The Set Mappings Generating Geometry and Physics

George. M. Van Treeck

Abstract. Countable distance and volume set mappings between sets of size c subintervals of domain and range intervals generate the properties of metric space, the Lp norms (for example, Manhattan and Euclidean distance), and the volume equation as c goes to zero. Countable volume is used to derive Newton's gravity force and Coulomb's charge force equations without using the inverse square law or Gauss's divergence theorem. The gravity and charge force derivations expose a proportionate interval principle that allows simple derivations of the inverse square law, spacetime, and general relativity equations. A symmetry constraint on totally ordered sets can limit distance and volume to 3 dimensions. All proofs are verified in Coq.

Contents

1. Introduction

The definitions of metric space, Euclidean distance, and area/volume in analysis [**Gol76**] [**Rud76**] are empirical models of Euclidean geometry [**Joy98**]. Deductive proofs that those definitions are derived from a set and limit-based foundation exposes properties of geometry that empirical modeling cannot provide, for example: the counting constraint between the infinitesimal members of domain and range sets that makes a space flat; the countable domain-to-range set mapping that makes Euclidean distance the smallest possible distance in flat space; the set operation and constraint generating the properties of metric space; and the symmetry constraint on totally ordered sets that can limit distance and volume to 3 dimensions.

2010 *Mathematics Subject Classification.* Primary 28A75, 28E15. Secondary 03E75, 51M99.

A definition of a dimension is where each countable domain set, x_i, has a corresponding range (distance) set, y_i. The countable distance spanning the domain sets is the cardinal of a constrained union of the range (distance) sets, $d_c = |\bigcup_{i=1}^{n} y_i|$, where vertical bars around a set, $|\{\cdots\}|$, or list, $|[\cdots]|$, indicates the cardinal (the number of members in the set or list). As the intersection of the range sets increases, more domain set members can map to a single range set member. Therefore, the cardinal of a constrained union range set, d_c, is a function of a constrained number of domain-to-range set mappings. Countable volume is the cardinal of the set of all possible countable distance unions, which is function of the number of distance-to-distance (range-to-range) set mappings.

Applying these abstract, countable set definitions of distance and volume to sets of size c subintervals of domain and range intervals generates the properties of metric space, all L_p norms (Minkowski distances, for example, Manhattan and Euclidean distance), and the volume equation as $c \to 0$.

Countable volume is used to derive Newton's gravity force and Coulomb's charge force equations without using the inverse square law or Gauss's divergence theorem. The gravity and charge force derivations expose a proportionate interval principle that allows simple derivations of the inverse square law, spacetime, and general relativity equations.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

A tool is needed to derive geometric relations from the number of possible mappings between the p_x number of size c subintervals in one interval and the p_y number of size c subintervals in another interval. A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals, where each subinterval has the *same* size, c.

DEFINITION 2.1. Ruler measure, M: $\forall\, c,\ s \in \mathbb{R},\ [a, b] \subset \mathbb{R},\ s = b - a \wedge c > 0 \wedge (p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \sum_{i=1}^{p} c = pc$.

THEOREM 2.2. *Ruler convergence:* $M = \lim_{c\to 0} pc = s$.

The proof is trivial but is included here for completeness. The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\ p = floor(s/c)\ \wedge\ 0 \le |floor(s/c) - s/c| < 1\ \Rightarrow\ 0 \le |p - s/c| < 1. \tag{2.1}$$

Multiply all sides of inequality 2.1 by c:

$$\forall\, c > 0, \quad 0 \le |p - s/c| < 1 \quad \Rightarrow \quad 0 \le |pc - s| < |c|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \\ &\quad \Rightarrow \quad \forall\, \epsilon = \delta :\ \ |c - 0| < \delta\ \ \wedge\ \ |pc - s| < \epsilon\ \ :=\ \ M = \lim_{c\to 0} pc = s. \end{aligned} \tag{2.3}$$

□

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}},\ 3.14_{c=10^{-2}},\ 3.141_{c=10^{-3}}, \ldots, \pi_{\lim_{c\to 0}}$.

3. Distance

3.1. Countable distance. Each disjoint domain set, x_i, has its own range (distance) set, y_i. The countable distance spanning the disjoint domain sets is the cardinal, d_c, of a union range (distance) set that is constrained by a relation between the number of members in the domain set and corresponding range set.

It will be shown in the next subsections that the constraint, $|x_i| = |y_i|$, generates Manhattan and Euclidean distance at the boundaries (generates flat space/rectilinear distances). Generalizing distance and volume beyond flat space is shown in the last section of this article.

DEFINITION 3.1. Countable distance, d_c, in flat space:

$$d_c = |\bigcup_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i|.$$

3.2. Union-Sum Inequality. The inequality, $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$, is used in this article. The proof is trivial but is included here for completeness.

The proof follows from the associative law of addition where the sum of set sizes is equal to the size of all the set members appended into a list and the commutative law of addition that allows sorting that list into a list of unique members (the *union* set) and a list of duplicates. For example, $y_1 = \{a, b, c\}$ and $y_2 = \{c, d, e\} \Rightarrow \bigcup_{i=1}^{2} |y_i| = |\{a, b, c, d, e\}| = 5 < \sum_{i=1}^{2} |y_i| = |\{a, b, c\}| + |\{c, d, e\}| = |[a, b, c, c, d, e]| = |\{a, b, c, d, e\}| + |[c]| = 6$.

LEMMA 3.2. *Union-Sum Inequality:* $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$.

PROOF. A formal proof, union_sum_inequality, using sorting into a set of unique members (*union* set) and a list of duplicates, is in the file euclidrelations.v.

$$\sum_{i=1}^{n} |y_i| = |append_{i=1}^{n} y_i| = |sort(append_{i=1}^{n} y_i)| \\ = |\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i|. \tag{3.1}$$

$$|\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| \quad \wedge \quad |duplicates_{i=1}^{n} y_i| \geq 0 \\ \Rightarrow \quad |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|. \quad \square \tag{3.2}$$

3.3. Countable distance range. From the countable distance constraint (3.1), where $|x_i| = |y_i| = p_i$, the countable distance, d_c, ranges from a function of the sum of 1-1 domain-to-range set correspondences, $d_c = \mathrm{f}(\sum_{i=1}^{n}(1 \cdot |y_i|)) = \mathrm{f}(\sum_{i=1}^{n} p_i)$, to a function of the sum of each-to-each (Cartesian product) domain-to-range set mappings, $d_c = \mathrm{f}(\sum_{i=1}^{n}(|x_i| \cdot |y_i|)) = \mathrm{f}(\sum_{i=1}^{n} p_i^2)$.

3.4. Manhattan distance.

THEOREM 3.3. *Manhattan (largest) distance, d, is the size of the range interval,* $[d_0, d_m]$, *corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = d_m - d_0, \quad s_i = b_i - a_i.$$

The formal proof, "taxicab_distance," is in the Coq file, euclidrelations.v.

PROOF.
From the countable distance definition (3.1) and the union-sum inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \wedge |x_i| = |y_i| = p_i \quad \Rightarrow \quad \exists\, p_i,\, d_c : d_c = \sum_{i=1}^{n} p_i. \tag{3.3}$$

Multiply both sides of equation 3.3 by c and take the limit:

$$(3.4)\quad d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \Rightarrow \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c).$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$(3.5)\qquad d = d_m - d_0 \quad \Rightarrow \quad \exists\, c\, d : \ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$(3.6)\quad s_i = b_i - a_i \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i.$$

Combine equations 3.5, 3.4, 3.6:

$$(3.7)\quad d = \lim_{c\to 0} d_c \cdot c \quad \wedge \quad \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) \quad \wedge$$
$$\lim_{c\to 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square$$

3.5. Euclidean distance.

THEOREM 3.4. *Euclidean (smallest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = d_m - d_0, \quad s_i = b_i - a_i.$$

The formal proof, "Euclidean_distance," is in the Coq file, euclidrelations.v.

PROOF.
Apply the rule of product to the largest number of domain-to-range set mappings, where all p_i number of range set members, y_i, map to each of the p_i number of members in the domain set, x_i, which, by the rule of product, is the Cartesian product, $|y_i| \cdot |x_i|$:

$$(3.8)\qquad |x_i| = |y_i| = p_i \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2.$$

From the countable distance definition (3.1) and the union-sum inequality (3.2), the smallest possible distance is the equality case:

$$(3.9)\quad d_c = |\bigcup_{i=1}^{n} y_i| \le \sum_{i=1}^{n} |y_i| \wedge |x_i| = |y_i| = p_i \quad \Rightarrow \quad \exists\, p_i,\ d_c : \ d_c = \sum_{i=1}^{n} p_i.$$

Square both sides of equation 3.9 ($x = y \Leftrightarrow \mathrm{f}(x) = \mathrm{f}(y)$):

$$(3.10)\qquad \exists\, p_i,\ d_c : \ d_c = \sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\ d_c : \ d_c^2 = (\sum_{i=1}^{n} p_i)^2.$$

Apply the square of sum inequality, $(\sum_{i=1}^{n} p_i)^2 \ge \sum_{i=1}^{n} p_i^2$, to equation 3.10 and select the smallest area (the equality) case:

$$(3.11)\quad d_c^2 = (\sum_{i=1}^{n} p_i)^2 = \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j$$
$$= \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} p_i \sum_{j=1, j\ne i}^{n} p_j \ge \sum_{i=1}^{n} p_i^2 \quad \Rightarrow \quad \exists\, p_i : \ d_c^2 = \sum_{i=1}^{n} p_i^2.$$

Multiply both sides of equation 3.11 by c^2, simplify, and take the limit.

$$(3.12)\quad d_c^2 = \sum_{i=1}^{n} p_i^2 \Rightarrow d_c^2 \cdot c^2 = \sum_{i=1}^{n} p_i^2 \cdot c^2 \Leftrightarrow (d_c \cdot c)^2 = \sum_{i=1}^{n} (p_i \cdot c)^2$$
$$\Rightarrow \quad \lim_{c\to 0} (d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.13)\quad \exists\, c\, d \in \mathbb{R} : \ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c\to 0} (d_c \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$(3.14)\quad s_i = b_i - a_i \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i.$$

Combine equations 3.13, 3.12, 3.14:

$$d^2 = \lim_{c\to 0}(d_c \cdot c)^2 \quad \wedge \quad \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 \quad \wedge$$
$$\lim_{c\to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \tag{3.15}$$ □

3.6. Metric Space. All distances, $d(u, w)$, satisfying the countable distance definition (3.1), where the ruler is applied, generates the properties of metric space. The formal proofs: triangle_inequality, non_negativity, identity_of_indiscernibles, and symmetry are in the Coq file, euclidrelations.v.

THEOREM 3.5. *Triangle Inequality:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \leq d(u, v) + d(v, w)$.

PROOF. Use the countable distance (3.1) and union-sum inequality (3.2) as conditions. And next apply the ruler measure (2.1) and ruler convergence (2.2).

$$\begin{aligned}
&\forall\, c > 0,\ d(u, w),\ d(u, v),\ d(v, w)\ : \\
&\quad |y_1| = floor(d(u, v)/c) \ \wedge\ |y_2| = floor(d(v, w)/c) \ \wedge \\
&\quad d_c = floor(d(u, w)/c) \quad \wedge \quad d_c = |y_1 \cup y_2| \leq |y_1| + |y_2| \\
&\quad \Rightarrow\ floor(d(u, w)/c) \leq floor(d(u, v)/c) + floor(d(v, w)/c) \\
&\quad \Rightarrow\ floor(d(u, w)/c) \cdot c \leq floor(d(u, v)/c) \cdot c + floor(d(v, w)/c) \cdot c \\
&\quad \Rightarrow\ \lim_{c\to 0} floor(d(u, w)/c) \cdot c \leq \lim_{c\to 0} floor(d(u, v)/c) \cdot c + \lim_{c\to 0} floor(d(v, w)/c) \cdot c \\
&\quad = d(u, w) \leq d(u, v) + d(v, w).
\end{aligned} \tag{3.16}$$ □

THEOREM 3.6. *Non-negativity:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \geq 0$.

PROOF. By definition, a set always has a size (cardinal) ≥ 0:

$$\begin{aligned}
&\forall\, c > 0,\ d(u, w)\ : \quad floor(d(u, w)/c) = d_c \quad \wedge \quad d_c = |y_1 \cup y_2| \geq 0 \\
&\quad \Rightarrow \quad floor(d(u, w)/c) = d_c \geq 0 \quad \Rightarrow \quad d(u, w) = \lim_{c\to 0} d_c \cdot c \geq 0.
\end{aligned} \tag{3.17}$$ □

THEOREM 3.7. *Identity of Indiscernibles:* $d(w, w) = 0$.

PROOF. Apply the triangle inequality property (3.5):

$$\forall\, d(u, v) = d(v, w) = 0 \ \wedge\ d(u, w) \leq d(u, v) + d(v, w) \ \Rightarrow\ d(u, w) \leq 0. \tag{3.18}$$

Combine the non-negativity property (3.6) and the previous inequality (3.18):

$$d(u, w) \geq 0 \ \wedge\ d(u, w) \leq 0 \ \Leftrightarrow\ 0 \leq d(u, w) \leq 0 \ \Rightarrow\ d(u, w) = 0. \tag{3.19}$$

Combine the result of step 3.19 and the condition, $d(u, v) = 0$, in step 3.18.

$$d(u, w) = 0 \ \wedge\ d(u, v) = 0 \ \Rightarrow\ w = v. \tag{3.20}$$

Combine the condition, $d(v, w) = 0$, in step 3.18 and the result of step 3.20.

$$d(v, w) = 0 \ \wedge\ w = v \ \Rightarrow\ d(w, w) = 0. \tag{3.21}$$ □

THEOREM 3.8. *Symmetry:* $d_c = |y_1 \cup y_2| \ \wedge\ |x_i| = |y_i| \ \Rightarrow\ d(u, v) = d(v, u)$.

PROOF. The number domain set, x_i, members mapping to a range set, y_i, member increases with the amount of range set intersection. Therefore, the range of countable distances (3.3) is a function of domain-to-range set members, under the constraint, $|x_i| = |y_i|$, is:

$$|x_i| = |y_i| = p_i \quad \Rightarrow \quad d_c = \mathrm{f}(\textstyle\sum_{i=1}^{n} |x_i| \cdot |y_i|^q) = \mathrm{f}(\sum_{i=1}^{n} p_i^{1+q}),\ q \in \{0,\ 1\}. \tag{3.22}$$

Where d_c is applied to sets of size c subintervals of intervals, the previous Manhattan distance proof (3.3), $d(x,y) = \mathrm{f}(\sum_{i=1}^{2} s_i^1)$, and Euclidean distance proof (3.4), $d(x,y) = \mathrm{f}(\sum_{i=1}^{2} s_i^2)$, show that distance is a function of domain interval sizes, s_i, where $x = s_1$ and $y = s_2$. Generalizing:

$$\forall\, p : p \geq 0, \quad d(x,y) = \mathrm{f}(\textstyle\sum_{i=1}^{2} s_i^p) = \mathrm{f}(x^p + y^p) \\ \Rightarrow \quad d(u,v) = \mathrm{f}(u^p + v^p) = \mathrm{f}(v^p + u^p) = d(v,u). \quad \square \tag{3.23}$$

4. Euclidean Volume

$\mathbb{R}^n$, the Lebesgue measure, Riemann integral, and Lebesgue integral define (assume) area/volume to be the product of domain interval lengths. The goal here is to derive the area/volume equation from an abstract, set-based definition of volume without assuming the product of interval lengths.

Countable volume is the cardinal of the set of all possible countable distance unions. Each possible union of a member of one distance (range) set is either an intersection or non-intersection with a member in each of the other range sets forming an n-tuple (Cartesian coordinate). By the rule of product, the total number of n-tuples (unions), v_c, is the Cartesian product of distance-to-distance (range-to-range) set mappings, $v_c = |\times_{i=1}^{n} y_i|$.

Definition 4.1. Euclidean (largest possible) Countable Volume in flat space:

$$v_c = |\times_{i=1}^{n} y_i| = \textstyle\prod_{i=1}^{n} |y_i| : \quad \bigcap_{i=1}^{n} x_i = \bigcap_{i=1}^{n} y_i = \emptyset \quad \wedge \quad |x_i| = |y_i|.$$

Theorem 4.2. *Euclidean volume, v, is length of the range interval, $[v_0, v_m]$, equal to product of domain interval lengths, $\{[a_1,b_1],[a_2,b_2],\ldots,[a_n,b_n]\}$:*

$$v = \textstyle\prod_{i=1}^{n} s_i, \ \ v = v_m - v_0, \ \ s_i = b_i - a_i.$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

Proof.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of subintervals.

$$\forall\, i\; n \in \mathbb{N}, \quad i \in [1,n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i| = |y_i|. \tag{4.1}$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0}(p_i \cdot c) = s_i. \tag{4.2}$$

$$v_c = \textstyle\prod_{i=1}^{n} |y_i| \quad \wedge \quad |y_i| = p_i \quad \Rightarrow \quad v_c = \prod_{i=1}^{n} p_i. \tag{4.3}$$

Multiply both sides of equation 4.3 by c^n:

$$v_c \cdot c^n = (\textstyle\prod_{i=1}^{n} p_i) \cdot c^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.4}$$

Apply the ruler (2.1) to the range interval, $[v_0, v_m]$ (where $v = v_m - v_0$). Combine with equation 4.4. Apply the ruler convergence (2.2) and equation 4.2.

$$\forall\, v_c, n \in \mathbb{N}, \ c >= 0\ \exists\, v \in \mathbb{R} : \ floor(v/c^n) = v_c \quad \wedge \quad v_c \cdot c^n = \textstyle\prod_{i=1}^{n} (p_i \cdot c) \\ \Rightarrow \quad v = \lim_{c\to 0} v_c \cdot c^n = \textstyle\prod_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^{n} s_i. \quad \square \tag{4.5}$$

5. Applications to physics

5.1. Newton's gravity force equation. m_1 and m_2, are the sizes of two independent mass intervals, where each size c subinterval of a mass interval exerts a force on each size c subinterval of the other mass interval. If p_1 and p_2 are the number of size c components in each mass interval, then the total force, F, is equal to the total number of forces, which is proportionate to the Cartesian product, $p_1 \cdot p_2$, and proportionate to the size, c, of each component. Applying the volume proof (4.2), the total size of the Cartesian product of size c components is:

$$p_1 = floor(m_1/c) \quad \wedge \quad p_2 = floor(m_2/c) \quad \wedge \quad F := m_0 a \propto p_1 c \cdot p_2 c \\ \Rightarrow \quad F := m_0 a \propto (\lim_{c\to 0} p_1 c \cdot \lim_{c \to 0} p_2 c) = m_1 m_2. \tag{5.1}$$

$$F := m_0 a = m_0 r/t_c^2 \propto m_1 m_2 \quad \wedge \quad m_0 = m_1 \quad \Rightarrow \quad r \propto m_1 \quad \Rightarrow \\ \exists\, m_G, r_c \in \mathbb{R}: \; r = (r_c/m_G) m_2, \tag{5.2}$$

where: m_0 is rest mass, a is acceleration, r is Euclidean distance, t_c is a unit of time, and r_c/m_G is a unit-factoring proportion ratio.

$$F := m_0 a = m_0 r/t_c^2 \propto m_1 m_2 \quad \wedge \quad m_0 = m_1 \quad \Rightarrow \quad r \propto m_1 \quad \Rightarrow \\ \exists\, m_G, r_c \in \mathbb{R}: \; r = (r_c/m_G) m_2, \tag{5.3}$$

where r_c/m_G is a unit-factoring proportion ratio.

$$m_0 = m_1 \quad \wedge \quad r = (m_G/r_c) m_2 \quad \wedge \quad F := m_0 a = m_0 r/t_c^2 \\ \Rightarrow \quad F := m_0 a = m_0 r/t_c^2 = (r_c/m_G) m_1 m_2/t_c^2. \tag{5.4}$$

From equation (5.2):

$$\exists\, t_c, r_c \in \mathbb{R} \quad \Rightarrow \quad \exists\, t \in \mathbb{R}: \; t = (t_c/r_c) r. \tag{5.5}$$

$$r = (r_c/t_c) t \quad \wedge \quad F := m_0 a = (r_c/m_G) m_1 m_2/t_c^2 \quad \Rightarrow \\ F := m_0 a = (r_c/m_G)(r_c^2/t_c^2) m_1 m_2/r^2 = (r_c^3/m_G t_c^2) m_1 m_2/r^2 = G m_1 m_2/r^2, \tag{5.6}$$

where the gravitational constant, $G = r_c^3/m_G t_c^2$, has the SI units: $m^3 kg^{-1} s^{-2}$.

5.2. Coulomb's charge force. q_1 and q_2, are the sizes of two independent charge intervals, where each size c subinterval of a charge interval exerts a force on each size c subinterval of the other charge interval. If p_1 and p_2 are the number of size c components in each charge interval, then the total force, F, is equal to the total number of forces, which is proportionate to the Cartesian product, $p_1 \cdot p_2$, and the size, c, of each component. Applying the volume proof (4.2), the total size of the Cartesian product of size c components is:

$$p_1 = floor(q_1/c) \quad \wedge \quad p_2 = floor(q_2/c) \quad \wedge \quad F \propto p_1 c \cdot p_2 c \\ \Rightarrow \quad F := m_0 a \propto (\lim_{c\to 0} p_1 c \cdot \lim_{c\to 0} p_2 c) = (q_1 q_2). \tag{5.7}$$

$$F := m_0 a = m_0 r/t_c^2 \propto q_1 q_2 \quad \wedge \quad m_0 = (m_C/q_C) q_1 \quad \Rightarrow \quad r \propto q_1 \quad \Rightarrow \\ \exists\, q_C, r_c \in \mathbb{R}: \; r = (r_c/q_C) q_2, \tag{5.8}$$

where: m_0 is rest mass, a is acceleration, r is Euclidean distance, t_c is a unit of time, m_C/q_C and q_C/r_c are unit-factoring proportion ratios.

$$m_0 = (m_C/q_C)q_1 \quad \wedge \quad r = (q_C/r_c)q_2 \quad \wedge \quad F := m_0 a = m_0 r/t_c^2 \\ \Rightarrow \quad F := m_0 a = m_0 r/t_c^2 = (m_C/q_C)(r_c/q_C)q_1 q_2/t_c^2 = (m_C r_c/q_C^2)q_1 q_2/t_c^2. \tag{5.9}$$

From equation (5.8):

$$\exists\, t_c, r_c \in \mathbb{R} \quad \Rightarrow \quad \exists\, t \in \mathbb{R} : \; t = (t_c/r_c)r. \tag{5.10}$$

$$r = (r_c/t_c)t \quad \wedge \quad a_C = r_c/t_c^2 \quad \wedge \quad F := m_0 a = (m_C r_c/q_C^2)q_1 q_2/t_c^2 \quad \Rightarrow \\ F := m_0 a = (r_c^2/t_c^2)(m_C r_c/q_C^2)q_1 q_2/r^2 = ((m_C a_C)r_c^2/q_C^2)q_1 q_2/r^2 = k_c q_1 q_2/r^2, \tag{5.11}$$

where the charge constant, $k_C = (m_C a_C)r_c^2/q_C^2$, has the SI units: Nm^2C^{-2}.

5.3. Spacetime equations. As shown in the derivations of Newton's gravity force (5.1) and Coulomb's charge force (5.2) equations, for any Euclidean distance interval having size, r, an interval having size, t, exists, where $r = (r_c/t_c)t = ct$, and $r_c/t_c = c$ is a unit-factoring proportion ratio.

Applying the ruler to two intervals, $[0, d_1]$ and $[0, d_2]$, in two inertial (independent, non-accelerating) frames of reference, the smallest distance (and time) spanning the two domain intervals converges to the Euclidean distance (3.4), r.

$$r^2 = d_1^2 + d_2^2 \quad \wedge \quad r = (r_c/t_c)t = ct \quad \Rightarrow \quad (ct)^2 = d_1^2 + d_2^2 \\ \Leftrightarrow \quad d_1^2 = (ct)^2 - (x^2 + y^2 + z^2), \tag{5.12}$$

where $d_2^2 = x^2 + y^2 + z^2$, which is one form of Minkowski's well-known flat spacetime interval equation [**Bru17**]. And the length contraction and time dilation equations also follow directly from $(ct)^2 = d_1^2 + d_2^2$, where $v = d/t$:

$$d_2^2 = c^2t^2 - d_1^2 \;\Rightarrow\; d_2^2/t^2 = c^2 - v_1^2 \;\Rightarrow\; d = t\sqrt{c^2 - v^2} = ct\sqrt{1 - (v/c)^2}. \tag{5.13}$$

$$d = ct\sqrt{1 - (v/c)^2} \quad \wedge \quad t' = d/c \quad \Rightarrow \quad t = t'/\sqrt{1 - (v/c)^2}. \tag{5.14}$$

5.4. General relativity without the complexity of tensors: Using the mass and spacetime ratios: $r = (r_c/m_G)m = ct \Rightarrow (r_c/m_G)m \cdot ct = r^2 \Rightarrow m = (m_G/r_c c)r^2/t = (m_G/r_c c)rv$. For a constant mass, m, a decrease in the distance, r, between two mass centers requires a decrease in time, t, (time slows down). v is the relativistic orbital velocity at distance, r. $F = ma = mr/t^2 = mc^2/r$. $E = mc^2 = (m_G/r_c)r^3/t^2$. And $KE = mv^2/2 = (m_G c^2/2r_c)r$. Likewise, for charge, $r = (r_c/q_C)q = ct \Rightarrow q = (q_C/r_c c)r^2/t = (q_C/r_c c)rv$, $E = qc^2 = (q_C/r_c)r^3/t^2$, and $KE = qv^2/2 = (q_C c^2/2r_c)r$. And so on.

5.5. 3 dimensional balls. Countable distance, $d_c = |\bigcup_{i=1}^n y_i|$, (3.1), countable volume, $v_c = \prod_{i=1}^n |y_i|$, (4.1), Manhattan distance (3.3), Euclidean distance (3.4), and volume (4.2) requires that a set of intervals/dimensions can be assigned a *total order* ($i = 1$ to n). And the commutative properties of union, multiplication, and addition allow sequencing through each interval (dimension) in every possible order. "Strict" sequencing (no jumping over other members) via the successor and predecessor relations of a totally ordered set in every possible order requires each set member to be sequentially adjacent (either a successor or predecessor) to every other member, herein referred to as a symmetry constraint.

It will now be proved that coexistence of the symmetry constraint on a sequentially ordered set defines a cyclic set that contains at most 3 members, in this case, 3 dimensions of ordered and symmetric distance and volume.

Definition 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i \;\in \{x_1, \ldots, x_n\},\\ \textit{successor}\; x_i = x_{i+1} \;\;\wedge\;\; \textit{predecessor}\; x_{i+1} = x_i.$$

Definition 5.2. Symmetry Constraint (every set member is sequentially adjacent to every other member):

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; \textit{successor}\; x_i = x_j \;\;\Leftrightarrow\;\; \textit{predecessor}\; x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$i = n \;\;\wedge\;\; j = 1 \;\;\Rightarrow\;\; \textit{successor}\; x_n = x_1 \;\;\wedge\;\; \textit{predecessor}\; x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

Proof. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry constraint (5.2):

$$i = n \;\;\wedge\;\; j = 1 \;\;\wedge\;\; \textit{successor}\; x_i = x_j \;\;\Rightarrow\;\; \textit{successor}\; x_n = x_1. \tag{5.15}$$

Applying the definition of the symmetry constraint (5.2) to conclusion 5.15:

$$\textit{successor}\; x_i = x_j \;\;\Rightarrow\;\; \textit{predecessor}\; x_j = x_i \;\;\Rightarrow\;\; \textit{predecessor}\; x_1 = x_n. \tag{5.16}$$

□

Theorem 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

Proof.
It was proved that an ordered and symmetric set is a cyclic set (5.3).

Definition 5.5. Successor of m is n:

$$\textit{Successor}(m, n, \textit{setsize}) \leftarrow (m = \textit{setsize} \wedge n = 1) \vee (n = m + 1 \leq \textit{setsize}). \tag{5.17}$$

Definition 5.6. Predecessor of m is n:

$$\textit{Predecessor}(m, n, \textit{setsize}) \leftarrow (m = 1 \wedge n = \textit{setsize}) \vee (n = m - q \geq 1). \tag{5.18}$$

Definition 5.7. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$\textit{Adjacent}(m, n, \textit{setsize}) \leftarrow \textit{Successor}(m, n, \textit{setsize}) \vee \textit{Predecessor}(m, n, \textit{setsize}). \tag{5.19}$$

Prove that every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (m = setsize \wedge n = 1). \tag{5.20}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (n = m+1 \leq setsize). \tag{5.21}$$

$$Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (n = setsize \wedge m = 1). \tag{5.22}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m+1 \leq setsize). \tag{5.23}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m - q \geq 1). \tag{5.24}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \wedge m = 1). \tag{5.25}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \wedge n = setsize). \tag{5.26}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m+1 \leq setsize). \tag{5.27}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m - q \geq 1). \tag{5.28}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\begin{aligned} \forall\, setsize > 3: \quad & \neg Successor(1,3,setsize > 3) \\ & \leftarrow Successor(1,2,setsize > 3) \leftarrow (n = m+1 \leq setsize). \end{aligned} \tag{5.29}$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\, setsize > 3: \quad & \neg Predecessor(1,3,setsize > 3) \\ & \leftarrow Predecessor(1,setsize,setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \end{aligned} \tag{5.30}$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\, setsize > 3: \quad & \neg Adjacent(1,3,setsize > 3) \\ & \leftarrow \neg Successor(1,3,setsize > 3) \wedge \neg Predecessor(1,3,setsize > 3). \quad \square \end{aligned} \tag{5.31}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

(1) The Manhattan and Euclidean distance proofs (3.3) (3.4) and the Euclidean volume proof (4.2) show the constraint that each domain set has a corresponding range (distance) set containing the same number of members, $|x_i| = |y_i|$, generates flat space (rectilinear distances and volume).

(2) Generalizing the flat space constraint on countable distance and volume, $|x_i| = |y_i|$, to $|x_i| = |y_i|^q$, $q \geq 0$, generates all the L^p norms (Minkowski distances), $||L||_p = (\sum_{i=1}^{n} s_i^p)^{1/p}$. For example, using the same proof pattern as for Euclidean distance (3.4): $|y_i| = p_i \Rightarrow |x_i| = p_i^q \Rightarrow \sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^q \cdot p_i = \sum_{i=1}^{n} p_i^{q+1} \leq d_c^{q+1}$. Choosing the equality case and applying the ruler: $d^{q+1} = \sum_{i=1}^{n} s_i^{q+1}$. And $p = q+1 \Rightarrow d^p = \sum_{i=1}^{n} s_i^p \Rightarrow d = (\sum_{i=1}^{n} s_i^p)^{1/p}$.

(3) Obviously, the L_p norms (Minkowski distances) also follow from Euclidean volume because a p-dimensional volume can only be equal to the sum of other p-dimensional volumes: $\forall\ V \in \mathbb{R}^p\ \exists\ v_1, \cdots, v_n \in \mathbb{R}^p : V = \sum_{i=1}^{n} v_i \Rightarrow \forall\ d^p = V\ \exists\ s_1, \cdots s_n \in \mathbb{R} : d^p = \sum_{i=1}^{n} s_i^p \Rightarrow d = (\sum_{i=1}^{n} s_i^p)^{1/p}$.

(4) The curvature of a space around a point is typically measured in terms of second order differential equations, e.g., the Laplacian. A set-based measure of the amount of curvature is how far q deviates from the value, 1, in the countable distance and volume constraint, $|x_i| = |y_i|^q$.

(5) The smallest possible countable distance (3.1), $d_c = |\bigcup_{i=1}^n y_i|$, is the case of the largest intersection of the range sets, which is also the case of the largest possible number (the Cartesian product) of domain-to-range set mappings, in flat space: $d_c = \mathrm{f}(\sum_{i=1}^{n} |x_i| \cdot |y_i|) = \mathrm{f}(\sum_{i=1}^{n} p_i^2)$. And applying the ruler to create countable sets of subintervals of domain and range intervals, the Cartesian product of domain-to-range set mappings yields the Euclidean distance equation.

(6) Manhattan (largest) distance and Euclidean (largest) volume are both cases of disjoint range sets, $\bigcap_{i=1}^n y_i = \emptyset$, in flat space (where $|x_i| = |y_i|$):

$$d_c = |\bigcup_{i=1}^n y_i| : \quad \bigcap_{i=1}^n x_i = \bigcap_{i=1}^n y_i = \emptyset \quad \wedge \quad |x_i| = |y_i|.$$

$$v_c = |\times_{i=1}^n y_i| : \quad \bigcap_{i=1}^n x_i = \bigcap_{i=1}^n y_i = \emptyset \quad \wedge \quad |x_i| = |y_i|.$$

(7) Applying the ruler (2.1), ruler convergence (2.2), and the volume proof (4.2) to the Cartesian product of same-sized, infinitesimal mass forces and charge forces to derive Newton's gravity force (5.1) and Coulomb's charge force (5.2) equations provide several new achievements and insights into physics.
   (a) The first deductive derivations. All other derivations have been empirical and inductive (not fully provable).
   (b) The first derivations not using other laws of physics, the inverse square law, or Gauss's divergence theorem.
   (c) The first derivations to show that time is proportionate to distance ($r = (r_c/t_c)t = ct$), which is then used to derive the spacetime equations (5.3) without any notion of the speed of light.
   (d) The first derivations to show that all Euclidean distance intervals having a size, r, have proportionately sized intervals of other types: $r = (r_c/q_C)q = (r_c/m_G)m = (r_c/t_c)t = ct$, which allows much simpler derivations of general relativity equations (5.4) without the need for vectors and tensors.
   (e) The first derivations to show that the definition of force, $F := ma$, containing acceleration, $a = r/t_c^2$, and combined with $r = (r_c/t_c)t$, generates the inverse square law, $F = m_0 a = m_0 r/t_c^2 = k x_1 x_2 / r^2$.
   (f) Some constants like charge, vacuum magnetic permeability, fine structure, etc. contain the value 4π because the creators assumed Gauss's flux divergence on the surface of a sphere having the area, $4\pi r^2$. The derivations here show that the inverse square law is the result of the definition of force and acceleration – not flux divergence. Therefore, those versions of the constants containing the value 4π are incorrect.

(g) The phenomenon of quantum charge and mass values is the result of proportion ratios (constant first derivatives) with respect to the same distance, r_c.
(h) If there are quantum values of mass, m_G, and charge, q_C, then there is a quantum distance, r_c, where the gravity and charge forces do not exist (are not defined) at smaller distances.
(i) Discrete, state values have no proportion ratio to varying distance and time interval lengths. For example, the change of spin values of two quantum entangled electrons and the change of polarization of two quantum entangled photons are independent of the distance and time between the entangled particles.

(8) It was proved that a totally ordered set with a symmetry constraint is a cyclic set with at most 3 members (5.3). And the definitions of distance and volume both require a total order and symmetry, which provides several insights.
(a) Using Occam's razor, a cyclic set of at most 3 members is the most parsimonious explanation of only observing 3 dimensions of distance and volume.
(b) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of three members (5.4), each member being an ordered and symmetric set of 3 dimensions (three balls), yielding a total of 9 ordered and symmetric dimensions of geometric space.
(c) Each ordered and symmetric ball can have at most 3 ordered and symmetric dimensions of discrete states of the same type, for example, a set of 3 binary values, 1 and -1, indicating vector orientation.
(d) Each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values of the same type per ball, for example, spin values: -1, 0, 1 per orthogonal plane in the ball.
(e) Each of the 3 possible ordered and symmetric dimensions of discrete physical states could contain an unordered collection (bag) of discrete state values. Bags are non-deterministic. For example, every time an unordered binary state is pulled from a bag, there is a 50 percent chance of getting one of the binary values.

References

[Bru17] P. Bruskiewich, *A very simple introduction to special relativity: Part two - four vectors, the lorentz transformation and group velocity (the new mathematics for the millions book 38)*, Pythagoras Publishing, 2017. ↑8
[CG15] W. Conradie and V. Goranko, *Logic and discrete mathematics*, Wiley, 2015. ↑
[Coq15] Coq, *Coq proof assistant*, 2015. `https://coq.inria.fr/documentation`. ↑2
[Gol76] R. R. Goldberg, *Methods of real analysis*, John Wiley and Sons, 1976. ↑1
[Joy98] D. E. Joyce, *Euclid's elements*, 1998. `http://aleph0.clarku.edu/~djoyce/java/elements/elements.html`. ↑1
[Rud76] W. Rudin, *Principles of mathematical analysis*, McGraw Hill Education, 1976. ↑1

George Van Treeck, 668 Westline Dr., Alameda, CA 94501