# The Set Properties Generating Geometry and Physics

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Abstract. Volume and distance equations (for example, Euclidean distance) are derived from a set and limit-based foundation, without referencing the primitives and relations of geometry. An Euclidean volume proof provides an alternative to second derivative-based methods for measuring the curvature of a space and provides simpler derivations of Newton's gravity force and Coulomb's charge force equations that do not use the inverse square law or Gauss's divergence theorem. The derivations of the gravity and charge forces exposes a ratio (constant first derivative) principle that allows simpler derivations of the spacetime and some general relativity equations. A symmetry property can limit an ordered set to 3 members (for example, 3 dimensions). All proofs are verified in Coq.

## Contents

## 1. Introduction

Mathematical (real) analysis can construct differential calculus from a set and limit-based foundation, without the need to reference the primitives and relations of Euclidean geometry, like side, angle, slope, etc. But the Riemann and Lebesgue integrals and measure theory (for example, Hilbert spaces and the Lebesgue measure) use the Euclidean volume equation as a definition. And the vector norm and metric space use Euclidean distance and its properties as definitions [**Gol76**] [**Rud76**]. Here, these definitions are derived from a set and limit-based foundation without referencing the primitives and relations Euclidean geometry.

"Countable volume", $v_c$, is defined as the cardinal of an ordered, countable set of n-tuples, where each n-tuple is a set of integer indexes into disjoint, countable, sets: $x_1, \cdots, x_n$. Therefore, the cardinal, $v_c$, is equal to the product of the domain set cardinals, $|x_i|$ : $v_c = \prod_{i=1}^{n} |x_i|$.

Where a real value, $v$, is proportionate to the countable number of n-tuples, $v_c$, $\exists\, c \in \mathbb{R} : v = v_c \cdot c$. Proving that $\lim_{c \to 0} v_c \cdot c = \lim_{c \to 0} (\prod_{i=1}^{n} |x_i|) \cdot c$ is the Euclidean volume equation provides an alternative to second derivative-based methods, like the Laplacian and tensors, for measuring the curvature of a space. And the proof has other applications to physics.

Every Euclidean and non-Euclidean n-dimensional volume (n-volume) is a magnitude, $v$, which has a corresponding cuboid volume, $v = d^n$, where each domain interval has the size, $d \in \mathbb{R}$. And all volumes being instances of countable volume implies that there is also a countable cuboid volume, $v_c = d_c^n$, where each domain set has the cardinal, $d_c$.

An n-volume can only be the sum of n-volumes. And a countable n-volume can only be the sum of countable n-volumes. It will be proved that the $L_p$ norms (Minkowski distances), $d = (\sum_{i=1}^{m} s_i^n)^{1/n}$, are instances of the sum of countable cuboid volumes, which have the metric space properties. All "geometric" distances are functions of the sum of n-volumes.

If a set of n number of domain intervals/dimensions have a strict linear order, then sequencing in all n-at-time permutations (orders) requires that each set member is sequentially adjacent, either a successor or predecessor, to every other member (a "symmetry" property), which limits the number of members in a strict linearly ordered set.

All the proofs in this article are trivial. But to ensure confidence, all the proofs have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

## 2. Ruler measure and convergence

In order to compute areas and volumes, Riemann and Lebesgue integrals divide all intervals into the *same* number subintervals (infinitesimals, for example: $dx$, $dy$, $dz$), where the size of the infinitesimals *vary* with the size of the intervals. The varying size of infinitesimals makes it difficult for integrals (and differential equations) to directly express the Cartesian mappings between the $p_x$ number of size $c$ infinitesimals in one domain interval and the $p_y$ number of the *same* size $c$ infinitesimals in a different-sized domain interval. Further, using integrals that define Euclidean volume to derive Euclidean volume would be circular logic.

Therefore, a different tool is used here. A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, $p$, of whole subintervals (infinitesimals), where each infinitesimal has the *same* size, $c$. The ruler is both an inner and outer measure of an interval.

DEFINITION 2.1. Ruler measure, $M$: $\forall\, [a, b] \subset \mathbb{R},\ \ s = b - a\ \ \wedge\ \ c > 0\ \ \wedge$ $(p = floor(s/c)\ \ \vee\ \ p = ceiling(s/c))\ \ \wedge\ \ M = \sum_{i=1}^{p} c = pc$.

THEOREM 2.2. *Ruler convergence:* $M = \lim_{c \to 0} pc = s$.

The formal proof, "limit_c_0_M_eq_exact_size," in the Coq file, euclidrelations.v.

PROOF. (epsilon-delta proof)

By definition of the floor function, $floor(x) = max(\{y : y \leq x, y \in \mathbb{Z}, x \in \mathbb{R}\})$:

(2.1)   $\forall c > 0, \; p = floor(s/c) \; \wedge \; 0 \leq |floor(s/c) - s/c| < 1 \; \Rightarrow \; |p - s/c| < 1.$

Multiply both sides of inequality 2.1 by $c$:

(2.2)   $\forall c > 0, \quad |p - s/c| < 1 \quad \Rightarrow \quad |pc - s| < |c| = |c - 0|.$

(2.3)   $\forall \epsilon = \delta \quad \wedge \quad |pc - s| < |c - 0| < \delta$

$\Rightarrow \quad |c - 0| < \delta \quad \wedge \quad |pc - s| < \delta = \epsilon \quad := \quad M = \lim_{c \to 0} pc = s. \quad \square$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}}, \; 3.14_{c=10^{-2}}, \; 3.141_{c=10^{-3}}, ..., \pi_{\lim_{c \to 0}}.$

LEMMA 2.3. $\forall n \geq 1, \quad 0 < c < 1 \quad \Rightarrow \quad \lim_{c \to 0} c^n = \lim_{c \to 0} c.$

PROOF. The formal proof, "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

(2.4)   $n \geq 1 \quad \wedge \quad 0 < c < 1 \quad \Rightarrow \quad 0 < c^n < c \quad \Rightarrow \quad |c - c^n| < |c| = |c - 0|.$

(2.5)   $\forall \epsilon = \delta \quad \wedge \quad |c - c^n| < |c - 0| < \delta$

$\Rightarrow \quad |c - 0| < \delta \quad \wedge \quad |c - c^n| < \delta = \epsilon \quad := \quad \lim_{c \to 0} c^n = 0.$

(2.6)   $\lim_{c \to 0} c^n = 0 \quad \wedge \quad \lim_{c \to 0} c = 0 \quad \Rightarrow \quad \lim_{c \to 0} c^n = \lim_{c \to 0} c. \quad \square$

## 3. Euclidean Volume

DEFINITION 3.1. Countable volume, $v_c$ is the number of Cartesian product mappings (n-tuples) between the members of $n$ number of disjoint, countable domain sets:

$$\exists \, n, v_c \in \mathbb{N}, \quad x_1, \cdots, x_n : \quad v_c = \prod_{i=1}^{n} |x_i|, \quad \bigcap_{i=1}^{n} x_i = \emptyset$$

THEOREM 3.2. *Euclidean volume,* $v = \prod_{i=1}^{n} s_i$, *where $v$ is the size of the range interval and $s_i$ is the size of a domain interval, is the case of the number of n-tuples, $v_c$, times an interval size, c.*

$$v_c = \prod_{i=1}^{n} |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^{n} s_i, \; v = v_a - v_b, \; s_i = b_i - a_i.$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.

Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, $x_i$, containing $|x_i|$ number of size $c$ subintervals and apply ruler convergence (2.2):

(3.1)   $\forall \, i \, n \in \mathbb{N}, \; i \in [1, n], \; c > 0 \; \wedge \; floor(s_i/c) = |x_i| \; \Rightarrow \; s_i = \lim_{c \to 0}(|x_i| \cdot c).$

(3.2)   $s_i = \lim_{c \to 0}(|x_i| \cdot c) \quad \Leftrightarrow \quad \prod_{i=1}^{n} s_i = \prod_{i=1}^{n} \lim_{c \to 0}(|x_i| \cdot c).$

(3.3)   $\prod_{i=1}^{n} s_i = \prod_{i=1}^{n} \lim_{c \to 0}(|x_i| \cdot c) \quad \Leftrightarrow \quad \prod_{i=1}^{n} s_i = \lim_{c \to 0}(\prod_{i=1}^{n} |x_i|) \cdot c^n.$

Apply lemma 2.3 to equation 3.3:

(3.4)   $\prod_{i=1}^{n} s_i = \lim_{c \to 0}(\prod_{i=1}^{n} |x_i|) \cdot c^n \quad \wedge \quad \lim_{c \to 0} c^n = \lim_{c \to 0} c$

$\Leftrightarrow \quad \prod_{i=1}^{n} s_i = \lim_{c \to 0}(\prod_{i=1}^{n} |x_i|) \cdot c.$

Apply the ruler (2.1) and ruler convergence (2.2):

$$(3.5) \qquad \exists\, v \in \mathbb{R} : v_c = floor(v/c) \quad \Leftrightarrow \quad v = \lim_{c \to 0} v_c \cdot c.$$

Apply the definition of the countable volume (3.1):

$$(3.6) \qquad v_c = \prod_{i=1}^{n} |x_i| \quad \Leftrightarrow \quad \lim_{c \to 0} v_c \cdot c = \lim_{c \to 0} \left(\prod_{i=1}^{n} |x_i|\right) \cdot c.$$

Combine equations 3.5, 3.6, and 3.4:

$$(3.7) \quad v = \lim_{c \to 0} v_c \cdot c \quad \wedge \quad \lim_{c \to 0} v_c \cdot c = \lim_{c \to 0}\left(\prod_{i=1}^{n} |x_i|\right) \cdot c \quad \wedge$$
$$\lim_{c \to 0}\left(\prod_{i=1}^{n} |x_i|\right) \cdot c = \prod_{i=1}^{n} s_i \quad \Leftrightarrow \quad v = \prod_{i=1}^{n} s_i. \quad \square$$

# 4. Distance

## 4.1. Countable cuboid volume.

DEFINITION 4.1. The countable cuboid volume, $d_c^n$, is the sum of m number of sets of countable cuboid volumes.

$$\forall\, n \in \mathbb{N}, \quad d_c \in \{0, \mathbb{N}\} \quad \exists\, m \in \mathbb{N}, \quad x_1, \cdots, x_m \in X, \quad \bigcap_{i=1}^{m} x_i = \emptyset :$$
$$d_c^n = \sum_{i=1}^{m} |x_i|^n.$$

## 4.2. Minkowski distance ($L_p$ norm).

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

THEOREM 4.2. *Minkowski distance ($L_p$ norm) is an instance of the countable cuboid volume (4.1).*

$$d_c^n = \sum_{i=1}^{m} |x_i|^n \quad \Rightarrow \quad \exists\, d, s_1, \cdots, s_m \in \mathbb{R} : \quad d = \left(\sum_{i=1}^{m} s_i^n\right)^{1/n}.$$

PROOF. Apply the ruler (2.1):

$$(4.1) \qquad \exists\, d, s_1, \cdots, s_m \in \mathbb{R} : d_c = floor(d/c) \quad \wedge \quad |x_i| = floor(s_i/c).$$

Apply the ruler convergence (2.2):

$$(4.2) \quad d_c^n = \sum_{i=1}^{m} |x_i|^n \Rightarrow d^n = \lim_{c \to 0}(d_c \cdot c)^n = \lim_{c \to 0} \sum_{i=1}^{m}(|x_i| \cdot c)^n = \sum_{i=1}^{m} s_i^n.$$

$$(4.3) \qquad d^n = \sum_{i=1}^{m} s_i^n \quad \Leftrightarrow \quad d = \left(\sum_{i=1}^{m} s_i^n\right)^{1/n}. \qquad \square$$

## 4.3. Distance inequality.
Proving that all Minkowski distances ($L_p$ norms) satisfy the metric space triangle inequality requires another inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N}, \ v_a, v_b \geq 0 : \ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand the n-volume, $(v_a^{1/n} + v_b^{1/n})^n$, using the binomial expansion:

$$(4.4) \quad \forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq (v_a + v_b +$$
$$\sum_{i=1}^{n} \binom{n}{k}(v_a^{1/n})^{n-k}(v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k}(v_a^{1/n})^k(v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n.$$

Take the $n^{th}$ root of both sides of the inequality:

$$(4.5) \ \forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad \square$$

**4.4. Distance sum inequality.** The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N},\ a_i, b_i \geq 0:\ (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$(4.6) \quad \forall\, m, n \in \mathbb{N},\ v_a, v_b \geq 0:\quad v_a = \textstyle\sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge$$
$$(v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad ((\textstyle\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} =$$
$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \quad \square$$

**4.5. Metric Space.** All Minkowski distances ($L_p$ norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$$d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k:$

$$(4.7) \qquad (u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}.$$

Apply the distance inequality (4.3) to the inequality 4.7:

$$(4.8) \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \ \wedge\ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}$$
$$\wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p$$
$$\Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p}$$
$$\Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq$$
$$(u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \quad \square$$

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$(4.9) \quad \forall\, p\ :\ p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p}$$
$$\Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \quad \square$$

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

PROOF. By definition, the length of an interval is always $\geq 0$:

$$(4.10) \qquad \forall\, [a_1, b_1], [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0.$$

$$(4.11) \qquad p \geq 1,\ u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \qquad \square$$

THEOREM 4.8. *Identity of Indiscernibles:* $d(u, u) = 0.$

PROOF. From the non-negativity property (4.7):

$$(4.12) \quad d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0$$
$$\Rightarrow \quad \exists \, d(u, w) = d(u, v) = d(v, w) = 0.$$

$$(4.13) \qquad\qquad d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v.$$

$$(4.14) \qquad\qquad d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \qquad \square$$

## 5. Applications to physics

**5.1. 3 dimensional balls.** The following proofs are in the physics section because limiting the domain intervals/dimensions to a strict linearly ordered set, where every interval is sequentially adjacent, is an additional restriction on volume and distance used to explain why physical space is limited to a cyclic set of 3 dimensions.

DEFINITION 5.1. Strict linearly ordered set:

$$\forall \, i \, n \in \mathbb{N}, \ i \in [1, n-1], \ \forall \, x_i \in \{x_1, \ldots, x_n\},$$
$$successor \ x_i = x_{i+1} \ \wedge \ predecessor \ x_{i+1} = x_i.$$

DEFINITION 5.2. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall \, i \, j \, n \in \mathbb{N}, \ \forall \, x_i \, x_j \in \{x_1, \ldots, x_n\}, \ successor \ x_i = x_j \ \Leftrightarrow \ predecessor \ x_j = x_i.$$

THEOREM 5.3. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n \ \wedge \ j = 1 \ \Rightarrow \ successor \ x_n = x_1 \ \wedge \ predecessor \ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of $x_n$ and the predecessor of $x_1$. Therefore, the only member that can be a successor of $x_n$, without creating a contradiction, is $x_1$. And the only member that can be a predecessor of $x_1$, without creating a contradiction, is $x_n$. Applying the symmetry property (5.2):

$$(5.1) \qquad i = n \ \wedge \ j = 1 \ \wedge \ successor \ x_i = x_j \ \Rightarrow \ successor \ x_n = x_1.$$

Applying the definition of the symmetry property (5.2) to conclusion 5.1:

$$(5.2) \quad successor \ x_i = x_j \ \Rightarrow \ predecessor \ x_j = x_i \ \Rightarrow \ predecessor \ x_1 = x_n. \qquad \square$$

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

Lemmas: `adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323,` and `not_all_mutually_adjacent_gt_3`.

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3).

DEFINITION 5.5. Successor of $m$ is $n$:

$$(5.3) \quad Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize).$$

DEFINITION 5.6. Predecessor of $m$ is $n$:

$$(5.4) \quad Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1).$$

DEFINITION 5.7. Adjacent: member $m$ is sequentially adjacent to member $n$ if the successor of $m$ is $n$ or the predecessor of $m$ is $n$. Notionally:
$$(5.5)$$
$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize).$$

Prove that every member is adjacent to every other member, where $setsize \in \{1, 2, 3\}$:

$$(5.6) \qquad Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1).$$

$$(5.7) \qquad Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize).$$
$$(5.8) \qquad Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1).$$

$$(5.9) \qquad Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize).$$
$$(5.10) \qquad Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1).$$
$$(5.11) \qquad Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1).$$
$$(5.12) \qquad Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize).$$
$$(5.13) \qquad Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize).$$
$$(5.14) \qquad Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1).$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$(5.15) \quad \forall \, setsize > 3 : \quad \neg Successor(1, 3, setsize > 3)$$
$$\leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize).$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$(5.16) \quad \forall \, setsize > 3 : \quad \neg Predecessor(1, 3, setsize > 3)$$
$$\leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$(5.17) \quad \forall \, setsize > 3 : \quad \neg Adjacent(1, 3, setsize > 3)$$
$$\leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

**5.2. Spacetime equations.** Equating different types of interval lengths requires unit-factoring conversion ratios. For example, the geometric interval length, $r$, can be related to the time interval length, $t$, by a unit-factoring conversion ratio: $r_c/t_c = c : r = (r_c/t_c)t = ct$. And the unit-factoring conversion ratio, $c$, allows the time volume, $(ct)^n$, to be equated to the geometric volume, $r^n$:

$$(5.18) \qquad r = ct \quad \Rightarrow \quad (ct)^n = r^n.$$

And from the Minkowski distance proof every cuboid volume is the sum of cuboid volumes (4.2). And Euclidean distance is a Minkowski distance. Therefore:

$$(5.19) \qquad (ct)^2 = r^2 \quad \wedge \quad r^2 = r_1^2 + r_2^2 \quad \Rightarrow \quad (ct)^2 = r_1^2 + r_2^2.$$

$$(5.20) \quad (ct)^2 = r_1^2 + r_2^2 \quad \wedge \quad r_2^2 = x^2 + y^2 + z^2 \quad \Rightarrow \quad r_1^2 = (ct)^2 - (x^2 + y^2 + z^2),$$

which is one form of Minkowski's flat spacetime interval equation [**Bru17**]. And the length contraction and time dilation equations also follow directly from $(ct)^2 = r_1^2 + r_2^2$, where $v = r_1/t$:

$$(5.21) \quad r_2^2 = (ct)^2 - r_1^2 \ \wedge \ L = r_2 \ \Rightarrow \ L^2 = c^2t^2 - r_1^2 \ \Rightarrow \ L = ct\sqrt{1 - (v/c)^2}.$$

$$(5.22) \qquad L = ct\sqrt{1 - (v/c)^2} \quad \wedge \quad L_0 = ct \quad \Rightarrow \quad L = L_0\sqrt{1 - (v/c)^2}.$$

$$(5.23) \qquad L = ct\sqrt{1 - (v/c)^2} \ \wedge \ t' = L/c \ \Rightarrow \ t' = t\sqrt{1 - (v/c)^2}.$$

**5.3. Newton's gravity force equation.** $m_1$ and $m_2$, are the sizes of two independent mass intervals, where each size $c$ component of a mass interval exerts a force on each size $c$ component of the other mass interval. If $p_1$ and $p_2$ are the number of size $c$ components in each mass interval, then the total force, $F$, is equal to the total number of forces, $p_1 \cdot p_2$, and proportionate to the size, $c$, of each component. Applying the ruler (2.1) and volume proof (3.2), where the force, $F$, is defined as the rest mass, $m_0$, times acceleration, $a$:

$$(5.24) \quad p_1 = floor(m_1/c) \quad \wedge \quad p_2 = floor(m_2/c) \quad \wedge \quad F := m_0 a \propto (p_1 \cdot p_2)c$$
$$\Rightarrow \quad F := m_0 a \propto \lim_{c \to 0}(p_1 \cdot p_2)c = \lim_{c \to 0}(p_1 \cdot p_2)c^2 = \lim_{c \to 0} p_1 c \cdot p_2 c = m_1 m_2,$$

$$(5.25) \quad F := m_0 a := m_0 r/t^2 \propto m_1 m_2 \ \wedge \ m_0 = m_1 \ \Rightarrow \ r \propto m_2 \ \Rightarrow$$
$$\exists \, m_G, r_c \in \mathbb{R} : \ r = (dr/dm)m_2 = (r_c/m_G)m_2,$$

where: $r$ is Euclidean distance, $t$ is time, and $r_c/m_G$ is a unit-factoring ratio.

$$(5.26) \quad m_0 = m_1 \quad \wedge \quad r = (m_G/r_c)m_2 \quad \wedge \quad F = m_0 r/t^2$$
$$\Rightarrow \quad F = m_0 r/t^2 = (r_c/m_G)m_1 m_2/t^2.$$

From the derivation of the spacetime equations:

$$(5.27) \quad \exists \, t_c, r_c \in \mathbb{R} : \ t = (t_c/r_c)r \quad \wedge \quad F = (r_c/m_G)m_1 m_2/t^2 \quad \Rightarrow$$
$$F = (r_c/m_G)(r_c^2/t_c^2)m_1 m_2/r^2 = (r_c^3/m_G t_c^2)m_1 m_2/r^2 = Gm_1 m_2/r^2,$$

where the gravitational constant, $G = r_c^3/m_G t_c^2$, has the SI units: $m^3 kg^{-1} s^{-2}$.

**5.4. Coulomb's charge force.** $q_1$ and $q_2$, are the sizes of two independent charge intervals, where each size $c$ component of a charge interval exerts a force on each size $c$ component of the other charge interval. If $p_1$ and $p_2$ are the number of size $c$ components in each charge interval, then the total force, $F$, is equal to the total number of forces, $p_1 \cdot p_2$, and proportionate to the size, $c$, of each component. Applying the ruler (2.1) and volume proof (3.2), where the force, $F$, is defined as the rest mass, $m_0$, times acceleration, $a$:

$$(5.28) \quad p_1 = floor(q_1/c) \quad \wedge \quad p_2 = floor(q_2/c) \quad \wedge \quad F \propto (p_1 \cdot p_2)c$$
$$\Rightarrow \quad F := m_0 a \propto \lim_{c \to 0}(p_1 \cdot p_2)c = \lim_{c \to 0}(p_1 \cdot p_2)c^2 = \lim_{c \to 0} p_1 c \cdot p_2 c = q_1 q_2,$$

$$(5.29) \quad F := m_0 a := m_0 r/t^2 \propto q_1 q_2 \quad \wedge$$
$$m_0 = (dm/dq)q_1 = (m_G/q_C)q_1 \quad \Rightarrow \quad r \propto q_2$$
$$\Rightarrow \quad \exists \, q_C, r_c \in \mathbb{R} : \, r = (dr/dq)q_2 = (r_c/q_C)q_2,$$

where: $r$ is Euclidean distance, $t$ is time, $m_G/q_C$ and $r_c/q_C$ are unit-factoring ratios.

$$(5.30) \quad m_0 = (m_G/q_C)q_1 \quad \wedge \quad r = (q_C/r_c)q_2 \quad \wedge \quad F = m_0 r/t^2$$
$$\Rightarrow \quad F = m_0 r/t^2 = (m_G/q_C)(r_c/q_C)q_1 q_2/t^2 = (m_G r_c/q_C^2)q_1 q_2/t^2.$$

From the derivation of the spacetime equations:

$$(5.31) \quad \exists \, t_c, r_c \in \mathbb{R} : \, t = (t_c/r_c)r \quad \wedge \quad a_G = r_c/t_c^2 \quad \wedge \quad F = (m_G r_c/q_C^2)q_1 q_2/t^2$$
$$\Rightarrow \quad F = (r_c^2/t_c^2)(m_G r_c/q_C^2)q_1 q_2/r^2 = ((m_G a_G)r_c^2/q_C^2)q_1 q_2/r^2 = k_c q_1 q_2/r^2,$$

where the charge constant, $k_C = (m_G a_G)r_c^2/q_C^2$, has the SI units: $Nm^2 C^{-2}$.

**5.5. Some general relativity equations:** Combining the ratio (constant first derivative) equations into partial differential equations: $r = (r_c/m_G)m = ct \Rightarrow (r_c/m_G)m \cdot ct = r^2 \Rightarrow m = (m_G/r_c c)r^2/t = (m_G/r_c c)rv$. For a constant mass, $m$, a decrease in the distance, $r$, between two mass centers causes a decrease in time, $t$, (time slows down). $v$ is the relativistic orbital velocity at distance, $r$. $(r_c/m_G)m \cdot (ct)^2 = r^3 \Rightarrow E = mc^2 = (m_G/r_c)r^3/t^2$. And $(ct)^2 = r^2 \Rightarrow c^2 = v^2 \Rightarrow (r_c/m_G)mv^2 = c^2 r \Rightarrow KE = mv^2/2 = (m_G c^2/2r_c)r$.

$c = r_c/t_c \approx 3 \cdot 10^8 ms^{-1}$ and $G = r_c^3/m_G t_c^2 = (r_c/m_G)(r_c/t_c)^2 \approx 6.7 \cdot 10^{-11}m^3 kg^{-1}s^{-2} \Rightarrow r_c/m_G \approx 6.7 \cdot 10^{-11}m^3 kg^{-1}s^{-2}/(3 \cdot 10^8 m \ s^{-1})^2 \approx 7.4 \cdot 10^{-28}m \ kg^{-1}$, which can be used to quantify the constants in the previously derived equations. For example, $m = (m_G/r_c c)rv \approx (1/((7.4 \cdot 10^{-28}m \ kg^{-1})(3 \cdot 10^8 m \ s^{-1})))rv \approx (4.5 \cdot 10^{18}kg \ s \ m^{-2})rv$.

Likewise, for charge, $r = (r_c/q_C)q = ct \Rightarrow q = (q_C/r_c c)r^2/t = (q_C/r_c c)rv$, $E = qc^2 = (q_C/r_c)r^3/t^2$, and $KE = qv^2/2 = (q_C c^2/2r_c)r$. And if the ratio of an electron's mass to charge is $m_G/q_C$, then $m_G/q_C \approx 9.1 \cdot 10^{-31}kg/1.6 \cdot 10^{-19}C \approx 5.7 \cdot 10^{-12}kgC^{-1}$. And using Coulomb's constant in ratio form: $k_C = (r_c/t_c)^2(m_G r_c/q_C^2) \approx 9 \cdot 10^9 Nm^2 C^{-2} \approx (3 \cdot 10^8 m \ s^{-1})^2(5.7 \cdot 10^{-12}kg \ C^{-1})(r_c/q_c) \Rightarrow r_c/q_C \approx 1.7 \cdot 10^5 m \ C^{-1}$. Therefore, $q = (q_C/r_c c)rv \approx (1/((1.7 \cdot 10^5 m \ C^{-1})(3 \cdot 10^8 m \ s^{-1})))rv \approx (1.9 \cdot 10^{-13}C \ s \ m^{-2})rv$.

## 6. Insights and implications

(1) The properties of both volume and distance are functions of countable sets of Cartesian n-tuples, where each n-tuple is a set of integer indexes into corresponding countable domain sets.

(2) The second derivative is the traditional way of measuring the curvature of a space, for example, the Laplacian and tensors. The Euclidean volume proof (3.2) suggests a measure of curvature is the ratio of adjacent range infinitesimals, where a set of n-tuples indexing into size $c$ domain infinitesimals map bijective to the set of range infinitesimals. Flat space is where the ratio of a range infinitesimal to a neighbor range infinitesimal is one and the second derivative is zero.

In other words, the derivative-based measures are the ratio of a range infinitesimal to a domain infinitesimal, where the domain infinitesimal size varies with the length of domain interval. And, here, the ratio of two range infinitesimals are measured, where the domain infinitesimals are all the same size, $c$, independent of the domain interval lengths.

(3) All volumes having corresponding cuboid volumes and all "geometric" distances being functions of the sum of volumes implies that all geometric distance functions have corresponding functions that are sum of cuboid volumes, which are the Minkowski distances (4.2). And it was proved that the Minkowski distances have the properties of metric space (4.5). Therefore, if the definition of a complete metric space allows functions are not a function of the sum of n-volumes, then the definition of a complete metric space is not a sufficient filter to obtain only geometric distances.

(4) Propositional logic proofs that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line (equation), where it is assumed that the straight line length is the smallest distance [**Joy98**]. And proofs that a straight line (equation) is the smallest distance have equated the straight line to the Euclidean distance.

All distance measures in Euclidean space have equivalent Minkowski distances (4.2). In the Minkowski distance equation, $d = (\sum_{i=1}^{m} s_i^n)^{1/n}$, if $m$ represents the number of domain intervals, one interval from each dimension, then $1 \le n \le m$. And $m = 2 \Rightarrow 1 \le n \le 2$, which constrains all Minkowski distances to a range from Manhattan distance (the largest distance) to Euclidean distance (the smallest distance) in Euclidean (flat) 2-space.

(5) Hilbert spaces allow fractional dimensions (fractals), which is the case of intersecting domain sets and requires generalizing the countable volume definition (3.1) to: $v_c = \prod_{i=1}^{n}(|x_i| - |x_i \bigcap(\bigcup_{j=1, i\ne j}^{n} x_j)|)$.

(6) Compare the distance sum inequality (4.4):

$$(\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \le (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

to Minkowski's sum inequality:

$$(\sum_{i=1}^{m}(a_i^n + b_i^n)^n)^{1/n} \le (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

Note the difference in the left side of the two inequalities:

$$(\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \quad vs. \quad (\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n}.$$

Minkowski's sum inequality proof assumes: convexity and the $L_p$ space inequalities (for example, Hölder's inequality or Mahler's inequality) or the triangle inequality. In contrast, the distance (sum) inequality is a more fundamental inequality that does not require the assumptions of the Minkowski sum inequality.

(7) Applying the ruler (2.1) and volume proof (3.2) to the Cartesian product of same-sized, infinitesimal mass forces and charge forces to derive Newton's gravity force (5.3) and Coulomb's charge force (5.4) equations provide several firsts and some insights into physics:

 (a) These are the first derivations to not use the inverse square law or Gauss's divergence theorem.

 (b) These are the first derivations to show that the definition of force, $F := m_0 a$, containing acceleration, $a = r/t^2$, where $r$ is a distance that is proportionate to time, $t$, generates the inverse square law. Using the same derivation steps as for Coulomb's charge force (5.4): $F := m_0 a := m_0 r/t^2 = (r_c/t_c)^2 (m_y r_c/x_y^2) x_1 x_2 / r^2 = k_y x_1 x_2 / r^2$.

 (c) Using Occam's razor, those versions of constants like: charge, vacuum magnetic permeability, fine structure, etc. that contain the value $4\pi$ might be incorrect because those constants are based on the less parsimonious assumption that the inverse square law is due to Gauss's flux divergence on a sphere having the surface area, $4\pi r^2$.

 (d) These are the first derivations to show that the gravity force, charge force, spacetime, and general relativity equations all depend on time being proportionate to distance: $r = (r_c/t_c)t = ct$.

 (e) Normally, time, mass, and charge are considered to be independent of distance. But derivations of the special relativity (5.2), gravity force (5.3), and charge force (5.4) equations show that they are dependent variables with respect to distance: $r = (r_c/t_c)t = ct$, $r = (r_c/m_G)m$, and $r = (r_c/q_C)q$. This implies that if the size of space is increasing, then time, mass, and charge are also increasing (and the converse).

 (f) The derivations of the special relativity, gravity and charge force equations expose a unit-factoring conversion ratio (constant first derivative) principle. Combining the ratios into equations allows simple algebraic derivations of some general relativity equations (5.5) without the need for solving second derivative (spacetime curvature) tensors.

 (g) A state is represented by a constant value. And a constant value, by definition, cannot vary with distance and time interval lengths. Therefore, the spin states of two quantum entangled electrons and the polarization states of two quantum entangled photons are independent of the amount of distance and time between the entangled particles.

(8) It was proved that sequencing through a set of n members, having a strict linear order, in all n-at-time permutations, a symmetric set, is a cyclic set with at most 3 members (5.3). And empirical observation indicates that the dimensions of physical space are such symmetric sets.

    (a) Using Occam's razor, a strict linear order and symmetry is the most parsimonious explanation for only observing 3 dimensions of geometric distance and volume.

    (b) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (5.4), each member being an ordered and symmetric set of 3 dimensions (three balls).

    (c) Each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values of the same type, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values of the same type per 3-dimensional ball, for example, vector orientation values: -1, 0, 1 per dimension in the ball.

    (d) The strict linear order and symmetry of the 3 dimensions of physical space and vector orientation is what makes physical space chiral (allows mirror images and clockwise/counter-clockwise rotations).

    (e) Each of the 3 possible ordered and symmetric dimensions of discrete physical states could contain an unordered collection (bag) of discrete state values. Bags are non-deterministic. For example, every time an unordered binary state is "pulled" from a bag, there is a 50 percent chance of getting one of the binary values.

(9) It was shown that some fundamental geometry (volume and the Minkowski distances/$L_p$ norms) and physics (gravity force and charge force) are derived from the combinatorial mappings between the infinitesimals of real-valued intervals. The proofs and derivations in this article show that the ruler (2.1) is a tool to directly express and solve some combinatorial relations in geometry, probability, physics, etc. that are difficult to directly express with differential equations and integrals.

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