SOME SET PROPERTIES UNDERLYING GEOMETRY AND PHYSICS

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Abstract. Some volume and distance equations are derived from ordered sets of combinations (n-tuples). A combinatorial property can limit distance to a set to 3 dimensions. Other dimensions have different types (not members of the distance set) with constant ratios of a distance unit interval length to unit interval lengths of time, mass, and charge. The proofs and ratios are used to: 1) derive well-known gravity, charge, electromagnetic equations, special relativity, Schwarzchild time dilation and metric equations, and quantum physics equations; 2) derive the gravity, charge, vacuum permittivity, permeability, Planck, and fine structure constants; 3) add quantum extensions to gravity and charge equations. All the proofs are verified in Rocq.

Contents

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1. Introduction

Many well-known physics equations either assume an Euclidean vector space or assume that the space near each coordinate point is an Euclidean vector space. For example, Newton's gravity force [14] and Coulomb's charge force [6] equations assume an Euclidean vector space. And, general relativity uses pseudo-Riemann geometry [20], where Riemann and pseudo-Riemann geometries define the space near each coordinate point to be an Euclidean vector space $\subset \mathbb{R}^n$ [13] [20].

Because Euclidean volume and distance permeate math and physics, understanding the combinatorial principles that generate the Euclidean volume and distance equations will provide a deeper understanding of physics. For example, many well-known physics equations will be derived from just the properties of physical space.

The Riemann integral, Lebesgue integral, and Lebesgue measure define Euclidean volume as the product of interval sizes $\subset \mathbb{R}^n$ [7] [16]. And integral calculus and vector analysis define Euclidean distance and the inner product [20] [7] [16]. Because Euclidean volume and distance equations are defined, integral calculus, the Lebesgue measure, and vector analysis cannot be used to derive the Euclidean volume and distance equations.

In this article, a "ruler" measure of intervals $\subset \mathbb{R}$, will be used to prove that abstract, countable, ordered sets of combinations (n-tuples) imply the Euclidean volume equation and some distance equations, including Euclidean distance. The ruler measure makes no Euclidean volume or distance assumptions.

All the proofs in this article have been verified using using the Rocq proof verification system [15]. The formal proofs are in the Rocq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

Where $|x_i|$ is the cardinal of (number of elements in) in a countable set, x_i, the countable number of ordered combinations (n-tuples) is v_c. The ruler measure will be used to prove the Euclidean volume relation:

$$\forall\, x_i.x_j \in \{x_1, \cdots, x_n\},\ i \neq j, \quad x_i \bigcap x_j = \emptyset, \quad v_c = \prod_{i=1}^{n} |x_i| \\ \Rightarrow \quad v = \prod_{i=1}^{n} s_i, \quad s_i = b_i - a_i, \quad [a_i, b_i] \subset \mathbb{R}. \tag{1.1}$$

For all $n > 1$, there are an infinite number of combinations of domain values, $s_1, \cdots, s_n$, that multiplied yield the same range value, v. Inferring a domain value, d, from a volume, v, requires an inverse (bijective) function of volume, where d : $d = f_n^{-1}(v)$ and $v = f_n(d)$. The simplest bijective case for all n extends the $n = 1$ case, $v_c = |x_1| = d_c$:

$$\exists\, d_c, v_c, |x_i| \in \{0, \mathbb{N}\} :\ v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n. \tag{1.2}$$

And all n-volumes, v_c and v, can be expressed as the sum of n-volumes, where the ruler measure will be used to prove that:

$$d_c^n = \sum_{i=1}^{m} v_{c_i} = \sum_{i=1}^{m} (\prod_{j=1}^{n} |x_{i,j}|) \ \Rightarrow\ d^n = \sum_{i=1}^{m} v_i = \sum_{i=1}^{m} (\prod_{j=1}^{n} s_{i,j}). \tag{1.3}$$

The $n = 2$ case is the basis of the inner product. Where each v_{c_i} is also the bijective function, $v_{c_i} = d_{c_i}^n$, the ruler measure will be used to prove that:

$$d_c^n = \sum_{i=1}^{m} d_{c_i}^n \quad \Rightarrow \quad d^n = \sum_{i=1}^{m} d_i^n. \tag{1.4}$$

$|d|$ is the p-norm (Minkowski distance) [12], which will be proved to imply the metric space properties [16]. The $n = 2$ case is, obviously, the Euclidean distance.

Volume and distance are derived from sets of ordered combinations (n-tuples). Volume and distance have another combinatorial (permutation) property.

Calculating volume and distance requires multiplication and addition. The commutative properties of multiplication and addition allows sequencing (multiplying or adding) an ordered set of n number of values in all $n!$ permutations.

Reliably re-sequencing a set of values in the same order requires assigning a sequential order to the values. Further, the *only* sequential order, where you can start with any value and sequence in a repeatable order, is a cyclic order.

Reliably re-sequencing of a cyclic set in all $n!$ permutations, is a symmetry, where every set member is either an *immediate* cyclic successor or an *immediate* cyclic predecessor to every other set member, which is, herein, referred to as an "immediate symmetric" cyclic set (ISCS). First-order logic will be used to prove an ISCS has $n \leq 3$ members.

Application to physics uses the following 3 hypotheses:

(1) **ISCS:** Physical distance is an ISCS, where more than 3 dimensions have non-distance types (are members of other sets), and $\{t\ (time), m\ (mass), q\ (charge)\}$ is the ISCS of "non-distance" dimensions, each dimension $\subseteq \mathbb{R}$.
(2) **Cartesian:** Each local coordinate point is the origin of a Cartesian grid (the space near each coordinate point is Euclidean), where for each unit interval length, r_p, of distance, there is a unit interval length: t_p of time; m_p of mass; and q_p of charge, such that: $r = (r_p/t_p)t = (r_p/m_p)m = (r_p/q_p)q$, where $r_p/t_p = c_t$, $r_p/m_p = c_m$, and $r_p/q_p = c_q$,
(3) **Maximum ratios** c_t, c_m, and c_q are the largest ratios. For example, the ratio, c_t, is the speed of light.

A consequence of the Cartesian hypothesis (2) is that all equations derived from combining the constant ratios are also the same at each coordinate point (covariance).

The Newton's gravity [14] and Coulomb's charge force [6] equations will, later, be derived from the ratios as:

$$F = (c_m c_t^2) m_1 m_2 / r^2 = G m_1 m_2 / r^2 \quad \text{and} \tag{1.5}$$

$$F = (c_q^2 c_t^2 / c_m) q_1 q_2 / r^2 = k_e q_1 q_2 / r^2. \tag{1.6}$$

The ratio, c_m, is calculated from the empirical values of G, and c_t (the speed of light) in Newton's equation. And the ratio, c_q, is calculated from the values of k_e, c_t, and c_m in Coulomb's equation. The derivations from ratios will show that G, k_e, ε_0, μ_0, and $\hbar$ are **not** fundamental (atomic) constants.

Algebraic manipulation of the 3 direct proportion ratios yields 3 inverse proportion ratios, $r = t_p r_p/t = m_p r_p/m = q_p r_p/q$, where $k_t = t_p r_p$, $k_m = m_p r_p$, and $k_q = q_p r_p$. The combination of the direct and inverse proportion ratios are used to derive the Planck relation and the reduced Planck constant, $\hbar = k_m c_t$. The values of k_t, k_m, and k_q are calculated from the values of $\hbar$, c_t, c_m, and c_q.

The values of the units r_p, t_p, m_p, and q_p are calculated from the direct and inverse ratios and found to be the Planck unit values (which is the reason for the "p" subscripts). The fine structure electron coupling constant, α, is derived as the ratio of two forces that reduce to the ratio of subtypes: $\alpha = q_e^2/q_p^2$, which is much simpler and more informative than the standard equation, $\alpha = q_e^2/4\pi\varepsilon_0\hbar c$ [3].

The proofs and the 3 direct proportion ratios are used to provide simple derivations of: the gravitational constant, G, the Newton, Gauss, and Poisson gravity equations, Coulomb's charge force and charge constant, k_e, the special relativity equations, the Schwarzschild time dilation and black hole metric equations (pointing to a simplified method of finding solutions to Einstein's general relativity equations), the Gauss, Lorentz, and Faraday electromagnetic equations, the vacuum permittivity, ε_0, and vacuum permeability, μ_0, constants.

The ratios and Planck relation are used to derive the Compton wavelength, the position-space Schrödenger, and the Dirac wave equations. The inverse proportion ratios are also used to add quantum extensions to some general relativity and classical physics equations. Finally, the derivations expose some incorrect assumptions commonly used in physics today.

2. Ruler measure and convergence

A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of size κ subintervals. The ruler is both an inner and outer measure of an interval.

Definition 2.1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\, [a, b] \subset \mathbb{R}$, $s = b - a \ \wedge\ 0 < \kappa \leq 1 \ \wedge\ (p = floor(s/\kappa) \ \vee\ p = ceiling(s/\kappa))$.

Theorem 2.2. *Ruler convergence:* $M = \lim_{\kappa\to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y:\ y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, 0 < \kappa \leq 1, p = floor(s/\kappa) \wedge 0 \leq |floor(s/\kappa) - s/\kappa| < 1 \Rightarrow |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by κ:

$$\forall\, 0 < \kappa \leq 1, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ &\quad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \delta = \epsilon \quad := \quad M = \lim_{\kappa\to 0} p\kappa = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \ \cdot \kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, ..., \pi_{\lim_{\kappa\to 0}}$.

Lemma 2.3. $\forall\, n \geq 1, \quad 0 < \kappa \leq 1 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa$.

Proof. The formal proof , "lim_c_to_n_eq_lim_c," is in the Rocq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa \leq 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta \\ &\quad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa\to 0} \kappa^n = 0. \end{aligned} \tag{2.5}$$

$$\lim_{\kappa\to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa\to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa. \tag{2.6}$$

$\square$

3. Volume

Definition 3.1. A countable n-volume, v_c is the number of ordered combinations (n-tuples) of n number of disjoint, countable domain sets:

$$\forall\, x_i.x_j \in \{x_1, \cdots, x_n\},\ i \neq j, \quad x_i \bigcap x_j = \emptyset, \quad v_c = \textstyle\prod_{i=1}^{n} |x_i|. \tag{3.1}$$

Theorem 3.2. *Euclidean volume,*

$$\begin{aligned} &\forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\ [v_a, v_b] \subset \mathbb{R},\ s_i = b_i - a_i,\ v = v_b - v_a : \\ &\qquad\qquad v_c = \textstyle\prod_{i=1}^{n} |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^{n} s_i. \end{aligned} \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Rocq file, euclidrelations.v.

Proof.

$$v_c = \prod_{i=1}^n |x_i| \;\Leftrightarrow\; v_c\kappa = (\prod_{i=1}^n |x_i|)\kappa \;\Leftrightarrow\; \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{3.3}$$

Apply the ruler (2.1) and ruler convergence (2.2) to equation 3.3:

$$\exists\, v,\kappa \in \mathbb{R}:\; v_c = floor(v/\kappa) \quad\Rightarrow\quad v = \lim_{\kappa\to 0} v_c\kappa \quad\wedge$$
$$\lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa \quad\Rightarrow\quad v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{3.4}$$

Apply lemma 2.3 to equation 3.4:

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa \quad\wedge\quad \lim_{\kappa\to 0}\kappa^n = \lim_{\kappa\to 0}\kappa$$
$$\Rightarrow\quad v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa^n = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa). \tag{3.5}$$

Apply the ruler (2.1) and ruler convergence (2.2) to s_i:

$$\exists\, s_i,\kappa \in \mathbb{R}:\; floor(s_i/\kappa) = |x_i| \quad\Rightarrow\quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i. \tag{3.6}$$

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa) \quad\wedge\quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i \quad\Leftrightarrow\quad v = \prod_{i=1}^n s_i \tag{3.7}$$

□

4. Distance

Definition 4.1. Countable distance,

$$\forall\, n \in \mathbb{N},\; v_c, d_c \in \{0,\ \mathbb{N}\},\;\; x_i.x_j \in \{x_1,\cdots,x_n\},\; i \neq j,\quad x_i \bigcap x_j = \emptyset \quad\wedge$$
$$v_c = \prod_{i=1}^n |x_i| = \prod_{i=1}^n d_c = d_c^n. \tag{4.1}$$

Lemma 4.2. *A volume is the sum of volumes,*

$$v_c = d_c^n = \sum_{i=1}^m v_{c_i} \quad\Rightarrow\quad v = \sum_{i=1}^m v_i,\quad v, v_i \in \mathbb{R}.$$

The formal proof, "sum_of _volumes," is in the Rocq file, euclidrelations.v.

Proof. From the condition of this theorem:

$$v_c = \sum_{i=1}^m v_{c_i} \quad\Leftrightarrow\quad \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}\sum_{j=1}^m (v_{c_i}\kappa). \tag{4.2}$$

Apply lemma 2.3 to equation 4.2:

$$\lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\sum_{j=1}^m v_{c_i})\kappa \quad\wedge\quad \lim_{\kappa\to 0}\kappa^n = \lim_{\kappa\to 0}\kappa$$
$$\Leftrightarrow\;\; \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}\sum_{j=1}^m (v_{c_i})\kappa^n \;\;\Leftrightarrow\;\; \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}\sum_{j=1}^m (v_{c_i}\kappa). \tag{4.3}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to equation 4.3:

$$\exists\, v, v_i:\; v = floor(d/\kappa),\; v = \lim_{\kappa\to 0} v_c\kappa$$
$$\wedge\quad v_{c_i} = floor(v_i/\kappa),\; v_i = \lim_{\kappa\to 0} v_{c_i}\kappa \quad\wedge\quad \lim_{\kappa\to 0}(d_c\kappa)^n = \lim_{\kappa\to 0}\sum_{j=1}^m (v_{c_i}\kappa)$$
$$\Leftrightarrow\quad v = \lim_{\kappa\to 0}(d_c\kappa)^n = \lim_{\kappa\to 0}\sum_{j=1}^m (v_{c_i}\kappa) = \sum_{j=1}^m v_i^n. \quad \square \tag{4.4}$$

4.1. Sum of volumes distance.

Theorem 4.3. *Sum of volumes distance:*

$$v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad \Rightarrow \quad d^n = \sum_{i=1}^m (\prod_{j=1}^n s_{i_j}).$$

The formal proof, "sum_of_volumes_distance," is in the Rocq file, euclidrelations.v.

Proof. From lemma 4.2 and the Euclidean volume theorem 3.2:

$$\begin{aligned} v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad &\Leftrightarrow \quad d^n = \textstyle\sum_{i=1}^m (\prod_{j=1}^n v_i) \quad \wedge \quad v_i = \prod_{j=1}^n s_{i_j} \\ v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad &\Leftrightarrow \quad d^n = \textstyle\sum_{i=1}^m (\prod_{j=1}^n s_{i_j}). \quad \square \end{aligned} \tag{4.5}$$

4.2. Minkowski distance (p-norm).

Theorem 4.4. *Minkowski distance (p-norm):*

$$v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} = \sum_{i=1}^m d_{c_i}^n \quad \Leftrightarrow \quad d^n = \sum_{i=1}^m d_i^n.$$

The formal proof, "Minkowski_distance," is in the Rocq file, euclidrelations.v.

Proof. From lemma 4.2 and the Euclidean volume theorem 3.2:

$$\begin{aligned} v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad &\Leftrightarrow \quad d^n = \textstyle\sum_{i=1}^m v_i \quad \wedge \quad v_i = \prod_{j=1}^n d_i = d_i^n \\ v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad &\Leftrightarrow \quad d^n = \textstyle\sum_{i=1}^m d^n. \quad \square \end{aligned} \tag{4.6}$$

4.3. Distance inequality.

The formal proof, distance_inequality, is in the Rocq file, euclidrelations.v.

Theorem 4.5. *Distance inequality*

$$\forall\, n \in \mathbb{N},\;\; v_a, v_b \geq 0 : \;\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

Proof. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$\begin{aligned} &\forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq v_a + v_b + \\ &\textstyle\sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n. \end{aligned} \tag{4.7}$$

Take the n^{th} root of both sides of the inequality 4.7:

$$\forall\, v_a, v_b \geq 0,\; n \in \mathbb{N} : \; v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \;\Rightarrow\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \tag{4.8}$$

$\square$

4.4. Distance sum inequality.

The formal proof, distance_sum_inequality, is in the Rocq file, euclidrelations.v.

Theorem 4.6. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N},\;\; a_i, b_i \geq 0 : \;\; (\textstyle\sum_{i=1}^m (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}.$$

Proof. Apply the distance inequality (4.5):

$$\begin{aligned} &\forall\, m, n \in \mathbb{N},\;\; v_a, v_b \geq 0 : \quad v_a = \textstyle\sum_{i=1}^m a_i^n \quad \wedge \quad v_b = \sum_{i=1}^m b_i^n \quad \wedge \\ &\qquad (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad ((\textstyle\sum_{i=1}^m a_i^n) + (\sum_{i=1}^m b_i^n))^{1/n} = \\ &\qquad\qquad (\textstyle\sum_{i=1}^m (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}. \quad \square \end{aligned} \tag{4.9}$$

4.5. **Metric Space.** All Minkowski distances (p-norms) imply the metric space properties. The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Rocq file, euclidrelations.v.

Theorem 4.7. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$

Proof. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.10}$$

Apply the distance inequality (4.5) to the inequality 4.10:

$$\begin{aligned} &(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \ \wedge \ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ &\qquad \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ \Rightarrow \quad &(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ &\qquad \Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq \\ &\qquad (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \quad \square \end{aligned} \tag{4.11}$$

Theorem 4.8. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

Proof. By the commutative law of addition:

$$\begin{aligned} &\forall\, p : p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ &\qquad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \quad \square \end{aligned} \tag{4.12}$$

Theorem 4.9. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

Proof. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0. \tag{4.13}$$

$$p \geq 1,\ \ u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \tag{4.14}$$

$\square$

Theorem 4.10. *Identity of Indiscernibles:* $d(u, u) = 0.$

Proof. From the non-negativity property (4.9):

$$\begin{aligned} &d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0 \\ &\qquad \Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0. \end{aligned} \tag{4.15}$$

$$d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v. \tag{4.16}$$

$$d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \tag{4.17}$$

$\square$

4.6. **Set properties limiting a set to at most 3 members.** The following definitions and proof use first order logic. A Horn clause-like expression is used, here, to make the proof easier to read. By convention, the proof goal is on the left side and supporting facts are on the right side of the implication sign ($\leftarrow$). The formal proofs in the Rocq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

Definition 4.11. Immediate Cyclic Successor of m is n:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\}: \\ Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize). \tag{4.18}$$

Definition 4.12. Immediate Cyclic Predecessor of m is n:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\}: \\ Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1). \tag{4.19}$$

Definition 4.13. Adjacent: Member m is sequentially adjacent to member n if the immediate cyclic successor of m is n or the immediate cyclic predecessor of m is n. Notionally:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\}: \\ Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{4.20}$$

Definition 4.14. Immediate Symmetric (every set member is sequentially adjacent to every other member):

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\}: \quad Adjacent(m, n, setsize). \tag{4.21}$$

Theorem 4.15. *An immediate symmetric cyclic set is limited to at most 3 members.*

Proof.
Every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \tag{4.22}$$
$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \tag{4.23}$$
$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1). \tag{4.24}$$
$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize). \tag{4.25}$$
$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \tag{4.26}$$
$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1). \tag{4.27}$$
$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize). \tag{4.28}$$
$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize). \tag{4.29}$$
$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \tag{4.30}$$

Member 2 is the only immediate successor of member 1 for all $setsize \geq 3$, which implies member 3 is not ($\neg$) an immediate successor of member 1 for all $setsize \geq 3$:

$$\neg Successor(1, 3, setsize \geq 3) \\ \leftarrow Successor(1, 2, setsize \geq 3) \leftarrow (n = m + 1 \leq setsize). \tag{4.31}$$

Member $n = setsize > 3$ is the only immediate predecessor of member 1, which implies member 3 is not ($\neg$) an immediate predecessor of member 1 for all $setsize > 3$:

$$\neg Predecessor(1, 3, setsize \geq 3) \\ \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \tag{4.32}$$

For all *setsize* > 3, some elements are not ($\neg$) sequentially adjacent to every other element (not immediate symmetric):

$$\neg Adjacent(1, 3, setsize > 3) \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \tag{4.33}$$

The Symmetric goal matches Adjacent goals 4.22 and fails for all "setsize" greater than three.

5. Applications to physics

Where distance is an immediate symmetric cyclic set (ISCS) of dimensions, the 3D proof (4.15) requires more dimensions to have non-distance types (members of other sets). Let $\tau = \{t\ (time),\ m\ (mass),\ q\ (charge)\}$ be the ISCS of type "non-distance" dimensions, where for each Cartesian unit length, r_p, of distance, r, there are unit lengths: t_p of time, t; m_p of mass, m; and q_p of charge, q, such that:

$$r = (r_p/t_p)t = (r_p/m_p)m = (r_p/q_p)q, \tag{5.1}$$

where:

$$c_t = r_p/t_p, \quad c_m = r_p/m_p, \quad c = c_t = r_p/q_p. \tag{5.2}$$

5.1. **Derivation of the constant, G, and the gravity laws of Newton, Gauss, and Poisson.** From equation 5.2:

$$r = c_m m \quad \wedge \quad r = c_t t \quad \Rightarrow \quad r/(c_t t)^2 = c_m m/r^2 \quad \Rightarrow \quad r/t^2 = (c_m c_t^2)m/r^2 = Gm/r^2, \tag{5.3}$$

where $G = c_m c_t^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$ [14].

Newton's law follows from multiplying both sides of equation 5.3 by m:

$$r/t^2 = Gm/r^2 \;\Leftrightarrow\; F := mr/t^2 = Gm^2/r^2. \tag{5.4}$$

$$F = Gm^2/r^2 \wedge \forall\, m \in \mathbb{R} : \exists\, m_1, m_2 \in \mathbb{R} : m_1 m_2 = m^2 \;\Rightarrow\; F = Gm_1 m_2/r^2. \tag{5.5}$$

Equation 5.3 relates linear acceleration, r/t^2, to mass and distance. Gauss's gravity field, $\mathbf{g}$, and Poisson's gravity field, $-\nabla\Phi(r,t)$, relates orbital acceleration, $2\pi r/t^2$, to mass and distance. Multiplying both sides of equation 5.3 by 2π and differentiating yields Gauss's and Poison's laws [6]:

$$\mathbf{g} = -\nabla\Phi(\vec{r},t) = 2\pi r/t^2 = 2\pi Gm/r^2 \;\Rightarrow\; \nabla\cdot\mathbf{g} = \nabla^2\Phi(\vec{r},t) = -4\pi Gm/r^3. \tag{5.6}$$

$$\nabla\cdot\mathbf{g} = \nabla^2\Phi(\vec{r},t) = -4\pi Gm/r^3 \wedge \rho = m/r^3 \;\Rightarrow\; \nabla\cdot\mathbf{g} = \nabla^2\Phi(\vec{r},t) = -4\pi G\rho. \tag{5.7}$$

5.2. **Derivation of Coulomb's charge constant, k_e, and charge force.** [6] From equation 5.2:

$$r = c_q q \quad \Rightarrow \quad r^2 = c_q^2 q^2 \quad \Rightarrow \quad c_q^2 q^2/r^2 = 1. \tag{5.8}$$

$$r = c_m m = c_t t \quad \Rightarrow \quad mr = ((1/c_m)r)(c_t t) = ((1/c_m)c_t t)(c_t t) = (1/c_m)(c_t t)^2 \quad \Rightarrow \quad (c_m/c_t^2)mr/t^2 = 1. \tag{5.9}$$

$$c_q^2 q^2/r^2 = 1 \quad \wedge \quad (c_m/c_t^2)mr/t^2 = 1 \quad \Rightarrow \quad F := mr/t^2 = (c_q^2 c_t^2/c_m)q^2/r^2 = k_e q^2/r^2, \tag{5.10}$$

and $k_e = c_q^2 c_t^2/c_m$, conforms to the SI units: $kg \cdot m^3 \cdot s^{-2} \cdot C^{-2} = N \cdot m^2 \cdot C^{-2}$ [6].

$$\exists\, q_1, q_2 \in \mathbb{R}:\ q_1 q_2 = q^2 \quad \wedge \quad F = k_e q^2/r^2 \quad \Rightarrow \quad F = k_e q_1 q_2/r^2. \tag{5.11}$$

5.3. **3 fundamental direct proportion ratios.** c_t, c_m, and c_q:

$$c_t = r_p/t_p \approx 2.99792458 \cdot 10^8 m\ s^{-1}. \tag{5.12}$$

$$G = (r_p/m_p)c_t^2 = c_m c_t^2 \quad \Rightarrow \quad c_m = r_p/m_p \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}. \tag{5.13}$$

$$k_e = c_q^2 c_t^2/c_m \quad \Rightarrow \quad c_q = r_p/q_p \approx 8.6175172023 \cdot 10^{-18} m\ \ C^{-1}. \tag{5.14}$$

5.4. **3 fundamental inverse proportion ratios.** k_t, k_m, and k_q:

$$\begin{aligned} r/t = r_p/t_p, \quad r/m = r_p/m_p \quad &\Rightarrow \quad (r/t)/(r/m) = (r_p/t_p)/(r_p/m_p) \quad \Rightarrow \\ (mr)/(tr) = (m_p r_p)/(t_p r_p) \quad &\Rightarrow \quad mr = m_p r_p = k_m, \quad tr = t_p r_p = k_t. \end{aligned} \tag{5.15}$$

$$\begin{aligned} r/t = r_p/t_p, \quad r/q = r_p/q_p \quad &\Rightarrow \quad (r/t)/(r/q) = (r_p/t_p)/(r_p/q_p) \quad \Rightarrow \\ (qr)/(tr) = (q_p r_p)/(t_p r_p) \quad &\Rightarrow \quad qr = q_p r_p = k_q, \quad tr = t_p r_p = k_t. \end{aligned} \tag{5.16}$$

5.5. **Planck relation and constant,** h**.** [9] Applying both the direct proportion ratio (5.12), and inverse proportion ratio (5.15):

$$r = ct \quad \wedge \quad m = k_m/r \quad \Rightarrow \quad m(ct)^2 = (k_m/r)r^2 = k_m r. \tag{5.17}$$

$$\begin{aligned} &m(ct)^2 = k_m r \quad \wedge \quad r/t = c \\ &\qquad \Rightarrow \quad E := mc^2 = k_m r/t^2 = (k_m c)(1/t) = \hbar\omega = \hbar\omega(2\pi/2\pi) = hf, \end{aligned} \tag{5.18}$$

where the reduced Planck constant, $\hbar = k_m c$, angular frequency, $\omega = 1/t$, the full Planck constant, $h = 2\pi\hbar$, and the cycles per second frequency (Hertz), $f = 1/2\pi t$.

$$k_m = m_p r_p = \hbar/c \approx 3.5176729162 \cdot 10^{-43}\ kg\ m. \tag{5.19}$$

$$k_t = t_p r_p = k_m c_m/c_t \approx 8.7136291599 \cdot 10^{-79}\ s\ m. \tag{5.20}$$

$$k_q = q_p r_p = k_t c_t/c_q \approx 3.0313607071 \cdot 10^{-52}\ C\ m. \tag{5.21}$$

5.6. **4 quantum (Planck) units.** Length (r_p), time (t_p), mass (m_p), charge (q_p):

$$r_p = \sqrt{r_p^2} = \sqrt{c_t k_t} = \sqrt{c_m k_m} = \sqrt{c_q k_q} \approx 1.6162550244 \cdot 10^{-35}\ m. \tag{5.22}$$

$$t_p = r_p/c_t \approx 5.3912464472 \cdot 10^{-44}\ s. \tag{5.23}$$

$$m_p = r_p/c_m \approx 2.176434343 \cdot 10^{-8}\ kg. \tag{5.24}$$

$$q_p = r_p/c_q \approx 1.875546038 \cdot 10^{-18}\ C. \tag{5.25}$$

5.7. **Subtype ratios.** The ratios of two subtypes of force implies ratios of the form: $\alpha_\tau = \frac{F_{\tau_1}}{F_{\tau_2}} = \frac{K\tau_1^2/r^2}{K\tau_2^2/r^2} = \frac{\tau_1^2}{\tau_2^2}$. For example, where q_e is the elementary (electron) charge ($1.60217663 \cdot 10^{-19}\ C$), and q_p is Planck charge unit, the fine structure electron coupling constant is:

$$\alpha_q = q_e^2/q_p^2 \approx 0.0072973526. \tag{5.26}$$

5.8. **Space-time-mass-charge.** Let r be an Euclidean distance. Then by the Minkowski distance theorem (4.4), $r^2 = \sum_{i=1}^{m} r_i^2$. Let, $r' = r_1$ and $r_v^2 = (\sum_{i=2}^{m} r_i^2)$. From the 3D theorem (4.15) and Cartesian hypothesis (2):

$$\forall\, \tau \in \{t, m, q\},\ r^2 = r'^2 + r_v^2,\ \exists\, \mu, \nu :\ r = \mu\tau \quad \wedge \quad r_v = \nu\tau$$
$$\Rightarrow \quad (\mu\tau)^2 = r'^2 + (\nu\tau)^2 \quad \Rightarrow \quad r' = \sqrt{(\mu\tau)^2 - (\nu\tau)^2} \ = \ \mu\tau\sqrt{1 - (\nu/\mu)^2}. \quad (5.27)$$

Rest frame distance, r', contracts relative to stationary frame distance, r, as $\nu \to \mu$:

$$r' = \mu\tau\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad \mu\tau = r \quad \Rightarrow \quad r' = r\sqrt{1 - (\nu/\mu)^2}. \quad (5.28)$$

Stationary frame type, τ, dilates relative to the rest frame type, τ', as $\nu \to \mu$:

$$\mu\tau = r'/\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad r' = \mu\tau' \quad \Rightarrow \quad \tau = \tau'/\sqrt{1 - (\nu/\mu)^2}. \quad (5.29)$$

Where τ is type, time, the space-like flat Minkowski spacetime event interval is:

$$\mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_v^2 \quad \wedge \quad \mathrm{d}r_v^2 = \mathrm{d}r_1^2 + \mathrm{d}r_2^2 + \mathrm{d}r_3^2 \quad \wedge \quad \mathrm{d}(\mu\tau) = \mathrm{d}r$$
$$\Rightarrow \quad \mathrm{d}r'^2 = \mathrm{d}(\mu\tau)^2 - \mathrm{d}r_1^2 - \mathrm{d}r_2^2 - \mathrm{d}r_3^2. \quad (5.30)$$

5.9. **Derivation of Schwarzschild's gravitational time dilation and black hole metric.** [18] [1] From equations 5.28 and 5.1:

$$\sqrt{1 - (v^2/c^2)} = \sqrt{1 - (v^2/c^2)(r/r)} \quad \wedge \quad c_m m/r = 1$$
$$\Rightarrow \quad \sqrt{1 - (v^2/c^2)} = \sqrt{1 - (c_m m)v^2/rc^2}. \quad (5.31)$$

Where v_{escape} is the escape velocity:

$$\sqrt{1 - (v^2/c^2)} = \sqrt{1 - (c_m m)v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv_{escape}^2$$
$$\Rightarrow \quad \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2c_m m v_{escape}^2/rc^2}. \quad (5.32)$$

$$\sqrt{1 - (v^2/c^2)} = \lim_{v_{escape} \to c} \sqrt{1 - 2c_m m v_{escape}^2/rc^2}$$
$$= \sqrt{1 - 2c_m mc^2/rc^2}. \quad (5.33)$$

Combining equation 5.33 with the derivation of G (5.5):

$$c_m c^2 = G \quad \wedge \quad \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2c_m mc^2/rc^2}$$
$$\Rightarrow \quad \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2Gm/rc^2}. \quad (5.34)$$

Combining equation 5.34 with equation 5.29 yields Schwarzschild's gravitational time dilation [18] [1]:

$$\sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2Gm/rc^2} \quad \wedge \quad t' = t\sqrt{1 - (v^2/c^2)}$$
$$\Rightarrow \quad t' = t\sqrt{1 - 2Gm/rc^2}. \quad (5.35)$$

Schwarzschild defined the black hole event horizon radius, $r_s := 2Gm/c^2$. From equations 5.28 and 5.35:

$$r' = r\sqrt{1 - (v/c)^2} \quad \wedge \quad \sqrt{1 - (v/c)^2} = \sqrt{1 - 2Gm/rc^2} \quad \wedge \quad r_s := 2Gm/c^2$$
$$\Rightarrow \quad r' = r\sqrt{1 - 2Gm/rc^2} = r\sqrt{1 - r_s/r}. \quad (5.36)$$

Using the time-like spacetime interval, where $\mathrm{d}s^2$ is negative:

$$r' = r\sqrt{1 - r_s/r} \quad \wedge \quad \mathrm{d}s^2 = \mathrm{d}r'^2 - \mathrm{d}r^2$$
$$\Rightarrow \quad \mathrm{d}s^2 = (\sqrt{1 - r_s/r}\mathrm{d}r')^2 - (\mathrm{d}r/\sqrt{1 - r_s/r})^2 = (1 - r_s/r)\mathrm{d}r'^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2. \tag{5.37}$$

$$\mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}r'^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2 \quad \wedge \quad \mathrm{d}r' = \mathrm{d}(ct) \quad \wedge \quad c = 1$$
$$\Rightarrow \quad \mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}t^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2. \tag{5.38}$$

Using spherical coordinates to translate from 2D to 4D yields Schwarzschild's black hole metric [18] [1]:

$$\mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}t^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2 = f(r, t)$$
$$\Rightarrow \quad \mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}t^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + sin^2\theta\mathrm{d}\phi^2) = f(r, t, \theta, \phi)$$
$$\Rightarrow \quad g_{\mu,\nu} = diag[1 - r_s/r),\ (1 - r_s/r)^{-1},\ r^2(\mathrm{d}\theta^2),\ r^2(sin^2\theta\mathrm{d}\phi^2)]. \tag{5.39}$$

5.10. **Simple derivations of solutions to Einstein's general relativity (field) equation.** Step 1) Use the ratios to define functions returning scalar values for each component of the metric, $g_{\nu,\mu}$, in Einstein's field equations [5] [20]: All functions derived from the ratios, where the units on each side of the equation balance, are valid metrics, for example, the previous Schwarzschild black hole metric derivation using the ratios (5.9).

Step 2) Express the EFE as 2D tensors: As shown in equation 5.39, the Schwarzschild metric was first derived as a 2D metric and then expanded to a 4D metric. Further, the 4D flat spacetime interval equation (5.30) is an instance of the 2D equation, $\mathrm{d}r'^2 = \mathrm{d}(ct)^2 - \mathrm{d}r_v^2$, where $\mathrm{d}r_v^2$ is the magnitude of a 3-dimensional vector.

The 2D metric tensor allows using the much simpler 2D Ricci curvature and scalar curvature. And the 2D tensors reduce the number of independent equations to solve.

Step 3) One simple method to translate from 2D to 4D is to use spherical coordinates, where r and t remain unchanged and two added dimensions are the angles, ϕ, and θ. For example, the 2D Schwarzschild metric was translated to 4D using this method in equation 5.39.

5.11. **Relativistic Lorentz law, and vacuum permeability, μ_0.** Coulomb's charge force equation 5.10 relates linear acceleration, r/t^2, to charge and distance. Gauss's electric field, $\mathbf{E}$, relates centripetal or centrifugal acceleration, $2\pi r/t^2$ (depending on whether the charges are attracting or repelling), to charge and distance:

$$F_C = mr/t^2 = k_e q^2/r^2 \quad \Rightarrow \quad \exists\, F_E \in \mathbb{R} : F_E = m(2\pi r/t^2) = 2\pi k_e q^2/r^2. \tag{5.40}$$

Applying the distance contraction equation 5.28 to equation 5.40, where r is the stationary frame of reference and r' is moving particle (rest) frame of reference.:

$$r = r'/\sqrt{1 - v^2/c^2} \quad \wedge \quad F = 2\pi k_e q^2/r^2 \quad \Rightarrow \quad F = 2\pi k_e q^2(1 - v^2/c^2)/r'^2. \tag{5.41}$$
$$E := 2\pi k_e q/r'^2 \quad \Rightarrow \quad F = q(E - v^2(2\pi k_e/c^2)q/r'^2). \tag{5.42}$$
$$B := (2\pi k_e/c^2)vq/r'^2 \quad \Rightarrow \quad F = q(E - vB). \tag{5.43}$$
$$F = q(E - vB) \quad \Rightarrow \quad \mathbf{F} = q(\mathbf{E} - \vec{\mathbf{v}} \times \mathbf{B}), \tag{5.44}$$

which is Lorentz law in the rest frame of reference. And

$$\mathbf{F} = q(\mathbf{E} + \vec{\mathbf{v}} \times \mathbf{B}), \tag{5.45}$$

is Lorentz law in the stationary frame of reference. The direction of rotation depends on your perspective (frame of reference).

The electric field, $E := 2\pi k_e q/r'^2$, conforms to the SI units $kg \cdot m \cdot s^{-2} \cdot C^{-1} = N \cdot C^{-1}$ and the magnetic field, $B = (2\pi k_e/c^2)vq/r'^2$, conforms to the base SI units: $kg \cdot s^{-1} \cdot C^{-1} = kg \cdot s^{-2} \cdot A^{-1} = T$.

$$B := (2\pi k_e/c^2)vq/r'^2 \quad\wedge\quad B := \mu_0 H \quad\wedge\quad \mu_0 := 4\pi k_e/c^2 \quad\Rightarrow\quad H = vq/2r'^2, \tag{5.46}$$

where $\mu_0 = 4\pi k_e/c^2$ conforms to the SI units $kg \cdot m \cdot C^{-2} = kg \cdot m \cdot s^{-2}A^{-2}$ and $H = vq/2r'^2$ conforms to the SI units $C \cdot s^{-1} \cdot m^{-1} = A \cdot m^{-1}$.

5.12. **Vacuum permittivity, ε_0, and Gauss's law for electric fields.** From equation 5.42:

$$E = 2\pi k_e q/r^2 \quad\Leftrightarrow\quad \mathbf{E} = 2\pi k_e q/\vec{\mathbf{r}}^2 \quad\Rightarrow\quad \nabla \cdot \mathbf{E} = -4\pi k_e q/\vec{\mathbf{r}}^3. \tag{5.47}$$

$$\nabla \cdot \mathbf{E} = -4\pi k_e q/\vec{\mathbf{r}}^3 \quad\wedge\quad \varepsilon_0 := 1/4\pi k_e \quad\wedge\quad \rho = q/\vec{\mathbf{r}}^3 \quad\Rightarrow\quad \nabla \cdot \mathbf{E} = -\rho/\varepsilon_0, \tag{5.48}$$

which is Gauss's electric field law [6].

5.13. **Derivation of Faraday's law.** From the magnetic field equation 5.43, where the velocity of light, $v = c$:

$$B = (2\pi k_e/c^2)qv/r^2 \quad\wedge\quad v = c \quad\wedge\quad r = ct \quad\Rightarrow\quad B = (2\pi k_e/c^3)q/t^2. \tag{5.49}$$

$$B = (2\pi k_e/c^3)q/t^2 \quad\Rightarrow\quad \partial B/\partial t = -(4\pi k_e/c^3)q/t^3. \tag{5.50}$$

$$\partial B/\partial t = -(4\pi k_e/c^3)q/t^3 \quad\wedge\quad r = ct \quad\Rightarrow\quad \partial B/\partial t = -4\pi k_e q/r^3. \tag{5.51}$$

From equation 5.47:

$$\mathbf{E} = 2\pi k_e q/\vec{\mathbf{r}}^2 \quad\Rightarrow\quad \nabla \times \mathbf{E} = 4\pi k_e q/\vec{\mathbf{r}}^3. \tag{5.52}$$

Combining equations 5.52 and 5.51 yields Faraday's law [6]:

$$\nabla \times \mathbf{E} = 4\pi k_e q/\vec{\mathbf{r}}^3 \quad\wedge\quad \partial\mathbf{B}/\partial t = -4\pi k_e q/\vec{\mathbf{r}}^3 \quad\Rightarrow\quad \nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t. \tag{5.53}$$

5.14. **Compton wavelength, λ.** [9] From equations 5.15 and 5.18:

$$r = k_m/m \quad\wedge\quad h = 2\pi k_m c \quad\Rightarrow\quad \lambda = 2\pi r = 2\pi k_m/m = (2\pi k_m/m)(c/c) = h/mc. \tag{5.54}$$

5.15. **Schrödenger's position-space equation.** Start with the previously derived Planck relation 5.18 and multiply the kinetic energy component by mc/mc:

$$mc^2 = \hbar\omega = \hbar/t \quad\Rightarrow\quad \exists\, V(r,t) :\ \hbar/t = \hbar/2t + V(r,t) \quad\Rightarrow\quad \hbar/t = \hbar mc/2mct + V(r,t). \tag{5.55}$$

And from the distance-to-time (speed of light) ratio (5.12):

$$\hbar/t = \hbar mc/2mct + V(r,t) \quad\wedge\quad r = ct \quad\Rightarrow\quad \hbar/t = \hbar mc^2/2mcr + V(r,t). \tag{5.56}$$

$$\hbar/t = \hbar mc^2/2mcr + V(r,t) \ \wedge\ \hbar/t = mc^2 \quad\Rightarrow\quad \hbar/t = \hbar^2/2mcrt + V(r,t). \tag{5.57}$$

$$\hbar/t = \hbar^2/2mcrt + V(r,t) \quad\wedge\quad r = ct \quad\Rightarrow\quad \hbar/t = \hbar^2/2mr^2 + V(r,t). \tag{5.58}$$

Multiply both sides of equation 5.58 by a function, $\Psi(r,t)$.

$$\hbar/t = \hbar^2/2mr^2 + V(r,t) \Rightarrow (\hbar/t)\Psi(r,t) = (\hbar^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t). \tag{5.59}$$

$$(\hbar/t)\Psi(r,t) = (\hbar^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t) \quad \wedge$$
$$\forall\, \Psi(r,t) : \partial^2\Psi(r,t)/\partial r^2 = (-1/r^2)\Psi(r,t) \quad \wedge \quad \partial\Psi(r,t)/\partial t = (i/t)\Psi(r,t)$$
$$\Rightarrow \quad i\hbar\partial\Psi(r,t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t), \tag{5.60}$$

which is the one-dimensional position-space Schrödenger's equation [17] [9].

$$i\hbar\partial\Psi(r,t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t) \quad \wedge \quad ||\vec{\mathbf{r}}|| = r$$
$$\Rightarrow \quad \exists\, \vec{\mathbf{r}} : i\hbar\partial\Psi(\vec{\mathbf{r}},t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(\vec{\mathbf{r}},t)/\partial \vec{\mathbf{r}}^2 + V(\vec{\mathbf{r}},t)\Psi(\vec{\mathbf{r}},t), \tag{5.61}$$

which is the 3-dimensional position-space Schrödenger's equation [17] [9].

5.16. **Dirac's wave equation.** Using the derived Planck relation 5.18:

$$mc^2 = \hbar/t \quad \Rightarrow \quad \exists\, V(r,t) : mc^2/2 + V(r,t) = \hbar/t$$
$$\Rightarrow \quad 2\hbar/t - 2V(r,t) = mc^2. \tag{5.62}$$

$$\forall\, V(r,t) : V(r,t) = i\hbar/t \quad \wedge \quad r = ct \quad \wedge \quad 2\hbar/t - 2V(r,t) = mc^2$$
$$\Rightarrow \quad 2\hbar/t - i2\hbar c/r = mc^2. \tag{5.63}$$

Use the charge ratio, $r = c_q q$, and time ratio, $r = ct$. to multiply each term on the left side of equation 5.63 by 1:

$$qc_q/r = qc_q/ct = 1 \quad \wedge \quad 2\hbar/t - i2\hbar c/r = mc^2$$
$$\Rightarrow \quad 2\hbar(qc_q/c)/t^2 - i2\hbar((qc_q/c)/r^2)c = mc^2. \tag{5.64}$$

Applying a quantum amplitude equation in complex form to equation 5.65, where τ is a unit time and ρ is a unit wavelength:

$$A_0 = (c_q/c)((1/t)\tau - i(1/r)\rho) \;\wedge\; 2\hbar(qc_q/c)/t^2 - i2\hbar((qc_q/c)/r^2)c = mc^2$$
$$\Rightarrow \quad 2\hbar\partial(-qA_0)/\partial t - i2\hbar(\partial(-qA_0)/\partial r)c = mc^2. \tag{5.65}$$

Translating equation 5.65 to moving (rest frame) coordinates via the Lorentz factor, $\gamma_0 = 1/\sqrt{1-(v/c)^2}$:

$$2\hbar\partial(-qA_0)/\partial t - i\hbar h(\partial(-qA_0)/\partial r)c = mc^2$$
$$\Rightarrow \quad \gamma_0 2\hbar\partial(-qA_0)/\partial t - \gamma_0 i2\hbar(\partial(-qA_0)/\partial r)c = mc^2. \tag{5.66}$$

Multiplying both sides of equation 5.66 by $\Psi(r,t)$:

$$\gamma_0 2\hbar\partial(-qA_0)/\partial t - \gamma_0 i2\hbar(\partial(-qA_0)/\partial r)c = mc^2$$
$$\Rightarrow \quad \gamma_0 2\hbar(\partial(-qA_0)/\partial t)\Psi(r,t) - \gamma_0 i2\hbar(\partial(-qA_0)/\partial r)c\Psi(r,t) = mc^2\Psi(r,t). \tag{5.67}$$

Applying the vectors to equation 5.67:

$$\gamma_0 2\hbar(\partial(-qA_0)/\partial t)\Psi(r,t) - \gamma_0 i2\hbar(\partial(-qA_0)/\partial r)c\Psi(r,t) = mc^2\Psi(r,t) \wedge$$
$$||\vec{\mathbf{r}}|| = r \quad \wedge \quad ||\vec{\mathbf{A}}|| = A_0 \quad \wedge \quad ||\vec{\gamma}|| = \gamma_0 \quad \wedge \quad \Leftrightarrow \quad \exists\, \vec{\mathbf{r}}, \vec{\mathbf{A}}, \vec{\gamma} :$$
$$\gamma_0 2\hbar(\partial(-qA_0)/\partial t)\Psi(r,t) - \vec{\gamma}\cdot i2\hbar(\partial(-q\vec{\mathbf{A}})/\partial r)c\Psi(\vec{\mathbf{r}},t) = mc^2\Psi(\vec{\mathbf{r}},t). \tag{5.68}$$

Adding a $\frac{1}{2}$ spin to equation 5.65 yields Dirac's wave equation [4] [9]:

$$\gamma_0 2\hbar(\partial(-qA_0)/\partial t)\Psi(r,t) - \vec{\gamma}\cdot i2\hbar(\partial(-q\vec{\mathbf{A}})/\partial r)c\Psi(\vec{\mathbf{r}},t) = mc^2\Psi(\vec{\mathbf{r}},t)$$
$$\wedge \quad A_0 = \frac{1}{2}(c_q/c)((1/t)\tau - i(1/r)\rho)$$
$$\Rightarrow \quad \gamma_0\hbar(\partial(-qA_0)/\partial t)\Psi(r,t) - \vec{\gamma}\cdot i\hbar(\partial(-q\vec{\mathbf{A}})/\partial r)c\Psi(\vec{\mathbf{r}},t) = mc^2\Psi(\vec{\mathbf{r}},t). \quad (5.69)$$

5.17. **Total mass.** The total mass of a particle is $m = \sqrt{m_0^2 + m_{ke}^2}$, where m_0 is the rest mass and m_{ke} is the kinetic energy-equivalent mass. Applying both the direct (5.12) and inverse proportion ratios (5.15):

$$m_0 = r/(r_p/m_p) = r/c_m \quad \wedge \quad m_{ke} = (m_p r_p)/r = k_m/r \quad \wedge$$
$$m = \sqrt{m_0^2 + m_{ke}^2} \quad \Rightarrow \quad m = \sqrt{(r/c_m)^2 + (k_m/r)^2}. \quad (5.70)$$

5.18. **Quantum extension to general relativity.** The simplest way to demonstrate how to add quantum physics to general relativity is by extending Schwarzschild's time dilation equation and black hole metric (5.9). Start by changing equation 5.31 in the Schwarzschild derivation:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-(v^2/c^2)(r/r)} \ \wedge \ r = \sqrt{(c_m m)^2 + (k_m/m)^2} = Q_m$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1 - Q_m v^2/rc^2}. \quad (5.71)$$

$$\sqrt{1-(v^2/c^2)} = \sqrt{1 - Q_m v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv_{escape}^2$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1 - 2Q_m v_{escape}^2/rc^2}. \quad (5.72)$$

$$\sqrt{1-(v^2/c^2)} = \lim_{v_{escape}\to c}\sqrt{1 - 2Q_m v_{escape}^2/rc^2}$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1 - 2Q_m c^2/rc^2} = \sqrt{1 - 2Q_m/r}. \quad (5.73)$$

Combining equation 5.73 with equation 5.29 yields Schwarzschild's gravitational time dilation with a quantum mass effect:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1 - 2Q_m/r} \quad \wedge \quad t' = t\sqrt{1-(v^2/c^2)}$$
$$\Rightarrow \quad t' = t\sqrt{1 - 2Q_m/r}. \quad (5.74)$$

Schwarzschild defined the black hole event horizon radius, $r_s := 2Gm/c^2$. The radius with the quantum extension is $r_s := 2Q_m$. At this point the exact same equations 5.36 through 5.39 yield what looks like the same Schwarzschild black hole metric.

5.19. **Quantum extension to Newton's gravity force.** The quantum mass effect is easier to understand in the context Newton's gravity equation than in general relativity, because the metric equations and solutions in the EFEs are much more complex. From equations 5.75 and 5.1:

$$m/\sqrt{(r/c_m)^2 + (k_m/r)^2} = 1 \quad \wedge \quad r^2/(ct)^2 = 1$$
$$\Rightarrow \quad r^2/(ct)^2 = m/\sqrt{(r/c_m)^2 + (k_m/r)^2}$$
$$\Rightarrow \quad r^2/t^2 = c^2 m/\sqrt{(r/c_m)^2 + (k_m/r)^2}. \quad (5.75)$$

$$r^2/t^2 = c^2m/\sqrt{(r/c_m)^2 + (k_m/r)^2}$$
$$\Rightarrow \quad (m/r)(r^2/t^2) = (m/r)(c^2m/\sqrt{(r/c_m)^2 + (k_m/r)^2})$$
$$\Rightarrow \quad F := mr/t^2 = c^2m^2/\sqrt{(r^4/c_m^2) + k_m^2}. \tag{5.76}$$

$$F = c^2m^2/\sqrt{(r^4/c_m^2) + k_m^2} \quad \wedge \quad \forall\, m \in \mathbb{R},\, \exists\, m_1, m_2 \in \mathbb{R} :\, m_1m_2 = m^2$$
$$\Rightarrow \quad F = c^2m_1m_2/\sqrt{(r^4/c_m^2) + k_m^2}. \tag{5.77}$$

5.20. **Quantum extension to Coulomb's force.**

$$q^2/((r/c_q)^2 + (k_q/r)^2) = 1 \quad \wedge \quad r^2/(ct)^2 = 1$$
$$\Rightarrow \quad r^2/(ct)^2 = q^2/((r/c_q)^2 + (k_q/r)^2)$$
$$\Rightarrow \quad r^2/t^2 = c^2q^2/((r/c_q)^2 + (k_q/r)^2). \tag{5.78}$$

$$(1/r)(r^2/t^2) = (1/r)(c^2q^2/((r/c_q)^2 + (k_q/r)^2))$$
$$\Rightarrow \quad r/t^2 = c^2q^2/(r^3/c_q^2 + k_q^2/r). \tag{5.79}$$

$$\forall\, q \in \mathbb{R} :\, \exists\, q_1, q_2 \in \mathbb{R} :\, q_1q_2 = q^2 \quad \wedge \quad r/t^2 = c^2q^2/(r^3/c_q^2 + k_q^2/r)$$
$$\Rightarrow \quad \exists\, q_1, q_2 \in \mathbb{R} :\, r^2/t^2 = c^2q_1q_2/(r^3/c_q^2 + k_q^2/r). \tag{5.80}$$

$$r^2/t^2 = c^2q_1q_2/(r^3/c_q^2 + k_q^2/r) \quad \wedge \quad m = r/c_m$$
$$\Rightarrow \quad F := mr/t^2 = (c^2/c_m)q_1q_2/(r^2/c_q^2 + k_q^2/r^2). \tag{5.81}$$

6. Insights and implications

(1) The ruler measure (2.1) and convergence theorem (2.2) were shown to be useful tools for proving that a countable sets of n-tuples imply a corresponding real-valued equation.

(2) Where the total n-volume is both the sum and subtraction of n-volumes, the $n = 2$ case of the sum of volumes equation (4.3) is the vector inner product. The distributive and associate laws of multiplication and addition allow the $\pm$ signed volumes to represented as each domain interval length multiplied by a $\pm$-signed unit values:
$$\alpha_i, \beta_i \in \{-1, 1\}, \quad d^2 = \textstyle\sum_{i=1}^{m} (a_i\alpha_i)(b_i\beta_i) := \mathbf{a} \cdot \mathbf{b}. \tag{6.1}$$

(3) Defining all Euclidean and non-Euclidean distance measures as
$$\forall\, n,\, d : \quad f_n(d) = v = \textstyle\sum_{i=1}^{m} v_i, \quad d = f_n^{-1}(v) = f_n^{-1}(\sum_{i=1}^{m} v_i) : \tag{6.2}$$

   (a) shows the intimate relation between distance and volume that definitions, like inner product space and metric space, ignore [20] [7] [16];

   (b) is a more simple and concise definition of a distance measure that includes the properties used in the definitions of inner product space and metric space [20] [7] [16].

(4) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equates Euclidean distance to a straight line [10]. And analytic proofs sum infinitesimal Euclidean distances, $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$, where the Euler-Lagrange equation is used find the minimum solution, which is the straight line equation, $y = mx + b$ [2]. In both cases, it is assumed that the straight line is the smallest distance.

Without using the notion of a straight line: The Minkowski distance, $d = (\sum_{i=1}^{m} d_i^n)^{1/n}$, was derived without the notion of straight lines (4.4). And the proof that all Minkowski distances imply the triangle inequality (4.11) is also a proof that Euclidean distance is the shortest distance:

$$d(u,w) \le d(u,v) + d(v,w)$$
$$\Rightarrow \quad d(u,w) = (a^2+b^2)^{1/2} \le (a^2)^{1/2} + (b^2)^{1/2} = a + b. \tag{6.3}$$

At this point, one might ask if there are curves shorter than the taxicab distance, $a+b$, that are also shorter than the Euclidean distance, $(a^2+b^2)^{1/2}$. The answer is that, in an Euclidean plane, there are only 2 distances, taxicab and Euclidean. Every curve is either the sum of infinitesimal taxicab distances or the sum of infinitesimal Euclidean distances.

(5) The left side of the distance sum inequality (4.6),

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \le (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}, \tag{6.4}$$

differs from the left side of Minkowski's sum inequality [12]:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \le (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \tag{6.5}$$

(a) The two inequalities are only the same where $n = 1$.

(b) The distance sum inequality (4.6) is a more fundamental inequality because the proof does not require the convexity and Hölder's inequality assumptions required to prove the Minkowski sum inequality [12].

(c) The distance sum inequality term, $\forall\, n > 1$, $v_i^n = a_i^n + b_i^n$: $d = v^{1/n} = (\sum_{i=1}^{m} v_i^n)^{1/n}$, is the Minkowski distance, which makes it directly related to geometry. For example, the $m = 1$ case of the distance sum inequality was used to prove that all Minkowski distances imply the triangle inequality (4.7), where the $n = 2$ case of the triangle inequality is also a proof that the Euclidean distance is always the shortest distance in an Euclidean plane.

But the Minkowski sum inequality term, $\forall\, n > 1$, $v > 0$: $d = v^{1/n} = (\sum_{i=1}^{m}((v_i^n)^{\mathbf{n}}))^{1/n} = (\sum_{i=1}^{m} v_i^{\mathbf{n^2}})^{1/n}$, is *not* a Minkowski distance.

(d) The distance sum inequality might be applicable to machine learning.

(6) **Combinatorics**, the ordered set of combinations of countable, disjoint sets (n-tuples), $v_c = \prod_{i=1}^{n} |x_i|$, was proven to imply: the Euclidean volume equation (3.2), the sum of volumes equation (4.3) (which includes the inner product), and the Minkowski distance equation (4.4) (which includes the Manhattan and Euclidean distance equations), without relying on the geometric primitives and relations in Euclidean geometry [10], axiomatic geometry [11] [8] [19], and vector analysis [20].

(7) **Combinatorics**, repeatable sequencing through an ordered set of n number of members to yield all $n!$ permutations of its members (without jumping around) was proved to be a cyclic set having $n \leq 3$ members (4.15). Higher dimensions must have different types (members of different sets).
   (a) For example, the vector inner product space can only be extended beyond 3 dimensions if and only if the higher dimensions have non-distance types, for example, time.
   (b) Order and symmetry probably limit the number of non-distance types $\subset \mathbb{R}$ to 3, for example: time, mass, and charge. As shown in the special relativity section (5.8), there is 6-dimensional space-time-mass-charge.
   (c) Each of 3 immediate symmetric cyclic dimensions of space can have at most 3 immediate symmetric cyclic state values of the same type, for example, an immediate symmetric cyclic set of 3 orientations, $\{-1, 0, 1\}$, 3 quark color charges, {red, green, blue}, 3 quark anti-color charges, and so on.
   (d) If the states are not ordered (a bag of states), then a state value is undetermined (or superimposed) until observed (like Schrödenger's poisoned cat being both alive and dead until the box is opened [17]).
   (e) A discrete (point) value has measure 0 (zero-length interval size). The ratio of a time or distance interval length to zero is undefined, which is the reason quantum entangled state values exist independent of time and distance.

(8) For each unit, r_p, of a 3-dimensional distance interval having a length, r, there are units of other types of intervals forming unit ratios (5.3): $c_t = r_p/t_p$, $c_m = r_p/m_p$, $c_q = r_p/q_p \Leftrightarrow$ the inverse proportion ratios (5.4): $k_t = r_p t_p$, $k_m = r_p m_p$, $k_t = r_p q_p$, where r_p, t_p, m_p, and q_p are the Planck units.

(9) Empirical laws *describe* relations. Deriving the laws from the ratios *explains* the relations. Further, the derivations of the gravity, charge, relativity, electromagnetic, and quantum physics equations from the ratios were much shorter and simpler than other derivations, which shows that the ratios are an important tool for physicists and engineers.

(10) As shown in subsection 5.10, the derivation of the Schwarzschild's time dilation and black hole metric (5.9) [18] [1] using ratios exposed a way of simplifying the finding of solutions to Einstein's field equations.

(11) The speed of light ratio, c_t, is a component of the constants: $G = c_m c_t^2$, $k_e = c_q^2 c_t^2/c_m$, $\varepsilon_0 = 1/4\pi k_e = 1/4\pi(c_q^2 c_t^2/c_m)$, $\hbar = k_m c_t$.

The only constant derived, in this article, that does not contain c_t is vacuum permeability: $\mu_0 = 4\pi k_e/c_t^2 = 4\pi c_q^2/c_m$.

(12) Using the quantum (Planck) units, r_p and t_p: $r_p/t_p^2 \approx 5.5607262989 \cdot 10^{51}\ m\ s^{-2}$, which suggests a maximum acceleration for masses.

(13) The simplification of μ_0 into the quantum units shows two interesting relationships:

$$
\begin{aligned}
\mu_0 &= 4\pi\frac{k_e}{c_t^2} = 4\pi\frac{c_q^2}{c_m} = 4\pi\frac{(r_p/q_p)^2}{r_p/m_p} = 4\pi\frac{m_p r_p}{q_p^2} = 4\pi\frac{k_m}{q_p^2} \\
&\approx 4\pi\frac{3.5176729162 \cdot 10^{-43}}{3.5176729162 \cdot 10^{-35}} = 4\pi \cdot 10^{-7}\ kg\ m\ C^{-2} = 4\pi \cdot 10^{-7}\ H\ m^{-1}.
\end{aligned} \tag{6.6}
$$

(a) The first time $k_m = m_p r_p$ appears is in the derivation of the Planck relation and Planck constant, $h = k_m c$ (5.5), the second time in the Compton wavelength, $r = k_m/m$ (5.14). And now, k_m appears as a components of k_e, ε_0, and μ_0.
(b) It is an open question why $\frac{c_q^2}{c_m} = \frac{(r_p/q_p)^2}{r_p/m_p} = \frac{k_m}{q_p^2} = 1.0 \cdot 10^{-7}$ exactly.

(14) The fine structure constant, α was derived from the ratio of two forces of two subtypes which reduces to ratio of the square of the subtypes (5.7).
(a) The CODATA electron coupling version of the fine structure constant, α is defined as: $\alpha = q_e^2/4\pi\varepsilon_0\hbar c) = q_e^2/2\varepsilon_0 hc$ [3].
(i) The derivation of α, in this article (5.7), is much simpler because it is the ratio of two subtypes of charge: elementary (electron) charge, q_e^2 and Planck charge, q_p^2: $\alpha = q_e^2/q_p^2 \approx 0.0072973526$, which is the empirical CODATA value [3].
(ii) The following steps show that the CODATA definition reduces to the ratio-derived equation:

$$\begin{aligned}\varepsilon_0 := 1/4\pi k_e = 1/(4\pi(c_q^2 c_t^2/c_m) \quad \wedge \quad \hbar = k_m c_t \quad \wedge \quad h = 2\pi\hbar \\ \Rightarrow \quad \varepsilon_0 hc = 2\pi k_m c_t^2/(4\pi(c_q^2/c_m)c_t^2) = k_m/(2(c_q^2/c_m)) \\ = m_p r_p/(2((r_p/q_p)^2/(r_p/m_p))) = q_p^2/2. \end{aligned} \tag{6.7}$$

$$\alpha = q_e^2/2\varepsilon_0 hc \quad \wedge \quad \varepsilon_0 hc = q_p^2/2 \quad \Rightarrow \quad \alpha = q_e^2/2(q_p^2/2) = q_e^2/q_p^2. \tag{6.8}$$

(b) Other fine structure constants can also be expressed more simply as the ratios of two subtypes of fields, for example, an electron gravity coupling constant can be expressed as the ratio of the rest electron mass to a Planck mass unit: $\alpha_{G_m} = m_e^2/m_p^2$.

(15) The derivations of empirical and hypothesized laws of physics from the ratios expose some **incorrect assumptions** currently used in physics:
(a) Empirical and hypothesized laws of physics use an *opaque* constant, K, that is defined to make an equation, where the units balance, $g = Kf(r, t, \cdots)$. The opacity has led to the *incorrect* assumptions of those constants being fundamental (atomic) constants.
In this article, some opaque constants are derived directly from (composed of) the ratios: gravity, $G = c_m c_t^2$ (5.5), charge, $k_e = c_q^2 c_t^2/c_m$ (5.10), and Planck $h = k_m c_t$ (5.18). $\varepsilon_0 = 1/4\pi k_e = 1/4\pi c_m/((c_q^2/c_m)c_t^2)$ (5.48) and $\mu_0 = 4\pi k_e/c_t^2 = 4\pi c_q^2/c_m$ (5.45).
And the quantum extensions to: Schwarzschild's time dilation (5.73) Newton's gravity force (5.77), and Coulomb's charge force show, that where the quantum effects become measurable, the constants G, k_e, ε_0, and μ_0 no longer exist (are no longer valid).
Therefore, G, k_e, ε_0, μ_0, and h are **not** fundamental constants.
(b) The derivations of: $\nabla \cdot \mathbf{g} = -4\pi G\rho$ from $\mathbf{g} = 2\pi Gm/r^2$ (5.6), $\nabla \cdot \mathbf{E} = -\rho/\varepsilon_0$ from $\mathbf{E} = 2\pi k_e q/r^2$ (5.48), and $\partial\mathbf{B}/\partial t = -\mu_0\rho$ from $\mathbf{B} = 2\pi k_e q/r^2$ (5.51), show that the use of mass and charge density, ρ, are unnecessary complications that obfuscates the pattern, $\nabla \cdot f(x, y, r) = -2k_{x,y}y/r^3$, being derived from the inverse square pattern, $f(x, y, r) = k_{x,y}y/r^2$. And the energy density in the stress-energy tensor, $T_{\mu,\nu}$,

in Einstein's field equations [20] also obfuscates the inverse square assumption.

(c) Einstein's relativity equations assume the Lorentz transformations, that the laws of physics are same at each coordinate point, and that the speed of light is the same at each coordinate point [5] [20]. The derivations, in this article, were made without those assumptions (does even require the notion of light). Assuming Cartesian coordinates at each coordinate point, creates unit ratios, where all equations (laws) derived from the unit ratios must be the same at each coordinate point. Deriving numeric values for the ratios assumes that the ratio, c_t, is the speed of light, but the equations themselves do not require the notion of light.

(d) The derivation of the magnetic field from special relativity (5.42) shows that magnetic field, $\mathbf{B}$, is the spacetime bend (curl) of the electric field, $\mathbf{E}$. The magnetic force is a torque caused by spacetime bending of the radial charge force.
Charged particles have an orientation that rotates, where a half-spin is a π radians rotation of the orientation (dipole moment). Positive and negative charges with orientations in the same direction have opposite spins (angular momentum).

(16) The quantum extensions to: Schwarzschild's time dilation (5.73) black hole metric (5.39), Newton's gravity force (5.77), and Coulomb's charge force (5.80) make quantifiable predictions:

(a) The gravitation and charge forces peak at finite amounts as $r \to 0$: for gravity, $\lim_{r\to 0} F = c^2 m_1 m_2 / k_m$, and for charge, $\lim_{r\to 0} F = 0$. Finite maximum gravity and charge forces: 1) allows radioactivity, finite sloped energy well walls; and 2) eliminates the problem of forces going to infinity as $r \to 0$, which might eliminate the need to hypothesize the existence of a weak force and strong force.

(b) The quantum-extended Schwarzschild time dilation and metric, gravity, and charge equations reduce to the classic equations, where the distance between masses and charges is sufficiently large or the masses and charges sufficiently large that the quantum effect is not measurable. **Note** that G, k_e, ε_0, μ_0, and κ (Einstein's constant, which contains G) do not exist (are not valid), where the quantum effects becomes measurable.

(c) And the covariant tensor components, in Einstein's field equations, that had the units $1/distance^2$, will now have the more complex units, $1/\sqrt{(distance^4/c_m^2) + k_m^2}$.

(d) $1/\sqrt{(distance^4/c_m^2) + k_m^2}$ implies that as distance $\to 0$, spacetime curvature peaks at a finite amount, which might imply that black holes have sizes > 0 (might not be singularities). Black hole evaporation might be possible. If there was a "big bang," then it might not have originated from a singularity.

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