Some Set Properties Underlying Geometry and Physics

George. M. Van Treeck

Abstract. Euclidean volume and the Minkowski distances (Manhattan, Euclidean, etc. distances) are derived from a set and limit-based foundation without referencing the primitives and relations of geometry. Sequencing a strict linearly ordered set in all n-at-a-time orders via successor/predecessor relations is proved to be a cyclic set of at most 3 members. A cyclic set of 3 distance domain interval lengths are related to other types of domain interval lengths by unit-factoring ratios. The ratios are used to provide simple and short derivations of: the gravity (G), charge (k_e), and Planck (h) constants, the spacetime, Schwarzchild's gravitational time dilation, Einstein's gravitational lens, Planck relation, Compton wavelength, de Broglie wavelength, and quantum-relativity equations. All the proofs are verified in Coq.

Contents

1. Introduction

Mathematical analysis can construct differential calculus from a set and limit-based foundation without referencing the primitives and relations of Euclidean geometry, like straight line, angle, etc., which provides a more rigorous foundation and deeper understanding of geometry and physics. But Euclidean volume in the Riemann integral, Lebesgue integral, measure theory, and distance in the vector magnitude and metric space criteria are definitions motivated by Euclidean geometry [Gol76] [Rud76] rather than derived from a set and limit-based foundation.

2010 *Mathematics Subject Classification.* Primary 28A75, 28E15. Secondary 03E75, 51M99.

An intuitive, set-based motivation of Euclidean volume is the number, v_c, of ordered combinations (n-tuples): $v_c = \prod_{i=1}^{n} |x_i|$, where $|x_i|$ is the cardinal of the countable, disjoint set, x_i. But, some well-known analysis textbooks do not provide a proof that, $\lim_{\kappa \to 0} v_c \cdot \kappa = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|) \cdot \kappa \Rightarrow v = \prod_{i=1}^{n} s_i$, where each set, x_i, is a set of size κ subintervals of each interval, $[a_i, b_i] \subset \mathbb{R}$, and where $s_i = b_i - a_i$. [Gol76] [Rud76]. In this article, that proof is provided.

$v_c = \prod_{i=1}^{n} |x_i| = f(|x_1|, \cdots, |x_n|, n)$. If f is a bijective function, then $\exists\ d_c : d_c = f^{-1}(v_c, n)$ and $v_c = f(d_c, n) = f(|x_1|, \cdots, |x_n|, n)$. If f is also isomorphic, then $\forall\ |x_i|,\ d_c = |x_1| = \cdots = |x_n|$ and $v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n$.

Where $v_c = f(|x_1|, \cdots, |x_n|, n)$ is a bijective and isomorphic function, it will be proved that $\lim_{\kappa \to 0} v_c \cdot \kappa = \lim_{\kappa \to 0}(\sum_{j=1}^{m} v_{c_i}) \cdot \kappa \Rightarrow d^n = \sum_{i=1}^{m} d_i^n$. d is the ρ-norm (Minkowski distance) [Min53], which will be proved to imply the metric space properties [Rud76].

Sequencing the domain sets, $x_1, \cdots, x_n$, from $i = 1$ to n, is a strict linear (total) order, where a total order is defined in terms of successor and predecessor relations [CG15]. Sequencing a set, via successor and predecessor relations, in all n-at-a-time orders, requires a "symmetry" constraint, where every set member is either a successor or predecessor to every other set member. A strict linearly ordered and symmetric set will be proved to be a cyclic set, where $n \leq 3$.

Therefore, if $\{s_1, s_2, s_3\}$ is a strict linearly ordered and symmetric set of 3 "distance" dimensions, then another dimension, s_4, must have a different type (is a member of different set). In an orthogonal Cartesian grid around a point, there are constant, unit-factoring ratios between a unit of a 3-dimensional distance, r, and units of other orthogonal dimensions: $r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q = \cdots$.

The ratio constants are used for simple and short derivations of: the gravity (G), charge (k_e), and Planck (h) constants. And the ratio constants are also used for simpler and shorter derivations of: spacetime, Schwarzchild gravitational time dilation, Einstein's gravitational lens, Planck relation, Compton wavelength, de Broglie wavelength, quantum-relativity gravity and charge equations.

All the proofs in this article have been verified using using the Coq proof verification system [Coq23]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of size κ subintervals. The ruler is both an inner and outer measure of an interval.

Definition 2.1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\ [a, b] \subset \mathbb{R}$, $s = b - a\ \wedge\ 0 < \kappa \leq 1\ \wedge\ (p = floor(s/\kappa)\ \vee\ p = ceiling(s/\kappa))$.

Theorem 2.2. *Ruler convergence:* $M = \lim_{\kappa \to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\ \kappa > 0,\ p = floor(s/\kappa)\ \wedge\ 0 \leq |floor(s/\kappa) - s/\kappa| < 1\ \Rightarrow\ |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by κ:

$$\forall\ \kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \epsilon \quad := \quad M = \lim_{\kappa \to 0} p\kappa = s. \quad \square \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, ..., \pi_{\lim_{\kappa \to 0}}$.

LEMMA 2.3. $\forall\, n \geq 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa$.

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta \\ \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa \to 0} \kappa^n = 0. \tag{2.5}$$

$$\lim_{\kappa \to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa \to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa. \qquad \square \tag{2.6}$$

3. Volume

DEFINITION 3.1. A countable n-volume is the number of ordered combinations (n-tuples), v_c, of the members of n number of disjoint, countable domain sets, x_i:

$$\exists\, n \in \mathbb{N},\ v_c \in \{0,\ \mathbb{N}\},\ \ x_i \in \{x_1, \cdots, x_n\} : \ \bigcap_{i=1}^{n} x_i = \emptyset \ \wedge \ v_c = \prod_{i=1}^{n} |x_i|. \tag{3.1}$$

THEOREM 3.2. *Euclidean volume,*

$$\forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\ \ [v_a, v_b] \subset \mathbb{R},\ \ s_i = b_i - a_i,\ \ v = v_b - v_a : \\ v_c = \prod_{i=1}^{n} |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^{n} s_i. \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.

$$v_c = \prod_{i=1}^{n} |x_i| \ \Leftrightarrow\ v_c\kappa = (\prod_{i=1}^{n} |x_i|)\kappa \ \Leftrightarrow\ \lim_{\kappa \to 0} v_c\kappa = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|)\kappa. \tag{3.3}$$

Apply the ruler (2.1) and ruler convergence (2.2) to equation 3.3:

$$\exists\, v, \kappa \in \mathbb{R} : \ v_c = floor(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa \to 0} v_c\kappa = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|)\kappa. \tag{3.4}$$

Apply lemma 2.3 to equation 3.4:

$$v = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|)\kappa = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|)\kappa^n = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|\kappa). \tag{3.5}$$

Apply the ruler (2.1) and ruler convergence (2.2) to s_i:

$$\exists\, s_i, \kappa \in \mathbb{R} : \ floor(s_i/\kappa) = |x_i| \quad \Rightarrow \quad \lim_{\kappa \to 0}(|x_i|\kappa) = s_i. \tag{3.6}$$

$$v = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|\kappa) \ \wedge \ \lim_{\kappa \to 0}(|x_i|\kappa) = s_i \ \Rightarrow \ v = \prod_{i=1}^{n} s_i \qquad \square \tag{3.7}$$

4. Distance

DEFINITION 4.1. Countable distance, $d_c = f(v_c, n) = f(|x_1|, \cdots, |x_n|, n) = \prod_{i=1}^{n} |x_i|$ is bijective and isomorphic: $v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n$.

4.1. Minkowski distance (ρ-norm).

THEOREM 4.2. *Minkowski distance (ρ-norm):*

$$v_c = \sum_{j=1}^{m} v_{c_i} \quad \Rightarrow \quad \exists\, d, d_i \in \mathbb{R} :\ d^n = \sum_{i=1}^{m} d_i^n.$$

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

PROOF. Apply the countable distance definition (4.1) to the assumption:

$$(4.1)\quad v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n \quad \wedge \quad v_{c_i} = \prod_{j=1}^{n} |x_{i_j}| = \prod_{i=1}^{n} d_{c_i} = d_{c_i}^n$$
$$\wedge \quad v_c = \sum_{j=1}^{m} v_{c_i} \quad \Rightarrow \quad d_c^n = \sum_{j=1}^{m} d_{c_i}^n.$$

Multiply both sides of equation 4.1 by κ and take the limit:

$$(4.2)\qquad d_c^n = \sum_{j=1}^{m} d_{c_i}^n \quad \Leftrightarrow \quad \lim_{\kappa\to 0} d_c^n \kappa = \lim_{\kappa\to 0} \sum_{j=1}^{m} d_{c_i}^n \kappa.$$

Apply lemma 2.3 to equation 4.1:

$$(4.3)\quad \lim_{\kappa\to 0} d_c^n \kappa = \lim_{\kappa\to 0} \sum_{j=1}^{m} d_{c_i}^n \kappa \quad \wedge \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa$$
$$\Leftrightarrow \lim_{\kappa\to 0} d_c^n \kappa^n = \lim_{\kappa\to 0} \sum_{j=1}^{m} d_{c_i}^n \kappa^n \Leftrightarrow \lim_{\kappa\to 0} (d_c\kappa)^n = \lim_{\kappa\to 0} \sum_{j=1}^{m} (d_{c_i}\kappa)^n.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to equation 4.3:

$$(4.4)\quad \exists\, d, d_i :\ d_c = floor(d/\kappa),\ d = \lim_{\kappa\to 0} d_c\kappa$$
$$\wedge \quad d_{c_i} = floor(d_i/\kappa),\ d_i = \lim_{\kappa\to 0} d_{c_i}\kappa \quad \Rightarrow$$
$$d^n = \lim_{\kappa\to 0} (d_c\kappa)^n = \lim_{\kappa\to 0} \sum_{j=1}^{m} (d_{c_i}\kappa)^n = \sum_{j=1}^{m} d_i^n. \quad \square$$

4.2. Distance inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N},\ \ v_a, v_b \geq 0 :\ \ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$(4.5)\quad \forall\, v_a, v_b \geq 0 :\quad v_a + v_b \leq v_a + v_b +$$
$$\sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n.$$

Take the n^{th} of both sides of the inequality 4.5:

$$(4.6)\ \forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad \square$$

4.3. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N},\ \ a_i, b_i \geq 0 :\ \ \left(\sum_{i=1}^{m} (a_i^n + b_i^n)\right)^{1/n} \leq \left(\sum_{i=1}^{m} a_i^n\right)^{1/n} + \left(\sum_{i=1}^{m} b_i^n\right)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$(4.7)\quad \forall\, m, n \in \mathbb{N},\ \ v_a, v_b \geq 0 :\quad v_a = \sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge$$
$$(v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad \left(\left(\sum_{i=1}^{m} a_i^n\right) + \left(\sum_{i=1}^{m} b_i^n\right)\right)^{1/n} =$$
$$\left(\sum_{i=1}^{m} (a_i^n + b_i^n)\right)^{1/n} \leq \left(\sum_{i=1}^{m} a_i^n\right)^{1/n} + \left(\sum_{i=1}^{m} b_i^n\right)^{1/n}. \quad \square$$

4.4. Metric Space. All Minkowski distances (ρ-norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.8}$$

Apply the distance inequality (4.3) to the inequality 4.8:

$$\begin{aligned}(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \;\wedge\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ \Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ \Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq \\ (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w).\end{aligned} \tag{4.9}$$

□

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$\begin{aligned}\forall\, p : p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u).\end{aligned} \tag{4.10}$$

□

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0. \tag{4.11}$$

$$p \geq 1,\ \ u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \tag{4.12}$$

□

THEOREM 4.8. *Identity of Indiscernibles:* $d(u, u) = 0$.

PROOF. From the non-negativity property (4.7):

$$\begin{aligned}d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0 \\ \Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0.\end{aligned} \tag{4.13}$$

$$d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v. \tag{4.14}$$

$$d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \tag{4.15}$$

□

4.5. Set properties limiting a set to at most 3 members.

DEFINITION 4.9. Totally ordered set:

$$\begin{aligned}\forall\, i\ n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i\ \in \{x_1, \ldots, x_n\}, \\ \textit{successor}\ x_i = x_{i+1} \ \wedge\ \textit{predecessor}\ x_{i+1} = x_i.\end{aligned}$$

DEFINITION 4.10. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i,\ j,\ n \in \mathbb{N},\ \forall\, x_i,\ x_j \in \{x_1, \ldots, x_n\},\ \textit{successor}\ x_i = x_j\ \Leftrightarrow\ \textit{predecessor}\ x_j = x_i.$$

THEOREM 4.11. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n \;\wedge\; j = 1 \;\Rightarrow\; \mathit{successor}\ x_n = x_1 \;\wedge\; \mathit{predecessor}\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (4.9) assigns a unique label to each set member and assigns unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (4.10):

$$i = n \;\wedge\; j = 1 \;\wedge\; \mathit{successor}\ x_i = x_j \;\Rightarrow\; \mathit{successor}\ x_n = x_1. \tag{4.16}$$

Applying the definition of the symmetry property (4.10) to conclusion 4.16:

$$\mathit{successor}\ x_i = x_j \;\Rightarrow\; \mathit{predecessor}\ x_j = x_i \;\Rightarrow\; \mathit{predecessor}\ x_1 = x_n. \tag{4.17}$$ □

THEOREM 4.12. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (4.11).

DEFINITION 4.13. (Cyclic) Successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (n = m + 1 \leq \mathit{setsize}). \tag{4.18}$$

DEFINITION 4.14. (Cyclic) Predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (n = m - 1 \geq 1). \tag{4.19}$$

DEFINITION 4.15. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$\mathit{Adjacent}(m, n, \mathit{setsize}) \leftarrow \mathit{Successor}(m, n, \mathit{setsize}) \vee \mathit{Predecessor}(m, n, \mathit{setsize}). \tag{4.20}$$

Every member is adjacent to every other member, where $\mathit{setsize} \in \{1,\ 2,\ 3\}$:

$$\mathit{Adjacent}(1,1,1) \leftarrow \mathit{Successor}(1,1,1) \leftarrow (m = \mathit{setsize} \wedge n = 1). \tag{4.21}$$

$$\mathit{Adjacent}(1,2,2) \leftarrow \mathit{Successor}(1,2,2) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{4.22}$$

$$\mathit{Adjacent}(2,1,2) \leftarrow \mathit{Successor}(2,1,2) \leftarrow (n = \mathit{setsize} \wedge m = 1). \tag{4.23}$$

$$\mathit{Adjacent}(1,2,3) \leftarrow \mathit{Successor}(1,2,3) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{4.24}$$

$$\mathit{Adjacent}(2,1,3) \leftarrow \mathit{Predecessor}(2,1,3) \leftarrow (n = m - 1 \geq 1). \tag{4.25}$$

$$\mathit{Adjacent}(3,1,3) \leftarrow \mathit{Successor}(3,1,3) \leftarrow (n = \mathit{setsize} \wedge m = 1). \tag{4.26}$$

$$\mathit{Adjacent}(1,3,3) \leftarrow \mathit{Predecessor}(1,3,3) \leftarrow (m = 1 \wedge n = \mathit{setsize}). \tag{4.27}$$

$$\mathit{Adjacent}(2,3,3) \leftarrow \mathit{Successor}(2,3,3) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{4.28}$$

$$\mathit{Adjacent}(3,2,3) \leftarrow \mathit{Predecessor}(3,2,3) \leftarrow (n = m - 1 \geq 1). \tag{4.29}$$

Member 2 is the only successor of member 1 for all $setsize \geq 3$, which implies member 3 is not ($\neg$) a successor of member 1 for all $setsize \geq 3$:

$$(4.30) \quad \neg Successor(1, 3, setsize \geq 3) \leftarrow Successor(1, 2, setsize \geq 3) \leftarrow (n = m + 1 \leq setsize).$$

Member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not ($\neg$) a predecessor of member 1 for all $setsize > 3$:

$$(4.31) \quad \neg Predecessor(1, 3, setsize \geq 3) \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).$$

For all $setsize \geq 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not symmetric):

$$(4.32) \quad \neg Adjacent(1, 3, setsize > 3) \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square$$

5. Applications to physics

From the volume proof (3.2), two disjoint distance intervals, $[0, r_1]$ and $[0, r_2]$, define a 2-volume. From the Minkowski distance proof (4.2), $\exists\, r : r^2 = r_1^2 + r_2^2$. And from the 3D proof (4.12), for some non-distance type, $\tau : \tau \in \{t\ (time),\ m\ (mass),\ q\ (charge), \cdots\}$, there exist unit-factoring ratios, $\mu,\ \nu_1,\ \nu_2$:

$$(5.1) \quad \forall\, r, r_1, r_2 :\ r^2 = r_1^2 + r_2^2 \quad \wedge \quad r = \mu\tau \quad \wedge \quad r_1 = \nu_1\tau \quad \wedge \quad r_2 = \nu_2\tau \quad \Rightarrow \quad (\mu\tau)^2 = (\nu_1\tau)^2 + (\nu_2\tau)^2 \quad \Rightarrow \quad \mu \geq \nu_1 \quad \wedge \quad \mu \geq \nu_2.$$

μ is the maximum-possible ($\mu \geq \nu_1,\ \nu_2$), constant, unit-factoring ratio, where:

$$(5.2) \quad \mu \in \{r_c/t_c,\ r_c/m_c,\ r_c/q_c, \cdots\} :\ r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q = \cdots .$$

5.1. Space-time-mass-charge equations.

Form equation 5.1:

$$(5.3) \quad \forall\, r, r', r_\nu, \mu, \nu :\ r^2 = r'^2 + r_\nu^2 \quad \wedge \quad r = \mu\tau \quad \wedge \quad r_\nu = \nu\tau \quad \Rightarrow \quad r' = \sqrt{(\mu\tau)^2 - (\nu\tau)^2} \ = \ \mu\tau\sqrt{1 - (\nu/\mu)^2}.$$

Rest distance, r', contracts relative to coordinate distance, r, as $\nu \to \mu$:

$$(5.4) \qquad r' = \mu\tau\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad \mu\tau = r \quad \Rightarrow \quad r' = r\sqrt{1 - (\nu/\mu)^2}.$$

Coordinate length, τ, dilates relative to the rest length, τ', as $\nu \to \mu$:

$$(5.5) \qquad \mu\tau = r'/\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad r' = \mu\tau' \quad \Rightarrow \quad \tau = \tau'/\sqrt{1 - (\nu/\mu)^2}.$$

Where τ is time, the space-like flat Minkowski spacetime event interval is:

$$(5.6) \quad \mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_\nu^2 \quad \wedge \quad \mathrm{d}r_\nu^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \quad \wedge \quad \mathrm{d}(\mu\tau) = \mathrm{d}r \quad \Rightarrow \quad \mathrm{d}r'^2 = \mathrm{d}(\mu\tau)^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2 - \mathrm{d}x_3^2.$$

5.2. Newton's gravity force and the constant, G.

From equation 5.2:

$$(5.7) \quad \forall\, m_1, m_2, m, r \in \mathbb{R} :\ m_1 m_2 = m^2 \quad \wedge \quad r = (r_c/m_c)m \quad \Rightarrow \quad m_1 m_2 = ((m_c/r_c)r)^2 \quad \Rightarrow \quad (r_c/m_c)^2 m_1 m_2 / r^2 = 1.$$

$$(5.8) \qquad r = (r_c/t_c)t = ct \ \Rightarrow \ mr = (m_c/r_c)(ct)^2 \ \Rightarrow \ ((r_c/m_c)/c^2)mr/t^2 = 1,$$

$$(5.9)\quad ((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/m_c)^2 m_1 m_2/r^2 = 1$$
$$\Rightarrow \quad F := mr/t^2 = ((r_c/m_c)c^2)m_1 m_2/r^2 = Gm_1 m_2/r^2,$$

where Newton's constant, $G = (r_c/m_c)c^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$.

5.3. Gauss's and Poisson's gravity equations. From 5.7 and 5.9:

$$(5.10)\quad m_1 m_2 = m^2 \;\Rightarrow\; mr/t^2 = Gm_1 m_2/r^2 = Gm^2/r^2 \quad \Rightarrow \quad r/t^2 = Gm/r^2.$$

In vector form, if $\mathbf{R}$ is the radial vector at distance, r, then Gauss's gravitational field, $\mathbf{g}$, and Poisson's scalar potential per unit mass, Φ, are defined as:

$$(5.11)\quad r/t^2 = Gm/r^2 \quad \Rightarrow \quad \mathbf{g} := \mathbf{R}/t^2 = -Gm/|\mathbf{R}|^2 = -Gm\mathbf{R}/|\mathbf{R}|^3 := \nabla\Phi.$$

The magnitude of $\mathbf{R}$ is the surface element size, $\mathrm{d}S$, on a sphere having the area, $4\pi|\mathbf{R}|^3$. And where ρ is the spherical mass density, yields the differential forms of Gauss's and Poisson's gravity equations:

$$(5.12)\quad \mathbf{g} = \nabla\Phi = -Gm\mathbf{R}/|\mathbf{R}|^3 = (-Gm\mathbf{R}/|\mathbf{R}|^3)(4\pi/4\pi) \quad \wedge \quad \rho = m/4\pi|\mathbf{R}|^3$$
$$\Rightarrow \quad \mathbf{g} = \nabla\Phi = -4\pi G\rho\mathbf{R} \quad \Rightarrow \quad \nabla\cdot\mathbf{g} = \nabla^2\Phi = -4\pi G\rho.$$

5.4. Simplifying Einstein's general relativity (field equations). The strategy for simplifying Einsteins' field equations (EFEs) is "divide and conquer." The derivation of Gauss's and Poisson's gravity equations (5.12) were derived from the simpler non-vector form of Newton's gravity equation (5.9). That is, the first step is to use simpler non-vector equations, where solutions to the vector equations are constrained by solutions to the simpler non-vector equations.

A simplification to the EFEs, $\mathbf{G}_{\mu,\nu} - g_{\mu,\nu} = k\mathbf{T}_{\mu,\nu}$, [**Wey52**] follows from the space-like spacetime interval equation being derived from a non-vector, 2-dimensional equation (5.6), which is generalized to: $\mathrm{d}r'^2 = \alpha_0\mathrm{d}(\mu\tau)^2 - \mathrm{d}r_\nu^2$, where $\mathrm{d}r_\nu^2 = \alpha_1\mathrm{d}x_1^2 + \alpha_2\mathrm{d}x_2^2 + \alpha_3\mathrm{d}x_3^2$. Therefore, the 4×4 time-like metric tensor, $g_{\mu,\nu} = diag(-\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ can be simplified to a 2×2 matrix, $g_{\mu,\nu} = diag(-\alpha_0, 1)$. The 2×2 matrix allows using a 2-dimensional Gaussian curvature in the matrix, $\mathbf{G}_{\mu,\nu}$, which is much simpler to calculate than the 4-dimensional Ricci tensor, $\mathbf{R}_{\mu,\nu}$, and scalar, R, where $\mathbf{G}_{\mu,\nu} = \mathbf{R}_{\mu,\nu} - g_{\mu,\nu}R/2$. And the 2×2 matrices reduce the number of independent equations to solve.

5.5. Coulomb's charge force and constant, k_e. From equation 5.2:

$$(5.13)\quad \forall\, q_1, q_2, q, r \in \mathbb{R}:\; q_1 q_2 = q^2 \quad \wedge \quad r = (r_c/q_c)q$$
$$\Rightarrow \quad q_1 q_2 = ((q_c/r_c)r)^2 \quad \Rightarrow \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1.$$

$$(5.14)\quad r = (r_c/t_c)t = ct \;\Rightarrow\; mr = (m_c/r_c)(ct)^2 \;\Rightarrow\; ((r_c/m_c)/c^2)mr/t^2 = 1.$$

$$(5.15)\quad ((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1$$
$$\Rightarrow \quad F := mr/t^2 = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2.$$

$$(5.16)\quad r_c/t_c = c \quad \wedge \quad F = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2$$
$$\Rightarrow \quad F = (m_c(r_c/t_c^2))(r_c/q_c)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2,$$

where Coulomb's constant, $k_e = (m_c(r_c/t_c^2))(r_c/q_c)^2$, conforms to the SI units: $N\cdot m^2\cdot C^{-2}$.

5.6. Schwarzchild's gravitational time dilation. [Che10] From equations 5.5 and 5.2:

$$(5.17)\quad t' = t\sqrt{1-(v^2/c^2)(r/r)} \quad\wedge\quad r = (r_c/m_c)m \quad\Rightarrow\quad t' = t\sqrt{1-((r_c/m_c)m)v^2/rc^2}.$$

$$(5.18)\quad t' = t\sqrt{1-(r_c/m_c)mv^2/rc^2} \quad\wedge\quad KE = mv^2/2 = mv_{escape}^2 \quad\Rightarrow\quad t' = t\sqrt{1-2(r_c/m_c)mv_{escape}^2/rc^2}.$$

Combine equation 5.18 with the derivation of G (5.9):

$$(5.19)\quad (r_c/m_c)c^2 = G \quad\wedge\quad t' = \lim_{v_{escape}\to c} t\sqrt{1-2(r_c/m_c)mv_{escape}^2/rc^2} = t\sqrt{1-2(r_c/m_c)mc^2/rc^2} \quad\Rightarrow\quad t' = t\sqrt{1-2Gm/rc^2}.$$

5.7. Einstein's gravitational lens. [Che10] Using the same steps top to derive Schwarzchild's gravitational time dilation equation (5.19), the gravitational distance dilation is:

$$(5.20)\qquad r' = r\sqrt{1-(v^2/c^2)(r/r)} \quad\Rightarrow\quad r' = r\sqrt{1-2Gm/rc^2}.$$

The incremental deflection of light (work), $\mathrm{d}s$: $\mathrm{d}s = 2Gm/rc^2$. But work is caused by a change in energy. Therefore, an increment of deflection, $\mathrm{d}s$, must correspond to an incremental distance, $\mathrm{d}r$: $r = \mathrm{d}r$.

$$(5.21)\qquad \mathrm{d}s = 2Gm/rc^2 \quad\wedge\quad \mathrm{d}r = r \quad\Rightarrow\quad \mathrm{d}s/\mathrm{d}r = 2Gm/r^2c^2.$$

There are two deflections: 1) deflection as the light approaches a mass, and 2) deflection as light passes the same mass. Therefore, the total deflection is doubled:

$$(5.22)\qquad \mathrm{d}s/\mathrm{d}r = 2(2Gm/r^2c^2) \quad\Rightarrow\quad s = \int \mathrm{d}s = \int 4Gm/r^2c^2\mathrm{d}r = 4Gm/rc^2.$$

5.8. 3 fundamental direct proportion ratios. c_t, c_m, and c_q:

$$(5.23)\qquad c_t = r_c/t_c \approx 2.99792458\cdot 10^8 m\ s^{-1}.$$

$$(5.24)\quad G = (r_c/m_c)c_t^2 = c_m c_t^2 \quad\Rightarrow\quad c_m = r_c/m_c \approx 7.4261602691\cdot 10^{-28} m\ kg^{-1}.$$

$$(5.25)\quad k_e = (c_t^2/c_m)(r_c/q_c)^2 \quad\Rightarrow\quad c_q = r_c/q_c \approx 8.6175172023\cdot 10^{-18} m\ C^{-1}.$$

5.9. 3 fundamental inverse proportion ratios. c_t, c_m, and c_q (5.8) $\Leftrightarrow$ k_t, k_m, and k_q:

$$(5.26)\quad r/t = r_c/t_c,\quad r/m = r_c/m_c \quad\Leftrightarrow\quad (r/t)/(r/m) = (r_c/t_c)/(r_c/m_c) \quad\Leftrightarrow\quad (mr)/(tr) = (m_cr_c)/(t_cr_c) \quad\Leftrightarrow\quad mr = m_cr_c = k_m,\quad tr = t_cr_c = k_t.$$

$$(5.27)\quad r/t = r_c/t_c,\quad r/q = r_c/q_c \quad\Leftrightarrow\quad (r/t)/(r/q) = (r_c/t_c)/(r_c/q_c) \quad\Leftrightarrow\quad (qr)/(tr) = (q_cr_c)/(t_cr_c) \quad\Leftrightarrow\quad qr = q_cr_c = k_q,\quad tr = t_cr_c = k_t.$$

$$(5.28)\qquad k_m = m_cr_c = h/c \approx 2.21022\cdot 10^{-42}\ kg\ m.$$

$$(5.29)\qquad k_t = t_cr_c = k_m/(c_t/c_m) \approx 5.47494\cdot 10^{-78}\ s\ m.$$

$$(5.30)\qquad k_q = q_cr_c = (c_t/c_q)k_t \approx 1.90466\cdot 10^{-52}\ C\ m.$$

5.10. Planck relation and constant, h. [Jai11] Applying both the direct proportion (5.23), $r/t = r_c/t_c = c$, and inverse proportion (5.26), $mr = m_c r_c = k_m$, ratios:

$$m(ct)^2 = mr^2 \ \wedge \ m = m_c r_c/r = k_m/r \quad \Rightarrow \quad m(ct)^2 = (k_m/r)r^2 = k_m r. \tag{5.31}$$

$$m(ct)^2 = k_m r \quad \wedge \quad r/t = r_c/t_c = c$$
$$\Rightarrow \quad E := mc^2 = k_m r/t^2 = (k_m(r/t))\,(1/t) = (k_m c)(1/t) = hf, \tag{5.32}$$

where the Planck constant, $h = k_m c$, and the frequency, $f = 1/t$.

5.11. Compton wavelength. [Jai11] From equations 5.26 and 5.32:

$$mr = k_m \quad \Rightarrow \quad r = k_m/m = (k_m/m)(c/c) = h/mc. \tag{5.33}$$

5.12. de Broglie wavelength. [Jai11] From equations 5.3 and 5.33:

$$v = r'/t = c\sqrt{1-(v'/c)^2} \quad \wedge \quad r = h/mc \quad \Rightarrow \quad r = (h/mv)\sqrt{1-(v'/c)^2}, \tag{5.34}$$

where $r_\nu/t = v'$ is the rest frame of reference velocity and $r'/t = v$ is the velocity observed from the stationary frame of reference.

5.13. Total mass. The total mass of a particle is $m = \sqrt{m_0^2 + m_{ke}^2}$, where m_0 is the rest mass and m_{ke} is the kinetic energy-equivalent mass. Applying both the direct (5.23) and inverse proportion ratios (5.26):

$$m_0 = (m_c/r_c)r \quad \wedge \quad m_{ke} = m_c r_c/r \quad \wedge \quad m = \sqrt{m_0^2 + m_{ke}^2}$$
$$\Rightarrow \quad m = \sqrt{((m_c/r_c)r)^2 + ((m_c r_c)/r)^2}. \tag{5.35}$$

The quantum effect, $((m_c r_c)/r)^2$, is easier to express and understand by extending Newtonian gravity than by extending general relativity.

5.14. Quantum-special relativity extensions to Newton's gravity force.

$$\exists\, m : \ m_1 m_2 = m^2 = ((m_c/r_c)r)^2 + ((m_c r_c)/r)^2$$
$$\Rightarrow \quad m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2) = 1. \tag{5.36}$$

Applying the spacetime equation 5.4 to equation 5.8:

$$r' = r\sqrt{1-(v/c)^2} \quad \wedge \quad ((r_c/m_c)/c^2)mr/t^2 = 1$$
$$\Rightarrow \quad ((r_c/m_c)/c^2\sqrt{1-(v/c)^2})mr'/t^2 = 1, \tag{5.37}$$

where r'/t^2 is the acceleration observed from a stationary frame of reference. Combining equations 5.36 and 5.37:

$$F = (c^2\sqrt{1-(v/c)^2}/(r_c/m_c))m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2). \tag{5.38}$$

5.15. Quantum-special relativity extensions to Coulomb's charge force.

$$F = (c^2\sqrt{1-(v/c)^2}/(r_c/m_c))(r_c/q_c)^2 q_1 q_2/(((q_c/r_c)r)^2 + ((q_c r_c)/r)^2). \tag{5.39}$$

6. Insights and implications

(1) Combinatorics, the ordered combinations of countable, disjoint sets (n-tuples), generates both Euclidean volume (3.2) and the Minkowski distances (4.2).
(2) Combinatorics, all n-at-time permutations of an ordered and symmetric set of distance dimensions, limits the set to 3 dimensions (4.12).
(3) Deriving Euclidean volume (3.2) and the Minkowski distances (4.2) from the same abstract, countable set of n-tuples (3.1) provides a single, unifying set and limit-based foundation under Euclidean geometry without relying on the geometric primitives and relations in Euclidean geometry [**Joy98**], axiomatic geometry [**Lee10**], and vector analysis [**Wey52**].
(4) The definition of a metric space [**Rud76**] ignores the intimate relation between distance and volume. A simpler and more sufficient definition that has the metric space properties (4.4) is: a distance measure is an inverse (bijective), isomorphic function of volume (4.1).
(5) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [**Joy98**]. And analytic proofs that the straight line length is the smallest distance equate the straight line length to Euclidean distance.

    Without using the notion of a straight line: Euclidean volume was derived from a set of n-tuples (3.2). And all distance measures (bijective, isomorphic functions of n-volumes) derived from Euclidean 2-volumes (areas) are Minkowski distances (4.2), where $n \in \{1, 2\}$: $n = 1$ is the Manhattan (largest) distance case, $d = \sum_{i=1}^{m} s_i$. $n = 2$ is the Euclidean (smallest) distance case, $d = (\sum_{i=1}^{m} s_i^2)^{1/2}$. For the case, $n \in \mathbb{R}$, $1 \leq n \leq 2$: d decreases monotonically as n goes from 1 to 2.
(6) The left side of the distance sum inequality (4.4),

$$(\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}, \tag{6.1}$$

    differs from the left side of Minkowski's sum inequality [**Min53**]:

$$(\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \tag{6.2}$$

    The two inequalities are only the same where $n = 1$. The distance sum inequality is a more fundamental inequality because the proof does not require the convexity and Hölder's inequality assumptions required to prove the Minkowski sum inequality. And the distance sum inequality term, $(a_i^n + b_i^n)^{1/n}$, is the Minkowski distance spanning the n-volume, $a_i^n + b_i^n$, which makes it directly related to geometry (for example, the metric space triangle inequality was derived from the $m = 1$ case (4.5)). But the Minkowski sum inequality term, $\forall\, n : (n \neq 1)$. $((a_i^n + b_i^n)^{\mathbf{n}})^{1/n}$, is **not** a distance spanning the same n-volume, $a_i^n + b_i^n$.
(7) The derivations of the spacetime equations, in this article (5.1), differ from other derivations:
    (a) The derivations, here, do not rely on the Lorentz transformations or Einsteins' postulates [**Ein15**]. The derivations do not even require the notion of light.

(b) The derivations, here, rely only on the Euclidean volume proof (3.2), the Minkowski distances proof (4.1), and the 3D proof (4.12), which provides the insight that the properties of physical space creates a maximum speed, the spacetime equations, and 3 dimensions of distance. For example, from the direct proportion equations 5.1 and 5.2, $\mu = r_c/t_c$ is always the maximum ratio (the speed of light).
(c) The same derivations are also valid for spacemass and spacecharge.
(d) **Both** special and general relativity result from equation 5.1, where the ratio of any 3-dimensional distance, r, to other types is always a constant in all inertial (special relativity) frames and all accelerating (general relativity) frames. For example, the special relativity time dilation equation 5.5 was derived from constant distance-to-time ratios and combined with a constant distance-to-mass ratio (5.8) yields Schwarzchild's gravitational time dilation equation (5.19) [**Che10**].

(8) A simplification to Einstein's field equations, $\mathbf{G}_{\mu,\nu} - g_{\mu,\nu} = k\mathbf{T}_{\mu,\nu}$, [**Wey52**] follows from the space-like spacetime interval equation being derived from a 2-dimensional equation (5.6), which is generalized to: $\mathrm{d}r'^2 = \alpha_0 \mathrm{d}(\mu\tau)^2 - \mathrm{d}r_\nu^2$, where $\mathrm{d}r_\nu^2 = \alpha_1 \mathrm{d}x_1^2 + \alpha_2 \mathrm{d}x_2^2 + \alpha_3 \mathrm{d}x_3^2$. Therefore, the 4×4 time-like metric tensor, $g_{\mu,\nu} = diag(-\alpha_0,\ \alpha_1, \alpha_2, \alpha_3)$ can be simplified to a 2×2 metric tensor, $g_{\mu,\nu} = diag(-\alpha_0, 1)$. The 2×2 metric tensor allows using a 2-dimensional Gaussian curvature in the Einstein tensor, $\mathbf{G}_{\mu,\nu}$, which is much simpler to calculate than the 4-dimensional Ricci tensor, $\mathbf{R}_{\mu,\nu}$, and scalar, R, where $\mathbf{G}_{\mu,\nu} = \mathbf{R}_{\mu,\nu} - g_{\mu,\nu}R/2$. And the 2×2 tensors reduce the number of independent equations to solve.

(9) The direct proportion ratios, $r_c/t_c = c_t$, $r_c/m_c = c_m$, $(r_c/q_c) = c_q \Leftrightarrow$ the inverse proportion ratios, $t_c r_c = k_t$, $m_c r_c = k_m$, and $q_c r_c = k_q$ (5.9) are the properties of a symmetry group.
(a) The inverse square law for gravity (5.7) and charge (5.13) is a result of the direct proportion ratios.
(b) The combination of direct and inverse proportion ratios create the particle-wave equations: Planck relation (5.10), Compton wavelength (5.33), and de Broglie wavelength (5.34).
(c) The gravity, G (5.9), charge k_e (5.16), and Planck h (5.32) constants were all derived from the constant proportion ratios. Therefore, G, k_e, and h are **not** "fundamental" constants.
(d) G, k_e, and h all depend on the speed of light ratio, c_t: $G = c_m c_t^2$, $k_e = (c_q^2/c_m)c_t^2$, and $h = k_m c_t$.
(e) $k_e = (c_q^2/c_m)c_t^2 = ((m_c/r_c)(r_c/t_c)^2)c_q^2 = (m_c(r_c/t_c^2))c_q^2$, where the term, r_c/t_c^2, suggests a maximum acceleration constant.
(f) The ratios used to derive k_e (5.16) do not contain the value, 4π, which indicates the current standard definitions of k_e in terms of vacuum permitivity, ε_0, and permeability, μ_0, where $k_e = 1/4\pi\varepsilon_0$ and $k_e = \mu_0 c^2/4\pi$, are **not** logically derived in orthogonal Cartesian coordinates. Likewise, the logic of the reduced Planck constant, $\hbar = h/2\pi$, in orthogonal coordinates, might need to be reconsidered.

(10) The quantum-special relativity extensions to Newton's gravity force (5.38) and Coulomb's charge force (5.39) make quantifiable predictions.

(a) $\lim_{r \to 0} F = 0$. The distance, r, where **both** the gravity and charge forces peak is: $r = \sqrt{c_m k_m} = \sqrt{c_q k_q} = \sqrt{r_c^2} \approx 4.05135 \cdot 10^{-35}\ m$.
  (i) Gravitational time dilation peaks at $r \approx 4.05135 \cdot 10^{-35}\ m$.
  (ii) Black holes might have measurable sizes > 0.
  (iii) The finite gravity-charge well allows radioactivity and quantum tunneling.
  (iv) As the kinetic energy (temperature) decreases, more particles will stay within their gravity-charge well distance, $r << 4.05135 \cdot 10^{-35}\ m$, allowing superconductivity and Bose-Einstein condensates.
  (v) $r_c \approx 4.05135 \cdot 10^{-35}\ m$ suggests a quantum distance, time, mass, and charge.

(b) The term, r'/t^2, in equation 5.38 is the acceleration observed from the stationary frame. From earth, as a stationary frame, equation 5.38 predicts the planet Mercury's orbiting velocity (angular acceleration) around the sun would appear to be slightly slower than predicted by Newton's gravity equation 5.9.

(11) A constant ratio cannot map a constant value to continuously varying values. Therefore, the discrete spin states of two quantum entangled particles and the polarization states of two quantum entangled photons are independent of continuously varying distance and time interval lengths.

(12) The set-based, first-order logic proof that a strict linearly ordered and symmetric set is a cyclic set of at most 3 members (4.12) is a simpler and more logically rigorous hypothesis for observing only 3 dimensions of physical space than: parallel dimensions that cannot be detected or extra dimensions rolled up into infinitesimal balls that are too small to detect.
  (a) Higher order dimensions must have different types (members of different sets), for example, dimensions of time, mass, and charge.
  (b) Each of 3 ordered and symmetric dimensions of space can have at most 3 sequentially ordered and symmetric state values, for example, an ordered and symmetric set of 3 vector orientations, $\{-1, 0, 1\}$, per dimension of space and at most 3 spin states per plane, etc.
  If the states are not sequentially ordered (a bag of states), then a state value is undetermined until observed (like Schrodinger's cat being both alive and dead until the box is opened). That is, there would be no "axiom of choice" that allows selecting a particular state.

References

[CG15] W. Conradie and V. Goranko, *Logic and discrete mathematics*, Wiley, 2015. ↑2

[Che10] T.P. Cheng, *Relativity, gravitation and cosmology: A basic introduction*, Oxford Master Series in Physics, OUP Oxford, 2010. ↑9, 12

[Coq23] Coq, *Coq proof assistant*, 2023. https://coq.inria.fr/documentation. ↑2

[Ein15] A. Einstein, *Relativity, the special and general theory*, Princeton University Press, 2015. ↑11

[Gol76] R. R. Goldberg, *Methods of real analysis*, John Wiley and Sons, 1976. ↑1, 2

[Jai11] M.C. Jain, *Quantum mechanics: A textbook for undergraduates*, PHI Learning Private Limited, New Delhi, India, 2011. ↑10

[Joy98] D. E. Joyce, *Euclid's elements*, 1998. http://aleph0.clarku.edu/~djoyce/java/elements/elements.html. ↑11

[Lee10] J. M. Lee, *Axiomatic geometry*, American Mathematical Society, 2010. ↑11

[Min53] H. Minkowski, *Geometrie der zahlen*, Chelsea, 1953. reprint. ↑2, 11
[Rud76] W. Rudin, *Principles of mathematical analysis*, McGraw Hill Education, 1976. ↑1, 2, 11
[Wey52] H. Weyl, *Space-time-matter*, Dover Publications Inc, 1952. ↑8, 11, 12

George Van Treeck, 668 Westline Dr., Alameda, CA 94501