SOME SET PROPERTIES UNDERLYING GEOMETRY AND PHYSICS

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Abstract. Volume and distance equations are proved to be instances of ordered sets of combinations (n-tuples). The combinatorial properties can limit volume and distance to a set of 3 dimensions. More dimensions have different types (are members of other sets), with ratios of a distance unit to units of time, mass, and charge. The proofs and ratios are used to: derive well-known gravity, charge, electromagnetic equations, special and general relativity equations, and quantum physics equations; derive the gravity, charge, vacuum permittivity, vacuum permeability, Planck, and fine structure constants; add quantum extensions to gravity and charge equations. All the proofs are verified in Rocq.

Contents

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1. Introduction

The Riemann integral, Lebesgue integral, and Lebesgue measure define Euclidean volume as the product of interval sizes $\subset \mathbb{R}^n$ [8] [15]. And Euclidean distance, vector magnitude, and the many "spaces": inner product, metric, Hausdorff, Cauchy, etc. are definitions, in analysis [18] [8] [15].

Justification of the definitions uses vigorous finger-pointing to geometry and physics because definitions can only *describe* aspects of volume and distance. Deriving volume and distance equations from an abstract, set and limit-based foundation exposes the principles that *explain*, cause, the volume and distance equations, which provides new insights and tools for analysis and physics.

All the proofs in this article have been verified using using the Rocq proof verification system [14]. The formal proofs are in the Rocq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

Using calculus and σ-algebras (for example, the Lebesgue measure) to prove the volume and distance theorems in this article would result in circular logic. Therefore, a "ruler" measure of intervals, $[a, b] \subset \mathbb{R}$, will be used to prove the theorems.

Where $|x_i|$ is the cardinal of (number of elements in) a countable set, x_i, the countable number of ordered combinations (n-tuples) is v_c. The ruler measure will be used to prove the Euclidean volume relation:

$$\forall\, x_i \in \{x_1, \cdots, x_n\} = X, \quad \bigcap_{x_i \in X} x_i = \emptyset : \quad v_c = \prod_{i=1}^{n} |x_i| \\ \Leftrightarrow \quad v = \prod_{i=1}^{n} s_i, \quad s_i = b_i - a_i, \quad [a_i, b_i] \subset \mathbb{R}. \tag{1.1}$$

Summing volumes is only useful if they are of the same subtype of volume, for example, a 2-volume can only be the sum of 2-volumes. Therefore, any volume measure must include the number of domain values (dimensions), n.

For all $n > 1$, there are an infinite number of combinations of domain values, s_i, that produce the same value, v. Therefore, a comparison measure should be in the form of a single, representative domain value, d, and n, which implies that for all Euclidean and non-Euclidean n-volumes, d, is an inverse (bijective) function of an n-volume: $d : v_n = f(d)$ and $d = f^{-1}(v_n)$.

For the case of countable n-volumes, the simplest case is:

$$\exists\, d_c, v_c, |x_i| \in \{0, \mathbb{N}\} : \; v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n, \tag{1.2}$$

And $d = f^{-1}(v)$ and $v = \sum_{i=1}^{m} v_i \Rightarrow d = f^{-1}(\sum_{i=1}^{m} v_i)$. The ruler measure will be used to prove that:

$$d_c^n = \sum_{i=1}^{m} v_{c_i} = \sum_{i=1}^{m} (\prod_{j=1}^{n} |x_{i,j}|) \;\Leftrightarrow\; d^n = \sum_{i=1}^{m} v_i = \sum_{i=1}^{m} (\prod_{j=1}^{n} s_{i,j}). \tag{1.3}$$

The $n = 2$ case is the basis of the inner product. Where each v_{c_i} is also a bijective function, $v_{c_i} = d_{c_i}^n$, the ruler measure will be used to prove that:

$$d_c^n = \sum_{i=1}^{m} d_{c_i}^n \quad \Leftrightarrow \quad d^n = \sum_{i=1}^{m} d_i^n. \tag{1.4}$$

$|d|$ is the p-norm (Minkowski distance) [12], which will be proved to imply the metric space properties [15]. The $n = 2$ case is, obviously, the Euclidean distance.

Volume and distance are derived from sets of ordered combinations (n-tuples). Volume and distance have another combinatorial (permutation) property.

Calculating volume requires multiplying a sequentially ordered set of domain values. And calculating distance requires summing a sequentially ordered set of values. The commutative properties of multiplication and addition allows sequencing an ordered set in all $n!$ permutations.

You cannot re-sequence the domain values in the same order, unless you place the values in a sequential order. Further, the *only* sequential order, where you can start with any value and sequence in a repeatable order, is a "cyclic" order.

The second sequenced member must be either the *immediate* cyclic successor or *immediate* cyclic predecessor. And repeatable, sequencing of a cyclic set in all $n!$ permutations, is a symmetry, where every set member is either an *immediate* cyclic successor or an *immediate* cyclic predecessor to every other set member, herein, referred to as an "immediate symmetric" cyclic set. An immediate symmetric cyclic set will be proved to have $n \leq 3$ members.

An immediate cyclic set of 3 "distance" dimensions requires more dimensions to have non-distance types (be members of other sets), where for each subinterval (unit) length, r_p, of distance interval length, r, there are unit lengths: t_p of time interval length, t; m_p of mass interval length, m; and q_p of charge interval length, q, such that: $r = (r_p/t_p)t = (r_p/m_p)m = (r_p/q_p)q$.

The proofs and the 3 direct proportion ratios are used to provide simple derivations of the Newton, Gauss, and Poisson gravity equations, Coulomb charge, Gauss, Lorentz, and Faraday electromagnetic equations [and the constants: gravity (G), charge (k_e), vacuum permittivity (ε_0), and vacuum permeability (μ_0)]. They are also used to derive all the special relativity equations, the Schwarzschild time dilation and black hole metric equations pointing to a simplified method of finding solutions to Einstein's general relativity equations.

Next, algebraic manipulation of the 3 direct proportion ratios yields 3 inverse proportion ratios, $r = t_p r_p/t = m_p r_p/m = q_p r_p/q$. The fine structure constant, α, is derived from ratio of subtypes: $\alpha = q_e^2/q_p^2$ where q_e is the elementary (electron) charge and q_p is the Planck charge unit.

The combination of the direct and inverse proportion ratios are used to derive the Planck relation, the Planck constant, $h = (m_p r_p)(r_p/t_p)$, the Compton, position-space Schrödenger, and Dirac wave equations. And finally, the inverse proportion ratios are used to add quantum extensions to some general relativity and classical physics equations.

2. Ruler measure and convergence

A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of size κ subintervals. The ruler is both an inner and outer measure of an interval.

Definition 2.1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\, [a, b] \subset \mathbb{R}$, $s = b - a \;\wedge\; 0 < \kappa \leq 1 \;\wedge\; (p = floor(s/\kappa) \;\vee\; p = ceiling(s/\kappa))$.

Theorem 2.2. *Ruler convergence:* $M = \lim_{\kappa\to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : \, y \leq x,\, y \in \mathbb{Z},\, x \in \mathbb{R}\})$:

$$\forall\, 0 < \kappa \leq 1, p = floor(s/\kappa) \wedge 0 \leq |floor(s/\kappa) - s/\kappa| < 1 \Rightarrow |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by κ:

$$\forall\, 0 < \kappa \leq 1, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ &\quad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \delta = \epsilon \quad := \quad M = \lim_{\kappa\to 0} p\kappa = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, ..., \pi_{\lim_{\kappa\to 0}}$.

Lemma 2.3. $\forall\, n \geq 1, \quad 0 < \kappa \leq 1 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa$.

Proof. The formal proof , "lim_c_to_n_eq_lim_c," is in the Rocq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa \leq 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa\to 0} \kappa^n = 0. \end{aligned} \tag{2.5}$$

$$\lim_{\kappa\to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa\to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa. \tag{2.6}$$

□

3. Volume

Definition 3.1. A countable n-volume is the number of ordered combinations (n-tuples), v_c, of the members of n number of disjoint, countable domain sets, x_i:

$$x_i \in \{x_1, \cdots, x_n\} = X,\ |x_i| \in \{0, \mathbb{N}\} : \ \bigcap_{x_i \in X} x_i = \emptyset \ \wedge \ v_c = \prod_{i=1}^n |x_i|. \tag{3.1}$$

Theorem 3.2. *Euclidean volume,*

$$\begin{aligned} &\forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\ [v_a, v_b] \subset \mathbb{R},\ s_i = b_i - a_i,\ v = v_b - v_a : \\ &\qquad v_c = \prod_{i=1}^n |x_i| \quad \Leftrightarrow \quad v = \prod_{i=1}^n s_i. \end{aligned} \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Rocq file, euclidrelations.v.

Proof.

$$v_c = \prod_{i=1}^n |x_i| \ \Leftrightarrow\ v_c\kappa = (\prod_{i=1}^n |x_i|)\kappa \ \Leftrightarrow\ \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{3.3}$$

Apply the ruler (2.1) and ruler convergence (2.2) to equation 3.3:

$$\begin{aligned} &\exists\, v, \kappa \in \mathbb{R} : \ v_c = floor(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa \quad \wedge \\ &\qquad \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa \quad \Rightarrow \quad v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \end{aligned} \tag{3.4}$$

Apply lemma 2.3 to equation 3.4:

$$\begin{aligned} &v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa \quad \wedge \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa \\ &\qquad \Rightarrow \quad v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa^n = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa). \end{aligned} \tag{3.5}$$

Apply the ruler (2.1) and ruler convergence (2.2) to s_i:

$$\exists\, s_i, \kappa \in \mathbb{R} : \ floor(s_i/\kappa) = |x_i| \quad \Rightarrow \quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i. \tag{3.6}$$

$$\begin{aligned} &v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa) \quad \wedge \quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i \\ &\qquad \Leftrightarrow \quad v = \lim_{\kappa\to 0}(|x_i|\kappa) = \prod_{i=1}^n s_i \quad \square \end{aligned} \tag{3.7}$$

4. Distance

Definition 4.1. Countable distance,

$$\begin{aligned} &\exists\, n \in \mathbb{N},\ v_c, d_c \in \{0,\ \mathbb{N}\},\ x_i \in \{x_1, \cdots, x_n\} = X : \ \bigcap_{x_i \in X} x_i = \emptyset \quad \wedge \\ &\qquad d_c = |x_1| = \cdots = |x_n| \quad \wedge \quad v_c = \prod_{i=1}^n |x_i| = \prod_{i=1}^n d_c = d_c^n. \end{aligned} \tag{4.1}$$

Lemma 4.2. *A volume is the sum of volumes,*

$$v_c = d_c^n = \sum_{i=1}^m v_{c_i} \quad \Leftrightarrow \quad v = \sum_{i=1}^m v_i, \quad v, v_i \in \mathbb{R}.$$

The formal proof, "sum_of _volumes," is in the Rocq file, euclidrelations.v.

Proof. From the condition of this theorem:

$$v_c = \sum_{i=1}^m v_{c_i} \quad \Leftrightarrow \quad \lim_{\kappa\to 0} v_c \kappa = \lim_{\kappa\to 0} \sum_{j=1}^m (v_{c_i}\kappa). \tag{4.2}$$

Apply lemma 2.3 to equation 4.2:

$$\begin{aligned} &\lim_{\kappa\to 0} v_c \kappa = \lim_{\kappa\to 0} (\textstyle\sum_{j=1}^m v_{c_i})\kappa \quad \wedge \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa \\ &\Leftrightarrow \lim_{\kappa\to 0} v_c \kappa = \lim_{\kappa\to 0} \textstyle\sum_{j=1}^m (v_{c_i})\kappa^n \;\Leftrightarrow\; \lim_{\kappa\to 0} v_c \kappa = \lim_{\kappa\to 0} \sum_{j=1}^m (v_{c_i}\kappa). \end{aligned} \tag{4.3}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to equation 4.3:

$$\begin{aligned} &\exists\, v, v_i :\; v = floor(d/\kappa),\; v = \lim_{\kappa\to 0} v_c \kappa \\ &\wedge \quad v_{c_i} = floor(v_i/\kappa),\; v_i = \lim_{\kappa\to 0} v_{c_i}\kappa \quad \wedge \quad \lim_{\kappa\to 0} (d_c\kappa)^n = \lim_{\kappa\to 0} \textstyle\sum_{j=1}^m (v_{c_i}\kappa) \\ &\Leftrightarrow \quad v = \lim_{\kappa\to 0} (d_c\kappa)^n = \lim_{\kappa\to 0} \textstyle\sum_{j=1}^m (v_{c_i}\kappa) = \sum_{j=1}^m v_i^n. \quad \square \end{aligned} \tag{4.4}$$

4.1. Sum of volumes distance.

Theorem 4.3. *Sum of volumes distance:*

$$v_c = d_c^n = \sum_{i=1}^m v_{c_i} \quad \Leftrightarrow \quad d^n = \sum_{i=1}^m (\prod_{j=1}^n s_{i_j}).$$

The formal proof, "sum_of_volumes_distance," is in the Rocq file, euclidrelations.v.

Proof. From lemma 4.2 and the Euclidean volume theorem 3.2:

$$\begin{aligned} &v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad \Leftrightarrow \quad d^n = \sum_{i=1}^m (\prod_{j=1}^n v_i) \quad \wedge \quad v_i = \prod_{j=1}^n s_{i_j} \\ &v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad \Leftrightarrow \quad d^n = \sum_{i=1}^m (\prod_{j=1}^n s_{i_j}). \quad \square \end{aligned} \tag{4.5}$$

4.2. Minkowski distance (p-norm).

Theorem 4.4. *Minkowski distance (p-norm):*

$$v_c = d_c^n = \sum_{i=1}^m v_{c_i} = \sum_{i=1}^m d_{c_i}^n \quad \Leftrightarrow \quad d^n = \sum_{i=1}^m d_i^n.$$

The formal proof, "Minkowski_distance," is in the Rocq file, euclidrelations.v.

Proof. From lemma 4.2 and the Euclidean volume theorem 3.2:

$$\begin{aligned} &v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad \Leftrightarrow \quad d^n = \sum_{i=1}^m v_i \quad \wedge \quad v_i = \prod_{j=1}^n d_i = d_i^n \\ &v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad \Leftrightarrow \quad d^n = \sum_{i=1}^m d^n. \quad \square \end{aligned} \tag{4.6}$$

4.3. Distance inequality.

The formal proof, distance_inequality, is in the Rocq file, euclidrelations.v.

Theorem 4.5. *Distance inequality*

$$\forall\, n \in \mathbb{N},\;\; v_a, v_b \geq 0 :\;\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

Proof. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$\begin{aligned} &\forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq v_a + v_b + \\ &\textstyle\sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n. \end{aligned} \tag{4.7}$$

Take the n^{th} root of both sides of the inequality 4.7:

$$\forall\, v_a, v_b \geq 0,\; n \in \mathbb{N} :\; v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \;\Rightarrow\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \tag{4.8}$$

$\square$

4.4. **Distance sum inequality.** The formal proof, distance_sum_inequality, is in the Rocq file, euclidrelations.v.

Theorem 4.6. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N},\;\; a_i, b_i \geq 0 :\;\; (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

Proof. Apply the distance inequality (4.5):

$$\begin{aligned} &\forall\, m, n \in \mathbb{N},\;\; v_a, v_b \geq 0 : \quad v_a = \textstyle\sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge \\ &\qquad (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad ((\textstyle\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} = \\ &\qquad\qquad (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \quad \square \end{aligned} \tag{4.9}$$

4.5. **Metric Space.** All Minkowski distances (p-norms) imply the metric space properties. The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Rocq file, euclidrelations.v.

Theorem 4.7. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$

Proof. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.10}$$

Apply the distance inequality (4.5) to the inequality 4.10:

$$\begin{aligned} &(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \;\;\wedge\;\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ &\qquad\qquad \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ &\Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ &\qquad\qquad \Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq \\ &\qquad\qquad\qquad (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \quad \square \end{aligned} \tag{4.11}$$

Theorem 4.8. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

Proof. By the commutative law of addition:

$$\begin{aligned} &\forall\, p \,:\, p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ &\qquad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \quad \square \end{aligned} \tag{4.12}$$

Theorem 4.9. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

Proof. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\; [a_2, b_2], \quad u = b_1 - a_1,\; v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\; v \geq 0. \tag{4.13}$$

$$p \geq 1,\;\; u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \tag{4.14}$$

$\square$

Theorem 4.10. *Identity of Indiscernibles:* $d(u, u) = 0$.

Proof. From the non-negativity property (4.9):

$$\begin{aligned} &d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0 \\ &\qquad\qquad \Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0. \end{aligned} \tag{4.15}$$

$$d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v. \tag{4.16}$$

$$d(u,v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u,u) = 0. \tag{4.17}$$

□

4.6. **Set properties limiting a set to at most 3 members.** The following definitions and proof use first order logic. A Horn clause-like expression is used, here, to make the proof easier to read. By convention, the proof goal is on the left side and supporting facts are on the right side of the implication sign (←). The formal proofs in the Rocq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

Definition 4.11. Immediate Cyclic Successor of m is n:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\} : \\ Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize). \tag{4.18}$$

Definition 4.12. Immediate Cyclic Predecessor of m is n:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\} : \\ Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1). \tag{4.19}$$

Definition 4.13. Adjacent: Member m is sequentially adjacent to member n if the immediate cyclic successor of m is n or the immediate cyclic predecessor of m is n. Notionally:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\} : \\ Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{4.20}$$

Definition 4.14. Immediate Symmetric (every set member is sequentially adjacent to every other member):

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\} : \quad Adjacent(m, n, setsize). \tag{4.21}$$

Theorem 4.15. *An immediate symmetric cyclic set is limited to at most 3 members.*

Proof.
Every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (m = setsize \wedge n = 1). \tag{4.22}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (n = m + 1 \leq setsize). \tag{4.23}$$

$$Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (n = setsize \wedge m = 1). \tag{4.24}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m + 1 \leq setsize). \tag{4.25}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m - 1 \geq 1). \tag{4.26}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \wedge m = 1). \tag{4.27}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \wedge n = setsize). \tag{4.28}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m + 1 \leq setsize). \tag{4.29}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m - 1 \geq 1). \tag{4.30}$$

Member 2 is the only immediate successor of member 1 for all $setsize \geq 3$, which implies member 3 is not ($\neg$) an immediate successor of member 1 for all $setsize \geq 3$:

$$\begin{aligned}&\neg Successor(1, 3, setsize \geq 3)\\ &\qquad\leftarrow Successor(1, 2, setsize \geq 3) \leftarrow (n = m + 1 \leq setsize). \end{aligned}\tag{4.31}$$

Member $n = setsize > 3$ is the only immediate predecessor of member 1, which implies member 3 is not ($\neg$) an immediate predecessor of member 1 for all $setsize > 3$:

$$\begin{aligned}&\neg Predecessor(1, 3, setsize \geq 3)\\ &\quad\leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \end{aligned}\tag{4.32}$$

For all $setsize > 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not immediate symmetric):

$$\begin{aligned}&\neg Adjacent(1, 3, setsize > 3)\\ &\quad\leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \end{aligned}\tag{4.33}$$

The Symmetric goal matches Adjacent goals 4.22 and fails for all "setsize" greater than three.

5. Applications to physics

Where distance is an immediate cyclic set of dimensions, the 3D proof requires more dimensions to have non-distance types, where for each unit length, r_p, of distance interval length, r, there are unit lengths: t_p of time interval length, t; m_p of mass interval length, m; and q_p of charge interval length, q, such that:

$$r = (r_p/t_p)t = (r_p/m_p)m = (r_p/q_p)q. \tag{5.1}$$

5.1. Derivation of the constant, G, and the gravity laws of Newton, Gauss, and Poisson.

From equation 5.1:

$$\begin{aligned}&r = (r_p/m_p)m \quad \wedge \quad r = (r_p/t_p)t = ct \quad \Rightarrow \quad r/(ct)^2 = (r_p/m_p)m/r^2\\ &\qquad\qquad \Rightarrow \quad r/t^2 = ((r_p/m_p)c^2)m/r^2 = Gm/r^2, \end{aligned}\tag{5.2}$$

where the constant, $G = (r_p/m_p)c^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$ [13].

Newton's law follows from multiplying both sides of equation 5.2 by m:

$$r/t^2 = Gm/r^2 \;\Leftrightarrow\; F := mr/t^2 = Gm^2/r^2. \tag{5.3}$$

$$F = Gm^2/r^2 \wedge \forall\, m \in \mathbb{R} : \exists\, m_1, m_2 \in \mathbb{R} : m_1 m_2 = m^2 \;\Rightarrow\; F = Gm_1 m_2/r^2. \tag{5.4}$$

From equation 5.2, Gauss's gravity field, $\mathbf{g}$ and Poisson's gravity field, $\nabla\Phi(r, t)$:

$$\begin{aligned}&\mathbf{g} = -\nabla\Phi(\vec{r}, t) := r/t^2 = Gm/r^2\\ &\quad\Rightarrow \quad \nabla \cdot \mathbf{g} = \nabla^2\Phi(\vec{r}, t) = -2Gm/r^3 = (-2Gm/r^3)(2\pi/2\pi) \quad \wedge \quad \rho = m/2\pi r^3\\ &\qquad\qquad \Rightarrow \quad \nabla \cdot \mathbf{g} = \nabla^2\Phi(\vec{r}, t) = -4\pi G\rho. \end{aligned}\tag{5.5}$$

5.2. Derivation of Coulomb's charge constant, k_e and charge force.

$$\forall\, q \in \mathbb{R}:\ \exists\, q_1, q_2 \in \mathbb{R}:\ q_1 q_2 = q^2 \quad \wedge \quad r = (r_p/q_p)q$$
$$\Rightarrow \quad \exists\, q_1, q_2 \in \mathbb{R}:\ q_1 q_2 = q^2 = ((q_p/r_p)r)^2 \quad \Rightarrow \quad (r_p/q_p)^2 q_1 q_2/r^2 = 1. \quad (5.6)$$

$$r = (r_p/t_p)t = ct \quad \wedge \quad r = (r_p/m_p)m = ct$$
$$\Rightarrow \quad mr = (m_p/r_p)rct = (m_p/r_p)(ct)^2 \quad \Rightarrow \quad ((r_p/m_p)/c^2)mr/t^2 = 1. \quad (5.7)$$

$$((r_p/m_p)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_p/q_p)^2 q_1 q_2/r^2 = 1$$
$$\Rightarrow \quad F := mr/t^2 = ((m_p/r_p)c^2)(r_p/q_p)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2. \quad (5.8)$$

where Coulomb's constant, $k_e = ((m_p/r_p)c^2)(r_p/q_p)^2$, conforms to the base SI units: $kg \cdot m^3 \cdot s^{-2} \cdot C^{-2}$, which is equivalent to the charge SI units: $N \cdot m^2 \cdot C^{-2}$ [7].

5.3. Vacuum permittivity, ε_0, and Gauss's law for electric fields.

From Coulomb's charge force equation 5.8:

$$\exists\, q, E \in \mathbb{R}:\ F = k_e q_1 q_2/r^2 = k_e q^2/r^2 := qE$$
$$\Rightarrow \quad E = k_e q/r^2 \quad \Leftrightarrow \quad \mathbf{E} = k_e q/\vec{\mathbf{r}}^2, \quad (5.9)$$

where E has the SI units $N \cdot C^{-1}$.

$$\mathbf{E} = k_e q/\vec{\mathbf{r}}^2 \quad \Rightarrow \quad \nabla \cdot \mathbf{E} = -2k_e q/\vec{\mathbf{r}}^3. \quad (5.10)$$

$$\nabla \cdot \mathbf{E} = -(2k_e q/\vec{\mathbf{r}}^3)(2\pi/2\pi) \quad \wedge \quad \rho = q/2\pi\vec{\mathbf{r}}^3 \quad \wedge \quad \varepsilon_0 := 1/4\pi k_e$$
$$\Rightarrow \quad \nabla \cdot \mathbf{E} = -4\pi k_e \rho = -\rho/\varepsilon_0, \quad (5.11)$$

which is Gauss's electric field law [7].

5.4. Magnetic field.

From Coulomb's charge force equation 5.8:

$$\exists\, q \in \mathbb{R}:\ F = k_e q_1 q_2/r^2 = k_e q^2/r^2 \quad \wedge \quad r = (r_p/t_p)t = ct$$
$$\Rightarrow \quad Fct = r(k_e q^2/r^2) \quad \Leftrightarrow \quad F = (r/ct)(k_e q^2/r^2). \quad (5.12)$$

For a unit radius around a point charge, the torque force, $F_T = F\ sin(\theta)$:

$$F = (r/ct)(k_e q^2/r^2) \quad \wedge \quad F_T = F\ sin(\theta) \quad \wedge \quad v = (r\ sin(\theta)/t)$$
$$\Rightarrow \quad F_T = (r/ct)(k_e q^2/r^2)\ sin(\theta) = (r\ sin(\theta)/t)((k_e/c)q^2/r^2)$$
$$= v(k_e/c)q^2/r^2 := qvB \quad \Leftrightarrow \quad \mathbf{F}_T = q\vec{\mathbf{v}} \times \mathbf{B}, \quad (5.13)$$

where v is the point charge speed, c is the speed of light, and $B = (k_e/c)q/r^2$, conforms to the base SI units: $kg \cdot s^{-1} \cdot C^{-1} \;=\; kg \cdot s^{-2} \cdot A^{-1} \;=\; T$.

5.5. Lorentz's force law.

Summing Gauss's electric force equation 5.9 plus the magnetic force equation 5.13 yields the Lorentz law:

$$\mathbf{F}_E = q\mathbf{E} \quad \wedge \quad \mathbf{F}_B = q\vec{\mathbf{v}} \times \mathbf{B} \quad \Rightarrow \quad \mathbf{F}_{total} = \mathbf{F}_E + \mathbf{F}_B = q(\mathbf{E} + \vec{\mathbf{v}} \times \mathbf{B}). \quad (5.14)$$

5.6. **Vacuum permeability, μ_0, and Faraday's law.** From the magnetic field equation 5.13:

$$\mathbf{B} = (k_e/c)q/r^2 \quad \wedge \quad r = ct \quad \Rightarrow \quad \mathbf{B} = (k_e/c^3)q/t^2. \tag{5.15}$$

$$\mathbf{B} = (k_e/c^3)q/t^2 \quad \Rightarrow \quad \partial\mathbf{B}/\partial t = -(2k_e/c^3)q/t^3. \tag{5.16}$$

$$\partial\mathbf{B}/\partial t = -(2k_e/c^3)q/t^3 \quad \wedge \quad r = ct \quad \Rightarrow \quad \partial\mathbf{B}/\partial t = -2k_e q/r^3. \tag{5.17}$$

From equation 5.9:

$$\mathbf{E} = k_e q/r^2 \quad \Rightarrow \quad \nabla \times \mathbf{E} = 2k_e q/r^3. \tag{5.18}$$

Combining equations 5.18 and 5.17 yields Faraday's law [7]:

$$\nabla \times \mathbf{E} = 2k_e q/r^3 \quad \wedge \quad \partial\mathbf{B}/\partial t = -2k_e q/r^3 \quad \Rightarrow \quad \nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t. \tag{5.19}$$

$$\partial\mathbf{B}/\partial t = -(2k_e q/r^3)(2\pi/2\pi) \quad \wedge \quad \rho = q/2\pi r^3 \quad \wedge \quad \mu_0 := 4\pi k_e/c^2$$
$$\Rightarrow \quad \partial\mathbf{B}/\partial t = -4\pi k_e \rho = -\mu_0 \rho. \tag{5.20}$$

5.7. **Space-time-mass-charge.** Let r be an Euclidean distance. Then by the Minkowski distance theorem (4.4), $r^2 = \sum_{i=1}^{m} r_i^2$. Let, $r' = r_1$ and $r_v = (\sum_{i=1}^{m-1} r_i^2)$. From equation 5.1, there are ratios μ and ν such that:

$$\forall\, \tau \in \{t, m, q\},\ r^2 = r'^2 + r_v^2,\ \exists\, \mu, \nu : \quad \wedge \quad r = \mu\tau \quad \wedge \quad r_v = \nu\tau$$
$$\Rightarrow \quad (\mu t)^2 = r'^2 + (\nu\tau)^2 \quad \Rightarrow \quad r' = \sqrt{(\mu\tau)^2 - (\nu\tau)^2} \ = \ \mu\tau\sqrt{1 - (\nu/\mu)^2}. \tag{5.21}$$

Rest frame distance, r', contracts relative to stationary frame distance, r, as $\nu \to \mu$:

$$r' = \mu\tau\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad \mu\tau = r \quad \Rightarrow \quad r' = r\sqrt{1 - (\nu/\mu)^2}. \tag{5.22}$$

Stationary frame type, τ, dilates relative to the rest frame type, τ', as $\nu \to \mu$:

$$\mu\tau = r'/\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad r' = \mu\tau' \quad \Rightarrow \quad \tau = \tau'/\sqrt{1 - (\nu/\mu)^2}. \tag{5.23}$$

Where τ is type, time, the space-like flat Minkowski spacetime event interval is:

$$\mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_v^2 \quad \wedge \quad \mathrm{d}r_v^2 = \mathrm{d}r_1^2 + \mathrm{d}r_2^2 + \mathrm{d}r_3^2 \quad \wedge \quad \mathrm{d}(\mu\tau) = \mathrm{d}r$$
$$\Rightarrow \quad \mathrm{d}r'^2 = \mathrm{d}(\mu\tau)^2 - \mathrm{d}r_1^2 - \mathrm{d}r_2^2 - \mathrm{d}r_3^2. \tag{5.24}$$

5.8. **Derivation of Schwarzschild's gravitational time dilation and black hole metric.** [17] [1] From equations 5.22 and 5.1:

$$\sqrt{1 - (v^2/c^2)} = \sqrt{1 - (v^2/c^2)(r/r)} \quad \wedge \quad r = (r_p/m_p)m$$
$$\Rightarrow \quad \sqrt{1 - (v^2/c^2)} = \sqrt{1 - ((r_p/m_p)m)v^2/rc^2}. \tag{5.25}$$

Where v_{escape} is the escape velocity:

$$\sqrt{1 - (v^2/c^2)} = \sqrt{1 - ((r_p/m_p)m)v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv_{escape}^2$$
$$\Rightarrow \quad \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2(r_p/m_p)mv_{escape}^2/rc^2}. \tag{5.26}$$

$$\sqrt{1 - (v^2/c^2)} = \lim_{v_{escape} \to c} \sqrt{1 - 2(r_p/m_p)mv_{escape}^2/rc^2}$$
$$= \sqrt{1 - 2(r_p/m_p)mc^2/rc^2}. \tag{5.27}$$

Combining equation 5.27 with the derivation of G (5.4):

$$(r_p/m_p)c^2 = G \quad \wedge \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2(r_p/m_p)mc^2/rc^2} \\ \Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2Gm/rc^2}. \quad (5.28)$$

Combining equation 5.28 with equation 5.23 yields Schwarzschild's gravitational time dilation:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-2Gm/rc^2} \quad \wedge \quad t' = t\sqrt{1-(v^2/c^2)} \\ \Rightarrow \quad t' = t\sqrt{1-2Gm/rc^2}. \quad (5.29)$$

Schwarzschild defined the black hole event horizon radius, $r_s := 2Gm/c^2$. From equations 5.22 and 5.30:

$$r' = r\sqrt{1-(v/c)^2} \quad \wedge \quad \sqrt{1-(v/c)^2} = \sqrt{1-2Gm/rc^2} \\ \Rightarrow \quad r' = r\sqrt{1-2Gm/rc^2} = r\sqrt{1-r_s/r}. \quad (5.30)$$

Using the time-like spacetime interval, where $\mathrm{d}s^2$ is negative:

$$r' = r\sqrt{1-r_s/r} \quad \wedge \quad \mathrm{d}s^2 = \mathrm{d}r'^2 - \mathrm{d}r^2 \\ \Rightarrow \quad \mathrm{d}s^2 = (\sqrt{1-r_s/r}\mathrm{d}r')^2 - (\mathrm{d}r/\sqrt{1-r_s/r})^2 = (1-r_s/r)\mathrm{d}r'^2 - (1-r_s/r)^{-1}\mathrm{d}r^2. \quad (5.31)$$

$$\mathrm{d}s^2 = (1-r_s/r)\mathrm{d}r'^2 - (1-r_s/r)^{-1}\mathrm{d}r^2 \quad \wedge \quad \mathrm{d}r' = \mathrm{d}(ct) \quad \wedge \quad c = 1 \\ \Rightarrow \quad \mathrm{d}s^2 = (1-r_s/r)\mathrm{d}t^2 - (1-r_s/r)^{-1}\mathrm{d}r^2. \quad (5.32)$$

Translating from 2D to 4D yields Schwarzschild's black hole metric:

$$\mathrm{d}s^2 = (1-r_s/r)\mathrm{d}t^2 - (1-r_s/r)^{-1}\mathrm{d}r^2 = f(r,t) \\ \Rightarrow \quad \mathrm{d}s^2 = (1-r_s/r)\mathrm{d}t^2 - (1-r_s/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + sin^2\theta\mathrm{d}\phi^2) = f(r,t,\theta,\phi) \\ \Rightarrow \quad g_{\mu,\nu} = diag[1-r_s/r),\ (1-r_s/r)^{-1},\ r^2(\mathrm{d}\theta^2),\ r^2(sin^2\theta\mathrm{d}\phi^2)]. \quad (5.33)$$

5.9. **Simplifying Einstein's general relativity (field) equation.** Step 1) Use the ratios to define functions returning scalar values for each component of the metric, $g_{\nu,\mu}$, in Einstein's field equations [6] [18]: All functions derived from the ratios, where the units on each side of the equation balance, are valid metrics, for example, the previous Schwarzschild black hole metric derivation using the ratios (5.8).

Step 2) Express the EFE as 2D tensors: As shown in equation 5.33, the Schwarzschild metric was first derived as a 2D metric and then expanded to a 4D metric. Further, the 4D flat spacetime interval equation (5.24) is an instance of the 2D equation, $\mathrm{d}r'^2 = \mathrm{d}(ct)^2 - \mathrm{d}r_v^2$, where $\mathrm{d}r_v^2$ is the magnitude of a 3-dimensional vector.

The 2D metric tensor allows using the much simpler 2D Ricci curvature and scalar curvature. And the 2D tensors reduce the number of independent equations to solve.

Step 3) One simple method to translate from 2D to 4D is to use spherical coordinates, where r and t remain unchanged and two added dimensions are the angles, ϕ, and θ. For example, the 2D Schwarzschild metric was translated to 4D using this method in equation 5.33.

5.10. **3 fundamental direct proportion ratios.** c_t, c_m, and c_q:

$$c_t = r_p/t_p \approx 2.99792458 \cdot 10^8 m\ s^{-1}. \tag{5.34}$$

$$G = (r_p/m_p)c_t^2 = c_m c_t^2 \quad \Rightarrow \quad c_m = r_p/m_p \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}. \tag{5.35}$$

$$k_e = (c_t^2/c_m)(r_p/q_p)^2 \quad \Rightarrow \quad c_q = r_p/q_p \approx 8.6175172023 \cdot 10^{-18} m\ C^{-1}. \tag{5.36}$$

5.11. **3 fundamental inverse proportion ratios.** k_t, k_m, and k_q:

$$r/t = r_p/t_p, \quad r/m = r_p/m_p \quad \Rightarrow \quad (r/t)/(r/m) = (r_p/t_p)/(r_p/m_p) \quad \Rightarrow$$
$$(mr)/(tr) = (m_p r_p)/(t_p r_p) \quad \Rightarrow \quad mr = m_p r_p = k_m, \quad tr = t_p r_p = k_t. \tag{5.37}$$

$$r/t = r_p/t_p, \quad r/q = r_p/q_p \quad \Rightarrow \quad (r/t)/(r/q) = (r_p/t_p)/(r_p/q_p) \quad \Rightarrow$$
$$(qr)/(tr) = (q_p r_p)/(t_p r_p) \quad \Rightarrow \quad qr = q_p r_p = k_q, \quad tr = t_p r_p = k_t. \tag{5.38}$$

$$k_m = m_p r_p = \hbar/c \approx 3.51767291 \cdot 10^{-43}\ kg\ m. \tag{5.39}$$
$$k_t = t_p r_p = k_m c_m/c_t \approx 8.71362910 \cdot 10^{-79}\ s\ m. \tag{5.40}$$
$$k_q = q_p r_p = k_t c_t/c_q \approx 3.03136069 \cdot 10^{-53}\ C\ m. \tag{5.41}$$

5.12. **4 quantum units.** Distance (r_p), time (t_p), mass (m_p), and charge (q_p):

$$r_p = \sqrt{r_p^2} = \sqrt{c_t k_t} = \sqrt{c_m k_m} = \sqrt{c_q k_q} \approx 1.6162550244 \cdot 10^{-35}\ m. \tag{5.42}$$
$$t_p = r_p/c_t \approx 5.3912464472 \cdot 10^{-44}\ s. \tag{5.43}$$
$$m_p = r_p/c_m \approx 2.176434343 \cdot 10^{-8}\ kg. \tag{5.44}$$
$$q_p = r_p/c_q \approx 1.875546038 \cdot 10^{-18}\ C. \tag{5.45}$$

5.13. **Subtype ratios.** $\frac{F_1}{F_2} = \frac{K\tau_1^2/r^2}{K\tau_2^2/r^2} = \frac{\tau_1^2}{\tau_2^2}$. Where q_e is the elementary (electron) charge ($1.60217663 \cdot 10^{-19}\ C$), the fine structure constant, α is:

$$\alpha = q_e^2/q_p^2 \approx 0.0072973526. \tag{5.46}$$

5.14. **Planck relation and constant, h.** [9] Applying both the direct proportion ratio (5.34), and inverse proportion ratio (5.37):

$$r = ct \quad \wedge \quad m = k_m/r \quad \Rightarrow \quad m(ct)^2 = (k_m/r)r^2 = k_m r. \tag{5.47}$$

$$m(ct)^2 = k_m r \quad \wedge \quad r/t = r_p/t_p = c$$
$$\Rightarrow \quad E := mc^2 = k_m r/t^2 = (k_m(r/t))\,(1/t) = (k_m c)(1/t) = \hbar f, \tag{5.48}$$

where $\hbar = k_m c$ is the *reduced* Planck constant and the frequency, $f = 1/t$.

5.15. **Compton wavelength.** [9] From equations 5.37 and 5.48:

$$h = k_m c/2\pi \ \wedge\ mr = 2\pi k_m \ \Rightarrow\ r = 2\pi k_m/m = (2\pi k_m/m)(c/c) = h/mc. \tag{5.49}$$

5.16. **Schrödenger's equation.** Start with the previously derived Planck relation 5.48 and multiply the kinetic energy component by mc/mc:

$$\hbar/t = mc^2 \Rightarrow \exists V(r,t) : \hbar/t = \hbar/2t + V(r,t) \Rightarrow \hbar/t = hmc/2mct + V(r,t). \quad (5.50)$$

And from the distance-to-time (speed of light) ratio (5.34):

$$\hbar/t = hmc/2mct + V(r,t) \ \wedge \ r = ct \ \Rightarrow \ \hbar/t = hmc^2/2mcr + V(r,t). \quad (5.51)$$

$$\hbar/t = \hbar mc^2/2mcr + V(r,t) \ \wedge \ \hbar/t = mc^2 \ \Rightarrow \ \hbar/t = \hbar^2/2mcrt + V(r,t). \quad (5.52)$$

$$\hbar/t = \hbar^2/2mcrt + V(r,t) \ \wedge \ r = ct \ \Rightarrow \ \hbar/t = \hbar^2/2mr^2 + V(r,t). \quad (5.53)$$

Multiply both sides of equation 5.53 by a function, $\Psi(r,t)$.

$$\hbar/t = \hbar^2/2mr^2 + V(r,t) \\ \Rightarrow \quad (\hbar/t)\Psi(r,t) = (\hbar^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t). \quad (5.54)$$

$$(\hbar/t)\Psi(r,t) = (\hbar^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t) \quad \wedge \\ \forall\ \Psi(r,t) :\ \partial^2\Psi(r,t)/\partial r^2 = (-1/r^2)\Psi(r,t) \quad \wedge \quad \partial\Psi(r,t)/\partial t = (i\ 1/t)\Psi(r,t) \\ \Rightarrow \quad i\hbar\partial\Psi(r,t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t), \quad (5.55)$$

which is the one dimensional position-space Schrödenger's equation [16] [9].

$$i\hbar\partial\Psi(r,t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t) \quad \wedge \quad ||\vec{\mathbf{r}}|| = r \\ \Rightarrow \quad \exists\ \vec{\mathbf{r}} :\ i\hbar\partial\Psi(\vec{\mathbf{r}},t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(\vec{\mathbf{r}},t)/\partial\vec{\mathbf{r}}^2 + V(\vec{\mathbf{r}},t)\Psi(\vec{\mathbf{r}},t), \quad (5.56)$$

which is the position-space Schrödenger's equation in 3 dimensions of space [16] [9].

5.17. **Dirac's wave equation.** Using the derived Planck relation 5.48:

$$mc^2 = \hbar/t \quad \Rightarrow \quad \exists\ V(r,t) :\ mc^2/2 + V(r,t) = \hbar/t \\ \Rightarrow \quad 2\hbar/t - 2V(r,t) = mc^2. \quad (5.57)$$

$$\forall\ V(r,t) :\ V(r,t) = i\hbar/t \quad \wedge \quad r = ct \quad \wedge \quad 2\hbar/t - 2V(r,t) = mc^2 \\ \Rightarrow \quad 2\hbar/t - i2\hbar c/r = mc^2. \quad (5.58)$$

Use the charge ratio, c_q, and time ratio, $c_t = c$ to multiply each term on the left side of equation 5.58 by 1:

$$qc_q/r = qc_q/ct = 1 \quad \wedge \quad 2\hbar/t - i2\hbar c/r = mc^2 \\ \Rightarrow \quad 2\hbar(-qc_q/c)/t^2 - i2\hbar((-qc_q/c)/r^2)c = mc^2. \quad (5.59)$$

where a negative sign is added to q to indicate an attractive force between an electron and a nucleus.

Applying a quantum amplitude equation in complex form to equation 5.60:

$$A_0 = (c_q/c)((1/t)) - i(1/r)) \ \wedge \ 2\hbar(-qc_q/c)/t^2 - i2\hbar((-qc_q/c)/r^2)c = mc^2 \\ \Rightarrow \quad 2\hbar\partial(-qA_0)/\partial t - i2\hbar(\partial(-qA_0)/\partial r)c = mc^2. \quad (5.60)$$

Translating equation 5.60 to moving coordinates via the Lorentz factor, $\gamma_0 = 1/\sqrt{1-(v/c)^2}$:

$$2\hbar\partial(-qA_0)/\partial t - i2\hbar(\partial(-qA_0)/\partial r)c = mc^2 \\ \Rightarrow \quad \gamma_0 2\hbar\partial(-qA_0)/\partial t - \gamma_0 i2\hbar(\partial(-qA_0)/\partial r)c = mc^2. \quad (5.61)$$

Multiplying both sides of equation 5.61 by $\Psi(r,t)$:

$$\gamma_0 2\hbar\partial(-qA_0)/\partial t - \gamma_0 i2\hbar(\partial(-qA_0)/\partial r)c = mc^2 \\ \Rightarrow \quad \gamma_0 2\hbar(\partial(-qA_0)/\partial t)\Psi(r,t) - \gamma_0 i2\hbar(\partial(-qA_0)/\partial r)c\Psi(r,t) = mc^2\Psi(r,t). \quad (5.62)$$

Applying the vectors to equation 5.62 yields Dirac's wave equation [5] [9]:

$$\gamma_0 2\hbar(\partial(-qA_0)/\partial t)\Psi(r,t) - \gamma_0 i2\hbar(\partial(-qA_0)/\partial r)c\Psi(r,t) = mc^2\Psi(r,t) \wedge \\ ||\vec{\mathbf{r}}|| = r \quad \wedge \quad ||\vec{\mathbf{A}}|| = A_0 \quad \wedge \quad ||\vec{\gamma}|| = \gamma_0 \quad \wedge \quad \Leftrightarrow \quad \exists\, \vec{\mathbf{r}}, \vec{\mathbf{A}}, \vec{\gamma} : \\ \gamma_0 2\hbar(\partial(-qA_0)/\partial t)\Psi(r,t) - \vec{\gamma}\cdot i2\hbar(\partial(-q\vec{\mathbf{A}})/\partial r)c\Psi(\vec{\mathbf{r}},t) = mc^2\Psi(\vec{\mathbf{r}},t). \quad (5.63)$$

5.18. **Total mass.** The total mass of a particle is $m = \sqrt{m_0^2 + m_{ke}^2}$, where m_0 is the rest mass and m_{ke} is the kinetic energy-equivalent mass. Applying both the direct (5.34) and inverse proportion ratios (5.37):

$$m_0 = r/(r_p/m_p) = r/c_m \quad \wedge \quad m_{ke} = (m_p r_p)/r = k_m/r \quad \wedge \\ m = \sqrt{m_0^2 + m_{ke}^2} \quad \Rightarrow \quad m = \sqrt{(r/c_m)^2 + (k_m/r)^2}. \quad (5.64)$$

5.19. **Quantum extension to general relativity.** The simplest way to demonstrate how to add quantum physics to general relativity is by extending Schwarzschild's time dilation equation and black hole metric (5.8). Start by changing equation 5.25 in the Schwarzschild derivation:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-(v^2/c^2)(r/r)} \wedge r = \sqrt{(c_m m)^2 + (k_m/m)^2} = Q_m \\ \Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1 - Q_m v^2/rc^2}. \quad (5.65)$$

$$\sqrt{1-(v^2/c^2)} = \sqrt{1 - Q_m v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv_{escape}^2 \\ \Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1 - 2Q_m v_{escape}^2/rc^2}. \quad (5.66)$$

$$\sqrt{1-(v^2/c^2)} = \lim_{v_{escape}\to c} \sqrt{1 - 2Q_m v_{escape}^2/rc^2} \\ \Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1 - 2Q_m c^2/rc^2} = \sqrt{1 - 2Q_m/r}. \quad (5.67)$$

Combining equation 5.67 with equation 5.23 yields Schwarzschild's gravitational time dilation with a quantum mass effect:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1 - 2Q_m/r} \quad \wedge \quad t' = t\sqrt{1-(v^2/c^2)} \\ \Rightarrow \quad t' = t\sqrt{1 - 2Q_m/r}. \quad (5.68)$$

Schwarzschild defined the black hole event horizon radius, $r_s := 2Gm/c^2$. The radius with the quantum extension is $r_s := 2Q_m$. At this point the exact same equations 5.30 through 5.33 yield what looks like the same Schwarzschild black hole metric.

5.20. **Quantum extension to Newton's gravity force.** The quantum mass effect is easier to understand in the context Newton's gravity equation than in general relativity, because the metric equations and solutions in the EFEs are much more complex. From equations 5.69 and 5.1:

$$\begin{aligned} &m/\sqrt{(r/c_m)^2+(k_m/r)^2}=1 \quad \wedge \quad r^2/(ct)^2=1 \\ &\qquad \Rightarrow \quad r^2/(ct)^2 = m/\sqrt{(r/c_m)^2+(k_m/r)^2} \\ &\qquad\qquad \Rightarrow \quad r^2/t^2 = c^2 m/\sqrt{(r/c_m)^2+(k_m/r)^2}. \end{aligned} \tag{5.69}$$

$$\begin{aligned} &r^2/t^2 = c^2 m/\sqrt{(r/c_m)^2+(k_m/r)^2} \\ &\qquad \Rightarrow \quad (m/r)(r^2/t^2) = (m/r)(c^2 m/\sqrt{(r/c_m)^2+(k_m/r)^2}) \\ &\Rightarrow \quad F := mr/t^2 = c^2m^2/(r\sqrt{(r/c_m)^2+(k_m/r)^2}) = c^2m^2/\sqrt{(r^4/c_m^2)+k_m^2}. \end{aligned} \tag{5.70}$$

$$\begin{aligned} &F = c^2m^2/\sqrt{(r^4/c_m^2)+k_m^2} \quad \wedge \quad \forall\, m\in\mathbb{R},\ \exists\, m_1,m_2\in\mathbb{R}:\ m_1m_2=m^2 \\ &\qquad\qquad \Rightarrow \quad F = c^2 m_1 m_2/\sqrt{(r^4/c_m^2)+k_m^2}. \end{aligned} \tag{5.71}$$

5.21. **Quantum extension to Coulomb's force.**

$$\begin{aligned} &q^2/((r/c_q)^2+(k_q/r)^2)=1 \quad \wedge \quad r^2/(ct)^2=1 \\ &\qquad \Rightarrow \quad r^2/(ct)^2 = q^2/((r/c_q)^2+(k_q/r)^2) \\ &\qquad\qquad \Rightarrow \quad r^2/t^2 = c^2q^2/((r/c_q)^2+(k_q/r)^2). \end{aligned} \tag{5.72}$$

$$\begin{aligned} &\forall\, q\in\mathbb{R}:\ \exists\, q_1,q_2\in\mathbb{R}:\ q_1q_2=q^2 \quad \wedge \quad r^2/t^2 = c^2q^2/((r/c_q)^2+(k_q/r)^2) \\ &\qquad \Rightarrow \quad \exists\, q_1,q_2\in\mathbb{R}:\ r^2/t^2 = c^2q_1q_2/((r/c_q)^2+(k_q/r)^2) \\ &\qquad\qquad \Rightarrow \quad r/t^2 = c^2q_1q_2/(r((r/c_q)^2+(k_q/r)^2)). \end{aligned} \tag{5.73}$$

$$\begin{aligned} &r/t^2 = c^2q_1q_2/(r((r/c_q)^2+(k_q/r)^2)) \quad \wedge \quad m=\sqrt{(r/c_m)^2+(k_m/r)^2} \\ &\quad \Rightarrow \quad F := mr/t^2 = c^2q_1q_2\sqrt{(r/c_m)^2+(k_m/r)^2}/(r((r/c_q)^2+(k_q/r)^2)) \\ &\qquad\qquad = c^2q_1q_2\sqrt{(r^4/c_m^2)+k_m^2}/((r^4/c_q^2)+k_q^2). \end{aligned} \tag{5.74}$$

6. Insights and implications

(1) The ruler measure (2.1) and convergence theorem (2.2) were shown to be useful tools for proving that a real-valued equation is the only instance of an abstract, countable set relation and that set relation is the only instance of that same real-valued equation.

(2) Combinatorics, the ordered set of combinations of countable, disjoint sets (n-tuples), $v_c = \prod_{i=1}^{n}|x_i|$, was proven to imply the Euclidean volume equation (3.2), the sum of volumes equation (4.3) (which includes the inner product), and the Minkowski distance equation (4.4) (which includes the

Manhattan and Euclidean distance equations), without relying on the geometric primitives and relations in Euclidean geometry [10], axiomatic geometry [11], and vector analysis [18].

(3) Where the total n-volume is both the sum and subtraction of n-volumes, the $n = 2$ case is the vector inner product. The distributive and associate laws of multiplication and addition allow the $\pm$ signed volumes to represented as each domain interval length multiplied by a $\pm$-signed unit values, $\alpha_i, \beta_i \in \{-1, 1\}$:

$$d^2 = \textstyle\sum_{i=1}^{m}(a_i\alpha_i)(b_i\beta_i) := \mathbf{a} \cdot \mathbf{b}. \tag{6.1}$$

(4) Defining all Euclidean and non-Euclidean distance measures as:

$$d: \quad f(d, n) = v = \textstyle\sum_{i=1}^{m} v_i, \quad d = f^{-1}(v, n) = f(\sum_{i=1}^{m} v_i)^{-1}. \tag{6.2}$$

   (a) shows the intimate relation between distance and volume that definitions, like inner product space and metric space, ignore [18] [8] [15];
   (b) is a more simple and concise definition of a distance measure that includes all the properties used in the definitions of inner product space and metric space [18] [8] [15];

(5) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [10]. And analytic proofs sum infinitesimal distances, $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$, where the Euler-Lagrange equation is used find the minimum solution, which is the straight line equation, $y = mx + b$ [2].

   Without using the notion of a straight line: All distance measures in an Euclidean volume have corresponding Minkowski distances (4.4). For all 2-volumes, all Minkowski distances are limited to $n \in \{1, 2\}$: $n = 1$ is the larger (Manhattan) distance case, $d = \sum_{i=1}^{m} s_i$. $n = 2$ is the smaller (Euclidean) distance case, $d = (\sum_{i=1}^{m} s_i^2)^{1/2}$. That is, by the Cauchy-Schwarz inequality: $\sum_{i=1}^{m} s_i \geq (\sum_{i=1}^{m} s_i^2)^{1/2}$. And where $1 \leq n \leq 2$, d decreases monotonically as $n \to 2$.

(6) The left side of the distance sum inequality (4.6),

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}, \tag{6.3}$$

differs from the left side of Minkowski's sum inequality [12]:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \tag{6.4}$$

   (a) The two inequalities are only the same where $n = 1$.
   (b) The distance sum inequality (4.6) is a more fundamental inequality because the proof does not require the convexity and Hölder's inequality assumptions required to prove the Minkowski sum inequality [12].
   (c) The distance sum inequality term, $\forall\, n > 1$: $d = v^{1/n} = (a_i^n + b_i^n)^{1/n}$, is the Minkowski distance spanning the n-volume, $v = a_i^n + b_i^n$, which makes it directly related to geometry (for example, the metric space triangle inequality was derived from the $m = 1$ case for all $n \geq 1$ (4.7)). But the Minkowski sum inequality term, $\forall\, n > 1$: $d = v^{1/n} \neq v = (a_i^n + b_i^n) = ((a_i^n + b_i^n)^{\mathbf{n}})^{1/n}$.

(7) Combinatorics, repeatable sequencing through an ordered set to yield all $n!$ permutations of its members (without jumping around) was proved to

be a cyclic set having $n \leq 3$ members (4.15). Higher dimensions must have different types (members of different sets).

(a) For example, the vector inner product space can only be extended beyond 3 dimensions if and only if the higher dimensions have non-distance types, for example, time.

(b) But order and symmetry probably limit the number of fundamental types to a very small number, for example: time, mass, and charge per dimension of distance.

(c) Each of 3 immediate symmetric cyclic dimensions of space can have at most 3 immediate symmetric cyclic state values, for example, an immediate symmetric cyclic set of 2 vector orientations, $\{-1, 1\}$, per dimension of space and at most 3 spin states per plane, $\{-1, 0, 1\}$.

(d) If the states are not ordered (a bag of states), then a state value is undetermined until observed (like Schrödenger's poisoned cat being both alive and dead until the box is opened [16]). For a bag of states, there is **no** "axiom of choice" [4], an axiom often used in math proofs that allows selecting a particular set element (in this case, selecting a particular state).

(e) A discrete value has measure 0 (no size). The ratio of a time or distance interval length to zero is undefined, which is the reason quantum entangled particles change discrete state values together independent of time and distance.

(8) For each unit, r_p, of a 3-dimensional distance interval having a length, r, there are units of other types of intervals forming unit ratios (5.10): $c_t = r_p/t_p$, $c_m = r_p/m_p$, $c_q = r_p/q_p \Leftrightarrow$ the inverse proportion ratios (5.11): $k_t = r_p t_p$, $k_m = r_p m_p$, $k_t = r_p q_p$.

(9) Empirical and hypothesized laws start with an assumption of the form, $y \propto f(x_1, \cdots, x_n)$, where an **opaque** constant, K, is defined to make an equation, where the units balance, $y = Kf(x_1, \cdots, x_n)$. The opaque constants have led to the assumptions of some constants being fundamental.

In this article, both the equations and constants are derived together from the ratios. As a result, the derivations show that constants, previously thought to be fundamental, are composed of the ratios.

(a) The gravity, $G = c_m c_t^2$ (5.4), charge, $k_e = (c_q^2/c_m)c_t^2$ (5.8), and Planck $h = k_m c_t$ (5.48) constants were all derived directly from (composed of) the ratios. And vacuum permittivity, ε_0 and vacuum permeability, μ_0, are both defined in terms of k_e: $\varepsilon_0 := 1/4\pi k_e$ (5.11) and $\mu_0 := 1/c_t^2\varepsilon_0 = 4\pi k_e/c_t^2$ (5.20).

(b) Further, it will be shown, later in this section, that where the quantum effects become measurable, that the constants G, k_e, ε_0, and μ_0 are no longer valid.

(c) Therefore, G, k_e, ε_0, μ_0, and h are **not** fundamental constants.

(d) Using the ratios instead of empirical constants in equations would show the shared principles underlying the different laws of physics. For example, the speed of light ratio, c_t, is a component of the constants: $G = c_m c_t^2$, $k_e = (c_q^2/c_m)c_t^2$, $\varepsilon_0 = 1/(4\pi(c_q^2/c_m)c_t^2)$, $h = k_m c_t$.

(e) Using the ratios instead of the opaque constants can sometimes simplify equations. For example the derivation of the Compton wavelength equation, $r = h/mc$, (5.15) shows that the computation of the wavelength, r, is overly complex (because it assumes the Planck constant is a fundamental constant) and can be simplified to $r = 2\pi k_m/m$.
(f) Empirical laws, like Newton's gravity law (5.3) and Coulomb's charge law (5.7), and the Planck relation (5.48) are *descriptions* of relations. Deriving the equations from the ratios *explains* the relations. Further, all the other derivations, in this article, were much shorter and simpler than all previous derivations, which shows that the ratios are an important new tool for physicists and engineers.

(10) The derivations of: $\nabla \cdot \mathbf{g} = -4\pi G\rho$ from $\nabla \cdot \mathbf{g} = -2Gm/r^3$ (5.5), $\nabla \cdot \mathbf{E} = -\rho/\varepsilon_0$ (5.11) from $\nabla \cdot \mathbf{E} = -2k_e q/r^3$ (5.10), and $\partial\mathbf{B}/\partial t = -\mu_0\rho$ from $\partial\mathbf{B}/\partial t = -2k_e q/r^3$ (5.20), show that the use of mass and charge density, ρ, and the definitions of ε_0 and μ_0 are unnecessary complications that obfuscate the pattern, $\partial f(x,y,r)/\partial r = -2k_x y/r^3$, and the inverse square pattern, $f(x,y,r) = k_x y/r^2$. Likewise, the $4\pi G$ in $\kappa = 2(4\pi G)/c^4$ and the energy density in the stress-energy tensor, $T_{\mu,\nu}$, in Einstein's field equations [18] also obfuscates the inverse square assumption.

(11) The derivation of the Lorentz (5.14) and Faraday laws (5.19) shows that: $F_E = qE$ and $F_B = qvB = q(v/c)E \Rightarrow c = E/B$.

(12) The mass-charge wavelength, $r = \sqrt{(k_m/m)^2 + (k_q/q)^2}$, has a the mass-charge oriented axis, $\vec{\mathbf{r}}$, is the direction of an elementary particle's translation motion. The following are consequences:
(a) A non-zero mass-charge wavelength implies a particle with a non-zero radius, which implies the possibility on a rotating elementary charged particle, rather than a point particle. Positively and negatively charged particles moving through a charge field in the same direction will experience opposite torque on their axes (like a compass needle).
(b) Particles with the same charge (for example, electrons on a wire) moving in the same direction would create aligned torques on a charged particle moving through the charge field.
(c) Elementary charged particles do not have spin and do not have an intrinsic angular momentum. The particle's mass gives resistance to rotation of the mass-charge axis. As shown in the derivation of Dirac's wave equation (5.17), instead of $\pm\frac{1}{2}$-spin, there is a $\pm\pi$-axis flip.
(d) Positive and negative charged particles moving through a torque (magnetic) field roll off to the right or left due to opposite torque fields, which explains Lorentz's right-hand and left-hand rules for positive and negative charges.

(13) The derivation of Schwarzschild's time dilation and black hole metric (5.8) [17] [1] using ratios:
(a) was much shorter and simpler than Schwarzschild's derivation;
(b) did not require the complexity of manipulating Christoffel symbols, calculating determinants, partial derivatives, etc.;
(c) points to a way of simplifying the finding of solutions to Einstein's field equations: 1) the ratios can be used to derive the components of the metric, $g_{\mu,\nu}$, independent of Einstein's field equations, and 2) the

field equations can be first solved as 2-dimensional tensors and then generalized to 4-dimensional tensors (5.9).

(14) Using the quantum units, r_p and t_p: $r_p/t_p^2 \approx 5.56072628 \cdot 10^{51}\ m\ s^{-2}$, which suggests a maximum acceleration for masses.

(15) The simplification of μ_0 into the quantum units shows two interesting relationships:

$$\mu_0 := \frac{1}{c_t^2 \varepsilon_0} = \frac{4\pi k_e}{c_t^2} = 4\pi \frac{c_q^2}{c_m} = 4\pi \frac{(r_p/q_p)^2}{r_p/m_p} = 4\pi \frac{m_p r_p}{q_p^2} = 4\pi \frac{k_m}{q_p^2}$$
$$\approx 4\pi \frac{3.51767291 \cdot 10^{-43}}{3.51767291 \cdot 10^{-36}} = 4\pi \cdot 10^{-7}\ kg\ m\ C^{-2} = 4\pi \cdot 10^{-7}\ H\ m^{-1}. \quad (6.5)$$

(a) The first time $k_m = m_p r_p$ appears is in the derivation of the Planck relation and Planck constant, $h = k_m c$ (5.14), the second time in the Compton wavelength, $r = k_m/m$ (5.15). And now, k_m appears as a components of k_e and μ_0.

(b) It is an open question why $\frac{c_q^2}{c_m} = \frac{(r_p/q_p)^2}{r_p/m_p} = \frac{k_m}{q_p^2} = 1.0 \cdot 10^{-7}$ exactly.

(16) Two subtypes are related via the ratios of the same super-type (5.13).

(a) The CODATA electron coupling version of the fine structure constant, α is defined as: $\alpha = q_e^2/4\pi\varepsilon_0\hbar c)$ [3].

(i) The derivation of α, in this article (5.13), is much simpler because it is the ratio of two subtypes of charge: elementary (electron) charge, q_e^2 and Planck charge, q_p^2: $\alpha = q_e^2/q_p^2 \approx 0.0072973526$, which is the empirical CODATA value [3].

(ii) The following steps show that the CODATA definition reduces to the ratio-derived equation:

$$\varepsilon_0 := 1/4\pi k_e = 1/(4\pi(c_q^2/c_m)c_t^2) \quad \wedge \quad \hbar = k_m c_t$$
$$\Rightarrow \quad \varepsilon_0 \hbar c = k_m c_t^2/(4\pi(c_q^2/c_m)c_t^2) = k_m/(4\pi(c_q^2/c_m))$$
$$= m_p r_p/(4\pi((r_p/q_p)^2/(r_p/m_p))) = q_p^2/4\pi = q_p^2/2. \quad (6.6)$$

$$\alpha = q_e^2/2\varepsilon_0\hbar c \quad \wedge \quad \varepsilon_0\hbar c = q_p^2/2 \quad \Rightarrow \quad \alpha = q_e^2/q_p^2. \quad (6.7)$$

(iii) As shown above, CODATA defines the fine structure constant in terms of a relationship to the Planck constant, hence, the ratio containing the reduced Planck unit, q_p: $\alpha = q_e^2/q_p^2$.

(iv) Other fine structure constants can also be expressed more simply as the ratios of two subtypes of fields, for example, an electron gravity coupling constant can be expressed as the ratio of a stationary electron mass to a quantum mass unit: $\alpha_m = m_e^2/m_p^2$.

(17) Special and general relativity assume covariance, which states that the laws of physics are invariant in every coordinate frame of reference [6]. Where the space near each coordinate point is Euclidean-like (for example, Riemann and pseudo-Riemann surfaces), the ratios are constant near every coordinate frame of reference. And because the laws of physics are derived from the ratios, the laws of physics are the same near every coordinate frame of reference.

(a) The ratio-based derivations of the spacetime equations, in this article (5.7), do not rely on the Lorentz transformations or Einsteins' postulates [6]. The derivations do not even require the notion of light.
(b) The ratio-based derivations are also valid for spacemass and spacecharge.
(c) The special relativity time dilation equation 5.23 was derived from the distance-to-time ratio, $r = (r_p/t_p)t$, and combined with the distance-to-mass ratio, $r = (r_p/m_p)m$, (5.10) yielded Schwarzschild's gravitational time dilation and black hole metric equations (5.30).

(18) The derivation of Schrödenger (5.16) and Dirac wave equations (5.17), in this article, differs from other derivations:
(a) The derivations in this article use energy-frequency (Planck relation) instead of the energy-momentum (Hamiltonian operator).
(b) $h^2/2m$, was derived, in this article, from the Planck relation (5.53), where the Planck relation was also rigorously derived from the ratios (5.14). Other derivations assume (define) the energy-momentum relation as: $(\mathbf{p} \cdot \mathbf{p})/2m = \hbar^2/2m$ [16] [7].

(19) The quantum extensions to: Schwarzschild's time dilation (5.67) black hole metric (5.33), Newton's gravity force (5.71), and Coulomb's charge force (5.74) make quantifiable predictions:
(a) The gravitation and charge forces peak at finite amounts as $r \to 0$: $\lim_{r \to 0} F = c^2 m_1 m_2 / k_m$ and $\lim_{r \to 0} F = c^2 q_1 q_2 k_m / k_q^2$. Finite maximum gravity and charge forces: 1) allows radioactivity, finite sloped energy well walls, and possibly black hole evaporation; 2) eliminates the problem of forces going to infinity as $r \to 0$.
(b) The Schwarzschild metric, gravity, and charge equations reduce to the classic equations, where the distance between masses and charges is sufficiently large or the masses and charges sufficiently large that the quantum effect is not measurable. **Note** that G, k_e, ε_0, μ_0, and κ (Einstein's constant, which contains G) are not valid, where the quantum effects becomes measurable.
(c) And the covariant tensor components, in Einstein's field equations, that had the units $1/distance^2$, will now have the more complex units, $1/\sqrt{(distance^4/c_m^2) + k_m^2}$.
(d) $1/\sqrt{(distance^4/c_m^2) + k_m^2}$ implies that as distance $\to 0$, spacetime curvature peaks at a finite amount, which predicts that black holes probably have sizes > 0 (are probably not singularities). If there was a "big bang", then it might not have originated from a singularity.

References

[1] S. Antoci and A. Loinger, *An english translation of k. schwarzschild's 1916 article: On the gravitational field of a mass point according to einstein's theory* (1999). https://arxiv.org/abs/physics/9905030. ↑10, 18

[2] F. W. Byron and R. W. Fuller, *Mathematics of classical and quantum physics (dover books on physics)*, Dover Publications, 1992. ↑16

[3] CODATA, *Fine struccture constant*, NIST, 2022. https://physics.nist.gov/cgi-bin/cuu/Value?alph|search_for=fine+structure. ↑19

[4] W. Conradie and V. Goranko, *Logic and discrete mathematics*, Wiley, 2015. ↑17

[5] P.A.M. Dirac, *The principles of quantum mechanics*, Snowball Publishing, 1957. ↑14

[6] A. Einstein, *Relativity, the special and general theory*, Princeton University Press, 2015. ↑11, 19, 20
[7] R. Feynman, R. Leighton, and M. Sands, *The feynman lectures on physics*, California Institute of Technology, 2010. ↑9, 10, 20
[8] R. R. Goldberg, *Methods of real analysis*, John Wiley and Sons, 1976. ↑1, 16
[9] M.C. Jain, *Quantum mechanics: A textbook for undergraduates*, PHI Learning Private Limited, New Delhi, India, 2011. ↑12, 13, 14
[10] D. E. Joyce, *Euclid's elements*, 1998. http://aleph0.clarku.edu/~djoyce/java/elements/elements.html. ↑16
[11] J. M. Lee, *Axiomatic geometry*, American Mathematical Society, 2010. ↑16
[12] H. Minkowski, *Geometrie der zahlen*, Chelsea, 1953. reprint. ↑2, 16
[13] I. Newton, *The mathematical principles of natural philosophy*, Patristic Publishing, 2019. reprint. ↑8
[14] Rocq, *Rocq proof assistant*, 2025. https://rocq-prover.org/docs. ↑2
[15] W. Rudin, *Principles of mathematical analysis*, McGraw Hill Education, 1976. ↑1, 2, 16
[16] E. Schrödenger, *Collected papers on wave mechanics*, Minkowski Institute Press, 2020. ↑13, 17, 20
[17] K. Schwarzschild, *Über das gravitationsfeld eines massenpunktes nach der einsteinschen theorie*, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften **7** (1916), 198–196. https://archive.org/stream/sitzungsberichte1916deutsch#page/188/mode/2up. ↑10, 18
[18] H. Weyl, *Space-time-matter*, Dover Publications Inc, 1952. ↑1, 11, 16, 18

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