The Set Properties Generating Geometry and Physics

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Abstract. Volume and the Minkowski distances/Lp norms (e.g., Manhattan and Euclidean distance) are derived from a set and limit-based foundation without referencing the primitives and relations of geometry. Sequencing a strict linearly ordered set in all n-at-a-time permutations via successor/predecessor relations is a cyclic set of at most 3 members. Therefore, all other interval lengths have different types from a cyclic set of 3 distance interval lengths. The linear transformations between different types of interval lengths and the set proofs provide simpler derivations of the spacetime, Lorentz, Newton's gravity, Coulomb's charge force, and Planck-Einstein equations and corresponding constants. All proofs are verified in Coq.

Contents

1. Introduction

Mathematical (real) analysis can construct differential calculus from a set and limit-based foundation without the need to reference the primitives and relations of Euclidean geometry, like straight line, angle, slope, etc. But the Riemann and Lebesgue integrals and measure theory define Euclidean volume as the product of interval lengths. The vector norm and the metric space axioms are definitions motivated by Euclidean distance. [Gol76] [Rud76] Here, the primitives and relations of geometry and some physics equations are shown to be instances of set properties and the relations between sets.

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The cardinal, v_c, of an abstract, countable set of Cartesian product n-tuples is its n-volume: $v_c = \prod_{i=1}^n |x_i|$, where $|x_i|$ is the cardinal of the countable, disjoint set, x_i. Where $|x_i|$ is the approximate number of same-sized partitions of $[a_i, b_i] \subset \mathbb{R}$ and $s_i = b_i - a_i$, it will be proved that Euclidean volume, $v = \prod_{i=1}^n s_i$, is an instance of $v_c = \prod_{i=1}^n |x_i|$. Generalizations to include some non-Euclidean volumes are: $v_c = a_c \prod_{i=1}^n |x_i|$ and $v = a \prod_{i=1}^n s_i$.

A set of n-tuples is the union of disjoint subsets, which results in an n-volume that is the sum of n-volumes: $v_c = \sum_{i=1}^m v_{c_i}$, where it will be proved that $\prod_{i=1}^n s_i = \sum_{j=1}^m (\prod_{i=1}^n s_{i,j})$ is an instance of $v_c = \sum_{i=1}^m v_{c_i}$. Note that where the interval lengths are signed ($\pm$-directed), the $n = 2$ case is the dot product, $a \cdot b = \sum_{j=1}^m a_j b_j$, where $a = s_1$, $b = s_2$, $a_j = s_{1,j}$ and $b_j = s_{2,j}$.

Where the countable domain set sizes are inverse functions of n-volumes: $\exists\ x, x_i, m : v_c = |x|^n = \sum_{i=1}^m |x_i|^n$, where it will be proved that $d^n = \sum_{i=1}^m d_i^n$ is an instance of $|x|^n = \sum_{i=1}^m |x_i|^n$. d is the L_p norm (Minkowski distance), which will be proved to imply the metric space axioms. Generalizations to include some non-Euclidean distances are: $|x|^n = \sum_{i=1}^m a_{c_i} |x_i|^n$ and $d^n = \sum_{i=1}^m a_i d_i^n$.

In the prior equations, sequencing a set, from $i = 1$ to n, is a strict linear (total) order that set theory defines in terms of successor and predecessor functions. But sequencing a strict linear order in all n-at-a-time orders requires an additional "symmetry" constraint, where every set member is either a successor or predecessor to every other set member, which will be proved to be a cyclic set, where $n \leq 3$.

Therefore, an element cannot be added to a set of 3 linearly ordered and symmetric "distance" elements and has a different type (is a member of a different set). A linear transformation from the size of one set element to the size of another element having a different type is a constant, type conversion ratio. The ratios combined with the set proofs provide simpler derivations of the spacetime, Lorentz, Newton's gravity, Coulomb's charge force, and Planck-Einstein equations and corresponding constants. Impacts on Einstein's field equations are also shown.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq23**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

Derivatives and integrals use a 1-1 correspondence between the infinitesimals of each interval, where the size of the infinitesimals in each interval are proportionate to the size of the containing interval, which precludes using derivatives and integrals to directly express many-to-many (Cartesian product) mappings between same-sized, size κ, infinitesimals in different-sized intervals. Further, using tools that define Euclidean volume and distance precludes using those tools to derive Euclidean volume and distance.

Therefore, a different tool is used here. A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals (infinitesimals), where each infinitesimal has the *same* size, κ, across all intervals. The ruler is both an inner and outer measure of an interval.

DEFINITION 2.1. Ruler measure, M: $\forall\ [a, b] \subset \mathbb{R},\ \ s = b - a\ \ \wedge\ \ \kappa > 0\ \ \wedge\ (p = floor(s/\kappa)\ \ \vee\ \ p = ceiling(s/\kappa))\ \ \wedge\ \ M = \sum_{i=1}^p \kappa = p\kappa$.

THEOREM 2.2. *Ruler convergence:* $M = \lim_{\kappa \to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, \kappa > 0,\ p = floor(s/\kappa) \ \wedge\ 0 \leq |floor(s/\kappa) - s/\kappa| < 1 \ \Rightarrow\ |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by κ:

$$\forall\, \kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \delta = \epsilon \quad := \quad M = \lim_{\kappa \to 0} p\kappa = s. \qquad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, \ldots, \pi_{\lim_{\kappa \to 0}}$.

LEMMA 2.3. $\forall\, n \geq 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa.$

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa \to 0} \kappa^n = 0. \end{aligned} \tag{2.5}$$

$$\lim_{\kappa \to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa \to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa. \qquad \square \tag{2.6}$$

3. Volume

DEFINITION 3.1. An n-volume is the number of ordered combinations (n-tuples), v_c, of the members of n number of disjoint, countable domain sets, x_i:

$$\exists\, n \in \mathbb{N}, a_c, v_c, \in \{0,\ \mathbb{N}\},\ x_i \in \{x_1, \cdots, x_n\},\ \bigcap_{i=1}^{n} x_i = \emptyset :\ v_c = a_c \prod_{i=1}^{n} |x_i|. \tag{3.1}$$

THEOREM 3.2. *Euclidean volume, $v = \prod_{i=1}^{n} s_i$, is the instance of the "flat" n-volume (3.1) case, $v_c = \prod_{i=1}^{n} |x_i|$, where each countable set, x_i, is the set of partitions of an interval case, $[a_i, b_i] \subset \mathbb{R}$.*

$$\begin{aligned} &\forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\ \ [v_a, v_b] \subset \mathbb{R},\ \ s_i = b_i - a_i,\ \ v = v_b - v_a, \\ &\qquad\qquad v_c = \prod_{i=1}^{n} |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^{n} s_i. \end{aligned} \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.

$$v_c = \prod_{i=1}^{n} |x_i| \ \Leftrightarrow\ v_c\kappa = (\prod_{i=1}^{n} |x_i|)\kappa \ \Leftrightarrow\ \lim_{\kappa \to 0} v_c\kappa = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|)\kappa. \tag{3.3}$$

Apply the ruler (2.1) and ruler convergence (2.2) to v:

$$\exists\, v, \kappa \in \mathbb{R} :\ v_c = floor(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa \to 0} v_c\kappa = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|)\kappa. \tag{3.4}$$

Apply lemma 2.3 to equation 3.4:

$$v = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|)\kappa = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|)\kappa^n = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|\kappa). \tag{3.5}$$

Apply the ruler (2.1) and ruler convergence (2.2) to s_i:

$$\exists\, s_i, \kappa \in \mathbb{R} :\ floor(s_i/\kappa) = |x_i| \quad \Rightarrow \quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i. \tag{3.6}$$

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|\kappa) \ \wedge\ \lim_{\kappa\to 0}(|x_i|\kappa) = s_i \ \Rightarrow\ v = \prod_{i=1}^{n} s_i \tag{3.7}$$ □

THEOREM 3.3. *Sum of volumes:*

$$\forall\, x_{i,j} \in \{x_{i_1}, \cdots, x_{i_m}\} = x_i :\ v_c = \prod_{i=1}^{n} |x_i| \quad \wedge \quad v_{c_{i,j}} = \prod_{i=1}^{n} |x_{i,j}| \ \wedge \\ v_c = \sum_{i=1}^{m} v_{c_i} \quad \Rightarrow \quad \exists\, s_i, s_{i,j} \in \mathbb{R} :\ \prod_{i=1}^{n} s_i = \sum_{j=1}^{m}(\prod_{i=1}^{n} s_{i,j}). \tag{3.8}$$

The formal proof, "sum_of_volumes," is in the Coq file, euclidrelations.v.

PROOF. From the Euclidean volume theorem (3.2):

$$v_c = \prod_{i=1}^{n} |x_i| \ \Rightarrow\ v = \prod_{i=1}^{n} s_i \ \wedge\ v_{c_{i,j}} = \prod_{i=1}^{n} |x_{i,j}| \ \Rightarrow\ v_i = \prod_{i=1}^{n} s_{i,j}. \tag{3.9}$$

Apply the ruler (2.1) and ruler convergence (2.2):

$$\exists\, v, v_i, \kappa \in R :\quad v_c = floor(v/\kappa) \quad \wedge \quad v_{c_{i,j}} = floor(v_i/\kappa) \\ \Rightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa \quad \wedge \quad v_i = \lim_{\kappa\to 0} v_{c_{i,j}}\kappa. \tag{3.10}$$

$$v_c = \sum_{i=1}^{m} v_{c_i} \quad \Leftrightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\sum_{i=1}^{m} v_{c_i})\kappa. \tag{3.11}$$

Apply lemma 2.3 to equation 3.11:

$$\lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa \ \wedge\ v = \lim_{\kappa\to 0}(\sum_{i=1}^{m} v_{c_i})\kappa \ \wedge\ v_i = \lim_{\kappa\to 0} v_{c_{i,j}}\kappa \\ \Rightarrow \quad v = \lim_{\kappa\to 0}(\sum_{i=1}^{m} v_{c_i})\kappa^n = \lim_{\kappa\to 0} \sum_{i=1}^{m}(v_{c_i}\kappa) = \sum_{i=1}^{m} v_i. \tag{3.12}$$

$$v = \prod_{i=1}^{n} s_i \wedge v_i = \prod_{i=1}^{n} s_{i,j} \wedge v = \sum_{i=1}^{m} v_i \Rightarrow \prod_{i=1}^{n} s_i = \sum_{j=1}^{m} \prod_{i=1}^{n} s_{i,j}. \tag{3.13}$$ □

4. Distance

4.1. Minkowski distance (L_p norm).

THEOREM 4.1. *The Minkowski distance (L_p norm), d, is the instance of the sum of (union of disjoint) n-volumes (3.3):*

$$\prod_{i=1}^{n} |x_i| = \sum_{j=1}^{m}(\prod_{i=1}^{n} |x_{i,j}|) \quad \Rightarrow \quad \exists\, d, d_i \in \mathbb{R} :\quad d^n = \sum_{i=1}^{m} d_i^n.$$

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

PROOF. From the Euclidean volume proof (3.2):

$$\prod_{i=1}^{n} |x_i| = \sum_{j=1}^{m}(\prod_{i=1}^{n} |x_{i,j}|) \quad \Rightarrow \quad \prod_{i=1}^{n} s_i = \sum_{j=1}^{m}(\prod_{i=1}^{n} s_{i,j}) \tag{4.1}$$

$$\exists\, d, d_i, s_i, s_{i,j} \in \mathbb{R} :\ s_1 = \cdots = s_n = d \quad \wedge \quad s_{i_1} = \cdots = s_{i_m} = d_i \quad \wedge \\ \prod_{i=1}^{n} s_i = \sum_{j=1}^{m}(\prod_{i=1}^{n} s_{i,j}) \quad \Rightarrow \quad d^n = \prod_{i=1}^{n} s_i = \sum_{j=1}^{m}(\prod_{i=1}^{n} s_{i,j}) = \sum_{i=1}^{m} d_i^n. \tag{4.2}$$ □

4.2. Distance inequality. Proving that all Minkowski distances (L_p norms) satisfy the metric space triangle inequality requires another inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.2. *Distance inequality*

$$\forall\, n \in \mathbb{N},\ \ v_a, v_b \geq 0:\ \ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand the n-volume, $(v_a^{1/n} + v_b^{1/n})^n$, using the binomial expansion:

$$\begin{aligned}(4.3)\quad & \forall\, v_a, v_b \geq 0: \quad v_a + v_b \leq v_a + v_b + \\ & \textstyle\sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n.\end{aligned}$$

Take the n^{th} root of both sides of the inequality:

$$(4.4)\ \forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad \square$$

4.3. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N},\ \ a_i, b_i \geq 0:\ \ (\textstyle\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.2):

$$\begin{aligned}(4.5)\quad & \forall\, m, n \in \mathbb{N},\ v_a, v_b \geq 0: \quad v_a = \textstyle\sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge \\ & (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad ((\textstyle\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} = \\ & (\textstyle\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \quad \square\end{aligned}$$

4.4. Metric Space. All Minkowski distances (L_p norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.4. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(4.6) \qquad (u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}.$$

Apply the distance inequality (4.2) to the inequality 4.6:

$$\begin{aligned}(4.7)\quad & (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \ \wedge \ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ & \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ \Rightarrow \quad & (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ & \Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq \\ & (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \quad \square\end{aligned}$$

THEOREM 4.5. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$\forall\, p \,:\, p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \qquad (4.8)$$
$$\Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \qquad \square$$

THEOREM 4.6. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0. \qquad (4.9)$$

$$p \geq 1,\ \ u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \qquad (4.10) \qquad \square$$

THEOREM 4.7. *Identity of Indiscernibles:* $d(u, u) = 0$.

PROOF. From the non-negativity property (4.6):

$$d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0 \qquad (4.11)$$
$$\Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0.$$

$$d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v. \qquad (4.12)$$

$$d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \qquad (4.13) \qquad \square$$

5. Applications to physics

5.1. The properties limiting a set to at most 3 members.

DEFINITION 5.1. Totally ordered set:

$$\forall\, i\, n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i\ \in \{x_1, \ldots, x_n\},$$
$$\textit{successor}\ x_i = x_{i+1}\ \wedge\ \textit{predecessor}\ x_{i+1} = x_i.$$

DEFINITION 5.2. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i,\ j,\ n \in \mathbb{N},\ \forall\, x_i,\ x_j \in \{x_1, \ldots, x_n\},\ \textit{successor}\ x_i = x_j\ \Leftrightarrow\ \textit{predecessor}\ x_j = x_i.$$

THEOREM 5.3. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n\ \wedge\ j = 1\ \Rightarrow\ \textit{successor}\ x_n = x_1\ \wedge\ \textit{predecessor}\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (5.1) assigns a unique label to each set member and assigns unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (5.2):

$$i = n\ \wedge\ j = 1\ \wedge\ \textit{successor}\ x_i = x_j\ \Rightarrow\ \textit{successor}\ x_n = x_1. \qquad (5.1)$$

Applying the definition of the symmetry property (5.2) to conclusion 5.1:

$$\textit{successor}\ x_i = x_j\ \Rightarrow\ \textit{predecessor}\ x_j = x_i\ \Rightarrow\ \textit{predecessor}\ x_1 = x_n. \qquad (5.2) \qquad \square$$

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

Proof.
It was proved that an ordered and symmetric set is a cyclic set (5.3).

Definition 5.5. (Cyclic) Successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize). \tag{5.3}$$

Definition 5.6. (Cyclic) Predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1). \tag{5.4}$$

Definition 5.7. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{5.5}$$

Prove that every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \tag{5.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \tag{5.7}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1). \tag{5.8}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.9}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1). \tag{5.11}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize). \tag{5.12}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.13}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \tag{5.14}$$

Member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not ($\neg$) a successor of member 1 for all $setsize > 3$:

$$\neg Successor(1, 3, setsize > 3) \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.15}$$

Member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not ($\neg$) a predecessor of member 1 for all $setsize > 3$:

$$\neg Predecessor(1, 3, setsize > 3) \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \tag{5.16}$$

For all $setsize > 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not symmetric):

$$\neg Adjacent(1, 3, setsize > 3) \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \tag{5.17}$$

□

From the 3D proof (5.4), the interval lengths: t (time), m (mass), and q (charge) have different types (are from different sets) from a 3-dimensional interval length, r, that can be related via constant, unit-factoring, conversion ratios:

$$r = (r_c/t_c)t = ct = (r_c/m_G)m = (r_c/q_C)q, \tag{5.18}$$

5.2. Spacetime and Lorentz equations. From the Euclidean volume proof (3.2), two disjoint intervals, $[0, r]$ and $[0, r']$, defines an Euclidean 2-space. From the Minkowski distance proof (4.1), the interval lengths, r and r', are inverse functions of 2 cuboid 2-volumes. Either $r' \geq r$ or $r \geq r'$ can be chosen. $r \geq r'$ is used here.

(5.19)

$$\forall\, r \geq r' \;\exists\, r_v \in \mathbb{R} : \; r^2 = r'^2 + r_v^2 \quad \wedge \quad \exists\, r_c,\; t_c,\; c,\; v \in \mathbb{R} : \; r = (r_c/t_c)t = ct$$

$$\wedge \quad r_v = vt \quad \Rightarrow \quad (ct)^2 = {r'}^2 + (vt)^2 \quad \Rightarrow \quad r' = \sqrt{(ct)^2 - (vt)^2} = ct\sqrt{1 - (v/c)^2}.$$

Local (proper) distance, r', contracts relative to coordinate distance, r, as $v \to c$:

$$r' = ct\sqrt{1 - (v/c)^2} \quad \wedge \quad ct = r \quad \Rightarrow \quad r' = r\sqrt{1 - (v/c)^2}. \tag{5.20}$$

From equation 5.19, coordinate time, t, dilates relative to local time, t', as $v \to c$:

$$ct = r'/\sqrt{1 - (v/c)^2} \quad \wedge \quad r' = ct' \quad \Rightarrow \quad t = t'/\sqrt{1 - (v/c)^2}. \tag{5.21}$$

Using $r^2 = r'^2 + r_v^2$ from equation 5.19, where r_v is a 3-dimensional distance, one form of the flat Minkowski's spacetime event interval is:

$$\begin{aligned} \mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_v^2 \quad \wedge \quad \mathrm{d}r_v^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \quad \wedge \quad \mathrm{d}(ct) = \mathrm{d}r \\ \Rightarrow \quad \mathrm{d}r'^2 = \mathrm{d}(ct)^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2 - \mathrm{d}x_3^2. \end{aligned} \tag{5.22}$$

The Lorentz transformations follow from equation 5.20 and Galilean transformation:

$$r' = r/\sqrt{1 - (v/c)^2} \quad \wedge \quad r = r' + vt \quad \Rightarrow \quad r' = (r - vt)/\sqrt{1 - (v/c)^2}. \tag{5.23}$$

$$\begin{aligned} r' = (r - vt)/\sqrt{1 - (v/c)^2} \quad \wedge \quad r = ct \quad \wedge \quad r' = ct' \\ \Rightarrow \quad t' = (t - (vt/c))/\sqrt{1 - (v/c)^2} = (t - (vr/c^2))/\sqrt{1 - (v/c)^2}. \end{aligned} \tag{5.24}$$

5.3. Einstein's field (general relativity) equations. The spacetime interval equation (5.22) is the "flat" case of $ds^2 = \sum_{i=0}^{3} g_{i,i} dx_i^2$, where the metric tensor is: $g_{\mu,\nu} = diag(g_{0,0}, g_{1,1}, g_{2,2}, g_{3,3})$ [**Ein15**] [**Wey52**]. However, the 4-D spacetime interval equation is derived from the 2-D equation, $ds^2 = \sum_{i=0}^{1} g_{i,i} dx_i^2$, where x_1 is a 3-dimensional distance. Each infinitesimal spacetime distance along the Levi-Civita connection between two point-masses is Euclidean-like. In that specific case, the 4×4 metric tensor, $g_{\mu,\nu}$, can be replaced by a 2×2 metric tensor, $g_{u,v} = diag(g_{0,0}, g_{1,1})$, which reduces the number of field equations and unknowns.

The Gaussian curvature of a surface is limited to, at most, 3-D. Einstein created a Gaussian-like curvature, $G_{\mu,\nu}$, induced by $g_{\mu,\nu}$, in 4-D via the Ricci tensor, $\mathbf{R}_{\mu,\nu}$, and it's scalar curvature, R, where $G_{\mu,\nu} = \mathbf{R}_{\mu,\nu} - (1/2)Rg_{\mu,\nu}$ [**Wey52**]. Where the the 2×2 metric tensor can be used, the 2-D Gaussian curvature can be used, which eliminates the need for the complicated calculations of the Ricci tensor and its scalar curvature.

Further, the sum of two tensors is a tensor, $T_{\mu,\nu} : G_{\mu,\nu} + g_{\mu,\nu} = kT_{\mu,\nu}$, where Einstein defined the conversion factor as $k = 8\pi G/c^4$ [**Wey52**]. k and Newton's gravitational constant, G, were defined (conjectured) rather than derived.

5.4. Newton's gravity force equation and constant. From equation 5.18:

$$\forall\, m_1 m_2 = m^2 = (m_G/r_c)^2 r^2 \quad \Rightarrow \quad (r_c/m_G)^2 m_1 m_2/r^2 = 1. \tag{5.25}$$

From equation 5.21, if r is the proper distance, then $r = ct\sqrt{1-(v/c)^2}$:

$$\begin{aligned} r = ct\sqrt{1-(v/c)^2} \quad \wedge \quad mr = ((m_G/r_c)r)(ct\sqrt{1-(v/c)^2}) \\ \Rightarrow \quad mr = (m_G/r_c)(ct)^2(1-(v/c)^2) = (m_G/r_c)(ct)^2 - (vt)^2). \end{aligned} \tag{5.26}$$

$$mr = (m_G/r_c)(ct)^2 - (vt)^2) \quad \Rightarrow \quad ((r_c/m_G)/(c^2-v^2))mr/t^2 = 1. \tag{5.27}$$

$$\begin{aligned} ((r_c/m_G)/(c^2-v^2))mr/t^2 = 1 \quad \wedge \quad (r_c/m_G)^2 m_1 m_2/r^2 = 1 \\ \Rightarrow \quad F := mr/t^2 = ((r_c/m_G)(c^2-v^2))m_1 m_2/r^2 = G m_1 m_2/r^2, \end{aligned} \tag{5.28}$$

where $G = ((r_c/m_G)(c^2-v^2))$, has the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$. G is Newton's constant, where $v = 0$.

5.5. Coulomb's charge force and constant. From equation 5.18:

$$r = (r_c/m_G)m = (r_c/q_C)q_1 \Rightarrow m = (m_G/q_C)q_1. \tag{5.29}$$

Substituting equations 5.29 and 5.18 into equation 5.28, where $v = 0$:

$$\begin{aligned} m = (m_G/q_C)q_1 \quad \wedge \quad r_c/t_c = c \quad \wedge \quad F = ((r_c/m_G)c^2)m_1 m_2/r^2 \\ \Rightarrow \quad F = (m_G/q_C)(r_c/q_C)(r_c/t_c)^2 q_1 q_2/r^2. \end{aligned} \tag{5.30}$$

$$\begin{aligned} a_G = r_c/t_c^2 \quad \wedge \quad F = (m_G/q_C)(r_c/q_C)(r_c/t_c)^2 q_1 q_2/r^2 \\ \Rightarrow \quad F = (m_G a_G)(r_c/q_C)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2, \end{aligned} \tag{5.31}$$

where $v = 0$, $k_e = (m_G a_G)(r_c/q_C)^2$, has the SI units: $N \cdot m^2 \cdot C^{-2}$. And where $|v| > 0$, $F = (m_G/q_C)(r_c/q_C)(c^2 - v^2)q_1 q_2/r^2$.

5.6. Work and Planck-Einstein equations: From the ratios 5.18:

$$\begin{aligned} r = (r_c/m_G)m \quad \wedge \quad r = (r_c/t_c)t = ct \quad \wedge \quad \exists\, m_p \in \mathbb{R} :\ m_p = (t/t_c)^2 m_G \\ \Rightarrow mr = (m_G/r_c)r^2 = (m_G/r_c)(r_c/t_c)^2 t^2 = (t/t_c)^2 m_G r_c = m_p r_c = k_W \\ \approx 2.2102190943 \cdot 10^{-42}\ kg\ m, \end{aligned} \tag{5.32}$$

where r is the displacement (Compton wavelength) of the mass, m.

$$m(ct)^2 = mr^2 \quad \wedge \quad mr = k_W \quad \Rightarrow \quad m(ct)^2 = k_W r. \tag{5.33}$$

$$m(ct)^2 = k_W r \Rightarrow E = mc^2 = k_W r/t^2 = k_W(r/t)(1/t) = (k_W c)(1/t) = hf, \tag{5.34}$$

where the Planck constant $h = k_W c$ and the frequency $f = 1/t$. The relativistic Planck-Einstein equation is: $m(c^2 - v^2) = hf$.

6. Insights and implications

(1) The function, $d = (\sum_{i=1}^{m} a_i s_i^n)^{1/n}$, where a_i, is a scalar value, inherits the properties of a metric space from the Minkowski distance (4.4). Therefore, proving that Euclidean volume and the Minkowski distance are instances of the same abstract, set-based definition of a countable n-volume provides a unifying set and limit-based foundation under volume and distance without using the geometric primitives and relations required in Euclidean geometry [**Joy98**], axiomatic geometry [**Lee10**], and vector analysis [**Wey52**].

(2) The interval length, $s = b - a$, in the ruler measure (2.1) can be replaced with a $\pm$-directed integer length: $s = (b \Leftarrow a = 0 : a \Leftarrow b = 0)$, where $a = 0$ or $b = 0$. This allows distinguishing between $[x, 0]$ and $[0, x]$. Substituting the $\pm$-directed integer length in the n-volume as the sum of n-volumes (3.3) gives the $n = 2$ case: $a \cdot b = \sum_{j=1}^{m} a_j b_j$, where $a = s_1$, $b = s_2$, $a_j = s_{1,j}$ and $b_j = s_{2,j}$, the properties of the dot product, without using the notions of angle, orthogonal, projection, etc.

(3) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [**Joy98**]. And proofs that a straight line is the smallest distance equate the straight line to the Euclidean distance.

Using the calculus of variations for a shortest distance proof would result in circular logic due to the Euclidean assumptions in the definition of the integral.

The set-based proofs provide a shortest distance proof without using the notion of a straight line. It was proved that all real-valued instances of "flat" countable n-volumes are Minkowski distances (4.1). Therefore, in flat 2-space, the Minkowski distances are constrained to a range from Manhattan distance (the largest distance, where $n = 1$), $d = (\sum_{i=1}^{m} s_i^1)^{1/1} = \sum_{i=1}^{m} s_i$, to Euclidean distance (the smallest distance, where $n = 2$), $d = (\sum_{i=1}^{m} s_i^2)^{1/2}$.

(4) Compare the distance sum inequality (4.3),

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}, \tag{6.1}$$

used to prove that all Minkowski distances satisfy the metric space triangle inequality property (4.4), to Minkowski's sum inequality:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n)^n)^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n} \tag{6.2}$$

[**Min53**]. Note the difference in the left side of each equation:

$$\forall\, n > 1,\ 0 < a_i^n, b_i^n < 1:\ (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \ > \ (\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n}. \tag{6.3}$$

$$\forall\, n > 1,\ a_i^n, b_i^n \geq 1:\ (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \ < \ (\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n}. \tag{6.4}$$

The distance sum inequality is a more fundamental inequality because its proof does not require the convexity and various inequality theorems required to prove the Minkowski sum inequality. And the distance sum inequality is derived from the definitions of volume and distance, which makes it more directly related to geometry.

(5) From the 3D proof (5.4), more intervals than the 3 dimensions of distance intervals must have different types with lengths that are related to a 3-dimensional distance interval length, r, via constant, unit-factoring, conversion ratios (both direct and inverse proportion ratios). In SI units:
$c_t = r_c/t_c \approx 2.99792458 \cdot 10^8 m\ s^{-1}$.
$c_m = r_c/m_G \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}$.
$c_q = r_c/q_C \approx 8.6175172023 \cdot 10^{-18} m\ C^{-1}$.
$k_W = m_p r_c \approx 2.2102190943 \cdot 10^{-42}\ kg\ m$.

(6) The derivations in this article show that the spacetime (5.2), gravity force (5.26), charge force (5.31), and Planck-Einstein (5.34) equations all depend on time being proportionate to distance: $r = (r_c/t_c)t = ct$. For example, where $v = 0$: $G = (r_c/m_G)c^2$, $k_e = (m_G/q_C)(r_c/q_C)c^2$, and $h = (m_p r_c)c = k_W c$.

(7) The ratios make all distance, wavelength, time, frequency, mass, charge, etc. sizes relative to each other. There are no absolute sizes > 0 – no Planck-like distance, time, etc.

(8) The derivations of the spacetime equations and Lorentz transformations, here (5.2), differ from other derivations.
  (a) The derivations, here, are much shorter and simpler.
  (b) The derivations of the spacetime equations, here, do not rely on the Lorentz transformations or Einsteins' postulates [**Ein15**]. The derivations do not even require the notion of light.
  (c) The derivations, here, rely only on geometry: the Euclidean volume proof (3.2), the Minkowski distances proof (4.1), and the 3D proof (5.4), which provides the insight that the geometry of physical space creates: 1) a maximum speed, c; 2) the spacetime equations; and 3) the Lorentz transformations.
  (d) The distance-to-mass ratio, $r = (r_c/m_G)m$, and distance-to-charge ratio, $r = (r_c/q_C)q$, can replace the distance-to-time ratio, $r = (r_c/t_c)t$, to derive spacemass and spacecharge equations.

(9) Applying the ratios to derive Newton's gravity force (5.4) and Coulomb's charge force (5.5) equations provide:
  (a) Derivations that do not assume the inverse square law or Gauss's flux divergence theorem. **Note:** the components of the Ricci and metric tensors in Einstein's field equations have the units, 1/distance^2 [**Wey52**], which is an assumption of the inverse square law.
  (b) The first derivations to show that the inverse square law and the property of force as mass times acceleration are the result of the conversion ratios, $r = (r_c/t_c)t = (r_c/m_G)m$.
  (c) Using Occam's razor, those versions of constants like: Gauss's gravity and charge constants, vacuum magnetic permeability, etc. that contain the value 4π might be incorrect because those constants are based on the less parsimonious assumption that the inverse square law is due to Gauss's flux divergence on a sphere having the surface area, $4\pi r^2$.
  (d) Einstein's gravity constant, $k = 8\pi G/c^4$ [**Wey52**], is only valid when the local velocity is 0. Otherwise, $k = 8\pi(r_G/m_G)(c^2 - v^2)/c^4$. As $v \to c \Rightarrow F \to 0$, implies a universe expanding faster than predicted

by a constant k and also predicts an accelerating expansion.

(e) From the spacetime equation 5.19, the distance r is a constant in the coordinate frame of reference, where $r = ct$. In Newton's gravity and Coulomb's charge equations, r, is in the local frame of reference. Therefore, $r' = ct$ is in the coordinate frame of reference, where $r'^2 = r^2 + r_v^2 \Rightarrow r^2 = r'^2 - r_v^2 \Rightarrow \lim_{r\to 0} r^2 = \lim_{r\to 0} r'^2 - r_v^2 = \lim_{r\to 0}(r'/t)^2 - (r_v/t)^2 = \lim_{r\to 0} c^2 - v^2 = 0$. Therefore, the inverse square law and flux divergence are approximations that only hold where $r >> 0$, which might have implications for particle physics and black holes.

(10) There is no constant ratio converting a constant state value to continuously varying distance, time, mass, and charge values. Therefore, the spin states of two quantum entangled particles and the polarization states of two quantum entangled photons are independent of the amount of distance between the particles and independent of time (no [zero] time).

(11) Linear algebra, vector analysis, differential geometry, etc. assume any number of possible dimensions. For example, the Gram-Schmidt process is a method to find an orthogonal vector for any n-dimensional vector [**Coh21**]. None of those disciplines are capable of exposing the properties that can limit a geometry to 3 dimensions. But the proof that a strict linearly ordered and symmetric set is a cyclic set of at most 3 members (5.4) is the simplest explanation for observing only 3 dimensions of physical space.

(a) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (5.4), each member being an ordered and symmetric set of 3 dimensions (three 3-dimensional balls).

(b) Each of 3 ordered and symmetric dimensions of space can have at most 3 sequentially ordered and symmetric state values, for example, an ordered and symmetric set of 3 vector orientations, $\{-1, 0, 1\}$, per dimension of space, at most 3 spin states per dimension, etc.

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