A Combinatorial Foundation for Analytic Geometry

George. M. Van Treeck

Abstract. A ruler-based measure of intervals allows a new class of combinatorial proofs providing new insights into both measure theory and analytic geometry. Applying the ruler to the definition of a countable distance range converges to the taxicab distance equation as the upper boundary of the range, the Euclidean distance equation as the lower boundary of the range, and the triangle inequality over the full range. A combinatorial definition of size (length/area/volume) converges to the product of interval sizes (Euclidean volume) used in the Lebesgue measure. Combinatorics limits a geometry having the properties of both order and symmetry to a cyclic set of at most 3 dimensions, which is the basis of the right-hand rule. Implications for non-Euclidean geometries and higher dimensional geometries are discussed. All the proofs are verified in Coq.

Contents

1. Introduction

Definitions like the metric space and Lebesgue measure and equations like Euclidean distance used in real analysis, calculus, and analytic geometry are imported from elementary geometry. Because the definitions and equations are imported rather than derived from set and number theory, the definitions and equations do not provide insight into the counting principles that generate geometry.

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For example, there has been no proof that a counting-based definition of smallest distance converges to the Euclidean distance equation. Without a formal understanding of the counting-based principles that converge to the triangle inequality, taxicab distance, and Euclidean distance, there is no formal understanding of the counting principles that generate the properties of a metric space and counting principles that can generate some elliptic and hyperbolic distances.

Further, there have been no published proof that the Cartesian product of the subintervals in a set of intervals converges to the product of the interval sizes (Euclidean volume). Euclidean integrals and the Lebesgue measure assume the volume equation rather than deriving it from a set-based, combinatorial definition.

The traditional indefinite integral (antiderivative) of calculus is used to prove that a **real-valued, continuous function** relating the **size** of the subintervals of domain intervals to the **size** of the subintervals of an image interval converges to a real-valued function. Whereas, what is needed for counting-based (combinatorial) proofs is an indefinite integral that proves that a **counting-based function** relating the **number** of the subintervals of domain intervals to the **number** of the subintervals in an image interval converges to a real-valued function.

Combinatorial proofs require a different method of dividing intervals into subintervals herein referred to as a ruler. A ruler measures each interval of a set of intervals to the nearest integer number of subintervals, each subinterval having the same size. The ruler is an approximate measure that ignores partial subintervals.

In the traditional method of dividing a set of intervals into subintervals, the number of subintervals is the same in each interval and the size of the subintervals in each interval varies depending on the function. In contrast, for the ruler method of dividing a set of intervals into subintervals, the size of the subintervals in each interval is the same size and the number of subintervals in each interval varies with the size of the interval.

Same-sized subintervals across both the set of domain intervals and image interval allows defining a countable relationship between the domain subintervals and image subintervals. As the subinterval size goes to zero, the combinatorial relationships that define smallest countable distance and countable size (length/area/volume) converge to the n-dimensional Euclidean distance and volume equations.

The purpose of this article is to show that: 1) A defined range of countable (combinatorial) relationships between the sets of subintervals in domain intervals to the set of subintervals of a "distance" interval converges to the triangle inequality, the upper boundary of the range converges to the taxicab distance equation, and the lower boundary of the range converges to the Euclidean distance equation. 2) A combinatorial definition of size (length, area, volume) as a relationship between the countable sets of subintervals in domain intervals and the subintervals of a "size" interval converges to the product of the interval sizes (Euclidean volume). 3) Combinatorics limits a geometry that has the properties of being both ordered and symmetric to a cyclic set of at most three dimensions, which is the basis of the right-hand rule used throughout mathematics, physics, and engineering.

The proofs in this article are verified formally using the Coq Proof Assistant [15] version 8.4p16. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at:
`https://github.com/treeck/CombinatorialGeometry`.

2. Ruler measure and convergence

DEFINITION 2.1. Ruler measure: A ruler measures the size of a closed, open, or semi-open interval as the nearest integer number of whole subintervals, p, times the subinterval size, c, where c is the independent variable. Notionally:

$$\forall\, c\, s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |b-a|\ \wedge\ c > 0\ \wedge \tag{2.1}$$

$$(p = floor(s/c)\ \vee\ p = ceiling(s/c)\ \wedge\ M = \lim_{c \to 0} \sum_{i=1}^{p} c = \lim_{c \to 0} pc.$$

The ruler has the three properties of measure in a σ-algebra:

(1) Non-negativity: $\forall\, E \in \Sigma,\ \mu(E) \geq 0: \quad s = |b-a|\ \wedge\ c > 0\ \Rightarrow$ $M = \lim_{c \to 0} pc \geq 0$.
(2) Zero-sized empty set: $\mu(\emptyset) = 0: \quad b = a\ \Rightarrow M = \lim_{c \to 0} pc = 0$.
(3) Countable additivity: $\forall\, \{E_i\}_{i \in \mathbb{N}},\ |\cap_{i=1}^{\infty} E_i| = \emptyset \wedge \mu(\cup_{i=1}^{\infty} E_i) = \mu(\Sigma_{i=1}^{\infty} E_i)$. $(c \to 0 \Rightarrow p \to \infty)\ \wedge\ \mu(E_i) = c\ \Rightarrow\ \mu(\Sigma_{i=1}^{\infty} E_i) = \Sigma_{p=1}^{\infty} c = \lim_{c \to 0} pc$.

For example showing convergence, using the interval, $[0,\pi]$, $s = |\pi - 0|$, $c = 10^{-i}, i \in \mathbb{N}$, and $p = floor(s/c)$, then, $p \cdot c = 3.1,\ 3.14,\ 3.141, ..., \pi$.

THEOREM 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\ \ s = |b-a|\ \ \Rightarrow\ \ M = \lim_{c \to 0} pc = s.$

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y:\ y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0, \quad p = floor(s/c) \quad \Rightarrow \quad 0 \leq |p - s/c| < 1. \tag{2.2}$$

Multiply all sides by $|c|$:

$$\forall\, c > 0, \quad 0 \leq |p - s/c| < 1 \quad \Rightarrow \quad 0 \leq |pc - s| < |c|. \tag{2.3}$$

$$\forall\, c > 0,\ \exists\, \delta,\ \epsilon\ :\ 0 \leq |pc - s| < |c| = |c - 0| < \delta = \epsilon \tag{2.4}$$

$$\Rightarrow \quad 0 < |c - 0| < \delta \quad \wedge \quad 0 \leq |pc - s| < \epsilon = \delta \quad := \quad M = \lim_{c \to 0} pc = s. \quad \square$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure).

3. Distance

Here, countable distance is the number of elements in an image (distance) set corresponding to an equal number of elements of a domain set. This notion of distance is extended across disjoint (non-intersecting) domain sets, where the distance sets may intersect with each other. This creates a range of possible distances:

DEFINITION 3.1. Countable distance range, d_c:

$$\forall\, i\, j \in [1,n],\ \ x_i \subseteq X,\ \ |\bigcup_{i=1}^{n} x_i| = \sum_{i=1}^{n} |x_i|\ \ \wedge\ \ \forall\, x_i\, \exists\, y_i \subseteq Y:\ \ |x_i| = |y_i|\ \ \wedge$$

$$d_c = |\bigcup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| - |\bigcap_{i=1}^{n} y_i|.$$

In the above definition of countable distance range (the range of distances between two elements), the vertical bars around a set is the standard notation for indicating the cardinal (number of elements in the set). To prevent too much overloading on the vertical bar, the symbol for "such that" is the colon.

To make the formal proof of Euclidean distance easier to understand, the notions of taxicab and Euclidean distance will be discussed using the example of a distance set of equal-valued coins, C, corresponding to a set of apples, A, and a set of bananas, B. To determine the real-valued amount of coin, C_r, as the distance corresponding to any number of apples, $|A|$, and bananas, $|B|$, use the ruler measure (2.1) to divide sets of apples, bananas, and coins into pieces:

$$
\begin{aligned}
(3.1)\quad & \forall\, c\ C_r \in \mathbb{R}, \quad c > 0 \quad \wedge \\
& p_1 = floor(|A|/c) \ \wedge \ p_2 = floor(|B|)/c) \ \wedge \ d_c = p_d = floor(C_r/c) \ \Rightarrow \\
& |\{applePiece_1, applePiece_2, \ldots, applePiece_{p_1}\}| = p_1 \quad \wedge \\
& |\{bananaPiece_1, bananaPiece_2, \ldots, bananaPiece_{p_2}\}| = p_2 \quad \wedge \\
& |\{coinPiece_1, coinPiece_2, \ldots, coinPiece_{p_d}\}| = p_d = d_c.
\end{aligned}
$$

From the countable distance range (3.1), for p_1 number of apple pieces there exists a set of p_1 number of coin pieces. And for p_2 number of banana pieces, there exists a set of p_2 number of coin pieces.

Definition 3.2. countable taxicab (largest spanning) distance measure:

$$
|\bigcap_{i=1}^{n} y_i| = 0 \quad \wedge \quad d_c = \sum_{i=1}^{n} |y_i| - |\bigcap_{i=1}^{n} y_i| \quad \Rightarrow \quad d_c = \sum_{i=1}^{n} |y_i|.
$$

In the taxicab distance case, the number of coin pieces is equal to the number of apple pieces plus banana pieces, $p_d = p_1 + p_2$, because there is zero intersection between the set of coin pieces corresponding to the apple pieces and the set of coin pieces corresponding to banana pieces. Multiplying both sides of the equation, $p_d = p_1 + p_2$, by c and applying the ruler convergence theorem (2.2) yields the real-valued amount of coinage, $C_r = \lim_{c \to 0} p_d \cdot c = \lim_{c \to 0}(p_1 \cdot c) + \lim_{c \to 0}(p_2 \cdot c) = |A| + |B|$.

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance."

Definition 3.3. countable Euclidean (smallest spanning) distance measure:

$$
|\bigcap_{i=1}^{n} y_i| = max\_intersects \quad \Rightarrow \quad d_c = \sum_{i=1}^{n} |y_i| - max\_intersects.
$$

The largest possible number of intersection correspondences, "*max_intersects*," is the case of the smallest (shortest) spanning distance.

Like most inclusion-exclusion principle problems, "*max_intersects*" and d_c can only be derived by indirect counting.

The maximum possible number of intersection correspondences is the set of the maximum possible number of $(applePiece, coinPiece)$ correspondences plus the maximum possible number of $(bananaPiece, coinPiece)$ correspondences. From the definition of a countable distance range (3.1), each of p_1 number of coin pieces can correspond to a maximum of p_1 number of apple pieces, yielding a maximum of $p_1 \times p_1 = p_1^2$ number of possible $(applePiece, coinPiece)$ correspondences, which is also equal to p_1^2 number of $(coinPiece, coinPiece)$ combinations. Likewise, there are a maximum of p_2^2 number of possible $(banana\ Piece, coinPiece)$ correspondences, which is also equal to p_2^2 number of $(coinPiece, coinPiece)$ combinations. Therefore, there are a maximum of $p_1^2 + p_2^2 = |\{(fruitPiece, coinPiece)\}| = |\{(coinPiece, coinPiece)\}|$ combinations.

Multiply both sides by c^2 and apply the ruler convergence theorem (2.2):

$$(p_1 \cdot c)^2 + (p_2 \cdot c)^2 = |\{(coinPiece, coinPiece)\}| \cdot c^2 \quad \wedge$$
$$|A| = \lim_{c \to 0} p_1 \cdot c \quad \wedge \quad |B| = \lim_{c \to 0} p_2 \cdot c$$
$$\Rightarrow \quad |A|^2 + |B|^2 = \lim_{c \to 0}(p_1 \cdot c)^2 + \lim_{c \to 0}(p_2 \cdot c)^2 = \lim_{c \to 0} |\{(coinPiece, coinPiece)\}| \cdot c^2.$$

Equation 3.1 divided the C_r amount of coin into p_d number of pieces. Therefore, there are a maximum possible $p_d^2 = |\{(coinPiece, coinPiece)\}|$ combinations.

Multiply both sides by c^2 and apply the ruler convergence theorem (2.2):

$$(p_d \cdot c)^2 = |\{(coinPiece, coinPiece)\}| \cdot c^2 \quad \wedge \quad C_r = \lim_{c \to 0} p_d \cdot c$$
$$\Rightarrow \quad C_r^2 = \lim_{c \to 0}(p_d \cdot c)^2 = \lim_{c \to 0} |\{(coinPiece, coinPiece)\}| \cdot c^2.$$

$$|A|^2 + |B|^2 = \lim_{c \to 0} |\{(coinPiece, coinPiece)\}| \cdot c^2 \quad \wedge$$
$$C_r^2 = \lim_{c \to 0} |\{(coinPiece, coinPiece)\}| \cdot c^2 \quad \Rightarrow \quad |A|^2 + |B|^2 = C_r^2.$$

THEOREM 3.4. *Euclidean (smallest) distance, d, is the size of the distance interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals, $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_2,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|, \quad i \in [1, n], \quad i, n \in \mathbb{N}.$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals. Next, apply the definition of the countable distance range (3.1) and the rule of product:

$$\begin{aligned} &\forall\, i \in [1, n], \quad c > 0 \quad \wedge \quad p_i = floor(s_i/c) \quad \wedge \\ &|\{x_i : x_i \in \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\}| = |\{y_i : y_i \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_i}\}\}| = p_i \quad \Rightarrow \\ &\forall\, i \in [1, n], \quad |\{(x_i, y_i)\}| = |\{(y_i, y_i)\}| = p_i^2. \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\forall\, i \in [1, n], \quad |\{(y_i, y_i)\}| = p_i^2 \quad \wedge \quad y \in \{y_1, \cdots, y_n\} \quad \Rightarrow \\ &|\sum_{i=1}^{n} \{(y_i, y_i)\}| = \sum_{i=1}^{n} p_i^2 = |\{(y, y)\}|. \end{aligned} \tag{3.3}$$

Multiply both sides of 3.3 by c^2 and apply the ruler convergence theorem (2.2):

$$\begin{aligned} &s_i = \lim_{c \to 0} p_i \cdot c \quad \wedge \quad \sum_{i=1}^{n} (p_i \cdot c)^2 = |\{(y, y)\}| \cdot c^2 \\ &\Rightarrow \quad \sum_{i=1}^{n} s_i^2 = \sum_{i=1}^{n} \lim_{c \to 0}(p_i \cdot c)^2 = \lim_{c \to 0} |\{(y, y)\}| \cdot c^2. \end{aligned} \tag{3.4}$$

Use the ruler to divide the exact size, $d = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_d, number of subintervals and apply the rule of product:

$$\forall\, i \in [1,n],\ c > 0\ \wedge\ p_d = floor(d/c)\ \wedge\ p_d = |\{y :\ y \in \{y_{1_i}, y_{2_i}, \ldots, y_{p_d}\}\}| \\ \Rightarrow\quad p_d^2 = |\{(y,y)\}|. \tag{3.5}$$

Multiply both sides of 3.5 by c^2 and apply the ruler convergence theorem (2.2):

$$d = \lim_{c\to 0} p_d \cdot c \quad\wedge\quad (p_d \cdot c)^2 = |\{(y,y)\}| \cdot c^2 \\ \Rightarrow\quad d^2 = \lim_{c\to 0}(p_d \cdot c)^2 = \lim_{c\to 0} |\{(y,y)\}| \cdot c^2. \tag{3.6}$$

Combine equations 3.4 and 3.6:

$$d^2 = \lim_{c\to 0} |\{(y,y)\}| \cdot c^2 \quad\wedge\quad \sum_{i=1}^{n} s_i^2 = \lim_{c\to 0} |\{(y,y)\}| \cdot c^2 \\ \Rightarrow\quad d^2 = \sum_{i=1}^{n} s_i^2. \quad \square \tag{3.7}$$

3.1. Triangle inequality. The definition of a metric in real analysis is based on the triangle inequality, $\mathbf{d}(\mathbf{u},\mathbf{w}) \le \mathbf{d}(\mathbf{u},\mathbf{v}) + \mathbf{d}(\mathbf{v},\mathbf{w})$, that has been intuitively motivated by the triangle [**Gol76**]. Applying the ruler (2.1) and convergence theorem (2.2) to the definition of a countable distance range (3.1) yields the real-valued triangle inequality:

$$d_c = |\bigcup_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| - |\bigcap_{i=1}^{n} y_i| \le \sum_{i=1}^{n} |y_i| \quad\wedge \\ d_c = floor(\mathbf{d}(\mathbf{u},\mathbf{w})/c)\ \wedge\ |y_1| = floor(\mathbf{d}(\mathbf{u},\mathbf{v})/c)\ \wedge\ |y_2| = floor(\mathbf{d}(\mathbf{v},\mathbf{w})/c) \\ \Rightarrow\ \mathbf{d}(\mathbf{u},\mathbf{w}) = \lim_{c\to 0} d_c \cdot c \le \sum_{i=1}^{2} \lim_{c\to 0} |y_i| \cdot c = \mathbf{d}(\mathbf{u},\mathbf{v}) + \mathbf{d}(\mathbf{v},\mathbf{w}). \tag{3.8}$$

4. Size (length/area/volume)

The countable size measure is the number of combinations (correspondences) between members of disjoint domain sets, which is the Cartesian product of the domain set sizes. For example, given $|A|$ number of apples and $|B|$ number of bananas, the size measure is: $|\{(apple, banana)\}| = |A| \times |B|$ number of combinations.

Definition 4.1. countable size (length/area/volume) measure, S_c:

$$\forall\, i \in [1,n],\ x_i \subseteq X, \quad |\bigcup_{i=1}^{n} x_i| = \sum_{i=1}^{n} |x_i| \quad\wedge\quad \{(x_1,\ldots,x_n)\} = y \quad\wedge \\ S_c = |y| = |\{(x_1,\ldots,x_n)\}| = \prod_{i=1}^{n} |x_i|.$$

Theorem 4.2. *Euclidean size (length/area/volume), S, is the size of an image interval, $[y_0, y_m]$, corresponding to a set of disjoint domain intervals:* $\{[x_{0,1}, x_{m_1,1}], [x_{0,2}, x_{m_2,2}], \ldots, [x_{0,n}, x_{m_n,n}]\}$, *where:*

$$S = \prod_{i=1}^{n} s_i, \quad S = |y_m - y_0|, \quad s_i = |x_{m_i,i} - x_{0,i}|, \quad i \in [1,n], \quad i,n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_size."

PROOF.
Use the ruler (2.1) to divide the exact size, $s_i = |x_{m_i,i} - x_{0,i}|$, of each of the domain intervals, $[x_{0,i}, x_{m_i,i}]$, into p_i number of subintervals.

$$\forall\, i \in [1,n],\, c > 0 \wedge p_i = floor(s_i/c) \Rightarrow |\{x_i : x_i \in \{x_{1_i}, x_{2_i}, \ldots, x_{p_i}\}\}| = p_i. \tag{4.1}$$

Use the ruler (2.1) to divide the exact size, $S = |y_m - y_0|$, of the image interval, $[y_0, y_m]$, into p_S^n subintervals, where p_S^n satisfies the definition a countable size measure, S_c.

$$\forall\, c > 0 \quad \wedge \quad \exists\, r \in \mathbb{R},\ S = r^n \quad \wedge \quad p_S = floor(r/c) \quad \wedge \quad p_S^n = S_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i. \tag{4.2}$$

Multiply both sides of equation 4.2 by c^n to get the ruler measures:

$$p_S^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3:

$$\begin{aligned} S = r^n = \lim_{c\to 0} (p_S \cdot c)^n \quad \wedge \quad \lim_{c\to 0} (p_i \cdot c) = s_i \quad \wedge \quad (p_S \cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c) \\ \Rightarrow \quad S = \lim_{c\to 0} (p_S \cdot c)^n = \prod_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) = \prod_{i=1}^{n} s_i. \end{aligned} \tag{4.4}$$ □

5. Derived geometric definitions

5.1. Derived geometric primitives. There are no new mathematics in this section of derived geometric primitives. The purpose of this section is to show a difference in perspective. In classical geometry, Euclidean distance is a product of lines and angles. Here, the perspective is reversed to show that lines and angles are non-primitive relationships generated from the primitive relationship, Euclidean distance.

DEFINITION 5.1. Straight line segment is the smallest (Euclidean) distance interval, $[y_0, y_m]$ (3.4).

DEFINITION 5.2. Straight line segment orientation (slope): $\mathrm{d}b/\mathrm{d}a = b/a$, where $a = x_{m_1,1} - x_{0,1}$ and $b = x_{m_2,2} - x_{0,2}$ are the signed sizes of two domain intervals, $[x_{0,1}, x_{m_1,1}]$ and $[x_{0,2}, x_{m_2,2}]$.

The signed sizes, a and b, of the two domain intervals can be calculated from a single parametric distance, θ, and Euclidean distance, d.

DEFINITION 5.3. Parametric distance (arc angle), θ:

$$b/a = \mathrm{d}b/\mathrm{d}a = \mathrm{d}b/\mathrm{d}\theta \cdot \mathrm{d}\theta/\mathrm{d}a = \sqrt{d^2 - a^2}/\sqrt{d^2 - b^2} \tag{5.1}$$

$$Case: \quad \mathrm{d}b/\mathrm{d}a = b/a = 1 \quad \Rightarrow \quad \mathrm{d}\theta/\mathrm{d}a = 1/\sqrt{d^2 - b^2} = 1/\sqrt{d^2 - a^2} \tag{5.2}$$

Applying Taylor's theorem [**Gol76**] and a table of integrals [**Wc11**]:

$$\int \mathrm{d}\theta = \int \mathrm{d}a/\sqrt{d^2 - a^2} \quad \Rightarrow \quad \theta = \sin^{-1}(a/d) = \cos^{-1}(b/d). \tag{5.3}$$

5.2. Vectors. Before discussing the implications of the proofs in this article on vector analysis for dimensions greater than three, the notions of vector, parallel, and orthogonal are defined here in terms of sets of intervals.

DEFINITION 5.4. Vector: A vector is the ordered set of the signed domain interval sizes, $\mathbf{s} = \{s_1, \ldots, s_n\}$, where $s_i = x_{m_i,i} - x_{0,i}$ for the domain interval, $[x_{0,i}, x_{m_i,i}]$.

DEFINITION 5.5. Parallel (congruent) vectors: Two vectors are parallel if each ratio of the signed sizes in one vector equals the ratio of the corresponding signed sizes in another vector (same rate of change in the same direction):

$$\frac{s_{1_i}}{s_{1_{i+1}}} = \frac{s_{2_i}}{s_{2_{i+1}}}, \quad i \in [1, n-1]. \tag{5.4}$$

DEFINITION 5.6. Orthogonal vectors: Two vectors are orthogonal if each ratio of the signed sizes in one vector is the inverse ratio and inverse sign of two corresponding signed sizes in another vector (inverse rate of change and inverse directions). Simplifying the equation yields the **dot (inner) product** equal to zero for any number of dimensions:

$$\frac{s_{1_i}}{s_{1_{i+1}}} = -\frac{s_{2_{i+1}}}{s_{2_i}}, \quad i \in [1, n-1] \quad \Leftrightarrow \quad \sum_{i=1}^{n} s_{1_i} \cdot s_{2_i} = 0. \tag{5.5}$$

6. Ordered and symmetric geometries

Euclidean size (area/volume) and distance are invariant for every order (permutation) of a set of intervals. A function (like size or distance) where every permutation of the arguments yields the same value(s) is called a symmetric function. Two sets of intervals with the same volume and spanning distance (for example, $\{[0,2],\ [0,1],\ [0,5]\}$ and $\{[0,5],\ [0,2],\ [0,1]\}$) can be distinguished by assigning an order (relative position) to the elements of the sets.

It will now be proved that any geometry, both Euclidean and non-Euclidean, that has both symmetry (every permutation of domain intervals yields the same distance and volume) and order (ability to discriminate distances and volumes by orientation), is a cyclic set limited to at most 3 domain intervals (dimensions), which is the basis for the right-hand rule. The implications with respect to vector operations and higher dimensioned geometries are discussed in the summary.

DEFINITION 6.1. Ordered geometry:

$$\forall\, i\, n \in \mathbb{N},\ \forall\, x_i \in \{x_1, \ldots, x_n\},\ \textit{successor}\ x_i = x_{i+1}\ \wedge\ \textit{predecessor}\ x_{i+1} = x_i.$$

If every element is sequentially adjacent to every other element, then traversing in both successor and predecessor order generates every possible permutation of elements (symmetry). This allows defining symmetry via successor and predecessor to be compatible with the notion of order.

DEFINITION 6.2. Symmetric geometry:

$$\forall\, i\, j\, n \in \mathbb{N},\ \forall\, x_i\, x_j \in \{x_1, \ldots, x_n\},\ \textit{successor}\ x_i = x_j\ \wedge\ \textit{predecessor}\ x_j = x_i.$$

THEOREM 6.3. *An ordered and symmetric geometry is a cyclic set.*

$$\forall\, i\, j\, n \in \mathbb{N},\ \forall\, x_i\, x_j \in \{x_1, \ldots, x_n\},\ i = n\ \wedge\ j = 1$$
$$\Rightarrow \quad \textit{successor}\ x_n = x_1\ \wedge\ \textit{predecessor}\ x_1 = x_n.$$

The theorem and formal Coq-based proof is "ordered_symmetric_is_cyclic," which is located in the file threed.v.

PROOF. The property of order (6.1) defines unique successors and predecessors for all elements except for the successor of x_n and the predecessor of x_1. From the properties of a symmetric geometry (6.2):

$$i = n \ \wedge \ j = 1 \ \wedge \ successor \ x_i = x_j \ \Rightarrow \ successor \ x_n = x_1. \tag{6.1}$$

$$i = n \ \wedge \ j = 1 \ \wedge \ predecessor \ x_j = x_i \ \Rightarrow \ predecessor \ x_1 = x_n. \tag{6.2}$$ □

For example, using the cyclic set with elements labeled, $\{1,\ 2,\ 3\}$, starting with each element and counting through 3 cyclic successors and counting through 3 cyclic predecessors yields all possible permutations: $(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(1, 3, 2)$, $(3, 2, 1)$, and $(2, 1, 3)$. That is, a cyclically ordered set preserves sequential order while allowing a set of n-at-a-time permutations. If all possible n-at-a-time permutations are generated, then the cyclic set is also symmetric.

THEOREM 6.4. *An ordered and symmetric geometry is limited to at most 3 elements. That is, each element is sequentially adjacent (a successor or predecessor) to every other element in a set only where the number of elements (set sizes) are less than or equal to 3.*

The Coq-based lemmas and proofs in the file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn-like clauses (a subset of first-order logic) with unification and resolution. Horn clauses make it clear which facts satisfy a goal.

PROOF.
Because an ordered and symmetric set is a cyclic set (6.3), the successors and predecessors are cyclic:

DEFINITION 6.5. Successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (m + 1 \leq setsize). \tag{6.3}$$

DEFINITION 6.6. Predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (m - 1 \geq 1). \tag{6.4}$$

DEFINITION 6.7. Adjacent: element m is adjacent to element n (an allowed permutation), if the cyclic successor of m is n or the cyclic predccessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{6.5}$$

Every element is adjacent to every other element, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{6.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (1 + 1 \leq 2). \tag{6.7}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (2 = 2 \wedge 1 = 1). \tag{6.8}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (1+1 \leq 2). \tag{6.9}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (2-1 \geq 1). \tag{6.10}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (3=3 \wedge 1=1). \tag{6.11}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (1=1 \wedge 3=3). \tag{6.12}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (2+1 \leq 3). \tag{6.13}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (3-1 \geq 1). \tag{6.14}$$

For all $n = setsize > 3$, there exist non-adjacent elements (not every permutation allowed):

$$\forall\, n > 3,\ Successor(1,2,n) \Rightarrow \forall\, n > 3,\ \neg Successor(1,3,n). \tag{6.15}$$

That is, 2 is the only successor of 1 for all $n > 3$, which implies 3 is not a successor of 1 for all $n > 3$.

$$\forall\, n > 3,\ Predecessor(1,n,n) \Rightarrow \forall\, n > 3,\ \neg Predecessor(1,3,n). \tag{6.16}$$

That is, n is the only predecessor of 1 for all $n > 3$, which implies 3 is not a predecessor of n for all $n > 3$.

$$\forall\, n > 3,\ \neg Adjacent(1,3,n) \leftarrow \neg Successor(1,3,n) \wedge \neg Predecessor(1,3,n). \tag{6.17}$$

□

7. Summary

A ruler-based measure of intervals is an analytic tool allowing combinatorial proofs that provides new insights into geometry:

(1) Combinatorial relations between the elements of sets converge to the Euclidean distance (3.4) and size (length/area/volume) (4.2) equations without notions of side, angle, and shape, and without motivation from diagrams.
(2) Taxicab distance (3.2), Euclidean distance (3.3), and the triangle inequality are derived from the definition of the countable distance range (3.1), where taxicab distance is the largest possible monotonic distance and Euclidean distance is the smallest distance.
(3) Because Euclidean distance relies on the number of distance set elements being equal to the number of elements in a domain set, elliptic distance would require fewer distance set elements and hyperbolic distance would require more distance set elements than in a domain set.
(4) Combinatorics limits a geometry having the properties of both order and symmetry is a cyclic set (6.3) of at most three elements (dimensions) (6.4), which is the basis of the right-hand rule.
(5) Vector orthogonality (inner product equal to zero) is valid for any number of dimensions (5.6) and only has the property of order. However, geometric orthogonality (perpendicular) requires the additional property of symmetry (6.2), which is limited to at most three dimensions (6.4).

The vector cross product and curl operations are based on the right-hand rule (a cyclic set of three dimensions) and can not be extended beyond three dimensions without losing either geometric order (orientation) or symmetry.

A means to preserve both geometric order and symmetry in a higher dimensioned geometry (for example, Clifford geometry) is to have a vector of three "root" dimensions of space, where size and distance in the three root dimensions are a function of other variables in separate vectors, forming a hierarchy of dimensions.

A cyclic set is a closed walk. An observer in the closed walk would only be able to detect higher dimensions indirectly via changes in the three closed walk dimensions (what physicists call "work").

Displaying higher dimensional manifolds in Euclidean coordinate diagrams (for example three dimensional Cartesian coordinates and spherical coordinates) is probably only meaningful for the case where three of the modeled dimensions are both geometrically ordered (6.2) and geometrically symmetric (6.2).

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George Van Treeck, 668 Westline Dr., Alameda, CA 94501