Some Set Properties Underlying Geometry and Physics

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ABSTRACT. Euclidean volume and the Minkowski distances (Manhattan, Euclidean, etc. distances) are derived from a set and limit-based foundation without referencing the primitives and relations of geometry. Sequencing a strict linearly ordered set in all n-at-a-time orders via successor/predecessor relations is proved to be a cyclic set of at most 3 members. A cyclic set of 3 distance domain interval lengths are related to other types of domain interval lengths by constant unit-factoring ratios. The geometry proofs and ratios are used to derive the gravity, charge, and Planck constants, and used to provide simpler, shorter, and more rigorous derivations of some well-known classical gravity and charge equations, special and general relativity equations, and quantum physics equations. All the proofs are verified in Coq.

CONTENTS

1. Introduction

Mathematical analysis constructs differential calculus from a set and limit-based foundation without referencing the primitives and relations of Euclidean geometry, like straight line, angle, etc., which provides a more rigorous foundation and deeper understanding of geometry and physics. But Euclidean volume in the Riemann integral, Lebesgue integral, measure theory, and distance in the vector magnitude and metric space criteria are definitions motivated by Euclidean geometry [**Gol76**] [**Rud76**] rather than derived from a set and limit-based foundation.

A simple, abstract, set-based definition of Euclidean volume is the number, v_c, of ordered combinations (n-tuples): $v_c = \prod_{i=1}^{n} |x_i|$, where $|x_i|$ is the cardinal of a countable, disjoint set, x_i. But real analysis text books and measure theory start with the definition of volume as the product of interval sizes, $v = \prod_{i=1}^{n} s_i$, where $s_i = b_i - a_i$, $[a_i, b_i] \subset \mathbb{R}$ [**Gol76**] [**Rud76**]. Where each set, x_i, is a set of size κ subintervals of the interval, $[a_i, b_i]$, it will be proved that, $\lim_{\kappa \to 0} v_c \cdot \kappa = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|) \cdot \kappa \Rightarrow v = \prod_{i=1}^{n} s_i$.

$v_c = \prod_{i=1}^{n} |x_i| = f(|x_1|, \cdots, |x_n|, n)$. If f is a bijective function, then $\exists\, d_c : d_c = f^{-1}(v_c, n)$ and $v_c = f(d_c, n) = f(|x_1|, \cdots, |x_n|, n)$. If f is also isomorphic, then $\forall\, |x_i| : d_c = |x_1| = \cdots = |x_n|$ and $v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n$.

Where $v_c = f(|x_1|, \cdots, |x_n|, n)$ is a bijective and isomorphic function, it will be proved that $\lim_{\kappa \to 0} v_c \cdot \kappa = \lim_{\kappa \to 0} (\sum_{j=1}^{m} v_{c_i}) \cdot \kappa \Rightarrow d^n = \sum_{i=1}^{m} d_i^n$. d is the ρ-norm (Minkowski distance) [**Min53**], which will be proved to imply the metric space properties [**Rud76**].

Sequencing the domain sets, $x_1, \cdots, x_n$, from $i = 1$ to n, is a strict linear (total) order, where a total order is defined in terms of successor and predecessor relations [**CG15**]. Sequencing a set, via successor and predecessor relations, in all n-at-a-time orders, requires a "symmetry" constraint, where every set member is either a successor or predecessor to every other set member. A strict linearly ordered and symmetric set will be proved to be a cyclic set, where $n \leq 3$.

Therefore, if $\{s_1, s_2, s_3\}$ is a strict linearly ordered and symmetric set of 3 "distance" dimensions, then another dimension, s_4, must have a different type (is a member of different set). In an orthogonal Cartesian grid around a local origin point, there are constant, unit-factoring ratios, eigenvalues, between a unit of a 3-dimensional distance, r, and the units of other types of dimensions: $r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q = \cdots$.

The geometry proofs and ratios are used to derive the gravity (G), charge (k_e), and Planck (h) constants, and used to provide simpler, shorter, and more rigorous derivations of some well-known classical gravity and charge equations, special and general relativity equations, and quantum physics equations.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq23**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of size κ subintervals. The ruler is both an inner and outer measure of an interval.

DEFINITION 2.1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\, [a, b] \subset \mathbb{R}$, $s = b - a\ \wedge\ 0 < \kappa \leq 1\ \wedge\ (p = floor(s/\kappa)\ \vee\ p = ceiling(s/\kappa))$.

THEOREM 2.2. *Ruler convergence:* $M = \lim_{\kappa \to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$(2.1)\quad \forall\, \kappa > 0,\ p = floor(s/\kappa)\ \wedge\ 0 \leq |floor(s/\kappa) - s/\kappa| < 1\ \Rightarrow\ |p - s/\kappa| < 1.$$

Multiply both sides of inequality 2.1 by κ:

$$(2.2) \qquad \forall\, \kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|.$$

$$(2.3) \quad \forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta$$
$$\Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \delta = \epsilon \quad := \quad M = \lim_{\kappa \to 0} p\kappa = s. \quad \square$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, ..., \pi_{\lim_{\kappa \to 0}}$.

LEMMA 2.3. $\forall\, n \geq 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa$.

PROOF. The formal proof, "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$(2.4) \quad n \geq 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|.$$

$$(2.5) \quad \forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta$$
$$\Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa \to 0} \kappa^n = 0.$$

$$(2.6) \qquad \lim_{\kappa \to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa \to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa. \qquad \square$$

3. Volume

DEFINITION 3.1. A countable n-volume is the number of ordered combinations (n-tuples), v_c, of the members of n number of disjoint, countable domain sets, x_i:

$$(3.1) \quad \exists\, n \in \mathbb{N},\ v_c \in \{0, \mathbb{N}\},\ x_i \in \{x_1, \cdots, x_n\} : \ \bigcap_{i=1}^{n} x_i = \emptyset \ \wedge \ v_c = \prod_{i=1}^{n} |x_i|.$$

THEOREM 3.2. *Euclidean volume,*

$$(3.2) \quad \forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\ [v_a, v_b] \subset \mathbb{R},\ s_i = b_i - a_i,\ v = v_b - v_a :$$
$$v_c = \prod_{i=1}^{n} |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^{n} s_i.$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.

$$(3.3) \quad v_c = \prod_{i=1}^{n} |x_i| \Leftrightarrow v_c \kappa = (\prod_{i=1}^{n} |x_i|)\kappa \Leftrightarrow \lim_{\kappa \to 0} v_c \kappa = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|)\kappa.$$

Apply the ruler (2.1) and ruler convergence (2.2) to equation 3.3:

$$(3.4) \quad \exists\, v, \kappa \in \mathbb{R} : v_c = floor(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa \to 0} v_c \kappa = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|)\kappa.$$

Apply lemma 2.3 to equation 3.4:

$$(3.5) \quad v = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|)\kappa = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|)\kappa^n = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|\kappa).$$

Apply the ruler (2.1) and ruler convergence (2.2) to s_i:

$$(3.6) \quad \exists\, s_i, \kappa \in \mathbb{R} : floor(s_i/\kappa) = |x_i| \quad \Rightarrow \quad \lim_{\kappa \to 0} (|x_i|\kappa) = s_i.$$

$$(3.7) \quad v = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|\kappa) \ \wedge \ \lim_{\kappa \to 0} (|x_i|\kappa) = s_i \ \Rightarrow \ v = \prod_{i=1}^{n} s_i \qquad \square$$

4. Distance

DEFINITION 4.1. Countable distance, $d_c = f(v_c, n) = f(|x_1|, \cdots, |x_n|, n) = \prod_{i=1}^{n} |x_i|$ is bijective and isomorphic: $v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n$.

4.1. Minkowski distance (ρ-norm).

THEOREM 4.2. *Minkowski distance (ρ-norm):*

$$v_c = \sum_{j=1}^{m} v_{c_i} \quad \Rightarrow \quad \exists\, d, d_i \in \mathbb{R} : d^n = \sum_{i=1}^{m} d_i^n.$$

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

PROOF. Apply the countable distance definition (4.1) to the assumption:

$$(4.1) \quad v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n \quad \wedge \quad v_{c_i} = \prod_{j=1}^{n} |x_{i_j}| = \prod_{i=1}^{n} d_{c_i} = d_{c_i}^n$$

$$\wedge \quad v_c = \sum_{j=1}^{m} v_{c_i} \quad \Rightarrow \quad d_c^n = \sum_{j=1}^{m} d_{c_i}^n.$$

Multiply both sides of equation 4.1 by κ and take the limit:

$$(4.2) \quad d_c^n = \sum_{j=1}^{m} d_{c_i}^n \quad \Leftrightarrow \quad \lim_{\kappa \to 0} d_c^n \kappa = \lim_{\kappa \to 0} \sum_{j=1}^{m} d_{c_i}^n \kappa.$$

Apply lemma 2.3 to equation 4.1:

$$(4.3) \quad \lim_{\kappa \to 0} d_c^n \kappa = \lim_{\kappa \to 0} \sum_{j=1}^{m} d_{c_i}^n \kappa \quad \wedge \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa$$

$$\Leftrightarrow \lim_{\kappa \to 0} d_c^n \kappa^n = \lim_{\kappa \to 0} \sum_{j=1}^{m} d_{c_i}^n \kappa^n \Leftrightarrow \lim_{\kappa \to 0} (d_c \kappa)^n = \lim_{\kappa \to 0} \sum_{j=1}^{m} (d_{c_i} \kappa)^n.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to equation 4.3:

$$(4.4) \quad \exists\, d, d_i : d_c = floor(d/\kappa), \ d = \lim_{\kappa \to 0} d_c \kappa$$

$$\wedge \quad d_{c_i} = floor(d_i/\kappa), \ d_i = \lim_{\kappa \to 0} d_{c_i} \kappa \quad \Rightarrow$$

$$d^n = \lim_{\kappa \to 0} (d_c \kappa)^n = \lim_{\kappa \to 0} \sum_{j=1}^{m} (d_{c_i} \kappa)^n = \sum_{j=1}^{m} d_i^n. \quad \square$$

4.2. Distance inequality.
The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N}, \ v_a, v_b \geq 0 : (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$(4.5) \quad \forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq v_a + v_b +$$

$$\sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n.$$

Take the nth of both sides of the inequality 4.5:

$$(4.6) \quad \forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad \square$$

4.3. Distance sum inequality.
The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N}, \ a_i, b_i \geq 0 : \left(\sum_{i=1}^{m} (a_i^n + b_i^n) \right)^{1/n} \leq \left(\sum_{i=1}^{m} a_i^n \right)^{1/n} + \left(\sum_{i=1}^{m} b_i^n \right)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$(4.7) \quad \forall\, m, n \in \mathbb{N}, \ v_a, v_b \geq 0 : \quad v_a = \sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge$$

$$(v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad \left(\left(\sum_{i=1}^{m} a_i^n \right) + \left(\sum_{i=1}^{m} b_i^n \right) \right)^{1/n} =$$

$$\left(\sum_{i=1}^{m} (a_i^n + b_i^n) \right)^{1/n} \leq \left(\sum_{i=1}^{m} a_i^n \right)^{1/n} + \left(\sum_{i=1}^{m} b_i^n \right)^{1/n}. \quad \square$$

4.4. Metric Space. All Minkowski distances (ρ-norms) imply the metric space properties.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$$d(s_1, s_2) = (\textstyle\sum_{i=1}^2 s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k:$

$$(4.8) \qquad (u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}.$$

Apply the distance inequality (4.3) to the inequality 4.8:

$$(4.9) \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \;\wedge\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}$$

$$\wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p$$

$$\Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p}$$

$$\Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq$$

$$(u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \quad \square$$

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$(4.10) \quad \forall\, p : p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^2 s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p}$$

$$\Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \quad \square$$

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$(4.11) \qquad \forall\, [a_1, b_1], [a_2, b_2], \quad u = b_1 - a_1,\, v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\, v \geq 0.$$

$$(4.12) \qquad p \geq 1,\; u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \qquad \square$$

THEOREM 4.8. *Identity of Indiscernibles:* $d(u, u) = 0.$

PROOF. From the non-negativity property (4.7):

$$(4.13) \quad d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0$$

$$\Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0.$$

$$(4.14) \qquad\qquad d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v.$$

$$(4.15) \qquad\qquad d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \qquad \square$$

4.5. Set properties limiting a set to at most 3 members.

DEFINITION 4.9. Totally ordered set:

$$\forall\, i\, n \in \mathbb{N},\; i \in [1, n-1],\; \forall\, x_i \in \{x_1, \ldots, x_n\},$$

$$\text{successor } x_i = x_{i+1} \;\wedge\; \text{predecessor } x_{i+1} = x_i.$$

DEFINITION 4.10. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i,\, j,\, n \in \mathbb{N},\; \forall\, x_i,\, x_j \in \{x_1, \ldots, x_n\},\; \text{successor } x_i = x_j \;\Leftrightarrow\; \text{predecessor } x_j = x_i.$$

THEOREM 4.11. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n \; \wedge \; j = 1 \; \Rightarrow \; successor \; x_n = x_1 \; \wedge \; predecessor \; x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (4.9) assigns a unique label to each set member and assigns unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (4.10):

(4.16) $\qquad i = n \; \wedge \; j = 1 \; \wedge \; successor \; x_i = x_j \; \Rightarrow \; successor \; x_n = x_1.$

Applying the definition of the symmetry property (4.10) to conclusion 4.16:

(4.17) $successor \; x_i = x_j \; \Rightarrow \; predecessor \; x_j = x_i \; \Rightarrow \; predecessor \; x_1 = x_n.$ $\qquad \square$

THEOREM 4.12. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (4.11).

DEFINITION 4.13. (Cyclic) Successor of m is n:

(4.18) $Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize).$

DEFINITION 4.14. (Cyclic) Predecessor of m is n:

(4.19) $Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1).$

DEFINITION 4.15. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:
(4.20)
$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize).$

Every member is adjacent to every other member, where $setsize \in \{1, \, 2, \, 3\}$:

(4.21) $\qquad Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1).$

(4.22) $\qquad Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize).$

(4.23) $\qquad Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1).$

(4.24) $\qquad Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize).$

(4.25) $\qquad Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1).$

(4.26) $\qquad Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1).$

(4.27) $\qquad Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize).$

(4.28) $\qquad Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize).$

(4.29) $\qquad Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1).$

Member 2 is the only successor of member 1 for all $setsize \geq 3$, which implies member 3 is not ($\neg$) a successor of member 1 for all $setsize \geq 3$:

(4.30) $\quad \neg Successor(1, 3, setsize \geq 3)$
$$\leftarrow Successor(1, 2, setsize \geq 3) \leftarrow (n = m + 1 \leq setsize).$$

Member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not ($\neg$) a predecessor of member 1 for all $setsize > 3$:

(4.31) $\quad \neg Predecessor(1, 3, setsize \geq 3)$
$$\leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).$$

For all $setsize \geq 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not symmetric):

(4.32) $\quad \neg Adjacent(1, 3, setsize > 3)$
$$\leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square$$

5. Applications to physics

From the volume proof (3.2), two disjoint distance intervals, $[0, r_1]$ and $[0, r_2]$, define a 2-volume. From the Minkowski distance proof (4.2), $\exists\, r : r^2 = r_1^2 + r_2^2$. And from the 3D proof (4.12), for some non-distance type, $\tau : \tau \in \{t\ (time),\ m\ (mass),\ q\ (charge), \cdots \}$, there exist unit-factoring ratios, $\mu,\ \nu_1,\ \nu_2$:

(5.1) $\quad \forall\, r, r_1, r_2 : r^2 = r_1^2 + r_2^2 \quad \wedge \quad r = \mu\tau \quad \wedge \quad r_1 = \nu_1\tau \quad \wedge \quad r_2 = \nu_2\tau$
$$\Rightarrow \quad (\mu\tau)^2 = (\nu_1\tau)^2 + (\nu_2\tau)^2 \quad \Rightarrow \quad \mu \geq \nu_1 \quad \wedge \quad \mu \geq \nu_2.$$

μ is the maximum-possible ($\mu \geq \nu_1,\ \nu_2$), constant, unit-factoring ratio, where:

(5.2) $\quad \mu \in \{r_c/t_c,\ r_c/m_c,\ r_c/q_c, \cdots \} : r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q = \cdots.$

5.1. Derivation of the constant G, and the gravity laws of Gauss, Poisson, and Newton. From equation 5.2:

(5.3) $\quad r = (r_c/m_c)m \quad \wedge \quad r = (r_c/t_c) = ct \quad \Rightarrow \quad r/(ct)^2 = (r_c/m_c)m/r^2$
$$\Rightarrow \quad r/t^2 = ((r_c/m_c)c^2)m/r^2 = Gm/r^2,$$

where Newton's constant, $G = (r_c/m_c)c^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$.

ρ is the mass field density (Gauss's flux divergence) on a sphere having the surface area $4\pi r^2$, which yields the differential forms of Gauss's flux divergence, $\nabla \cdot \mathbf{g}$ and Poisson's curl per unit mass, $\nabla^2 \Phi(r, t)$:

(5.4) $\quad \mathbf{g} = \nabla\Phi(r, t) := r/t^2 = (-Gm/r^2)(4\pi/4\pi) \quad \wedge \quad \rho = m/4\pi r$
$$\Rightarrow \quad \nabla \cdot \mathbf{g} = \nabla^2 \Phi(\vec{r}, t) = -4\pi G\rho.$$

Newton's law follows from multiplying both sides equation 5.3 by m:

(5.5) $\quad mr/t^2 = Gm^2/r^2 \quad \wedge \quad \forall\, m_1, m_2, m \in \mathbb{R} : m_1 m_2 = m^2$
$$\Rightarrow \quad F := mr/t^2 = Gm^2/r^2 = Gm_1 m_2/r^2.$$

5.2. Derivation of the constant, k_e and Coulomb's charge force.

$$(5.6) \quad \forall\, q_1, q_2, q, r \in \mathbb{R} : \quad q_1 q_2 = q^2 \quad \wedge \quad r = (r_c/q_c)q$$
$$\Rightarrow \quad q_1 q_2 = ((q_c/r_c)r)^2 \quad \Rightarrow \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1.$$

$$(5.7) \quad r = (r_c/t_c)t = ct \Rightarrow mr = (m_c/r_c)(ct)^2 \Rightarrow ((r_c/m_c)/c^2)mr/t^2 = 1.$$

$$(5.8) \quad ((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1$$
$$\Rightarrow \quad F := mr/t^2 = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2.$$

$$(5.9) \quad r_c/t_c = c \quad \wedge \quad F = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2$$
$$\Rightarrow \quad F = (m_c(r_c/t_c^2))(r_c/q_c)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2,$$

where Coulomb's constant, $k_e = (m_c(r_c/t_c^2))(r_c/q_c)^2$, conforms to the SI units: $N \cdot m^2 \cdot C^{-2}$.

5.3. Vacuum permitivity, ε_0, and Gauss's law for electric fields. From equation 5.2:

$$(5.10) \quad r = (r_c/q_c)q \quad \wedge \quad r = (r_c/t_c) = ct \quad \Rightarrow \quad r/(ct)^2 = (r_c/q_c)q/r^2$$
$$\Rightarrow \quad r/t^2 = ((r_c/q_c)/c^2)q/r^2,$$

ρ is the charge (electric) field density (Gauss's flux divergence) on a sphere having the surface area $4\pi r^2$, which yields the differential form of Gauss's flux divergence, $\nabla \cdot \mathbf{E}$:

$$(5.11) \quad r/t^2 = (((r_c/q_c)/c^2)q/r^2)(4\pi/4\pi) \quad \wedge \quad \rho = q/4\pi r$$
$$\Rightarrow \quad \nabla \cdot \mathbf{q} := r/t^2 = 4\pi((r_c/q_c)c^2)\rho.$$

Multiply both sides of equation 5.11 by $(m_c/r_c)(r_c/q_c)$ and apply the derivation of k_e in equation 5.8 and the of vacuum permitivity, $\varepsilon_0 = 1/4\pi k_e$:

$$(5.12) \quad \nabla \cdot \mathbf{E} = ((m_c/r_c)(r_c/q_c))\nabla \cdot \mathbf{q} = ((m_c/r_c)(r_c/q_c))4\pi((r_c/q_c)/c^2)\rho$$
$$= 4\pi((m_c/r_c)c^2)(r_c/q_c)^2\rho = 4\pi k_e \rho = \rho/\varepsilon_0,$$

which is the Gauss's law for electric fields that is used in Maxwell's equations.

5.4. Space-time-mass-charge equations. Form equation 5.1:

$$(5.13) \quad \forall\, r, r', r_v, \mu, \nu : \quad r^2 = r'^2 + r_v^2 \quad \wedge \quad r = \mu\tau \quad \wedge \quad r_v = \nu\tau$$
$$\Rightarrow \quad r' = \sqrt{(\mu\tau)^2 - (\nu\tau)^2} = \mu\tau\sqrt{1 - (\nu/\mu)^2}.$$

Rest (local) distance, r', contracts relative to stationary distance, r, as $\nu \to \mu$:

$$(5.14) \quad r' = \mu\tau\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad \mu\tau = r \quad \Rightarrow \quad r' = r\sqrt{1 - (\nu/\mu)^2}.$$

Interval length, τ, dilates relative to the rest interval length, τ', as $\nu \to \mu$:

$$(5.15) \quad \mu\tau = r'/\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad r' = \mu\tau' \quad \Rightarrow \quad \tau = \tau'/\sqrt{1 - (\nu/\mu)^2}.$$

Where τ is time, the space-like flat Minkowski spacetime event interval is:

$$(5.16) \quad \mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_v^2 \quad \wedge \quad \mathrm{d}r_v^2 = \mathrm{d}r_1^2 + \mathrm{d}r_2^2 + \mathrm{d}r_3^2 \quad \wedge \quad \mathrm{d}(\mu\tau) = \mathrm{d}r$$
$$\Rightarrow \quad \mathrm{d}r'^2 = \mathrm{d}(\mu\tau)^2 - \mathrm{d}r_1^2 - \mathrm{d}r_2^2 - \mathrm{d}r_3^2.$$

5.5. Simplifying Einstein's general relativity (field) equation. In this subsection, the previous spacetime equations and ratios are used to simplify finding solutions to Einstein's field (general relativity) equation (EFE) is outlined. And in the next subsection an example is provided for deriving the Schwarzchild metric.

Einstein was motivated by Gaussian curvature, hence the terms, $\mathbf{G}_{\mu,\nu}$, and $g_{\mu,\nu}$, in the EFE, $\mathbf{G}_{\mu,\nu} + \Lambda g_{\mu,\nu} = k\mathbf{T}_{\mu,\nu}$, [**Ein15**] [**Wey52**]. Also, the $4\pi G$ in Einstein's constant, $k = (2/c^4)4\pi G$, and energy density in $\mathbf{T}_{\mu,\nu}$ were motivated by Gauss's gravity law. But, Gaussian curvature cannot be extended beyond 3 dimensions. Therefore, the much more complex 4-dimensional (4D) Ricci curvature, $\mathbf{R}$, and scalar curvature, R, are used, where $\mathbf{G}_{\mu,\nu} = \mathbf{R}_{\mu,\nu} - g_{\mu,\nu}R/2$.

Step 1) Express the EFE as 2D tensors: The 4D flat spacetime interval equation (5.16) is an instance of the 2D equation, $\mathrm{d}r'^2 = \mathrm{d}(ct)^2 - \mathrm{d}r_v^2$, where $\mathrm{d}r_v^2 = \mathrm{d}r_1^2 + \mathrm{d}r_2^2 + \mathrm{d}r_3^2$. Generalizing to curved spacetime: $\mathrm{d}r'^2 = g_{0,0}\mathrm{d}(ct)^2 - g_{1_v,1_v}\mathrm{d}r_v^2$, where $g_{1_v,1_v}\mathrm{d}r_v^2 = g_{1,1}\mathrm{d}r_1^2 + g_{2,2}\mathrm{d}r_2^2 + g_{3,3}\mathrm{d}r_3^2$, and the $g_{\mu,\nu}$ terms are functions returning scalar values.

The 2D metric tensor allows using the 2D Gaussian curvature, $K = det(\mathbf{H})$, in the Einstein tensor, $\mathbf{G}_{\mu,\nu}$, such that $\mathbf{G}_{\mu,\nu} = \mathbf{H} - g_{\mu,\nu}K$, where $\mathbf{H}$ is the 2×2 Hessian matrix:

$$(5.17) \qquad \mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 g_{i,i}}{\partial t^2} & \dfrac{\partial^2 g_{i,i}}{\partial t\, \partial r} \\ \dfrac{\partial^2 g_{i,i}}{\partial t\, \partial r} & \dfrac{\partial^2 g_{i,i}}{\partial r^2} \end{bmatrix}.$$

$$(5.18) \qquad K = det(\mathbf{H}) = \frac{\partial^2 g_{i,i}}{\partial t^2}\frac{\partial^2 g_{i,i}}{\partial r^2} - \left(\frac{\partial^2 g_{i,i}}{\partial t\, \partial r}\right)^2.$$

The 2D Gaussian curvature, K, is much simpler to calculate than the 4D Ricci curvature, $\mathbf{R}_{\mu,\nu}$, and scalar, R. And the 2D tensors reduce the number of independent equations to solve.

Step 2) Use the ratios to define relations: All functions derived from the ratios, $r = (r_c/t_c)t = ct$ and $r = (r_c/m_c)m$ (and $r = (r_c/q_c)q$ for cases using charge), where the units balance on both sides of the 2D EFE, are valid solutions. For example, $r^3/r^3 = (r_c/m_c)mc^2t^2/rc^2t^2 = Gm/rc^2 = f(r,t,m)$.

Step 3) Translate the 2D solutions to the 4D EFE solutions by expanding the spatial component, $g_{1_v,1_v}\mathrm{d}r_v^2$ to $g_{1,1}\mathrm{d}r_1^2 + g_{2,2}\mathrm{d}r_2^2 + g_{3,3}\mathrm{d}r_3^2$. One simple method is to use spherical coordinates, where: $g_{1_v,1_v}\mathrm{d}r_1^2 = g_{1,1}\mathrm{d}r_v^2$, and $g_{2,2}\mathrm{d}r_2^2 + g_{3,3}\mathrm{d}r_3^2 = r^2(\mathrm{d}\theta^2 + sin^2\theta\mathrm{d}\phi^2)$.

The $\mathbf{T}_{0,0}$ component in the 4D stress-energy tensor is set to the $\mathbf{T}_{0,0}$ component in the 2D energy-stress tensor and $\mathbf{T}_{1,1}$ component in the 4D stress-energy tensor is set to the 2D $\mathbf{T}_{1,1}$ component. And so on.

5.6. Derivation of Schwarzchild's gravitational time dilation and black hole metric. [**Che10**] From equations 5.14 and 5.2:

$$(5.19) \quad t' = t\sqrt{1 - (v^2/c^2)(r/r)} \quad \wedge \quad r = (r_c/m_c)m$$

$$\Rightarrow \quad t' = t\sqrt{1 - ((r_c/m_c)m)v^2/rc^2}.$$

(5.20) $t' = t\sqrt{1 - (r_c/m_c)mv^2/rc^2}$ $\wedge$ $KE = mv^2/2 = mv_{escape}^2$

$$\Rightarrow \quad t' = t\sqrt{1 - 2(r_c/m_c)mv_{escape}^2/rc^2}.$$

Combining equation 5.20 with the derivation of G (5.5) yields gravitational time dilation:

(5.21) $(r_c/m_c)c^2 = G$ $\wedge$ $t' = \lim_{v_{escape} \to c} t\sqrt{1 - 2(r_c/m_c)mv_{escape}^2/rc^2}$

$$= t\sqrt{1 - 2(r_c/m_c)mc^2/rc^2} \quad \Rightarrow \quad t' = t\sqrt{1 - 2Gm/rc^2}.$$

Schwarzchild defined the black hole event horizon radius, $r_s := 2Gm/c^2$.

(5.22) $r_s = 2Gm/c^2$ $\wedge$ $t' = t\sqrt{1 - 2Gm/rc^2}$ $\Rightarrow$ $t' = t\sqrt{1 - r_s/r}.$

From equations 5.14 and 5.22:

(5.23) $r' = r\sqrt{1 - (v/c)^2} = r\sqrt{1 - 2Gm/rc^2} = r\sqrt{1 - r_s/r}.$

From equation 5.13:

(5.24) $r_v^2 = r^2 - r'^2$ $\Rightarrow$ $r_v \propto r/r' = r/r\sqrt{1 - r_s/r} = 1/\sqrt{1 - r_s/r}$

(5.25) $ds^2 = (\sqrt{1 - r_s/r}d(ct))^2 - (dr_v/\sqrt{1 - r_s/r})^2$ $\wedge$ $c = 1$ $\wedge$ $dr_v \propto dr$

$$\Rightarrow \quad ds^2 = (1 - r_s/r)dt^2 - (1 - r_s/r)^{-1}dr^2.$$

Expanding to 4D spherical coordinates yields Schwarzchild's black hole metric:

(5.26) $ds^2 = (1 - r_s/r)dt^2 - (1 - r_s/r)^{-1}dr^2 = f(r, t)$

$\rightarrow$ $ds^2 = (1 - r_s/r)dt^2 - (1 - r_s/r)^{-1}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) = f(r, t, \theta, \phi).$

5.7. Derivation of Einstein's gravitational lens. [Che10] Using the same steps to derive Schwarzchild's gravitational time dilation equation (5.21):

(5.27) $r' = r\sqrt{1 - (v^2/c^2)(r/r)}$ $\Rightarrow$ $r' = r\sqrt{1 - 2Gm/rc^2}.$

The incremental deflection of light (work), $ds = f(r - r') = 2Gm/rc^2$. Therefore, an increment of deflection, ds, must correspond to an incremental distance, $dr : r = dr$.

(5.28) $ds = 2Gm/rc^2$ $\wedge$ $dr = r$ $\Rightarrow$ $ds/dr = 2Gm/r^2c^2.$

There are two deflections: 1) deflection as the light approaches a mass, and 2) deflection as light passes the same mass. Therefore, the total deflection is doubled:

(5.29) $2ds/dr = 2(2Gm/r^2c^2)$ $\Rightarrow$ $s = 2\int ds = 2\int 2Gm/r^2c^2 dr = 4Gm/rc^2.$

5.8. 3 fundamental direct proportion ratios. c_t, c_m, and c_q:

(5.30) $c_t = r_c/t_c \approx 2.99792458 \cdot 10^8 m\ s^{-1}.$

(5.31) $G = (r_c/m_c)c_t^2 = c_m c_t^2$ $\Rightarrow$ $c_m = r_c/m_c \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}.$

(5.32) $k_e = (c_t^2/c_m)(r_c/q_c)^2$ $\Rightarrow$ $c_q = r_c/q_c \approx 8.6175172023 \cdot 10^{-18} m\ C^{-1}.$

5.9. 3 fundamental inverse proportion ratios. c_t, c_m, and c_q (5.8) $\Leftrightarrow$ k_t, k_m, and k_q:

$$(5.33) \quad r/t = r_c/t_c, \quad r/m = r_c/m_c \quad \Leftrightarrow \quad (r/t)/(r/m) = (r_c/t_c)/(r_c/m_c) \quad \Leftrightarrow$$
$$(mr)/(tr) = (m_c r_c)/(t_c r_c) \quad \Leftrightarrow \quad mr = m_c r_c = k_m, \quad tr = t_c r_c = k_t.$$

$$(5.34) \quad r/t = r_c/t_c, \quad r/q = r_c/q_c \quad \Leftrightarrow \quad (r/t)/(r/q) = (r_c/t_c)/(r_c/q_c) \quad \Leftrightarrow$$
$$(qr)/(tr) = (q_c r_c)/(t_c r_c) \quad \Leftrightarrow \quad qr = q_c r_c = k_q, \quad tr = t_c r_c = k_t.$$

$$(5.35) \quad E = hf(2\pi/2\pi) = (h/2\pi)/(2\pi f) = \hbar\omega \quad \Rightarrow \quad \hbar \approx 1.05457181 \cdot 10^{-34}.$$

$$(5.36) \quad k_m = m_c r_c = \hbar/c \approx 3.51767294 \cdot 10^{-42} \; kg \; m.$$

$$(5.37) \quad k_t = t_c r_c = k_m/(c_t/c_m) \approx 8.71362916 \cdot 10^{-79} \; s \; m.$$

$$(5.38) \quad k_q = q_c r_c = (c_t/c_q)k_t \approx 3.03136071 \cdot 10^{-523} \; C \; m.$$

5.10. 4 quantum units. The Planck units for: distance (r_c), time (t_c), mass (m_c), plus an additional quantum unit for charge (q_c):

$$(5.39) \quad r_c = \sqrt{r_c^2} = \sqrt{c_t k_t} = \sqrt{c_m k_m} = \sqrt{c_q k_q} \approx 1.61625502 \cdot 10^{-35} \; m.$$

$$(5.40) \quad t_c = r_c/c_t \approx 5.39124645 \cdot 10^{-43} \; s.$$

$$(5.41) \quad m_c = r_c/c_m \approx 2.1764343 \cdot 10^{-8} \; kg.$$

$$(5.42) \quad q_c = r_c/c_q \approx 1.87554604 \cdot 10^{-18} \; C.$$

5.11. Planck relation and constant, h. [**Jai11**] Applying both the direct proportion (5.30), $r/t = r_c/t_c = c$, and inverse proportion (5.33), $mr = m_c r_c = k_m$, ratios:

$$(5.43) \quad m(ct)^2 = mr^2 \;\wedge\; m = m_c r_c/r = k_m/r \quad \Rightarrow \quad m(ct)^2 = (k_m/r)r^2 = k_m r.$$

$$(5.44) \quad m(ct)^2 = k_m r \quad \wedge \quad r/t = r_c/t_c = c$$
$$\Rightarrow \quad E := mc^2 = k_m r/t^2 = (k_m(r/t))\,(1/t) = (k_m c)(1/t) = hf,$$

where the Planck constant, $h = k_m c$, and the frequency, $f = 1/t$.

5.12. Compton wavelength. [**Jai11**] From equations 5.33 and 5.44:

$$(5.45) \quad mr = k_m \quad \wedge \quad h = k_m c \quad \Rightarrow \quad r = k_m/m = (k_m/m)(c/c) = h/mc.$$

5.13. de Broglie wavelength. [**Jai11**] From equations 5.14 and 5.45:

$$(5.46) \quad r_v = ((r_v/t)/(r/t))r = (v/c)r \;\wedge\; mr = k_m \quad \Rightarrow \quad r_v = k_m(c/v)/m = h/mv.$$

Note that r_v is the wavelength in the frame of reference of the moving mass (for example, wavelength as measured by an observer riding in a moving vehicle).

5.14. Schrödenger's wave equation. There have been no rigorous derivations of Schrödenger's wave equation because they all are riddled with "reasonable" assumptions. For example, Schrödenger and all other derivations assume a Gaussian (normal) distribution of kinetic energies. Further, even Gauss had to make some unproven assumptions in deriving the normal distribution: 1) There exists a true value. 2) The mean value of measurements approaches the true value. 3) The measures closest to the mean value have the smallest error. It was later that each measured value was defined a true value with a probability of occurrence based on distance from the mean value.

Here, the derivation of Schrödenger's equation does not use the notion of a mean value and normal distribution. Using the ratio-derived Planck relation 5.44:

$$(5.47) \qquad E = mc^2 = hf = h(1/t) \quad \Rightarrow \quad \exists\, V(r,t): h/t = mc^2/2 + V(r,t),$$

where $V(r,t)$, is the potential energy.

$$(5.48)\ \ h/t = mc^2/2 + V(r,t) \wedge mc^2/2 = (mc)^2/2m \Rightarrow h/t = (mc)^2/2m + V(r,t).$$

Let $\nu(r,t)$ be the difference between a measured wavelength (or position), r, and the probable next measured value, $f(r)$, in the complex plane: $r - f(r) + ig(t)$. And as time, t, increases, the probability of any given value $f(r)$ decreases inversely:

$$(5.49)\ \ \nu(r,t) = r - (f(r) + ig(t)) \quad \wedge \quad d\nu(r,t)/dt = 1/t \quad \Rightarrow \quad dg(t)/dt = 1/t.$$

And, from the unit-factoring ratio 5.2, $r = ct$:

$$(5.50) \qquad d\nu(r,t)/dt = 1/t \quad \wedge \quad r = ct \quad \Rightarrow \quad df(r)/dr = 1/r.$$

$$(5.51)\ \ \ d\nu(r,t)/dr = 1 - 1/r \wedge d\nu(r,t)/dt = 1/t \Rightarrow \nu(r,t) = r - \ln|r| - i\ln|t|.$$

As r and t decrease (frequency/energy increases), the range of possible probable values of $f(r)$ and $g(t)$, $\Psi(r,t)$, decreases inversely:

$$(5.52) \qquad d\nu(r,t)/d\Psi(r,t) = 1/\Psi(r,t). \quad \Rightarrow \quad \nu(r,t) = \ln|\Psi(r,t)|.$$

Combining equations 5.50 and 5.52:

$$(5.53)\ \ \ \nu(r,t) = r - \ln|r| - i\ln|t| = z \quad \wedge \quad \nu(r,t) = \ln|\Psi(r,t)|$$
$$\Rightarrow \quad \ln|\Psi(r,t)| = r - \ln|r| - i\ln|t| \quad \Rightarrow \quad \Psi(r,t) = e^{r-\ln|r|-i\ln|t|} = e^z.$$

$$(5.54)\ \ \ d\Psi(r,t)/dz = de^z/dz = ze^z$$
$$\Rightarrow \quad d^2\Psi(r,t)/dr^2 = -(1/r^2)\Psi(r,t) \quad \wedge \quad d\Psi(r,t)/dt = i(1/t)\Psi(r,t).$$

Multiplying both sides of equation 5.48 by $\Psi(r,t)$ and applying equations 5.54:

$$(5.55)\ \ \ (h^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t) = (h/t)\Psi(r,t) \quad \wedge$$
$$d^2\Psi(r,t)/dr^2 = -(1/r^2)\Psi(r,t) \quad \wedge \quad d\Psi(r,t)/dt = i(1/t)\Psi(r,t)$$
$$\Rightarrow \quad -(h^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t) = ih\partial\Psi(r,t)/\partial t,$$

which is Schrödenger's spacetime wave equation using the **full** Planck constant. Using the **reduced** Planck constant, $\hbar = h/2\pi$, requires adding a compensating angular momentum (spin) factor, 2π, to $\Psi(r,t): \hat{\Psi}(r,t) = e^{2\pi(r-\ln|r|-i\ln|t|)}$, yielding:

$$(5.56) \qquad -(\hbar^2/2m)\partial^2\hat{\Psi}(r,t)/\partial r^2 + V(r,t)\hat{\Psi}(r,t) = i\hbar\partial\hat{\Psi}(r,t)/\partial t.$$

5.15. Dirac's wave equation. Using the ratio-derived Planck relation 5.44:

$$(5.57) \quad E = mc^2 = h/t \quad \Rightarrow \quad \exists \, V(r,t): \, mc^2/2 + V(r,t)/2 = h/t$$
$$\Rightarrow \quad 2h/t - 2V(r,t) = mc^2.$$

$$(5.58) \quad \exists \, V(r,t): \, V(r,t) = ih/t \quad \wedge \quad 2h/t - 2V(r,t) = mc^2 \quad \wedge \quad r = ct$$
$$\Rightarrow \quad 2h/t - i2hc/r = mc^2.$$

Translating equation 5.58 to moving coordinates via the Lorentz factor, $\gamma_0 = 1/\sqrt{1-(v/c)^2}$:

$$(5.59) \qquad\qquad 2\gamma_0 2h/t - i2\gamma_0 2hc/r = mc^2.$$

Applying the charge ratio, c_q, (5.32) to equation 5.59:

$$(5.60) \quad q(r_c/q_c)c/r = qc_q c/r = 1 \quad \wedge \quad 2\gamma_0 2h/t - i2\gamma_0 2hc/r = mc^2$$
$$\Rightarrow \quad \gamma_0 2h(-qc_q)/t^2 - \gamma_0 i2h(-qc_q)c/rt = mc^2.$$

where a negative sign is added to q to indicate an attractive force between an electron and a nucleus.

$$(5.61) \quad \gamma_0 2h(-qc_q)/t^2 - \gamma_0 i2h(-qc_q)c/rt = mc^2 \quad \wedge \quad r = ct$$
$$\Rightarrow \quad \gamma_0 2h(-qc_q)/t^2 - \gamma_0 i2h(-qc_q)c^2/r^2 = mc^2.$$

The wave amplitude, Ψ, is proportionate to the charge, $-q$, times the wave amplitude equation at the space-time point, (ρ, τ):

$$(5.62) \qquad \Psi = cos((-qc_q)(k\rho + \omega\tau)) + i\,sin((-qc_q)(k\rho + \omega\tau)).$$

And applying Gauss's theorem to the quantum wave amplitude:

$$(5.63) \quad \Psi = cos((-qc_q)(k\rho + \omega\tau)) + i\,sin((-qc_q)(k\rho + \omega\tau)) \quad \wedge$$
$$cos(\theta) + i\,sin(\theta) = e^{i\theta} \quad \Rightarrow \quad \Psi = e^{i\,(-qc_q)(k\rho + \omega\tau)}.$$

Normalizing ρ and τ, where the wave number, $k = c/r(= 1/t)$, and the angular frequency, $\omega = 1/t$:

$$(5.64) \quad \Psi = e^{i(-qc_q)((c/r)\rho + (1/t)\tau)}, \quad \rho = 1, \; \tau = 1 \quad \Rightarrow \quad \Psi(r,t) = e^{i(-qc_q)((c/r)+(1/t))}.$$

$$(5.65) \quad \Psi(r,t) = e^{i(-qc_q)((c/r)+(1/t))} \wedge A_0 = c_q((c/r)+(1/t)) \Rightarrow \Psi(r,t) = e^{i(-qA_0)}.$$

Multiplying both sides of equation 5.61 by $\Psi(r,t)$:

$$(5.66) \quad \gamma_0 2h(-qc_q)/t^2 - \gamma_0 i2h(-qc_q)c^2/r^2 = mc^2$$
$$\Rightarrow \quad \gamma_0 2h(-qc_q)/t^2 \Psi(r,t) - \gamma_0 i2h(-qc_q)c^2/r^2 \Psi(r,t) = mc^2 \Psi(r,t).$$

$$(5.67) \quad \partial\Psi(r,t)/\partial r \;=\; \partial e^{i(-qA_0)}/\partial r \;=\; (\partial i(-qA_0)/\partial r)e^{i(-qA_0)}$$
$$=\; (\partial i(-qA_0)/\partial r)\Psi(r,t) \;=\; (-qc_q)c/r^2 \Psi(r,t).$$

$$(5.68) \quad \partial\Psi(r,t)/\partial t \;=\; \partial e^{i(-qA_0)}/\partial t \;=\; (\partial i(-qA_0)/\partial t)e^{i(-qA_0)}$$
$$=\; (\partial i(-qA_0)/\partial t)\Psi(r,t) \;=\; (-qc_q)/t^2\Psi(r,t).$$

Substituting equations 5.67, and 5.68 into 5.66:

$$(5.69) \quad (\partial i(-qA_0)/\partial t)\Psi(r,t) = (-qc_q)/t^2\Psi(r,t) \quad \wedge$$
$$(\partial i(-qA_0)/\partial r)\Psi(r,t) = (-qc_q)c/r^2\Psi(r,t) \quad \wedge$$
$$\gamma_0 2h(-qc_q)1/t^2\Psi(r,t) - \gamma_0 i2h(-qc_q)c^2/r^2\Psi(r,t) = mc^2\Psi(r,t)$$
$$\Rightarrow \quad \gamma_0 2h(\partial i(-qA_0)/\partial t)\Psi(r,t) - \gamma_0 i2h(\partial i(-qA_0)/\partial r)c\Psi(r,t) = mc^2\Psi(r,t).$$

The distance, r, is the magnitude of a 3-dimensional vector, $\overrightarrow{\mathbf{r}}$, A_0 is magnitude of a 3-dimensional vector amplitude, $\overrightarrow{\mathbf{A}}$, and γ_0 is magnitude of a 3-dimensional Lorentz factor, $\overrightarrow{\gamma}$. Applying the vectors to equation 5.69:

$$(5.70) \quad \gamma_0 2hc(\partial i(-qA_0)/\partial t)\Psi(r,t) - \gamma_0 i2h(\partial i(-qA_0)/\partial r)c\Psi(r,t) = mc^2\Psi(r,t) \wedge$$
$$\overrightarrow{\mathbf{r}} \Rightarrow r \quad \wedge \quad \overrightarrow{\mathbf{A}} \Rightarrow A_0 \quad \wedge \quad \overrightarrow{\gamma} \Rightarrow \gamma_0 \quad \wedge \quad \Leftrightarrow \quad \exists\, \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{A}}, \overrightarrow{\gamma} :$$
$$\gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \overrightarrow{\gamma} \cdot i2h(\partial(-q\overrightarrow{\mathbf{A}})/\partial r)c\Psi(r,t) = mc^2\Psi(r,t).$$

Adding a $\frac{1}{2}$ angular rotation ($\frac{1}{2}$ spin) of π to equation 5.65 allows substituting the reduced Planck constant, $\hbar = h/2\pi$, into equation 5.70, which yields Dirac's wave equation:

$$(5.71) \quad \gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \overrightarrow{\gamma} \cdot i2h(\partial(-q\overrightarrow{\mathbf{A}})/\partial r)c\Psi(r,t) = mc^2\Psi(r,t) \quad \wedge$$
$$A_0 = \pi c_q((c/r) + (1/t)) \quad \wedge \quad \Psi(r,t) = e^{i(-qA_0)}$$
$$\Rightarrow \quad \gamma_0 \hbar(\partial(-qA_0)/\partial t)\Psi(r,t) - \overrightarrow{\gamma} \cdot i\hbar(\partial(-q\overrightarrow{\mathbf{A}})/\partial r)c\Psi(r,t) = mc^2\Psi(r,t).$$

5.16. Total mass. The total mass of a particle is $m = \sqrt{m_0^2 + m_{ke}^2}$, where m_0 is the rest mass and m_{ke} is the kinetic energy-equivalent mass. Applying both the direct (5.30) and inverse proportion ratios (5.33):

$$(5.72) \quad m_0 = (m_c/r_c)r \quad \wedge \quad m_{ke} = m_cr_c/r \quad \wedge \quad m = \sqrt{m_0^2 + m_{ke}^2}$$
$$\Rightarrow \quad m = \sqrt{((m_c/r_c)r)^2 + ((m_cr_c)/r)^2}.$$

The quantum effect, $((m_cr_c)/r)^2$, is easier to express and understand by extending Newtonian gravity than by extending general relativity.

5.17. Quantum-special relativity extensions to Newton's gravity force.

$$(5.73) \quad \exists\, m : m_1m_2 = m^2 = ((m_c/r_c)r)^2 + ((m_cr_c)/r)^2$$
$$\Rightarrow \quad m_1m_2/(((m_c/r_c)r)^2 + ((m_cr_c)/r)^2) = 1.$$

Applying the ratio-derived spacetime equation 5.14 to equation 5.4:

$$(5.74) \quad r' = r\sqrt{1 - (v/c)^2} \quad \wedge \quad ((r_c/m_c)/c^2)mr/t^2 = 1$$
$$\Rightarrow \quad ((r_c/m_c)/c^2\sqrt{1 - (v/c)^2})mr'/t^2 = 1,$$

where r'/t^2 is the acceleration observed from a stationary frame of reference.

Combining equations 5.73 and 5.74:

$$(5.75) \quad F \;=\; (c^2\sqrt{1-(v/c)^2}/(r_c/m_c))m_1m_2/(((m_c/r_c)r)^2 \;+\; ((m_cr_c)/r)^2).$$

Using the ratio-derived gravitational time dilation (5.24):

$$(5.76) \quad F \;=\; (c^2\sqrt{1-2Gm/rc^2}/(r_c/m_c))m_1m_2/(((m_c/r_c)r)^2 \;+\; ((m_cr_c)/r)^2).$$

5.18. Quantum-special relativity extensions to Coulomb's charge force.

$$(5.77) \quad F = (c^2\sqrt{1-(v/c)^2}/(r_c/m_c))(r_c/q_c)^2 q_1q_2/(((q_c/r_c)r)^2 + ((q_cr_c)/r)^2).$$

6. Insights and implications

(1) Combinatorics, the ordered combinations of countable, disjoint sets (n-tuples), generates both Euclidean volume (3.2) and the Minkowski distances (4.2).

(2) Combinatorics, all n-at-time permutations of an ordered and symmetric set of distance dimensions, limits the set to 3 dimensions (4.12).

(3) Deriving Euclidean volume (3.2) and the Minkowski distances (4.2) from the same abstract, countable set of n-tuples (3.1) provides a single, unifying set and limit-based foundation under Euclidean geometry without relying on the geometric primitives and relations in Euclidean geometry [**Joy98**], axiomatic geometry [**Lee10**], and vector analysis [**Wey52**].

(4) The definition of a metric space [**Rud76**] ignores the intimate relation between distance and volume. A simpler and more sufficient definition that has the metric space properties (4.4) is: a distance measure is an inverse (bijective), isomorphic function of volume (4.1).

(5) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [**Joy98**]. And analytic proofs that the straight line length is the smallest distance equate the straight line length to Euclidean distance.

Without using the notion of a straight line: Euclidean volume was derived from a set of n-tuples (3.2). And all distance measures (bijective, isomorphic functions of n-volumes) derived from Euclidean 2-volumes (areas) are Minkowski distances (4.2), where $n \in \{1,2\}$: $n = 1$ is the Manhattan (largest) distance case, $d = \sum_{i=1}^{m} s_i$. $n = 2$ is the Euclidean (smallest) distance case, $d = (\sum_{i=1}^{m} s_i^2)^{1/2}$. For the case, $n \in \mathbb{R}$, $1 \leq n \leq 2$: d decreases monotonically as n goes from 1 to 2.

(6) The left side of the distance sum inequality (4.4),

$$(6.1) \quad (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n},$$

differs from the left side of Minkowski's sum inequality [**Min53**]:

$$(6.2) \quad (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

The two inequalities are only the same where $n = 1$. The distance sum inequality is a more fundamental inequality because the proof does not require the convexity and Hölder's inequality assumptions required to prove the Minkowski sum inequality. And the distance sum inequality term, $(a_i^n + b_i^n)^{1/n}$, is the Minkowski distance spanning the n-volume, $a_i^n + b_i^n$, which makes it directly related to geometry (for example, the metric space

triangle inequality was derived from the $m = 1$ case (4.5)). But the Minkowski sum inequality term, $\forall\, n : (n \neq 1).\ ((a_i^n + b_i^n)^{\mathbf{n}})^{1/n}$, is **not** a distance spanning the same n-volume, $a_i^n + b_i^n$.

(7) The derivations of the spacetime equations, in this article (5.4), differ from other derivations:

    (a) The derivations, here, do not rely on the Lorentz transformations or Einsteins' postulates [**Ein15**]. The derivations do not even require the notion of light.

    (b) The derivations, here, rely only on the Euclidean volume proof (3.2), the Minkowski distances proof (4.1), and the 3D proof (4.12), which provides the insight that the properties of physical space creates a maximum speed, the spacetime equations, and 3 dimensions of distance. For example, from the direct proportion equations 5.1 and 5.2, $\mu = r_c/t_c$ is always the maximum ratio (the speed of light).

    (c) The same derivations are also valid for spacemass and spacecharge.

    (d) In special and general relativity, covariance, states that the equations expressing the laws of physics are invariant in every frame of reference. Here, a more fundamental covariance assertion is made: The ratio of any 3-dimensional distance, r, to another type remains constant in all frames of reference.

    For example, the special relativity time dilation equation 5.15 was derived from constant distance-to-time ratios and combined with a constant distance-to-mass ratio (5.8) yields Schwarzschild's gravitational time dilation equation (5.24) [**Che10**].

(8) Combinations of the direct proportion ratios, $r_c/t_c = c_t$, $r_c/m_c = c_m$, $(r_c/q_c) = c_q \Leftrightarrow$ the inverse proportion ratios, $t_c r_c = k_t$, $m_c r_c = k_m$, and $q_c r_c = k_q$ (5.9) generate most of the well known physical constants.

    (a) The inverse square law for gravity (5.3) and charge (5.6) is a result of the direct proportion ratios.

    (b) The combination of direct and inverse proportion ratios create the particle-wave equations: Planck relation (5.11), Compton wavelength (5.45), and de Broglie wavelength (5.46).

    (c) The gravity, G (5.5), charge k_e (5.9), vacuum permitivity, ε_0, and Planck h (5.44) constants were all derived from the constant proportion ratios. Therefore, G, k_e, ε_0, and h are **not** "fundamental" constants.

    (d) G, k_e, and h all depend on the speed of light ratio, c_t: $G = c_m c_t^2$, $k_e = (c_q^2/c_m)c_t^2$, and $h = k_m c_t$.

    (e) $k_e = (c_q^2/c_m)c_t^2 = ((m_c/r_c)(r_c/t_c)^2)c_q^2 = (m_c(r_c/t_c^2))c_q^2$, where the term, r_c/t_c^2, suggests a maximum acceleration constant, which agrees with the MOND theories of gravity.

(9) The quantum-special relativity extensions to Newton's gravity force (5.75) and Coulomb's charge force (5.77) make quantifiable predictions. Specifically: $\lim_{r \to 0} F = 0$, and **both** the gravity and charge forces peak at the Planck length: $r_c = \sqrt{r_c^2} = \sqrt{c_t k_t} = \sqrt{c_m k_m} = \sqrt{c_q k_q} \approx 1.61625502 \cdot 10^{-35}\ m$ (5.39).

    (a) Gravitational time dilation peaks at r_c.

    (b) Black holes have measurable sizes > 0 (are not singularities).

    (c) The finite gravity-charge well allows radioactivity and quantum tunneling.

    (d) As the kinetic energy (temperature) decreases, more particles will stay within their gravity-charge well distance, r_c, allowing superconductivity and Bose-Einstein condensates.

(10) A constant value cannot be mapped to continuously varying values. Therefore, the discrete spin states of two quantum entangled particles and the polarization states of two quantum entangled photons are independent of continuously varying distance and time interval lengths.

(11) The set-based, first-order logic proof that a strict linearly ordered and symmetric set is a cyclic set of at most 3 members (4.12) is a simpler and more logically rigorous hypothesis for observing only 3 dimensions of physical space than: parallel dimensions that cannot be detected or extra dimensions rolled up into infinitesimal balls that are too small to detect.

    (a) Higher order dimensions must have different types (members of different sets), for example, dimensions of time, mass, and charge.

    (b) Each of 3 ordered and symmetric dimensions of space can have at most 3 sequentially ordered and symmetric state values, for example, an ordered and symmetric set of 3 vector orientations, $\{-1, 0, 1\}$, per dimension of space and at most 3 spin states per plane, etc.

    If the states are not sequentially ordered (a bag of states), then a state value is undetermined until observed (like Schrʹodinger's poisoned cat being both alive and dead until the box is opened). That is, there would be no "axiom of choice" that allows selecting a particular state.

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