A Combinatorial Foundation for Analytic Geometry

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Abstract. Using a ruler-like measure of intervals with real analysis allows new proofs that provide insights into the combinatorial principles generating geometry: A set-based definition of a countable distance range applied to sets of subintervals of intervals converges to the taxicab distance equation as the upper boundary, the Euclidean distance equation as the lower boundary, and the triangle inequality over the full range, which provides an analytic motivation for the definition of metric space independent of elementary geometry. The Cartesian product of the subintervals of intervals converges to the product of interval sizes used in the Lebesgue measure and Euclidean integrals. Also combinatorics limits a geometry that is both ordered and symmetric to a cyclic set of at most 3 dimensions, which is the basis of the right-hand rule. Implications for non-Euclidean geometries and higher dimensional geometries are discussed. All the proofs are verified in Coq.

Contents

1. Introduction

The triangle inequality of a metric space, Euclidean distance metric, and the volume equation (product of interval sizes) of the Lebesgue measure and Euclidean integrals are imported into mathematical analysis from Euclidean geometry [Gol76]. Because the geometric relations are imported as primitives rather than derived from set and number theory, mathematical analysis has provided no insight into the counting principles that motivate and generate those geometric relations.

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For example, a simple basis for measuring distance is “correspondence”, where each element in a distance measure corresponds to one element in a domain set, like one pebble for each step walked. If one distance element can correspond to multiple domain set elements, then the distance set will contain fewer elements than the domain set. The constraint that for every disjoint domain subset there exists a distance subset with the same number of elements results in a defined range of possible distance set sizes as a function of the number of correspondences.

This “countable distance range” constraint can be applied to the correspondences (combinations) of the set of subintervals in a distance interval to the set of subintervals in domain intervals. As the subinterval size go to zero, the countable distance range converges to the real-valued triangle inequality, which provides a motivation for the definition of a metric space independent of Euclidean geometry.

The upper boundary of the range converges to the taxicab distance equation. And the lower boundary of the range converges to the Euclidean distance equation, which provides a new insight into the combinatorial principle generating the smallest distance spanning disjoint sets.

However, there have been no combinatorial derivations of the real-valued triangle inequality and Euclidean distance equations. Further, there has been no proof that the Cartesian product (combinations) of the subintervals of intervals converges to the product of interval sizes, the Euclidean volume equation, used in the Lebesgue measure and Euclidean integrals. Attempts at a proof using the Lebesgue measure or Euclidean integrals would be circular logic.

The various traditional indefinite integrals (antiderivatives) derive a real-valued equation from a **real-valued, continuous function** relating the **sizes** of the subintervals. In contrast, what is needed for counting-based (combinatorial) proofs is an indefinite integration that derives a real-valued equation from a **combinatorial function** relating the integer **number** of same-sized subintervals of domain intervals to the integer **number** of same-sized subintervals in an image interval.

Combinatorial integration requires measuring the number of same-sized subintervals in both the domain and image intervals similar to using a ruler (measuring stick). The ruler is an approximate measure that ignores partial subintervals. In contrast, the traditional method of dividing a set of intervals into subintervals, the number of subintervals is the same in both the domain and image intervals and the size of some subintervals can vary.

Same-sized subintervals across both the set of domain intervals and image interval allows defining a countable relationship between the number of domain subintervals and the number of image subintervals. For example, as the subinterval size goes to zero, the combinatorial relationships that define smallest countable distance and countable size (length/area/volume) converge to the n-dimensional Euclidean distance and volume equations.

Differentiating between two sets of domain intervals (for example, $\{[0, 2],\ [0, 1], [0, 5]\}$ and $\{[0, 5],\ [0, 2],\ [0, 1]\}$) having the same spanning distance and volume, both Euclidean and non-Euclidean, requires each set of domain intervals to be ordered. However, the total number of distance and volume subinterval combinations is independent of the order of the combinations. Therefore, distance and volume are the same for any ordering of the set of domain intervals corresponding to those combinations (a symmetric geometry).

Combinatorics limits a geometry that is both ordered and symmetric to a cyclic set of at most three dimensions, which is the basis of the right-hand rule.

The proofs in this article are verified formally using the Coq Proof Assistant [**Coq15**] version 8.4p16. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at: https://github.com/treeck/CombinatorialGeometry.

2. Ruler measure and convergence

DEFINITION 2.1. Ruler measure: A ruler measures the size of a closed, open, or semi-open interval as the nearest integer number of whole subintervals, p, times the subinterval size, c, where c is the independent variable. Notionally:

$$\begin{aligned} (2.1) \quad & \forall\, c\, s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |b-a| \ \ \wedge\ \ c > 0\ \wedge \\ & (p = floor(s/c)\ \ \vee\ \ p = ceiling(s/c))\ \ \wedge\ \ M = \lim_{c\to 0} \Sigma_{p=1}^{\infty} c = \lim_{c\to 0} pc. \end{aligned}$$

THEOREM 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\ \ s = |b-a| \ \ \Rightarrow\ \ M = \lim_{c\to 0} pc = s.$

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$(2.2) \qquad \forall\, c > 0, \quad p = floor(s/c) \quad \Rightarrow \quad 0 \le |p - s/c| < 1.$$

Multiply all sides by $|c|$:

$$(2.3) \qquad \forall\, c > 0, \quad 0 \le |p - s/c| < 1 \quad \Rightarrow \quad 0 \le |pc - s| < |c|.$$

$$\begin{aligned} (2.4) \quad & \forall\, c > 0,\ \exists\, \delta,\ \epsilon\ :\ 0 \le |pc - s| < |c| = |c - 0| < \delta = \epsilon \\ & \Rightarrow \quad 0 < |c - 0| < \delta \quad \wedge \quad 0 \le |pc - s| < \epsilon = \delta \quad := \quad M = \lim_{c\to 0} pc = s. \quad \square \end{aligned}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure).

For example, showing convergence using the interval, $[0, \pi]$, $s = |\pi - 0|$, $c = 10^{-i}, i \in \mathbb{N}$, and $p = floor(s/c)$, then, $p \cdot c = 3.1,\ 3.14,\ 3.141, ..., \pi$.

3. Distance

A simple basis for measuring distance is "correspondence", where each element in a distance measure set corresponds to one element in a domain set, like one pebble for each step walked. If one distance element can correspond to multiple domain set elements, then the distance set will contain fewer elements than the domain set. The constraint that for each i^{th} disjoint domain subset containing p_i number of elements there exists a distance subset with the same p_i number of elements results in a defined range of possible distance set sizes as function of the number of correspondences.

DEFINITION 3.1. Countable distance range, d_c:

$$\forall\, i\, n \in \mathbb{N},\ \ x_i \subseteq X,\ \ \bigcap_{i=1}^{n} x_i = \emptyset,\ \ \forall\, x_i\ \exists\, y_i \subseteq Y :\ \ |x_i| = |y_i| \quad \wedge \quad d_c = |Y|.$$

Notation conventions: In the definition of countable distance range (3.1), the vertical bars around a set is the standard notation for indicating the cardinal (number of elements in the set). To prevent too much overloading on the vertical bar, the symbol for "such that" is the colon.

From the definition of countable distance range (3.1), the amount of intersection of distance subsets is not defined, which results in a range of possible distance set sizes. Notionally:

$$|\cap_{i=1}^{n} y_i| \geq 0 \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| \geq |\cup_{i=1}^{n} y_i| = |Y|. \tag{3.1}$$

The countable distance range property, $|x_i| = |y_i|$, implies a limitation on the number of possible correspondences of a distance subset element to domain subset elements. If each of the p_i number of elements of the i^{th} distance set has a one-to-one (bijective) correspondence to a domain subset element, then the number of correspondences per distance set is: $1 \cdot |x_i| = 1 \cdot p_i = p_i = |y_i|$. And therefore, the distance, $d_c = |Y| = |\cup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i$, is the largest possible distance and the upper bound of the distance range.

If each of the p_i number of elements of the i^{th} distance subset corresponds to all p_i number of domain subset elements, then the largest number of correspondences per distance subset is: $|y_i| \cdot |x_i| = p_i \cdot p_i = p_i^2$. The largest number of possible correspondences implies the smallest possible distance and is the lower bound of the distance range,

$$\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} |\{(y_a, y_b) : y_a\ y_b \in y_i\}|, \tag{3.2}$$

where each pair, (y_a, y_b), represents a combination (correspondence) between two elements in the distance set, y_i. From combining equations 3.1 and 3.2:

$$\begin{aligned} &\sum_{i=1}^{n} |y_i| \geq |\cup_{i=1}^{n} y_i| = |Y| \quad \Rightarrow \\ &\qquad \sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} |\{(y_a, y_b) : y_a\ y_b \in y_i\}| \geq |\{(y_a, y_b) : y_a\ y_b \in Y\}|. \end{aligned} \tag{3.3}$$

From the relation 3.3, there exist the cases of equality:

$$\begin{aligned} &\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |y_i|^2 \\ &\qquad = \sum_{i=1}^{n} |\{(y_a, y_b) : y_a\ y_b \in y_i\}| = |\{(y_a, y_b) : y_a\ y_b \in Y\}|. \end{aligned} \tag{3.4}$$

This is the case where are distance subsets are disjoint, which is the largest possible sum of unique combinations, which is therefore the smallest distance relationship.

The ruler (2.1) and ruler convergence theorem (2.2) can be applied to real-valued intervals to show the shortest distance case converges to the real-valued Euclidean distance equation (always a real-valued square root).

The proof of the taxicab and Euclidean distance equations requires the strategy of showing that the right and left sides of a proposed counting-based equation both converge to the same real value and therefore are equal. That is, the propositional logic, $A = B \wedge C = B \Rightarrow A = C$, is used.

THEOREM 3.2. *Taxicab (largest) distance, d, is the size of the distance interval,* $[y_0, y_m]$, *corresponding to a set of disjoint domain intervals,* $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, *where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|.$$

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals. Next, apply the definition of the countable distance range (3.1) and the rule of product:

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1,n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\ |\{x_i : x_i \in \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\}| = |\{y_i : y_i \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_i}\}\}| = p_i. \tag{3.5}$$

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1,n], \quad y \in y_i \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i = |\{y\}|. \tag{3.6}$$

Multiply both sides of 3.6 by c and apply the ruler convergence theorem (2.2):

$$s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n} (p_i \cdot c) = |\{y\}| \cdot c \\ \Rightarrow \quad \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) = \lim_{c\to 0} |\{y\}| \cdot c. \tag{3.7}$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$\forall\, c > 0, \quad p_d = floor(d/c) = |\{y : \; y \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_d}\}\}|. \tag{3.8}$$

Multiply both sides of 3.8 by c and apply the ruler convergence theorem (2.2):

$$d = \lim_{c\to 0} p_d \cdot c \;\wedge\; p_d \cdot c = |\{y\}| \cdot c \;\Rightarrow\; d = \lim_{c\to 0} p_d \cdot c = \lim_{c\to 0} |\{y\}| \cdot c. \tag{3.9}$$

Combine equations 3.9 and 3.7:

$$d = \lim_{c\to 0} |\{y\}| \cdot c \quad \wedge \quad \sum_{i=1}^{n} s_i = \lim_{c\to 0} |\{y\}| \cdot c \quad \Rightarrow \quad d = \sum_{i=1}^{n} s_i. \tag{3.10}$$ □

THEOREM 3.3. *Euclidean (smallest) distance, d, is the size of the distance interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals, $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}].$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals. Next, apply the definition of the countable distance range (3.1) and the rule of product:

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1,n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\ x_i = \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\} \quad \wedge \quad y_i = \{y_{1_i}, y_{2_i}, \ldots, y_{p_i}\}\}|. \tag{3.11}$$

$$\textstyle\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} |y_i|^2 = \sum_{i=1}^{n} |\{(y_a, y_b) : y_a\; y_b \in y_i\}|, \tag{3.12}$$

where each pair, (y_a, y_b), represents a combination (correspondence) between two elements in the distance subset, y_i. From definition of countable distance range (3.1), the amount of intersection of distance subsets is not defined, which results in a range of possible distance set sizes. Notionally:

$$|\cap_{i=1}^{n} y_i| \geq 0 \quad \Rightarrow \quad \textstyle\sum_{i=1}^{n} |y_i| \geq |\cup_{i=1}^{n} y_i| = |Y|. \tag{3.13}$$

From combining equations 3.12 and 3.13:

$$\begin{aligned}(3.14)\quad & \textstyle\sum_{i=1}^{n}|y_i| \geq |\cup_{i=1}^{n} y_i| = |Y| \quad \Rightarrow \\ & \textstyle\sum_{i=1}^{n}|y_i|^2 = \sum_{i=1}^{n}|\{(y_a, y_b) : y_a\, y_b \in y_i\}| \geq |\{(y_a, y_b) : y_a\, y_b \in Y\}|.\end{aligned}$$

From the relation 3.14, there exist the cases of equality:

$$\begin{aligned}(3.15)\quad & \textstyle\sum_{i=1}^{n}|y_i|\cdot|x_i| = \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n}|y_i|^2 \\ & = \textstyle\sum_{i=1}^{n}|\{(y_a, y_b) : y_a\, y_b \in \{y_i\}\}| = |\{(y_a, y_b) : y_a\, y_b \in Y\}|.\end{aligned}$$

Multiply both sides of equation 3.15 by c^2 and apply the ruler convergence theorem.

$$\begin{aligned}(3.16)\quad & s_i = \lim_{c\to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n}(p_i \cdot c)^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\|\cdot c^2 \\ & \Rightarrow \quad \sum_{i=1}^{n} s_i^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\| \cdot c^2.\end{aligned}$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$\begin{aligned}(3.17)\quad & \forall\, c > 0, \quad p_d = floor(d/c) = |\{y_{1_i}, y_{2_i}, \ldots, y_{p_d}\}| = |Y| \\ & \Rightarrow \quad p_d^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\}|,\end{aligned}$$

where $\{(y_a, y_b)\}$ is the set of all combination pairs of elements of Y. Multiply both sides of 3.16 by c^2 and apply the ruler convergence theorem (2.2):

$$\begin{aligned}(3.18)\quad & d = \lim_{c\to 0} p_d \cdot c \quad \wedge \quad (p_d \cdot c)^2 = |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \\ & \Rightarrow \quad d^2 = \lim_{c\to 0}(p_d \cdot c)^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2.\end{aligned}$$

Combine equations 3.16 and 3.18:

$$\begin{aligned}(3.19)\quad & d^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \quad \wedge \\ & \sum_{i=1}^{n} s_i^2 = \lim_{c\to 0} |\{(y_a, y_b) : y_a\, y_b \in Y\}| \cdot c^2 \quad \Rightarrow \quad d^2 = \sum_{i=1}^{n} s_i^2. \quad \square\end{aligned}$$

3.1. Triangle inequality. The definition of a metric in real analysis is based on the triangle inequality, $\mathbf{d}(\mathbf{u}, \mathbf{w}) \leq \mathbf{d}(\mathbf{u}, \mathbf{v}) + \mathbf{d}(\mathbf{v}, \mathbf{w})$, that has been intuitively motivated by the triangle [**Gol76**]. Applying the inclusion-exclusion principle, ruler (2.1), and ruler convergence theorem (2.2) to the definition of a countable distance range (3.1) yields the real-valued triangle inequality:

$$\begin{aligned}(3.20)\quad & d_c = |Y| = |\bigcup_{i=1}^{2} y_i| \leq \sum_{i=1}^{2} |y_i| \quad \wedge \\ & d_c = floor(\mathbf{d}(\mathbf{u}, \mathbf{w})/c) \;\wedge\; |y_1| = floor(\mathbf{d}(\mathbf{u}, \mathbf{v})/c) \;\wedge\; |y_2| = floor(\mathbf{d}(\mathbf{v}, \mathbf{w})/c) \\ & \Rightarrow \; \mathbf{d}(\mathbf{u}, \mathbf{w}) = \lim_{c\to 0} d_c \cdot c \leq \sum_{i=1}^{2} \lim_{c\to 0} |y_i| \cdot c = \mathbf{d}(\mathbf{u}, \mathbf{v}) + \mathbf{d}(\mathbf{v}, \mathbf{w}).\end{aligned}$$

4. Size (length/area/volume)

Until now, there has not been a proof that the Cartesian product of the subintervals of intervals converges to the product of the interval sizes, the Euclidean volume equation, used by the Lebesgue measure and Euclidean integrals. This section will use the ruler (2.1) and ruler convergence theorem (2.2) to prove that the Cartesian product of the subintervals of intervals converges to the product of interval sizes.

The countable size measure is the number of combinations between members of disjoint domain sets, which is the Cartesian product of the domain set sizes.

Definition 4.1. Countable size (length/area/volume) measure, S_c:

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1,n], \quad x_i \subseteq X, \quad |\bigcap_{i=1}^{n} x_i| = \emptyset \quad \wedge \quad \{(x_1, \ldots, x_n)\} = y \quad \wedge$$
$$S_c = |y| = |\{(x_1, \ldots, x_n)\}| = \prod_{i=1}^{n} |x_i|.$$

Theorem 4.2. *Euclidean size (length/area/volume), S, is the size of an image interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals:* $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_n,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, *where:*

$$S = \prod_{i=1}^{n} s_i, \quad S = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|, \quad i \in [1,n], \quad i,n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_size."

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1,n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge$$
$$x_i = \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\} \quad \Rightarrow |x_i| = p_i. \tag{4.1}$$

Use the ruler (2.1) to divide the exact size, $S = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_S^n subintervals, where p_S^n satisfies the definition a countable size measure, S_c (4.1).

$$\forall\, c > 0 \quad \wedge \quad \exists\, r \in \mathbb{R},\; S = r^n \quad \wedge \quad p_S = floor(r/c) \quad \wedge$$
$$p_S^n = S_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i. \tag{4.2}$$

Multiply both sides of equation 4.2 by c^n to get the ruler measures:

$$p_S^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3:

$$S = r^n = \lim_{c \to 0} (p_S \cdot c)^n \quad \wedge \quad \lim_{c \to 0} (p_i \cdot c) = s_i \quad \wedge \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c)$$
$$\Rightarrow \quad S = \lim_{c \to 0} (p_S \cdot c)^n = \prod_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square \tag{4.4}$$

5. Ordered and symmetric geometries

Two sets of intervals with the same volume and spanning distance (for example, $\{[0,2],\ [0,1],\ [0,5]\}$ and $\{[0,5],\ [0,2],\ [0,1]\}$) can only be distinguished by assigning a sequential order (orientation) to the elements of the sets. Notionally:

Definition 5.1. Ordered geometry:

$$\forall\, i\ n \in \mathbb{N},\ \ \forall\, x_i \in \{x_1, \ldots, x_{n-1}\},\ \ \textit{successor}\ x_i = x_{i+1}\ \ \wedge\ \ \textit{predecessor}\ x_{i+1} = x_i.$$

However, the total number of distance and volume subinterval combinations is independent of the order of the combinations. Therefore, distance and volume are the same for any ordering of the set of domain intervals corresponding to those combinations. A function, like size or distance, where every permutation of the arguments to the function yields the same value is called a symmetric function. "Symmetric" means that all permutations are valid and yield the same result, where each "permutation" is a different ordering of a set.

For all elements, $x_i\ x_j \in \{x_1, \ldots, x_n\}$: traversing the ordered set in successor order yields the permutation, $(\ldots, x_i, x_j, \ldots)$, and traversing in predecessor order yields the permutation, $(\ldots, x_j, x_j, \ldots)$. If all permutations are valid (a symmetric geometry), then every element has a sequentially adjacent successor and every element has a sequentially adjacent predecessor. Notionally:

Definition 5.2. Symmetric geometry:

$$\forall\, i\ j\ n \in \mathbb{N},\ \ \forall\, x_i\ x_j \in \{x_1, \ldots, x_n\},\ \ \textit{successor}\ x_i = x_j\ \ \wedge\ \ \textit{predecessor}\ x_j = x_i.$$

Theorem 5.3. *A ordered and symmetric geometry is a cyclic set.*

$$\forall\, i\ j\ n \in \mathbb{N},\ \ i = n\ \ \wedge\ \ j = 1\ \ \Rightarrow\ \ \textit{successor}\ x_n = x_1\ \ \wedge\ \ \textit{predecessor}\ x_1 = x_n.$$

The theorem and formal Coq-based proof is "ordered_symmetric_is_cyclic," which is located in the file threed.v.

Proof. The property of order (5.1) defines unique successors and predecessors for all elements except for the successor of x_n and the predecessor of x_1. From the properties of a symmetric geometry (5.2):

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \textit{successor}\ x_i = x_j\ \ \Rightarrow\ \ \textit{successor}\ x_n = x_1. \tag{5.1}$$

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \textit{predecessor}\ x_j = x_i\ \ \Rightarrow\ \ \textit{predecessor}\ x_1 = x_n. \tag{5.2}$$

□

For example, using the cyclic set with elements labeled, $\{1,\ 2,\ 3\}$, starting with each element and counting through 3 cyclic successors and counting through 3 cyclic predecessors yields all possible permutations: $(1,2,3)$, $(2,3,1)$, $(3,1,2)$, $(1,3,2)$, $(3,2,1)$, and $(2,1,3)$. That is, a cyclically ordered set preserves sequential order while allowing some n-at-a-time permutations. If all possible n-at-a-time permutations are generated, then the cyclic set is also symmetric.

Theorem 5.4. *An ordered and symmetric geometry is limited to at most 3 elements. That is, each element is sequentially adjacent (a successor or predecessor) to every other element in a set only where the number of elements (set sizes) are less than or equal to 3.*

The Coq-based lemmas and proofs in the file threed.v are:

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Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213,
adj313, adj323, and not_all_mutually_adjacent_gt_3.
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The following proof uses Horn-like clauses (a subset of first-order logic) with unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
Because a symmetric and ordered set is a cyclic set (5.3), the successors and predecessors are cyclic:

DEFINITION 5.5. Successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (m + 1 \leq setsize). \tag{5.3}$$

DEFINITION 5.6. Predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: element m is adjacent to element n (an allowed permutation), if the cyclic successor of m is n or the cyclic predccessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{5.5}$$

Every element is adjacent to every other element, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{5.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (1 + 1 \leq 2). \tag{5.7}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (2 = 2 \wedge 1 = 1). \tag{5.8}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (1 + 1 \leq 2). \tag{5.9}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (2 - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (3 = 3 \wedge 1 = 1). \tag{5.11}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (1 = 1 \wedge 3 = 3). \tag{5.12}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (2 + 1 \leq 3). \tag{5.13}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (3 - 1 \geq 1). \tag{5.14}$$

For all $n = setsize > 3$, there exist non-adjacent elements (not every permutation allowed):

$$\forall\, n > 3,\ Successor(1, 2, n) \Rightarrow \forall\, n > 3,\ \neg Successor(1, 3, n). \tag{5.15}$$

That is, 2 is the only successor of 1 for all $n > 3$, which implies 3 is not a successor of 1 for all $n > 3$.

$$\forall\, n > 3,\ Predecessor(1, n, n) \Rightarrow \forall\, n > 3,\ \neg Predecessor(1, 3, n). \tag{5.16}$$

That is, n is the only predecessor of 1 for all $n > 3$, which implies 3 is not a predecessor of n for all $n > 3$.

$$\forall\, n > 3,\ \neg Adjacent(1, 3, n) \leftarrow \neg Successor(1, 3, n) \wedge \neg Predecessor(1, 3, n). \tag{5.17}$$

□

6. Summary

In the past, the triangle inequality of the metric space, Euclidean distance metric, and volume equation of the Lebesgue measure and Euclidean integrals were imported from Euclidean geometry. Importing from elementary geometry allows geometry to motivate real analysis, calculus, and measure theory. But, analysis has been unable to motivate the notions of elementary geometry and unable to provide insights into the properties that generate the definitions imported from elementary geometry.

Using the ruler measure of intervals with real analysis is a tool allowing a new class of proofs that provides insights into combinatorial principles generating the triangle inequality, Euclidean distance, and volume of elementary geometry:

(1) The property that every disjoint domain subset has a corresponding distance subset with the same number of elements constrains the maximum number of possible correspondences from each distance subset element to domain subset elements. The the maximum possible number correspondences from each distance element to domain set elements yields the smallest possible distance set. Using the ruler measure with real analysis the case of the largest number of correspondences converges to Euclidean distance, which provides insights into the principles generating the smallest distance spanning disjoint domain sets.
(2) Combinatorial relations between the elements of sets converge to the Euclidean distance (3.3) and size (length/area/volume) (4.2) equations without notions of angle, and shape, and without motivation from diagrams. In particular, the notion of arc angle as the parametric parameter relating the sizes of two domain intervals, given the Euclidean distance, can be easily derived using using calculus to generate the sine and cosine functions of the parametric parameter. In other words, Euclidean distance is a primitive from which the notion of arc angle is derived, which is very different from what elementary geometry teaches.
(3) The triangle inequality (3.1) is derived from the definition of the countable distance range (3.1), which provides a counting-based motivation for the definition of the metric space without the need for Euclidean geometry.
(4) The Euclidean volume (product of interval sizes) of the Lebesgue measure is derived from use of the more fundamental ruler measure.

Combinatorics combined with symmetric functions for distance and size (length/area/volume) provides insights into other geometric constraints.

(1) Combinatorics limits a geometry (both Euclidean and non-Euclidean) having the properties of both order (5.1) and symmetry (5.2) to a cyclic set (5.3) of at most three elements (dimensions) (5.4), which is the basis of the right-hand rule that permeates mathematics, physics, and engineering.
(2) A cyclic set is a closed walk. An observer in a closed walk of three dimensions would only be able to detect higher, non-closed walk dimensions (other variables) indirectly via distance and size changes in the three closed walk dimensions, where a change in distance is what physicists call "work."

References

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