The Two Set Relations Generating Geometry

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ABSTRACT. A ruler (measuring stick) divides both domain and range intervals approximately into the nearest integer number of same-sized subintervals. As the subinterval size converges to zero: 1) Where each range interval has a size greater than or equal to the corresponding domain interval generates the properties of metric space and all Lp norms, in particular Manhattan and Euclidean distances. 2) The Cartesian product of the number of members in each domain set converges to the product of interval interval sizes (Euclidean area/volume). The ruler measure-based proofs of Euclidean distance and area/volume are used to derive the charge force, Newtonian gravity force, and spacetime equations. Time limits physical geometry to 3 dimensions. All proofs are verified in Coq.

CONTENTS

1. Introduction

Introduction to mathematical analysis textbooks start with a rigorous, step-by-step building of a foundation of ordered set, convergence, limit, continuity, etc. from set definitions and field axioms. But, metric space, Euclidean distance (metric/vector norm), and Euclidean area/volume (as the product of interval sizes used in measure and integration) are defined in mathematical analysis [**Gol76**] [**Rud76**] rather than rigorously derived from two fundamental set-based definitions.

A "ruler" measure of intervals applies convergence to two types of set relations to derive the respective properties of distance and volume, which provides some new

insights into geometry and physics, for example: the single set relation generating the triangle inequality, non-negativity, and identity of indiscernibles properties of metric space; the mapping between sets that makes Euclidean distance the smallest possible distance between two distinct points in $\mathbb{R}^n$; the mapping between sets that makes distance different from area/volume; the set-based reason the forces of charge and gravity vary inversely with the square of the distance between two objects; how time places an additional constraint on the set-based definitions of distance and volume, which limits physical geometry to 3 dimensions.

All the proofs in this article have been formally verified using using the Coq proof verification system [**Coq15**]. The formal proofs are located in the Coq files, "euclidrelations.v" and "threed.v," at:

https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

A ruler (measuring stick) partitions both domain and range intervals *approximately* into the nearest integer number of *same-sized*, c, subintervals, where different-sized intervals have a *different* number of subintervals. In contrast, the Riemann and Lebesgue integrals partition both domain and range intervals into the *same* number of subintervals, where different-sized intervals have *different-sized* subintervals [**Gol76**] [**Rud76**]. Ruler vs. integral: different number vs. same number of subintervals and same-sized vs. different-sized subintervals.

The ruler measure allows counting the number of mappings, ranging from a one-to-one correspondence to a many-to-many mapping, between the set of subintervals having size c in one interval and the set of subintervals having the same size, c, in another interval. The mapping (combinatorial) relations converge to continuous, bijective relations as the subinterval size, c, converges to zero.

DEFINITION 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, p, each subinterval having the same size, c. Notionally:

$$(2.1) \quad \forall\, c\, s \in \mathbb{R},\ [a,b] \subset \mathbb{R},\ s = |a - b|\ \wedge\ c > 0\ \wedge$$
$$(p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \sum_{i=1}^{p} c = pc.$$

THEOREM 2.2. *Ruler convergence:*
$$\forall\, [a,b] \subset \mathbb{R},\ s = |a - b|\ \Rightarrow\ M = \lim_{c \to 0} pc = s.$$

The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$(2.2)\ \forall\, c > 0,\ p = floor(s/c)\ \wedge\ 0 \le |floor(s/c) - s/c| < 1\ \Rightarrow\ 0 \le |p - s/c| < 1.$$

Multiply all sides of inequality 2.2 by $|c|$:

$$(2.3) \qquad \forall\, c > 0,\quad 0 \le |p - s/c| < 1\quad \Rightarrow\quad 0 \le |pc - s| < |c|.$$

$$(2.4) \quad \forall\, \delta\, :\, |pc - s| < |c| = |c - 0| < \delta$$
$$\Rightarrow\quad \forall\, \epsilon = \delta\, :\, |c - 0| < \delta\ \wedge\ |pc - s| < \epsilon\ :=\ M = \lim_{c \to 0} pc = s.\quad \square$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = |0 - \pi|$, $c = 10^{-i}$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{i=1}, 3.14_{i=2}, 3.141_{i=3}, ..., \pi_{\lim_{i \to \infty} \Rightarrow c \to 0}$.

3. Distance

Notation convention: Curly brackets, $\{\cdots\}$, delimit a set; square brackets, $[\cdots]$, delimit a list; and vertical bars around a set or list, $|\cdots|$, indicates the cardinal (number of members in the set or list).

3.1. Countable distance space. A countable measure of distance is the number of steps walked, which corresponds to an equal number of pieces of land. Generalizing, the countable distance is the number of members in a range set, y_i, which equals the number of members in a corresponding domain set, x_i: $|x_i| = |y_i|$. And the countable distance spanning multiple, disjoint, domain sets, $\bigcap_{i=1}^{n} x_i = \emptyset$, is the number of members, d_c, in the union range set: $d_c = |\bigcup_{i=1}^{n} y_i|$.

DEFINITION 3.1. *Countable distance space, d_c:*
$$\bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad d_c = |\bigcup_{i=1}^{n} y_i| \quad \wedge \quad |x_i| \geq |y_i|.$$

THEOREM 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$.

This well-known inequality follows from the inclusion-exclusion principle [**CG15**]. But, a more intuitive and simple proof follows from the associative law of addition, which requires the sum of the set sizes to equal the size of all the set members appended into a list. And, by the commutative law of addition, the list can be sorted into a list of unique members (the union set) and a list of duplicate members. For example, $|\{a, b, c\}| + |\{c, d, e\}| = |[a, b, c, c, d, e]| = |\{a, b, c, d, e\}| + |[c]| \Rightarrow |\{a, b, c, d, e\}| = |\{a, b, c\}| + |\{c, d, e\}| - |[c]|$. The list of duplicates being ≥ 0 implies the union set size is always $\leq$ the sum of the set sizes.

A formal proof, inclusion_exclusion_inequality, using sorting into a set of unique members (union set) and a list of duplicate members, is in the file euclidrelations.v.

PROOF. By the associative law of addition, append the sets into a list. Next, by the commutative law of addition, sort the list into uniques and duplicates, and then subtract duplicates from both sides:

(3.1) $\quad \sum_{i=1}^{n} |y_i| = |append_{i=1}^{n} y_i| = |sort(append_{i=1}^{n} y_i)|$
$= |\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i| \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| - |duplicates_{i=1}^{n} y_i| = |\bigcup_{i=1}^{n} y_i|$.

(3.2) $\quad |\bigcup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| - |duplicates_{i=1}^{n} y_i| \quad \wedge \quad |duplicates_{i=1}^{n} y_i| \geq 0$
$\Rightarrow \quad |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|. \quad \square$

3.2. Metric Space. All function range intervals, $d(u, w)$, satisfying the countable distance space definition, $d_c = |\bigcup_{i=1}^{n} y_i|$, where the ruler is applied, generates the three metric space properties: triangle inequality, non-negativity, and identity of indecernables. The fourth property of metric space, symmetry $[d(u, v) = d(v, u)]$, is motivated by Manhattan and Euclidean distance. The formal proofs: triangle_inequality, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 3.3. *Triangle Inequality:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \leq d(u, v) + d(v, w)$.

PROOF. Apply the ruler measure (2.1), the countable distance space condition (3.1), inclusion-exclusion inequality (3.2), and then ruler convergence (2.2).

(3.3) $\quad \forall\, c > 0,\ d(u, w),\ d(u, v),\ d(v, w)\ :$

$$|y_1| = floor(d(u, v)/c)\ \wedge\ |y_2| = floor(d(v, w)/c)\ \wedge$$
$$d_c = floor(d(u, w)/c)\ \wedge\ d_c = |y_1 \cup y_2| \leq |y_1| + |y_2|$$
$$\Rightarrow\ floor(d(u, w)/c) \leq floor(d(u, v)/c) + floor(d(v, w)/c)$$
$$\Rightarrow\ floor(d(u, w)/c) \cdot c \leq floor(d(u, v)/c) \cdot c + floor(d(v, w)/c) \cdot c$$
$$\Rightarrow\ \lim_{c \to 0} floor(d(u, w)/c) \cdot c \leq \lim_{c \to 0} floor(d(u, v)/c) \cdot c + \lim_{c \to 0} floor(d(v, w)/c) \cdot c$$
$$\Rightarrow\ d(u, w) \leq d(u, v) + d(v, w). \quad \square$$

THEOREM 3.4. *Non-negativity:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \geq 0$.

PROOF.
By definition, a set always has a size (cardinal) ≥ 0:
(3.4) $\quad \forall\, c > 0,\ d(u, w)\ :\quad floor(d(u, w)/c) = d_c\ \wedge\ d_c = |y_1 \cup y_2| \geq 0$
$$\Rightarrow\quad floor(d(u, w)/c) = d_c \geq 0\quad \Rightarrow\quad d(u, w) = \lim_{c \to 0} d_c \cdot c \geq 0. \quad \square$$

THEOREM 3.5. *Identity of Indiscernibles:* $d(w, w) = 0$.

PROOF. Apply the triangle inequality property (3.3):

(3.5) $\quad \forall\, d(u, v) = d(v, w) = 0\ \wedge\ d(u, w) \leq d(u, v) + d(v, w)\ \Rightarrow\ d(u, w) \leq 0.$

Combine the non-negativity property (3.4) and the previous inequality (3.5):

(3.6) $\qquad d(u, w) \geq 0\ \wedge\ d(u, w) \leq 0\ \Leftrightarrow\ 0 \leq d(u, w) \leq 0\ \Rightarrow\ d(u, w) = 0.$

(3.7) $\qquad\qquad\qquad d(u, w) = 0\ \wedge\ d(u, v) = 0\ \Rightarrow\ w = v.$

(3.8) $\qquad\qquad\qquad d(v, w) = 0\ \wedge\ w = v\ \Rightarrow\ d(w, w) = 0. \qquad\qquad \square$

3.3. Distance space range. From the countable distance space definition, $d_c = |\bigcup_{i=1}^{n} y_i|$. As the amount of intersection increases, a single range set member can map to more domain set members. Therefore, d_c is a function of range-to-domain set member mappings.

From the countable distance space property (3.1), where $|x_i| = |y_i| = p_i$, the total number of range-to-domain set member mappings vary from the sum of one-to-one correspondences (no intersection and largest distance), $\sum_{i=1}^{n} x_i \cdot 1 = \sum_{i=1}^{n} p_i$, to the sum of the most many-to-many correspondences (largest intersection and smallest possible distance), $\sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^2$. Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of range-to-domain set mapping cases converges to the real-valued Manhattan and Euclidean distance relations.

3.4. Manhattan distance.

THEOREM 3.6. *Manhattan (largest) distance, d, is the size of the range interval,* $[d_0, d_m]$, *corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*
$$d = \sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "taxicab_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, d_c, is the equality case:

(3.9) $\quad d_c = |\bigcup_{i=1}^{n} y_i| \le \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i$
$$\Rightarrow \quad d_c \le \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c : \ d_c = \sum_{i=1}^{n} p_i.$$

Multiply both sides of equation 3.11 by c and take the limit:

(3.10) $\quad d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \Rightarrow \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c).$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

(3.11) $\qquad d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d : \ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c.$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

(3.12) $\quad s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c \to 0} p_i \cdot c = s_i.$

Combine equations 3.11, 3.10, 3.12:

(3.13) $\quad d = \lim_{c \to 0} d_c \cdot c \quad \wedge \quad \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) \quad \wedge$
$\lim_{c \to 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square$

3.5. Euclidean distance.

THEOREM 3.7. *The smallest Euclidean distance, d, is the size of the range interval,* $[d_0, d_m]$, *corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*
$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "Euclidean_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
Apply the rule of product to the largest number of range-to-domain set mappings, where all p_i number of range set members, y_i, map to each of the p_i number of members in the domain set, x_i, and where $|x_i| = |y_i| = p_i$:

(3.14) $\qquad\qquad |x_i| = |y_i| = p_i \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2.$

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

(3.15) $\quad d_c = |\bigcup_{i=1}^{n} y_i| \le \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i$
$$\Rightarrow \quad d_c \le \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c : \ d_c = \sum_{i=1}^{n} p_i.$$

Square both sides of equation 3.15 $(x = y \Leftrightarrow f(x) = f(y))$:

(3.16) $\qquad \exists\, p_i,\, d_c : \ d_c = \sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\, d_c : \ d_c^2 = \left(\sum_{i=1}^{n} p_i\right)^2.$

Apply the square of sum inequality, $(\sum_{i=1}^n p_i)^2 \geq \sum_{i=1}^n p_i^2$, to equation 3.16 and select the smallest area (the equality) case:

$$(3.17) \quad d_c = \sum_{i=1}^n p_i, \ p_i \geq 0 \quad \Rightarrow \quad d_c^2 = (\sum_{i=1}^n p_i)^2 = \sum_{i=1}^n \sum_{j=1}^n p_i p_j$$
$$= \sum_{i=1}^n p_i^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n p_i p_j \geq \sum_{i=1}^n p_i^2$$
$$\Rightarrow \quad \exists \, p_i : \ d_c^2 = \sum_{i=1}^n p_i^2.$$

Multiply both sides of equation 3.17 by c^2, simplify, and take the limit.

$$(3.18) \quad d_c^2 = \sum_{i=1}^n p_i^2 \ \Rightarrow \ d_c^2 \cdot c^2 = \sum_{i=1}^n p_i^2 \cdot c^2 \ \Leftrightarrow \ (d_c \cdot c)^2 = \sum_{i=1}^n (p_i \cdot c)^2$$
$$\Rightarrow \quad \lim_{c \to 0}(d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \to 0}(p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.19) \quad \exists \, c \, d : \ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c \to 0}(d_c \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$(3.20) \quad s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c \to 0} p_i \cdot c = s_i.$$

Combine equations 3.19, 3.18, 3.20:

$$(3.21) \quad d^2 = \lim_{c \to 0}(d_c \cdot c)^2 \quad \wedge \quad \lim_{c \to 0}(d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \to 0}(p_i \cdot c)^2 \quad \wedge$$
$$\lim_{c \to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d^2 = \lim_{c \to 0}(d_c \cdot c)^2 = \sum_{i=1}^n \lim_{c \to 0}(p_i \cdot c)^2 = \sum_{i=1}^n s_i^2. \quad \square$$

4. Euclidean Volume

The number of all possible combinations (n-tuples) taking one member from each disjoint set is the Cartesian product of the number of members in each set. Notionally:

DEFINITION 4.1. Countable Volume, V_c:
$$\bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad V_c = \prod_{i=1}^n |x_i|.$$

THEOREM 4.2. *Euclidean volume, V, is size of the range interval, $[v_0, v_m]$, corresponding to all the possible combinations of the members of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$. Notionally:*
$$V = \prod_{i=1}^n s_i, \ V = |v_0 - v_m|, \ s_i = |a_i - b_i|.$$

The theorem, "Euclidean_volume," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, x_i of p_i number of subintervals.

$$(4.1) \quad \forall \, i \, n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i|.$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$(4.2) \quad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c \to 0}(p_i \cdot c) = s_i.$$

Use the ruler (2.1) to divide the exact size, $V = |v_0 - v_m|$, of the range interval, $[v_0, v_m]$, into p^n subintervals. Use those cases, where V_c has an integer n^{th} root.

$$(4.3) \quad \forall \, p^n = V_c \in \mathbb{N}, \ \exists \, V \in \mathbb{R}, \ x_i : floor(V/c^n) = V_c = p^n = \prod_{i=1}^n |x_i| = \prod_{i=1}^n p_i.$$

Apply the ruler convergence theorem (2.2) to equation 4.3 and simplify:

$$(4.4) \qquad floor(V/c^n) = p^n \quad \Rightarrow \quad V = \lim_{c \to 0} p^n \cdot c^n = \lim_{c \to 0} (p \cdot c)^n.$$

Multiply both sides of equation 4.3 by c^n and simplify:

$$(4.5) \quad p^n = \prod_{i=1}^n p_i \quad \Rightarrow \quad p^n \cdot c^n = (\prod_{i=1}^n p_i) \cdot c^n \quad \Leftrightarrow \quad (p \cdot c)^n = \prod_{i=1}^n (p_i \cdot c)$$
$$\Rightarrow \quad \lim_{c \to 0} (p \cdot c)^n = \prod_{i=1}^n \lim_{c \to 0} (p_i \cdot c)$$

Combine equations 4.4, 4.5, and 4.2:

$$(4.6) \quad V = \lim_{c \to 0} (p \cdot c)^n \quad \wedge \quad \lim_{c \to 0} (p \cdot c)^n = \prod_{i=1}^n \lim_{c \to 0} (p_i \cdot c) \quad \wedge$$
$$\lim_{c \to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad V = \lim_{c \to 0}(p \cdot c)^n = \prod_{i=1}^n \lim_{c \to 0}(p_i \cdot c) = \prod_{i=1}^n s_i. \quad \square$$

5. Applications to physics

5.1. Charge force equation. The sizes, q_1 and q_2, of two charges are independent domain variables, where each size, c, component of a charge exerts a force on each same size, c, component of the other charge. The total force, F, is proportionate to the total number of forces (the Cartesian product of the number of infinitesimal components) multiplied times a quantum charge force, $m_C a_C$. From the volume proof (4.2), the relation between to domain variables is:

$$(5.1) \qquad F \propto (m_C a_C)(\lim_{c \to 0} floor(q_1/c) \cdot c)(\lim_{c \to 0} floor(q_2/c) \cdot c) = (m_C a_C)(q_1 q_2).$$

For any interval, $[0, q]$, a proportionately sized interval, $[0, r]$, can be defined with a proportion ratio, $q_C/r_C \in \mathbb{R}$, such that $r(q_C/r_C) = q$:

$$(5.2) \quad \forall\, q_1, q_2 \in \mathbb{R}\, \exists\, q \in \mathbb{R} : q^2 = q_1 q_2 \,\wedge\, r(q_C/r_C) = q \,\Rightarrow\, (r(q_C/r_C))^2 = q_1 q_2.$$

$$(5.3) \quad (r(q_C/r_C))^2 = q_1 q_2 \,\wedge\, F \propto (m_C a_C)(q_1 q_2)$$
$$\Rightarrow \quad F \propto (m_C a_C)(r(q_C/r_C))^2 = (m_C a_C)(q_1 q_2).$$

$$(5.4) \quad m_C a_C = (x/x)(m_C a_C) = (m_C x)(a_C/x)$$
$$\wedge \quad \exists\, m_0, a \in \mathbb{R} : m_0 = m_C x,\, a = a_C/x \quad \Rightarrow \quad \exists\, m_0, a \in \mathbb{R} : m_0 a = m_C a_C.$$

Combine equations 5.4 and 5.3:

$$(5.5) \quad m_0 a = m_C a_C \quad \wedge \quad F \propto (m_C a_C)(r(q_C/r_C))^2 = (m_C a_C)(q_1 q_2)$$
$$\Rightarrow \quad F \propto (m_0 a)(r(q_C/r_C))^2 = (m_C a_C)(q_1 q_2)$$
$$\Rightarrow \quad \exists\, k_C \in \mathbb{R} : \quad F = m_0 a = (m_C a_C r_C^2/q_C^2) q_1 q_2/r^2 = k_C q_1 q_2/r^2,$$

where $k_C = m_C a_C r_C^2/q_C^2$ corresponds to the accepted standard units: $Nm^2 C^{-2}$.

5.2. Newtonian gravity force equation. The sizes, m_1 and m_2, of two masses are independent domain variables, where each size, c, component of a mass exerts a force on each same size, c, component of the other mass. The total force, F, is proportionate to the total number of forces (the Cartesian product of the number of infinitesimal components) multiplied times a quantum gravity force, $m_G a_G$. From the volume proof (4.2), the relation between to domain variables is:

$$(5.6) \quad F \propto (m_G a_G)(\lim_{c \to 0} floor(m_1/c) \cdot c)(\lim_{c \to 0} floor(m_2/c) \cdot c) = (m_G a_G)(m_1 m_2).$$

For any interval, $[0, m]$, a proportionately sized interval, $[0, r]$, can be defined with a proportion ratio, $m_G/r_G \in \mathbb{R}$, such that $r(m_G/r_G) = m$:

(5.7) $\forall\, m_1, m_2 \in \mathbb{R}\, \exists\, m \in \mathbb{R} : \quad m^2 = m_1 m_2 \quad \wedge \quad r(m_G/r_G) = m$

$$\Rightarrow \quad (r(m_G/r_G))^2 = m_1 m_2.$$

(5.8) $(r(m_G/r_G))^2 = m_1 m_2 \;\wedge\; F \propto (m_G a_G)(m_1 m_2)$

$$\Rightarrow \quad F \propto (m_G a_G)(r(m_G/r_G))^2 = (m_G a_G)(m_1 m_2).$$

(5.9) $m_G a_G = (x/x)(m_G a_G) = (m_G x)(a_G/x)$

$\wedge \quad \exists\, m_0, a \in \mathbb{R} : m_0 = m_G x,\; a = a_G/x \quad \Rightarrow \quad \exists\, m_0, a \in \mathbb{R} : m_0 a = m_G a_G.$

Combine equations 5.9 and 5.8:

(5.10) $m_0 a = m_G a_G \quad \wedge \quad F \propto (m_G a_G)(r(m_G/r_G))^2 = (m_G a_G)(m_1 m_2)$

$$\Rightarrow \quad F \propto (m_0 a)(r(m_G/r_G))^2 = (m_G a_G)(m_1 m_2)$$

$$\Rightarrow \quad \exists\, G \in \mathbb{R} : \quad F = m_0 a = (m_G a_G r_G^2/m_G^2)m_1 m_2/r^2 = G m_1 m_2/r^2,$$

where $G = r_G^3/m_G t_G^2$ corresponds to the accepted standard units: $m^3 kg^{-1} s^{-2}$.

5.3. Spacetime equations. The purpose of the derivations of the spacetime dilation and interval equations is to show the same pattern of an Euclidean distance interval having a proportionate interval of any other type, in this case, time. Any interval, $[0, t]$, proportionate to the Euclidean range interval, $[0, r]$, is measured in terms of a conversion constant, c, that is the ratio of some value, r_c to some value, t_c, such that $r = (r_c/t_c)t = ct$.

(5.11) $r^2 = d_1^2 + d_2^2 \quad \wedge \quad r = (r_c/t_c)t = ct$

$$\Rightarrow \quad (ct)^2 = d_1^2 + d_2^2 \quad \Rightarrow \quad d_2 = \sqrt{(ct)^2 - d_1^2}.$$

(5.12) $d_2 = \sqrt{(ct)^2 - d_1^2} \quad \wedge \quad d = d_2 \quad \wedge \quad d_1 = vt$

$$\Rightarrow \quad d = \sqrt{(ct)^2 - (vt)^2} = ct\sqrt{1 - (v^2/c^2)},$$

which is the spacetime dilation equation. [**Bru17**].

(5.13) $d_2^2 = (ct)^2 - d_1^2 \quad \wedge \quad s = d_2 \quad \wedge \quad d_1^2 = x^2 + y^2 + z^2$

$$\Rightarrow \quad s^2 = (ct)^2 - (x^2 + y^2 + z^2),$$

which is the spacetime interval equation [**Bru17**].

5.4. 3 dimensions of physical geometry. The set and arithmetic operations used to calculate distance and volume requires sequencing through a totally ordered set of dimensions, for example, the countable distance space: $d_c = |\bigcup_{i=1}^n y_i|$, Euclidean distance: $d^2 = \sum_{i=1}^n s_i^2$, countable volume: $V_c = \prod_{i=1}^n |x_i|$, and Euclidean volume: $V = \prod_{i=1}^n s_i$. The commutative property of the union, addition, and multiplication operations also allows sequencing through a set of n number of dimensions in all $n!$ number of possible orders.

But, a *physical*, deterministic sequencer requires a *physical* set to have a single total order, at most one successor and at most one predecessor per set member, during the *time* of sequencing. A total order is the only way to *physically determine* that a sequencer traversed in the order, $[x_2, x_1, \cdots]$, and next sequenced in the order, $[x_1, x_2, \cdots]$, during some *time* period. Deterministic sequencing in every

possible order via the same successor/predecessor relations (same total order) requires each set member to be either a successor or predecessor to every other set member (sequentially adjacent), herein referred to as a symmetric geometry.

It will now be proved that a set satisfying the constraints of a single total order and also symmetric defines a cyclic set containing at most 3 members, in this case, 3 dimensions of physical space.

DEFINITION 5.1. Ordered geometry:

$$\forall\, i\, n \in \mathbb{N},\ i \in [1, n-1],\ \forall\, x_i \in \{x_1, \ldots, x_n\},$$
$$successor\ x_i = x_{i+1}\ \wedge\ predecessor\ x_{i+1} = x_i.$$

DEFINITION 5.2. Symmetric geometry (every set member is sequentially adjacent to any other member):

$$\forall\, i\, j\, n \in \mathbb{N},\ \forall\, x_i\, x_j \in \{x_1, \ldots, x_n\},\ successor\ x_i = x_j\ \Leftrightarrow\ predecessor\ x_j = x_i.$$

THEOREM 5.3. *An ordered and symmetric set is a cyclic set.*

$$successor\ x_n = x_1\ \wedge\ predecessor\ x_1 = x_n.$$

The theorem, "ordered_symmetric_is_cyclic," and formal proof is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$(5.14) \qquad i = n\ \wedge\ j = 1\ \wedge\ successor\ x_i = x_j\ \Rightarrow\ successor\ x_n = x_1.$$

$$(5.15) \quad i = n\ \wedge\ j = 1\ \wedge\ predecessor\ x_j = x_i\ \Rightarrow\ predecessor\ x_1 = x_n. \qquad \square$$

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

DEFINITION 5.5. Cyclic successor of m is n:

$$(5.16)\quad Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m+1 \leq setsize).$$

DEFINITION 5.6. Cyclic predecessor of m is n:

$$(5.17)\quad Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1).$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:
(5.18)
$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize).$$

Prove that every member is adjacent to every other member, where $setsize \in \{1, 2, 3\}$:

$$(5.19) \qquad Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1).$$

$$(5.20) \qquad Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize).$$
$$(5.21) \qquad Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1).$$

$$(5.22) \qquad Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize).$$
$$(5.23) \qquad Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1).$$
$$(5.24) \qquad Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1).$$
$$(5.25) \qquad Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize).$$
$$(5.26) \qquad Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize).$$
$$(5.27) \qquad Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1).$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$(5.28) \quad \forall \, setsize > 3 : \quad \neg Successor(1, 3, setsize > 3)$$
$$\leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize).$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$(5.29) \quad \forall \, setsize > 3 : \quad \neg Predecessor(1, 3, setsize > 3)$$
$$\leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$(5.30) \quad \forall \, setsize > 3 : \quad \neg Adjacent(1, 3, setsize > 3)$$
$$\leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

Applying the ruler measure (2.1) and ruler convergence (2.2) to the set relations, countable distance space (3.1) and countable volume (4.1) yields the following insights and implications:

  (1) The properties of metric space, Euclidean distance and area/volume can be derived from two set relations without using the notions of Euclidean geometry [**Joy98**] like plane, side, angle, perpendicular, congruence, intersection, etc.

(2) The ruler measure-based proofs provide the insight that distance is a function of the combinatorial *range*-to-domain set member mappings. Whereas, area/volume is a function of the combinatorial *domain*-to-domain set member mappings.

(3) The equality case, $|x_i| = |y_i| = p_i$, of the countable distance space (3.1) constraint, $|x_i| \geq |y_i|$, limits the largest total number of range-to-domain set mappings (largest intersection, $d_c = |\bigcup_{i=1}^{n} y_i|$, and smallest distance) to $\sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^2$, which is the set-based reason Euclidean distance (3.7) is the smallest possible distance between two distinct points in $\mathbb{R}^n$. The cases of $|x_i| > |y_i|$ generate the L^p norms, $||L||_p = (\sum_{i=1}^{n} s_i^p)^{1/p}$, where $p > 2$, which are shorter distances than Euclidean distance, L^2, but distances where $p > 2$ do not exist in "flat (Euclidean) space."

(4) The equality case, $|x_i| = |y_i|$, of the countable distance space (3.1) constraint, $|x_i| \geq |y_i|$, generated Manhattan distance (3.6) (the L^1 norm) at the upper boundary, which is also the upper boundary (equality case) of the triangle inequality, $d(u, w) \leq d(u, v) + d(v, w)$, of a metric space. Therefore, the cases of $|x_i| < |y_i|$ are not part of the countable distance space definition because such cases would generate functions that violate the triangle inequality.

(5) The distance spanning multiple, disjoint, domain sets being proportionate to the number of members, d_c, in the union range set: $d_c = |\bigcup_{i=1}^{n} y_i|$ (3.1) generates the triangle inequality, non-negativity, and identity of indiscernibles properties of metric space (3.2). Manhattan and Euclidean distance are the intuitive motivation the fourth property of metric space, symmetry: $d(u, v) = d(v, u)$. But, the formal reason for the property of symmetry is that the type of combinatorial range-to-domain set mapping is the same for every domain-range set pair.

(6) All Euclidean distance range intervals having a size, r, can have proportionately sized intervals of other types, for example: $r = (r_C/q_C)q = (r_G/m_G)m = (r_c/t_c)/t$. Applying the ruler to those intervals allows application of the Euclidean area/volume and distance proofs to derive the charge force, Newtonian gravity force, and spacetime equations.

   (a) $q_1 q_2 = q^2 = (q_C/r_C)^2 r^2$ and $m_1 m_2 = m^2 = (m_G/r_G)^2 r^2$, explains why the charge and gravity forces vary inversely with the square of the distance.

   (b) If there are quantum values of charge, q_C, and mass, m_G, then there are quantum distances, r_C and r_G, where the forces do not exist (not defined) at smaller distances. The maximum quantum force: F, between the two quantum masses, $m_1 = m_2 = m_G$, at the distance, r_G, is: $F = m_G a_G = 6.67430(15) \times 10^{-11}$ kg m/s^2. Quantum charge and mass eliminate the need to invent stronger forces to override the charge and gravity forces at distances smaller than r_C and r_G. This would imply that the smallest distance, r_G, between quantum two masses at the center of all black holes is the same distance as between two quantum masses at the surface of all black holes. All black holes are uniformly dense. And all black holes, regardless of mass, have the same density.

(c) Discrete valued variables (discrete states like spin) would not have proportionate continuous distance intervals. Therefore, discrete value changes with respect to time could appear at multiple locations, at the same time, independent of the distance between locations (for example, the change in spin of two quantum coupled particles).

(7) Relativity theory assumes that only 3 dimensions of space exist [**Bru17**]. The proof in this article (5.4) explains why time constrains physical distance and volume to at most three dimensions. There may be more dimensions of "space," but those dimensions have different "types" than the geometric dimensions (for example, time).

(8) The proof of at most 3 dimensions of any set of ordered and symmetric members (5.4), implies that each infinitesimal volume (ball) can have at most 3 ordered and symmetric dimensions of discrete values of the same type. And each dimension of discrete values can have at most 3 ordered and symmetric discrete values, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values corresponding to 27 possible "types" of infinitesimal balls.

(9) If the three possible ordered and symmetric dimensions of discrete values contained an unordered set of discrete values, for example, 3 dimensions of unordered binary values, then there would be $2 \cdot 2 \cdot 2 = 8$ possible combinations of values. These unordered values would be non-deterministic. For example, any time a value is physically measured, there would be a 50-50 chance of having one of the binary values.

(10) Where infinitesimal balls intersect, an algebra of the interactions of the discrete values needs to be developed. The interaction of the discrete values associated with overlapping infinitesimal balls might result in what we perceive as particles, waves, and motion.

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