The Set Properties Generating Geometry and Physics

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Abstract. Volume and the Minkowski distances/Lp norms (e.g., Euclidean distance) are derived from a set and limit-based foundation without referencing the primitives of geometry. Sequencing a linearly ordered set in all n-at-time permutations via successor/predecessor relations is a cyclic set limiting n to at most 3, for example, 3 dimensions of physical space. Therefore, all other interval lengths have different types that can only be related to a 3-dimensional distance interval length via conversion ratios. The ratios and geometry proofs provide simpler derivations of the spacetime, Newton's gravity, Coulomb's charge force, and Einstein-Planck equations and exposes the ratios composing their corresponding constants. All proofs are verified in Coq.

Contents

1. Introduction

Mathematical (real) analysis can construct differential calculus from a set and limit-based foundation without the need to reference the primitives of Euclidean geometry, like straight line, angle, slope, etc. But the Riemann and Lebesgue integrals and measure theory (for example, Hilbert spaces and the Lebesgue measure) define Euclidean volume as the product of interval lengths. And the vector norm and metric space use Euclidean distance and its properties as definitions [**Gol76**] [**Rud76**]. Here, volume and distance are derived from a simple set and limit-based foundation without the hand-waving references to side, angle, triangle, rectangle, etc. for justification.

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The usual Cartesian definition of volume is the set of all ordered combinations, n-tuples, of members from the disjoint sets, $x_i \in \{x_1, \cdots, x_n\}$. Therefore, the size of a "countable" volume, v_c, is the cardinal of a set of n-tuples, $v_c = \prod_{i=1}^{n} |x_i|$, where $|x_i|$ is the cardinal of the countable set, x_i. But proving that the Euclidean volume equation is an instance of this countable volume, where $|x_i|$ is the number of partitions of the interval, $[a_i, b_i] \subset \mathbb{R}$, requires avoiding the circular logic of assuming that each n-tuple corresponds to a product of interval sizes.

Further, the equality constraint requires the same operation to be applied to both sides of the countable volume size equation, $v_c = \prod_{i=1}^{n} |x_i|$, which will only allow Euclidean volume to be derived. A generalization to include non-Euclidean volumes is: $v_c = \prod_{i=1}^{n} f_i(|x_i|)$.

For each non-Euclidean and Euclidean n-volume size, $v \in \mathbb{R} \, \exists \, d \in \mathbb{R} : \, v = d^n$. And an n-volume can only be sum of n-volumes. Therefore, the sum of disjoint non-Euclidean and Euclidean n-volumes has a corresponding sum of cuboid n-volumes: $d^n = v = \sum_{i=1}^{m} v_i = \sum_{i=1}^{m} d_i^n \Rightarrow d = (\sum_{i=1}^{m} d_i^n)^{1/n}$, which are the L_p norms (Minkowski distances) that have the properties of a metric space.

It will be proved that the Minkowski distances are derived the sum of countable set-based n-volumes, which provides a coherent set and limit-based foundation for both volume and distance. All "geometric" distance measures are inverse functions of the sum of disjoint n-volumes of the same type (all cuboid, all spherical, etc.).

The permutation values of the Levi-Civita pseudo-tensor [**Ari89**] are: $\epsilon_{ijk} = 1$, where 3 dimensions of space are sequenced in the cyclic-successor order, $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$; $\epsilon_{ijk} = -1$, where sequenced in cyclic-predecessor order, $(i, j, k) \in \{(3, 2, 1), (2, 1, 3), (1, 3, 2)\}$; and otherwise $\epsilon_{ijk} = 0$. The Levi-Civita pseudo-tensor is an example of the common assumption that the dimensions of space can be sequenced in any n-at-time order.

But sequencing a set from 1 to n (for example, a set of n number domain intervals or dimensions) implies that each set member can be uniquely labeled, counted, and sequenced in a repeatable order, which is a strict linear order that set theory defines in terms of successor and predecessor functions. It will be proved that a strict linearly ordered set that can be sequenced in all n-at-a-time permutations *only* via the successor/predecessor relations is a cyclic set, where n $\leq$ 3.

Therefore, an interval length that is not in a cyclic set of 3 "distance" interval lengths has a different type (member of a different set) that can only be related to a 3-dimensional distance via unit-factoring, conversion ratios. The ratios combined with the geometry proofs provide simpler derivations of the spacetime, Newton's gravity, Coulomb's charge force, and Einstein-Planck equations and exposes the ratios that compose the gravity, charge, and Planck constants.

All the proofs in this article are trivial. But to ensure confidence, all the proofs have been verified using using the Coq proof verification system [**Coq23**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

Derivatives and integrals use a 1-1 correspondence between the infinitesimals of each interval, where the size of the infinitesimals in each interval are proportionate to the size of the interval, which precludes using derivatives and integrals to directly express many-to-one, one-to-many, and many-to-many (Cartesian product) mappings between same-sized, size κ, infinitesimals in different-sized intervals. Further, using tools that define Euclidean volume and distance precludes using those tools to derive Euclidean volume and distance.

Therefore, a different tool is used here. A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals (infinitesimals), where each infinitesimal has the *same* size, κ. The ruler is both an inner and outer measure of an interval.

DEFINITION 2.1. Ruler measure, M: $\forall\, [a,b] \subset \mathbb{R},\ \ s = b - a \ \wedge\ \kappa > 0 \ \wedge$ $(p = floor(s/\kappa) \ \vee\ p = ceiling(s/\kappa)) \ \wedge\ M = \sum_{i=1}^{p} \kappa = p\kappa.$

THEOREM 2.2. *Ruler convergence:* $M = \lim_{\kappa\to 0} p\kappa = s.$

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, \kappa > 0,\ p = floor(s/\kappa)\ \wedge\ 0 \le |floor(s/\kappa) - s/\kappa| < 1 \ \Rightarrow\ |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by κ:

$$\forall\, \kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ &\quad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \delta = \epsilon \quad := \quad M = \lim_{\kappa\to 0} p\kappa = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0,\pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p\cdot\kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, \ldots, \pi_{\lim_{\kappa\to 0}}.$

LEMMA 2.3. $\forall\, n \ge 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa.$

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$n \ge 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta \\ &\quad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa\to 0} \kappa^n = 0. \end{aligned} \tag{2.5}$$

$$\lim_{\kappa\to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa\to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa. \quad \square \tag{2.6}$$

3. Volume

DEFINITION 3.1. Countable volume size, v_c, is the number of ordered combinations (n-tuples) of the members of n number of disjoint, countable domain sets, x_i:

$$\exists\, n \in \mathbb{N}, v_c, \in \{0,\ \mathbb{N}\},\ x_i \in \{x_1, \cdots, x_n\},\ \bigcap_{i=1}^{n} x_i = \emptyset : \quad v_c = \prod_{i=1}^{n} f_i(|x_i|). \tag{3.1}$$

THEOREM 3.2. *Euclidean volume size,* $v = \prod_{i=1}^{n} s_i$*, is the equality case of countable volume size,* $v_c = \prod_{i=1}^{n} |x_i|$*, where each countable set,* x_i*, is the set of partitions of an interval,* $[a_i, b_i] \subset \mathbb{R}$*.*

$$\begin{aligned} &\forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\ [v_a, v_b] \subset \mathbb{R},\ \ s_i = b_i - a_i,\ \ v = v_a - v_b, \\ &\qquad v_c = \prod_{i=1}^{n} |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^{n} s_i. \end{aligned} \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing $|x_i|$ number of size κ partitions and apply ruler convergence (2.2):

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n],\; \kappa > 0 \;\;\wedge\;\; floor(s_i/\kappa) = |x_i| \;\;\Rightarrow\;\; s_i = \lim_{\kappa\to 0}(|x_i| \cdot \kappa). \tag{3.3}$$

$$s_i = \lim_{\kappa\to 0}(|x_i| \cdot \kappa) \quad\Leftrightarrow\quad \prod_{i=1}^{n} s_i = \prod_{i=1}^{n} \lim_{\kappa\to 0}(|x_i| \cdot \kappa). \tag{3.4}$$

$$\prod_{i=1}^{n} s_i = \prod_{i=1}^{n} \lim_{\kappa\to 0}(|x_i| \cdot \kappa) \quad\Leftrightarrow\quad \prod_{i=1}^{n} s_i = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|) \cdot \kappa^n. \tag{3.5}$$

Apply lemma 2.3 to equation 3.5:

$$\begin{aligned}\prod_{i=1}^{n} s_i = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|) \cdot \kappa^n \quad\wedge\quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa \\ \Rightarrow\quad \prod_{i=1}^{n} s_i = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|) \cdot \kappa.\end{aligned} \tag{3.6}$$

Apply the ruler (2.1) and ruler convergence (2.2) to v:

$$\exists\, v \in \mathbb{R} :\; v_c = floor(v/\kappa) \quad\Leftrightarrow\quad v = \lim_{\kappa\to 0} v_c \cdot \kappa. \tag{3.7}$$

Multiply both sides of the countable volume equation 3.1 by κ:

$$v_c = \prod_{i=1}^{n} |x_i| \quad\Leftrightarrow\quad v_c \cdot \kappa = (\prod_{i=1}^{n} |x_i|) \cdot \kappa \tag{3.8}$$

$$v_c \cdot \kappa = (\prod_{i=1}^{n} |x_i|) \cdot \kappa \quad\Leftrightarrow\quad \lim_{\kappa\to 0} v_c \cdot \kappa = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|) \cdot \kappa. \tag{3.9}$$

Combine equations 3.7, 3.9, and 3.6:

$$\begin{aligned}v = \lim_{\kappa\to 0} v_c \cdot \kappa \quad\wedge\quad \lim_{\kappa\to 0} v_c \cdot \kappa = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|) \cdot \kappa \quad\wedge \\ \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|) \cdot \kappa = \prod_{i=1}^{n} s_i \quad\Rightarrow\quad v = \prod_{i=1}^{n} s_i.\end{aligned} \tag{3.10}$$ □

4. Distance

4.1. Countable cuboid n-volume size.

DEFINITION 4.1. The countable cuboid volume size, d_c^n, is the sum of m number of disjoint countable cuboid volume sizes.

$$\forall\, n \in \mathbb{N},\quad d_c \in \{0, \mathbb{N}\} \quad \exists\, m \in \mathbb{N},\quad x_i \in \{x_1, \cdots, x_m\},\quad \bigcap_{i=1}^{m} x_i = \emptyset :$$
$$d_c^n = \sum_{i=1}^{m} |x_i|^n.$$

4.2. Minkowski distance (L_p norm).

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

THEOREM 4.2. *The Minkowski distances (L_p norms) are derived from the sum of countable cuboid n-volume sizes* (4.1).

$$d_c^n = \sum_{i=1}^{m} |x_i|^n \quad\Rightarrow\quad \exists\, d, s_1, \cdots, s_m \in \mathbb{R} :\quad d = (\sum_{i=1}^{m} s_i^n)^{1/n}.$$

PROOF. Apply the ruler (2.1):

$$\exists\, d, s_1, \cdots, s_m \in \mathbb{R} :\; d_c = floor(d/\kappa) \quad\wedge\quad |x_i| = floor(s_i/\kappa). \tag{4.1}$$

Apply the ruler convergence (2.2):

$$|x_i| = floor(s_i/\kappa) \quad\Rightarrow\quad s_i = \lim_{\kappa\to 0} |x_i| \cdot \kappa. \tag{4.2}$$

$$d_c^n = \sum_{i=1}^{m} |x_i|^n \quad\Rightarrow\quad d^n = \lim_{\kappa\to 0}(d_c \cdot \kappa)^n = \lim_{\kappa\to 0}(\sum_{i=1}^{m}(|x_i|^n) \cdot \kappa. \tag{4.3}$$

Apply lemma 2.3 to equation 4.3 and substitute equation 4.2:

$$d^n = \lim_{\kappa\to 0}(\textstyle\sum_{i=1}^m(|x_i|^n)\cdot\kappa \quad \wedge \quad \lim_{\kappa\to 0}\kappa^n = \lim_{\kappa\to 0}\kappa \quad \Rightarrow \quad d^n = \lim_{\kappa\to 0}\sum_{i=1}^m(|x_i|^n)\cdot\kappa^n = \lim_{\kappa\to 0}\sum_{i=1}^m(|x_i|\cdot\kappa)^n. \tag{4.4}$$

Apply equation 4.2 to equation 4.4:

$$d^n = \lim_{\kappa\to 0}\textstyle\sum_{i=1}^m(|x_i|\cdot\kappa)^n \quad \wedge \quad s_i = \lim_{\kappa\to 0}|x_i|\cdot\kappa \quad \Rightarrow \quad d^n = \sum_{i=1}^m s_i^n. \tag{4.5}$$

$$d^n = \textstyle\sum_{i=1}^m s_i^n \quad \Leftrightarrow \quad d = (\sum_{i=1}^m s_i^n)^{1/n}. \tag{4.6}$$

□

4.3. Distance inequality. Proving that all Minkowski distances (L_p norms) satisfy the metric space triangle inequality requires another inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance inequality*

$$\forall\, n\in\mathbb{N},\ \ v_a, v_b \geq 0:\ \ (v_a+v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand the n-volume, $(v_a^{1/n}+v_b^{1/n})^n$, using the binomial expansion:

$$\forall\, v_a, v_b \geq 0: \quad v_a + v_b \leq v_a + v_b + \textstyle\sum_{i=1}^n \binom{n}{k}(v_a^{1/n})^{n-k}(v_b^{1/n})^k + \sum_{i=1}^n \binom{n}{k}(v_a^{1/n})^k(v_b^{1/n})^{n-k} = (v_a^{1/n}+v_b^{1/n})^n. \tag{4.7}$$

Take the n^{th} root of both sides of the inequality:

$$\forall\, v_a, v_b \geq 0, n\in\mathbb{N}: v_a+v_b \leq (v_a^{1/n}+v_b^{1/n})^n \Rightarrow (v_a+v_b)^{1/n} \leq v_a^{1/n}+v_b^{1/n}. \tag{4.8}$$

□

4.4. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m, n\in\mathbb{N},\ \ a_i, b_i \geq 0:\ \ (\textstyle\sum_{i=1}^m(a_i^n+b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$\forall\, m, n\in\mathbb{N},\ v_a, v_b \geq 0: \quad v_a = \textstyle\sum_{i=1}^m a_i^n \quad \wedge \quad v_b = \sum_{i=1}^m b_i^n \quad \wedge \quad (v_a+v_b)^{1/n} \leq v_a^{1/n}+v_b^{1/n} \quad \Rightarrow \quad ((\sum_{i=1}^m a_i^n) + (\sum_{i=1}^m b_i^n))^{1/n} = (\sum_{i=1}^m(a_i^n+b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}. \tag{4.9}$$

□

4.5. Metric Space. All Minkowski distances (L_p norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.10}$$

Apply the distance inequality (4.3) to the inequality 4.10:

$$
\begin{aligned}
(4.11)\quad & (u^p + w^p)^{1/p} \le ((u^p + v^p) + (v^p + w^p))^{1/p} \;\wedge\; (v_a + v_b)^{1/n} \le v_a^{1/n} + v_b^{1/n} \\
& \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\
\Rightarrow \quad & (u^p + w^p)^{1/p} \le ((u^p + v^p) + (v^p + w^p))^{1/p} \le (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\
& \Rightarrow \quad d(u,w) = (u^p + w^p)^{1/p} \le \\
& (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u,v) + d(v,w). \quad \square
\end{aligned}
$$

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,v) = d(v,u).$

PROOF. By the commutative law of addition:

$$
\begin{aligned}
(4.12)\quad & \forall\, p : p \ge 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\
& \Rightarrow \quad d(u,v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v,u). \quad \square
\end{aligned}
$$

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,w) \ge 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \ge 0,\ v \ge 0. \tag{4.13}$$

$$p \ge 1,\ \ u, v \ge 0 \quad \Rightarrow \quad d(u,v) = (u^p + v^p)^{1/p} \ge 0. \tag{4.14}$$ $\square$

THEOREM 4.8. *Identity of Indiscernibles:* $d(u,u) = 0$.

PROOF. From the non-negativity property (4.7):

$$
\begin{aligned}
(4.15)\quad & d(u,w) \ge 0 \quad \wedge \quad d(u,v) \ge 0 \quad \wedge \quad d(v,w) \ge 0 \\
& \Rightarrow \quad \exists\, d(u,w) = d(u,v) = d(v,w) = 0.
\end{aligned}
$$

$$d(u,w) = d(v,w) = 0 \quad \Rightarrow \quad u = v. \tag{4.16}$$

$$d(u,v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u,u) = 0. \tag{4.17}$$ $\square$

4.6. At most 3 dimensions of space.

DEFINITION 4.9. Totally ordered set:

$$
\begin{aligned}
& \forall\, i\ n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i \ \in \{x_1, \ldots, x_n\}, \\
& \qquad \textit{successor}\ x_i = x_{i+1} \ \wedge\ \textit{predecessor}\ x_{i+1} = x_i.
\end{aligned}
$$

DEFINITION 4.10. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i\ j\ n \in \mathbb{N},\ \ \forall\, x_i\ x_j \in \{x_1, \ldots, x_n\},\ \ \textit{successor}\ x_i = x_j \ \Leftrightarrow\ \textit{predecessor}\ x_j = x_i.$$

THEOREM 4.11. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n \ \wedge\ j = 1 \ \Rightarrow\ \textit{successor}\ x_n = x_1 \ \wedge\ \textit{predecessor}\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

Proof. A total order (4.9) assigns a unique label to each set member and assigns unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (4.10):

$$i = n \;\wedge\; j = 1 \;\wedge\; \mathit{successor}\; x_i = x_j \;\Rightarrow\; \mathit{successor}\; x_n = x_1. \tag{4.18}$$

Applying the definition of the symmetry property (4.10) to conclusion 4.18:

$$\mathit{successor}\; x_i = x_j \;\Rightarrow\; \mathit{predecessor}\; x_j = x_i \;\Rightarrow\; \mathit{predecessor}\; x_1 = x_n. \tag{4.19}$$ □

Theorem 4.12. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

Proof.
It was proved that an ordered and symmetric set is a cyclic set (4.11).

Definition 4.13. (Cyclic) Successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (n = m + 1 \leq \mathit{setsize}). \tag{4.20}$$

Definition 4.14. (Cyclic) Predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (n = m - 1 \geq 1). \tag{4.21}$$

Definition 4.15. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$\mathit{Adjacent}(m, n, \mathit{setsize}) \leftarrow \mathit{Successor}(m, n, \mathit{setsize}) \vee \mathit{Predecessor}(m, n, \mathit{setsize}). \tag{4.22}$$

Prove that every member is adjacent to every other member, where $\mathit{setsize} \in \{1,\ 2,\ 3\}$:

$$\mathit{Adjacent}(1, 1, 1) \leftarrow \mathit{Successor}(1, 1, 1) \leftarrow (m = \mathit{setsize} \wedge n = 1). \tag{4.23}$$

$$\mathit{Adjacent}(1, 2, 2) \leftarrow \mathit{Successor}(1, 2, 2) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{4.24}$$

$$\mathit{Adjacent}(2, 1, 2) \leftarrow \mathit{Successor}(2, 1, 2) \leftarrow (n = \mathit{setsize} \wedge m = 1). \tag{4.25}$$

$$\mathit{Adjacent}(1, 2, 3) \leftarrow \mathit{Successor}(1, 2, 3) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{4.26}$$

$$\mathit{Adjacent}(2, 1, 3) \leftarrow \mathit{Predecessor}(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \tag{4.27}$$

$$\mathit{Adjacent}(3, 1, 3) \leftarrow \mathit{Successor}(3, 1, 3) \leftarrow (n = \mathit{setsize} \wedge m = 1). \tag{4.28}$$

$$\mathit{Adjacent}(1, 3, 3) \leftarrow \mathit{Predecessor}(1, 3, 3) \leftarrow (m = 1 \wedge n = \mathit{setsize}). \tag{4.29}$$

$$\mathit{Adjacent}(2, 3, 3) \leftarrow \mathit{Successor}(2, 3, 3) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{4.30}$$

$$\mathit{Adjacent}(3, 2, 3) \leftarrow \mathit{Predecessor}(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \tag{4.31}$$

Member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not ($\neg$) a successor of member 1 for all $setsize > 3$:

$$\neg Successor(1, 3, setsize > 3) \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \tag{4.32}$$

Member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not ($\neg$) a predecessor of member 1 for all $setsize > 3$:

$$\neg Predecessor(1, 3, setsize > 3) \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \tag{4.33}$$

For all $setsize > 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not symmetric):

$$\neg Adjacent(1, 3, setsize > 3) \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \tag{4.34}$$

5. Applications to physics

5.1. Spacetime and Lorentz equations. From the Euclidean volume proof (3.2), two independent (disjoint) intervals, $[0, r]$ and $[0, r']$, defines an Euclidean 2-space. From the Minkowski distance proof (4.2), the interval length, r, is an inverse function of a cuboid 2-volume, and the interval length, r', is an inverse function of a cuboid 2-volume, which sum to a cuboid 2-volume: $r_v^2 = r^2 + r'^2$. And from the 3D proof (4.12), if r is a 3-Dimensional distance, then any other interval length, t, must have a different type (from a different set) that is related to r via a constant, unit-factoring, conversion ratio: $c : \ r/t = r_c/t_c = c$.

$$r_v^2 = r^2 + r'^2 \quad \wedge \quad \exists\, r_c,\ t_c,\ c,\ v \in \mathbb{R} : \ r/t = r_c/t_c = c \quad \wedge \quad r_v/t = v \quad \Rightarrow \quad r' = \sqrt{(ct)^2 - (vt)^2} \ = \ ct\sqrt{1 - (v/c)^2}. \tag{5.1}$$

Local (proper) distance, r', contracts relative to coordinate distance, r, as $v \to c$:

$$r' = ct\sqrt{1 - (v/c)^2} \quad \wedge \quad ct = r \quad \Rightarrow \quad r' = r\sqrt{1 - (v/c)^2}. \tag{5.2}$$

The Lorentz transformations follow from equation 5.2 and Galilean transformation:

$$r' = r/\sqrt{1 - (v/c)^2} \quad \wedge \quad r = r' + vt \quad \Rightarrow \quad r' = (r - vt)/\sqrt{1 - (v/c)^2}. \tag{5.3}$$

$$r' = (r - vt)/\sqrt{1 - (v/c)^2} \quad \wedge \quad r = ct \quad \wedge \quad r' = ct' \quad \Rightarrow \quad t' = (t - (vt/c))/\sqrt{1 - (v/c)^2} = (t - (vr/c^2))/\sqrt{1 - (v/c)^2}. \tag{5.4}$$

Coordinate time, t, dilates relative to local (proper) time, t, as $v \to c$:

$$ct = r'/\sqrt{1 - (v/c)^2} \quad \wedge \quad r' = ct' \quad \Rightarrow \quad t = t'/\sqrt{1 - (v/c)^2}. \tag{5.5}$$

r is a 3-dimensional distance. Therefore, the "$- + + +$" form of Minkowski's spacetime event interval [**Ein15**] is:

$$ct = r \quad \wedge \quad r^2 = x^2 + y^2 + z^2 \quad \Rightarrow \quad 0 = -(ct)^2 + x^2 + y^2 + z^2. \tag{5.6}$$

5.2. Newton's gravity force equation. From the 3D proof (4.12), where r is a 3-dimensional distance, a mass interval length, m, must have a different type that is related to the distance via a unit-factoring, conversion ratio, $r = (r_c/m_G)m$:

$$F := m_1 a := m_1 r/t^2 \quad \wedge \quad \exists\, m_G, r_c, m_1 \in \mathbb{R} : \ r = (r_c/m_G)m_2 \\ \Rightarrow F := m_1 r/t^2 = (r_c/m_G)m_1 m_2/t^2, \tag{5.7}$$

where a constant mass, m_1, and force implies an acceleration, $a := r/t^2$.

From equation 5.3, the proper distance, $r = ct\sqrt{1-(v/c)^2}$, and where $v = 0$:

$$r = ct \quad \wedge \quad F = (r_c/m_G)m_1 m_2/t^2 \quad \Rightarrow \\ F = ((r_c/m_G)c^2)m_1 m_2/r^2 = G m_1 m_2/r^2, \tag{5.8}$$

where the constant, $G = (r_c/m_G)c^2$, has the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$. And where $|v| > 0$, $F = (r_c/m_G)(c^2 - v^2)m_1 m_2/r^2$.

5.3. Coulomb's charge force. From the 3D proof (4.12), where r is a 3-dimensional distance, a charge interval length, q, must have a different type that is related to the distance via a unit-factoring, conversion ratio, $r = (r_c/q_C)q$:

$$r = (m_G/r_c)m = (r_c/q_C)q_1 \Rightarrow m = (m_C/q_C)q_1. \tag{5.9}$$

$$F := ma := mr/t^2 \ \wedge \ m = (m_C/q_C)q_1 \ \wedge \ r = (r_c/q_C)q_2 \\ \Rightarrow F := mr/t^2 = (m_C/q_C)(r_c/q_C)q_1 q_2/t^2, \tag{5.10}$$

where a constant mass, m, and force implies an acceleration, $a := r/t^2$.

From equation 5.3, the proper distance, $r = ct\sqrt{1-(v/c)^2}$, and where $v = 0$:

$$r = ct = (r_c/t_c)t \quad \wedge \quad F = (m_C/q_C)(r_c/q_C)q_1 q_2/t^2 \\ \Rightarrow \quad F = (m_C/q_C)(r_c/q_C)(r_c/t_c)^2 q_1 q_2/r^2. \tag{5.11}$$

$$a_G = r_c/t_c^2 \quad \wedge \quad F = (m_C/q_C)(r_c/q_C)(r_c/t_c)^2 q_1 q_2/r^2 \\ \Rightarrow \quad F = (m_C a_G)(r_c/q_C)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2, \tag{5.12}$$

where the predicted charge constant, $k_e = (m_C a_G)(r_c/q_C)^2$, has the SI units: $N \cdot m^2 \cdot C^{-2}$. And where $|v| > 0$, $F = (m_C/q_C)(r_c/q_C)(c^2 - v^2)q_1 q_2/r^2$.

5.4. Einstein-Planck and energy-charge equations: $m = (m_p/r_p)r$ and $r/t = r_c/t_c = c \Rightarrow m(ct)^2 = ((m_p/r_p)r)r^2$. Dividing both sides by t^2 yields Einstein's energy: $E = mc^2 = ((m_p/r_p)r)(r/t)^2 = ((m_p/r_p)r)(r_c/t_c)^2 = ((m_p r_c/r_p t_c)c)r = (m_p r_c c)(r/(r_p t_c)) = hf$, which is the Einstein-Planck equation, where the Planck constant is, $h = m_p r_c c$, and $f = r/(r_p t_c)$ is the frequency in cycles per second. $h = (m_p r_c)c = k_W c$, such that $m_0 r = k_W \approx 2.2102190943 \cdot 10^{-42}\ kg\ m$, where r is the work displacement (Compton wavelength) on the rest mass, m_0.

Likewise, for charge, $r = (r_C/q_C)q = (r_p/m_p)m \Rightarrow m = (m_p/r_p)(r_C/q_C)q \Rightarrow E = mc^2 = (m_p/r_p)(r_C/q_C)qc^2 = (m_p r_c c) \cdot (r_C q/r_p q_C t_c) = hf_q$.

6. Insights and implications

(1) The definition of a complete metric space is insufficient because it is only useful as a filter criteria and fails to teach what type of function causes the properties that define metric space. Deriving Euclidean volume (3.2) and the Minkowski distances (4.2) from a common set-based definition of countable volume (3.1), teaches that all "geometric" distances are inverse functions of the sum of disjoint n-volumes of the same type (sum of cuboid, sum of spherical, etc.).

(2) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [**Joy98**]. And proofs that a straight line is the smallest distance equate the straight line to the Euclidean distance.

The calculus of variations cannot be used to prove that the smallest distance is the Euclidean distance in Euclidean space because the integrals make Euclidean assumptions, which would result in circular logic.

However, every "geometric" distance measure is an inverse function of the sum of disjoint n-volumes with a corresponding Minkowski distance (4.2), $d = (\sum_{i=1}^{m} s_i^n)^{1/n}$. If m represents the number of dimensions, then $m = 2 \Rightarrow 1 \leq n \leq 2$, which constrains the Minkowski distances to a range from Manhattan distance (the largest distance, $d = \sum_{i=1}^{2} s_i$) to Euclidean distance (the smallest distance, $d = (\sum_{i=1}^{2} s_i^2)^{1/2}$).

(3) Hilbert spaces allow fractional dimensions (fractals), which is the case of intersecting distance sets and requires generalizing the countable volume definition (3.1) from $v_c = \prod_{i=1}^{n} f_i(|x_i|)$ to:
$v_c = \prod_{i=1}^{n} f_i(|x_i| - |x_i \bigcap (\bigcup_{j=1,i\neq j}^{n} x_j)|))$.

Distance measures are used in shortest and least cost path search algorithm, and machine learning. Intersecting domain sets allow neural networks to generalize a response across domains.

(4) Compare the distance sum inequality (4.4),

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n},$$

used to prove that all Minkowski distances satisfy the metric space triangle inequality property (4.5), to Minkowski's sum inequality:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n)^n)^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

Note the exponent difference in the left side of the two inequalities:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \quad vs. \quad (\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n}.$$

The proof of Minkowski's sum inequality assumes: convexity and the L_p space inequalities (for example, H́older's inequality or Mahler's inequality) or the triangle inequality. In contrast, the distance (sum) inequality is a more fundamental inequality because its proof does not require the assumptions of the Minkowski sum inequality.

(5) From the 3D proof (4.12), more intervals than the 3 dimensions of distance intervals must have different types with lengths that are related to a 3-dimensional distance interval length, r, via constant, unit-factoring, conversion ratios (both direct and inverse proportion ratios). The direct proportion ratios for time, mass, and charge are: $r = (r_c/t_c)t =$

$ct = (r_c/m_G)m = (r_c/q_C)q$. An inverse proportion ratio is the mass-displacement ratio: $m_0 r = (m_p r_c) = k_W$. The speed of light, c, the gravity constant, G, the charge constant, k_e, and the Planck constant, h, were all derived from these constant ratios.

(6) The derivations in this article show that the spacetime, gravity force, charge force, and Einstein-Planck equations all depend on time being proportionate to distance: $r = (r_c/t_c)t = ct$. For example, from the derivation of Newton's gravity equation (5.8), where $v = 0$: $G = (r_c/m_G)c^2$. Likewise, from the derivation of Coulomb's charge force equation (5.12) the constant, where $v = 0$: $k_e = (m_G/q_C)(r_c/q_C)c^2$. And from the derivation of the Planck constant (5.4): $h = (m_p r_c)c = k_W c$.

(7) The gravity, charge, and Planck constants are not fundamental constants because the constants are derived from other (conversion ratio) constants.

(8) The derivation of the Planck constant (5.4),
$E = mc^2 = \cdots = (m_p r_c c)(r/(r_p t_c)) = hf$, shows that all wave and particle sizes are relative to the relations between conversion ratios. That is, just as there is no absolute inertial frame of reference, there also is no absolute size.

(9) The derivations of the spacetime equations and Lorentz transformations, here (5.1), differ from all other derivations and provide insights that the other derivations cannot provide.
   (a) The derivations, here, are much shorter and simpler.
   (b) The derivations of the spacetime equations, here, do not rely on the Lorentz transformations or Einsteins' postulates [**Ein15**]. The derivations do not even require the notion of light.
   (c) The derivations, here, rely only on geometry: the Euclidean volume proof (3.2), the Minkowski distances proof (4.2), and the 3D proof (4.12), which provides the insight that the geometry of physical space alone creates: a constant, unit-factoring conversion ratio, c, with respect to a 3-dimensional distance, the spacetime equations, and Lorentz transformations.

(10) Applying the ratios to derive Newton's gravity force (5.2) and Coulomb's charge force (5.3) equations provide some firsts and some new insights into physics:
   (a) These are the first derivations to not assume the inverse square law or Gauss's flux divergence theorem.
   (b) These are the first derivations to show that the definition of force, $F := ma$, containing acceleration, $a = r/t^2$, where r is a distance that is proportionate to time, t, generates the inverse square law.
   (c) Using Occam's razor, those versions of constants like: charge, vacuum magnetic permeability, etc. that contain the value 4π might be incorrect because those constants are based on the less parsimonious assumption that the inverse square law is due to Gauss's flux divergence on a sphere having the surface area, $4\pi r^2$.
   (d) These are the first derivations to predict that G and k_e are constants, in the local frame of reference, only where the local velocity is zero. The derived relativistic gravity and charge force equations are:
   $F = (r_c/m_G)(c^2 - v^2)m_1 m_2/r^2$ (5.8) and

$F = (m_C/q_C)(r_c/q_C)(c^2 - v^2)q_1q_2/r^2$ (5.12).
Therefore, Einstein's gravity constant, $k = 8\pi G/c^4$ [**Ein15**], is only valid when the local velocity is 0. Otherwise, $k = 8\pi(r_G/m_G)(c^2 - v^2)/c^4$, which implies that as $v \to c \Rightarrow F \to 0$, which implies a universe expanding faster than predicted by a constant k and also predicts an accelerating expansion.

(11) There is no unit-factoring ratio converting a state, a single value, to a continuously varying distance, time, mass, and charge interval lengths. Therefore, the spin states of two quantum entangled particles and the polarization states of two quantum entangled photons are independent of the amount of distance between the particles and independent of time (instantaneous).

(12) Sequencing a set of n number physical dimensions from 1 to n implies that each set member can be uniquely labeled (for example, "length", "width", and "height"), counted, and sequenced in a repeatable order, which is a strict linear order that set theory defines in terms of successor and predecessor functions. It was proved that sequencing through a set, having a strict linear order via the successor/predecessor relations in all n-at-a-time permutations, is a cyclic set with n $\leq$ 3 (4.12), which is why there are only 3 dimensions of physical space.
   (a) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (4.12), each member being an ordered and symmetric set of 3 dimensions (three 3-dimensional balls).
   (b) Each of 3 ordered and symmetric dimensions of space can have only 3 sequentially ordered and symmetric state values. For example, the ordered and symmetric set of the 3 vector orientations, $\{-1, 0, 1\}$, per dimension.
   (c) Each of the 3 ordered and symmetric dimensions of space could correspond to an unordered collection (bag) of discrete state values. The lack of order makes bags non-deterministic. For example, every time a binary state is "pulled" from a bag (for example, a bag of coin tosses), there is a 50 percent chance of getting one of the binary values.

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