Some Set Properties Underlying Geometry and Physics

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Abstract. Euclidean volume and the Minkowski distances (Manhattan, Euclidean, etc. distances) are derived from a set and limit-based foundation without referencing the primitives and relations of geometry. Sequencing a strict linearly ordered set in all n-at-a-time orders via successor/predecessor relations is proved to be a cyclic set of at most 3 members. A cyclic set of 3 distance domain interval lengths are related to other types of domain interval lengths by unit-factoring ratios. The ratios provide simple and short derivations of the: spacetime, Planck-Einstein, Compton wavelength, de Broglie wavelength equations, Newton's gravity and Coulomb's charge force with quantum-relativity extensions and derivations of the gravity (G), charge (k_e), and Planck (h) constants. All the proofs are verified in Coq.

Contents

1. Introduction

Mathematical analysis can construct differential calculus from a set and limit-based foundation without referencing the primitives and relations of Euclidean geometry, like straight line, angle, etc., which provides a more rigorous foundation and deeper understanding of geometry and physics. But Euclidean volume in the Riemann integral, Lebesgue integral, measure theory, and distance in the vector magnitude and metric space criteria are definitions motivated by Euclidean geometry [**Gol76**] [**Rud76**], rather than derived from a set and limit-based foundation.

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An intuitive, set-based motivation of Euclidean volume is the number, v_c, of ordered combinations (n-tuples): $v_c = \prod_{i=1}^{n} |x_i|$, where $|x_i|$ is the cardinal of the countable, disjoint set, x_i. But, some well-known analysis textbooks do not provide proofs that $v_c = \prod_{i=1}^{n} |x_i| \Rightarrow v = \prod_{i=1}^{n} s_i$, where each set, x_i, is a set of subintervals of the interval, $[a_i, b_i] \subset \mathbb{R}$, and $s_i = b_i - a_i$. [**Gol76**] [**Rud76**]. In this article, a simple proof is provided that: $v_c = \prod_{i=1}^{n} |x_i| \Rightarrow v = \prod_{i=1}^{n} s_i$.

$v_c = \prod_{i=1}^{n} |x_i| = f(|x_1|, \cdots, |x_n|, n)$. If f is a bijective function, then $\exists\ d_c : d_c = f^{-1}(v_c, n)$ and $v_c = f(d_c, n) = f(|x_1|, \cdots, |x_n|, n)$. If f is isomorphic, then $d_c = |x_1| = \cdots = |x_n|$ and $v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n$ for all $|x_i|$.

Where $v_c = f(|x_1|, \cdots, |x_n|, n)$ is a bijective and isomorphic function, it will be proved that $v_c = \sum_{j=1}^{m} v_{c_i} \Rightarrow d^n = \sum_{i=1}^{m} d_i^n$. d is the ρ-norm (Minkowski distance) [**Min53**], which will be proved to imply the metric space properties [**Rud76**].

Sequencing the domain sets, $x_1, \cdots, x_n$, from $i = 1$ to n, is a strict linear (total) order, where a total order is defined in terms of successor and predecessor relations [**CG15**]. Sequencing a set, via successor and predecessor relations, in all n-at-a-time orders, requires a "symmetry" constraint, where every set member is either a successor or predecessor to every other set member. A strict linearly ordered and symmetric set will be proved to be a cyclic set, where $n \leq 3$.

Therefore, if $\{s_1, s_2, s_3\}$ is a strict linearly ordered and symmetric set of 3 "distance" domain interval lengths, then another domain interval length, s_4, must have a different type (is a member of different set). The distance and 3D proofs will show that there are constant, maximum, unit-factoring ratios between a distance domain interval length, r, and other types of domain interval lengths: $r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q$.

The constant ratios are used to provide simple and short derivations of the: spacetime, Planck-Einstein, Compton wavelength, de Broglie wavelength equations, Newton's gravity and Coulomb's charge force with quantum-relativity extensions and derivations of the gravity (G), charge (k_e), and Planck (h) constants. The derivation of the spacetime interval equation allows simplifying Einstein's field (general relativity) equations.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq23**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of size κ subintervals. The ruler is both an inner and outer measure of an interval.

Definition 2.1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\ [a, b] \subset \mathbb{R}$, $s = b - a \ \wedge\ 0 < \kappa \leq 1 \ \wedge\ (p = floor(s/\kappa) \ \vee\ p = ceiling(s/\kappa))$.

Theorem 2.2. *Ruler convergence:* $M = \lim_{\kappa \to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\ \kappa > 0,\ p = floor(s/\kappa)\ \wedge\ 0 \leq |floor(s/\kappa) - s/\kappa| < 1 \ \Rightarrow\ |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by κ:

$$\forall\, \kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\begin{aligned}&\forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \epsilon \quad := \quad M = \lim_{\kappa\to 0} p\kappa = s. \quad \square\end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, \ldots, \pi_{\lim_{\kappa\to 0}}$.

LEMMA 2.3. $\forall\, n \geq 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa.$

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\begin{aligned}&\forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa\to 0} \kappa^n = 0.\end{aligned} \tag{2.5}$$

$$\lim_{\kappa\to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa\to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa. \qquad \square \tag{2.6}$$

3. Volume

DEFINITION 3.1. A countable n-volume is the number of ordered combinations (n-tuples), v_c, of the members of n number of disjoint, countable domain sets, x_i:

$$\exists\, n \in \mathbb{N},\ v_c \in \{0,\ \mathbb{N}\},\ \ x_i \in \{x_1, \cdots, x_n\} : \ \bigcap_{i=1}^n x_i = \emptyset \ \wedge \ v_c = \prod_{i=1}^n |x_i|. \tag{3.1}$$

THEOREM 3.2. *Euclidean volume,*

$$\begin{aligned}&\forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\ \ [v_a, v_b] \subset \mathbb{R},\ \ s_i = b_i - a_i,\ \ v = v_b - v_a : \\ &\qquad v_c = \prod_{i=1}^n |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^n s_i.\end{aligned} \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.

$$v_c = \prod_{i=1}^n |x_i| \ \Leftrightarrow \ v_c\kappa = (\prod_{i=1}^n |x_i|)\kappa \ \Leftrightarrow \ \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{3.3}$$

Apply the ruler (2.1) and ruler convergence (2.2) to equation 3.3:

$$\exists\, v, \kappa \in \mathbb{R} :\ v_c = floor(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{3.4}$$

Apply lemma 2.3 to equation 3.4:

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa^n = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa). \tag{3.5}$$

Apply the ruler (2.1) and ruler convergence (2.2) to s_i:

$$\exists\, s_i, \kappa \in \mathbb{R} :\ floor(s_i/\kappa) = |x_i| \quad \Rightarrow \quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i. \tag{3.6}$$

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa) \ \wedge \ \lim_{\kappa\to 0}(|x_i|\kappa) = s_i \ \Rightarrow \ v = \prod_{i=1}^n s_i \qquad \square \tag{3.7}$$

4. Distance

DEFINITION 4.1. Countable distance, $d_c = f(v_c, n) = f(|x_1|, \cdots, |x_n|, n) = \prod_{i=1}^n |x_i|$ is bijective and isomorphic: $v_c = \prod_{i=1}^n |x_i| = \prod_{i=1}^n d_c = d_c^n$.

4.1. Minkowski distance (ρ-norm).

THEOREM 4.2. *Minkowski distance (ρ-norm):*

$$v_c = \sum_{j=1}^{m} v_{c_i} \quad \Rightarrow \quad \exists\, d, d_i \in \mathbb{R} : \; d^n = \sum_{i=1}^{m} d_i^n.$$

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

PROOF. Apply the countable distance definition (4.1) to the assumption:

$$v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n \quad \wedge \quad v_{c_i} = \prod_{j=1}^{n} |x_{i_j}| = \prod_{i=1}^{n} d_{c_i} = d_{c_i}^n \quad \wedge \quad v_c = \sum_{j=1}^{m} v_{c_i} \quad \Rightarrow \quad d_c^n = \sum_{j=1}^{m} d_{c_i}^n. \tag{4.1}$$

Multiply both sides of equation 4.1 by κ and take the limit:

$$d_c^n = \sum_{j=1}^{m} d_{c_i}^n \quad \Leftrightarrow \quad \lim_{\kappa\to 0} d_c^n \kappa = \lim_{\kappa\to 0} \sum_{j=1}^{m} d_{c_i}^n \kappa. \tag{4.2}$$

Apply lemma 2.3 to equation 4.1:

$$\lim_{\kappa\to 0} d_c^n \kappa = \lim_{\kappa\to 0} \sum_{j=1}^{m} d_{c_i}^n \kappa \quad \wedge \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa$$
$$\Leftrightarrow \lim_{\kappa\to 0} d_c^n \kappa^n = \lim_{\kappa\to 0} \sum_{j=1}^{m} d_{c_i}^n \kappa^n \Leftrightarrow \lim_{\kappa\to 0} (d_c\kappa)^n = \lim_{\kappa\to 0} \sum_{j=1}^{m} (d_{c_i}\kappa)^n. \tag{4.3}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to equation 4.3:

$$\exists\, d, d_i : \; d_c = floor(d/\kappa), \; d = \lim_{\kappa\to 0} d_c \kappa \quad \wedge \quad d_{c_i} = floor(d_i/\kappa), \; d_i = \lim_{\kappa\to 0} d_{c_i}\kappa \quad \Rightarrow \quad d^n = \lim_{\kappa\to 0} (d_c\kappa)^n = \lim_{\kappa\to 0} \sum_{j=1}^{m} (d_{c_i}\kappa)^n = \sum_{j=1}^{m} d_i^n. \tag{4.4}$$

□

4.2. Distance inequality.

The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N}, \;\; v_a, v_b \geq 0 : \;\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$\forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq v_a + v_b + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n. \tag{4.5}$$

Take the n^{th} of both sides of the inequality 4.5:

$$\forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \tag{4.6}$$

□

4.3. Distance sum inequality.

The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N}, \;\; a_i, b_i \geq 0 : \;\; \left(\sum_{i=1}^{m} (a_i^n + b_i^n)\right)^{1/n} \leq \left(\sum_{i=1}^{m} a_i^n\right)^{1/n} + \left(\sum_{i=1}^{m} b_i^n\right)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$\forall\, m, n \in \mathbb{N}, \;\; v_a, v_b \geq 0 : \quad v_a = \sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge \quad (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad \left(\left(\sum_{i=1}^{m} a_i^n\right) + \left(\sum_{i=1}^{m} b_i^n\right)\right)^{1/n} = \left(\sum_{i=1}^{m} (a_i^n + b_i^n)\right)^{1/n} \leq \left(\sum_{i=1}^{m} a_i^n\right)^{1/n} + \left(\sum_{i=1}^{m} b_i^n\right)^{1/n}. \tag{4.7}$$

□

4.4. Metric Space. All Minkowski distances (ρ-norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.8}$$

Apply the distance inequality (4.3) to the inequality 4.8:

$$\begin{aligned} (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \ \wedge \ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ \Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ \Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq \\ (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \end{aligned} \tag{4.9}$$ □

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$\begin{aligned} \forall\, p\, :\, p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \end{aligned} \tag{4.10}$$ □

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\, [a_2, b_2], \quad u = b_1 - a_1,\; v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\; v \geq 0. \tag{4.11}$$

$$p \geq 1,\; u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \tag{4.12}$$ □

THEOREM 4.8. *Identity of Indiscernibles:* $d(u, u) = 0$.

PROOF. From the non-negativity property (4.7):

$$\begin{aligned} d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0 \\ \Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0. \end{aligned} \tag{4.13}$$

$$d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v. \tag{4.14}$$

$$d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \tag{4.15}$$ □

4.5. The properties limiting a set to at most 3 members.

DEFINITION 4.9. Totally ordered set:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i\; \in \{x_1, \ldots, x_n\},$$
$$\mathit{successor}\; x_i = x_{i+1} \;\;\wedge\;\; \mathit{predecessor}\; x_{i+1} = x_i.$$

DEFINITION 4.10. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i,\; j,\; n \in \mathbb{N},\; \forall\, x_i,\; x_j \in \{x_1, \ldots, x_n\},\; \mathit{successor}\; x_i = x_j \;\Leftrightarrow\; \mathit{predecessor}\; x_j = x_i.$$

THEOREM 4.11. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n \;\;\wedge\;\; j = 1 \;\;\Rightarrow\;\; \mathit{successor}\; x_n = x_1 \;\;\wedge\;\; \mathit{predecessor}\; x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (4.9) assigns a unique label to each set member and assigns unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (4.10):

$$i = n \;\;\wedge\;\; j = 1 \;\;\wedge\;\; \mathit{successor}\; x_i = x_j \;\;\Rightarrow\;\; \mathit{successor}\; x_n = x_1. \tag{4.16}$$

Applying the definition of the symmetry property (4.10) to conclusion 4.16:

$$\mathit{successor}\; x_i = x_j \;\;\Rightarrow\;\; \mathit{predecessor}\; x_j = x_i \;\;\Rightarrow\;\; \mathit{predecessor}\; x_1 = x_n. \tag{4.17}$$ □

THEOREM 4.12. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (4.11).

DEFINITION 4.13. (Cyclic) Successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (n = m + 1 \leq \mathit{setsize}). \tag{4.18}$$

DEFINITION 4.14. (Cyclic) Predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (n = m - 1 \geq 1). \tag{4.19}$$

DEFINITION 4.15. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$\mathit{Adjacent}(m, n, \mathit{setsize}) \leftarrow \mathit{Successor}(m, n, \mathit{setsize}) \vee \mathit{Predecessor}(m, n, \mathit{setsize}). \tag{4.20}$$

Every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

(4.21) $\quad Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (m = setsize \wedge n = 1).$

(4.22) $\quad Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (n = m+1 \leq setsize).$

(4.23) $\quad Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (n = setsize \wedge m = 1).$

(4.24) $\quad Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m+1 \leq setsize).$

(4.25) $\quad Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m-1 \geq 1).$

(4.26) $\quad Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \wedge m = 1).$

(4.27) $\quad Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \wedge n = setsize).$

(4.28) $\quad Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m+1 \leq setsize).$

(4.29) $\quad Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m-1 \geq 1).$

Member 2 is the only successor of member 1 for all $setsize \geq 3$, which implies member 3 is not ($\neg$) a successor of member 1 for all $setsize \geq 3$:

(4.30) $$\neg Successor(1,3,setsize \geq 3) \leftarrow Successor(1,2,setsize \geq 3) \leftarrow (n = m+1 \leq setsize).$$

Member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not ($\neg$) a predecessor of member 1 for all $setsize > 3$:

(4.31) $$\neg Predecessor(1,3,setsize \geq 3) \leftarrow Predecessor(1,setsize,setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).$$

For all $setsize \geq 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not symmetric):

(4.32) $$\neg Adjacent(1,3,setsize > 3) \leftarrow \neg Successor(1,3,setsize > 3) \wedge \neg Predecessor(1,3,setsize > 3). \quad \square$$

5. Applications to physics

From the volume proof (3.2), two disjoint 3D distance intervals, $[0, r_1]$ and $[0, r_2]$, define a 2-volume. From the Minkowski distance proof (4.2), $\exists\ r : r^2 = r_1^2 + r_2^2$. And from the 3D proof (4.12), for some non-distance type, $\tau : \tau \in \{t\ (time),\ m\ (mass),\ q\ (charge), \cdots\}$, there exist unit-factoring ratios, $\mu,\ \nu_1,\ \nu_2$:

(5.1) $$\forall\, r, r_1, r_2 :\ r^2 = r_1^2 + r_2^2 \quad \wedge \quad r = \mu\tau \quad \wedge \quad r_1 = \nu_1\tau \quad \wedge \quad r_2 = \nu_2\tau \quad \Rightarrow \quad (\mu\tau)^2 = (\nu_1\tau)^2 + (\nu_2\tau)^2 \quad \Rightarrow \quad \mu \geq \nu_1 \quad \wedge \quad \mu \geq \nu_2.$$

For a constant r, μ is a constant, maximum, unit-factoring ratio, where:

(5.2) $$\mu \in \{r_c/t_c,\ r_c/m_c,\ r_c/q_c, \cdots\} :\ r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q = \cdots .$$

5.1. Spacetime equations. Form equation 5.1:

(5.3) $$\forall\, r, r', r_\nu, \mu, \nu :\ r^2 = r'^2 + r_\nu^2 \quad \wedge \quad r = \mu\tau \quad \wedge \quad r_\nu = \nu\tau \quad \Rightarrow \quad r' = \sqrt{(\mu\tau)^2 - (\nu\tau)^2}\ =\ \mu\tau\sqrt{1-(\nu/\mu)^2}.$$

Local (proper) distance, r', contracts relative to coordinate distance, r, as $\nu \to \mu$:

(5.4) $$r' = \mu\tau\sqrt{1-(\nu/\mu)^2} \quad \wedge \quad \mu\tau = r \quad \Rightarrow \quad r' = r\sqrt{1-(\nu/\mu)^2}.$$

Coordinate length, τ, dilates relative to local length, τ', as $\nu \to \mu$:

$$\mu\tau = r'/\sqrt{1-(\nu/\mu)^2} \quad \wedge \quad r' = \mu\tau' \quad \Rightarrow \quad \tau = \tau'/\sqrt{1-(\nu/\mu)^2}. \tag{5.5}$$

Where τ is time, one form of the flat Minkowski spacetime event interval is:

$$\mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_\nu^2 \quad \wedge \quad \mathrm{d}r_\nu^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \quad \wedge \quad \mathrm{d}(\mu\tau) = \mathrm{d}r \\ \Rightarrow \quad \mathrm{d}r'^2 = \mathrm{d}(\mu\tau)^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2 - \mathrm{d}x_3^2. \tag{5.6}$$

5.2. Newton's gravity force and the constant, G. From equation 5.2:

$$\forall\, m_1, m_2, m, r \in \mathbb{R}:\ m_1 m_2 = m^2 \quad \wedge \quad r = (r_c/m_c)m \\ \Rightarrow \quad m_1 m_2 = ((m_c/r_c)r)^2 \quad \Rightarrow \quad (r_c/m_c)^2 m_1 m_2/r^2 = 1. \tag{5.7}$$

$$r = (r_c/t_c)t = ct \ \Rightarrow \ mr = (m_c/r_c)(ct)^2 \ \Rightarrow \ ((r_c/m_c)/c^2)mr/t^2 = 1, \tag{5.8}$$

$$((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/m_c)^2 m_1 m_2/r^2 = 1 \\ \Rightarrow \quad F := mr/t^2 = ((r_c/m_c)c^2)m_1 m_2/r^2 = G m_1 m_2/r^2, \tag{5.9}$$

where Newton's constant, $G = (r_c/m_c)c^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$.

5.3. Coulomb's charge force and constant, k_e. From equation 5.2:

$$\forall\, q_1, q_2, q, r \in \mathbb{R}:\ q_1 q_2 = q^2 \quad \wedge \quad r = (r_c/q_c)q \\ \Rightarrow \quad q_1 q_2 = ((q_c/r_c)r)^2 \quad \Rightarrow \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1. \tag{5.10}$$

$$r = (r_c/t_c)t = ct \ \Rightarrow \ mr = (m_c/r_c)(ct)^2 \ \Rightarrow \ ((r_c/m_c)/c^2)mr/t^2 = 1. \tag{5.11}$$

$$((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1 \\ \Rightarrow \quad F := mr/t^2 = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2. \tag{5.12}$$

$$r_c/t_c = c \quad \wedge \quad F = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2 \\ \Rightarrow \quad F = (m_c(r_c/t_c^2))(r_c/q_c)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2, \tag{5.13}$$

where Coulomb's constant, $k_e = (m_c(r_c/t_c^2))(r_c/q_c)^2$, conforms to the SI units: $N \cdot m^2 \cdot C^{-2}$.

5.4. 3 fundamental direct proportion ratios.

$$c_t = r_c/t_c \approx 2.99792458 \cdot 10^8 m\ s^{-1}. \tag{5.14}$$

$$G = (r_c/m_c)c_t^2 = c_m c_t^2 \quad \Rightarrow \quad c_m = r_c/m_c \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}. \tag{5.15}$$

$$k_e = (c_t^2/c_m)(r_c/q_c)^2 \quad \Rightarrow \quad c_q = r_c/q_c \approx 8.6175172023 \cdot 10^{-18} m\ C^{-1}. \tag{5.16}$$

5.5. 3 fundamental inverse proportion ratios. The 3 direct proportion ratios c_t, c_m, and c_q (5.4) $\Leftrightarrow$ 3 inverse proportion ratios, k_t, k_m, and k_q:

$$r/t = r_c/t_c, \quad r/m = r_c/m_c \quad \Leftrightarrow \quad (r/t)/(r/m) = (r_c/t_c)/(r_c/m_c) \quad \Leftrightarrow \\ (mr)/(tr) = (m_c r_c)/(t_c r_c) \quad \Leftrightarrow \quad mr = m_c r_c = k_m, \quad tr = t_c r_c = k_t. \tag{5.17}$$

$$r/t = r_c/t_c, \quad r/q = r_c/q_c \quad \Leftrightarrow \quad (r/t)/(r/q) = (r_c/t_c)/(r_c/q_c) \quad \Leftrightarrow \\ (qr)/(tr) = (q_c r_c)/(t_c r_c) \quad \Leftrightarrow \quad qr = q_c r_c = k_q, \quad tr = t_c r_c = k_t. \tag{5.18}$$

5.6. Planck-Einstein equation. [Lan78] Applying both the direct (5.14), $r/t = r_c/t_c = c$, and inverse (5.17), $mr = m_c r_c = k_m$, proportion ratios:

$$m(ct)^2 = mr^2 \ \wedge \ m = m_c r_c/r = k_m/r \quad \Rightarrow \quad m(ct)^2 = (k_m/r)r^2 = k_m r. \tag{5.19}$$

$$m(ct)^2 = k_m r \quad \wedge \quad r/t = r_c/t_c = c$$
$$\Rightarrow \quad E := mc^2 = k_m r/t^2 = (k_m(r/t))\,(1/t) = (k_m c)(1/t) = hf, \tag{5.20}$$

where the Planck constant, $h = k_m c$, and the frequency, $f = 1/t$.

5.7. Compton wavelength, r. [Jai11] From equations 5.17 and 5.20:

$$mr = k_m \quad \Rightarrow \quad r = k_m/m = k_m c/mc = h/mc. \tag{5.21}$$

5.8. de Broglie wavelength, r. [Jai11] From equations 5.3 and 5.21:

$$\exists\, v, v' : \ v = r'/t = c\sqrt{1-(v'/c)^2} \quad \wedge \quad r = h/mc$$
$$\Rightarrow \quad r = (h/mv)\sqrt{1-(v'/c)^2}, \tag{5.22}$$

where $v' = 0 \Rightarrow r = h/mv$.

5.9. Inverse proportion ratio values:

$$k_m = m_c r_c = h/c \approx 2.21022 \cdot 10^{-42}\ kg\ m. \tag{5.23}$$

$$k_t = t_c r_c = k_m/(c_t/c_m) \approx 5.47494 \cdot 10^{-78}\ s\ m. \tag{5.24}$$

$$k_q = q_c r_c = (c_t/c_q)k_t \approx 1.90466 \cdot 10^{-52}\ C\ m. \tag{5.25}$$

The total mass of a particle is $m = \sqrt{m_0^2 + m_{ke}^2}$, where m_0 is the rest mass and m_{ke} is the kinetic energy-equivalent mass. Applying both the direct (5.14) and inverse proportion ratios (5.17):

$$m_0 = (m_c/r_c)r \quad \wedge \quad m_{ke} = m_c r_c/r \quad \wedge \quad m = \sqrt{m_0^2 + m_{ke}^2}$$
$$\Rightarrow \quad m = \sqrt{((m_c/r_c)r)^2 + ((m_c r_c)/r)^2}. \tag{5.26}$$

5.10. Quantum-general relativity. Inserting the quantum effect into Einstein's field equations can be done by replacing component units, $1/distance^2$ in the Ricci and metric tensors with $1/(distance^2 + 1/distance^2)$.

5.11. Quantum-special relativity extensions to Newton's gravity force.

$$\exists\, m : \ m_1 m_2 = m^2 = ((m_c/r_c)r)^2 + ((m_c r_c)/r)^2$$
$$\Rightarrow \quad m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2) = 1. \tag{5.27}$$

Newton's gravity force in the local frame of reference comes from applying the spacetime equations, 5.4 and 5.5 to equation 5.9:

$$r' = r\sqrt{1-(v/c)^2} \quad \wedge \quad t' = t\sqrt{1-(v/c)^2} \quad \wedge \quad ((r_c/m_c)/c^2)mr/t^2 = 1$$
$$\Rightarrow \quad \sqrt{1-(v/c)^2}((r_c/m_c)/c^2)mr/t^2 = 1. \tag{5.28}$$

$$\sqrt{1-(v/c)^2}((r_c/m_c)/c^2)mr/t^2 = 1 = m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2)$$
$$\Rightarrow \quad F := mr'/t'^2 = (G/\sqrt{1-(v/c)^2})m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2). \tag{5.29}$$

5.12. Quantum-special relativity extensions to Coulomb's charge force.

$$F = (k_e/\sqrt{1-(v/c)^2})q_1q_2/(((q_c/r_c)r)^2 + ((q_cr_c)/r)^2). \tag{5.30}$$

6. Insights and implications

(1) Deriving volume and distance from the same abstract, countable set of n-tuples provides a single, unifying set and limit-based foundation under Euclidean geometry without relying on the geometric primitives and relations in Euclidean geometry [**Joy98**], axiomatic geometry [**Lee10**], and vector analysis [**Wey52**].

(2) The definition of a complete metric space [**Rud76**] ignores the intimate relation between distance and volume. A more sufficient definition is: a distance measure is the inverse (bijective) and isomorphic function of volume (4.1).

(3) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [**Joy98**]. And proofs that the straight line length is the smallest distance equate the straight line length to Euclidean distance.

Without using the notion of a straight line: Euclidean volume was derived from a set of n-tuples (3.2). And all distance measures (bijective, isomorphic functions of n-volumes) derived from Euclidean 2-volumes (areas) are Minkowski distances (4.2), where $n \in \{1, 2\}$: $n = 1$ is the Manhattan (largest) distance case, $d = \sum_{i=1}^{m} s_i$. $n = 2$ is the Euclidean (smallest) distance case, $d = (\sum_{i=1}^{m} s_i^2)^{1/2}$. For the case, $n \in \mathbb{R}$, $1 \le n \le 2$, d decreases monotonically as n goes from 1 to 2.

(4) The left side of the distance sum inequality (4.4),

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \le (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}, \tag{6.1}$$

differs from the left side of Minkowski's sum inequality [**Min53**]:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \le (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \tag{6.2}$$

The two inequalities are only the same where $n = 1$. The distance sum inequality is a more fundamental inequality because its proof does not require the convexity and Hölder's inequality assumptions required to prove the Minkowski sum inequality. And the distance sum inequality is derived from volume and distance, which makes it directly related to geometry.

(5) The derivations of the spacetime equations, in this article (5.1), differ from other derivations:
   (a) The derivations, here, do not rely on the Lorentz transformations or Einsteins' postulates [**Ein15**]. The derivations do not even require the notion of light.
   (b) The derivations, here, rely only on the Euclidean volume proof (3.2), the Minkowski distances proof (4.1), and the 3D proof (4.12), which provides the insight that the properties of physical space creates a maximum speed and the spacetime equations. For example, from the direct proportion equations 5.1 and 5.2, $\mu = r_c/t_c$ is always the maximum ratio (the speed of light).
   (c) The same derivations are also valid for spacemass and spacecharge.

(6) The flat spacetime interval equation was derived from a 2-dimensional equation (5.6), which is generalized to: $dr'^2 = \alpha_1 d(\mu\tau)^2 - dr_\nu^2$, where $dr_\nu^2 = \alpha_2 dx_1^2 + \alpha_3 dx_2^2 + \alpha_4 dx_3^2$. Therefore, the 4×4 metric tensor, $g_{\mu,\nu} = diag(\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4)$, in Einstein's field equations [**Ein15**], can be simplified to a 2×2 metric tensor, $g_{\mu,\nu} = diag(\alpha_1, -1)$. The 2×2 metric tensor allows using a 2-dimensional Gaussian curvature, which is much simpler to calculate than the 4-dimensional Ricci curvature. And the 2×2 tensors reduce the number of independent equations to solve.

(7) The direct proportion ratios, $r_c/t_c = c_t$, $r_c/m_c = c_m$, $(r_c/q_c) = c_q \Leftrightarrow$ the inverse proportion ratios, $t_c r_c = k_t$, $m_c r_c = k_m$, and $q_c r_c = k_q$ (5.5).
   (a) The direct proportion ratios are the units in a Cartesian grid. And the inverse proportion ratios are conservation units (You do not get something for nothing – a change in one thing causes an inverse change in something else).
   (b) The combination of direct and inverse proportion ratios create the particle-wave equations: Planck-Einstein (5.6), Compton wavelength (5.21), and de Broglie wavelength (5.22).
   (c) The gravity, G (5.9), charge k_e (5.13), and Planck h (5.20) constants were all derived from the direct and inverse proportion ratios. Therefore, G, k_e, and h are **not** "fundamental" constants.
   (d) The ratios used to derive k_e (5.13) do not contain the value, 4π, which indicates the current "standard' definitions of k_e in terms of permitivity, ε_0, and permeability, μ_0, where $k_e = 1/4\pi\varepsilon_0$ and $k_e = \mu_0 c^2/4\pi$, are **not logical** in Euclidean (rectangular) coordinates. Likewise, the logic of the reduced Planck constant, $\hbar = h/2\pi$, in rectangular coordinates, needs to be reconsidered.
   (e) G, k_e, and h all depend on the speed of light ratio, c_t: $G = c_m c_t^2$, $k_e = (c_q^2/c_m)c_t^2$, and $h = k_m c_t$.
   (f) $k_e = (c_q^2/c_m)c_t^2 = ((m_c/r_c)(r_c/t_c)^2)c_q^2 = (m_c(r_c/t_c^2))c_q^2$, where the term, r_c/t_c^2, which suggests a maximum acceleration constant.

(8) Applying the direct proportion ratios (5.4) to derive Newton's gravity force (5.2) and Coulomb's charge force (5.3) equations provide:
   (a) Derivations that do not assume the inverse square law or Gauss's flux divergence theorem as a cause of the inverse square law. **Note:** In Einstein's field equations, the components of the Ricci and metric tensors have the units, $1/\text{distance}^2$, Einstein's constant, $k = (4\pi G)2/c^4$, and the mass density component of the energy-stress tensor [**Wey52**] are assumptions of the inverse square law and Gauss's flux divergence.
   (b) The first derivations to show that the inverse square law and the property of force as mass times acceleration are the result of the direct proportion ratios, $r = (r_c/t_c)t = (r_c/m_c)m$.

(9) The quantum-special relativity extensions to Newton's gravity force (5.29) and Coulomb's charge force (5.30) make quantifiable predictions.
   (a) $\lim_{r\to 0} F = 0$. The classical-quantum gravity and charge boundaries are where: $r^2/c_m^2 = k_m^2/r^2$ and $r^2/c_q^2 = k_q^2/r^2$. The classical-quantum boundary distance is the **same** for both gravity and charge: $r \approx 4.05135 \cdot 10^{-35}\ m$, where the forces between two point-like particles peak. If the quantum effects exists, then:

   (i) Black holes have finite sizes – are not point-like singularities.
   (ii) The finite gravity and charge wells allow radioactivity and quantum tunneling.
   (iii) As the kinetic energy (temperature) decreases, more particles will stay within a single gravity and charge well, forming a Bose-Einstein condensate.
  (b) The relativity component would cause a star orbiting a galaxy at high velocity to experience a smaller force of attraction to the center of that galaxy.

(10) There is no constant ratio mapping a discrete value to a continuously varying value. Therefore, the discrete spin states of two quantum entangled particles and the polarization states of two quantum entangled photons are independent of continuously varying distance and time interval lengths.

(11) The set-based, first-order logic proof that a strict linearly ordered and symmetric set is a cyclic set of at most 3 members (4.12) is the simplest and most logically rigorous explanation for observing only 3 dimensions of physical space, less contrived and more rigorous than: parallel dimensions that cannot be detected or extra dimensions rolled up into infinitesimal balls that are too small to detect.
  (a) Higher order dimensions must have different types (members of different sets), for example, dimensions of time, mass, and charge.
  (b) Each of 3 ordered and symmetric dimensions of space can have at most 3 sequentially ordered and symmetric state values, for example, an ordered and symmetric set of 3 vector orientations, $\{-1, 0, 1\}$, per dimension of space and at most 3 spin states per plane, etc.
  If the states are not sequentially ordered (a bag of states), then a state value is undetermined until observed (like Schrodinger's cat being both alive and dead until the box is opened). That is, there is no axiom of choice that allows selecting a particular state.

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