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Some Set Properties Underlying Geometry and Physics

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Abstract

Euclidean volume and some distance equations are proved to be instances of abstract sets of ordered combinations (n-tuples). A permutation property of a set of unordered domain elements is proved to constrain a list of those elements to at most 3, for example, 3 distance elements (height, width, and depth) and 3 non-distance elements (time, mass, and charge). Where the space near each local coordinate point is Euclidean-like, each 3-dimensional distance unit corresponds to units of time, mass, and charge. Unit ratios are used for shorter and simpler derivations of gravity, charge, electromagnetic, relativity, quantum physics equations, and constants. The ratios are used for simple quantum extensions to classical and relativity equations. The proofs are verified in Rocq.

[1]

CONTENTS

Introduction	3
Ruler measure and convergence	6
Volume	8
Euclidean volume	8
Distance	9
Vector inner product	9
Minkowski distance (p -norm)	10
Metric Space	11
Properties limiting a set to at most 3 elements	12
Applications to physics	14
Derivation of 3 direct proportion ratios: c_t , c_m , and c_q	14
Derivation of G , and the Newton, Gauss, and Poisson gravity laws	14
Derivation of the special relativity equations	15
Derivation of Schwarzschild's gravitational time dilation and black hole metric	15
Simple method to find general relativity solutions	16
Derivation of k_e , Coulomb charge law, ϵ_0 , E, and Gauss electric field law	17

Derivation of B, μ_0 , and Lorentz's law	18
Derivation of Maxwell-Faraday's law	19
Derivation of the 3 direct proportion ratio values: c_t , c_m , and c_q	20
Derivation of the fine structure constant, α	20
Derivation of the 4 reduced Planck unit values: r_p , t_p , m_p , q_p	20
Derivation of 3 inverse proportion ratios: k_t , k_m , and k_q	21
Derivation of \hbar , h , and the Planck relation	21
Derivation of the Compton wavelength, λ	22
Derivation of Schrödinger's position-space wave equation	22
Derivation of Dirac's wave equation	23
Total of a type	24
Quantum extension to general relativity	24
Quantum extension to Newton's gravity force	25
Quantum extension to Coulomb's charge force	25
Insights and implications	25
References	31

Keywords: mathematical physics, combinatorics, set theory, distance measure, inner product, gravity, charge, electromagnetism, relativity, quantum physics.

INTRODUCTION

Derivations of the gravity, charge, electromagnetic, relativity, and quantum physics equations and constants in this article are much shorter, simpler, and more rigorous than other derivations because the equations and constants are derived from math proof-motivated “first principles” (a small set of simple relations) and assumptions. In contrast, current textbooks derive empirical laws from other empirical laws and use assumptions without math proof-motivation.

Examples of deriving an empirical law from other empirical laws are: Deriving Newton’s law of gravity from Kepler’s laws [2] or Gauss’s law [3] (and vice versa), or deriving Newton’s law as a special case of Einstein’s general relativity equations [4] [5] and Maxwell deriving the electromagnetic wave equation from a collection of several empirical laws [3].

An example of an assumption without mathematical motivation is relativity theory assuming 3 dimensions of space (and 1 dimension of time) [6][5]. Integration, topology, measure theory, Hilbert spaces, linear algebra, (pseudo-)Riemann geometry, etc. have no restriction on the number of dimensions [7][8][9][5][10]. If there were a proof that sets with certain properties are limited to at most 3 elements and the set of physical space dimensions had those same properties, then the assumption would have a mathematical motivation.

The “problem” is mathematical analysis has not provided a rigorous (set, sequence, limit, and combinatorial) foundation that can motivate some common physics assumptions and expose some first principles from which many physics equations can be directly derived. Therefore, the obvious first step is prove that volume and distance equations are instances of abstract sets with combinatorial properties applicable to geometry and physics.

The proofs in this article are informal, where each informal (normal human-readable) proof has a corresponding formal proof, which has been verified using the Rocq proof verification system [11]. The formal proofs are in the Rocq files, “euclidrelations.v” and “threed.v,” which are included as ancillary files.

Let $|x_i|$ be the cardinal of (number of elements in) the countable set, $x_i \in \{x_1, \dots, x_n\}$. Where the number of ordered combinations (n -tuples) of x_1, \dots, x_n is the countable range value, v_c , the Euclidean volume equation will be proved to be an instance v_c :

$$\forall v_c, |x_i| \in \{0, \mathbb{N}\}, x_i \in \{x_1, \dots, x_n\}, v_c = \prod_{i=1}^n |x_i| \Rightarrow v = \prod_{i=1}^n s_i, s_i, v \in \mathbb{R}. \quad (1)$$

For all $n > 1$, there are many cases where different domain values, $|x_1|, \dots, |x_n|$, multiplied yield the same range value, v_c . Inferring a domain value, d_c , from v_c , requires an inverse (bijective) function, $d_c = f_n^{-1}(v_c)$ and $v_c = f_n(d_c)$. The simplest bijective case for $n = 1$ is: $v_c = |x_1| = d_c$. The simplest bijective case for all n that includes the $n = 1$ case is:

$$\exists d_c, v_c, |x_i| \in \{0, \mathbb{N}\} : v_c = \prod_{i=1}^n |x_i| = \prod_{i=1}^n d_c = d_c^n. \quad (2)$$

A set of n -tuples being the union of disjoint subsets of n -tuples implies that the domain value, d_c , is also the inverse function of the sum of n -tuples, where it will be proved that:

$$d_c^n = v_c = \sum_{i=1}^m v_{c_i} = \sum_{i=1}^m (\prod_{j=1}^n |x_{i,j}|) \Rightarrow d^n = \sum_{i=1}^m (\prod_{j=1}^n s_{i,j}). \quad (3)$$

Where each $s_{i,j}$ is \pm -signed and m is the number of dimensions, the $n = 2$ case is the dot product.

Where each v_{c_i} also is the bijective function, $v_{c_i} = d_{c_i}^n$:

$$d_c^n = \sum_{i=1}^m d_{c_i}^n \Rightarrow d^n = \sum_{i=1}^m d_i^n. \quad (4)$$

$|d|$ is the p -norm (Minkowski distance) [12], which will be proved to imply the metric space properties [8]. The $n = 2$ case is the Euclidean distance.

Many sets contain unordered elements, where each element does not have an intrinsic property making it the first, second, \dots , or last (n -th), for example, the set, {height, width, depth} \equiv {depth, width, height}, and the set, {time, mass, charge} \equiv {mass, charge, time}. A set of unordered elements can be placed in a “list,” where the list is sequenced one-by-one in a total order via successor and predecessor functions, $\text{successor}(x_i) = x_{i+1}$ and $\text{predecessor}(x_i) = x_{i-1}$, thereby allowing calculation of the number of n -tuples, volumes, and distances, for example, $v = \prod_{i=1}^n s_i$ and $v = \prod_{i=n}^1 s_i$.

But the union, intersection, multiplication, and addition operations defining the total number of n -tuples and the corresponding volume and distance equations are commutative. The commutative property allows sequencing the *same* totally ordered list of n number of elements in each of the $n!$ possible unique sequences (permutations).

Only a cyclic list, where $\text{successor}(x_n) = x_1$ and $\text{predecessor}(x_1) = x_n$, allows sequencing 1 through n (and n through 1), and also allows each element to be the first element of some of those $n!$ permutations. For example, where $n = 3$, the element s_2 is the first element of the cyclic successor sequence, $v = s_2 \times s_3 \times s_1$, and the cyclic predecessor sequence, $v = s_2 \times s_1 \times s_3$.

The only cyclic lists that allows all $n!$ permutations using the cyclic successor and predecessor functions are where each list element is sequentially adjacent (either an *immediate* cyclic successor or an *immediate* cyclic predecessor) to every other element, herein, referred to as an immediate symmetric cyclic list (ISCL). An ISCL will be proved to have $n \leq 3$ elements.

Application to physics uses the ISCL and Euclidean space properties:

1. **ISCL:** $\{r_1, r_2, r_3\}$ is an ISCL of 3 “distance” domain values, each value $\in \mathbb{R}$, and $\{t \text{ (time)}, m \text{ (mass)}, q \text{ (charge)}\}$ is the ISCL of 3 “non-distance” domain values, each value $\in \mathbb{R}$. Physical space is 6-dimensional: $r_1-r_2-r_3-t-m-q$.
2. **Cartesian:** The Euclidean volume and distance equations are derived from abstract sets of n -tuples, where each n -tuple has a corresponding value, $\kappa \in \mathbb{R}$ (each κ the

same size). Where $|x| = |y|$, each κ_x , has a corresponding κ_y . Therefore, near each Euclidean-like local coordinate point, each unit length, r_p , of $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$ corresponds to unit lengths: t_p of time, t ; m_p of mass, m ; and q_p of charge, q .

A consequence of the 1-1 correspondence of units is the direct proportion ratios: $r = (r_p/t_p)t$, $r = (r_p/m_p)m$, and $r = (r_p/q_p)q$. Letting: $c_t = r_p/t_p$, $c_m = r_p/m_p$, and $c_q = r_p/q_p$, the proofs and the 3 direct proportion ratios, c_t , c_m , and c_q , are used to provide simple and short derivations of: the gravitational constant, $G = c_m c_t^2$, the Newton, Gauss, and Poisson gravity equations [2][3], the special relativity equations [6][5], the Schwarzschild gravitational time dilation and black hole metric equations [13][14] (illustrating a simple method of finding solutions to Einstein's general relativity equations), the Coulomb, Lorentz, and Faraday laws [3] and their related constants $k_e = c_q^2 c_t^2 / c_m$, $\varepsilon_0 = 1/4\pi k_e$, $\mu_0 = 4\pi k_e/c^2$. The values of c_t , c_m , and c_q are derived from the empirical values: $c = c_t$, G , and k_e .

A new rationale for the fine structure electron coupling constant, α , is presented, where α is the ratio of two subtypes of Coulomb's charge force that reduces to the unit-less ratio, $\alpha = q_e^2/q_p^2$, which is simpler and more elucidating than the CODATA standard equation, $\alpha = q_e^2/4\pi\varepsilon_0\hbar c$ [15]. $\alpha = q_e^2/4\pi\varepsilon_0\hbar c$ simplifies to $\alpha = q_e^2/q_p^2$. And the gravitational fine structure constant, $\alpha_G = Gm_e^2/\hbar c$. simplifies to: $\alpha_G = m_e^2/m_p^2$, the ratio of two subtypes of Newton's gravity force.

The reduced Planck constant, q_p , value is derived from the fine structure constant equation, $\alpha = q_e^2/q_p^2$. Using the value of q_p , and the values of the ratios, c_t , c_m , and c_q , the values of the other reduced Planck constants, r_p , t_p , and m_p are derived.

Algebraic manipulation of the 3 direct proportion ratios yields 3 inverse proportion ratios, $r = t_p r_p/t$, $r = m_p r_p/m$. and $r = q_p r_p/q$. And let $k_t = t_p r_p$, $k_m = m_p r_p$, and $k_q = q_p r_p$.

The Planck constants, \hbar , h , and Planck relation are derived from the ratios $c_t = r_p/t_p$ and $r = m_p r_p/m$. The ratios and Planck relation are used to derive the Compton wavelength, the Schrödinger position-space wave equation and Dirac wave equation [3][16][17]. And the inverse proportion ratios are used to add quantum extensions to some general relativity and classical physics equations.

RULER MEASURE AND CONVERGENCE

Definition .1. Ruler measure, $M = \sum_{i=1}^p \kappa = p\kappa$, where $\forall s, \kappa \in \mathbb{R}$, $0 < |\kappa| \leq 1$,

$$(p = \text{floor}(s/\kappa) \vee p = \text{ceiling}(s/\kappa)).$$

Theorem .2. *Ruler convergence:* $M = \lim_{\kappa \rightarrow 0} p\kappa = s$.

The formal proof, “limit_c_0_M_eq_exact_size,” is in the file, euclidrelations.v.

Proof. (epsilon-delta proof)

$\text{floor}(x)$ is the integer part of x . Therefore:

$$\text{floor}(x) = \max(\{y : y \leq x, y \in \mathbb{Z}, x \in \mathbb{R}\}) \Rightarrow |\text{floor}(x) - x| < 1. \quad (5)$$

$$|\text{floor}(s/\kappa) - s/\kappa| < 1 \wedge p = \text{floor}(s/\kappa) \Rightarrow |p - s/\kappa| < 1. \quad (6)$$

Multiply both sides of inequality 6 by $|\kappa|$:

$$|p - s/\kappa| < 1 \Rightarrow |p\kappa - s| < |\kappa| = |\kappa - 0|. \quad (7)$$

$$\begin{aligned} \forall \epsilon = p \wedge |p\kappa - s| < |\kappa - 0| < p \Rightarrow |\kappa - 0| < p \wedge |p\kappa - s| < \epsilon \\ := M = \lim_{\kappa \rightarrow 0} p\kappa = s. \quad \square \end{aligned} \quad (8)$$

The following is an example of ruler convergence where, $s = \pi \Rightarrow p = \text{floor}(s/\kappa) \Rightarrow p\kappa = 3.1_{\kappa=10^{-1}}, 3.14_{\kappa=10^{-2}}, 3.141_{\kappa=10^{-3}}, \dots, \pi_{\lim_{\kappa \rightarrow 0}}$.

Lemma .3. $\forall n \geq 1, 0 < |\kappa| < 1 : \lim_{\kappa \rightarrow 0} \kappa^n = \lim_{\kappa \rightarrow 0} \kappa$.

Proof. The formal proof , “lim_c_to_n_eq_lim_c,” is in the Rocq file, euclidrelations.v.

$$n \geq 1 \wedge 0 < \kappa \leq 1 \Rightarrow 0 < \kappa^n \leq \kappa \Rightarrow |\kappa - \kappa^n| \leq |\kappa| = |\kappa - 0|. \quad (9)$$

$$\begin{aligned} \forall \epsilon = p \wedge |\kappa - \kappa^n| \leq |\kappa - 0| < p \Rightarrow |\kappa - 0| < p \wedge |\kappa - \kappa^n| < p = \epsilon \\ := \lim_{\kappa \rightarrow 0} \kappa^n = 0. \end{aligned} \quad (10)$$

$$\lim_{\kappa \rightarrow 0} \kappa^n = 0 \wedge \lim_{\kappa \rightarrow 0} \kappa = 0 \Rightarrow \lim_{\kappa \rightarrow 0} \kappa^n = \lim_{\kappa \rightarrow 0} \kappa. \quad (11)$$

□

VOLUME

Euclidean volume

Theorem .4. *Euclidean volume,*

$$\forall v_c, d_c, |x_i| \in \{0, \mathbb{N}\}, \quad x_i \in \{x_1, \dots, x_n\}, \quad v_c = \prod_{i=1}^n |x_i| \Rightarrow v = \prod_{i=1}^n s_i, \quad s_i, v \in \mathbb{R}. \quad (12)$$

The formal proof, “Euclidean_volume,” is in the Rocq file, euclidrelations.v.

Proof.

$$v_c = \prod_{i=1}^n |x_i| \Leftrightarrow v_c \kappa = (\prod_{i=1}^n |x_i|) \kappa \Leftrightarrow \lim_{\kappa \rightarrow 0} v_c \kappa = \lim_{\kappa \rightarrow 0} (\prod_{i=1}^n |x_i|) \kappa. \quad (13)$$

Apply the ruler (.1) and ruler convergence (.2) to equation 13:

$$\begin{aligned} \exists v, \kappa \in \mathbb{R} : v_c = \text{floor}(v/\kappa) \Rightarrow v = \lim_{\kappa \rightarrow 0} v_c \kappa \wedge \lim_{\kappa \rightarrow 0} v_c \kappa = \lim_{\kappa \rightarrow 0} (\prod_{i=1}^n |x_i|) \kappa \\ \Rightarrow v = \lim_{\kappa \rightarrow 0} (\prod_{i=1}^n |x_i|) \kappa. \end{aligned} \quad (14)$$

Apply lemma .3 to equation 14:

$$\begin{aligned} v = \lim_{\kappa \rightarrow 0} (\prod_{i=1}^n |x_i|) \kappa \wedge \lim_{\kappa \rightarrow 0} \kappa^n = \lim_{\kappa \rightarrow 0} \kappa \\ \Rightarrow v = \lim_{\kappa \rightarrow 0} (\prod_{i=1}^n |x_i|) \kappa^n = \lim_{\kappa \rightarrow 0} (\prod_{i=1}^n (|x_i| \kappa)). \end{aligned} \quad (15)$$

Apply the ruler (.1) and ruler convergence (.2) to s_i :

$$\exists s_i, \kappa \in \mathbb{R} : \text{floor}(s_i/\kappa) = |x_i| \Rightarrow \lim_{\kappa \rightarrow 0} (|x_i| \kappa) = s_i. \quad (16)$$

$$v = \lim_{\kappa \rightarrow 0} (\prod_{i=1}^n |x_i| \kappa) \wedge \lim_{\kappa \rightarrow 0} (|x_i| \kappa) = s_i \Leftrightarrow v = \prod_{i=1}^n s_i \quad (17)$$

□

Lemma .5. *The number of n-tuples, v_c , is the sum of the number of n-tuples, v_{c_i} , in each subset of n-tuples, implies a volume is the sum of volumes,*

$$v_c = \sum_{i=1}^m v_{c_i} \Rightarrow v = \sum_{i=1}^m v_i, \quad v, v_i \in \mathbb{R}.$$

The formal proof, “sum_of_volumes,” is in the Rocq file, euclidrelations.v.

Proof. From the condition of this theorem:

$$v_c = \sum_{i=1}^m v_{c_i} \Leftrightarrow \lim_{\kappa \rightarrow 0} v_c \kappa = \lim_{\kappa \rightarrow 0} \sum_{j=1}^m (v_{c_i} \kappa). \quad (18)$$

Apply lemma .3 to equation 18:

$$\begin{aligned} \lim_{\kappa \rightarrow 0} v_c \kappa &= \lim_{\kappa \rightarrow 0} (\sum_{j=1}^m v_{c_i}) \kappa \quad \wedge \quad \lim_{\kappa \rightarrow 0} \kappa^n = \lim_{\kappa \rightarrow 0} \kappa \\ &\Leftrightarrow \lim_{\kappa \rightarrow 0} v_c \kappa = \lim_{\kappa \rightarrow 0} \sum_{j=1}^m (v_{c_i} \kappa). \end{aligned} \quad (19)$$

Apply the ruler (.1) and ruler convergence theorem (.2) to equation 19:

$$\begin{aligned} \exists v, v_i : v = \lim_{\kappa \rightarrow 0} v_c \kappa \quad \wedge \quad \lim_{\kappa \rightarrow 0} v_c \kappa = \lim_{\kappa \rightarrow 0} \sum_{j=1}^m (v_{c_i} \kappa) \\ \Rightarrow v == \lim_{\kappa \rightarrow 0} \sum_{j=1}^m (v_{c_i} \kappa). \end{aligned} \quad (20)$$

Apply the ruler (.1) and ruler convergence theorem (.2) to equation 20:

$$v == \lim_{\kappa \rightarrow 0} \sum_{j=1}^m (v_{c_i} \kappa) \quad \wedge \quad \exists v_i, v_{c_i} : v_i = \lim_{\kappa \rightarrow 0} v_{c_i} \kappa \Rightarrow v = \sum_{j=1}^m v_i. \quad (21)$$

□

DISTANCE

Definition .6. Bijective, countable domain value, d_c :

$$\exists v_c, d_c, |x_i| \in \{0, \mathbb{N}\}, x_i \in \{x_1, \dots, x_n\}, v_c = \prod_{i=1}^n |x_i| = \prod_{i=1}^n d_c = d_c^n. \quad (22)$$

Vector inner product

Theorem .7. *Sum of volumes distance:*

$$d_c^n = v_c = \sum_{i=1}^m v_{c_i} \Rightarrow d^n = \sum_{i=1}^m (\prod_{j=1}^n s_{i_j}).$$

The formal proof, “sum_of_volumes_distance,” is in the Rocq file, euclidrelations.v.

Proof. From the sum of volumes lemma .5 and the Euclidean volume theorem .4:

$$\begin{aligned} d_c^n = \sum_{i=1}^m v_{c_i} \Rightarrow d^n = \sum_{i=1}^m v_i \quad \wedge \quad v_i = \prod_{j=1}^n s_{i_j} \\ \Rightarrow d^n = \sum_{i=1}^m v_i = \sum_{i=1}^m (\prod_{j=1}^n s_{i_j}). \quad \square \end{aligned} \quad (23)$$

Note: In the volume proof (.4), where κ is negative, volume is negative. From the lemma (.3), $\forall n \geq 1, 0 < |\kappa| < 1 : \lim_{\kappa \rightarrow 0} \kappa^n = \lim_{\kappa \rightarrow 0} \kappa, \exists \kappa_i \in \{\kappa_1, \dots, \kappa_n\}$. Therefore, κ_i can have negative values, where $\prod_{i=1}^n \kappa_i = \kappa^n$. And where $\kappa_i < 0$, the corresponding $s_{i,j} < 0$. Therefore, the $n = 2$ case of the sum of volumes distance is the dot product.

Minkowski distance (p -norm)

Theorem .8. *Minkowski distance (p -norm):*

$$d_c^n = v_c = \sum_{i=1}^m v_{c_i} = \sum_{i=1}^m d_{c_i}^n \Leftrightarrow d^n = \sum_{i=1}^m d_i^n.$$

The formal proof, “Minkowski_distance,” is in the Rocq file, euclidrelations.v.

Proof. From the sum of volumes distance theorem .7 and the Euclidean volume theorem .4:

$$\begin{aligned} d_c^n = v_c = \sum_{i=1}^m v_{c_i} &\Rightarrow d^n = v = \sum_{i=1}^m v_i \wedge v_i = \prod_{j=1}^n d_i = d_i^n \\ &\Rightarrow d^n = \sum_{i=1}^m d_i^n \quad \square \end{aligned} \quad (24)$$

Theorem .9. *Distance triangle inequality*

$$\forall n \in \mathbb{N}, v_a, v_b \geq 0 : (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

The formal proof, distance_inequality, is in the Rocq file, euclidrelations.v.

Proof. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$\begin{aligned} \forall v_a, v_b \geq 0 : v_a + v_b &\leq v_a + v_b + \sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \\ &\sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n. \end{aligned} \quad (25)$$

Take the n^{th} root of both sides of the inequality 25:

$$\forall v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad (26)$$

□

Theorem .10. *Distance sum inequality*

$$\forall m, n \in \mathbb{N}, a_i, b_i \geq 0 : (\sum_{i=1}^m (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}.$$

The formal proof, distance_sum_inequality, is in the Rocq file, euclidrelations.v.

Proof. Apply the distance triangle inequality (9):

$$\begin{aligned} \forall m, n \in \mathbb{N}, v_a, v_b \geq 0 : v_a = \sum_{i=1}^m a_i^n \wedge v_b = \sum_{i=1}^m b_i^n \wedge (v_a + v_b)^{1/n} &\leq v_a^{1/n} + v_b^{1/n} \\ \Rightarrow ((\sum_{i=1}^m a_i^n) + (\sum_{i=1}^m b_i^n))^{1/n} &= (\sum_{i=1}^m (a_i^n + b_i^n))^{1/n} \leq \\ &(\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}. \end{aligned} \quad \square \quad (27)$$

Metric Space

All Minkowski distances (p -norms) imply the metric space properties. The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Rocq file, euclidrelations.v.

Theorem .11. *Triangle Inequality:*

$$\begin{aligned} d(s_1, s_2) &= (\sum_{i=1}^2 s_i^p)^{1/p}, \quad p \geq 1, \quad u = s_1, \quad w = s_2, \quad v = w/k \\ \Rightarrow \quad d(u, w) &= (u^p + w^p)^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \end{aligned}$$

Proof. $\forall p \geq 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \quad (28)$$

Apply the distance triangle inequality (.9) to the inequality 28:

$$\begin{aligned} (u^p + w^p)^{1/p} &\leq ((u^p + v^p) + (v^p + w^p))^{1/p} \quad \wedge \quad (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ &\quad \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ \Rightarrow \quad (u^p + w^p)^{1/p} &\leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ \Rightarrow \quad d(u, w) &= (u^p + w^p)^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \quad \square \quad (29) \end{aligned}$$

Theorem .12. *Symmetry:*

$$d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p}, \quad p \geq 1, \quad u = s_1, \quad v = s_2 \quad \Rightarrow \quad d(u, v) = d(v, u).$$

Proof. By the commutative law of addition:

$$\begin{aligned} \forall p : p \geq 1, \quad d(s_1, s_2) &= (\sum_{i=1}^2 s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ \Rightarrow \quad d(u, v) &= (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \quad \square \quad (30) \end{aligned}$$

Theorem .13. *Non-negativity:*

$$d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p}, \quad p \geq 1, \quad u = s_1, \quad w = s_2, \quad v = w/k \quad \Rightarrow \quad d(u, w) \geq 0.$$

Proof. By definition, the length of an interval is always ≥ 0 :

$$\forall [a_1, b_1], [a_2, b_2], \quad u = b_1 - a_1, \quad v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0, \quad v \geq 0. \quad (31)$$

$$p \geq 1, \quad u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \quad (32)$$

\square

Theorem .14. *Identity of Indiscernibles:* $d(u, u) = 0$.

Proof. From the non-negativity property (.13):

$$\begin{aligned} d(u, w) \geq 0 & \wedge d(u, v) \geq 0 \wedge d(v, w) \geq 0 \\ \Rightarrow \exists d(u, w) = d(u, v) = d(v, w) = 0. \end{aligned} \quad (33)$$

$$d(u, w) = d(v, w) = 0 \Rightarrow u = v. \quad (34)$$

$$d(u, v) = 0 \wedge u = v \Rightarrow d(u, u) = 0. \quad (35)$$

□

Properties limiting a set to at most 3 elements

The following definitions and proof use first order logic. A Horn clause-like expression is used, here, to make the proof easier to read. By convention, the proof goal is on the left side and supporting facts are on the right side of the implication sign (\leftarrow). The formal proofs in the Rocq file threed.v are: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.

Definition .15. Immediate Cyclic Successor of m is n :

$$\begin{aligned} \forall x_m, x_n \in \{x_1, \dots, x_{\text{setszie}}\} : \\ \text{Successor}(m, n, \text{setszie}) \leftarrow (m = \text{setszie} \wedge n = 1) \vee (n = m + 1 \leq \text{setszie}). \end{aligned} \quad (36)$$

Definition .16. Immediate Cyclic Predecessor of m is n :

$$\begin{aligned} \forall x_m, x_n \in \{x_1, \dots, x_{\text{setszie}}\} : \\ \text{Predecessor}(m, n, \text{setszie}) \leftarrow (m = 1 \wedge n = \text{setszie}) \vee (n = m - 1 \geq 1). \end{aligned} \quad (37)$$

Definition .17. Adjacent: element m is sequentially adjacent to element n if the immediate cyclic successor of m is n or the immediate cyclic predecessor of m is n . Notionally:

$$\begin{aligned} \forall x_m, x_n \in \{x_1, \dots, x_{\text{setszie}}\} : \\ \text{Adjacent}(m, n, \text{setszie}) \leftarrow \text{Successor}(m, n, \text{setszie}) \vee \text{Predecessor}(m, n, \text{setszie}). \end{aligned} \quad (38)$$

Definition .18. Immediate Symmetric (every set element is sequentially adjacent to every other element):

$$\forall x_m, x_n \in \{x_1, \dots, x_{\text{setszie}}\} : \text{Adjacent}(m, n, \text{setszie}). \quad (39)$$

Theorem .19. An immediate symmetric cyclic list (ISCL) is limited to at most 3 elements.

Proof.

Every element is adjacent to every other element, where $setsize \in \{1, 2, 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \quad (40)$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \quad (41)$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize). \quad (42)$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \quad (43)$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1). \quad (44)$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize). \quad (45)$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize). \quad (46)$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \quad (47)$$

Element 2 is the only immediate successor of element 1 for all $setsize \geq 3$, which implies element 3 is not (\neg) an immediate successor of element 1 for all $setsize \geq 3$:

$$\neg Successor(1, 3, setsize \geq 3) \leftarrow Successor(1, 2, setsize \geq 3) \leftarrow (n = m + 1 \leq setsize). \quad (48)$$

Element $n = setsize > 3$ is the only immediate predecessor of element 1, which implies element 3 is not (\neg) an immediate predecessor of element 1 for all $setsize > 3$:

$$\neg Predecessor(1, 3, setsize \geq 3) \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \quad (49)$$

For all $setsize > 3$, some elements are not (\neg) sequentially adjacent to every other element (not immediate symmetric):

$$\begin{aligned} \neg Adjacent(1, 3, setsize > 3) \leftarrow \\ \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \end{aligned} \quad (50)$$

The Symmetric goal matches the Adjacent goals 40 and fails for all “setsize” greater than three.

APPLICATIONS TO PHYSICS

Derivation of 3 direct proportion ratios: c_t , c_m , and c_q

Application to physics uses the ISCL (.19) and Euclidean space properties:

1. **ISCL:** Let: $\{r_1, r_2, r_3\}$ is an ISCL of 3 “distance” domain values, each value $\in \mathbb{R}$, and $\{t \text{ (time)}, m \text{ (mass)}, q \text{ (charge)}\}$ is the ISCL of 3 “non-distance” domain values, each value $\in \mathbb{R}$.
2. **Cartesian:** The Euclidean volume and distance equations are derived from abstract sets of n-tuples (.4) (.8), where each n-tuple has a corresponding value, $\kappa \in \mathbb{R}$ (each κ the same size). Where $|x| = |y|$, each κ_x , has a corresponding κ_y . Therefore, near each Euclidean-like local coordinate point, each unit length, r_p , of $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$ corresponds to unit lengths: t_p of time, t ; m_p of mass, m ; and q_p of charge, q .

A corresponding number of units can be expressed as, $r/r_p = \tau/\tau_p$, where $\tau \in \{t, m, q\}$. Multiplying both sides by r_p yields 3 direct proportion ratios:

$$\begin{aligned} \forall r_p, \tau_p, r, \tau \subseteq \mathbb{R}, r_p, \tau_p &\text{ constants, } \tau \in \{t, m, q\} : r/r_p = \tau/\tau_p \Rightarrow r = (r_p/\tau_p)/\tau \\ &\Rightarrow r = (r_p/t_p)t \wedge r = (r_p/m_p)m \wedge r = (r_p/q_p)q. \end{aligned} \quad (51)$$

And for conciseness, let:

$$r_p/t_p = c_t \wedge r_p/m_p = c_m \wedge r_p/q_p = c_q. \quad (52)$$

Derivation of G , and the Newton, Gauss, and Poisson gravity laws

From equations 52, linear acceleration, r/t^2 , can be related mass, m , and distance, r :

$$r = c_m m \wedge r = c_t t \Rightarrow r/(c_t)^2 = c_m m/r^2 \Rightarrow r/t^2 = (c_m c_t^2) m/r^2 = Gm/r^2, \quad (53)$$

where $G = c_m c_t^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$ [2].

Newton’s law follows from multiplying both sides of equation 53 by m :

$$r/t^2 = Gm/r^2 \Leftrightarrow F := mr/t^2 = Gm^2/r^2. \quad (54)$$

$$F = Gm^2/r^2 \wedge \forall m \in \mathbb{R} : \exists m_1, m_2 \in \mathbb{R} : m_1 m_2 = m^2 \Rightarrow F = Gm_1 m_2/r^2. \quad (55)$$

A new interpretation of Gauss's and Poisson's gravity laws is presented here. Equation 53 relates linear acceleration, r/t^2 , to mass and distance. Gauss's gravity field, \mathbf{g} , and Poisson's gravity field, $-\nabla\Phi(\tilde{\mathbf{r}}, t)$ is the angular (orbital) acceleration, $2\pi r/t^2$. Applying equation 53:

$$r/t^2 = Gm/|\tilde{\mathbf{r}}|^2 \quad \wedge \quad \mathbf{g} = -\nabla\Phi(\tilde{\mathbf{r}}, t) = 2\pi r/t^2 \quad \Rightarrow \quad \mathbf{g} = -\nabla\Phi(\tilde{\mathbf{r}}, t) = 2\pi Gm/|\tilde{\mathbf{r}}|^2 \quad (56)$$

$$\mathbf{g} = -\nabla\Phi(\tilde{\mathbf{r}}, t) = 2\pi Gm/|\tilde{\mathbf{r}}|^2 \quad \Rightarrow \quad \nabla \cdot \mathbf{g} = \nabla^2\Phi(\tilde{\mathbf{r}}, t) = -4\pi Gm/|\tilde{\mathbf{r}}|^3 = -4\pi G\rho, \quad (57)$$

where $\rho = m/|\tilde{\mathbf{r}}|^3$ is the mass density:

Derivation of the special relativity equations

$$\begin{aligned} \forall \tau \in \{t, m, q\}, r^2 = r'^2 + r_v^2, \exists \mu, \nu : r = \mu\tau \quad \wedge \quad r_v = \nu\tau \quad \Rightarrow \quad (\mu\tau)^2 = r'^2 + (\nu\tau)^2 \\ \Rightarrow \quad r' = \sqrt{(\mu\tau)^2 - (\nu\tau)^2} = \mu\tau\sqrt{1 - (\nu/\mu)^2}. \end{aligned} \quad (58)$$

Local frame distance, r' , contracts relative to a distant observer frame distance, r , as $\nu \rightarrow \mu$:

$$r' = \mu\tau\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad \mu\tau = r \quad \Rightarrow \quad r' = r\sqrt{1 - (\nu/\mu)^2}. \quad (59)$$

A distant observer frame type, τ , dilates relative to the local observer frame type, τ' , as $\nu \rightarrow \mu$:

$$\mu\tau = r'/\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad r' = \mu\tau \quad \Rightarrow \quad \tau = \tau'/\sqrt{1 - (\nu/\mu)^2}. \quad (60)$$

Where τ is type, time, the space-like flat Minkowski spacetime event interval is:

$$\begin{aligned} dr^2 = dr'^2 + dr_v^2 \quad \wedge \quad dr_v^2 = dr_1^2 + dr_2^2 + dr_3^2 \quad \wedge \quad d(\mu\tau) = dr \\ \Rightarrow \quad dr'^2 = d(\mu\tau)^2 - dr_1^2 - dr_2^2 - dr_3^2, \quad \text{or in 6 dimensions :} \\ dr'^2 = d(c_t t)^2 + d(c_m m)^2 + d(c_q q)^2 - dr_1^2 - dr_2^2 - dr_3^2. \end{aligned} \quad (61)$$

Derivation of Schwarzschild's gravitational time dilation and black hole metric

From equations 59 and 51:

$$\sqrt{1 - (v^2/c^2)} = \sqrt{1 - (1)(v^2/c^2)} \wedge c_m m/r = 1 \Rightarrow \sqrt{1 - (v^2/c^2)} = \sqrt{1 - c_m m v^2 / r c^2}. \quad (62)$$

Where v_{escape} is the escape velocity:

$$\begin{aligned} \sqrt{1 - (v^2/c^2)} = \sqrt{1 - c_m m v^2 / r c^2} \quad \wedge \quad KE = mv^2/2 = mv_{\text{escape}}^2 \\ \Rightarrow \quad \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2c_m m v_{\text{escape}}^2 / r c^2}. \end{aligned} \quad (63)$$

For a photon, the escape velocity, $v_{escape} = c$.

$$\sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2c_m m v_{escape}^2 / r c^2} \wedge v_{escape} = c \\ \Rightarrow \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2c_m m c^2 / r c^2}. \quad (64)$$

Combining equation 64 with the derivation of G (55):

$$c_m c^2 = G \wedge \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2c_m m c^2 / r c^2} \Rightarrow \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2Gm/r c^2}. \quad (65)$$

Combining equation 65 with equation 60 yields Schwarzschild's gravitational time dilation:

$$\sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2Gm/r c^2} \wedge t' = t \sqrt{1 - (v^2/c^2)} \Rightarrow t' = t \sqrt{1 - 2Gm/r c^2}. \quad (66)$$

From equations 59 and 65, where Schwarzschild's defined the black hole event horizon radius as $\alpha := 2Gm/c^2$:

$$r' = r \sqrt{1 - (v/c)^2} = r \sqrt{1 - 2Gm/r c^2} \wedge \alpha := 2Gm/c^2 \Rightarrow r' = r \sqrt{1 - \alpha/r}. \quad (67)$$

Applying equation 67 to the time-like spacetime interval equation 61:

$$r' = r \sqrt{1 - \alpha/r} \wedge ds^2 = dr'^2 - dr^2 \\ \Rightarrow ds^2 = (\sqrt{1 - \alpha/r} dr)^2 - (dr'/\sqrt{1 - \alpha/r})^2 = (1 - \alpha/r) dr^2 - (1 - \alpha/r)^{-1} dr'^2. \quad (68)$$

$$ds^2 = (1 - \alpha/r) dr^2 - (1 - \alpha/r)^{-1} dr'^2 \wedge dr = d(ct) \wedge c = 1 \wedge \lim_{ds \rightarrow 0} dr' = dr \\ \Rightarrow \lim_{ds \rightarrow 0} ds^2 = (1 - \alpha/r) dt^2 - (1 - \alpha/r)^{-1} dr^2. \quad (69)$$

Using spherical coordinates to translate from 2D to 4D yields the $+ - --$ form of Schwarzschild's black hole metric [13] [14]:

$$ds^2 = (1 - \alpha/r) dt^2 - (1 - \alpha/r)^{-1} dr^2 \\ \Rightarrow ds^2 = (1 - \alpha/r) dt^2 - (1 - \alpha/r)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ \Rightarrow g_{\mu,\nu} = diag[(1 - \alpha/r), -(1 - \alpha/r)^{-1}, -r^2(d\theta^2), -r^2(\sin^2\theta d\phi^2)]. \quad (70)$$

Simple method to find general relativity solutions

Einstein's field equation is:

$$G_{\mu,\nu} = \mathbf{R} + \frac{1}{2} R g_{\mu,\nu} = \kappa T_{\mu,\nu}, \quad (71)$$

where: $G_{\mu,\nu}$ is Einstein's tensor, \mathbf{R} is the Ricci curvature, R is the scalar curvature, $g_{\mu,\nu}$ is the metric tensor, $\kappa = 8\pi G/c^4$, and $T_{\mu,\nu}$ is the stress-energy tensor [6][5].

The goal of the field equation is to determine the geodesic path and acceleration of a particle caused by the distribution of mass and energy as specified in the stress-energy tensor, $T_{\mu,\nu}$. This requires solving the equation for the metric tensor, $g_{\mu,\nu}$. But the metric tensor has a complex nonlinear relation to the Ricci and scalar curvature, which makes it complicated to determine the metric. Often there are no exact solutions.

In this article, the metric, $g_{\mu,\nu}$, is determined independent of Einstein's field equation. The infinitesimal space near every coordinate point on a pseudo-Riemann surface is Euclidean, where all physics equations derived from the ratios and special relativity equations (58) are valid. This leads to the following steps to solve for the metric tensor, $g_{\mu,\nu}$, independent of Einstein's field equation.

Step 1) Use the ratios and relativity equations to define functions returning scalar values for each component of the metric, $g_{\nu,\mu}$, in Einstein's field equations [6][5]: An example is the previous Schwarzschild black hole metric derivation using the ratios and special relativity equations (62).

Step 2) Express the Einstein field equation as 2D tensors: As shown in equation 70, the Schwarzschild metric was first derived as a 2D metric and then expanded to a 4D metric. Further, the 4D flat spacetime interval equation (61) is an instance of the 2D equation, $dr'^2 = d(ct)^2 - dr_v^2$.

(Optional) The 2D metric tensor allows using the much simpler 2D Ricci curvature and scalar curvature. And the 2D tensors reduce the number of independent equations to solve, which can next be used to set constraints on the solutions in the 4D tensors.

Step 3) One simple method to translate from 2D to 4D is to use spherical coordinates, where r and t remain unchanged and two added dimensions are the angles, ϕ , and θ . For example, the 2D Schwarzschild metric was translated to 4D using this method in equation 70. The spherical coordinates can then be translated to other types of coordinates.

Derivation of k_e , Coulomb charge law, ε_0 , \mathbf{E} , and Gauss electric field law

From equations 52, linear acceleration, r/t^2 , can be related charge, q , and distance, r :

$$r = c_q q \wedge r = c_t t \Rightarrow r/(c_t t)^2 = c_q q/r^2 \Rightarrow r/t^2 = (c_q c_t^2) q/r^2, \quad (72)$$

$$r = c_m m = c_q q \Rightarrow m = (c_q/c_m)q. \quad (73)$$

Combining equations 72 and 73 yields Coulombs's law:

$$r/t^2 = (c_q c_t^2)q/r^2 \wedge m = (c_q/c_m)q \Rightarrow F := mr/t^2 = (c_q^2 c_t^2/c_m)q^2/r^2 = k_e q^2/r^2. \quad (74)$$

where $k_e = c_q^2 c_t^2/c_m$, conforms to the SI units: $kg \cdot m^3 \cdot s^{-2} \cdot C^{-2} = N \cdot m^2 \cdot C^{-2}$ [3].

$$\forall q \in \mathbb{R} \exists q_1, q_2 \in \mathbb{R} : q_1 q_2 = q^2 \wedge F = k_e q^2/r^2 \Rightarrow F = k_e q_1 q_2/r^2. \quad (75)$$

A new interpretation of Gauss's electric field law is presented here. Equation 72 relates the linear acceleration, r/t^2 , to charge and distance. Gauss's electric field, \mathbf{E} , relates angular (orbital) acceleration, $2\pi r/t^2$ to charge and distance. Applying equation 74:

$$F_C = mr/t^2 = k_e q^2/r^2 \Rightarrow \exists F_E \in \mathbb{R} : F_E = 2\pi(mr/t^2) = 2\pi k_e q^2/r^2 \quad (76)$$

$$F_E = 2\pi k_e q^2/r^2 = q(2\pi(k_e q/r^2)) \wedge \exists E \in R : E = 2\pi k_e q/r^2 \Rightarrow F_E = qE. \quad (77)$$

The electric field, $E := 2\pi k_e q/r^2$, conforms to the SI units $kg \cdot m \cdot s^{-2} \cdot C^{-1} = N \cdot C^{-1}$.

$$\mathbf{E} = 2\pi k_e q/|\vec{\mathbf{r}}|^2 \Rightarrow \nabla \cdot \mathbf{E} = -4\pi k_e q/|\vec{\mathbf{r}}|^3. \quad (78)$$

$$\nabla \cdot \mathbf{E} = -4\pi k_e q/|\vec{\mathbf{r}}|^3 \wedge \varepsilon_0 := 1/4\pi k_e \wedge \rho = q/|\vec{\mathbf{r}}|^3 \Rightarrow \nabla \cdot \mathbf{E} = -\rho/\varepsilon_0, \quad (79)$$

which is Gauss's electric field law [3].

Derivation of \mathbf{B} , μ_0 , and Lorentz's law

Applying the distance contraction equation, 59, to equation 77, where r is the distant observer frame of reference and r' is moving particle local frame of reference:

$$r = r'/\sqrt{1 - v^2/c^2} \wedge F = 2\pi k_e q^2/r^2 \Rightarrow F = 2\pi k_e q^2(1 - v^2/c^2)/r'^2. \quad (80)$$

From equation 77:

$$E = 2\pi k_e q/r'^2 \wedge F = 2\pi k_e q^2(1 - v^2/c^2)/r'^2 \Rightarrow F = q(E - ((2\pi k_e/c^2)q/r'^2)v^2). \quad (81)$$

$$F = q(E - ((2\pi k_e/c^2)q/r'^2)v^2) \Rightarrow \exists B : B = (2\pi k_e/c^2)vq/r'^2 \wedge F = q(E - Bv). \quad (82)$$

$$F = q(E - Bv) \Rightarrow \mathbf{F} = q(\mathbf{E} - \mathbf{B} \times \vec{\mathbf{v}}). \quad (83)$$

$$\mathbf{B} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \mathbf{B}) \quad \wedge \quad \mathbf{F} = q(\mathbf{E} - \mathbf{B} \times \vec{\mathbf{v}}) \quad \Rightarrow \quad \mathbf{F} = q(\mathbf{E} + \vec{\mathbf{v}} \times \mathbf{B}), \quad (84)$$

which is Lorentz law in the rest (observer on the moving particle) frame of reference, where the magnetic field, $B = (2\pi k_e/c^2)vq/r'^2$, conforms to the base SI units: $kg \cdot s^{-1} \cdot C^{-1} = kg \cdot s^{-2} \cdot A^{-1} = T$.

The magnetic field divergence outward, $\nabla \cdot \mathbf{B}$, at a point, P , normal to a surface equals the magnetic field divergence inward, $\nabla \cdot \mathbf{B}$, at the same point, P :

$$\mathbf{B} = (2\pi k_e/c^2)vq/|\vec{\mathbf{r}}|^2 \quad \Rightarrow \quad \nabla \cdot \mathbf{B} = -(4\pi k_e/c^2)q/|\vec{\mathbf{r}}|^3 + (4\pi k_e/c^2)q/|\vec{\mathbf{r}}|^3 = 0. \quad (85)$$

$$\nabla \cdot \mathbf{B} = -(4\pi k_e/c^2)q/|\vec{\mathbf{r}}|^3 + (4\pi k_e/c^2)q/|\vec{\mathbf{r}}|^3 = 0 \quad \Rightarrow \quad \mu_0 = (4\pi k_e/c^2), \quad (86)$$

where $\mu_0 = 4\pi k_e/c^2$ conforms to the SI units $kg \cdot m \cdot C^{-2} = kg \cdot m \cdot s^{-2}A^{-2}$.

Note: In the last section of this article, it will be shown that the CODATA definition, $\mu_0 = 2h\alpha/cq_e^2$ [15], and $\mu_0 = 4\pi k_e/c^2$ both reduce to the same equation.

Derivation of Maxwell-Faraday's law

From the magnetic field equation 83, where the electric and magnetic fields are propagating at the speed, $v = c$:

$$B = (2\pi k_e/c^2)qv/r^2 \quad \wedge \quad v = c \quad \wedge \quad r = ct \quad \Rightarrow \quad B = (2\pi k_e/c^3)q/t^2. \quad (87)$$

$$B = (2\pi k_e/c^3)q/t^2 \quad \Rightarrow \quad \partial B / \partial t = -(4\pi k_e/c^3)q/t^3. \quad (88)$$

$$\partial B / \partial t = -(4\pi k_e/c^3)q/t^3 \quad \wedge \quad r = ct \quad \Rightarrow \quad \partial B / \partial t = -4\pi k_e q/r^3. \quad (89)$$

Faraday's law is the case, where the electric field, \mathbf{E} , is a single radial vector, $\mathbf{E}_{\tilde{\mathbf{r}}_1}$ acting on some point, P . Therefore, at point, P , $\mathbf{E}_{\tilde{\mathbf{r}}_2} = 0$ and $\mathbf{E}_{\tilde{\mathbf{r}}_3} = 0$. From the Lorentz equation 84, the magnetic field, \mathbf{B} at point P , is perpendicular to $\mathbf{E}_{\tilde{\mathbf{r}}_1}$ (aligned on $\tilde{\mathbf{r}}_2$). For a constant force, the Lorentz equation makes the electric field, $\mathbf{E}_{\tilde{\mathbf{r}}_1}$, a function of $\mathbf{B}_{\tilde{\mathbf{r}}_2}$. Combining with equation 78:

$$\begin{aligned} \mathbf{E}_{\tilde{\mathbf{r}}_1} &= 2\pi k_e q/|\tilde{\mathbf{r}}_2|^2 = f(\mathbf{B}_{\tilde{\mathbf{r}}_2}) \quad \wedge \quad \mathbf{E}_{\tilde{\mathbf{r}}_2} = 0 \quad \wedge \quad \mathbf{E}_{\tilde{\mathbf{r}}_3} = 0 \quad \Rightarrow \quad \nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial \tilde{\mathbf{r}}_1^2} & \frac{\partial}{\partial \tilde{\mathbf{r}}_2^2} & \frac{\partial}{\partial \tilde{\mathbf{r}}_3^2} \\ \mathbf{E}_{\tilde{\mathbf{r}}_1} & 0 & 0 \end{vmatrix} \\ &= \left(\frac{\partial \mathbf{E}_{r_3}}{\partial \tilde{\mathbf{r}}_2^2} - \frac{\partial \mathbf{E}_{r_2}}{\partial \tilde{\mathbf{r}}_3^2} \right) \hat{\mathbf{i}} + \left(\frac{\partial \mathbf{E}_{r_1}}{\partial \tilde{\mathbf{r}}_3^2} - \frac{\partial \mathbf{E}_{r_3}}{\partial \tilde{\mathbf{r}}_1^2} \right) \hat{\mathbf{j}} + \left(\frac{\partial \mathbf{E}_{r_2}}{\partial \tilde{\mathbf{r}}_1^2} - \frac{\partial \mathbf{E}_{r_1}}{\partial \tilde{\mathbf{r}}_2^2} \right) \hat{\mathbf{k}} \\ &= 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + \left(0 - \frac{\partial 2\pi k_e q/|\tilde{\mathbf{r}}_2|^2}{\partial |\tilde{\mathbf{r}}_2|^2} \right) \hat{\mathbf{k}} = 4\pi k_e q/|\tilde{\mathbf{r}}_2|^3 \hat{\mathbf{k}} \equiv \nabla \times \mathbf{E} = 4\pi k_e q/|\tilde{\mathbf{r}}_2|^3 \hat{\mathbf{k}}. \quad (90) \end{aligned}$$

Combining equations 90 and 89 yields Maxwell-Faraday's law [3]:

$$\nabla \times \mathbf{E} = 4\pi k_e q / |\vec{\mathbf{r}}|^3 \quad \wedge \quad \partial \mathbf{B} / \partial t = -4\pi k_e q / |\vec{\mathbf{r}}|^3 \quad \Rightarrow \quad \nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t. \quad (91)$$

Derivation of the 3 direct proportion ratio values: c_t , c_m , and c_q

$$c_t = c \approx 2.99792458 \cdot 10^8 m s^{-1}. \quad (92)$$

$$G = c_m c_t^2 \quad \wedge \quad G \approx 6.67430(15) \cdot 10^{-11} m^3 / kg s^2 \quad \Rightarrow \quad c_m = G / c_t^2 \approx 7.4261619381 \cdot 10^{-28} m kg^{-1}. \quad (93)$$

$$k_e = c_q^2 c_t^2 / c_m \quad \wedge \quad k_e \approx 8.9875517923(13) \cdot 10^9 N m^2 / C^2 \quad \Rightarrow \quad c_q = \sqrt{k_e c_m / c_t^2} \approx 8.6175181707 \cdot 10^{-18} m C^{-1}. \quad (94)$$

Derivation of the fine structure constant, α

The ratio of two subtypes of force implies ratios of the form:

$$\alpha_\tau = \frac{F_{\tau_1}}{F_{\tau_2}} = \frac{K\tau_1^2/r^2}{K\tau_2^2/r^2} = \frac{\tau_1^2}{\tau_2^2}. \quad (95)$$

For example, where q_e is the elementary (electron) charge ($1.60217663 \cdot 10^{-19} C$), and q_p is reduced Planck charge unit, and the fine structure electron coupling constant has the empirical value $7.2973525705 \cdot 10^{-3}$:

$$\alpha_q = q_e^2 / q_p^2 \approx 7.2973525705 \cdot 10^{-3}. \quad (96)$$

The gravitational fine structure constant is: $\alpha_G = m_e^2 / m_p^2$.

Derivation of the 4 reduced Planck unit values: r_p , t_p , m_p , q_p

: From the fine structure equation 96:

$$\alpha = q_e^2 / q_p^2 \quad \Rightarrow \quad q_p = \sqrt{q_e^2 / \alpha}. \quad (97)$$

$$q_p = q_p = \sqrt{q_e^2/\alpha} \quad \wedge \quad q_e = 1.60217663 \cdot 10^{-19} C \quad \wedge \quad \alpha \approx 7.2973525705 \cdot 10^{-3} \quad (98)$$

$$\Rightarrow q_p \approx 1.8755460376 \cdot 10^{-18} C. \quad (99)$$

$$r_p = q_p c_q \approx 1.6162550243 \cdot 10^{-35} m. \quad (100)$$

$$t_p = r_p/c_t \approx 5.3912464479 \cdot 10^{-44} s. \quad (101)$$

$$m_p = r_p/c_m \approx 2.1764338534 \cdot 10^{-8} kg. \quad (102)$$

Derivation of 3 inverse proportion ratios: k_t , k_m , and k_q

$$\begin{aligned} r/t &= r_p/t_p \quad \wedge \quad r/m = r_p/m_p \quad \Rightarrow \quad (r/t)/(r/m) = (r_p/t_p)/(r_p/m_p) \\ \Rightarrow \quad (mr)/(tr) &= (m_p r_p)/(t_p r_p) \quad \Rightarrow \quad mr = m_p r_p = k_m, \quad tr = t_p r_p = k_t. \end{aligned} \quad (103)$$

$$\begin{aligned} r/t &= r_p/t_p \quad \wedge \quad r/q = r_p/q_p \quad \Rightarrow \quad (r/t)/(r/q) = (r_p/t_p)/(r_p/q_p) \\ \Rightarrow \quad (qr)/(tr) &= (q_p r_p)/(t_p r_p) \quad \Rightarrow \quad qr = q_p r_p = k_q, \quad tr = t_p r_p = k_t. \end{aligned} \quad (104)$$

From the previously calculated reduced Planck unit values:

$$k_m = m_p r_p \approx 3.5176721506 \cdot 10^{43} kg m \quad (105)$$

$$k_t = t_p r_p \approx 8.713629159 \cdot 10^{-79} s m \quad (106)$$

$$k_q = q_p r_p \approx 3.0313607066 \cdot 10^{-53} C m. \quad (107)$$

Derivation of \hbar , h , and the Planck relation

[3][18] Applying both the direct proportion ratio (92), and inverse proportion ratio (103):

$$m = k_m/r \quad \wedge \quad r = ct \quad \Rightarrow \quad m(ct)^2 = (k_m/r)r^2 = k_m r. \quad (108)$$

$$m(ct)^2 = k_m r \quad \Rightarrow \quad E := mc^2 = k_m r / t^2. \quad (109)$$

$$E = mc^2 = k_m r / t^2 \quad \wedge \quad r/t = c \quad \Rightarrow \quad E = mc^2 = (k_m c)(1/t). \quad (110)$$

$$\begin{aligned} E = mc^2 &= (k_m c)(1/t) \quad \wedge \quad k_m c \approx 1.054571 \cdot 10^{-34} = \hbar \\ \Rightarrow \quad E = mc^2 &\approx \hbar(1/t) = \hbar f \quad \Rightarrow \quad \exists E_{full} : E_{full} = 2\pi\hbar f = hf. \end{aligned} \quad (111)$$

where the reduced Planck constant, $\hbar = k_m c$, and frequency, $f = 1 \text{ cycle/seconds}$.

Derivation of the Compton wavelength, λ

[3][18] From equations 105 and 111:

$$\lambda = 2\pi r = 2\pi k_m/m = 2\pi(k_m/m)(c/c) \quad \wedge \quad h = 2\pi\hbar = 2\pi k_m c \quad \Rightarrow \quad \lambda = h/mc. \quad (112)$$

Derivation of Schrödinger's position-space wave equation

Start with the previously derived Planck relation 111 and multiply the kinetic energy component by mc/mc :

$$\begin{aligned} mc^2 = \hbar/t &\Rightarrow \exists V(r, t) : \hbar/t = \hbar/2t + V(r, t) \\ &\Rightarrow \hbar/t = \hbar mc/2mct + V(r, t). \end{aligned} \quad (113)$$

And from the distance-to-time (speed of light) ratio (92):

$$\hbar/t = \hbar mc/2mct + V(r, t) \quad \wedge \quad r = ct \quad \Rightarrow \quad \hbar/t = \hbar mc^2/2mcr + V(r, t). \quad (114)$$

$$\hbar/t = \hbar mc^2/2mcr + V(r, t) \quad \wedge \quad \hbar/t = mc^2 \quad \Rightarrow \quad \hbar/t = \hbar^2/2mcrt + V(r, t). \quad (115)$$

$$\hbar/t = \hbar^2/2mcrt + V(r, t) \quad \wedge \quad r = ct \quad \Rightarrow \quad \hbar/t = \hbar^2/2mr^2 + V(r, t). \quad (116)$$

Multiply both sides of equation 116 by a position-space distribution function, $\Psi(r, t)$.

$$\hbar/t = \hbar^2/2mr^2 + V(r, t) \quad \Rightarrow \quad (\hbar/t)\Psi(r, t) = (\hbar^2/2mr^2)\Psi(r, t) + V(r, t)\Psi(r, t). \quad (117)$$

Set constraints on $\Psi(r, t)$:

$$\begin{aligned} (\hbar/t)\Psi(r, t) &= (\hbar^2/2mr^2)\Psi(r, t) + V(r, t)\Psi(r, t) \quad \wedge \\ \forall \Psi(r, t) : \partial^2\Psi(r, t)/\partial r^2 &= (-1/r^2)\Psi(r, t) \quad \wedge \quad \partial\Psi(r, t)/\partial t = (i/1/t)\Psi(r, t) \\ \Rightarrow i\hbar\partial\Psi(r, t)/\partial t &= -(\hbar^2/2m)\partial^2\Psi(r, t)/\partial r^2 + V(r, t)\Psi(r, t), \end{aligned} \quad (118)$$

which is the one-dimensional position-space Schrödinger's equation [16][18].

$$\begin{aligned} i\hbar\partial\Psi(r, t)/\partial t &= -(\hbar^2/2m)\partial^2\Psi(r, t)/\partial r^2 + V(r, t)\Psi(r, t) \quad \wedge \quad \|\vec{r}\| = r \\ \Rightarrow \exists \vec{r} : i\hbar\partial\Psi(\vec{r}, t)/\partial t &= -(\hbar^2/2m)\partial^2\Psi(\vec{r}, t)/\partial \vec{r}^2 + V(\vec{r}, t)\Psi(\vec{r}, t), \end{aligned} \quad (119)$$

which is the 3-dimensional position-space Schrödinger's equation [16] [18].

Derivation of Dirac's wave equation

Using the derived Planck relation 111:

$$mc^2 = \hbar/t \Rightarrow \exists V(r, t) : mc^2/2 + V(r, t) = \hbar/t \Rightarrow 2\hbar/t - 2V(r, t) = mc^2. \quad (120)$$

$$\begin{aligned} \forall V(r, t) : V(r, t) = i\hbar/t \wedge r = ct \wedge 2\hbar/t - 2V(r, t) = mc^2 \\ \Rightarrow 2\hbar/t - i2hc/r = mc^2. \end{aligned} \quad (121)$$

Use the ratios, $r = c_q q$, and, $r = ct$. to multiply each term on the left side of equation 121 by 1:

$$qc_q/r = qc_q/ct = 1 \wedge 2\hbar/t - i2hc/r = mc^2 \Rightarrow 2\hbar(qc_q/c)/t^2 - i2\hbar((qc_q/c)/r^2)c = mc^2. \quad (122)$$

Applying a quantum amplitude equation in complex form to equation 121:

$$\begin{aligned} A_0 = 2(c_q/c)((1/t) - i(1/r)) \wedge 2\hbar(qc_q/c)/t^2 - i2\hbar((qc_q/c)/r^2)c = mc^2 \\ \Rightarrow \hbar\partial(-qA_0)/\partial t - i\hbar(\partial(-qA_0)/\partial r)c = mc^2. \end{aligned} \quad (123)$$

Translating equation 123 to moving (rest frame) coordinates via the Lorentz factor, $\gamma_0 = 1/\sqrt{1 - (v/c)^2}$:

$$\begin{aligned} \hbar\partial(-qA_0)/\partial t - i\hbar(\partial(-qA_0)/\partial r)c = mc^2 \\ \Rightarrow \gamma_0\hbar\partial(-qA_0)/\partial t - \gamma_0i\hbar(\partial(-qA_0)/\partial r)c = mc^2. \end{aligned} \quad (124)$$

Translating equation 124 to vector form:

$$\begin{aligned} \gamma_0\hbar(\partial(-qA_0)/\partial t) - \gamma_0i\hbar(\partial(-qA_0)/\partial r)c = mc^2 \wedge \\ ||\vec{\mathbf{r}}|| = r \wedge ||\vec{\mathbf{A}}|| = A_0 \wedge ||\vec{\gamma}|| = \gamma_0 \\ \Leftrightarrow \exists \vec{\mathbf{r}}, \vec{\mathbf{A}}, \vec{\gamma} : \gamma_0\hbar(\partial(-qA_0)/\partial t) - \vec{\gamma} \cdot i\hbar(\partial(-q\vec{\mathbf{A}})/\partial r)c = mc^2. \end{aligned} \quad (125)$$

Multiplying both sides of equation 125 by the spinor, Ψ , yields: Dirac's equation:

$$\begin{aligned} \gamma_0\hbar(\partial(-qA_0)/\partial t) - \vec{\gamma} \cdot i\hbar(\partial(-q\vec{\mathbf{A}})/\partial r)c = mc^2 \Rightarrow \\ \gamma_0\hbar(\partial(-qA_0)/\partial t)\Psi - \vec{\gamma} \cdot i\hbar(\partial(-q\vec{\mathbf{A}})/\partial r)c\Psi = mc^2\Psi \end{aligned} \quad (126)$$

Total of a type

Applying both the direct (92) and inverse proportion ratios (104) to the Euclidean distance:

$$\begin{aligned} r &= \sqrt{r_1^2 + r_2^2}, \quad r_1 = c_\tau \tau, \quad r_2 = k_\tau / \tau, \quad \tau \in \{t, m, q\} \\ &\Rightarrow r = \sqrt{(c_\tau \tau)^2 + (k_\tau / \tau)^2} \Leftrightarrow \tau = \sqrt{(r/c_\tau)^2 + (k_\tau/r)^2}. \end{aligned} \quad (127)$$

Quantum extension to general relativity

The simplest way to demonstrate how to add quantum physics to general relativity is by extending Schwarzschild's gravitational time dilation equation and black hole metric. Apply the total of a type equation 127 to the derivation of Schwarzschild's time dilation and metric (62):

$$\begin{aligned} \sqrt{1 - (v^2/c^2)} &= \sqrt{1 - (v^2/c^2)(r/r)} \quad \wedge \quad r = \sqrt{(c_m m)^2 + (k_m/m)^2} = Q_m \\ &\Rightarrow \sqrt{1 - (v^2/c^2)} = \sqrt{1 - Q_m v^2 / r c^2}. \end{aligned} \quad (128)$$

$$\begin{aligned} \sqrt{1 - (v^2/c^2)} &= \sqrt{1 - Q_m v^2 / r c^2} \quad \wedge \quad KE = mv^2/2 = mv_{\text{escape}}^2 \\ &\Rightarrow \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2Q_m v_{\text{escape}}^2 / r c^2}. \end{aligned} \quad (129)$$

For a photon, the escape velocity, $v_{\text{escape}} = c$.

$$\begin{aligned} \sqrt{1 - (v^2/c^2)} &= \sqrt{1 - 2Q_m v_{\text{escape}}^2 / r c^2}. \quad \wedge \quad v_{\text{escape}} = c \\ &\Rightarrow \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2Q_m c^2 / r c^2} = \sqrt{1 - 2Q_m / r}. \end{aligned} \quad (130)$$

Combining equation 130 with equation 60 yields Schwarzschild's gravitational time dilation with a quantum mass effect:

$$\sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2Q_m / r} \quad \wedge \quad t' = t \sqrt{1 - (v^2/c^2)} \Rightarrow t' = t \sqrt{1 - 2Q_m / r}. \quad (131)$$

Schwarzschild defined the black hole event horizon radius, $\alpha := 2Gm/c^2$. The radius with the quantum extension is $\alpha := 2Q_m$. At this point the exact same equations 67 through 70 yield what looks like the same Schwarzschild black hole metric.

Quantum extension to Newton's gravity force

The quantum mass effect is easier to understand in the context Newton's gravity equation than in general relativity, because the metric equations and solutions in the EFEs are much more complex. From equations 127 and 51:

$$\begin{aligned} m/\sqrt{(r/c_m)^2 + (k_m/r)^2} = 1 \quad \wedge \quad r^2/(ct)^2 = 1 \quad \Rightarrow \quad r^2/(ct)^2 &= m/\sqrt{(r/c_m)^2 + (k_m/r)^2} \\ \Rightarrow \quad r^2/t^2 &= c^2 m/\sqrt{(r/c_m)^2 + (k_m/r)^2}. \end{aligned} \quad (132)$$

$$\begin{aligned} r^2/t^2 = c^2 m/\sqrt{(r/c_m)^2 + (k_m/r)^2} \quad \Rightarrow \quad (m/r)(r^2/t^2) &= (m/r)(c^2 m/\sqrt{(r/c_m)^2 + (k_m/r)^2}) \\ \Rightarrow \quad F := mr/t^2 &= c^2 m^2/\sqrt{(r^4/c_m^2) + k_m^2}. \end{aligned} \quad (133)$$

$$\begin{aligned} F = c^2 m^2/\sqrt{(r^4/c_m^2) + k_m^2} \quad \wedge \quad \forall m \in \mathbb{R}, \exists m_1, m_2 \in \mathbb{R} : m_1 m_2 = m^2 \\ \Rightarrow \quad F = c^2 m_1 m_2/\sqrt{(r^4/c_m^2) + k_m^2}. \end{aligned} \quad (134)$$

Quantum extension to Coulomb's charge force

$$\begin{aligned} q^2/((r/c_q)^2 + (k_q/r)^2) = 1 \quad \wedge \quad r^2/(ct)^2 = 1 \quad \Rightarrow \quad r^2/(ct)^2 &= q^2/((r/c_q)^2 + (k_q/r)^2) \\ \Rightarrow \quad r^2/t^2 &= c^2 q^2/((r/c_q)^2 + (k_q/r)^2). \end{aligned} \quad (135)$$

$$(1/r)(r^2/t^2) = (1/r)(c^2 q^2/((r/c_q)^2 + (k_q/r)^2)) \quad \Rightarrow \quad r/t^2 = c^2 q^2/(r^3/c_q^2 + k_q^2/r). \quad (136)$$

$$\begin{aligned} \forall q \in \mathbb{R} : \exists q_1, q_2 \in \mathbb{R} : q_1 q_2 = q^2 \quad \wedge \quad r/t^2 &= c^2 q^2/(r^3/c_q^2 + k_q^2/r) \quad \Rightarrow \\ \exists q_1, q_2 \in \mathbb{R} : r/t^2 &= c^2 q_1 q_2/(r^3/c_q^2 + k_q^2/r). \end{aligned} \quad (137)$$

$$r/t^2 = c^2 q_1 q_2/(r^3/c_q^2 + k_q^2/r) \quad \wedge \quad m = r/c_m \quad \Rightarrow \quad F := mr/t^2 = (c^2/c_m) q_1 q_2/(r^2/c_q^2 + k_q^2/r^2). \quad (138)$$

INSIGHTS AND IMPLICATIONS

1. The ruler measure (.1) and convergence theorem (.2) were shown to be useful tools for proving that a countable sets of n-tuples imply a corresponding real-valued equation.

2. The notion of distance was derived as the inverse function of the sum of Euclidean volumes (.7), which suggests defining a distance measures as the inverse function of the sum of subset Euclidean and non-Euclidean volumes,

$$\forall n \in \mathbb{N}, \quad f_n, g \in C^2, \quad d = f_n^{-1}(\sum_{i=1}^m f_n(v_i)) = f_n^{-1}(\sum_{i=1}^m g(s_{i,1}, \dots, s_{i,n})) : \quad (139)$$

- (a) shows the intimate relation between distance and volume that definitions, like inner product space and metric space, ignore [7][8];
 - (b) is a more simple and concise definition of a distance measure, having the metric space properties [7][8].
3. The left side of the distance sum inequality (.10),

$$(\sum_{i=1}^m (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}, \quad (140)$$

differs from the left side of Minkowski's sum inequality [12]:

$$(\sum_{i=1}^m (a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}. \quad (141)$$

- (a) The two inequalities are only the same where $n = 1$.
 - (b) The distance sum inequality (.10) is a more fundamental inequality because the proof does not require the convexity and Hölder's inequality assumptions required to prove the Minkowski sum inequality [12].
 - (c) $\forall n > 1, v > 0$: The distance sum inequality term, $v_i^n = a_i^n + b_i^n$: $d = v^{1/n} = (\sum_{i=1}^m v_i^n)^{1/n}$, is the Minkowski distance, which makes it directly related to geometry. But the Minkowski sum inequality term, $d = v^{1/n} = (\sum_{i=1}^m ((v_i^n)^{\mathbf{n}}))^{1/n} = (\sum_{i=1}^m v_i^{\mathbf{n}^2})^{1/n}$, is *not* a Minkowski distance.
 - (d) The distance sum inequality might be applicable to machine learning.
4. **Combinatorics.** The number of n-tuples, $v_c = \prod_{i=1}^n |x_i|$, was proven to imply: the Euclidean volume equation (.4). And the notion of distance was derived as the inverse function of the sum of the number of n-tuples (.7) (which includes the inner product) and the Minkowski distance equation (.8) (which includes the Manhattan and Euclidean distance equations), without relying on the geometric primitives and relations in Euclidean geometry [19][20], axiomatic geometry [21][22][23] [24][25], trigonometry [26] [27] calculus [28][26] [29], and vector analysis [5].

5. **Combinatorics.** The commutative property of the operations defining volume and distance equations limits a domain set to $n \leq 3$ elements (.19). Other elements must have different types (are elements of different sets).
 - (a) For example, the vector inner product space, in physical space, can only be extended beyond 3 dimensions if and only if the other dimensions have non-distance types, for example, dimensions of time, mass, and charge.
 - (b) If each type of quantum state is an ISCL, then there are at most 3 states of the same type: 3 orientations per dimension of space (+1,0-1), 3 quark color charges, {red, green, blue}, 3 quark anti-color charges, and so on.
6. A discrete (point) valued state has measure 0 (zero-length interval). The ratio of a time or distance interval length to zero is undefined, which is the reason quantum entangled (discrete) state values exist independent of (non-deterministic with respect to) distance (position), time, mass, and charge. Therefore, a state value in space and time is undetermined until observed, like Schrödinger's poisoned cat being both alive and dead until the box is opened [16].
7. For each Cartesian axis unit, r_p , of a 3-dimensional distance interval having a length, r , there are Cartesian axis units of other types of intervals forming unit ratios (92): $c_t = r_p/t_p$, $c_m = r_p/m_p$, $c_q = r_p/q_p \Leftrightarrow$ the inverse proportion ratios (103): $k_t = r_p t_p$, $k_m = r_p m_p$, $k_t = r_p q_p$, where r_p , t_p , m_p , and q_p are the reduced Planck units (98).
8. Empirical laws *describe* relations. Deriving empirical laws from the ratios *explains* the relations. Further, all the derivations of the physics equations from the ratios are much shorter and simpler than other derivations, which shows that the ratios are an important tool for physicists and engineers.
9. As shown in the subsection deriving the Schwarzschild's gravitational time dilation and black hole metric (62) [13][14] using ratios illustrates a simple way of finding solutions to Einstein's field equations.
10. c_t , c_m , and c_q are the **maximum** ratios: $r = \sqrt{r_1^2 + r_2^2 + r_3^2} \Rightarrow r_1, r_2, r_3 \leq r \Rightarrow \forall \tau \in \{t, m, q\} : r_1/\tau, r_2/\tau, r_3/\tau \leq r/\tau = r_p/\tau_p = c_\tau$. For example, the speeds of gravity and light waves are limited to the speed, c_t .

11. c_t is a component of the constants: $G = c_m c_t^2$, $k_e = c_q^2 c_t^2 / c_m$, $\varepsilon_0 = 1/4\pi k_e = 1/4\pi(c_q^2 c_t^2 / c_m)$, and $\hbar = k_m c_t$. The only constant, derived in this article, that does not contain c_t is vacuum permeability: $\mu_0 = 4\pi k_e / c_t^2 = 4\pi c_q^2 / c_m$.
12. The fine structure electron coupling constant, α was derived from the ratio of two subtypes of charge force that reduces to the unit-less ratio $\alpha = q_e^2 / q_p^2 \approx 0.0072973526$ (96), which is the empirical CODATA value [15].
- (a) The CODATA electron coupling version of the fine structure constant, α is defined as: $\alpha = q_e^2 / 4\pi\varepsilon_0\hbar c = q_e^2 / 2\varepsilon_0 hc$ [15]. The following steps show that the CODATA definition reduces to the ratio-derived equation:
- $$\begin{aligned} \varepsilon_0 := 1/4\pi k_e &= 1/(4\pi(c_q^2 c_t^2 / c_m)) \quad \wedge \quad \hbar = k_m c_t \quad \wedge \quad h = 2\pi\hbar \\ \Rightarrow \quad \varepsilon_0 hc &= 2\pi k_m c_t^2 / (4\pi(c_q^2 / c_m)c_t^2) = k_m / (2(c_q^2 / c_m)) \\ &= m_p r_p / (2((r_p / q_p)^2 / (r_p / m_p))) = q_p^2 / 2. \end{aligned} \quad (142)$$
- $$\alpha = q_e^2 / 2\varepsilon_0 hc \quad \wedge \quad \varepsilon_0 hc = q_p^2 / 2 \quad \Rightarrow \quad \alpha = q_e^2 / 2(q_p^2 / 2) = q_e^2 / q_p^2. \quad (143)$$
- (b) The fine structure electron coupling constant is the ratio of electron static charge force to the propagating quantum charge (photon/electromagnetic) wave force, caused by a moving charged particle.
- (c) The gravitational fine structure electron coupling constant, α_G , can be expressed as the ratio of two subtypes of Newton's gravity force, where m_e is the rest electron mass and m_p is the reduced Planck mass unit: $\alpha_G = m_e^2 / m_p^2$. The currently used gravitational fine structure constant, $\alpha_G = Gm_e^2 / \hbar c = (c_m c_t^2) m_e^2 / (k_m c_t) c_t = c_m m_e^2 / k_m = (r_p / m_p) m_e^2 / m_p r_p = m_e^2 / m_p^2$.
13. The following will show that the NIST/CODATA standard definition of $\mu_0 = 2h\alpha/cq_e^2$ [15] and the derivation of $\mu_0 = 4\pi k_e / c_t^2 = 1/\varepsilon c^2$, in this article, are equivalent. Using the ratios to reduce the CODATA definition:

$$h = 2\pi k_m c_t, \quad \alpha = q_e^2 / q_p^2 \Rightarrow \mu_0 = 2h\alpha/cq_e^2 = 4\pi k_m c_t (q_e^2 / q_p^2) / c_t q_e^2 = 4\pi k_m / q_p^2. \quad (144)$$

Using the ratios to reduce the definition of μ_0 derived in this article:

$$\mu_0 = 4\pi k_e / c_t^2 = 4\pi c_q^2 / c_m = 4\pi(r_p / q_p)^2 / (r_p / m_p) = 4\pi m_p r_p / q_p^2 = 4\pi k_m / q_p^2. \quad (145)$$

Both equations reducing to the same base equation shows that the old definition, $\mu_0 = 4\pi k_e/c_t^2 = 1/\varepsilon c^2$, which differs slightly in value from the NIST/CODATA definition, $\mu_0 = 2h\alpha/cq_e^2$, is due to differences in measurement methods.

14. Empirical and hypothesized laws of physics use an *opaque* constant to make the units in an equation balance. The opacity has led to the *incorrect* assumptions of those constants being fundamental (atomic) constants.

In this article, some opaque constants are derived directly from (composed of) the ratios: gravity, $G = c_m c_t^2$ (55), charge, $k_e = c_q^2 c_t^2 / c_m$ (74), and reduced Planck $\hbar = k_m c_t$ (111). $\varepsilon_0 = 1/4\pi k_e = 1/4\pi c_m / ((c_q^2/c_m)c_t^2)$ (79) and $\mu_0 = 4\pi k_e/c_t^2 = 4\pi c_q^2/c_m$ (85).

And the quantum extensions to: Schwarzschild's gravitational time dilation (130) Newton's gravity force (134), and Coulomb's charge force show, that where the quantum effects become measurable, the constants G , k_e , ε_0 , and μ_0 no longer exist (are no longer valid).

Therefore, G , k_e , ε_0 , μ_0 , and h are **not** fundamental constants.

15. Constants that use notions of temperature and pressure for example, the Boltzmann constant [15], are probably not possible to define solely in terms of Planck units.
16. The derivations of: $\nabla \cdot \mathbf{g} = -4\pi G\rho$ from $\mathbf{g} = 2\pi Gm/|\tilde{\mathbf{r}}|^2$ (56), and $\nabla \cdot \mathbf{E} = -\rho/\varepsilon_0$ from $\mathbf{E} = 2\pi k_e q/|\tilde{\mathbf{r}}|^2$ (79), show that the use of mass and charge density, ρ show that:

- (a) Mass and charge density as motivations are unnecessary complexities. Equation 53 used to derive Newton's law and equation 72 used to derive Coulomb's law relate linear acceleration, r/t^2 to mass and charge. It is small step to recognize Gauss's \mathbf{g} , Poisson's $-\nabla\Phi(\tilde{\mathbf{r}}, t)$, and Gauss's \mathbf{E} relate the angular (orbital) acceleration, $2\pi r/t^2$, to mass and charge, where differentiation yields the gravity field and electric field laws.

The derivations are much simpler and shorter using angular (orbital) acceleration, $2\pi r/t^2$. The orbit length, $2\pi r$, corresponds to the Gauss's line integral length of a curve circumscribing a sphere containing the mass or charge.

- (b) Mass and charge density obfuscates the pattern, $\nabla \cdot f(x, r) = -4\pi k_x x / |\tilde{\mathbf{r}}|^3$, being derived from the inverse square pattern, $f(x, r) = 2\pi k_x x / |\tilde{\mathbf{r}}|^2$. The constant, G

(derived from the inverse square relation (53)), is used in Einstein's constant, $\kappa = 8\pi G/c^4$, and the use of energy density in the stress energy tensor, $T_{\mu,\nu}$, in Einstein's field equations [5] also obfuscate the inverse square pattern.

17. Einstein's relativity equations: 1) assume the Lorentz transformations, 2) assume the laws of physics are same at each coordinate point, 3) assume the notion of light, and 4) assume that the speed of light is the same at each coordinate point [6][5].

The derivations, in this article, were made without those assumptions (does not even require the notion of light). Assuming Euclidean space near each coordinate point creates unit ratios, where all equations (laws) derived from the unit ratios must be the same at each coordinate point.

18. The derivation of the magnetic field, \mathbf{B} , (82) shows that the magnetic field is a space-time bend caused by relativistic charged particle linear and angular (orbital or spin) velocities. Molecular arrangements in the construction of permanent magnets, photo-voltaic cells, and superconductors are optimized by accurately controlling the space-time bend via the arrangement of orbitals, bond oscillations, and spin alignments.
19. Using the quantum units, r_p and t_p : r_p/t_p^2 , suggests a maximum linear acceleration for masses. And $2\pi r_p/t_p^2$ suggests a maximum orbital or rotational acceleration.
20. The quantum extensions to: Schwarzschild's gravitational time dilation (130), black hole metric (70), Newton's gravity force (134), and Coulomb's charge force (137) make quantifiable predictions:
 - (a) For gravity, $\lim_{r \rightarrow 0} F = c^2 m_1 m_2 / k_m$, and for charge, $\lim_{r \rightarrow 0} F = 0$. Finite maximum gravity and charge forces: 1) allows radioactivity without the need for a weak force, 2) finite sloped energy well walls reduce the need for quantum tunneling, and 3) eliminates the problem of forces going to infinity as $r \rightarrow 0$.
 - (b) The quantum-extended relativity and classic equations reduce to the non-extended relativity and classic equations and constants, where the distance between masses and charges is sufficiently large or the masses and charges sufficiently large that the quantum effects are not measurable. **Note** that G , k_e , ε_0 ,

μ_0 , and κ (Einstein's constant, which contains G) do not exist (are not valid), where the quantum effects becomes measurable.

- (c) The covariant tensor components, in Einstein's field equations, that had the units $1/distance^2$, will now have the more complex units, $1/\sqrt{(distance^4/c_\tau^2) + k_\tau^2}$, $\tau \in \{t, m\}$.
 - (d) $1/\sqrt{(distance^4/c_\tau^2) + k_\tau^2}$ implies that as distance $\rightarrow 0$, spacetime curvature peaks at the Planck unit distance, r_p (98). The ratio of quantum units, m_p/r_p^3 , might indicate a maximum mass density. A finite force (spacetime curvature) and finite mass density would imply that black holes have sizes > 0 (are not singularities). Black hole evaporation might be possible. If there was a “big bang,” then it might not have originated from a singularity.
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