Some Set Properties Underlying Geometry and Physics

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Abstract. Euclidean volume and the Minkowski distances (Manhattan, Euclidean, etc. distances) are derived from a set and limit-based foundation without referencing the primitives and relations of geometry. Sequencing a strict linearly ordered set in all n-at-a-time orders via successor/predecessor relations is proved to be a cyclic set of at most 3 members. Therefore, other dimensions beyond an ordered and cyclic set of 3 dimensions of distance must have other types related by geometric, unit-factoring ratios. The geometry proofs and geometric ratios are used to derive the gravity, charge, and Planck constants, and used to provide simpler, shorter, and more rigorous derivations of some well-known classical gravity and charge equations, special and general relativity equations, and quantum physics equations. All the proofs are verified in Coq.

Contents

1. Introduction

Mathematical analysis constructs differential calculus from a set and limit-based foundation without referencing the primitives and relations of Euclidean geometry, like straight line, angle, etc., which provides a more rigorous foundation and deeper understanding of geometry and physics. But Euclidean volume in the Riemann integral, Lebesgue integral, measure theory, and distance in the vector magnitude and metric space criteria are definitions motivated by Euclidean geometry [**Gol76**] [**Rud76**] rather than derived from a set and limit-based foundation.

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A simple, abstract, set-based definition of Euclidean volume is the number, v_c, of ordered combinations (n-tuples): $v_c = \prod_{i=1}^{n} |x_i|$, where $|x_i|$ is the cardinal of a countable, disjoint set, x_i. But real analysis text books and measure theory start with the definition of volume as the product of interval sizes, $v = \prod_{i=1}^{n} s_i$, where $s_i = b_i - a_i$, $[a_i, b_i] \subset \mathbb{R}$ [**Gol76**] [**Rud76**]. Where each set, x_i, is a set of size κ subintervals of the interval, $[a_i, b_i]$, it will be proved that, $\lim_{\kappa\to 0} v_c \cdot \kappa = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|) \cdot \kappa \Rightarrow v = \prod_{i=1}^{n} s_i$.

$v_c = \prod_{i=1}^{n} |x_i| = f(|x_1|, \cdots, |x_n|, n)$. If f is a bijective function, then $\exists\ d_c : d_c = f^{-1}(v_c, n)$ and $v_c = f(d_c, n) = f(|x_1|, \cdots, |x_n|, n)$. If f is also isomorphic, then $\forall\ |x_i| : d_c = |x_1| = \cdots = |x_n|$ and $v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n$.

Where $v_c = f(|x_1|, \cdots, |x_n|, n)$ is a bijective and isomorphic function, it will be proved that $\lim_{\kappa\to 0} v_c \cdot \kappa = \lim_{\kappa\to 0}(\sum_{j=1}^{m} v_{c_i}) \cdot \kappa \Rightarrow d^n = \sum_{i=1}^{m} d_i^n$. d is the ρ-norm (Minkowski distance) [**Min53**], which will be proved to imply the metric space properties [**Rud76**].

Sequencing the domain sets, $x_1, \cdots, x_n$, from $i = 1$ to n, is a strict linear (total) order, where a total order is defined in terms of successor and predecessor relations [**CG15**]. Sequencing a set, via successor and predecessor relations, in all n-at-a-time orders, requires a "symmetry" constraint, where every set member is either a successor or predecessor to every other set member. A strict linearly ordered and symmetric set will be proved to be a cyclic set, where $n \leq 3$.

Therefore, if $\{s_1, s_2, s_3\}$ is a strict linearly ordered and symmetric set of 3 "distance" dimensions, then another dimension, s_4, must have a different type (is a member of different set). Therefore, a Cartesian grid near every point in a Riemann and pseudo-Riemann space, has constant, unit-factoring ratios, eigenvalues, between a unit of a 3-dimensional distance, r, and the units of other types of dimensions: $r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q = \cdots$.

The geometry proofs and geometric ratios are used to derive the gravity (G), charge (k_e), and Planck (h) constants, and used to provide simpler, shorter, and more rigorous derivations of some well-known classical gravity and charge equations, special and general relativity equations, and quantum physics equations.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq23**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of size κ subintervals. The ruler is both an inner and outer measure of an interval.

Definition 2.1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\ [a, b] \subset \mathbb{R}$, $s = b - a \ \wedge\ 0 < \kappa \leq 1 \ \wedge\ (p = floor(s/\kappa) \ \vee\ p = ceiling(s/\kappa))$.

Theorem 2.2. *Ruler convergence:* $M = \lim_{\kappa\to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\ \kappa > 0,\ p = floor(s/\kappa) \ \wedge\ 0 \leq |floor(s/\kappa) - s/\kappa| < 1 \ \Rightarrow\ |p - s/\kappa| < 1. \quad (2.1)$$

Multiply both sides of inequality 2.1 by κ:

$$\forall\, \kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \delta = \epsilon \quad := \quad M = \lim_{\kappa \to 0} p\kappa = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, ..., \pi_{\lim_{\kappa\to 0}}$.

LEMMA 2.3. $\forall\, n \geq 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa.$

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa\to 0} \kappa^n = 0. \end{aligned} \tag{2.5}$$

$$\lim_{\kappa\to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa\to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa. \qquad \square \tag{2.6}$$

3. Volume

DEFINITION 3.1. A countable n-volume is the number of ordered combinations (n-tuples), v_c, of the members of n number of disjoint, countable domain sets, x_i:

$$\exists\, n \in \mathbb{N},\ v_c \in \{0,\ \mathbb{N}\},\ \ x_i \in \{x_1, \cdots, x_n\} : \ \bigcap_{i=1}^n x_i = \emptyset \ \wedge \ v_c = \prod_{i=1}^n |x_i|. \tag{3.1}$$

THEOREM 3.2. *Euclidean volume,*

$$\begin{aligned} &\forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\ \ [v_a, v_b] \subset \mathbb{R},\ \ s_i = b_i - a_i,\ \ v = v_b - v_a : \\ &\qquad\qquad v_c = \prod_{i=1}^n |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^n s_i. \end{aligned} \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.

$$v_c = \prod_{i=1}^n |x_i| \ \Leftrightarrow \ v_c\kappa = (\prod_{i=1}^n |x_i|)\kappa \ \Leftrightarrow \ \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{3.3}$$

Apply the ruler (2.1) and ruler convergence (2.2) to equation 3.3:

$$\exists\, v, \kappa \in \mathbb{R} : \ v_c = floor(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{3.4}$$

Apply lemma 2.3 to equation 3.4:

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa^n = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa). \tag{3.5}$$

Apply the ruler (2.1) and ruler convergence (2.2) to s_i:

$$\exists\, s_i, \kappa \in \mathbb{R} : \ floor(s_i/\kappa) = |x_i| \quad \Rightarrow \quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i. \tag{3.6}$$

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa) \ \wedge \ \lim_{\kappa\to 0}(|x_i|\kappa) = s_i \ \Rightarrow \ v = \prod_{i=1}^n s_i \qquad \square \tag{3.7}$$

4. Distance

DEFINITION 4.1. Countable distance, $d_c = f(v_c, n) = f(|x_1|, \cdots, |x_n|, n) = \prod_{i=1}^n |x_i|$ is bijective and isomorphic: $v_c = \prod_{i=1}^n |x_i| = \prod_{i=1}^n d_c = d_c^n$.

4.1. Minkowski distance (ρ-norm).

THEOREM 4.2. *Minkowski distance (ρ-norm):*

$$v_c = \sum_{j=1}^m v_{c_i} \quad \Rightarrow \quad \exists\, d, d_i \in \mathbb{R} :\ d^n = \sum_{i=1}^m d_i^n.$$

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

PROOF. Apply the countable distance definition (4.1) to the assumption:

$$(4.1) \quad v_c = \prod_{i=1}^n |x_i| = \prod_{i=1}^n d_c = d_c^n \quad \wedge \quad v_{c_i} = \prod_{j=1}^n |x_{i_j}| = \prod_{i=1}^n d_{c_i} = d_{c_i}^n \\ \wedge \quad v_c = \sum_{j=1}^m v_{c_i} \quad \Rightarrow \quad d_c^n = \sum_{j=1}^m d_{c_i}^n.$$

Multiply both sides of equation 4.1 by κ and take the limit:

$$(4.2) \qquad d_c^n = \sum_{j=1}^m d_{c_i}^n \quad \Leftrightarrow \quad \lim_{\kappa\to 0} d_c^n \kappa = \lim_{\kappa\to 0} \sum_{j=1}^m d_{c_i}^n \kappa.$$

Apply lemma 2.3 to equation 4.1:

$$(4.3) \quad \lim_{\kappa\to 0} d_c^n \kappa = \lim_{\kappa\to 0} \sum_{j=1}^m d_{c_i}^n \kappa \quad \wedge \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa \\ \Leftrightarrow \lim_{\kappa\to 0} d_c^n \kappa^n = \lim_{\kappa\to 0} \sum_{j=1}^m d_{c_i}^n \kappa^n \Leftrightarrow \lim_{\kappa\to 0} (d_c\kappa)^n = \lim_{\kappa\to 0} \sum_{j=1}^m (d_{c_i}\kappa)^n.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to equation 4.3:

$$(4.4) \quad \exists\, d, d_i :\ d_c = floor(d/\kappa),\ d = \lim_{\kappa\to 0} d_c\kappa \\ \wedge \quad d_{c_i} = floor(d_i/\kappa),\ d_i = \lim_{\kappa\to 0} d_{c_i}\kappa \quad \Rightarrow \\ d^n = \lim_{\kappa\to 0} (d_c\kappa)^n = \lim_{\kappa\to 0} \sum_{j=1}^m (d_{c_i}\kappa)^n = \sum_{j=1}^m d_i^n. \quad \square$$

4.2. Distance inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N},\ v_a, v_b \geq 0 :\ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$(4.5) \quad \forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq v_a + v_b + \\ \sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n.$$

Take the n^{th} of both sides of the inequality 4.5:

$$(4.6)\ \forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad \square$$

4.3. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N},\ a_i, b_i \geq 0 :\ (\sum_{i=1}^m (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$(4.7) \quad \forall\, m, n \in \mathbb{N},\ v_a, v_b \geq 0 : \quad v_a = \sum_{i=1}^m a_i^n \quad \wedge \quad v_b = \sum_{i=1}^m b_i^n \quad \wedge \\ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad ((\sum_{i=1}^m a_i^n) + (\sum_{i=1}^m b_i^n))^{1/n} = \\ (\sum_{i=1}^m (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}. \quad \square$$

4.4. Metric Space. All Minkowski distances (ρ-norms) imply the metric space properties.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.8}$$

Apply the distance inequality (4.3) to the inequality 4.8:

$$\begin{aligned} &(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \;\wedge\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ &\qquad \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ \Rightarrow \quad &(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ &\qquad \Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq \\ &\qquad\qquad (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \end{aligned} \tag{4.9}$$ □

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$\begin{aligned} &\forall\, p : p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ &\qquad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \end{aligned} \tag{4.10}$$ □

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0. \tag{4.11}$$

$$p \geq 1,\ \ u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \tag{4.12}$$ □

THEOREM 4.8. *Identity of Indiscernibles:* $d(u, u) = 0$.

PROOF. From the non-negativity property (4.7):

$$\begin{aligned} &d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0 \\ &\qquad \Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0. \end{aligned} \tag{4.13}$$

$$d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v. \tag{4.14}$$

$$d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \tag{4.15}$$ □

4.5. Set properties limiting a set to at most 3 members.

DEFINITION 4.9. Totally ordered set:

$$\begin{aligned} &\forall\, i\ n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i \ \in \{x_1, \ldots, x_n\}, \\ &\qquad\qquad \textit{successor } x_i = x_{i+1} \ \wedge\ \textit{predecessor } x_{i+1} = x_i. \end{aligned}$$

DEFINITION 4.10. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i,\ j,\ n \in \mathbb{N},\ \forall\, x_i,\ x_j \in \{x_1, \ldots, x_n\},\ \textit{successor } x_i = x_j \ \Leftrightarrow\ \textit{predecessor } x_j = x_i.$$

THEOREM 4.11. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n \ \wedge \ j = 1 \ \Rightarrow \ \mathit{successor}\ x_n = x_1 \ \wedge \ \mathit{predecessor}\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (4.9) assigns a unique label to each set member and assigns unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (4.10):

$$i = n \ \wedge \ j = 1 \ \wedge \ \mathit{successor}\ x_i = x_j \ \Rightarrow \ \mathit{successor}\ x_n = x_1. \tag{4.16}$$

Applying the definition of the symmetry property (4.10) to conclusion 4.16:

$$\mathit{successor}\ x_i = x_j \ \Rightarrow \ \mathit{predecessor}\ x_j = x_i \ \Rightarrow \ \mathit{predecessor}\ x_1 = x_n. \tag{4.17}$$ □

THEOREM 4.12. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (4.11).

DEFINITION 4.13. (Cyclic) Successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (n = m + 1 \leq \mathit{setsize}). \tag{4.18}$$

DEFINITION 4.14. (Cyclic) Predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (n = m - 1 \geq 1). \tag{4.19}$$

DEFINITION 4.15. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$\mathit{Adjacent}(m, n, \mathit{setsize}) \leftarrow \mathit{Successor}(m, n, \mathit{setsize}) \vee \mathit{Predecessor}(m, n, \mathit{setsize}). \tag{4.20}$$

Every member is adjacent to every other member, where $\mathit{setsize} \in \{1,\ 2,\ 3\}$:

$$\mathit{Adjacent}(1, 1, 1) \leftarrow \mathit{Successor}(1, 1, 1) \leftarrow (m = \mathit{setsize} \wedge n = 1). \tag{4.21}$$

$$\mathit{Adjacent}(1, 2, 2) \leftarrow \mathit{Successor}(1, 2, 2) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{4.22}$$

$$\mathit{Adjacent}(2, 1, 2) \leftarrow \mathit{Successor}(2, 1, 2) \leftarrow (n = \mathit{setsize} \wedge m = 1). \tag{4.23}$$

$$\mathit{Adjacent}(1, 2, 3) \leftarrow \mathit{Successor}(1, 2, 3) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{4.24}$$

$$\mathit{Adjacent}(2, 1, 3) \leftarrow \mathit{Predecessor}(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \tag{4.25}$$

$$\mathit{Adjacent}(3, 1, 3) \leftarrow \mathit{Successor}(3, 1, 3) \leftarrow (n = \mathit{setsize} \wedge m = 1). \tag{4.26}$$

$$\mathit{Adjacent}(1, 3, 3) \leftarrow \mathit{Predecessor}(1, 3, 3) \leftarrow (m = 1 \wedge n = \mathit{setsize}). \tag{4.27}$$

$$\mathit{Adjacent}(2, 3, 3) \leftarrow \mathit{Successor}(2, 3, 3) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{4.28}$$

$$\mathit{Adjacent}(3, 2, 3) \leftarrow \mathit{Predecessor}(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \tag{4.29}$$

Member 2 is the only successor of member 1 for all $setsize \geq 3$, which implies member 3 is not ($\neg$) a successor of member 1 for all $setsize \geq 3$:

$$(4.30)\quad \neg Successor(1,3,setsize \geq 3) \leftarrow Successor(1,2,setsize \geq 3) \leftarrow (n = m+1 \leq setsize).$$

Member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not ($\neg$) a predecessor of member 1 for all $setsize > 3$:

$$(4.31)\quad \neg Predecessor(1,3,setsize \geq 3) \leftarrow Predecessor(1,setsize,setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).$$

For all $setsize \geq 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not symmetric):

$$(4.32)\quad \neg Adjacent(1,3,setsize > 3) \leftarrow \neg Successor(1,3,setsize > 3) \wedge \neg Predecessor(1,3,setsize > 3). \quad \square$$

5. Applications to physics

From the volume proof (3.2), two disjoint distance intervals, $[0,r_1]$ and $[0,r_2]$, define a 2-volume. From the Minkowski distance proof (4.2), $\exists\, r : r^2 = r_1^2 + r_2^2$. And from the 3D proof (4.12), for some non-distance type, $\tau : \tau \in \{t\ (time),\ m\ (mass),\ q\ (charge), \cdots\}$, there exist unit-factoring ratios, $\mu,\ \nu_1,\ \nu_2$:

$$(5.1)\quad \forall\, r, r_1, r_2 :\ r^2 = r_1^2 + r_2^2 \quad \wedge \quad r = \mu\tau \quad \wedge \quad r_1 = \nu_1\tau \quad \wedge \quad r_2 = \nu_2\tau \\ \Rightarrow \quad (\mu\tau)^2 = (\nu_1\tau)^2 + (\nu_2\tau)^2 \quad \Rightarrow \quad \mu \geq \nu_1 \quad \wedge \quad \mu \geq \nu_2.$$

μ is the maximum-possible ($\mu \geq \nu_1,\ \nu_2$), constant, unit-factoring ratio, where:

$$(5.2)\quad \mu \in \{r_c/t_c,\ r_c/m_c,\ r_c/q_c, \cdots\} :\ r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q = \cdots .$$

5.1. Derivation of the constant G, and the gravity laws of Newton, Gauss, and Poisson.

From equation 5.2:

$$(5.3)\quad r = (r_c/m_c)m \quad \wedge \quad r = (r_c/t_c) = ct \quad \Rightarrow \quad r/(ct)^2 = (r_c/m_c)m/r^2 \\ \Rightarrow \quad r/t^2 = ((r_c/m_c)c^2)m/r^2 = Gm/r^2,$$

where Newton's constant, $G = (r_c/m_c)c^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$.

Newton's law follows from multiplying both sides equation 5.3 by m:

$$(5.4)\quad r/t^2 = Gm^2/r^2 \quad \wedge \quad \forall\, m \in \mathbb{R} :\ \exists\, m_1, m_2 \in \mathbb{R} :\ m_1 m_2 = m^2 \\ \Rightarrow \quad \exists\, m_1, m_2 \in \mathbb{R} :\ F := mr/t^2 = Gm^2/r^2 = Gm_1 m_2/r^2.$$

Again, starting with equation 5.3 and using ρ as the mass field density (Gauss's flux divergence) on a sphere having the surface area $4\pi r^2$ yields the differential forms of Gauss's flux divergence, $\nabla \cdot \mathbf{g}$ and Poisson's curl per unit mass, $\nabla^2\Phi(r,t)$:

$$(5.5)\quad \mathbf{g} = \nabla\Phi(r,t) := r/t^2 = (-Gm/r^2)(4\pi/4\pi) \quad \wedge \quad \rho = m/4\pi r^2 \\ \Rightarrow \quad \nabla \cdot \mathbf{g} = \nabla^2\Phi(\overrightarrow{r},t) = -4\pi G\rho.$$

5.2. Derivation of Coulomb's charge constant, k_e and charge force.

$$\forall\, q \in \mathbb{R}:\ \exists\, q_1, q_2 \in \mathbb{R}:\ q_1 q_2 = q^2 \quad\wedge\quad r = (r_c/q_c)q \tag{5.6}$$
$$\Rightarrow\quad \exists\, q_1, q_2 \in \mathbb{R}:\ q_1 q_2 = q^2 = ((q_c/r_c)r)^2 \quad\Rightarrow\quad (r_c/q_c)^2 q_1 q_2/r^2 = 1.$$

$$r = (r_c/t_c)t = ct \;\Rightarrow\; mr = (m_c/r_c)(ct)^2 \;\Rightarrow\; ((r_c/m_c)/c^2)mr/t^2 = 1. \tag{5.7}$$

$$((r_c/m_c)/c^2)mr/t^2 = 1 \quad\wedge\quad (r_c/q_c)^2 q_1 q_2/r^2 = 1 \tag{5.8}$$
$$\Rightarrow\quad F := mr/t^2 = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2.$$

$$r_c/t_c = c \quad\wedge\quad F = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2 \tag{5.9}$$
$$\Rightarrow\quad F = (m_c(r_c/t_c^2))(r_c/q_c)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2,$$

where Coulomb's constant, $k_e = (m_c(r_c/t_c^2))(r_c/q_c)^2$, conforms to the SI units: $N \cdot m^2 \cdot C^{-2}$.

5.3. Vacuum permitivity, ε_0, and Gauss's law for electric fields. From equation 5.2:

$$r = (r_c/q_c)q \quad\wedge\quad r = (r_c/t_c) = ct \quad\Rightarrow\quad r/(ct)^2 = (r_c/q_c)q/r^2 \tag{5.10}$$
$$\Rightarrow\quad r/t^2 = ((r_c/q_c)/c^2)q/r^2,$$

Using ρ as the charge (electric) field density (Gauss's flux divergence) on a sphere having the surface area $4\pi r^2$ in equation 5.10 yields the differential form of Gauss's flux divergence, $\nabla \cdot \mathbf{E}$:

$$r/t^2 = (((r_c/q_c)/c^2)q/r^2)(4\pi/4\pi) \quad\wedge\quad \rho = q/4\pi r^2 \tag{5.11}$$
$$\Rightarrow\quad \nabla \cdot \mathbf{q} := r/t^2 = 4\pi((r_c/q_c)c^2)\rho.$$

Multiply both sides of equation 5.11 by $(m_c/r_c)(r_c/q_c)$ and apply the derivation of k_e (5.9) in equation 5.8 and the of vacuum permitivity, $\varepsilon_0 = 1/4\pi k_e$:

$$\nabla \cdot \mathbf{E} = ((m_c/r_c)(r_c/q_c))\nabla \cdot \mathbf{q} = ((m_c/r_c)(r_c/q_c))4\pi((r_c/q_c)/c^2)\rho \tag{5.12}$$
$$= 4\pi((m_c/r_c)c^2)(r_c/q_c)^2\rho = 4\pi k_e \rho = \rho/\varepsilon_0,$$

which is the Gauss's law for electric fields that is used in Maxwell's equations.

5.4. Space-time-mass-charge equations. Form equation 5.1:

$$\forall\, r, r', r_v, \mu, \nu:\ r^2 = r'^2 + r_v^2 \quad\wedge\quad r = \mu\tau \quad\wedge\quad r_v = \nu\tau \tag{5.13}$$
$$\Rightarrow\quad r' = \sqrt{(\mu\tau)^2 - (\nu\tau)^2} \;=\; \mu\tau\sqrt{1-(\nu/\mu)^2}.$$

Rest (local) distance, r', contracts relative to stationary distance, r, as $\nu \to \mu$:

$$r' = \mu\tau\sqrt{1-(\nu/\mu)^2} \quad\wedge\quad \mu\tau = r \quad\Rightarrow\quad r' = r\sqrt{1-(\nu/\mu)^2}. \tag{5.14}$$

Interval length, τ, dilates relative to the rest interval length, τ', as $\nu \to \mu$:

$$\mu\tau = r'/\sqrt{1-(\nu/\mu)^2} \quad\wedge\quad r' = \mu\tau' \quad\Rightarrow\quad \tau = \tau'/\sqrt{1-(\nu/\mu)^2}. \tag{5.15}$$

Where τ is time, the space-like flat Minkowski spacetime event interval is:

$$\mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_v^2 \quad\wedge\quad \mathrm{d}r_v^2 = \mathrm{d}r_1^2 + \mathrm{d}r_2^2 + \mathrm{d}r_3^2 \quad\wedge\quad \mathrm{d}(\mu\tau) = \mathrm{d}r \tag{5.16}$$
$$\Rightarrow\quad \mathrm{d}r'^2 = \mathrm{d}(\mu\tau)^2 - \mathrm{d}r_1^2 - \mathrm{d}r_2^2 - \mathrm{d}r_3^2.$$

5.5. Simplifying Einstein's general relativity (field) equation. In this subsection, the previous spacetime equations and ratios are used to simplify finding solutions to Einstein's field (general relativity) equation (EFE) is outlined. As an example, a simplified derivation of the Schwarzchild metric is in the next subsection.

Einstein was motivated by Gaussian curvature, hence the terms, $\mathbf{G}_{\mu,\nu}$, and $g_{\mu,\nu}$, in the EFE, $\mathbf{G}_{\mu,\nu} + \Lambda g_{\mu,\nu} = k\mathbf{T}_{\mu,\nu}$, [**Ein15**] [**Wey52**]. Also, the $4\pi G$ in Einstein's constant, $k = 8\pi G/c^4 = (2/c^4)4\pi G$, and energy density in $\mathbf{T}_{\mu,\nu}$ were motivated by Gauss's gravity law. But, Gaussian curvature cannot be extended beyond 3 dimensions. Therefore, the much more complex 4-dimensional (4D) Ricci curvature, $\mathbf{R}$, and scalar curvature, R, are used, where $\mathbf{G}_{\mu,\nu} = \mathbf{R}_{\mu,\nu} - g_{\mu,\nu}R/2$.

Simplification step 1) Express the EFE as 2D tensors: The 4D flat spacetime interval equation (5.16) is an instance of the 2D equation, $\mathrm{d}r'^2 = \mathrm{d}(ct)^2 - \mathrm{d}r_v^2$, where $\mathrm{d}r_v^2$ is the magnitude of a 3-dimensional vector. Generalizing to curved spacetime: $\mathrm{d}r'^2 = g_{0,0}\mathrm{d}(ct)^2 - g_{1_v,1_v}\mathrm{d}r_v^2$, where $g_{1_v,1_v}\mathrm{d}r_v^2 = g_{1,1}\mathrm{d}r_1^2 + g_{2,2}\mathrm{d}r_2^2 + g_{3,3}\mathrm{d}r_3^2$, and the $g_{\mu,\nu}$ terms are functions returning scalar values.

The 2D metric tensor allows using the 2D Gaussian curvature, $K = det(\mathbf{H})$, in the Einstein tensor, $\mathbf{G}_{\mu,\nu}$, such that $\mathbf{G}_{\mu,\nu} = \mathbf{H} - g_{\mu,\nu}K$, where $\mathbf{H}$ is the 2×2 Hessian matrix:

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 g_{i,i}}{\partial t^2} & \dfrac{\partial^2 g_{i,i}}{\partial t\,\partial r} \\ \dfrac{\partial^2 g_{i,i}}{\partial t\,\partial r} & \dfrac{\partial^2 g_{i,i}}{\partial r^2} \end{bmatrix}. \tag{5.17}$$

$$K = det(\mathbf{H}) = \frac{\partial^2 g_{i,i}}{\partial t^2}\frac{\partial^2 g_{i,i}}{\partial r^2} - \left(\frac{\partial^2 g_{i,i}}{\partial t\,\partial r}\right)^2. \tag{5.18}$$

The 2D Gaussian curvature, K, is much simpler to calculate than the 4D Ricci curvature, $\mathbf{R}_{\mu,\nu}$, and scalar, R. And the 2D tensors reduce the number of independent equations to solve.

Step 2) Use the ratios to define relations: All functions derived from the ratios, $r = (r_c/t_c)t = ct$ and $r = (r_c/m_c)m$ (and $r = (r_c/q_c)q$ for cases using charge), where the units balance on both sides of the 2D EFE, are valid solutions. For example, $r^3/r^3 = (r_c/m_c)mc^2t^2/rc^2t^2 = Gm/rc^2 = f(r,t,m)$.

Step 3) Translate the 2D solutions to the 4D EFE solutions by expanding the spatial component, $g_{1_v,1_v}\mathrm{d}r_v^2$ to $g_{1,1}\mathrm{d}r_1^2 + g_{2,2}\mathrm{d}r_2^2 + g_{3,3}\mathrm{d}r_3^2$. One simple method is to use spherical coordinates, where: $g_{1_v,1_v}\mathrm{d}r_1^2 = g_{1,1}\mathrm{d}r_v^2$, and $g_{2,2}\mathrm{d}r_2^2 + g_{3,3}\mathrm{d}r_3^2 = r^2(\mathrm{d}\theta^2 + sin^2\theta\mathrm{d}\phi^2)$.

The $\mathbf{T}_{0,0}$ component in the 4D stress-energy tensor is set to the $\mathbf{T}_{0,0}$ component in the 2D energy-stress tensor and $\mathbf{T}_{1,1}$ component in the 4D stress-energy tensor is set to the 2D $\mathbf{T}_{1,1}$ component. And so on.

5.6. Derivation of Schwarzchild's gravitational time dilation and black hole metric. [**Che10**] From equations 5.14 and 5.2:

$$t' = t\sqrt{1-(v^2/c^2)(r/r)} \quad \wedge \quad r = (r_c/m_c)m \tag{5.19}$$
$$\Rightarrow \quad t' = t\sqrt{1-((r_c/m_c)m)v^2/rc^2}.$$

$$t' = t\sqrt{1-(r_c/m_c)mv^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv_{escape}^2 \\ \Rightarrow \quad t' = t\sqrt{1-2(r_c/m_c)mv_{escape}^2/rc^2}. \tag{5.20}$$

Combining equation 5.20 with the derivation of G (5.4) yields gravitational time dilation:

$$(r_c/m_c)c^2 = G \quad \wedge \quad t' = \lim_{v_{escape}\to c} t\sqrt{1-2(r_c/m_c)mv_{escape}^2/rc^2} \\ = t\sqrt{1-2(r_c/m_c)mc^2/rc^2} \quad \Rightarrow \quad t' = t\sqrt{1-2Gm/rc^2}. \tag{5.21}$$

Schwarzchild defined the black hole event horizon radius, $r_s := 2Gm/c^2$.

$$r_s = 2Gm/c^2 \quad \wedge \quad t' = t\sqrt{1-2Gm/rc^2} \quad \Rightarrow \quad t' = t\sqrt{1-r_s/r}. \tag{5.22}$$

From equations 5.14 and 5.22:

$$r' = r\sqrt{1-(v/c)^2} = r\sqrt{1-2Gm/rc^2} = r\sqrt{1-r_s/r}. \tag{5.23}$$

From equation 5.13:

$$r_v^2 = r^2 - r'^2 \quad \Rightarrow \quad r_v \propto r/r' = r/r\sqrt{1-r_s/r} = 1/\sqrt{1-r_s/r} \tag{5.24}$$

$$\mathrm{d}s^2 = (\sqrt{1-r_s/r}\mathrm{d}(ct))^2 - (\mathrm{d}r_v/\sqrt{1-r_s/r})^2 \quad \wedge \quad c=1 \quad \wedge \quad \mathrm{d}r_v \propto \mathrm{d}r \\ \Rightarrow \quad \mathrm{d}s^2 = (1-r_s/r)\mathrm{d}t^2 - (1-r_s/r)^{-1}\mathrm{d}r^2. \tag{5.25}$$

Expanding to 4D spherical coordinates yields Schwarzchild's black hole metric:

$$\mathrm{d}s^2 = (1-r_s/r)\mathrm{d}t^2 - (1-r_s/r)^{-1}\mathrm{d}r^2 = f(r,t) \\ \rightarrow \quad \mathrm{d}s^2 = (1-r_s/r)\mathrm{d}t^2 - (1-r_s/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + sin^2\theta\mathrm{d}\phi^2) = f(r,t,\theta,\phi). \tag{5.26}$$

5.7. Derivation of Einstein's gravitational lens. [Che10] Using the same steps to derive Schwarzchild's gravitational time dilation equation (5.21):

$$r' = r\sqrt{1-(v^2/c^2)(r/r)} \quad \Rightarrow \quad r' = r\sqrt{1-2Gm/rc^2}. \tag{5.27}$$

The incremental deflection of light (work), $\mathrm{d}s = f(r-r') = 2Gm/rc^2$. Therefore, an increment of deflection, $\mathrm{d}s$, must correspond to an incremental distance, $\mathrm{d}r : r = \mathrm{d}r$.

$$\mathrm{d}s = 2Gm/rc^2 \quad \wedge \quad \mathrm{d}r = r \quad \Rightarrow \quad \mathrm{d}s/\mathrm{d}r = 2Gm/r^2c^2. \tag{5.28}$$

There are two deflections: 1) deflection as the light approaches a mass, and 2) deflection as light passes the same mass. Therefore, the total deflection is doubled:

$$2\mathrm{d}s/\mathrm{d}r = 2(2Gm/r^2c^2) \quad \Rightarrow \quad s = 2\int \mathrm{d}s = 2\int 2Gm/r^2c^2\mathrm{d}r = 4Gm/rc^2. \tag{5.29}$$

5.8. 3 fundamental direct proportion ratios. c_t, c_m, and c_q:

$$c_t = r_c/t_c \approx 2.99792458 \cdot 10^8 m\ s^{-1}. \tag{5.30}$$

$$G = (r_c/m_c)c_t^2 = c_m c_t^2 \quad \Rightarrow \quad c_m = r_c/m_c \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}. \tag{5.31}$$

$$k_e = (c_t^2/c_m)(r_c/q_c)^2 \quad \Rightarrow \quad c_q = r_c/q_c \approx 8.6175172023 \cdot 10^{-18} m\ C^{-1}. \tag{5.32}$$

5.9. 3 fundamental inverse proportion ratios. k_t, k_m, and k_q:

$$\begin{aligned}(5.33)\quad & r/t = r_c/t_c, \quad r/m = r_c/m_c \quad \Rightarrow \quad (r/t)/(r/m) = (r_c/t_c)/(r_c/m_c) \quad \Rightarrow \\ & (mr)/(tr) = (m_c r_c)/(t_c r_c) \quad \Rightarrow \quad mr = m_c r_c = k_m, \quad tr = t_c r_c = k_t.\end{aligned}$$

$$\begin{aligned}(5.34)\quad & r/t = r_c/t_c, \quad r/q = r_c/q_c \quad \Rightarrow \quad (r/t)/(r/q) = (r_c/t_c)/(r_c/q_c) \quad \Rightarrow \\ & (qr)/(tr) = (q_c r_c)/(t_c r_c) \quad \Rightarrow \quad qr = q_c r_c = k_q, \quad tr = t_c r_c = k_t.\end{aligned}$$

5.10. Planck relation and constant, h. [**Jai11**] Applying both the direct proportion (5.30), $r/t = r_c/t_c = c$, and inverse proportion (5.33), $mr = m_c r_c = k_m$, ratios:

$$(5.35)\quad m(ct)^2 = mr^2 \ \wedge \ m = m_c r_c / r = k_m / r \quad \Rightarrow \quad m(ct)^2 = (k_m/r) r^2 = k_m r.$$

$$\begin{aligned}(5.36)\quad & m(ct)^2 = k_m r \quad \wedge \quad r/t = r_c/t_c = c \\ & \Rightarrow \quad E := mc^2 = k_m r/t^2 = (k_m(r/t))\,(1/t) = (k_m c)(1/t) = hf,\end{aligned}$$

where the Planck constant, $h = k_m c$, and the frequency, $f = 1/t$.

$$(5.37)\qquad k_m = m_c r_c = h/c \approx 2.2102190943 \cdot 10^{-42}\ kg\ m.$$

$$(5.38)\qquad k_t = t_c r_c = k_m/(c_t/c_m) \approx 5.4749346710 \cdot 10^{-78}\ s\ m.$$

$$(5.39)\qquad k_q = q_c r_c = (c_t/c_q) k_t \approx 1.9046601056 \cdot 10^{-52}\ C\ m.$$

5.11. 4 quantum units. Distance (r_c), time (t_c), mass (m_c), and charge (q_c):

$$(5.40)\qquad r_c = \sqrt{r_c^2} = \sqrt{c_t k_t} = \sqrt{c_m k_m} = \sqrt{c_q k_q} \approx 4.0513505432 \cdot 10^{-35}\ m.$$

$$(5.41)\qquad t_c = r_c/c_t \approx 1.3513850782 \cdot 10^{-43}\ s.$$

$$(5.42)\qquad m_c = r_c/c_m \approx 5.4555118613 \cdot 10^{-8}\ kg.$$

$$(5.43)\qquad q_c = r_c/c_q \approx 4.7012967286 \cdot 10^{-18}\ C.$$

Planck length $= r_c/\sqrt{2\pi}$, time $= t_c/\sqrt{2\pi}$, mass $= m_c/\sqrt{2\pi}$, charge $= q_c/\sqrt{2\pi}$.

5.12. Compton wavelength. [**Jai11**] From equations 5.33 and 5.36:

$$(5.44)\qquad mr = k_m \quad \wedge \quad h = k_m c \quad \Rightarrow \quad r = k_m/m = (k_m/m)(c/c) = h/mc.$$

5.13. de Broglie wavelength. [**Jai11**] From equations 5.14 and 5.44:

$$(5.45)\quad r_v = ((r_v/t)/(r/t))r = (v/c)r \ \wedge \ mr = k_m \ \Rightarrow \ r_v = k_m(c/v)/m = h/mv.$$

Note that r_v is the wavelength in the frame of reference of the moving mass (for example, wavelength as measured by an observer riding in a moving vehicle).

5.14. Schrödenger's equation. Start with the previously derived Planck relation 5.36 and multiply by mc/mc:

$$h/t = mc^2 \Rightarrow \exists V(r,t) : h/t = h/2t + V(r,t) \Rightarrow h/t = hmc/2mct + V(r,t). \tag{5.46}$$

And from the distance-to-time (speed of light) ratio (5.30):

$$h/t = hmc/2mct + V(r,t) \ \wedge \ r = ct \ \Rightarrow \ h/t = hmc^2/2mcr + V(r,t). \tag{5.47}$$

$$h/t = hmc^2/2mcr + V(r,t) \ \wedge \ h/t = mc^2 \ \Rightarrow \ h/t = h^2/2mcrt + V(r,t). \tag{5.48}$$

$$h/t = h^2/2mcrt + V(r,t) \ \wedge \ r = ct \ \Rightarrow \ h/t = h^2/2mr^2 + V(r,t). \tag{5.49}$$

Translate to the reduced Planck constant:

$$h/t = h^2/2mr^2 + V(r,t) \wedge \hbar = h/2\pi \ \Rightarrow \ 2\pi\hbar/t = (2\pi)^2\hbar^2/2mr^2 + V(r,t). \tag{5.50}$$

Multiply both sides by any real-valued function, $\Psi(r,t)$.

$$\begin{aligned} &2\pi\hbar/t = (2\pi)^2\hbar^2/2mr^2 + V(r,t) \\ &\qquad \Rightarrow \quad (2\pi\hbar/t)\Psi(r,t) = ((2\pi)^2\hbar^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t). \end{aligned} \tag{5.51}$$

$$\begin{aligned} &(2\pi\hbar/t)\Psi(r,t) = ((2\pi)^2\hbar^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t) \quad \wedge \\ &\forall\, \Psi(r,t) :\ \partial^2\Psi(r,t)/\partial r^2 = (-(2\pi)^2/r^2)\Psi(r,t) \quad \wedge \quad \partial\Psi(r,t)/\partial t = (i\ 2\pi/t)\Psi(r,t) \\ &\qquad \Rightarrow \quad -(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t) = i\hbar\partial\Psi(r,t)/\partial t, \end{aligned} \tag{5.52}$$

which is Schrödenger's equation in one dimension of space.

$$\begin{aligned} &-(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t) = i\hbar\partial\Psi(r,t)/\partial t \quad \wedge \quad ||\overrightarrow{\mathbf{r}}|| = r \\ &\quad \Rightarrow \quad \exists\ \overrightarrow{\mathbf{r}} :\ -(\hbar^2/2m)\partial^2\Psi(\overrightarrow{\mathbf{r}},t)/\partial\overrightarrow{\mathbf{r}}^2 + V(\overrightarrow{\mathbf{r}},t)\Psi(\overrightarrow{\mathbf{r}},t) = i\hbar\partial\Psi(\overrightarrow{\mathbf{r}},t)/\partial t, \end{aligned} \tag{5.53}$$

which is Schrödenger's equation in three dimensions of space.

5.15. Dirac's wave equation. Using the derived Planck relation 5.36:

$$\begin{aligned} &mc^2 = h/t \quad \Rightarrow \quad \exists\ V(r,t) :\ mc^2/2 + V(r,t) = h/t \\ &\qquad\qquad \Rightarrow \quad 2h/t - 2V(r,t) = mc^2. \end{aligned} \tag{5.54}$$

$$\begin{aligned} &\forall\ V(r,t) :\ V(r,t) = h/t \quad \wedge \quad r = ct \quad \wedge \quad 2h/t - 2V(r,t) = mc^2 \\ &\qquad\qquad \Rightarrow \quad 2h/t - 2hc/r = mc^2. \end{aligned} \tag{5.55}$$

Applying the charge ratio, c_q, (5.31) to multiply each term on the left side of equation 5.55 by 1:

$$\begin{aligned} &q(r_c/q_c)c/r = qc_qc/r = qc_q/t = 1 \quad \wedge \quad 2h/t - \gamma_0 2hc/r = mc^2 \\ &\qquad\qquad \Rightarrow \quad 2h(-qc_q)/t^2 - 2h(-qc_q)c/rt = mc^2. \end{aligned} \tag{5.56}$$

where a negative sign is added to q to indicate an attractive force between an electron and a nucleus.

$$\begin{aligned} &2h(-qc_q)/t^2 - 2h(-qc_q)c/rt = mc^2 \quad \wedge \quad r = ct \\ &\qquad\qquad \Rightarrow \quad 2h(-qc_q)/t^2 - 2h(-qc_q)c^2/r^2 = mc^2. \end{aligned} \tag{5.57}$$

Applying a quantum amplitude equation in complex form to equation 5.57:

$$A_0 = c_q((1/t)) + i(c/r)) \quad \wedge \quad 2h(-qc_q)/t^2 - 2h(-qc_q)c^2/r^2 = mc^2 \\ \Rightarrow \quad 2h\partial(-qA_0)/\partial t - i2h(\partial(-qA_0)/\partial r)c = mc^2. \tag{5.58}$$

Translating equation 5.58 to moving coordinates via the Lorentz factor, $\gamma_0 = 1/\sqrt{1-(v/c)^2}$:

$$2h\partial(-qA_0)/\partial t - i2h(\partial(-qA_0)/\partial r)c = mc^2 \\ \Rightarrow \quad \gamma_0 2h\partial(-qA_0)/\partial t - \gamma_0 i2h(\partial(-qA_0)/\partial r)c = mc^2. \tag{5.59}$$

Multiplying both sides of equation 5.59 by $\Psi(r,t)$:

$$\gamma_0 2h\partial(-qA_0)/\partial t - \gamma_0 i2h(\partial(-qA_0)/\partial r)c = mc^2 \\ \Rightarrow \quad \gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \gamma_0 i2h(\partial(-qA_0)/\partial r)c\Psi(r,t) = mc^2\Psi(r,t). \tag{5.60}$$

Applying the vectors to equation 5.60:

$$\gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \gamma_0 i2h(\partial(-qA_0)/\partial r)c\Psi(r,t) = mc^2\Psi(r,t) \wedge \\ ||\overrightarrow{\mathbf{r}}|| = r \quad \wedge \quad ||\overrightarrow{\mathbf{A}}|| = A_0 \quad \wedge \quad ||\overrightarrow{\gamma}|| = \gamma_0 \quad \wedge \quad \Leftrightarrow \quad \exists\, \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{A}}, \overrightarrow{\gamma} : \\ \gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \overrightarrow{\gamma} \cdot i2h(\partial(-q\overrightarrow{\mathbf{A}})/\partial r)c\Psi(\overrightarrow{\mathbf{r}},t) = mc^2\Psi(\overrightarrow{\mathbf{r}},t). \tag{5.61}$$

Adding a $\frac{1}{2}$ angular rotation (spin-$\frac{1}{2}$) of π to equation 5.58 allows substituting the reduced Planck constant, $\hbar = h/2\pi$, into equation 5.61, which yields Dirac's wave equation:

$$\gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \overrightarrow{\gamma} \cdot i2h(\partial(-q\overrightarrow{\mathbf{A}})/\partial r)c\Psi(r,t) = mc^2\Psi(r,t) \quad \wedge \\ A_0 = \pi c_q((1/t) + (c/r)) \\ \Rightarrow \quad \gamma_0 \hbar(\partial(-qA_0)/\partial t)\Psi(r,t) - \overrightarrow{\gamma} \cdot i\hbar(\partial(-q\overrightarrow{\mathbf{A}})/\partial r)c\Psi(\overrightarrow{\mathbf{r}},t) = mc^2\Psi(\overrightarrow{\mathbf{r}},t). \tag{5.62}$$

5.16. Total mass. The total mass of a particle is $m = \sqrt{m_0^2 + m_{ke}^2}$, where m_0 is the rest mass and m_{ke} is the kinetic energy-equivalent mass. Applying both the direct (5.30) and inverse proportion ratios (5.33):

$$m_0 = (m_c/r_c)r \quad \wedge \quad m_{ke} = m_c r_c/r \quad \wedge \quad m = \sqrt{m_0^2 + m_{ke}^2} \\ \Rightarrow \quad m = \sqrt{((m_c/r_c)r)^2 + ((m_c r_c)/r)^2}. \tag{5.63}$$

Before discussing the unification of general relativity and quantum physics, the quantum effect, $((m_c r_c)/r)^2$, is easier to express and understand by first extending Newtonian gravity with the quantum effect.

5.17. Quantum-special relativity extensions to Newton's gravity force.

$$\exists\, m :\ m_1 m_2 = m^2 = ((m_c/r_c)r)^2 + ((m_c r_c)/r)^2 \\ \Rightarrow \quad m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2) = 1. \tag{5.64}$$

Applying the ratio-derived spacetime equation 5.14 to equation 5.5:

$$r' = r\sqrt{1-(v/c)^2} \quad \wedge \quad ((r_c/m_c)/c^2)mr/t^2 = 1 \\ \Rightarrow \quad ((r_c/m_c)/c^2\sqrt{1-(v/c)^2})mr'/t^2 = 1, \tag{5.65}$$

where r'/t^2 is the acceleration observed from a stationary frame of reference. Combining equations 5.64 and 5.65:

$$F = r'/t^2 = (c^2\sqrt{1-(v/c)^2}/(r_c/m_c))m_1m_2/(((m_c/r_c)r)^2+((m_cr_c)/r)^2). \tag{5.66}$$

Using the ratio-derived gravitational time dilation (5.24):

$$F = (c^2\sqrt{1-2Gm/rc^2}/(r_c/m_c))m_1m_2/(((m_c/r_c)r)^2 + ((m_cr_c)/r)^2). \tag{5.67}$$

5.18. Quantum-special relativity extensions to Coulomb's charge force.

$$F = (c^2\sqrt{1-(v/c)^2}/(r_c/m_c))(r_c/q_c)^2q_1q_2/(((q_c/r_c)r)^2 + ((q_cr_c)/r)^2). \tag{5.68}$$

5.19. Unifying general relativity and quantum physics. In Einstein's constant, $k = 8\pi G/c^4$, G is Newton's gravity constant. But as shown in the quantum extension to Newton's gravity equation, at very small distances, where quantum effects are measurable, (5.66), the constant, G, is replaced by multiple ratio constants. Also, the components of the metric, $g_{\nu,\mu}$, etc. would need to changed similar to how it was done in the quantum extension to Newton's gravity equation.

Note that the derivation of Schwarzchild's black hole metric (5.6), Schrödenger's equation (5.14), and Dirac's equation (5.15) were all derived more simply, in this article, by deriving the equations in 2D spacetime and then translating back into 4D spacetime. The difference is that Schwarzchild's black hole metric was derived in the real-valued Euclidean plane, whereas Schrödenger's and Dirac's equations were derived in the complex plane.

The fundamental theorem of algebra states that all real-valued equations (for example, Einstein's field equations) have complex-valued solutions. Therefore, the simplest way to unify general relativity and quantum physics to find relativity solutions that incorporate the quantum effect in the 2D complex plane. Even this simplification is very complex and beyond the scope of this article (which is show how set and number theory provide a foundation under geometry and physics and provide new insights).

6. Insights and implications

(1) Combinatorics, the ordered combinations of countable, disjoint sets (n-tuples), generates both Euclidean volume (3.2) and the Minkowski distances (4.2), which includes Manhattan and Euclidean distances.
(2) Combinatorics, all n-at-time permutations of an ordered and symmetric set of distance dimensions, limits the set to 3 dimensions (4.12).
(3) Deriving Euclidean volume (3.2) and the Minkowski distances (4.2) from the same abstract, countable set of n-tuples (3.1) provides a single, unifying set and limit-based foundation under Euclidean geometry without relying on the geometric primitives and relations in Euclidean geometry [**Joy98**], axiomatic geometry [**Lee10**], and vector analysis [**Wey52**].
(4) The definition of a metric space [**Rud76**] ignores the intimate relation between distance and volume. A more sufficient definition is: a distance measure is an inverse (bijective), isomorphic function of a volume equal to the sum of bijective, isomorphic functions of volumes (4.1). This definition implies the metric space properties.

(5) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [**Joy98**]. And analytic proofs that the straight line length is the smallest distance equate the straight line length to Euclidean distance.

Without using the notion of a straight line: Euclidean volume was derived from a set of n-tuples (3.2). And all distance measures (bijective, isomorphic functions of n-volumes) derived from Euclidean 2-volumes (areas) are Minkowski distances (4.2), where $n \in \{1, 2\}$: $n = 1$ is the Manhattan (largest monotonic) distance case, $d = \sum_{i=1}^{m} s_i$. $n = 2$ is the Euclidean (smallest monotonic) distance case, $d = (\sum_{i=1}^{m} s_i^2)^{1/2}$. For the case, $n \in \mathbb{R}$, $1 \leq n \leq 2$: d decreases monotonically as n goes from 1 to 2.

(6) The left side of the distance sum inequality (4.4),

$$(\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}, \tag{6.1}$$

differs from the left side of Minkowski's sum inequality [**Min53**]:

$$(\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \tag{6.2}$$

The two inequalities are only the same where $n = 1$.

(a) The distance sum inequality (4.4) is a more fundamental inequality because the proof does not require the convexity and Hölder's inequality assumptions of the Minkowski sum inequality proof [**Min53**].

(b) The Minkowski sum inequality term, $\forall\ n > 1 : ((a_i^n + b_i^n)^{\mathbf{n}})^{1/n}$, is **not** a Minkowski distance spanning the n-volume, $a_i^n + b_i^n$. But the distance sum inequality term, $(a_i^n + b_i^n)^{1/n}$, is the Minkowski distance spanning the n-volume, $a_i^n + b_i^n$, which makes it directly related to geometry (for example, the metric space triangle inequality was derived from the $m = 1$ case for all $n \geq 1$ (4.5)).

(7) The set-based, first-order logic proof that a strict linearly ordered and symmetric set is a cyclic set of at most 3 members (4.12) is a simpler and more logically rigorous hypothesis for observing only 3 dimensions of physical space than parallel dimensions that cannot be detected or extra dimensions rolled up into infinitesimal balls that are too small to detect.

(a) Higher order dimensions must have different types (members of different sets), for example, types/dimensions of time, mass, and charge. Order and symmetry probably limit the number of fundamental types/dimensions to a very small number. For example, temperature, measured in Kelvins, is not a true dimension because temperature is more correctly a measure of energy, frequency, or momentum, where entropy is a drop in frequency or momentum. An oscillation in charge at a position causes a spacetime ripple that does not require a magnetic field to propagate. If magnetic force is a pseudo (fictitious) force that is a function of distance, time. and charge, then "magnet" (or pole) also would not be a fundamental type/dimension. Likewise, one should not immediately assume other fundamental types/dimensions that would correspond to the strong force, weak force, etc. For example, quantum effects might radioactivity without a weak force.

(b) Each of 3 ordered and symmetric dimensions of space can have at most 3 sequentially ordered and symmetric state values, for example,

an ordered and symmetric set of 3 vector orientations, $\{-1, 0, 1\}$, per dimension of space and at most 3 spin states per plane, etc.
If the states are not sequentially ordered (a bag of states), then a state value is undetermined until observed (like Schrödenger's poisoned cat being both alive and dead until the box is opened). That is, for a bag of states, there is no "axiom of choice", an axiom often used in math proofs that allows selecting a particular set element (state).

(8) A direct consequence of the Minkowski distance proof (4.1), and the 3D proof (4.12) is that any dimensions beyond 3 dimensions of space must have other types that, near any point in Euclidean and Euclidean-like spaces (for example, Riemann and pseudo-Riemann spaces), are related by constant direct proportion ratios (5.8): $c_t = r_c/t_c$, $c_m = r_c/m_c$, $c_q = r_c/q_c$ $\Leftrightarrow$ the inverse proportion ratios (5.9): $k_t = r_c t_c$, $k_m = r_c m_c$, $k_t = r_c q_c$, where the combination of the direct and inverse ratios implies the quantum units (5.11): r_c, t_c, m_c, q_c. These ratios and quantum units were shown to be the basis of most physics.
   (a) The gravity, G (5.4), charge k_e (5.9), vacuum permitivity, ε_0, and Planck h (5.36) constants were all derived from the ratios (5.9). Therefore, G, k_e, ε_0, and h are **not** "fundamental" constants.
   (b) Planck length $= r_c/\sqrt{2\pi}$, time $= t_c/\sqrt{2\pi}$, mass $= m_c/\sqrt{2\pi}$, and charge $= q_c/\sqrt{2\pi}$.
   (c) The inverse square law for gravity (5.3) and charge (5.6) were shown to be a result of the direct proportion ratios.
   (d) The proofs and constant ratios are the basis of relativity theory.
      (i) There is always a maximum ratio (for example, the speed of light, $c_t = r_c/t_c$).
      (ii) Special and general relativity assume covariance, which states that the equations expressing the laws of physics are invariant in every frame of reference. The laws of physics being derived from the constant ratios near any point within Riemann and pseudo-Riemann spaces causes the covariance.
      For example, the special relativity time dilation equation 5.15 was derived from the ratio, $c_t = r_c/t_c$ (the speed of light), and combined with a ratio, $c_m = r_c/m_c$, (5.8) yielded Schwarzchild's general relativity gravitational time dilation equation (5.24).
   (e) The combination of direct and inverse proportion ratios was shown to create the particle-wave equations: Planck relation (5.10), Compton wavelength (5.44), de Broglie wavelength (5.45), Schrödenger (5.14) and Dirac equations (5.15).
   (f) G, k_e, and h all depend on the speed of light ratio, c_t: $G = c_m c_t^2$, $k_e = (c_q^2/c_m)c_t^2$, and $h = k_m c_t$.
   (g) $k_e = (c_q^2/c_m)c_t^2 = ((m_c/r_c)(r_c/t_c)^2)c_q^2 = (m_c(r_c/t_c^2))c_q^2$, where the term, r_c/t_c^2, suggests a maximum acceleration constant, which agrees with the MOND theories of gravity.

(9) The derivations of the spacetime equations, in this article (5.4), differ from other derivations:
   (a) The derivations, here, do not rely on the Lorentz transformations or Einsteins' postulates [**Ein15**]. The derivations do not even require

the notion of light.
   (b) The same derivations are also valid for spacemass and spacecharge.
   (c) The derivations, here, rely only on the Euclidean volume proof (3.2), the Minkowski distances proof (4.1), and the 3D proof (4.12), which provides the insight that the properties of physical space creates maximum ratios, the spacetime equations, and 3 dimensions of distance. For example, the ratio, $c = c_t = r_c/t_c$ is always the maximum speed (of light) ratio.
(10) The derivation Schrödenger's equation (5.14) and Dirac's equation (5.15), in this article, differs from other derivations:
   (a) Other derivations are based on Hamiltonian (energy-momentum) relations. In contrast, the derivations here rely on the Planck (energy-frequency) relation.
   (b) The derivations here are more rigorous because:
      (i) The derivations here start with a foundation of geometric ratios to derive the Planck constant (5.10), whereas other derivations assume that the Planck constant is a fundamental constant.
      (ii) The Planck relation (5.10) was derived here, whereas other derivations **incorrectly** assume (define) the energy-momentum relation as: $\hat{T} = (\mathbf{p} \cdot \mathbf{p})/2m = \hbar^2/2m$. However, the derivation here shows that the reduced Planck constant, $\hbar$, should be replaced with the full Planck constant, $h = 2\pi\hbar$).
      (iii) Other derivations assume the probability distribution has a mean value, where values closer to the mean are more probable. The derivation here makes no such assumptions.
      (iv) Schrödenger's equation uses the reduced Planck constant, $\hbar = h/2\pi$. But the derivation, in this article, shows the reduced Planck constant is only valid if the partial derivatives of the probability distribution function, Ψ, contains compensating 2π terms: $\partial^2\Psi(r,t)/\partial r^2 = (-(2\pi)^2/r^2)\Psi(r,t)$ and $\partial\Psi(r,t)/\partial t = (i\ 2\pi/t)\Psi(r,t)$. Finding solutions to Schrödenger's equation would be simpler if the full Planck constant is used because it would reduce the complexity of $\Psi(r,t)$.
(11) The quantum-special relativity extensions to Newton's gravity force (5.66) and Coulomb's charge force (5.68) make quantifiable predictions. Specifically: $\lim_{r\to 0} F = 0$, and **both** the gravity and charge forces peak at the quantum length: $r_c = \sqrt{r_c^2} = \sqrt{c_t k_t} = \sqrt{c_m k_m} = \sqrt{c_q k_q} \approx 4.0513505432 \cdot 10^{-35}\ m$ (5.40). Note that the Planck length is $r_c/\sqrt{2\pi}$.
   (a) Gravitational time dilation peaks at r_c.
   (b) Black holes have measurable sizes > 0 (are not singularities).
   (c) The finite gravity-charge well allows radioactivity, quantum tunneling, and possibly black hole evaporation.
   (d) As the kinetic energy (temperature) decreases, more particles will stay within their gravity-charge well distance, r_c, allowing superconductivity and Bose-Einstein condensates.
(12) A constant value cannot be mapped to continuously varying values. Therefore, the discrete spin states of two quantum entangled particles and the polarization states of two quantum entangled photons are independent of

continuously varying distance and time interval lengths.

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