The Set Properties Generating Geometry and Physics

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Abstract. Volume and the Minkowski distances/Lp norms (e.g., Manhattan and Euclidean distance) are derived from a set and limit-based foundation without referencing the primitives and relations of geometry. Sequencing a strict linearly ordered set in all n-at-a-time permutations via successor/predecessor relations is a cyclic set of at most 3 members. Therefore, all other interval lengths have different types from a cyclic set of 3 distance interval lengths. Unit-factoring ratios between different types of interval lengths and the set proofs provide simpler derivations of the spacetime, Newton's gravity, Coulomb's charge force, Planck-Einstein, quantum-relativity gravity equations and corresponding constants. All proofs are verified in Coq.

Contents

1. Introduction

Mathematical (real) analysis can construct differential calculus from a set and limit-based foundation without the need to reference the primitives and relations of Euclidean geometry, like straight line, angle, shape, etc., providing a more rigorous foundation to calculus. But volume in the Riemann integral, Lebesgue integral, measure theory and distance in the vector norm and metric space criteria are all definitions motivated by Euclidean geometry. [**Gol76**] [**Rud76**] Here, volume and distance are motivated and derived from a set and limit-based foundation.

A well-known set-based motivation of Euclidean volume is the number of members (cardinal), v_c, of an abstract, countable set of Cartesian product n-tuples:

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$v_c = \prod_{i=1}^{n} |x_i|$, where $|x_i|$ is the cardinal of the countable, disjoint set, x_i. Where each x_i is a set of size κ subintervals of $[a_i, b_i] \subset \mathbb{R}$ and $s_i = b_i - a_i$, It will be proved that $v_c = \prod_{i=1}^{n} |x_i| \Rightarrow v = \prod_{i=1}^{n} s_i$. Non-Euclidean volumes are $v_c = \prod_{i=1}^{n} \alpha_i |x_i|$ and $v = \prod_{i=1}^{n} \alpha_i s_i$, where α_i is a function returning a scalar value $\neq 1$.

Comparing n-volumes of the same class, n, is a function, $d_c = f(v_c, n)$, where f is a bijective ($v_c = f^{-1}(d_c, n)$), monotonic increasing function of d_c. d_c is, herein, defined as the distance measure of an n-volume. Comparing n-volumes that are the sum of n-volumes, $v_c = \sum_{j=1}^{m} v_{c_i}$, yields a distance measure of the form: $d_c = f(v_c, n) = (\sum_{j=1}^{m} f_i(v_{c_i}, n))$. It will be proved that $d_c^n = (\sum_{j=1}^{m} v_{c_i}^n)$ $\Rightarrow d^n = \sum_{i=1}^{m} d_i^n$. d is the L_p norm (Minkowski distance), which will be proved to imply the metric space criteria.

In the prior equations, sequencing a set, $\{x_1 \cdots x_n\}$, from $i = 1$ to n, is a strict linear (total) order that set theory defines in terms of successor and predecessor functions. But, if the sets can be sequenced in all n-at-a-time orders, then an additional "symmetry" constraint is required, where every set member is either a successor or predecessor to every other set member. A strict linearly ordered and symmetric set is proved to be a cyclic set, where $n \leq 3$.

Therefore, if $\{x, y, z\}$ is a strict linearly ordered and symmetric set of 3 "distance" dimensions, then a fourth dimension, t, must have a different type (is a member of different set). Where derivatives and integrals divide the domain intervals into the same number of subintervals, each subinterval of a distance domain interval maps to a proportionate-sized subinterval of some other type of domain interval, which is expressed by a unit-factoring ratio, for example, *meters/second.*

Simpler and shorter derivations of the: spacetime, Newton's gravity, Coulomb's charge force, Planck-Einstein, quantum-relativity gravity equations, and their corresponding constants are provided using the ratios, $r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q$, combined with the results of the volume, distance, and 3D proofs. Impacts on Einstein's field equations are also discussed.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq23**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

Derivatives and integrals divide each interval into the same number of subintervals. Therefore, the size of the subintervals in each domain interval are proportionate to the size of the containing interval, which precludes using derivatives and integrals to directly express many-to-many (Cartesian product) mappings between same-sized, size κ, subintervals in different-sized domain intervals. Further, tools that define Euclidean volume and distance cannot be used to to derive Euclidean volume and distance.

Therefore, a different tool is used here. A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of size κ subintervals. The ruler is both an inner and outer measure of an interval.

Definition 2.1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\, [a, b] \subset \mathbb{R}$, $s = b - a \;\wedge\; 0 < \kappa \leq 1 \;\wedge\; (p = floor(s/\kappa) \;\vee\; p = ceiling(s/\kappa))$.

Theorem 2.2. *Ruler convergence:* $M = \lim_{\kappa \to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : y \leq x, y \in \mathbb{Z}, x \in \mathbb{R}\})$:

$$\forall\, \kappa > 0,\ p = floor(s/\kappa)\ \wedge\ 0 \leq |floor(s/\kappa) - s/\kappa| < 1\ \Rightarrow\ |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by κ:

$$\forall\, \kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \epsilon \quad := \quad M = \lim_{\kappa \to 0} p\kappa = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, \ldots, \pi_{\lim_{\kappa\to 0}}$.

LEMMA 2.3. $\forall\, n \geq 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa$.

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa \to 0} \kappa^n = 0. \end{aligned} \tag{2.5}$$

$$\lim_{\kappa \to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa \to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa. \qquad \square \tag{2.6}$$

3. Volume

DEFINITION 3.1. An n-volume is the number of ordered combinations (n-tuples), v_c, of the members of n number of disjoint, countable domain sets, x_i:

$$\exists\, n \in \mathbb{N},\ v_c, \in \{0,\ \mathbb{N}\},\ \ x_i \in \{x_1, \cdots, x_n\} :\ \bigcap_{i=1}^{n} x_i = \emptyset \ \wedge\ v_c = \prod_{i=1}^{n} |x_i|. \tag{3.1}$$

THEOREM 3.2. *Euclidean volume,*

$$\begin{aligned} &\forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\ \ [v_a, v_b] \subset \mathbb{R},\ \ s_i = b_i - a_i,\ \ v = v_b - v_a : \\ &\qquad v_c = \prod_{i=1}^{n} |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^{n} s_i. \end{aligned} \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.

$$v_c = \prod_{i=1}^{n} |x_i| \ \Leftrightarrow\ v_c\kappa = (\prod_{i=1}^{n} |x_i|)\kappa \ \Leftrightarrow\ \lim_{\kappa \to 0} v_c\kappa = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|)\kappa. \tag{3.3}$$

Apply the ruler (2.1) and ruler convergence (2.2) to equation 3.3:

$$\exists\, v, \kappa \in \mathbb{R} :\ v_c = floor(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa \to 0} v_c\kappa = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|)\kappa. \tag{3.4}$$

Apply lemma 2.3 to equation 3.4:

$$v = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|)\kappa = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|)\kappa^n = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|\kappa). \tag{3.5}$$

Apply the ruler (2.1) and ruler convergence (2.2) to s_i:

$$\exists\, s_i, \kappa \in \mathbb{R} :\ floor(s_i/\kappa) = |x_i| \quad \Rightarrow \quad \lim_{\kappa \to 0}(|x_i|\kappa) = s_i. \tag{3.6}$$

$$v = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|\kappa) \ \wedge\ \lim_{\kappa \to 0}(|x_i|\kappa) = s_i \ \Rightarrow\ v = \prod_{i=1}^{n} s_i \qquad \square \tag{3.7}$$

THEOREM 3.3. *Sum of volumes:*

$$\forall\, x_{i,j} \in \{x_{i_1}, \cdots, x_{i_m}\} = x_i :\ v_c = \textstyle\prod_{i=1}^{n} |x_i| \quad \wedge \quad v_{c_j} = \prod_{i=1}^{n} |x_{i,j}| \quad \wedge \quad v_c = \sum_{j=1}^{m} v_{c_j} \quad \Rightarrow \quad \exists\, s_i, s_{i,j} \in \mathbb{R} :\ \prod_{i=1}^{n} s_i = \sum_{j=1}^{m} (\prod_{i=1}^{n} s_{i,j}). \tag{3.8}$$

The formal proof, "sum_of_volumes," is in the Coq file, euclidrelations.v.

PROOF. From the Euclidean volume theorem (3.2):

$$v_c = \textstyle\prod_{i=1}^{n} |x_i| \ \Rightarrow\ v = \prod_{i=1}^{n} s_i \ \wedge\ v_{c_j} = \prod_{i=1}^{n} |x_{i,j}| \ \Rightarrow\ v_j = \prod_{i=1}^{n} s_{i,j}. \tag{3.9}$$

Apply the ruler (2.1) and ruler convergence (2.2):

$$\exists\, v, v_j, \kappa \in R :\quad v_c = floor(v/\kappa) \quad \wedge \quad v_{c_j} = floor(v_i/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa \to 0} v_c \kappa \quad \wedge \quad v_i = \lim_{\kappa \to 0} v_{c_j} \kappa. \tag{3.10}$$

$$v_c = \textstyle\sum_{j=1}^{m} v_{c_j} \quad \Leftrightarrow \quad v = \lim_{\kappa \to 0} v_c \kappa = \lim_{\kappa \to 0} (\sum_{j=1}^{m} v_{c_j}) \kappa. \tag{3.11}$$

Apply lemma 2.3 to equation 3.11:

$$\lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa \ \wedge\ v = \lim_{\kappa \to 0} (\textstyle\sum_{j=1}^{m} v_{c_j}) \kappa \ \wedge\ v_i = \lim_{\kappa \to 0} v_{c_j} \kappa \quad \Rightarrow \quad v = \lim_{\kappa \to 0} (\sum_{j=1}^{m} v_{c_j}) \kappa^n = \lim_{\kappa \to 0} \sum_{j=1}^{m} (v_{c_j} \kappa) = \sum_{j=1}^{m} v_j. \tag{3.12}$$

$$v = \textstyle\prod_{i=1}^{n} s_i \quad \wedge \quad v_j = \prod_{i=1}^{n} s_{i,j} \quad \wedge \quad v = \sum_{j=1}^{m} v_j \quad \Rightarrow \quad \prod_{i=1}^{n} s_i = \sum_{j=1}^{m} \prod_{i=1}^{n} s_{i,j}. \tag{3.13}$$

□

4. Distance

DEFINITION 4.1. The distance measure (metric), d_c of a countable n-volume, $v_c = \prod_{i=1}^{n} \alpha_i |x_i| = \sum_{j=1}^{m} (\prod_{i=1}^{n} \alpha_{i,j} |x_{i,j}|) = \sum_{j=1}^{m} v_{c_i}$, is defined as the cardinal of the set $x :\ d_c = |x| = f(v_c, n) = f(\sum_{j=1}^{m} f_i(v_{c_i}, n))$, where f and f_i are bijective, monotonic increasing functions of v_c and v_{c_i}.

4.1. Minkowski distance (L_p norm).

THEOREM 4.2. *Minkowski distance (L_p norm):*

$$v_c = \textstyle\prod_{i=1}^{n} |x_i| = \sum_{j=1}^{m} (\prod_{i=1}^{n} |x_{i,j}|) = \sum_{j=1}^{m} v_{c_i} \quad \Rightarrow \quad d^n = \sum_{i=1}^{m} d_i^n.$$

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

PROOF. From the sum of volumes proof (3.3), where all subintervals of all intervals are the same size, κ:

$$\textstyle\prod_{i=1}^{n} |x_i| = \sum_{j=1}^{m} (\prod_{i=1}^{n} |x_{i,j}|) \quad \Rightarrow \quad \prod_{i=1}^{n} s_i = \sum_{j=1}^{m} (\prod_{i=1}^{n} s_{i,j}) \tag{4.1}$$

$$\forall\, s_i, s_{i,j} \in \mathbb{R}\ \exists\, d, d_i \in \mathbb{R} :\ d = (\textstyle\sum_{i=1}^{n} s_i)/n \quad \wedge \quad d_i = (\sum_{j=1}^{m} s_{i,j})/n \quad \Rightarrow \quad d^n = \prod_{i=1}^{n} s_i = \sum_{j=1}^{m} (\prod_{i=1}^{n} s_{i,j}) = \sum_{i=1}^{m} d_i^n. \tag{4.2}$$

□

4.2. Distance inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N},\;\; v_a, v_b \geq 0 :\;\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$\forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq v_a + v_b + \sum_{i=1}^{n}\binom{n}{k}(v_a^{1/n})^{n-k}(v_b^{1/n})^k + \sum_{i=1}^{n}\binom{n}{k}(v_a^{1/n})^k(v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n. \tag{4.3}$$

Take the n^{th} of both sides of the inequality 4.3:

$$\forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad \square \tag{4.4}$$

4.3. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N},\;\; a_i, b_i \geq 0 :\;\; (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$\begin{aligned}&\forall\, m, n \in \mathbb{N},\;\; v_a, v_b \geq 0 : \quad v_a = \textstyle\sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge \\ &(v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad ((\textstyle\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} = \\ &(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \quad \square\end{aligned} \tag{4.5}$$

4.4. Metric Space. All Minkowski distances (L_p norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.6}$$

Apply the distance inequality (4.3) to the inequality 4.6:

$$\begin{aligned}&(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \;\wedge\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ &\qquad \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ &\Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ &\qquad \Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq \\ &\qquad (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \quad \square\end{aligned} \tag{4.7}$$

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$\begin{aligned}&\forall\, p : p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ &\qquad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \quad \square\end{aligned} \tag{4.8}$$

Theorem 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

Proof. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0. \tag{4.9}$$

$$p \geq 1,\ u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \tag{4.10}$$ □

Theorem 4.8. *Identity of Indiscernibles:* $d(u, u) = 0$.

Proof. From the non-negativity property (4.7):

$$\begin{aligned} d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0 \\ \Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0. \end{aligned} \tag{4.11}$$

$$d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v. \tag{4.12}$$

$$d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \tag{4.13}$$ □

4.5. The properties limiting a set to at most 3 members.

Definition 4.9. Totally ordered set:

$$\begin{aligned} &\forall\, i\ n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i\ \in \{x_1, \ldots, x_n\}, \\ &\qquad\qquad \textit{successor}\ x_i = x_{i+1}\ \wedge\ \textit{predecessor}\ x_{i+1} = x_i. \end{aligned}$$

Definition 4.10. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i,\ j,\ n \in \mathbb{N},\ \forall\, x_i,\ x_j \in \{x_1, \ldots, x_n\},\ \textit{successor}\ x_i = x_j\ \Leftrightarrow\ \textit{predecessor}\ x_j = x_i.$$

Theorem 4.11. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n\ \wedge\ j = 1\ \Rightarrow\ \textit{successor}\ x_n = x_1\ \wedge\ \textit{predecessor}\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

Proof. A total order (4.9) assigns a unique label to each set member and assigns unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (4.10):

$$i = n\ \wedge\ j = 1\ \wedge\ \textit{successor}\ x_i = x_j\ \Rightarrow\ \textit{successor}\ x_n = x_1. \tag{4.14}$$

Applying the definition of the symmetry property (4.10) to conclusion 4.14:

$$\textit{successor}\ x_i = x_j\ \Rightarrow\ \textit{predecessor}\ x_j = x_i\ \Rightarrow\ \textit{predecessor}\ x_1 = x_n. \tag{4.15}$$ □

Theorem 4.12. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (4.11).

DEFINITION 4.13. (Cyclic) Successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize). \tag{4.16}$$

DEFINITION 4.14. (Cyclic) Predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1). \tag{4.17}$$

DEFINITION 4.15. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{4.18}$$

Prove that every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \tag{4.19}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \tag{4.20}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1). \tag{4.21}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize). \tag{4.22}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \tag{4.23}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1). \tag{4.24}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize). \tag{4.25}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize). \tag{4.26}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \tag{4.27}$$

Member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not ($\neg$) a successor of member 1 for all $setsize > 3$:

$$\begin{aligned}&\neg Successor(1, 3, setsize > 3)\\ &\qquad\leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize).\end{aligned} \tag{4.28}$$

Member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not ($\neg$) a predecessor of member 1 for all $setsize > 3$:

$$\begin{aligned}&\neg Predecessor(1, 3, setsize > 3)\\ &\qquad\leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).\end{aligned} \tag{4.29}$$

For all $setsize > 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not symmetric):

$$\begin{aligned}&\neg Adjacent(1, 3, setsize > 3)\\ &\qquad\leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3).\end{aligned} \tag{4.30}$$

□

5. Applications to physics

From the 3D proof (4.12), dividing a set of domain intervals into the same number of subintervals, a 3-dimensional distance subinterval length, r, maps to proportionately sized subinterval lengths of other types, t, m, and q, where:

$$r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q. \tag{5.1}$$

5.1. Spacetime equations. From the volume proof (3.2), in a local Euclidean space, two disjoint distance intervals, $[0, r]$ and $[0, r']$, define a 2-volume (area). From the Minkowski distance proof (4.2), the distance interval lengths, r and r', are functions of two areas, v and v' having the sizes, $v = r^2$ and $v' = {r'}^2$. And $\forall\, r \geq r' \,\exists\, r_\nu \in \mathbb{R} : r^2 = {r'}^2 + r_\nu^2$. Combined with the 3D proof (4.12):

$$\begin{aligned} &\exists\, \mu,\ \nu \in \mathbb{R} :\ r = \mu t \quad \wedge \quad r_\nu = \nu t \quad \wedge \quad \forall\, r \geq r' \,\exists\, r_\nu \in \mathbb{R} :\ r^2 = r'^2 + r_\nu^2 \\ &\qquad \Rightarrow \quad (\mu t)^2 = {r'}^2 + (\nu t)^2 \quad \Rightarrow \quad r' = \sqrt{(\mu t)^2 - (\nu t)^2} \ = \ \mu t\sqrt{1 - (\nu/\mu)^2}. \end{aligned} \tag{5.2}$$

Local (proper) distance, r', contracts relative to coordinate distance, r, as $\nu \to \mu$:

$$r' = \mu t\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad \mu t = r \quad \Rightarrow \quad r' = r\sqrt{1 - (\nu/\mu)^2}. \tag{5.3}$$

From equation 5.2, coordinate length, t, dilates relative to local length, t', as $\nu \to \mu$:

$$\mu t = r'/\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad r' = \mu t' \quad \Rightarrow \quad t = t'/\sqrt{1 - (\nu/\mu)^2}. \tag{5.4}$$

Using $r^2 = r'^2 + r_\nu^2$ from equation 5.2, where r_ν is a 3-dimensional distance, one form of the flat Minkowski's spacetime event interval is:

$$\begin{aligned} &\mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_\nu^2 \quad \wedge \quad \mathrm{d}r_\nu^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \quad \wedge \quad \mathrm{d}(\mu t) = \mathrm{d}r \\ &\qquad \Rightarrow \quad \mathrm{d}r'^2 = \mathrm{d}(\mu t)^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2 - \mathrm{d}x_3^2. \end{aligned} \tag{5.5}$$

5.2. Newton's gravity force and the constant, G. From equation 5.1:

$$\begin{aligned} &\forall\, m_1, m_2, m, r \in \mathbb{R} :\ m_1 m_2 = m^2 \quad \wedge \quad m = (m_c/r_c)r \\ &\qquad \Rightarrow \quad m_1 m_2 = ((m_c/r_c)r)^2 \quad \Rightarrow \quad (r_c/m_c)^2 m_1 m_2/r^2 = 1. \end{aligned} \tag{5.6}$$

$$r = (r_c/t_c)t = ct \ \Rightarrow \ mr = (m_c/r_c)(ct)^2 \ \Rightarrow \ ((r_c/m_c)/c^2)mr/t^2 = 1. \tag{5.7}$$

$$\begin{aligned} &((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/m_c)^2 m_1 m_2/r^2 = 1 \\ &\qquad \Rightarrow \quad F := mr/t^2 = ((r_c/m_c)c^2)m_1 m_2/r^2 = G m_1 m_2/r^2, \end{aligned} \tag{5.8}$$

where Newton's constant, $G = (r_c/m_c)c^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$.

5.3. Coulomb's charge force and constant. From equation 5.1:

$$\begin{aligned} &\forall\, q_1, q_2, q, r \in \mathbb{R} :\ q_1 q_2 = q^2 \quad \wedge \quad q = (q_c/r_c)r \\ &\qquad \Rightarrow \quad q_1 q_2 = ((q_c/r_c)r)^2 \quad \Rightarrow \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1. \end{aligned} \tag{5.9}$$

$$r = (r_c/t_c)t = ct \ \Rightarrow \ mr = (m_c/r_c)(ct)^2 \ \Rightarrow \ ((r_c/m_c)/c^2)mr/t^2 = 1. \tag{5.10}$$

$$\begin{aligned} &((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1 \\ &\qquad \Rightarrow \quad F := mr/t^2 = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2. \end{aligned} \tag{5.11}$$

$$(5.12)\quad r_c/t_c = c \quad \wedge \quad F = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2$$
$$\Rightarrow \quad F = (m_c(r_c/t_c^2))(r_c/q_c)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2,$$

where Coulomb's constant, $k_e = (m_c(r_c/t_c^2))(r_c/q_c)^2$, conforms to the SI units: $N \cdot m^2 \cdot C^{-2}$.

5.4. 3 fundamental constants. c_t, c_m, and c_q.

$$(5.13)\qquad c_t = r_c/t_c \approx 2.99792458 \cdot 10^8 m\ s^{-1}.$$

$$(5.14)\qquad G = (r_c/m_c)c_t^2 \quad \Rightarrow \quad c_m = r_c/m_c \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}.$$

$$(5.15)\qquad k_e = ((m_c/r_c)c_t^2)(r_c/q_c)^2 \Rightarrow \quad c_q = r_c/q_c \approx 8.6175172023 \cdot 10^{-18} m\ C^{-1}.$$

5.5. Principle of conservation. A change in distance corresponds to an inversely proportionate change in another type of measure. The ratios of c_t/c_m and c_t/c_q yields 3 conservation constants, k_t, k_m, and k_q that are the basis of particle-wave behavior:

$$(5.16)\qquad c_t/c_m = (m_c/r_c)(r_c/t_c) = (m_c r_c)/(t_c r_c) = k_m/k_t.$$

$$(5.17)\qquad c_t/c_q = (q_c/r_c)(r_c/t_c) = (q_c r_c)/(t_c r_c) = k_q/k_t.$$

5.6. Planck-Einstein equation: Applying both the relative measure ratios 5.1 and the conservation ratios 5.5:

$$(5.18)\qquad m(ct)^2 = mr^2 \quad \wedge \quad mr = m_c r_c = k_m \quad \Rightarrow \quad m(ct)^2 = k_m r.$$

$$(5.19)\quad m(ct)^2 = k_m r \quad \wedge \quad r_c/t_c = r/t = c$$
$$\Rightarrow \quad E := mc^2 = k_m r/t^2 = (k_m(r/t))\,(1/t) = (k_m c)(1/t) = hf,$$

where the Planck constant $h = k_m c$ and the frequency $f = 1/t$.

$$(5.20)\qquad k_m = m_c r_c = h/c \approx 2.21022 \cdot 10^{-42}\ kg\ m.$$

$$(5.21)\qquad k_t = t_c r_c = k_m/(c_t/c_m) \approx 5.47493 \cdot 10^{-78}\ s\ m.$$

$$(5.22)\qquad k_q = q_c r_c = (c_t/c_q)k_t \approx 1.90466 \cdot 10^{-52}\ C\ m.$$

5.7. Quantum-special relativity gravity force. The total mass of a particle is $m = \sqrt{m_0^2 + m_{ke}^2}$, where m_0 is the rest mass and m_{ke} is the kinetic energy-equivalent mass. Applying both the relative measure ratios 5.1 and the conservation ratios 5.5:

$$(5.23)\quad m_0 = (m_c/r_c)r \quad \wedge \quad m_{ke} = m_c r_c/r \quad \wedge \quad m = \sqrt{m_0^2 + m_{ke}^2}$$
$$\Rightarrow \quad m = \sqrt{((m_c/r_c)r)^2 + ((m_c r_c)/r)^2}.$$

Applying equation 5.23 to equation 5.6:

$$(5.24)\quad \exists\, m:\ m_1 m_2 = m^2 = ((m_c/r_c)r)^2 + ((m_c r_c)/r)^2$$
$$\Rightarrow \quad m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2) = 1,$$

where r is the distance in the coordinate frame of reference. But, the experienced force in the proper (local) frame of reference is, from equation 5.2, $r' = \sqrt{(ct)^2 - (vt)^2}$:

$$r' = \sqrt{(ct)^2 - (vt)^2} \quad \Rightarrow \quad m_0 r' = (m_c/r_c)((ct)^2 - (vt)^2). \tag{5.25}$$

$$m_0 r' = (m_c/r_c)((ct)^2 - (vt)^2) \quad \Rightarrow \quad ((r_c/m_c)/(c^2 - v^2))m_0 r'/t^2 = 1. \tag{5.26}$$

$$\begin{aligned} &((r_c/m_c)/(c^2 - v^2))m_0 r'/t^2 = 1 \\ &\qquad \wedge \quad m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2) = 1 \\ \Rightarrow \quad &F := m_0 r'/t^2 = ((m_c/r_c)(c^2 - v^2))m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2). \end{aligned} \tag{5.27}$$

5.8. Quantum-special relativity charge force. Applying $m = (m_c/q_c)q$ to the quantum-relativistic gravity equation (5.7):

$$F = (m_c/q_c)^2(m_c/r_c)(c^2 - v^2)q_1 q_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2). \tag{5.28}$$

6. Insights and implications

(1) Proving that volume and distance are derived from the same abstract, set-based definition of a countable n-volume provides a unifying and more rigorous set and limit-based foundation under integration, measure theory, the vector norm, and the metric space axioms without using the geometric primitives and relations required in Euclidean geometry [**Joy98**], axiomatic geometry [**Lee10**], and vector analysis [**Wey52**].

(2) Conjecture: A distance measure as an interval length, $d = f(v, n) = f(\sum_{j=1}^{m} f_i(v, n))$, where f and f_i are continuous, bijective, monotonic increasing functions of a v_c-derived n-volume, v, is a restriction on the definition of a complete metric space.

(3) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [**Joy98**]. And proofs that a straight line is the smallest distance equate the straight line to the Euclidean distance.

Using the calculus of variations for a shortest distance proof would result in circular logic due to the Euclidean assumptions in the definitions of the Riemann, Lebesgue, and line integrals.

All distance measures of an Euclidean 2-volume (area) are Minkowski distances (4.2), where $1 \le n \le 2$. $n = 1$ is the Manhattan (largest) distance case, $d = \sum_{i=1}^{m} s_i$. And $n = 2$ is the Euclidean (smallest) distance case, $d = (\sum_{i=1}^{m} s_i^2)^{1/2}$. Here, there is no need for geometric primitives, like straight line, triangle, etc.

(4) Compare the distance sum inequality (4.4),

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \le (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}, \tag{6.1}$$

to Minkowski's sum inequality:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n)^n)^{1/n} \le (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n} \tag{6.2}$$

[**Min53**]. Note the exponent difference in the left side of each equation:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \quad \textbf{vs.} \quad (\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n}. \tag{6.3}$$

The two inequalities only intersect where $n = 1$. The distance sum inequality is a more fundamental inequality because its proof does not require the convexity and various inequality theorems required to prove the Minkowski sum inequality. And the distance sum inequality is derived from the definitions of volume and distance, which makes it more directly related to geometry.

(5) The gravity (5.8), charge (5.12), and Planck (5.19) constants were all derived from more fundamental constants, $(r_c/t_c) = c_t$, $(r_c/m_c) = c_m$, $(r_c/q_c) = c_q$, and $m_c r_c = k_m$. And all depend on the speed of light constant, c_t: For example, $G = c_m c_t^2$, $k_e = (c_q^2/c_m)c_t^2$, and $h = k_m c_t$.

(6) Algebraic manipulation of Coulomb's constant, $k_e = (r_c/q_c)^2((m_c/r_c)c^2) = (m_c(r_c/t_c^2))(r_c/q_c)^2$, yields the constant acceleration term, r_c/t_c^2, which suggests there might be a maximum acceleration constant.

(7) The derivations of the spacetime equations, here (5.1), differ from other derivations.
   (a) The derivations, here, are shorter and simpler.
   (b) The derivations, here, do not rely on the Lorentz transformations or Einsteins' postulates [**Ein15**]. The derivations do not even require the notion of light.
   (c) The derivations, here, rely only on geometry: the Euclidean volume proof (3.2), the Minkowski distances proof (4.1), and the 3D proof (4.12), which provides the insight that the properties of physical space creates a maximum speed, and the spacetime equations.
   (d) The derivations are valid for spacetime, spacemass, and spacecharge.

(8) Applying the ratios to derive Newton's gravity force (5.2) and Coulomb's charge force (5.3) equations provide:
   (a) Derivations that do not assume the inverse square law or Gauss's flux divergence theorem. **Note:** the components of the Ricci and metric tensors in Einstein's field equations have the units, $1/\text{distance}^2$ [**Wey52**], which is an assumption of the inverse square law.
   (b) The first derivations to show that the inverse square law and the property of force as mass times acceleration are the result of the conversion ratios, $r = (r_c/t_c)t = (r_c/m_c)m$. And the derivation of the inverse square law does not rely on Gauss's flux divergence on the surface of a sphere.

The quantum-special relativity extension to Newton's gravity equation (5.26) makes empirically verifiable predictions.
   (a) In Newton's gravity force, Gauss's gravity law, and Einstein's field (general relativity) equations, the force, $F \to \infty$ as the distance, $r \to 0$. But, in the quantum-special relativity extension to Newton's gravity equation, $F \to 0$ as $r \to 0$. Where the distance between point-like particles is less than approximately 10^{-4} m, the gravity force should measurably decrease, which implies larger black hole radii and maybe allows black hole evaporation.
   (b) Further, the quantum-special relativity gravity equation indicates that Newton's gravity constant, G, Gauss's constant, $4\pi G$, and Einstein's gravity constant, $k = 8\pi G/c^4$, [**Wey52**], are only valid where the local velocity, $v = 0$.

(c) Adapting the quantum-relativistic gravity equation to Einsteins field (general relativity) equations requires replacing G in the constant k with "$((m_c/r_c)(c^2 - v^2))$", and the components in the metric and Ricci tensor must have the units, "$1/(distance^2 + 1/distance^2)$".

(9) There is no unit-factoring ratio converting a discrete state value to a continuously varying interval length. Therefore, the spin states of two quantum entangled particles and the polarization states of two quantum entangled photons are independent of varying distance and time interval lengths.

(10) Linear algebra, vector analysis, differential geometry, etc. assume any number of possible dimensions. For example, the Gram-Schmidt process is a method to find an orthogonal vector for any n-dimensional vector [**Coh21**]. None of those disciplines have exposed the properties that can limit a geometry to 3 dimensions. But the proof that a strict linearly ordered and symmetric set is a cyclic set of at most 3 members (4.12) is the simplest explanation for observing only 3 dimensions of physical space.

(a) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (4.12), each member being an ordered and symmetric set of 3 dimensions (three 3-dimensional balls).

(b) Each of 3 ordered and symmetric dimensions of space can have at most 3 sequentially ordered and symmetric state values, for example, an ordered and symmetric set of 3 vector orientations, $\{-1, 0, 1\}$, per dimension of space and at most 3 spin states per dimension, etc.

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