The Countable Set Mappings Generating Geometry

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Abstract. Where each countable, disjoint domain set has a corresponding range (distance) set: the countable distance spanning the domain sets is a function of the number of domain-to-distance set mappings; and countable volume is a function of the number of distance-to-distance set mappings. The countable distance and volume mappings between sets of size c subintervals of domain and range intervals generate the properties of metric space, the Lp norms (for example, Manhattan and Euclidean distance), and the volume equation as c goes to 0. The volume proof is used to derive Coulomb's charge force and Newton's gravity force equations without using other laws of physics or Gauss's divergence theorem. A symmetry constraint on a strict totally ordered set limits the set to at most 3 members, for example, 3 dimensions of distance and volume. All proofs are verified in Coq.

Contents

1. Introduction

The definitions of metric space, Euclidean distance, and area/volume in analysis [Gol76] [Rud76] have been motivated by Euclidean geometry [Joy98]. Here, a set and limit-based foundation is used to motivate and derive the properties of metric space, Euclidean distance, volume, and generalize to non-Euclidean spaces, without relying on geometry primitives, like: side, angle, triangle, rectangle, etc.

The set and limit-based foundation also exposes other properties of geometry that point-set topology has failed to provide, for example: the counting constraint

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between domain and range sets that makes a space flat; the countable domain-to-range set mapping that makes Euclidean distance the smallest possible distance in flat space; the set operation and constraint generating the properties of metric space; and the symmetry constraint on sets that can limit a set to at most 3 members, for example, 3 dimensions of distance and volume.

Where each disjoint, countable domain set, x_i, has a corresponding range (distance) set, y_i, the countable distance spanning the domain sets is the cardinal of the union of the range (distance) sets: $d_c = |\bigcup_{i=1}^n y_i|$. As the intersection of the range sets increases, more domain set members can map to a single range set member. Therefore, the cardinal of the union range set, d_c, is a function of the number of domain-to-range set mappings. Countable volume is the cardinal of the set of all possible range (distance) set intersections, which is a function of the number of distance-to-distance (range-to-range) set mappings.

Applying these abstract, countable set definitions of distance and volume to sets of size c subintervals of domain and range intervals generates the properties of metric space, all L_p norms (Minkowski distances, for example, Manhattan and Euclidean distance), and the volume equation as $c \to 0$. Some applications to physics are also shown.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals, where each subinterval has the *same* size, c. The ruler makes it easy to derive geometric relations from the number of possible mappings between the p_x number of size c subintervals in one interval and the p_y number of size c subintervals in another interval.

Definition 2.1. Ruler measure, M: $\forall\, c,\ s \in \mathbb{R},\ [a, b] \subset \mathbb{R},\ s = b - a \wedge c > 0 \wedge (p = floor(s/c) \ \vee\ p = ceiling(s/c)) \ \wedge\ M = \sum_{i=1}^{p} c = pc.$

Theorem 2.2. *Ruler convergence:* $M = \lim_{c \to 0} pc = s.$

The proof is trivial but is included here for completeness. The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : \ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\ p = floor(s/c) \ \wedge\ 0 \le |floor(s/c) - s/c| < 1 \ \Rightarrow\ 0 \le |p - s/c| < 1. \tag{2.1}$$

Multiply all sides of inequality 2.1 by c:

$$\forall\, c > 0, \quad 0 \le |p - s/c| < 1 \quad \Rightarrow \quad 0 \le |pc - s| < |c|. \tag{2.2}$$

$$\begin{aligned} \forall\, \delta :\ & |pc - s| < |c| = |c - 0| < \delta \\ & \Rightarrow \quad \forall\, \epsilon = \delta :\ |c - 0| < \delta \ \wedge\ |pc - s| < \epsilon \ := \ M = \lim_{c \to 0} pc = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}},\ 3.14_{c=10^{-2}},\ 3.141_{c=10^{-3}}, ..., \pi_{\lim_{c \to 0}}.$

3. Distance

Notation conventions: Vertical bars around a set, $|\{\cdots\}|$, or list, $|[\cdots]|$, indicates the cardinal (the number of members in the set or list).

3.1. Countable distance. Each disjoint domain set, x_i, has its own independent range (distance) set, y_i. The countable distance spanning the disjoint domain sets is the cardinal, d_c, of the union range (distance) set.

It will be shown in the next subsections that the constraint, $|x_i| = |y_i|$, generates Manhattan and Euclidean distance at the boundaries (generates flat space/rectilinear distances). Generalizing distance and volume beyond flat space is shown in the last section of this article.

Definition 3.1. Countable distance, d_c, in flat space:

$$d_c = |\bigcup_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i|.$$

3.2. Union-Sum Inequality. The inequality, $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$, is used in this article. The proof is trivial but is included here for completeness.

The proof follows from the associative law of addition where the sum of set sizes is equal to the size of all the set members appended into a list and the commutative law of addition that allows sorting that list into a list of unique members (the *union* set) and a list of duplicates. For example, $y_1 = \{a, b, c\}$ and $y_2 = \{c, d, e\} \Rightarrow \bigcup_{i=1}^{2} |y_i| = |\{a, b, c, d, e\}| = 5 < \sum_{i=1}^{2} |y_i| = |\{a, b, c\}| + |\{c, d, e\}| = |[a, b, c, c, d, e]| = |\{a, b, c, d, e\}| + |[c]| = 6$.

Lemma 3.2. *Union-Sum Inequality:* $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$.

Proof. A formal proof, union_sum_inequality, using sorting into a set of unique members (*union* set) and a list of duplicates, is in the file euclidrelations.v.

$$\sum_{i=1}^{n} |y_i| = |append_{i=1}^{n} y_i| = |sort(append_{i=1}^{n} y_i)| = |\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i|. \tag{3.1}$$

$$|\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| \quad \wedge \quad |duplicates_{i=1}^{n} y_i| \geq 0 \Rightarrow |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|. \quad \square \tag{3.2}$$

3.3. Countable distance range. Each domain set, x_i has its own independent range set, y_i. From the countable distance constraint (3.1), where $|x_i| = |y_i| = p_i$, the countable distance, d_c, ranges from a function of the sum of 1-1 correspondence mappings, $d_c = \mathrm{f}(\sum_{i=1}^{n}(1 \cdot |y_i|)) = \mathrm{f}(\sum_{i=1}^{n} p_i)$, to a function of the sum of all-to-each (Cartesian product) mappings, $d_c = \mathrm{f}(\sum_{i=1}^{n}(|x_i| \cdot |y_i|)) = \mathrm{f}(\sum_{i=1}^{n} p_i^2)$.

Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of domain-to-range set mapping cases converges to the real-valued Manhattan and Euclidean distance relations.

3.4. Manhattan distance.

Theorem 3.3. *Manhattan (largest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:

$$d = \sum_{i=1}^{n} s_i, \quad d = d_m - d_0, \quad s_i = b_i - a_i.$$

The formal proof, "taxicab_distance," is in the Coq file, euclidrelations.v.

PROOF.
From the countable distance definition (3.1) and the union-sum inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n}|y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \quad \Rightarrow \quad d_c \leq \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i. \tag{3.3}$$

Multiply both sides of equation 3.3 by c and take the limit:

$$d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n}(p_i \cdot c) \Rightarrow \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c). \tag{3.4}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$d = d_m - d_0 \quad \Rightarrow \quad \exists\, c\, d :\ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c. \tag{3.5}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$s_i = b_i - a_i \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i. \tag{3.6}$$

Combine equations 3.5, 3.4, 3.6:

$$d = \lim_{c\to 0} d_c \cdot c \quad \wedge \quad \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) \quad \wedge \quad \lim_{c\to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c) = \sum_{i=1}^{n} s_i. \tag{3.7}$$ □

3.5. Euclidean distance.

THEOREM 3.4. *Euclidean (smallest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals, $\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = d_m - d_0, \quad s_i = b_i - a_i.$$

The formal proof, "Euclidean_distance," is in the Coq file, euclidrelations.v.

PROOF.
Apply the rule of product to the largest number of domain-to-range set mappings, where all p_i number of range set members, y_i, map to each of the p_i number of members in the domain set, x_i, which, by the rule of product, is the Cartesian product, $|y_i| \cdot |x_i|$:

$$|x_i| = |y_i| = p_i \quad \Rightarrow \quad \sum_{i=1}^{n}|y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2. \tag{3.8}$$

From the countable distance definition (3.1) and the union-sum inequality (3.2), the smallest possible distance is the equality case:

$$d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n}|y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \quad \Rightarrow \quad d_c \leq \sum_{i=1}^{n} p_i \quad \Rightarrow \quad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i. \tag{3.9}$$

Square both sides of equation 3.9 ($x = y \Leftrightarrow f(x) = f(y)$):

$$\exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\ d_c :\ d_c^2 = (\sum_{i=1}^{n} p_i)^2. \tag{3.10}$$

Apply the square of sum inequality, $(\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$, to equation 3.10 and select the smallest area (the equality) case:

$$d_c^2 = (\sum_{i=1}^{n} p_i)^2 = \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j = \sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} p_i \sum_{j=1, j\neq i}^{n} p_j \geq \sum_{i=1}^{n} p_i^2 \quad \Rightarrow \quad \exists\, p_i :\ d_c^2 = \sum_{i=1}^{n} p_i^2. \tag{3.11}$$

Multiply both sides of equation 3.11 by c^2, simplify, and take the limit.

$$(3.12)\quad d_c^2 = \sum_{i=1}^{n} p_i^2 \;\Rightarrow\; d_c^2 \cdot c^2 = \sum_{i=1}^{n} p_i^2 \cdot c^2 \;\Leftrightarrow\; (d_c \cdot c)^2 = \sum_{i=1}^{n} (p_i \cdot c)^2 \;\Rightarrow\; \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$(3.13)\quad \exists\, c\, d \in \mathbb{R} : \; floor(d/c) = d_c \quad\Rightarrow\quad d = \lim_{c\to 0} d_c \cdot c \quad\Rightarrow\quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$(3.14)\quad s_i = b_i - a_i \quad\wedge\quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad\Rightarrow\quad \lim_{c\to 0} p_i \cdot c = s_i.$$

Combine equations 3.13, 3.12, 3.14:

$$(3.15)\quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 \quad\wedge\quad \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 \quad\wedge$$
$$\lim_{c\to 0}(p_i \cdot c) = s_i \quad\Rightarrow\quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 = \sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \quad\square$$

3.6. Metric Space. All distances, $d(u,w)$, satisfying the countable distance definition (3.1), where the ruler is applied, generates the properties of metric space. The formal proofs: triangle_inequality, non_negativity, identity_of_ indiscernibles, and symmetry are in the Coq file, euclidrelations.v.

THEOREM 3.5. *Triangle Inequality:* $d_c = |y_1 \cup y_2| \Rightarrow d(u,w) \le d(u,v) + d(v,w)$.

PROOF. Apply the ruler measure (2.1), the countable distance condition (3.1), union-sum inequality (3.2), and then ruler convergence (2.2).

$$\begin{aligned}
(3.16)\quad & \forall\, c > 0,\; d(u,w),\; d(u,v),\; d(v,w)\; : \\
& |y_1| = floor(d(u,v)/c) \;\wedge\; |y_2| = floor(d(v,w)/c) \;\wedge \\
& d_c = floor(d(u,w)/c) \quad\wedge\quad d_c = |y_1 \cup y_2| \le |y_1| + |y_2| \\
& \Rightarrow\; floor(d(u,w)/c) \le floor(d(u,v)/c) + floor(d(v,w)/c) \\
& \Rightarrow\; floor(d(u,w)/c)\cdot c \le floor(d(u,v)/c)\cdot c + floor(d(v,w)/c)\cdot c \\
& \Rightarrow\; \lim_{c\to 0} floor(d(u,w)/c)\cdot c \le \lim_{c\to 0} floor(d(u,v)/c)\cdot c + \lim_{c\to 0} floor(d(v,w)/c)\cdot c \\
& \Rightarrow\; d(u,w) \le d(u,v) + d(v,w). \quad\square
\end{aligned}$$

THEOREM 3.6. *Non-negativity:* $d_c = |y_1 \cup y_2| \Rightarrow d(u,w) \ge 0$.

PROOF. By definition, a set always has a size (cardinal) ≥ 0:

$$\begin{aligned}
(3.17)\quad & \forall\, c > 0,\; d(u,w)\; : \quad floor(d(u,w)/c) = d_c \quad\wedge\quad d_c = |y_1 \cup y_2| \ge 0 \\
& \Rightarrow\quad floor(d(u,w)/c) = d_c \ge 0 \quad\Rightarrow\quad d(u,w) = \lim_{c\to 0} d_c \cdot c \ge 0. \quad\square
\end{aligned}$$

THEOREM 3.7. *Identity of Indiscernibles:* $d(w,w) = 0$.

PROOF. Apply the triangle inequality property (3.5):

$$(3.18)\quad \forall\, d(u,v) = d(v,w) = 0 \;\wedge\; d(u,w) \le d(u,v) + d(v,w) \;\Rightarrow\; d(u,w) \le 0.$$

Combine the non-negativity property (3.6) and the previous inequality (3.18):

$$(3.19)\quad d(u,w) \ge 0 \;\wedge\; d(u,w) \le 0 \;\Leftrightarrow\; 0 \le d(u,w) \le 0 \;\Rightarrow\; d(u,w) = 0.$$

Combine the result of step 3.19 and the condition, $d(u,v) = 0$, in step 3.18.

$$(3.20)\quad d(u,w) = 0 \;\wedge\; d(u,v) = 0 \;\Rightarrow\; w = v.$$

Combine the condition, $d(v,w) = 0$, in step 3.18 and the result of step 3.20.

$$(3.21)\quad d(v,w) = 0 \;\wedge\; w = v \;\Rightarrow\; d(w,w) = 0. \qquad\square$$

THEOREM 3.8. *Symmetry:* $d_c = |y_1 \cup y_2| \;\wedge\; |x_i| = |y_i| \;\Rightarrow\; d(u,v) = d(v,u)$.

PROOF. The number of mapping of domain set, x_i, members to a range set, y_i, member increases with the amount of range set intersection. Therefore, the range of countable distances (3.3) is a function of domain-to-range set members, under the constraint, $|x_i| = |y_i|$, is:

$$|x_i| = |y_i| = p_i \quad \Rightarrow \quad d_c = \mathrm{f}(\textstyle\sum_{i=1}^{n} |x_i| \cdot |y_i|^q) = \mathrm{f}(\sum_{i=1}^{n} p_i^{1+q}),\ q \in \{0,\ 1\}. \tag{3.22}$$

Where d_c is applied to sets of size c subintervals of intervals, the previous Manhattan distance proof (3.3), $d(x,y) = \mathrm{f}(\sum_{i=1}^{2} s_i^1)$, and Euclidean distance proof (3.4), $d(x,y) = \mathrm{f}(\sum_{i=1}^{2} s_i^2)$, show that distance is a function of domain interval sizes, s_i, where $x = s_1$ and $y = s_2$. Generalizing:

$$\begin{aligned} \forall\, p : p \geq 0, \quad & d(x,y) = \mathrm{f}(\textstyle\sum_{i=1}^{2} s_i^p) = \mathrm{f}(x^p + y^p) \\ & \Rightarrow \quad d(u,v) = \mathrm{f}(u^p + v^p) = \mathrm{f}(v^p + u^p) = d(v,u). \quad \square \end{aligned} \tag{3.23}$$

4. Euclidean Volume

$\mathbb{R}^n$, the Lebesgue measure, Riemann integral, and Lebesgue integral define (assume) area/volume to be the product of domain interval lengths. The goal here is to derive the area/volume equation from an abstract, set-based definition of volume without assuming the product of interval lengths.

Countable volume is the cardinal of the set of all possible range (distance) set intersections, which is the set of n-tuples of one member from each range set. And the number of n-tuples is the number of distance-to-distance (range-to-range) set mappings. By the rule of product, the largest number of mappings is the Cartesian product of the number of members in each disjoint, range set:

DEFINITION 4.1. Euclidean (largest possible) Countable Volume in flat space:

$$v_c = |\times_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \bigcap_{i=1}^{n} y_i = \emptyset \quad \wedge \quad |x_i| = |y_i|.$$

THEOREM 4.2. *Euclidean volume, v, is length of the range interval, $[v_0, v_m]$, equal to product of domain interval lengths, $\{[a_1,b_1],[a_2,b_2],\ldots,[a_n,b_n]\}$:*

$$v = \prod_{i=1}^{n} s_i, \;\; v = v_m - v_0, \;\; s_i = b_i - a_i.$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of subintervals.

$$\forall\, i\, n \in \mathbb{N}, \quad i \in [1,n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i| = |y_i|. \tag{4.1}$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c \to 0}(p_i \cdot c) = s_i. \tag{4.2}$$

Apply the associative law of multiplication to derive the countable volume (4.1) in terms of p_i:

$$v_c = |\times_{i=1}^{n} y_i| = \prod_{i=1}^{n} |y_i| \quad \wedge \quad |y_i| = p_i \quad \Rightarrow \quad v_c = \prod_{i=1}^{n} p_i. \tag{4.3}$$

Multiply both sides of equation 4.3 by c^n:

$$v_c \cdot c^n = (\prod_{i=1}^{n} p_i) \cdot c^n = \prod_{i=1}^{n} (p_i \cdot c). \tag{4.4}$$

(4.5) $\forall\, n, v_c \,\in \mathbb{N} \,\exists\, p \in \mathbb{R} \,:\; p^n = v_c \;\Rightarrow\; v_c \cdot c^n = p^n \cdot c^n = (p \cdot c)^n = \prod_{i=1}^n (p_i \cdot c).$

Apply the ruler (2.1) and ruler convergence (2.2) to the range interval, $[v_0, v_m]$ (where $v = v_m - v_0$), and then combine with equations 4.5 and 4.2:

(4.6) $floor(v/c^n) = p^n \Rightarrow v = \lim_{c\to 0}(p \cdot c)^n = \prod_{i=1}^n \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^n s_i.$ □

5. Applications to physics

5.1. Coulomb's charge force. q_1 and q_2, are the sizes of two independent charge intervals, where each infinitesimal size c subinterval of a charge interval exerts an infinitesimal force, $m_C a_C$, on each size c subinterval of the other charge interval. The total force, F, is equal to the total number of forces, which is proportionate to the Cartesian product of the infinitesimal size c components:

(5.1) $$p_1 = floor(q_1/c) \quad\wedge\quad p_2 = floor(q_2/c)$$
$$\Rightarrow \quad F = (m_C a_C/q_C^2)(\lim_{c\to 0} p_1 c \cdot \lim_{c\to 0} p_2 c) =$$
$$(m_C a_C/q_C^2) \int_0^{q_2} \int_0^{q_1} \mathrm{d}^2 c = (m_C a_C/q_C^2)(q_1 q_2),$$

where $m_C a_C/q_C^2$ is a unit-factoring conversion ratio relating the force, F, to the charge area $q_1 q_2$.

By the definition of force, $F = m_0 a$, for a constant mass, m_0, an increase of charge, q, causes the distance, r, moved during the acceleration to increase proportionately. Therefore, every size, q, of a charge interval, has a proportionate, size r, distance interval, where $r \,:\; q = (q_C/r_C)r$ and q_C/r_C is a unit-factoring conversion ratio.

(5.2) $\forall\, q_1, q_2 \geq 0 \;\exists\, q \in \mathbb{R} \,:\; q^2 = q_1 q_2 \;\wedge\; (q_C/r_C)r = q \;\Rightarrow\; ((q_C/r_C)r)^2 = q_1 q_2.$

(5.3) $$((q_C/r_C)r)^2 = q_1 q_2 \;\wedge\; F = (m_C a_C/q_C^2)(q_1 q_2)$$
$$\Rightarrow \quad F = (m_C a_C/q_C^2)((q_C/r_C)r)^2 = (m_C a_C/r_C^2)r^2 = (m_C a_C/q_C^2)(q_1 q_2)$$
$$\Rightarrow \quad F = m_C a_C = (m_C a_C r_C^2/q_C^2) q_1 q_2/r^2 = k_c q_1 q_2/r^2.$$

where $k_C = m_C a_C r_C^2/q_C^2$ corresponds to the SI units: Nm^2C^{-2}. And multiplying both sides of equation 5.3 by xy:

(5.4) $\forall\, x,\; y \geq 1, m_0 = x \cdot m_C,\, a = y \cdot a_C,\, d^2 = r^2/(x \cdot y) \Rightarrow F = m_0 a = k_c q_1 q_2/d^2.$

5.2. Newton's gravity force equation. m_1 and m_2, are the sizes of two independent mass intervals, where each infinitesimal size c subinterval of a mass interval exerts an infinitesimal force, $m_G a_G$, on each size c subinterval of the other mass interval. The total force, F, is equal to the total number of forces, which is proportionate to the Cartesian product of the infinitesimal size c components:

(5.5) $$p_1 = floor(m_1/c) \quad\wedge\quad p_2 = floor(m_2/c)$$
$$\Rightarrow \quad F = (m_G a_G/m_G^2)(\lim_{c\to 0} p_1 c \cdot \lim_{c\to 0} p_2 c) = a_G/m_G(\lim_{c\to 0} p_1 c \cdot \lim_{c\to 0} p_2 c) =$$
$$a_G/m_G \int_0^{m_2} \int_0^{m_1} \mathrm{d}^2 c = a_G/m_G(m_1 m_2),$$

where $m_G a_G/m_G^2 = a_G/m_G$ is a unit-factoring conversion ratio relating the force, F, to the mass area $m_1 m_2$.

By the definition of force, $F = m_0 a$, for a constant mass, m_0, an increase of masses, m_1 or m_2, causes the distance, r, moved during the acceleration to increase proportionately. Therefore, every size, m, of a mass interval, has a proportionate, size r, distance interval, where $r : m = (m_G/r_G)r$ and m_G/r_G is a unit-factoring conversion ratio.

$$\forall\, m_1, m_2 \geq 0\, \exists\, m \in \mathbb{R} : \quad m^2 = m_1 m_2 \quad \wedge \quad (m_G/r_G)r = m \\ \Rightarrow \quad ((m_G/r_G)r)^2 = m_1 m_2. \tag{5.6}$$

$$((m_G/r_G)r)^2 = m_1 m_2 \;\wedge\; F = (a_G/m_G)(m_1 m_2) \\ \Rightarrow \quad F = (a_G/m_G)((m_G/r_G)r)^2 = (a_G/m_G)(m_1 m_2) \\ \Rightarrow \quad F = (m_G a_G/r_G^2)r^2 = (a_G/m_G)(m_1 m_2) \\ \Rightarrow \quad F = m_G a_G = (a_G r_G^2/m_G)m_1 m_2/r^2. \tag{5.7}$$

$$\exists\, t_G \in \mathbb{R} : r_G/t_G^2 = a_G \quad \wedge \quad F = m_G a_G = (a_G r_G^2/m_G)m_1 m_2/r^2 \\ \Rightarrow \quad F = m_G a_G = (r_G^3/m_G t_G^2)m_1 m_2/r^2 = G m_1 m_2/r^2, \tag{5.8}$$

where $G = r_G^3/m_G t_G^2$ corresponds to the SI units: $m^3 kg^{-1} s^{-2}$. And multiplying both sides of equation 5.8 by xy:

$$\forall\, x,\; y \geq 1, m_0 = x \cdot m_G, a = y \cdot a_G, d^2 = r^2/(x \cdot y) \Rightarrow F = m_0 a = G m_1 m_2/d^2. \tag{5.9}$$

5.3. Spacetime equations. For any Euclidean distance interval having size, r, a interval having size, t, can be defined, where $r = (r_c/t_c)t = ct$, and $r_c/t_c = c$ is a unit-factoring conversion ratio.

Applying the ruler to two intervals, $[0, d_1]$ and $[0, d_2]$, in two inertial (independent, non-accelerating) frames of reference, the distance (and time) spanning the two domain intervals converges to a range of distances (and times) from Manhattan (3.3) to Euclidean distance (3.4).

$$r^2 = d_1^2 + d_2^2 \quad \wedge \quad r = (r_c/t_c)t = ct \quad \Rightarrow \quad (ct)^2 = d_1^2 + d_2^2 \\ \Leftrightarrow \quad d_1^2 = (ct)^2 - (x^2 + y^2 + z^2), \tag{5.10}$$

where $d_2^2 = x^2 + y^2 + z^2$, which is one form of Minkowski's well-known flat spacetime interval equation [**Bru17**]. And, the time dilation and length contraction equations also follow directly by dividing both sides of $(ct)^2 = d_1^2 + d_2^2$ by t^2 and using $v = d/t$.

5.4. 3 dimensional balls. Countable distance, $d_c = |\bigcup_{i=1}^{n} y_i|$, (3.1), countable volume, $v_c = |\times_{i=1}^{n} y_i|$, (4.1), Manhattan distance (3.3), Euclidean distance (3.4), and volume (4.2) requires that a set of intervals/dimensions can be assigned a *strict total order* ($i = 1$ to n). And the commutative properties of union, multiplication, and addition allow sequencing through each interval (dimension) in every possible order. **Note** that "jumping" from member 1 to member m of a set requires calculating an offset that is an implicit traversal of successor/predecessor relations. Therefore, "strict" sequencing (no jumping over other members) via the successor and predecessor relations of a strict totally ordered set in every possible order requires each set member to be sequentially adjacent (either a successor or predecessor) to every other member, herein referred to as a symmetry constraint.

It will now be proved that (coexistence) of the symmetry constraint on a sequentially ordered set defines a cyclic set that contains at most 3 members, in this case, 3 dimensions of ordered and symmetric distance and volume. If there are higher dimension of space, then the cyclic property prevents sequencing from the 3 lower, cyclic set of dimensions to any higher dimensions.

Definition 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i\;\; \in \{x_1, \ldots, x_n\},$$
$$\mathit{successor}\; x_i = x_{i+1} \;\;\wedge\;\; \mathit{predecessor}\; x_{i+1} = x_i.$$

Definition 5.2. Symmetry Constraint (every set member is sequentially adjacent to every other member):

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; \mathit{successor}\; x_i = x_j \;\;\Leftrightarrow\;\; \mathit{predecessor}\; x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$i = n \;\;\wedge\;\; j = 1 \;\;\Rightarrow\;\; \mathit{successor}\; x_n = x_1 \;\;\wedge\;\; \mathit{predecessor}\; x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

Proof. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry constraint (5.2):

$$i = n \;\;\wedge\;\; j = 1 \;\;\wedge\;\; \mathit{successor}\; x_i = x_j \;\;\Rightarrow\;\; \mathit{successor}\; x_n = x_1. \tag{5.11}$$

Applying the definition of the symmetry constraint (5.2) to conclusion 5.11:

$$\mathit{successor}\; x_i = x_j \;\;\Rightarrow\;\; \mathit{predecessor}\; x_j = x_i \;\;\Rightarrow\;\; \mathit{predecessor}\; x_1 = x_n. \tag{5.12}$$ □

Theorem 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

Proof.
It was proved that an ordered and symmetric set is a cyclic set (5.3).

Definition 5.5. Successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (n = m + 1 \leq \mathit{setsize}). \tag{5.13}$$

Definition 5.6. Predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (n = m - q \geq 1). \tag{5.14}$$

Definition 5.7. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$\mathit{Adjacent}(m, n, \mathit{setsize}) \leftarrow \mathit{Successor}(m, n, \mathit{setsize}) \vee \mathit{Predecessor}(m, n, \mathit{setsize}). \tag{5.15}$$

Prove that every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (m = setsize \wedge n = 1). \tag{5.16}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (n = m + 1 \leq setsize). \tag{5.17}$$

$$Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (n = setsize \wedge m = 1). \tag{5.18}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m + 1 \leq setsize). \tag{5.19}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m - q \geq 1). \tag{5.20}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \wedge m = 1). \tag{5.21}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \wedge n = setsize). \tag{5.22}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m + 1 \leq setsize). \tag{5.23}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m - q \geq 1). \tag{5.24}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Successor(1,3,setsize > 3) \\ & \leftarrow Successor(1,2,setsize > 3) \leftarrow (n = m + 1 \leq setsize). \end{aligned} \tag{5.25}$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Predecessor(1,3,setsize > 3) \\ & \leftarrow Predecessor(1,setsize,setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \end{aligned} \tag{5.26}$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Adjacent(1,3,setsize > 3) \\ & \leftarrow \neg Successor(1,3,setsize > 3) \wedge \neg Predecessor(1,3,setsize > 3). \quad \square \end{aligned} \tag{5.27}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

(1) The set-based derivations of Manhattan distance (3.3) and Euclidean distance (3.4) show that the correct interpretation of a metric space-satisfying function of the form, $d(x, y)$, is the distance spanning two disjoint domain intervals having the lengths x and y.

(2) The smallest possible countable distance (3.1), $d_c = |\bigcup_{i=1}^{n} y_i|$, is the case of largest intersection of range sets, which is also the case of the largest possible number (the Cartesian product) of domain-to-range set mappings, in flat space: $d_c = \mathrm{f}(\sum_{i=1}^{n}|x_i| \cdot |y_i|) = \mathrm{f}(\sum_{i=1}^{n} p_i^2)$. And applying the ruler the Cartesian product of domain-to-range set mappings yields the Euclidean distance equation.

(3) Generalizing the countable distance and volume constraint, $|x_i| = |y_i|$, to $|x_i| = |y_i|^q$, $q \geq 0$, generates all the L^p norms (Minkowski distances), $||L||_p = (\sum_{i=1}^{n} s_i^p)^{1/p}$. For example, using the same proof pattern as for Euclidean distance (3.4): $|y_i| = p_i \Rightarrow |x_i| = p_i^q \Rightarrow \sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^q \cdot p_i = \sum_{i=1}^{n} p_i^{q+1} \leq d_c^{q+1}$. Choosing the equality case and applying the ruler: $d^{q+1} = \sum_{i=1}^{n} s_i^{q+1}$.

(4) $-1 \leq q < 0 \Rightarrow 0 \leq p < 1$ and generates functions that do not satisfy the metric space triangle inequality property.

(5) The curvature of a space around a point is typically measured in terms of second order differential equations, e.g., the Laplacian. A set-based measure of the amount of curvature is how far q deviates from the value, 1, in the countable distance and volume constraint, $|x_i| = |y_i|^q$.

(6) Manhattan (largest) distance and Euclidean (largest) volume are both cases of disjoint range sets, $\bigcap_{i=1}^{n} y_i = \emptyset$, in flat space (where $|x_i| = |y_i|$):

$$d_c = |\bigcup_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset.$$

$$v_c = | \times_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset.$$

(7) There are combinatorial relationships between countable sets of subintervals of intervals in statistics, probability, physics, etc., where the ruler is an applicable tool. For example, applying the ruler (2.1) and ruler convergence (2.2) to the Cartesian product of same-sized, infinitesimal charge forces and mass forces allowed deriving Coulomb's charge force (5.1) and Newton's gravity force (5.5) equations in a few steps each, without using other laws of physics or Gauss's divergence theorem.

(8) **The Proportionate Interval Principle:** The derivations of the charge force, gravity force, and spacetime equations shows that all Euclidean distance intervals having a size, r, have proportionately sized intervals of other types: $r = (r_C/q_C)q = (r_G/m_G)m = (r_c/t_c)t = ct$, where the conversion ratios are for unit-factor analysis.
   (a) Some versions of the charge constant, vacuum magnetic permeability constant, fine structure constant, etc. contain the value 4π because the creators assumed flux divergence on the surface of a sphere, $4\pi r^2$. Using Occam's razor, the mapping of rectangular geometric area to rectangular charge and mass areas provides more parsimonious derivations of the inverse square law, charge, and gravity force equations than flux divergence. Therefore, those versions of the constants containing the value 4π might be incorrect.
   (b) $(r_G/m_G)m \cdot ct = r^2 \Rightarrow m = (m_G/r_G c)r^2/t = (m_G/r_G c)rv$. For a constant mass, m, as the distance, r, to the mass center decreases, then time, t, must also decrease (time slows down), which agrees with relativity theory and observation, and where v is orbital velocity at distance, r, from the mass, m. Also, $(r_G/m_G)m \cdot (ct)^2 = r^3 \Rightarrow E = mc^2 = (m_G/r_G)r^3/t^2$. Likewise, for charge, $q = (q_C/r_C c)r^2/t = (q_C/r_C c)rv$ and $E = qc^2 = (q_C/r_C)r^3/t^2$.
   (c) The phenomenon of quantum values is the result of multiple conversion ratios that are related to each other.

   (d) If there are quantum values of charge, q_C, and mass, m_G, then there are quantum distances (wavelengths), r_C and r_G, where the charge and gravity forces do not exist (are not defined) at smaller distances.
   (e) There is no proportion relationship between a constant, finite value and the varying interval lengths of distance. For example, the change of spin values of two quantum entangled electrons and the change of polarization of two quantum entangled photons are independent the distance between the entangled particles.

(9) Any higher dimensions of space not being sequentially reachable from the lower 3 dimensions because the lower 3 dimensions are a cyclic set (5.4) is a more parsimonious explanation of not seeing any higher dimensions than the higher dimensions being rolled into infinitesimal balls, which requires an additional explanation of what causes the higher dimensions to be rolled up and additional equations describing the rollups.

(10) If there are higher dimensions of ordered and symmetric space, then there is a set of three members, each member being an ordered and symmetric set of 3 dimensions (three boxes), yielding a total of 9 ordered and symmetric dimensions of space.

(11) Each ordered and symmetric ball can have at most 3 ordered and symmetric dimensions of discrete states of the same type, for example, a set of 3 binary values, 1 and -1, indicating vector orientation.

(12) Each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values of the same type per ball, for example, spin values: -1, 0, 1 per orthogonal plane in the ball.

(13) Each of the 3 possible ordered and symmetric dimensions of discrete physical states could contain an unordered collection (bag) of discrete state values. Bags (of states) are non-deterministic. For example, every time an unordered binary state is measured, there is a 50 percent chance of having one of the binary values.

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