Some Set Properties Underlying Geometry and Physics

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Abstract. Volume and distance are proved to be derived from a countable set and limit-based foundation. A symmetry constraint on a totally ordered set of distances is proved to limit the set to at most 3 members. Other dimensions have different types (are members of different sets), with constant geometric ratios between a unit of 3-dimensional distance and units of other continuous types (time, mass, and charge). The geometry proofs and ratios are used to derive the gravity, charge, permitivity, Planck, and fine structure constants. And the proofs and ratios are used to provide simpler, more rigorous derivations of some well-known classical gravity and charge equations, special and general relativity equations, and quantum physics equations. All the proofs are verified in Coq.

Contents

1. Introduction

Mathematical analysis constructs differential calculus from a set and limit-based foundation without referencing the primitives and relations of Euclidean geometry, like straight line, angle, etc., which provides a more rigorous foundation and deeper understanding of geometry and physics. But the definition of Euclidean volume in the Riemann integral, Lebesgue integral, and measure theory are motivated by Euclidean geometry. And the definitions of metric space criteria, vector magnitude, and inner product are definitions motivated by Euclidean distance [**Gol76**] [**Rud76**] rather than derived from a set and limit-based foundation.

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A simple, abstract, set-based definition of Euclidean volume is the number, v_c, of ordered combinations (n-tuples): $v_c = \prod_{i=1}^{n} |x_i|$, where $|x_i|$ is the cardinal of a countable, disjoint set, x_i. But real analysis text books and measure theory start with the definition of volume as the product of interval sizes, $v = \prod_{i=1}^{n} s_i$, where $s_i = b_i - a_i$, $[a_i, b_i] \subset \mathbb{R}$ [**Gol76**] [**Rud76**]. Where each set, x_i, is a set of size κ subintervals of the interval, $[a_i, b_i]$, it will be proved that, $\lim_{\kappa \to 0} v_c \cdot \kappa = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|) \cdot \kappa \Rightarrow v = \prod_{i=1}^{n} s_i$.

Where $v_c = \prod_{i=1}^{n} |x_i| = f(|x_1|, \cdots, |x_n|, n)$, if f is a bijective function, then $\exists\, d_c : d_c = f^{-1}(v_c, n)$ and $v_c = f(d_c, n) = f(|x_1|, \cdots, |x_n|, n)$. If f is also isomorphic, then $\forall\, |x_i| : d_c = |x_1| = \cdots = |x_n|$ and $v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n$.

Where $v_c = f(|x_1|, \cdots, |x_n|, n)$ is a bijective and isomorphic function, it will be proved that $\lim_{\kappa \to 0} v_c \cdot \kappa = \lim_{\kappa \to 0}(\sum_{j=1}^{m} v_{c_i}) \cdot \kappa \Rightarrow d^n = \sum_{i=1}^{m} d_i^n$. d is the ρ-norm (Minkowski distance) [**Min53**], which will be proved to imply the metric space properties [**Rud76**].

Sequencing the domain sets, $x_1, \cdots, x_n$, from $i = 1$ to n, is a strict linear (total) order, where a total order is defined in terms of successor and predecessor relations [**CG15**]. Sequencing a set, via successor and predecessor relations, in all n-at-a-time orders, is a "symmetry" property, where every set member is either a successor or predecessor to every other set member. A totally ordered and symmetric set will be proved to be a cyclic set, where $n \leq 3$.

Therefore, if $\{s_1, s_2, s_3\}$ is a totally ordered and symmetric set of 3 "distance" dimensions, then another dimension, s_4, must have a different type (member of different set), where for each unit of a distance dimension, there are units other types of dimensions, with constant geometric ratios between a unit of distance and a unit of other continuous dimensions: $r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q = \cdots$.

The geometry proofs and ratios are used to derive the gravity (G), charge (k_e), permitivity (ε_0), Planck (h), and fine structure (α) constants. And the proofs and ratios are used to provide simpler, more rigorous derivations of some well-known classical gravity and charge equations, special and general relativity equations, and quantum physics equations. The ratios are also used to easily add quantum effects to general relativity and classical physics equations.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq23**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of size κ subintervals. The ruler is both an inner and outer measure of an interval.

DEFINITION 2.1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\, [a, b] \subset \mathbb{R}$, $s = b - a \;\wedge\; 0 < \kappa \leq 1 \;\wedge\; (p = floor(s/\kappa) \;\vee\; p = ceiling(s/\kappa))$.

THEOREM 2.2. *Ruler convergence:* $M = \lim_{\kappa \to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : y \leq x,\, y \in \mathbb{Z},\, x \in \mathbb{R}\})$:

$$\forall\, \kappa > 0,\, p = floor(s/\kappa) \;\wedge\; 0 \leq |floor(s/\kappa) - s/\kappa| < 1 \;\Rightarrow\; |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by κ:

$$\forall\ \kappa > 0,\quad |p - s/\kappa| < 1 \quad\Rightarrow\quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\forall\ \epsilon = \delta \quad\wedge\quad |p\kappa - s| < |\kappa - 0| < \delta$$
$$\Rightarrow\quad |\kappa - 0| < \delta \quad\wedge\quad |p\kappa - s| < \delta = \epsilon \quad := \quad M = \lim_{\kappa\to 0} p\kappa = s. \quad \square \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p\cdot\kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, \ldots, \pi_{\lim_{\kappa\to 0}}$.

Lemma 2.3. $\forall\ n \geq 1,\quad 0 < \kappa < 1 \quad\Rightarrow\quad \lim_{\kappa\to 0}\kappa^n = \lim_{\kappa\to 0}\kappa.$

Proof. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$n \geq 1 \quad\wedge\quad 0 < \kappa < 1 \quad\Rightarrow\quad 0 < \kappa^n < \kappa \quad\Rightarrow\quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\forall\ \epsilon = \delta \quad\wedge\quad |\kappa - \kappa^n| < |\kappa - 0| < \delta$$
$$\Rightarrow\quad |\kappa - 0| < \delta \quad\wedge\quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa\to 0}\kappa^n = 0. \tag{2.5}$$

$$\lim_{\kappa\to 0}\kappa^n = 0 \quad\wedge\quad \lim_{\kappa\to 0}\kappa = 0 \quad\Rightarrow\quad \lim_{\kappa\to 0}\kappa^n = \lim_{\kappa\to 0}\kappa. \qquad \square \tag{2.6}$$

3. Volume

Definition 3.1. A countable n-volume is the number of ordered combinations (n-tuples), v_c, of the members of n number of disjoint, countable domain sets, x_i:

$$\exists\ n \in \mathbb{N},\ v_c \in \{0,\ \mathbb{N}\},\ \ x_i \in \{x_1, \cdots, x_n\} :\ \bigcap_{i=1}^n x_i = \emptyset \ \wedge\ v_c = \prod_{i=1}^n |x_i|. \tag{3.1}$$

Theorem 3.2. *Euclidean volume,*

$$\forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\ \ [v_a, v_b] \subset \mathbb{R},\ \ s_i = b_i - a_i,\ \ v = v_b - v_a :$$
$$v_c = \prod_{i=1}^n |x_i| \quad\Rightarrow\quad v = \prod_{i=1}^n s_i. \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

Proof.

$$v_c = \prod_{i=1}^n |x_i| \ \Leftrightarrow\ v_c\kappa = (\prod_{i=1}^n |x_i|)\kappa \ \Leftrightarrow\ \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{3.3}$$

Apply the ruler (2.1) and ruler convergence (2.2) to equation 3.3:

$$\exists\ v, \kappa \in \mathbb{R} :\ v_c = floor(v/\kappa) \quad\Rightarrow\quad v = \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{3.4}$$

Apply lemma 2.3 to equation 3.4:

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa^n = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa). \tag{3.5}$$

Apply the ruler (2.1) and ruler convergence (2.2) to s_i:

$$\exists\ s_i, \kappa \in \mathbb{R} :\ floor(s_i/\kappa) = |x_i| \quad\Rightarrow\quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i. \tag{3.6}$$

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa) \ \wedge\ \lim_{\kappa\to 0}(|x_i|\kappa) = s_i \ \Rightarrow\ v = \prod_{i=1}^n s_i \qquad \square \tag{3.7}$$

4. Distance

Definition 4.1. Countable distance, $d_c = f(v_c, n) = f(|x_1|, \cdots, |x_n|, n) = \prod_{i=1}^n |x_i|$ is bijective and isomorphic: $v_c = \prod_{i=1}^n |x_i| = \prod_{i=1}^n d_c = d_c^n$.

4.1. Minkowski distance (ρ-norm).

THEOREM 4.2. *Minkowski distance (ρ-norm):*

$$v_c = \sum_{j=1}^{m} v_{c_i} \quad \Rightarrow \quad \exists\, d, d_i \in \mathbb{R} : \; d^n = \sum_{i=1}^{m} d_i^n.$$

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

PROOF. Apply the countable distance definition (4.1) to the assumption:

$$v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n \quad \wedge \quad v_{c_i} = \prod_{j=1}^{n} |x_{i_j}| = \prod_{i=1}^{n} d_{c_i} = d_{c_i}^n \\ \wedge \quad v_c = \sum_{j=1}^{m} v_{c_i} \quad \Rightarrow \quad d_c^n = \sum_{j=1}^{m} d_{c_i}^n. \tag{4.1}$$

Multiply both sides of equation 4.1 by κ and take the limit:

$$d_c^n = \sum_{j=1}^{m} d_{c_i}^n \quad \Leftrightarrow \quad \lim_{\kappa\to 0} d_c^n \kappa = \lim_{\kappa\to 0} \sum_{j=1}^{m} d_{c_i}^n \kappa. \tag{4.2}$$

Apply lemma 2.3 to equation 4.1:

$$\lim_{\kappa\to 0} d_c^n \kappa = \lim_{\kappa\to 0} \sum_{j=1}^{m} d_{c_i}^n \kappa \quad \wedge \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa \\ \Leftrightarrow \lim_{\kappa\to 0} d_c^n \kappa^n = \lim_{\kappa\to 0} \sum_{j=1}^{m} d_{c_i}^n \kappa^n \Leftrightarrow \lim_{\kappa\to 0} (d_c \kappa)^n = \lim_{\kappa\to 0} \sum_{j=1}^{m} (d_{c_i} \kappa)^n. \tag{4.3}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to equation 4.3:

$$\exists\, d, d_i : \; d_c = floor(d/\kappa), \; d = \lim_{\kappa\to 0} d_c \kappa \\ \wedge \quad d_{c_i} = floor(d_i/\kappa), \; d_i = \lim_{\kappa\to 0} d_{c_i} \kappa \quad \Rightarrow \\ d^n = \lim_{\kappa\to 0} (d_c \kappa)^n = \lim_{\kappa\to 0} \sum_{j=1}^{m} (d_{c_i} \kappa)^n = \sum_{j=1}^{m} d_i^n. \tag{4.4}$$

□

4.2. Distance inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N}, \;\; v_a, v_b \geq 0 : \;\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$\forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq v_a + v_b + \\ \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n. \tag{4.5}$$

Take the n^{th} root of both sides of the inequality 4.5:

$$\forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \tag{4.6}$$

□

4.3. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N}, \;\; a_i, b_i \geq 0 : \;\; (\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$\forall\, m, n \in \mathbb{N}, \;\; v_a, v_b \geq 0 : \quad v_a = \sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge \\ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad ((\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} = \\ (\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \tag{4.7}$$

□

4.4. Metric Space. All Minkowski distances (ρ-norms) imply the metric space properties. The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.8}$$

Apply the distance inequality (4.3) to the inequality 4.8:

$$\begin{aligned} &(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \ \wedge \ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ &\qquad \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ &\Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ &\qquad \Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq \\ &\qquad\qquad (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \end{aligned} \tag{4.9}$$ □

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$\begin{aligned} &\forall\, p : p \geq 1, \quad d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ &\qquad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \end{aligned} \tag{4.10}$$ □

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0. \tag{4.11}$$

$$p \geq 1,\ u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \tag{4.12}$$ □

THEOREM 4.8. *Identity of Indiscernibles:* $d(u, u) = 0$.

PROOF. From the non-negativity property (4.7):

$$\begin{aligned} &d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0 \\ &\qquad \Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0. \end{aligned} \tag{4.13}$$

$$d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v. \tag{4.14}$$

$$d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \tag{4.15}$$ □

4.5. Set properties limiting a set to at most 3 members.

DEFINITION 4.9. Totally ordered set:

$$\begin{aligned} &\forall\, i\, n \in \mathbb{N}, \ \ i \in [1, n-1], \ \ \forall\, x_i \ \in \{x_1, \ldots, x_n\}, \\ &\qquad \textit{successor}\ x_i = x_{i+1} \ \wedge \ \textit{predecessor}\ x_{i+1} = x_i. \end{aligned}$$

DEFINITION 4.10. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i,\ j,\ n \in \mathbb{N},\ \forall\, x_i,\ x_j \in \{x_1, \ldots, x_n\},\ \textit{successor}\ x_i = x_j \ \Leftrightarrow\ \textit{predecessor}\ x_j = x_i.$$

THEOREM 4.11. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n \ \wedge \ j = 1 \ \Rightarrow \ successor\ x_n = x_1 \ \wedge \ predecessor\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (4.9) assigns a unique label to each set member and assigns unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (4.10):

$$i = n \ \wedge \ j = 1 \ \wedge \ successor\ x_i = x_j \ \Rightarrow \ successor\ x_n = x_1. \tag{4.16}$$

Applying the definition of the symmetry property (4.10) to conclusion 4.16:

$$successor\ x_i = x_j \ \Rightarrow \ predecessor\ x_j = x_i \ \Rightarrow \ predecessor\ x_1 = x_n. \tag{4.17}$$ □

THEOREM 4.12. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (4.11).

DEFINITION 4.13. (Cyclic) Successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m+1 \leq setsize). \tag{4.18}$$

DEFINITION 4.14. (Cyclic) Predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1). \tag{4.19}$$

DEFINITION 4.15. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{4.20}$$

Every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \tag{4.21}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \tag{4.22}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1). \tag{4.23}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize). \tag{4.24}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \tag{4.25}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1). \tag{4.26}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize). \tag{4.27}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize). \tag{4.28}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \tag{4.29}$$

Member 2 is the only successor of member 1 for all *setsize* ≥ 3, which implies member 3 is not ($\neg$) a successor of member 1 for all *setsize* ≥ 3:

$$(4.30) \quad \neg Successor(1, 3, setsize \geq 3) \\ \leftarrow Successor(1, 2, setsize \geq 3) \leftarrow (n = m + 1 \leq setsize).$$

Member n = *setsize* > 3 is the only predecessor of member 1, which implies member 3 is not ($\neg$) a predecessor of member 1 for all *setsize* > 3:

$$(4.31) \quad \neg Predecessor(1, 3, setsize \geq 3) \\ \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).$$

For all *setsize* ≥ 3, some elements are not ($\neg$) sequentially adjacent to every other element (not symmetric):

$$(4.32) \quad \neg Adjacent(1, 3, setsize > 3) \\ \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square$$

5. Applications to physics

From the volume proof (3.2), two disjoint distance intervals, $[0, r_1]$ and $[0, r_2]$, define a 2-volume. From the Minkowski distance proof (4.2), $\exists\, r : r^2 = r_1^2 + r_2^2$. And from the 3D proof (4.12), for some non-distance type, $\tau : \tau \in \{t\ (time),\ m\ (mass),\ q\ (charge), \cdots\}$, there exist constant, unit-factoring ratios, $\mu,\ \nu_1,\ \nu_2$:

$$(5.1) \quad \forall\, r, r_1, r_2 :\ r^2 = r_1^2 + r_2^2 \quad \wedge \quad r = \mu\tau \quad \wedge \quad r_1 = \nu_1\tau \quad \wedge \quad r_2 = \nu_2\tau \\ \Rightarrow \quad (\mu\tau)^2 = (\nu_1\tau)^2 + (\nu_2\tau)^2 \quad \Rightarrow \quad \mu \geq \nu_1 \quad \wedge \quad \mu \geq \nu_2.$$

μ is the maximum-possible ($\mu \geq \nu_1,\ \nu_2$), constant, unit-factoring ratio, where:

$$(5.2) \quad \mu \in \{r_c/t_c,\ r_c/m_c,\ r_c/q_c, \cdots\} :\ r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q = \cdots .$$

5.1. Derivation of the constant G, and the gravity laws of Newton, Gauss, and Poisson. From equation 5.2:

$$(5.3) \quad r = (r_c/m_c)m \quad \wedge \quad r = (r_c/t_c) = ct \quad \Rightarrow \quad r/(ct)^2 = (r_c/m_c)m/r^2 \\ \Rightarrow \quad r/t^2 = ((r_c/m_c)c^2)m/r^2 = Gm/r^2,$$

where Newton's constant, $G = (r_c/m_c)c^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$.

Newton's law follows from multiplying both sides equation 5.3 by m:

$$(5.4) \quad r/t^2 = Gm^2/r^2 \quad \wedge \quad \forall\, m \in \mathbb{R} :\ \exists\, m_1, m_2 \in \mathbb{R} :\ m_1 m_2 = m^2 \\ \Rightarrow \quad \exists\, m_1, m_2 \in \mathbb{R} :\ F := mr/t^2 = Gm^2/r^2 = Gm_1 m_2/r^2.$$

Again, starting with equation 5.3 and using ρ as the mass field density (Gauss's flux divergence) on a sphere having the surface area $4\pi r^2$ yields the differential forms of Gauss's flux divergence, $\nabla \cdot \mathbf{g}$ and Poisson's curl per unit mass, $\nabla^2\Phi(r, t)$:

$$(5.5) \quad \mathbf{g} = \nabla\Phi(r, t) := r/t^2 = (-Gm/r^2)(4\pi/4\pi) \quad \wedge \quad \rho = m/4\pi r^2 \\ \Rightarrow \quad \nabla \cdot \mathbf{g} = \nabla^2\Phi(\overrightarrow{r}, t) = -4\pi G\rho.$$

5.2. Derivation of Coulomb's charge constant, k_e and charge force.

$$\forall\, q \in \mathbb{R} :\ \exists\, q_1, q_2 \in \mathbb{R} :\ q_1 q_2 = q^2 \quad \wedge \quad r = (r_c/q_c)q \qquad \Rightarrow \quad \exists\, q_1, q_2 \in \mathbb{R} :\ q_1 q_2 = q^2 = ((q_c/r_c)r)^2 \quad \Rightarrow \quad (r_c/q_c)^2 q_1 q_2 / r^2 = 1. \tag{5.6}$$

$$r = (r_c/t_c)t = ct \ \Rightarrow\ mr = (m_c/r_c)(ct)^2 \ \Rightarrow\ ((r_c/m_c)/c^2)mr/t^2 = 1. \tag{5.7}$$

$$((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1 \qquad \Rightarrow \quad F := mr/t^2 = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2. \tag{5.8}$$

$$r_c/t_c = c \quad \wedge \quad F = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2 \qquad \Rightarrow \quad F = (m_c(r_c/t_c^2))(r_c/q_c)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2, \tag{5.9}$$

where Coulomb's constant, $k_e = (m_c(r_c/t_c^2))(r_c/q_c)^2$, conforms to the SI units: $N \cdot m^2 \cdot C^{-2}$.

5.3. Vacuum permitivity, ε_0, and Gauss's law for electric fields.

From equation 5.2:

$$r = (r_c/q_c)q \quad \wedge \quad r = (r_c/t_c) = ct \quad \Rightarrow \quad r/(ct)^2 = (r_c/q_c)q/r^2 \qquad \Rightarrow \quad r/t^2 = ((r_c/q_c)/c^2)q/r^2, \tag{5.10}$$

From equation 5.10, the charge flux divergence, $\nabla \cdot \mathbf{q} \propto 1/r^2$. Where ρ is the charge field density on the sphere surface area, $4\pi r^2$:

$$r/t^2 = (((r_c/q_c)/c^2)q/r^2)(4\pi/4\pi) \quad \wedge \quad \rho = q/4\pi r^2 \qquad \Rightarrow \quad \nabla \cdot \mathbf{q} := r/t^2 = 4\pi((r_c/q_c)c^2)\rho. \tag{5.11}$$

And Gauss's electric flux divergence, $\nabla \cdot \mathbf{E} \propto \nabla \cdot \mathbf{q}$. Multiply both sides of equation 5.11 by $(m_c/r_c)(r_c/q_c)$ and apply the derivation of k_e (5.9) in equation 5.8 and the of vacuum permitivity, $\varepsilon_0 = 1/4\pi k_e$:

$$\nabla \cdot \mathbf{E} = ((m_c/r_c)(r_c/q_c))\nabla \cdot \mathbf{q} = ((m_c/r_c)(r_c/q_c))4\pi((r_c/q_c)/c^2)\rho \qquad = 4\pi((m_c/r_c)c^2)(r_c/q_c)^2\rho = 4\pi k_e \rho = \rho/\varepsilon_0, \tag{5.12}$$

which is the Gauss's law for electric fields that is used in Maxwell's equations, etc.

5.4. Space-time-mass-charge equations.

Form equation 5.1:

$$\forall\, r, r', r_v, \mu, \nu :\ r^2 = r'^2 + r_v^2 \quad \wedge \quad r = \mu\tau \quad \wedge \quad r_v = \nu\tau \qquad \Rightarrow \quad r' = \sqrt{(\mu\tau)^2 - (\nu\tau)^2} \ = \ \mu\tau\sqrt{1 - (\nu/\mu)^2}. \tag{5.13}$$

Rest frame distance, r', contracts relative to stationary frame distance, r, as $\nu \to \mu$:

$$r' = \mu\tau\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad \mu\tau = r \quad \Rightarrow \quad r' = r\sqrt{1 - (\nu/\mu)^2}. \tag{5.14}$$

Stationary frame type, τ, dilates relative to the rest frame type, τ', as $\nu \to \mu$:

$$\mu\tau = r'/\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad r' = \mu\tau' \quad \Rightarrow \quad \tau = \tau'/\sqrt{1 - (\nu/\mu)^2}. \tag{5.15}$$

Where τ is type, time, the space-like flat Minkowski spacetime event interval is:

$$\mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_v^2 \quad \wedge \quad \mathrm{d}r_v^2 = \mathrm{d}r_1^2 + \mathrm{d}r_2^2 + \mathrm{d}r_3^2 \quad \wedge \quad \mathrm{d}(\mu\tau) = \mathrm{d}r \qquad \Rightarrow \quad \mathrm{d}r'^2 = \mathrm{d}(\mu\tau)^2 - \mathrm{d}r_1^2 - \mathrm{d}r_2^2 - \mathrm{d}r_3^2. \tag{5.16}$$

5.5. Derivation of Schwarzchild's gravitational time dilation and black hole metric. [Che10] From equations 5.14 and 5.2:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-(v^2/c^2)(r/r)} \quad \wedge \quad r = (r_c/m_c)m \\ \Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-((r_c/m_c)m)v^2/rc^2}. \tag{5.17}$$

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-((r_c/m_c)m)v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv^2_{escape} \\ \Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2(r_c/m_c)mv^2_{escape}/rc^2}. \tag{5.18}$$

$$\sqrt{1-(v^2/c^2)} = \lim_{v_{escape}\to c} \sqrt{1-2(r_c/m_c)mv^2_{escape}/rc^2} \\ \Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2(r_c/m_c)mc^2/rc^2}. \tag{5.19}$$

Combining equation 5.19 with the derivation of G (5.4):

$$(r_c/m_c)c^2 = G \quad \wedge \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2(r_c/m_c)mc^2/rc^2} \\ \Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2Gm/rc^2}. \tag{5.20}$$

Combining equation 5.20 with equation 5.15 yields Schwarzchild's gravitational time dilation:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-2Gm/rc^2} \quad \wedge \quad t' = t\sqrt{1-(v^2/c^2)} \\ \Rightarrow \quad t' = t\sqrt{1-2Gm/rc^2}. \tag{5.21}$$

Schwarzchild defined the black hole event horizon radius, $r_s := 2Gm/c^2$.

$$r_s = 2Gm/c^2 \quad \wedge \quad t' = t\sqrt{1-2Gm/rc^2} \quad \Rightarrow \quad t' = t\sqrt{1-r_s/r}. \tag{5.22}$$

From equations 5.14 and 5.22:

$$r' = r\sqrt{1-(v/c)^2} \quad \wedge \quad \sqrt{1-(v/c)^2} = \sqrt{1-2Gm/rc^2} \\ \Rightarrow \quad r' = r\sqrt{1-2Gm/rc^2} = r\sqrt{1-r_s/r}. \tag{5.23}$$

Using the time-like spacetime interval, where $\mathrm{d}s^2$ is negative:

$$r' = r\sqrt{1-r_s/r} \quad \wedge \quad \mathrm{d}s^2 = \mathrm{d}r'^2 - \mathrm{d}r^2 \\ \Rightarrow \quad \mathrm{d}s^2 = (\sqrt{1-r_s/r}\mathrm{d}r')^2 - (\mathrm{d}r/\sqrt{1-r_s/r})^2 = (1-r_s/r)\mathrm{d}r'^2 - (1-r_s/r)^{-1}\mathrm{d}r^2. \tag{5.24}$$

$$\mathrm{d}s^2 = (1-r_s/r)\mathrm{d}r'^2 - (1-r_s/r)^{-1}\mathrm{d}r^2 \quad \wedge \quad \mathrm{d}r' = \mathrm{d}(ct) \quad \wedge \quad c = 1 \\ \Rightarrow \quad \mathrm{d}s^2 = (1-r_s/r)\mathrm{d}t^2 - (1-r_s/r)^{-1}\mathrm{d}r^2. \tag{5.25}$$

Translating from 2D to 4D yields Schwarzchild's black hole metric:

$$\mathrm{d}s^2 = (1-r_s/r)\mathrm{d}t^2 - (1-r_s/r)^{-1}\mathrm{d}r^2 = f(r,t) \\ \Rightarrow \quad \mathrm{d}s^2 = (1-r_s/r)\mathrm{d}t^2 - (1-r_s/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + sin^2\theta\mathrm{d}\phi^2) = f(r,t,\theta,\phi). \tag{5.26}$$

5.6. Simplifying Einstein's general relativity (field) equation. Simplification step 1) Use the geometric ratios to define functions returning values for each component of the metric, $g_{\nu,\mu}$, in Einstein's field equations [**Ein15**] [**Wey52**]: All functions derived from the ratios are valid metrics, as an example, the previous Schwarzchild black hole metric derivation using the geometric ratios (5.5).

Simplification step 2) Express the EFE as 2D tensors: As shown in equation 5.26, the Schwarzchild metric was first derived as a 2D metric and then expanded to a 4D metric. Further, the 4D flat spacetime interval equation (5.16) is an instance of the 2D equation, $\mathrm{d}r'^2 = \mathrm{d}(ct)^2 - \mathrm{d}r_v^2$, where $\mathrm{d}r_v^2$ is the magnitude of a 3-dimensional vector.

The 2D metric tensor allows using the much simpler 2D Ricci curvature and scalar curvature. And the 2D tensors reduce the number of independent equations to solve.

Simplification step 3) One simple method to translate from 2D to 4D is to use spherical coordinates, where r and t remain unchanged and two added dimensions are the angles, ϕ, and θ. For example, the 2D Schwarzchild metric was translated to 4D using this method in equation 5.26.

5.7. 3 fundamental direct proportion ratios. c_t, c_m, and c_q:

$$c_t = r_c/t_c \approx 2.99792458 \cdot 10^8 m\ s^{-1}. \tag{5.27}$$

$$G = (r_c/m_c)c_t^2 = c_m c_t^2 \quad \Rightarrow \quad c_m = r_c/m_c \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}. \tag{5.28}$$

$$k_e = (c_t^2/c_m)(r_c/q_c)^2 \quad \Rightarrow \quad c_q = r_c/q_c \approx 8.6175172023 \cdot 10^{-18} m\ C^{-1}. \tag{5.29}$$

5.8. 3 fundamental inverse proportion ratios. k_t, k_m, and k_q:

$$\begin{aligned} r/t = r_c/t_c, \quad r/m = r_c/m_c \quad &\Rightarrow \quad (r/t)/(r/m) = (r_c/t_c)/(r_c/m_c) \quad \Rightarrow \\ (mr)/(tr) = (m_c r_c)/(t_c r_c) \quad &\Rightarrow \quad mr = m_c r_c = k_m, \quad tr = t_c r_c = k_t. \end{aligned} \tag{5.30}$$

$$\begin{aligned} r/t = r_c/t_c, \quad r/q = r_c/q_c \quad &\Rightarrow \quad (r/t)/(r/q) = (r_c/t_c)/(r_c/q_c) \quad \Rightarrow \\ (qr)/(tr) = (q_c r_c)/(t_c r_c) \quad &\Rightarrow \quad qr = q_c r_c = k_q, \quad tr = t_c r_c = k_t. \end{aligned} \tag{5.31}$$

5.9. Planck relation and constant, h. [**Jai11**] Applying both the direct proportion ratio (5.27), and inverse proportion ratio (5.30):

$$m(ct)^2 = mr^2 \ \wedge \ m = m_c r_c/r = k_m/r \quad \Rightarrow \quad m(ct)^2 = (k_m/r)r^2 = k_m r. \tag{5.32}$$

$$\begin{aligned} &m(ct)^2 = k_m r \quad \wedge \quad r/t = r_c/t_c = c \\ &\quad \Rightarrow \quad E := mc^2 = k_m r/t^2 = (k_m(r/t))\,(1/t) = (k_m c)(1/t) = hf, \end{aligned} \tag{5.33}$$

where the Planck constant, $h = k_m c$, and the frequency, $f = 1/t$.

$$k_m = m_c r_c = h/c \approx 2.2102190943 \cdot 10^{-42}\ kg\ m. \tag{5.34}$$

$$k_t = t_c r_c = k_m c_m/c_t \approx 5.4749346710 \cdot 10^{-78}\ s\ m. \tag{5.35}$$

$$k_q = q_c r_c = k_t c_t/c_q \approx 1.9046601056 \cdot 10^{-52}\ C\ m. \tag{5.36}$$

5.10. Compton wavelength. [**Jai11**] From equations 5.30 and 5.33:

$$mr = k_m \quad \wedge \quad h = k_m c \quad \Rightarrow \quad r = k_m/m = (k_m/m)(c/c) = h/mc. \tag{5.37}$$

5.11. 4 quantum units. Distance (r_c), time (t_c), mass (m_c), and charge (q_c):

$$r_c = \sqrt{r_c^2} = \sqrt{c_t k_t} = \sqrt{c_m k_m} = \sqrt{c_q k_q} \approx 4.0513505432 \cdot 10^{-35}\ m. \tag{5.38}$$

$$t_c = r_c/c_t \approx 1.3513850782 \cdot 10^{-43}\ s. \tag{5.39}$$

$$m_c = r_c/c_m \approx 5.4555118613 \cdot 10^{-8}\ kg. \tag{5.40}$$

$$q_c = r_c/c_q \approx 4.7012967286 \cdot 10^{-18}\ C. \tag{5.41}$$

Planck length $= r_c/\sqrt{2\pi}$, time $= t_c/\sqrt{2\pi}$, mass $= m_c/\sqrt{2\pi}$, charge $= q_c/\sqrt{2\pi}$.

5.12. Fine structure constant. The fine structure constant, α, is the ratio of two types (dimensions) of charge fields: 1) the *stationary* elementary particle charge field, F_e, and 2) the *moving* elementary charge (electromagnetic) wave field (the reduced Planck charge unit), F_p. From the ratio-derived Coulomb's law equation 5.8:

$$\exists\ \alpha \in \mathbb{R} :\ \alpha = \frac{F_e}{F_p} = \frac{k_e q_e^2/r^2}{k_e q_p^2/r^2} = q_e^2/q_p^2 = q_e^2/(q_c/\sqrt{2\pi})^2 \approx 0.0072973526. \tag{5.42}$$

5.13. Schrödenger's equation. Start with the previously derived Planck relation 5.33 and multiply the kinetic energy component by mc/mc:

$$h/t = mc^2\ \Rightarrow\ \exists V(r,t) : h/t = h/2t + V(r,t)\ \Rightarrow\ h/t = hmc/2mct + V(r,t). \tag{5.43}$$

And from the distance-to-time (speed of light) ratio (5.27):

$$h/t = hmc/2mct + V(r,t)\ \wedge\ r = ct\ \Rightarrow\ h/t = hmc^2/2mcr + V(r,t). \tag{5.44}$$

$$h/t = hmc^2/2mcr + V(r,t)\ \wedge\ h/t = mc^2\ \Rightarrow\ h/t = h^2/2mcrt + V(r,t). \tag{5.45}$$

$$h/t = h^2/2mcrt + V(r,t)\ \wedge\ r = ct\ \Rightarrow\ h/t = h^2/2mr^2 + V(r,t). \tag{5.46}$$

Replace the Planck constant in equation 5.46 to the reduced Planck constant:

$$h/t = h^2/2mr^2 + V(r,t)\ \wedge\ \hbar = h/2\pi\ \Rightarrow\ 2\pi\hbar/t = (2\pi)^2\hbar^2/2mr^2 + V(r,t). \tag{5.47}$$

Multiply both sides of equation 5.47 by a function, $\Psi(r,t)$.

$$\begin{aligned} & 2\pi\hbar/t = (2\pi)^2\hbar^2/2mr^2 + V(r,t) \\ & \qquad \Rightarrow\ (2\pi\hbar/t)\Psi(r,t) = ((2\pi)^2\hbar^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t). \end{aligned} \tag{5.48}$$

$$\begin{aligned} & (2\pi\hbar/t)\Psi(r,t) = ((2\pi)^2\hbar^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t)\quad \wedge \\ & \forall\ \Psi(r,t) :\ \partial^2\Psi(r,t)/\partial r^2 = (-(2\pi)^2/r^2)\Psi(r,t)\quad \wedge\quad \partial\Psi(r,t)/\partial t = (i\ 2\pi/t)\Psi(r,t) \\ & \qquad \Rightarrow\ i\hbar\partial\Psi(r,t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t), \end{aligned} \tag{5.49}$$

which is Schrödenger's equation in one dimension of space.

$$\begin{aligned} & -(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t) = i\hbar\partial\Psi(r,t)/\partial t\quad \wedge\quad ||\overrightarrow{\mathbf{r}}|| = r \\ & \quad \Rightarrow\ \exists\ \overrightarrow{\mathbf{r}} :\ i\hbar\partial\Psi(\overrightarrow{\mathbf{r}},t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(\overrightarrow{\mathbf{r}},t)/\partial\overrightarrow{\mathbf{r}}^2 + V(\overrightarrow{\mathbf{r}},t)\Psi(\overrightarrow{\mathbf{r}},t), \end{aligned} \tag{5.50}$$

which is Schrödenger's equation in three dimensions of space.

5.14. Dirac's wave equation. Using the derived Planck relation 5.33:

$$mc^2 = h/t \quad \Rightarrow \quad \exists\, V(r,t) : \; mc^2/2 + V(r,t) = h/t \quad \Rightarrow \quad 2h/t - 2V(r,t) = mc^2. \tag{5.51}$$

$$\forall\, V(r,t) : \; V(r,t) = h/t \quad \wedge \quad r = ct \quad \wedge \quad 2h/t - 2V(r,t) = mc^2 \quad \Rightarrow \quad 2h/t - 2hc/r = mc^2. \tag{5.52}$$

Applying the charge ratio, c_q, (5.28) to multiply each term on the left side of equation 5.52 by 1:

$$q(r_c/q_c)c/r = qc_q c/r = qc_q/t = 1 \quad \wedge \quad 2h/t - 2hc/r = mc^2 \quad \Rightarrow \quad 2h(-qc_q)/t^2 - 2h(-qc_q)c/rt = mc^2. \tag{5.53}$$

where a negative sign is added to q to indicate an attractive force between an electron and a nucleus.

$$2h(-qc_q)/t^2 - 2h(-qc_q)c/rt = mc^2 \quad \wedge \quad r = ct \quad \Rightarrow \quad 2h(-qc_q)/t^2 - 2h(-qc_q)c^2/r^2 = mc^2. \tag{5.54}$$

Applying a quantum amplitude equation in complex form to equation 5.54:

$$A_0 = c_q((1/t)) + i(c/r)) \quad \wedge \quad 2h(-qc_q)/t^2 - 2h(-qc_q)c^2/r^2 = mc^2 \quad \Rightarrow \quad 2h\partial(-qA_0)/\partial t - i2h(\partial(-qA_0)/\partial r)c = mc^2. \tag{5.55}$$

Translating equation 5.55 to moving coordinates via the Lorentz factor, $\gamma_0 = 1/\sqrt{1-(v/c)^2}$:

$$2h\partial(-qA_0)/\partial t - i2h(\partial(-qA_0)/\partial r)c = mc^2 \quad \Rightarrow \quad \gamma_0 2h\partial(-qA_0)/\partial t - \gamma_0 i2h(\partial(-qA_0)/\partial r)c = mc^2. \tag{5.56}$$

Multiplying both sides of equation 5.56 by $\Psi(r,t)$:

$$\gamma_0 2h\partial(-qA_0)/\partial t - \gamma_0 i2h(\partial(-qA_0)/\partial r)c = mc^2 \quad \Rightarrow \quad \gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \gamma_0 i2h(\partial(-qA_0)/\partial r)c\Psi(r,t) = mc^2\Psi(r,t). \tag{5.57}$$

Applying the vectors to equation 5.57:

$$\begin{gathered} \gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \gamma_0 i2h(\partial(-qA_0)/\partial r)c\Psi(r,t) = mc^2\Psi(r,t) \wedge \\ ||\overrightarrow{\mathbf{r}}|| = r \quad \wedge \quad ||\overrightarrow{\mathbf{A}}|| = A_0 \quad \wedge \quad ||\overrightarrow{\gamma}|| = \gamma_0 \quad \wedge \quad \Leftrightarrow \quad \exists\, \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{A}}, \overrightarrow{\gamma} : \\ \gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \overrightarrow{\gamma} \cdot i2h(\partial(-q\overrightarrow{\mathbf{A}})/\partial r)c\Psi(\overrightarrow{\mathbf{r}},t) = mc^2\Psi(\overrightarrow{\mathbf{r}},t). \end{gathered} \tag{5.58}$$

Adding a $\frac{1}{2}$ angular rotation (spin-$\frac{1}{2}$) of π to equation 5.55 allows substituting the reduced Planck constant, $\hbar = h/2\pi$, into equation 5.58, which yields Dirac's wave equation:

$$\begin{gathered} \gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \overrightarrow{\gamma} \cdot i2h(\partial(-q\overrightarrow{\mathbf{A}})/\partial r)c\Psi(\overrightarrow{\mathbf{r}},t) = mc^2\Psi(\overrightarrow{\mathbf{r}},t) \quad \wedge \\ A_0 = \pi c_q((1/t) + (c/r)) \\ \Rightarrow \quad \gamma_0\hbar(\partial(-qA_0)/\partial t)\Psi(r,t) - \overrightarrow{\gamma} \cdot i\hbar(\partial(-q\overrightarrow{\mathbf{A}})/\partial r)c\Psi(\overrightarrow{\mathbf{r}},t) = mc^2\Psi(\overrightarrow{\mathbf{r}},t). \end{gathered} \tag{5.59}$$

5.15. Total mass. The total mass of a particle is $m = \sqrt{m_0^2 + m_{ke}^2}$, where m_0 is the rest mass and m_{ke} is the kinetic energy-equivalent mass. Applying both the direct (5.27) and inverse proportion ratios (5.30):

(5.60)
$$m_0 = r/(r_c/m_c) = r/c_m \ \wedge \ m_{ke} = (m_c r_c)/r = k_m/r \ \wedge \ m = \sqrt{m_0^2 + m_{ke}^2} \Rightarrow \ m = \sqrt{(r/c_m)^2 + (k_m/r)^2}.$$

5.16. Quantum extension to general relativity. The simplest way to demonstrate how to add quantum physics to general relativity is by extending the Schwarzchild's black hole metric (5.5). Start by changing equation 5.17 in the Schwarzchild derivation:

$$(5.61)\quad \sqrt{1-(v^2/c^2)} = \sqrt{1-(v^2/c^2)(r/r)} \ \wedge \ r = \sqrt{(c_m m)^2 + (k_m/m)^2} = Q_m \Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1 - Q_m v^2/rc^2}.$$

$$(5.62)\quad \sqrt{1-(v^2/c^2)} = \sqrt{1 - Q_m v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv_{escape}^2 \Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1 - 2Q_m v_{escape}^2/rc^2}.$$

$$(5.63)\quad \sqrt{1-(v^2/c^2)} = \lim_{v_{escape}\to c} \sqrt{1 - 2Q_m v_{escape}^2/rc^2} \Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1 - 2Q_m c^2/rc^2} = \sqrt{1 - 2Q_m/r}.$$

Combining equation 5.63 with equation 5.15 yields Schwarzchild's gravitational time dilation with a quantum mass effect:

$$(5.64)\quad \sqrt{1-(v^2/c^2)} = \sqrt{1 - 2Q_m/r} \quad \wedge \quad t' = t\sqrt{1-(v^2/c^2)} \Rightarrow \quad t' = t\sqrt{1 - 2Q_m/r}.$$

Schwarzchild defined the black hole event horizon radius, $r_s := 2Gm/c^2$. The radius with the quantum extension is $r_s := 2Q_m$:

$$(5.65)\quad r_s = 2Q_m \quad \wedge \quad t' = t\sqrt{1 - 2Q_m/r} \quad \Rightarrow \quad t' = t\sqrt{1 - r_s/r}.$$

From equations 5.14 and 5.65:

$$(5.66)\quad r' = r\sqrt{1-(v/c)^2} \ \wedge \ \sqrt{1-(v/c)^2} = \sqrt{1 - r_s/r} \ \Rightarrow \ r' = r\sqrt{1 - r_s/r}.$$

Using the time-like spacetime interval, where ds is negative:

$$(5.67)\quad r' = r\sqrt{1 - r_s/r} \quad \wedge \quad \mathrm{d}s^2 = \mathrm{d}r'^2 - \mathrm{d}r^2$$
$$\Rightarrow \quad \mathrm{d}s^2 = (\sqrt{1 - r_s/r}\mathrm{d}r')^2 - (\mathrm{d}r/\sqrt{1 - r_s/r})^2 = (1 - r_s/r)\mathrm{d}r'^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2.$$

$$(5.68)\quad \mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}r'^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2 \quad \wedge \quad \mathrm{d}r' = \mathrm{d}(ct) \quad \wedge \quad c = 1 \Rightarrow \quad \mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}t^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2.$$

Expanding to 4D spherical coordinates yields Schwarzchild's black hole metric:

$$(5.69)\quad \mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}t^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2 = f(r,t)$$
$$\Rightarrow \quad \mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}t^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + sin^2\theta \mathrm{d}\phi^2) = f(r,t,\theta,\phi).$$

5.17. Quantum extension to Newton's gravity force. The quantum mass effect is easier to understand in the context Newton's gravity equation than in general relativity, because the metric equations and solutions in the EFEs are much more complex. From equation 5.2:

$$
\begin{aligned}
(5.70)\quad & m/\sqrt{(r/c_m)^2+(k_m/r)^2}=1 \quad \wedge \quad r^2/(ct)^2=1 \\
& \Rightarrow \quad r^2/(ct)^2 = m/\sqrt{(r/c_m)^2+(k_m/r)^2} \\
& \Rightarrow \quad r^2/t^2 = c^2m/\sqrt{(r/c_m)^2+(k_m/r)^2}.
\end{aligned}
$$

$$
\begin{aligned}
(5.71)\quad & r^2/t^2 = c^2m/\sqrt{(r/c_m)^2+(k_m/r)^2} \\
& \Rightarrow \quad (m/r)(r^2/t^2 = (m/r)(c^2m/\sqrt{(r/c_m)^2+(k_m/r)^2}) \\
& \Rightarrow \quad F := mr/t^2 = c^2m^2/(r\sqrt{(r/c_m)^2+(k_m/r)^2}) = c^2m^2/\sqrt{(r^4/c_m^2)+k_m^2}.
\end{aligned}
$$

$$
\begin{aligned}
(5.72)\quad & F = c^2m^2/\sqrt{(r^4/c_m^2)+k_m^2} \quad \wedge \quad \forall\, m\in\mathbb{R},\, \exists\, m_1,m_2\in\mathbb{R}:\ m_1m_2=m^2 \\
& \Rightarrow \quad F = c^2m_1m_2/\sqrt{(r^4/c_m^2)+k_m^2}.
\end{aligned}
$$

6. Insights and implications

(1) Combinatorics, the ordered combinations of countable, disjoint sets (n-tuples), generates both Euclidean volume (3.2) and the Minkowski distances (4.2), which includes Manhattan and Euclidean distances.

(2) Deriving Euclidean volume (3.2) and the Minkowski distances (4.2) from the same abstract, countable set of n-tuples (3.1) provides a single, unifying set and limit-based foundation under Euclidean geometry without relying on the geometric primitives and relations in Euclidean geometry [**Joy98**], axiomatic geometry [**Lee10**], and vector analysis [**Wey52**].

(3) The definition of metric space allows functions that do not have an intuitive geometric interpretation. The definition of a metric space [**Rud76**] ignores the intimate relation between distance and volume.

A sufficient definition is: a distance measure is an inverse (bijective), isomorphic function of a volume equal to the sum of bijective, isomorphic functions of volumes (4.1). This definition has the metric space properties and only allows functions having an intuitive geometric interpretation.

(4) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [**Joy98**]. And analytic proofs that the straight line length is the smallest distance equate the straight line length to Euclidean distance.

Without using the notion of a straight line: All distance measures (bijective, isomorphic functions of n-volumes) derived from Euclidean 2-volumes (areas) are Minkowski distances (4.2), where $n\in\{1,2\}$: $n=1$ is the Manhattan (largest monotonic) distance case, $d=\sum_{i=1}^m s_i$. $n=2$ is the Euclidean (smallest) distance case, $d=(\sum_{i=1}^m s_i^2)^{1/2}$. For the case, $n\in\mathbb{R}$, $1\le n\le 2$: d decreases monotonically as n goes from 1 to 2.

(5) The left side of the distance sum inequality (4.4),

$$(6.1)\qquad (\textstyle\sum_{i=1}^m(a_i^n+b_i^n))^{1/n} \le (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n},$$

differs from the left side of Minkowski's sum inequality [**Min53**]:

$$(\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \tag{6.2}$$

The two inequalities are only the same where $n = 1$.

(a) The distance sum inequality (4.4) is a more fundamental inequality because the proof does not require the convexity and Hölder's inequality assumptions of the Minkowski sum inequality proof [**Min53**].

(b) The Minkowski sum inequality term, $\forall\ n > 1 : ((a_i^n + b_i^n)^{\mathbf{n}})^{1/n}$, is **not** a Minkowski distance spanning the n-volume, $a_i^n + b_i^n$. But the distance sum inequality term, $(a_i^n + b_i^n)^{1/n}$, is the Minkowski distance spanning the n-volume, $a_i^n + b_i^n$, which makes it directly related to geometry (for example, the metric space triangle inequality was derived from the $m = 1$ case for all $n \geq 1$ (4.5)).

(6) Combinatorics, all n-at-time permutations of an ordered and symmetric set, limits the set to 3 members (4.12). This set-based, first-order logic proof is a simpler and more logically rigorous hypothesis for observing only 3 dimensions of physical space than parallel dimensions that cannot be detected or extra dimensions rolled up into infinitesimal balls that are too small to detect.

(a) Higher order dimensions must have different types (members of different sets), for example, types/dimensions of time, mass, and charge. Order and symmetry probably limit the number of fundamental types/dimensions to a very small number. For example, temperature, measured in Kelvins, is not a true dimension because temperature is more correctly a measure of frequency or kinetic energy, where entropy is a drop in frequency or kinetic energy. The magnetic force might be a pseudo (fictitious) force that is a function of distance, time. charge, and spin. Likewise, one should not immediately assume other fundamental types/dimensions that would correspond to the strong force, weak force, etc. For example, quantum effects might allow radioactivity without a weak force.

(b) Each of 3 ordered and symmetric dimensions of space can have at most 3 sequentially ordered and symmetric state values, for example, an ordered and symmetric set of 3 vector orientations, $\{-1, 0, 1\}$, per dimension of space and at most 3 spin states per plane, etc.
If the states are not sequentially ordered (a bag of states), then a state value is undetermined until observed (like Schrödenger's poisoned cat being both alive and dead until the box is opened). That is, for a bag of states, there is no "axiom of choice", an axiom often used in math proofs that allows selecting a particular set element (state).

(c) A discrete value cannot form a constant geometric ratio with continuous dimensions of distance and time. Therefore, the discrete spin states of two quantum entangled particles and the polarization states of two quantum entangled photons are independent of continuously varying distance and time interval lengths. This is reason entangled particles change states at the same time over any distance.

(d) For each 3-dimensional distance dimension unit there are units of other types of dimensions, which implies constant geometric ratios between a unit of distance and a unit of other types (5.7): $c_t = r_c/t_c$,

$c_m = r_c/m_c$, $c_q = r_c/q_c \Leftrightarrow$ the inverse proportion ratios (5.8): $k_t = r_c t_c$, $k_m = r_c m_c$, $k_t = r_c q_c$, where the combination of the direct and inverse ratios implies the quantum units (5.11): r_c, t_c, m_c, q_c. These ratios and quantum units were shown to be the basis of much physics:

(i) The gravity, G (5.4), charge k_e (5.9), vacuum permitivity, ε_0, and Planck h (5.33) constants were all derived from the ratios. Therefore, G, k_e, ε_0, and h are **not** "fundamental" constants.

(ii) Planck length $= r_c/\sqrt{2\pi}$, time $= t_c/\sqrt{2\pi}$, mass $= m_c/\sqrt{2\pi}$, and charge $= q_c/\sqrt{2\pi}$.

(iii) The inverse square law for gravity (5.3) and charge (5.6) were shown to be a result of the direct proportion ratios.

(iv) The geometric ratios are the basis of relativity theory.

(A) From equation 5.1, there is always a maximum ratio (for example, the speed of light, $c_t = r_c/t_c$).

(B) Special and general relativity assume covariance, which states that the laws of physics are invariant in every frame of reference. Covariance is the result of the same geometric ratios in every frame of reference.

For example, the special relativity time dilation equation 5.15 was derived from the ratio, $c_t = r_c/t_c$ (the speed of light), and combined with the ratio, $c_m = r_c/m_c$, (5.7) yielded Schwarzchild's general relativity gravitational time dilation and black hole metric equations (5.22).

(v) The combination of direct and inverse proportion ratios was shown to create the particle-wave equations: Planck relation (5.9), Compton wavelength (5.37), Schrödenger (5.13), and Dirac equations (5.14).

(vi) G, k_e, and h all depend on the speed of light ratio, c_t: $G = c_m c_t^2$, $k_e = (c_q^2/c_m)c_t^2$, and $h = k_m c_t$.

(vii) $k_e = (c_q^2/c_m)c_t^2 = ((m_c/r_c)(r_c/t_c)^2)c_q^2 = (m_c(r_c/t_c^2))c_q^2$, where the term, r_c/t_c^2, suggests a maximum acceleration constant, which agrees with the MOND theories of gravity.

(viii) The derivation of the Compton wavelength equation (5.10) shows that the computation of the wavelength is overly complex (because it assumes the Planck constant is a fundamental constant) and can be simplified to $r = k_m/m$.

(ix) The fine structure constant, α, has been an empirical constant with defined equations. For example, the CODATA definition is: $\alpha = q_e^2/4\pi\varepsilon\hbar c)$, where α is assumed to be dimensionless.

However, the derivation of α, in this article (5.42), shows that α does have dimensions because it is the ratio of two types (dimensions) of charge fields: 1) the *stationary* elementary particle charge field, F_e, and 2) the *moving* elementary charge (electromagnetic) wave field (the reduced Planck charge unit), F_p, which yields the more parsimonious equation: $\alpha = q_e^2/q_p^2$.

(7) The derivations of the spacetime equations, in this article (5.4), differ from other derivations:

(a) The derivations, here, do not rely on the Lorentz transformations or Einsteins' postulates [**Ein15**]. The derivations do not even require the notion of light.
(b) The same derivations are also valid for spacemass and spacecharge.
(c) The derivations, here, rely only on the Minkowski distances proof (4.1), and the 3D proof (4.12), which provides the insight that the properties of physical space creates constant maximum ratios, the spacetime equations, and 3 dimensions of distance.

(8) The derivation of Schrödenger's equation (5.13) and Dirac's equation (5.14), in this article, differs from other derivations:
(a) Other derivations are based on the Hamiltonian (energy-momentum) operator. In contrast, the derivations here rely on the Planck (energy-frequency) relation.
(b) The derivations here are more rigorous because:
(i) The derivations here start with a foundation of geometric ratios to derive the Planck constant (5.9), whereas other derivations assume that the Planck constant is a fundamental constant.
(ii) The energy-momentum term, $h^2/2m$, was derived, in this article, from the Planck relation (5.46), where the Planck relation was also rigorously derived (5.9). Other derivations **incorrectly** assume (define) the energy-momentum relation as: $(\mathbf{p}\cdot\mathbf{p})/2m = \hbar^2/2m$. The reduced Planck constant is only valid if the partial derivatives of the probability distribution function, $\Psi(r,t)$, contains compensating 2π terms: $\partial^2\Psi(r,t)/\partial r^2 = (-(2\pi)^2/r^2)\Psi(r,t)$ and $\partial\Psi(r,t)/\partial t = (i\ 2\pi/t)\Psi(r,t)$. Finding solutions to Schrödenger's equation would be simpler if the full Planck constant is used because it would reduce the complexity of $\Psi(r,t)$.
(iii) Other derivations assume the probability distribution has a mean value, where values closer to the mean are more probable. The derivation here makes no such assumptions.

(9) The quantum extensions to: Schwarzchild's black hole metric (5.66) and Newton's gravity force (5.72) make quantifiable predictions. Specifically: $\lim_{r\to 0} F = c^2 m_1 m_2/k_m$, and **both** the gravity and charge forces peak at the quantum length: $r_c = \sqrt{r_c^2} = \sqrt{c_t k_t} = \sqrt{c_m k_m} = \sqrt{c_q k_q} \approx 4.0513505432 \cdot 10^{-35}\ m$ (5.38).
(a) Newton's gravitational constant, G, and Coulomb's constant, k_e, are not valid, where the distance, r, is sufficiently small that the quantum effects becomes measurable.
(b) Finding solutions to Einstein's field equations becomes more difficult because the covariant components that had the units $1/distance^2$, will now have the more complex units, $1/\sqrt{(distance^4/c_m^2) + k_m^2}$ and Einstein's constant (which contains G) is no longer valid, where the distance, r, is sufficiently small that the quantum effects becomes measurable.
(c) Gravitational time dilation peaks at r_c (no time singularity and no event horizon).
(d) Black holes have sizes > 0 (no size singularities).

(e) The finite gravity-charge well allows radioactivity, quantum tunneling, and possibly black hole evaporation.

(f) As the kinetic energy (temperature) decreases, more particles will stay within their gravity-charge well distance, r_c, allowing superconductivity and Bose-Einstein condensates.

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