The Set Properties Generating Geometry and Physics

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Abstract. Volume and the Minkowski distances/Lp norms (e.g., Manhattan and Euclidean distance) are derived from a set and limit-based foundation without referencing the primitives and relations of geometry. Sequencing a strict linearly ordered set in all n-at-a-time permutations via successor/predecessor relations is a cyclic set of at most 3 members. Therefore, all other interval lengths have different types from a cyclic set of 3 distance interval lengths. Constant ratios between different types of interval lengths and the set proofs provide simpler derivations of the spacetime, Lorentz, Newton's gravity, Coulomb's charge force, Planck-Einstein, quantum-relativity gravity equations and corresponding constants. All proofs are verified in Coq.

Contents

1. Introduction

Mathematical (real) analysis can construct differential calculus from a set and limit-based foundation without the need to reference the primitives and relations of Euclidean geometry, like straight line, angle, slope, etc. But volume in the Riemann integral, Lebesgue integral, and measure theory and distance in the vector norm and metric space axioms are all definitions motivated by Euclidean geometry. [Gol76] [Rud76] Here, volume and distance are motivated and derived from a set and limit-based foundation.

A well-known set-based motivation of Euclidean volume is the number of members (cardinal), v_c, of an abstract, countable set of Cartesian product n-tuples:

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$v_c = \prod_{i=1}^{n} |x_i|$, where $|x_i|$ is the cardinal of the countable, disjoint set, x_i. Yet, real analysis books have not provided a simple proof that the Euclidean volume equation, $v = \prod_{i=1}^{n} s_i$, is an instance of $v_c = \prod_{i=1}^{n} |x_i|$, where $|x_i|$ is the approximate number of same-sized partitions of $[a_i, b_i] \subset \mathbb{R}$ and $s_i = b_i - a_i$. [**Gol76**] [**Rud76**] In this article, a proof is provided to fill that gap.

Every real-valued n-volume, v, can be partitioned into the sum of m number of sub-n-volumes, $v = \sum_{i=1}^{m} v_i$. And, where the domain set sizes, d and d_i, are inverse functions of n-volumes: $v = d^n = \sum_{i=1}^{m} d_i^n = \sum_{i=1}^{m} v_i$. It will be proved that $d^n = \sum_{i=1}^{m} d_i^n$ is an instance of $|x|^n = \sum_{i=1}^{m} |x_i|^n$, where $|x|$ and $|x_i|$ are the approximate number of same-sized partitions of the intervals $[a, b]$ and $[a_i, b_i]$. d is the L_p norm (Minkowski distance), which will be proved to imply the metric space axioms.

In the prior equations, sequencing a set, from $i = 1$ to n, is a strict linear (total) order that set theory defines in terms of successor and predecessor functions. But sequencing a strict linear order in all n-at-a-time orders requires an additional "symmetry" constraint, where every set member is either a successor or predecessor to every other set member, which will be proved to be a cyclic set, where $n \leq 3$.

Therefore, where $\mathbb{R}^3$ is a strict linearly ordered and symmetric set of 3 "distance" dimensions, $\{x, y, z\}$, a fourth dimension of $\mathbb{R}$ must have a different type (is a member of different set). At coordinate, (t_0, x_0, y_0, z_0), the intervals, $[x_0, x_1]$ and $[t_0, t_1]$ are mapped to each other via a constant, unit-factoring, conversion ratio (linear transformation). The ratios combined with the set proofs provide simpler derivations of the spacetime, Lorentz, Newton's gravity, Coulomb's charge force, Planck-Einstein, quantum-relativistic gravity equations and corresponding constants. Impacts on Einstein's field equations are also shown.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq23**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

Derivatives and integrals use a 1-1 correspondence between the infinitesimals of each interval, where the size of the infinitesimals in each interval are proportionate to the size of the containing interval, which precludes using derivatives and integrals to directly express many-to-many (Cartesian product) mappings between same-sized, size κ, infinitesimals in different-sized intervals. Further, using tools that define Euclidean volume and distance precludes using those tools to derive Euclidean volume and distance.

Therefore, a different tool is used here. A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals (infinitesimals), where each infinitesimal has the *same* size, κ, across all intervals. The ruler is both an inner and outer measure of an interval.

Definition 2.1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\ [a, b] \subset \mathbb{R}$, $s = b - a \ \wedge\ \kappa > 0 \ \wedge\ (p = \mathit{floor}(s/\kappa) \ \vee\ p = \mathit{ceiling}(s/\kappa))$.

Theorem 2.2. *Ruler convergence:* $M = \lim_{\kappa \to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\,\kappa > 0,\ p = floor(s/\kappa)\ \wedge\ 0 \leq |floor(s/\kappa) - s/\kappa| < 1\ \Rightarrow\ |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by κ:

$$\forall\,\kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\begin{aligned} &\forall\,\epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ &\quad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \delta = \epsilon \quad := \quad M = \lim_{\kappa \to 0} p\kappa = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, \ldots, \pi_{\lim_{\kappa\to 0}}$.

LEMMA 2.3. $\forall\, n \geq 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa.$

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\begin{aligned} &\forall\,\epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta \\ &\quad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa \to 0} \kappa^n = 0. \end{aligned} \tag{2.5}$$

$$\lim_{\kappa \to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa \to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa. \qquad \square \tag{2.6}$$

3. Volume

DEFINITION 3.1. An n-volume is the number of ordered combinations (n-tuples), v_c, of the members of n number of disjoint, countable domain sets, x_i:

$$\exists\, n \in \mathbb{N}, a_c, v_c, \in \{0,\ \mathbb{N}\},\ x_i \in \{x_1, \cdots, x_n\}, \textstyle\bigcap_{i=1}^n x_i = \emptyset :\ v_c = a_c \prod_{i=1}^n |x_i|. \tag{3.1}$$

THEOREM 3.2. *Euclidean volume, $v = \prod_{i=1}^n s_i$, is the instance of the "flat" n-volume (3.1) case, $v_c = \prod_{i=1}^n |x_i|$, where each countable set, x_i, is the set of partitions of an interval case, $[a_i, b_i] \subset \mathbb{R}$.*

$$\begin{aligned} &\forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\ \ [v_a, v_b] \subset \mathbb{R},\ \ s_i = b_i - a_i,\ \ v = v_b - v_a, \\ &\qquad\qquad v_c = \textstyle\prod_{i=1}^n |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^n s_i. \end{aligned} \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.

$$v_c = \textstyle\prod_{i=1}^n |x_i| \ \Leftrightarrow\ v_c\kappa = (\prod_{i=1}^n |x_i|)\kappa \ \Leftrightarrow\ \lim_{\kappa \to 0} v_c\kappa = \lim_{\kappa \to 0} (\prod_{i=1}^n |x_i|)\kappa. \tag{3.3}$$

Apply the ruler (2.1) and ruler convergence (2.2) to v:

$$\exists\, v, \kappa \in \mathbb{R} :\ v_c = floor(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa \to 0} v_c\kappa = \lim_{\kappa \to 0} (\textstyle\prod_{i=1}^n |x_i|)\kappa. \tag{3.4}$$

Apply lemma 2.3 to equation 3.4:

$$v = \lim_{\kappa \to 0} (\textstyle\prod_{i=1}^n |x_i|)\kappa = \lim_{\kappa \to 0} (\prod_{i=1}^n |x_i|)\kappa^n = \lim_{\kappa \to 0} (\prod_{i=1}^n |x_i|\kappa). \tag{3.5}$$

Apply the ruler (2.1) and ruler convergence (2.2) to s_i:

$$\exists\, s_i, \kappa \in \mathbb{R} :\ floor(s_i/\kappa) = |x_i| \quad \Rightarrow \quad \lim_{\kappa \to 0} (|x_i|\kappa) = s_i. \tag{3.6}$$

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|\kappa) \;\wedge\; \lim_{\kappa\to 0}(|x_i|\kappa) = s_i \;\Rightarrow\; v = \prod_{i=1}^{n} s_i \tag{3.7}$$ □

THEOREM 3.3. *Sum of volumes:*

$$\begin{aligned} &\forall\, x_{i,j} \in \{x_{i_1}, \cdots, x_{i_m}\} = x_i : \; v_c = \prod_{i=1}^{n} |x_i| \quad \wedge \quad v_{c_j} = \prod_{i=1}^{n} |x_{i,j}| \quad \wedge \\ &\qquad v_c = \sum_{j=1}^{m} v_{c_j} \quad \Rightarrow \quad \exists\, s_i, s_{i,j} \in \mathbb{R} : \; \prod_{i=1}^{n} s_i = \sum_{j=1}^{m}(\prod_{i=1}^{n} s_{i,j}). \end{aligned} \tag{3.8}$$

The formal proof, "sum_of_volumes," is in the Coq file, euclidrelations.v.

PROOF. From the Euclidean volume theorem (3.2):

$$v_c = \prod_{i=1}^{n} |x_i| \;\Rightarrow\; v = \prod_{i=1}^{n} s_i \;\wedge\; v_{c_j} = \prod_{i=1}^{n} |x_{i,j}| \;\Rightarrow\; v_j = \prod_{i=1}^{n} s_{i,j}. \tag{3.9}$$

Apply the ruler (2.1) and ruler convergence (2.2):

$$\begin{aligned} &\exists\, v, v_j, \kappa \in R : \quad v_c = floor(v/\kappa) \quad \wedge \quad v_{c_j} = floor(v_i/\kappa) \\ &\qquad \Rightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa \quad \wedge \quad v_i = \lim_{\kappa\to 0} v_{c_j}\kappa. \end{aligned} \tag{3.10}$$

$$v_c = \sum_{j=1}^{m} v_{c_j} \quad \Leftrightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\sum_{j=1}^{m} v_{c_j})\kappa. \tag{3.11}$$

Apply lemma 2.3 to equation 3.11:

$$\begin{aligned} &\lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa \;\wedge\; v = \lim_{\kappa\to 0}(\sum_{j=1}^{m} v_{c_j})\kappa \;\wedge\; v_i = \lim_{\kappa\to 0} v_{c_j}\kappa \\ &\qquad \Rightarrow \quad v = \lim_{\kappa\to 0}(\sum_{j=1}^{m} v_{c_j})\kappa^n = \lim_{\kappa\to 0} \sum_{j=1}^{m}(v_{c_j}\kappa) = \sum_{j=1}^{m} v_j. \end{aligned} \tag{3.12}$$

$$\begin{aligned} &v = \prod_{i=1}^{n} s_i \quad \wedge \quad v_j = \prod_{i=1}^{n} s_{i,j} \quad \wedge \quad v = \sum_{j=1}^{m} v_j \\ &\qquad \Rightarrow \quad \prod_{i=1}^{n} s_i = \sum_{j=1}^{m} \prod_{i=1}^{n} s_{i,j}. \end{aligned} \tag{3.13}$$ □

4. Distance

4.1. Minkowski distance (L_p norm).

THEOREM 4.1. *The Minkowski distance (L_p norm), d, is the instance of the sum of (union of disjoint) n-volumes (3.3), and where distance is an inverse function of volume:*

$$\prod_{i=1}^{n} |x_i| = \sum_{j=1}^{m}(\prod_{i=1}^{n} |x_{i,j}|) \quad \Rightarrow \quad \exists\, d, d_i \in \mathbb{R} : \quad d^n = \sum_{i=1}^{m} d_i^n.$$

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

PROOF. From the Euclidean volume proof (3.2):

$$\prod_{i=1}^{n} |x_i| = \sum_{j=1}^{m}(\prod_{i=1}^{n} |x_{i,j}|) \quad \Rightarrow \quad \prod_{i=1}^{n} s_i = \sum_{j=1}^{m}(\prod_{i=1}^{n} s_{i,j}) \tag{4.1}$$

$$\begin{aligned} &\exists\, d, d_i, s_i, s_{i,j} \in \mathbb{R} : \; s_1 = \cdots = s_n = d \quad \wedge \quad s_{i_1} = \cdots = s_{i_m} = d_i \quad \wedge \\ &\prod_{i=1}^{n} s_i = \sum_{j=1}^{m}(\prod_{i=1}^{n} s_{i,j}) \quad \Rightarrow \quad d^n = \prod_{i=1}^{n} s_i = \sum_{j=1}^{m}(\prod_{i=1}^{n} s_{i,j}) = \sum_{i=1}^{m} d_i^n. \end{aligned} \tag{4.2}$$ □

4.2. Distance inequality. Proving that all Minkowski distances (L_p norms) satisfy the metric space triangle inequality requires another inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.2. *Distance inequality*

$$\forall\, n \in \mathbb{N},\ \ v_a, v_b \geq 0:\ \ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand the n-volume, $(v_a^{1/n} + v_b^{1/n})^n$, using the binomial expansion:

$$\forall\, v_a, v_b \geq 0: \quad v_a + v_b \leq v_a + v_b + \sum_{i=1}^{n}\binom{n}{k}(v_a^{1/n})^{n-k}(v_b^{1/n})^k + \sum_{i=1}^{n}\binom{n}{k}(v_a^{1/n})^k(v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n. \tag{4.3}$$

Take the n^{th} root of both sides of the inequality:

$$\forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \tag{4.4}$$

□

4.3. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N},\ \ a_i, b_i \geq 0:\ \ (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.2):

$$\forall\, m, n \in \mathbb{N},\ \ v_a, v_b \geq 0: \quad v_a = \textstyle\sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge \quad (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad ((\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} = (\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \tag{4.5}$$

□

4.4. Metric Space. All Minkowski distances (L_p norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.4. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.6}$$

Apply the distance inequality (4.2) to the inequality 4.6:

$$\begin{aligned} &(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \ \wedge \ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ &\qquad \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ \Rightarrow \quad &(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ &\qquad \Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq \\ &\qquad\qquad (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \end{aligned} \tag{4.7}$$

□

THEOREM 4.5. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$\forall\, p \,:\, p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ \Rightarrow \quad d(u,v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v,u). \quad \square \tag{4.8}$$

THEOREM 4.6. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0. \tag{4.9}$$

$$p \geq 1,\ u, v \geq 0 \quad \Rightarrow \quad d(u,v) = (u^p + v^p)^{1/p} \geq 0. \tag{4.10}$$

$\square$

THEOREM 4.7. *Identity of Indiscernibles:* $d(u,u) = 0$.

PROOF. From the non-negativity property (4.6):

$$d(u,w) \geq 0 \quad \wedge \quad d(u,v) \geq 0 \quad \wedge \quad d(v,w) \geq 0 \\ \Rightarrow \quad \exists\, d(u,w) = d(u,v) = d(v,w) = 0. \tag{4.11}$$

$$d(u,w) = d(v,w) = 0 \quad \Rightarrow \quad u = v. \tag{4.12}$$

$$d(u,v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u,u) = 0. \tag{4.13}$$

$\square$

5. Applications to physics

5.1. The properties limiting a set to at most 3 members.

DEFINITION 5.1. Totally ordered set:

$$\forall\, i\, n \in \mathbb{N},\ \ i \in [1, n-1],\ \forall\, x_i\ \in \{x_1, \ldots, x_n\}, \\ \textit{successor}\ x_i = x_{i+1}\ \wedge\ \textit{predecessor}\ x_{i+1} = x_i.$$

DEFINITION 5.2. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i,\ j,\ n \in \mathbb{N},\ \forall\, x_i,\ x_j \in \{x_1, \ldots, x_n\},\ \textit{successor}\ x_i = x_j\ \Leftrightarrow\ \textit{predecessor}\ x_j = x_i.$$

THEOREM 5.3. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n\ \wedge\ j = 1\ \Rightarrow\ \textit{successor}\ x_n = x_1\ \wedge\ \textit{predecessor}\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (5.1) assigns a unique label to each set member and assigns unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (5.2):

$$i = n\ \wedge\ j = 1\ \wedge\ \textit{successor}\ x_i = x_j\ \Rightarrow\ \textit{successor}\ x_n = x_1. \tag{5.1}$$

Applying the definition of the symmetry property (5.2) to conclusion 5.1:

$$\textit{successor}\ x_i = x_j\ \Rightarrow\ \textit{predecessor}\ x_j = x_i\ \Rightarrow\ \textit{predecessor}\ x_1 = x_n. \quad \square \tag{5.2}$$

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3).

DEFINITION 5.5. (Cyclic) Successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize). \tag{5.3}$$

DEFINITION 5.6. (Cyclic) Predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{5.5}$$

Prove that every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \tag{5.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \tag{5.7}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1). \tag{5.8}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.9}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \tag{5.10}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1). \tag{5.11}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize). \tag{5.12}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.13}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \tag{5.14}$$

Member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not ($\neg$) a successor of member 1 for all $setsize > 3$:

$$\neg Successor(1, 3, setsize > 3) \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.15}$$

Member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not ($\neg$) a predecessor of member 1 for all $setsize > 3$:

$$\neg Predecessor(1, 3, setsize > 3) \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \tag{5.16}$$

For all $setsize > 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not symmetric):

$$\neg Adjacent(1, 3, setsize > 3) \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \tag{5.17}$$

□

From the 3D proof (5.4), the interval lengths: t (time), m (mass), and q (charge) have different types (are from different sets) from a 3-dimensional distance interval length, r, that can be related via constant, unit-factoring, conversion ratios:

$$r = (r_c/t_c)t = ct = (r_c/m_G)m = (r_c/q_C)q, \tag{5.18}$$

5.2. Spacetime and Lorentz equations. From the 3D proof (5.4), let c and v be constant, unit-factoring conversion ratios from time, t, to distance, r. From the Euclidean volume proof (3.2), two disjoint intervals, $[0, r]$ and $[0, r']$, defines an Euclidean 2-space. From the Minkowski distance proof (4.1), the interval lengths, r and r', are inverse functions of 2 cuboid 2-volumes. Either $r' \geq r$ or $r \geq r'$ can be chosen. $r \geq r'$ is used here.

(5.19)
$$\forall\, r \geq r' \,\exists\, r_v \in \mathbb{R} : \; r^2 = r'^2 + r_v^2 \quad \wedge \quad \exists\, r_c,\; t_c,\; c,\; v \in \mathbb{R} : \; r = (r_c/t_c)t = ct$$
$$\wedge \quad r_v = vt \quad \Rightarrow \quad (ct)^2 = {r'}^2 + (vt)^2 \quad \Rightarrow \quad r' = \sqrt{(ct)^2 - (vt)^2} = ct\sqrt{1 - (v/c)^2}.$$

Local (proper) distance, r', contracts relative to coordinate distance, r, as $v \to c$:

$$r' = ct\sqrt{1 - (v/c)^2} \quad \wedge \quad ct = r \quad \Rightarrow \quad r' = r\sqrt{1 - (v/c)^2}. \tag{5.20}$$

From equation 5.19, coordinate time, t, dilates relative to local time, t', as $v \to c$:

$$ct = r'/\sqrt{1 - (v/c)^2} \quad \wedge \quad r' = ct' \quad \Rightarrow \quad t = t'/\sqrt{1 - (v/c)^2}. \tag{5.21}$$

Using $r^2 = r'^2 + r_v^2$ from equation 5.19, where r_v is a 3-dimensional distance, one form of the flat Minkowski's spacetime event interval is:

$$\begin{aligned} \mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_v^2 \quad \wedge \quad \mathrm{d}r_v^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \quad \wedge \quad \mathrm{d}(ct) = \mathrm{d}r \\ \Rightarrow \quad \mathrm{d}r'^2 = \mathrm{d}(ct)^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2 - \mathrm{d}x_3^2. \end{aligned} \tag{5.22}$$

The Lorentz transformations follow from equation 5.20 and the Galilean transformation, r = r' + vt:

$$r' = r/\sqrt{1 - (v/c)^2} \quad \wedge \quad r = r' + vt \quad \Rightarrow \quad r' = (r - vt)/\sqrt{1 - (v/c)^2}. \tag{5.23}$$

$$\begin{aligned} r' = (r - vt)/\sqrt{1 - (v/c)^2} \quad \wedge \quad r = ct \quad \wedge \quad r' = ct' \\ \Rightarrow \quad t' = (t - (vt/c))/\sqrt{1 - (v/c)^2} = (t - (vr/c^2))/\sqrt{1 - (v/c)^2}. \end{aligned} \tag{5.24}$$

5.3. Newton's gravity force and the constant, G. From equation 5.18:

$$\begin{aligned} \forall\, m_1, m_2, m, r \in \mathbb{R} : \; m_1 m_2 = m^2 \quad \wedge \quad m = (m_G/r_c)r \\ \Rightarrow \quad m_1 m_2 = m^2 = ((m_G/r_c)r)^2 \quad \Rightarrow \quad (r_c/m_G)^2 m_1 m_2/r^2 = 1. \end{aligned} \tag{5.25}$$

$$r = r_c/t_c = ct \quad \wedge \quad mr = ((m_G/r_c)r)(ct) \quad \Rightarrow \quad mr = (m_G/r_c)(ct)^2. \tag{5.26}$$

$$mr = (m_G/r_c)(ct)^2 \quad \Rightarrow \quad ((r_c/m_G)/c^2)mr/t^2 = 1. \tag{5.27}$$

$$\begin{aligned} ((r_c/m_G)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/m_G)^2 m_1 m_2/r^2 = 1 \\ \Rightarrow \quad F := mr/t^2 = ((r_c/m_G)c^2)m_1 m_2/r^2 = G m_1 m_2/r^2, \end{aligned} \tag{5.28}$$

where Newton's constant, $G = (r_c/m_G)c^2$, has the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$.

5.4. Coulomb's charge force and constant. From equation 5.18:

$$r = (r_c/m_G)m = (r_c/q_C)q_1 \Rightarrow m = (m_G/q_C)q_1. \tag{5.29}$$

Substituting equations 5.29 and 5.18 into equation 5.28, where $v = 0$:

$$m = (m_G/q_C)q \quad \wedge \quad r_c/t_c = c \quad \wedge \quad F = ((r_c/m_G)c^2)m_1m_2/r^2 \\ \Rightarrow \quad F = (m_G/q_C)(r_c/q_C)(r_c/t_c)^2 q_1q_2/r^2. \tag{5.30}$$

$$a_G = r_c/t_c^2 \quad \wedge \quad F = (m_G/q_C)(r_c/q_C)(r_c/t_c)^2 q_1q_2/r^2 \\ \Rightarrow \quad F = (m_Ga_G)(r_c/q_C)^2 q_1q_2/r^2 = k_e q_1q_2/r^2, \tag{5.31}$$

where $v = 0$, $k_e = (m_Ga_G)(r_c/q_C)^2$, has the SI units: $N \cdot m^2 \cdot C^{-2}$.

5.5. Work constant and Planck-Einstein equation: From the ratios 5.18:

$$r = (r_c/m_G)m \quad \wedge \quad r = (r_c/t_c)t = ct \quad \wedge \quad \exists\, m_p \in \mathbb{R} : \ m_p = (t/t_c)^2 m_G \\ \Rightarrow mr = (m_G/r_c)r^2 = (m_G/r_c)(r_c/t_c)^2t^2 = (t/t_c)^2 m_G r_c = m_p r_c := k_m \\ \approx 2.2102190943 \cdot 10^{-42}\ kg\ m, \tag{5.32}$$

where r is the displacement (Compton wavelength) of the mass, m, and k_m, is a work constant.

Likewise, using the charge ratio, $r = (r_c/q_C)q$, there is a charge-work constant, $k_q :\ qr = q_p r_c = k_q$.

$$m(ct)^2 = mr^2 \quad \wedge \quad mr = k_m \quad \Rightarrow \quad m(ct)^2 = k_m r. \tag{5.33}$$

$$m(ct)^2 = k_m r \quad \wedge \quad r_c/t_c = r/t = c \\ \Rightarrow \quad E = mc^2 = k_m r/t^2 = (k_m(r/t))\,(1/t) = (k_m c)(1/t) = hf, \tag{5.34}$$

where the Planck constant $h = k_m c$ and the frequency $f = 1/t$.

5.6. Quantum-relativistic gravity. The total mass of a particle is $m = \sqrt{m_0^2 + m_p^2}$, where m_0 is the rest mass and m_p is the photon-momentum equivalent mass. Applying the work constant (5.32), $m = \sqrt{((m_G/r_c)r)^2 + ((m_p r_c)/r)^2}$ to equation 5.25:

$$\forall\, m_1m_2 = m^2 = ((m_G/r_c)r)^2 + ((m_pr_c)/r)^2 \\ \Rightarrow \quad m_1m_2/(((m_G/r_c)r)^2 + ((m_pr_c)/r)^2) = 1. \tag{5.35}$$

From equation 5.19, if r is the proper distance, then $r = \sqrt{(ct)^2 - (vt)^2}$:

$$r = \sqrt{(ct)^2 - (vt)^2} \quad \Rightarrow \quad mr = (m_G/r_c)((ct)^2 - (vt)^2) + m_pr_c. \tag{5.36}$$

$$mr = (m_G/r_c)((ct)^2 - (vt)^2) + m_pr_c \\ \Rightarrow \quad ((r_c/m_G)/(c^2 - v^2))(mr - m_pr_c)/t^2 = 1. \tag{5.37}$$

$$((r_c/m_G)/(c^2 - v^2))(mr - m_pr_c)/t^2 = 1 \\ \wedge \quad m_1m_2/(((m_G/r_c)r)^2 + ((m_pr_c)/r)^2) = 1 \\ \Rightarrow \quad F = (mr - m_pr_c)/t^2 = ((m_G/r_c)(c^2 - v^2))m_1m_2/(((m_G/r_c)r)^2 + ((m_pr_c)/r)^2). \tag{5.38}$$

5.7. Quantum-relativistic charge. Applying $m = (m_G/q_C)q$ to the quantum-relativistic gravity equation (5.6):

$$F = (r_c/m_G)(c^2 - v^2)(m_G/q_C)^2 q_1 q_2/(((m_G/r_c)r)^2 + ((m_p r_c)/r)^2). \tag{5.39}$$

6. Insights and implications

(1) The function, $d = (\sum_{i=1}^{m} a_i s_i^n)^{1/n}$, where a_i is a function returning a scalar value, inherits the properties of a metric space from the Minkowski distance (4.4). For example, the a_i function values correspond to the $g(i,i)$ components in the metric tensor, $g(\mu,\nu)$, used in Einstein's field (general relativity) equations [**Wey52**]. Proving that Euclidean volume and the Minkowski distance are instances of the same abstract, set-based definition of a countable n-volume provides a unifying set and limit-based foundation under volume and distance without using the geometric primitives and relations required in Euclidean geometry [**Joy98**], axiomatic geometry [**Lee10**], and vector analysis [**Wey52**].

(2) The interval length, $s = b - a$, in the ruler measure (2.1) can be replaced with a $\pm$-signed integer length: $s = (b - a \Leftarrow a = \omega \,:\, -(b-a) \Leftarrow b = \omega)$, where either $a = \omega$ or $b = \omega$ and ω is the local origin value. The $\pm$-signed interval lengths, s_i, (3.2), that specify a $\pm$-signed Euclidean volume (3.2), are the components of a vector.

(3) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [**Joy98**]. And proofs that a straight line is the smallest distance equate the straight line to the Euclidean distance.

Using the calculus of variations for a shortest distance proof would result in circular logic due to the Euclidean assumptions in the definition of the integral.

It was proved that all "flat" distances, where distance is an inverse function of an n-volume, are Minkowski distances (4.1). In an Euclidean 2-volume (area), the Minkowski distances, range from Manhattan distance (the largest distance, where $n = 1$), $d = \sum_{i=1}^{m} s_i$, to Euclidean distance (the smallest distance, where $n = 2$), $d = (\sum_{i=1}^{m} s_i^2)^{1/2}$.

(4) Compare the distance sum inequality (4.3),

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \le (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}, \tag{6.1}$$

used to prove that all Minkowski distances satisfy the metric space triangle inequality property (4.4), to Minkowski's sum inequality:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n)^n)^{1/n} \le (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n} \tag{6.2}$$

[**Min53**]. Note the difference in the left side of each equation:

$$\forall\, n > 1,\ 0 < a_i^n, b_i^n < 1:\ (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \ > \ (\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n}. \tag{6.3}$$

$$\forall\, n > 1,\ a_i^n, b_i^n \ge 1:\ (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \ < \ (\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n}. \tag{6.4}$$

The distance sum inequality is a more fundamental inequality because its proof does not require the convexity and various inequality theorems required to prove the Minkowski sum inequality. And the distance sum inequality is derived from the definitions of volume and distance, which makes it more directly related to geometry.

(5) From the 3D proof (5.4), more intervals than the 3 dimensions of distance intervals must have different types with lengths that are related to a 3-dimensional distance interval length, r, via constant, unit-factoring, conversion ratios (both direct and inverse proportion ratios). In SI units:
   $c_t = r_c/t_c \approx 2.99792458 \cdot 10^8 m\ s^{-1}$.
   $c_m = r_c/m_G \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}$.
   $c_q = r_c/q_C \approx 8.6175172023 \cdot 10^{-18} m\ C^{-1}$.
   $k_m = m_p r_c \approx 2.2102190943 \cdot 10^{-42}\ kg\ m$.
   $k_q = q_p r_c \approx Too\ be\ Determined$.

(6) Using the ratios, $m(ct)^2 = (m_G/r_c)r^3 \Rightarrow E = mc^2 = (m_G/r_c)r^3/t^2$, it appears that an "empty" volume of space, r^3, corresponds to a potential (dark) mass and potential (dark) energy.

(7) The derivations in this article show that the spacetime (5.2), gravity force (5.26), charge force (5.31), and Planck (5.34) constants all depend on time being proportionate to distance: $r = (r_c/t_c)t = ct$. For example, $G = (r_c/m_G)c^2$, $k_e = (m_G/q_C)(r_c/q_C)c^2$, and $h = (m_p r_c)c = k_m c$.

(8) The ratios make all distance, wavelength, time, frequency, mass, charge, etc. sizes relative to each other. There are no absolute sizes > 0. That is, there is no Planck-like (quantum) distance, time, mass, charge, etc.

(9) The derivations of the spacetime equations and Lorentz transformations, here (5.2), differ from other derivations.
   (a) The derivations, here, are much shorter and simpler.
   (b) The derivations of the spacetime equations, here, do not rely on the Lorentz transformations or Einsteins' postulates [**Ein15**]. The derivations do not even require the notion of light.
   (c) The derivations, here, rely only on geometry: the Euclidean volume proof (3.2), the Minkowski distances proof (4.1), and the 3D proof (5.4), which provides the insight that the geometry of physical space creates: 1) a maximum speed, c; 2) the spacetime equations; and 3) the Lorentz transformations.
   (d) The distance-to-mass ratio, $r = (r_c/m_G)m$, and distance-to-charge ratio, $r = (r_c/q_C)q$, can replace the distance-to-time ratio, $r = (r_c/t_c)t$, in the spacetime derivations, to derive corresponding spacemass and spacecharge equations.

(10) Applying the ratios to derive Newton's gravity force (5.3) and Coulomb's charge force (5.4) equations provide:
   (a) Derivations that do not assume the inverse square law or Gauss's flux divergence theorem. **Note:** the components of the Ricci and metric tensors in Einstein's field equations have the units, $1/\text{distance}^2$ [**Wey52**], which is an assumption of the inverse square law.
   (b) The first derivations to show that the inverse square law and the property of force as mass times acceleration are the result of the conversion ratios, $r = (r_c/t_c)t = (r_c/m_G)m$.
   (c) In Newton's gravity force, Gauss's gravity law, and Einstein's field (general relativity) equations, the force, $F \to \infty$ as the distance, $r \to 0$. But, in the quantum-relativistic extension to Newton's gravity

equation (5.38), $F \to 0$ as $r \to 0$. The quantum-relativistic gravity equation indicates that Newton's gravity constant, G, and Einstein's gravity constant, $k = 8\pi G/c^4$, [**Wey52**], are only valid where the local velocity, $v = 0$. $v > 0$ implies a universe expanding faster than predicted by k and also predicts an accelerating expansion.
Further, where the distance between particles is less than approximately 10^{-3} m, the gravity force starts to measurably deviate from the inverse square law. And the gravity force starts decreasing at distances less than approximately 10^{-8} m, which implies larger black hole radii and maybe allows black hole evaporation.

(d) The quantum-relativistic extension to Newton's gravity equation (5.38) might also have a third, graviton-momentum component.

(11) There is no constant ratio converting a constant value to a continuously varying value. Therefore, the spin states of two quantum entangled particles and the polarization states of two quantum entangled photons are independent of a varying distance interval length between the particles and independent of a varying time interval length (instantaneous).

(12) Linear algebra, vector analysis, differential geometry, etc. assume any number of possible dimensions. For example, the Gram-Schmidt process is a method to find an orthogonal vector for any n-dimensional vector [**Coh21**]. None of those disciplines have exposed the properties that can limit a geometry to 3 dimensions. But the proof that a strict linearly ordered and symmetric set is a cyclic set of at most 3 members (5.4) is the simplest explanation for observing only 3 dimensions of physical space.

(a) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (5.4), each member being an ordered and symmetric set of 3 dimensions (three 3-dimensional balls).

(b) Each of 3 ordered and symmetric dimensions of space can have at most 3 sequentially ordered and symmetric state values, for example, an ordered and symmetric set of 3 vector orientations, $\{-1, 0, 1\}$, per dimension of space and at most 3 spin states per dimension, etc.

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