The Two Set Relations Generating Geometry

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Abstract. A ruler (measuring stick) partitions both domain and range intervals approximately into sets of same-sized subintervals. As the subinterval size converges to zero: 1) The union of range sets of subintervals, where each domain set maps to a less or equal-sized range set, generates the properties of metric space and all Lp norms, in particular, the Manhattan and Euclidean distance equations. 2) The Cartesian product of domain sets of subintervals converges to the product of interval interval sizes (Euclidean area/volume). The proofs allow much simpler derivations of Coulomb's charge force, Newton's gravity force, and spacetime equations without using other physical laws or Gauss's divergence theorem. Time limits physical distance and volume to 3 dimensions. All proofs are verified in Coq.

Contents

1. Introduction

Metric space and the L^p norms, like the Euclidean distance metric/vector norm, have been defined in mathematical analysis [**Gol76**] [**Rud76**] rather than derived from a more fundamental set-based definition of distance. Further, all functions having the metric space properties are defined as distance measures. In contrast, area/volume is a function definition that is part the Lebesgue and Borel measures.

Topology and measure theory have lacked coherent, set-based definitions of both distance and volume and have lacked a single measure that applies to both

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distance and volume. In this article, both distance and volume are abstract countable set definitions, where a "ruler" measure of intervals creates sets of subintervals conforming to the abstract set definitions of distance and volume.

The derivations of metric space, Euclidean distance, and volume equation provide some insights into geometry and physics, for example: the single set operation generating the triangle inequality ($d(u,w) \leq d(u,v) + d(v,w)$), non-negativity ($d(u,w) \geq 0$), and identity of indiscernibles ($d(w,w) = 0$) properties of metric space; the mapping between sets that makes Euclidean distance the smallest possible distance between two distinct points in $\mathbb{R}^n$; the mapping between sets that makes distance different from area/volume; why the derivations of Coulomb's charge force, Newton's gravity force, and spacetime equations do not require using other laws of physics or Gauss's divergence theorem; the proportionate interval principle generating the inverse square law and flux divergence; how time places an additional constraint on physical sets, which limits physical distance and volume to 3 dimensions.

A ruler (measuring stick) partitions an interval into the nearest integer number of same-sized subintervals, where the ruler measure is the sum of the subinterval sizes. All L^p norms and the Euclidean volume equation are functions of the total number of mappings, ranging from a one-to-one correspondence to a many-to-many (Cartesian product) mapping, between the set of subintervals having size c in one interval and the set of subintervals having the same size, c, in another interval. The mapping (combinatorial) relations converge to continuous relations as the subinterval size, c, converges to zero.

All the proofs in this article have been formally verified using using the Coq proof verification system [**Coq15**]. The formal proofs are located in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

Definition 2.1. Ruler measure: A ruler measures the size, M, of a closed, open, or semi-open interval as the sum of the sizes of the nearest integer number of whole subintervals, p, each subinterval having the same size, c. Notionally:

$$\forall\, c,\ s \in \mathbb{R},\ \ [a,b] \subset \mathbb{R},\ \ s = |a-b| \ \wedge\ c > 0\ \wedge \\ (p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \textstyle\sum_{i=1}^{p} c = pc. \tag{2.1}$$

Theorem 2.2. *Ruler convergence:*
$\forall\, [a,b] \subset \mathbb{R},\ \ s = |a-b| \ \Rightarrow\ M = \lim_{c \to 0} pc = s.$

The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\ p = floor(s/c)\ \wedge\ 0 \leq |floor(s/c) - s/c| < 1\ \Rightarrow\ 0 \leq |p - s/c| < 1. \tag{2.2}$$

Multiply all sides of inequality 2.2 by $|c|$:

$$\forall\, c > 0,\quad 0 \leq |p - s/c| < 1 \quad\Rightarrow\quad 0 \leq |pc - s| < |c|. \tag{2.3}$$

$$(2.4) \quad \forall\, \delta \,:\, |pc - s| < |c| = |c - 0| < \delta$$
$$\Rightarrow \quad \forall\, \epsilon = \delta:\ |c - 0| < \delta \ \wedge\ |pc - s| < \epsilon \ := \ M = \lim_{c \to 0} pc = s. \quad \square$$

The proof steps using the ceiling function (the outer measure) are the same as the steps in the previous proof using the floor function (the inner measure). The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = |0 - \pi|$, $c = 10^{-i}$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{i=1},\ 3.14_{i=2},\ 3.141_{i=3}, ..., \pi_{\lim_{i\to\infty \Rightarrow c \to 0}}$.

3. Distance

Notation convention: Vertical bars around a set or list, $|\cdots|$, indicates the cardinal (number of members in the set or list).

3.1. Countable distance space. Countable distance is a set operation-based abstraction of distance, that allows derivation of the properties of metric space, and the L^p norms, in particular, Manhattan and Euclidean distance. An example of a countable distance is the number of members in a range set, y_i, which equals the number of members in a corresponding domain set, x_i: $|x_i| = |y_i|$. And the countable distance spanning multiple, disjoint, domain sets, $\bigcap_{i=1}^n x_i = \emptyset$, is the number of members, d_c, in the union range set: $d_c = |\bigcup_{i=1}^n y_i|$. Generalizing:

Definition 3.1. Countable distance space, d_c :

$$\bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad d_c = |\bigcup_{i=1}^n y_i| \quad \wedge \quad |x_i| = |y_i|^q,\ q \geq 1.$$

Theorem 3.2. *Inclusion-exclusion Inequality:* $|\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|$.

The inclusion-exclusion inequality follows from the inclusion-exclusion principle [CG15]. But, a more intuitive and simple proof follows from the associative law of addition where the sum of set sizes is equal to the size of all the set members appended into a list and the commutative law of addition that allows sorting that list into a list of unique members (the union set) and a list of duplicates. The list of duplicates being ≥ 0 implies the union size is always $\leq$ the sum of set sizes.

A formal proof, inclusion_exclusion_inequality, using sorting into a set of unique members (union set) and a list of duplicate members, is in the file euclidrelations.v.

Proof.

$$(3.1) \quad \sum_{i=1}^n |y_i| = |append_{i=1}^n y_i| = |sort(append_{i=1}^n y_i)|$$
$$= |\bigcup_{i=1}^n y_i| + |duplicates_{i=1}^n y_i|.$$

$$(3.2) \quad |\bigcup_{i=1}^n y_i| + |duplicates_{i=1}^n y_i| = \sum_{i=1}^n |y_i| \quad \wedge \quad |duplicates_{i=1}^n y_i| \geq 0$$
$$\Rightarrow \quad |\bigcup_{i=1}^n y_i| \leq \sum_{i=1}^n |y_i|. \quad \square$$

3.2. Metric Space. All function range intervals, $d(u, w)$, satisfying the countable distance space definition, $d_c = |\bigcup_{i=1}^n y_i|$, where the ruler is applied, generates three of the four metric space properties: triangle inequality, non-negativity, and identity of indiscernibles. The set-based reason for the fourth property of metric space, symmetry $[d(u, v) = d(v, u)]$, will be identified in the last section of this article. The formal proofs: triangle_inequality, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

Theorem 3.3. *Triangle Inequality:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \leq d(u, v) + d(v, w)$.

Proof. Apply the ruler measure (2.1), the countable distance space condition (3.1), inclusion-exclusion inequality (3.2), and then ruler convergence (2.2).

$$
\begin{aligned}
(3.3)\quad & \forall\, c > 0,\ d(u,w),\ d(u,v),\ d(v,w)\ : \\
& |y_1| = floor(d(u,v)/c) \ \wedge\ |y_2| = floor(d(v,w)/c) \ \wedge \\
& d_c = floor(d(u,w)/c) \ \wedge\ d_c = |y_1 \cup y_2| \le |y_1| + |y_2| \\
& \Rightarrow\ floor(d(u,w)/c) \le floor(d(u,v)/c) + floor(d(v,w)/c) \\
& \Rightarrow\ floor(d(u,w)/c)\cdot c \le floor(d(u,v)/c)\cdot c + floor(d(v,w)/c)\cdot c \\
& \Rightarrow\ \lim_{c\to 0} floor(d(u,w)/c)\cdot c \le \lim_{c\to 0} floor(d(u,v)/c)\cdot c + \lim_{c\to 0} floor(d(v,w)/c)\cdot c \\
& \Rightarrow\ d(u,w) \le d(u,v) + d(v,w). \qquad \square
\end{aligned}
$$

Theorem 3.4. *Non-negativity:* $d_c = |y_1 \cup y_2| \Rightarrow d(u,w) \ge 0$.

Proof. By definition, a set always has a size (cardinal) ≥ 0:

$$
\begin{aligned}
(3.4)\quad & \forall\, c > 0,\ d(u,w)\ :\quad floor(d(u,w)/c) = d_c \quad\wedge\quad d_c = |y_1 \cup y_2| \ge 0 \\
& \Rightarrow\quad floor(d(u,w)/c) = d_c \ge 0 \quad\Rightarrow\quad d(u,w) = \lim_{c\to 0} d_c \cdot c \ge 0. \qquad \square
\end{aligned}
$$

Theorem 3.5. *Identity of Indiscernibles:* $d(w,w) = 0$.

Proof. Apply the triangle inequality property (3.3):

$$\forall\, d(u,v) = d(v,w) = 0\ \wedge\ d(u,w) \le d(u,v) + d(v,w)\ \Rightarrow\ d(u,w) \le 0. \tag{3.5}$$

Combine the non-negativity property (3.4) and the previous inequality (3.5):

$$d(u,w) \ge 0\ \wedge\ d(u,w) \le 0\ \Leftrightarrow\ 0 \le d(u,w) \le 0\ \Rightarrow\ d(u,w) = 0. \tag{3.6}$$

Combine the result of step 3.6 and the condition, $d(u,v) = 0$, in step 3.5.

$$d(u,w) = 0\ \wedge\ d(u,v) = 0\ \Rightarrow\ w = v. \tag{3.7}$$

Combine the condition, $d(v,w) = 0$, in step 3.5 and the result of step 3.7.

$$d(v,w) = 0\ \wedge\ w = v\ \Rightarrow\ d(w,w) = 0. \tag{3.8}$$

$\square$

3.3. Distance space range. From the countable distance space definition, $d_c = |\bigcup_{i=1}^{n} y_i|$, as the amount of intersection increases, more domain set members can map to a single range set member. Therefore, the number of domain-to-range set member mappings is a function of the amount of range set intersection.

From the countable distance space property (3.1), where $|x_i| = |y_i| = p_i = 1$, each domain set member maps one-to-one to a unique range set member (no intersection and largest distance), $|y_i| \cdot 1 = p_i = 1$ mapping, and also each domain member maps to every range set member (largest intersection and smallest possible distance), $|x_i| \cdot |y_i| = p_i^2 = 1$ mapping. The types of domain-to-range set mappings that are true for one range set member are true for all members in a range set. Therefore, the total number of domain-to-range set mappings varies from $\sum_{i=1}^{n} |y_i| \cdot 1 = \sum_{i=1}^{n} p_i$ to $\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2$ mappings. Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of domain-to-range set mapping cases converges to the real-valued Manhattan and Euclidean distance relations.

3.4. Manhattan distance.

THEOREM 3.6. *Manhattan (largest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:

$$d = \sum_{i=1}^{n} s_i, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "taxicab_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$\begin{aligned} d_c = |\textstyle\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad &\wedge \quad |x_i| = |y_i| = p_i \\ &\Rightarrow \quad d_c \leq \textstyle\sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c :\ d_c = \sum_{i=1}^{n} p_i. \end{aligned} \tag{3.9}$$

Multiply both sides of equation 3.11 by c and take the limit:

$$d_c = \textstyle\sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \Rightarrow \lim_{c \to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c). \tag{3.10}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$d = |d_0 - d_m| \quad \Rightarrow \quad \exists\, c\, d :\ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c. \tag{3.11}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c \to 0} p_i \cdot c = s_i. \tag{3.12}$$

Combine equations 3.11, 3.10, 3.12:

$$\begin{aligned} &d = \lim_{c \to 0} d_c \cdot c \quad \wedge \quad \lim_{c \to 0} d_c \cdot c = \textstyle\sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) \quad \wedge \\ &\lim_{c \to 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c \to 0} d_c \cdot c = \textstyle\sum_{i=1}^{n} \lim_{c \to 0} (p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square \end{aligned} \tag{3.13}$$

3.5. Euclidean distance.

THEOREM 3.7. *Euclidean (smallest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = |d_0 - d_m|, \quad s_i = |a_i - b_i|.$$

The theorem, "Euclidean_distance," and formal proof is in the Coq file, euclidrelations.v.

PROOF.
Apply the rule of product to the largest number of domain-to-range set mappings, where all p_i number of range set members, y_i, map to each of the p_i number of members in the domain set, x_i, which is the Cartesian product, $|y_i| \cdot |x_i|$:

$$|x_i| = |y_i| = p_i \quad \Rightarrow \quad \textstyle\sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2. \tag{3.14}$$

From the countable distance space definition (3.1) and the inclusion-exclusion inequality (3.2), choose the equality case:

$$\begin{aligned} d_c = |\textstyle\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad &\wedge \quad |x_i| = |y_i| = p_i \\ &\Rightarrow \quad d_c \leq \textstyle\sum_{i=1}^{n} |y_i| = \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c :\ d_c = \sum_{i=1}^{n} p_i. \end{aligned} \tag{3.15}$$

Square both sides of equation 3.15 ($x = y \Leftrightarrow f(x) = f(y)$):

$$\exists\, p_i,\, d_c :\ d_c = \textstyle\sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\, d_c :\ d_c^2 = (\sum_{i=1}^{n} p_i)^2. \tag{3.16}$$

Apply the square of sum inequality, $(\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$, to equation 3.16 and select the smallest area (the equality) case:

$$(3.17) \quad d_c^2 = (\sum_{i=1}^{n} p_i)^2 = \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j$$
$$= \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} p_i \sum_{j=1, j\neq i}^{n} p_j \geq \sum_{i=1}^{n} p_i^2 \quad \Rightarrow \quad \exists\, p_i : \, d_c^2 = \sum_{i=1}^{n} p_i^2.$$

Multiply both sides of equation 3.17 by c^2, simplify, and take the limit.

$$(3.18) \quad d_c^2 = \sum_{i=1}^{n} p_i^2 \;\Rightarrow\; d_c^2 \cdot c^2 = \sum_{i=1}^{n} p_i^2 \cdot c^2 \;\Leftrightarrow\; (d_c \cdot c)^2 = \sum_{i=1}^{n} (p_i \cdot c)^2$$
$$\Rightarrow \quad \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.19) \quad \exists\, c\, d \in \mathbb{R} : \; floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$(3.20) \quad s_i = |a_i - b_i| \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i.$$

Combine equations 3.19, 3.18, 3.20:

$$(3.21) \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 \quad \wedge \quad \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 \quad \wedge$$
$$\lim_{c\to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \quad \square$$

4. Euclidean Volume

The Lebesgue and Borel measures of an n-volume [**Gol76**] [**Rud76**] and the volume integral, $\iiint dxdydz$, as the product of interval sizes are based on the product σ-algebra, the Cartesian product of countable sets of intervals. The ruler measure derivation of volume from the Cartesian product of countable sets of *same-sized* subintervals has two advantages: 1) Combined with the countable distance definition (3.1) provides a coherent set operation-based perspective and way of measuring across both distance and volume; 2) Allows simpler and more intuitive derivations of some equations in physics (like the charge force and gravity force equations). Notionally:

Definition 4.1. Countable Volume, v_c:

$$\bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad v_c = |\times_{i=1}^{n} x_i|.$$

Theorem 4.2. *Euclidean volume, v, is size of the range interval, $[v_0, v_m]$, corresponding to the Cartesian product of all the members of the domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$. Notionally:*

$$v = \prod_{i=1}^{n} s_i, \;\; v = |v_0 - v_m|, \;\; s_i = |a_i - b_i|.$$

The theorem, "Euclidean_volume," and formal proof is in the Coq file, euclidrelations.v.

Proof.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of subintervals.

$$(4.1) \qquad \forall\, i\; n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i|.$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$(4.2) \qquad floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0}(p_i \cdot c) = s_i.$$

State the countable volume (4.1) in terms of p_i:

$$(4.3)\quad v_c = |\times_{i=1}^n x_i| \quad \wedge \quad |x_i| = p_i \quad \Rightarrow \quad v_c = |\times_{i=1}^n x_i| = \prod_{i=1}^n |x_i| = \prod_{i=1}^n p_i.$$

Multiply both sides of equation (4.3) by c^n:

$$(4.4)\qquad v_c \cdot c^n = (\prod_{i=1}^n p_i) \cdot c^n = \prod_{i=1}^n (p_i \cdot c).$$

Use those cases, where v_c has an integer n^{th} root.

$$(4.5)\qquad \forall\, p^n = v_c \in \mathbb{N} \;:\; v_c \cdot c^n = p^n \cdot c^n = (p \cdot c)^n = \prod_{i=1}^n (p_i \cdot c).$$

Use the ruler (2.1) to partition the range interval, $[v_0, v_m]$, into $floor(v/c^n) = p^n$ subintervals and then apply the ruler convergence theorem (2.2) to equation 4.5:

$$(4.6)\qquad v = \lim_{c\to 0}(p \cdot c)^n = \prod_{i=1}^n \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^n s_i. \qquad \square$$

5. Applications to physics

5.1. Coulomb's charge force. The sizes, q_1 and q_2, of two charges are independent domain variables, where each size, c, component of a charge exerts a force on each same size, c, component of the other charge. The total force, F, is proportionate to the total number of forces (the Cartesian product of the number of *same-sized*, infinitesimal components) multiplied times a quantum charge force, $m_C a_C$. From the volume proof (4.2), the Cartesian product converges to $q_1 q_2$:

$$(5.1)\qquad F \propto (m_C a_C)(\lim_{c\to 0} floor(q_1/c) \cdot c)(\lim_{c\to 0} floor(q_2/c) \cdot c) = (m_C a_C)(q_1 q_2).$$

To solve for $F = m_C a_C$, there must be another variable, r, such that ratio, $q_1 q_2 / f(r) = 1$. From equation 5.1, an increase in charge, q, causes a proportionate increase in force, F. Therefore, to maintain the ratio $q_1 q_2/f(r) = 1$, r must be proportionate to q. $r \propto q \Rightarrow \exists\, q_C/r_C \in \mathbb{R} \;:\; r(q_C/r_C) = q$:

$$(5.2)\quad \forall\, q_1, q_2 \geq 0 \,\exists\, q \in \mathbb{R} \;: q^2 = q_1 q_2 \;\wedge\; r(q_C/r_C) = q \;\Rightarrow\; (r(q_C/r_C))^2 = q_1 q_2.$$

$$\begin{aligned}(5.3)\quad (r(q_C/r_C))^2 = q_1 q_2 \;\wedge\; F \propto (m_C a_C)(q_1 q_2)&\\ \Rightarrow\quad F \propto (m_C a_C)(r(q_C/r_C))^2 = (m_C a_C)(q_1 q_2)&\\ \Rightarrow\quad F = m_C a_C = (m_C a_C r_C^2/q_C^2) q_1 q_2/r^2 = k_c q_1 q_2/r^2.&\end{aligned}$$

where $k_C = m_C a_C r_C^2/q_C^2$ corresponds to the SI units: Nm^2C^{-2}.

$$\begin{aligned}(5.4)\quad & m_C a_C = (m_C a_C)(x/x) = (m_C x)(a_C/x)\\ & \wedge \quad \exists\, m_0,\; a \in \mathbb{R} :\; m_0 = m_C x,\; a = a_C/x \quad \Rightarrow \quad \exists\, m_0,\; a \in \mathbb{R} :\; m_0 a = m_C a_C,\end{aligned}$$

Combine equations 5.4 and 5.3 to obtain the force in terms of the rest mass:

$$(5.5)\quad m_0 a = m_C a_C \quad \wedge \quad F = m_C a_C = k_c q_1 q_2/r^2 \quad \Rightarrow \quad F = m_0 a = k_c q_1 q_2/r^2.$$

5.2. Newton's gravity force equation. The sizes, m_1 and m_2, of two masses are independent domain variables, where each size, c, component of a mass exerts a force on each same size, c, component of the other mass. The total force, F, is proportionate to the total number of forces (the Cartesian product of the number of *same-sized*, infinitesimal components) multiplied times a quantum gravity force, $m_G a_G$. From the volume proof (4.2), the Cartesian product converges to $m_1 m_2$:

$$(5.6)\quad F \propto (m_G a_G)(\lim_{c\to 0} floor(m_1/c) \cdot c)(\lim_{c\to 0} floor(m_2/c) \cdot c) = (m_G a_G)(m_1 m_2).$$

To solve for $F = m_G a_G$, there must be another variable, r, such that ratio, $m_1 m_2/f(r) = 1$. From equation 5.6, an increase in mass, m, causes a proportionate increase in force, F. Therefore, to maintain the ratio $m_1 m_2/f(r) = 1$, r must be proportionate to m. $r \propto m \Rightarrow \exists\, m_G/r_G \in \mathbb{R} \,:\, r(m_G/r_G) = m$:

$$\forall\, m_1, m_2 \geq 0 \,\exists\, m \in \mathbb{R} \;: \quad m^2 = m_1 m_2 \quad \wedge \quad r(m_G/r_G) = m \\ \Rightarrow \quad (r(m_G/r_G))^2 = m_1 m_2. \tag{5.7}$$

$$(r(m_G/r_G))^2 = m_1 m_2 \;\wedge\; F \propto (m_G a_G)(m_1 m_2) \\ \Rightarrow \quad F \propto (m_G a_G)(r(m_G/r_G))^2 = (m_G a_G)(m_1 m_2) \\ \Rightarrow \quad F = m_G a_G = (m_G a_G r_G^2/q_G^2) m_1 m_2/r^2. \tag{5.8}$$

$$\exists\, t_G \in \mathbb{R} :\; r_G/t_G^2 = a_G \quad \wedge \quad F = m_G a_G = (m_G a_G r_G^2/m_G^2) m_1 m_2/r^2 \\ \Rightarrow \quad F = m_G a_G = (r_G^3/m_G t_G^2) m_1 m_2/r^2 = G m_1 m_2/r^2, \tag{5.9}$$

where $G = r_G^3/m_G t_G^2$ corresponds to the SI units: $m^3 kg^{-1} s^{-2}$.

$$m_G a_G = (m_G a_G)(x/x) = (m_G x)(a_G/x) \\ \wedge \quad \exists\, m_0,\; a \in \mathbb{R} :\; m_0 = m_G x,\; a = a_G/x \quad \Rightarrow \quad \exists\, m_0,\; a \in \mathbb{R} :\; m_0 a = m_G a_G. \tag{5.10}$$

Combine equations 5.10 and 5.9 to obtain the force in terms of the rest mass:

$$m_0 a = m_G a_G \;\wedge\; F = m_G a_G = G m_1 m_2/r^2 \;\Rightarrow\; F = m_0 a = G m_1 m_2/r^2, \tag{5.11}$$

5.3. Spacetime equations. Applying the ruler measure, if sequencing across each same-sized subinterval of a *physical*, Euclidean distance (range) interval, $[0, r]$, corresponds to a proportionate number of same-sized subintervals of a time interval, $[0, t]$, then, as the subinterval size converges to zero, the interval, $[0, t]$, is proportionate to the range interval, $[0, r]$, where there is a conversion constant, c, that is the ratio of some value, r_c to some value, t_c, such that $r = (r_c/t_c)t = ct$.

Applying the ruler, to two intervals, $[0, d_1]$ and $[0, d_2]$, in two inertial (independent, non-accelerating) frames of reference, the distance and time to sequence over the subintervals the two intervals converges to a range of distances (and times) from Manhattan (3.6) to Euclidean distance (3.7).

$$r^2 = d_1^2 + d_2^2 \quad \wedge \quad r = (r_c/t_c)t = ct \\ \Rightarrow \quad (ct)^2 = d_1^2 + d_2^2 \quad \Rightarrow \quad d_2 = \sqrt{(ct)^2 - d_1^2}. \tag{5.12}$$

$$d_2 = \sqrt{(ct)^2 - d_1^2} \quad \wedge \quad d = d_2 \quad \wedge \quad d_1 = vt \\ \Rightarrow \quad d \;=\; \sqrt{(ct)^2 - (vt)^2} \;=\; ct\sqrt{1 - (v^2/c^2)}, \tag{5.13}$$

which is the spacetime dilation equation. [**Bru17**].

$$d_2^2 = (ct)^2 - d_1^2 \quad \wedge \quad s = d_2 \quad \wedge \quad d_1^2 = x^2 + y^2 + z^2 \\ \Rightarrow \quad s^2 = (ct)^2 - (x^2 + y^2 + z^2), \tag{5.14}$$

which is one form of the spacetime interval equation [**Bru17**].

5.4. 3 dimensions of physical geometry. The set and arithmetic operations used to calculate distance and volume requires sequencing through a totally ordered set of dimensions, for example, the countable distance space: $d_c = |\bigcup_{i=1}^{n} y_i|$, Euclidean distance: $d^2 = \sum_{i=1}^{n} s_i^2$, countable volume: $v_c = |\times_{i=1}^{n} x_i|$, and Euclidean volume: $v = \prod_{i=1}^{n} s_i$. The commutative property of the union, addition, and multiplication operations also allows sequencing through a set of n number of dimensions in all $n!$ number of possible orders.

Physical sets have the additional constraint that sequencing across the members of a non-empty set takes some greater than zero amount of *time*. *Determining* that a *physical* sequencer sequenced a physical set in the order, $[x_5,\ x_4,\ \cdots\ ,\ x_1]$, and next sequenced in the order, $[x_1,\ x_2,\ \cdots\ ,\ x_5]$, requires the total order, at most one successor and at most one predecessor per set member, to not change during the *time* of the sequencing. Deterministic sequencing of a totally ordered set via successor/predecessor links in each possible order requires each set member to be either a successor or predecessor to every other set member (sequentially adjacent) during the *time* of sequencing, herein referred to as a symmetric geometry.

It will now be proved that a set satisfying the constraints of a single total order and also symmetric defines a cyclic set containing at most 3 members, in this case, 3 dimensions of physical space.

Definition 5.1. Ordered geometry:

$$\forall\ i\ n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\ x_i\ \in \{x_1, \ldots, x_n\},$$
$$\textit{successor}\ x_i = x_{i+1}\ \ \wedge\ \ \textit{predecessor}\ x_{i+1} = x_i.$$

Definition 5.2. Symmetric geometry (every set member is sequentially adjacent to every other member):

$$\forall\ i\ j\ n \in \mathbb{N},\ \ \forall\ x_i\ x_j \in \{x_1, \ldots, x_n\},\ \ \textit{successor}\ x_i = x_j\ \ \Leftrightarrow\ \ \textit{predecessor}\ x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$\textit{successor}\ x_n = x_1\ \ \wedge\ \ \textit{predecessor}\ x_1 = x_n.$$

The theorem, "ordered_symmetric_is_cyclic," and formal proof is in the Coq file, threed.v.

Proof. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \textit{successor}\ x_i = x_j\ \ \Rightarrow\ \ \textit{successor}\ x_n = x_1. \tag{5.15}$$

Applying the definition of a symmetric geometry (5.2) to conclusion 5.15:

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \textit{predecessor}\ x_j = x_i\ \ \Rightarrow\ \ \textit{predecessor}\ x_1 = x_n. \tag{5.16}$$ □

Theorem 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

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Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213,
adj313, adj323, and not_all_mutually_adjacent_gt_3.
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The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

DEFINITION 5.5. Cyclic successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (n = m + 1 \leq \mathit{setsize}). \tag{5.17}$$

DEFINITION 5.6. Cyclic predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (n = m - q \geq 1). \tag{5.18}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:

$$\mathit{Adjacent}(m, n, \mathit{setsize}) \leftarrow \mathit{Successor}(m, n, \mathit{setsize}) \vee \mathit{Predecessor}(m, n, \mathit{setsize}). \tag{5.19}$$

Prove that every member is adjacent to every other member, where $\mathit{setsize} \in \{1,\ 2,\ 3\}$:

$$\mathit{Adjacent}(1, 1, 1) \leftarrow \mathit{Successor}(1, 1, 1) \leftarrow (m = \mathit{setsize} \wedge n = 1). \tag{5.20}$$

$$\mathit{Adjacent}(1, 2, 2) \leftarrow \mathit{Successor}(1, 2, 2) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{5.21}$$

$$\mathit{Adjacent}(2, 1, 2) \leftarrow \mathit{Successor}(2, 1, 2) \leftarrow (n = \mathit{setsize} \wedge m = 1). \tag{5.22}$$

$$\mathit{Adjacent}(1, 2, 3) \leftarrow \mathit{Successor}(1, 2, 3) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{5.23}$$

$$\mathit{Adjacent}(2, 1, 3) \leftarrow \mathit{Predecessor}(2, 1, 3) \leftarrow (n = m - q \geq 1). \tag{5.24}$$

$$\mathit{Adjacent}(3, 1, 3) \leftarrow \mathit{Successor}(3, 1, 3) \leftarrow (n = \mathit{setsize} \wedge m = 1). \tag{5.25}$$

$$\mathit{Adjacent}(1, 3, 3) \leftarrow \mathit{Predecessor}(1, 3, 3) \leftarrow (m = 1 \wedge n = \mathit{setsize}). \tag{5.26}$$

$$\mathit{Adjacent}(2, 3, 3) \leftarrow \mathit{Successor}(2, 3, 3) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{5.27}$$

$$\mathit{Adjacent}(3, 2, 3) \leftarrow \mathit{Predecessor}(3, 2, 3) \leftarrow (n = m - q \geq 1). \tag{5.28}$$

Must prove that for all $\mathit{setsize} > 3$, there exist non-adjacent members. For example, the first and third members are not $(\neg)$ adjacent:

$$\begin{aligned} \forall\ \mathit{setsize} > 3 : \quad & \neg\mathit{Successor}(1, 3, \mathit{setsize} > 3) \\ & \leftarrow \mathit{Successor}(1, 2, \mathit{setsize} > 3) \leftarrow (n = m + 1 \leq \mathit{setsize}). \end{aligned} \tag{5.29}$$

That is, member 2 is the only successor of member 1 for all $\mathit{setsize} > 3$, which implies member 3 is not a successor of member 1 for all $\mathit{setsize} > 3$.

$$\begin{aligned} \forall\ \mathit{setsize} > 3 : \quad & \neg\mathit{Predecessor}(1, 3, \mathit{setsize} > 3) \\ & \leftarrow \mathit{Predecessor}(1, \mathit{setsize}, \mathit{setsize} > 3) \leftarrow (m = 1 \wedge n = \mathit{setsize} > 3). \end{aligned} \tag{5.30}$$

That is, member $n = \mathit{setsize} > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $\mathit{setsize} > 3$.

$$\begin{aligned} \forall\ \mathit{setsize} > 3 : \quad & \neg\mathit{Adjacent}(1, 3, \mathit{setsize} > 3) \\ & \leftarrow \neg\mathit{Successor}(1, 3, \mathit{setsize} > 3) \wedge \neg\mathit{Predecessor}(1, 3, \mathit{setsize} > 3). \end{aligned} \tag{5.31}$$ □

That is, for all *setsize* > 3, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

Applying the ruler measure (2.1) and ruler convergence (2.2) to the set relations, countable distance space (3.1) and countable volume (4.1) yields the following insights and implications:

(1) The properties of metric space, Euclidean distance and area/volume can be derived from two set relations without using the notions of Euclidean geometry [**Joy98**] like plane, side, angle, perpendicular, congruence, intersection, etc.

(2) The ruler measure-based proofs provide the insight that distance is a function of the combinatorial domain-to-*range* set member mappings. Whereas, area/volume is a function of the combinatorial domain-to-*domain* set member mappings.

(3) The largest intersection and smallest distance, $d_c = |\bigcup_{i=1}^{n} y_i|$, has the most domain set members mapping to each range set member. The equality case, $|x_i| = |y_i| = p_i$, of the countable distance space constraint, $|x_i| = |y_i|^q$, $q \geq 1$, (3.1) limits the largest total number of domain-to-range set mappings to $\sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^2$, which is the set-based reason Euclidean distance (3.7) is the smallest possible distance between two distinct points in $\mathbb{R}^n$.

(4) It was shown that the countable distance set operation, $d_c = |\bigcup_{i=1}^{n} y_i|$, (3.1) generates three of the metric space properties. The countable distance constraint, $|x_i| = |y_i|^q$, $q \geq 1$, (3.1) is reason for the fourth property of metric space, symmetry: $d(u,v) = d(v,u)$, where the combinatorial domain-to-range set mapping is the same for every domain-range set pair.

(5) $|x_i| = |y_i|^q$, $q \geq 1$, (3.1) generates all the L^p norms, $||L||_p = (\sum_{i=1}^{n} s_i^p)^{1/p}$. For example, using the same proof pattern as for Euclidean distance (3.7): $p_i = |y_i| \Rightarrow |x_i| = p_i^q \Rightarrow \sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^{q+1} \leq d_c^{q+1} \ \ldots$.

(6) The Euclidean volume proof was used to derive the Coulomb's charge force (5.1) and Newton's gravity force (5.6) without other physics laws or Gauss's divergence theorem. The Euclidean distance proof was used to derive the spacetime equations (5.12) without a constant speed of light assumption or even the notion of light.

(7) **The Proportionate Interval Principle:** The derivations of the charge force, gravity force, and spacetime equations exposes the principle that all Euclidean distance range intervals having a size, r, have proportionately sized intervals of other types, for example: $r = (r_C/q_C)q = (r_G/m_G)m = (r_c/t_c)t$.
   (a) The proportionate interval principle combined with the Cartesian product of infinitesimal forces is the cause of the inverse square law and flux divergence.
   (b) If there are quantum values of charge, q_C, and mass, m_G, then there are quantum distances, r_C and r_G, where the forces do not exist (not defined) at smaller distances, which might have implications for high-energy particle collisions and the density of black holes.

(c) Discrete valued variables (discrete states like spin) do not have proportionate continuous distance intervals. Therefore, discrete value changes with respect to time are independent of distance (for example, the change in spin of two quantum coupled particles).

(8) Relativity theory assumes that only 3 dimensions of geometric space exist [**Bru17**]. The proof in this article (5.4) shows that time constrains physical distance and volume to at most three dimensions. Theories of higher dimensions of *physical* distance and volume would cause logical inconsistencies.

(9) The proof of at most 3 members in any set of ordered and symmetric members (5.4), implies that each infinitesimal volume (ball) can have at most 3 ordered and symmetric dimensions of discrete values of the same type. And each dimension of discrete values can have at most 3 ordered and symmetric discrete values, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values corresponding to 27 possible "types" of infinitesimal balls.

(10) If each of the three possible ordered and symmetric dimensions of discrete values contained unordered sets of discrete values, for example, unordered binary values, then there would be $2 \cdot 2 \cdot 2 = 8$ possible combinations of values. These unordered values would be non-deterministic. For example, every time a value is physically measured, there would be a 50-50 chance of having one of the binary values.

(11) Where infinitesimal balls intersect, an algebra of the interactions of the discrete values needs to be developed. The interaction of the discrete values associated with overlapping infinitesimal balls might result in what we perceive as waves, motion, mass, charge, etc.

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