The Real Analysis and Set Relations of Geometry

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Abstract. A range from 1-to-1 to many-to-many mappings between each disjoint domain set and a corresponding range set containing the same number of members, where the range sets in some cases intersect and the set members are the same-sized subintervals of intervals, converges to: the triangle inequality, Manhattan distance at the upper boundary, and Euclidean distance at the lower boundary, which provides countable set-based definitions of: metric space, longest, and shortest distances spanning disjoint sets. A many-to-many relation between sets of same-sized subintervals of intervals converges to the product of interval sizes (Euclidean area/volume). The total ordering and symmetry properties of these set-based relations limit the number of domain dimensions to 3. All ordered and symmetric, higher-dimensional geometries, like the spacetime four-vector, collapse into hierarchical 2 or 3-dimensional geometries. Proofs are verified in Coq.

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1. Introduction

Metric space, Manhattan distance, Euclidean distance, and Euclidean area/volume are motivated by geometry and assumed in measure and integration [Gol76] rather than motivated and derived from more fundamental relations between countable sets. This article will use some very simple real analysis to motivate and derive distance and volume from relationships between countable sets.

The relationships between countable sets generating distance and volume provides a formal, set-based understanding of geometry, for example, why (without

2010 *Mathematics Subject Classification.* Primary 28A75, 28E15.

any notions of side, angle, and shapes) Euclidean distance is the smallest distance between two distinct points in $\mathbb{R}^n$. It will also be shown that these set-based relations have properties that constrain the number of domain dimensions (variables) and also constrain how a range variable, for example, time, relates to the domain variables of another type, for example, space (length, width, and height).

The proofs in this article are verified formally using the Coq Proof Assistant [**Coq15**] version 8.7.0. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at:
https://github.com/treeck/CombinatorialGeometry.

2. Ruler measure and convergence

Euclidean distance and volume are derived from many-to-many relations. But, a function only allows each domain set member to map to one range set member. Therefore, deriving distance and volume requires a different type of measure that does not have Euclidean assumptions and also allows the full range of mappings from a one-to-one correspondence to a many-to-many mapping.

A ruler (measuring stick) measures a real-valued interval as the nearest integer number of same-sized subintervals (units), where the partial subintervals are ignored. The ruler measure allows defining relations, for example a many-to-many relation, between the set of same-sized subintervals in one interval and the set of same-sized subintervals in another interval. The countable relations converge to continuous, bijective functions as the subinterval size converges to zero.

Definition 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, p, each subinterval having the same size, c. Notionally:

$$\begin{aligned}(2.1)\quad &\forall\, c\; s \in \mathbb{R},\;\; [a,b] \subset \mathbb{R},\;\; s = |a-b| \;\;\wedge\;\; c > 0 \;\wedge \\ &\qquad (p = floor(s/c) \;\;\vee\;\; p = ceiling(s/c)) \;\;\wedge\;\; M = \textstyle\sum_{i=1}^{p} c = pc.\end{aligned}$$

Theorem 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\;\; s = |a-b| \;\;\Rightarrow\;\; M = \lim_{c\to 0} pc = s.$

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\; y \leq x,\; y \in \mathbb{Z},\; x \in \mathbb{R}\})$:

$$(2.2)\;\; \forall\, c > 0,\, p = floor(s/c) \;\wedge\; 0 \leq |floor(s/c) - s/c| < 1 \;\Rightarrow\; 0 \leq |p - s/c| < 1.$$

Multiply all sides of inequality 2.2 by $|c|$:

$$(2.3)\qquad \forall\, c > 0, \quad 0 \leq |p - s/c| < 1 \quad \Rightarrow \quad 0 \leq |pc - s| < |c|.$$

$$\begin{aligned}(2.4)\quad &\forall\, \delta \;:\; |pc - s| < |c| = |c - 0| < \delta \\ &\qquad \Rightarrow \quad \forall\, \epsilon = \delta :\;\; |c - 0| < \delta \;\;\wedge\;\; |pc - s| < \epsilon \;\;:=\;\; M = \lim_{c\to 0} pc = s. \qquad \square\end{aligned}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence, where: $[0, \pi]$, $s = |0 - \pi|$, $c = 10^{-i}$, and $p = floor(s/c) \Rightarrow\; p \cdot c = 3.1_{i=1},\; 3.14_{i=2},\; 3.141_{i=3}, ..., \pi.$

3. Distance

Notation conventions: The vertical bars around a set is the standard notation for indicating the cardinal (number of members in the set). To prevent over use of the vertical bar, the symbol for "such that" is the colon.

3.1. Countable metric space. A simple countable distance measure is that a range (distance) set has the same number of members as a corresponding domain set. For example, the number of steps walked in a distance set must equal the number pieces of land traversed. Generalizing, for each distance set, y_i, containing p_i number of members there exists a corresponding domain set, x_i, with the same p_i number of members. Notionally, $|x_i| = |y_i| = p_i$.

It is assumed here that the domain sets are "countably" disjoint. Notionally: $|\bigcup_{i=1}^n x_i| = \sum_{i=1}^n |x_i| \Leftrightarrow \bigcap_{i=1}^n x_i = \emptyset$. And the number of members in the union of the distance sets, d_c, depends on the amount of intersection of the distance sets, where: $|\bigcup_{i=1}^n y_i| < \sum_{i=1}^n |y_i| \Leftrightarrow \bigcap_{i=1}^n y_i \neq \emptyset$.

Definition 3.1. Countable metric space:

$$|\textstyle\bigcup_{i=1}^n x_i| = \sum_{i=1}^n |x_i| \quad \wedge \quad d_c = |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i.$$

3.2. Real-valued metric Space. Applying the ruler (2.1), and ruler convergence theorem (2.2) to the definition of a countable metric space (3.1) yields the real-valued triangle inequality and non-negativity properties of metric space:

$$\begin{aligned} &d_c = |\textstyle\bigcup_{i=1}^{2} y_i| \leq \sum_{i=1}^{2} |y_i| \quad \wedge \\ &d_c = floor(d(u,w)/c) \;\wedge\; |y_1| = floor(d(u,v)/c) \;\wedge\; |y_2| = floor(d(v,w)/c) \\ &\quad\Rightarrow\; d(u,w) = \lim_{c\to 0} d_c \cdot c \leq \textstyle\sum_{i=1}^{2} \lim_{c\to 0} |y_i| \cdot c = d(u,v) + d(v,w). \end{aligned} \tag{3.1}$$

The number of members in any countable set is always non-negative. And the product of two non-negative numbers, $d_c \cdot c$, is always a non-negative number:

$$\begin{aligned} &\forall\, c > 0,\; d_c = floor(d(u,w)/c) \quad \wedge \quad d_c = |\textstyle\bigcup_{i=1}^n y_i| \geq 0 \\ &\quad\Rightarrow \quad floor(d(u,w)/c) = d_c \geq 0 \quad \Rightarrow \quad d(u,w) = \lim_{c\to 0} d_c \cdot c \geq 0. \end{aligned} \tag{3.2}$$

3.3. Metric space range. Where the range (distance) sets intersect, multiple domain set members map to a single distance member. Therefore, the union distance, d_c, is related to the number of domain-to-distance member mappings.

Consider the trivial case of the countable metric space (3.1), where a domain set has only one member: $|x_i| = |y_i| = p_i = 1$: 1) Each member of x_i maps to only one member of y_i, yielding $|x_i| \cdot |y_i| = |x_i| \cdot 1 = p_i = 1$ number of domain-to-distance member mappings. 2) All p_i number of members of x_i map to each member in y_i, yielding $|x_i| \cdot |y_i| = p_i^2 = 1$ number of domain-to-distance member mappings.

The range of domain-to-distance mappings, p_i to p_i^2, that is true for one non-empty set size is true for all non-empty set sizes. Therefore, $d_c = \sum_{i=1}^{n} p_i$ is the largest possible distance because it is the case of the smallest number of domain-to-distance mappings (no intersection of the distance sets). And $\exists\, \mathbf{f} : d_c = \mathbf{f}(\sum_{i=1}^{n} p_i^2)$ is the smallest possible distance because it is the case of the largest number of domain-to-distance mappings (largest allowed intersection of distance sets). Applying the ruler (2.1) and ruler convergence theorem (2.2) to the longest and shortest distance cases yields the real-valued, Manhattan and Euclidean distance functions.

3.4. Manhattan distance.

THEOREM 3.2. *Manhattan (longest) distance, d, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:

$$d = \sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance."

PROOF.
From the countable metric space definition (3.1), the largest possible countable distance, d_c, is the equality case:

$$\begin{aligned}(3.3)\quad & d_c \le \sum_{i=1}^{n} |y_i| \quad \wedge \quad |y_i| = p_i \\ & \Rightarrow \quad d_c \le \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c :\ d_c = \sum_{i=1}^{n} p_i.\end{aligned}$$

Multiply both sides of equation 3.5 by c and take the limit:

$$(3.4)\quad d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \Rightarrow \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c).$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$(3.5)\qquad d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d :\ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of s_i:

$$(3.6)\qquad \forall\, i \in [1, n],\ s_i = |a_i - b_i| \quad \Rightarrow \quad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i.$$

Combine equations 3.5, 3.4, 3.6:

$$\begin{aligned}(3.7)\quad & d = \lim_{c\to 0} d_c \cdot c \quad \wedge \quad \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) \quad \wedge \\ & \lim_{c\to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square\end{aligned}$$

3.5. Euclidean distance.

THEOREM 3.3. *Euclidean (shortest) distance, d, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

PROOF.
Apply the rule of product to the largest number of domain-to-distance set mappings, where all p_i number of domain set members, x_i, map to each of the p_i number of members in the distance set, y_i:

$$(3.8)\qquad \sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2.$$

From the countable metric space definition (3.1), choose the equality case:

$$\begin{aligned}(3.9)\quad & d_c \le \sum_{i=1}^{n} |y_i| \quad \wedge \quad |y_i| = p_i \\ & \Rightarrow \quad d_c \le \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c :\ d_c = \sum_{i=1}^{n} p_i.\end{aligned}$$

Square both sides of equation 3.9 ($x = y \Leftrightarrow f(x) = f(y)$):

$$(3.10)\qquad \exists\, p_i,\, d_c :\ d_c = \sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\, d_c :\ d_c^2 = (\sum_{i=1}^{n} p_i)^2.$$

Choose the equality case of the Cauchy-Schwartz inequality:

$$(3.11)\quad (\sum_{i=1}^{n} p_i)^2 = (\sum_{i=1}^{n} p_i)\cdot(\sum_{j=1}^{n} p_j) = \sum_{i=1}^{n} p_i^2 + (\sum_{i=1}^{n} p_i)\cdot(\sum_{j=1,\, j\neq i}^{n} p_j) \geq \sum_{i=1}^{n} p_i^2 \Rightarrow \exists\, p_i : (\sum_{i=1}^{n} p_i)^2 = \sum_{i=1}^{n} p_i^2.$$

Combine equations 3.10 and and 3.11:

$$(3.12)\quad \exists\, p_i,\, d_c : d_c^2 = (\sum_{i=1}^{n} p_i)^2 \quad\wedge\quad \exists\, p_i : (\sum_{i=1}^{n} p_i)^2 = \sum_{i=1}^{n} p_i^2 \Rightarrow \exists\, p_i,\, d_c : d_c^2 = \sum_{i=1}^{n} p_i^2.$$

Multiply both sides of equation 3.12 by c^2, simplify, and take the limit.

$$(3.13)\quad d_c^2 = \sum_{i=1}^{n} p_i^2 \Rightarrow d_c^2\cdot c^2 = \sum_{i=1}^{n} p_i^2\cdot c^2 = (d_c\cdot c)^2 = \sum_{i=1}^{n}(p_i\cdot c)^2 \Rightarrow \lim_{c\to 0}(d_c\cdot c)^2 = \sum_{i=1}^{n}\lim_{c\to 0}(p_i\cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.14)\quad \exists\, c\, d : floor(d/c) = d_c \quad\Rightarrow\quad d = \lim_{c\to 0} d_c\cdot c \quad\Rightarrow\quad d^2 = \lim_{c\to 0}(d_c\cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.15)\quad \forall\, i\in[1,n],\ floor(s_i/c) = p_i \quad\Rightarrow\quad \lim_{c\to 0} p_i\cdot c = s_i \Rightarrow \lim_{c\to 0}(p_i\cdot c)^2 = s_i^2 \quad\Rightarrow\quad \sum_{i=1}^{n}\lim_{c\to 0}(p_i\cdot c)^2 = \sum_{i=1}^{n} s_i^2.$$

Combine equations 3.14, 3.13, 3.15:

$$(3.16)\quad d^2 = \lim_{c\to 0}(d_c\cdot c)^2 \quad\wedge\quad \lim_{c\to 0}(d_c\cdot c)^2 = \sum_{i=1}^{n}\lim_{c\to 0}(p_i\cdot c)^2 \quad\wedge\quad \sum_{i=1}^{n}\lim_{c\to 0}(p_i\cdot c)^2 = \sum_{i=1}^{n} s_i^2 \Rightarrow d^2 = \lim_{c\to 0}(d_c\cdot c)^2 = \sum_{i=1}^{n}\lim_{c\to 0}(p_i\cdot c)^2 = \sum_{i=1}^{n} s_i^2. \quad\square$$

4. Euclidean Volume

The number of all possible combinations (all many-to-many mappings) between members in a countable set x_1 and a countable set x_2 is the Cartesian product, $|x_1|\cdot|x_2|$. This section will use the ruler (2.1) and ruler convergence theorem (2.2) to prove that the Cartesian product of the same-sized subintervals of intervals converges to the product of interval sizes as the subinterval size converges to zero. The first step is to define a countable set-based measure of area/volume as the Cartesian product (many-to-many mappings) of disjoint domain set members.

DEFINITION 4.1. Countable volume measure, V_c:

$$\sum_{i=1}^{n}|x_i| = |\bigcup_{i=1}^{n} x_i|, \quad V_c = \prod_{i=1}^{n}|x_i|.$$

THEOREM 4.2. *Euclidean volume, V, is the size of a range interval, $[v_0, v_m]$, corresponding to a set of disjoint intervals:*
$\{[a_1,b_1],[a_2,b_2],\ldots,[a_n,b_n]\}$, *where:*

$$V = \prod_{i=1}^{n} s_i, \quad V = |v_0 - v_m|, \quad s_i = |a_i - b_i|, \quad i\in[1,n], \quad i,n\in\mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_volume."

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, x_i of p_i number of subintervals.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i|. \tag{4.1}$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c \to 0}(p_i \cdot c) = s_i. \tag{4.2}$$

Use the ruler (2.1) to divide the exact size, $V = |v_0 - v_m|$, of the range interval, $[v_0, v_m]$, into p^n subintervals. Every integer number, V_c, does **not** have an integer n^{th} root. However, for those cases where V_c does have an integer n^{th} root, there is a p^n that satisfies the definition a countable volume measure, V_c (4.1). Notionally:

$$\forall\, p^n = V_c \in \mathbb{N},\, \exists\, V \in \mathbb{R},\, x_i\; : floor(V/c^n) = V_c = p^n = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i. \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3 and simplify:

$$floor(V/c^n) = p^n \quad \Rightarrow \quad V = \lim_{c \to 0} p^n \cdot c^n = \lim_{c \to 0} (p \cdot c)^n. \tag{4.4}$$

Multiply both sides of equation 4.3 by c^n and simplify:

$$p^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad p^n \cdot c^n = (\prod_{i=1}^{n} p_i) \cdot c^n \quad \Leftrightarrow \quad (p \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.5}$$

Combine equations 4.4, 4.5, and 4.2:

$$V = \lim_{c \to 0} (p \cdot c)^n \quad \wedge \quad (p \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c) \quad \wedge \quad \lim_{c \to 0}(p_i \cdot c) = s_i \tag{4.6}$$

$$\Rightarrow \quad V = \lim_{c \to 0} (p \cdot c)^n = \prod_{i=1}^{n} \lim_{c \to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square$$

5. Ordered and symmetric geometries

Inspecting the equations of distance and volume, there is no reason to assume any limit to the number dimensions. If there are any limitations to the number dimensions, then those limitations probably come from the underlying principles that generate distance and volume.

The union operations in the countable metric space principle (3.1) generating the properties of real-valued metric space and the countable volume principle (4.1) generating Euclidean space requires being able to iterate sequentially through each set (dimension), which implies a total ordering exists. The commutative property of union also allows each set (dimension) to be sequentially adjacent to any other dimension (herein, referred to as a symmetric geometry).

Asserting that a specific order exists (is true), for example, $\{x_1, x_2, x_3, x_4\}$, contradicts the assertion that x_1 is allowed to be sequentially adjacent to any other element, for example, x_3. It will now be proved that satisfying the *simultaneous* constraints of both total ordering and symmetry limits the number of dimensions of distance and volume.

Definition 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N}, \;\; i \in [1, n-1], \;\; \forall\, x_i \;\in \{x_1, \ldots, x_n\},$$
$$successor\; x_i = x_{i+1} \;\; \wedge \;\; predecessor\; x_{i+1} = x_i,$$

where each $x_i \in \{x_1, \ldots, x_n\}$ is a set of subintervals of a real-valued domain interval (dimension).

Definition 5.2. Symmetric geometry (every member is sequentally adjacent to every other member):

$$\forall\, i\ j\ n \in \mathbb{N},\ \forall\, x_i\ x_j \in \{x_1, \ldots, x_n\},\ \ \mathit{successor}\ x_i = x_j\ \ \wedge\ \ \mathit{predecessor}\ x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric geometry is a cyclic set.*

$$\mathit{successor}\ x_n = x_1\ \ \wedge\ \ \mathit{predecessor}\ x_1 = x_n.$$

The theorem and formal Coq-based proof is “ordered_symmetric_is_cyclic,” which is located in the file threed.v.

Proof. The property of order (5.1) defines unique successors and predecessors for all members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecssor of x_1, without creating a constradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \mathit{successor}\ x_i = x_j\ \ \Rightarrow\ \ \mathit{successor}\ x_n = x_1. \tag{5.1}$$

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \mathit{predecessor}\ x_j = x_i\ \ \Rightarrow\ \ \mathit{predecessor}\ x_1 = x_n. \tag{5.2}$$

□

Theorem 5.4. *An ordered and symmetric geometry is limited to at most 3 members.*

The Coq-based lemmas and proofs in the file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

Proof.
Because a symmetric and ordered set is a cyclic set (5.3), the successors and predecessors are cyclic:

Definition 5.5. Successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (m + 1 \leq \mathit{setsize}). \tag{5.3}$$

Definition 5.6. Predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (m - 1 \geq 1). \tag{5.4}$$

Definition 5.7. Adjacent: member m is sequentially adjacent to member n (required for a ”symmetric” set (5.2)), if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:

$$\mathit{Adjacent}(m, n, \mathit{setsize}) \leftarrow \mathit{Successor}(m, n, \mathit{setsize}) \vee \mathit{Predecessor}(m, n, \mathit{setsize}). \tag{5.5}$$

Every member is adjacent to every other member, where $\mathit{setsize} \in \{1,\ 2,\ 3\}$:

$$\mathit{Adjacent}(1, 1, 1) \leftarrow \mathit{Successor}(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{5.6}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (1+1 \leq 2). \tag{5.7}$$
$$Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (2=2 \wedge 1=1). \tag{5.8}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (1+1 \leq 2). \tag{5.9}$$
$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (2-1 \geq 1). \tag{5.10}$$
$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (3=3 \wedge 1=1). \tag{5.11}$$
$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (1=1 \wedge 3=3). \tag{5.12}$$
$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (2+1 \leq 3). \tag{5.13}$$
$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (3-1 \geq 1). \tag{5.14}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not adjacent:

$$\forall\ setsize > 3: \quad \neg Successor(1,3,setsize) \leftarrow Successor(1,2,setsize) \leftarrow (1+1 \leq setsize). \tag{5.15}$$

That is, 2 is the only successor of 1 for all $setsize > 3$, which implies 3 is not a successor of 1 for all $setsize > 3$.

$$\forall\ setsize > 3: \quad \neg Predecessor(1,3,setsize) \leftarrow Predecessor(1,n,setsize) \leftarrow (1=1 \wedge n = setsize). \tag{5.16}$$

That is, $n = setsize$ is the only predecessor of 1 for all $setsize > 3$, which implies 3 is not a predecessor of 1 for all $setsize > 3$.

$$\forall\ setsize > 3: \quad \neg Adjacent(1,3,setsize) \leftarrow \neg Successor(1,3,setsize) \wedge \neg Predecessor(1,3,setsize). \quad \square \tag{5.17}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (violates the symmetry property).

6. Summary

Applying some very simple real analysis, in the form of the ruler measure (2.1) and ruler convergence proof (2.2), to a set of real-valued domain intervals and a range interval yields some new insights into geometry and physics.

(1) Discrete, relations between countable sets converge to the continuous, bijective relations: triangle inequality, Manhattan distance, Euclidean distance and volume. Other types of measures do not have that capability.
(2) Ruler measure-based proofs expose the difference between distance and volume measures: Distance is a function of the number of mappings between the members of each disjoint domain set and a corresponding range set. In contrast, volume is a function of the many-to-many mapping between the members of disjoint domain sets. Other types of measures, like Borel, Hausdorff, and Lebesgue, do not provide that insight.
(3) Applying the ruler measure to the countable metric space (3.1) provides the insight that all notions of distance are based on the principle that for each disjoint domain set there exists a corresponding range set containing the same number of members, where the range sets in some cases intersect:

(a) Applying the ruler and ruler convergence to the countable metric space principle (3.1) generates the real-valued triangle inequality and non-negativity properties of metric space (3.2) with Manhattan distance at the upper boundary and Euclidean distance at the lower boundary, which provides the insight that a "flat" geometry is one where for each disjoint, countable domain set of same-sized members there exists a corresponding range (distance) set with same number of same-sized members.

(b) All L^p norms ($||\mathbf{x}||_p = (\sum_{i=1}^{n} x_i^p)^{1/p}$), where $p < 1$ and $p > 2$, violate the principle that each range set has the same number of same-sized members as the corresponding domain set. Therefore, all L^p norms, where $p < 1$ and $p > 2$, are not flat distances. All L^p norms, where $p < 1$, are distances in open (hyperbolic) space. All L^p norms, where $p > 2$, are distances in closed (elliptic) space.

(c) The upper bound of the countable metric space converging to Manhattan distance (3.2) provides the insight that the largest (longest) monotonic distance path is the case of disjoint range (distance) sets, where there is a 1-1 correspondence between the domain and range set members. Therefore, the countable Manhattan distance is $d_c = \sum_{i=1}^{n} p_i$, where p_i is the i^{th} number of domain-to-range mappings.

(d) The lower bound of the countable metric space converging to Euclidean distance (3.3) provides the insight that the smallest (shortest) possible monotonic distance path is the case of the intersecting range sets, where the maximum number of domain-to-range set mappings is the many-to-many relation. And the total number of many-to-mapping mappings is $\sum_{i=1}^{n} p_i^2$. From equation 3.11 of the Euclidean distance proof: $d_c^2 = (\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$. The smallest possible distance is the equality case: $d_c^2 = \sum_{i=1}^{n} p_i^2$. And applying the ruler measure and convergence theorem yields the real-valued smallest possible distance relation: $d^2 = \sum_{i=1}^{n} s_i^2$.

(e) Euclidean distance 3.3 was derived from a set-based, many-to-many relation without any notions of side, angle, or shape. A parametric variable relating the sizes of two domain intervals can be easily derived using using calculus (Taylor series) and the Euclidean distance equation to generate the arc sine and arc cosine functions of the parametric variable (arc angle). In other words, the notions of side and angle are derived from Euclidean distance, which is the reverse perspective of classical geometry [**Joy98**] [**Loo68**] [**Ber88**] and axiomatic foundations for geometry [**Bir32**] [**Hil80**] [**TG99**].

(4) Applying the ruler measure and ruler convergence proof to the countable volume definition (4.1) allows a proof that the Cartesian product of same-sized subintervals of real-valued intervals converges to the product of the interval sizes (Euclidean length/area/volume):

(a) Euclidean volume was derived from a combinatorial relation without notions of sides, angles, and shape.

(b) The Lebesgue and Hausdorff measures, Riemann and Lebesgue integration, and vector analysis can only assume Euclidean space ($\mathbb{R}^n$).

(5) The set-based relations of countable metric space (3.1) and countable volume (4.1) that generate metric space, Manhattan distance, Euclidean distance, and volume equations have the properties of total ordering (5.1) and symmetry (5.2). A geometry that is simultaneously both ordered and symmetric limits distance and volume to a cyclic set (5.3) of three dimensions (5.4), which explains why there are only three dimensions of physical space (length, width, and height).

(6) All valid higher dimensional theories of physics must collapse into hierarchical 2 or 3-dimensional geometries, where all domain dimensions at each level in the hierarchy are the same type. The four-vectors common in physics are 2-dimensional geometries that have been "flattened." For example, the spacetime four-vector length, $d = \sqrt{(ct)^2 - (x^2 + y^2 + z^2)}$, where c is the speed of light and t is time, can be expressed in a form like, $d_2 = \sqrt{(ct)^2 - d_1^2}$, where $d_1 = \sqrt{x^2 + y^2 + z^2}$ and $d_2 = d$.

    Applying the Euclidean distance proof (3.3) to the 2-dimensional spacetime equation, $(ct)^2 = d_1^2 + d_2^2$, provides the perspective that d_1 and d_2 are lengths in two frames of reference (the sizes of two domain intervals) and the size of each range subinterval is the same size (same speed of light) in both frames of reference.

    Likewise, the 2-dimensional length of the energy-momentum four-vector is: $E^2 = (mv^2)^2 + (pc)^2$, where E is energy, m is the resting mass, v is the three-velocity, p is the relativistic three-momentum ($p = \gamma mv$, where $\gamma = (1/(1-(v/c)^2))^{1/2}$ is the Lorentz factor), and c is the speed of light. This equation is used in relativistic mechanics, relativistic quantum mechanics, relativistic quantum field theory, and particle physics. The Euclidean distance proof (3.3) provides the perspective that the units of energy (subintervals of the range interval) are the same size in both the Newtonian frame of reference (the domain interval, having size mv^2), and relativistic frame of reference (the domain interval, having size γmvc).

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