The Set Relations Generating Euclidean Geometry

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Abstract. Applying a ruler-like measure to the definition of a countable distance space, where for each disjoint domain set there exists a corresponding same-sized range set and the range sets in some cases intersect, converges to: the triangle inequality and non-negativity properties of metric space, Manhattan distance at the upper boundary, and Euclidean distance at the lower boundary. Applying the ruler-like measure to a set-based definition of countable volume converges to the product of interval interval sizes (Euclidean area/volume). The total order and symmetry properties of these two set-based countable distance and volume definitions limit an Euclidean geometry to 3 dimensions. Proofs are verified in Coq.

Contents

1. Introduction

Most of mathematical analysis is derived from a foundation of set-based axioms, except for: triangle inequality, Manhattan distance, Euclidean distance, and Euclidean area/volume, which are motivated by Euclidean geometry and used as definitions in measure (for example, metric space, Hausdorff, and Lebesgue) and integration (Lebesgue and Riemann) [Gol76]. This article will use some very simple real analysis to motivate and derive triangle inequality, Manhattan and Euclidean distance from a single set-based axiom and derive volume from another set-based axiom.

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The relationships between countable sets generating distance and volume provides new insights into Euclidean geometry, metric space, and physics. For example, (*without any notions of side, angle, and shapes*) the set-based reason: 1) Euclidean distance is the smallest distance between two distinct points in Euclidean space; 2) for the triangle inequality and non-negativity properties of metric space; 3) metric space might not be a sufficient condition for a distance metric; 4) physical, Euclidean geometry is limited to 3 dimensions.

The proofs in this article are verified formally using the Coq Proof Assistant [**Coq15**] version 8.7.0. The Coq-based definitions, theorems, and proofs are in the files "euclidrelations.v" and "threed.v" located at:
https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

Deriving distance and volume from set and number theory requires a measure that does not have Euclidean assumptions and also allows the full range of mappings from a one-to-one correspondence to a many-to-many mapping. A ruler (measuring stick) measures a real-valued interval as the nearest integer number of same-sized subintervals (units), where the partial subintervals are ignored.

The ruler measure allows defining relations, for example a many-to-many relation, between the set of same-sized subintervals in one interval and the set of same-sized subintervals in another interval. The countable relations converge to continuous, bijective functions as the subinterval size converges to zero.

Definition 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, p, each subinterval having the same size, c. Notionally:

$$\forall\, c\, s \in \mathbb{R},\;\; [a,b] \subset \mathbb{R},\;\; s = |a-b| \;\;\wedge\;\; c > 0 \;\;\wedge \\ (p = floor(s/c) \;\;\vee\;\; p = ceiling(s/c)) \;\;\wedge\;\; M = \textstyle\sum_{i=1}^{p} c = pc. \tag{2.1}$$

Theorem 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\;\; s = |a-b| \;\;\Rightarrow\;\; M = \lim_{c\to 0} pc = s.$

The Coq-based theorem and proof in the file euclidrelations.v is "limit_c_0_M_eq_exact_size."

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\; y \leq x,\; y \in \mathbb{Z},\; x \in \mathbb{R}\})$:

$$\forall\, c > 0,\; p = floor(s/c) \;\wedge\; 0 \leq |floor(s/c) - s/c| < 1 \;\Rightarrow\; 0 \leq |p - s/c| < 1. \tag{2.2}$$

Multiply all sides of inequality 2.2 by $|c|$:

$$\forall\, c > 0, \quad 0 \leq |p - s/c| < 1 \quad \Rightarrow \quad 0 \leq |pc - s| < |c|. \tag{2.3}$$

$$\forall\, \delta \;:\; |pc - s| < |c| = |c - 0| < \delta \\ \Rightarrow \quad \forall\, \epsilon = \delta :\;\; |c - 0| < \delta \;\;\wedge\;\; |pc - s| < \epsilon \;\;:=\;\; M = \lim_{c\to 0} pc = s. \quad \square \tag{2.4}$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence, where: $[0, \pi]$, $s = |0 - \pi|$, $c = 10^{-i}$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{i=1},\; 3.14_{i=2},\; 3.141_{i=3}, ..., \pi$.

3. Distance

Notation conventions: The vertical bars around a set is the standard notation for indicating the cardinal (number of members in the set). To prevent over use of the vertical bar, the symbol for "such that" is the colon.

3.1. Countable distance space. The most fundamental notion of distance is that for each disjoint domain set, x_i, there exists a corresponding range set, x_i containing the same number of members, p_i: $|x_i| = |y_i| = p_i$. For example, there should be as many steps walked in the range set, y_i, as there are pieces of traversed land in the corresponding domain set, x_i.

Where the range sets are disjoint (no intersection), the size of the union is equal to the sum of the set sizes: $|\bigcup_{i=1}^n y_i| = \sum_{i=1}^n |y_i| \Leftrightarrow \bigcap_{i=1}^n y_i = \emptyset$. Where the range sets intersect, the size of the union is less than the sum of the set sizes: $|\bigcup_{i=1}^n y_i| < \sum_{i=1}^n |y_i| \Leftrightarrow \bigcap_{i=1}^n y_i \neq \emptyset$.

Definition 3.1. Countable distance space, d_c:

$$|\bigcup_{i=1}^n x_i| = \sum_{i=1}^n |x_i| \quad \wedge \quad d_c = |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i.$$

3.2. Metric Space. Applying the ruler (2.1) and ruler convergence (2.2) to the countable distance space property, $d_c = |\bigcup {}_{i=1}^2 y_i| \leq \sum {}_{i=1}^2 |y_i|$, (3.1) generates the real-valued triangle inequality and non-negativity properties of metric space:

$$\begin{gathered}(3.1)\quad d_c = |\bigcup {}_{i=1}^2 y_i| \leq \sum {}_{i=1}^2 |y_i| \quad \wedge \quad \forall\, c > 0 \ : \\ d_c = floor(d(u,w)/c) \ \wedge \ |y_1| = floor(d(u,v)/c) \ \wedge \ |y_2| = floor(d(v,w)/c) \\ \Rightarrow \ d(u,w) = \lim_{c\to 0} d_c \cdot c \leq \sum {}_{i=1}^2 \lim_{c\to 0} |y_i| \cdot c = d(u,v) + d(v,w).\end{gathered}$$

The number of members in any countable set is always non-negative. And the product of two non-negative numbers, $d_c \cdot c$, is always a non-negative number:

$$\begin{gathered}(3.2)\quad \forall\, c > 0 \ : \quad d_c = floor(d(u,w)/c) \quad \wedge \quad d_c = |\bigcup_{i=1}^n y_i| \geq 0 \\ \Rightarrow \quad floor(d(u,w)/c) = d_c \geq 0 \quad \Rightarrow \quad d(u,w) = \lim_{c\to 0} d_c \cdot c \geq 0.\end{gathered}$$

3.3. Distance space range. Where the range sets intersect, multiple domain set members map to a single range set member. Therefore, the union distance, d_c, is function of the number of domain-to-range set member mappings.

The property, $|x_i| = |y_i| = p_i$, (3.1) constrains the range of possible domain-to-range set member mapping from $|x_i| \cdot 1 = p_i$ (where members of x_i corresponds 1-1 to members of y_i) to $|x_i| \cdot |y_i| = p_i^2$ (where all p_i number of members of x_i map to each member in y_i, which is a many-to-many relation).

Therefore, $\exists\, \mathbf{f} : d_c = \mathbf{f}(\sum {}_{i=1}^n p_i)$ is the largest possible distance because it is the case of the smallest number of domain-to-range mappings (no intersection of the range sets). And $\exists\, \mathbf{f} : d_c = \mathbf{f}(\sum {}_{i=1}^n p_i^2)$ is the smallest possible distance because it is the case of the largest number of domain-to-range mappings (largest allowed intersection of range sets). Applying the ruler (2.1) and ruler convergence theorem (2.2) to the longest and shortest countable distance cases yields the real-valued, Manhattan and Euclidean distance functions.

3.4. Manhattan distance.

Theorem 3.2. *Manhattan (longest) distance, d, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The formal Coq-based theorem and proof in file euclidrelations.v is "taxicab_distance."

PROOF.
From the countable distance space definition (3.1), the largest possible countable distance, d_c, is the equality case:

$$(3.3) \quad d_c \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |y_i| = p_i \\ \Rightarrow \quad d_c \leq \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c :\ d_c = \sum_{i=1}^{n} p_i.$$

Multiply both sides of equation 3.5 by c and take the limit:

$$(3.4) \quad d_c = \sum_{i=1}^{n} p_i \ \Rightarrow\ d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \ \Rightarrow\ \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c).$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$(3.5) \qquad d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d :\ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of s_i:

$$(3.6) \qquad \forall\, i \in [1, n],\ s_i = |a_i - b_i| \quad \Rightarrow \quad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c \to 0} p_i \cdot c = s_i.$$

Combine equations 3.5, 3.4, 3.6:

$$(3.7) \quad d = \lim_{c \to 0} d_c \cdot c \quad \wedge \quad \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) \quad \wedge \\ \lim_{c \to 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square$$

3.5. Euclidean distance.

THEOREM 3.3. *Euclidean (shortest) distance, d, is the size of the distance interval, $[d_0, d_m]$, mapping to a set of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The formal Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_distance."

PROOF.
Apply the rule of product to the largest number of domain-to-range set mappings, where all p_i number of domain set members, x_i, map to each of the p_i number of members in the range set, y_i:

$$(3.8) \qquad \sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2.$$

From the countable distance space definition (3.1), choose the equality case:

$$(3.9) \quad d_c \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |y_i| = p_i \\ \Rightarrow \quad d_c \leq \sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c :\ d_c = \sum_{i=1}^{n} p_i.$$

Square both sides of equation 3.9 ($x = y \Leftrightarrow f(x) = f(y)$):

$$(3.10) \qquad \exists\, p_i,\, d_c :\ d_c = \sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\, d_c :\ d_c^2 = (\sum_{i=1}^{n} p_i)^2.$$

Apply the Cauchy-Schwartz inequality to equation 3.10 and select the smallest (equality) case:

$$(3.11) \qquad d_c^2 = (\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2 \quad \Rightarrow \quad \exists\, p_i :\ d_c^2 = \sum_{i=1}^{n} p_i^2.$$

Multiply both sides of equation 3.11 by c^2, simplify, and take the limit.

$$(3.12)\quad d_c^2 = \sum_{i=1}^{n} p_i^2 \;\Rightarrow\; d_c^2 \cdot c^2 = \sum_{i=1}^{n} p_i^2 \cdot c^2 \;\Leftrightarrow\; (d_c \cdot c)^2 = \sum_{i=1}^{n} (p_i \cdot c)^2$$
$$\Rightarrow\quad \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.13)\qquad \exists\, c\, d:\; floor(d/c) = d_c \quad\Rightarrow\quad d = \lim_{c\to 0} d_c \cdot c \quad\Rightarrow\quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.14)\quad \forall\, i \in [1,n],\; floor(s_i/c) = p_i \quad\Rightarrow\quad \lim_{c\to 0} p_i \cdot c = s_i$$
$$\Rightarrow\quad \lim_{c\to 0}(p_i \cdot c)^2 = s_i^2 \quad\Rightarrow\quad \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2.$$

Combine equations 3.13, 3.12, 3.14:

$$(3.15)\quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 \quad\wedge\quad \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 \quad\wedge$$
$$\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2$$
$$\Rightarrow\quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \qquad \square$$

4. Euclidean Volume

The number of all possible combinations (all many-to-many mappings) between members in a countable set x_1 and a countable set x_2 is the Cartesian product, $|x_1| \cdot |x_2|$. This section will use the ruler (2.1) and ruler convergence theorem (2.2) to prove that the Cartesian product of the same-sized subintervals of intervals converges to the product of interval sizes as the subinterval size converges to zero. The first step is to define a countable set-based measure of area/volume as the Cartesian product (many-to-many mappings) of disjoint domain set members.

Definition 4.1. Countable volume measure, V_c:

$$\textstyle\sum_{i=1}^{n} |x_i| = |\bigcup_{i=1}^{n} x_i|, \qquad V_c = \prod_{i=1}^{n} |x_i|.$$

Theorem 4.2. *Euclidean volume, V, is the size of a range interval, $[v_0, v_m]$, corresponding to a set of disjoint intervals:*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$V = \prod_{i=1}^{n} s_i, \quad V = |v_0 - v_m|, \quad s_i = |a_i - b_i|, \quad i \in [1,n], \quad i, n \in \mathbb{N}.$$

The Coq-based theorem and proof in the file euclidrelations.v is "Euclidean_volume."

Proof.
Use the ruler (2.1) to divide the exact size, $s_i = |a_i - b_i|$, of each of the domain intervals, $[a_i, b_i]$, into a set, x_i of p_i number of subintervals.

$$(4.1)\qquad \forall\, i\, n \in \mathbb{N}, \quad i \in [1,n], \quad c > 0 \quad\wedge\quad floor(s_i/c) = p_i = |x_i|.$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$(4.2)\qquad floor(s_i/c) = p_i \quad\Rightarrow\quad \lim_{c\to 0}(p_i \cdot c) = s_i.$$

Use the ruler (2.1) to divide the exact size, $V = |v_0 - v_m|$, of the range interval, $[v_0, v_m]$, into p^n subintervals. Every integer number, V_c, does **not** have an integer n^{th} root. However, for those cases where V_c does have an integer n^{th} root, there is a p^n that satisfies the definition a countable volume measure, V_c (4.1). Notionally:

$$\forall\, p^n = V_c \in \mathbb{N},\ \exists\, V \in \mathbb{R},\ x_i\ : floor(V/c^n) = V_c = p^n = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} p_i. \tag{4.3}$$

Apply the ruler convergence theorem (2.2) to equation 4.3 and simplify:

$$floor(V/c^n) = p^n \quad \Rightarrow \quad V = \lim_{c\to 0} p^n \cdot c^n = \lim_{c\to 0} (p\cdot c)^n. \tag{4.4}$$

Multiply both sides of equation 4.3 by c^n and simplify:

$$p^n = \prod_{i=1}^{n} p_i \quad \Rightarrow \quad p^n \cdot c^n = (\prod_{i=1}^{n} p_i)\cdot c^n \quad \Leftrightarrow \quad (p\cdot c)^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.5}$$

Combine equations 4.4, 4.5, and 4.2:

$$V = \lim_{c\to 0} (p\cdot c)^n \quad \wedge \quad (p\cdot c)^n = \prod_{i=1}^{n} (p_i\cdot c) \quad \wedge \quad \lim\nolimits_{c\to 0}(p_i\cdot c) = s_i \tag{4.6}$$
$$\Rightarrow \quad V = \lim_{c\to 0}(p\cdot c)^n = \prod_{i=1}^{n} \lim\nolimits_{c\to 0}(p_i\cdot c) = \prod_{i=1}^{n} s_i. \qquad \square$$

5. Ordered and symmetric geometries

Calculating the union and addition operations of distance and the union and multiplication operations of volume requires iterating sequentially through each set (dimension of sets), which implies a total order of the sets. The commutative property of the union, addition, and multiplication also allows each set to be sequentially adjacent to any other set, herein referred to as a symmetric geometry.

But, a set can have only one order *at a point in time* because each member of a sequentially ordered set can have at most one successor and at most one predecessor. It will now be proved that a set satisfying the *simultaneous* constraints of a single total order and symmetry defines a cyclic set containing at most 3 members.

Definition 5.1. Ordered geometry:

$$\forall\, i\ n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i\ \in \{x_1, \ldots, x_n\},$$
$$successor\ x_i = x_{i+1}\ \ \wedge\ \ predecessor\ x_{i+1} = x_i,$$

where each $x_i\ \in \{x_1, \ldots, x_n\}$ is a set of subintervals of a real-valued domain interval (from a dimension of intervals).

Definition 5.2. Symmetric geometry (every member of a set is sequentially adjacent to every other member):

$$\forall\, i\ j\ n \in \mathbb{N},\ \ \forall\, x_i\ x_j \in \{x_1, \ldots, x_n\},\ \ successor\ x_i = x_j\ \ \wedge\ \ predecessor\ x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$successor\ x_n = x_1\ \ \wedge\ \ predecessor\ x_1 = x_n.$$

The theorem and formal Coq-based proof is "ordered_symmetric_is_cyclic," which is located in the file threed.v.

Proof. The property of order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecssor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ successor\ x_i = x_j\ \ \Rightarrow\ \ successor\ x_n = x_1. \tag{5.1}$$

$$i = n \;\wedge\; j = 1 \;\wedge\; \mathit{predecessor}\; x_j = x_i \;\Rightarrow\; \mathit{predecessor}\; x_1 = x_n. \tag{5.2}$$ □

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The Coq-based lemmas and proofs in the file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set of sets (5.3). In other words, the successors and predecessors of ordered and symmetric set are cyclic:

DEFINITION 5.5. Cyclic successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (m + 1 \leq \mathit{setsize}). \tag{5.3}$$

DEFINITION 5.6. Cyclic predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n (required for a "symmetric" set (5.2)), if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:

$$\mathit{Adjacent}(m, n, \mathit{setsize}) \leftarrow \mathit{Successor}(m, n, \mathit{setsize}) \vee \mathit{Predecessor}(m, n, \mathit{setsize}). \tag{5.5}$$

Every member is adjacent to every other member, where $\mathit{setsize} \in \{1,\ 2,\ 3\}$:

$$\mathit{Adjacent}(1,1,1) \leftarrow \mathit{Successor}(1,1,1) \leftarrow (1 = 1 \wedge 1 = 1). \tag{5.6}$$

$$\mathit{Adjacent}(1,2,2) \leftarrow \mathit{Successor}(1,2,2) \leftarrow (1 + 1 \leq 2). \tag{5.7}$$

$$\mathit{Adjacent}(2,1,2) \leftarrow \mathit{Successor}(2,1,2) \leftarrow (2 = 2 \wedge 1 = 1). \tag{5.8}$$

$$\mathit{Adjacent}(1,2,3) \leftarrow \mathit{Successor}(1,2,3) \leftarrow (1 + 1 \leq 2). \tag{5.9}$$

$$\mathit{Adjacent}(2,1,3) \leftarrow \mathit{Predecessor}(2,1,3) \leftarrow (2 - 1 \geq 1). \tag{5.10}$$

$$\mathit{Adjacent}(3,1,3) \leftarrow \mathit{Successor}(3,1,3) \leftarrow (3 = 3 \wedge 1 = 1). \tag{5.11}$$

$$\mathit{Adjacent}(1,3,3) \leftarrow \mathit{Predecessor}(1,3,3) \leftarrow (1 = 1 \wedge 3 = 3). \tag{5.12}$$

$$\mathit{Adjacent}(2,3,3) \leftarrow \mathit{Successor}(2,3,3) \leftarrow (2 + 1 \leq 3). \tag{5.13}$$

$$\mathit{Adjacent}(3,2,3) \leftarrow \mathit{Predecessor}(3,2,3) \leftarrow (3 - 1 \geq 1). \tag{5.14}$$

Must prove that for all $\mathit{setsize} > 3$, there exist non-adjacent members. For example, the first and third members are not adjacent:

$$\forall\, \mathit{setsize} > 3 : \quad \neg \mathit{Successor}(1, 3, \mathit{setsize}) \leftarrow \mathit{Successor}(1, 2, \mathit{setsize}) \leftarrow (1 + 1 \leq \mathit{setsize}). \tag{5.15}$$

That is, 2 is the only successor of 1 for all $setsize > 3$, which implies 3 is not a successor of 1 for all $setsize > 3$.

$$\forall\, setsize > 3: \quad \neg Predecessor(1, 3, setsize) \\ \leftarrow Predecessor(1, n, setsize) \leftarrow (1 = 1 \wedge n = setsize). \tag{5.16}$$

That is, $n = setsize$ is the only predecessor of 1 for all $setsize > 3$, which implies 3 is not a predecessor of 1 for all $setsize > 3$.

$$\forall\, setsize > 3: \quad \neg Adjacent(1, 3, setsize) \\ \leftarrow \neg Successor(1, 3, setsize) \wedge \neg Predecessor(1, 3, setsize). \quad \square \tag{5.17}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (violates the symmetry property).

6. Summary

Applying the ruler measure (2.1) and ruler convergence proof (2.2), to a set of real-valued domain intervals and a range interval yields some new insights into geometry and physics.

(1) Distance is a function of the number of domain-to-*range* set member mappings. Area/volume is a function of the number of domain-to-*domain* set member mappings. Other types of measure, like metric space, Borel, Hausdorff, and Lebesgue, do not provide that insight. Metric space allowing area/volume as a metric calls into question whether metric space is a sufficient condition for distance metrics.

(2) Applying the ruler measure to the countable distance space (3.1) provides the insight that all notions of distance are derived from the principle that for each domain set, x_i, there exists a corresponding range set, y_i, containing the same number of members, p_i: $|x_i| = |y_i| = p_i$ (3.1). For example, there should be as many steps walked in the range set, y_i, as there are pieces of traversed land in the corresponding domain set, x_i.

And the union size depends on the amount of intersection of range sets: $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$.

  (a) Applying the ruler to the set relation, $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$, (3.1) generates the real-valued triangle inequality and non-negativity properties of metric space (3.2).

  (b) Where the range sets intersect, multiple domain set members can map to a single range set member. Therefore, distance is a function of domain-to-range set member mappings. The property, $|x_i| = |y_i| = p_i$, constrains the range of possible domain-to-range set member mapping from $\sum_{i=1}^{n} p_i$ to $\sum_{i=1}^{n} p_i^2$. Applying the ruler to the fewest mappings case converges to Manhattan distance (3.2) and the most mappings case converges to Euclidean distance (3.3).

  (c) The countable distance space only allows those L^p norms ($||x|| = (\sum_{i=1}^{n} x_i^p)^{1/p}$), where $1 \leq p \leq 2$. All distance measures, where $1 \leq p \leq 2$, coexist *simultaneously* between any two distinct points (all true at the same time), which is a logically consistent space of distance measures. For example, one can draw both the Manhattan and Euclidean distances between the same two points.

But, for every $p > 2$, the distance measure can *not* coexist simultaneously with *some* other distance measures between the same two points. For example, one can *not* draw the case of the distance corresponding to $p = 3$ between those same two points that is also spanned by Euclidean distance.
Allowing L^p norms that can not all be true at the same time between two distinct points is another case indicating the definition of metric space might not be a sufficient condition for distance metrics.

(d) Euclidean distance (3.3) was derived from a many-to-many relation between each disjoint domain set and a corresponding same-sized range set, where the range sets intersect *without any notions of side, angle, or shape.* A parametric variable relating the sizes of two domain intervals can be easily derived using using calculus (Taylor series) and the Euclidean distance equation to generate the arc sine and arc cosine functions of the parametric variable (arc angle). In other words, the notions of side and angle are derived from Euclidean distance, which is the reverse perspective of classical geometry [**Joy98**] [**Loo68**] [**Ber88**] and axiomatic foundations for geometry [**Bir32**] [**Hil80**] [**TG99**].

(3) Applying the ruler measure and ruler convergence proof to the countable volume definition (4.1) allows a proof that the Cartesian product of same-sized subintervals of real-valued intervals converges to the product of the interval sizes (Euclidean length/area/volume):
  (a) Euclidean volume was derived from a many-to-many (combinatorial) relation without notions of sides, angles, and shape.
  (b) The Lebesgue measure, Riemann and Lebesgue integration use Euclidean volume/space ($\mathbb{R}^n$) as a definition rather than deriving Euclidean volume from more fundamental set-based relations.

(4) The set-based relations generating distance and volume have the *simultaneous* properties of a single total order of dimensions (5.1) and symmetry (5.2), which limits distance and volume to a cyclic set (5.3) of three dimensions (5.4), which explains why there are only three dimensions of physical space (width, height, and depth).

(5) The *simultaneously* ordered and symmetric properties that generate the three dimensions of geometric space constrain all higher dimensional theories of physics to *hierarchical* 2 or 3-dimensional geometries to prevent contradictions. For example, the four-vectors common in physics [**Bru17**] are hierarchical, 2-dimensional geometries that have been "flattened."

The spacetime four-vector length, $d = \sqrt{(ct)^2 - (x^2 + y^2 + z^2)}$, where c is the speed of light and t is time, can be expressed in a form like, $(ct)^2 = d_1^2 + d_2^2$, where $d_1^2 = x^2 + y^2 + z^2$ and $d_2 = d$. Likewise, the energy-momentum four-vector has the 2-dimensional length: $E^2 = (mv^2)^2 + (pc)^2$, where E is energy, m is the resting mass, v is the 3-dimensional velocity, c is the speed of light, and p is the relativistic momentum ($p = \gamma mv$, where $\gamma = (1/(1 - (v/c)^2))^{1/2}$ is the Lorentz factor).

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