The Two Set Relations Generating Geometry

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ABSTRACT. Where each countable domain set has a corresponding range set, countable distance is defined as the cardinal of the union of the range sets and countable volume as the cardinal of the set of n-tuples of members from the disjoint range sets. The countable distance and volume set operations applied to sets of size c subintervals of domain and range intervals generate the properties of metric space, all Lp norms (for example, Manhattan and Euclidean distance), and the volume equation as c goes to 0. The volume proof is used to derive Coulomb's charge force and Newton's gravity force equations without using other laws of physics or Gauss's divergence theorem. A symmetry constraint on a totally ordered set limits the set to at most 3 members, for example, 3 dimensions. All proofs are verified in Coq.

CONTENTS

1. Introduction

Metric space, Euclidean distance/norm, and Euclidean area/volume have been definitions in mathematical analysis [**Gol76**] [**Rud76**] motivated by Euclidean geometry [**Joy98**], rather than derived from an abstract set and limit-based foundation. A consequence of no set and limit-based foundation is that analysis has not been able to identify: the constraint between domain and range sets that generates flat space; the domain-to-range set relation that makes Euclidean distance the smallest possible distance between two distinct points in flat space; the set relations generating the metric space properties, etc.

Where each countable domain set has a corresponding range set, countable distance is defined as the cardinal of the union of the range sets and countable volume as the cardinal of the set of n-tuples of members from the disjoint range sets. The countable distance and volume set operations applied to sets of size c subintervals of domain and range intervals generate the properties of metric space, all L_p norms (Minkowski distances, for example, Manhattan and Euclidean distance), and the volume equation as $c \to 0$.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

A ruler (measuring stick) measures the size, M, of an interval, $[a, b]$, *approximately* as the sum of the nearest integer number (floor or ceiling) of whole subintervals, p, each subinterval having the same size, c. Although the ruler is equivalent to an integral in $\mathbb{R}$, $\lim_{c \to 0} pc = b - a = \int_a^b dc$, the ruler format, $\lim_{c \to 0} pc$, makes it easier to derive geometric relations from the relative number of size c subintervals in each interval.

DEFINITION 2.1. Ruler measure: $\forall\, c,\ s \in \mathbb{R},\ [a, b] \subset \mathbb{R},\ s = b - a\ \wedge\ c > 0\ \wedge$ $(p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \sum_{i=1}^{p} c = pc.$

THEOREM 2.2. *Ruler convergence:* $M = \lim_{c \to 0} pc = s.$

The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

(2.1) $\forall\, c > 0,\ p = floor(s/c)\ \wedge\ 0 \leq |floor(s/c) - s/c| < 1\ \Rightarrow\ 0 \leq |p - s/c| < 1.$

Multiply all sides of inequality 2.1 by c:

(2.2) $\qquad\qquad \forall\, c > 0,\quad 0 \leq |p - s/c| < 1\quad \Rightarrow\quad 0 \leq |pc - s| < |c|.$

(2.3) $\quad \forall\, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta$
$\qquad\qquad \Rightarrow\quad \forall\, \epsilon = \delta\ :\ |c - 0| < \delta\ \wedge\ |pc - s| < \epsilon\ :=\ M = \lim_{c \to 0} pc = s.\ \ \square$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}},\ 3.14_{c=10^{-2}},\ 3.141_{c=10^{-3}}, ..., \pi_{\lim_{c \to 0}}.$

3. Distance

Notation convention: Vertical bars around a set or list, $|\cdots|$, indicates the cardinal (number of members in the set or list).

3.1. Countable distance. A concrete example of a countable distance is the number of same-sized steps walked in a range (distance) set, y, which equals the number of same-sized pieces of land in a corresponding domain set, x: $|x| = |y|$. Generalizing, each disjoint domain set, x_i, has a corresponding range (distance) set, y_i. The countable distance spanning the disjoint domain sets is the number of members, d_c, in the union range set:

DEFINITION 3.1. Countable distance, d_c :
$$d_c = |\textstyle\bigcup_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i|.$$

It will be shown in the next subsections that the constraint, $|x_i| = |y_i|$, generates Manhattan and Euclidean distance at the boundaries (generates flat space). Extending distance beyond flat space is shown in the last section of this article.

3.2. Inclusion-Exclusion Inequality. The well-known inequality, $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$, is used often in this article. Therefore, to be thorough, a trivial proof is included here. The inequality follows directly from the inclusion-exclusion principle [**CG15**]. But, a more intuitive and simple proof follows from the associative law of addition where the sum of set sizes is equal to the size of all the set members appended into a list and the commutative law of addition that allows sorting that list into a list of unique members (the *union* set) and a list of duplicates. For example, for $y_1 = \{a, b, c\}$ and $y_2 = \{c, d, e\}$: $\sum_{i=1}^{2} |y_i| = 6 = |[a, b, c, c, d, e]| = |\{a, b, c, d, e\}| + |[c]|$. The duplicates being ≥ 0 implies the union size is always $\leq$ the sum of set sizes.

LEMMA 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$.

PROOF. A formal proof, inclusion_exclusion_inequality, using sorting into a set of unique members (*union* set) and a list of duplicates, is in the file euclidrelations.v.

$$(3.1) \quad \sum_{i=1}^{n} |y_i| = |append_{i=1}^{n} y_i| = |sort(append_{i=1}^{n} y_i)|$$
$$= |\textstyle\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i|.$$

$$(3.2) \quad |\textstyle\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| \quad \wedge \quad |duplicates_{i=1}^{n} y_i| \geq 0$$
$$\Rightarrow \quad |\textstyle\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|. \quad \square$$

3.3. Countable distance range. From the countable distance definition (3.1), $d_c = |\bigcup_{i=1}^{n} y_i|$, as the amount of intersection increases, more domain set members can map to a single range set member. Therefore, the countable distance, d_c, is a function of the number of domain-to-range set member mappings.

From the countable distance constraint (3.1), where $|x_i| = |y_i| = p_i$: 1) The equality case of the inclusion-exclusion inequality (3.2), $d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$, is the case of no intersection, where there is a 1-1 domain-to-range set correspondence, $d_c = \sum_{i=1}^{n}(1 \cdot |y_i|) = \sum_{i=1}^{n} p_i$ number of mappings; 2) Each domain set member can map to as many as all range set members (the Cartesian product), $\sum_{i=1}^{n}(|y_i| \cdot |x_i|) = \sum_{i=1}^{n} p_i^2$ number of mappings and $\exists\, f : d_c = f(\sum_{i=1}^{n} p_i^2)$.

Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of domain-to-range set mapping cases converges to the real-valued Manhattan and Euclidean distance relations.

3.4. Manhattan distance.

THEOREM 3.3. *Manhattan (largest) distance, d, is the size of the range interval,* $[d_0, d_m]$, *corresponding to a set of disjoint domain intervals,* $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*
$$d = \sum_{i=1}^{n} s_i, \quad d = d_m - d_0, \quad s_i = b_i - a_i.$$

The theorem, "taxicab_distance," and formal proof are in the Coq file, euclidrelations.v.

PROOF.

From the countable distance definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$(3.3) \quad d_c = |\textstyle\bigcup_{i=1}^n y_i| \leq \textstyle\sum_{i=1}^n |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \quad \Rightarrow \quad d_c \leq \textstyle\sum_{i=1}^n p_i$$
$$\Rightarrow \quad \exists\, p_i,\, d_c :\; d_c = \textstyle\sum_{i=1}^n p_i.$$

Multiply both sides of equation 3.3 by c and take the limit:

$$(3.4) \quad d_c = \textstyle\sum_{i=1}^n p_i \Rightarrow d_c \cdot c = \textstyle\sum_{i=1}^n (p_i \cdot c) \Rightarrow \lim_{c \to 0} d_c \cdot c = \textstyle\sum_{i=1}^n \lim_{c \to 0} (p_i \cdot c).$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$(3.5) \qquad d = d_m - d_0 \quad \Rightarrow \quad \exists\, c\, d :\; floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$(3.6) \quad s_i = b_i - a_i \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c \to 0} p_i \cdot c = s_i.$$

Combine equations 3.5, 3.4, 3.6:

$$(3.7) \quad d = \lim_{c \to 0} d_c \cdot c \quad \wedge \quad \lim_{c \to 0} d_c \cdot c = \textstyle\sum_{i=1}^n \lim_{c \to 0} (p_i \cdot c) \quad \wedge$$
$$\lim_{c \to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c = \textstyle\sum_{i=1}^n \lim_{c \to 0}(p_i \cdot c) = \textstyle\sum_{i=1}^n s_i. \quad \square$$

3.5. Euclidean distance.

THEOREM 3.4. *Euclidean (smallest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:
$$d^2 = \textstyle\sum_{i=1}^n s_i^2, \quad d = d_m - d_0, \quad s_i = b_i - a_i.$$

The theorem, "Euclidean_distance," and formal proof are in the Coq file, euclidrelations.v.

PROOF.

Apply the rule of product to the largest number of domain-to-range set mappings, where all p_i number of range set members, y_i, map to each of the p_i number of members in the domain set, x_i, which is the Cartesian product, $|y_i| \cdot |x_i|$:

$$(3.8) \qquad |x_i| = |y_i| = p_i \quad \Rightarrow \quad \textstyle\sum_{i=1}^n |y_i| \cdot |x_i| = \textstyle\sum_{i=1}^n p_i^2.$$

From the countable distance definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$(3.9) \quad d_c = |\textstyle\bigcup_{i=1}^n y_i| \leq \textstyle\sum_{i=1}^n |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \quad \Rightarrow \quad d_c \leq \textstyle\sum_{i=1}^n p_i$$
$$\Rightarrow \quad \exists\, p_i,\, d_c :\; d_c = \textstyle\sum_{i=1}^n p_i.$$

Square both sides of equation 3.9 ($x = y \Leftrightarrow f(x) = f(y)$):

$$(3.10) \qquad \exists\, p_i,\, d_c :\; d_c = \textstyle\sum_{i=1}^n p_i \quad \Leftrightarrow \quad \exists\, p_i,\, d_c :\; d_c^2 = (\textstyle\sum_{i=1}^n p_i)^2.$$

Apply the square of sum inequality, $(\sum_{i=1}^n p_i)^2 \geq \sum_{i=1}^n p_i^2$, to equation 3.10 and select the smallest area (the equality) case:

$$(3.11) \quad d_c^2 = (\textstyle\sum_{i=1}^n p_i)^2 = \textstyle\sum_{i=1}^n p_i \textstyle\sum_{j=1}^n p_j$$
$$= \textstyle\sum_{i=1}^n p_i^2 + \textstyle\sum_{i=1}^n p_i \textstyle\sum_{j=1, j \neq i}^n p_j \geq \textstyle\sum_{i=1}^n p_i^2 \quad \Rightarrow \quad \exists\, p_i :\; d_c^2 = \textstyle\sum_{i=1}^n p_i^2.$$

Multiply both sides of equation 3.11 by c^2, simplify, and take the limit.

(3.12) $\quad d_c^2 = \sum_{i=1}^{n} p_i^2 \quad \Rightarrow \quad d_c^2 \cdot c^2 = \sum_{i=1}^{n} p_i^2 \cdot c^2 \quad \Leftrightarrow \quad (d_c \cdot c)^2 = \sum_{i=1}^{n} (p_i \cdot c)^2$

$\Rightarrow \quad \lim_{c \to 0} (d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c)^2.$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

(3.13) $\quad \exists \, c \, d \in \mathbb{R} : \; floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c \to 0} (d_c \cdot c)^2.$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

(3.14) $\quad s_i = b_i - a_i \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c \to 0} p_i \cdot c = s_i.$

Combine equations 3.13, 3.12, 3.14:

(3.15) $\quad d^2 = \lim_{c \to 0} (d_c \cdot c)^2 \quad \wedge \quad \lim_{c \to 0} (d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c)^2 \quad \wedge$

$\lim_{c \to 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d^2 = \lim_{c \to 0} (d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \quad \square$

3.6. Metric Space. All function range intervals, $d(u, w)$, satisfying the countable distance definition (3.1), where the ruler is applied, generates the properties of metric space. The formal proofs: triangle_inequality, non_negativity, identity_of_indiscernibles, and symmetry are in the Coq file, euclidrelations.v.

THEOREM 3.5. *Triangle Inequality:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \le d(u, v) + d(v, w).$

PROOF. Apply the ruler measure (2.1), the countable distance condition (3.1), inclusion-exclusion inequality (3.2), and then ruler convergence (2.2).

(3.16) $\quad \forall \, c > 0, \; d(u, w), \; d(u, v), \; d(v, w) :$

$|y_1| = floor(d(u, v)/c) \quad \wedge \quad |y_2| = floor(d(v, w)/c) \quad \wedge$

$d_c = floor(d(u, w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \le |y_1| + |y_2|$

$\Rightarrow \quad floor(d(u, w)/c) \le floor(d(u, v)/c) + floor(d(v, w)/c)$

$\Rightarrow \quad floor(d(u, w)/c) \cdot c \le floor(d(u, v)/c) \cdot c + floor(d(v, w)/c) \cdot c$

$\Rightarrow \quad \lim_{c \to 0} floor(d(u, w)/c) \cdot c \le \lim_{c \to 0} floor(d(u, v)/c) \cdot c + \lim_{c \to 0} floor(d(v, w)/c) \cdot c$

$\Rightarrow \quad d(u, w) \le d(u, v) + d(v, w). \quad \square$

THEOREM 3.6. *Non-negativity:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \ge 0.$

PROOF. By definition, a set always has a size (cardinal) ≥ 0:

(3.17) $\quad \forall \, c > 0, \; d(u, w) : \quad floor(d(u, w)/c) = d_c \quad \wedge \quad d_c = |y_1 \cup y_2| \ge 0$

$\Rightarrow \quad floor(d(u, w)/c) = d_c \ge 0 \quad \Rightarrow \quad d(u, w) = \lim_{c \to 0} d_c \cdot c \ge 0. \quad \square$

THEOREM 3.7. *Identity of Indiscernibles:* $d(w, w) = 0.$

PROOF. Apply the triangle inequality property (3.5):

(3.18) $\quad \forall \, d(u, v) = d(v, w) = 0 \; \wedge \; d(u, w) \le d(u, v) + d(v, w) \; \Rightarrow \; d(u, w) \le 0.$

Combine the non-negativity property (3.6) and the previous inequality (3.18):

(3.19) $\quad d(u, w) \ge 0 \; \wedge \; d(u, w) \le 0 \; \Leftrightarrow \; 0 \le d(u, w) \le 0 \; \Rightarrow \; d(u, w) = 0.$

Combine the result of step 3.19 and the condition, $d(u, v) = 0$, in step 3.18.

(3.20) $\quad\quad\quad\quad d(u, w) = 0 \; \wedge \; d(u, v) = 0 \; \Rightarrow \; w = v.$

Combine the condition, $d(v, w) = 0$, in step 3.18 and the result of step 3.20.

(3.21) $\quad\quad\quad\quad d(v, w) = 0 \; \wedge \; w = v \; \Rightarrow \; d(w, w) = 0. \quad\quad\quad \square$

THEOREM 3.8. *Symmetry:* $|x_i| = |y_i| \Rightarrow d(u, v) = d(v, u)$.

PROOF. The range of countable distances (3.3) is a function of domain-to-range set members, under the constraint, $|x_i| = |y_i|$:

$$(3.22) \quad |x_i| = |y_i| = p_i \quad \Rightarrow \quad d_c = \text{f}(\textstyle\sum_{i=1}^{n} |x_i| \cdot |y_i|^q) = \text{f}(\textstyle\sum_{i=1}^{n} p_i^{1+q}), \ 0 \le q \le 1.$$

From the Manhattan distance (3.3), where $d_c = \text{f}(\sum_{i=1}^{n} p_i^1) \Rightarrow d(x, y) = \text{f}(x^1 + y^1)$, and Euclidean distance (3.4), where $d_c = \text{f}(\sum_{i=1}^{n} p_i^2) \Rightarrow d(x, y) = \text{f}(x^2 + y^2)$, proofs:

$$(3.23) \qquad d_c = \text{f}(\textstyle\sum_{i=1}^{n} p_i^{1+q})) \quad \Rightarrow \quad \forall \, p \, : \, 1 \le p \le 2, \quad d(x, y) = \text{f}(x^p + y^p).$$

By the commutative law of addition:

$$(3.24) \quad d(u, v) = \text{f}(u^p + v^p) \quad \wedge \quad d(v, u) = \text{f}(v^p + u^p)$$
$$\Rightarrow \quad d(u, v) = \text{f}(u^p + v^p) = \text{f}(v^p + u^p) = d(v, u). \quad \square$$

4. Euclidean Volume

The Lebesgue area/volume measure, Riemann integral, and Lebesgue integral define (assume) area/volume to be the product of disjoint, domain interval lengths. The hand-waving argument for the volume equation has been the Cartesian product generated n-tuples of points/coordinates. The goal, here, is to *prove* the validity of that argument.

Manhattan distance is the largest distance and Euclidean volume the largest volume in flat space (where $|x_i| = |y_i|$) because both are the case of disjoint range sets, $\bigcap_{i=1}^{n} y_i = \emptyset$.

$$d_c = |\textstyle\bigcup_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset.$$

DEFINITION 4.1. Countable Volume, v_c, is the cardinal of the set of n-tuples of members from countable, disjoint, range sets:

$$v_c = |\textstyle\times_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset.$$

THEOREM 4.2. *Euclidean volume, v, is length of the range interval,* $[v_0, v_m]$, *equal to product of domain interval lengths,* $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$:

$$v = \textstyle\prod_{i=1}^{n} s_i, \ \ v = v_m - v_0, \ \ s_i = b_i - a_i.$$

The theorem, "Euclidean_volume," and formal proof are in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of subintervals.

$$(4.1) \qquad \forall \, i \, n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i| = |y_i|.$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$(4.2) \qquad\qquad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c \to 0}(p_i \cdot c) = s_i.$$

Apply the associative law of multiplication to derive the countable volume (4.1) in terms of p_i:

$$(4.3) \qquad v_c = |\textstyle\times_{i=1}^{n} y_i| = \prod_{i=1}^{n} |y_i| \quad \wedge \quad |y_i| = p_i \quad \Rightarrow \quad v_c = \prod_{i=1}^{n} p_i.$$

Multiply both sides of equation 4.3 by c^n:

$$(4.4) \qquad\qquad v_c \cdot c^n = (\textstyle\prod_{i=1}^{n} p_i) \cdot c^n = \prod_{i=1}^{n}(p_i \cdot c).$$

Use those cases, where v_c has an integer n^{th} root.

(4.5) $\quad \forall\, n,\, p,\, v_c\, \in \mathbb{N}\, :\, p^n = v_c\, \Rightarrow\, v_c \cdot c^n = p^n \cdot c^n = (p \cdot c)^n = \prod_{i=1}^{n}(p_i \cdot c).$

Apply the ruler (2.1) and ruler convergence (2.2) to the range interval, $[v_0, v_m]$ (where $v = v_m - v_0$), and then combine with equations 4.5 and 4.2:

(4.6) $\ floor(v/c^n) = p^n \Rightarrow v = \lim_{c \to 0}(p \cdot c)^n = \prod_{i=1}^{n} \lim_{c \to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i.$ $\quad\square$

5. Applications to physics

5.1. Coulomb's charge force. The sizes, q_1 and q_2, of two charges are independent domain variables, where each infinitesimal size c component of a charge exerts an infinitesimal force, $m_C a_C$, on each infinitesimal size c component of the other charge. The total force, F, is proportionate to the total number of forces (the Cartesian product of the infinitesimal size c components) multiplied times the infinitesimal charge force, $m_C a_C$. Applying the ruler, $p_1 = floor(q_1/c)$ and $p_2 = floor(q_2/c)$, and the Cartesian product, $p_1 \times p_2$, of size c components yields:

(5.1) $\quad F \propto m_C a_C(\lim_{c \to 0} p_1 c \cdot \lim_{c \to 0} p_2 c) = m_C a_C \int_0^{q_2} \int_0^{q_1} \mathrm{d}^2 c = m_C a_C(q_1 q_2).$

From equation 5.1, a change in charge, q, causes a proportionate change in force, F. Solving for the infinitesimal force, $F = m_C a_C$, requires a proportionate variable, r, to offset the effect of a change in q. $r \propto q \Rightarrow \exists\, q_C,\, r_C \in \mathbb{R}\, :\, r(q_C/r_C) = q$, where q_C/r_C is a unit-factoring conversion ratio:

(5.2) $\ \forall\, q_1, q_2 \geq 0\, \exists\, q \in \mathbb{R}\, :\, q^2 = q_1 q_2\ \wedge\ r(q_C/r_C) = q\ \Rightarrow\ (r(q_C/r_C))^2 = q_1 q_2.$

(5.3) $\quad (r(q_C/r_C))^2 = q_1 q_2\ \wedge\ F \propto m_C a_C(q_1 q_2)$

$\Rightarrow \quad F \propto m_C a_C(r(q_C/r_C))^2 = m_C a_C(q_1 q_2)$

$\Rightarrow \quad F = m_C a_C = (m_C a_C r_C^2/q_C^2)q_1 q_2/r^2 = k_c q_1 q_2/r^2.$

where $k_C = m_C a_C r_C^2/q_C^2$ corresponds to the SI units: $Nm^2 C^{-2}$.

5.2. Newton's gravity force equation. The sizes, m_1 and m_2, of two masses are independent domain variables, where each infinitesimal size c component of a mass exerts an infinitesimal force, $m_G a_G$, on each infinitesimal size c component of the other mass. The total force, F, is proportionate to the total number of forces (the Cartesian product of the size c components) multiplied times the infinitesimal gravity force, $m_G a_G$. Applying the ruler, $p_1 = floor(m_1/c)$ and $p_2 = floor(m_2/c)$, and the Cartesian product, $p_1 \times p_2$, of size c components yields:

(5.4) $\ F \propto m_G a_G(\lim_{c \to 0} p_1 c \cdot \lim_{c \to 0} p_2 c) = m_G a_G \int_0^{m_2} \int_0^{m_1} \mathrm{d}^2 c = m_G a_G(m_1 m_2).$

From equation 5.4, an change in mass, m, causes a proportionate change in force, F. Solving for the infinitesimal force, $F = m_G a_G$, requires a proportionate variable, r, to offset the effect of a change in m. $r \propto m \Rightarrow \exists\, m_G,\, r_G \in \mathbb{R}\, :\, r(m_G/r_G) = m$, where m_G/r_G is a unit-factoring conversion ratio:

(5.5) $\quad \forall\, m_1, m_2 \geq 0\, \exists\, m \in \mathbb{R}\, :\quad m^2 = m_1 m_2\quad \wedge\quad r(m_G/r_G) = m$

$\Rightarrow \quad (r(m_G/r_G))^2 = m_1 m_2.$

(5.6) $(r(m_G/r_G))^2 = m_1m_2 \ \wedge \ F \propto m_Ga_G(m_1m_2)$
$$\Rightarrow \quad F \propto m_Ga_G(r(m_G/r_G))^2 = m_Ga_G(m_1m_2)$$
$$\Rightarrow \quad F = m_Ga_G = (a_Gr_G^2/m_G)m_1m_2/r^2.$$

(5.7) $\exists\, t_G \in \mathbb{R}: r_G/t_G^2 = a_G \quad \wedge \quad F = (a_Gr_G^2/m_G)m_1m_2/r^2$
$$\Rightarrow \quad F = (r_G^3/m_Gt_G^2)m_1m_2/r^2 = Gm_1m_2/r^2,$$

where $G = r_G^3/m_Gt_G^2$ corresponds to the SI units: $m^3kg^{-1}s^{-2}$.

5.3. Spacetime equation. The charge (5.3) and gravity (5.7) force equations were derived from the principle that charge and mass are proportionate to distance: $r = (r_C/q_C)q = (r_G/m_G)m$. If time is also proportionate to distance, then $r = (r_c/t_c)t = ct$, where the unit-factoring conversion ratio, $r_c/t_c = c$, is some constant speed.

Applying the ruler to two intervals, $[0, d_1]$ and $[0, d_2]$, in two inertial (independent, non-accelerating) frames of reference, the distance (and time) spanning the two domain intervals converges to a range of distances (and times) from Manhattan (3.3) to Euclidean distance (3.4).

(5.8) $r^2 = d_1^2 + d_2^2 \quad \wedge \quad r = (r_c/t_c)t = ct \quad \Rightarrow \quad (ct)^2 = d_1^2 + d_2^2,$

which is a form of the well-known spacetime interval equation (in flat space), where d_1 and d_2 are 3-dimensional Euclidean distances [**Bru17**].

5.4. 3 dimensional balls. The countable distance, $d_c = |\bigcup_{i=1}^n y_i|$, (3.1) and volume, $d_c = |\times_{i=1}^n y_i|$, (4.1) generating geometric relations require being able to assign a sequential order (a total order, 1 through n) to a set of intervals (dimensions). And the commutative properties of union, addition, and product allow sequencing through each interval (dimension) in every possible order. But, sequencing via successor and predecessor relations in every possible order through a sequentially ordered set requires each set member to be sequentially adjacent (either a successor or predecessor) to every other member, herein referred to as a symmetric geometry.

It will now be proved that the constraint (coexistence) of symmetry on a sequentially ordered set defines a cyclic set that contains at most 3 members, in this case, 3 dimensions of ordered and symmetric distance and volume. If there are higher dimension of space, then those higher dimensions can not be "reached" by sequencing from one of the 3 lower dimension to one of the higher dimensions.

DEFINITION 5.1. Ordered geometry:

$\forall\, i\, n \in \mathbb{N}, \ i \in [1, n-1], \ \forall\, x_i \in \{x_1, \ldots, x_n\},$
$$successor\ x_i = x_{i+1} \ \wedge \ predecessor\ x_{i+1} = x_i.$$

The case of metric space functions that are not of the form,

DEFINITION 5.2. Symmetric geometry (every set member is sequentially adjacent to every other member):

$\forall\, i\, j\, n \in \mathbb{N}, \ \forall\, x_i\, x_j \in \{x_1, \ldots, x_n\}, \ successor\ x_i = x_j \ \Leftrightarrow \ predecessor\ x_j = x_i.$

THEOREM 5.3. *An ordered and symmetric set is a cyclic set.*

$$i = n \ \wedge \ j = 1 \ \Rightarrow \ successor\ x_n = x_1 \ \wedge \ predecessor\ x_1 = x_n.$$

The theorem, "ordered_symmetric_is_cyclic," and formal proof is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$(5.9) \qquad i = n \ \wedge \ j = 1 \ \wedge \ successor \ x_i = x_j \ \Rightarrow \ successor \ x_n = x_1.$$

Applying the definition of a symmetric geometry (5.2) to conclusion 5.9:

$$(5.10) \ successor \ x_i = x_j \ \Rightarrow \ predecessor \ x_j = x_i \ \Rightarrow \ predecessor \ x_1 = x_n. \qquad \square$$

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

DEFINITION 5.5. Cyclic successor of m is n:

$$(5.11) \ Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize).$$

DEFINITION 5.6. Cyclic predecessor of m is n:

$$(5.12) \quad Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - q \geq 1).$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:
$$(5.13)$$
$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize).$$

Prove that every member is adjacent to every other member, where $setsize \in \{1, 2, 3\}$:

$$(5.14) \qquad Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1).$$

$$(5.15) \qquad Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize).$$

$$(5.16) \qquad Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1).$$

(5.17) $Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m + 1 \leq setsize)$.

(5.18) $Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m - q \geq 1)$.

(5.19) $Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \wedge m = 1)$.

(5.20) $Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \wedge n = setsize)$.

(5.21) $Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m + 1 \leq setsize)$.

(5.22) $Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m - q \geq 1)$.

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

(5.23) $\forall\, setsize > 3:$ $\neg Successor(1,3, setsize > 3)$
$$\leftarrow Successor(1,2, setsize > 3) \leftarrow (n = m + 1 \leq setsize).$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

(5.24) $\forall\, setsize > 3:$ $\neg Predecessor(1,3, setsize > 3)$
$$\leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

(5.25) $\forall\, setsize > 3:$ $\neg Adjacent(1,3, setsize > 3)$
$$\leftarrow \neg Successor(1,3, setsize > 3) \wedge \neg Predecessor(1,3, setsize > 3). \quad \square$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

(1) The countable distance (3.1), d_c, is a function of the domain-to-range set mappings, where the constraint, $|x_i| = |y_i| = p_i$, allows a range of domain-to-range set mappings from Manhattan distance, $d_c = \sum_{i=1}^{n} 1 \cdot |y_i| = \sum_{i=1}^{n} p_i$ (3.3) to Euclidean distance, $d_c = f(\sum_{i=1}^{n} |x_i| \cdot |y_i|) = f(\sum_{i=1}^{n} p_i^2)$ (3.4). The case where both domain-to-range set mappings, $\sum_{i=1}^{n} p_i$ and $\sum_{i=1}^{n} p_i^2$, coexist is: $d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c^2 = (\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$. The equality case is where the smallest distance, $d_c^2 = \sum_{i=1}^{n} p_i^2$, coexists with the largest (Manhattan) distance, $d_c = \sum_{i=1}^{n} p_i$, in flat space ($|x_i| = |y_i|$), which is the set-based reason Euclidean distance (3.4) is the smallest possible distance between two distinct points in $\mathbb{R}^n$.

(2) Generalizing the countable distance and volume constraint, $|x_i| = |y_i|$, to $|x_i| = |y_i|^q$, $q > -1$, generates all the L^p norms (Minkowski distances), $||L||_p = (\sum_{i=1}^{n} s_i^p)^{1/p}$. For example, using the same proof pattern as for Euclidean distance (3.4): $|y_i| = p_i \Rightarrow |x_i| = p_i^q \Rightarrow \sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^q \cdot p_i = \sum_{i=1}^{n} p_i^{q+1} \leq d_c^{q+1} \ldots d = (\sum_{i=1}^{n} s_i^{q+1})^{1/(q+1)}$.

(3) The countable distance constraint (3.1) generates Minkowski distances of the form, $d(x,y) = f(x^p + y^p)$. Clearly there are metrics that are not Minkowski distances (for example, the absolute value metric, $d(x,y) = |x - y|$), which raises the question of whether the definition of metric space is sufficiently restrictive to be useful for applied mathematics.

(4) There are combinatorial relationships between countable sets of subintervals of intervals in statistics, probability, physics, etc., where the ruler is an applicable tool. For example, applying the ruler (2.1) and ruler convergence (2.2) to the Cartesian product of same-sized, infinitesimal charge forces and mass forces allowed deriving Coulomb's charge force (5.1) and Newton's gravity force (5.4) equations in a few steps each, without using other laws of physics or Gauss's divergence theorem.

(5) **The Proportionate Interval Principle:** The derivations of the charge force, gravity force, and spacetime equations shows that all Euclidean distance intervals having a size, r, have proportionately sized intervals of other types: $r = (r_C/q_C)q = (r_G/m_G)m = (r_c/t_c)t$, where the conversion ratios are for unit-factor analysis.

   (a) The derivations of charge and gravity forces requiring the conversion ratios, $q = (q_C/r_C)r$ and $m = (m_G/r_G)r$, implies that if there are quantum values of charge, q_C, and mass, m_G, then there are quantum distances, r_C and r_G, where the charge and gravity forces do not exist (are not defined) at smaller distances, which agrees with the theories of the Planck length, l_P, and the Schwarzschild radius, r_s.

   (b) The charge and gravity force derivations show that the proportionate interval principal generates the inverse square law, where rectangular geometric area (r^2) maps to rectangular charge area $(q_1 q_2)$ and mass area $(m_1 m_2)$. But, some versions of the charge constant, vacuum magnetic permeability constant, fine structure constant, etc. contain the value 4π because the creators assumed geometric dilution (flux divergence on the surface of a sphere, $4\pi r^2$). Using Occam's razor, the proportionate interval principle is a more parsimonious derivation of the inverse square law than flux divergence. Therefore, those versions of the constants containing the value 4π might be incorrect.

   (c) Time proportionate to Euclidean distance, $r = ct \Rightarrow r^2 = (ct)^2 = d_1^2 + d_2^2$, (5.8) is an alternative to the four-vector inner product as the motivation for the spacetime interval equation.

   (d) A countable set of values has measure 0. And because 0 times any distance is 0, there is no proportion relationship of a countable set of values to distance. Therefore, a countable set of state value changes with respect to time are independent of distance. For example, the change in the spin values of two quantum coupled particles and the change in polarization of two quantum coupled photons are independent of the distance between the coupled particles.

(6) The proof that ordered set where every member is sequentially adjacent is a cyclic set of a most three members (5.4) provides a more parsimonious explanation of higher dimensions not being sequentially reachable than the higher dimensions being rolled up into infinitesimal balls. This is analogous to being locked in a box, where you can not seeing, hearing, or feeling any boxes that might be outside your box.

(7) If there are higher dimensions of space, then there is most likely an ordered and symmetric set of three members, each member being an ordered and symmetric set of dimensions (three boxes), yielding a total of 9 dimensions.

(8) The proof of at most 3 members in any ordered and symmetric set (5.4) has implications beyond only 3 ordered and symmetric dimensions of space.

   (a) Each *physical* infinitesimal volume (ball) can have at most 3 ordered and symmetric dimensions of discrete *physical* states of the same type, for example, a set of 3 binary values, 1 and -1, indicating vector orientation.

   (b) And each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values of the same type per ball.

   (c) If each of the three possible ordered and symmetric dimensions of discrete physical states contained unordered sets (bags) of discrete state values, for example, unordered binary values. Bags (states) are non-deterministic. For example, every time that an unordered state is physically measured, there is a 50 percent chance of having one of the binary values. Bags of discrete values might be a way to model quantum physics.

   (d) Where infinitesimal (or perhaps Schwarzschild radius) balls intersect, an arithmetic of the interactions of the discrete states with respect to time needs to be developed. The interaction of the discrete states associated with intersecting balls with respect to time might result in what we perceive as motion, waves, particles, spin, polarization, work, force, mass, charge, etc.

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