Some Set Properties Underlying Geometry and Physics

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Abstract. Euclidean volume and the Minkowski distances are proved to instances of a set of n-tuples. A commutative property is proved to limit distance to a set of 3 dimensions. Other compact and continuous dimensions have different types (are members of different sets), with unit-factoring ratios of a distance unit to units of other types (time, mass, and charge). The proofs and ratios are used to derive the gravity, charge, permitivity, Planck, and fine structure constants, and used to provide simple derivations of some well-known classical gravity and charge equations, special and general relativity equations, and quantum physics equations. All the proofs are verified in Coq.

Contents

1. Introduction

The Riemann integral, Lebesgue integral, and Lebesgue measure define Euclidean volume as the product of interval sizes. And analysis defines Euclidean distance, inner product, and metric space, etc. [Gol76] [Rud76].

In an effort to make analysis self-contained, analysis: 1) defines the product of interval sizes to be a subset of $\mathbb{R}^n$, where each n-tuple corresponds to a real value, and 2) every real value returned by a function defining a line corresponds to an n-tuple of domain values. Both volume and distance expressed in terms of a single, abstract, combinatorial foundation, a foundation of an ordered set of combinations (n-tuples), might provide a deeper understanding of some aspects of geometry and physics.

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Here, proofs using n-tuples will use the calculus method of summing a countable infinity of infinitesimal intervals $\in \mathbb{R}$. Where $|x_i|$ is the cardinal of a countable set, x_i, the Euclidean number of n-tuples, v_c, is where the sets are disjoint:

$$\forall\, x_i \in \{x_1, \cdots, x_n\}, \quad \bigcap_{i=1}^{n} x_i = \emptyset : \quad v_c = \prod_{i=1}^{n} |x_i|. \tag{1.1}$$

The goal, now, is to prove that the *only* equation implied by the countable set of n-tuples is the Euclidean volume equation and the *only* countable set implied by the Euclidean distance equation is the countable set of n-tuples:

$$\forall\, v_c = \prod_{i=1}^{n} |x_i| \Leftrightarrow v = \prod_{i=1}^{n} s_i, \quad [a_i, b_i] \in \mathbb{R}, \quad s_i = b_i - a_i. \tag{1.2}$$

But the calculus method of dividing all the domain and range intervals into the same number, p, of subintervals results in the circular logic of an n-volume defined as the sum of n-volumes. Therefore, a "ruler" measure will be used to prove proposition 1.2.

If $f(|x_1, \cdots, |x_n|)$ is a bijective and isomorphic function, then:

$$\exists\, d_c \in \{0, \mathbb{N}\} : d_c = |x_1| = \cdots = |x_n| \quad \Rightarrow \quad v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n. \tag{1.3}$$

Where f is bijective and isomorphic, the ruler measure will be used to prove that:

$$v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n \quad \Rightarrow \quad d^n = \sum_{i=1}^{m} d_i^n. \tag{1.4}$$

d is the ρ-norm (Minkowski distance) [**Min53**], which will be proved to imply the metric space properties [**Rud76**].

The commutative law of multiplication allows any order of multiplication of $|x_i|$, s_i, and d_i. And the commutative law of summation allows any order of adding volumes. Where n is known, **only** a cyclic order allows starting with any set member for multiplication or addition. And the commutative laws allow any other set member to be sequenced as the second member for multiplication or addition, etc. Sequencing a cyclic set in every possible order (permutation) for multiplication or addition is a "symmetry" property, where every set member is either an *immediate* successor or an *immediate* predecessor to every other set member. A symmetric cyclic set will be proved to have $n \leq 3$ members.

Therefore, if $\{s_1, s_2, s_3\}$ is a symmetric cyclic set of 3 "distances," then another element, s_4, must have a different type (member of different set), with constant, unit-factoring ratios of a unit of distance, r, to units of other compact and continuous types. For example: $r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q = \cdots$.

The ratios are used to derive the gravity (G), charge (k_e), vacuum permitivity (ε_0), Planck (h), and fine structure (α) constants. In other words, the ratios (and quantum units: r_c, t_c, m_c, and q_c) are fundamental but G, k_e, ε_0, h, and α are **not** fundamental constants.

The proofs and ratios are used to provide simpler, more rigorous derivations of some well-known classical gravity and charge equations, special and general relativity equations, and quantum physics equations. The ratios are also used to add quantum effects to general relativity and classical physics equations.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq23**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of size κ subintervals. The ruler is both an inner and outer measure of an interval.

DEFINITION 2.1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\, [a, b] \subset \mathbb{R}$, $s = b - a \;\wedge\; 0 < \kappa \leq 1 \;\wedge\; (p = floor(s/\kappa) \;\vee\; p = ceiling(s/\kappa))$.

THEOREM 2.2. *Ruler convergence:* $M = \lim_{\kappa \to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y:\; y \leq x,\, y \in \mathbb{Z},\, x \in \mathbb{R}\})$:

$$\forall\, \kappa > 0,\, p = floor(s/\kappa) \;\wedge\; 0 \leq |floor(s/\kappa) - s/\kappa| < 1 \;\Rightarrow\; |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by κ:

$$\forall\, \kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \delta = \epsilon \quad := \quad M = \lim_{\kappa \to 0} p\kappa = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}},\; 3.14_{\kappa=10^{-2}},\; 3.141_{\kappa=10^{-3}}, \ldots, \pi_{\lim_{\kappa \to 0}}$.

LEMMA 2.3. $\forall\, n \geq 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa$.

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa \to 0} \kappa^n = 0. \end{aligned} \tag{2.5}$$

$$\lim_{\kappa \to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa \to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa. \qquad \square \tag{2.6}$$

3. Volume

DEFINITION 3.1. A countable n-volume is the number of ordered combinations (n-tuples), v_c, of the members of n number of disjoint, countable domain sets, x_i:

$$\exists\, n \in \mathbb{N},\, v_c \in \{0,\, \mathbb{N}\},\; x_i \in \{x_1, \cdots, x_n\} :\; \bigcap_{i=1}^{n} x_i = \emptyset \;\wedge\; v_c = \prod_{i=1}^{n} |x_i|. \tag{3.1}$$

THEOREM 3.2. *Euclidean volume,*

$$\begin{aligned} &\forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\; [v_a, v_b] \subset \mathbb{R},\; s_i = b_i - a_i,\; v = v_b - v_a : \\ &\qquad\qquad v_c = \prod_{i=1}^{n} |x_i| \quad \Leftrightarrow \quad v = \prod_{i=1}^{n} s_i. \end{aligned} \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.

$$v_c = \prod_{i=1}^n |x_i| \Leftrightarrow v_c\kappa = (\prod_{i=1}^n |x_i|)\kappa \Leftrightarrow \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{3.3}$$

Apply the ruler (2.1) and ruler convergence (2.2) to equation 3.3:

$$\exists\, v, \kappa \in \mathbb{R} : \ v_c = floor(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa \quad \wedge \quad \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa \quad \Rightarrow \quad v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{3.4}$$

Apply lemma 2.3 to equation 3.4:

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa \quad \wedge \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa \quad \Rightarrow \quad v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa^n = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa). \tag{3.5}$$

Apply the ruler (2.1) and ruler convergence (2.2) to s_i:

$$\exists\, s_i, \kappa \in \mathbb{R} : \ floor(s_i/\kappa) = |x_i| \quad \Rightarrow \quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i. \tag{3.6}$$

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa) \quad \wedge \quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i \quad \Leftrightarrow \quad v = \lim_{\kappa\to 0}(|x_i|\kappa) = \prod_{i=1}^n s_i \quad \square \tag{3.7}$$

4. Distance

DEFINITION 4.1. Countable distance,

$$\exists\, n \in \mathbb{N},\ v_c, d_c \in \{0,\ \mathbb{N}\},\ \ x_i \in \{x_1, \cdots, x_n\} : \ \bigcap_{i=1}^n x_i = \emptyset \quad \wedge \quad d_c = |x_1| = \cdots = |x_n| \quad \wedge \quad v_c = \prod_{i=1}^n |x_i| = \prod_{i=1}^n d_c = d_c^n. \tag{4.1}$$

4.1. Minkowski distance (ρ-norm).

THEOREM 4.2. *Minkowski distance (ρ-norm):*

$$v_c = \prod_{i=1}^n |x_i| = \prod_{i=1}^n d_c = d_c^n \quad \Rightarrow \quad d^n = \sum_{i=1}^m d_i^n, \quad d, d_i \in \mathbb{R}.$$

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

PROOF. Apply the countable distance definition (4.1) to the assumption:

$$v_c = \prod_{i=1}^n |x_i| = \prod_{i=1}^n d_c = d_c^n \quad \wedge \quad v_{c_i} = \prod_{j=1}^n |x_{i_j}| = \prod_{i=1}^n d_{c_i} = d_{c_i}^n \quad \wedge \quad v_c = \sum_{j=1}^m v_{c_i} \quad \Rightarrow \quad d_c^n = \sum_{j=1}^m d_{c_i}^n. \tag{4.2}$$

Multiply both sides of equation 4.2 by κ and take the limit:

$$d_c^n = \sum_{j=1}^m d_{c_i}^n \quad \Leftrightarrow \quad \lim_{\kappa\to 0} d_c^n\kappa = \lim_{\kappa\to 0} \sum_{j=1}^m d_{c_i}^n\kappa. \tag{4.3}$$

Apply lemma 2.3 to equation 4.2:

$$\lim_{\kappa\to 0} d_c^n\kappa = \lim_{\kappa\to 0} \sum_{j=1}^m d_{c_i}^n\kappa \quad \wedge \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa \Leftrightarrow \lim_{\kappa\to 0} d_c^n\kappa^n = \lim_{\kappa\to 0} \sum_{j=1}^m d_{c_i}^n\kappa^n \Leftrightarrow \lim_{\kappa\to 0}(d_c\kappa)^n = \lim_{\kappa\to 0} \sum_{j=1}^m (d_{c_i}\kappa)^n. \tag{4.4}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to equation 4.4:

$$\exists\, d, d_i : \ d_c = floor(d/\kappa),\ d = \lim_{\kappa\to 0} d_c\kappa \quad \wedge \quad d_{c_i} = floor(d_i/\kappa),\ d_i = \lim_{\kappa\to 0} d_{c_i}\kappa \quad \Rightarrow \quad d^n = \lim_{\kappa\to 0}(d_c\kappa)^n = \lim_{\kappa\to 0} \sum_{j=1}^m (d_{c_i}\kappa)^n = \sum_{j=1}^m d_i^n. \quad \square \tag{4.5}$$

4.2. Distance inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N},\;\; v_a, v_b \geq 0 :\;\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$\forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq v_a + v_b + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n. \tag{4.6}$$

Take the n^{th} root of both sides of the inequality 4.6:

$$\forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \tag{4.7}$$ ☐

4.3. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N},\;\; a_i, b_i \geq 0 :\;\; (\textstyle\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$\begin{gathered} \forall\, m, n \in \mathbb{N},\;\; v_a, v_b \geq 0 : \quad v_a = \textstyle\sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge \\ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad ((\textstyle\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} = \\ (\textstyle\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \end{gathered} \tag{4.8}$$ ☐

4.4. Metric Space. All Minkowski distances (ρ-norms) imply the metric space properties. The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.9}$$

Apply the distance inequality (4.3) to the inequality 4.9:

$$\begin{gathered} (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \;\wedge\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ \Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ \Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq \\ (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \end{gathered} \tag{4.10}$$ ☐

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$\begin{gathered} \forall\, p \,:\, p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \end{gathered} \tag{4.11}$$ ☐

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0. \tag{4.12}$$

$$p \geq 1,\ u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \tag{4.13}$$

□

THEOREM 4.8. *Identity of Indiscernibles:* $d(u, u) = 0$.

PROOF. From the non-negativity property (4.7):

$$\begin{aligned} d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0 \\ \Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0. \end{aligned} \tag{4.14}$$

$$d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v. \tag{4.15}$$

$$d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \tag{4.16}$$

□

4.5. Set properties limiting a set to at most 3 members. The following definitions and proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal. By convention, the proof goal is on the left side and supporting facts are on the right side of the implication sign ($\leftarrow$).

DEFINITION 4.9. Immediate Cyclic Successor of m is n:

$$\begin{aligned} &\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\} : \\ &Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize). \end{aligned} \tag{4.17}$$

DEFINITION 4.10. Immediate Cyclic Predecessor of m is n:

$$\begin{aligned} &\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\} : \\ &Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1). \end{aligned} \tag{4.18}$$

DEFINITION 4.11. Adjacent: Member m is sequentially adjacent to member n if the immediate cyclic successor of m is n or the immediate cyclic predecessor of m is n. Notionally:

$$\begin{aligned} &\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\} : \\ &Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \end{aligned} \tag{4.19}$$

DEFINITION 4.12. Symmetric (every set member is sequentially adjacent to every other member):

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\} : \quad Adjacent(m, n, setsize). \tag{4.20}$$

THEOREM 4.13. *A cyclic and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

PROOF.
Every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (m = setsize \wedge n = 1). \tag{4.21}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (n = m+1 \leq setsize). \tag{4.22}$$

$$Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (n = setsize \wedge m = 1). \tag{4.23}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m+1 \leq setsize). \tag{4.24}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m-1 \geq 1). \tag{4.25}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \wedge m = 1). \tag{4.26}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \wedge n = setsize). \tag{4.27}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m+1 \leq setsize). \tag{4.28}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m-1 \geq 1). \tag{4.29}$$

Member 2 is the only immediate successor of member 1 for all $setsize \geq 3$, which implies member 3 is not ($\neg$) an immediate successor of member 1 for all $setsize \geq 3$:

$$\neg Successor(1,3,setsize \geq 3) \\ \leftarrow Successor(1,2,setsize \geq 3) \leftarrow (n = m+1 \leq setsize). \tag{4.30}$$

Member $n = setsize > 3$ is the only immediate predecessor of member 1, which implies member 3 is not ($\neg$) an immediate predecessor of member 1 for all $setsize > 3$:

$$\neg Predecessor(1,3,setsize \geq 3) \\ \leftarrow Predecessor(1,setsize,setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \tag{4.31}$$

For all $setsize > 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not symmetric):

$$\neg Adjacent(1,3,setsize > 3) \\ \leftarrow \neg Successor(1,3,setsize > 3) \wedge \neg Predecessor(1,3,setsize > 3). \quad \square \tag{4.32}$$

5. Applications to physics

From the volume proof (3.2), two disjoint distance intervals, $[0,r_1]$ and $[0,r_2]$, define a 2-volume. From the Minkowski distance proof (4.2), $\exists\, r : r^2 = r_1^2 + r_2^2$. And from the 3D proof (4.13), for some non-distance type, $\tau : \tau \in \{t\ (time),\ m\ (mass),\ q\ (charge), \cdots\}$, there exist constant, unit-factoring ratios, $\mu,\ \nu_1,\ \nu_2$:

$$\forall\, r, r_1, r_2 :\ r^2 = r_1^2 + r_2^2 \quad \wedge \quad r = \mu\tau \quad \wedge \quad r_1 = \nu_1\tau \quad \wedge \quad r_2 = \nu_2\tau \\ \Rightarrow \quad (\mu\tau)^2 = (\nu_1\tau)^2 + (\nu_2\tau)^2 \quad \Rightarrow \quad \mu \geq \nu_1 \quad \wedge \quad \mu \geq \nu_2. \tag{5.1}$$

μ is the maximum-possible ($\mu \geq \nu_1,\ \nu_2$), constant, unit-factoring ratio, where:

$$\mu \in \{r_c/t_c,\ r_c/m_c,\ r_c/q_c, \cdots\} :\ r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q = \cdots . \tag{5.2}$$

5.1. Derivation of the constant G, and the gravity laws of Newton, Gauss, and Poisson. From equation 5.2:

$$r = (r_c/m_c)m \quad \wedge \quad r = (r_c/t_c) = ct \quad \Rightarrow \quad r/(ct)^2 = (r_c/m_c)m/r^2 \\ \Rightarrow \quad r/t^2 = ((r_c/m_c)c^2)m/r^2 = Gm/r^2, \tag{5.3}$$

where Newton's constant, $G = (r_c/m_c)c^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$.

Newton's law follows from multiplying both sides equation 5.3 by m:

$$r/t^2 = Gm/r^2 \quad \wedge \quad \forall\, m \in \mathbb{R} :\ \exists\, m_1, m_2 \in \mathbb{R} :\ m_1 m_2 = m^2 \\ \Rightarrow \quad \exists\, m_1, m_2 \in \mathbb{R} :\ F := mr/t^2 = Gm^2/r^2 = Gm_1 m_2/r^2. \tag{5.4}$$

Gauss's flux divergence, $\nabla \cdot \mathbf{g}$ and Poisson's curl per unit mass, $\nabla^2\Phi(r,t)$ are measures of acceleration, r/t^2. Again, starting with equation 5.3 and using ρ as the mass field density (Gauss's flux divergence) on a sphere having the surface area $4\pi r^2$ yields the differential forms of:

$$\nabla \cdot \mathbf{g} = \nabla^2\Phi(\overrightarrow{r}, t) = r/t^2 \quad \wedge \quad r/t^2 = (-Gm/r^2)(4\pi/4\pi) \quad \wedge \quad \rho = m/4\pi r^2 \\ \Rightarrow \quad \nabla \cdot \mathbf{g} = \nabla^2\Phi(\overrightarrow{r}, t) = -4\pi G\rho. \tag{5.5}$$

5.2. Derivation of Coulomb's charge constant, k_e and charge force.

$$\forall\, q \in \mathbb{R} :\ \exists\, q_1, q_2 \in \mathbb{R} :\ q_1 q_2 = q^2 \quad \wedge \quad r = (r_c/q_c)q \\ \Rightarrow \quad \exists\, q_1, q_2 \in \mathbb{R} :\ q_1 q_2 = q^2 = ((q_c/r_c)r)^2 \quad \Rightarrow \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1. \tag{5.6}$$

$$r = (r_c/t_c)t = ct \ \Rightarrow \ mr = (m_c/r_c)(ct)^2 \ \Rightarrow \ ((r_c/m_c)/c^2)mr/t^2 = 1. \tag{5.7}$$

$$((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1 \\ \Rightarrow \quad F := mr/t^2 = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2. \tag{5.8}$$

$$r_c/t_c = c \quad \wedge \quad F = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2 \\ \Rightarrow \quad F = (m_c(r_c/t_c^2))(r_c/q_c)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2, \tag{5.9}$$

where Coulomb's constant, $k_e = (m_c(r_c/t_c^2))(r_c/q_c)^2$, conforms to the SI units: $N \cdot m^2 \cdot C^{-2}$.

5.3. Vacuum permitivity, ε_0, and Gauss's law for electric fields. From equation 5.2:

$$r = (r_c/q_c)q \quad \wedge \quad r = (r_c/t_c) = ct \quad \Rightarrow \quad r/(ct)^2 = (r_c/q_c)q/r^2 \\ \Rightarrow \quad r/t^2 = ((r_c/q_c)/c^2)q/r^2, \tag{5.10}$$

And where ρ is the charge field density on the sphere surface area, $4\pi r^2$:

$$r/t^2 = (((r_c/q_c)/c^2)q/r^2)(4\pi/4\pi) \quad \wedge \quad \rho = q/4\pi r^2 \\ \Rightarrow \quad r/t^2 = 4\pi((r_c/q_c)c^2)\rho. \tag{5.11}$$

Multiply both sides by of equation 5.11 by $(m_c/r_c)(r_c/q_c)$ and use the derivation of k_e in equations 5.8 and 5.9:

$$r/t^2 = 4\pi((r_c/q_c)c^2)\rho \quad \wedge \quad k_e = ((m_c/r_c)c^2)(r_c/q_c)^2 \\ \Rightarrow \quad (m_c/r_c)(r_c/q_c)r/t^2 = 4\pi(m_c/r_c)(r_c/q_c)((r_c/q_c)c^2)\rho = 4\pi k_e\rho. \tag{5.12}$$

Gauss's electric flux divergence, $\nabla \cdot \mathbf{E}$ is a measure of force per charge:

$$\nabla \cdot \mathbf{E} := (m_c/q_c)r/t^2 \ \wedge \ (m_c/r_c)(r_c/q_c)r/t^2 = 4\pi k_e\rho \ \wedge \ \varepsilon_0 := 1/4\pi k_e \\ \Rightarrow \ \nabla \cdot \mathbf{E} := (m_c/q_c)r/t^2 = (m_c/r_c)(r_c/q_c)r/t^2 = 4\pi k_e\rho = \rho/\varepsilon_0. \tag{5.13}$$

5.4. Space-time-mass-charge equations. From equation 5.1:

$$\forall\, r, r', r_v, \mu, \nu :\; r^2 = r'^2 + r_v^2 \quad\wedge\quad r = \mu\tau \quad\wedge\quad r_v = \nu\tau \quad\Rightarrow\quad r' = \sqrt{(\mu\tau)^2 - (\nu\tau)^2} \;=\; \mu\tau\sqrt{1-(\nu/\mu)^2}. \tag{5.14}$$

Rest frame distance, r', contracts relative to stationary frame distance, r, as $\nu \to \mu$:

$$r' = \mu\tau\sqrt{1-(\nu/\mu)^2} \quad\wedge\quad \mu\tau = r \quad\Rightarrow\quad r' = r\sqrt{1-(\nu/\mu)^2}. \tag{5.15}$$

Stationary frame type, τ, dilates relative to the rest frame type, τ', as $\nu \to \mu$:

$$\mu\tau = r'/\sqrt{1-(\nu/\mu)^2} \quad\wedge\quad r' = \mu\tau' \quad\Rightarrow\quad \tau = \tau'/\sqrt{1-(\nu/\mu)^2}. \tag{5.16}$$

Where τ is type, time, the space-like flat Minkowski spacetime event interval is:

$$\mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_v^2 \quad\wedge\quad \mathrm{d}r_v^2 = \mathrm{d}r_1^2 + \mathrm{d}r_2^2 + \mathrm{d}r_3^2 \quad\wedge\quad \mathrm{d}(\mu\tau) = \mathrm{d}r \quad\Rightarrow\quad \mathrm{d}r'^2 = \mathrm{d}(\mu\tau)^2 - \mathrm{d}r_1^2 - \mathrm{d}r_2^2 - \mathrm{d}r_3^2. \tag{5.17}$$

5.5. Derivation of Schwarzchild's gravitational time dilation and black hole metric. [Che10] From equations 5.15 and 5.2:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-(v^2/c^2)(r/r)} \quad\wedge\quad r = (r_c/m_c)m \quad\Rightarrow\quad \sqrt{1-(v^2/c^2)} = \sqrt{1-((r_c/m_c)m)v^2/rc^2}. \tag{5.18}$$

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-((r_c/m_c)m)v^2/rc^2} \quad\wedge\quad KE = mv^2/2 = mv_{escape}^2 \quad\Rightarrow\quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2(r_c/m_c)mv_{escape}^2/rc^2}. \tag{5.19}$$

$$\sqrt{1-(v^2/c^2)} = \lim_{v_{escape}\to c}\sqrt{1-2(r_c/m_c)mv_{escape}^2/rc^2} = \sqrt{1-2(r_c/m_c)mc^2/rc^2}. \tag{5.20}$$

Combining equation 5.20 with the derivation of G (5.4):

$$(r_c/m_c)c^2 = G \quad\wedge\quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2(r_c/m_c)mc^2/rc^2} \quad\Rightarrow\quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2Gm/rc^2}. \tag{5.21}$$

Combining equation 5.21 with equation 5.16 yields Schwarzchild's gravitational time dilation:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-2Gm/rc^2} \quad\wedge\quad t' = t\sqrt{1-(v^2/c^2)} \quad\Rightarrow\quad t' = t\sqrt{1-2Gm/rc^2}. \tag{5.22}$$

Schwarzchild defined the black hole event horizon radius, $r_s := 2Gm/c^2$.

$$r_s = 2Gm/c^2 \quad\wedge\quad t' = t\sqrt{1-2Gm/rc^2} \quad\Rightarrow\quad t' = t\sqrt{1-r_s/r}. \tag{5.23}$$

From equations 5.15 and 5.23:

$$r' = r\sqrt{1-(v/c)^2} \quad\wedge\quad \sqrt{1-(v/c)^2} = \sqrt{1-2Gm/rc^2} \quad\Rightarrow\quad r' = r\sqrt{1-2Gm/rc^2} = r\sqrt{1-r_s/r}. \tag{5.24}$$

Using the time-like spacetime interval, where $\mathrm{d}s^2$ is negative:

$$r' = r\sqrt{1 - r_s/r} \quad \wedge \quad \mathrm{d}s^2 = \mathrm{d}r'^2 - \mathrm{d}r^2 \tag{5.25}$$
$$\Rightarrow \quad \mathrm{d}s^2 = (\sqrt{1 - r_s/r}\mathrm{d}r')^2 - (\mathrm{d}r/\sqrt{1 - r_s/r})^2 = (1 - r_s/r)\mathrm{d}r'^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2.$$

$$\mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}r'^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2 \quad \wedge \quad \mathrm{d}r' = \mathrm{d}(ct) \quad \wedge \quad c = 1 \tag{5.26}$$
$$\Rightarrow \quad \mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}t^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2.$$

Translating from 2D to 4D yields Schwarzchild's black hole metric:

$$\mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}t^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2 = f(r, t) \tag{5.27}$$
$$\Rightarrow \quad \mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}t^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + sin^2\theta\mathrm{d}\phi^2) = f(r, t, \theta, \phi).$$

5.6. Simplifying Einstein's general relativity (field) equation. Simplification step 1) Use the unit-factoring ratios to define functions returning values for each component of the metric, $g_{\nu,\mu}$, in Einstein's field equations [**Ein15**] [**Wey52**]: All functions derived from the ratios are valid metrics, as an example, the previous Schwarzchild black hole metric derivation using the unit-factoring ratios (5.5).

Simplification step 2) Express the EFE as 2D tensors: As shown in equation 5.27, the Schwarzchild metric was first derived as a 2D metric and then expanded to a 4D metric. Further, the 4D flat spacetime interval equation (5.17) is an instance of the 2D equation, $\mathrm{d}r'^2 = \mathrm{d}(ct)^2 - \mathrm{d}r_v^2$, where $\mathrm{d}r_v^2$ is the magnitude of a 3-dimensional vector.

The 2D metric tensor allows using the much simpler 2D Ricci curvature and scalar curvature. And the 2D tensors reduce the number of independent equations to solve.

Simplification step 3) One simple method to translate from 2D to 4D is to use spherical coordinates, where r and t remain unchanged and two added dimensions are the angles, ϕ, and θ. For example, the 2D Schwarzchild metric was translated to 4D using this method in equation 5.27.

5.7. 3 fundamental direct proportion ratios. c_t, c_m, and c_q:

$$c_t = r_c/t_c \approx 2.99792458 \cdot 10^8 m\ s^{-1}. \tag{5.28}$$

$$G = (r_c/m_c)c_t^2 = c_m c_t^2 \quad \Rightarrow \quad c_m = r_c/m_c \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}. \tag{5.29}$$

$$k_e = (c_t^2/c_m)(r_c/q_c)^2 \quad \Rightarrow \quad c_q = r_c/q_c \approx 8.6175172023 \cdot 10^{-18} m\ C^{-1}. \tag{5.30}$$

5.8. 3 fundamental inverse proportion ratios. k_t, k_m, and k_q:

$$r/t = r_c/t_c, \quad r/m = r_c/m_c \quad \Rightarrow \quad (r/t)/(r/m) = (r_c/t_c)/(r_c/m_c) \quad \Rightarrow \tag{5.31}$$
$$(mr)/(tr) = (m_c r_c)/(t_c r_c) \quad \Rightarrow \quad mr = m_c r_c = k_m, \quad tr = t_c r_c = k_t.$$

$$r/t = r_c/t_c, \quad r/q = r_c/q_c \quad \Rightarrow \quad (r/t)/(r/q) = (r_c/t_c)/(r_c/q_c) \quad \Rightarrow \tag{5.32}$$
$$(qr)/(tr) = (q_c r_c)/(t_c r_c) \quad \Rightarrow \quad qr = q_c r_c = k_q, \quad tr = t_c r_c = k_t.$$

5.9. Planck relation and constant, h. [Jai11] Applying both the direct proportion ratio (5.28), and inverse proportion ratio (5.31):

$$m(ct)^2 = mr^2 \;\wedge\; m = m_c r_c/r = k_m/r \quad\Rightarrow\quad m(ct)^2 = (k_m/r)r^2 = k_m r. \tag{5.33}$$

$$\begin{aligned} & m(ct)^2 = k_m r \quad \wedge \quad r/t = r_c/t_c = c \\ & \qquad \Rightarrow \quad E := mc^2 = k_m r/t^2 = (k_m(r/t))\,(1/t) = (k_m c)(1/t) = hf, \end{aligned} \tag{5.34}$$

where the Planck constant, $h = k_m c$, and the frequency, $f = 1/t$.

$$k_m = m_c r_c = h/c \approx 2.2102190943 \cdot 10^{-42}\ kg\ m. \tag{5.35}$$

$$k_t = t_c r_c = k_m c_m/c_t \approx 5.4749346710 \cdot 10^{-78}\ s\ m. \tag{5.36}$$

$$k_q = q_c r_c = k_t c_t/c_q \approx 1.9046601056 \cdot 10^{-52}\ C\ m. \tag{5.37}$$

5.10. Compton wavelength. [Jai11] From equations 5.31 and 5.34:

$$mr = k_m \quad \wedge \quad h = k_m c \quad \Rightarrow \quad r = k_m/m = (k_m/m)(c/c) = h/mc. \tag{5.38}$$

5.11. 4 quantum units. Distance (r_c), time (t_c), mass (m_c), and charge (q_c):

$$r_c = \sqrt{r_c^2} = \sqrt{c_t k_t} = \sqrt{c_m k_m} = \sqrt{c_q k_q} \approx 4.0513505432 \cdot 10^{-35}\ m. \tag{5.39}$$

$$t_c = r_c/c_t \approx 1.3513850782 \cdot 10^{-43}\ s. \tag{5.40}$$

$$m_c = r_c/c_m \approx 5.4555118613 \cdot 10^{-8}\ kg. \tag{5.41}$$

$$q_c = r_c/c_q \approx 4.7012967286 \cdot 10^{-18}\ C. \tag{5.42}$$

Planck length $= r_c/\sqrt{2\pi}$, time $= t_c/\sqrt{2\pi}$, mass $= m_c/\sqrt{2\pi}$, charge $= q_c/\sqrt{2\pi}$.

5.12. Fine structure constant. The fine structure constant, α, is the ratio of two types of charge fields: 1) the *stationary* elementary particle charge field, F_e, and 2) the *moving* elementary charge (electromagnetic) wave field (the reduced Planck charge unit), F_p. From the ratio-derived Coulomb's law equation 5.8:

$$\exists\, \alpha \in \mathbb{R}:\ \alpha = \frac{F_e}{F_p} = \frac{k_e q_e^2/r^2}{k_e q_p^2/r^2} = q_e^2/q_p^2 = q_e^2/(q_c/\sqrt{2\pi})^2 \approx 0.0072973526. \tag{5.43}$$

5.13. Schrödenger's equation. Start with the previously derived Planck relation 5.34 and multiply the kinetic energy component by mc/mc:

$$h/t = mc^2 \Rightarrow \exists V(r,t) : h/t = h/2t + V(r,t) \Rightarrow h/t = hmc/2mct + V(r,t). \tag{5.44}$$

And from the distance-to-time (speed of light) ratio (5.28):

$$h/t = hmc/2mct + V(r,t) \;\wedge\; r = ct \;\Rightarrow\; h/t = hmc^2/2mcr + V(r,t). \tag{5.45}$$

$$h/t = hmc^2/2mcr + V(r,t) \;\wedge\; h/t = mc^2 \;\Rightarrow\; h/t = h^2/2mcrt + V(r,t). \tag{5.46}$$

$$h/t = h^2/2mcrt + V(r,t) \;\wedge\; r = ct \;\Rightarrow\; h/t = h^2/2mr^2 + V(r,t). \tag{5.47}$$

Replace the Planck constant in equation 5.47 with the reduced Planck constant:

$$h/t = h^2/2mr^2 + V(r,t) \;\wedge\; \hbar = h/2\pi \;\Rightarrow\; 2\pi\hbar/t = (2\pi)^2\hbar^2/2mr^2 + V(r,t). \tag{5.48}$$

Multiply both sides of equation 5.48 by a function, $\Psi(r,t)$.

$$2\pi\hbar/t = (2\pi)^2\hbar^2/2mr^2 + V(r,t) \Rightarrow (2\pi\hbar/t)\Psi(r,t) = ((2\pi)^2\hbar^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t). \tag{5.49}$$

$$(2\pi\hbar/t)\Psi(r,t) = ((2\pi)^2\hbar^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t) \quad \wedge \quad \forall\, \Psi(r,t) :\ \partial^2\Psi(r,t)/\partial r^2 = (-(2\pi)^2/r^2)\Psi(r,t) \quad \wedge \quad \partial\Psi(r,t)/\partial t = (i\ 2\pi/t)\Psi(r,t) \Rightarrow i\hbar\partial\Psi(r,t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t), \tag{5.50}$$

which is Schrödenger's equation in one dimension of space.

$$-(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t) = i\hbar\partial\Psi(r,t)/\partial t \quad \wedge \quad ||\overrightarrow{\mathbf{r}}|| = r \Rightarrow \exists\, \overrightarrow{\mathbf{r}} :\ i\hbar\partial\Psi(\overrightarrow{\mathbf{r}},t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(\overrightarrow{\mathbf{r}},t)/\partial\overrightarrow{\mathbf{r}}^2 + V(\overrightarrow{\mathbf{r}},t)\Psi(\overrightarrow{\mathbf{r}},t), \tag{5.51}$$

which is Schrödenger's equation in three dimensions of space.

5.14. Dirac's wave equation. Using the derived Planck relation 5.34:

$$mc^2 = h/t \quad \Rightarrow \quad \exists\, V(r,t) :\ mc^2/2 + V(r,t) = h/t \Rightarrow 2h/t - 2V(r,t) = mc^2. \tag{5.52}$$

$$\forall\, V(r,t) :\ V(r,t) = h/t \quad \wedge \quad r = ct \quad \wedge \quad 2h/t - 2V(r,t) = mc^2 \Rightarrow 2h/t - 2hc/r = mc^2. \tag{5.53}$$

Use the charge ratio, c_q, and time ratio, $c_t = c$ to multiply each term on the left side of equation 5.53 by 1:

$$qc_q/r = qc_q/ct = 1 \quad \wedge \quad 2h/t - 2hc/r = mc^2 \Rightarrow 2h(-qc_q/c)/t^2 - 2h((-qc_q/c)/r^2)c = mc^2. \tag{5.54}$$

where a negative sign is added to q to indicate an attractive force between an electron and a nucleus.

Applying a quantum amplitude equation in complex form to equation 5.55:

$$A_0 = (c_q/c)((1/t)) + i(1/r)) \quad \wedge \quad 2h(-qc_q/c)/t^2 - 2h((-qc_q/c)/r^2)c = mc^2 \Rightarrow 2h\partial(-qA_0)/\partial t - i2h(\partial(-qA_0)/\partial r)c = mc^2. \tag{5.55}$$

Translating equation 5.55 to moving coordinates via the Lorentz factor, $\gamma_0 = 1/\sqrt{1-(v/c)^2}$:

$$2h\partial(-qA_0)/\partial t - i2h(\partial(-qA_0)/\partial r)c = mc^2 \Rightarrow \gamma_0 2h\partial(-qA_0)/\partial t - \gamma_0 i2h(\partial(-qA_0)/\partial r)c = mc^2. \tag{5.56}$$

Multiplying both sides of equation 5.56 by $\Psi(r,t)$:

$$\gamma_0 2h\partial(-qA_0)/\partial t - \gamma_0 i2h(\partial(-qA_0)/\partial r)c = mc^2 \Rightarrow \gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \gamma_0 i2h(\partial(-qA_0)/\partial r)c\Psi(r,t) = mc^2\Psi(r,t). \tag{5.57}$$

Applying the vectors to equation 5.57:

$$(5.58)\quad \gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \gamma_0 i2h(\partial(-qA_0)/\partial r)c\Psi(r,t) = mc^2\Psi(r,t) \wedge$$
$$||\overrightarrow{\mathbf{r}}|| = r \quad \wedge \quad ||\overrightarrow{\mathbf{A}}|| = A_0 \quad \wedge \quad ||\overrightarrow{\gamma}|| = \gamma_0 \quad \wedge \quad \Leftrightarrow \quad \exists\, \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{A}}, \overrightarrow{\gamma} :$$
$$\gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \overrightarrow{\gamma}\cdot i2h(\partial(-q\overrightarrow{\mathbf{A}})/\partial r)c\Psi(\overrightarrow{\mathbf{r}},t) = mc^2\Psi(\overrightarrow{\mathbf{r}},t).$$

Adding a $\frac{1}{2}$ angular rotation (spin-$\frac{1}{2}$) of π to equation 5.55 allows substituting the reduced Planck constant, $\hbar = h/2\pi$, into equation 5.58, which yields Dirac's wave equation:

$$(5.59)\quad \gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \overrightarrow{\gamma}\cdot i2h(\partial(-q\overrightarrow{\mathbf{A}})/\partial r)c\Psi(\overrightarrow{\mathbf{r}},t) = mc^2\Psi(\overrightarrow{\mathbf{r}},t)$$
$$\wedge\, A_0 = \pi(c_q/c)((1/t)+(1/r))$$
$$\Rightarrow \quad \gamma_0\hbar(\partial(-qA_0)/\partial t)\Psi(r,t) - \overrightarrow{\gamma}\cdot i\hbar(\partial(-q\overrightarrow{\mathbf{A}})/\partial r)c\Psi(\overrightarrow{\mathbf{r}},t) = mc^2\Psi(\overrightarrow{\mathbf{r}},t).$$

5.15. Total mass. The total mass of a particle is $m = \sqrt{m_0^2 + m_{ke}^2}$, where m_0 is the rest mass and m_{ke} is the kinetic energy-equivalent mass. Applying both the direct (5.28) and inverse proportion ratios (5.31):

$$(5.60)\quad m_0 = r/(r_c/m_c) = r/c_m \quad \wedge \quad m_{ke} = (m_c r_c)/r = k_m/r \quad \wedge$$
$$m = \sqrt{m_0^2 + m_{ke}^2} \quad \Rightarrow \quad m = \sqrt{(r/c_m)^2 + (k_m/r)^2}.$$

5.16. Quantum extension to general relativity. The simplest way to demonstrate how to add quantum physics to general relativity is by extending the Schwarzchild's black hole metric (5.5). Start by changing equation 5.18 in the Schwarzchild derivation:

$$(5.61)\quad \sqrt{1-(v^2/c^2)} = \sqrt{1-(v^2/c^2)(r/r)} \ \wedge\ r = \sqrt{(c_m m)^2 + (k_m/m)^2} = Q_m$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1 - Q_m v^2/rc^2}.$$

$$(5.62)\quad \sqrt{1-(v^2/c^2)} = \sqrt{1 - Q_m v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv_{escape}^2$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1 - 2Q_m v_{escape}^2/rc^2}.$$

$$(5.63)\quad \sqrt{1-(v^2/c^2)} = \lim_{v_{escape}\to c}\sqrt{1 - 2Q_m v_{escape}^2/rc^2}$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2Q_m c^2/rc^2} = \sqrt{1-2Q_m/r}.$$

Combining equation 5.63 with equation 5.16 yields Schwarzchild's gravitational time dilation with a quantum mass effect:

$$(5.64)\quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2Q_m/r} \quad \wedge \quad t' = t\sqrt{1-(v^2/c^2)}$$
$$\Rightarrow \quad t' = t\sqrt{1-2Q_m/r}.$$

Schwarzchild defined the black hole event horizon radius, $r_s := 2Gm/c^2$. The radius with the quantum extension is $r_s := 2Q_m$:

$$(5.65)\qquad r_s = 2Q_m \quad \wedge \quad t' = t\sqrt{1-2Q_m/r} \quad \Rightarrow \quad t' = t\sqrt{1-r_s/r}.$$

From equations 5.15 and 5.65:

$$(5.66)\quad r' = r\sqrt{1-(v/c)^2} \ \wedge\ \sqrt{1-(v/c)^2} = \sqrt{1-r_s/r} \ \Rightarrow\ r' = r\sqrt{1-r_s/r}.$$

Using the time-like spacetime interval, where $\mathrm{d}s$ is negative:

$$r' = r\sqrt{1 - r_s/r} \quad \wedge \quad \mathrm{d}s^2 = \mathrm{d}r'^2 - \mathrm{d}r^2 \tag{5.67}$$
$$\Rightarrow \quad \mathrm{d}s^2 = (\sqrt{1 - r_s/r}\mathrm{d}r')^2 - (\mathrm{d}r/\sqrt{1 - r_s/r})^2 = (1 - r_s/r)\mathrm{d}r'^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2.$$

$$\mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}r'^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2 \quad \wedge \quad \mathrm{d}r' = \mathrm{d}(ct) \quad \wedge \quad c = 1 \tag{5.68}$$
$$\Rightarrow \quad \mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}t^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2.$$

Expanding to 4D spherical coordinates yields Schwarzchild's black hole metric:

$$\mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}t^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2 = f(r, t) \tag{5.69}$$
$$\Rightarrow \quad \mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}t^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + sin^2\theta\mathrm{d}\phi^2) = f(r, t, \theta, \phi).$$

5.17. Quantum extension to Newton's gravity force. The quantum mass effect is easier to understand in the context Newton's gravity equation than in general relativity, because the metric equations and solutions in the EFEs are much more complex. From equation 5.2:

$$m/\sqrt{(r/c_m)^2 + (k_m/r)^2} = 1 \quad \wedge \quad r^2/(ct)^2 = 1 \tag{5.70}$$
$$\Rightarrow \quad r^2/(ct)^2 = m/\sqrt{(r/c_m)^2 + (k_m/r)^2}$$
$$\Rightarrow \quad r^2/t^2 = c^2m/\sqrt{(r/c_m)^2 + (k_m/r)^2}.$$

$$r^2/t^2 = c^2m/\sqrt{(r/c_m)^2 + (k_m/r)^2} \tag{5.71}$$
$$\Rightarrow \quad (m/r)(r^2/t^2 = (m/r)(c^2m/\sqrt{(r/c_m)^2 + (k_m/r)^2})$$
$$\Rightarrow \quad F := mr/t^2 = c^2m^2/(r\sqrt{(r/c_m)^2 + (k_m/r)^2}) = c^2m^2/\sqrt{(r^4/c_m^2) + k_m^2}.$$

$$F = c^2m^2/\sqrt{(r^4/c_m^2) + k_m^2} \quad \wedge \quad \forall\, m \in \mathbb{R}, \exists\, m_1, m_2 \in \mathbb{R} : \; m_1m_2 = m^2 \tag{5.72}$$
$$\Rightarrow \quad F = c^2m_1m_2/\sqrt{(r^4/c_m^2) + k_m^2}.$$

6. Insights and implications

(1) The ruler measure (2.1) and convergence theorem (2.2) are tools for proving that a real-valued (and possibly a complex-valued) equation is the only instance of an abstract, countable set relation.

(2) Combinatorics, the ordered combinations of countable, disjoint sets (n-tuples), generates both Euclidean volume (3.2) and the Minkowski distances (4.2), which includes Manhattan and Euclidean distances, without relying on the geometric primitives and relations in Euclidean geometry [**Joy98**], axiomatic geometry [**Lee10**], and vector analysis [**Wey52**].

(3) It was proved that Minkowski distances have the metric space properties (4.4). But the Minkowski distances (Manhattan distance, Euclidean distance, etc.) were proved to be the *only* distance functions implied by the countable set of n-tuples. Therefore, the definition of metric space as the criteria for a distance measure might not be sufficiently restrictive to be useful for most geometry and physics.

(4) Note that the basis of the inner product follows immediately from the $n = 2$ case of the Minkowski distance (the Euclidean distance):

$$\forall\, d_i \in \mathbb{R}, \exists\, a_i, b_i \in \mathbb{R} : a_i b_i = d_i^2 \quad \wedge \quad d^2 = \textstyle\sum_{i=1}^m d_i^2 \quad \Rightarrow \quad d^2 = \sum_{i=1}^m a_i b_i. \tag{6.1}$$

If a_i and b_i can have negative values indicating negative-sized intervals (direction), then the inner product vector space is also derived from the same combinatorial (n-tuple-based) definition of distance (4.1).

(5) An area for further study is how to use only set operation-based modifications to the n-tuple equation, $v_c = \prod_{i=1}^n |x_i|$, to generate specific elliptic and hyperbolic equations.

(6) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [**Joy98**]. And analytic proofs that the straight line length is the smallest distance equate the straight line length to Euclidean distance.

Without using the notion of a straight line: All distance measures derived from the set-based countable distance (4.1) are Minkowski distances (4.2). For all 2-volumes, all Minkowski distances are limited to $n \in \{1, 2\}$: $n = 1$ is the Manhattan (largest monotonic) distance case, $d = \sum_{i=1}^m s_i$. $n = 2$ is the Euclidean (smallest) distance case, $d = (\sum_{i=1}^m s_i^2)^{1/2}$. For the case, $n \in \mathbb{R}$, $1 \leq n \leq 2$: d decreases monotonically as n goes from 1 to 2.

(7) The left side of the distance sum inequality (4.4),

$$(\textstyle\sum_{i=1}^m (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}, \tag{6.2}$$

differs from the left side of Minkowski's sum inequality [**Min53**]:

$$(\textstyle\sum_{i=1}^m (a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}. \tag{6.3}$$

The two inequalities are only the same where $n = 1$.

(a) The distance sum inequality (4.4) is a more fundamental inequality because the proof does not require the convexity and Hölder's inequality assumptions of the Minkowski sum inequality proof [**Min53**].

(b) The Minkowski sum inequality term, $\forall\ n > 1 : ((a_i^n + b_i^n)^{\mathbf{n}})^{1/n}$, is **not** a Minkowski distance spanning the n-volume, $a_i^n + b_i^n$. But the distance sum inequality term, $(a_i^n + b_i^n)^{1/n}$, is the Minkowski distance spanning the n-volume, $a_i^n + b_i^n$, which makes it directly related to geometry (for example, the metric space triangle inequality was derived from the $m = 1$ case for all $n \geq 1$ (4.5)).

(8) Combinatorics, all n-at-time permutations of a cyclic and symmetric set, limits the set to 3 members (4.13), where the permutations are due to commutative properties of multiplication and addition of the domain sizes generating distance and volume. This set-based, first-order logic proof is a simpler and more logically rigorous hypothesis for observing only 3 dimensions of physical space than parallel dimensions that cannot be detected or extra dimensions rolled up into infinitesimal balls.

(a) Higher order dimensions must have different types (members of different sets), for example, types/dimensions of time, mass, and charge. Order and symmetry probably limit the number of fundamental types to a very small number. For example, temperature, measured in Kelvins, is not a true type because temperature is more correctly a measure of energy, where entropy is a drop in (kinetic or electromagnetic) energy. The magnetic field might be a pseudo (fictitious) field that is a function of distance, time. charge, and spin. Likewise, one should not immediately assume the strong force field, weak force field, etc. are types. For example, quantum effects might allow radioactivity without a weak force.

(b) The definition of inner product can only be extended beyond 3 dimensions of distance if and only if the higher dimensions have types different from distance for example time as in Minkowki's spacetime interval.

(c) Each of 3 cyclic and symmetric dimensions of space can have at most 3 cyclic and symmetric state values, for example, a cyclic and symmetric set of 3 vector orientations, $\{-1, 0, 1\}$, per dimension of space and at most 3 spin states per plane, etc.

(d) If the states are not ordered (a bag of states), then a state value is undetermined until observed (like Schrödenger's poisoned cat being both alive and dead until the box is opened). That is, for a bag of states, there is no "axiom of choice", an axiom often used in math proofs that allows selecting a particular set element (state).

(e) A discrete value has measure 0 (no size). The ratio of a time or distance interval size greater than zero to zero is undefined (infinite). This is the reason quantum entangled particles change discrete state values at the same time and change independent distance.

(f) For each unit of a 3-dimensional, compact and continuous distance, unit there are units of other compact and continuous types of elements (5.7): $c_t = r_c/t_c$, $c_m = r_c/m_c$, $c_q = r_c/q_c \Leftrightarrow$ the inverse proportion ratios (5.8): $k_t = r_c t_c$, $k_m = r_c m_c$, $k_t = r_c q_c$, where the combination of the direct and inverse ratios implies the quantum units (5.11): r_c, t_c, m_c, q_c. These ratios and quantum units were shown to be the basis of much physics:

   (i) The gravity, G (5.4), charge k_e (5.9), vacuum permitivity, ε_0, and Planck h (5.34) constants were all derived from the ratios. Therefore, G, k_e, ε_0, and h are **not** "fundamental" constants.

   (ii) Planck length $= r_c/\sqrt{2\pi}$, time $= t_c/\sqrt{2\pi}$, mass $= m_c/\sqrt{2\pi}$, and charge $= q_c/\sqrt{2\pi}$. The quantum units, r_c, t_c, m_c, and q_c are more fundamental than the Planck units because the quantum units combined into the SI units of the constants that were previously considered fundamental (G, k_e, ε_0, and h) yield the empirical values of those constants.

   (iii) G, k_e, and h all depend on the speed of light ratio, c_t: $G = c_m c_t^2$, $k_e = (c_q^2/c_m)c_t^2$, $\varepsilon_0 := 1/4\pi k_e = 1/(4\pi(c_q^2/c_m)c_t^2)$ and $h = k_m c_t$.

   (iv) $r_c/t_c^2 \approx 2.2184088232 \cdot 10^{51}\ m\ s^{-2}$ suggests a maximum acceleration constant for both gravity and charge.

   (v) The inverse square law for gravity (5.3) and charge (5.6) were shown to be a result of the direct proportion ratios.

   (vi) The unit-factoring ratios are the basis of relativity theory.

(A) From equation 5.1, there is always a maximum ratio (for example, the speed of light, $c_t = r_c/t_c$).

(B) Special and general relativity assume covariance, which states that the laws of physics are invariant in every frame of reference. Covariance is the result of the same unit-factoring ratios in every frame of reference.
For example, the special relativity time dilation equation 5.16 was derived from the ratio, $c_t = r_c/t_c$, and combined with the ratio, $c_m = r_c/m_c$, (5.7) yielded Schwarzchild's general relativity gravitational time dilation and black hole metric equations (5.23).

(vii) The combination of direct and inverse proportion ratios was shown to create the particle-wave equations: Planck relation (5.9), Compton wavelength (5.38), Schrödenger (5.13), and Dirac equations (5.14).
This seems to support the empirical notion of particle-wave duality. But consider the moral about the four blind men experiencing an elephant for the first time (paraphrased): The first man feels the tail and says, "An elephant is a rope." The second man feels the leg and says, "You must be feeling a branch, because I feel a large tree trunk." The third man feels the body of the elephant and says, "You are feeling a column in front of a wall." The fourth man feels the trunk that wraps around his arm and screams, "Run for your lives it's giant snake!" What exists at the microcosmic scale, is probably not a particle, not a wave, and not a particle-wave.

(viii) The derivation of the Compton wavelength equation (5.10) shows that the computation of the wavelength, r, is overly complex (because it assumes the Planck constant is a fundamental constant) and can be simplified to $r = k_m/m$.

(ix) The fine structure constant, α, has been an empirical constant with defined equations. For example, the CODATA definition is: $\alpha = q_e^2/4\pi\varepsilon\hbar c)$, where α is assumed to be dimensionless.
However, the derivation of α, in this article (5.43), shows that α does have dimensions because it is the ratio of two types of charge fields: 1) the *stationary* elementary particle charge field, F_e, and 2) the *moving* elementary charge (electromagnetic) wave field, F_p, which yields the more parsimonious equation: $\alpha = q_e^2/q_p^2$, where q_p is the Planck charge unit.
There may be other fine structure constants, for example, the ratio of a stationary elementary particle gravity field to a moving elementary gravity field: $\alpha_m = F_e/F_p = m_e^2/m_p^2$.

(9) The derivations of the spacetime equations, in this article (5.4), differ from other derivations:

(a) The derivations, here, do not rely on the Lorentz transformations or Einsteins' postulates [**Ein15**]. The derivations do not even require the notion of light.

(b) The same derivations are also valid for spacemass and spacecharge.

   (c) The derivations, here, rely only on the Minkowski distances proof (4.1), and the 3D proof (4.13), which provides the insight that the properties of physical space creates constant maximum ratios, the spacetime equations, and 3 dimensions of distance.
(10) The derivation of Schrödenger's equation (5.13) and Dirac's equation (5.14), in this article, differs from other derivations:
   (a) Other derivations are based on the Hamiltonian (energy-momentum) operator. In contrast, the derivations, in this article, rely on the Planck (energy-frequency) relation.
   (b) The derivations here are more rigorous because:
      (i) The energy-momentum term, $h^2/2m$, was derived, in this article, from the Planck relation (5.47), where the Planck relation was also rigorously derived (5.9). Other derivations **incorrectly** assume (define) the energy-momentum relation as: $(\mathbf{p}\cdot\mathbf{p})/2m = \hbar^2/2m$. The reduced Planck constant is only valid if the partial derivatives of the probability distribution function, $\Psi(r,t)$, contains compensating 2π terms: $\partial^2\Psi(r,t)/\partial r^2 = (-(2\pi)^2/r^2)\Psi(r,t)$ and $\partial\Psi(r,t)/\partial t = (i\ 2\pi/t)\Psi(r,t)$. Finding solutions to Schrödenger's equation would be simpler if the full Planck constant is used because it would reduce the complexity of $\Psi(r,t)$.
      (ii) Other derivations assume the probability distribution has a mean value, where values closer to the mean are more probable. The derivation here makes no such assumptions.
(11) The quantum extensions to: Schwarzchild's black hole metric (5.66) and Newton's gravity force (5.72) make quantifiable predictions. Specifically: $\lim_{r\to 0} F = c^2 m_1 m_2/k_m$, and **both** the gravity and charge forces peak at the quantum length: $r_c = \sqrt{r_c^2} = \sqrt{c_t k_t} = \sqrt{c_m k_m} = \sqrt{c_q k_q} \approx 4.0513505432 \cdot 10^{-35}\ m$ (5.39).
   (a) Newton's gravitational constant, G, and Coulomb's constant, k_e, are not valid, where the distance, r, is sufficiently small that the quantum effects becomes measurable.
   (b) Finding solutions to Einstein's field equations becomes more difficult because the covariant components that had the units $1/distance^2$, will now have the more complex units, $1/\sqrt{(distance^4/c_m^2) + k_m^2}$, and Einstein's constant (which contains G) is no longer valid, where the distance is sufficiently small that the quantum effects becomes measurable.
   (c) Gravitational time dilation peaks at r_c (no black hole time singularity, no event horizon).
   (d) Black holes have sizes > 0 (no size singularities).
   (e) A finite maximum gravity-charge allows radioactivity, quantum tunneling, and possibly black hole evaporation.

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