The Set Properties Generating Geometry and Physics

George. M. Van Treeck

Abstract. Volume and the Minkowski distances/Lp norms (e.g., Euclidean distance) are derived from a set and limit-based foundation without referencing the primitives and relations of geometry. Sequencing a set of n number of intervals/dimensions in all n-at-a-time permutations via a strict linear order of successor/predecessor relations is a cyclic set limiting n to at most 3. Therefore, all other interval lengths have different types that can only be related to a distance interval length via unit-factoring, conversion ratio constants. The ratios and geometry proofs provide simpler derivations and new insights into the spacetime, gravity, charge force, and Einstein-Planck equations and their corresponding constants. All proofs are verified in Coq.

Contents

1. Introduction

Mathematical (real) analysis can construct differential calculus from a set and limit-based foundation without the need to reference the primitives and relations of Euclidean geometry, like side, angle, slope, etc. But the Riemann and Lebesgue integrals and measure theory (for example, Hilbert spaces and the Lebesgue measure) use the Euclidean volume equation as a definition. And the inner product, vector norm, and metric space use Euclidean distance and its properties as definitions [Gol76] [Rud76]. Here, these definitions are derived from a set and limit-based foundation without referencing the primitives and relations Euclidean geometry.

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In measure theory, a function is a measure if it satisfies some abstract criteria (metric space, Borel, Lebesgue, product σ-measure, etc.). A similar approach is used here, where a function is a volume equation if it satisfies the criteria of an abstract "countable volume", v_c, defined as the Cartesian product of the cardinals (number of members) of the countable sets, $x_i \in \{x_1, \cdots, x_n\} : v_c = \prod_{i=1}^{n} |x_i|$. Using the Cartesian product of cardinals avoids defining volume as the product of interval sizes and allows both Euclidean and non-Euclidean volume equations. Euclidean volume is derived from the case of countable sets of same-sized, size c, infinitesimals in each domain interval and the range interval, which provides a new measure of flatness/curvature.

Every Euclidean and non-Euclidean n-dimensional volume (n-volume) is a real-valued scalar, v, that has a corresponding cuboid n-volume, $v = d^n$, with a corresponding countable cuboid n-volume. And an n-volume can only be the sum of n-volumes. It will be proved that the L_p norms (Minkowski distances), $d = (\sum_{i=1}^{m} d_i^n)^{1/n}$, are the functions derived from the sum of countable cuboid n-volumes. And all Minkowski distances have the properties of a metric space. Therefore, all "geometric" distance measures are reducible to Minkowski distances.

Proving that a set of n number of independent domain intervals/dimensions sequenced in all n-at-a-time permutations (orders) via a strict linear order of successor/predecessor relations is a cyclic set limiting $n \leq 3$, implies that all additional intervals have different types that can only be related via unit-factoring, conversion ratio constants. The ratios combined with the Minkowski distance and volume proofs provide simpler derivations of the spacetime, Newton's gravity, Coulomb's charge force, and Einstein-Planck equations and exposes the ratio constants that generate the gravity, charge, and Planck constants.

All the proofs in this article are trivial. But to ensure confidence, all the proofs have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

Derivatives and anti-derivative integrals divide the domain intervals and range interval such that $\forall\ \Delta x_i \subset [x_a, x_b]\ \exists\ \Delta f_i \subset [f(x_a), f(x_b)]$. The size the infinitesimals in an interval are proportionate the size of that interval, which makes it difficult for differential equations and integrals to directly express the Cartesian mappings between the p_x number of size c infinitesimals in one interval and the p_y number of the *same* size c infinitesimals in a different-sized interval. Further, using integrals and measures that define Euclidean volume (for example, Riemann and Lebesgue) to derive Euclidean volume would be circular logic.

Therefore, a different tool is used here. A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals (infinitesimals), where each infinitesimal has the *same* size, c. The ruler is both an inner and outer measure of an interval.

DEFINITION 2.1. Ruler measure, M: $\forall\ [a, b] \subset \mathbb{R},\ \ s = b - a\ \wedge\ c > 0\ \wedge\ (p = floor(s/c)\ \vee\ p = ceiling(s/c))\ \wedge\ M = \sum_{i=1}^{p} c = pc$.

THEOREM 2.2. *Ruler convergence:* $M = \lim_{c \to 0} pc = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : \ y \leq x, \ y \in \mathbb{Z}, \ x \in \mathbb{R}\})$:

$$\forall\, c > 0, \ p = floor(s/c) \ \wedge \ 0 \leq |floor(s/c) - s/c| < 1 \ \Rightarrow \ |p - s/c| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by c:

$$\forall\, c > 0, \quad |p - s/c| < 1 \quad \Rightarrow \quad |pc - s| < |c| = |c - 0|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |pc - s| < |c - 0| < \delta \\ &\qquad \Rightarrow \quad |c - 0| < \delta \quad \wedge \quad |pc - s| < \delta = \epsilon \quad := \quad M = \lim_{c \to 0} pc = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}}, \ 3.14_{c=10^{-2}}, \ 3.141_{c=10^{-3}}, ..., \pi_{\lim_{c \to 0}}$.

Lemma 2.3. $\forall\, n \geq 1, \quad 0 < c < 1 \quad \Rightarrow \quad \lim_{c \to 0} c^n = \lim_{c \to 0} c.$

Proof. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < c < 1 \quad \Rightarrow \quad 0 < c^n < c \quad \Rightarrow \quad |c - c^n| < |c| = |c - 0|. \tag{2.4}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |c - c^n| < |c - 0| < \delta \\ &\qquad \Rightarrow \quad |c - 0| < \delta \quad \wedge \quad |c - c^n| < \delta = \epsilon \quad := \quad \lim_{c \to 0} c^n = 0. \end{aligned} \tag{2.5}$$

$$\lim_{c \to 0} c^n = 0 \quad \wedge \quad \lim_{c \to 0} c = 0 \quad \Rightarrow \quad \lim_{c \to 0} c^n = \lim_{c \to 0} c. \tag{2.6}$$

$\square$

3. Euclidean Volume

Definition 3.1. Countable volume, v_c, is the number of Cartesian product mappings (n-tuples) between the members of n number of disjoint, countable domain sets:

$$\exists\, n, v_c \in \mathbb{N}, \quad x_1, \cdots, x_n : \quad v_c = \prod_{i=1}^{n} |x_i|, \quad \bigcap_{i=1}^{n} x_i = \emptyset$$

Theorem 3.2. *Euclidean volume, $v = \prod_{i=1}^{n} s_i$, is the countable volume case, $v_c = \prod_{i=1}^{n} |x_i|$, of countable sets of same-sized, size c, infinitesimals in each domain interval, $[a_i, b_i]$, and the range interval, $[v_a, v_b]$.*

$$v_c \cdot c = (\prod_{i=1}^{n} |x_i|) \cdot c \quad \Rightarrow \quad v = \prod_{i=1}^{n} s_i, \ \ v = v_a - v_b, \ \ s_i = b_i - a_i.$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

Proof.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing $|x_i|$ number of size c subintervals and apply ruler convergence (2.2):

$$\forall\, i\ n \in \mathbb{N}, \ \ i \in [1, n], \ c > 0 \ \wedge \ floor(s_i/c) = |x_i| \ \Rightarrow \ s_i = \lim_{c \to 0}(|x_i| \cdot c). \tag{3.1}$$

$$s_i = \lim_{c \to 0}(|x_i| \cdot c) \quad \Leftrightarrow \quad \prod_{i=1}^{n} s_i = \prod_{i=1}^{n} \lim_{c \to 0}(|x_i| \cdot c). \tag{3.2}$$

$$\prod_{i=1}^{n} s_i = \prod_{i=1}^{n} \lim_{c \to 0}(|x_i| \cdot c) \quad \Leftrightarrow \quad \prod_{i=1}^{n} s_i = \lim_{c \to 0}(\prod_{i=1}^{n} |x_i|) \cdot c^n. \tag{3.3}$$

Apply lemma 2.3 to equation 3.3:

$$\begin{aligned} &\prod_{i=1}^{n} s_i = \lim_{c \to 0}(\prod_{i=1}^{n} |x_i|) \cdot c^n \quad \wedge \quad \lim_{c \to 0} c^n = \lim_{c \to 0} c \\ &\qquad \Rightarrow \quad \prod_{i=1}^{n} s_i = \lim_{c \to 0}(\prod_{i=1}^{n} |x_i|) \cdot c. \end{aligned} \tag{3.4}$$

Apply the ruler (2.1) and ruler convergence (2.2):

$$\exists\, v \in \mathbb{R} :\ v_c = floor(v/c) \quad \Leftrightarrow \quad v = \lim_{c\to 0} v_c \cdot c. \tag{3.5}$$

Apply the definition of the countable volume (3.1):

$$v_c = \prod_{i=1}^{n} |x_i| \quad \Leftrightarrow \quad \lim_{c\to 0} v_c \cdot c = \lim_{c\to 0}(\prod_{i=1}^{n} |x_i|) \cdot c. \tag{3.6}$$

Combine equations 3.5, 3.6, and 3.4:

$$v = \lim_{c\to 0} v_c \cdot c \quad \wedge \quad \lim_{c\to 0} v_c \cdot c = \lim_{c\to 0}(\prod_{i=1}^{n} |x_i|) \cdot c \quad \wedge \\ \lim_{c\to 0}(\prod_{i=1}^{n} |x_i|) \cdot c = \prod_{i=1}^{n} s_i \quad \Leftrightarrow \quad v = \prod_{i=1}^{n} s_i. \tag{3.7}$$ □

4. Distance

4.1. Countable cuboid n-volume.

DEFINITION 4.1. The countable cuboid volume, d_c^n, is the sum of m number of sets of countable cuboid volumes.

$$\forall\, n \in \mathbb{N}, \quad d_c \in \{0, \mathbb{N}\} \quad \exists\, m \in \mathbb{N}, \quad x_1, \cdots, x_m \in X, \quad \bigcap_{i=1}^{m} x_i = \emptyset : \\ d_c^n = \sum_{i=1}^{m} |x_i|^n.$$

4.2. Minkowski distance (L_p norm).

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

THEOREM 4.2. *The Minkowski distances (L_p norms) are derived from the sum of countable cuboid n-volumes (4.1).*

$$d_c^n = \sum_{i=1}^{m} |x_i|^n \quad \Rightarrow \quad \exists\, d, s_1, \cdots, s_m \in \mathbb{R} : \quad d = (\sum_{i=1}^{m} s_i^n)^{1/n}.$$

PROOF. Apply the ruler (2.1):

$$\exists\, d, s_1, \cdots, s_m \in \mathbb{R} :\ d_c = floor(d/c) \quad \wedge \quad |x_i| = floor(s_i/c). \tag{4.1}$$

Apply the ruler convergence (2.2):

$$d_c^n = \sum_{i=1}^{m} |x_i|^n \Rightarrow d^n = \lim_{c\to 0}(d_c \cdot c)^n = \lim_{c\to 0} \sum_{i=1}^{m}(|x_i| \cdot c)^n = \sum_{i=1}^{m} s_i^n. \tag{4.2}$$

$$d^n = \sum_{i=1}^{m} s_i^n \quad \Leftrightarrow \quad d = (\sum_{i=1}^{m} s_i^n)^{1/n}. \tag{4.3}$$ □

4.3. Distance inequality. Proving that all Minkowski distances (L_p norms) satisfy the metric space triangle inequality requires another inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N}, \ v_a, v_b \geq 0 : \ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand the n-volume, $(v_a^{1/n} + v_b^{1/n})^n$, using the binomial expansion:

$$\forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq v_a + v_b + \\ \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n. \tag{4.4}$$

Take the n^{th} root of both sides of the inequality:

$$\forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \tag{4.5}$$ □

4.4. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N},\ \ a_i, b_i \geq 0:\ \ (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$\begin{aligned}(4.6)\quad &\forall\, m, n \in \mathbb{N},\ v_a, v_b \geq 0:\quad v_a = \textstyle\sum_{i=1}^{m} a_i^n \quad\wedge\quad v_b = \sum_{i=1}^{m} b_i^n \quad\wedge\\ &(v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad\Rightarrow\quad ((\textstyle\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} =\\ &(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.\end{aligned}$$

□

4.5. Metric Space. All Minkowski distances (L_p norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad\Rightarrow\quad d(u, w) \leq d(u, v) + d(v, w).$

PROOF. $\forall\, p \geq 1,\quad k > 1,\quad u = s_1,\quad w = s_2,\quad v = w/k$:

$$(4.7)\qquad (u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}.$$

Apply the distance inequality (4.3) to the inequality 4.7:

$$\begin{aligned}(4.8)\quad &(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \ \wedge\ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}\\ &\wedge\quad v_a = u^p + v^p \quad\wedge\quad v_b = v^p + w^p\\ \Rightarrow\quad &(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p}\\ &\Rightarrow\quad d(u, w) = (u^p + w^p)^{1/p} \leq\\ &(u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w).\end{aligned}$$

□

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad\Rightarrow\quad d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$\begin{aligned}(4.9)\quad &\forall\, p\ :\ p \geq 1,\quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p}\\ &\Rightarrow\quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u).\end{aligned}$$

□

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad\Rightarrow\quad d(u, w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$(4.10)\qquad \forall\, [a_1, b_1],\ [a_2, b_2],\quad u = b_1 - a_1,\ v = b_2 - a_2,\quad\Rightarrow\quad u \geq 0,\ v \geq 0.$$

$$(4.11)\qquad p \geq 1,\ \ u, v \geq 0 \quad\Rightarrow\quad d(u, v) = (u^p + v^p)^{1/p} \geq 0.$$

□

THEOREM 4.8. *Identity of Indiscernibles:* $d(u, u) = 0$.

PROOF. From the non-negativity property (4.7):

$$d(u,w) \geq 0 \quad \wedge \quad d(u,v) \geq 0 \quad \wedge \quad d(v,w) \geq 0 \\ \Rightarrow \quad \exists\, d(u,w) = d(u,v) = d(v,w) = 0. \tag{4.12}$$

$$d(u,w) = d(v,w) = 0 \quad \Rightarrow \quad u = v. \tag{4.13}$$

$$d(u,v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u,u) = 0. \tag{4.14}$$

□

5. Applications to physics

5.1. At most 3 dimensions of physical space. The following two proofs are in the physics section because limiting the domain intervals/dimensions to a strict linearly ordered set that can be sequenced in all n-at-a-time orders, is an additional restriction on volume and distance used to explain why physical space is limited to a cyclic set of 3 dimensions.

DEFINITION 5.1. Strict linearly ordered set:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i \;\in \{x_1, \ldots, x_n\}, \\ \mathit{successor}\ x_i = x_{i+1} \;\;\wedge\;\; \mathit{predecessor}\ x_{i+1} = x_i.$$

DEFINITION 5.2. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; \mathit{successor}\ x_i = x_j \;\;\Leftrightarrow\;\; \mathit{predecessor}\ x_j = x_i.$$

THEOREM 5.3. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n \;\;\wedge\;\; j = 1 \;\;\Rightarrow\;\; \mathit{successor}\ x_n = x_1 \;\;\wedge\;\; \mathit{predecessor}\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (5.2):

$$i = n \;\;\wedge\;\; j = 1 \;\;\wedge\;\; \mathit{successor}\ x_i = x_j \;\;\Rightarrow\;\; \mathit{successor}\ x_n = x_1. \tag{5.1}$$

Applying the definition of the symmetry property (5.2) to conclusion 5.1:

$$\mathit{successor}\ x_i = x_j \;\;\Rightarrow\;\; \mathit{predecessor}\ x_j = x_i \;\;\Rightarrow\;\; \mathit{predecessor}\ x_1 = x_n. \tag{5.2}$$

□

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3).

DEFINITION 5.5. Successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize). \tag{5.3}$$

DEFINITION 5.6. Predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{5.5}$$

Prove that every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \tag{5.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \tag{5.7}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1). \tag{5.8}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.9}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1). \tag{5.11}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize). \tag{5.12}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.13}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \tag{5.14}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\begin{aligned} \forall\, setsize > 3 : \quad & \neg Successor(1, 3, setsize > 3) \\ & \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \end{aligned} \tag{5.15}$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\, setsize > 3 : \quad & \neg Predecessor(1, 3, setsize > 3) \\ & \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \end{aligned} \tag{5.16}$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\, setsize > 3 : \quad & \neg Adjacent(1, 3, setsize > 3) \\ & \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \end{aligned} \tag{5.17}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

Therefore, any higher dimensions of intervals must have a different type that can only be related to the 3 geometric distance and volume dimensions via unit-factoring, conversion constants (both direct and inverse proportionate constants), for example, $r = (r_c/t_c)t = ct$ and $r = (m_p r_c)/m_0 = k_W/m_0$.

5.2. Spacetime geometry. The many other derivations of the spacetime equations assume Einstein's two postulates: 1) the laws of physics are the same in every inertial frame of reference and 2) the speed of light is a constant in every inertial frame of reference. Here, the properties of physical space generate the spacetime equations.

A distance, r, viewed from two independent frames of reference have the two local apparent distances, r_1 and r_2, which implies that r_1 and r_2 are dependent on r. From the Minkowski distance proof (4.2), where r_1 and r_2 are distances in two dimensions (independent frames of reference), $r^2 = r_1^2 + r_2^2$. And from the 3D proof (5.4), if the distances are Euclidean distances in 3-space, then a higher dimension interval size, t, must have a different type that is related to physical distance, r, via a unit-factoring, conversion ratio constant, $r = (r_c/t_c)t = ct$:

$$r^2 = r_1^2 + r_2^2 \quad \wedge \quad \exists\, r_c, t_c, c \in \mathbb{N} : \ (r_c/t_c)t = ct = r \quad \Rightarrow \quad (ct)^2 = r_1^2 + r_2^2. \tag{5.18}$$

$$r_1^2 = (ct)^2 - r_2^2 \quad \wedge \quad v = r_2/t \quad \Rightarrow \quad r_1 = ct\sqrt{1 - (v/c)^2}. \tag{5.19}$$

Distance contraction by using $ct = r$ from equation 5.19:

$$r_1 = ct\sqrt{1 - (v/c)^2} \quad \wedge \quad ct = r \quad \Rightarrow \quad r_1 = r\sqrt{1 - (v/c)^2}. \tag{5.20}$$

Time dilation by dividing both sides of equation 5.19 by c:

$$r_1 = ct\sqrt{1 - (v/c)^2} \quad \wedge \quad t' = r_1/c \quad \Rightarrow \quad t' = t\sqrt{1 - (v/c)^2}. \tag{5.21}$$

If r_1 and r_2 are Euclidean distances, then by the 3D proof (5.4), the Euclidean distances are limited to at most 3 dimensions. Using the (-+++) form of Minkowski's flat spacetime [**Bru17**], the size of the "spatial" event separation interval, r_2, is:

$$\begin{aligned} r^2 = r_1^2 + r_2^2 \ \wedge \ r^2 = x^2 + y^2 + z^2 \ \wedge \ \exists\, r_c, t_c, c \in \mathbb{N} : \ (r_c/t_c)t = ct = r_1 \\ \Rightarrow \quad r_2^2 = -(ct)^2 + r^2 = -(ct)^2 + (x^2 + y^2 + z^2). \end{aligned} \tag{5.22}$$

5.3. Newton's gravity force equation. m_1 and m_2, are the sizes of two independent masses, where each size c component of a mass exerts a force on each size c component of the other mass. If p_1 and p_2 are the number of size c components in each mass, then the total force, F, is equal to the total number of forces, $p_1 \cdot p_2$, and proportionate to the size, c, of each component. Applying the ruler (2.1) and volume proof (3.2), where the force, F, is defined as the rest mass, m_0, times acceleration, a:

$$\begin{aligned} p_1 = floor(m_1/c) \quad \wedge \quad p_2 = floor(m_2/c) \quad \wedge \quad F := m_0 a \propto (p_1 \cdot p_2)c \\ \Rightarrow \quad F := m_0 a \propto \lim_{c \to 0}(p_1 \cdot p_2)c = \lim_{c \to 0}(p_1 \cdot p_2)c^2 = \lim_{c \to 0} p_1 c \cdot p_2 c = m_1 m_2. \end{aligned} \tag{5.23}$$

$$\begin{aligned} F := m_0 a := m_0 r/t^2 \propto m_1 m_2 \ \wedge \ m_0 = m_1 \ \Rightarrow \ r \propto m_2 \ \Rightarrow \\ \exists\, m_G, r_G \in \mathbb{R} : \ r = (r_G/m_G)m_2, \end{aligned} \tag{5.24}$$

where: r is Euclidean distance, t is time, and r_G/m_G is a unit-factoring ratio.

$$\begin{aligned} m_0 = m_1 \quad \wedge \quad r = (m_G/r_G)m_2 \quad \wedge \quad F = m_0 r/t^2 \\ \Rightarrow \quad F = m_0 r/t^2 = (r_G/m_G)m_1 m_2/t^2. \end{aligned} \tag{5.25}$$

Using the local frame of reference, $r = ct\sqrt{1-(v/c)^2}$, and $v = 0$:

$$(5.26)\quad r = ct \quad \wedge \quad F = (r_G/m_G)m_1m_2/t^2 \quad \Rightarrow \\ F = ((r_G/m_G)c^2m_1m_2/r^2 = Gm_1m_2/r^2,$$

where the constant, $G = (r_G/m_G)c^2$, has the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$. And where $|v| > 0$, $F = (r_G/m_G)(c^2 - v^2)q_1q_2/r^2$.

5.4. Coulomb's charge force. q_1 and q_2, are the sizes of two independent charges, where each size c component of a charge exerts a force on each size c component of the other charge. If p_1 and p_2 are the number of size c components in each charge, then the total force, F, is equal to the total number of forces, $p_1 \cdot p_2$, and proportionate to the size, c, of each component. Applying the ruler (2.1) and volume proof (3.2), where the force, F, is defined as the rest mass, m_0, times acceleration, a:

$$(5.27)\quad p_1 = floor(q_1/c) \quad \wedge \quad p_2 = floor(q_2/c) \quad \wedge \quad F \propto (p_1 \cdot p_2)c \\ \Rightarrow \quad F := m_0a \propto \lim_{c\to 0}(p_1 \cdot p_2)c = \lim_{c\to 0}(p_1 \cdot p_2)c^2 = \lim_{c\to 0} p_1c \cdot p_2c = q_1q_2.$$

$$(5.28)\quad F := m_0a := m_0r/t^2 \propto q_1q_2 \quad \wedge \\ m_0 = (m_G/r_G)(r_C/q_C)q_1 = (m_Gr_C/q_Cr_G)q_1 \quad \Rightarrow \quad r \propto q_2 \\ \Rightarrow \quad \exists\, q_C, r_C \in \mathbb{R} : \ r = (r_C/q_C)q_2,$$

where: r is Euclidean distance, t is time, m_G/q_C and r_C/q_C are unit-factoring ratios.

$$(5.29)\quad m_0 = (m_Gr_C/q_Cr_G)q_1 \quad \wedge \quad r = (r_C/q_C)q_2 \quad \wedge \quad F = m_0r/t^2 \\ \Rightarrow \quad F = m_0r/t^2 = (m_G/r_G)(r_C/q_C)^2q_1q_2/t^2.$$

Using the local frame of reference, $r = ct\sqrt{1-(v/c)^2}$, and $v = 0$:

$$(5.30)\quad r = ct = (r_c/t_c)t \quad \wedge \quad a_G = r_c/t_c^2 \quad \wedge \quad F = (m_G/r_G)(r_C/q_C)^2q_1q_2/t^2 \\ \Rightarrow \quad F = ((m_G/r_G)(r_C/q_C)^2(r_c/t_c)^2)q_1q_2/r^2 = \\ (m_Ga_G)(r_c/r_G)(r_C/q_C)^2q_1q_2/r^2 = k_eq_1q_2/r^2,$$

where the predicted charge constant, $k_e = (m_Ga_G)(r_c/r_G)(r_C/q_C)^2$, has the SI units: $N \cdot m^2 \cdot C^{-2}$.

And where $|v| > 0$, $F = ((m_G/r_G)(r_C/q_C)^2)(c^2 - v^2)q_1q_2/r^2$.

5.5. Einstein-Planck and energy-charge equations: Combining the ratio (constant first derivative) equations: $m = (m_p/r_p)r$ and $r/t = r_c/t_c = c \Rightarrow m(ct)^2 = (m_p/r_p)r \cdot r^2$. Dividing both sides by t^2: $E = mc^2 = (m_p/r_p)r \cdot (r/t)^2 = (m_p/r_p)r \cdot (r_c/t_c)^2 = (m_pr_c/r_pt_c)c \cdot r = (m_pr_cc) \cdot (r/(r_pt_c)) = h \cdot \nu$, which is the Einstein-Planck equation, where the Planck constant is, $h = m_pr_cc$, and $\nu = r/(r_pt_c)$ is the frequency in cycles per second. $h/c = m_pr_c = k_W \Rightarrow h = k_Wc$, where r_c is the work displacement (Compton wavelength) of a particle with the rest mass, m_p.

Likewise, for charge, $r = (r_C/q_C)q = (r_p/m_p)m \Rightarrow m = (m_p/r_p)(r_C/q_C)q \Rightarrow E = mc^2 = (m_p/r_p)(r_C/q_C)qc^2 = (m_pr_cc(r_C/r_p)) \cdot (q/(q_Ct_c)) = h(r_C/r_p) \cdot \nu$, where the frequency is, $\nu = q/(q_Ct_c)$.

6. Insights and implications

(1) The Euclidean volume proof (3.2) shows that "flat" space is where the ratio of a range infinitesimal size to a neighbor range infinitesimal size is one and the domain infinitesimals (for example: $\mathrm{d}t$, $\mathrm{d}m$, $\mathrm{d}q$, $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$, etc.) are all the same size, c, independent of the corresponding domain interval sizes (for example, $[0,t]$, $[0,m]$, $[0,q]$, $[0,x], \cdots$). Whereas, the infinitesimal sizes in second derivative-based measures of curvature vary with their corresponding interval sizes.

(2) All geometric distance functions are functions of the sum of cuboid n-volumes, which are the Minkowski distances (4.2). And it was proved that the Minkowski distances have the properties of metric space (4.5). Therefore, if the definition of a complete metric space allows functions that are not a function of the sum of n-volumes, then the definition of a complete metric space is not a sufficient filter to obtain only geometric distances.

(3) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line (equation), where it is assumed that the straight line length is the smallest distance [**Joy98**]. And proofs that a straight line (equation) is the smallest distance have equated the straight line to the Euclidean distance.

All geometric distance measures are functions that can be reduced to a function of the sum of cuboid n-volumes, which are the Minkowski distances (4.2), $d = (\sum_{i=1}^{m} s_i^n)^{1/n}$. If m represents the number of domain intervals, one interval from each dimension, then $1 \leq n \leq m$. And $m = 2 \Rightarrow 1 \leq n \leq 2$, which constrains all Minkowski distances to a range from Manhattan distance (the largest distance) to Euclidean distance (the smallest distance) in Euclidean (flat) 2-space.

(4) Hilbert spaces allow fractional dimensions (fractals), which is the case of intersecting domain sets and requires generalizing the countable volume definition (3.1), $v_c = \prod_{i=1}^{n} |x_i|$, to: $v_c = \prod_{i=1}^{n}(|x_i| - |x_i \bigcap (\bigcup_{j=1, i \neq j}^{n} x_j)|)$.

(5) Compare the distance sum inequality (4.4),

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n},$$

used to prove that all Minkowski distances satisfy the metric space triangle inequality property (4.5), to Minkowski's sum inequality:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n)^n)^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

Note the exponent difference in the left side of the two inequalities:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \quad vs. \quad (\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n}.$$

Minkowski's sum inequality proof assumes: convexity and the L_p space inequalities (for example, Hölder's inequality or Mahler's inequality) or the triangle inequality. In contrast, the distance (sum) inequality is a more fundamental inequality that does not require the assumptions of the Minkowski sum inequality.

(6) The derivations of the spacetime, gravity force, charge force, and Einstein-Planck equations expose a unit-factoring conversion ratio (constant first derivative) principle: $r = (r_c/t_c)t = ct$, $r = (r_G/m_G)m$, and $r = (r_C/q_C)q$.

The gravity constant, G, the charge constant, k_e, and the Planck constant, h were all derived from these constant ratios.

(7) The derivations in this article show that the spacetime, gravity force, charge force, and Einstein-Planck equations all depend on time being proportionate to distance: $r = (r_c/t_c)t = ct$. For example, from the derivation of Newton's gravity equation (5.26), where $v = 0$: $G = (r_G/m_G)c^2$. Likewise, from the derivation of Coulomb's charge force equation (5.30) the constant, where $v = 0$: $k_e = (m_G/r_G)(r_C/q_C)^2c^2$. And from the derivation of the Planck constant (5.5), $h = (m_p r_c)c$.

(8) The derivation of the Planck-Einstein equation (5.5) shows that the Planck constant, h, is the product of two fundamental constants, the work constant, k_W, and the speed of light constant, c: $h = k_W c$, where $k_W = m_p r_c \approx 2.2102190943 \cdot 10^{-42}\ kg \cdot m$, where r_c is the work displacement (Compton wavelength) of a particle having the rest mass, m_p. For example, using the rest mass of an electron, $m_{e^-} \approx 9.1093837 \cdot 10^{-31}\ kg \Rightarrow r_c = k_W/m_{e^-} \approx 2.426310239 \cdot 10^{-12} m$, which is the accepted value of the electron Compton wavelength. **Note:** the mass-displacement equation, $k_W = m_p r_c$, is analogous to the light frequency-wave equation: $c = f\lambda$, where λ is the wavelength and f is the frequency. The mass-displacement equation simplifies the many physics equations that contain both c and h. The mass-displacement equation says that, for a given amount of energy, a larger mass will have a smaller displacement, which makes those simplified equations more intuitive.

(9) A state is represented by a constant value. And a constant value, by definition, cannot vary with distance and time interval lengths. Therefore, the spin states of two quantum entangled electrons and the polarization states of two quantum entangled photons are independent of the amount of distance and time between the entangled particles.

(10) Applying the ruler (2.1) and volume proof (3.2) to the Cartesian product of same-sized, infinitesimal mass forces and charge forces to derive Newton's gravity force (5.3) and Coulomb's charge force (5.4) equations provide some firsts and some insights into physics:
   (a) These are the first derivations to not assume the inverse square law or Gauss's divergence theorem.
   (b) These are the first derivations to show that the definition of force, $F := m_0 a$, containing acceleration, $a = r/t^2$, where r is a distance that is proportionate to time, t, generates the inverse square law.
   (c) Using Occam's razor, those versions of constants like: charge, vacuum magnetic permeability, etc. that contain the value 4π might be incorrect because those constants are based on the less parsimonious assumption that the inverse square law is due to Gauss's flux divergence on a sphere having the surface area, $4\pi r^2$.
   (d) The derived spacetime relativistic equations for gravity and charge force are: $F = (r_G/m_G)(c^2 - v^2)m_1 m_2/r^2$ (5.26) and $F = ((m_G/r_G)(r_C/q_C)^2)(c^2 - v^2)q_1 q_2/r^2$ (5.30). Therefore, $v \to c \Rightarrow F \to 0$.

(11) It was proved that sequencing through a set of n members, having a strict linear order, in all n-at-time permutations, a symmetric set, is a cyclic set

with at most 3 members (5.3).

(a) Using Occam's razor, a strict linear order and symmetry is the most parsimonious explanation for only observing 3 dimensions of physical distance and volume.

(b) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (5.4), each member being an ordered and symmetric set of 3 dimensions (three balls).

(c) Each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values of the same type, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values of the same type per 3-dimensional ball, for example, vector orientation values: -1, 0, 1 per dimension in the ball.

(d) Only 3 dimensions of physical space and 3 vector orientations (-1, 0, 1) per dimension is what makes physical space chiral (allows mirror images and clockwise/counter-clockwise rotations).

(e) Each of the 3 possible ordered and symmetric dimensions of discrete physical states could contain an unordered collection (bag) of discrete state values. Bags are non-deterministic. For example, every time an unordered binary state is "pulled" from a bag, there is a 50 percent chance of getting one of the binary values.

(12) It was shown that some fundamental geometry (volume and the Minkowski distances/L_p norms) and physics (gravity force and charge force) are derived from the combinatorial (Cartesian product) mappings between the infinitesimals of real-valued domain intervals. The proofs and derivations in this article show that the ruler (2.1) is a tool to directly express and solve such combinatorial relations that occur often in geometry, probability, physics, etc. that, in some cases, might be difficult to directly express with differential equations and integrals.

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George Van Treeck, 668 Westline Dr., Alameda, CA 94501