The Set Mappings Generating Geometry and Physics

George. M. Van Treeck

Abstract. The Euclidean volume equation is derived from a set and limit-based foundation. Distance as a function of volume is used for simple derivations of the Minkowski distances (for example, Manhattan and Euclidean distance) and the properties of properties metric space. The Euclidean volume proof provides simpler and more rigorous derivations of Newton's gravity force and Coulomb's charge force equations (without using the inverse square law or Gauss's divergence theorem). The derivations of the gravity and charge force equations exposes a ratio (constant first derivative) principle that allows simpler derivations of the spacetime equations and some general relativity equations. A symmetry property can limit geometric distance and volume to 3 dimensions per ball and a limit of 3 3-dimensional balls. All proofs are verified in Coq.

Contents

1. Introduction

Metric space, Euclidean distance, and area/volume are opaque definitions in mathematical analysis [Gol76] [Rud76] motivated by Euclidean geometry [Joy98]. Deriving those definitions from a set and limit-based foundation, without relying on the primitives and relations of Euclidean geometry, explains aspects of geometry and physics that opaque definitions and point-set topology cannot provide, for example, the countable set mappings that makes a space flat and also makes Euclidean distance is the smallest distance in flat space.

2010 *Mathematics Subject Classification.* Primary 28A75, 28E15. Secondary 03E75, 51M99.

Cartesian geometry motivates the idea of volume as the Cartesian product of mappings (coordinates/n-tuples) between the members of domain intervals, which converges to the product of interval lengths. Even though the proof of convergence is trivial, it results in each n-tuple corresponding to an infinitesimal volume, where volume is "proved" to be the sum of infinitesimal volumes, which not useful.

Therefore, $\mathbb{R}^n$, Riemann and Lebesgue integrals and measure theory use the opaque definition of volume as the product disjoint domain interval lengths. But, the value in proving that volume as the Cartesian product of mappings, even if trivial, is that it can provide insights into distance and physics. Therefore, a trivial, set, and limit-based proof is presented.

If countable volume is the Cartesian product of mappings (n-tuples) and countable distance is the number of a sequence (subset) of volume n-tuples, then the number of distance n-tuples is constrained by the number of volume n-tuples. Therefore, some distance measures are inverse functions of volume. Distance as a function of volume is used for simple derivations of the Minkowski distances/L_p norms (for example, Manhattan and Euclidean distance) and the properties of properties metric space without relying on the notions of line, angle, triangle, rectangle, etc.

The Euclidean volume proof is used to provide the first, rigorous derivations of Newton's gravity force and Coulomb's charge force equations (derivations that do not use the inverse square law or Gauss's divergence theorem). The derivations of the gravity and charge forces expose a ratio (constant first derivative) principle that generates the spacetime equations and some general relativity equations.

A symmetry constraint on the mapping between a set of integers and a set of domain intervals/dimensions (a totally ordered set) can limit geometric distance and volume to 3 dimensions per ball and a limit of 3 3-dimensional balls.

All the proofs in this article are trivial. But, to ensure confidence, all the proofs have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

Note: In order to compute areas and volumes, integrals divide all intervals into the *same* number subintervals (infinitesimals), where the size of the infinitesimals in each interval can *vary*, which makes it difficult for integrals to directly express the number of mappings between the p_x number of size c infinitesimals in one interval and the p_y number of size c infinitesimals in another interval.

In contrast to the integral, a ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals (infinitesimals), where each infinitesimal has the *same* size, c.

DEFINITION 2.1. Ruler measure, M: $\forall\, [a, b] \subset \mathbb{R},\ \ s = b - a\ \wedge\ c > 0\ \wedge$ $(p = \mathit{floor}(s/c)\ \vee\ p = \mathit{ceiling}(s/c))\ \wedge\ M = \sum_{i=1}^{p} c = pc.$

THEOREM 2.2. *Ruler convergence:* $M = \lim_{c\to 0} pc = s.$

The proof is trivial but is included here for completeness. The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : y \leq x, y \in \mathbb{Z}, x \in \mathbb{R}\})$:

$$\forall\, c > 0,\, p = floor(s/c) \;\wedge\; 0 \leq |floor(s/c) - s/c| < 1 \;\Rightarrow\; 0 \leq |p - s/c| < 1. \tag{2.1}$$

Multiply all sides of inequality 2.1 by c:

$$\forall\, c > 0, \quad 0 \leq |p - s/c| < 1 \quad \Rightarrow \quad 0 \leq |pc - s| < |c|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \delta : |pc - s| < |c| = |c - 0| < \delta \\ &\qquad \Rightarrow \quad \forall\, \epsilon = \delta : \;\; |c - 0| < \delta \;\wedge\; |pc - s| < \epsilon \;\; := \;\; M = \lim_{c \to 0} pc = s. \end{aligned} \tag{2.3}$$

$\square$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}},\; 3.14_{c=10^{-2}},\; 3.141_{c=10^{-3}}, ..., \pi_{\lim_{c \to 0}}$.

3. Euclidean Volume

Definition 3.1. Countable Volume, v_c is the number of all possible correspondences (n-tuples) between the members of n number of disjoint, countable domain sets, $x_1, \cdots, x_n$:

$$v_c = |\times_{i=1}^{n} x_i| = \prod_{i=1}^{n} |x_i|, \quad \bigcap_{i=1}^{n} x_i = \emptyset$$

Theorem 3.2. *Euclidean volume, v, is length of the range interval, $[v_u, v_w]$, which is equal to product of domain interval lengths, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$:*

$$v = \prod_{i=1}^{n} s_i, \;\; v = v_w - v_u, \;\; s_i = b_i - a_i.$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

Proof.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of size c subintervals and apply ruler convergence (2.2):

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n],\; c > 0 \;\wedge\; floor(s_i/c) = |x_i| \;\Rightarrow\; \lim_{c \to 0}(|x_i| \cdot c) = s_i. \tag{3.1}$$

$$\lim_{c \to 0}(|x_i| \cdot c) = s_i \quad \Rightarrow \quad \lim_{c \to 0} \prod_{i=1}^{n}(|x_i| \cdot c) = \prod_{i=1}^{n} s_i. \tag{3.2}$$

Combine the countable volume definition (3.1) with equation 3.2:

$$\begin{aligned} &v_c = \prod_{i=1}^{n} |x_i| \quad \wedge \quad \lim_{c \to 0} \prod_{i=1}^{n}(|x_i| \cdot c) = \prod_{i=1}^{n} s_i \\ &\qquad \Rightarrow \quad \exists\, v \in \mathbb{R},\;\, \mathrm{f}: \;\; v = lim_{c \to 0} \mathrm{f}(v_c, c) = \lim_{c \to 0} \prod_{i=1}^{n}(|x_i| \cdot c) = \prod_{i=1}^{n} s_i. \end{aligned} \tag{3.3}$$

By ruler convergence (2.2):

$$\exists\, v \in \mathbb{R} : \; v_c = floor(v/c) \quad \Rightarrow \quad v = \lim_{c \to 0} v_c \cdot c. \tag{3.4}$$

Combine equations 3.4 and 3.3:

$$\begin{aligned} &v = \lim_{c \to 0} v_c \cdot c \quad \wedge \quad v = \lim_{c \to 0} \mathrm{f}(v_c, c) = \prod_{i=1}^{n} s_i \\ &\qquad\qquad \Rightarrow \quad v = \lim_{c \to 0} v_c \cdot c = \prod_{i=1}^{n} s_i. \end{aligned} \tag{3.5}$$

$\square$

4. Distance

4.1. n-distance.

Definition 4.1. n-distance, d:

$$v = \prod_{i=1}^{n} d = d^n \quad \Leftrightarrow \quad d = v^{1/n}.$$

4.2. Minkowski distance (L_p norm). Only like types can be added together. For example, only scalars can be added to a scalar and only vectors can be added to a vector. Likewise, an n-volume can only be the sum of n-volumes.

THEOREM 4.2. *Minkowski distance (L_p norm): All distances that are a function of n-volumes are Minkowski distances (L_p norms).*

$$v = \prod_{i=1}^{n} d = d^n \quad \Rightarrow \quad d = (\sum_{i=1}^{m} s_i^n)^{1/n}$$

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

PROOF.

$$\forall\, v, v_1, \cdots, v_m : \; v = \sum_{i=1}^{m} v_i \quad \wedge \quad v = d^n \quad \Rightarrow \quad d^n = v = \sum_{i=1}^{m} v_i. \tag{4.1}$$

An n-volume can only be the sum of n-volumes:

$$v = \sum_{i=1}^{m} v_i \quad \wedge \quad \exists\, s_i \in \mathbb{R} : \; s_i^n = v_i \quad \Rightarrow \quad v = d^n = \sum_{i=1}^{m} s_i^n \quad \Leftrightarrow \quad d = (\sum_{i=1}^{m} s_i^n)^{1/n}. \quad \square \tag{4.2}$$

4.3. Countable distance. Applying the ruler to countable distance, d_c, generates the Minkowski distances.

DEFINITION 4.3. Countable distance, d_c:

$$d_c^n = \sum_{i=1}^{m} |x_i|^n.$$

LEMMA 4.4. *Countable distance generates Minkowski distance.*

$$d_c^n = \sum_{i=1}^{m} |x_i|^n \quad \Rightarrow \quad d = (\sum_{i=1}^{m} s_i^n)^{1/n}.$$

PROOF. Apply the ruler (2.1) and ruler convergence (2.2):

$$\exists\, d, s_1, \cdots, s_m \in \mathbb{R} : \; d_c = floor(d/c) \quad \wedge \quad |x_i| = floor(s_i/c) \quad \wedge$$
$$d_c^n = \sum_{i=1}^{m} |x_i|^n \quad \Rightarrow \quad d^n = \lim_{c\to 0}(d_c \cdot c)^n = \lim_{c\to 0} \sum_{i=1}^{m}(|x_i| \cdot c)^n = \sum_{i=1}^{m} s_i^n. \quad \square \tag{4.3}$$

4.4. Volume inequality. Proving that the Minkowski distance (L_p norm) satisfies the metric space triangle inequality uses another inequality. And for completeness, that inequality must be proved.

THEOREM 4.5. *Volume inequality*
The formal proof, volume_inequality, is in the Coq file, euclidrelations.v.

$$\forall\, n \in \mathbb{N}, \;\; v_a, v_b \geq 0 : \;\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand the n-volume, $(v_a^{1/n} + v_b^{1/n})^n$, using the binomial expansion:

$$\forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq (v_a + v_b + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k}) = (v_a^{1/n} + v_b^{1/n})^n. \tag{4.4}$$

Take the n^{th} root of both sides of an equation ($\forall\, n, x, y \geq 0 : x \leq y \Leftrightarrow x^{1/n} \leq y^{1/n}$):

$$\forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad \square \tag{4.5}$$

4.5. Volume sum inequality.

THEOREM 4.6. *Volume sum inequality*
The formal proof, volume_sum_inequality, is in the Coq file, euclidrelations.v.

$$\forall\, m,n \in \mathbb{N},\ \ v_a, v_b \geq 0: \ \ (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

PROOF. Apply the volume inequality (4.5):

$$\begin{aligned}(4.6)\quad & \forall\, m,n \in \mathbb{N},\ v_a, v_b \geq 0: \quad v_a = \textstyle\sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge \\ & (v_a+v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad ((\textstyle\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} = \\ & (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \quad \square\end{aligned}$$

4.6. Metric Space. The Minkowski distances (L_p norms) generate the properties of metric space.

The formal proofs: symmetry, triangle_inequality, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.7. *Symmetry:* $d(s_1,s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,v) = d(v,u).$

PROOF. By the commutative law of addition:

$$\begin{aligned}(4.7)\quad & \forall\, p\ :\ 1 \leq p \leq 2, \quad d(s_1,s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ & \Rightarrow \quad d(u,v) = (u^p+v^p)^{1/p} = (v^p+u^p)^{1/p} = d(v,u). \quad \square\end{aligned}$$

THEOREM 4.8. *Triangle Inequality:*
$d(s_1,s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,w) \leq d(u,v) + d(v,w).$

PROOF. $\forall\, p \geq 1, \quad k > 0, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(4.8)\qquad (u^p+w^p)^{1/p} \leq ((u^p+w^p) + 2v^p)^{1/p} = ((u^p+v^p) + (v^p+w^p))^{1/p}.$$

Apply the volume inequality (4.5) to the inequality 4.8:

$$\begin{aligned}(4.9)\quad & (u^p+w^p)^{1/p} \leq ((u^p+v^p)+(v^p+w^p))^{1/p} \quad \wedge \quad (a+b)^{1/p} \leq a^{1/p} + b^{1/p} \\ & \wedge \quad a = u^p + v^p \quad \wedge \quad b = v^p + w^p) \\ \Rightarrow \quad & (u^p+w^p)^{1/p} \leq ((u^p+v^p)+(v^p+w^p))^{1/p} \leq (u^p+v^p)^{1/p} + (v^p+w^p)^{1/p} \\ & \Rightarrow \quad d(u,w) = (u^p+w^p)^{1/p} \leq \\ & (u^p+v^p)^{1/p} + (v^p+w^p)^{1/p} = d(u,v) + d(v,w). \quad \square\end{aligned}$$

THEOREM 4.9. *Non-negativity:* $d(s_1,s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$(4.10)\qquad \forall\, [a_1,b_1],\ [a_2,b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0.$$

$$(4.11)\qquad p \geq 1,\ \ u,v \geq 0 \quad \Rightarrow \quad d(u,v) = (u^p+v^p)^{1/p} \geq 0. \qquad \square$$

THEOREM 4.10. *Identity of Indiscernibles:* $d(w,w) = 0$.

PROOF. Apply the triangle inequality property (4.8):

$$(4.12)\qquad \forall\, d(u,v) = d(v,w) = 0\ \wedge\ d(u,w) \leq d(u,v) + d(v,w) \ \Rightarrow\ d(u,w) \leq 0.$$

Combine the non-negativity property (4.9) and the previous inequality (4.12):

$$(4.13)\qquad d(u,w) \geq 0\ \wedge\ d(u,w) \leq 0\ \Leftrightarrow\ 0 \leq d(u,w) \leq 0\ \Rightarrow\ d(u,w) = 0.$$

Combine the result of step 4.13 and the condition, $d(u,v) = 0$, in step 4.12.

$$d(u,w) = 0 \ \wedge \ d(u,v) = 0 \ \Rightarrow \ w = v. \tag{4.14}$$

Combine the condition, $d(v,w) = 0$, in step 4.12 and the result of step 4.14.

$$d(v,w) = 0 \ \wedge \ w = v \ \Rightarrow \ d(w,w) = 0. \tag{4.15}$$

□

5. Applications to physics

5.1. Newton's gravity force equation. m_1 and m_2, are the sizes of two independent mass intervals, where each size c component of a mass interval exerts a force on each size c component of the other mass interval. If p_1 and p_2 are the number of size c components in each mass interval, then the total force, F, is equal to the total number of forces, which is proportionate to the Cartesian product, $p_1 \cdot p_2$, and proportionate to the size, c, of each component. Applying the ruler (2.1) and volume proof (3.2):

$$\begin{aligned} p_1 = floor(m_1/c) \quad \wedge \quad p_2 = floor(m_2/c) \quad \wedge \quad F := m_0 a \propto (p_1 c \cdot p_2 c) \\ \Rightarrow \quad F := m_0 a \propto \lim_{c \to 0}(p_1 c \cdot p_2 c) = m_1 m_2, \end{aligned} \tag{5.1}$$

where the force, F, is defined as the rest mass, m_0, times acceleration, a.

Note that integrals have no means of directly specifying the p_1 and p_2 of size c infinitesimals. Therefore, it is difficult to use integrals to rigorously derive: $\lim_{c\to 0}(p_1 c \cdot p_2 c) = m_1 m_2$.

$$\begin{aligned} F := m_0 a = m_0 r/t^2 \propto m_1 m_2 \ \wedge \ m_0 = m_1 \ \Rightarrow \ r \propto m_1 \ \Rightarrow \\ \exists\, m_G, r_c \in \mathbb{R} :\ r = (\mathrm{d}r/\mathrm{d}m) m_2 = (r_c/m_G) m_2, \end{aligned} \tag{5.2}$$

where: r is Euclidean distance, t is time, and r_c/m_G is a unit-factoring proportion ratio.

$$\begin{aligned} m_0 = m_1 \quad \wedge \quad r = (m_G/r_c) m_2 \quad \wedge \quad F = m_0 r/t^2 \\ \Rightarrow \quad F = m_0 r/t^2 = (r_c/m_G) m_1 m_2/t^2. \end{aligned} \tag{5.3}$$

From equation the definition of force, $F := m_0 a$:

$$\textstyle\int_0^t a\mathrm{d}t = r/t \ \Rightarrow \ \exists\, t_c, r_c \in \mathbb{R} :\ t/r = (\mathrm{d}t/\mathrm{d}r) = t_c/r_c \ \Rightarrow \ t = (t_c/r_c) r. \tag{5.4}$$

$$\begin{aligned} t = (t_c/r_c) r \quad \wedge \quad F = (r_c/m_G) m_1 m_2/t^2 \quad \Rightarrow \\ F = (r_c/m_G)(r_c^2/t_c^2) m_1 m_2/r^2 = (r_c^3/m_G t_c^2) m_1 m_2/r^2 = G m_1 m_2/r^2, \end{aligned} \tag{5.5}$$

where the gravitational constant, $G = r_c^3/m_G t_c^2$, has the SI units: $m^3 kg^{-1} s^{-2}$.

5.2. Coulomb's charge force. q_1 and q_2, are the sizes of two independent charge intervals, where each size c component of a charge interval exerts a force on each size c component of the other charge interval. If p_1 and p_2 are the number of size c components in each charge interval, then the total force, F, is equal to the total number of forces, which is proportionate to the Cartesian product, $p_1 \cdot p_2$, and proportionate to the size, c, of each component. Applying the ruler (2.1) and volume proof (3.2):

$$\begin{aligned} p_1 = floor(q_1/c) \quad \wedge \quad p_2 = floor(q_2/c) \quad \wedge \quad F \propto (p_1 p_2) c \\ \Rightarrow \quad F := m_0 a \propto \lim_{c \to 0}(p_1 c \cdot p_2 c) = q_1 q_2, \end{aligned} \tag{5.6}$$

where the force, F, is defined as the rest mass, m_0, times acceleration, a.

$$(5.7)\quad F := m_0 a = m_0 r/t^2 \propto q_1 q_2 \quad \wedge \quad m_0 = (\mathrm{d}m/\mathrm{d}q)q_1 = (m_G/q_C)q_1 \quad \Rightarrow \quad r \propto q_1 \quad \Rightarrow \quad \exists\, q_C, r_c \in \mathbb{R}:\ r = (\mathrm{d}r/\mathrm{d}q)q_2 = (r_c/q_C)q_2,$$

where: r is Euclidean distance, t is time, m_G/q_C and r_c/q_C are unit-factoring proportion ratios.

$$(5.8)\quad m_0 = (m_G/q_C)q_1 \quad \wedge \quad r = (q_C/r_c)q_2 \quad \wedge \quad F = m_0 r/t^2 \quad \Rightarrow \quad F = m_0 r/t^2 = (m_G/q_C)(r_c/q_C)q_1q_2/t^2 = (m_G r_c/q_C^2)q_1q_2/t^2.$$

From equation the definition of force, $F := m_0 a$:

$$(5.9)\quad \textstyle\int_0^t a\mathrm{d}t = r/t \;\Rightarrow\; \exists\, t_c, r_c \in \mathbb{R}:\ t/r = (\mathrm{d}t/\mathrm{d}r) = t_c/r_c \;\Rightarrow\; t = (t_c/r_c)r.$$

$$(5.10)\quad t = (t_c/r_c)r \quad \wedge \quad a_G = r_c/t_c^2 \quad \wedge \quad F = (m_G r_c/q_C^2)q_1q_2/t^2 \quad \Rightarrow \quad F = (r_c^2/t_c^2)(m_G r_c/q_C^2)q_1q_2/r^2 = ((m_G a_G)r_c^2/q_C^2)q_1q_2/r^2 = k_c q_1 q_2/r^2,$$

where the charge constant, $k_C = (m_G a_G)r_c^2/q_C^2$, has the SI units: Nm^2C^{-2}.

5.3. Spacetime equations. As shown in the derivations of Newton's gravity force (5.1) and Coulomb's charge force (5.2) equations: $r = (r_c/t_c)t = ct$, where $r_c/t_c = c$ is a unit-factoring proportion ratio. And, the smallest distance (and time) spanning the two inertial (independent, non-accelerating) frames of reference, $[0, r_1]$ and $[0, r_2]$, is the Euclidean distance, r.

$$(5.11)\quad r = ct \quad \Rightarrow \quad (ct)^2 = r_1^2 + r_2^2 \quad \Leftrightarrow \quad r_1^2 = (ct)^2 - (x^2 + y^2 + z^2),$$

where $r_2^2 = x^2 + y^2 + z^2$, which is one form of Minkowski's flat spacetime interval equation [**Bru17**]. And the length contraction and time dilation equations also follow directly from $(ct)^2 = r_1^2 + r_2^2$, where $v = r_1/t$:

$$(5.12)\quad r_2^2 = (ct)^2 - r_1^2 \;\wedge\; L = r_2 \;\Rightarrow\; L^2 = c^2t^2 - v^2 \;\Rightarrow\; L = ct\sqrt{1-(v/c)^2}.$$

$$(5.13)\quad L = ct\sqrt{1-(v/c)^2} \quad \wedge \quad L_0 = ct \quad \Rightarrow \quad L = L_0\sqrt{1-(v/c)^2}.$$

$$(5.14)\quad L = ct\sqrt{1-(v/c)^2} \;\wedge\; t' = L/c \;\Rightarrow\; t' = t\sqrt{1-(v/c)^2}.$$

5.4. Some general relativity equations: Combining the ratio (constant first derivative) equations into partial differential equations: $r = (r_c/m_G)m = ct \Rightarrow (r_c/m_G)m \cdot ct = r^2 \Rightarrow m = (m_G/r_c c)r^2/t = (m_G/r_c c)rv$. For a constant mass, m, a decrease in the distance, r, between two mass centers causes a decrease in time, t, (time slows down). v is the relativistic orbital velocity at distance, r. $(r_c/m_G)m \cdot (ct)^2 = r^3 \Rightarrow E = mc^2 = (m_G/r_c)r^3/t^2$. And $(ct)^2 = r^2 \Rightarrow c^2 = v^2 \Rightarrow (r_c/m_G)mv^2 = c^2 r \Rightarrow KE = mv^2/2 = (m_G c^2/2r_c)r$.

Given that $c = r_c/t_c \approx 3 \cdot 10^8 ms^{-1}$ and $G = r_c^3/m_G t_c^2 = (r_c/m_G)(r_c/t_c)^2 \approx 6.7 \cdot 10^{-11} m^3kg^{-1}s^{-2} \Rightarrow r_c/m_G \approx (6.7 \cdot 10^{-11} m^3kg^{-1}s^{-2}/(3 \cdot 10^8 m\ s^{-1})^2 \approx 7.4 \cdot 10^{-28} m\ kg^{-1}$, which can be used to quantify the constants in the previously derived equations. For example, $m = (m_G/r_c c)rv \approx (1/((7.4 \cdot 10^{-28} m\ kg^{-1})(3 \cdot 10^8 m\ s^{-1})))rv \approx (4.5 \cdot 10^{18} kg\ s\ m^{-2})rv$.

Likewise, for charge, $r = (r_c/q_C)q = ct \Rightarrow q = (q_C/r_c c)r^2/t = (q_C/r_c c)rv$, $E = qc^2 = (q_C/r_c)r^3/t^2$, and $KE = qv^2/2 = (q_C c^2/2r_c)r$. And if the ratio of an electron's mass to charge is m_G/q_C, then $m_G/q_C \approx 9.1 \cdot 10^{-31}kg/1.6 \cdot 10^{-19}C \approx 5.7 \cdot 10^{-12}kgC^{-1}$. And using Coulomb's constant in ratio form: $k_C = (r_c/t_c)^2(m_G r_c/q_C^2) \approx 9 \cdot 10^9 Nm^2C^{-2} \approx (3 \cdot 10^8 m\ s^{-1})^2(5.7 \cdot 10^{-12}kg\ C^{-1})(r_c/q_c) \Rightarrow r_c/q_C \approx 1.7 \cdot 10^5 m\ C^{-1}$. Therefore, $q = (q_C/r_c c)rv \approx (1/((1.7 \cdot 10^5 m\ C^{-1})(3 \cdot 10^8 m\ s^{-1})))rv \approx (1.9 \cdot 10^{-13}C\ s\ m^{-2})rv$.

5.5. 3 dimensional balls. Countable volume, $v_c = \prod_{i=1}^{n} |x_i|$, Euclidean volume, $v = \prod_{i=1}^{n} s_i$, and all Minkowski distances, $d = (\sum_{i=1}^{n} s_i^n)^{1/n}$, require that a set of domain intervals/dimensions can be assigned a *total order*. A total order is defined in terms of successor and predecessor relations, where, in this case, the successor and predecessor relations are specified by the integers $i = 1$ to n that map to a set of domain intervals/dimensions.

But the commutative properties of union, multiplication, and addition allow sequencing through each interval (dimension) in every possible order. And "jumping" (indexing) over set members to another member requires calculating an offset, which is implicitly sequencing via the successor and predecessor relations.

Therefore, sequencing directly via the successor and predecessor relations from one set member to every other member requires each set member to be sequentially adjacent (either a successor or predecessor) to every other member, herein referred to as a symmetry constraint. It will now be proved that coexistence of the symmetry constraint on a sequentially ordered set defines a cyclic set that contains at most 3 members, in this case, 3 dimensions per ball and 3 3-dimensional balls.

DEFINITION 5.1. Ordered geometry:

$$\forall\, i\, n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i\ \in \{x_1, \ldots, x_n\},$$
$$successor\ x_i = x_{i+1}\ \ \wedge\ \ predecessor\ x_{i+1} = x_i.$$

DEFINITION 5.2. Symmetry Constraint (every set member is sequentially adjacent to every other member):

$$\forall\, i\, j\, n \in \mathbb{N},\ \ \forall\, x_i\, x_j \in \{x_1, \ldots, x_n\},\ \ successor\ x_i = x_j\ \ \Leftrightarrow\ \ predecessor\ x_j = x_i.$$

THEOREM 5.3. *An ordered and symmetric set is a cyclic set.*

$$i = n\ \wedge\ j = 1\ \Rightarrow\ successor\ x_n = x_1\ \ \wedge\ predecessor\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry constraint (5.2):

$$i = n\ \wedge\ j = 1\ \wedge\ successor\ x_i = x_j\ \ \Rightarrow\ \ successor\ x_n = x_1. \tag{5.15}$$

Applying the definition of the symmetry constraint (5.2) to conclusion 5.15:

$$successor\ x_i = x_j\ \ \Rightarrow\ \ predecessor\ x_j = x_i\ \ \Rightarrow\ \ predecessor\ x_1 = x_n. \tag{5.16}$$ □

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3).

DEFINITION 5.5. Successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (n = m+1 \leq \mathit{setsize}). \tag{5.17}$$

DEFINITION 5.6. Predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (n = m - q \geq 1). \tag{5.18}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$\mathit{Adjacent}(m, n, \mathit{setsize}) \leftarrow \mathit{Successor}(m, n, \mathit{setsize}) \vee \mathit{Predecessor}(m, n, \mathit{setsize}). \tag{5.19}$$

Prove that every member is adjacent to every other member, where $\mathit{setsize} \in \{1,\ 2,\ 3\}$:

$$\mathit{Adjacent}(1, 1, 1) \leftarrow \mathit{Successor}(1, 1, 1) \leftarrow (m = \mathit{setsize} \wedge n = 1). \tag{5.20}$$

$$\mathit{Adjacent}(1, 2, 2) \leftarrow \mathit{Successor}(1, 2, 2) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{5.21}$$

$$\mathit{Adjacent}(2, 1, 2) \leftarrow \mathit{Successor}(2, 1, 2) \leftarrow (n = \mathit{setsize} \wedge m = 1). \tag{5.22}$$

$$\mathit{Adjacent}(1, 2, 3) \leftarrow \mathit{Successor}(1, 2, 3) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{5.23}$$

$$\mathit{Adjacent}(2, 1, 3) \leftarrow \mathit{Predecessor}(2, 1, 3) \leftarrow (n = m - q \geq 1). \tag{5.24}$$

$$\mathit{Adjacent}(3, 1, 3) \leftarrow \mathit{Successor}(3, 1, 3) \leftarrow (n = \mathit{setsize} \wedge m = 1). \tag{5.25}$$

$$\mathit{Adjacent}(1, 3, 3) \leftarrow \mathit{Predecessor}(1, 3, 3) \leftarrow (m = 1 \wedge n = \mathit{setsize}). \tag{5.26}$$

$$\mathit{Adjacent}(2, 3, 3) \leftarrow \mathit{Successor}(2, 3, 3) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{5.27}$$

$$\mathit{Adjacent}(3, 2, 3) \leftarrow \mathit{Predecessor}(3, 2, 3) \leftarrow (n = m - q \geq 1). \tag{5.28}$$

Must prove that for all $\mathit{setsize} > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\begin{aligned} \forall\, \mathit{setsize} > 3 : \quad & \neg \mathit{Successor}(1, 3, \mathit{setsize} > 3) \\ & \leftarrow \mathit{Successor}(1, 2, \mathit{setsize} > 3) \leftarrow (n = m + 1 \leq \mathit{setsize}). \end{aligned} \tag{5.29}$$

That is, member 2 is the only successor of member 1 for all $\mathit{setsize} > 3$, which implies member 3 is not a successor of member 1 for all $\mathit{setsize} > 3$.

$$\begin{aligned} \forall\, \mathit{setsize} > 3 : \quad & \neg \mathit{Predecessor}(1, 3, \mathit{setsize} > 3) \\ & \leftarrow \mathit{Predecessor}(1, \mathit{setsize}, \mathit{setsize} > 3) \leftarrow (m = 1 \wedge n = \mathit{setsize} > 3). \end{aligned} \tag{5.30}$$

That is, member $n = \mathit{setsize} > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $\mathit{setsize} > 3$.

$$\forall\, \mathit{setsize} > 3: \quad \neg \mathit{Adjacent}(1, 3, \mathit{setsize} > 3) \\ \leftarrow \neg \mathit{Successor}(1, 3, \mathit{setsize} > 3) \wedge \neg \mathit{Predecessor}(1, 3, \mathit{setsize} > 3). \quad \square \tag{5.31}$$

That is, for all $\mathit{setsize} > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

(1) It was shown that all distances that are an inverse function of an n-volume are Minkowski distances (4.2). And the Minkowski distances have the properties defining metric space (4.6). Therefore, the criteria of a distance measure being a function equivalent to a Minkowski distance (or all functions derived from an n-volume) might model geometric distance more completely than the definition of metric space by filtering out some contrived functions that satisfy the criteria of a metric space.

(2) A new inequality, the volume inequality (4.5), was used to help derive the metric space triangle inequality (4.8), which is another indicator that distance is a function of volume.

(3) The volume inequality extended to the sum of n-volumes (4.6):

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

Minkowski's sum inequality:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n)^n)^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

Note the difference in the left side of the two equations:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m}(a_i^n + b_i^n)^n)^{1/n}.$$

Derivation of the volume sum inequality is much simpler and shorter than the derivation of Minkowski's sum inequality. Unlike Minkowski's sum inequality proof, the volume inequality and volume sum inequality proofs do not depend on: convexity, L_p spaces, Hölder's inequality, Mahler's inequality, or the triangle inequality, which indicates that the volume (sum) inequality is a fundamental inequality.

(4) Proofs that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line (equation), where it is assumed that the straight line is the smallest distance [**Joy98**]. And proofs that a straight line (equation) is the smallest distance equate the straight line to the Euclidean distance, which is shallow logic. There have been no set and limit-based explanations of why the Euclidean distance/straight line is the smallest distance.

Countable distance (4.4), $d_c^n = \sum_{i=1}^{m} |x_i|^n$, generates Minkowski distance (4.2), which exposes the countable domain-to-self set mappings that generate distance. The domain-to-self set mapping that generate flat space (rectilinear distances) is where: 1) each member of domain set, x_i, maps to itself once, and 2) each member of domain set, x_i, maps at most once to each member of domain set, x_i. Therefore, the countable distance, d_c, in flat space, ranges:

   (a) from the sum of bijective mappings (the sum of 1-1 correspondences), which converges to Manhattan distance (by lemma 4.4), for example, $d = a + b + c$,
   (b) to the sum of the Cartesian product mappings, which converges to Euclidean distance (by lemma 4.4): $d = (a^2 + b^2 + c^2)^{1/2}$.

   $\forall\, a, b, c > 0,\ 1 \leq p < 2 :\ (a^2 + b^2 + c^2)^{1/2} < (a^p + b^p + c^p)^{1/p}$, where the largest number of domain-to-self set mappings (the Cartesian product) makes Euclidean distance is the smallest distance in flat space.

(5) Applying the ruler (2.1) and volume proof (3.2) to the Cartesian product of same-sized, infinitesimal mass forces and charge forces to derive Newton's gravity force (5.1) and Coulomb's charge force (5.2) equations provide several firsts and some insights into physics:
   (a) These are the first deductive derivations of the gravity and charge forces. All other derivations have been empirical and use Newton's induction, which is not fully provable, for example, assumes the inverse square law based on empirical observation.
   (b) These are the first derivations to not use the inverse square law or Gauss's divergence theorem.
   (c) These are the first derivations to show that the definition of force, $F := m_0 a$, containing acceleration, $a : \int_0^t a dt = r/t \ \Rightarrow\ \exists\, t_c, r_c \in \mathbb{R} :\ t/r = t_c/r_c \ \Rightarrow\ t = (t_c/r_c)r$, generates the inverse square law: $F := m_0 a = m_0 r/t^2 = (r_c/t_c)^2(m_x r_c/x_x^2)x_1 x_2/r^2 = k_x x_1 x_2/r^2$.
   (d) Using Occam's razor, those versions of constants like: charge, vacuum magnetic permeability, fine structure, etc. that contain the value 4π are probably incorrect because those constants are based on the less parsimonious assumption that the inverse square law is due to Gauss's flux divergence on a sphere having the surface area, $4\pi r^2$.
   (e) These are the first derivations to show that time is proportionate to distance: $r = (r_c/t_c)t = ct$, which is used to derive the spacetime equations (5.3) without the notion of the speed of light.
   (f) The derivations show for the first time that gravity force, charge force, spacetime, and general relativity all depend on time being proportionate to distance.
   (g) Combining the constant first derivatives (ratios) into partial differential equations allows simple algebraic derivations of some general relativity equations (5.4) without the need for integrating second derivative (spacetime curvature) tensors.
   (h) A state is represented by a constant value. And a constant value, by definition, cannot vary with distance and time interval lengths. Therefore, the spin values of two quantum entangled electrons and the polarization of two quantum entangled photons are independent of the amount of distance and time between the entangled particles.

(6) It was proved that a totally ordered set with a symmetry constraint is a cyclic set with at most 3 members (5.3). And the definitions of distance and volume both require a total order and symmetry, which provides several insights:
   (a) Using Occam's razor, a cyclic set of at most 3 members is the most parsimonious explanation of only observing 3 dimensions of geometric

distance and volume.

   (b) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (5.4), each member being an ordered and symmetric set of 3 dimensions (three balls), yielding a total of at most 9 ordered and symmetric dimensions of geometric space.
   (c) Each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values of the same type, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values of the same type per 3-dimensional ball, for example, vector orientation values: -1, 0, 1 per orthogonal direction in the ball.
   (d) Each of the 3 possible ordered and symmetric dimensions of discrete physical states could contain an unordered collection (bag) of discrete state values. Bags are non-deterministic. For example, every time an unordered binary state is "pulled" from a bag, there is a 50 percent chance of getting one of the binary values.

(7) Functions that area a bijective mapping (1-1 correspondence) between the elements of real-valued intervals is a primary tool in mathematics. But, in this article, it was shown that some fundamental geometry (volume and the Minkowski distances/L_p norms) and physics (gravity force and charge force) are derived from the combinatorial mappings between the elements of real-valued intervals. And only 3 dimensions of geometric space visible is also due to a combinatorics.

(8) The proofs and derivations in this article show that the ruler (2.1) is a tool to directly express some combinatorial relations in geometry, probability, physics, etc. that are difficult to directly express with differential equations and integrals.

References

[Bru17] P. Bruskiewich, *A very simple introduction to special relativity: Part two - four vectors, the lorentz transformation and group velocity (the new mathematics for the millions book 38)*, Pythagoras Publishing, 2017. ↑7

[Coq15] Coq, *Coq proof assistant*, 2015. https://coq.inria.fr/documentation. ↑2

[Gol76] R. R. Goldberg, *Methods of real analysis*, John Wiley and Sons, 1976. ↑1

[Joy98] D. E. Joyce, *Euclid's elements*, 1998. http://aleph0.clarku.edu/~djoyce/java/elements/elements.html. ↑1, 10

[Rud76] W. Rudin, *Principles of mathematical analysis*, McGraw Hill Education, 1976. ↑1

George Van Treeck, 668 Westline Dr., Alameda, CA 94501