 1)$, and $(2, 1, 3)$. That is, a cyclically ordered set preserves sequential order while allowing some n-at-a-time permutations. If all possible n-at-a-time permutations are generated, then the cyclic set is also symmetric.

THEOREM 5.4. *An ordered and symmetric geometry is limited to at most 3 elements. That is, each element is sequentially adjacent (a successor or predecessor) to every other element in a set only where the number of elements (set sizes) are less than or equal to 3.*

The Coq-based lemmas and proofs in the file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn-like clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.

Because a symmetric and ordered set is a cyclic set (5.3), the successors and predecessors are cyclic:

DEFINITION 5.5. Successor of m is n:

$$\textit{Successor}(m, n, \textit{setsize}) \leftarrow (m = \textit{setsize} \wedge n = 1) \vee (m + 1 \leq \textit{setsize}). \tag{5.3}$$

DEFINITION 5.6. Predecessor of m is n:

$$\textit{Predecessor}(m, n, \textit{setsize}) \leftarrow (m = 1 \wedge n = \textit{setsize}) \vee (m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: element m is adjacent to element n (an allowed permutation), if the cyclic successor of m is n or the cyclic predccessor of m is n. Notionally:

$$\textit{Adjacent}(m, n, \textit{setsize}) \leftarrow \textit{Successor}(m, n, \textit{setsize}) \vee \textit{Predecessor}(m, n, \textit{setsize}). \tag{5.5}$$

Every element is adjacent to every other element, where *setsize* $\in \{1,\ 2,\ 3\}$:

$$Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{5.6}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (1 + 1 \leq 2). \tag{5.7}$$

$$Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (2 = 2 \wedge 1 = 1). \tag{5.8}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (1 + 1 \leq 2). \tag{5.9}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (2 - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (3 = 3 \wedge 1 = 1). \tag{5.11}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (1 = 1 \wedge 3 = 3). \tag{5.12}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (2 + 1 \leq 3). \tag{5.13}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (3 - 1 \geq 1). \tag{5.14}$$

Must prove that for all *setsize* > 3, there exist non-adjacent elements (not every permutation allowed). For example, the first and third elements are not adjacent:

$$\begin{aligned} \forall\ setsize > 3: \quad & \neg Successor(1,3,setsize) \\ & \leftarrow Successor(1,2,setsize) \leftarrow (1 + 1 \leq setsize). \end{aligned} \tag{5.15}$$

That is, 2 is the only successor of 1 for all *setsize* > 3, which implies 3 is not a successor of 1 for all *setsize* > 3.

$$\begin{aligned} \forall\ setsize > 3: \quad & \neg Predecessor(1,3,setsize) \\ & \leftarrow Predecessor(1,n,setsize) \leftarrow (1 = 1 \wedge n = setsize). \end{aligned} \tag{5.16}$$

That is, $n = setsize$ is the only predecessor of 1 for all *setsize* > 3, which implies 3 is not a predecessor of 1 for all *setsize* > 3.

$$\begin{aligned} \forall\ setsize > 3: \quad & \neg Adjacent(1,3,setsize) \\ & \leftarrow \neg Successor(1,3,setsize) \wedge \neg Predecessor(1,3,setsize). \end{aligned} \tag{5.17}$$

□

6. Summary

The ruler measure (2.1) of real-valued intervals is a tool allowing a class of proofs that provide insights into counting principles underlying all geometry:

(1) The countable distance range (3.1) specifies that every disjoint domain subset has a corresponding distance subset with the same number of elements, which constrains the range of possible correspondences from each distance subset element to domain subset elements, a range of countable distances. Using the ruler measure and ruler convergence theorem, the countable distance range converges to the triangle inequality (3.1), which provides a counting-based motivation for both the triangle inequality and notion of a metric space. Importing the triangle inequality from Euclidean geometry does not provide these counting-based insights.

(2) The property that every disjoint domain subset containing p_i elements has a corresponding distance subset with the same number of elements constrains the maximum number of possible correspondences from each distance subset element to domain subset elements to the sum of squares, p_i^2. The maximum possible number correspondences from each distance

element to domain set elements yields the smallest countable distance set. Using the ruler measure with real analysis the case of the smallest countable distance (largest number of correspondences) converges to the Euclidean distance equation (3.3). Again, importing the Euclidean distance metric into mathematical analysis as a primitive does not provide this counting-based insight into the notion of smallest distance and Euclidean distance.

(3) Any distance measure where for each of p_i number of distance elements there are **not** p_i number of correspondences to domain set elements is a non-Euclidean distance measure.
(4) Euclidean distance and volume were derived starting with a single real-valued, domain interval rather than two domain intervals (corresponding to two sides of a right triangle). That is, Euclidean distance and volume were derived without any of the Euclidean geometry notions of "side", "angle", "side-angle-size", and congruence.
(5) Arc angle defined as a parametric variable relating the sizes of two domain intervals can be easily derived using using calculus and the Euclidean distance equation to generate the sine and cosine functions of the parametric parameter. In other words, the notions of side and arc angle are derived from the counting primitive, Euclidean distance.
(6) The Euclidean volume (product of interval sizes) of the Lebesgue measure is derived from use of the more fundamental ruler measure.
(7) Derivation of the triangle inequality, taxicab distance and Euclidean distance from a single counting-based definition of a distance range indicates a set-based definition of "orthogonal" (perpendicular) sets that does not rely on the notions of angle or vector spaces. Notionally:

$$A \perp B \Leftrightarrow (A \cup \overline{A}) \cap (B \cup \overline{B}) = \emptyset.$$

For example, $A = [0,1] \subset \mathbb{R}_1 \ \wedge \ B = [0,1] \subset \mathbb{R}_2 \ \wedge \ (A \cup \overline{A}) \cap (B \cup \overline{B}) = \emptyset \ \Leftrightarrow \ A \perp B$.

Beyond combinatorics on infinitesimal subintervals generating the properties of distance and volume, combinatorics also generates some of the macro properties of geometry.

(1) Combinatorics limits both Euclidean and non-Euclidean geometries having the properties of both order (5.1) and symmetry (5.2) to a cyclic set (5.3) of at most three elements (dimensions) (5.4), which is the basis of the right-hand rule that permeates mathematics, physics, and engineering.
(2) A cyclic set is a closed walk. An observer in a closed walk of three dimensions would only be able to detect higher dimensions (other variables) indirectly via distance and size changes in the three closed walk dimensions, where a change in distance is what physicists call "work."

Just as the properties of distance, volume, three dimensions of space, and the right-hand rule are generated by combinatorics, combinatorics on real-valued intervals might also generate "particles", "waves", and "forces".

References

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