The Set Mappings Generating Geometry and Physics

George. M. Van Treeck

ABSTRACT. The Euclidean volume equation is derived from a set and limit-based foundation. Distance as a function of volume is used for simple derivations of the Minkowski distances (for example, Manhattan and Euclidean distance) and the properties of properties metric space. The Euclidean volume proof provides simpler and more rigorous derivations of Newton's gravity force and Coulomb's charge force equations (without using the inverse square law or Gauss's divergence theorem). The derivations of the gravity and charge forces exposes a ratio (constant first derivative) principle that allows simpler derivations of the spacetime equations and some general relativity equations. A symmetry property can limit geometric distance and volume to 3 dimensions per ball and a limit of 3 3-dimensional balls. All proofs are verified in Coq.

CONTENTS

1. Introduction

In mathematical analysis, metric space, Euclidean distance, and Euclidean area/volume are definitions [**Gol76**] [**Rud76**] motivated by Euclidean geometry [**Joy98**] rather than derived from set theory. Assigning a total order to sets of disjoint, real-valued intervals (dimensions) and a total order to the infinitesimals in each interval generate: volume, the L_p norms/Minkowski distances (for example, Manhattan and Euclidean distance), and the properties of metric space. And constraining a totally ordered set of dimensions to only allow *direct* sequencing in every possible order restricts distance and volume to a cyclic set of at most 3 dimensions.

Cartesian geometry motivates volume and distance in terms of a coordinate (an n-tuple of domain set points or infinitesimals) corresponding to a range set point or infinitesimal. But the set of n-tuples is countable because constructing an n-tuple requires a totally ordered (integer indexed) set of domain intervals, where each domain interval contains a totally ordered (integer indexed) set of points or infinitesimals.

A countable infinity of n-tuples implies a countable infinity of range set points, which always has measure 0. And if each n-tuple corresponds to an infinitesimal, then volume is defined as the sum of infinitesimal volumes and distance as the sum of infinitesimal distances, which is circular logic.

In order to avoid the measure 0 and circular logic problems, countable volume and countable distance are defined as the the cardinals of countable sets of Cartesian product mappings (countable sets of n-tuples). Proving that the Euclidean volume equation is an instance of the countable volume definition provides an alternative perspective into what makes a space flat versus curved and has applications to physics. Proving that all L_p norms/Minkowski distances (for example, Manhattan and Euclidean distance) are instances of countable distance shows that all "geometric" distance equations are inverse functions of volume and also explains why Euclidean distance the smallest distance in Euclidean space.

All the proofs in this article are trivial. But to ensure confidence, all the proofs have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

In order to compute areas and volumes, integrals divide all intervals into the *same* number subintervals (infinitesimals), where the size of the infinitesimals in different-sized intervals can *vary*, which makes it difficult for integrals to directly express the number of mappings between the p_x number of size c infinitesimals in one interval and the p_y number of size c infinitesimals in a different-sized interval.

Therefore, a different tool is used here. A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals (infinitesimals), where each infinitesimal has the *same* size, c. The ruler is both an inner and outer (floor and ceiling) measure of an interval.

DEFINITION 2.1. Ruler measure, M: $\forall [a,b] \subset \mathbb{R}, \ s = b - a \ \wedge \ c > 0 \ \wedge$ $(p = floor(s/c) \ \vee \ p = ceiling(s/c)) \ \wedge \ M = \sum_{i=1}^{p} c = pc.$

THEOREM 2.2. *Ruler convergence:* $M = \lim_{c \to 0} pc = s.$

The formal proof, "limit_c_0_M_eq_exact_size," in the Coq file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : \ y \leq x, \ y \in \mathbb{Z}, \ x \in \mathbb{R}\})$:

(2.1) $\forall c > 0, \ p = floor(s/c) \ \wedge \ 0 \leq |floor(s/c) - s/c| < 1 \ \Rightarrow \ |p - s/c| < 1.$

Multiply both sides of inequality 2.1 by c:

(2.2) $\forall c > 0, \quad |p - s/c| < 1 \quad \Rightarrow \quad |pc - s| < |c| = |c - 0|.$

(2.3) $\quad \forall \, \epsilon = \delta \quad \wedge \quad |pc - s| < |c - 0| < \delta$

$\Rightarrow \quad |c - 0| < \delta \quad \wedge \quad |pc - s| < \delta = \epsilon \quad := \quad M = \lim_{c \to 0} pc = s. \quad \square$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}}, \, 3.14_{c=10^{-2}}, \, 3.141_{c=10^{-3}}, ..., \pi_{\lim_{c \to 0}}$.

LEMMA 2.3. $\forall \, n \geq 1, \quad 0 < c < 1 \quad \Rightarrow \quad \lim_{c \to 0} c^n = \lim_{c \to 0} c.$

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

(2.4) $\quad n \geq 1 \quad \wedge \quad 0 < c < 1 \quad \Rightarrow \quad 0 < c^n < c \quad \Rightarrow \quad |c - c^n| < |c| = |c - 0|.$

(2.5) $\quad \forall \, \epsilon = \delta \quad \wedge \quad |c - c^n| < |c - 0| < \delta$

$\Rightarrow \quad |c - 0| < \delta \quad \wedge \quad |c - c^n| < \delta = \epsilon \quad := \quad \lim_{c \to 0} c^n = 0.$

(2.6) $\quad \lim_{c \to 0} c^n = 0 \quad \wedge \quad \lim_{c \to 0} c = 0 \quad \Rightarrow \quad \lim_{c \to 0} c^n = \lim_{c \to 0} c. \quad \square$

3. Euclidean Volume

DEFINITION 3.1. Countable volume, v_c is the number of Cartesian product mappings (n-tuples) between the members of n number of disjoint, countable domain sets:

$$\exists \, n, v_c \in \mathbb{N}, \quad x_1, \cdots, x_n : \quad v_c = \prod_{i=1}^{n} |x_i|, \quad \bigcap_{i=1}^{n} x_i = \emptyset$$

THEOREM 3.2. *Euclidean volume, v, defined as length of the range interval,* $[v_a, v_b]$, *that is equal to product of domain interval lengths,* $\{[a_1, b_1], \ldots, [a_n, b_n]\}$, *is an instance of the countable volume* (3.1):

$$v_c = \prod_{i=1}^{n} |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^{n} s_i, \, v = v_a - v_b, \, s_i = b_i - a_i.$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.

Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of size c subintervals and apply ruler convergence (2.2):

(3.1) $\quad \forall \, i \, n \in \mathbb{N}, \, i \in [1, n], \, c > 0 \, \wedge \, floor(s_i/c) = |x_i| \quad \Rightarrow \quad s_i = \lim_{c \to 0}(|x_i| \cdot c).$

Decompose the right-hand side of the volume equation into the right-hand side of the countable n-volume definition (3.1).

(3.2) $\quad s_i = \lim_{c \to 0}(|x_i| \cdot c) \quad \Leftrightarrow \quad \prod_{i=1}^{n} s_i = \lim_{c \to 0} \prod_{i=1}^{n}(|x_i| \cdot c).$

(3.3) $\quad \prod_{i=1}^{n} s_i = \lim_{c \to 0} \prod_{i=1}^{n}(|x_i| \cdot c) \quad \Leftrightarrow \quad \prod_{i=1}^{n} s_i = \lim_{c \to 0}(\prod_{i=1}^{n} |x_i|) \cdot c^n.$

(3.4) $\quad v_c = \prod_{i=1}^{n} |x_i| \quad \Leftrightarrow \quad \lim_{c \to 0} v_c \cdot c^n = \lim_{c \to 0}(\prod_{i=1}^{n} |x_i|) \cdot c^n = \prod_{i=1}^{n} s_i.$

Decompose the left-hand side of the volume equation into the left-hand side of the countable n-volume definition (3.1).

(3.5) $\quad \exists \, v \in \mathbb{R} : v_c = floor(v/c) \quad \Leftrightarrow \quad v = \lim_{c \to 0} v_c \cdot c.$

Apply lemma 2.3 to equation 3.5:

(3.6) $\quad v = \lim_{c \to 0} v_c \cdot c \quad \wedge \quad \lim_{c \to 0} c^n = \lim_{c \to 0} c \quad \Leftrightarrow \quad v = \lim_{c \to 0} v_c \cdot c^n.$

Combine equations 3.6 and 3.4:

(3.7) $\quad v = \lim_{c \to 0} v_c \cdot c^n \quad \wedge \quad \lim_{c \to 0} v_c \cdot c^n = \prod_{i=1}^{n} s_i \quad \Leftrightarrow \quad v = \prod_{i=1}^{n} s_i. \quad \square$

4. Distance

4.1. Countable distance. N-volumes have three key properties:

(1) Every n-volume (for example, triangle, sphere, torus, etc.) has a corresponding cuboid n-volume, d^n, with the same magnitude. For example, a $2cm$ by $8cm$ rectangle has the corresponding cuboid 2-volume, $4cm^2$.

(2) From the volume proof (3.2), every n-volume is a function of the number of n-tuples. And the number of n-tuples is the Cartesian product of n number of countable sets. In this case, $d^n = \lim_{c \to 0} |x|^n \cdot c^n$.

(3) Only quantities having the same type can be summed. Therefore, an n-volume can only be the sum of n-volumes:
$d^n = \sum_{i=1}^m d_i^n = \lim_{c \to 0} \sum_{i=1}^m |x_i|^n \cdot c^n$.

DEFINITION 4.1. The countable, cuboid n-volume, d_c^n, is the sum of m number of countable n-volumes.

$$\forall\, n \in \mathbb{N}, \quad d_c \in \{0, \mathbb{N}\} \quad \exists\, m \in \mathbb{N}, \quad x_1, \cdots, x_m \in X, \quad \bigcap_{i=1}^m x_i = \emptyset :$$
$$d_c^n = \sum_{i=1}^m |x_i|^n.$$

4.2. Minkowski distance (L_p norm).
The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

THEOREM 4.2. *Minkowski distance (L_p norm) is an instance of the countable cuboid n-volume* (4.1).

$$d_c^n = \sum_{i=1}^m |x_i|^n \quad \Rightarrow \quad \exists\, d, s_1, \cdots, s_m \in \mathbb{R} : \quad d = \left(\sum_{i=1}^m s_i^n \right)^{1/n}.$$

PROOF. Apply the ruler (2.1) and ruler convergence (2.2):

(4.1) $\exists\, d, s_1, \cdots, s_m \in \mathbb{R} : d_c = floor(d/c) \quad \wedge \quad |x_i| = floor(s_i/c) \quad \wedge$
$d_c^n = \sum_{i=1}^m |x_i|^n \quad \Rightarrow \quad d^n = \lim_{c \to 0} (d_c \cdot c)^n = \lim_{c \to 0} \sum_{i=1}^m (|x_i| \cdot c)^n = \sum_{i=1}^m s_i^n.$

(4.2) $$d^n = \sum_{i=1}^m s_i^n \quad \Leftrightarrow \quad d = \left(\sum_{i=1}^m s_i^n \right)^{1/n}. \qquad \square$$

4.3. Distance inequality. Proving that all Minkowski distances (L_p norms) satisfy the metric space triangle inequality requires another inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N}, \; v_a, v_b \geq 0 : \; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand the n-volume, $(v_a^{1/n} + v_b^{1/n})^n$, using the binomial expansion:

(4.3) $\forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq (v_a + v_b +$
$\sum_{i=1}^n \binom{n}{k}(v_a^{1/n})^{n-k}(v_b^{1/n})^k + \sum_{i=1}^n \binom{n}{k}(v_a^{1/n})^k(v_b^{1/n})^{n-k}) = (v_a^{1/n} + v_b^{1/n})^n.$

Take the nth root of both sides of the inequality:

(4.4) $\forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad \square$

4.4. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N}, \;\; a_i, b_i \geq 0 : \; \left(\sum_{i=1}^{m}(a_i^n + b_i^n)\right)^{1/n} \leq \left(\sum_{i=1}^{m} a_i^n\right)^{1/n} + \left(\sum_{i=1}^{m} b_i^n\right)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$(4.5) \quad \forall\, m, n \in \mathbb{N}, \; v_a, v_b \geq 0 : \quad v_a = \sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge$$

$$(v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad \left(\left(\sum_{i=1}^{m} a_i^n\right) + \left(\sum_{i=1}^{m} b_i^n\right)\right)^{1/n} =$$

$$\left(\sum_{i=1}^{m}(a_i^n + b_i^n)\right)^{1/n} \leq \left(\sum_{i=1}^{m} a_i^n\right)^{1/n} + \left(\sum_{i=1}^{m} b_i^n\right)^{1/n}. \quad \square$$

4.5. Metric Space. All Minkowski distances (L_p norms) have the properties of metric space.

The formal proofs: symmetry, triangle_inequality, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Symmetry:* $d(s_1, s_2) = \left(\sum_{i=1}^{2} s_i^p\right)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$(4.6) \quad \forall\, p : p \geq 1, \quad d(s_1, s_2) = \left(\sum_{i=1}^{2} s_i^p\right)^{1/p} = (s_1^p + s_2^p)^{1/p}$$

$$\Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \quad \square$$

THEOREM 4.6. *Triangle Inequality:*
$d(s_1, s_2) = \left(\sum_{i=1}^{2} s_i^p\right)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k:$

$$(4.7) \qquad (u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}.$$

Apply the distance inequality (4.3) to the inequality 4.7:

$$(4.8) \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \; \wedge \; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}$$

$$\wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p$$

$$\Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p}$$

$$\Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq$$

$$(u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \quad \square$$

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = \left(\sum_{i=1}^{2} s_i^p\right)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$(4.9) \qquad \forall\, [a_1, b_1], [a_2, b_2], \quad u = b_1 - a_1, \; v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0, \; v \geq 0.$$

$$(4.10) \qquad p \geq 1, \; u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \qquad \square$$

THEOREM 4.8. *Identity of Indiscernibles:* $d(u, u) = 0.$

PROOF. From the non-negativity property (4.7):

$$(4.11) \quad d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0$$

$$\Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0.$$

$$(4.12) \qquad d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v.$$

$$(4.13) \qquad d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \qquad \square$$

5. Applications to physics

5.1. Newton's gravity force equation. m_1 and m_2, are the sizes of two independent mass intervals, where each size c component of a mass interval exerts a force on each size c component of the other mass interval. If p_1 and p_2 are the number of size c components in each mass interval, then the total force, F, is equal to the total number of forces, $p_1 \cdot p_2$, and proportionate to the size, c, of each component. Applying the ruler (2.1) and volume proof (3.2), where the force, F, is defined as the rest mass, m_0, times acceleration, a:

$$(5.1) \quad p_1 = floor(m_1/c) \quad \wedge \quad p_2 = floor(m_2/c) \quad \wedge \quad F := m_0 a \propto (p_1 \cdot p_2)c$$
$$\Rightarrow \quad F := m_0 a \propto \lim_{c \to 0}(p_1 \cdot p_2)c = \lim_{c \to 0}(p_1 \cdot p_2)c^2 = \lim_{c \to 0} p_1 c \cdot p_2 c = m_1 m_2,$$

$$(5.2) \quad F := m_0 a := m_0 r/t^2 \propto m_1 m_2 \ \wedge \ m_0 = m_1 \ \Rightarrow \ r \propto m_2 \ \Rightarrow$$
$$\exists \, m_G, r_c \in \mathbb{R}: \ r = (dr/dm)m_2 = (r_c/m_G)m_2,$$

where: r is Euclidean distance, t is time, and r_c/m_G is a unit-factoring ratio.

$$(5.3) \quad m_0 = m_1 \quad \wedge \quad r = (m_G/r_c)m_2 \quad \wedge \quad F = m_0 r/t^2$$
$$\Rightarrow \quad F = m_0 r/t^2 = (r_c/m_G)m_1 m_2/t^2.$$

From the definition of force, $F := m_0 a$:

$$(5.4) \quad \int_0^t a \, dt = r/t \ \Rightarrow \ \exists \, t_c, r_c \in \mathbb{R}: \ r/t = (dr/dt) = r_c/t_c \ \Rightarrow \ t = (t_c/r_c)r.$$

$$(5.5) \quad t = (t_c/r_c)r \quad \wedge \quad F = (r_c/m_G)m_1 m_2/t^2 \quad \Rightarrow$$
$$F = (r_c/m_G)(r_c^2/t_c^2)m_1 m_2/r^2 = (r_c^3/m_G t_c^2)m_1 m_2/r^2 = Gm_1 m_2/r^2,$$

where the gravitational constant, $G = r_c^3/m_G t_c^2$, has the SI units: $m^3 kg^{-1} s^{-2}$.

5.2. Coulomb's charge force. q_1 and q_2, are the sizes of two independent charge intervals, where each size c component of a charge interval exerts a force on each size c component of the other charge interval. If p_1 and p_2 are the number of size c components in each charge interval, then the total force, F, is equal to the total number of forces, $p_1 \cdot p_2$, and proportionate to the size, c, of each component. Applying the ruler (2.1) and volume proof (3.2), where the force, F, is defined as the rest mass, m_0, times acceleration, a:

$$(5.6) \quad p_1 = floor(q_1/c) \quad \wedge \quad p_2 = floor(q_2/c) \quad \wedge \quad F \propto (p_1 \cdot p_2)c$$
$$\Rightarrow \quad F := m_0 a \propto \lim_{c \to 0}(p_1 \cdot p_2)c = \lim_{c \to 0}(p_1 \cdot p_2)c^2 = \lim_{c \to 0} p_1 c \cdot p_2 c = q_1 q_2,$$

$$(5.7) \quad F := m_0 a := m_0 r/t^2 \propto q_1 q_2 \quad \wedge$$
$$m_0 = (dm/dq)q_1 = (m_G/q_C)q_1 \quad \Rightarrow \quad r \propto q_2$$
$$\Rightarrow \quad \exists \, q_C, r_c \in \mathbb{R}: \ r = (dr/dq)q_2 = (r_c/q_C)q_2,$$

where: r is Euclidean distance, t is time, m_G/q_C and r_c/q_C are unit-factoring ratios.

(5.8) $\quad m_0 = (m_G/q_C)q_1 \quad \wedge \quad r = (q_C/r_c)q_2 \quad \wedge \quad F = m_0 r/t^2$
$$\Rightarrow \quad F = m_0 r/t^2 = (m_G/q_C)(r_c/q_C)q_1 q_2/t^2 = (m_G r_c/q_C^2)q_1 q_2/t^2.$$

From the definition of force, $F := m_0 a$:

(5.9) $\quad \int_0^t a \, dt = r/t \ \Rightarrow \ \exists \, t_c, r_c \in \mathbb{R} : \ r/t = (dr/dt) = r_c/t_c \ \Rightarrow \ t = (t_c/r_c)r.$

(5.10) $\quad t = (t_c/r_c)r \quad \wedge \quad a_G = r_c/t_c^2 \quad \wedge \quad F = (m_G r_c/q_C^2)q_1 q_2/t^2 \quad \Rightarrow$
$$F = (r_c^2/t_c^2)(m_G r_c/q_C^2)q_1 q_2/r^2 = ((m_G a_G)r_c^2/q_C^2)q_1 q_2/r^2 = k_c q_1 q_2/r^2,$$

where the charge constant, $k_C = (m_G a_G)r_c^2/q_C^2$, has the SI units: $Nm^2 C^{-2}$.

5.3. Spacetime equations. As shown in the derivations of Newton's gravity force (5.1) and Coulomb's charge force (5.2) equations: $r = (r_c/t_c)t = ct$, where r is the Euclidean distance and $r_c/t_c = c$ is a unit-factoring proportion ratio. And, the smallest distance (and time) spanning the two inertial (independent, non-accelerating) frames of reference, $[0, r_1]$ and $[0, r_2]$, is the Euclidean distance, r.

(5.11) $\qquad r = ct \quad \Rightarrow \quad (ct)^2 = r_1^2 + r_2^2 \quad \Leftrightarrow \quad r_1^2 = (ct)^2 - (x^2 + y^2 + z^2),$

where $r_2^2 = x^2 + y^2 + z^2$, which is one form of Minkowski's flat spacetime interval equation [**Bru17**]. And the length contraction and time dilation equations also follow directly from $(ct)^2 = r_1^2 + r_2^2$, where $v = r_1/t$:

(5.12) $\quad r_2^2 = (ct)^2 - r_1^2 \ \wedge \ L = r_2 \ \Rightarrow \ L^2 = c^2 t^2 - r_1^2 \ \Rightarrow \ L = ct\sqrt{1 - (v/c)^2}.$

(5.13) $\qquad L = ct\sqrt{1 - (v/c)^2} \quad \wedge \quad L_0 = ct \quad \Rightarrow \quad L = L_0\sqrt{1 - (v/c)^2}.$

(5.14) $\qquad L = ct\sqrt{1 - (v/c)^2} \ \wedge \ t' = L/c \ \Rightarrow \ t' = t\sqrt{1 - (v/c)^2}.$

5.4. Some general relativity equations: Combining the ratio (constant first derivative) equations into partial differential equations: $r = (r_c/m_G)m = ct \Rightarrow (r_c/m_G)m \cdot ct = r^2 \Rightarrow m = (m_G/r_c c)r^2/t = (m_G/r_c c)rv$. For a constant mass, m, a decrease in the distance, r, between two mass centers causes a decrease in time, t, (time slows down). v is the relativistic orbital velocity at distance, r. $(r_c/m_G)m \cdot (ct)^2 = r^3 \Rightarrow E = mc^2 = (m_G/r_c)r^3/t^2$. And $(ct)^2 = r^2 \Rightarrow c^2 = v^2 \Rightarrow (r_c/m_G)mv^2 = c^2 r \Rightarrow KE = mv^2/2 = (m_G c^2/2r_c)r$.

$c = r_c/t_c \approx 3 \cdot 10^8 ms^{-1}$ and $G = r_c^3/m_G t_c^2 = (r_c/m_G)(r_c/t_c)^2 \approx 6.7 \cdot 10^{-11} m^3 kg^{-1} s^{-2} \Rightarrow r_c/m_G \approx (6.7 \cdot 10^{-11} m^3 kg^{-1} s^{-2}/(3 \cdot 10^8 m \ s^{-1})^2 \approx 7.4 \cdot 10^{-28} m \ kg^{-1}$, which can be used to quantify the constants in the previously derived equations. For example, $m = (m_G/r_c c)rv \approx (1/((7.4 \cdot 10^{-28} m \ kg^{-1})(3 \cdot 10^8 m \ s^{-1})))rv \approx (4.5 \cdot 10^{18} kg \ s \ m^{-2})rv$.

Likewise, for charge, $r = (r_c/q_C)q = ct \Rightarrow q = (q_C/r_c c)r^2/t = (q_C/r_c c)rv$, $E = qc^2 = (q_C/r_c)r^3/t^2$, and $KE = qv^2/2 = (q_C c^2/2r_c)r$. And if the ratio of an electron's mass to charge is m_G/q_C, then $m_G/q_C \approx 9.1 \cdot 10^{-31} kg/1.6 \cdot 10^{-19}C \approx 5.7 \cdot 10^{-12} kgC^{-1}$. And using Coulomb's constant in ratio form: $k_C = (r_c/t_c)^2(m_G r_c/q_C^2) \approx 9 \cdot 10^9 Nm^2 C^{-2} \approx (3 \cdot 10^8 m \ s^{-1})^2(5.7 \cdot 10^{-12} kg \ C^{-1})(r_c/q_c) \Rightarrow r_c/q_C \approx 1.7 \cdot 10^5 m \ C^{-1}$. Therefore, $q = (q_C/r_c c)rv \approx (1/((1.7 \cdot 10^5 m \ C^{-1})(3 \cdot 10^8 m \ s^{-1})))rv \approx (1.9 \cdot 10^{-13}C \ s \ m^{-2})rv$.

5.5. 3 dimensional balls. Countable volume, $v_c = \prod_{i=1}^{n} |x_i|$, Euclidean volume, $v = \prod_{i=1}^{n} s_i$, and all Minkowski distances, $d = (\sum_{i=1}^{n} s_i^n)^{1/n}$, require that a set of domain intervals/dimensions can be assigned a *total order*. A total order is defined in terms of successor and predecessor relations, where, in this case, the successor and predecessor relations are specified by the integers $i = 1$ to n that map to a set of domain intervals/dimensions.

The commutative properties of union, multiplication, and addition allow sequencing through each interval (dimension) in every possible order. But, deterministic sequencing requires that "jumping" (indexing) over set members to a specific member requires calculating an offset, which is implicitly sequencing via the successor and predecessor relations.

Therefore, deterministic sequencing directly via the successor and predecessor relations from one set member to every other member requires each set member to be sequentially adjacent (either a successor or predecessor) to every other member, herein referred to as a symmetry constraint. It will now be proved that coexistence of the symmetry constraint on a sequentially ordered set defines a cyclic set that contains at most 3 members, in this case, 3 dimensions per ball and 3 3-dimensional balls.

DEFINITION 5.1. Ordered geometry:

$$\forall\, i\, n \in \mathbb{N},\ i \in [1, n-1],\ \forall\, x_i \in \{x_1, \ldots, x_n\},$$
$$successor\ x_i = x_{i+1}\ \wedge\ predecessor\ x_{i+1} = x_i.$$

DEFINITION 5.2. Symmetry Constraint (every set member is sequentially adjacent to every other member):

$$\forall\, i\, j\, n \in \mathbb{N},\ \forall\, x_i\, x_j \in \{x_1, \ldots, x_n\},\ successor\ x_i = x_j\ \Leftrightarrow\ predecessor\ x_j = x_i.$$

THEOREM 5.3. *An ordered and symmetric set is a cyclic set.*

$$i = n\ \wedge\ j = 1\ \Rightarrow\ successor\ x_n = x_1\ \wedge\ predecessor\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry constraint (5.2):

(5.15) $\qquad i = n\ \wedge\ j = 1\ \wedge\ successor\ x_i = x_j\ \Rightarrow\ successor\ x_n = x_1.$

Applying the definition of the symmetry constraint (5.2) to conclusion 5.15:

(5.16) $successor\ x_i = x_j\ \Rightarrow\ predecessor\ x_j = x_i\ \Rightarrow\ predecessor\ x_1 = x_n.$ $\qquad \square$

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

Lemmas: `adj111`, `adj122`, `adj212`, `adj123`, `adj133`, `adj233`, `adj213`, `adj313`, `adj323`, and `not_all_mutually_adjacent_gt_3`.

The following proof uses Horn clauses (a subset of first order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3).

DEFINITION 5.5. Successor of m is n:

$$(5.17) \quad Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m+1 \leq setsize).$$

DEFINITION 5.6. Predecessor of m is n:

$$(5.18) \quad Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1).$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:
(5.19)
$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize).$$

Prove that every member is adjacent to every other member, where $setsize \in \{1, 2, 3\}$:

$$(5.20) \qquad Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1).$$

$$(5.21) \qquad Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize).$$
$$(5.22) \qquad Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1).$$

$$(5.23) \qquad Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize).$$
$$(5.24) \qquad Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1).$$
$$(5.25) \qquad Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1).$$
$$(5.26) \qquad Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize).$$
$$(5.27) \qquad Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize).$$
$$(5.28) \qquad Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1).$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$(5.29) \quad \forall\, setsize > 3: \quad \neg Successor(1, 3, setsize > 3)$$
$$\leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize).$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$(5.30) \quad \forall\, setsize > 3: \quad \neg Predecessor(1, 3, setsize > 3)$$
$$\leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$(5.31) \quad \forall\, setsize > 3: \quad \neg Adjacent(1, 3, setsize > 3)$$
$$\leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

(1) The second derivative is the traditional way of measuring the flatness/ curvature of space, for example, the Laplacian and tensors. Proving that Euclidean area/volume is an instance of the countable n-volume provides the alternative perspective that "flat" space is where each Cartesian n-tuple corresponds to an infinitesimal having the same size as all other infinitesimals (3.2) and otherwise the space is curved (non-Euclidean).

(2) Newton's gravity equation (5.1) and Coulomb's charge equation (5.2) are examples of non-Euclidean (curved) space, where each force n-tuple corresponds to an infinitesimal, ΔF, and the size of the infinitesimal varies inversely with the area, r^2. For example, $r_b = r_a + \Delta r$, $r_c = r_b + \Delta r \Rightarrow$ $(r_b^2 - r_a^2) < (r_c^2 - r_b^2) \Rightarrow \Delta F_{a,b} > \Delta F_{b,c}$.

(3) Every n-volume can only be expressed as the sum of n-volumes, which constrains the size of a distance measures within each n-volume. Therefore, all "geometric" distance measures are inverse functions of the n-volumes If the definition of a complete metric space allows equations that are not inverse functions of volume, then the definition of a complete metric space is not a sufficient filter to obtain only geometric distances.

(4) Proofs that Euclidean distance is the smallest distance between two distinct points have equated Euclidean distance to a straight line (equation), where it is assumed that the straight line length is the smallest distance [**Joy98**]. And proofs that a straight line (equation) is the smallest distance have equated the straight line to the Euclidean distance. There have been no set and limit-based explanations of why the Euclidean distance/straight line length is the smallest distance.

  All "geometric" distance measures reduce to corresponding Minkowski distances (4.2). In the Minkowski equation, $d = (\sum_{i=1}^{m} s_i^n)^{1/n}$, if m represents the number of domain intervals/countable sets (one from each dimension), then $1 \leq n \leq m$. And $m = 2 \Rightarrow 1 \leq n \leq 2$, which constrains all Minkowski distances to a range from Manhattan distance (the largest distance) to Euclidean distance (the smallest distance).

(5) Compare the distance sum inequality (4.4):

$$(\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

to the Minkowski's sum inequality:

$$(\sum_{i=1}^{m}(a_i^n + b_i^n)^n)^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

Note the difference in the left side of the two inequalities:

$$(\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \quad vs. \quad (\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n}.$$

The derivation of the distance sum inequality is much simpler and shorter than the derivation of Minkowski's sum inequality. Unlike Minkowski's sum inequality proof, the distance (sum) inequality proofs do not depend on: convexity, L_p space inequalities (for example, Hölder's inequality or Mahler's inequality), or the triangle inequality, which indicates that the distance (sum) inequality is a more fundamental inequality than the other distance-related inequalities.

(6) Applying the ruler (2.1) and volume proof (3.2) to the Cartesian product of same-sized, infinitesimal mass forces and charge forces to derive

Newton's gravity force (5.1) and Coulomb's charge force (5.2) equations provide several firsts and some insights into physics:

(a) These are the first deductive derivations of the gravity and charge forces. All other derivations have been empirical and use Newton's induction, which is not fully provable, for example, assumes the inverse square law, which is based on empirical observation. And sometimes also assumes the inverse square law is due to Gauss' flux divergence.

(b) These are the first derivations to not use the inverse square law or Gauss's divergence theorem.

(c) These are the first derivations to show that the definition of force, $F := m_0 a$, containing acceleration, a, generates the inverse square law: $\int_0^t a\,dt = r/t \Rightarrow \exists\, t_c, r_c \in \mathbb{R} : t = (t_c/r_c)r$. Using the same derivation steps as for Coulomb's charge force:
$F := m_0 a = m_0 r/t^2 = (r_c/t_c)^2 (m_x r_c/x_x^2) x_1 x_2 / r^2 = k_x x_1 x_2 / r^2$.

(d) Using Occam's razor, those versions of constants like: charge, vacuum magnetic permeability, fine structure, etc. that contain the value 4π might be incorrect because those constants are based on the less parsimonious assumption that the inverse square law is due to Gauss's flux divergence on a sphere having the surface area, $4\pi r^2$.

(e) These are the first derivations to show that the gravity force, charge force, spacetime, and general relativity all depend on time being proportionate to distance: $r = (r_c/t_c)t = ct$.

(f) The derivations of the gravity and charge force equations expose a ratio (constant first derivative) principle. Combining the constant first derivatives (ratios) into partial differential equations allows simple algebraic derivations of some general relativity equations (5.4) without the need for integrating second derivative (spacetime curvature) tensors.

(g) A state is represented by a constant value. And a constant value, by definition, cannot vary with distance and time interval lengths. Therefore, the spin states of two quantum entangled electrons and the polarization states of two quantum entangled photons are independent of the amount of distance and time between the entangled particles.

(7) It was proved that deterministic sequencing in a totally ordered set from one set member directly to every other set member requires a cyclic set with at most 3 members (5.3). And empirical observation indicates that geometric space is a totally orderable set of dimensions and allows deterministic sequencing from one dimension directly (without jumping over other members) to every other dimension.

(a) Using Occam's razor, a cyclic set of at most 3 members is the most parsimonious explanation of only observing 3 dimensions of geometric distance and volume.

(b) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (5.4), each member being an ordered and symmetric set of 3 dimensions (three balls).

(c) Each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values of the same type, which allows

$3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values of the same type per 3-dimensional ball, for example, vector orientation values: -1, 0, 1 per orthogonal direction in the ball.

(d) Each of the 3 possible ordered and symmetric dimensions of discrete physical states could contain an unordered collection (bag) of discrete state values. Bags are non-deterministic. For example, every time an unordered binary state is "pulled" from a bag, there is a 50 percent chance of getting one of the binary values.

(8) It was shown that some fundamental geometry (volume and the Minkowski distances/L_p norms) and physics (gravity force and charge force) are derived from the combinatorial mappings between the infinitesimals of real-valued intervals. The proofs and derivations in this article show that the ruler (2.1) is a tool to directly express some combinatorial relations in geometry, probability, physics, etc. that are difficult to directly express with differential equations and integrals.

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GEORGE VAN TREECK, 668 WESTLINE DR., ALAMEDA, CA 94501