The Set Properties Generating Geometry and Physics

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Abstract. Volume and the Minkowski distances/Lp norms (e.g., Euclidean distance) are derived from a set and limit-based foundation without referencing the primitives and relations of geometry. Adding the restrictions of a strict linear order on a set of n number of independent domain intervals that can be sequenced in all n-at-a-time orders limits the set to 3 intervals. Therefore, all other interval lengths have different types that are related to a distance interval length via unit-factoring, conversion ratios. The ratios and volume proof allow simpler derivations of Newton's gravity force and Coulomb's charge force equations without using the inverse square law or Gauss's divergence theorem. And the ratios allow simpler derivations of the Einstein-Planck and spacetime equations. All proofs are verified in Coq.

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1. Introduction

Mathematical (real) analysis can construct differential calculus from a set and limit-based foundation without the need to reference the primitives and relations of Euclidean geometry, like side, angle, slope, etc. But the Riemann and Lebesgue integrals and measure theory (for example, Hilbert spaces and the Lebesgue measure) use the Euclidean volume equation as a definition. And the vector norm and metric space use Euclidean distance and its properties as definitions [**Gol76**] [**Rud76**]. Here, these definitions are derived from a set and limit-based foundation without referencing the primitives and relations Euclidean geometry.

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"Countable volume", v_c, is defined as the cardinal of an ordered, countable set of n-tuples, where each n-tuple is a set of integer indexes into disjoint, countable, sets: $x_1, \cdots, x_n$. Therefore, the cardinal, v_c, is equal to the product of the domain set cardinals, $|x_i|$: $v_c = \prod_{i=1}^{n} |x_i|$.

Where a real value, v, is proportionate to the countable number of n-tuples, v_c, $\exists\, c \in \mathbb{R} : v = v_c \cdot c$. Proving that $\lim_{c \to 0} v_c \cdot c = \lim_{c \to 0} (\prod_{i=1}^{n} |x_i|) \cdot c$ is the Euclidean volume equation provides an alternative to second derivative-based methods, like the Laplacian and tensors, for measuring the curvature of a space. And the proof has other applications to physics.

Every Euclidean and non-Euclidean n-dimensional volume (n-volume) is a magnitude, v, which has a corresponding cuboid volume, $v = d^n$, where each domain interval has the size, $d \in \mathbb{R}$. And all volumes being instances of countable volume implies that there is also a countable cuboid volume, $v_c = d_c^n$, where each domain set has the cardinal, d_c.

An n-volume can only be the sum of n-volumes. And a countable n-volume can only be the sum of countable n-volumes. It will be proved that the L_p norms (Minkowski distances), $d = (\sum_{i=1}^{m} s_i^n)^{1/n}$, are instances of the sum of countable cuboid volumes, which have the metric space properties. All "geometric" spaces are functions of the sum of n-volumes.

Proving that if a set of n number of independent domain intervals/dimensions have the additional restriction of having a strict linear order that can sequenced in all n-at-a-time permutations (orders) limits the number of members in the set, implies that all additional intervals have a different type and are related by unit-factoring, conversion ratios. The conversion ratios allow simpler derivations of the spacetime, gravity force, charge force, and Einstein-Planck equations.

All the proofs in this article are trivial. But to ensure confidence, all the proofs have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

In order to compute areas and volumes, Riemann and Lebesgue integrals divide all intervals into the *same* number subintervals (infinitesimals, for example: dx, dy, dz), where the size of the infinitesimals *vary* with the size of the intervals. The varying size of infinitesimals makes it difficult for integrals (and differential equations) to directly express the Cartesian mappings between the p_x number of size c infinitesimals in one domain interval and the p_y number of the *same* size c infinitesimals in a different-sized domain interval. Further, using integrals that define Euclidean volume to derive Euclidean volume would be circular logic.

Therefore, a different tool is used here. A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals (infinitesimals), where each infinitesimal has the *same* size, c. The ruler is both an inner and outer measure of an interval.

Definition 2.1. Ruler measure, M: $\forall\, [a, b] \subset \mathbb{R},\ \ s = b - a\ \wedge\ c > 0\ \wedge$ $(p = \mathit{floor}(s/c)\ \vee\ p = \mathit{ceiling}(s/c))\ \wedge\ M = \sum_{i=1}^{p} c = pc$.

Theorem 2.2. *Ruler convergence:* $M = \lim_{c \to 0} pc = s$.

The formal proof, "limit_c_0_M_eq_exact_size," in the Coq file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : y \leq x, y \in \mathbb{Z}, x \in \mathbb{R}\})$:

$$\forall c > 0, p = floor(s/c) \wedge 0 \leq |floor(s/c) - s/c| < 1 \Rightarrow |p - s/c| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by c:

$$\forall c > 0, \quad |p - s/c| < 1 \quad \Rightarrow \quad |pc - s| < |c| = |c - 0|. \tag{2.2}$$

$$\begin{aligned} &\forall \epsilon = \delta \quad \wedge \quad |pc - s| < |c - 0| < \delta \\ &\quad \Rightarrow \quad |c - 0| < \delta \quad \wedge \quad |pc - s| < \delta = \epsilon \quad := \quad M = \lim_{c \to 0} pc = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}},\ 3.14_{c=10^{-2}},\ 3.141_{c=10^{-3}}, \ldots, \pi_{\lim_{c\to 0}}$.

LEMMA 2.3. $\forall n \geq 1, \quad 0 < c < 1 \quad \Rightarrow \quad \lim_{c \to 0} c^n = \lim_{c \to 0} c.$

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < c < 1 \quad \Rightarrow \quad 0 < c^n < c \quad \Rightarrow \quad |c - c^n| < |c| = |c - 0|. \tag{2.4}$$

$$\begin{aligned} &\forall \epsilon = \delta \quad \wedge \quad |c - c^n| < |c - 0| < \delta \\ &\quad \Rightarrow \quad |c - 0| < \delta \quad \wedge \quad |c - c^n| < \delta = \epsilon \quad := \quad \lim_{c \to 0} c^n = 0. \end{aligned} \tag{2.5}$$

$$\lim_{c \to 0} c^n = 0 \quad \wedge \quad \lim_{c \to 0} c = 0 \quad \Rightarrow \quad \lim_{c \to 0} c^n = \lim_{c \to 0} c. \tag{2.6}$$

$\square$

3. Euclidean Volume

DEFINITION 3.1. Countable volume, v_c is the number of Cartesian product mappings (n-tuples) between the members of n number of disjoint, countable domain sets:

$$\exists\, n, v_c \in \mathbb{N}, \quad x_1, \cdots, x_n : \quad v_c = \prod_{i=1}^n |x_i|, \quad \bigcap_{i=1}^n x_i = \emptyset$$

THEOREM 3.2. *Euclidean volume, $v = \prod_{i=1}^n s_i$, where v is the size of the range interval and s_i is the size of a domain interval, is the case of the number of n-tuples, v_c, times an interval size, c.*

$$v_c = \prod_{i=1}^n |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^n s_i, \ v = v_a - v_b, \ s_i = b_i - a_i.$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing $|x_i|$ number of size c subintervals and apply ruler convergence (2.2):

$$\forall\, i\, n \in \mathbb{N}, \ i \in [1, n], \ c > 0 \ \wedge \ floor(s_i/c) = |x_i| \ \Rightarrow \ s_i = \lim_{c \to 0}(|x_i| \cdot c). \tag{3.1}$$

$$s_i = \lim_{c \to 0}(|x_i| \cdot c) \quad \Leftrightarrow \quad \prod_{i=1}^n s_i = \prod_{i=1}^n \lim_{c \to 0}(|x_i| \cdot c). \tag{3.2}$$

$$\prod_{i=1}^n s_i = \prod_{i=1}^n \lim_{c \to 0}(|x_i| \cdot c) \quad \Leftrightarrow \quad \prod_{i=1}^n s_i = \lim_{c \to 0}\left(\prod_{i=1}^n |x_i|\right) \cdot c^n. \tag{3.3}$$

Apply lemma 2.3 to equation 3.3:

$$\begin{aligned} &\prod_{i=1}^n s_i = \lim_{c \to 0}\left(\prod_{i=1}^n |x_i|\right) \cdot c^n \quad \wedge \quad \lim_{c \to 0} c^n = \lim_{c \to 0} c \\ &\qquad\qquad \Leftrightarrow \quad \prod_{i=1}^n s_i = \lim_{c \to 0}\left(\prod_{i=1}^n |x_i|\right) \cdot c. \end{aligned} \tag{3.4}$$

Apply the ruler (2.1) and ruler convergence (2.2):

$$\exists\, v \in \mathbb{R} :\ v_c = floor(v/c) \quad \Leftrightarrow \quad v = \lim_{c\to 0} v_c \cdot c. \tag{3.5}$$

Apply the definition of the countable volume (3.1):

$$v_c = \prod_{i=1}^{n} |x_i| \quad \Leftrightarrow \quad \lim_{c\to 0} v_c \cdot c = \lim_{c\to 0}(\prod_{i=1}^{n} |x_i|) \cdot c. \tag{3.6}$$

Combine equations 3.5, 3.6, and 3.4:

$$v = \lim_{c\to 0} v_c \cdot c \quad \wedge \quad \lim_{c\to 0} v_c \cdot c = \lim_{c\to 0}(\prod_{i=1}^{n} |x_i|) \cdot c \quad \wedge \\ \lim_{c\to 0}(\prod_{i=1}^{n} |x_i|) \cdot c = \prod_{i=1}^{n} s_i \quad \Leftrightarrow \quad v = \prod_{i=1}^{n} s_i. \quad \square \tag{3.7}$$

4. Distance

4.1. Countable cuboid volume.

Definition 4.1. The countable cuboid volume, d_c^n, is the sum of m number of sets of countable cuboid volumes.

$$\forall\, n \in \mathbb{N}, \quad d_c \in \{0, \mathbb{N}\} \quad \exists\, m \in \mathbb{N}, \quad x_1, \cdots, x_m \in X, \quad \bigcap_{i=1}^{m} x_i = \emptyset : \\ d_c^n = \sum_{i=1}^{m} |x_i|^n.$$

4.2. Minkowski distance (L_p norm).

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

Theorem 4.2. *Minkowski distance (L_p norm) is an instance of the countable cuboid volume* (4.1).

$$d_c^n = \sum_{i=1}^{m} |x_i|^n \quad \Rightarrow \quad \exists\, d, s_1, \cdots, s_m \in \mathbb{R} : \quad d = (\sum_{i=1}^{m} s_i^n)^{1/n}.$$

Proof. Apply the ruler (2.1):

$$\exists\, d, s_1, \cdots, s_m \in \mathbb{R} :\ d_c = floor(d/c) \quad \wedge \quad |x_i| = floor(s_i/c). \tag{4.1}$$

Apply the ruler convergence (2.2):

$$d_c^n = \sum_{i=1}^{m} |x_i|^n \Rightarrow d^n = \lim_{c\to 0}(d_c \cdot c)^n = \lim_{c\to 0} \sum_{i=1}^{m}(|x_i| \cdot c)^n = \sum_{i=1}^{m} s_i^n. \tag{4.2}$$

$$d^n = \sum_{i=1}^{m} s_i^n \quad \Leftrightarrow \quad d = (\sum_{i=1}^{m} s_i^n)^{1/n}. \quad \square \tag{4.3}$$

4.3. Distance inequality. Proving that all Minkowski distances (L_p norms) satisfy the metric space triangle inequality requires another inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

Theorem 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N}, \ v_a, v_b \geq 0 : \ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

Proof. Expand the n-volume, $(v_a^{1/n} + v_b^{1/n})^n$, using the binomial expansion:

$$\forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq (v_a + v_b + \\ \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k}) = (v_a^{1/n} + v_b^{1/n})^n. \tag{4.4}$$

Take the n^{th} root of both sides of the inequality:

$$\forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad \square \tag{4.5}$$

4.4. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m,n \in \mathbb{N},\;\; a_i, b_i \geq 0: \;\; (\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$\begin{aligned}(4.6)\quad & \forall\, m,n \in \mathbb{N},\;\; v_a, v_b \geq 0: \quad v_a = \sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge \\ & (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad ((\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} = \\ & (\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \quad \square\end{aligned}$$

4.5. Metric Space. All Minkowski distances (L_p norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,w) \leq d(u,v) + d(v,w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(4.7)\qquad (u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}.$$

Apply the distance inequality (4.3) to the inequality 4.7:

$$\begin{aligned}(4.8)\quad & (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \;\wedge\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ & \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ \Rightarrow \quad & (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ & \Rightarrow \quad d(u,w) = (u^p + w^p)^{1/p} \leq \\ & (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u,v) + d(v,w). \quad \square\end{aligned}$$

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,v) = d(v,u).$

PROOF. By the commutative law of addition:

$$\begin{aligned}(4.9)\quad & \forall\, p : p \geq 1, \quad d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ & \Rightarrow \quad d(u,v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v,u). \quad \square\end{aligned}$$

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$(4.10)\qquad \forall\, [a_1, b_1],\; [a_2, b_2], \quad u = b_1 - a_1,\; v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\; v \geq 0.$$

$$(4.11)\qquad p \geq 1,\;\; u, v \geq 0 \quad \Rightarrow \quad d(u,v) = (u^p + v^p)^{1/p} \geq 0. \qquad \square$$

THEOREM 4.8. *Identity of Indiscernibles:* $d(u,u) = 0$.

PROOF. From the non-negativity property (4.7):

$$d(u,w) \geq 0 \quad \wedge \quad d(u,v) \geq 0 \quad \wedge \quad d(v,w) \geq 0 \\ \Rightarrow \quad \exists\, d(u,w) = d(u,v) = d(v,w) = 0. \tag{4.12}$$

$$d(u,w) = d(v,w) = 0 \quad \Rightarrow \quad u = v. \tag{4.13}$$

$$d(u,v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u,u) = 0. \tag{4.14}$$

□

5. Applications to physics

5.1. At most 3 dimensions of physical space. The following two proofs are in the physics section because limiting the domain intervals/dimensions to a strict linearly ordered set that can be sequenced in all n-at-a-time orders, is an additional restriction on volume and distance used to explain why physical space is limited to a cyclic set of 3 dimensions.

DEFINITION 5.1. Strict linearly ordered set:

$$\forall\, i\, n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i\ \in \{x_1, \ldots, x_n\}, \\ \textit{successor}\ x_i = x_{i+1}\ \ \wedge\ \textit{predecessor}\ x_{i+1} = x_i.$$

DEFINITION 5.2. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i\, j\, n \in \mathbb{N},\ \ \forall\, x_i\, x_j \in \{x_1, \ldots, x_n\},\ \ \textit{successor}\ x_i = x_j\ \ \Leftrightarrow\ \ \textit{predecessor}\ x_j = x_i.$$

THEOREM 5.3. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n\ \ \wedge\ \ j = 1\ \ \Rightarrow\ \ \textit{successor}\ x_n = x_1\ \ \wedge\ \textit{predecessor}\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (5.2):

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \textit{successor}\ x_i = x_j\ \ \Rightarrow\ \ \textit{successor}\ x_n = x_1. \tag{5.1}$$

Applying the definition of the symmetry property (5.2) to conclusion 5.1:

$$\textit{successor}\ x_i = x_j\ \ \Rightarrow\ \ \textit{predecessor}\ x_j = x_i\ \ \Rightarrow\ \ \textit{predecessor}\ x_1 = x_n. \tag{5.2}$$

□

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3).

DEFINITION 5.5. Successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize). \tag{5.3}$$

DEFINITION 5.6. Predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{5.5}$$

Prove that every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \tag{5.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \tag{5.7}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1). \tag{5.8}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.9}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1). \tag{5.11}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize). \tag{5.12}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.13}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \tag{5.14}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Successor(1, 3, setsize > 3) \\ & \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \end{aligned} \tag{5.15}$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Predecessor(1, 3, setsize > 3) \\ & \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \end{aligned} \tag{5.16}$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Adjacent(1, 3, setsize > 3) \\ & \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \end{aligned} \tag{5.17}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

5.2. Spacetime geometry. The many other derivations of the spacetime equations assume, without proof, Einstein's two postulates: 1) the laws of physics are the same in every inertial frame of reference and 2) the speed of light is a constant in every inertial frame of reference. Here, the set properties that generate physical space also generate Einstein's postulates and the spacetime equations.

From the Minkowski distance proof (4.2), every Minkowski distance, r, in 2-space is a function of two geometric distances, r_1 and r_2, one from each dimension (independent frame of reference), where: $r^2 = r_1^2 + r_2^2$. And the same applies for time, mass, charge distances (interval sizes). For example, $t^2 = t_1^2 + t_2^2$.

And it was proved that physical geometric space is limited to at most 3 dimensions (5.4). Therefore, any additional dimensions/intervals related to geometric distance and volume must have a different type, which requires a unit-factoring, conversion ratio (for example, $r = (r_c/t_c)t = (r_c/m_G)m = (r_c/q_C)q$, etc.).

$$r^2 = r_1^2 + r_2^2 \quad \wedge \quad t^2 = t_1^2 + t_2^2 \quad \wedge \quad \exists\, r_c, t_c, c \in \mathbb{N}:\ (r_c/t_c)t = ct = r \\ \Rightarrow \quad (ct)^2 = (ct_1)^2 + (ct_2)^2 = r_1^2 + r_2^2, \tag{5.18}$$

which proves that the conversion ratios are constants for distance, time, mass, charge, etc. in both frames of reference and also proves both Einstein's first and second postulates. Further, from equation 5.18:

$$r_1^2 = (ct)^2 - r_2^2 \quad \wedge \quad v = r_2/t \quad \Rightarrow \quad r_1 = ct\sqrt{1-(v/c)^2}. \tag{5.19}$$

Distance contraction by using $ct = r$ from equation 5.19:

$$r_1 = ct\sqrt{1-(v/c)^2} \quad \wedge \quad ct = r \quad \Rightarrow \quad r_1 = r\sqrt{1-(v/c)^2}. \tag{5.20}$$

Time dilation by dividing both sides of equation 5.19 by c:

$$r_1 = ct\sqrt{1-(v/c)^2} \quad \wedge \quad t' = r_1/c \quad \Rightarrow \quad t' = t\sqrt{1-(v/c)^2}. \tag{5.21}$$

If r_1 and r_2 are Euclidean distances, then by the previous set-based proof (5.4), the Euclidean distances are limited to at most 3 dimensions. And using equation 5.18:

$$(ct)^2 = r_1^2 + r_2^2 \quad \wedge \quad r_2^2 = x^2+y^2+z^2 \quad \Rightarrow \quad -r_1^2 = -(ct)^2 + (x^2+y^2+z^2). \tag{5.22}$$

Using the (-+++) form of Minkowski's flat spacetime, the size of the "event separation" interval, s, [**Bru17**] follows from equation 5.22:

$$s^2 = -r_1^2 = -(ct)^2 + (x^2+y^2+z^2). \tag{5.23}$$

5.3. Newton's gravity force equation. m_1 and m_2, are the sizes of two independent mass intervals, where each size c component of a mass interval exerts a force on each size c component of the other mass interval. If p_1 and p_2 are the number of size c components in each mass interval, then the total force, F, is equal to the total number of forces, $p_1 \cdot p_2$, and proportionate to the size, c, of each component. Applying the ruler (2.1) and volume proof (3.2), where the force, F, is defined as the rest mass, m_0, times acceleration, a:

$$p_1 = floor(m_1/c) \quad \wedge \quad p_2 = floor(m_2/c) \quad \wedge \quad F := m_0 a \propto (p_1 \cdot p_2)c \\ \Rightarrow \quad F := m_0 a \propto \lim_{c\to 0}(p_1 \cdot p_2)c = \lim_{c\to 0}(p_1 \cdot p_2)c^2 = \lim_{c\to 0} p_1 c \cdot p_2 c = m_1 m_2, \tag{5.24}$$

$$F := m_0 a := m_0 r/t^2 \propto m_1 m_2 \ \wedge \ m_0 = m_1 \ \Rightarrow \ r \propto m_2 \ \Rightarrow \\ \exists\, m_G, r_c \in \mathbb{R}:\ r = (\mathrm{d}r/\mathrm{d}m)m_2 = (r_c/m_G)m_2, \tag{5.25}$$

where: r is Euclidean distance, t is time, and r_c/m_G is a unit-factoring ratio.

$$(5.26) \quad m_0 = m_1 \quad \wedge \quad r = (m_G/r_c)m_2 \quad \wedge \quad F = m_0 r/t^2 \\ \Rightarrow \quad F = m_0 r/t^2 = (r_c/m_G)m_1 m_2/t^2.$$

Using proper (local) time in a non-accelerating frame of reference:

$$(5.27) \quad r = ct\sqrt{1-(v/c)^2} \quad \wedge \quad F = (r_c/m_G)m_1 m_2/t^2 \quad \Rightarrow \\ F = ((r_c/m_G)(c^2 - v^2))m_1 m_2/r^2 = Gm_1 m_2/r^2,$$

where the predicted gravitational constant at $v = 0$, is $G = (r_c/m_G)c^2$, which is compatible with the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$.

5.4. Coulomb's charge force. q_1 and q_2, are the sizes of two independent charge intervals, where each size c component of a charge interval exerts a force on each size c component of the other charge interval. If p_1 and p_2 are the number of size c components in each charge interval, then the total force, F, is equal to the total number of forces, $p_1 \cdot p_2$, and proportionate to the size, c, of each component. Applying the ruler (2.1) and volume proof (3.2), where the force, F, is defined as the rest mass, m_0, times acceleration, a:

$$(5.28) \quad p_1 = floor(q_1/c) \quad \wedge \quad p_2 = floor(q_2/c) \quad \wedge \quad F \propto (p_1 \cdot p_2)c \\ \Rightarrow \quad F := m_0 a \propto \lim_{c \to 0}(p_1 \cdot p_2)c = \lim_{c \to 0}(p_1 \cdot p_2)c^2 = \lim_{c \to 0} p_1 c \cdot p_2 c = q_1 q_2,$$

$$(5.29) \quad F := m_0 a := m_0 r/t^2 \propto q_1 q_2 \quad \wedge \\ m_0 = (\mathrm{d}m/\mathrm{d}q)q_1 = (m_G/q_C)q_1 \quad \Rightarrow \quad r \propto q_2 \\ \Rightarrow \quad \exists\, q_C, r_c \in \mathbb{R} : \; r = (\mathrm{d}r/\mathrm{d}q)q_2 = (r_c/q_C)q_2,$$

where: r is Euclidean distance, t is time, m_G/q_C and r_c/q_C are unit-factoring ratios.

$$(5.30) \quad m_0 = (m_G/q_C)q_1 \quad \wedge \quad r = (q_C/r_c)q_2 \quad \wedge \quad F = m_0 r/t^2 \\ \Rightarrow \quad F = m_0 r/t^2 = (m_G/q_C)(r_c/q_C)q_1 q_2/t^2 = (m_G r_c/q_C^2)q_1 q_2/t^2.$$

Using the local frame of reference, $r = ct\sqrt{1-(v/c)^2}$, and $v = 0$:

$$(5.31) \quad r = ct = (r_c/t_c)t \quad \wedge \quad a_G = r_c/t_c^2 \quad \wedge \quad F = (m_G r_c/q_C^2)q_1 q_2/t^2 \\ \Rightarrow \quad F = ((m_G r_c/q_C^2)(r_c/t_c)^2)q_1 q_2/r^2 = (m_G a_G)r_c^2/q_C^2)q_1 q_2/r^2 = k_e q_1 q_2/r^2,$$

where the predicted charge constant, $k_e = (m_G a_G)r_c^2/q_C^2$, which is compatible with the SI units: $N \cdot m^2 \cdot C^{-2}$. And where $v > 0$, $F = ((m_G r_c/q_C^2)(c^2 - v^2)q_1 q_2/r^2$.

5.5. Einstein-Planck and energy-charge equations: Combining the ratio (constant first derivative) equations: $m = (m_p/r_c)r$ and $r/t = r_c/t_c = c \Rightarrow m(ct)^2 = (m_p/r_c)r \cdot r^2$. Dividing both sides by t^2: $E = mc^2 = (m_p/r_c)r \cdot (r/t)^2 = (m_p/r_c)r \cdot (r_c/t_c)^2 = (m_p r_c c) \cdot (r/(r_c t_c)) = h \cdot \nu$, which is the Einstein-Planck equation, where, in the Planck constant, $h = m_p r_c c$, and m_p is the particle mass, and r_c is the Compton wavelength of the particle. And $\nu = r/(r_c t_c)$ is the frequency in cycles per second.

Likewise, for charge, $r = (r_c/q_C)q = (r_c/m_p)m \Rightarrow m = (m_p/q_C)q \Rightarrow E = mc^2 = (m_p/q_C)qc^2 = (m_p r_c c) \cdot (q/(q_C t_c) = h \cdot \nu$, where the frequency, $\nu = q/(q_C t_c)$.

6. Insights and implications

(1) The Euclidean volume proof (3.2) shows that "flat" space is where the ratio of a range infinitesimal size to a neighbor range infinitesimal size is one and the domain infinitesimals (for example: $\mathrm{d}t$, $\mathrm{d}m$, $\mathrm{d}q$, $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$, etc.) are all the same size, c, independent of the corresponding domain interval sizes.

(2) All volumes having corresponding cuboid volumes and all "geometric" distances being functions of the sum of volumes implies that all geometric distance functions have corresponding functions that are the sum of cuboid volumes, which are the Minkowski distances (4.2). And it was proved that the Minkowski distances have the properties of metric space (4.5). Therefore, if the definition of a complete metric space allows functions that are not a function of the sum of n-volumes, then the definition of a complete metric space is not a sufficient filter to obtain only geometric distances.

(3) Propositional logic-based proofs that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line (equation), where it is assumed that the straight line length is the smallest distance [**Joy98**]. And proofs that a straight line (equation) is the smallest distance have equated the straight line to the Euclidean distance.

    All distance measures in Euclidean space have equivalent Minkowski distances (4.2). In the Minkowski distance equation, $d = (\sum_{i=1}^{m} s_i^n)^{1/n}$, if m represents the number of domain intervals, one interval from each dimension, then $1 \le n \le m$. And $m = 2 \Rightarrow 1 \le n \le 2$, which constrains all Minkowski distances to a range from Manhattan distance (the largest distance) to Euclidean distance (the smallest distance) in Euclidean (flat) 2-space.

(4) Hilbert spaces allow fractional dimensions (fractals), which is the case of intersecting domain sets and requires generalizing the countable volume definition (3.1) to: $v_c = \prod_{i=1}^{n}(|x_i| - |x_i \bigcap (\bigcup_{j=1, i\neq j}^{n} x_j)|)$.

(5) Compare the distance sum inequality (4.4):

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \le (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

to Minkowski's sum inequality:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n)^n)^{1/n} \le (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

Note the exponent difference in the left side of the two inequalities:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \quad vs. \quad (\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n}.$$

Minkowski's sum inequality proof assumes: convexity and the L_p space inequalities (for example, Hölder's inequality or Mahler's inequality) or the triangle inequality. In contrast, the distance (sum) inequality is a more fundamental inequality that does not require the assumptions of the Minkowski sum inequality.

(6) The derivations in this article show that the gravity force, charge force, spacetime, and Einstein-Planck equations all depend on time being proportionate to distance: $r = (r_c/t_c)t = ct$ and the equations all contain

the constant, c^2. For example, from the derivation of Newton's gravity equation (5.27), where $v = 0$: $G = (r_c/m_G)c^2$. Likewise, from the derivation of Coulomb's charge force equation (5.31) the constant, where $v = 0$: $k_e = (m_G r_c/q_C^2)c^2$. And from the derivation of the Planck-Einstein equation (5.5), $E = h \cdot \nu = (m_p r_c c) \cdot (r/(r_c t_c)) = (m_p/r_c)rc(r_c/t_c) = (m_p/r_c)rc^2 = mc^2$.

(7) Applying the ruler (2.1) and volume proof (3.2) to the Cartesian product of same-sized, infinitesimal mass forces and charge forces to derive Newton's gravity force (5.3) and Coulomb's charge force (5.4) equations provide some firsts and some insights into physics:
   (a) These are the first derivations to not assume the inverse square law or Gauss's divergence theorem.
   (b) These are the first derivations to show that the definition of force, $F := m_0 a$, containing acceleration, $a = r/t^2$, where r is a distance that is proportionate to time, t, generates the inverse square law. Using the same derivation steps as for Coulomb's charge force (5.4), the inverse square law is generalized to: $F := m_0 a := m_0 r/t^2 = c^2(m_y r_c/x_y^2)x_1 x_2/r^2 = k_y x_1 x_2/r^2$.
   (c) The derived spacetime relativistic equations for gravity and charge force are: $F = G(c^2 - v^2)m_1 m_2/r^2$ (5.27) and $F = ((m_G r_c/q_C^2)(c^2 - v^2)q_1 q_2/r^2$ (5.31). Therefore, $v \to c \Rightarrow F \to 0$.
   (d) Using Occam's razor, those versions of constants like: charge, vacuum magnetic permeability, fine structure, etc. that contain the value 4π might be incorrect because those constants are based on the less parsimonious assumption that the inverse square law is due to Gauss's flux divergence on a sphere having the surface area, $4\pi r^2$.
   (e) A state is represented by a constant value. And a constant value, by definition, cannot vary with distance and time interval lengths. Therefore, the spin states (spin directions) of two quantum entangled electrons and the polarization states of two quantum entangled photons are independent of the amount of distance and time between the entangled particles.

(8) The derivations of the special relativity, gravity and charge force equations expose a unit-factoring conversion ratio (constant first derivative) principle: $r = (r_c/t_c)t = ct$, $r = (r_c/m_G)m$, and $r = (r_c/q_C)q$. Combining the ratios into equations allows a simple algebraic derivation of the Einstein-Planck equation (5.5).

(9) It was proved that sequencing through a set of n members, having a strict linear order, in all n-at-time permutations, a symmetric set, is a cyclic set with at most 3 members (5.3). And empirical observation indicates that physical space is a cyclic set of 3 dimensions.
   (a) Using Occam's razor, a strict linear order and symmetry is the most parsimonious explanation for only observing 3 dimensions of geometric distance and volume.
   (b) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (5.4), each member being an ordered and symmetric set of 3 dimensions (three balls).

(c) Each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values of the same type, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values of the same type per 3-dimensional ball, for example, vector orientation values: -1, 0, 1 per dimension in the ball.
(d) The strict linear order and symmetry of the 3 dimensions of physical space and 3 vector orientations per dimension is what makes physical space chiral (allows mirror images and clockwise/counter-clockwise rotations).
(e) Each of the 3 possible ordered and symmetric dimensions of discrete physical states could contain an unordered collection (bag) of discrete state values. Bags are non-deterministic. For example, every time an unordered binary state is "pulled" from a bag, there is a 50 percent chance of getting one of the binary values.

(10) It was shown that some fundamental geometry (volume and the Minkowski distances/L_p norms) and physics (gravity force and charge force) are derived from the combinatorial (Cartesian product) mappings between the infinitesimals of real-valued domain intervals. The proofs and derivations in this article show that the ruler (2.1) is a tool to directly express and solve such combinatorial relations that occur often in geometry, probability, physics, etc. that, in some cases, might be difficult to directly express with differential equations and integrals.

References

[Bru17] P. Bruskiewich, *A very simple introduction to special relativity: Part two - four vectors, the lorentz transformation and group velocity (the new mathematics for the millions book 38)*, Pythagoras Publishing, 2017. ↑8

[CG15] W. Conradie and V. Goranko, *Logic and discrete mathematics*, Wiley, 2015. ↑

[Coq15] Coq, *Coq proof assistant*, 2015. https://coq.inria.fr/documentation. ↑2

[Gol76] R. R. Goldberg, *Methods of real analysis*, John Wiley and Sons, 1976. ↑1

[Joy98] D. E. Joyce, *Euclid's elements*, 1998. http://aleph0.clarku.edu/~djoyce/java/elements/elements.html. ↑10

[Rud76] W. Rudin, *Principles of mathematical analysis*, McGraw Hill Education, 1976. ↑1

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