The Set Mappings Generating Geometry and Physics

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Abstract. The Euclidean volume equation is derived from a set and limit-based foundation. It is proved that that all distances that are a function of volume are Minkowski distances (for example, Manhattan and Euclidean distance), which generate the properties metric space. A simpler proof of the Minkowski inequality is also presented. The Euclidean volume proof provides simpler and more rigorous derivations of Newton's gravity force and Coulomb's charge force equations (without using the inverse square law or Gauss's divergence theorem). The derivations of the gravity and charge force equations exposes a ratio (constant first derivative) principle that allows simpler derivations of the spacetime equations and some general relativity equations. A symmetry property can limit geometric distance and volume to 3 dimensions per ball and a limit of 3 3-dimensional balls. All proofs are verified in Coq.

Contents

1. Introduction

The definitions of metric space, Euclidean distance, and area/volume in analysis [Gol76] [Rud76] mimic Euclidean geometry [Joy98]. Deriving those definitions from a set and limit-based foundation, without relying on any of the primitives and relations of Euclidean geometry, explains aspects of geometry and physics that mimicking cannot provide.

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Countable volume, v_c, is the cardinal (number of members) of the set of all possible mappings between disjoint, countable, domain sets, x_i (the number of n-tuples): $v_c = |\times_{i=1}^{n} x_i| = \prod_{i=1}^{n} |x_i|$ (where vertical bars around a set indicates the cardinal). Applying the countable set definition of volume to sets of size c infinitesimals of domain intervals generates the volume equation as $c \to 0$.

It will be shown that all distances that are a function of n-volumes are Minkowski distances (for example, Manhattan and Euclidean distance), which generate the properties metric space. The proof of the metric space triangle inequality relies on the Minkowski inequality. A simpler proof of the Minkowski inequality is presented.

A symmetry constraint on the mapping between a set of integers and a set of domain intervals/dimensions (a totally ordered set) can limit geometric distance and volume to 3 dimensions per ball and a limit of 3 3-dimensional balls.

The Euclidean volume proof is used to provide simpler and more rigorous derivations of Newton's gravity force and Coulomb's charge force equations (without using the inverse square law or Gauss's divergence theorem). The derivations of the gravity and charge forces expose a ratio (constant first derivative) principle that generates the spacetime equations and some general relativity equations.

All the proofs in this article are trivial. But, to ensure confidence in the correctness, all the proofs in this article have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

Note: In order to compute areas and volumes, integrals divide all intervals into the *same* number subintervals (infinitesimals), where the size of infinitesimals in each interval can *vary*, which makes it difficult for integrals to directly express the number of mappings between the p_x number of size c infinitesimals in one interval and the p_y number of size c infinitesimals in another interval.

In contrast to the integral, a ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals (infinitesimals), where each infinitesimal has the *same* size, c.

Definition 2.1. Ruler measure, M: $\forall\, c,\ s \in \mathbb{R},\ [a,b] \subset \mathbb{R},\ s = b-a \wedge c > 0 \wedge (p = \mathit{floor}(s/c)\ \vee\ p = \mathit{ceiling}(s/c))\ \wedge\ M = \sum_{i=1}^{p} c = pc.$

Theorem 2.2. *Ruler convergence:* $M = \lim_{c\to 0} pc = s$.

The proof is trivial but is included here for completeness. The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $\mathit{floor}(x) = \mathit{max}(\{y :\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\ p = \mathit{floor}(s/c)\ \wedge\ 0 \le |\mathit{floor}(s/c) - s/c| < 1\ \Rightarrow\ 0 \le |p - s/c| < 1. \tag{2.1}$$

Multiply all sides of inequality 2.1 by c:

$$\forall\, c > 0,\quad 0 \le |p - s/c| < 1 \quad \Rightarrow \quad 0 \le |pc - s| < |c|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \\ &\qquad \Rightarrow \quad \forall\, \epsilon = \delta :\ |c-0| < \delta\ \wedge\ |pc - s| < \epsilon\ \ :=\ \ M = \lim_{c\to 0} pc = s. \qquad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}},\ 3.14_{c=10^{-2}},\ 3.141_{c=10^{-3}}, \ldots, \pi_{\lim_{c\to 0}}$.

3. Euclidean Volume

There have been no published proofs deriving the Euclidean area/volume equation from the Cartesian product of elements (n-tuples), which is why $\mathbb{R}^n$, the Lebesgue measure, Riemann integral, Lebesgue integral, etc. define rather than derive area/volume.

The goal here is to derive the Euclidean area/volume equation from the Cartesian product of elements (n-tuples) from countable domain sets without assuming the product of interval lengths and without using the notions of a unit area/volume, line, angle, rectangle, etc.

DEFINITION 3.1. Countable Volume:

$$v_c = |\times_{i=1}^{n} x_i| = \prod_{i=1}^{n} |x_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset$$

LEMMA 3.2. $\forall\, 0 < c < 1$, $\lim_{c\to 0} c^n = \lim_{c\to 0} c$.

PROOF.

$$q > 1 \quad \wedge \quad n \geq 1 \quad \Rightarrow \quad q^n > q > 1 \quad \Rightarrow \quad 0 < 1/q^n < 1/q \tag{3.1}$$

$$0 < 1/q^n < 1/q \quad \wedge \quad c = 1/q \quad \Rightarrow \quad 0 < c^n < c. \tag{3.2}$$

$$0 < c^n < c \quad \Rightarrow \quad 0 < |c - c^n| < |c| = |c - 0|. \tag{3.3}$$

$$\begin{aligned} 0 < |c - c^n| < |c - 0| \quad &\Rightarrow \quad \forall\, \delta : \ |c - c^n| < |c - 0| < \delta \\ &\Rightarrow \quad \forall\, \epsilon = \delta : \ |c - 0| < \delta \quad \wedge \quad |c - c^n| < \epsilon := \lim_{c\to 0} c^n = 0. \end{aligned} \tag{3.4}$$

$$\lim_{c\to 0} c^n = 0 \quad \wedge \quad \lim_{c\to 0} c = 0 \quad \Rightarrow \quad \lim_{c\to 0} c^n = \lim_{c\to 0} c. \tag{3.5}$$

□

THEOREM 3.3. *Euclidean volume, v, is length of the range interval, $[v_u, v_w]$, which is equal to product of domain interval lengths, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$:*

$$v = \prod_{i=1}^{n} s_i, \ \ v = v_w - v_u, \ \ s_i = b_i - a_i.$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of subintervals.

$$\forall \ i \ n \ \in \ \mathbb{N}, \quad i \ \in \ [1, n], \quad c \ > \ 0 \quad \wedge \quad floor(s_i/c) \ = \ p_i \ = \ |x_i|. \tag{3.6}$$

Apply the ruler convergence theorem (2.2) to equation 3.6:

$$floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0}(p_i \cdot c) = s_i. \tag{3.7}$$

$$\begin{aligned} v_c = \prod_{i=1}^{n} |x_i| \quad \wedge \quad |x_i| = p_i \quad &\Rightarrow \quad v_c = \prod_{i=1}^{n} p_i \\ &\Rightarrow \quad \lim_{c\to 0} v_c \cdot c = \lim_{c\to 0}(\prod_{i=1}^{n} p_i) \cdot c. \end{aligned} \tag{3.8}$$

Note: that multiplying both sides of $v_c = \prod_{i=1}^n p_i$ by c^n in the previous step would make the logic circular. Therefore, apply lemma 3.2 to equation 3.8:

$$\lim_{c\to 0} c^n = \lim_{c\to 0} c \quad \wedge \quad \lim_{c\to 0} v_c \cdot c = \lim_{c\to 0}(\textstyle\prod_{i=1}^n p_i) \cdot c \tag{3.9}$$
$$\Rightarrow \quad \lim_{c\to 0}(v_c \cdot c) = \lim_{c\to 0}(\textstyle\prod_{i=1}^n p_i)\cdot c = \lim_{c\to 0}(\prod_{i=1}^n p_i)\cdot c^n = \lim_{c\to 0}\prod_{i=1}^n (p_i \cdot c).$$

By ruler convergence (2.2):

$$\exists\, v \in \mathbb{R} :\ v_c = floor(v/c) \quad \Rightarrow \quad v = \lim_{c\to 0}(v_c \cdot c). \tag{3.10}$$

Combine equation 3.10 with equation 3.9:

$$v = \lim_{c\to 0}(v_c \cdot c) \quad \wedge \quad \lim_{c\to 0}(v_c \cdot c) = \lim_{c\to 0}\textstyle\prod_{i=1}^n (p_i \cdot c) \tag{3.11}$$
$$\Rightarrow \quad v = \lim_{c\to 0}\textstyle\prod_{i=1}^n (p_i \cdot c).$$

Combine equation 3.7 and equation 3.11:

$$\lim_{c\to 0}(p_i \cdot c) = s_i \quad \wedge \quad v = \lim_{c\to 0}\textstyle\prod_{i=1}^n (p_i \cdot c) \quad \Rightarrow \quad v = \prod_{i=1}^n s_i. \tag{3.12}$$

□

4. Distance

Only like types can be added together. For example, only scalars can be added to a scalar and vectors added to a vector. Likewise, an n-dimensional volume (an n-volume) can only be the sum of n-volumes.

4.1. n-distance.

DEFINITION 4.1. n-distance

$$v = \textstyle\prod_{i=1}^n d = d^n \quad \Leftrightarrow \quad d = v^{1/n}.$$

4.2. Minkowski distance.

THEOREM 4.2. *Minkowski distance: All distances that are a function of volume are Minkowski distances.*

$$v = \textstyle\prod_{i=1}^n d = d^n \quad \Rightarrow \quad d = (\sum_{i=1}^n s_i^n)^{1/n}$$

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

PROOF.

$$\forall\, v, v_1, \cdots, v_m :\ v = \textstyle\sum_{i=1}^m v_i \quad \wedge \quad d = v^{1/n} \quad \Rightarrow \quad d = (\sum_{i=1}^m v_i)^{1/n}. \tag{4.1}$$

An n-volume can only be the sum of n-volumes:

$$d = (\textstyle\sum_{i=1}^m v_i)^{1/n} \ \wedge\ \exists\, s_i \in \mathbb{R} :\ s_i^n = v_i \ \Rightarrow\ d = (\sum_{i=1}^m s_i^n)^{1/n}. \tag{4.2}$$

□

4.3. Countable distance. Applying the ruler to Minkowski distances yields a countable distance, d_c.

DEFINITION 4.3. Countable distance, d_c:

$$d = (\textstyle\sum_{i=1}^m s_i^n)^{1/n} \quad \wedge \quad d_c = floor(v/c) \quad \wedge \quad |x_i| = floor(s_i/c)$$
$$\wedge \quad d_c = |x_1| = \cdots = |x_n| \quad \Rightarrow \quad d_c^n = \textstyle\sum_{i=1}^n |x_i|^n$$
$$\Leftrightarrow \quad d^n = \lim_{c\to 0}(d_c \cdot c)^n = \lim_{c\to 0}\textstyle\sum_{i=1}^m (|x_i| \cdot c)^n = \sum_{i=1}^m s_i^n$$
$$\Leftrightarrow \quad d = (\textstyle\sum_{i=1}^m s_i^n)^{1/n}.$$

4.4. Minkowski inequality. The Minkowski inequality is used in the derivation of the metric space triangle inequality. Some proofs assume the triangle inequality and, therefore, cannot be used to derive the triangle inequality. The other proofs of the inequality are complex and require a lot of effort to understand. In order to provide confidence in the derivation of the metric space triangle inequality, a simpler and easier to understand proof was developed for this article.

THEOREM 4.4. *Minkowski inequality*

$$\forall\, n \in \mathbb{N},\ a, b \geq 0:\ (a+b)^{1/n} \leq a^{1/n} + b^{1/n}$$

PROOF. Expand the denominator using the binomial expansion:

$$\begin{aligned}(4.3)\quad & (1+x)/(1+x^{1/n})^n = (1+x)/(1+x+\textstyle\sum_{i=1}^{n}\binom{n}{k}(x^{1/n})^{n-k}) \ \wedge \\ & \textstyle\sum_{i=1}^{n}\binom{n}{k}(x^{1/n})^{n-k} \geq 0 \quad \Rightarrow \quad (1+x)/(1+x+\sum_{i=1}^{n}\binom{n}{k}(x^{1/n})^{n-k}) \leq 1 \\ & \Rightarrow \quad 1+x \leq (1+x+\textstyle\sum_{i=1}^{n}\binom{n}{k}(x^{1/n})^{n-k}) = (1+x^{1/n})^n.\end{aligned}$$

$$\begin{aligned}(4.4)\quad & 1+x \leq (1+x^{1/n})^n \quad \wedge \quad x = b/a \quad \Rightarrow \quad 1+b/a \leq (1+(b/a)^{1/n})^n \\ & \Rightarrow \quad (1+b/a)^{1/n} \leq 1+(b/a)^{1/n} \\ & \Rightarrow \quad a^{1/n}(1+b/a)^{1/n} \leq a^{1/n}(1+(b/a)^{1/n}) \\ & \Rightarrow \quad (a+b)^{1/n} \leq a^{1/n} + b^{1/n}. \quad \square\end{aligned}$$

4.5. Metric Space. The properties of metric space have been motivated by Euclidean distance. But, all Minkowski distances generate the properties of metric space. The formal proofs: symmetry, triangle_inequality, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,v) = d(v,u)$.

PROOF. By the commutative law of addition:

$$\begin{aligned}(4.5)\quad & \forall\, p : 1 \leq p \leq 2, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ & \Rightarrow \quad d(u,v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v,u). \quad \square\end{aligned}$$

THEOREM 4.6. *Triangle Inequality:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,w) \leq d(u,v) + d(v,w)$.

PROOF. $\forall\, p \geq 1, \quad k > 0, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(4.6)\qquad (u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}.$$

Using Minkowski's inequality (4.4), $(a+b)^{1/p} \leq a^{1/p} + b^{1/p}$:

$$(4.7)\quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p}. \quad \square$$

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,w) \geq 0$.

PROOF. By definition, the length of an interval is always ≥ 0:

$$(4.8)\qquad \forall\, [a_1, b_1],\ [a_2, b_2], \quad s_1 = b_1 - a_1,\ s_2 = b_2 - a_2, \quad \Rightarrow \quad s_1 \geq 0,\ s_2 \geq 0.$$

$$(4.9)\qquad s_1 \geq 0,\ s_2 \geq 0 \quad \Rightarrow \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} \geq 0. \quad \square$$

THEOREM 4.8. *Identity of Indiscernibles:* $d(w,w) = 0$.

PROOF. Apply the triangle inequality property (4.6):

$$\forall\, d(u,v) = d(v,w) = 0 \;\wedge\; d(u,w) \leq d(u,v) + d(v,w) \;\Rightarrow\; d(u,w) \leq 0. \tag{4.10}$$

Combine the non-negativity property (4.7) and the previous inequality (4.10):

$$d(u,w) \geq 0 \;\wedge\; d(u,w) \leq 0 \;\Leftrightarrow\; 0 \leq d(u,w) \leq 0 \;\Rightarrow\; d(u,w) = 0. \tag{4.11}$$

Combine the result of step 4.11 and the condition, $d(u,v) = 0$, in step 4.10.

$$d(u,w) = 0 \;\wedge\; d(u,v) = 0 \;\Rightarrow\; w = v. \tag{4.12}$$

Combine the condition, $d(v,w) = 0$, in step 4.10 and the result of step 4.12.

$$d(v,w) = 0 \;\wedge\; w = v \;\Rightarrow\; d(w,w) = 0. \tag{4.13}$$ □

5. Applications to physics

5.1. Newton's gravity force equation. m_1 and m_2, are the sizes of two independent mass intervals, where each size c component of a mass interval exerts a force on each size c component of the other mass interval. If p_1 and p_2 are the number of size c components in each mass interval, then the total force, F, is equal to the total number of forces, which is proportionate to the Cartesian product, $p_1 \cdot p_2$, and proportionate to the size, c, of each component. Applying the volume proof (3.3), the total size of the Cartesian product of size c components is $p_1c \cdot p_2c$.

$$\begin{aligned} p_1 = floor(m_1/c) \quad \wedge \quad p_2 = floor(m_2/c) \quad \wedge \quad F := m_0 a \propto p_1 c \cdot p_2 c \\ \Rightarrow \quad F := m_0 a \propto \lim_{c \to 0}(p_1 c \cdot p_2 c) = m_1 m_2, \end{aligned} \tag{5.1}$$

where the force, F, is defined as the rest mass, m_0, times acceleration, a.

Note that integrals have no means of directly specifying the p_1 and p_2 of size c infinitesimals. Therefore, it is difficult to use integrals to rigorously derive: $\lim_{c\to 0}(p_1c \cdot p_2c) = m_1m_2$.

$$\begin{aligned} F := m_0 a = m_0 \mathrm{d}r/\mathrm{d}t^2 = m_0 r/t^2 \propto m_1 m_2 \;\wedge\; m_0 = m_1 \;\Rightarrow\; r \propto m_1 \;\Rightarrow \\ \exists\, m_G, r_c \in \mathbb{R} :\; r = (\mathrm{d}r/\mathrm{d}m) m_2 = (r_c/m_G) m_2, \end{aligned} \tag{5.2}$$

where: r is Euclidean distance, t is time, and r_c/m_G is a unit-factoring proportion ratio.

$$\begin{aligned} m_0 = m_1 \quad \wedge \quad r = (m_G/r_c) m_2 \quad \wedge \quad F = m_0 r/t^2 \\ \Rightarrow \quad F = m_0 r/t^2 = (r_c/m_G) m_1 m_2/t^2. \end{aligned} \tag{5.3}$$

From equation (5.3):

$$a = \mathrm{d}r/\mathrm{d}t^2 = r/t^2 \;\Rightarrow\; \exists\, t_c, r_c \in \mathbb{R} :\; t/r = (\mathrm{d}t/\mathrm{d}r) = t_c/r_c \;\Rightarrow\; t = (t_c/r_c) r. \tag{5.4}$$

$$\begin{aligned} t = (t_c/r_c) r \quad \wedge \quad F = (r_c/m_G) m_1 m_2/t^2 \quad \Rightarrow \\ F = (r_c/m_G)(r_c^2/t_c^2) m_1 m_2/r^2 = (r_c^3/m_G t_c^2) m_1 m_2/r^2 = G m_1 m_2/r^2, \end{aligned} \tag{5.5}$$

where the gravitational constant, $G = r_c^3/m_G t_c^2$, has the SI units: $m^3 kg^{-1} s^{-2}$.

5.2. Coulomb's charge force. q_1 and q_2, are the sizes of two independent charge intervals, where each size c component of a charge interval exerts a force on each size c component of the other charge interval. If p_1 and p_2 are the number of size c components in each charge interval, then the total force, F, is equal to the total number of forces, which is proportionate to the Cartesian product, $p_1 \cdot p_2$, and the size, c, of each component. Applying the volume proof (3.3), the total size of the Cartesian product of size c components is $p_1 c \cdot p_2 c$.

$$(5.6) \quad p_1 = floor(q_1/c) \quad \wedge \quad p_2 = floor(q_2/c) \quad \wedge \quad F \propto p_1 c \cdot p_2 c \\ \Rightarrow \quad F := m_0 a \propto (\lim_{c\to 0} p_1 c \cdot \lim_{c\to 0} p_2 c) = (q_1 q_2),$$

where the force, F, is defined as the rest mass, m_0, times acceleration, a.

$$(5.7) \quad F := m_0 a = m_0 \mathrm{d}r/\mathrm{d}t^2 = m_0 r/t^2 \propto q_1 q_2 \quad \wedge \\ m_0 = (\mathrm{d}m/\mathrm{d}q)q_1 = (m_G/q_C)q_1 \quad \Rightarrow \quad r \propto q_1 \\ \Rightarrow \quad \exists\, q_C, r_c \in \mathbb{R} : \; r = (\mathrm{d}r/\mathrm{d}q)q_2 = (r_c/q_C)q_2,$$

where: r is Euclidean distance, t is time, m_G/q_C and r_c/q_C are unit-factoring proportion ratios.

$$(5.8) \quad m_0 = (m_G/q_C)q_1 \quad \wedge \quad r = (q_C/r_c)q_2 \quad \wedge \quad F = m_0 r/t^2 \\ \Rightarrow \quad F = m_0 r/t^2 = (m_G/q_C)(r_c/q_C)q_1 q_2/t^2 = (m_G r_c/q_C^2)q_1 q_2/t^2.$$

From equation (5.7):

$$(5.9) \quad a = \mathrm{d}r/\mathrm{d}t^2 = r/t^2 \;\Rightarrow\; \exists\, t_c, r_c \in \mathbb{R} : \; t/r = (\mathrm{d}t/\mathrm{d}r) = t_c/r_c \;\Rightarrow\; t = (t_c/r_c)r.$$

$$(5.10) \quad t = (t_c/r_c)r \quad \wedge \quad a_G = r_c/t_c^2 \quad \wedge \quad F = (m_G r_c/q_C^2)q_1 q_2/t^2 \quad \Rightarrow \\ F = (r_c^2/t_c^2)(m_G r_c/q_C^2)q_1 q_2/r^2 = ((m_G a_G)r_c^2/q_C^2)q_1 q_2/r^2 = k_c q_1 q_2/r^2,$$

where the charge constant, $k_C = (m_G a_G)r_c^2/q_C^2$, has the SI units: Nm^2C^{-2}.

5.3. Spacetime equations. As shown in the derivations of Newton's gravity force (5.1) and Coulomb's charge force (5.2) equations: $r = (r_c/t_c)t = ct$, where $r_c/t_c = c$ is a unit-factoring proportion ratio. And, the smallest distance (and time) spanning the two inertial (independent, non-accelerating) frames of reference, $[0, r_1]$ and $[0, r_2]$, is the Euclidean distance, r.

$$(5.11) \qquad r = ct \quad \Rightarrow \quad (ct)^2 = r_1^2 + r_2^2 \quad \Leftrightarrow \quad r_1^2 = (ct)^2 - (x^2 + y^2 + z^2),$$

where $r_2^2 = x^2 + y^2 + z^2$, which is one form of Minkowski's flat spacetime interval equation [**Bru17**]. And the length contraction and time dilation equations also follow directly from $(ct)^2 = r_1^2 + r_2^2$, where $v = r_1/t$:

$$(5.12) \quad r_2^2 = (ct)^2 - r_1^2 \;\wedge\; r' = r_2 \;\Rightarrow\; r'^2/t^2 = c^2 - v^2 \;\Rightarrow\; r' = ct\sqrt{1-(v/c)^2}.$$

$$(5.13) \qquad r' = ct\sqrt{1-(v/c)^2} \quad \wedge \quad r = ct \quad \Rightarrow \quad r' = r\sqrt{1-(v/c)^2}.$$

$$(5.14) \quad r' = ct\sqrt{1-(v/c)^2} \;\Rightarrow\; t' = r'/c = t\sqrt{1-(v/c)^2} \;\Rightarrow\; t = t'/\sqrt{1-(v/c)^2}.$$

5.4. Some general relativity equations: Combining the ratio (constant first derivative) equations into partial differential equations: $r = (r_c/m_G)m = ct \Rightarrow (r_c/m_G)m \cdot ct = r^2 \Rightarrow m = (m_G/r_c c)r^2/t = (m_G/r_c c)rv$. For a constant mass, m, a decrease in the distance, r, between two mass centers causes a decrease in time, t, (time slows down). v is the relativistic orbital velocity at distance, r. $(r_c/m_G)m \cdot (ct)^2 = r^3 \Rightarrow E = mc^2 = (m_G/r_c)r^3/t^2$. And $(ct)^2 = r^2 \Rightarrow c^2 = v^2 \Rightarrow (r_c/m_G)mv^2 = c^2 r \Rightarrow KE = mv^2/2 = (m_G c^2/2r_c)r$.

Given that $c = r_c/t_c \approx 3 \cdot 10^8 ms^{-1}$ and $G = r_c^3/m_G t_c^2 = (r_c/m_G)(r_c/t_c)^2 \approx 6.7 \cdot 10^{-11} m^3 kg^{-1} s^{-2} \Rightarrow r_c/m_G \approx (6.7 \cdot 10^{-11} m^3 kg^{-1} s^{-2}/(3 \cdot 10^8 m\ s^{-1})^2 \approx 7.4 \cdot 10^{-28} m\ kg^{-1}$, which can be used to quantify the constants in the previously derived equations. For example, $m = (m_G/r_c c)rv \approx (1/((7.4 \cdot 10^{-28} m\ kg^{-1})(3 \cdot 10^8 m\ s^{-1})))rv \approx (4.5 \cdot 10^{18} kg\ s\ m^{-2})rv$.

Likewise, for charge, $r = (r_c/q_C)q = ct \Rightarrow q = (q_C/r_c c)r^2/t = (q_C/r_c c)rv$, $E = qc^2 = (q_C/r_c)r^3/t^2$, and $KE = qv^2/2 = (q_C c^2/2r_c)r$. And if the ratio of an electron's mass to charge is m_G/q_C, then $m_G/q_C \approx 9.1 \cdot 10^{-31} kg/1.6 \cdot 10^{-19} C \approx 5.7 \cdot 10^{-12} kgC^{-1}$. And using Coulomb's constant in ratio form: $k_C = (r_c/t_c)^2(m_G r_c/q_C^2) \approx 9 \cdot 10^9 Nm^2C^{-2} \approx (3 \cdot 10^8 m\ s^{-1})^2(5.7 \cdot 10^{-12} kg\ C^{-1})(r_c/q_c) \Rightarrow r_c/q_C \approx 1.7 \cdot 10^5 m\ C^{-1}$. Therefore, $q = (q_C/r_c c)rv \approx (1/((1.7 \cdot 10^5 m\ C^{-1})(3 \cdot 10^8 m\ s^{-1})))rv \approx (1.9 \cdot 10^{-13} C\ s\ m^{-2})rv$.

5.5. 3 dimensional balls. Countable volume, $v_c = \prod_{i=1}^n |x_i|$, Euclidean volume, $v = \prod_{i=1}^n s_i$, and all Minkowski distances, $d = (\sum_{i=1}^n s_i^n)^{1/n}$, require that a set of domain intervals/dimensions can be assigned a *total order*. A total order is defined in terms of successor and predecessor relations, where, in this case, the successor and predecessor relations are specified by the integers $i = 1$ to n that map to a set of domain intervals/dimensions.

But the commutative properties of union, multiplication, and addition allow sequencing through each interval (dimension) in every possible order. And "jumping" (indexing) over set members to another member requires calculating an offset, which is implicitly sequencing via the successor and predecessor relations.

Therefore, sequencing directly via the successor and predecessor relations from one set member to every other member requires each set member to be sequentially adjacent (either a successor or predecessor) to every other member, herein referred to as a symmetry constraint. It will now be proved that coexistence of the symmetry constraint on a sequentially ordered set defines a cyclic set that contains at most 3 members, in this case, 3 dimensions per ball and 3 3-dimensional balls.

Definition 5.1. Ordered geometry:

$$\forall\, i\, n \in \mathbb{N},\ i \in [1, n-1],\ \forall\, x_i\ \in \{x_1, \ldots, x_n\},$$
$$\textit{successor}\ x_i = x_{i+1}\ \wedge\ \textit{predecessor}\ x_{i+1} = x_i.$$

Definition 5.2. Symmetry Constraint (every set member is sequentially adjacent to every other member):

$$\forall\, i\, j\, n \in \mathbb{N},\ \forall\, x_i\, x_j \in \{x_1, \ldots, x_n\},\ \textit{successor}\ x_i = x_j\ \Leftrightarrow\ \textit{predecessor}\ x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$i = n\ \wedge\ j = 1\ \Rightarrow\ \textit{successor}\ x_n = x_1\ \wedge\ \textit{predecessor}\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry constraint (5.2):

$$i = n \ \wedge \ j = 1 \ \wedge \ \textit{successor}\ x_i = x_j \ \Rightarrow \ \textit{successor}\ x_n = x_1. \tag{5.15}$$

Applying the definition of the symmetry constraint (5.2) to conclusion 5.15:

$$\textit{successor}\ x_i = x_j \ \Rightarrow \ \textit{predecessor}\ x_j = x_i \ \Rightarrow \ \textit{predecessor}\ x_1 = x_n. \tag{5.16}$$ □

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3).

DEFINITION 5.5. Successor of m is n:

$$\textit{Successor}(m, n, \textit{setsize}) \leftarrow (m = \textit{setsize} \wedge n = 1) \vee (n = m + 1 \leq \textit{setsize}). \tag{5.17}$$

DEFINITION 5.6. Predecessor of m is n:

$$\textit{Predecessor}(m, n, \textit{setsize}) \leftarrow (m = 1 \wedge n = \textit{setsize}) \vee (n = m - q \geq 1). \tag{5.18}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$\textit{Adjacent}(m, n, \textit{setsize}) \leftarrow \textit{Successor}(m, n, \textit{setsize}) \vee \textit{Predecessor}(m, n, \textit{setsize}). \tag{5.19}$$

Prove that every member is adjacent to every other member, where $\textit{setsize} \in \{1,\ 2,\ 3\}$:

$$\textit{Adjacent}(1, 1, 1) \leftarrow \textit{Successor}(1, 1, 1) \leftarrow (m = \textit{setsize} \wedge n = 1). \tag{5.20}$$

$$\textit{Adjacent}(1, 2, 2) \leftarrow \textit{Successor}(1, 2, 2) \leftarrow (n = m + 1 \leq \textit{setsize}). \tag{5.21}$$

$$\textit{Adjacent}(2, 1, 2) \leftarrow \textit{Successor}(2, 1, 2) \leftarrow (n = \textit{setsize} \wedge m = 1). \tag{5.22}$$

$$\textit{Adjacent}(1, 2, 3) \leftarrow \textit{Successor}(1, 2, 3) \leftarrow (n = m + 1 \leq \textit{setsize}). \tag{5.23}$$

$$\textit{Adjacent}(2, 1, 3) \leftarrow \textit{Predecessor}(2, 1, 3) \leftarrow (n = m - q \geq 1). \tag{5.24}$$

$$\textit{Adjacent}(3, 1, 3) \leftarrow \textit{Successor}(3, 1, 3) \leftarrow (n = \textit{setsize} \wedge m = 1). \tag{5.25}$$

$$\textit{Adjacent}(1, 3, 3) \leftarrow \textit{Predecessor}(1, 3, 3) \leftarrow (m = 1 \wedge n = \textit{setsize}). \tag{5.26}$$

$$\textit{Adjacent}(2, 3, 3) \leftarrow \textit{Successor}(2, 3, 3) \leftarrow (n = m + 1 \leq \textit{setsize}). \tag{5.27}$$

$$\textit{Adjacent}(3, 2, 3) \leftarrow \textit{Predecessor}(3, 2, 3) \leftarrow (n = m - q \geq 1). \tag{5.28}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\forall\ setsize > 3: \quad \neg Successor(1, 3, setsize > 3) \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.29}$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$\forall\ setsize > 3: \quad \neg Predecessor(1, 3, setsize > 3) \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \tag{5.30}$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$\forall\ setsize > 3: \quad \neg Adjacent(1, 3, setsize > 3) \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \tag{5.31}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

(1) It was shown that all distances that are a function of volume are Minkowski distances (4.2). And the Minkowski distances have the properties defining metric space (4.5). The criteria of a distance measures being functions equivalent to a Minkowski distance (or all functions derived from volume) filters out functions that the less strict criteria of metric space would allow.

(2) A metric in the form $\mathrm{d}(x, y)$ is usually interpreted as the distance between two points, x and y. The derivation of the properties of metric space from the Minkowski distances indicates a more correct interpretation of $\mathrm{d}(x, y)$ is the distance spanning the domain sets (intervals) having the sizes, x and y.

(3) The derivations of Euclidean volume and Minkowski distances use integer numbers of dimensions because the domain sets are disjoint. But, non-integer numbers of dimensions (fractals) are accepted in Hilbert spaces. And fractal distances have interesting geometric properties. Intersecting domain sets would generate fractal dimensions of volume and distance.

(4) Proofs that Euclidean distance is the smallest distance between two distinct points have relied on equating Euclidean distance to the length of a straight line [**Joy98**], where it is assumed that the length of a straight line is the smallest distance between two distinct points (or to the length of the range set, $\Delta y : \Delta y = m\Delta x + b$, where it is assumed that $y = mx + b$ is the equation of a straight line). The derivation of the Minkowski distances provides a set and limit-based explanation of why Euclidean distance is smallest distance between two distinct points.

The derivation of countable distance, d_c, from the Minkowski distance (4.3) exposes the countable domain-to-domain set mappings that generate distance. Each countable domain set, x_i, has a corresponding range set having the size of that domain set's n-volume, $|x_i|^n$.

Flat space (rectilinear distance) ranges from a bijective mapping (1-1 correspondence) of each domain set with itself that generates Manhattan distance, $d = (a^1+b^1)^{1/1} = a+b$, to the Cartesian product of mappings of each domain set with itself that generates Euclidean distance, $d = (a^2 + b^2)^{1/2}$, where the exponent, 1/2 represents the amount of intersection of the n-volumes. The largest possible intersection of n-volumes in flat space is why the Euclidean distance is the smallest possible distance spanning domain sets.

(5) As shown in the derivations of Euclidean volume, Newton's gravity force, and Coulomb's charge force, the ruler (2.1) is a tool to directly express some counting relations in geometry, probability, physics, etc. that is difficult with integrals. And the ruler convergence (2.2) allows more rigorous derivation of real-valued relations from those counting relations.

(6) Applying the volume proof (3.3) to the Cartesian product of same-sized, infinitesimal mass forces and charge forces to derive Newton's gravity force (5.1) and Coulomb's charge force (5.2) equations provide several firsts and some insights into physics.
   (a) These are the first deductive derivations of the gravity and charge forces. All other derivations have been empirical and inductive (not fully provable).
   (b) These are the first derivations not using the inverse square law or Gauss's divergence theorem.
   (c) These are the first derivations to show that the definition of force, $F := m_0 a$, containing acceleration, $a = \mathrm{d}r/\mathrm{d}t^2 = r/t^2 \;\Rightarrow\; \exists\, t_c, r_c \in \mathbb{R} : t/r = (\mathrm{d}t/\mathrm{d}r) = t_c/r_c \;\Rightarrow\; r = (\mathrm{d}r/\mathrm{d}t)t = (r_c/t_c)t$, generates the inverse square law:
   $F := m_0 a = m_0 r/t^2 = (r_c/t_c)^2(m_x r_c/x_x^2)x_1 x_2/r^2 = k_x x_1 x_2/r^2$.
   (d) Therefore, those versions of constants like: charge, vacuum magnetic permeability, fine structure, etc. that contain the value 4π are probably incorrect because those constants are based on the less parsimonious assumption that the inverse square law is due to Gauss's flux divergence on a sphere having the surface area, $4\pi r^2$.
   (e) These are the first derivations to show that time is proportionate to distance: $r = (\mathrm{d}r/\mathrm{d}t)t = (r_c/t_c)t = ct$, which is used to derive the spacetime equations (5.3) without the notion of the speed of light. The derivations show for the first time that gravity, charge force, spacetime, and general relativity all depend on time being proportionate to distance.
   (f) The derivations to show that all Euclidean distance intervals having a size, r, have proportionately sized intervals of other types (constant first derivative equations): $r = (\mathrm{d}r/\mathrm{d}q)q = (r_c/q_C)q = (\mathrm{d}r/\mathrm{d}m)m = (r_c/m_G)m = (\mathrm{d}r/\mathrm{d}t)t = (r_c/t_c)t = ct$, where combining the constant first derivatives (ratios) into partial differential equations allows simple derivations of some general relativity equations (5.4) without the need for integrating second derivative (spacetime curvature) tensors.
   (g) A state is represented by a constant value that does not vary with distance and time interval lengths. For example, the change of spin values of two quantum entangled electrons and the change of polarization of two quantum entangled photons are independent of the amount of distance and time between the entangled particles.

(7) It was proved that a totally ordered set with a symmetry constraint is a cyclic set with at most 3 members (5.3). And the definitions of geometric distance and volume both require a total order and symmetry, which provides several insights.
   (a) Using Occam's razor, a cyclic set of at most 3 members is the most parsimonious explanation of only observing 3 dimensions of geometric distance and volume.
   (b) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (5.4), each member being an ordered and symmetric set of 3 dimensions (three balls), yielding a total of at most 9 ordered and symmetric dimensions of geometric space.
   (c) Each ordered and symmetric ball can have at most 3 ordered and symmetric dimensions of discrete states of the same type.
   (d) Each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values of the same type per ball, for example, vector orientation values: -1, 0, 1 per orthogonal direction in the ball.
   (e) Each of the 3 possible ordered and symmetric dimensions of discrete physical states could contain an unordered collection (bag) of discrete state values. Bags are non-deterministic. For example, every time an unordered binary state is "pulled" from a bag, there is a 50 percent chance of getting one of the binary values.

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