Some Set Properties Underlying Geometry and Physics

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Abstract. The Euclidean volume, inner product, and Minkowski distance equations are proved to be instances of a set of n-tuples. A symmetry property is proved to limit a cyclic set to at most 3 members. Where distance is a symmetric cyclic set, higher dimensions have different types (are members of different sets), with unit-factoring ratios of a distance unit to units of other types (time, mass, and charge). The proofs and ratios are used to derive the gravity, charge, vacuum permittivity, vacuum permeability, Planck, and fine structure constants, and used to provide simple derivations of well-known gravity, charge, electromagnetic equations, special and general relativity equations, and quantum physics equations. All the proofs are verified in Rocq.

Contents

1. Introduction

The Riemann integral, Lebesgue integral, and Lebesgue measure define Euclidean volume as the Cartesian product of interval sizes. And analysis defines Euclidean distance, inner product, vector, and metric space, etc. [**Gol76**] [**Rud76**]

If volume and distance equations can be derived from an abstract, set-based foundation, then that foundation might explain: 1) the properties that generate those equations in a way that geometry and measure theory have not explained, 2) show whether heuristic definitions, like metric space, are sufficient or flawed, and 3) provide new insights into both geometry and physics. In this article, volume and distance equations are derived from a single, abstract, combinatorial foundation, a

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foundation of countable, ordered sets of combinations (n-tuples) and permutations of set members.

Where $|x_i|$ is the cardinal of a countable set, x_i, the countable number of n-tuples, v_c, of disjoint sets x_i is:

$$\forall\, x_i \in \{x_1, \cdots, x_n\}, \quad \bigcap_{i=1}^{n} x_i = \emptyset : \quad v_c = \prod_{i=1}^{n} |x_i|. \tag{1.1}$$

Step 1: Prove that the *only* equation implied by the countable set of n-tuples is the Euclidean volume equation and the *only* countable set implied by the Euclidean distance equation is the countable set of n-tuples:

$$\forall\, v_c = \prod_{i=1}^{n} |x_i| \quad \Leftrightarrow \quad v = \prod_{i=1}^{n} s_i, \quad [a_i, b_i] \subset \mathbb{R}, \quad s_i = b_i - a_i. \tag{1.2}$$

A "ruler" measure of all $[a_i, b_i]$ will be used to prove proposition 1.2.

Step 2: Prove that, where the *total* countable volume, v_c, is a bijective function (for each n, $\exists!\, d_c : v_c = f(d_c)$ and $d_c = f^{-1}(v_c)$), v_c implies specific real-valued distance equations in an Euclidean volume. Where:

$$\exists\, d_c, v_c, |x_i| \in \{0, \mathbb{N}\} : \ v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n, \tag{1.3}$$

the ruler measure will be used to prove that:

$$d_c^n = \sum_{i=1}^{m} v_{c_i} = \sum_{i=1}^{m} \left(\prod_{j=1}^{n} |x_{i,j}|\right) \ \Leftrightarrow \ d^n = \sum_{i=1}^{m} v_i = \sum_{i=1}^{m} \left(\prod_{j=1}^{n} s_{i,j}\right). \tag{1.4}$$

Where the total n-volume is both the sum and subtraction of n-volumes, the domain values are $\pm$ signed. The $n = 2$ case is the vector inner product:

$$d^2 = \sum_{i=1}^{m} a_i b_i := \mathbf{a} \cdot \mathbf{b}. \tag{1.5}$$

Where $v_{c_i} = d_{c_i}^n$, the ruler measure will be used to prove that:

$$d_c^n = \sum_{i=1}^{m} v_{c_i} = \sum_{i=1}^{m} d_{c_i}^n \quad \Leftrightarrow \quad d^n = \sum_{i=1}^{m} d_i^n. \tag{1.6}$$

d is the ρ-norm (Minkowski distance) [**Min53**], which will be proved to imply the metric space properties [**Rud76**].

Step 3: Prove there are combinatorial properties that can limit a domain set to $n \leq 3$ members:

The commutative properties of multiplication of domain values (for example, $s_1 \cdot s_2 = s_2 \cdot s_1$) allow sequencing an ordered set of domain values in all $n!$ permutations. And there is no intrinsic property of a domain value that gives it a particular position of first, second, $\cdots$, last in the multiplication sequence.

The *only* sequentially ordered set, where any member can be selected first, is a cyclic set. Sequencing a cyclic set in all $n!$ permutations, is defined, here, as a "symmetric" cyclic set, where every set member is either an *immediate* cyclic successor or an *immediate* cyclic predecessor to every other set member. Using first-order logic, a symmetric cyclic set will be proved to have $n \leq 3$ members.

If $\{s_1, s_2, s_3\}$ is a symmetric cyclic set of 3 intervals $\subset \mathbb{R}$, then another interval $\subset \mathbb{R}$, must have a different type (s_4 is a member of a different set). If s_1 having length r is divided into μ-sized subintervals (units) and s_4 having length τ is divided into ν-sized subintervals (units), then: $r = (\mu/\nu)\tau$.

Step 4: Apply the previous results to physics:

For each unit length, r_c, of distance interval length, r, there are unit lengths: t_c of time interval length, t; m_c of mass interval length, m; and q_c of charge interval length, q, such that: $r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q$. For example, $r_c/t_c = c$ is the speed of light ratio.

The proofs and the 3 direct proportion ratios are used to provide simple derivations of the Newton, Gauss, and Poisson gravity equations, Coulomb charge, Gauss, and Faraday electromagnetic equations [and the constants: gravity (G), charge (k_e), vacuum permittivity (ε_0), and vacuum permeability (μ_0)]. They are also used to derive all the special relativity equations, the Schwarzchild time dilation and black hole metric equations pointing to a simplified method of finding solutions to Einstein's general relativity equations.

Next, algebraic manipulation of the 3 direct proportion ratios yields 3 inverse proportion ratios: $r = t_c r_c/t = m_c r_c/m = q_c r_c/q$. The direct and inverse proportion ratios are combined to derive quantitative values for the 4 quantum units: r_c, t_c, m_c, and q_c. And the Planck units and the fine structure constant, α, are each derived from the 4 quantum units as the ratios of subtypes.

The combination of the direct and inverse proportion ratios are used to derive the Planck relation, the Planck constant, h, the Compton, Schrödenger, and Dirac wave equations. And finally, the inverse proportion ratios are also used to add quantum extensions to some general relativity and classical physics equations.

All the proofs in this article have been verified using using the Rocq proof verification system [**Roc25**]. The formal proofs are in the Rocq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of size κ subintervals. The ruler is both an inner and outer measure of an interval.

Definition 2.1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\, [a, b] \subset \mathbb{R}$, $s = b - a \;\; \wedge \;\; 0 < \kappa \leq 1 \;\; \wedge \;\; (p = floor(s/\kappa) \;\; \vee \;\; p = ceiling(s/\kappa))$.

Theorem 2.2. *Ruler convergence:* $M = \lim_{\kappa \to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : \; y \leq x, \; y \in \mathbb{Z}, \; x \in \mathbb{R}\})$:

$$\forall\, \kappa > 0,\, p = floor(s/\kappa) \;\; \wedge \;\; 0 \leq |floor(s/\kappa) - s/\kappa| < 1 \;\; \Rightarrow \;\; |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by κ:

$$\forall\, \kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \delta = \epsilon \quad := \quad M = \lim_{\kappa \to 0} p\kappa = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}},\; 3.14_{\kappa=10^{-2}},\; 3.141_{\kappa=10^{-3}}, ..., \pi_{\lim_{\kappa \to 0}}$.

Lemma 2.3. $\forall\, n \geq 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa$.

Proof. The formal proof , "lim_c_to_n_eq_lim_c," is in the Rocq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta$$
$$\Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa \to 0} \kappa^n = 0. \tag{2.5}$$

$$\lim_{\kappa \to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa \to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa. \tag{2.6}$$ □

3. Volume

Definition 3.1. A countable n-volume is the number of ordered combinations (n-tuples), v_c, of the members of n number of disjoint, countable domain sets, x_i:

$$\exists\, n \in \mathbb{N},\ v_c \in \{0,\ \mathbb{N}\},\ \ x_i \in \{x_1, \cdots, x_n\} :\ \bigcap_{i=1}^{n} x_i = \emptyset\ \wedge\ v_c = \prod_{i=1}^{n} |x_i|. \tag{3.1}$$

Theorem 3.2. *Euclidean volume,*

$$\forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\ \ [v_a, v_b] \subset \mathbb{R},\ \ s_i = b_i - a_i,\ \ v = v_b - v_a :$$
$$v_c = \prod_{i=1}^{n} |x_i| \quad \Leftrightarrow \quad v = \prod_{i=1}^{n} s_i. \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Rocq file, euclidrelations.v.

Proof.

$$v_c = \prod_{i=1}^{n} |x_i|\ \Leftrightarrow\ v_c\kappa = (\prod_{i=1}^{n} |x_i|)\kappa\ \Leftrightarrow\ \lim_{\kappa \to 0} v_c\kappa = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|)\kappa. \tag{3.3}$$

Apply the ruler (2.1) and ruler convergence (2.2) to equation 3.3:

$$\exists\, v, \kappa \in \mathbb{R} :\ v_c = floor(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa \to 0} v_c\kappa \quad \wedge$$
$$\lim_{\kappa \to 0} v_c\kappa = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|)\kappa \quad \Rightarrow \quad v = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|)\kappa. \tag{3.4}$$

Apply lemma 2.3 to equation 3.4:

$$v = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|)\kappa \quad \wedge \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa$$
$$\Rightarrow \quad v = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|)\kappa^n = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|\kappa). \tag{3.5}$$

Apply the ruler (2.1) and ruler convergence (2.2) to s_i:

$$\exists\, s_i, \kappa \in \mathbb{R} :\ floor(s_i/\kappa) = |x_i| \quad \Rightarrow \quad \lim_{\kappa \to 0}(|x_i|\kappa) = s_i. \tag{3.6}$$

$$v = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|\kappa) \quad \wedge \quad \lim_{\kappa \to 0}(|x_i|\kappa) = s_i$$
$$\Leftrightarrow \quad v = \lim_{\kappa \to 0}(|x_i|\kappa) = \prod_{i=1}^{n} s_i \tag{3.7}$$ □

4. Distance

Definition 4.1. Countable distance,

$$\exists\, n \in \mathbb{N},\ v_c, d_c \in \{0,\ \mathbb{N}\},\ \ x_i \in \{x_1, \cdots, x_n\} :\ \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge$$
$$d_c = |x_1| = \cdots = |x_n| \quad \wedge \quad v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n. \tag{4.1}$$

Lemma 4.2. *A volume is the sum of volumes,*

$$v_c = d_c^n = \sum_{i=1}^{m} v_{c_i} \quad \Leftrightarrow \quad v = \sum_{i=1}^{m} v_i, \quad v, v_i \in \mathbb{R}.$$

The formal proof, "sum_of_volumes," is in the Rocq file, euclidrelations.v.

Proof. From the condition of this theorem:

$$v_c = \sum_{i=1}^{m} v_{c_i} \quad \Leftrightarrow \quad \lim_{\kappa \to 0} v_c\kappa = \lim_{\kappa \to 0} \sum_{j=1}^{m}(v_{c_i}\kappa). \tag{4.2}$$

Apply lemma 2.3 to equation 4.2:

$$\text{(4.3)}\quad \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\textstyle\sum_{j=1}^m v_{c_i})\kappa \quad\wedge\quad \lim_{\kappa\to 0}\kappa^n = \lim_{\kappa\to 0}\kappa$$
$$\Leftrightarrow\ \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}\textstyle\sum_{j=1}^m (v_{c_i})\kappa^n \ \Leftrightarrow\ \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}\sum_{j=1}^m (v_{c_i}\kappa).$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to equation 4.3:

$$\text{(4.4)}\quad \exists\, v, v_i :\ v = floor(d/\kappa),\ v = \lim_{\kappa\to 0} v_c\kappa$$
$$\wedge\quad v_{c_i} = floor(v_i/\kappa),\ v_i = \lim_{\kappa\to 0} v_{c_i}\kappa \quad\wedge\quad \lim_{\kappa\to 0}(d_c\kappa)^n = \lim_{\kappa\to 0}\textstyle\sum_{j=1}^m (v_{c_i}\kappa)$$
$$\Leftrightarrow\quad v = \lim_{\kappa\to 0}(d_c\kappa)^n = \lim_{\kappa\to 0}\textstyle\sum_{j=1}^m (v_{c_i}\kappa) = \sum_{j=1}^m v_i^n. \quad \square$$

4.1. Sum of volumes distance.

THEOREM 4.3. *Sum of volumes distance:*

$$v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad\Leftrightarrow\quad d^n = \sum_{i=1}^m (\prod_{j=1}^n s_{i_j}).$$

The formal proof, "sum_of _volumes _distance," is in the Rocq file, euclidrelations.v.

PROOF. From lemma 4.2 and the Euclidean volume theorem 3.2:

$$\text{(4.5)}\quad v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad\Leftrightarrow\quad d^n = \sum_{i=1}^m (\prod_{j=1}^n v_i) \quad\wedge\quad v_i = \prod_{j=1}^n s_{i_j}$$
$$v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad\Leftrightarrow\quad d^n = \sum_{i=1}^m (\prod_{j=1}^n s_{i_j}). \quad \square$$

4.2. Minkowski distance (ρ-norm).

THEOREM 4.4. *Minkowski distance (ρ-norm):*

$$v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} = \sum_{i=1}^m d_{c_i}^n \quad\Leftrightarrow\quad d^n = \sum_{i=1}^m d_i^n.$$

The formal proof, "Minkowski_distance," is in the Rocq file, euclidrelations.v.

PROOF. From lemma 4.2 and the Euclidean volume theorem 3.2:

$$\text{(4.6)}\quad v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad\Leftrightarrow\quad d^n = \sum_{i=1}^m v_i \quad\wedge\quad v_i = \prod_{j=1}^n d_i = d_i^n$$
$$v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad\Leftrightarrow\quad d^n = \sum_{i=1}^m d^n. \quad \square$$

4.3. Distance inequality. The formal proof, distance_inequality, is in the Rocq file, euclidrelations.v.

THEOREM 4.5. *Distance inequality*

$$\forall\, n \in \mathbb{N},\ v_a, v_b \geq 0 :\ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$\text{(4.7)}\quad \forall\, v_a, v_b \geq 0 :\quad v_a + v_b \leq v_a + v_b +$$
$$\textstyle\sum_{i=1}^n \binom{n}{k}(v_a^{1/n})^{n-k}(v_b^{1/n})^k + \sum_{i=1}^n \binom{n}{k}(v_a^{1/n})^k(v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n.$$

Take the n^{th} root of both sides of the inequality 4.7:

$$\text{(4.8)}\ \forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad \square$$

4.4. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Rocq file, euclidrelations.v.

THEOREM 4.6. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N},\ a_i, b_i \geq 0 :\ (\textstyle\sum_{i=1}^m (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}.$$

Proof. Apply the distance inequality (4.5):

$$\forall\, m,n \in \mathbb{N},\ v_a, v_b \geq 0 : \quad v_a = \textstyle\sum_{i=1}^m a_i^n \quad \wedge \quad v_b = \sum_{i=1}^m b_i^n \quad \wedge$$
$$(v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad ((\textstyle\sum_{i=1}^m a_i^n) + (\sum_{i=1}^m b_i^n))^{1/n} =$$
$$(\textstyle\sum_{i=1}^m (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}. \quad \square \tag{4.9}$$

4.5. Metric Space. All Minkowski distances (ρ-norms) imply the metric space properties. The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Rocq file, euclidrelations.v.

Theorem 4.7. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} \quad \Rightarrow \quad d(u,w) \leq d(u,v) + d(v,w)$.

Proof. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.10}$$

Apply the distance inequality (4.5) to the inequality 4.10:

$$(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \ \wedge\ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}$$
$$\wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p$$
$$\Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p}$$
$$\Rightarrow \quad d(u,w) = (u^p + w^p)^{1/p} \leq$$
$$(u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u,v) + d(v,w). \quad \square \tag{4.11}$$

Theorem 4.8. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} \quad \Rightarrow \quad d(u,v) = d(v,u)$.

Proof. By the commutative law of addition:

$$\forall\, p : p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^2 s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p}$$
$$\Rightarrow \quad d(u,v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v,u). \quad \square \tag{4.12}$$

Theorem 4.9. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} \quad \Rightarrow \quad d(u,w) \geq 0$.

Proof. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0. \tag{4.13}$$

$$p \geq 1,\ u, v \geq 0 \quad \Rightarrow \quad d(u,v) = (u^p + v^p)^{1/p} \geq 0. \tag{4.14}$$

$\square$

Theorem 4.10. *Identity of Indiscernibles:* $d(u,u) = 0$.

Proof. From the non-negativity property (4.9):

$$d(u,w) \geq 0 \quad \wedge \quad d(u,v) \geq 0 \quad \wedge \quad d(v,w) \geq 0$$
$$\Rightarrow \quad \exists\, d(u,w) = d(u,v) = d(v,w) = 0. \tag{4.15}$$

$$d(u,w) = d(v,w) = 0 \quad \Rightarrow \quad u = v. \tag{4.16}$$

$$d(u,v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u,u) = 0. \tag{4.17}$$

$\square$

4.6. Set properties limiting a set to at most 3 members. The following definitions and proof use first order logic. A Horn clause-like expression is used, here, to make the proof easier to read. By convention, the proof goal is on the left side and supporting facts are on the right side of the implication sign ($\leftarrow$). The formal proofs in the Rocq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

DEFINITION 4.11. Immediate Cyclic Successor of m is n:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\}: \quad Successor(m,n,setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m+1 \leq setsize). \tag{4.18}$$

DEFINITION 4.12. Immediate Cyclic Predecessor of m is n:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\}: \quad Predecessor(m,n,setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m-1 \geq 1). \tag{4.19}$$

DEFINITION 4.13. Adjacent: Member m is sequentially adjacent to member n if the immediate cyclic successor of m is n or the immediate cyclic predecessor of m is n. Notionally:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\}: \quad Adjacent(m,n,setsize) \leftarrow Successor(m,n,setsize) \vee Predecessor(m,n,setsize). \tag{4.20}$$

DEFINITION 4.14. Symmetric (every set member is sequentially adjacent to every other member):

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\}: \quad Adjacent(m,n,setsize). \tag{4.21}$$

THEOREM 4.15. *A cyclic and symmetric set is limited to at most 3 members.*

PROOF.
Every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (m = setsize \wedge n = 1). \tag{4.22}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (n = m+1 \leq setsize). \tag{4.23}$$

$$Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (n = setsize \wedge m = 1). \tag{4.24}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m+1 \leq setsize). \tag{4.25}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m-1 \geq 1). \tag{4.26}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \wedge m = 1). \tag{4.27}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \wedge n = setsize). \tag{4.28}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m+1 \leq setsize). \tag{4.29}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m-1 \geq 1). \tag{4.30}$$

Member 2 is the only immediate successor of member 1 for all $setsize \geq 3$, which implies member 3 is not ($\neg$) an immediate successor of member 1 for all $setsize \geq 3$:

$$\neg Successor(1,3,setsize \geq 3) \leftarrow Successor(1,2,setsize \geq 3) \leftarrow (n = m+1 \leq setsize). \tag{4.31}$$

Member $n = setsize > 3$ is the only immediate predecessor of member 1, which implies member 3 is not ($\neg$) an immediate predecessor of member 1 for all $setsize >$

3:

$$(4.32)\quad \neg Predecessor(1,3,setsize \geq 3) \leftarrow Predecessor(1,setsize,setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).$$

For all $setsize > 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not symmetric):

$$(4.33)\quad \neg Adjacent(1,3,setsize > 3) \leftarrow \neg Successor(1,3,setsize > 3) \wedge \neg Predecessor(1,3,setsize > 3). \quad \square$$

The Symmetric goal matches Adjacent goals 4.22 and fails for all "setsize" greater than three.

5. Applications to physics

From the volume proof (3.2), two disjoint distance intervals, $[0, r_1]$ and $[0, r_2]$, define a 2-volume. From the Minkowski distance proof (4.4), $\exists\, r : r^2 = r_1^2 + r_2^2$. And from the 3D proof (4.15), for some non-distance type, $\tau : \tau \in \{t\ (time),\ m\ (mass),\ q\ (charge), \cdots\}$, there exist constant, unit-factoring ratios, $\mu,\ \nu_1,\ \nu_2$:

$$(5.1)\quad \forall\, r, r_1, r_2 :\ r^2 = r_1^2 + r_2^2 \quad \wedge \quad r = \mu\tau \quad \wedge \quad r_1 = \nu_1\tau \quad \wedge \quad r_2 = \nu_2\tau \quad \Rightarrow \quad (\mu\tau)^2 = (\nu_1\tau)^2 + (\nu_2\tau)^2 \quad \Rightarrow \quad \mu \geq \nu_1 \quad \wedge \quad \mu \geq \nu_2.$$

μ is the maximum-possible ($\mu \geq \nu_1,\ \nu_2$), constant, unit-factoring ratio, where:

$$(5.2)\quad \mu \in \{r_c/t_c,\ r_c/m_c,\ r_c/q_c, \cdots\} :\ r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q = \cdots .$$

5.1. Derivation of the constant, G, and the gravity laws of Newton, Gauss, and Poisson.

From equation 5.2:

$$(5.3)\quad r = (r_c/m_c)m \quad \wedge \quad r = (r_c/t_c)t = ct \quad \Rightarrow \quad r/(ct)^2 = (r_c/m_c)m/r^2 \quad \Rightarrow \quad r/t^2 = ((r_c/m_c)c^2)m/r^2 = Gm/r^2,$$

where Newton's constant, $G = (r_c/m_c)c^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$.

Newton's law follows from multiplying both sides of equation 5.3 by m:

$$(5.4)\qquad r/t^2 = Gm/r^2 \ \Leftrightarrow\ F := mr/t^2 = Gm^2/r^2.$$

$$(5.5)\quad F = Gm^2/r^2 \wedge \forall\, m \in \mathbb{R} : \exists\, m_1, m_2 \in \mathbb{R} : m_1 m_2 = m^2 \ \Rightarrow\ F = Gm_1m_2/r^2.$$

Gauss's flux divergence, $\nabla \cdot \mathbf{g}$ and Poisson's curl per unit mass, $\nabla^2\Phi(r,t)$ are measures of acceleration, r/t^2. Again, starting with equation 5.3 and using ρ as the mass field density (Gauss's flux divergence) on a sphere having the surface area $4\pi r^2$ yields the differential forms:

$$(5.6)\quad \nabla \cdot \mathbf{g} = \nabla^2\Phi(\overrightarrow{r}, t) = r/t^2 = (-Gm/r^2)(4\pi/4\pi) \quad \wedge \quad \rho = m/4\pi r^2 \quad \Rightarrow \quad \nabla \cdot \mathbf{g} = \nabla^2\Phi(\overrightarrow{r}, t) = -4\pi G\rho.$$

5.2. Derivation of Coulomb's charge constant, k_e and charge force.

$$(5.7)\quad \forall\, q \in \mathbb{R} :\ \exists\, q_1, q_2 \in \mathbb{R} :\ q_1q_2 = q^2 \quad \wedge \quad r = (r_c/q_c)q \quad \Rightarrow \quad \exists\, q_1, q_2 \in \mathbb{R} :\ q_1q_2 = q^2 = ((q_c/r_c)r)^2 \quad \Rightarrow \quad (r_c/q_c)^2 q_1q_2/r^2 = 1.$$

$$(5.8)\quad r = (r_c/t_c)t = ct \quad \wedge \quad r = (r_c/m_c)m = ct \\ \Rightarrow \quad mr = (m_c/r_c)rct = (m_c/r_c)(ct)^2 \quad \Rightarrow \quad ((r_c/m_c)/c^2)mr/t^2 = 1.$$

$$(5.9)\quad ((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1 \\ \Rightarrow \quad F := mr/t^2 = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2.$$

where Coulomb's constant, $k_e = ((m_c/r_c)c^2)(r_c/q_c)^2$, has the units $kg{\cdot}m^3{\cdot}s^{-2}{\cdot}C^{-2}$, which is equivalent to the SI units: $N \cdot m^2 \cdot C^{-2}$.

5.3. Vacuum permittivity, ε_0, and Gauss's law for electric fields. From Coulomb's charge force equation 5.9, where $r = \overrightarrow{\mathbf{r_1}}$, and $\overrightarrow{\mathbf{r_2}}, \overrightarrow{\mathbf{r_3}} = 0$:

$$(5.10)\qquad \exists\, q \in \mathbb{R} : \ F = k_e q_1 q_2/r^2 = k_e q^2/r^2 := q\mathbf{E} \quad \Rightarrow \quad \mathbf{E} = k_e q/r^2,$$

where $\mathbf{E}$ has the SI units $N \cdot C^{-1}$.

$$(5.11)\qquad \mathbf{E} = k_e q/r^2 \quad \Rightarrow \quad \nabla \cdot \mathbf{E} = -2k_e q/r^3.$$

$$(5.12)\quad \nabla \cdot \mathbf{E} = -(2k_e q/r^3)(2\pi/2\pi) \quad \wedge \quad \rho = q/2\pi r^3 \quad \wedge \quad \varepsilon_0 := 1/4\pi k_e \\ \Rightarrow \quad \nabla \cdot \mathbf{E} = -4\pi k_e \rho = -\rho/\varepsilon_0.$$

5.4. Vacuum permeability, μ_0, and Faraday's law. $\mathbf{B} = \mathbf{E}/c$ has the base SI units: $kg \cdot s^{-1} \cdot C^{-1} = kg \cdot s^{-2} \cdot A^{-1} = T$. From equation 5.10:

$$(5.13)\qquad \mathbf{B} := \mathbf{E}/c = (k_e/c)q/r^2 \quad \wedge \quad r = ct \quad \Rightarrow \quad \mathbf{B} = (k_e/c^3)q/t^2.$$

$$(5.14)\qquad \mathbf{B} = (k_e/c^3)q/t^2 \quad \Rightarrow \quad \partial\mathbf{B}/\partial t = -(2k_e/c^3)q/t^3.$$

$$(5.15)\qquad \partial\mathbf{B}/\partial t = -(2k_e/c^3)q/t^3 \quad \wedge \quad r = ct \quad \Rightarrow \quad \partial\mathbf{B}/\partial t = -2k_e q/r^3.$$

$\nabla\times$ has the opposite signed (orthogonal) direction of $\nabla\cdot$. From equation 5.11:

$$(5.16)\qquad \nabla \cdot \mathbf{E} = -2k_e q/r^3 \quad \Rightarrow \quad \nabla \times \mathbf{E} = 2k_e q/r^3.$$

Combining equations 5.14 and 5.16 yields Faraday's law:

$$(5.17)\qquad \nabla \times \mathbf{E} = 2k_e q/r^3 \quad \wedge \quad \partial\mathbf{B}/\partial t = -2k_e q/r^3 \quad \Rightarrow \quad \nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t.$$

$$(5.18)\quad \partial\mathbf{B}/\partial t = -(2k_e q/r^2)(2\pi/2\pi) \quad \wedge \quad \rho = q/2\pi r^3 \quad \wedge \quad \mu_0 := 4\pi k_e/c^2 \\ \Rightarrow \quad \partial\mathbf{B}/\partial t = -4\pi k_e \rho = -\mu_0 \rho.$$

5.5. Space-time-mass-charge equations. From equation 5.1:

$$(5.19)\quad \forall\, r, r', r_v, \mu, \nu : \ r^2 = r'^2 + r_v^2 \quad \wedge \quad r = \mu\tau \quad \wedge \quad r_v = \nu\tau \\ \Rightarrow \quad r' = \sqrt{(\mu\tau)^2 - (\nu\tau)^2} \ = \ \mu\tau\sqrt{1 - (\nu/\mu)^2}.$$

Rest frame distance, r', contracts relative to stationary frame distance, r, as $\nu \to \mu$:

$$(5.20)\qquad r' = \mu\tau\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad \mu\tau = r \quad \Rightarrow \quad r' = r\sqrt{1 - (\nu/\mu)^2}.$$

Stationary frame type, τ, dilates relative to the rest frame type, τ', as $\nu \to \mu$:

$$(5.21)\qquad \mu\tau = r'/\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad r' = \mu\tau' \quad \Rightarrow \quad \tau = \tau'/\sqrt{1 - (\nu/\mu)^2}.$$

Where τ is type, time, the space-like flat Minkowski spacetime event interval is:

$$(5.22)\quad \mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_v^2 \quad \wedge \quad \mathrm{d}r_v^2 = \mathrm{d}r_1^2 + \mathrm{d}r_2^2 + \mathrm{d}r_3^2 \quad \wedge \quad \mathrm{d}(\mu\tau) = \mathrm{d}r \\ \Rightarrow \quad \mathrm{d}r'^2 = \mathrm{d}(\mu\tau)^2 - \mathrm{d}r_1^2 - \mathrm{d}r_2^2 - \mathrm{d}r_3^2.$$

5.6. Derivation of Schwarzchild's gravitational time dilation and black hole metric. [**Che10**] From equations 5.20 and 5.2:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-(v^2/c^2)(r/r)} \quad \wedge \quad r = (r_c/m_c)m \\ \Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-((r_c/m_c)m)v^2/rc^2}. \tag{5.23}$$

Where v_{escape} is the escape velocity:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-((r_c/m_c)m)v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv^2_{escape} \\ \Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2(r_c/m_c)mv^2_{escape}/rc^2}. \tag{5.24}$$

$$\sqrt{1-(v^2/c^2)} = \lim_{v_{escape}\to c}\sqrt{1-2(r_c/m_c)mv^2_{escape}/rc^2} \\ = \sqrt{1-2(r_c/m_c)mc^2/rc^2}. \tag{5.25}$$

Combining equation 5.25 with the derivation of G (5.5):

$$(r_c/m_c)c^2 = G \quad \wedge \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2(r_c/m_c)mc^2/rc^2} \\ \Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2Gm/rc^2}. \tag{5.26}$$

Combining equation 5.26 with equation 5.21 yields Schwarzchild's gravitational time dilation:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-2Gm/rc^2} \quad \wedge \quad t' = t\sqrt{1-(v^2/c^2)} \\ \Rightarrow \quad t' = t\sqrt{1-2Gm/rc^2}. \tag{5.27}$$

Schwarzchild defined the black hole event horizon radius, $r_s := 2Gm/c^2$.

$$r_s = 2Gm/c^2 \quad \wedge \quad t' = t\sqrt{1-2Gm/rc^2} \quad \Rightarrow \quad t' = t\sqrt{1-r_s/r}. \tag{5.28}$$

From equations 5.20 and 5.28:

$$r' = r\sqrt{1-(v/c)^2} \quad \wedge \quad \sqrt{1-(v/c)^2} = \sqrt{1-2Gm/rc^2} \\ \Rightarrow \quad r' = r\sqrt{1-2Gm/rc^2} = r\sqrt{1-r_s/r}. \tag{5.29}$$

Using the time-like spacetime interval, where $\mathrm{d}s^2$ is negative:

$$r' = r\sqrt{1-r_s/r} \quad \wedge \quad \mathrm{d}s^2 = \mathrm{d}r'^2 - \mathrm{d}r^2 \\ \Rightarrow \quad \mathrm{d}s^2 = (\sqrt{1-r_s/r}\mathrm{d}r')^2 - (\mathrm{d}r/\sqrt{1-r_s/r})^2 = (1-r_s/r)\mathrm{d}r'^2 - (1-r_s/r)^{-1}\mathrm{d}r^2. \tag{5.30}$$

$$\mathrm{d}s^2 = (1-r_s/r)\mathrm{d}r'^2 - (1-r_s/r)^{-1}\mathrm{d}r^2 \quad \wedge \quad \mathrm{d}r' = \mathrm{d}(ct) \quad \wedge \quad c = 1 \\ \Rightarrow \quad \mathrm{d}s^2 = (1-r_s/r)\mathrm{d}t^2 - (1-r_s/r)^{-1}\mathrm{d}r^2. \tag{5.31}$$

Translating from 2D to 4D yields Schwarzchild's black hole metric:

$$\mathrm{d}s^2 = (1-r_s/r)\mathrm{d}t^2 - (1-r_s/r)^{-1}\mathrm{d}r^2 = f(r,t) \\ \Rightarrow \quad \mathrm{d}s^2 = (1-r_s/r)\mathrm{d}t^2 - (1-r_s/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + sin^2\theta\mathrm{d}\phi^2) = f(r,t,\theta,\phi). \tag{5.32}$$

5.7. Simplifying Einstein's general relativity (field) equation. Simplification step 1) Use the unit-factoring ratios to define functions returning values for each component of the metric, $g_{\nu,\mu}$, in Einstein's field equations [**Ein15**] [**Wey52**]:

All functions derived from the ratios are valid metrics, for example, the previous Schwarzchild black hole metric derivation using the unit-factoring ratios (5.6).

Simplification step 2) Express the EFE as 2D tensors: As shown in equation 5.32, the Schwarzchild metric was first derived as a 2D metric and then expanded to a 4D metric. Further, the 4D flat spacetime interval equation (5.22) is an instance of the 2D equation, $\mathrm{d}r'^2 = \mathrm{d}(ct)^2 - \mathrm{d}r_v^2$, where $\mathrm{d}r_v^2$ is the magnitude of a 3-dimensional vector.

The 2D metric tensor allows using the much simpler 2D Ricci curvature and scalar curvature. And the 2D tensors reduce the number of independent equations to solve.

Simplification step 3) One simple method to translate from 2D to 4D is to use spherical coordinates, where r and t remain unchanged and two added dimensions are the angles, ϕ, and θ. For example, the 2D Schwarzchild metric was translated to 4D using this method in equation 5.32.

5.8. 3 fundamental direct proportion ratios. c_t, c_m, and c_q:

$$c_t = r_c/t_c \approx 2.99792458 \cdot 10^8 m\ s^{-1}. \tag{5.33}$$

$$G = (r_c/m_c)c_t^2 = c_m c_t^2 \quad \Rightarrow \quad c_m = r_c/m_c \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}. \tag{5.34}$$

$$k_e = (c_t^2/c_m)(r_c/q_c)^2 \quad \Rightarrow \quad c_q = r_c/q_c \approx 8.6175172023 \cdot 10^{-18} m\ C^{-1}. \tag{5.35}$$

5.9. 3 fundamental inverse proportion ratios. k_t, k_m, and k_q:

$$\begin{aligned} r/t = r_c/t_c, \quad r/m = r_c/m_c \quad &\Rightarrow \quad (r/t)/(r/m) = (r_c/t_c)/(r_c/m_c) \quad \Rightarrow \\ (mr)/(tr) = (m_c r_c)/(t_c r_c) \quad &\Rightarrow \quad mr = m_c r_c = k_m, \quad tr = t_c r_c = k_t. \end{aligned} \tag{5.36}$$

$$\begin{aligned} r/t = r_c/t_c, \quad r/q = r_c/q_c \quad &\Rightarrow \quad (r/t)/(r/q) = (r_c/t_c)/(r_c/q_c) \quad \Rightarrow \\ (qr)/(tr) = (q_c r_c)/(t_c r_c) \quad &\Rightarrow \quad qr = q_c r_c = k_q, \quad tr = t_c r_c = k_t. \end{aligned} \tag{5.37}$$

5.10. Planck relation and constant, h. [**Jai11**] Applying both the direct proportion ratio (5.33), and inverse proportion ratio (5.36):

$$r = ct \quad \wedge \quad m = k_m/r \quad \Rightarrow \quad m(ct)^2 = (k_m/r)r^2 = k_m r. \tag{5.38}$$

$$\begin{aligned} &m(ct)^2 = k_m r \quad \wedge \quad r/t = r_c/t_c = c \\ &\quad \Rightarrow \quad E := mc^2 = k_m r/t^2 = (k_m(r/t))\,(1/t) = (k_m c)(1/t) = hf, \end{aligned} \tag{5.39}$$

where the Planck constant, $h = k_m c$, and the frequency, $f = 1/t$.

$$k_m = m_c r_c = h/c \approx 2.2102190943 \cdot 10^{-42}\ kg\ m. \tag{5.40}$$

$$k_t = t_c r_c = k_m c_m/c_t \approx 5.4749346710 \cdot 10^{-78}\ s\ m. \tag{5.41}$$

$$k_q = q_c r_c = k_t c_t/c_q \approx 1.9046601056 \cdot 10^{-52}\ C\ m. \tag{5.42}$$

5.11. Compton wavelength. [**Jai11**] From equations 5.36 and 5.39:

$$mr = k_m \quad \wedge \quad h = k_m c \quad \Rightarrow \quad r = k_m/m = (k_m/m)(c/c) = h/mc. \tag{5.43}$$

5.12. 4 quantum units. Distance (r_c), time (t_c), mass (m_c), and charge (q_c):

$$r_c = \sqrt{r_c^2} = \sqrt{c_t k_t} = \sqrt{c_m k_m} = \sqrt{c_q k_q} \approx 4.0513505432 \cdot 10^{-35}\ m. \tag{5.44}$$

$$t_c = r_c/c_t \approx 1.3513850782 \cdot 10^{-43}\ s. \tag{5.45}$$

$$m_c = r_c/c_m \approx 5.4555118613 \cdot 10^{-8}\ kg. \tag{5.46}$$

$$q_c = r_c/c_q \approx 4.7012967286 \cdot 10^{-18}\ C. \tag{5.47}$$

5.13. Subtype ratios. The ratio of two subtypes of direct proportion ratio constants, $(x_{\tau_1}/x_{\tau_2})/(x_{\tau_1}/x_{\tau_2}) = 1$. The ratio of two subtypes of inverse proportion ratios, $\forall\ x_{\tau_1}/x_{\tau_2} > 0 : = (x_{\tau_1}/x_{\tau_2})(x_{\tau_1}/x_{\tau_2}), = (x_{\tau_1}x_{\tau_1})/(x_{\tau_2}x_{\tau_2}) = x_{\tau_1}^2/x_{\tau_2}^2 > 0$:

Planck length, r_p: $r_c^2/r_p^2 = 2\pi \Rightarrow r_p = r_c/\sqrt{2\pi} \approx 1.6162550244 \cdot 10^{-35}\ m.$

Planck time, t_p: $t_c^2/t_p^2 = 2\pi \Rightarrow t_p = t_c/\sqrt{2\pi} \approx 5.3912464472 \cdot 10^{-44}\ s.$

Planck mass, m_p: $m_c^2/m_p^2 = 2\pi \Rightarrow m_p = m_c/\sqrt{2\pi} \approx 2.176434343 \cdot 10^{-8}\ kg.$

Planck charge, q_p: $q_c^2/q_p^2 = 2\pi \Rightarrow q_p = q_c/\sqrt{2\pi} \approx 1.875546038 \cdot 10^{-18}\ C.$

Where q_e is the elementary (electron) charge ($1.60217663 \cdot 10^{-19}\ C$), the fine structure constant, α, is also the ratio of two inverse proportion ratios:

$$q_c^2/q_e^2 = 2\pi/\alpha \Rightarrow \alpha = 2\pi q_e^2/q_c^2 = q_e^2/(q_c/\sqrt{2\pi})^2 = q_e^2/q_p^2 \approx 0.0072973526. \tag{5.48}$$

5.14. Schrödenger's equation. Start with the previously derived Planck relation 5.39 and multiply the kinetic energy component by mc/mc:

$$h/t = mc^2 \Rightarrow \exists V(r,t) : h/t = h/2t + V(r,t) \Rightarrow h/t = hmc/2mct + V(r,t). \tag{5.49}$$

And from the distance-to-time (speed of light) ratio (5.33):

$$h/t = hmc/2mct + V(r,t) \ \wedge\ r = ct \ \Rightarrow\ h/t = hmc^2/2mcr + V(r,t). \tag{5.50}$$

$$h/t = hmc^2/2mcr + V(r,t) \ \wedge\ h/t = mc^2 \ \Rightarrow\ h/t = h^2/2mcrt + V(r,t). \tag{5.51}$$

$$h/t = h^2/2mcrt + V(r,t) \ \wedge\ r = ct \ \Rightarrow\ h/t = h^2/2mr^2 + V(r,t). \tag{5.52}$$

Replace the Planck constant in equation 5.52 with the reduced Planck constant:

$$h/t = h^2/2mr^2 + V(r,t) \wedge \hbar = h/2\pi \Rightarrow 2\pi\hbar/t = (2\pi)^2\hbar^2/2mr^2 + V(r,t). \tag{5.53}$$

Multiply both sides of equation 5.53 by a function, $\Psi(r,t)$.

$$\begin{aligned} &2\pi\hbar/t = (2\pi)^2\hbar^2/2mr^2 + V(r,t) \\ &\qquad \Rightarrow\ (2\pi\hbar/t)\Psi(r,t) = ((2\pi)^2\hbar^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t). \end{aligned} \tag{5.54}$$

$$\begin{aligned} &(2\pi\hbar/t)\Psi(r,t) = ((2\pi)^2\hbar^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t) \quad \wedge \\ &\forall\ \Psi(r,t) :\ \partial^2\Psi(r,t)/\partial r^2 = (-(2\pi)^2/r^2)\Psi(r,t) \quad \wedge \quad \partial\Psi(r,t)/\partial t = (i\ 2\pi/t)\Psi(r,t) \\ &\qquad \Rightarrow\ i\hbar\partial\Psi(r,t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t), \end{aligned} \tag{5.55}$$

which is Schrödenger's equation in one dimension of space.

$$\begin{aligned} &i\hbar\partial\Psi(r,t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t) \quad \wedge \quad ||\overrightarrow{\mathbf{r}}|| = r \\ &\Rightarrow\ \exists\ \overrightarrow{\mathbf{r}} :\ i\hbar\partial\Psi(\overrightarrow{\mathbf{r}},t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(\overrightarrow{\mathbf{r}},t)/\partial\overrightarrow{\mathbf{r}}^2 + V(\overrightarrow{\mathbf{r}},t)\Psi(\overrightarrow{\mathbf{r}},t), \end{aligned} \tag{5.56}$$

which is Schrödenger's equation in three dimensions of space.

5.15. Dirac's wave equation. Using the derived Planck relation 5.39:

$$mc^2 = h/t \quad \Rightarrow \quad \exists\, V(r,t) :\, mc^2/2 + V(r,t) = h/t \\ \Rightarrow \quad 2h/t - 2V(r,t) = mc^2. \tag{5.57}$$

$$\forall\, V(r,t) :\, V(r,t) = ih/t \quad \wedge \quad r = ct \quad \wedge \quad 2h/t - 2V(r,t) = mc^2 \\ \Rightarrow \quad 2h/t - i2hc/r = mc^2. \tag{5.58}$$

Use the charge ratio, c_q, and time ratio, $c_t = c$ to multiply each term on the left side of equation 5.58 by 1:

$$qc_q/r = qc_q/ct = 1 \quad \wedge \quad 2h/t - i2hc/r = mc^2 \\ \Rightarrow \quad 2h(-qc_q/c)/t^2 - i2h((-qc_q/c)/r^2)c = mc^2. \tag{5.59}$$

where a negative sign is added to q to indicate an attractive force between an electron and a nucleus.

Applying a quantum amplitude equation in complex form to equation 5.60:

$$A_0 = (c_q/c)((1/t)) - i(1/r)) \wedge 2h(-qc_q/c)/t^2 - i2h((-qc_q/c)/r^2)c = mc^2 \\ \Rightarrow \quad 2h\partial(-qA_0)/\partial t - i2h(\partial(-qA_0)/\partial r)c = mc^2. \tag{5.60}$$

Translating equation 5.60 to moving coordinates via the Lorentz factor, $\gamma_0 = 1/\sqrt{1-(v/c)^2}$:

$$2h\partial(-qA_0)/\partial t - i2h(\partial(-qA_0)/\partial r)c = mc^2 \\ \Rightarrow \quad \gamma_0 2h\partial(-qA_0)/\partial t - \gamma_0 i2h(\partial(-qA_0)/\partial r)c = mc^2. \tag{5.61}$$

Multiplying both sides of equation 5.61 by $\Psi(r,t)$:

$$\gamma_0 2h\partial(-qA_0)/\partial t - \gamma_0 i2h(\partial(-qA_0)/\partial r)c = mc^2 \\ \Rightarrow \quad \gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \gamma_0 i2h(\partial(-qA_0)/\partial r)c\Psi(r,t) = mc^2\Psi(r,t). \tag{5.62}$$

Applying the vectors to equation 5.62:

$$\gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \gamma_0 i2h(\partial(-qA_0)/\partial r)c\Psi(r,t) = mc^2\Psi(r,t) \wedge \\ ||\overrightarrow{\mathbf{r}}|| = r \quad \wedge \quad ||\overrightarrow{\mathbf{A}}|| = A_0 \quad \wedge \quad ||\overrightarrow{\gamma}|| = \gamma_0 \quad \wedge \quad \Leftrightarrow \quad \exists\, \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{A}}, \overrightarrow{\gamma} : \\ \gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \overrightarrow{\gamma} \cdot i2h(\partial(-q\overrightarrow{\mathbf{A}})/\partial r)c\Psi(\overrightarrow{\mathbf{r}},t) = mc^2\Psi(\overrightarrow{\mathbf{r}},t). \tag{5.63}$$

Adding a $\frac{1}{2}$ angular rotation (spin-$\frac{1}{2}$) of π to equation 5.60 allows substituting the reduced Planck constant, $\hbar = h/2\pi$, into equation 5.63, which yields Dirac's wave equation:

$$\gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \overrightarrow{\gamma} \cdot i2h(\partial(-q\overrightarrow{\mathbf{A}})/\partial r)c\Psi(\overrightarrow{\mathbf{r}},t) = mc^2\Psi(\overrightarrow{\mathbf{r}},t) \\ \wedge\, A_0 = \pi(c_q/c)((1/t) - i(1/r)) \\ \Rightarrow \quad \gamma_0\hbar(\partial(-qA_0)/\partial t)\Psi(r,t) - \overrightarrow{\gamma} \cdot i\hbar(\partial(-q\overrightarrow{\mathbf{A}})/\partial r)c\Psi(\overrightarrow{\mathbf{r}},t) = mc^2\Psi(\overrightarrow{\mathbf{r}},t). \tag{5.64}$$

5.16. Total mass. The total mass of a particle is $m = \sqrt{m_0^2 + m_{ke}^2}$, where m_0 is the rest mass and m_{ke} is the kinetic energy-equivalent mass. Applying both the direct (5.33) and inverse proportion ratios (5.36):

$$m_0 = r/(r_c/m_c) = r/c_m \quad \wedge \quad m_{ke} = (m_c r_c)/r = k_m/r \quad \wedge$$
$$m = \sqrt{m_0^2 + m_{ke}^2} \quad \Rightarrow \quad m = \sqrt{(r/c_m)^2 + (k_m/r)^2}. \tag{5.65}$$

5.17. Quantum extension to general relativity. The simplest way to demonstrate how to add quantum physics to general relativity is by extending the Schwarzchild's black hole metric (5.6). Start by changing equation 5.23 in the Schwarzchild derivation:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-(v^2/c^2)(r/r)} \;\wedge\; r = \sqrt{(c_m m)^2 + (k_m/m)^2} = Q_m$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-Q_m v^2/rc^2}. \tag{5.66}$$

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-Q_m v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv_{escape}^2$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2Q_m v_{escape}^2/rc^2}. \tag{5.67}$$

$$\sqrt{1-(v^2/c^2)} = \lim_{v_{escape}\to c} \sqrt{1-2Q_m v_{escape}^2/rc^2}$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2Q_m c^2/rc^2} = \sqrt{1-2Q_m/r}. \tag{5.68}$$

Combining equation 5.68 with equation 5.21 yields Schwarzchild's gravitational time dilation with a quantum mass effect:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-2Q_m/r} \quad \wedge \quad t' = t\sqrt{1-(v^2/c^2)}$$
$$\Rightarrow \quad t' = t\sqrt{1-2Q_m/r}. \tag{5.69}$$

Schwarzchild defined the black hole event horizon radius, $r_s := 2Gm/c^2$. The radius with the quantum extension is $r_s := 2Q_m$. At this point the exact same equations 5.28 through 5.32 yield what looks like the same Schwarzchild black hole metric.

5.18. Quantum extension to Newton's gravity force. The quantum mass effect is easier to understand in the context Newton's gravity equation than in general relativity, because the metric equations and solutions in the EFEs are much more complex. From equation 5.2:

$$m/\sqrt{(r/c_m)^2 + (k_m/r)^2} = 1 \quad \wedge \quad r^2/(ct)^2 = 1$$
$$\Rightarrow \quad r^2/(ct)^2 = m/\sqrt{(r/c_m)^2 + (k_m/r)^2}$$
$$\Rightarrow \quad r^2/t^2 = c^2 m/\sqrt{(r/c_m)^2 + (k_m/r)^2}. \tag{5.70}$$

$$r^2/t^2 = c^2 m/\sqrt{(r/c_m)^2 + (k_m/r)^2}$$
$$\Rightarrow \quad (m/r)(r^2/t^2 = (m/r)(c^2 m/\sqrt{(r/c_m)^2 + (k_m/r)^2})$$
$$\Rightarrow \quad F := mr/t^2 = c^2 m^2/(r\sqrt{(r/c_m)^2 + (k_m/r)^2}) = c^2 m^2/\sqrt{(r^4/c_m^2) + k_m^2}. \tag{5.71}$$

$$F = c^2 m^2/\sqrt{(r^4/c_m^2) + k_m^2} \quad \wedge \quad \forall\, m \in \mathbb{R},\, \exists\, m_1, m_2 \in \mathbb{R} :\; m_1 m_2 = m^2$$
$$\Rightarrow \quad F = c^2 m_1 m_2/\sqrt{(r^4/c_m^2) + k_m^2}. \tag{5.72}$$

5.19. Quantum extension to Coulomb's force.

$$
\begin{aligned}
(5.73)\quad & q^2/((r/c_q)^2+(k_q/r)^2)=1 \quad\wedge\quad r^2/(ct)^2=1 \\
&\quad\Rightarrow\quad r^2/(ct)^2=q^2/((r/c_q)^2+(k_q/r)^2) \\
&\qquad\quad\Rightarrow\quad r^2/t^2=c^2q^2/((r/c_q)^2+(k_q/r)^2).
\end{aligned}
$$

$$
\begin{aligned}
(5.74)\quad & \forall\, q\in\mathbb{R}:\ \exists\, q_1,q_2\in\mathbb{R}:\ q_1q_2=q^2 \quad\wedge\quad r^2/t^2=c^2q^2/((r/c_q)^2+(k_q/r)^2) \\
&\quad\Rightarrow\quad \exists\, q_1,q_2\in\mathbb{R}:\ r^2/t^2=c^2q_1q_2/((r/c_q)^2+(k_q/r)^2) \\
&\qquad\quad\Rightarrow\quad r/t^2=c^2q_1q_2/(r((r/c_q)^2+(k_q/r)^2)).
\end{aligned}
$$

$$
\begin{aligned}
(5.75)\quad & r/t^2=c^2q_1q_2/(r((r/c_q)^2+(k_q/r)^2)) \quad\wedge\quad m=\sqrt{(r/c_m)^2+(k_m/r)^2} \\
&\Rightarrow\quad F:=mr/t^2=c^2q_1q_2\sqrt{(r/c_m)^2+(k_m/r)^2}/(r((r/c_q)^2+(k_q/r)^2)) \\
&\qquad =c^2q_1q_2\sqrt{(r^4/c_m^2)+k_m^2}/((r^4/c_q^2)+k_q^2).
\end{aligned}
$$

6. Insights and implications

(1) The ruler measure (2.1) and convergence theorem (2.2) were shown to be useful tools for proving the bidirectional implication that a real-valued equation is the only instance of an abstract, countable set relation and that set relation is the only instance of that same equation.

(2) Combinatorics, the ordered combinations of countable, disjoint sets (n-tuples), $v_c=\sum_{i=1}^m v_{c_i}$), was proven to imply the Euclidean volume equation 3.2.

(3) Combinatorics, the bijective function constraint on v_c, where $v_c=\sum_{i=1}^m v_{c_i}$, was proven to bidirectionally imply the sum of volumes equation 4.3 (which includes the inner product), and the Minkowski distance equation 4.4 (which includes the Manhattan and Euclidean distance equations), without relying on the geometric primitives and relations in Euclidean geometry [**Joy98**], axiomatic geometry [**Lee10**], and vector analysis [**Wey52**].

(4) The axiomatic definitions of inner product space $\subset$ normed vector space $\subset$ metric space $\subset$ topological space, all ignore the intimate (bijective) relation between distance and total volume. Distance as the inverse function of the total volume, $d_c=f^{-1}v_c$, where $v_c=\sum_{i=1}^m v_{c_i}$:
   (a) is a simpler and more concise definition than the various "space" definitions;
   (b) bidirectionally implies well-known distance equations, whereas, the "spaces" are heuristic, filter criteria that are only useful to specify whether some function satisfies those heuristic criteria.

(5) Every Euclidean and non-Euclidean n-volume $v\in\mathbb{R}$ has a corresponding value $d:\ d^n=v$, which implies every Euclidean and non-Euclidean distance measure has a corresponding Minkowski distance (4.4). And it was proved that Minkowski distances have the metric space properties (4.5).

(6) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [**Joy98**].

And analytic proofs that the straight line length is the smallest distance equate the straight line length to Euclidean distance.

Without using the notion of a straight line: All distance measures in an Euclidean volume have corresponding Minkowski distances (4.4). For all 2-volumes, all Minkowski distances are limited to $n \in \{1, 2\}$: $n = 1$ is the Manhattan (largest monotonic) distance case, $d = \sum_{i=1}^{m} s_i$. $n = 2$ is the Euclidean (smallest) distance case, $d = (\sum_{i=1}^{m} s_i^2)^{1/2}$. That is: $\sum_{i=1}^{m} s_i \le (\sum_{i=1}^{m} s_i^2)^{1/2}$. For the case, $n \in \mathbb{R}$, $1 \le n \le 2$: d decreases monotonically as n goes from 1 to 2.

(7) The left side of the distance sum inequality (4.6),

$$(\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \le (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}, \tag{6.1}$$

differs from the left side of Minkowski's sum inequality [**Min53**]:

$$(\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \le (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \tag{6.2}$$

The two inequalities are only the same where $n = 1$.

   (a) The distance sum inequality (4.6) is a more fundamental inequality because the proof does not require the convexity and Hölder's inequality assumptions of the Minkowski sum inequality proof [**Min53**].
   (b) The Minkowski sum inequality term, $\forall\ n > 1 : ((a_i^n + b_i^n)^{\mathbf{n}})^{1/n}$, is **not** a Minkowski distance spanning the n-volume, $a_i^n + b_i^n$. But the distance sum inequality term, $(a_i^n + b_i^n)^{1/n}$, is the Minkowski distance spanning the n-volume, $a_i^n + b_i^n$, which makes it directly related to geometry (for example, the metric space triangle inequality was derived from the $m = 1$ case for all $n \ge 1$ (4.7)).

(8) Combinatorics, the sequencing through an ordered set to yield all $n!$ permutations of its members (without jumping around) was proved to be a cyclic set having $n \le 3$ members (4.15). Higher dimensions must have different types (members of different sets).
   (a) For example, the vector inner product space (which includes Riemann and pseudo-Riemann spaces) can only be extended beyond 3 dimensions if and only if the higher dimensions have non-distance types, for example, time.
   (b) But order and symmetry probably limit the number of fundamental types to a very small number. For example, temperature, measured in Kelvins, is not a true type because temperature is more correctly a measure of (kinetic or electromagnetic) energy which is a function of distance, time, mass, and charge. The derivation of Faraday's law (5.14) shows that the magnetic force is the charge force with respect to time. Likewise, one should not immediately assume the strong force field, weak force field, etc. are types. As will be discussed later, quantum effects might allow radioactivity without a weak force.
   (c) Each of 3 cyclic and symmetric dimensions of space can have at most 3 cyclic and symmetric state values, for example, a cyclic and symmetric set of 3 vector orientations, $\{-1, 0, 1\}$, per dimension of space and at most 3 spin states per plane, etc.
   (d) If the states are not ordered (a bag of states), then a state value is undetermined until observed (like Schrödenger's poisoned cat being both alive and dead until the box is opened). For a bag of states,

there is no "axiom of choice", an axiom often used in math proofs that allows selecting a particular set element (in this case, selecting a particular state).

(e) A discrete value has measure 0 (no size). The ratio of a time or distance interval length to zero is undefined (infinite), which is the reason quantum entangled particles change discrete state values together with no propagation delay and independent of distance.

(9) For each unit of a 3-dimensional, compact and continuous distance, there are units of other compact and continuous types of elements (5.8): $c_t = r_c/t_c$, $c_m = r_c/m_c$, $c_q = r_c/q_c \Leftrightarrow$ the inverse proportion ratios (5.9): $k_t = r_c t_c$, $k_m = r_c m_c$, $k_t = r_c q_c$, where the combination of the direct and inverse ratios implies the quantum units (5.12): r_c, t_c, m_c, q_c.

(10) The gravity, G (5.5), charge k_e (5.9), and Planck h (5.39) constants were all derived directly from the ratios. And vacuum permittivity, ε_0 and vacuum permeability, μ_0, are both definable in terms of k_e: $\varepsilon_0 := 1/4\pi k_e$ and $\mu_0 := 1/c_t^2\varepsilon_0 = 4\pi k_e/c_t^2$.
  (a) Therefore, G, k_e, ε_0, μ_0, and h are **not** "fundamental" constants.
  (b) Using the ratios instead of those constants in equations would show the shared principles underlying the different laws of physics. For example, G, k_e, ε_0, and h all depend on the speed of light ratio, c_t, and mass, c_m or k_m: $G = c_m c_t^2$, $k_e = (c_q^2/c_m)c_t^2$, $\varepsilon_0 = 1/(4\pi(c_q^2/c_m)c_t^2)$, $h = k_m c_t$, and $\mu_0 = 4\pi c_q^2/c_m = 4\pi k_m/q_c^2$.

(11) The derivations of the gravity and charge laws show that the inverse square laws are due to the ratios. The use of mass and charge density, ρ, and the definitions of ε_0 and μ_0 are unnecessary complications that obfuscate the ratios generating the inverse square law. For example, $\mathbf{E} = k_e q/r^2$ shows the inverse square relation more clearly than $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$.

(12) The derivation of Faraday's law (5.14) shows that:
  (a) The electric force, $\mathbf{E}$, is the relation of charge to distance. And the magnetic force, $\mathbf{B}$, is relation of charge to time, where $c = \mathbf{E}/\mathbf{B}$.
  (b) The derivation here does not rely on Gauss's divergence theorem, which indicates that the notion of magnetic dipoles on elementary charged particles is unnecessary. All attempts to measure the magnetic dipole of free electrons have failed. "Dipoles" are only found when measuring electrons in chemical bonds, which is the interaction of positive and negative charges.

(13) The Planck relation has been a hypothesized law. In this article, the Planck relation was derived from the ratios (5.10).

(14) The derivation of the Compton wavelength equation, using the ratios (5.43), is much simpler than deriving from momentum and conservation of energy.

(15) The derivation of the Compton wavelength equation, $r = h/mc$, (5.11) shows that the computation of the wavelength, r, is overly complex (because it assumes the Planck constant is a fundamental constant) and can be simplified to $r = k_m/m$.

(16) Using the quantum units, r_c and t_c: $r_c/t_c^2 \approx 2.2184088232 \cdot 10^{51}\ m\ s^{-2}$, which suggests a maximum acceleration for masses.

(17) The simplification of μ_0 into the quantum units shows two interesting relationships:

$$\mu_0 := \frac{1}{c_t^2 \varepsilon_0} = \frac{4\pi k_e}{c_t^2} = 4\pi \frac{c_q^2}{c_m} = 4\pi \frac{(r_c/q_c)^2}{r_c/m_c} = 4\pi \frac{m_c r_c}{q_c^2} = 4\pi \frac{k_m}{q_c^2}$$
$$\approx 4\pi \frac{2.2102190930 \cdot 10^{-42}}{2.2102190930 \cdot 10^{-35}} \approx 4\pi \cdot 10^{-7}\ kg\ m\ C^{-2} = 4\pi \cdot 10^{-7}\ H\ m^{-1}. \tag{6.3}$$

(a) The first time $k_m = m_c r_c$ appears is in the derivation of the Planck relation and Planck constant, $h = k_m c$ (5.10), the second time in the Compton wavelength, $r = k_m/m$ (5.11). And now, k_m appears in the definition of μ_0, which is further evidence that k_m is a fundamental ratio and that h, k_e, ε_0, and μ_0 are **not** fundamental constants.

(b) At least the first 10 significant digits of k_m and q_c^2 being equal is not a coincidence. The term, 10^{-7}, is an artifact of the relative scales of the units of measurement.

(18) Two subtypes are related via the ratios of two inverse proportion ratios (5.13).

(a) For example, the quantum charge and reduced Planck charge units are related via the ratio: $q_c^2/q_p^2 = 2\pi \Rightarrow q_p = q_c/\sqrt{2\pi}$.

(b) The CODATA electron coupling version of the fine structure constant, α is defined as: $\alpha = q_e^2/4\pi\varepsilon_0 \hbar c) = q_e^2/2\varepsilon_0 hc$ [**COD22**].

(i) The derivation of α, in this article (5.13), is much simpler because it is the ratio of two subtypes: elementary (electron) charge ratio constant, q_e^2 and charge wave (Planck) ratio, q_p^2: $\alpha = 2\pi q_e^2/q_c^2 = q_e^2/q_p^2 \approx 0.0072973526$, which is the empirical CODATA value [**COD22**].

(ii) The following steps show that the CODATA definition reduces to the ratio-derived equation:

$$\begin{aligned} \varepsilon_0 &:= 1/4\pi k_e = 1/(4\pi(c_q^2/c_m)c_t^2) \quad \wedge \quad h = k_m c_t \\ &\Rightarrow \quad \varepsilon_0 hc = k_m c_t^2/(4\pi(c_q^2/c_m)c_t^2) = k_m/(4\pi(c_q^2/c_m)) \\ &\qquad = m_c r_c/(4\pi((r_c/q_c)^2/(r_c/m_c))) = q_c^2/4\pi. \end{aligned} \tag{6.4}$$

$$\alpha = q_e^2/2\varepsilon_0 hc \quad \wedge \quad \varepsilon_0 hc = q_c^2/4\pi = q_p^2/2 \quad \Rightarrow \quad \alpha = q_e^2/q_p^2. \tag{6.5}$$

(iii) As shown above, CODATA defines the fine structure constant in terms of a relationship to the Planck constant, hence, the ratio containing the reduced Planck unit, q_p: $\alpha = q_e^2/q_p^2$. The quantum unit, q_c, appears naturally in the derivation of k_e, where $\varepsilon_0 := 1/4\pi k_e$. Therefore, a better definition to describe particle interaction with a charge (electromagnetic) wave is: $\alpha = q_e^2/q_c^2$, where the current CODATA value would be divided by 2π.

(iv) Other fine structure constants can also be expressed more simply as the ratios of two subtypes of fields, for example, an electron gravity coupling constant can be expressed as the ratio of a stationary electron mass to a quantum mass unit: $\alpha_m = m_e^2/m_p^2$ or $\alpha_m = m_e^2/m_c^2$.

(19) The combination of direct and inverse proportion ratios was shown to create the particle-wave equations: Planck relation (5.10), Compton wavelength (5.43), Schrödenger (5.14), and Dirac (5.15) equations.

The equations seem to agree with the physical observations of particle-wave duality. But consider the moral about the four blind men experiencing an elephant for the first time: The first man feels the tail and says, "An elephant is a rope." The second man feels the leg and says, "You must be feeling a branch, because I feel a large tree trunk." The third man feels the body of the elephant and says, "You are feeling a tree in front of a wall." The fourth man feels the trunk that wraps around his arm and screams, "Run for your lives it's giant snake!" A particle-wave electron is as insightful as a rope-tree-wall-snake elephant.

(20) Special and general relativity assume covariance, which states that the laws of physics are invariant in every coordinate frame of reference [**Ein15**]. The infinitesimal volume around every point on Riemann and pseudo-Riemann surfaces is Euclidean-like. Therefore, the same ratios exist near every coordinate point on the surfaces, which causes the same laws of physics at every coordinate frame of reference.
   (a) The ratio-based derivations of the spacetime equations, in this article (5.5), do not rely on the Lorentz transformations or Einsteins' postulates [**Ein15**]. The derivations do not even require the notion of light.
   (b) The ratio-based derivations are also valid for spacemass and spacecharge.
   (c) The special relativity time dilation equation 5.21 was derived from the distance-to-time ratio, $r = (r_c/t_c)t$, and combined with the distance-to-mass ratio, $r = (r_c/m_c)m$, (5.8) yielded Schwarzchild's gravitational time dilation and black hole metric equations (5.28).

(21) The derivation of Schrödenger (5.14) and Dirac wave equations (5.15), in this article, differs from other derivations:
   (a) Other derivations are based on the Hamiltonian (energy-momentum) operator, which is defined rather than derived. In contrast, the derivations, in this article, rely on the ratio-derived Planck (energy-frequency) relation.
   (b) The derivations here are more rigorous because the energy-momentum term, $h^2/2m$, was derived, in this article, from the Planck relation (5.52), where the Planck relation was also rigorously derived (5.10). Other derivations **incorrectly** assume (define) the energy-momentum relation as: $(\mathbf{p} \cdot \mathbf{p})/2m = \hbar^2/2m$. But the more rigorous derivation, in this article, shows that the reduced Planck constant is only valid if the equations contain compensating π based terms. For example, in Schrödenger's equation, the compensating 2π terms: $\partial^2\Psi(r,t)/\partial r^2 = (-(2\pi)^2/r^2)\Psi(r,t)$ and $\partial\Psi(r,t)/\partial t = (i\ 2\pi/t)\Psi(r,t)$. And in Dirac's equation, the compensating π term: $A_0 = \pi(c_q/c)((1/t)+(1/r))$. Finding solutions to Schrödenger's equation would be simpler if the full Planck constant is used because it would reduce the complexity of $\Psi(r,t)$.

(22) The quantum extensions to: Schwarzchild's time dilation 5.69 black hole metric (5.32), Newton's gravity force (5.72), and Coulomb's charge force (5.75) make quantifiable predictions:
   (a) The gravitation and charge forces peak at finite amounts as $r \to 0$: $\lim_{r\to 0} F = c^2 m_1 m_2 / k_m$ and $\lim_{r\to 0} F = c^2 q_1 q_2 k_m / k_q^2$. Finite maximum gravity and charge forces allows radioactivity, finite sloped energy well walls, and possibly black hole evaporation.
   (b) Understanding the quantum effect, where the mass is constant, and distance varies, is easiest to understand using the quantum extensions to Newton's gravity and Coulomb's charge equations. Both equations reduce to the classic equations, where the distance between masses and charges is sufficiently large or the masses and charges sufficiently large that the quantum effect is not measurable. **Note** that G, k_e, ε_0, μ_0, and k (Einstein's constant, which contains G) are not valid, where the quantum effects becomes measurable.
   (c) And the covariant tensor components, in Einstein's field equations, that had the units $1/distance^2$, will now have the more complex units, $1/\sqrt{(distance^4/c_m^2) + k_m^2}$.
   (d) $1/\sqrt{(distance^4/c_m^2) + k_m^2}$ implies that as distance $\to 0$, spacetime curvature peaks at a finite amount, which predicts that black holes probably have sizes > 0 (are probably not singularities). The big bang might not have originated from a singularity.
   (e) Schwarzchild defined the black hole event horizon radius, $r_s := 2Gm/c^2$, where $2Gm/c^2 = 2(c_m c^2)m/c^2 = 2c_m m$. The event horizon radius with the quantum extension is $r_s := 2Q_m = 2/\sqrt{(c_m m)^2 + (k_m/m)^2}$. Where the mass is sufficiently large that the quantum effect, k_m/m, is not measurable, the two equations are the same.

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