The Two Set Relations Generating Euclidean Geometry

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Abstract. A ruler-like measure divides sets of real-valued domain and range intervals into same-sized subintervals. The case, where for each disjoint domain set of subintervals there exists a corresponding same-sized range set and the range sets in some cases intersect, converges to: the triangle inequality, Manhattan distance at the upper boundary and Euclidean distance at the lower boundary of the triangle inequality. The Cartesian product of the number of subintervals in each domain interval converges to the product of interval interval sizes (Euclidean area/volume). All proofs are verified in Coq.

Contents

1. Introduction

Triangle inequality, distance non-negativity, Euclidean distance, and Euclidean area/volume are all motivated by Euclidean geometry and used as primitives in real analysis (metric space, Hausdorff and Lebesgue measures, and Riemann and Lebesgue integration) [**Gol76**] [**Rud76**]. In this article, a "ruler" measure of intervals is used to derive the properties of metric space, Manhattan distance, and Euclidean distance from a single, countable set-based axiom and derive Euclidean volume from another countable set-based axiom.

The ruler measure is a tool that allows epsilon-delta proofs of notions that have historically been used as definitions. For example, the ruler measure allows a short and simple proof that the size of the Cartesian product of all the real-values in a set of disjoint intervals is the product of the interval sizes (Euclidean area/volume).

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The derivations of geometric relations from set relations (*without any notions of plane, line, angle, etc.*) provide some new insights: 1) the set-based reasons Manhattan distance is largest, direct distance and Euclidean distance is the smallest distance between two distinct points in $\mathbb{R}^n$; 2) the single set relation generating the properties of metric space. 3) the set-based reason the properties of metric space might not be sufficient criteria for a distance measure; 4) how time places additional constraints on physical sets.

To give the reader of this article confidence that the proofs in this article are correct, all the proofs have corresponding formal proofs in the Coq files, "euclidrelations.v" and "threed.v," located at: https://github.com/treeck/RASRGeometry. Mathematicians all over the world use the Coq Proof Assistant [**Coq15**] to verify their proofs because proofs accepted by the Coq logic engine have a very high probability of being correct.

2. Ruler measure and convergence

A ruler (measuring stick) partitions both domain and range intervals *approximately* to the nearest integer number of same-sized subintervals, where the partial subintervals are ignored. The ruler measure allows counting the number of mappings, ranging from a one-to-one correspondence to a many-to-many mapping, between the set of same-sized subintervals in one interval and the set of same-sized subintervals in another interval. The mapping (combinatorial) relations converge to continuous, bijective relations as the subinterval size converges to zero.

Definition 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, p, each subinterval having the same size, c. Notionally:

$$\forall\, c\, s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |a-b|\ \wedge\ c > 0\ \wedge \\ (p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \textstyle\sum_{i=1}^{p} c = pc. \tag{2.1}$$

Theorem 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\ \ s = |a-b|\ \Rightarrow\ M = \lim_{c\to 0} pc = s.$

The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\, p = floor(s/c)\ \wedge\ 0 \le |floor(s/c) - s/c| < 1\ \Rightarrow\ 0 \le |p - s/c| < 1. \tag{2.2}$$

Multiply all sides of inequality 2.2 by $|c|$:

$$\forall\, c > 0,\quad 0 \le |p - s/c| < 1\quad \Rightarrow\quad 0 \le |pc - s| < |c|. \tag{2.3}$$

$$\forall\, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \\ \Rightarrow\quad \forall\, \epsilon = \delta :\ \ |c - 0| < \delta\ \wedge\ |pc - s| < \epsilon\ \ :=\ \ M = \lim_{c\to 0} pc = s.\quad \square \tag{2.4}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence, where: $[0,\pi]$, $s = |0-\pi|$, $c = 10^{-i}$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{i=1},\ 3.14_{i=2},\ 3.141_{i=3}, ..., \pi$.

3. Distance

Notation convention: In set theory, vertical bars around a set is the standard notation indicating the cardinal (number of members in the set).

3.1. Countable distance space. The most fundamental notion of distance is that for each disjoint domain set, x_i, there exists a corresponding range set, y_i, containing the same number of members: $|x_i| = |y_i|$. For example, there should be as many steps walked in the range set, y_i, as there are pieces of traversed land in the corresponding domain set, x_i. And the distance, d_c, spanning one or more disjoint domain sets is the size of the union of the range sets.

DEFINITION 3.1. Countable distance space, d_c:

$$\bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad d_c = |\bigcup_{i=1}^{n} y_i| \quad \wedge \quad |x_i| = |y_i|.$$

THEOREM 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$.

PROOF. This well-known inequality and proof is derived from the inclusion-exclusion principle [CG15] and the axiom, $u = v - w,\ w \geq 0 \Rightarrow u \leq v$, as shown at a high-level here for completeness. A formal proof, inclusion_exclusion_inequality, using partitioning instead of the da Silva formula, is in the file euclidrelations.v.

$$\begin{aligned} |\bigcup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| - \sum_{1\leq i<j\leq n} |y_i \cap y_j| + \cdots + (-1)^{n-1}|\bigcap_{i=1}^{n} y_i|) \quad \wedge \\ \sum_{1\leq i<j\leq n} |y_i \cap y_j| + \cdots + (-1)^{n-1}|\bigcap_{i=1}^{n} y_i|) \geq 0 \\ \Rightarrow \quad |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|. \quad \square \end{aligned} \tag{3.1}$$

3.2. Metric Space. Applying the ruler (2.1) and ruler convergence (2.2) to three range intervals having sizes, $d(u,w)$, $d(u,v)$, and $d(v,w)$, and using the inequality, $d_c = |\bigcup_{i=1}^{2} y_i| \leq \sum_{i=1}^{2} |y_i|$ generates the properties of metric space. The formal proofs: triangle_inequality, non_negativity, identity_of_indiscernibles, and symmetry, are in the Coq file, euclidrelations.v.

THEOREM 3.3. *Triangle Inequality:* $d(u,w) \leq d(u,v) + d(v,w)$*:*

PROOF.

$$\begin{aligned} \forall\, c > 0,\ |y_1| = floor(d(u,v)/c) \ \wedge\ |y_2| = floor(d(v,w)/c) \quad \wedge \\ d_c = floor(d(u,w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \leq |y_1| + |y_2| \\ \Rightarrow\ floor(d(u,w)/c) \leq floor(d(u,v)/c) + floor(d(v,w)/c) \\ \Rightarrow\ floor(d(u,w)/c)\cdot c \leq floor(d(u,v)/c)\cdot c + floor(d(v,w)/c)\cdot c \\ \Rightarrow\ \lim_{c\to 0} floor(d(u,w)/c)\cdot c \leq \lim_{c\to 0} floor(d(u,v)/c)\cdot c + \lim_{c\to 0} floor(d(v,w)/c)\cdot c \\ \Rightarrow\ d(u,w) \leq d(u,v) + d(v,w). \quad \square \end{aligned} \tag{3.2}$$

THEOREM 3.4. *Non-negativity:* $d(u,w) \geq 0$.

PROOF.

$$\begin{aligned} \forall\, c > 0 : \quad d_c = floor(d(u,w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \geq 0 \\ \Rightarrow \quad floor(d(u,w)/c) = d_c \geq 0 \quad \Rightarrow \quad d(u,w) = \lim_{c\to 0} d_c \cdot c \geq 0. \quad \square \end{aligned} \tag{3.3}$$

THEOREM 3.5. *Identity of Indiscernibles:* $d(w,w) = 0$.

PROOF.

$$\forall\, d(u,v) = d(v,w) = 0 \;\wedge\; d(u,w) \le d(u,v) + d(v,w) \;\wedge\; d(u,w) \ge 0 \;\Rightarrow\; d(u,w) = 0. \tag{3.4}$$

$$d(u,w) = 0 \;\wedge\; d(u,v) = 0 \;\Rightarrow\; w = v. \tag{3.5}$$

$$d(v,w) = 0 \;\wedge\; w = v \;\Rightarrow\; d(w,w) = 0. \tag{3.6}$$

□

THEOREM 3.6. *Symmetry:* $d(v,w) = d(w,v)$.

PROOF.

$$w = v \;\Rightarrow\; d(w,w) = d(v,w) \;\wedge\; d(w,w) = d(w,v) \;\Rightarrow\; d(v,w) = d(w,v). \tag{3.7}$$

□

3.3. Distance space range. Where the range sets intersect, multiple domain set members map to a single range set member. Therefore, the union set size, d_c, is function of the number of domain-to-range set member mappings.

The property, $|x_i| = |y_i| = p_i$, (3.1) constrains the range of domain-to-range set member mappings. Two facts are immediately obvious from the case, where $p_i = 1$: 1) Each of the p_i number of members in x_i corresponds 1-1 to a member in y_i, yielding $|x_i| \cdot 1 = p_i = 1$ number of domain-to-range mappings. 2) Each of the p_i number of members in x_i map to *each* of the p_i number of members in y_i, yielding $|x_i| \cdot |y_i| = p_i^2 = 1$ number of domain-to-range mappings.

Therefore, $\exists\, \mathbf{f} : d_c = \mathbf{f}(\sum_{i=1}^{n} p_i)$ is the largest possible distance because it is the case of the smallest number of domain-to-range mappings (no intersection of the range sets). And $\exists\, \mathbf{f} : d_c = \mathbf{f}(\sum_{i=1}^{n} p_i^2)$ is the smallest possible distance because it is the case of the largest number of domain-to-range mappings (largest allowed intersection of range sets). Applying the ruler (2.1) and ruler convergence theorem (2.2) to the largest and smallest countable distance cases yields the real-valued, Manhattan and Euclidean distance functions.

3.4. Manhattan distance.

THEOREM 3.7. *Manhattan (longest) distance, d, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "taxicab_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$d_c = |\bigcup_{i=1}^{n} y_i| \le \sum_{i=1}^{n} |y_i| \;\wedge\; |y_i| = p_i \;\Rightarrow\; d_c \le \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \;\Rightarrow\; \exists\, p_i,\, d_c :\; d_c = \sum_{i=1}^{n} p_i. \tag{3.8}$$

Multiply both sides of equation 3.10 by c and take the limit:

$$d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \Rightarrow \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c). \tag{3.9}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$d = |d_0 - d_m| \;\Rightarrow\; \exists\, c\, d :\; floor(d/c) = d_c \;\Rightarrow\; d = \lim_{c \to 0} d_c \cdot c. \tag{3.10}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of s_i:

$$\forall\, i \in [1,n],\ s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i. \tag{3.11}$$

Combine equations 3.10, 3.9, 3.11:

$$d = \lim_{c\to 0} d_c \cdot c \quad \wedge \quad \lim_{c\to 0} d_c \cdot c = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) \quad \wedge \tag{3.12}$$
$$\lim_{c\to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square$$

3.5. Euclidean distance.

THEOREM 3.8. *Euclidean (shortest) distance, d, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$d^2 = \textstyle\sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "Euclidean_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
Apply the rule of product to the largest number of domain-to-range set mappings, where all p_i number of domain set members, x_i, map to each of the p_i number of members in the range set, y_i:

$$\textstyle\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2. \tag{3.13}$$

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$d_c = |\textstyle\bigcup_{i=1}^{n} y_i| \le \sum_{i=1}^{n} |y_i| \quad \wedge \quad |y_i| = p_i \tag{3.14}$$
$$\Rightarrow \quad d_c \le \textstyle\sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i.$$

Square both sides of equation 3.14 ($x = y \Leftrightarrow f(x) = f(y)$):

$$\exists\, p_i,\ d_c :\ d_c = \textstyle\sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\ d_c :\ d_c^2 = (\sum_{i=1}^{n} p_i)^2. \tag{3.15}$$

Apply the Cauchy-Schwartz inequality to equation 3.15 and select the smallest distance (equality) case:

$$d_c^2 = (\textstyle\sum_{i=1}^{n} p_i)^2 \ge \sum_{i=1}^{n} p_i^2 \quad \Rightarrow \quad \exists\, p_i :\ d_c^2 = \sum_{i=1}^{n} p_i^2. \tag{3.16}$$

Multiply both sides of equation 3.16 by c^2, simplify, and take the limit.

$$d_c^2 = \textstyle\sum_{i=1}^{n} p_i^2 \ \Rightarrow\ d_c^2 \cdot c^2 = \sum_{i=1}^{n} p_i^2 \cdot c^2 \ \Leftrightarrow\ (d_c \cdot c)^2 = \sum_{i=1}^{n} (p_i \cdot c)^2 \tag{3.17}$$
$$\Rightarrow \quad \lim_{c\to 0}(d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$\exists\, c\, d :\ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2. \tag{3.18}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$\forall\, i \in [1,n],\ s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i \tag{3.19}$$
$$\Rightarrow \quad \lim_{c\to 0}(p_i \cdot c)^2 = s_i^2 \quad \Rightarrow \quad \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2.$$

Combine equations 3.18, 3.17, 3.19:

$$d^2 = \lim_{c\to 0}(d_c \cdot c)^2 \quad \wedge \quad \lim_{c\to 0}(d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 \quad \wedge \tag{3.20}$$
$$\textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2$$
$$\Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \quad \square$$

4. Euclidean Volume

A location is a range set member that corresponds 1-1 to a combination (tuple) of one member from each domain set. The number of all possible locations is the Cartesian product of the number members in each domain set. Notionally:

DEFINITION 4.1. All Possible Locations, V_c:

$$\bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad V_c = \prod_{i=1}^{n} |x_i|.$$

THEOREM 4.2. *Euclidean volume is the largest possible set of all real-valued locations, V, corresponding to a disjoint set of real-valued domain intervals:* $\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$, *where:*

$$V = \prod_{i=1}^{n} s_i, \quad V = |v_0 - v_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "Euclidean_volume," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, x_i of p_i number of subintervals.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i|. \tag{4.1}$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c \to 0}(p_i \cdot c) = s_i. \tag{4.2}$$

Use the ruler (2.1) to divide the exact size, $V = |v_0 - v_m|$, of the range interval, $[v_0, v_m]$, into p^n subintervals. Use those cases, where V_c has an integer n^{th} root.

$$\forall\, p^n = V_c \in \mathbb{N},\ \exists\, V \in \mathbb{R},\ x_i\ : floor(V/c^n) = V_c = p^n = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i. \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3 and simplify:

$$floor(V/c^n) = p^n \quad \Rightarrow \quad V = \lim_{c \to 0} p^n \cdot c^n = \lim_{c \to 0}(p \cdot c)^n. \tag{4.4}$$

Multiply both sides of equation 4.3 by c^n and simplify:

$$p^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad p^n \cdot c^n = (\prod_{i=1}^{n} p_i) \cdot c^n \quad \Leftrightarrow \quad (p \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.5}$$

Combine equations 4.4, 4.5, and 4.2:

$$\begin{aligned} V = \lim_{c \to 0}(p \cdot c)^n \quad \wedge \quad (p \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c) \quad \wedge \quad \lim\nolimits_{c \to 0}(p_i \cdot c) = s_i \\ \Rightarrow \quad V = \lim_{c \to 0}(p \cdot c)^n = \prod_{i=1}^{n} \lim\nolimits_{c \to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square \end{aligned} \tag{4.6}$$

5. Ordered and symmetric geometries

The set operations of countable distance range (3.1) and all possible locations (4.1) requires sequencing through each set. The commutative property of the set operations also allows sequencing, where each set can be sequentially adjacent to any other set, herein referred to as a symmetric geometry.

From a combinatoric perspective, there are $n!$ number of sequential arrangements of any n number of sets, where there are two arrangements having a set,

x_i, that is sequentially adjacent (once as a predecessor and once as a successor) to a set, x_j. Where all arrangements exist, the properties of sequential order and symmetry are satisfied for any number of sets (dimensions).

But, time places an additional constraint on physical sets. A physical set can have only one sequential order *at a time* because each set member can have at most one successor and at most one predecessor *at a time*. It will now be proved that a set (of physical sets of subintervals or physical dimensions) satisfying the constraints of a single sequential (total) order and symmetric *at the same time* defines a cyclic set containing at most 3 members (in this case, 3 dimensions of physical space).

Definition 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i\; \in \{x_1, \ldots, x_n\},$$
$$\textit{successor}\ x_i = x_{i+1}\;\; \wedge\;\; \textit{predecessor}\ x_{i+1} = x_i.$$

Definition 5.2. Symmetric geometry (every set member is sequentially adjacent to any other member):

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; \textit{successor}\ x_i = x_j\;\; \wedge\;\; \textit{predecessor}\ x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$\textit{successor}\ x_n = x_1\;\; \wedge\;\; \textit{predecessor}\ x_1 = x_n.$$

The theorem, "ordered_symmetric_is_cyclic," and formal proof is in the Coq file, threed.v.

Proof. The property of order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$i = n\;\; \wedge\;\; j = 1\;\; \wedge\;\; \textit{successor}\ x_i = x_j\;\; \Rightarrow\;\; \textit{successor}\ x_n = x_1. \tag{5.1}$$

$$i = n\;\; \wedge\;\; j = 1\;\; \wedge\;\; \textit{predecessor}\ x_j = x_i\;\; \Rightarrow\;\; \textit{predecessor}\ x_1 = x_n. \tag{5.2}$$ □

Theorem 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

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Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213,
adj313, adj323, and not_all_mutually_adjacent_gt_3.
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The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

Proof.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

Definition 5.5. Cyclic successor of m is n:

$$\textit{Successor}(m, n, \textit{setsize}) \leftarrow (m = \textit{setsize} \wedge n = 1) \vee (m + 1 \leq \textit{setsize}). \tag{5.3}$$

Definition 5.6. Cyclic predecessor of m is n:

$$\textit{Predecessor}(m, n, \textit{setsize}) \leftarrow (m = 1 \wedge n = \textit{setsize}) \vee (m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \lor Predecessor(m, n, setsize). \tag{5.5}$$

Every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (1 = 1 \land 1 = 1). \tag{5.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (1 + 1 \leq 2). \tag{5.7}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (2 = 2 \land 1 = 1). \tag{5.8}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (1 + 1 \leq 2). \tag{5.9}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (2 - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (3 = 3 \land 1 = 1). \tag{5.11}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (1 = 1 \land 3 = 3). \tag{5.12}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (2 + 1 \leq 3). \tag{5.13}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (3 - 1 \geq 1). \tag{5.14}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not adjacent:

$$\forall\ setsize > 3 : \quad \neg Successor(1, 3, setsize) \\ \leftarrow Successor(1, 2, setsize) \leftarrow (1 + 1 \leq setsize). \tag{5.15}$$

That is, 2 is the only successor of 1 for all $setsize > 3$, which implies 3 is not a successor of 1 for all $setsize > 3$.

$$\forall\ setsize > 3 : \quad \neg Predecessor(1, 3, setsize) \\ \leftarrow Predecessor(1, n, setsize) \leftarrow (1 = 1 \land n = setsize). \tag{5.16}$$

That is, $n = setsize$ is the only predecessor of 1 for all $setsize > 3$, which implies 3 is not a predecessor of 1 for all $setsize > 3$.

$$\forall\ setsize > 3 : \quad \neg Adjacent(1, 3, setsize) \\ \leftarrow \neg Successor(1, 3, setsize) \land \neg Predecessor(1, 3, setsize). \quad \square \tag{5.17}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (violates the symmetry property).

6. Conclusions and open questions

Applying the ruler measure (2.1) and ruler convergence (2.2) to the set realations, countable distance space (3.1) and all possible locations (4.1) yields the following conclusions and open questions:

(1) Distance is a function of the number of domain-to-*range* set member mappings. Area/volume is a function of the number of domain-to-*domain* set member mappings.
(2) Area and volume satisfying the criteria of metric space, while not being a function of domain-to-range set mappings, opens the question whether the definition of metric space is a sufficient criteria for a distance measure.

(3) All notions of distance are derived from the principle: $|x_i| = |y_i|$. And the countable distance spanning disjoint domain sets is: $d_c = |\bigcup_{i=1}^{n} y_i|$ (4.1).
   (a) A direct consequence of the inclusion-exclusion principle [CG15] is the set relation, $d_c = |\bigcup_{i=1}^{2} y_i| \leq \sum_{i=1}^{2} |y_i|$ (3.2), which generates the properties of metric space (3.2).
   (b) The countable distance space property, $|x_i| = |y_i|$ (3.1), constrains the range of domain-to-range set member mappings from $\sum_{i=1}^{n} p_i$ to $\sum_{i=1}^{n} p_i^2$. $d_c = \sum_{i=1}^{n} p_i \Leftrightarrow d_c^2 = (\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$. The smallest possible distance case is the equality case: $d_c^2 = \sum_{i=1}^{n} p_i^2$, where applying the ruler measure converges to the smallest possible real-valued distance, Euclidean distance (3.8).
   (c) The constraints: $|x_i| < |y_i|$, $|x_i| = |y_i|$, and $|x_i| > |y_i|$ yields three types of distance spaces: open, flat, and closed.
   (d) Do any open and closed distances satisfy the definition of metric space?
   (e) All hyperbolic distances, where d: Euclidean distance $< d \leq$ Manhattan distance: 1) co-exist with Euclidean distance, and 2) are all direct paths between two distinct points (where starting from one end point, each step along the path is a step closer to the other end point). But, are hyperbolic distances from open distance space, where $d >$ Manhattan distance, direct distances in Euclidean space? Or are open space distances direct paths only in non-Euclidean space (open volume space)?
   (f) If one draws the Manhattan and Euclidean distance between two points in Euclidean space, then it is not possible to also draw a closed distance between those two points (a distance smaller than Euclidean distance). Does this mean that closed distances only exist in a closed volume?

(4) The notion of a location (a point) as a 1-1 correspondence to a tuple of domain set members is the notion of a coordinate in geometry. The ruler measure allows proof that the set of all possible locations (4.1) generates Euclidean length/area/volume.
   (a) Euclidean volume has as many range set elements, V_r, as domain set combinations, V_c. The constraints: $V_c < V_r$, $V_c = V_r$, and $V_c > V_r$ yields three types of volume spaces: open, flat, and closed.
   (b) Do the open, flat, and closed volume spaces correspond to the open, flat, and closed distance spaces? For example, do open distances only exist in open volumes?
   (c) How do open and closed volume spaces map onto volumes in hyperbolic and elliptic spaces?

(5) Using the Taylor series and the Euclidean distance equation with two domain intervals sizes yields the arc sine and arc cosine functions. In other words, the parametric variable equating arc sine and arc cosine maps to the notion of angle, where the two domain intervals map to the notion of two line segments or two sides.

Other axiomatic foundations for geometry either use notions of line and angle as undefined primitives (for example, Birkhoff [Bir32] and Hilbert [Hil80]) or as definitions in terms of other undefined primitives (for example, Veblen [**Veb04**] and Tarski [**TG99**]).

(6) The proof showing that more than 3 dimensions of physical space would lead to contradictions (5.4) constrains all higher dimensional physics theories to *hierarchical* 2 or 3-dimensional geometries. For example, the four-vectors common in physics [**Bru17**] are hierarchical, 2-dimensional geometries that have been "flattened."

The spacetime four-vector length, $d = \sqrt{(ct)^2 - (x^2 + y^2 + z^2)}$, where c is the speed of light and t is time, can be expressed in a form like, $(ct)^2 = d_1^2 + d_2^2$, where $d_1^2 = x^2 + y^2 + z^2$ and $d_2 = d$. Likewise, the energy-momentum four-vector has the 2-dimensional form: $E^2 = (mv^2)^2 + (pc)^2$, where E is energy, m is the resting mass, v is the *3-dimensional* velocity, c is the speed of light, and p is the relativistic momentum ($p = \gamma mv$, where $\gamma = (1/(1-(v/c)^2))^{1/2}$ is the Lorentz factor).

References

[Bir32] G. D. Birkhoff, *A set of postulates for plane geometry (based on scale and protractors)*, Annals of Mathematics **33** (1932). ↑9

[Bru17] P. Bruskiewich, *A very simple introduction to special relativity: Part two - four vectors, the lorentz transformation and group velocity (the new mathematics for the millions book 38)*, Pythagoras Publishing, 2017. ↑10

[CG15] W. Conradie and V. Goranko, *Logic and discrete mathematics*, Wiley, 2015. ↑3, 9

[Coq15] Coq, *Coq proof assistant*, 2015. https://coq.inria.fr/documentation. ↑2

[Gol76] R. R. Goldberg, *Methods of real analysis*, John Wiley and Sons, 1976. ↑1

[Hil80] D. Hilbert, *The foundations of geometry (2cd ed)*, Chicago: Open Court, 1980. http://www.gutenberg.org/ebooks/17384. ↑9

[Rud76] W. Rudin, *Principles of mathematical analysis*, McGraw Hill Education, 1976. ↑1

[TG99] A. Tarski and S. Givant, *Tarski's system of geometry*, The Bulletin of Symbolic Logic **5** (1999), no. 2, 175–214. ↑10

[Veb04] O. Veblen, *A system of axioms for geometry*, Trans. Amer. Math. Soc **5** (1904), 343–384. https://doi.org/10.1090/S0002-9947-1904-1500678-X. ↑10

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