Some Set Properties Underlying Geometry and Physics

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Abstract

Euclidean volume and some distance equations are proved to be instances of abstract sets of ordered combinations (n-tuples). The commutative property of the operations defining volume and distance is proved to constrain sets of unordered domain values to at most 3 values, e.g., 3 distance values and 3 non-distance values (time, mass, and charge). Where the space near each local coordinate point is Euclidean-like, each 3-dimensional distance unit corresponds to units of time, mass, and charge. Unit ratios are used for shorter and simpler derivations of gravity, charge, electromagnetic, relativity, quantum physics equations, and constants. The ratios are also used for simple quantum extensions to classical and relativity equations. The proofs are verified in Rocq.

[1]

CONTENTS

Keywords: mathematical physics, combinatorics, set theory, distance measure, inner product, gravity, charge, electromagnetism, relativity, quantum physics.

INTRODUCTION

Classical (Galilean transformation) physics assumes Euclidean space. And general relativity assumes a pseudo-Riemann surface (a type of inner product space), where the space near each local coordinate point is Euclidean-like [2][3].

But integration, topology, measure theory, and vector analysis define volume, distance, and inner product, where the definitions (axioms) model geometry [4][5][6][3]. Deriving volume and distance equations without modeling the geometry notions of: point, straight line, side, angle, length, area, and volume, provides an analytic (set, sequence, limit, and combinatorial) perspective, which has applications to geometry and physics.

The proofs, in this article, have been verified using using the Rocq proof verification system [7]. The formal proofs are in the Rocq files, "euclidrelations.v" and "threed.v," which are included as ancillary files.

Let $|x_i|$ be the cardinal of (number of elements in) the countable set, $x_i \in \{x_1, \cdots, x_n\}$. Where the number of ordered combinations (n-tuples) of $x_1, \cdots, x_n$ is the countable range value, v_c, the Euclidean volume equation will be proved to be an instance v_c:

$$\forall\, v_c, |x_i| \in \{0,\ \mathbb{N}\},\ \ x_i \in \{x_1, \cdots, x_n\}, v_c = \textstyle\prod_{i=1}^{n} |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^{n} s_i,\ \ s_i, v \in \mathbb{R}. \tag{1}$$

For all $n > 1$, there are many cases where different domain values, $|x_1|, \cdots, |x_n|$, multiplied yield the same range value, v_c. Inferring a domain value, d_c, from v_c, requires an inverse (bijective) function, $d_c = f_n^{-1}(v_c)$ and $v_c = f_n(d_c)$. The simplest bijective case for $n = 1$ is: $v_c = |x_1| = d_c$. The simplest bijective case for all n that includes the $n = 1$ case is:

$$\exists\, d_c, v_c, |x_i| \in \{0, \mathbb{N}\} :\ v_c = \textstyle\prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n. \tag{2}$$

A set of n-tuples being the union of disjoint subsets of n-tuples implies that the domain value, d_c, is also the inverse function of the sum of n-tuples, where it will be proved that:

$$d_c^n = v_c = \textstyle\sum_{i=1}^{m} v_{c_i} = \sum_{i=1}^{m} (\prod_{j=1}^{n} |x_{i,j}|) \quad \Rightarrow \quad d^n = \sum_{i=1}^{m} (\prod_{j=1}^{n} s_{i,j}). \tag{3}$$

Where each $s_{i,j}$ is $\pm$-signed, the $n = 2$ case is the vector inner product.

Where each v_{c_i} also is the bijective function, $v_{c_i} = d_{c_i}^n$:

$$d_c^n = \textstyle\sum_{i=1}^{m} d_{c_i}^n \quad \Rightarrow \quad d^n = \sum_{i=1}^{m} d_i^n. \tag{4}$$

$|d|$ is the p-norm (Minkowski distance) [8], which will be proved to imply the metric space properties [5]. The $n = 2$ case is the Euclidean distance.

Many sets contain unordered elements, where each element does not have an intrinsic property making it the first, second, $\cdots$, or last (n-th), for example, the set, {height, width, depth} $\equiv$ {depth, width, height}. A set of unordered elements can be placed in a "list," where the list is sequenced one-by-one in a total order via successor and predecessor functions, $successor(x_i) = x_{i+1}$ and $predecessor(x_i) = x_{i-1}$, thereby allowing calculation of the number of n-tuples, volumes, and distances, for example, $v = \prod_{i=1}^{n} s_i = \prod_{i=n}^{1} s_i$.

But the union, intersection, multiplication, and addition operations defining the total number of n-tuples and the corresponding volume and distance equations are commutative.

The commutative property allows sequencing the *same* totally ordered list of n number of elements in each of the $n!$ possible unique sequences (permutations).

Only a cyclic list, where $successor(x_n) = x_1$ and $predecessor(x_1) = x_n$, allows sequencing 1 through n (and n through 1), and also allows each element to be the first element of some those $n!$ sequences in the *same* list. For example, where $n = 3$, the element s_1 is the first element of the cyclic successor sequence, $v = s_1 \times s_2 \times s_3$, and the cyclic predecessor sequence, $v = s_1 \times s_3 \times s_2$. And s_2 is the first element of the cyclic successor sequence, $v = s_2 \times s_3 \times s_1$, and the cyclic predecessor sequence, $v = s_2 \times s_1 \times s_3$. And so on.

The only cyclic list that allows all $n!$ permutations using the cyclic successor and predecessor functions is where each list element is sequentially adjacent (either an *immediate* cyclic successor or an *immediate* cyclic predecessor) to every other element, herein, referred to as an immediate symmetric cyclic list (ISCL). An ISCL will be proved to have $n \leq 3$ elements.

Application to physics uses the ISCL and Euclidean space properties:

1. **ISCL:** $\{r_1, r_2, r_3\}$ is an ISCL of 3 "distance" domain values, each value $\in \mathbb{R}$, and $\{t\ (time), m\ (mass),\ q\ (charge)\}$ is the ISCL of 3 "non-distance" domain values, each value $\in \mathbb{R}$. Physical space is 6-dimensional: r_1-r_2-r_3-t-m-q.

2. **Cartesian:** The Euclidean volume and distance equations are derived sets of n-tuples, where for each n-tuple $\in x$ there is a corresponding value, $\kappa_x \in \mathbb{R}$ (each κ_x the same size). And for each n-tuple $\in y$ there is a corresponding value, $\kappa_y \in \mathbb{R}$ (each κ_y the same size). This implies there are constant ratios between an n-tuple value κ_x and the n-tuple value, κ_y. Near each Euclidean-like local coordinate point, there is a Cartesian grid, where each unit length, r_δ, of a 3-dimensional length, $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$, corresponds to unit lengths: t_δ of time, t; m_δ of mass, m; and q_δ of charge, q.

A consequence of the 1-1 correspondence of units is the direct proportion ratios: $r = (r_\delta/t_\delta)t$, $r = (r_\delta/m_\delta)m$, and $r = (r_\delta/q_\delta)q$. Letting: $c_t = r_\delta/t_\delta$, $c_m = r_\delta/m_\delta$, and $c_q = r_\delta/q_\delta$, the proofs and the 3 direct proportion ratios, c_t, c_m, and c_q, are used to provide simpler and shorter derivations of: the special relativity equations[2][3], the gravitational constant, $G = c_m c_t^2$, the Newton, Gauss, and Poisson gravity equations [9][10], the Schwarzschild gravitational time dilation and black hole metric equations [11][12] (illustrating a simple method of finding solutions to Einstein's general relativity equations), the Coulomb, Lorentz, and Faraday laws [10] and their related constants $k_e = c_q^2 c_t^2 / c_m$, $\varepsilon_0 = 1/4\pi k_e$, $\mu_0 = 4\pi c_q^2 / c_m$. The values of c_t, c_m, and c_q are derived from the empirical values: $c = c_t$, G, and k_e.

One of the most fundamental rules of the physical world is, "You don't get something for nothing." Algebraic manipulation of the 3 direct proportion ratios yields 3 inverse proportion ratios, $r = t_\delta r_\delta / t$, $r = m_\delta r_\delta / m.$ amd $r = q_\delta r_\delta / q$, where an increase of one thing corresponds to a decrease in something else. And let $k_t = t_\delta r_\delta$, $k_m = m_\delta r_\delta$, and $k_q = q_\delta r_\delta$.

The direct and inverse proportion ratios are used to derive the Planck relation [10][13] and the Planck constant, $h = k_m c_t$. The values of k_t, k_m, and k_q are derived from h, c_t, c_m, and c_q. The quantum unit values, $\{r_\delta$, t_δ, m_δ, $q_\delta\}$, and the reduced Planck unit values, $\{r_p$, t_p, m_p, $q_p\}$, are derived from the ratios. Derivations from the ratios show that G, k_e, ε_0, μ_0, and h are **not** fundamental (atomic) constants.

After Coulomb's charge law and the value of the reduced Planck charge unit, q_p, have been derived, a new rationale for the fine structure electron coupling constant, α, is presented, where α is the ratio of two subtypes of Coulomb's charge force that reduces to the unitless ratio, $\alpha = q_e^2 / q_p^2$, which is simpler and more elucidating than the CODATA standard equation, $\alpha = q_e^2 / 4\pi\varepsilon_0 \hbar c$ [14].

The ratios and Planck relation are used to derive the Compton wavelength, the Schrödenger position-space wave equation and Dirac wave equation [10][15][16]. And the inverse proportion ratios are used to add quantum extensions to some general relativity and classical physics equations.

RULER MEASURE AND CONVERGENCE

Definition .1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\, s, \kappa \in \mathbb{R}, \quad 0 < |\kappa| \leq 1,$ $(p = floor(s/\kappa) \quad \vee \quad p = ceiling(s/\kappa))$.

Theorem .2. *Ruler convergence:* $M = \lim_{\kappa \to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

Proof. (epsilon-delta proof)

$floor(x)$ is the integer part of x. Therefore:

$$floor(x) = max(\{y : \; y \leq x, \; y \in \mathbb{Z}, \; x \in \mathbb{R}\}) \quad \Rightarrow \quad |floor(x) - x| < 1. \tag{5}$$

$$|floor(s/\kappa) - s/\kappa| < 1 \quad \wedge \quad p = floor(s/\kappa) \quad \Rightarrow \quad |p - s/\kappa| < 1. \tag{6}$$

Multiply both sides of inequality 6 by $|\kappa|$:

$$|p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{7}$$

$$\forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \quad \Rightarrow \quad |\kappa - 0| < \delta \ \wedge \ |p\kappa - s| < \epsilon$$
$$:= \quad M = \lim_{\kappa \to 0} p\kappa = s. \quad \square \tag{8}$$

The following is an example of ruler convergence where, $s = \pi \ \Rightarrow \ p = floor(s/\kappa) \Rightarrow p\kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, ..., \ \pi_{\lim_{\kappa\to 0}}$.

Lemma .3. $\forall\, n \geq 1,\ 0 < |\kappa| < 1 : \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa \to 0} \kappa.$

Proof. The formal proof , “lim_c_to_n_eq_lim_c,” is in the Rocq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa \leq 1 \quad \Rightarrow \quad 0 < \kappa^n \leq \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| \leq |\kappa| = |\kappa - 0|. \tag{9}$$

$$\forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| \leq |\kappa - 0| < \delta \quad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon$$
$$:= \quad \lim_{\kappa\to 0} \kappa^n = 0. \tag{10}$$

$$\lim_{\kappa\to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa\to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa. \tag{11}$$

$\square$

VOLUME

Euclidean volume

Theorem .4. *Euclidean volume,*

$$\forall\, v_c, d_c, |x_i| \in \{0,\ \mathbb{N}\},\ \ x_i \in \{x_1, \cdots, x_n\},\ v_c = \textstyle\prod_{i=1}^n |x_i| \ \Rightarrow \ v = \prod_{i=1}^n s_i, \quad s_i, v \in \mathbb{R}. \tag{12}$$

The formal proof, “Euclidean_volume,” is in the Rocq file, euclidrelations.v.

Proof.

$$v_c = \textstyle\prod_{i=1}^n |x_i| \quad \Leftrightarrow \quad v_c\kappa = (\prod_{i=1}^n |x_i|)\kappa \quad \Leftrightarrow \quad \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{13}$$

Apply the ruler (.1) and ruler convergence (.2) to equation 13:

$$\exists\, v, \kappa \in \mathbb{R} :\, v_c = floor(v/\kappa) \quad\Rightarrow\quad v = \lim_{\kappa\to 0} v_c\kappa \quad\wedge\quad \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\textstyle\prod_{i=1}^{n}|x_i|)\kappa$$
$$\Rightarrow\quad v = \lim_{\kappa\to 0}(\textstyle\prod_{i=1}^{n}|x_i|)\kappa. \quad (14)$$

Apply lemma .3 to equation 14:

$$v = \lim_{\kappa\to 0}(\textstyle\prod_{i=1}^{n}|x_i|)\kappa \quad\wedge\quad \lim_{\kappa\to 0}\kappa^n = \lim_{\kappa\to 0}\kappa$$
$$\Rightarrow\quad v = \lim_{\kappa\to 0}(\textstyle\prod_{i=1}^{n}|x_i|)\kappa^n = \lim_{\kappa\to 0}(\prod_{i=1}^{n}(|x_i|\kappa)). \quad (15)$$

Apply the ruler (.1) and ruler convergence (.2) to s_i:

$$\exists\, s_i, \kappa \in \mathbb{R} :\ floor(s_i/\kappa) = |x_i| \quad\Rightarrow\quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i. \quad (16)$$

$$v = \lim_{\kappa\to 0}(\textstyle\prod_{i=1}^{n}|x_i|\kappa) \quad\wedge\quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i \quad\Leftrightarrow\quad v = \prod_{i=1}^{n} s_i \quad (17)$$

□

Lemma .5. *The number of n-tuples, v_c, is the sum of the number of n-tuples, v_{c_i}, in each subset of n-tuples, implies a volume is the sum of volumes,*

$$v_c = \textstyle\sum_{i=1}^{m} v_{c_i} \quad\Rightarrow\quad v = \sum_{i=1}^{m} v_i, \quad v, v_i \in \mathbb{R}.$$

The formal proof, "sum_of _volumes," is in the Rocq file, euclidrelations.v.

Proof. From the condition of this theorem:

$$v_c = \textstyle\sum_{i=1}^{m} v_{c_i} \quad\Leftrightarrow\quad \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}\sum_{j=1}^{m}(v_{c_i}\kappa). \quad (18)$$

Apply lemma .3 to equation 18:

$$\lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\textstyle\sum_{j=1}^{m} v_{c_i})\kappa \quad\wedge\quad \lim_{\kappa\to 0}\kappa^n = \lim_{\kappa\to 0}\kappa$$
$$\Leftrightarrow\ \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}\textstyle\sum_{j=1}^{m}(v_{c_i}\kappa). \quad (19)$$

Apply the ruler (.1) and ruler convergence theorem (.2) to equation 19:

$$\exists\, v, v_i :\ v = \lim_{\kappa\to 0} v_c\kappa \quad\wedge\quad \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}\textstyle\sum_{j=1}^{m}(v_{c_i}\kappa)$$
$$\Rightarrow\quad v == \lim_{\kappa\to 0}\textstyle\sum_{j=1}^{m}(v_{c_i}\kappa). \quad (20)$$

Apply the ruler (.1) and ruler convergence theorem (.2) to equation 20:

$$v == \lim_{\kappa\to 0}\textstyle\sum_{j=1}^{m}(v_{c_i}\kappa) \quad\wedge\quad \exists\, v_i, v_{c_i} :\ v_i = \lim_{\kappa\to 0} v_{c_i}\kappa \quad\Rightarrow\quad v = \sum_{j=1}^{m} v_i. \quad (21)$$

□

DISTANCE

Definition .6. Bijective, countable domain value, d_c:

$$\exists\, v_c, d_c, |x_i| \in \{0,\ \mathbb{N}\},\ \ x_i \in \{x_1, \cdots, x_n\}, \quad v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n. \tag{22}$$

Vector inner product

Theorem .7. *Sum of volumes distance:*

$$d_c^n = v_c = \sum_{i=1}^{m} v_{c_i} \quad \Rightarrow \quad d^n = \sum_{i=1}^{m} (\prod_{j=1}^{n} s_{i_j}).$$

The formal proof, "sum_of _volumes _distance," is in the Rocq file, euclidrelations.v.

Proof. From the sum of volumes lemma .5 and the Euclidean volume theorem .4:

$$d_c^n = \sum_{i=1}^{m} v_{c_i} \quad \Rightarrow \quad d^n = \sum_{i=1}^{m} v_i \quad \wedge \quad v_i = \prod_{j=1}^{n} s_{i_j}$$
$$\Rightarrow \quad d^n = \sum_{i=1}^{m} v_i = \sum_{i=1}^{m} (\prod_{j=1}^{n} s_{i_j}). \quad \square \tag{23}$$

Note: In the volume proof (.4), where κ is negative, volume is negative. From the lemma (.3), $\forall\, n \geq 1,\ 0 < |\kappa| < 1 : \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa,\ \exists\, \kappa_i \in \{\kappa_1, \cdots, \kappa_n\}$. Therefore, κ_i can have negative values, where $\prod_{i=1}^{n} \kappa_i = \kappa^n$. And where $\kappa_i < 0$, the corresponding $s_{i,j} < 0$. Therefore, the $n = 2$ case of the sum of volumes distance is the vector inner product.

Minkowski distance (p-norm)

Theorem .8. *Minkowski distance (p-norm):*

$$d_c^n = v_c = \sum_{i=1}^{m} v_{c_i} = \sum_{i=1}^{m} d_{c_i}^n \quad \Leftrightarrow \quad d^n = \sum_{i=1}^{m} d_i^n.$$

The formal proof, "Minkowski_distance," is in the Rocq file, euclidrelations.v.

Proof. From the sum of volumes distance theorem .7 and the Euclidean volume theorem .4:

$$d_c^n = v_c = \sum_{i=1}^{m} v_{c_i} \quad \Rightarrow \quad d^n = v = \sum_{i=1}^{m} v_i \quad \wedge \quad v_i = \prod_{j=1}^{n} d_i = d_i^n$$
$$\Rightarrow \quad d^n = \sum_{i=1}^{m} d_i^n \quad \square \tag{24}$$

Theorem .9. *Distance triangle inequality*

$$\forall\, n \in \mathbb{N},\;\; v_a, v_b \geq 0 :\;\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

The formal proof, distance_inequality, is in the Rocq file, euclidrelations.v.

Proof. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$\forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq v_a + v_b + \textstyle\sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \textstyle\sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n. \quad (25)$$

Take the n^{th} root of both sides of the inequality 25:

$$\forall\, v_a, v_b \geq 0,\;\; n \in \mathbb{N} : \quad v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \quad \Rightarrow \quad (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad (26)$$

□

Theorem .10. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N},\;\; a_i, b_i \geq 0 : (\textstyle\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \leq (\textstyle\sum_{i=1}^{m} a_i^n)^{1/n} + (\textstyle\sum_{i=1}^{m} b_i^n)^{1/n}.$$

The formal proof, distance_sum_inequality, is in the Rocq file, euclidrelations.v.

Proof. Apply the distance triangle inequality (.9):

$$\forall\, m, n \in \mathbb{N},\;\; v_a, v_b \geq 0 :\;\; v_a = \textstyle\sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \textstyle\sum_{i=1}^{m} b_i^n \quad \wedge \quad (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}$$
$$\Rightarrow \quad ((\textstyle\sum_{i=1}^{m} a_i^n) + (\textstyle\sum_{i=1}^{m} b_i^n))^{1/n} = (\textstyle\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \leq (\textstyle\sum_{i=1}^{m} a_i^n)^{1/n} + (\textstyle\sum_{i=1}^{m} b_i^n)^{1/n}. \quad \square \quad (27)$$

Metric Space

All Minkowski distances (p-norms) imply the metric space properties. The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Rocq file, euclidrelations.v.

Theorem .11. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \Rightarrow d(u, w) \leq d(u, v) + d(v, w).$

Proof. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{28}$$

Apply the distance triangle inequality (.9) to the inequality 28:

$$\begin{aligned}(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \quad \wedge \quad (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ \Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ \Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \quad \square\end{aligned} \tag{29}$$

Theorem .12. *Symmetry:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

Proof. By the commutative law of addition:

$$\begin{aligned}\forall\, p \,:\, p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \quad \square\end{aligned} \tag{30}$$

Theorem .13. *Non-negativity:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \Rightarrow d(u, w) \geq 0.$

Proof. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0. \tag{31}$$

$$p \geq 1,\ \ u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \tag{32}$$

$\square$

Theorem .14. *Identity of Indiscernibles:* $d(u, u) = 0$.

Proof. From the non-negativity property (.13):

$$\begin{aligned}d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0 \\ \Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0.\end{aligned} \tag{33}$$

$$d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v. \tag{34}$$

$$d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \tag{35}$$

$\square$

Properties limiting a set to at most 3 elements

The following definitions and proof use first order logic. A Horn clause-like expression is used, here, to make the proof easier to read. By convention, the proof goal is on the left side and supporting facts are on the right side of the implication sign ($\leftarrow$). The formal proofs in the Rocq file threed.v are: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.

Definition .15. Immediate Cyclic Successor of m is n:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\} :$$
$$Successor(m, n, setsize) \quad \leftarrow \quad (m = setsize \wedge n = 1) \quad \vee \quad (n = m + 1 \leq setsize). \tag{36}$$

Definition .16. Immediate Cyclic Predecessor of m is n:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\} :$$
$$Predecessor(m, n, setsize) \quad \leftarrow \quad (m = 1 \wedge n = setsize) \quad \vee \quad (n = m - 1 \geq 1). \tag{37}$$

Definition .17. Adjacent: element m is sequentially adjacent to element n if the immediate cyclic successor of m is n or the immediate cyclic predecessor of m is n. Notionally:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\} :$$
$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \ \vee\ Predecessor(m, n, setsize). \tag{38}$$

Definition .18. Immediate Symmetric (every set element is sequentially adjacent to every other element):

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\} : \quad Adjacent(m, n, setsize). \tag{39}$$

Theorem .19. *An immediate symmetric cyclic list (ISCL) is limited to at most 3 elements.*

Proof.
Every element is adjacent to every other element, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \tag{40}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \tag{41}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m + 1 \le setsize). \tag{42}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m - 1 \ge 1). \tag{43}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \land m = 1). \tag{44}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \land n = setsize). \tag{45}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m + 1 \le setsize). \tag{46}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m - 1 \ge 1). \tag{47}$$

Element 2 is the only immediate successor of element 1 for all $setsize \ge 3$, which implies element 3 is not ($\neg$) an immediate successor of element 1 for all $setsize \ge 3$:

$$\neg Successor(1,3,setsize \ge 3) \leftarrow Successor(1,2,setsize \ge 3) \leftarrow (n = m + 1 \le setsize). \tag{48}$$

Element $n = setsize > 3$ is the only immediate predecessor of element 1, which implies element 3 is not ($\neg$) an immediate predecessor of element 1 for all $setsize > 3$:

$$\neg Predecessor(1,3,setsize \ge 3) \leftarrow Predecessor(1,setsize,setsize > 3) \leftarrow (m = 1 \land n = setsize > 3). \tag{49}$$

For all $setsize > 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not immediate symmetric):

$$\neg Adjacent(1,3,setsize > 3) \leftarrow \neg Successor(1,3,setsize > 3) \quad \land \quad \neg Predecessor(1,3,setsize > 3). \quad \square \tag{50}$$

The Symmetric goal matches Adjacent goals 40 through 47 and fails for all "setsize" greater than three.

APPLICATIONS TO PHYSICS

Derivation of 3 direct proportion ratios: c_t, c_m, and c_q

Application to physics uses the ISCL (.19) and Euclidean space properties:

1. **ISCL:** Let: $\{r_1, r_2, r_3\}$ is an ISCL of 3 "distance" domain values, each value $\in \mathbb{R}$, and $\{t\ (time), m\ (mass),\ q\ (charge)\}$ is the ISCL of 3 "non-distance" domain values, each value $\in \mathbb{R}$.

2. **Cartesian:** The Euclidean volume and distance equations are derived sets of n-tuples (.4) (.8), where for each n-tuple $\in x$ there is a corresponding value, $\kappa_x \in \mathbb{R}$ (each κ_x the same size). And for each n-tuple $\in y$ there is a corresponding value, $\kappa_y \in \mathbb{R}$ (each κ_y the same size). This implies there are constant ratios between an n-tuple value κ_x and the n-tuple value, κ_y. Near each Euclidean-like local coordinate point, there is a Cartesian grid, where each unit length, r_δ, of a 3-dimensional length, $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$, corresponds to unit lengths: t_δ of time, t; m_δ of mass, m; and q_δ of charge, q.

A correspoinding number of units can be expressed as, $r/r_\delta = \tau/\tau_\delta$, where $\tau \in \{t, m, q\}$. Multiplying both sides by r_δ yields 3 direct proportion ratios:

$$\forall\, r_\delta, \tau_\delta, r, \tau \subseteq \mathbb{R},\ r_\delta, \tau_\delta\ \textbf{constants},\ \tau \in \{t, m, q\} :\ r/r_\delta = \tau/\tau_\delta \quad \Rightarrow \quad r = (r_\delta/\tau_\delta)/\tau$$
$$\Rightarrow \quad r = (r_\delta/t_\delta)t \quad \wedge \quad r = (r_\delta/m_\delta)m \quad \wedge \quad r = (r_\delta/q_\delta)q. \quad (51)$$

And for conciseness, let:

$$r_\delta/t_\delta = c_t \quad \wedge \quad r_\delta/m_\delta = c_m \quad \wedge \quad r_\delta/q_\delta = c_q. \quad (52)$$

Derivation of space-time-mass-charge

$$\forall\, \tau \in \{t, m, q\},\ r^2 = r'^2 + r_v^2,\ \exists\, \mu, \nu :\ r = \mu\tau \quad \wedge \quad r_v = \nu\tau \quad \Rightarrow \quad (\mu\tau)^2 = r'^2 + (\nu\tau)^2$$
$$\Rightarrow \quad r' = \sqrt{(\mu\tau)^2 - (\nu\tau)^2} \;=\; \mu\tau\sqrt{1 - (\nu/\mu)^2}. \quad (53)$$

Local frame distance, r', contracts relative to a distant observer frame distance, r, as $\nu \to \mu$:

$$r' = \mu\tau\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad \mu\tau = r \quad \Rightarrow \quad r' = r\sqrt{1 - (\nu/\mu)^2}. \quad (54)$$

A distant observer frame type, τ, dilates relative to the local observer frame type, τ', as $\nu \to \mu$:

$$\mu\tau = r'/\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad r' = \mu\tau \quad \Rightarrow \quad \tau = \tau'/\sqrt{1 - (\nu/\mu)^2}. \quad (55)$$

Where τ is type, time, the space-like flat Minkowski spacetime event interval is:

$$\mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_v^2 \quad \wedge \quad \mathrm{d}r_v^2 = \mathrm{d}r_1^2 + \mathrm{d}r_2^2 + \mathrm{d}r_3^2 \quad \wedge \quad \mathrm{d}(\mu\tau) = \mathrm{d}r$$
$$\Rightarrow \; \mathrm{d}r'^2 = \mathrm{d}(\mu\tau)^2 - \mathrm{d}r_1^2 - \mathrm{d}r_2^2 - \mathrm{d}r_3^2, \quad \text{or in 6 dimensions :}$$
$$\mathrm{d}r'^2 = \mathrm{d}(c_t t)^2 + \mathrm{d}(c_m m)^2 + \mathrm{d}(c_q q)^2 - \mathrm{d}r_1^2 - \mathrm{d}r_2^2 - \mathrm{d}r_3^2. \quad (56)$$

Derivation of G, and the Newton, Gauss, and Poisson gravity laws

From equations 52:

$$r = c_m m \;\wedge\; r = c_t t \;\Rightarrow\; r/(c_t t)^2 = c_m m/r^2 \;\Rightarrow\; r/t^2 = (c_m c_t^2)m/r^2 = Gm/r^2, \quad (57)$$

where $G = c_m c_t^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$ [9].

Newton's law follows from multiplying both sides of equation 57 by m:

$$r/t^2 = Gm/r^2 \;\Leftrightarrow\; F := mr/t^2 = Gm^2/r^2. \quad (58)$$

$$F = Gm^2/r^2 \;\wedge\; \forall\, m \in \mathbb{R} :\, \exists\, m_1, m_2 \in \mathbb{R} :\, m_1 m_2 = m^2 \quad \Rightarrow \quad F = Gm_1 m_2/r^2. \quad (59)$$

$2\pi r$ is the length of a curve circumscribing a sphere containing a Gauss gravity field, $\mathbf{g}$, and Poisson's gravity field, $-\nabla\Phi(\tilde{\mathbf{r}}, t)$. Multiplying both sides of equation 57 by 2π:

$$r/t^2 = Gm/|\tilde{\mathbf{r}}|^2 \quad \wedge \quad \mathbf{g} = -\nabla\Phi(\tilde{\mathbf{r}}, t) = 2\pi r/t^2 \quad \Rightarrow \quad \mathbf{g} = -\nabla\Phi(\tilde{\mathbf{r}}, t) = 2\pi Gm/|\tilde{\mathbf{r}}|^2 \quad (60)$$

$$\mathbf{g} = -\nabla\Phi(\tilde{\mathbf{r}}, t) = 2\pi Gm/|\tilde{\mathbf{r}}|^2 \quad \Rightarrow \quad \nabla \cdot \mathbf{g} = \nabla^2\Phi(\tilde{\mathbf{r}}, t) = -4\pi Gm/|\tilde{\mathbf{r}}|^3 = -4\pi G\rho, \quad (61)$$

where $\rho = m/|\tilde{\mathbf{r}}|^3$ is the mass density:

Derivation of Schwarzschild's gravitational time dilation and black hole metric

From equations 54 and 51:

$$\sqrt{1 - (v^2/c^2)} = \sqrt{1 - (1)(v^2/c^2)} \wedge c_m m/r = 1 \;\Rightarrow\; \sqrt{1 - (v^2/c^2)} = \sqrt{1 - c_m m v^2/rc^2}. \quad (62)$$

Where v_{escape} is the escape velocity:

$$\sqrt{1 - (v^2/c^2)} = \sqrt{1 - c_m m v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv_{escape}^2$$
$$\Rightarrow \quad \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2c_m m v_{escape}^2/rc^2}. \quad (63)$$

For a photon, the escape velocity, $v_{escape} = c$.

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-2c_m m v_{escape}^2/rc^2}. \wedge v_{escape} = c$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2c_m mc^2/rc^2}. \quad (64)$$

Combining equation 64 with the derivation of G (59):

$$c_m c^2 = G \wedge \sqrt{1-(v^2/c^2)} = \sqrt{1-2c_m mc^2/rc^2} \Rightarrow \sqrt{1-(v^2/c^2)} = \sqrt{1-2Gm/rc^2}. \quad (65)$$

Combining equation 65 with equation 55 yields Schwarzschild's gravitational time dilation [11] [12]:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-2Gm/rc^2} \wedge t' = t\sqrt{1-(v^2/c^2)} \Rightarrow t' = t\sqrt{1-2Gm/rc^2}. \quad (66)$$

From equations 54 and 65, where Schwarzschild's defined the black hole event horizon radius as $\alpha := 2Gm/c^2$:

$$r' = r\sqrt{1-(v/c)^2} = r\sqrt{1-2Gm/rc^2} \quad \wedge \quad \alpha := 2Gm/c^2 \quad \Rightarrow \quad r' = r\sqrt{1-\alpha/r}. \quad (67)$$

Applying equation 67 to the time-like spacetime interval equation 56:

$$r' = r\sqrt{1-\alpha/r} \quad \wedge \quad \mathrm{d}s^2 = \mathrm{d}r'^2 - \mathrm{d}r^2$$
$$\Rightarrow \quad \mathrm{d}s^2 = (\sqrt{1-\alpha/r}\mathrm{d}r)^2 - (\mathrm{d}r'/\sqrt{1-\alpha/r})^2 = (1-\alpha/r)\mathrm{d}r^2 - (1-\alpha/r)^{-1}\mathrm{d}r'^2. \quad (68)$$

$$\mathrm{d}s^2 = (1-\alpha/r)\mathrm{d}r^2 - (1-\alpha/r)^{-1}\mathrm{d}r'^2 \quad \wedge \quad \mathrm{d}r = \mathrm{d}(ct) \quad \wedge \quad c = 1 \quad \wedge \quad \lim_{\mathrm{d}s\to 0}\mathrm{d}r' = \mathrm{d}r$$
$$\Rightarrow \quad \lim_{\mathrm{d}s\to 0}\mathrm{d}s^2 = (1-\alpha/r)\mathrm{d}t^2 - (1-\alpha/r)^{-1}\mathrm{d}r^2. \quad (69)$$

Using spherical coordinates to translate from 2D to 4D yields the $+ - --$ form of Schwarzschild's black hole metric [11] [12]:

$$\mathrm{d}s^2 = (1-\alpha/r)\mathrm{d}t^2 - (1-\alpha/r)^{-1}\mathrm{d}r^2$$
$$\Rightarrow \quad \mathrm{d}s^2 = (1-\alpha/r)\mathrm{d}t^2 - (1-\alpha/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + sin^2\theta\mathrm{d}\phi^2)$$
$$\Rightarrow \quad g_{\mu,\nu} = diag[(1-\alpha/r),\ -(1-\alpha/r)^{-1},\ -r^2(\mathrm{d}\theta^2),\ -r^2(sin^2\theta\mathrm{d}\phi^2)]. \quad (70)$$

Simple method to find general relativity solutions

Einstein's field equation is:

$$G_{\mu,\nu} = \mathbf{R} + \frac{1}{2}Rg_{\mu,\nu} = \kappa T_{\mu,\nu}, \tag{71}$$

where: $G_{\mu,\nu}$ is Einstein's tensor, $\mathbf{R}$ is the Ricci curvatature, R is the scalar curvature, $g_{\mu,\nu}$ is the metric tensor, $\kappa = 8\pi G/c^4$, and $T_{\mu,\nu}$ is the stress-energy tensor [2][3].

The goal of the field equation is to determine the geodesic path and acceleration of a particle caused by the distribution of mass and energy as specified in the stress-energy tensor, $T_{\mu,\nu}$. This requires solving the equation for the metric tensor, $g_{\mu,\nu}$. But the metric tensor has a complex nonlinear relation to the Ricci and scalar curvature, which makes it complicated to determine the metric. Often there are no exact solutions.

In this article, the metric, $g_{\mu,\nu}$, is determined independent of Einstein's field equation. The infinitesimal space near every coordinate point on a pseudo-Riemann surface is Euclidean, where all physics equations derived from the ratios and special relativity equations (53) are valid. This leads to the following steps to solve for the metric tensor, $g_{\mu,\nu}$, independent of Einstein's field equation.

Step 1) Use the ratios and relativity equations to define functions returning scalar values for each component of the metric, $g_{\nu,\mu}$, in Einstein's field equations [2][3]: An example is the previous Schwarzschild black hole metric derivation using the ratios and special relativity equations (62).

Step 2) Express the Einstein field equation as 2D tensors: As shown in equation 70, the Schwarzschild metric was first derived as a 2D metric and then expanded to a 4D metric. Further, the 4D flat spacetime interval equation (56) is an instance of the 2D equation, $\mathrm{d}r'^2 = \mathrm{d}(ct)^2 - \mathrm{d}r_v^2$.

(Optional) The 2D metric tensor allows using the much simpler 2D Ricci curvature and scalar curvature. And the 2D tensors reduce the number of independent equations to solve, which can next be used to set constraints on the solutions in the 4D tensors.

Step 3) One simple method to translate from 2D to 4D is to use spherical coordinates, where r and t remain unchanged and two added dimensions are the angles, ϕ, and θ. For example, the 2D Schwarzschild metric was translated to 4D using this method in equation 70. The spherical coordinates can then be translated to other types of coordinates.

Derivation of k_e, Coulomb charge law, ε_0, E, and Gauss electric field law

From equations 52:

$$r = c_q q \ \wedge \ r = c_t t \ \Rightarrow \ r/(c_t t)^2 = c_q q/r^2 \ \Rightarrow \ r/t^2 = (c_q c_t^2) q/r^2, \tag{72}$$

$$r = c_m m = c_q q \quad \Rightarrow \quad m = (c_q/c_m) q. \tag{73}$$

Combining equations 72 and 73 yields Coulombs's law:

$$r/t^2 = (c_q c_t^2) q/r^2 \ \wedge \ m = (c_q/c_m) q \ \Rightarrow \ F := mr/t^2 = (c_q^2 c_t^2/c_m) q^2/r^2 = k_e q^2/r^2, \tag{74}$$

where $k_e = c_q^2 c_t^2/c_m$, conforms to the SI units: $kg \cdot m^3 \cdot s^{-2} \cdot C^{-2} = N \cdot m^2 \cdot C^{-2}$ [10].

$$\forall\, q \in \mathbb{R}\, \exists\, q_1, q_2 \in \mathbb{R} : \ q_1 q_2 = q^2 \quad \wedge \quad F = k_e q^2/r^2 \quad \Rightarrow \quad F = k_e q_1 q_2/r^2. \tag{75}$$

$2\pi r$ is the length of a curve circumscribing a sphere containing the electric field, **E**. Multiplying both sides of equation 72 by 2π:

$$F_C = mr/t^2 = k_e q^2/r^2 \quad \Rightarrow \quad \exists\, F_E \in \mathbb{R} : \ F_E = 2\pi(mr/t^2) = 2\pi k_e q^2/r^2 \tag{76}$$

$$F_E = 2\pi k_e q^2/r^2 = q(2\pi(k_e q/r^2)) \quad \wedge \quad \exists\, E \in R : \ E = 2\pi k_e q/r^2 \quad \Rightarrow \quad F_E = qE. \tag{77}$$

The electric field, $E := 2\pi k_e q/r^2$, conforms to the SI units $kg \cdot m \cdot s^{-2} \cdot C^{-1} = N \cdot C^{-1}$.

$$\mathbf{E} = 2\pi k_e q/|\vec{\mathbf{r}}|^2 \quad \Rightarrow \quad \nabla \cdot \mathbf{E} = -4\pi k_e q/|\vec{\mathbf{r}}|^3. \tag{78}$$

$$\nabla \cdot \mathbf{E} = -4\pi k_e q/|\vec{\mathbf{r}}|^3 \quad \wedge \quad \varepsilon_0 := 1/4\pi k_e \quad \wedge \quad \rho = q/|\vec{\mathbf{r}}|^3 \quad \Rightarrow \quad \nabla \cdot \mathbf{E} = -\rho/\varepsilon_0, \tag{79}$$

which is Gauss's electric field law [10].

Derivation of B, μ_0, and Lorentz's law

Applying the distance contraction equation, 54, to equation 77, where r is the distant observer frame of reference and r' is moving particle local frame of reference:

$$r = r'/\sqrt{1 - v^2/c^2} \quad \wedge \quad F = 2\pi k_e q^2/r^2 \quad \Rightarrow \quad F = 2\pi k_e q^2 (1 - v^2/c^2)/r'^2. \tag{80}$$

From equation 77:

$$E = 2\pi k_e q/r'^2 \ \wedge \ F = 2\pi k_e q^2 (1 - v^2/c^2)/r'^2 \ \Rightarrow \ F = q(E - (((2\pi k_e/c^2) q/r'^2)) v^2). \tag{81}$$

$$F = q(E - (((2\pi k_e/c^2)q/r'^2))v^2) \Rightarrow \exists\, B:\ B = (2\pi k_e/c^2)vq/r'^2 \ \wedge\ F = q(E - Bv). \quad (82)$$

$$F = q(E - Bv) \quad \Rightarrow \quad \mathbf{F} = q(\mathbf{E} - \mathbf{B} \times \vec{\mathbf{v}}). \quad (83)$$

$$\mathbf{B} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \mathbf{B})) \quad \wedge \quad \mathbf{F} = q(\mathbf{E} - \mathbf{B} \times \vec{\mathbf{v}}) \quad \Rightarrow \quad \mathbf{F} = q(\mathbf{E} + \vec{\mathbf{v}} \times \mathbf{B}), \quad (84)$$

which is Lorentz law in the rest (observer on the moving particle) frame of reference, where the magnetic field, $B = (2\pi k_e/c^2)vq/r'^2$, conforms to the base SI units: $kg \cdot s^{-1} \cdot C^{-1} = kg \cdot s^{-2} \cdot A^{-1} = T$.

$$B = (2\pi k_e/c^2)vq/r'^2 \quad \wedge \quad B := \mu_0 H \quad \wedge \quad \mu_0 := 4\pi k_e/c^2 \quad \Rightarrow \quad H = vq/2r'^2, \quad (85)$$

where $\mu_0 = 4\pi k_e/c^2 = 4\pi c_q^2/c_m$ conforms to the SI units $kg \cdot m \cdot C^{-2} = kg \cdot m \cdot s^{-2} A^{-2}$ and $H = vq/2r'^2$ conforms to the SI units $C \cdot s^{-1} \cdot m^{-1} = A \cdot m^{-1}$.

Derivation of Maxwell-Faraday's law

From the magnetic field equation 83, where the electric and magnetic fields are propagating at the speed, $v = c$:

$$B = (2\pi k_e/c^2)qv/r^2 \quad \wedge \quad v = c \quad \wedge \quad r = ct \quad \Rightarrow \quad B = (2\pi k_e/c^3)q/t^2. \quad (86)$$

$$B = (2\pi k_e/c^3)q/t^2 \quad \Rightarrow \quad \partial B/\partial t = -(4\pi k_e/c^3)q/t^3. \quad (87)$$

$$\partial B/\partial t = -(4\pi k_e/c^3)q/t^3 \quad \wedge \quad r = ct \quad \Rightarrow \quad \partial B/\partial t = -4\pi k_e q/r^3. \quad (88)$$

Faraday's law is the case, where the electric field, $\mathbf{E}$, is a single radial vector, $\mathbf{E}_{\tilde{\mathbf{r}}_1}$ acting on some point, P. Therefore, at point, P, $\mathbf{E}_{\tilde{\mathbf{r}}_2} = 0$ and $\mathbf{E}_{\tilde{\mathbf{r}}_3} = 0$. From the Lorentz equation 84, the magnetic field, $\mathbf{B}$ at point P, is aligned on $\tilde{\mathbf{r}}_2$, where the electric field, $\mathbf{E}_{\tilde{\mathbf{r}}_1}$, is a function of $\mathbf{B}_{\tilde{\mathbf{r}}_2}$. Combining with equation 78:

$$\mathbf{E}_{r_1} = 2\pi k_e q/|\tilde{\mathbf{r}}_2|^2 = f(\mathbf{B}_{\tilde{\mathbf{r}}_2}) \quad \wedge \quad \mathbf{E}_{r_2} = 0 \quad \wedge \quad \mathbf{E}_{r_3} = 0 \quad \Rightarrow \quad \nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial \tilde{\mathbf{r}}_1^2} & \frac{\partial}{\partial \tilde{\mathbf{r}}_2^2} & \frac{\partial}{\partial \tilde{\mathbf{r}}_3^2} \\ \mathbf{E}_{r_1} & 0 & 0 \end{vmatrix}$$

$$= (\frac{\partial \mathbf{E}_{r_3}}{\partial \tilde{\mathbf{r}}_2^2} - \frac{\partial \mathbf{E}_{r_2}}{\partial \tilde{\mathbf{r}}_3^2})\hat{\mathbf{i}} + (\frac{\partial \mathbf{E}_{r_1}}{\partial \tilde{\mathbf{r}}_3^2} - \frac{\partial \mathbf{E}_{r_3}}{\partial \tilde{\mathbf{r}}_1^2})\hat{\mathbf{j}} + (\frac{\partial \mathbf{E}_{r_2}}{\partial \tilde{\mathbf{r}}_1^2} - \frac{\partial \mathbf{E}_{r_1}}{\partial \tilde{\mathbf{r}}_2^2})\hat{\mathbf{k}}$$

$$= 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + (0 - \frac{\partial 2\pi k_e q/|\tilde{\mathbf{r}}_2|^2}{\partial \tilde{\mathbf{r}}_2^2})\hat{\mathbf{k}} \quad = \quad 4\pi k_e q/|\vec{\mathbf{r}}_2|^3\hat{\mathbf{k}} \quad \equiv \quad \nabla \times \mathbf{E} = 4\pi k_e q/|\vec{\mathbf{r}}|^3. \quad (89)$$

Combining equations 89 and 88 yields Maxwell-Faraday's law [10]:

$$\nabla \times \mathbf{E} = 4\pi k_e q/|\vec{\mathbf{r}}|^3 \quad \wedge \quad \partial \mathbf{B}/\partial t = -4\pi k_e q/|\vec{\mathbf{r}}|^3 \quad \Rightarrow \quad \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t. \quad (90)$$

Derivation of the 3 direct proportion ratio values: c_t, c_m, and c_q

$$c_t = c \approx 2.99792458 \cdot 10^8 m\ s^{-1}. \tag{91}$$

$$G = c_m c_t^2 \quad \wedge \quad G \approx 6.67418478 \cdot 10^{-11} m^3/kg/s^2 \quad \Rightarrow$$
$$c_m = G/c_t^2 \approx 7.42603211 \cdot 10^{-28} m\ kg^{-1}. \tag{92}$$

$$k_e = c_q^2 c_t^2/c_m \quad \wedge \quad k_e \approx 8.9875517923 \cdot 10^9 Nm^2/C^2 \quad \Rightarrow$$
$$c_q = \sqrt{k_e c_m/c_t^2} \approx 8.61744282 \cdot 10^{-18} m\ C^{-1}. \tag{93}$$

Derivation of 3 inverse proportion ratios: k_t, k_m, and k_q

$$r/t = r_\delta/t_\delta \quad \wedge \quad r/m = r_\delta/m_\delta \quad \Rightarrow \quad (r/t)/(r/m) = (r_\delta/t_\delta)/(r_\delta/m_\delta)$$
$$\Rightarrow \quad (mr)/(tr) = (m_\delta r_\delta)/(t_\delta r_\delta) \quad \Rightarrow \quad mr = m_\delta r_\delta = k_m, \quad tr = t_\delta r_\delta = k_t. \tag{94}$$

$$r/t = r_\delta/t_\delta \quad \wedge \quad r/q = r_\delta/q_\delta \quad \Rightarrow \quad (r/t)/(r/q) = (r_\delta/t_\delta)/(r_\delta/q_\delta)$$
$$\Rightarrow \quad (qr)/(tr) = (q_\delta r_\delta)/(t_\delta r_\delta) \quad \Rightarrow \quad qr = q_\delta r_\delta = k_q, \quad tr = t_\delta r_\delta = k_t. \tag{95}$$

Derivation of h and the Planck relation

[10][13] Applying both the direct proportion ratio (91), and inverse proportion ratio (94):

$$m = k_m/r \quad \wedge \quad r = ct \quad \Rightarrow \quad m(ct)^2 = (k_m/r)r^2 = k_m r. \tag{96}$$

$$m(ct)^2 = k_m r \quad \Rightarrow \quad E := mc^2 = k_m r/t^2. \tag{97}$$

$$E = mc^2 = k_m r/t^2 \ \wedge \ r/t = c \ \Rightarrow \ E = mc^2 = (k_m c)(1/t) = hf, \tag{98}$$

where the full Planck constant is $h = k_m c$ and frequency in cycles per second is $1/t$.

Derivation of the 3 inverse proportion ratio values: k_t, k_m, and k_q

Using $h \approx 6.62607015 \cdot 10^{-34}$:

$$k_m = \hbar/c_t \approx 2.2102190943 \cdot 10^{-42}\ kg\ m. \tag{99}$$

$$k_t = k_m c_m/c_t \approx 5.4749346710 \cdot 10^{-78}\ s\ m. \tag{100}$$

$$k_q = k_t c_t/c_q \approx 1.9046601056 \cdot 10^{-52}\ C\ m. \tag{101}$$

Derivation of the 4 quantum unit values: r_δ, t_δ, m_δ, q_δ

$$: \quad r_\delta = \sqrt{r_\delta^2} = \sqrt{c_t k_t} = \sqrt{c_m k_m} = \sqrt{c_q k_q} \approx 4.0513505432 \cdot 10^{-35}\ m. \tag{102}$$

$$t_\delta = r_\delta / c_t \approx 1.3513850782 \cdot 10^{-43}\ s. \tag{103}$$

$$m_\delta = r_\delta / c_m \approx 5.4555118613 \cdot 10^{-8}\ kg. \tag{104}$$

$$q_\delta = r_\delta / c_q \approx 4.701296728 \cdot 10^{-18}\ C. \tag{105}$$

Derivation of the 4 reduced Planck unit values: r_p, t_p, m_p, q_p

$$: \quad r_p = r_\delta / \sqrt{2\pi} = \sqrt{r_\delta^2 / 2\pi} = \sqrt{c_t (k_t / 2\pi)} = \sqrt{c_m (k_m / 2\pi)} = \sqrt{c_q (k_q / 2\pi)} \tag{106}$$

$$r_p \approx 1.61624107 \cdot 10^{-35}\ m. \tag{107}$$

$$t_p = r_p / c_t = t_\delta / \sqrt{2\pi} \approx 5.39119991 \cdot 10^{-44}\ s. \tag{108}$$

$$m_p = r_p / c_m = m_\delta / \sqrt{2\pi} \approx 2.17645313 \cdot 10^{-8}\ kg. \tag{109}$$

$$q_p = r_p / c_q = q_\delta / \sqrt{2\pi} \approx 1.87554604 \cdot 10^{-18}\ C. \tag{110}$$

Derivation of the fine structure constant, α

The ratios of two subtypes of force implies ratios of the form:

$$\alpha_\tau = \frac{F_{\tau_1}}{F_{\tau_2}} = \frac{K\tau_1^2 / r^2}{K\tau_2^2 / r^2} = \frac{\tau_1^2}{\tau_2^2}. \tag{111}$$

For example, where q_e is the elementary (electron) charge ($1.60217663 \cdot 10^{-19}\ C$), and q_p is the reduced Planck charge unit, the fine structure electron coupling constant is:

$$\alpha_q = q_e^2 / q_p^2 \approx 0.0072973526. \tag{112}$$

Derivation of the Compton wavelength, λ

[10][13] From equations 94 and 98:

$$\lambda = r = k_m / m = (k_m / m)(c/c) \quad \wedge \quad h = k_m c \quad \Rightarrow \quad \lambda = h / mc. \tag{113}$$

Derivation of Schrödenger's position-space wave equation

Start with the previously derived Planck relation 98 and multiply the kinetic energy component by mc/mc:

$$mc^2 = h/t \quad \Rightarrow \quad \exists\, V(r,t) :\ h/t = h/2t + V(r,t)$$
$$\Rightarrow \quad h/t = hmc/2mct + V(r,t). \quad (114)$$

And from the distance-to-time (speed of light) ratio (91):

$$h/t = hmc/2mct + V(r,t) \quad \wedge \quad r = ct \quad \Rightarrow \quad h/t = hmc^2/2mcr + V(r,t). \quad (115)$$

$$h/t = hmc^2/2mcr + V(r,t) \ \wedge \ h/t = mc^2 \quad \Rightarrow \quad h/t = h^2/2mcrt + V(r,t). \quad (116)$$

$$h/t = h^2/2mcrt + V(r,t) \quad \wedge \quad r = ct \quad \Rightarrow \quad h/t = h^2/2mr^2 + V(r,t). \quad (117)$$

Multiply both sides of equation 117 by a position-space distribution function, $\Psi(r,t)$.

$$h/t = h^2/2mr^2 + V(r,t) \quad \Rightarrow \quad (h/t)\Psi(r,t) = (h^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t). \quad (118)$$

Using the reduced Planck constant, $\hbar = 2\pi h$:

$$(2\pi\hbar/t)\Psi(r,t) = ((2\pi\hbar)^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t) \quad \wedge$$
$$\forall\ \Psi(r,t) :\ \partial^2\Psi(r,t)/\partial r^2 = (-2\pi/r^2)\Psi(r,t) \quad \wedge \quad \partial\Psi(r,t)/\partial t = (i\ 2\pi/t)\Psi(r,t)$$
$$\Rightarrow \quad i\hbar\partial\Psi(r,t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t), \quad (119)$$

which is the one-dimensional position-space Schrödenger's equation [15][13].

$$i\hbar\partial\Psi(r,t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t) \quad \wedge \quad ||\vec{\mathbf{r}}|| = r$$
$$\Rightarrow \quad \exists\ \vec{\mathbf{r}} :\ i\hbar\partial\Psi(\vec{\mathbf{r}},t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(\vec{\mathbf{r}},t)/\partial\vec{\mathbf{r}}^2 + V(\vec{\mathbf{r}},t)\Psi(\vec{\mathbf{r}},t), \quad (120)$$

which is the 3-dimensional position-space Schrödenger's equation [15] [13].

Derivation of Dirac's wave equation

Using the derived Planck relation 98:

$$mc^2 = h/t \quad \Rightarrow \quad \exists\, V(r,t) :\ mc^2/2 + V(r,t) = h/t \quad \Rightarrow \quad 2h/t - 2V(r,t) = mc^2. \quad (121)$$

$$\forall\, V(r,t) :\ V(r,t) = ih/t \quad \wedge \quad r = ct \quad \wedge \quad 2h/t - 2V(r,t) = mc^2$$
$$\Rightarrow \quad 2h/t - i2hc/r = mc^2. \quad (122)$$

Use the ratios, $r = c_q q$, and, $r = ct$. to multiply each term on the left side of equation 122 by 1:

$$qc_q/r = qc_q/ct = 1 \wedge 2h/t - i2hc/r = mc^2 \Rightarrow 2h(qc_q/c)/t^2 - i2h((qc_q/c)/r^2)c = mc^2. \quad (123)$$

Using the reduced Planck constant, $h = 2\pi\hbar$:

$$2h(qc_q/c)/t^2 - i2h((qc_q/c)/r^2)c = mc^2 \quad \wedge \quad h = 2\pi\hbar$$
$$\Rightarrow \quad 4\pi\hbar(qc_q/c)/t^2 - i4\pi\hbar((qc_q/c)/r^2)c = mc^2. \quad (124)$$

Applying a quantum amplitude equation in complex form to equation 124:

$$A_0 = 4\pi(c_q/c)((1/t) - i(1/r)) \quad \wedge \quad 4\pi\hbar(qc_q/c)/t^2 - i4\pi\hbar((qc_q/c)/r^2)c = mc^2$$
$$\Rightarrow \ \hbar\partial(-qA_0)/\partial t - i\hbar(\partial(-qA_0)/\partial r)c = mc^2. \quad (125)$$

Translating equation 125 to moving (rest frame) coordinates via the Lorentz factor, $\gamma_0 = 1/\sqrt{1-(v/c)^2}$:

$$\hbar\partial(-qA_0)/\partial t - i\hbar(\partial(-qA_0)/\partial r)c = mc^2$$
$$\Rightarrow \quad \gamma_0\hbar\partial(-qA_0)/\partial t - \gamma_0 i\hbar(\partial(-qA_0)/\partial r)c = mc^2. \quad (126)$$

Translating equation 126 to vector form:

$$\gamma_0\hbar(\partial(-qA_0)/\partial t) - \gamma_0 i\hbar(\partial(-qA_0)/\partial r)c = mc^2 \quad \wedge$$
$$||\vec{\mathbf{r}}|| = r \quad \wedge \quad ||\vec{\mathbf{A}}|| = A_0 \quad \wedge \quad ||\vec{\gamma}|| = \gamma_0$$
$$\Leftrightarrow \quad \exists\, \vec{\mathbf{r}}, \vec{\mathbf{A}}, \vec{\gamma} : \gamma_0\hbar(\partial(-qA_0)/\partial t) - \vec{\gamma}\cdot i\hbar(\partial(-q\vec{\mathbf{A}})/\partial r)c = mc^2. \quad (127)$$

Multiplying both sides of equation 127 by the spinor, Ψ, yields:Dirac's equation:

$$\gamma_0\hbar(\partial(-qA_0)/\partial t) - \vec{\gamma}\cdot i\hbar(\partial(-q\vec{\mathbf{A}})/\partial r)c = mc^2 \quad \Rightarrow$$
$$\gamma_0\hbar(\partial(-qA_0)/\partial t)\Psi - \vec{\gamma}\cdot i\hbar(\partial(-q\vec{\mathbf{A}})/\partial r)c\Psi = mc^2\Psi \quad (128)$$

Total of a type

Applying both the direct (91) and inverse proportion ratios (95) to the Euclidean distance:

$$r = \sqrt{r_1^2 + r_2^2}, \quad r_1 = c_\tau \tau, \quad r_2 = k_\tau/\tau, \quad \tau \in \{t, m, q\}$$
$$\Rightarrow \quad r = \sqrt{(c_\tau \tau)^2 + (k_\tau/\tau)^2} \quad \Leftrightarrow \quad \tau = \sqrt{(r/c_\tau)^2 + (k_\tau/r)^2}. \quad (129)$$

Quantum extension to general relativity

The simplest way to demonstrate how to add quantum physics to general relativity is by extending Schwarzschild's gravitational time dilation equation and black hole metric. Apply the total of a type equation 129 to the derivation of Schwarzschild's time dilation and metric (62):

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-(v^2/c^2)(r/r)} \quad \wedge \quad r = \sqrt{(c_m m)^2 + (k_m/m)^2} = Q_m$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-Q_m v^2/rc^2}. \quad (130)$$

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-Q_m v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv_{escape}^2$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2Q_m v_{escape}^2/rc^2}. \quad (131)$$

For a photon, the escape velocity, $v_{escape} = c$.

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-2Q_m v_{escape}^2/rc^2}. \wedge v_{escape} = c$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2Q_m c^2/rc^2} = \sqrt{1-2Q_m/r}. \quad (132)$$

Combining equation 132 with equation 55 yields Schwarzschild's gravitational time dilation with a quantum mass effect:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-2Q_m/r} \quad \wedge \quad t' = t\sqrt{1-(v^2/c^2)} \quad \Rightarrow \quad t' = t\sqrt{1-2Q_m/r}. \quad (133)$$

Schwarzschild defined the black hole event horizon radius, $\alpha := 2Gm/c^2$. The radius with the quantum extension is $\alpha := 2Q_m$. At this point the exact same equations 67 through 70 yield what looks like the same Schwarzschild black hole metric.

Quantum extension to Newton's gravity force

The quantum mass effect is easier to understand in the context Newton's gravity equation than in general relativity, because the metric equations and solutions in the EFEs are much more complex. From equations 129 and 51:

$$m/\sqrt{(r/c_m)^2 + (k_m/r)^2} = 1 \quad \wedge \quad r^2/(ct)^2 = 1 \quad \Rightarrow \quad r^2/(ct)^2 = m/\sqrt{(r/c_m)^2 + (k_m/r)^2}$$
$$\Rightarrow \quad r^2/t^2 = c^2 m/\sqrt{(r/c_m)^2 + (k_m/r)^2}. \quad (134)$$

$$r^2/t^2 = c^2 m/\sqrt{(r/c_m)^2 + (k_m/r)^2} \quad \Rightarrow \quad (m/r)(r^2/t^2) = (m/r)(c^2 m/\sqrt{(r/c_m)^2 + (k_m/r)^2})$$
$$\Rightarrow \quad F := mr/t^2 = c^2 m^2/\sqrt{(r^4/c_m^2) + k_m^2}. \quad (135)$$

$$F = c^2 m^2/\sqrt{(r^4/c_m^2) + k_m^2} \quad \wedge \quad \forall\, m \in \mathbb{R},\ \exists\, m_1, m_2 \in \mathbb{R} :\ m_1 m_2 = m^2$$
$$\Rightarrow \quad F = c^2 m_1 m_2/\sqrt{(r^4/c_m^2) + k_m^2}. \quad (136)$$

Quantum extension to Coulomb's charge force

$$q^2/((r/c_q)^2 + (k_q/r)^2) = 1 \quad \wedge \quad r^2/(ct)^2 = 1 \quad \Rightarrow \quad r^2/(ct)^2 = q^2/((r/c_q)^2 + (k_q/r)^2)$$
$$\Rightarrow \quad r^2/t^2 = c^2 q^2/((r/c_q)^2 + (k_q/r)^2). \quad (137)$$

$$(1/r)(r^2/t^2) = (1/r)(c^2 q^2/((r/c_q)^2 + (k_q/r)^2)) \quad \Rightarrow \quad r/t^2 = c^2 q^2/(r^3/c_q^2 + k_q^2/r). \quad (138)$$

$$\forall\, q \in \mathbb{R} :\ \exists\, q_1, q_2 \in \mathbb{R} :\ q_1 q_2 = q^2 \quad \wedge \quad r/t^2 = c^2 q^2/(r^3/c_q^2 + k_q^2/r) \quad \Rightarrow$$
$$\exists\, q_1, q_2 \in \mathbb{R} :\ r/t^2 = c^2 q_1 q_2/(r^3/c_q^2 + k_q^2/r). \quad (139)$$

$$r/t^2 = c^2 q_1 q_2/(r^3/c_q^2 + k_q^2/r) \quad \wedge \quad m = r/c_m \quad \Rightarrow \quad F := mr/t^2 = (c^2/c_m) q_1 q_2/(r^2/c_q^2 + k_q^2/r^2). \quad (140)$$

INSIGHTS AND IMPLICATIONS

1. The ruler measure (.1) and convergence theorem (.2) were shown to be useful tools for proving that a countable sets of n-tuples imply a corresponding real-valued equation.

2. Defining all Euclidean and non-Euclidean distance measures as the inverse function of the sum of subset volumes:

$$\forall\ n,\ d,\ v_i : \quad d = f_n^{-1}(\textstyle\sum_{i=1}^m v_i) : \tag{141}$$

   (a) shows the intimate relation between distance and volume that definitions, like inner product space and metric space, ignore [3][4][5];

   (b) is a more simple and concise definition of a distance measure, where the $n = 2$ case can define an inner product space and the case, $d = f_n^{-1}(\sum_{i=1}^m f_n^{-1}(v_i))$, can define a metric space (.11) [3][4][5].

3. The left side of the distance sum inequality (.10),

$$(\textstyle\sum_{i=1}^m (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}, \tag{142}$$

   differs from the left side of Minkowski's sum inequality [8]:

$$(\textstyle\sum_{i=1}^m (a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}. \tag{143}$$

   (a) The two inequalities are only the same where $n = 1$.

   (b) The distance sum inequality (.10) is a more fundamental inequality because the proof does not require the convexity and Hölder's inequality assumptions required to prove the Minkowski sum inequality [8].

   (c) $\forall\ n > 1,\ v > 0$: The distance sum inequality term, $v_i^n = a_i^n + b_i^n$: $d = v^{1/n} = (\sum_{i=1}^m v_i^n)^{1/n}$, is the Minkowski distance, which makes it directly related to geometry. But the Minkowski sum inequality term, $d = v^{1/n} = (\sum_{i=1}^m ((v_i^n)^{\mathbf{n}}))^{1/n}$ $= (\sum_{i=1}^m v_i^{\mathbf{n^2}})^{1/n}$, is *not* a Minkowski distance.

   (d) The distance sum inequality might be applicable to machine learning.

4. **Combinatorics**. The number of n-tuples, $v_c = \prod_{i=1}^n |x_i|$, was proven to imply: the Euclidean volume equation (.4). And the notion of distance was derived as a domain value that is the inverse function of the sum of n-tuples (.7) (which includes the inner product) and the Minkowski distance equation (.8) (which includes the Manhattan and Euclidean distance equations), without relying on the geometric primitives and relations in Euclidean geometry [17][18], axiomatic geometry [19][20][21] [22][23], trigonometry [24] [25] calculus [26][24] [27], and vector analysis [3].

5. **Combinatorics**. The commutative property of the operations defining volume and distance equations limits a physical set to $n \leq 3$ elements (.19). Other physical dimensions must have different types (elements of different sets).

   (a) For example, the vector inner product space, in physical space, can only be extended beyond 3 dimensions if and only if the other dimensions have non-distance types, for example, dimensions of time, mass, and charge.

   (b) If each type of quantum state is an ISCL, then there are at most 3 states of the same type: 3 orientations per dimension of space (+1,0-1), 3 quark color charges, {red, green, blue}, 3 quark anti-color charges, and so on.

6. If quantum states are not ordered, then a state value is undetermined until observed, like Schrödenger's poisoned cat being both alive and dead until the box is opened [15].

7. A discrete (point) valued state has measure 0 (zero-length interval). The ratio of a time or distance interval length to zero is undefined, which is the reason quantum entangled (discrete) state values exist independent of distance, time, mass, and charge.

8. For each Cartesian axis unit, r_δ, of a 3-dimensional distance interval having a length, r, there are Cartesian axis units of other types of intervals forming unit ratios (91): $c_t = r_\delta/t_\delta$, $c_m = r_\delta/m_\delta$, $c_q = r_\delta/q_\delta \Leftrightarrow$ the inverse proportion ratios (94): $k_t = r_\delta t_\delta$, $k_m = r_\delta m_\delta$, $k_t = r_\delta q_\delta$, where r_δ, t_δ, m_δ, and q_δ are quantum units (102) and the reduced Planck units are derived from the quantum units (106).

9. Empirical laws *describe* relations. Deriving empirical laws from the ratios *explains* the relations. Further, all the derivations of the physics equations from the ratios are much shorter and simpler than other derivations, which shows that the ratios are an important tool for physicists and engineers.

10. As shown in the subsection deriving the Schwarzschild's gravitational time dilation and black hole metric (62) [11][12] using ratios illustrates a simple way of finding solutions to Einstein's field equations.

11. c_t, c_m, and c_q are the **maximum** ratios: $r = \sqrt{r_1^2 + r_2^2 + r_3^2} \;\Rightarrow\; r_1,\ r_2,\ r_3 \leq r$ $\Rightarrow\ \forall\, \tau \in \{t, m, q\} : r_1/\tau,\ r_2/\tau,\ r_3/\tau \leq r/\tau = r_\delta/\tau_\delta = c_\tau$. For example, the speeds of gravity and light waves are limited to the speed, c_t.

12. c_t is a component of the constants: $G = c_m c_t^2$, $k_e = c_q^2 c_t^2/c_m$, $\varepsilon_0 = 1/4\pi k_e = 1/4\pi(c_q^2 c_t^2/c_m)$, and $\hbar = k_m c_t$. The only constant, derived in this article, that does not contain c_t is vacuum permeability: $\mu_0 = 4\pi k_e/c_t^2 = 4\pi c_q^2/c_m$.

13. Using the quantum units, r_δ and t_δ: r_δ/t_δ^2, suggests a maximum linear acceleration for masses. And $2\pi r_\delta/t_\delta^2$ suggests a maximum orbital or rotational acceleration.

14. The simplification of μ_0 into the quantum units shows two interesting relationships:

$$\mu_0 = 4\pi\frac{k_e}{c_t^2} = 4\pi\frac{c_q^2}{c_m} = 4\pi\frac{(r_\delta/q_\delta)^2}{r_\delta/m_\delta} = 4\pi\frac{m_\delta r_\delta}{q_\delta^2} = 4\pi\frac{k_m}{q_\delta^2}$$
$$\approx 4\pi\frac{2.21021909 \cdot 10^{-42}}{2.21021909 \cdot 10^{-35}} = 4\pi \cdot 10^{-7}\ kg\ m\ C^{-2} = 4\pi \cdot 10^{-7}\ H\ m^{-1}. \quad (144)$$

   (a) The first time $k_m = m_\delta r_\delta$ appears is in the derivation of the Planck relation and Planck constant, $h = k_m c$ (96), the second time in the Compton wavelength, $r = k_m/m$ (113). And now, k_m appears as a component of k_e, and, therefore, also appears in ε_0 and μ_0, which are defined i terms of k_e.

   (b) An open question is why $\frac{k_m}{q_\delta^2}$ seems to equal $1.0 \cdot 10^{-7}$ exactly.

15. The fine structure electron coupling constant, α was derived from the ratio of two subtypes of charge force that reduces to the unit-less ratio $\alpha = q_e^2/q_p^2 \approx 0.0072973526$ (112), which is the empirical CODATA value [14].

   (a) The CODATA electron coupling version of the fine structure constant, α is defined as: $\alpha = q_e^2/4\pi\varepsilon_0\hbar c) = q_e^2/2\varepsilon_0 hc$ [14]. The following steps show that the CODATA definition reduces to the ratio-derived equation:

$$\varepsilon_0 := 1/4\pi k_e = 1/(4\pi(c_q^2 c_t^2/c_m) \quad \wedge \quad h = k_m c_t \quad \wedge \quad h = 2\pi\hbar$$
$$\Rightarrow \quad \varepsilon_0 hc = k_m c_t^2/(4\pi(c_q^2/c_m)c_t^2) = k_m/(2(c_q^2/c_m))$$
$$= m_\delta r_\delta/(4\pi((r_\delta/q_\delta)^2/(r_\delta/m_\delta))) = q_\delta^2/4\pi. \quad (145)$$

$$\varepsilon_0 hc = q_\delta^2/4\pi \quad \wedge \quad q_p = q_\delta/\sqrt{2\pi} \quad \Rightarrow \quad \varepsilon_0 hc = q_p^2/2. \quad (146)$$

$$\alpha = q_e^2/2\varepsilon_0 hc \quad \wedge \quad \varepsilon_0 hc = q_p^2/2 \quad \Rightarrow \quad \alpha = q_e^2/2(q_p^2/2) = q_e^2/q_p^2. \quad (147)$$

   (b) The fine structure electron coupling constant is the ratio of electron static charge force to the propagating quantum charge (photon/electromagnetic) wave force, caused by a moving charged particle.

(c) Other fine structure constants can also be expressed more simply as the ratios of two subtypes of forces, for example, a fine structure electron gravity coupling constant can be expressed as the ratio of the rest electron mass to the reduced Planck mass unit: $\alpha_m = m_e^2/m_p^2$.

16. Empirical and hypothesized laws of physics use an *opaque* constant to make the units in an equation balance. The opacity has led to the *incorrect* assumptions of those constants being fundamental (atomic) constants.

    In this article, some opaque constants are derived directly from (composed of) the ratios: gravity, $G = c_m c_t^2$ (59), charge, $k_e = c_q^2 c_t^2/c_m$ (74), and Planck $h = k_m c_t$ (98). $\varepsilon_0 = 1/4\pi k_e = 1/4\pi c_m/((c_q^2/c_m)c_t^2)$ (79) and $\mu_0 = 4\pi k_e/c_t^2 = 4\pi c_q^2/c_m$ (85).

    And the quantum extensions to: Schwarzschild's gravitational time dilation (132) Newton's gravity force (136), and Coulomb's charge force show, that where the quantum effects become measurable, the constants G, k_e, ε_0, and μ_0 no longer exist (are no longer valid).

    Therefore, G, k_e, ε_0, μ_0, and h are **not** fundamental constants.

17. Constants that use notions of temperature and pressure for example, the Boltzmann constant [14], are probably not possible to define solely in terms of the quantum or Planck units.

18. The derivations of: $\nabla \cdot \mathbf{g} = -4\pi G\rho$ from $\mathbf{g} = 2\pi G m/|\tilde{\mathbf{r}}|^2$ (60), and $\nabla \cdot \mathbf{E} = -\rho/\varepsilon_0$ from $\mathbf{E} = 2\pi k_e q/|\tilde{\mathbf{r}}|^2$ (79), show that the use of mass and charge density, ρ, are unnecessary complications that obfuscates the pattern, $\nabla \cdot f(x,r) = -4\pi k_x x/|\tilde{\mathbf{r}}|^3$, being derived from the inverse square pattern, $f(x,r) = 2\pi k_x x/|\tilde{\mathbf{r}}|^2$. The use of the constant, G (derived from the inverse square assumption (57)), in Einstein's constant, κ and use of energy density in the stress energy tensor, $T_{\mu,\nu}$, in Einstein's field equations [3] also obfuscate the inverse square assumption.

19. Einstein's relativity equations: 1) assume the Lorentz transformations, 2) assume the laws of physics are same at each coordinate point, 3) assume the notion of light, and 4) assume that the speed of light is the same at each coordinate point [2][3].

    The derivations, in this article, were made without those assumptions (does not even require the notion of light). Assuming Euclidean space near each coordinate point creates unit ratios, where all equations (laws) derived from the unit ratios must be the same at each coordinate point.

20. The derivation of the magnetic field, **B**, (82) shows that the magnetic field is a spacetime bend caused by relativistic charged particle linear and angular (orbital or spin) velocities. Molecular arrangements in the construction of permanent magnets, photovoltaic cells, and superconductors are optimized by accurately controlling the spacetime bend via the arrangement of orbitals, bond oscillations, and spin alignments.

21. The quantum extensions to: Schwarzschild's gravitational time dilation (132), black hole metric (70), Newton's gravity force (136), and Coulomb's charge force (139) make quantifiable predictions:

    (a) For gravity, $\lim_{r\to 0} F = c^2 m_1 m_2 / k_m$, and for charge, $\lim_{r\to 0} F = 0$. Finite maximum gravity and charge forces: 1) allows radioactivity without the need for a weak force, 2) finite sloped energy well walls reduce the need for quantum tunneling, and 3) eliminates the problem of forces going to infinity as $r \to 0$.

    (b) The quantum-extended relativity and classic equations reduce to the non-extended relativity and classic equations and constants, where the distance between masses and charges is sufficiently large or the masses and charges sufficiently large that the quantum effects are not measurable. **Note** that G, k_e, ε_0, μ_0, and κ (Einstein's constant, which contains G) do not exist (are not valid), where the quantum effects becomes measurable.

    (c) The covariant tensor components, in Einstein's field equations, that had the units $1/distance^2$, will now have the more complex units, $1/\sqrt{(distance^4/c_\tau^2) + k_\tau^2}$, $\tau \in \{t, m\}$.

    (d) $1/\sqrt{(distance^4/c_\tau^2) + k_\tau^2}$ implies that as distance $\to 0$, spacetime curvature peaks at the Planck distance, r_δ (102). The ratio of quantum units, m_δ / r_δ^3, might indicate a maximum mass density. A finite force (spacetime curvature) and finite mass density would imply that black holes have sizes > 0 (are not singularities). Black hole evaporation might be possible. If there was a "big bang," then it might not have originated from a singularity.

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