The Set Properties Generating Geometry and Physics

George. M. Van Treeck

Abstract. Euclidean volume is proved to be an instance of a countable, set-based volume. The Lp norms/Minkowski distances (for example, Manhattan and Euclidean distance) are proved to be instances of a set-based, countable distance. The Euclidean volume proof provides an alternative perspective on what makes a space flat versus curved and provides simpler and more rigorous derivations of Newton's gravity force and Coulomb's charge force equations (without using the inverse square law or Gauss's divergence theorem). The derivations of the gravity and charge forces exposes a ratio (constant first derivative) principle that allows simpler derivations of the spacetime and some general relativity equations. A symmetry property on a totally ordered set can limit distance and volume to 3 dimensions. All proofs are verified in Coq.

Contents

1. Introduction

Mathematical (real) analysis text books start by constructing differential calculus from a foundation of set and number theory, without the need to reference geometry. But the development of the vector norm, integrals, and measure theory leap to the Euclidean geometry-based definitions: Euclidean distance, Euclidean area/volume, and metric space respectively [Gol76] [Rud76]. This article constructs volume and distance from a set and limit-based foundation.

In this article, "countable volume" is defined as the cardinal of the set of Cartesian coordinates (n-tuples) corresponding to the members of countable domain sets.

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Dividing domain intervals *approximately* into countable sets of same-sized, size c, infinitesimals, allows constructing n-tuples corresponding to the countable sets of size c infinitesimals.

Proving that the Euclidean volume equation is an instance of the countable volume definition provides an alternative to second derivative-based methods (Laplacian, tensors, etc.) for specifying the flatness (curvature) a space. And the proof also has applications to classical physics.

If volume is a bijective function of distance, then distance is a bijective (inverse) function of volume. And all n-volumes are the sum of n-volumes, which implies that distance is also a function of the sum of n-volumes. Here, distance is defined as a function of the sum of countable volume cardinals. From this definition the expected equations and properties of distance can be derived.

Deterministic construction of an n-tuple from a set of size c infinitesimals in each domain interval, requires a total order of the domain intervals (dimensions). But, the members of a totally ordered set can only be sequenced via the order's defined successor and predecessor relations. Therefore, sequencing through a set of n number of members in all n-at-time permutations requires that each set member is sequentially adjacent, either a successor or predecessor, to every other member (a "symmetry" property), which limits the number of members in a totally ordered set.

All the proofs in this article are trivial. But to ensure confidence, all the proofs have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

In order to compute areas and volumes, integrals divide all intervals into the *same* number subintervals (infinitesimals, for example: dx, dy, dz), where the size of the infinitesimals *vary* with the size of the intervals. The varying size of infinitesimals makes it difficult for integrals (and differential equations) to directly express the Cartesian mappings between the p_x number of size c infinitesimals in one domain interval and the p_y number of the *same* size, c, infinitesimals in a different-sized domain interval.

Therefore, a different tool is used here. A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals (infinitesimals), where each infinitesimal has the *same* size, c. The ruler is both an inner and outer (floor and ceiling) measure of an interval.

Definition 2.1. Ruler measure, M: $\forall\, [a,b] \subset \mathbb{R},\;\; s = b - a \;\wedge\; c > 0 \;\wedge\; (p = \mathit{floor}(s/c) \;\vee\; p = \mathit{ceiling}(s/c)) \;\wedge\; M = \sum_{i=1}^{p} c = pc.$

Theorem 2.2. *Ruler convergence:* $M = \lim_{c \to 0} pc = s.$

The formal proof, "limit_c_0_M_eq_exact_size," in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $\mathit{floor}(x) = \mathit{max}(\{y :\; y \le x,\; y \in \mathbb{Z},\; x \in \mathbb{R}\})$:

$$\forall\, c > 0,\; p = \mathit{floor}(s/c) \;\wedge\; 0 \le |\mathit{floor}(s/c) - s/c| < 1 \;\Rightarrow\; |p - s/c| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by c:

$$\forall\, c > 0, \quad |p - s/c| < 1 \quad \Rightarrow \quad |pc - s| < |c| = |c - 0|. \tag{2.2}$$

$$(2.3)\quad \forall\, \epsilon = \delta \quad \wedge \quad |pc - s| < |c - 0| < \delta$$
$$\Rightarrow \quad |c - 0| < \delta \quad \wedge \quad |pc - s| < \delta = \epsilon \quad := \quad M = \lim_{c\to 0} pc = s. \quad \square$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}},\ 3.14_{c=10^{-2}},\ 3.141_{c=10^{-3}}, ..., \pi_{\lim_{c\to 0}}$.

LEMMA 2.3. $\forall\, n \geq 1, \quad 0 < c < 1 \quad \Rightarrow \quad \lim_{c\to 0} c^n = \lim_{c\to 0} c.$

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$(2.4)\quad n \geq 1 \quad \wedge \quad 0 < c < 1 \quad \Rightarrow \quad 0 < c^n < c \quad \Rightarrow \quad |c - c^n| < |c| = |c - 0|.$$

$$(2.5)\quad \forall\, \epsilon = \delta \quad \wedge \quad |c - c^n| < |c - 0| < \delta$$
$$\Rightarrow \quad |c - 0| < \delta \quad \wedge \quad |c - c^n| < \delta = \epsilon \quad := \quad \lim_{c\to 0} c^n = 0.$$

$$(2.6)\quad \lim_{c\to 0} c^n = 0 \quad \wedge \quad \lim_{c\to 0} c = 0 \quad \Rightarrow \quad \lim_{c\to 0} c^n = \lim_{c\to 0} c. \quad \square$$

3. Euclidean Volume

DEFINITION 3.1. Countable volume, v_c is the number of Cartesian product mappings (n-tuples) between the members of n number of disjoint, countable domain sets:

$$\exists\, n, v_c \in \mathbb{N}, \quad x_1, \cdots, x_n : \quad v_c = \textstyle\prod_{i=1}^{n} |x_i|, \quad \bigcap_{i=1}^{n} x_i = \emptyset$$

THEOREM 3.2. *Euclidean volume, v, defined as length of the range interval, $[v_a, v_b]$, that is equal to product of domain interval lengths, $\{[a_1, b_1], \ldots, [a_n, b_n]\}$, is an instance of the countable volume (3.1):*

$$v_c = \textstyle\prod_{i=1}^{n} |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^{n} s_i,\ \ v = v_a - v_b,\ \ s_i = b_i - a_i.$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of size c subintervals and apply ruler convergence (2.2):

$$(3.1)\quad \forall\, i\ n \in \mathbb{N},\ \ i \in [1, n],\ c > 0 \ \wedge\ floor(s_i/c) = |x_i| \ \Rightarrow\ s_i = \lim_{c\to 0}(|x_i| \cdot c).$$

Decompose the right-hand side of the volume equation into the right-hand side of the countable n-volume definition (3.1).

$$(3.2)\quad s_i = \lim_{c\to 0}(|x_i| \cdot c) \quad \Leftrightarrow \quad \textstyle\prod_{i=1}^{n} s_i = \prod_{i=1}^{n} \lim_{c\to 0}(|x_i| \cdot c).$$

$$(3.3)\quad \textstyle\prod_{i=1}^{n} s_i = \prod_{i=1}^{n} \lim_{c\to 0}(|x_i| \cdot c) \quad \Leftrightarrow \quad \prod_{i=1}^{n} s_i = \lim_{c\to 0}(\prod_{i=1}^{n} |x_i|) \cdot c^n.$$

$$(3.4)\quad v_c = \textstyle\prod_{i=1}^{n} |x_i| \quad \Leftrightarrow \quad \lim_{c\to 0} v_c \cdot c^n = \lim_{c\to 0}(\prod_{i=1}^{n} |x_i|) \cdot c^n = \prod_{i=1}^{n} s_i.$$

Decompose the left-hand side of the volume equation into the left-hand side of the countable n-volume definition (3.1).

$$(3.5)\quad \exists\, v \in \mathbb{R} :\ v_c = floor(v/c) \quad \Leftrightarrow \quad v = \lim_{c\to 0} v_c \cdot c.$$

Apply lemma 2.3 to equation 3.5:

$$(3.6)\quad v = \lim_{c\to 0} v_c \cdot c \quad \wedge \quad \lim_{c\to 0} c^n = \lim_{c\to 0} c \quad \Leftrightarrow \quad v = \lim_{c\to 0} v_c \cdot c^n.$$

Combine equations 3.6 and 3.4:

$$(3.7)\quad v = \lim_{c\to 0} v_c \cdot c^n \quad \wedge \quad \lim_{c\to 0} v_c \cdot c^n = \textstyle\prod_{i=1}^{n} s_i \quad \Leftrightarrow \quad v = \prod_{i=1}^{n} s_i. \quad \square$$

4. Distance

4.1. Countable distance. N-volumes have three key properties:

(1) Every shape of n-volume (for example, triangle, sphere, torus, etc.) in both Euclidean and non-Euclidean space is a non-negative magnitude that has a corresponding cuboid n-volume, d^n, with the same magnitude. For example, a rectangle with 2-volume magnitude, $2 \times 8 = 16$, has the corresponding cuboid 2-volume magnitude, $d^2 = 4^2 = 16$.

(2) Only quantities having the same type can be summed. Therefore, an n-volume can only be the sum of n-volumes, which all reduce to the corresponding sum of cuboid n-volumes: $d^n = \sum_{i=1}^{m} d_i^n$.

(3) From the volume proof (3.2), every n-volume is a function of the number of n-tuples. And the number of n-tuples is the Cartesian product of n number of countable sets. In this case, $d^n = \lim_{c \to 0}(|x| \cdot c)^n$.

Definition 4.1. The countable distance, d_c, is a function of the sum of m number of countable n-volumes.

$$\forall\, n \in \mathbb{N}, \quad d_c \in \{0, \mathbb{N}\} \quad \exists\, m \in \mathbb{N}, \quad x_1, \cdots, x_m \in X, \quad \bigcap_{i=1}^{m} x_i = \emptyset : \\ d_c^n = \sum_{i=1}^{m} |x_i|^n.$$

4.2. Minkowski distance (L_p norm).
The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

Theorem 4.2. *Minkowski distance (L_p norm) is an instance of the countable distance (4.1).*

$$d_c^n = \sum_{i=1}^{m} |x_i|^n \quad \Rightarrow \quad \exists\, d, s_1, \cdots, s_m \in \mathbb{R} : \quad d = (\sum_{i=1}^{m} s_i^n)^{1/n}.$$

Proof. Apply the ruler (2.1):

$$\exists\, d, s_1, \cdots, s_m \in \mathbb{R} : \ d_c = floor(d/c) \quad \wedge \quad |x_i| = floor(s_i/c). \tag{4.1}$$

Apply the ruler convergence (2.2):

$$d_c^n = \sum_{i=1}^{m} |x_i|^n \Rightarrow d^n = \lim_{c \to 0}(d_c \cdot c)^n = \lim_{c \to 0} \sum_{i=1}^{m}(|x_i| \cdot c)^n = \sum_{i=1}^{m} s_i^n. \tag{4.2}$$

$$d^n = \sum_{i=1}^{m} s_i^n \quad \Leftrightarrow \quad d = (\sum_{i=1}^{m} s_i^n)^{1/n}. \tag{4.3}$$

□

4.3. Distance inequality. Proving that all Minkowski distances (L_p norms) satisfy the metric space triangle inequality requires another inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

Theorem 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N}, \ v_a, v_b \geq 0 : \ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

Proof. Expand the n-volume, $(v_a^{1/n} + v_b^{1/n})^n$, using the binomial expansion:

$$\forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq (v_a + v_b + \\ \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k}) = (v_a^{1/n} + v_b^{1/n})^n. \tag{4.4}$$

Take the n^{th} root of both sides of the inequality:

$$\forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \tag{4.5}$$

□

4.4. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N},\ \ a_i, b_i \geq 0 :\ \ (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$\begin{aligned} &\forall\, m, n \in \mathbb{N},\ \ v_a, v_b \geq 0 :\quad v_a = \textstyle\sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge \\ &\quad (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad ((\textstyle\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} = \\ &\qquad (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \end{aligned} \tag{4.6}$$ □

4.5. Metric Space. All Minkowski distances (L_p norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.7}$$

Apply the distance inequality (4.3) to the inequality 4.7:

$$\begin{aligned} &(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \ \wedge\ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ &\qquad \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ \Rightarrow \quad &(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ &\qquad \Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq \\ &\qquad\quad (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \end{aligned} \tag{4.8}$$ □

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$\begin{aligned} &\forall\, p : p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ &\qquad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \end{aligned} \tag{4.9}$$ □

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0. \tag{4.10}$$

$$p \geq 1,\ \ u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \tag{4.11}$$ □

THEOREM 4.8. *Identity of Indiscernibles:* $d(u, u) = 0.$

PROOF. From the non-negativity property (4.7):

$$d(u,w) \geq 0 \quad \wedge \quad d(u,v) \geq 0 \quad \wedge \quad d(v,w) \geq 0 \quad \Rightarrow \quad \exists\, d(u,w) = d(u,v) = d(v,w) = 0. \tag{4.12}$$

$$d(u,w) = d(v,w) = 0 \quad \Rightarrow \quad u = v. \tag{4.13}$$

$$d(u,v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u,u) = 0. \tag{4.14}$$

□

5. Applications to physics

5.1. Newton's gravity force equation. m_1 and m_2, are the sizes of two independent mass intervals, where each size c component of a mass interval exerts a force on each size c component of the other mass interval. If p_1 and p_2 are the number of size c components in each mass interval, then the total force, F, is equal to the total number of forces, $p_1 \cdot p_2$, and proportionate to the size, c, of each component. Applying the ruler (2.1) and volume proof (3.2), where the force, F, is defined as the rest mass, m_0, times acceleration, a:

$$p_1 = floor(m_1/c) \quad \wedge \quad p_2 = floor(m_2/c) \quad \wedge \quad F := m_0 a \propto (p_1 \cdot p_2)c \quad \Rightarrow \quad F := m_0 a \propto \lim_{c \to 0} (p_1 \cdot p_2)c = \lim_{c \to 0} (p_1 \cdot p_2)c^2 = \lim_{c \to 0} p_1 c \cdot p_2 c = m_1 m_2, \tag{5.1}$$

$$F := m_0 a := m_0 r/t^2 \propto m_1 m_2 \ \wedge \ m_0 = m_1 \ \Rightarrow \ r \propto m_2 \ \Rightarrow \ \exists\, m_G, r_c \in \mathbb{R} :\ r = (\mathrm{d}r/\mathrm{d}m) m_2 = (r_c/m_G) m_2, \tag{5.2}$$

where: r is Euclidean distance, t is time, and r_c/m_G is a unit-factoring ratio.

$$m_0 = m_1 \quad \wedge \quad r = (m_G/r_c) m_2 \quad \wedge \quad F = m_0 r/t^2 \quad \Rightarrow \quad F = m_0 r/t^2 = (r_c/m_G) m_1 m_2/t^2. \tag{5.3}$$

From the definition of force, $F := m_0 a$:

$$\int_0^t a\mathrm{d}t = r/t \ \Rightarrow \ \exists\, t_c, r_c \in \mathbb{R} :\ r/t = (\mathrm{d}r/\mathrm{d}t) = r_c/t_c \ \Rightarrow \ t = (t_c/r_c) r. \tag{5.4}$$

$$t = (t_c/r_c) r \quad \wedge \quad F = (r_c/m_G) m_1 m_2/t^2 \quad \Rightarrow \quad F = (r_c/m_G)(r_c^2/t_c^2) m_1 m_2/r^2 = (r_c^3/m_G t_c^2) m_1 m_2/r^2 = G m_1 m_2/r^2, \tag{5.5}$$

where the gravitational constant, $G = r_c^3/m_G t_c^2$, has the SI units: $m^3 kg^{-1} s^{-2}$.

5.2. Coulomb's charge force. q_1 and q_2, are the sizes of two independent charge intervals, where each size c component of a charge interval exerts a force on each size c component of the other charge interval. If p_1 and p_2 are the number of size c components in each charge interval, then the total force, F, is equal to the total number of forces, $p_1 \cdot p_2$, and proportionate to the size, c, of each component. Applying the ruler (2.1) and volume proof (3.2), where the force, F, is defined as the rest mass, m_0, times acceleration, a:

$$p_1 = floor(q_1/c) \quad \wedge \quad p_2 = floor(q_2/c) \quad \wedge \quad F \propto (p_1 \cdot p_2)c \quad \Rightarrow \quad F := m_0 a \propto \lim_{c \to 0} (p_1 \cdot p_2)c = \lim_{c \to 0} (p_1 \cdot p_2)c^2 = \lim_{c \to 0} p_1 c \cdot p_2 c = q_1 q_2, \tag{5.6}$$

$$(5.7)\quad F := m_0 a := m_0 r/t^2 \propto q_1 q_2 \quad \wedge$$
$$m_0 = (\mathrm{d}m/\mathrm{d}q)q_1 = (m_G/q_C)q_1 \quad \Rightarrow \quad r \propto q_2$$
$$\Rightarrow \quad \exists\, q_C, r_c \in \mathbb{R}:\ r = (\mathrm{d}r/\mathrm{d}q)q_2 = (r_c/q_C)q_2,$$

where: r is Euclidean distance, t is time, m_G/q_C and r_c/q_C are unit-factoring ratios.

$$(5.8)\quad m_0 = (m_G/q_C)q_1 \quad \wedge \quad r = (q_C/r_c)q_2 \quad \wedge \quad F = m_0 r/t^2$$
$$\Rightarrow \quad F = m_0 r/t^2 = (m_G/q_C)(r_c/q_C)q_1 q_2/t^2 = (m_G r_c/q_C^2)q_1 q_2/t^2.$$

From the definition of force, $F := m_0 a$:

$$(5.9)\quad \textstyle\int_0^t a\mathrm{d}t = r/t \ \Rightarrow\ \exists\, t_c, r_c \in \mathbb{R}:\ r/t = (\mathrm{d}r/\mathrm{d}t) = r_c/t_c \ \Rightarrow\ t = (t_c/r_c)r.$$

$$(5.10)\quad t = (t_c/r_c)r \quad \wedge \quad a_G = r_c/t_c^2 \quad \wedge \quad F = (m_G r_c/q_C^2)q_1 q_2/t^2 \quad \Rightarrow$$
$$F = (r_c^2/t_c^2)(m_G r_c/q_C^2)q_1 q_2/r^2 = ((m_G a_G)r_c^2/q_C^2)q_1 q_2/r^2 = k_c q_1 q_2/r^2,$$

where the charge constant, $k_C = (m_G a_G)r_c^2/q_C^2$, has the SI units: Nm^2C^{-2}.

5.3. Spacetime equations. As shown in the derivations of Newton's gravity force (5.1) and Coulomb's charge force (5.2) equations: $r = (r_c/t_c)t = ct$, where r is the Euclidean distance and $r_c/t_c = c$ is a unit-factoring proportion ratio. And, the smallest distance (and time) spanning the two inertial (independent, non-accelerating) frames of reference, $[0, r_1]$ and $[0, r_2]$, is the Euclidean distance, r.

$$(5.11)\qquad r = ct \quad \Rightarrow \quad (ct)^2 = r_1^2 + r_2^2 \quad \Leftrightarrow \quad r_1^2 = (ct)^2 - (x^2 + y^2 + z^2),$$

where $r_2^2 = x^2 + y^2 + z^2$, which is one form of Minkowski's flat spacetime interval equation [**Bru17**]. And the length contraction and time dilation equations also follow directly from $(ct)^2 = r_1^2 + r_2^2$, where $v = r_1/t$:

$$(5.12)\quad r_2^2 = (ct)^2 - r_1^2 \ \wedge\ L = r_2 \ \Rightarrow\ L^2 = c^2t^2 - r_1^2 \ \Rightarrow\ L = ct\sqrt{1-(v/c)^2}.$$

$$(5.13)\qquad L = ct\sqrt{1-(v/c)^2} \quad \wedge \quad L_0 = ct \quad \Rightarrow \quad L = L_0\sqrt{1-(v/c)^2}.$$

$$(5.14)\qquad L = ct\sqrt{1-(v/c)^2} \ \wedge\ t' = L/c \ \Rightarrow\ t' = t\sqrt{1-(v/c)^2}.$$

5.4. Some general relativity equations: Combining the ratio (constant first derivative) equations into partial differential equations: $r = (r_c/m_G)m = ct \Rightarrow (r_c/m_G)m \cdot ct = r^2 \Rightarrow m = (m_G/r_c c)r^2/t = (m_G/r_c c)rv$. For a constant mass, m, a decrease in the distance, r, between two mass centers causes a decrease in time, t, (time slows down). v is the relativistic orbital velocity at distance, r. $(r_c/m_G)m \cdot (ct)^2 = r^3 \Rightarrow E = mc^2 = (m_G/r_c)r^3/t^2$. And $(ct)^2 = r^2 \Rightarrow c^2 = v^2 \Rightarrow (r_c/m_G)mv^2 = c^2 r \Rightarrow KE = mv^2/2 = (m_G c^2/2r_c)r$.

$c = r_c/t_c \approx 3 \cdot 10^8 ms^{-1}$ and $G = r_c^3/m_G t_c^2 = (r_c/m_G)(r_c/t_c)^2 \approx 6.7 \cdot 10^{-11}m^3kg^{-1}s^{-2} \Rightarrow r_c/m_G \approx (6.7 \cdot 10^{-11}m^3kg^{-1}s^{-2}/(3 \cdot 10^8 m\ s^{-1})^2 \approx 7.4 \cdot 10^{-28}m\ kg^{-1}$, which can be used to quantify the constants in the previously derived equations. For example, $m = (m_G/r_c c)rv \approx (1/((7.4 \cdot 10^{-28}m\ kg^{-1})(3 \cdot 10^8 m\ s^{-1})))rv \approx (4.5 \cdot 10^{18}kg\ s\ m^{-2})rv$.

Likewise, for charge, $r = (r_c/q_C)q = ct \Rightarrow q = (q_C/r_c c)r^2/t = (q_C/r_c c)rv$, $E = qc^2 = (q_C/r_c)r^3/t^2$, and $KE = qv^2/2 = (q_C c^2/2r_c)r$. And if the ratio of an electron's mass to charge is m_G/q_C, then $m_G/q_C \approx 9.1 \cdot 10^{-31}kg/1.6 \cdot 10^{-19}C \approx 5.7 \cdot 10^{-12}kgC^{-1}$. And using Coulomb's constant in ratio form: $k_C = (r_c/t_c)^2(m_G r_c/q_C^2) \approx 9 \cdot 10^9 Nm^2C^{-2} \approx (3 \cdot 10^8 m\ s^{-1})^2(5.7 \cdot 10^{-12}kg\ C^{-1})(r_c/q_c) \Rightarrow$

$r_c/q_C \approx 1.7 \cdot 10^5 m\ C^{-1}$. Therefore, $q = (q_C/r_c c) rv \approx (1/((1.7 \cdot 10^5 m\ C^{-1})(3 \cdot 10^8 m\ s^{-1}))) rv \approx (1.9 \cdot 10^{-13} C\ s\ m^{-2}) rv$.

5.5. 3 dimensional balls.

Definition 5.1. Totally ordered set:

$$\forall\, i\, n \in \mathbb{N},\ i \in [1, n-1],\ \forall\, x_i\ \in \{x_1, \ldots, x_n\},$$
$$\textit{successor } x_i = x_{i+1}\ \wedge\ \textit{predecessor } x_{i+1} = x_i.$$

Definition 5.2. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i\, j\, n \in \mathbb{N},\ \forall\, x_i\, x_j \in \{x_1, \ldots, x_n\},\ \textit{successor } x_i = x_j\ \Leftrightarrow\ \textit{predecessor } x_j = x_i.$$

Theorem 5.3. *A totally ordered and symmetric set is a cyclic set.*

$$i = n\ \wedge\ j = 1\ \Rightarrow\ \textit{successor } x_n = x_1\ \wedge\ \textit{predecessor } x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

Proof. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (5.2):

$$i = n\ \wedge\ j = 1\ \wedge\ \textit{successor } x_i = x_j\ \Rightarrow\ \textit{successor } x_n = x_1. \tag{5.15}$$

Applying the definition of the symmetry property (5.2) to conclusion 5.15:

$$\textit{successor } x_i = x_j\ \Rightarrow\ \textit{predecessor } x_j = x_i\ \Rightarrow\ \textit{predecessor } x_1 = x_n. \tag{5.16}$$ □

Theorem 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

Proof.
It was proved that an ordered and symmetric set is a cyclic set (5.3).

Definition 5.5. Successor of m is n:

$$\textit{Successor}(m, n, \textit{setsize}) \leftarrow (m = \textit{setsize} \wedge n = 1) \vee (n = m + 1 \leq \textit{setsize}). \tag{5.17}$$

Definition 5.6. Predecessor of m is n:

$$\textit{Predecessor}(m, n, \textit{setsize}) \leftarrow (m = 1 \wedge n = \textit{setsize}) \vee (n = m - 1 \geq 1). \tag{5.18}$$

Definition 5.7. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$\textit{Adjacent}(m, n, \textit{setsize}) \leftarrow \textit{Successor}(m, n, \textit{setsize}) \vee \textit{Predecessor}(m, n, \textit{setsize}). \tag{5.19}$$

Prove that every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (m = setsize \wedge n = 1). \tag{5.20}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (n = m + 1 \leq setsize). \tag{5.21}$$

$$Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (n = setsize \wedge m = 1). \tag{5.22}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m + 1 \leq setsize). \tag{5.23}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m - 1 \geq 1). \tag{5.24}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \wedge m = 1). \tag{5.25}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \wedge n = setsize). \tag{5.26}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m + 1 \leq setsize). \tag{5.27}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m - 1 \geq 1). \tag{5.28}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\begin{aligned} \forall\ setsize > 3: \quad & \neg Successor(1, 3, setsize > 3) \\ & \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \end{aligned} \tag{5.29}$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\ setsize > 3: \quad & \neg Predecessor(1, 3, setsize > 3) \\ & \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \end{aligned} \tag{5.30}$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\ setsize > 3: \quad & \neg Adjacent(1, 3, setsize > 3) \\ & \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \end{aligned} \tag{5.31}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

(1) There are 4 set properties that generate geometry:
   (a) The compactness of the set of real numbers in $\mathbb{R}$.
   (b) The continuity of the set of real numbers in $\mathbb{R}$.
   (c) Total order:
      (i) The set of real values in an interval are totally ordered, which causes the set of infinitesimals (compact and continuous subsets of the real values) in an interval to be totally ordered.
      (ii) The set of dimensions of $\mathbb{R}$ are totally ordered.
   (d) The set of dimensions are symmetric (5.2).

(2) If volume is a bijective function of a distance measure, then the distance measure is also a bijective function of volume. And an n-volume is the sum of n-volumes. Therefore, if the definition of a complete metric space allows equations that are not bijective functions of the sum of n-volumes,

then the definition of a complete metric space is not a sufficient filter to obtain only "geometric" distances.

(3) The second derivative is the traditional way of measuring the flatness (curvature) of space, for example, the Laplacian and tensors. Proving that Euclidean area/volume is an instance of the countable n-volume provides the alternative perspective that "flat" space is where each Cartesian n-tuple of same-sized (size c) domain infinitesimals corresponds to a range infinitesimal having the same size as all the neighbor range infinitesimals (3.2) and otherwise the space is curved (non-Euclidean).

(4) Newton's gravity equation (5.1) and Coulomb's charge equation (5.2) are examples of non-Euclidean (curved) space, where mass and charge are held constant. In those cases, each area n-tuple corresponds to an infinitesimal area, ΔA. And, where r is the distance between two masses or two charges, $r_b = r_a + \Delta r,\ r_c = r_b + \Delta r \ \Rightarrow\ (r_b^2 - r_a^2) < (r_c^2 - r_b^2) \ \Rightarrow\ \Delta A_{a,b} < \Delta A_{b,c}$. The space is curved (non-Euclidean) because $\Delta A_{a,b} \neq \Delta A_{b,c}$.

(5) Proofs that Euclidean distance is the smallest distance between two distinct points have equated Euclidean distance to a straight line (equation), where it is assumed that the straight line length is the smallest distance [**Joy98**]. And proofs that a straight line (equation) is the smallest distance have equated the straight line to the Euclidean distance. There have been no purely analytic explanations of why the Euclidean distance/straight line length is the smallest distance.

All distance measures in Euclidean space reduce to corresponding Minkowski distances (4.2). In the Minkowski equation, $d = (\sum_{i=1}^{m} s_i^n)^{1/n}$, if m represents the number of domain intervals/countable sets (one interval from each dimension), then $1 \leq n \leq m$. And $m = 2 \Rightarrow 1 \leq n \leq 2$, which constrains all Minkowski distances to a range from Manhattan distance (the largest distance) to Euclidean distance (the smallest distance) in Euclidean (flat) 2-space.

(6) The derivations of volume and distance used disjoint domain sets. Hilbert spaces allow fractional (fractal) dimensions, which is the case of intersecting domain sets. Therefore, Hilbert spaces would require generalizing the countable volume definition (3.1) to: $v_c = \prod_{i=1}^{n}(|x_i| - |x_i \bigcap (\bigcup_{j=1, i\neq j}^{n} x_j)|)$.

(7) Compare the distance sum inequality (4.4):

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

to Minkowski's sum inequality:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n)^n)^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

Note the difference in the left side of the two inequalities:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \quad vs. \quad (\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n}.$$

Minkowski's sum inequality proof depends on: convexity and the L_p space inequalities (for example, Hölder's inequality or Mahler's inequality) or the triangle inequality. In contrast, the distance (sum) inequality is a more fundamental inequality that does not require the assumptions of the Minkowski sum inequality.

(8) Applying the ruler (2.1) and volume proof (3.2) to the Cartesian product of same-sized, infinitesimal mass forces and charge forces to derive

Newton's gravity force (5.1) and Coulomb's charge force (5.2) equations provide several firsts and some insights into physics:

(a) These are the first deductive derivations of the gravity and charge forces. All other derivations have been empirical and use Newton's induction, which is not fully provable, for example, assumes the inverse square law, which is based on empirical observation. Sometimes it is assumed that the inverse square law is due to Gauss' flux divergence.

(b) These are the first derivations to not use the inverse square law or Gauss's divergence theorem.

(c) These are the first derivations to show that the definition of force, $F := m_0 a$, containing acceleration, a, generates the inverse square law: $\int_0^t a\mathrm{d}t = r/t \;\; \Rightarrow \;\; \exists\, t_c, r_c \in \mathbb{R} : \; t = (t_c/r_c)r$. Using the same derivation steps as for Coulomb's charge force:
$F := m_0 a = m_0 r/t^2 = (r_c/t_c)^2(m_y r_c/x_1^2)x_1 x_2/r^2 = k_y x_1 x_2/r^2$.

(d) Using Occam's razor, those versions of constants like: charge, vacuum magnetic permeability, fine structure, etc. that contain the value 4π might be incorrect because those constants are based on the less parsimonious assumption that the inverse square law is due to Gauss's flux divergence on a sphere having the surface area, $4\pi r^2$.

(e) These are the first derivations to show that the gravity force, charge force, spacetime, and general relativity all depend on time being proportionate to distance: $r = (r_c/t_c)t = ct$.

(f) The derivations of the gravity and charge force equations expose a ratio (constant first derivative) principle. Combining the constant first derivatives (ratios) into equations allows simple algebraic derivations of some general relativity equations (5.4) without the need for integrating second derivative (spacetime curvature) tensors.

(g) A state is represented by a constant value. And a constant value, by definition, cannot vary with distance and time interval lengths. Therefore, the spin states of two quantum entangled electrons and the polarization states of two quantum entangled photons are independent of the amount of distance and time between the entangled particles.

(9) It was proved that sequencing through a totally ordered set of n members in all n-at-time permutations, a symmetric set, requires a cyclic set with at most 3 members (5.3). And empirical observation indicates that geometric space is a totally ordered set of dimensions and allows sequencing from one dimension directly (without jumping over other members) to every other dimension. **Note:** Not all totally ordered sets are symmetric. For example, the set of subintervals in an interval are totally ordered but not symmetric.

(a) Using Occam's razor, a cyclic set of at most 3 members is the most parsimonious explanation of only observing 3 dimensions of geometric distance and volume.

(b) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (5.4), each member being an ordered and symmetric set of 3 dimensions (three balls).

(c) Each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values of the same type, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values of the same type per 3-dimensional ball, for example, vector orientation values: -1, 0, 1 per orthogonal direction in the ball.

(d) Each of the 3 possible ordered and symmetric dimensions of discrete physical states could contain an unordered collection (bag) of discrete state values. Bags are non-deterministic. For example, every time an unordered binary state is "pulled" from a bag, there is a 50 percent chance of getting one of the binary values.

(10) It was shown that some fundamental geometry (volume and the Minkowski distances/L_p norms) and physics (gravity force and charge force) are derived from the combinatorial mappings between the infinitesimals of real-valued intervals. The proofs and derivations in this article show that the ruler (2.1) is a tool to directly express and solve some combinatorial relations in geometry, probability, physics, etc. that are difficult to directly express with differential equations and integrals.

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George Van Treeck, 668 Westline Dr., Alameda, CA 94501