The Set Properties Generating Geometry and Physics

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Abstract. Volume and the Minkowski distances/Lp norms (e.g., Euclidean distance) are derived from a set and limit-based foundation without referencing the primitives of geometry. Sequencing a set of n number of distance intervals/dimensions in all non-zero Levi-Civita permutations via a strict linear order of successor/predecessor relations is a cyclic set limiting n to at most 3. Therefore, all other interval lengths have different types that can only be related to a distance interval length via conversion ratios. The ratios and geometry proofs provide simpler derivations of the spacetime, Newton's gravity, Coulomb's charge force, and Einstein-Planck equations and new insights into their corresponding constants. All proofs are verified in Coq.

Contents

1. Introduction

Mathematical (real) analysis can construct differential calculus from a set and limit-based foundation without the need to reference the primitives of Euclidean geometry, like straight line, angle, slope, etc. But the Riemann and Lebesgue integrals and measure theory (for example, Hilbert spaces and the Lebesgue measure) define Euclidean volume as the product of interval lengths. And the inner product, vector norm, and metric space use Euclidean distance and its properties as definitions [Gol76] [Rud76]. Here, volume and distance are derived from a very simple set and limit-based foundation without the hand-waving references to side, angle, triangle, rectangle, etc. for justification.

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An intuitive set-based definition of volume is the cardinal, v_c, of the set of all possible n-tuples from the countable disjoint sets, $x_i \in \{x_1, \cdots, x_n\}$: $v_c = \prod_{i=1}^{n} |x_i|$, where the cardinal of x_i, $|x_i|$, is a countable distance. It will be proved that Euclidean volume is derived from the countable volume case, where each countable set, x_i, is a set of *approximate* partitions of an interval $\subset \mathbb{R}$, each partition having the same size, κ, in each set. That is: $v = \lim_{\kappa\to 0} v_c \cdot \kappa = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|) \cdot \kappa = \cdots = \prod_{i=1}^{n} d_i$, where d_i is the length (distance measure) of an interval $\subset \mathbb{R}$.

Each Euclidean and non-Euclidean n-volume, v, has a corresponding cuboid n-volume, which is an instance of a cuboid countable n-volume: $v = \lim_{\kappa\to 0} v_c \cdot \kappa = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x|) \cdot \kappa = \cdots = d^n$. And an n-volume can only be the sum of n-volumes: $v = \sum_{i=1}^{m} v_i = \sum_{i=1}^{m} \lim_{\kappa\to 0}(\prod_{j=1}^{n} |x_i|) \cdot \kappa = \cdots = \sum_{i=1}^{m} d_i^n \to d = (\sum_{i=1}^{m} d_i^n)^{1/n}$, which are the L_p norms (Minkowski distances) that have the properties of a metric space. That is, every "geometric" distance measure is an inverse function of an n-volume that has a corresponding Minkowski distance.

The permutation values of the Levi-Civita pseudo-tensor are: $\epsilon_{ijk} = 1$, where the 3 dimensions of space are sequenced in the cyclic successor order, $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$; $\epsilon_{ijk} = -1$, where sequenced in cyclic predecessor order, $(i, j, k) \in \{(3, 2, 1), (2, 1, 3), (1, 3, 2)\}$; and otherwise $\epsilon_{ijk} = 0$. Proving that a set of intervals that can be sequenced in all n-at-a-time permutations via a strict linear order of successor/predecessor relations is a cyclic set limited to $n \leq 3$, implies that all additional interval lengths have different types that can only be related to a 3-dimensional distance via unit-factoring, conversion ratios. The ratios combined with the distance and volume proofs provide simpler derivations of the spacetime, Newton's gravity, Coulomb's charge force, and Einstein-Planck equations and exposes the ratios that generate the gravity, charge, and Planck constants.

All the proofs in this article are trivial. But to ensure confidence, all the proofs have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

Derivatives and integrals use a 1-1 correspondence between the infinitesimals of each interval, where the size of the infinitesimals in each interval are proportionate to the size of the interval, which precludes using derivatives and integrals to directly express many-to-one, one-to-many, and many-to-many (Cartesian product) mappings between same-sized, size κ, infinitesimals in different-sized intervals. Further, using tools that define Euclidean volume and distance precludes using those tools to derive Euclidean volume and distance.

Therefore, a different tool is used here. A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals (infinitesimals), where each infinitesimal has the *same* size, κ. The ruler is both an inner and outer measure of an interval.

Definition 2.1. Ruler measure, M: $\forall\, [a, b] \subset \mathbb{R},\ \ s = b - a \ \wedge\ \kappa > 0 \ \wedge\ (p = floor(s/\kappa) \ \vee\ p = ceiling(s/\kappa)) \ \wedge\ M = \sum_{i=1}^{p} \kappa = p\kappa$.

Theorem 2.2. *Ruler convergence:* $M = \lim_{\kappa\to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : \ y \leq x, \ y \in \mathbb{Z}, \ x \in \mathbb{R}\})$:

$$\forall\, \kappa > 0, \ p = floor(s/\kappa) \ \wedge \ 0 \leq |floor(s/\kappa) - s/\kappa| < 1 \ \Rightarrow \ |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by κ:

$$\forall\, \kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \delta = \epsilon \quad := \quad M = \lim_{\kappa \to 0} p\kappa = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}}, \ 3.14_{\kappa=10^{-2}}, \ 3.141_{\kappa=10^{-3}}, ..., \pi_{\lim_{\kappa\to 0}}$.

LEMMA 2.3. $\forall\, n \geq 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa$.

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa \to 0} \kappa^n = 0. \end{aligned} \tag{2.5}$$

$$\lim_{\kappa \to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa \to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa. \tag{2.6}$$

$\square$

3. Volume

DEFINITION 3.1. Countable volume, v_c, is the number of Cartesian product mappings (n-tuples) between the members of n number of disjoint, countable domain sets, x_i, where the cardinal, $|x_i|$, is a countable distance:

$$\exists\, n \in \mathbb{N}, \ v_c, \in \{0, \ \mathbb{N}\}, \ \ x_i \in \{x_1, \cdots, x_n\}, \ \bigcap_{i=1}^n x_i = \emptyset : \quad v_c = \prod_{i=1}^n |x_i|. \tag{3.1}$$

THEOREM 3.2. *Euclidean volume, $v = \prod_{i=1}^n s_i$, is the countable volume case, where each countable set, x_i, is a set of same-sized, size κ, partitions of an interval, $[a_i, b_i] \subset \mathbb{R}$.*

$$\begin{aligned} &\forall [a_i, b_i], \ [v_a, v_b] \subset \mathbb{R}, \ \ s_i = b_i - a_i, \ \ v = v_a - v_b, \ \ v_c = \prod_{i=1}^n |x_i| \\ &\qquad \Rightarrow \quad v = \prod_{i=1}^n s_i. \end{aligned} \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing $|x_i|$ number of size κ partitions and apply ruler convergence (2.2):

$$\forall\, i \ n \in \mathbb{N}, \ \ i \in [1, n], \ \kappa > 0 \ \wedge \ floor(s_i/\kappa) = |x_i| \ \Rightarrow \ s_i = \lim_{\kappa \to 0}(|x_i| \cdot \kappa). \tag{3.3}$$

$$s_i = \lim_{\kappa \to 0}(|x_i| \cdot \kappa) \quad \Leftrightarrow \quad \prod_{i=1}^n s_i = \prod_{i=1}^n \lim_{\kappa \to 0}(|x_i| \cdot \kappa). \tag{3.4}$$

$$\prod_{i=1}^n s_i = \prod_{i=1}^n \lim_{\kappa \to 0}(|x_i| \cdot \kappa) \quad \Leftrightarrow \quad \prod_{i=1}^n s_i = \lim_{\kappa \to 0}(\prod_{i=1}^n |x_i|) \cdot \kappa^n. \tag{3.5}$$

Apply lemma 2.3 to equation 3.5:

$$\prod_{i=1}^{n} s_i = \lim_{\kappa\to 0}(\prod_{i=1}^{n}|x_i|)\cdot\kappa^n \quad\wedge\quad \lim_{\kappa\to 0}\kappa^n = \lim_{\kappa\to 0}\kappa \quad\Rightarrow\quad \prod_{i=1}^{n} s_i = \lim_{\kappa\to 0}(\prod_{i=1}^{n}|x_i|)\cdot\kappa. \tag{3.6}$$

Apply the ruler (2.1) and ruler convergence (2.2) to v:

$$\exists\, v\in\mathbb{R}:\; v_c = floor(v/\kappa) \quad\Leftrightarrow\quad v = \lim_{\kappa\to 0} v_c\cdot\kappa. \tag{3.7}$$

Multiply both sides of the countable volume equation 3.1 by κ:

$$v_c = \prod_{i=1}^{n}|x_i| \quad\Leftrightarrow\quad v_c\cdot\kappa = (\prod_{i=1}^{n}|x_i|)\cdot\kappa \tag{3.8}$$

$$v_c\cdot\kappa = (\prod_{i=1}^{n}|x_i|)\cdot\kappa \quad\Leftrightarrow\quad \lim_{\kappa\to 0} v_c\cdot\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^{n}|x_i|)\cdot\kappa. \tag{3.9}$$

Combine equations 3.7, 3.9, and 3.6:

$$v = \lim_{\kappa\to 0} v_c\cdot\kappa \quad\wedge\quad \lim_{\kappa\to 0} v_c\cdot\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^{n}|x_i|)\cdot\kappa \quad\wedge\quad \lim_{\kappa\to 0}(\prod_{i=1}^{n}|x_i|)\cdot\kappa = \prod_{i=1}^{n} s_i \quad\Leftrightarrow\quad v = \prod_{i=1}^{n} s_i. \tag{3.10}$$ □

4. Distance

4.1. Countable cuboid n-volume.

Definition 4.1. The countable cuboid volume, d_c^n, is the sum of m number of sets of countable cuboid volumes.

$$\forall\, n\in\mathbb{N},\quad d_c\in\{0,\mathbb{N}\}\quad \exists\, m\in\mathbb{N},\quad x_1,\cdots,x_m\in X,\quad \bigcap_{i=1}^{m} x_i = \emptyset:\quad d_c^n = \sum_{i=1}^{m}|x_i|^n.$$

4.2. Minkowski distance (L_p norm).

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

Theorem 4.2. *The Minkowski distances (L_p norms) are derived from the sum of countable cuboid n-volumes* (4.1).

$$d_c^n = \sum_{i=1}^{m}|x_i|^n \quad\Rightarrow\quad \exists\, d, s_1,\cdots,s_m\in\mathbb{R}:\quad d = (\sum_{i=1}^{m} s_i^n)^{1/n}.$$

Proof. Apply the ruler (2.1):

$$\exists\, d, s_1,\cdots,s_m\in\mathbb{R}:\; d_c = floor(d/\kappa) \quad\wedge\quad |x_i| = floor(s_i/\kappa). \tag{4.1}$$

Apply the ruler convergence (2.2):

$$|x_i| = floor(s_i/\kappa) \quad\Rightarrow\quad s_i = \lim_{\kappa\to 0}|x_i|\cdot\kappa. \tag{4.2}$$

$$d_c^n = \sum_{i=1}^{m}|x_i|^n \quad\Rightarrow\quad d^n = \lim_{\kappa\to 0}(d_c\cdot\kappa)^n = \lim_{\kappa\to 0}(\sum_{i=1}^{m}(|x_i|^n)\cdot\kappa. \tag{4.3}$$

Apply lemma 2.3 to equation 4.3 and substitute equation 4.2:

$$d^n = \lim_{\kappa\to 0}(\sum_{i=1}^{m}(|x_i|^n)\cdot\kappa \quad\wedge\quad \lim_{\kappa\to 0}\kappa^n = \lim_{\kappa\to 0}\kappa \quad\Rightarrow\quad d^n = \lim_{\kappa\to 0}\sum_{i=1}^{m}(|x_i|^n)\cdot\kappa^n = \lim_{\kappa\to 0}\sum_{i=1}^{m}(|x_i|\cdot\kappa)^n. \tag{4.4}$$

Apply equation 4.2 to equation 4.4:

$$d^n = \lim_{\kappa\to 0}\sum_{i=1}^{m}(|x_i|\cdot\kappa)^n \quad\wedge\quad s_i = \lim_{\kappa\to 0}|x_i|\cdot\kappa \quad\Rightarrow\quad d^n = \sum_{i=1}^{m} s_i^n. \tag{4.5}$$

$$d^n = \sum_{i=1}^{m} s_i^n \quad\Leftrightarrow\quad d = (\sum_{i=1}^{m} s_i^n)^{1/n}. \tag{4.6}$$ □

4.3. Distance inequality. Proving that all Minkowski distances (L_p norms) satisfy the metric space triangle inequality requires another inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N},\ \ v_a, v_b \geq 0:\ \ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand the n-volume, $(v_a^{1/n} + v_b^{1/n})^n$, using the binomial expansion:

$$\begin{aligned}(4.7)\quad &\forall\, v_a, v_b \geq 0: \quad v_a + v_b \leq v_a + v_b + \\ &\textstyle\sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n.\end{aligned}$$

Take the n^{th} root of both sides of the inequality:

$$(4.8)\quad \forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad \square$$

4.4. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N},\ \ a_i, b_i \geq 0:\ \ (\textstyle\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$\begin{aligned}(4.9)\quad &\forall\, m, n \in \mathbb{N},\ v_a, v_b \geq 0: \quad v_a = \textstyle\sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge \\ &(v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad ((\textstyle\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} = \\ &(\textstyle\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \quad \square\end{aligned}$$

4.5. Metric Space. All Minkowski distances (L_p norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(4.10)\qquad (u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}.$$

Apply the distance inequality (4.3) to the inequality 4.10:

$$\begin{aligned}(4.11)\quad &(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \ \wedge\ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ &\qquad \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ \Rightarrow \quad &(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ &\Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq \\ &\qquad (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \quad \square\end{aligned}$$

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

Proof. By the commutative law of addition:

$$\forall\, p\ :\ p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \tag{4.12}$$
$$\Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \quad \square$$

Theorem 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

Proof. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0. \tag{4.13}$$

$$p \geq 1,\ \ u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \tag{4.14}$$

$\square$

Theorem 4.8. *Identity of Indiscernibles:* $d(u, u) = 0$.

Proof. From the non-negativity property (4.7):

$$d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0 \tag{4.15}$$
$$\Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0.$$

$$d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v. \tag{4.16}$$

$$d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \tag{4.17}$$

$\square$

5. Applications to physics

5.1. At most 3 dimensions of physical space. The following two proofs are in the physics section because limiting the distance intervals/dimensions to a strict linearly ordered set that can be sequenced in all n-at-a-time permutations, is an additional restriction on volume and distance used to explain why physical space is limited to a cyclic set of 3 dimensions.

Definition 5.1. Strict linearly ordered set:

$$\forall\, i\ n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i\ \in \{x_1, \ldots, x_n\},$$
$$\textit{successor}\ x_i = x_{i+1} \ \ \wedge\ \ \textit{predecessor}\ x_{i+1} = x_i.$$

Definition 5.2. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i\ j\ n \in \mathbb{N},\ \ \forall\, x_i\ x_j \in \{x_1, \ldots, x_n\},\ \ \textit{successor}\ x_i = x_j \ \ \Leftrightarrow\ \ \textit{predecessor}\ x_j = x_i.$$

Theorem 5.3. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n\ \ \wedge\ \ j = 1\ \ \Rightarrow\ \ \textit{successor}\ x_n = x_1\ \ \wedge\ \ \textit{predecessor}\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

Proof. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (5.2):

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \textit{successor}\ x_i = x_j\ \ \Rightarrow\ \ \textit{successor}\ x_n = x_1. \tag{5.1}$$

Applying the definition of the symmetry property (5.2) to conclusion 5.1:

$$\textit{successor}\ x_i = x_j\ \ \Rightarrow\ \ \textit{predecessor}\ x_j = x_i\ \ \Rightarrow\ \ \textit{predecessor}\ x_1 = x_n. \tag{5.2}$$

$\square$

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3).

DEFINITION 5.5. Successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize). \tag{5.3}$$

DEFINITION 5.6. Predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{5.5}$$

Prove that every member is adjacent to every other member, where $setsize \in \{1, 2, 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \tag{5.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \tag{5.7}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1). \tag{5.8}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.9}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1). \tag{5.11}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize). \tag{5.12}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.13}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \tag{5.14}$$

Member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not ($\neg$) a successor of member 1 for all $setsize > 3$:

$$\begin{aligned} \forall\, setsize > 3 : \quad & \neg Successor(1, 3, setsize > 3) \\ & \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \end{aligned} \tag{5.15}$$

Member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$:

$$\begin{aligned} \forall\, setsize > 3 : \quad & \neg Predecessor(1, 3, setsize > 3) \\ & \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \end{aligned} \tag{5.16}$$

For all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric):

$$(5.17)\quad \forall\, setsize > 3: \quad \neg Adjacent(1, 3, setsize > 3) \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square$$

5.2. Spacetime geometry. From the Euclidean volume proof (3.2), two independent (disjoint) intervals, $[0, r_1]$ and $[0, r_2]$, defines an Euclidean 2-space. From the Minkowski distance proof (4.2), the disjoint intervals each have corresponding disjoint cuboid 2-volumes that sum to a cuboid 2-volume: $r^2 = r_1^2 + r_2^2$. And from the 3D proof (5.4), where r, r_1, and r_2 are 3-dimensional distances, any other interval length, t, must have a different type that is related to the distances via unit-factoring, conversion ratios:

$$(5.18)\quad r^2 = r_1^2 + r_2^2 \quad \wedge \quad \exists\, r_c, t_c, c \in \mathbb{R}: (r_c/t_c) = c = r/t \quad \Rightarrow \quad (ct)^2 = r_1^2 + r_2^2.$$

$$(5.19)\quad (ct)^2 = r_1^2 + r_2^2 \quad \wedge \quad \exists\, r_v, t_v, v \in \mathbb{R}: (r_v/t_v) = v = r_2/t \quad \Rightarrow \quad r_1 = \sqrt{(ct)^2 - (vt)^2} = ct\sqrt{1 - (v/c)^2}.$$

Local distance, r_1, contracts relative to r as $v \to c$:

$$(5.20)\quad r_1 = ct\sqrt{1 - (v/c)^2} \quad \wedge \quad ct = r \quad \Rightarrow \quad r_1 = r\sqrt{1 - (v/c)^2}.$$

Local time, t_1, dilates (gets smaller) relative to t as $v \to c$:

$$(5.21)\quad r_1 = ct\sqrt{1 - (v/c)^2} \quad \wedge \quad t_1 = r_1/c \quad \Rightarrow \quad t_1 = t\sqrt{1 - (v/c)^2}.$$

Using the (-+++) form of Minkowski's flat spacetime [**Bru17**], the size of the event separation interval, r_1^2, is:

$$(5.22)\quad r_1^2 = -r_2^2 + r^2 \ \wedge \ ct = r_2 \ \wedge \ r^2 = x^2 + y^2 + z^2 \quad \Rightarrow \quad r_1^2 = -(ct)^2 + x^2 + y^2 + z^2.$$

5.3. Newton's gravity force equation. From the 3D proof (5.4), where r is a 3-dimensional distance, a mass interval length, m, must have a different type that is related to the distance via a unit-factoring, conversion ratio, $r = (r_G/m_G)m$:

$$(5.23)\quad F := m_0 a := m_0 r/t^2 \ \wedge \ m_0 = m_1 \ \wedge \ \exists\, m_G, r_c, m_2 \in \mathbb{R}: \ r = (r_G/m_G)m_2 \Rightarrow F := m_0 r/t^2 = (r_c/m_G)m_1 m_2/t^2,$$

where m_0 is a rest mass, $a := r/t^2$, is acceleration.

Using the local frame of reference, $r = ct\sqrt{1 - (v/c)^2}$, and $v = 0$:

$$(5.24)\quad r = ct \quad \wedge \quad F = (r_c/m_G)m_1 m_2/t^2 \quad \Rightarrow \quad F = ((r_c/m_G)c^2)m_1 m_2/r^2 = G m_1 m_2/r^2,$$

where the constant, $G = (r_c/m_G)c^2$, has the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$. And where $|v| > 0$, $F = (r_c/m_G)(c^2 - v^2)m_1 m_2/r^2$.

5.4. Coulomb's charge force. From the 3D proof (5.4), where r is a 3-dimensional distance, a charge interval length, q, must have a different type that is related to the distance via a unit-factoring, conversion ratio, $r = (r_C/q_C)q$:

$$F := m_0 a := m_0 r/t^2 \;\wedge\; m_0 = (m_C/q_C)q_1 \;\wedge\; r = (r_c/q_C)q_2 \\ \Rightarrow F := m_0 r/t^2 = (m_C/q_C)(r_c/q_C)q_1 q_2/t^2, \tag{5.25}$$

where m_0 is a rest mass, $a := r/t^2$, is acceleration.

Using the local frame of reference, $r = ct\sqrt{1-(v/c)^2}$, and $v = 0$:

$$r = ct = (r_c/t_c)t \quad \wedge \quad F = (m_C/q_C)(r_c/q_C)q_1 q_2/t^2 \\ \Rightarrow \quad F = (m_C/q_C)(r_c/q_C)(r_c/t_c)^2 q_1 q_2/r^2. \tag{5.26}$$

$$a_G = r_c/t_c^2 \quad \wedge \quad F = (m_C/q_C)(r_c/q_C)(r_c/t_c)^2 q_1 q_2/r^2 \\ \Rightarrow \quad F = (m_C a_G)(r_c/q_C)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2, \tag{5.27}$$

where the predicted charge constant, $k_e = (m_C a_G)(r_c/q_C)^2$, has the SI units: $N \cdot m^2 \cdot C^{-2}$. And where $|v| > 0$, $F = (m_C/q_C)(r_c/q_C)(c^2 - v^2)q_1 q_2/r^2$.

5.5. Einstein-Planck and energy-charge equations: Combining the unit-factoring conversion ratios: $m = (m_p/r_p)r$ and $r/t = r_c/t_c = c \Rightarrow m(ct)^2 = (m_p/r_p)r \cdot r^2$. Dividing both sides by t^2: $E = mc^2 = (m_p/r_p)r \cdot (r/t)^2 = (m_p/r_p)r \cdot (r_c/t_c)^2 = (m_p r_c/r_p t_c)c \cdot r = (m_p r_c c) \cdot (r/(r_p t_c)) = h \cdot f$, which is the Einstein-Planck equation, where the Planck constant is, $h = m_p r_c c$, and $f = r/(r_p t_c)$ is the frequency in cycles per second. $h/c = m_p r_c = k_W$, where r_c is the work displacement (Compton wavelength) of a particle with the rest mass, m_p.

Likewise, for charge, $r = (r_C/q_C)q = (r_p/m_p)m \Rightarrow m = (m_p/r_p)(r_C/q_C)q \Rightarrow E = mc^2 = (m_p/r_p)(r_C/q_C)qc^2 = (m_p r_c c) \cdot (r_C q/r_p q_C t_c) = h \cdot f'$.

5.6. General relativity (Einstein Field Equations): Einstein's gravity constant contains Newton's constant, $k = 8\pi G/c^4$. But, the derivation of G is only valid when the local velocity is 0. Otherwise, $k = 8\pi(r_G/m_G)(c^2 - v^2)/c^4$. EFE solutions using numerical analysis might be simpler and faster using ruler-based (same-sized) infinitesimals, κ, where (using Einstein notation): $\partial f(x_{\mu\nu})/\partial x^i = lim_{\kappa\to 0}\,(f(x_{\mu\nu}+\kappa) - f(x_{\mu\nu}))/\kappa$, because the size of the infinitesimal only needs to be computed once across all domain intervals for each iteration.

6. Insights and implications

(1) It was shown that every distance measure that is a inverse function of an n-volume has a corresponding Minkowski distance (4.2), which have the properties of metric space (4.5). Therefore, if the definition of a complete metric space allows functions that cannot be reduced to a Minkowski distance, then the definition of a complete metric space is not a sufficient filter to obtain only "geometric" distance measures within an n-volume.

(2) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line (equation), where it is assumed that the straight line length is the smallest distance [**Joy98**]. And proofs that a straight line (equation) is the smallest distance have equated the straight line to the Euclidean distance.

The calculus of variations cannot be used to prove that the smallest distance is the Euclidean distance because the integrals assume Euclidean distance infinitesimals, which would result in circular logic.

However, it was shown that all geometric distance measures are functions that can be reduced to a function of the sum of cuboid n-volumes, which are the Minkowski distances (4.2), $d = (\sum_{i=1}^{m} s_i^n)^{1/n}$. If m represents the number of domain intervals, one interval from each dimension, then $1 \leq n \leq m$. And $m = 2 \Rightarrow 1 \leq n \leq 2$, which constrains all Minkowski distances to a range from Manhattan distance (the largest distance) to Euclidean distance (the smallest distance) in Euclidean (flat) 2-space.

(3) Hilbert spaces allow fractional dimensions (fractals), which is the case of intersecting domain sets and requires generalizing the countable volume definition (3.1), $v_c = \prod_{i=1}^{n} |x_i|$, to: $v_c = \prod_{i=1}^{n}(|x_i| - |x_i \bigcap (\bigcup_{j=1,i\neq j}^{n} x_j)|)$.

(4) Compare the distance sum inequality (4.4),

$$(\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n},$$

used to prove that all Minkowski distances satisfy the metric space triangle inequality property (4.5), to Minkowski's sum inequality:

$$(\sum_{i=1}^{m}(a_i^n + b_i^n)^n)^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

Note the exponent difference in the left side of the two inequalities:

$$(\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \quad vs. \quad (\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n}.$$

Minkowski's sum inequality proof assumes: convexity and the L_p space inequalities (for example, Hölder's inequality or Mahler's inequality) or the triangle inequality. In contrast, the distance (sum) inequality is a more fundamental inequality that does not require the assumptions of the Minkowski sum inequality.

(5) From the 3D proof (5.4), more intervals than the 3 dimensions of distance intervals must have different types with lengths that are related to a distance interval length, r, via constant, unit-factoring, conversion ratios (both direct and inverse proportion ratios). The direct proportion ratios for time, mass, and charge are: $r = (r_c/t_c)t = ct = (r_c/m_G)m = (r_c/q_C)q$. An inverse proportion ratio is the mass-displacement ratio: $mr = (m_p r_c) = k_W$. The speed of light, c, the gravity constant, G, the charge constant, k_e, and the Planck constant, h were all derived from these constant ratios.

(6) The derivations of the spacetime equations, here, differ from all other derivations and provide insights that the other derivations cannot provide.
   (a) The derivations, here, do not rely on the Lorentz transformations or Einsteins' postulates [1) The laws of physics are the same in every frame of reference; 2) The speed of light is a constant in every frame of reference]. The derivations, here, do not even rely on the notions of time, velocity, and light!
   (b) The derivations, here, rely only on the set-based proofs of the Minkowski distances (4.2) and the 3D proof (5.4) to show the notion of velocity comes from the unit-factoring conversion ratios of an interval length, t, to the distance interval lengths r_1, r_2, and r.

   (c) From equation 5.18: $r^2 = r_1^2 + r_2^2 \Rightarrow r_1, r_2 \leq r \Rightarrow r_1/t, r_2/t \leq r/t$. That is, any ratios (velocities) in the local frames of reference, $v_1 = r_1/t$ and $v_2 = r_2/t$, will always be less than or equal to the ratio (velocity) in the common frame of reference, $c = r/t$.

(7) The derivations in this article show that the spacetime, gravity force, charge force, and Einstein-Planck equations all depend on time being proportionate to distance: $r = (r_c/t_c)t = ct$. For example, from the derivation of Newton's gravity equation (5.24), where $v = 0$: $G = (r_c/m_G)c^2$. Likewise, from the derivation of Coulomb's charge force equation (5.27) the constant, where $v = 0$: $k_e = (m_G/r_c)(r_c/q_C)^2 c^2$. And from the derivation of the Planck constant (5.5), $h = (m_p r_c)c$.

(8) The derivation of the Einstein-Planck equation (5.5), $h = (m_p r_c)c = k_W c$, shows that the Planck constant, h, is the product of two fundamental constants, the work constant, $h/c = m_p r_c = k_W \approx 2.2102190943 \cdot 10^{-42}\ kg\ m$, and the speed of light constant, c.

   The mass-displacement constant, $k_W = m_p r_c$, simplifies physics equations that contain both c and h. The equation says that, for a given amount of energy, a larger mass will have a smaller displacement (Compton wavelength), which makes those simplified equations more intuitive.

(9) Applying the ratios to derive Newton's gravity force (5.3) and Coulomb's charge force (5.4) equations provide some firsts and some new insights into physics:
   (a) These are the first derivations to not assume the inverse square law or Gauss's divergence theorem.
   (b) These are the first derivations to show that the definition of force, $F := m_0 a$, containing acceleration, $a = r/t^2$, where r is a distance that is proportionate to time, t, generates the inverse square law.
   (c) Using Occam's razor, those versions of constants like: charge, vacuum magnetic permeability, etc. that contain the value 4π might be incorrect because those constants are based on the less parsimonious assumption that the inverse square law is due to Gauss's flux divergence on a sphere having the surface area, $4\pi r^2$.
   (d) The derived relativistic gravity and charge force equations are: $F = (r_c/m_G)(c^2 - v^2)m_1 m_2/r^2$ (5.24) and
   $F = (m_C/q_C)(r_c/q_C)(c^2 - v^2)q_1 q_2/r^2$ (5.27).
   Note that $v \to c \Rightarrow F \to 0$. That is, gravity and charge have less effect on a (charged) mass with a higher velocity.

(10) A state is a constant value. Therefore, there are no unit-factoring conversion ratios to distance and time. For example, the spin states of two quantum entangled particles and the polarization states of two quantum entangled photons do not vary with the amount of distance and time between the entangled particles.

(11) It was proved that sequencing through a set, having a strict linear order via the successor/predecessor relations in all n-at-a-time permutations, is a cyclic set with $n \leq 3$ (5.4), which is the most parsimonious explanation for observing only 3 dimensions of physical distance and volume.
   (a) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (5.4), each member

being an ordered and symmetric set of 3 dimensions (three balls).

(b) Each of 3 ordered and symmetric dimensions of space can have only 3 sequentially ordered and symmetric state values. For example, the ordered and symmetric set of the 3 vector orientations, $\{-1, 0, 1\}$, per dimension, which makes physical space chiral (allows mirror images, the right-hand rule for the vector cross product, etc.).

(c) Each of the 3 ordered and symmetric dimensions of space could correspond to an unordered collection (bag) of discrete state values. Bags are non-deterministic. For example, every time an unordered binary state is "pulled" from a bag, there is a 50 percent chance of getting one of the binary values.

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