The Set Mappings Generating Geometry and Physics

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Abstract. All countable set mappings between sets of size c infinitesimals of domain intervals (n-tuples) generate the Euclidean volume equation as c goes to 0. Distance as a function of volume generates all the Minkowski distances (for example, Manhattan and Euclidean distance) and the properties of metric space. The Euclidean volume proof provides simpler and more rigorous derivations of Newton's gravity force and Coulomb's charge force equations (without using the inverse square law or Gauss's divergence theorem). The derivations of the gravity and charge force equations exposes a ratio (constant first derivative) principle that allows simpler derivations of the spacetime equations and some general relativity equations. A symmetry property can limit geometric distance and volume to 3 dimensions per ball and a limit of 3 3-dimensional balls. All proofs are verified in Coq.

Contents

1. Introduction

The definitions of metric space, Euclidean distance, and area/volume in analysis [Gol76] [Rud76] mimic Euclidean geometry [Joy98]. Proofs that those definitions are derived from a set and limit-based foundation without relying on any of the primitives and relations of Euclidean geometry (like line, congruence, angle, triangle, rectangle, etc.) exposes properties of geometry and physics that mimicking cannot provide.

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Countable volume, v_c, is the cardinal (number of members) of the set of all possible mappings between disjoint, countable, domain sets (the number of n-tuples): $v_c = | \times_{i=1}^{n} x_i|$ (where vertical bars around a set indicates the cardinal). Where all the domain sets have the same cardinal, countable distance, d_c, is the cardinal of one of the domain sets: $d_c^n = v_c = | \times_{i=1}^{n} x_i|$.

Applying the countable set definition of volume to sets of size c infinitesimals of domain and range intervals generates the volume equation as $c \to 0$. Distance as a function of volume generates all the L_p norms (Minkowski distances, for example, Manhattan and Euclidean distance) and the properties metric space.

The Euclidean volume proof is used to provide simpler and more rigorous derivations of: Newton's gravity force and Coulomb's charge force equations (without using the inverse square law or Gauss's divergence theorem). The derivations of the gravity and charge forces expose a ratio (constant first derivative) principle that generates the spacetime equations and some general relativity equations.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

Note: Integrals, for example, $\int_0^b \int_0^a \mathrm{d}^2 c$, do not constrain the infinitesimals across multiple intervals to be the same size, for example, does not constrain the infinitesimals in the domain interval, $[0, a] = [0, 1]$, to be the same size in the domain interval, $[0, b] = [0, \pi]$. Therefore, integrals cannot directly express the Cartesian product of mappings, $p_x \times p_y$, of size c infinitesimals of domain intervals.

A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals, where each subinterval has the *same* size, c. Therefore, the ruler can explicitly express express the number of mappings between the p_x number of size c subintervals in one interval and the p_y number of size c subintervals in another interval.

Definition 2.1. Ruler measure, M: $\forall\, c,\ s \in \mathbb{R},\ [a, b] \subset \mathbb{R},\ s = b-a \wedge c > 0 \wedge (p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \sum_{i=1}^{p} c = pc.$

Theorem 2.2. *Ruler convergence:* $M = \lim_{c\to 0} pc = s$.

The proof is trivial but is included here for completeness. The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\ p = floor(s/c)\ \wedge\ 0 \le |floor(s/c) - s/c| < 1\ \Rightarrow\ 0 \le |p - s/c| < 1. \tag{2.1}$$

Multiply all sides of inequality 2.1 by c:

$$\forall\, c > 0, \quad 0 \le |p - s/c| < 1 \quad \Rightarrow \quad 0 \le |pc - s| < |c|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \\ &\quad \Rightarrow \quad \forall\, \epsilon = \delta :\ |c - 0| < \delta\ \wedge\ |pc - s| < \epsilon\ := \ M = \lim_{c\to 0} pc = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}},\ 3.14_{c=10^{-2}},\ 3.141_{c=10^{-3}}, \ldots, \pi_{\lim_{c\to 0}}$.

3. Euclidean Volume

$\mathbb{R}^n$, the Lebesgue measure, Riemann integral, and Lebesgue integral define (assume) area/volume to be the product of domain interval lengths. The goal here is to derive the product of interval lengths from an abstract, set-based definition of volume without assuming the product of interval lengths, without the notion of a unit area/volume, and without the notions of line, angle, rectangle, etc.

Countable volume is the cardinal (number of members) of the set of all possible mappings between disjoint, countable, domain sets (the number of n-tuples), which by the rule of product, is the Cartesian product of disjoint, countable, domain set mappings.

DEFINITION 3.1. Countable Volume:

$$v_c = |\times_{i=1}^n x_i| = \prod_{i=1}^n |x_i| : \quad \bigcap_{i=1}^n x_i = \emptyset$$

THEOREM 3.2. *Euclidean volume, v, is length of the range interval, $[v_u, v_w]$, which is equal to product of domain interval lengths, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$:*

$$v = \prod_{i=1}^n s_i, \;\; v = v_w - v_u, \;\; s_i = b_i - a_i.$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of subintervals.

$$\forall\, i\, n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i|. \tag{3.1}$$

Apply the ruler convergence theorem (2.2) to equation 3.1:

$$floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0}(p_i \cdot c) = s_i. \tag{3.2}$$

$$v_c = \prod_{i=1}^n |x_i| \quad \wedge \quad |x_i| = p_i \quad \Rightarrow \quad v_c = \prod_{i=1}^n p_i. \tag{3.3}$$

Multiply both sides of equation 3.3 by c^n:

$$v_c \cdot c^n = (\prod_{i=1}^n p_i) \cdot c^n = \prod_{i=1}^n (p_i \cdot c). \tag{3.4}$$

Apply the ruler (2.1) to the range interval, $[v_u, v_w]$ (where $v = v_w - v_u$). Combine with equation 3.4. Apply the ruler convergence (2.2) and equation 3.2.

$$\begin{aligned} &\forall\, v_c, n \in \mathbb{N},\; c >= 0\; \exists\, v \in \mathbb{R} : \; floor(v/c^n) = v_c \quad \wedge \quad v_c \cdot c^n = \prod_{i=1}^n (p_i \cdot c) \\ &\quad \Rightarrow \quad v = \lim_{c\to 0} v_c \cdot c^n = \prod_{i=1}^n \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^n s_i. \quad \square \end{aligned} \tag{3.5}$$

4. Minkowski Distance

4.1. Minkowski distance.

THEOREM 4.1. *Minkowski distance: The Minkowski distance, d, is the one unique distance corresponding to an Euclidean volume, where d is the length of the range interval, $[d_u, d_v]$, which is a function of the domain interval lengths, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$:*

$$d = (\sum_{i=1}^n s_i^n)^{1/n}, \;\; d = d_v - d_u, \;\; s_i = b_i - a_i.$$

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

PROOF.
From the Euclidean volume proof (3.2):

$$\forall\, d, s_1, \cdots, s_n : \; d = s_1 = \cdots = s_n \quad \wedge \quad v = \textstyle\prod_{i=1}^n s_i \\ \Rightarrow \quad v = \textstyle\prod_{i=1}^n d = d^n \quad \Rightarrow \quad d = v^{1/n}. \tag{4.1}$$

$$\forall\, v, v_1, \cdots, v_n : \; v = \textstyle\sum_{i=1}^n v_i \quad \wedge \quad d = v^{1/n} \quad \Rightarrow \quad d = (\textstyle\sum_{i=1}^n v_i)^{1/n}. \tag{4.2}$$

$$d = (\textstyle\sum_{i=1}^n v_i)^{1/n} \;\wedge\; d = v^{1/n} \;\Rightarrow\; \exists\, s_1, \cdots, s_n : \; d = (\textstyle\sum_{i=1}^n s_i^n)^{1/n}. \tag{4.3}$$ □

4.2. Countable distance.

THEOREM 4.2. *Countable distance, d_c, is the one unique distance corresponding to a countable volume (number of all domain set mappings):*

$$d_c^n = |\times_{i=1}^n x_i| = \textstyle\prod_{i=1}^n |x_i| : \quad \bigcap_{i=1}^n x_i = \emptyset$$

PROOF. From the Minkowski distance proof 4.1:

$$d = v^{1/n} \quad \Rightarrow \quad d^n = v. \tag{4.4}$$

From the Euclidean volume proof (3.2):

$$v = \textstyle\prod_{i=1}^n s_i \quad \wedge \quad d^n = v \quad \Rightarrow \quad d^n = \textstyle\prod_{i=1}^n s_i. \tag{4.5}$$

$$d^n = \textstyle\prod_{i=1}^n s_i \quad \wedge \quad \forall\, c > 0 : \; d_c = floor(v/c), \; |x_i| = floor(s_i/c \\ \Rightarrow \quad d_c^n = \textstyle\prod_{i=1}^n |x_i|. \tag{4.6}$$ □

4.3. Metric Space. It remains to show that the Minkowski distances have all the properties of metric space. The formal proofs: symmetry, triangle_inequality, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.3. *Symmetry: $d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u)$.*

PROOF. By the commutative law of addition:

$$\forall\, p \,:\, 1 \le p \le 2, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^2 s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \tag{4.7}$$ □

THEOREM 4.4. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \le d(u, v) + d(v, w)$.

PROOF. $\forall\, p \ge 1, \quad k > 0, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \le ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.8}$$

Using Minkowski's inequality, $(a + b)^{1/p} \le a^{1/p} + b^{1/p}$:

$$(u^p + w^p)^{1/p} \le ((u^p + v^p) + (v^p + w^p))^{1/p} \le (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p}. \tag{4.9}$$ □

THEOREM 4.5. *Non-negativity: $d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \ge 0$.*

PROOF. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1], \; [a_2, b_2], \quad s_1 = b_1 - a_1, \; s_2 = b_2 - a_2, \quad \Rightarrow \quad s_1 \ge 0, \; s_2 \ge 0. \tag{4.10}$$

$$s_1 \ge 0, \; s_2 \ge 0 \quad \Rightarrow \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^2 s_i^p)^{1/p} \ge 0. \tag{4.11}$$ □

THEOREM 4.6. *Identity of Indiscernibles: $d(w, w) = 0$.*

PROOF. Apply the triangle inequality property (4.4):

$$\forall\, d(u,v) = d(v,w) = 0 \;\wedge\; d(u,w) \le d(u,v) + d(v,w) \;\Rightarrow\; d(u,w) \le 0. \tag{4.12}$$

Combine the non-negativity property (4.5) and the previous inequality (4.12):

$$d(u,w) \ge 0 \;\wedge\; d(u,w) \le 0 \;\Leftrightarrow\; 0 \le d(u,w) \le 0 \;\Rightarrow\; d(u,w) = 0. \tag{4.13}$$

Combine the result of step 4.13 and the condition, $d(u,v) = 0$, in step 4.12.

$$d(u,w) = 0 \;\wedge\; d(u,v) = 0 \;\Rightarrow\; w = v. \tag{4.14}$$

Combine the condition, $d(v,w) = 0$, in step 4.12 and the result of step 4.14.

$$d(v,w) = 0 \;\wedge\; w = v \;\Rightarrow\; d(w,w) = 0. \tag{4.15}$$

□

5. Applications to physics

5.1. Newton's gravity force equation. m_1 and m_2, are the sizes of two independent mass intervals, where each size c component of a mass interval exerts a force on each size c component of the other mass interval. If p_1 and p_2 are the number of size c components in each mass interval, then the total force, F, is equal to the total number of forces, which is proportionate to the Cartesian product, $p_1 \cdot p_2$, and proportionate to the size, c, of each component. Applying the volume proof (3.2), the total size of the Cartesian product of size c components is $p_1 c \cdot p_2 c$.

$$\begin{aligned} p_1 = floor(m_1/c) \quad \wedge \quad p_2 = floor(m_2/c) \quad \wedge \quad F := m_0 a \propto p_1 c \cdot p_2 c \\ \Rightarrow \quad F := m_0 a \propto \lim_{c \to 0} (p_1 c \cdot p_2 c) = m_1 m_2, \end{aligned} \tag{5.1}$$

where the force, F, is defined as the rest mass, m_0, times acceleration, a. **Note** that integrals have no means of directly specifying the p_1 and p_2 of size c infinitesimals. Therefore, it is difficult to use integrals to rigorously derive: $\lim_{c \to 0}(p_1 c \cdot p_2 c) = m_1 m_2$.

$$\begin{aligned} F := m_0 a = m_0 r/t_c^2 \propto m_1 m_2 \quad \wedge \quad m_0 = m_1 \quad \Rightarrow \quad r \propto m_1 \quad \Rightarrow \\ \exists\, m_G, r_c \in \mathbb{R} :\; r = (r_c/m_G) m_2, \end{aligned} \tag{5.2}$$

where: r is Euclidean distance, t_c is a unit of time, and r_c/m_G is a unit-factoring proportion ratio.

$$\begin{aligned} m_0 = m_1 \quad \wedge \quad r = (m_G/r_c) m_2 \quad \wedge \quad F = m_0 r/t_c^2 \\ \Rightarrow \quad F = m_0 r/t_c^2 = (r_c/m_G) m_1 m_2/t_c^2. \end{aligned} \tag{5.3}$$

From equation (5.2):

$$r \propto r_c \quad \wedge \quad \exists\, t \propto t_c \quad \Rightarrow \quad \exists\, t :\; t/r = t_c/r_c \quad \Rightarrow \quad t = (t_c/r_c) r. \tag{5.4}$$

$$\begin{aligned} t = (t_c/r_c) r \quad \wedge \quad F = (r_c/m_G) m_1 m_2/t_c^2 \quad \Rightarrow \\ F = (r_c/m_G)(r_c^2/t_c^2) m_1 m_2/r^2 = (r_c^3/m_G t_c^2) m_1 m_2/r^2 = G m_1 m_2/r^2, \end{aligned} \tag{5.5}$$

where the gravitational constant, $G = r_c^3/m_G t_c^2$, has the SI units: $m^3 kg^{-1} s^{-2}$.

5.2. Coulomb's charge force. q_1 and q_2, are the sizes of two independent charge intervals, where each size c component of a charge interval exerts a force on each size c component of the other charge interval. If p_1 and p_2 are the number of size c components in each charge interval, then the total force, F, is equal to the total number of forces, which is proportionate to the Cartesian product, $p_1 \cdot p_2$, and the size, c, of each component. Applying the volume proof (3.2), the total size of the Cartesian product of size c components is $p_1 c \cdot p_2 c$.

$$p_1 = floor(q_1/c) \quad \wedge \quad p_2 = floor(q_2/c) \quad \wedge \quad F \propto p_1 c \cdot p_2 c \\ \Rightarrow \quad F := m_0 a \propto (\lim_{c\to 0} p_1 c \cdot \lim_{c\to 0} p_2 c) = (q_1 q_2), \tag{5.6}$$

where the force, F, is defined as the rest mass, m_0, times acceleration, a.

$$F := m_0 a = m_0 r/t_c^2 \propto q_1 q_2 \quad \wedge \quad m_0 = (m_G/q_C)q_1 \quad \Rightarrow \quad r \propto q_1 \quad \Rightarrow \\ \exists\, q_C, r_c \in \mathbb{R} :\ r = (r_c/q_C)q_2, \tag{5.7}$$

where: r is Euclidean distance, t_c is a unit of time, m_G/q_C and q_C/r_c are unit-factoring proportion ratios.

$$m_0 = (m_G/q_C)q_1 \quad \wedge \quad r = (q_C/r_c)q_2 \quad \wedge \quad F = m_0 r/t_c^2 \\ \Rightarrow \quad F = m_0 r/t_c^2 = (m_G/q_C)(r_c/q_C)q_1 q_2/t_c^2 = (m_G r_c/q_C^2)q_1 q_2/t_c^2. \tag{5.8}$$

From equation (5.7):

$$r \propto r_c \quad \wedge \quad \exists\, t \propto t_c \quad \Rightarrow \quad \exists\, t :\ t/r = t_c/r_c \quad \Rightarrow \quad t = (t_c/r_c)r. \tag{5.9}$$

$$t = (t_c/r_c)r \quad \wedge \quad a_G = r_c/t_c^2 \quad \wedge \quad F = (m_G r_c/q_C^2)q_1 q_2/t_c^2 \quad \Rightarrow \\ F = (r_c^2/t_c^2)(m_G r_c/q_C^2)q_1 q_2/r^2 = ((m_G a_G)r_c^2/q_C^2)q_1 q_2/r^2 = k_c q_1 q_2/r^2, \tag{5.10}$$

where the charge constant, $k_C = (m_G a_G)r_c^2/q_C^2$, has the SI units: Nm^2C^{-2}.

5.3. Spacetime equations. As shown in the derivations of Newton's gravity force (5.1) and Coulomb's charge force (5.2) equations: $r = (r_c/t_c)t = ct$, where $r_c/t_c = c$ is a unit-factoring proportion ratio.

Applying the ruler to two intervals, $[0, r_1]$ and $[0, r_2]$, in two inertial (independent, non-accelerating) frames of reference, the smallest distance (and time) spanning the two intervals converges to the Euclidean distance (**??**), r.

$$r^2 = r_1^2 + r_2^2 \quad \wedge \quad r = (r_c/t_c)t = ct \quad \Rightarrow \quad (ct)^2 = r_1^2 + r_2^2 \\ \Leftrightarrow \quad r_1^2 = (ct)^2 - (x^2 + y^2 + z^2), \tag{5.11}$$

where $r_2^2 = x^2 + y^2 + z^2$, which is one form of Minkowski's flat spacetime interval equation [**Bru17**]. And the length contraction and time dilation equations also follow directly from $(ct)^2 = r_1^2 + r_2^2$, where $v = r_1/t$:

$$r_2^2 = (ct)^2 - r_1^2 \ \wedge \ r' = r_2 \ \Rightarrow \ r'^2/t^2 = c^2 - v^2 \ \Rightarrow \ r' = ct\sqrt{1 - (v/c)^2}. \tag{5.12}$$

$$r' = ct\sqrt{1 - (v/c)^2} \quad \wedge \quad r = ct \quad \Rightarrow \quad r' = r\sqrt{1 - (v/c)^2}. \tag{5.13}$$

$$r' = ct\sqrt{1 - (v/c)^2} \ \Rightarrow \ r'/c = t\sqrt{1 - (v/c)^2} \ \Rightarrow \ t = t'/\sqrt{1 - (v/c)^2}. \tag{5.14}$$

5.4. Some general relativity equations: Combining the ratio (first derivative) equations into partial differential equations: $r = (r_c/m_G)m = ct \Rightarrow (r_c/m_G)m \cdot ct = r^2 \Rightarrow m = (m_G/r_c c)r^2/t = (m_G/r_c c)rv$. For a constant mass, m, a decrease in the distance, r, between two mass centers causes a decrease in time, t, (time slows down). v is the relativistic orbital velocity at distance, r. $E = mc^2 = (m_G/r_c)r^3/t^2$. And $KE = mv^2/2 = (m_G c^2/2r_c)r$.

Given that $c = r_c/t_c \approx 3 \cdot 10^8 ms^{-1}$ and $G = r_c^3/m_G t_c^2 = (r_c/m_G)(r_c/t_c)^2 \approx 6.7 \cdot 10^{-11} m^3 kg^{-1} s^{-2} \Rightarrow r_c/m_G \approx (6.7 \cdot 10^{-11} m^3 kg^{-1} s^{-2}/(3 \cdot 10^8 m\ s^{-1})^2 \approx 7.4 \cdot 10^{-28} m\ kg^{-1}$, which can be used to quantify the constants in the previously derived equations. For example, $m = (m_G/r_c c)rv \approx (1/((7.4 \cdot 10^{-28} m\ kg^{-1})(3 \cdot 10^8 m\ s^{-1})))rv \approx (4.5 \cdot 10^{18} kg\ s\ m^{-2})rv$.

Likewise, for charge, $r = (r_c/q_C)q = ct \Rightarrow q = (q_C/r_c c)r^2/t = (q_C/r_c c)rv$, $E = qc^2 = (q_C/r_c)r^3/t^2$, and $KE = qv^2/2 = (q_C c^2/2r_c)r$. And if the ratio of an electron's mass to charge is m_G/q_C, then $m_G/q_C \approx 9.1 \cdot 10^{-31} kg/1.6 \cdot 10^{-19} C \approx 5.7 \cdot 10^{-12} kgC^{-1}$. And using Coulomb's constant in ratio form: $k_C = (r_c/t_c)^2(m_G r_c/q_C^2) \approx 9 \cdot 10^9 Nm^2C^{-2} \approx (3 \cdot 10^8 m\ s^{-1})^2(5.7 \cdot 10^{-12} kg\ C^{-1})(r_c/q_c) \Rightarrow r_c/q_C \approx 1.7 \cdot 10^5 m\ C^{-1}$. Therefore, $q = (q_C/r_c c)rv \approx (1/((1.7 \cdot 10^5 m\ C^{-1})(3 \cdot 10^8 m\ s^{-1})))rv \approx (1.9 \cdot 10^{-13} C\ s\ m^{-2})rv$.

5.5. 3 dimensional balls. Countable volume, $v_c = \prod_{i=1}^n |x_i|$, Euclidean volume, $v = \prod_{i=1}^n s_i$, countable distance, $d_c = (\prod_{i=1}^n |x_i|)^{1/n}$, and all Minkowski distances, $d = (\sum_{i=1}^n s_i^n)^{1/n}$, require that a set of domain intervals/dimensions can be assigned a *total order* ($i = 1$ to n). And the commutative properties of union, multiplication, and addition allow sequencing through each interval (dimension) in every possible order.

But, a total order is defined in terms of successor and predecessor relations. And "jumping" over set members to another member requires calculating an offset, which is implicitly sequencing via the successor and predecessor relations.

Therefore, sequencing directly via the successor and predecessor relations from one set member to every other member requires each set member to be sequentially adjacent (either a successor or predecessor) to every other member, herein referred to as a symmetry constraint. It will now be proved that coexistence of the symmetry constraint on a sequentially ordered set defines a cyclic set that contains at most 3 members, in this case, 3 ordered and symmetric dimensional balls and 3 3-dimensional balls.

Definition 5.1. Ordered geometry:

$$\forall\, i\, n \in \mathbb{N},\ i \in [1, n-1],\ \forall\, x_i \in \{x_1, \ldots, x_n\},$$
$$\mathit{successor}\ x_i = x_{i+1} \ \wedge\ \mathit{predecessor}\ x_{i+1} = x_i.$$

Definition 5.2. Symmetry Constraint (every set member is sequentially adjacent to every other member):

$$\forall\, i\, j\, n \in \mathbb{N},\ \forall\, x_i\, x_j \in \{x_1, \ldots, x_n\},\ \mathit{successor}\ x_i = x_j \ \Leftrightarrow\ \mathit{predecessor}\ x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$i = n \ \wedge\ j = 1 \ \Rightarrow\ \mathit{successor}\ x_n = x_1 \ \wedge\ \mathit{predecessor}\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry constraint (5.2):

$$i = n \ \wedge \ j = 1 \ \wedge \ \mathit{successor}\ x_i = x_j \ \Rightarrow \ \mathit{successor}\ x_n = x_1. \tag{5.15}$$

Applying the definition of the symmetry constraint (5.2) to conclusion 5.15:

$$\mathit{successor}\ x_i = x_j \ \Rightarrow \ \mathit{predecessor}\ x_j = x_i \ \Rightarrow \ \mathit{predecessor}\ x_1 = x_n. \tag{5.16}$$ □

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3).

DEFINITION 5.5. Successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (n = m+1 \leq \mathit{setsize}). \tag{5.17}$$

DEFINITION 5.6. Predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (n = m - q \geq 1). \tag{5.18}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$\mathit{Adjacent}(m, n, \mathit{setsize}) \leftarrow \mathit{Successor}(m, n, \mathit{setsize}) \vee \mathit{Predecessor}(m, n, \mathit{setsize}). \tag{5.19}$$

Prove that every member is adjacent to every other member, where $\mathit{setsize} \in \{1,\ 2,\ 3\}$:

$$\mathit{Adjacent}(1,1,1) \leftarrow \mathit{Successor}(1,1,1) \leftarrow (m = \mathit{setsize} \wedge n = 1). \tag{5.20}$$

$$\mathit{Adjacent}(1,2,2) \leftarrow \mathit{Successor}(1,2,2) \leftarrow (n = m+1 \leq \mathit{setsize}). \tag{5.21}$$

$$\mathit{Adjacent}(2,1,2) \leftarrow \mathit{Successor}(2,1,2) \leftarrow (n = \mathit{setsize} \wedge m = 1). \tag{5.22}$$

$$\mathit{Adjacent}(1,2,3) \leftarrow \mathit{Successor}(1,2,3) \leftarrow (n = m+1 \leq \mathit{setsize}). \tag{5.23}$$

$$\mathit{Adjacent}(2,1,3) \leftarrow \mathit{Predecessor}(2,1,3) \leftarrow (n = m - q \geq 1). \tag{5.24}$$

$$\mathit{Adjacent}(3,1,3) \leftarrow \mathit{Successor}(3,1,3) \leftarrow (n = \mathit{setsize} \wedge m = 1). \tag{5.25}$$

$$\mathit{Adjacent}(1,3,3) \leftarrow \mathit{Predecessor}(1,3,3) \leftarrow (m = 1 \wedge n = \mathit{setsize}). \tag{5.26}$$

$$\mathit{Adjacent}(2,3,3) \leftarrow \mathit{Successor}(2,3,3) \leftarrow (n = m+1 \leq \mathit{setsize}). \tag{5.27}$$

$$\mathit{Adjacent}(3,2,3) \leftarrow \mathit{Predecessor}(3,2,3) \leftarrow (n = m - q \geq 1). \tag{5.28}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\begin{aligned}(5.29)\quad \forall\, setsize > 3: \quad & \neg Successor(1, 3, setsize > 3) \\ & \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize).\end{aligned}$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$\begin{aligned}(5.30)\quad \forall\, setsize > 3: \quad & \neg Predecessor(1, 3, setsize > 3) \\ & \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).\end{aligned}$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$\begin{aligned}(5.31)\quad \forall\, setsize > 3: \quad & \neg Adjacent(1, 3, setsize > 3) \\ & \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square\end{aligned}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

(1) It was shown that the sums of n-volumes generate the Minkowski distances (4.1), which have the properties defining metric space (4.3). All functions derived from the sums of Euclidean n-volumes is probably a simpler and more thorough criteria of a distance metric than metric space because it would exclude more non-geometry motivated functions from being accepted as metrics.

(2) The derivations of Euclidean volume and Minkowski distances use integer numbers of dimensions. But, non-integer numbers of dimensions (fractals) are accepted in Hilbert spaces. And fractal distances have interesting geometric properties. A simple function-based definition of a dimension is where each disjoint domain set has its own independent (disjoint) range set. Would intersecting domain or range sets generate non-integer dimensions of volume and distance?

(3) The ruler (2.1) and ruler convergence (2.2) are tools to provide simpler and more rigorous derivations of some types of equations than possible with integrals, for example, the derivations of Euclidean volume, Newton's charge force, and Coulomb's charge force.

(4) Applying the volume proof (3.2) to the Cartesian product of same-sized, infinitesimal mass forces and charge forces to derive Newton's gravity force (5.1) and Coulomb's charge force (5.2) equations provide several firsts and several insights into physics.
   (a) These are the first deductive derivations. All other derivations have been empirical and inductive (not fully provable).
   (b) These are the first derivations not using the inverse square law or Gauss's divergence theorem.
   (c) These are the first derivations to show that time is proportionate to distance ($r = (r_c/t_c)t = ct$), which, combined with Euclidean distance, is used to derive the spacetime equations (5.3) without the notion of the speed of light. The derivations show for the first time

that gravity, charge force, and spacetime relativity all depend on time being proportionate to distance.

(d) These are the first derivations to show that all Euclidean distance intervals having a size, r, have proportionately sized intervals of other types (first derivative equations): $r = (r_c/q_C)q = (r_c/m_G)m = (r_c/t_c)t = ct$, where combining the first derivatives into partial differential equations allows simple derivations of some general relativity equations (5.4) without the need for integrating second derivative (spacetime curvature) tensors.

(e) These are the first derivations to show that the definition of force, $F := ma$, containing acceleration, $a = r/t_c^2$, and combined with $t = (t_c/r_c)r$, generates the inverse square law: $F = m_0 a = m_0 r/t_c^2 = (r_c/t_c)^2(m_x r_c/x_x^2)x_1 x_2/r^2 = k_x x_1 x_2/r^2$.

(f) Therefore, those versions of constants like: charge, vacuum magnetic permeability, fine structure, etc. that contain the value 4π might be incorrect because the creators assumed that the inverse square law was due to Gauss's flux divergence on the surface of a sphere having the area, $4\pi r^2$. For example, rigorously deriving a charge force constant containing 4π would require rigorously deriving $t = 2\pi^{1/2}(t_c/r_c)r$. Because combinatorics (Cartesian product mappings) generate Euclidean volume, there might be some combinatorial relation associated with distance that generates the $2\pi^{1/2}$, for example, the gamma function, $\Gamma(1/2) = \pi^{1/2}$. However, rectangular time area mapping to a proportionate rectangular geometric area, $t^2 = ((t_c/r_c)r)^2$, is a more parsimonious mapping.

(g) A state is represented by a constant value that does not vary with distance and time interval lengths. For example, the change of spin values of two quantum entangled electrons and the change of polarization of two quantum entangled photons are independent of the amount of distance and time between the entangled particles.

(5) It was proved that a totally ordered set with a symmetry constraint is a cyclic set with at most 3 members (5.3). And the definitions of geometric distance and volume both require a total order and symmetry, which provides several insights.

(a) Using Occam's razor, a cyclic set of at most 3 members is the most parsimonious explanation of only observing 3 dimensions of geometric distance and volume.

(b) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of three members (5.4), each member being an ordered and symmetric set of 3 dimensions (three balls), yielding a total of 9 ordered and symmetric dimensions of geometric space.

(c) Each ordered and symmetric ball can have at most 3 ordered and symmetric dimensions of discrete states of the same type, for example, a set of 3 binary values, 1 and -1, indicating vector orientation.

(d) Each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values of the same type per ball, for example, spin values: -1, 0, 1 per orthogonal plane in the ball.

(e) Each of the 3 possible ordered and symmetric dimensions of discrete physical states could contain an unordered collection (bag) of discrete state values. Bags are non-deterministic. For example, every time an unordered binary state is "pulled" from a bag, there is a 50 percent chance of getting one of the binary values.

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