Some Set Properties Underlying Geometry and Physics

George. M. Van Treeck

ABSTRACT. Volume and the Minkowski distances/p-norms (e.g., Manhattan and Euclidean distance) are derived from a set and limit-based foundation without referencing the primitives and relations of geometry. Sequencing a strict linearly ordered set in all n-at-a-time orders via successor/predecessor relations is a cyclic set of at most 3 members. Therefore, all other domain interval lengths have different types from a cyclic set of 3 distance domain interval lengths. Constant ratios between different types of interval lengths and the set proofs provide simpler derivations of the: spacetime, Newton's gravity, Coulomb's charge force, Planck-Einstein, Compton wavelength, de Broglie wavelength, quantum-relativity gravity equations, and provide simple derivation of the gravity (G), charge (k_e), and Planck (h) constants. All proofs are verified in Coq.

CONTENTS

1. Introduction

Mathematical analysis can construct differential calculus from a set and limit-based foundation without referencing the primitives and relations of Euclidean geometry, like straight line, angle, shape, etc., which provides a more rigorous foundation to calculus. But Euclidean volume in the Riemann integral, Lebesgue integral, measure theory, and distance in the vector magnitude and metric space criteria are definitions motivated by Euclidean geometry. [**Gol76**] [**Rud76**] Here, volume and distance is derived from a set and limit-based foundation.

An intuitive, set-based motivation of Euclidean volume is the number, v_c, of ordered combinations (n-tuples): $v_c = \prod_{i=1}^{n} |x_i|$, where $|x_i|$ is the cardinal of the countable, disjoint set, x_i. But, some well-known analysis textbooks do not provide proofs that $v_c = \prod_{i=1}^{n} |x_i| \Rightarrow v = \prod_{i=1}^{n} s_i$, where each set, x_i, is a set of subintervals of the interval, $[a_i, b_i] \subset \mathbb{R}$, and $s_i = b_i - a_i$. [**Gol76**] [**Rud76**]. In this article, a simple proof is provided that: $v_c = \prod_{i=1}^{n} |x_i| \Rightarrow v = \prod_{i=1}^{n} s_i$.

$v_c = \prod_{i=1}^{n} |x_i| = f(|x_1|, \cdots, |x_n|, n)$. If f is a bijective function, then $\exists\, d_c : d_c = f^{-1}(v_c, n)$ and $v_c = f(d_c, n) = f(|x_1|, \cdots, |x_n|, n)$. If $d_c = |x_1| = \cdots = |x_n|$, then $v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n$.

Where f is a bijective function, and where $d_c = |x_1| = \cdots = |x_n|$, it will be proved that $v_c = \sum_{j=1}^{m} v_{c_i} \Rightarrow d^n = \sum_{i=1}^{m} d_i^n$. d is the ρ-norm (Minkowski distance), which will be proved to imply the metric space properties.

Sequencing the domain sets, $x_1, \cdots, x_n$, from $i = 1$ to n, is a strict linear (total) order, where a total order is defined in terms of successor and predecessor functions. Being able to sequence a set via successor and predecessor functions, in all n-at-a-time orders, requires a "symmetry" constraint, where every set member is either a successor or predecessor to every other set member. A strict linearly ordered and symmetric set will be proved to be a cyclic set, where $n \leq 3$.

Therefore, if $\{s_1, s_2, s_3\}$ is a strict linearly ordered and symmetric set of 3 "distance" domain interval lengths, then another domain interval length, s_4, must have a different type (is a member of different set). The distance domain interval lengths are related to other types of domain interval lengths by unit-factoring ratios, for example, *meters/second*.

The set-based proofs will show that there are some relative maximum unit-factoring ratios: $r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q$, (for example, r_c/t_c is the speed of light). The ratios combined with the results of the set-based proofs are used to provide simple and short derivations of the: spacetime, Newton's gravity, Coulomb's charge force, Planck-Einstein, Compton wavelength, de Broglie wavelength, quantum-relativity gravity equations, and provide simple derivations of the gravity (G), charge (k_e), and Planck (h) constants. The derivation of spacetime interval equation allows simplifying Einstein's field (general relativity) equations.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq23**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of size κ subintervals. The ruler is both an inner and outer measure of an interval.

DEFINITION 2.1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\, [a, b] \subset \mathbb{R}$, $s = b - a \,\wedge\, 0 < \kappa \leq 1 \,\wedge\, (p = floor(s/\kappa) \,\vee\, p = ceiling(s/\kappa))$.

THEOREM 2.2. *Ruler convergence:* $M = \lim_{\kappa \to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : y \leq x, y \in \mathbb{Z}, x \in \mathbb{R}\})$:

(2.1) $\forall\, \kappa > 0,\; p = floor(s/\kappa) \,\wedge\, 0 \leq |floor(s/\kappa) - s/\kappa| < 1 \;\Rightarrow\; |p - s/\kappa| < 1.$

Multiply both sides of inequality 2.1 by κ:

$$(2.2) \qquad \forall\, \kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|.$$

$$(2.3) \quad \forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta$$
$$\Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \epsilon \quad := \quad M = \lim_{\kappa \to 0} p\kappa = s. \quad \square$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, ..., \pi_{\lim_{\kappa \to 0}}$.

LEMMA 2.3. $\forall\, n \geq 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa$.

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$(2.4) \quad n \geq 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|.$$

$$(2.5) \quad \forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta$$
$$\Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa \to 0} \kappa^n = 0.$$

$$(2.6) \qquad \lim_{\kappa \to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa \to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa. \qquad \square$$

3. Volume

DEFINITION 3.1. A countable n-volume is the number of ordered combinations (n-tuples), v_c, of the members of n number of disjoint, countable domain sets, x_i:

$$(3.1) \quad \exists\, n \in \mathbb{N},\ v_c \in \{0, \mathbb{N}\},\ x_i \in \{x_1, \cdots, x_n\} : \ \bigcap_{i=1}^{n} x_i = \emptyset \ \wedge \ v_c = \prod_{i=1}^{n} |x_i|.$$

THEOREM 3.2. *Euclidean volume,*

$$(3.2) \quad \forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\ [v_a, v_b] \subset \mathbb{R},\ s_i = b_i - a_i,\ v = v_b - v_a :$$
$$v_c = \prod_{i=1}^{n} |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^{n} s_i.$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.

$$(3.3) \quad v_c = \prod_{i=1}^{n} |x_i| \Leftrightarrow v_c \kappa = (\prod_{i=1}^{n} |x_i|)\kappa \Leftrightarrow \lim_{\kappa \to 0} v_c \kappa = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|)\kappa.$$

Apply the ruler (2.1) and ruler convergence (2.2) to equation 3.3:

$$(3.4) \quad \exists\, v, \kappa \in \mathbb{R} : v_c = floor(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa \to 0} v_c \kappa = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|)\kappa.$$

Apply lemma 2.3 to equation 3.4:

$$(3.5) \quad v = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|)\kappa = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|)\kappa^n = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|\kappa).$$

Apply the ruler (2.1) and ruler convergence (2.2) to s_i:

$$(3.6) \quad \exists\, s_i, \kappa \in \mathbb{R} : floor(s_i/\kappa) = |x_i| \quad \Rightarrow \quad \lim_{\kappa \to 0} (|x_i|\kappa) = s_i.$$

$$(3.7) \quad v = \lim_{\kappa \to 0} (\prod_{i=1}^{n} |x_i|\kappa) \ \wedge \ \lim_{\kappa \to 0} (|x_i|\kappa) = s_i \ \Rightarrow \ v = \prod_{i=1}^{n} s_i \qquad \square$$

4. Distance

DEFINITION 4.1. Countable distance, $d_c : d_c \in \mathbb{Z}_{\geq 0},\ f(d_c, n) = v_c \Leftrightarrow f^{-1}(v_c, n) = d_c$, where v_c a countable volume (3.1) and $d_c = |x_1| = \cdots = |x_n|$.

4.1. Minkowski distance (ρ-norm).

THEOREM 4.2. *Minkowski distance (ρ-norm):*

$$v_c = \sum_{j=1}^{m} v_{c_i} \quad \Rightarrow \quad \exists\, d, d_i \in \mathbb{R} : \; d^n = \sum_{i=1}^{m} d_i^n.$$

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

PROOF. Apply the countable distance definition (4.1) to the assumption:

$$(4.1) \quad v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n \quad \wedge \quad v_{c_i} = \prod_{j=1}^{n} |x_{i_j}| = \prod_{i=1}^{n} d_{c_i} = d_{c_i}^n$$
$$\wedge \quad v_c = \sum_{j=1}^{m} v_{c_i} \quad \Rightarrow \quad d_c^n = \sum_{j=1}^{m} d_{c_i}^n.$$

Multiply both sides of equation 4.1 by κ and take the limit:

$$(4.2) \qquad d_c^n = \sum_{j=1}^{m} d_{c_i}^n \quad \Leftrightarrow \quad \lim_{\kappa \to 0} d_c^n \kappa = \lim_{\kappa \to 0} \sum_{j=1}^{m} d_{c_i}^n \kappa.$$

Apply lemma 2.3 to equation 4.1:

$$(4.3) \quad \lim_{\kappa \to 0} d_c^n \kappa = \lim_{\kappa \to 0} \sum_{j=1}^{m} d_{c_i}^n \kappa \quad \wedge \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa$$
$$\Leftrightarrow \lim_{\kappa \to 0} d_c^n \kappa^n = \lim_{\kappa \to 0} \sum_{j=1}^{m} d_{c_i}^n \kappa^n \Leftrightarrow \lim_{\kappa \to 0} (d_c \kappa)^n = \lim_{\kappa \to 0} \sum_{j=1}^{m} (d_{c_i} \kappa)^n.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to equation 4.3:

$$(4.4) \quad \exists\, d, d_i : \; d_c = floor(d/\kappa), \; d = \lim_{\kappa \to 0} d_c \kappa$$
$$\wedge \quad d_{c_i} = floor(d_i/\kappa), \; d_i = \lim_{\kappa \to 0} d_{c_i} \kappa \quad \Rightarrow$$
$$d^n = \lim_{\kappa \to 0} (d_c \kappa)^n = \lim_{\kappa \to 0} \sum_{j=1}^{m} (d_{c_i} \kappa)^n = \sum_{j=1}^{m} d_i^n. \quad \square$$

4.2. Distance inequality.
The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N}, \; v_a, v_b \geq 0 : \; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$(4.5) \quad \forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq v_a + v_b +$$
$$\sum_{i=1}^{n} \binom{n}{k}(v_a^{1/n})^{n-k}(v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k}(v_a^{1/n})^k(v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n.$$

Take the nth of both sides of the inequality 4.5:

$$(4.6) \; \forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad \square$$

4.3. Distance sum inequality.
The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N}, \; a_i, b_i \geq 0 : \; \left(\sum_{i=1}^{m}(a_i^n + b_i^n)\right)^{1/n} \leq \left(\sum_{i=1}^{m} a_i^n\right)^{1/n} + \left(\sum_{i=1}^{m} b_i^n\right)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$(4.7) \quad \forall\, m, n \in \mathbb{N}, \; v_a, v_b \geq 0 : \quad v_a = \sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge$$
$$(v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad \left(\left(\sum_{i=1}^{m} a_i^n\right) + \left(\sum_{i=1}^{m} b_i^n\right)\right)^{1/n} =$$
$$\left(\sum_{i=1}^{m}(a_i^n + b_i^n)\right)^{1/n} \leq \left(\sum_{i=1}^{m} a_i^n\right)^{1/n} + \left(\sum_{i=1}^{m} b_i^n\right)^{1/n}. \quad \square$$

4.4. Metric Space. All Minkowski distances (ρ-norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k:$

$$(4.8) \qquad (u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}.$$

Apply the distance inequality (4.3) to the inequality 4.8:

$$(4.9) \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \;\wedge\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}$$
$$\wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p$$
$$\Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p}$$
$$\Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq$$
$$(u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \quad \square$$

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$(4.10) \quad \forall\, p : p \geq 1, \quad d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p}$$
$$\Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \quad \square$$

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$(4.11) \qquad \forall\, [a_1, b_1], [a_2, b_2], \quad u = b_1 - a_1, \, v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0, \, v \geq 0.$$

$$(4.12) \qquad p \geq 1, \; u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \qquad \square$$

THEOREM 4.8. *Identity of Indiscernibles:* $d(u, u) = 0.$

PROOF. From the non-negativity property (4.7):

$$(4.13) \quad d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0$$
$$\Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0.$$

$$(4.14) \qquad\qquad d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v.$$

$$(4.15) \qquad\qquad d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \qquad \square$$

4.5. The properties limiting a set to at most 3 members.

DEFINITION 4.9. Totally ordered set:

$$\forall\, i\, n \in \mathbb{N},\ i \in [1, n-1],\ \forall\, x_i \in \{x_1, \ldots, x_n\},$$
$$successor\ x_i = x_{i+1}\ \wedge\ predecessor\ x_{i+1} = x_i.$$

DEFINITION 4.10. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i,\, j,\, n \in \mathbb{N},\ \forall\, x_i,\, x_j \in \{x_1, \ldots, x_n\},\ successor\ x_i = x_j \Leftrightarrow predecessor\ x_j = x_i.$$

THEOREM 4.11. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n\ \wedge\ j = 1\ \Rightarrow\ successor\ x_n = x_1\ \wedge\ predecessor\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (4.9) assigns a unique label to each set member and assigns unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (4.10):

(4.16) $i = n\ \wedge\ j = 1\ \wedge\ successor\ x_i = x_j\ \Rightarrow\ successor\ x_n = x_1.$

Applying the definition of the symmetry property (4.10) to conclusion 4.16:

(4.17) $successor\ x_i = x_j\ \Rightarrow\ predecessor\ x_j = x_i\ \Rightarrow\ predecessor\ x_1 = x_n.$ □

THEOREM 4.12. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

Lemmas: `adj111`, `adj122`, `adj212`, `adj123`, `adj133`, `adj233`, `adj213`, `adj313`, `adj323`, and `not_all_mutually_adjacent_gt_3`.

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (4.11).

DEFINITION 4.13. (Cyclic) Successor of m is n:

(4.18) $Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize).$

DEFINITION 4.14. (Cyclic) Predecessor of m is n:

(4.19) $Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1).$

DEFINITION 4.15. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:
(4.20)
$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize).$

Every member is adjacent to every other member, where $setsize \in \{1, 2, 3\}$:

(4.21) $Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (m = setsize \wedge n = 1).$

(4.22) $Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (n = m + 1 \leq setsize).$

(4.23) $Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (n = setsize \wedge m = 1).$

(4.24) $Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m + 1 \leq setsize).$

(4.25) $Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m - 1 \geq 1).$

(4.26) $Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \wedge m = 1).$

(4.27) $Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \wedge n = setsize).$

(4.28) $Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m + 1 \leq setsize).$

(4.29) $Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m - 1 \geq 1).$

Member 2 is the only successor of member 1 for all $setsize \geq 3$, which implies member 3 is not ($\neg$) a successor of member 1 for all $setsize \geq 3$:

(4.30) $\neg Successor(1, 3, setsize \geq 3)$

$$\leftarrow Successor(1, 2, setsize \geq 3) \leftarrow (n = m + 1 \leq setsize).$$

Member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not ($\neg$) a predecessor of member 1 for all $setsize > 3$:

(4.31) $\neg Predecessor(1, 3, setsize \geq 3)$

$$\leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).$$

For all $setsize \geq 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not symmetric):

(4.32) $\neg Adjacent(1, 3, setsize > 3)$

$$\leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square$$

5. Applications to physics

5.1. Spacetime equations. From the volume proof (3.2), two disjoint distance intervals, $[0, r]$ and $[0, r']$, define a 2-volume. From the Minkowski distance proof (4.2), the distance interval lengths, r and r', are inverse functions of two 2-volumes, v and v' having the sizes, $v = r^2$ and $v' = r'^2$. And if r and r' are dependent, then $\exists\, r_\nu : r^2 = r'^2 + r_\nu^2$ or $r'^2 = r^2 + r_\nu^2$. For traditional notation, the $r^2 = r'^2 + r_\nu^2$ case is chosen. And from the 3D proof (4.12), for some non-distance type, τ ($\tau \in \{time(t),\ mass(m),\ charge(q), \cdots\}$) and conversion ratios, μ and ν:

(5.1) $\mu \in \{r_c/t_c,\ r_c/m_c,\ r_c/q_c\} : r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q.$

(5.2) $\exists\, \mu,\ \nu \in \mathbb{R} : r = \mu\tau \quad \wedge \quad r_\nu = \nu\tau \quad \wedge \quad \exists\, r_\nu \in \mathbb{R} : r^2 = r'^2 + r_\nu^2$

$$\Rightarrow \quad (\mu\tau)^2 = r'^2 + (\nu\tau)^2 \quad \Rightarrow \quad r' = \sqrt{(\mu\tau)^2 - (\nu\tau)^2} = \mu\tau\sqrt{1 - (\nu/\mu)^2}.$$

Local (proper) distance, r', contracts relative to coordinate distance, r, as $\nu \to \mu$:

(5.3) $r' = \mu\tau\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad \mu\tau = r \quad \Rightarrow \quad r' = r\sqrt{1 - (\nu/\mu)^2}.$

Coordinate length, τ, dilates relative to local length, τ', as $\nu \to \mu$:

(5.4) $\mu\tau = r'/\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad r' = \mu\tau' \quad \Rightarrow \quad \tau = \tau'/\sqrt{1 - (\nu/\mu)^2}.$

Where τ is time, one form of the flat Minkowski spacetime event interval is:

(5.5) $\quad dr^2 = dr'^2 + dr_\nu^2 \quad \wedge \quad dr_\nu^2 = dx_1^2 + dx_2^2 + dx_3^2 \quad \wedge \quad d(\mu\tau) = dr$
$$\Rightarrow \quad dr'^2 = d(\mu\tau)^2 - dx_1^2 - dx_2^2 - dx_3^2.$$

5.2. Newton's gravity force and the constant, G. From equation 5.1:

(5.6) $\quad \forall\, m_1, m_2, m, r \in \mathbb{R} : \; m_1 m_2 = m^2 \quad \wedge \quad m = (m_c/r_c)r$
$$\Rightarrow \quad m_1 m_2 = ((m_c/r_c)r)^2 \quad \Rightarrow \quad (r_c/m_c)^2 m_1 m_2/r^2 = 1.$$

(5.7) $\quad r = (r_c/t_c)t = ct \;\Rightarrow\; mr = (m_c/r_c)(ct)^2 \;\Rightarrow\; ((r_c/m_c)/c^2)mr/t^2 = 1.$

(5.8) $\quad ((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/m_c)^2 m_1 m_2/r^2 = 1$
$$\Rightarrow \quad F := mr/t^2 = ((r_c/m_c)c^2)m_1 m_2/r^2 = Gm_1 m_2/r^2,$$

where Newton's constant, $G = (r_c/m_c)c^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$.

5.3. Coulomb's charge force and constant, k_e. From equation 5.1:

(5.9) $\quad \forall\, q_1, q_2, q, r \in \mathbb{R} : \; q_1 q_2 = q^2 \quad \wedge \quad q = (q_c/r_c)r$
$$\Rightarrow \quad q_1 q_2 = ((q_c/r_c)r)^2 \quad \Rightarrow \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1.$$

(5.10) $\quad r = (r_c/t_c)t = ct \;\Rightarrow\; mr = (m_c/r_c)(ct)^2 \;\Rightarrow\; ((r_c/m_c)/c^2)mr/t^2 = 1.$

(5.11) $\quad ((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1$
$$\Rightarrow \quad F := mr/t^2 = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2.$$

(5.12) $\quad r_c/t_c = c \quad \wedge \quad F = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2$
$$\Rightarrow \quad F = (m_c(r_c/t_c^2))(r_c/q_c)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2,$$

where Coulomb's constant, $k_e = (m_c(r_c/t_c^2))(r_c/q_c)^2$, conforms to the SI units: $N \cdot m^2 \cdot C^{-2}$.

5.4. 3 fundamental constants. c_t, c_m, and c_q.

(5.13) $$c_t = r_c/t_c \approx 2.99792458 \cdot 10^8 m\ s^{-1}.$$

(5.14) $\quad G = (r_c/m_c)c_t^2 \quad \Rightarrow \quad c_m = r_c/m_c \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}.$

(5.15) $\quad k_e = ((m_c/r_c)c_t^2)(r_c/q_c)^2 \Rightarrow \quad c_q = r_c/q_c \approx 8.6175172023 \cdot 10^{-18} m\ C^{-1}.$

5.5. 3 work constants. The ratios c_t/c_m and c_t/c_q yields 3 inverse proportion (work) constants, k_t, k_m, and k_q that are the basis of particle-wave behavior:

(5.16) $\quad r/t = r_c/t_c, \quad r/m = r_c/m_c \quad \Leftrightarrow \quad (r/t)/(r/m) = (r_c/t_c)/(r_c/m_c) \quad \Leftrightarrow$
$$(mr)/(tr) = (m_c r_c)/(t_c r_c) \quad \Leftrightarrow \quad mr = m_c r_c = k_m, \quad tr = t_c r_c = k_t.$$

(5.17) $\quad r/t = r_c/t_c, \quad r/q = r_c/q_c \quad \Leftrightarrow \quad (r/t)/(r/q) = (r_c/t_c)/(r_c/q_c) \quad \Leftrightarrow$
$$(qr)/(tr) = (q_c r_c)/(t_c r_c) \quad \Leftrightarrow \quad qr = q_c r_c = k_q, \quad tr = t_c r_c = k_t.$$

5.6. Planck-Einstein equation: Applying both the direct proportion ratio (5.1), $r/t = r_c/t_c = c$, and the work ratio (5.16), $mr = m_c r_c = k_m$:

(5.18) $m(ct)^2 = mr^2 \ \wedge \ m = m_c r_c/r = k_m/r \quad \Rightarrow \quad m(ct)^2 = (k_m/r)r^2 = k_m r.$

(5.19) $m(ct)^2 = k_m r \quad \wedge \quad r/t = r_c/t_c = c$
$$\Rightarrow \quad E := mc^2 = k_m r/t^2 = (k_m(r/t))\,(1/t) = (k_m c)(1/t) = hf,$$

where the Planck constant, $h = k_m c$, and the frequency, $f = 1/t$.

5.7. Compton wavelength, r: From equations 5.16 and 5.19:

(5.20) $$mr = k_m \quad \Rightarrow \quad r = k_m/m = k_m c/mc = h/mc.$$

5.8. de Broglie wavelength, r: From equations 5.2 and 5.20:

(5.21) $v = r'/t = c\sqrt{1 - (v/c)^2} \quad \wedge \quad r = h/mc \quad \Rightarrow \quad r = (h/mv)\sqrt{1 - (v/c)^2}.$

5.9. Work constant values:

(5.22) $$k_m = m_c r_c = h/c \approx 2.21022 \cdot 10^{-42} \ kg \ m.$$

(5.23) $$k_t = t_c r_c = k_m/(c_t/c_m) \approx 5.47493 \cdot 10^{-78} \ s \ m.$$

(5.24) $$k_q = q_c r_c = (c_t/c_q)k_t \approx 1.90466 \cdot 10^{-52} \ C \ m.$$

5.10. Quantum-special relativity extensions to Newton's gravity force. The total mass of a particle is $m = \sqrt{m_0^2 + m_{ke}^2}$, where m_0 is the rest mass and m_{ke} is the kinetic energy-equivalent mass. Applying both the direct proportion ratios (5.1) and the work ratios (5.16):

(5.25) $m_0 = (m_c/r_c)r \quad \wedge \quad m_{ke} = m_c r_c/r \quad \wedge \quad m = \sqrt{m_0^2 + m_{ke}^2}$
$$\Rightarrow \quad m = \sqrt{((m_c/r_c)r)^2 + ((m_c r_c)/r)^2}.$$

(5.26) $\exists \, m : m_1 m_2 = m^2 = ((m_c/r_c)r)^2 + ((m_c r_c)/r)^2$
$$\Rightarrow \quad m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2) = 1.$$

Newton's gravity force in the local frame of reference comes from the spacetime equation, 5.2:

(5.27) $r' = \sqrt{(ct)^2 - (vt)^2} \quad \Rightarrow \quad m_0 r' = (m_c/r_c)((ct)^2 - (vt)^2).$

(5.28) $m_0 r' = (m_c/r_c)((ct)^2 - (vt)^2) \quad \Rightarrow \quad ((r_c/m_c)/(c^2 - v^2))m_0 r'/t^2 = 1.$

(5.29) $((r_c/m_c)/(c^2 - v^2))m_0 r'/t^2 = 1$
$$\wedge \quad m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2) = 1$$
$$\Rightarrow \quad F := m_0 r'/t^2 = ((m_c/r_c)(c^2 - v^2))m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2).$$

5.11. Quantum-special relativity extensions to Coulomb's charge force.

(5.30) $$F = ((m_c/r_c)(c^2 - v^2))q_1 q_2/(((q_c/r_c)r)^2 + ((q_c r_c)/r)^2).$$

6. Insights and implications

(1) Deriving volume and distance from the same abstract, countable set of n-tuples provides a single, unifying set and limit-based foundation under Euclidean geometry without relying on the geometric primitives and relations in Euclidean geometry [**Joy98**], axiomatic geometry [**Lee10**], and vector analysis [**Wey52**].

(2) The definition of a complete metric space [**Rud76**] ignores the intimate relation between distance and volume. A more sufficient definition is: a distance measure is the inverse (bijective) function of volume (4.1).

(3) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [**Joy98**]. And proofs that the straight line length is the smallest distance equate the straight line length to Euclidean distance.

Without using the notion of a straight line: Euclidean volume was derived from a set of n-tuples (3.2). And all distance measures (inverse functions of n-volumes) derived from Euclidean 2-volumes (areas) are Minkowski distances (4.2), where $n \in \{1, 2\}$: $n = 1$ is the Manhattan (largest) distance case, $d = \sum_{i=1}^{m} s_i$. $n = 2$ is the Euclidean (smallest) distance case, $d = (\sum_{i=1}^{m} s_i^2)^{1/2}$. Hilbert spaces allow non-integer values of n (fractals). In that case, $1 \leq n \leq 2$ and d decreases monotonically as n goes from 1 to 2.

(4) The left side of the distance sum inequality (4.4),

$$(6.1) \qquad (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n},$$

differs from the left side of Minkowski's sum inequality [**Min53**]:

$$(6.2) \qquad (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

The two inequalities are only the same where $n = 1$. The distance sum inequality is a more fundamental inequality because its proof does not require the convexity and Hölder's inequality assumptions required to prove the Minkowski sum inequality. And the distance sum inequality is derived from volume and distance, which makes it directly related to geometry.

(5) The derivations of the spacetime equations, in this article (5.1), differ from other derivations:

  (a) The derivations, here, do not rely on the Lorentz transformations or Einsteins' postulates [**Ein15**]. The derivations do not even require the notion of light.

  (b) The derivations, here, rely only on the Euclidean volume proof (3.2), the Minkowski distances proof (4.1), and the 3D proof (4.12), which provides the insight that the properties of physical space creates a maximum speed and the spacetime equations. For example, from the spacetime equation 5.1, the unit-factoring ratios $\mu, \nu : (\mu\tau)^2 = r'^2 + (\nu\tau)^2 \Rightarrow \mu \geq \nu$. Therefore, $\forall \mu : \mu \in \{r_c/t_c, \ r_c/m_c, \ r_c/q_c\}$, r_c/t_c is the always maximum possible speed.

  (c) The same derivations are also valid for spacemass and spacecharge.

(6) The flat spacetime interval equation was derived from a 2-dimensional equation (5.5), which is generalized to: $dr'^2 = \alpha_1 d(\mu\tau)^2 - dr_\nu^2$, where

$dr_\nu^2 = \alpha_2 dx_1^2 + \alpha_3 dx_2^2 + \alpha_4 dx_3^2$. The 4×4 metric tensor, $g_{\mu,\nu} = diag(\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4)$, in Einstein's field equations [**Ein15**], can be simplified to a 2×2 metric tensor, $g_{i,j} = diag(\alpha_1, -1)$. The 2×2 metric tensor allows using a 2-dimensional Gaussian curvature, which is much simpler to calculate than the 4-dimensional Ricci curvature. And the 2×2 tensors reduce the number of independent equations to solve.

(7) The constant, direct proportion ratios, $r_c/t_c = c_t$, $r_c/m_c = c_m$, $(r_c/q_c) = c_q$, (5.4) were used to derive the constant, inverse proportion (work) ratios, $t_c r_c = k_t$, $m_c r_c = k_m$, and $q_c r_c = k_q$ (5.5). But k_t, k_m, and k_q also derive c_t, c_m, and c_q (bi-directional implication at each derivation step).

  (a) The gravity, G (5.8), charge k_e (5.12), and Planck h (5.19) constants were all derived from the constant ratios. And all depend on the speed of light ratio, c_t: $G = c_m c_t^2$, $k_e = (c_q^2/c_m)c_t^2$, and $h = k_m c_t$.

  (b) The ratios used to derive k_e (5.12) do not contain the value, 4π, which indicates the current "standard" definitions of permitivity, ε_0 and permeability, μ_0, where $k_e = 1/4\pi\varepsilon_0$ and $k_e = \mu_0 c^2/4\pi$, are **incorrect**.

  (c) Multiplying and dividing direct and inverse proportion ratios create the particle-wave equations: Planck-Einstein (5.6), Compton wavelength (5.20), and de Broglie wavelength (5.21).

  (d) $k_e = ((m_c/r_c)c^2)(r_c/q_c)^2 = (m_c(r_c/t_c^2))(r_c/q_c)^2$, contains the term, r_c/t_c^2, which suggests a maximum acceleration constant.

(8) Applying the constant ratios to derive Newton's gravity force (5.2) and Coulomb's charge force (5.3) equations provide:

  (a) Derivations that do not assume the inverse square law or Gauss's flux divergence theorem as a cause of the inverse square law. **Note:** In Einstein's field equations, the components of the Ricci and metric tensors have the units, $1/distance^2$, Einstein's constant, $k = (4\pi G)2/c^4$, and the mass density component of the energy-stress tensor [**Wey52**] are assumptions of the inverse square law and Gauss's flux divergence.

  (b) The first derivations to show that the inverse square law and the property of force as mass times acceleration are the result of the constant ratios, $r = (r_c/t_c)t = (r_c/m_c)m$.

(9) The quantum-special relativity extensions to Newton's gravity force equation (5.28) and Coulomb's charge force equation (5.30) make quantifiable predictions.

  (a) In Newton's gravity force, Gauss's gravity law, Einstein's field equations, and Coulomb's charge force equations the force, $F \to \infty$ as the distance, $r \to 0$, which forces the hypothesizing of other forces to keep atoms from collapsing into point-like singularities where the gravitation and charge forces would be infinite. But, the quantum components of the gravity and charge equations cause $F \to 0$ as $r \to 0$. The gravity force between two point-like particles starts to decrease where the distance is less than approximately $6.5 \cdot 10^{-8}$ m. And the charge force starts to decrease at distances less than approximately $6 \cdot 10^{-18}$ m. If the quantum effects exists, then:

    (i) There are no point-like singularities in black holes.

(ii) The finite gravity and charge wells allow radioactivity and quantum tunneling.

(b) Newton's gravity constant, G, Gauss's gravity constant, $4\pi G$, and Einstein's gravity constant, $k = 8\pi G/c^4$, [**Wey52**], are only valid where the local velocity, $v = 0$. The special relativity component replaces G with "$(m_c/r_c)(c^2 - v^2) = c_m(c_t^2 - v^2)$" in the local frame of reference.

For example, in the local frame of reference, an observer on a star orbiting a galaxy at relativistic speeds will measure a lower gravitational force, $F = m_0 r'/t^2$, toward the galaxy center than predicted by a constant G. But an observer on earth would measure a higher gravitational force, $F = (m_0 r'/t^2)/(c^2 - v^2)$, toward the galaxy center of that same star orbiting a distant galaxy than predicted by G.

(10) There is no constant ratio mapping a discrete value to a continuously varying value. Therefore, the discrete spin states of two quantum entangled particles and the polarization states of two quantum entangled photons are independent of continuously varying distance and time interval lengths.

(11) The set-based, first-order logic proof that a strict linearly ordered and symmetric set is a cyclic set of at most 3 members (4.12) is the simplest and most logically rigorous explanation for observing only 3 dimensions of physical space, less contrived and more rigorous than: parallel dimensions that cannot be detected or extra dimensions rolled up into infinitesimal balls that are too small to detect.

(a) Higher order dimensions must have different types (members of different sets), for example, dimensions of time, mass, and charge.

(b) Each of 3 ordered and symmetric dimensions of space can have at most 3 sequentially ordered and symmetric state values, for example, an ordered and symmetric set of 3 vector orientations, $\{-1, 0, 1\}$, per dimension of space and at most 3 spin states per plane, etc.

If the states are not sequentially ordered (a bag of states), then a state value is undetermined until observed (like Schrodinger's cat being both alive and dead until the box is opened).

References

[Coq23] Coq, *Coq proof assistant*, 2023. https://coq.inria.fr/documentation. ↑2

[Ein15] A. Einstein, *Relativity, the special and general theory*, Princeton University Press, 2015. ↑10, 11

[Gol76] R. R. Goldberg, *Methods of real analysis*, John Wiley and Sons, 1976. ↑1, 2

[Joy98] D. E. Joyce, *Euclid's elements*, 1998. http://aleph0.clarku.edu/~djoyce/java/elements/elements.html. ↑10

[Lee10] J. M. Lee, *Axiomatic geometry*, American Mathematical Society, 2010. ↑10

[Min53] H. Minkowski, *Geometrie der zahlen*, Chelsea, 1953. reprint. ↑10

[Rud76] W. Rudin, *Principles of mathematical analysis*, McGraw Hill Education, 1976. ↑1, 2, 10

[Wey52] H. Weyl, *Space-time-matter*, Dover Publications Inc, 1952. ↑10, 11, 12

GEORGE VAN TREECK, 668 WESTLINE DR., ALAMEDA, CA 94501