The Set Properties Generating Geometry and Physics

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ABSTRACT. Volume and the Minkowski distances/Lp norms (e.g., Euclidean distance) are derived from a set and limit-based foundation without referencing the primitives of geometry. Sequencing a linearly ordered set in all n-at-time permutations via successor/predecessor relations is a cyclic set limiting n to at most 3, for example, 3 dimensions of physical space. Therefore, all other interval lengths have different types that can only be related to a 3-dimensional distance interval length via conversion ratios. The ratios and geometry proofs provide simpler derivations of the spacetime, Newton's gravity, Coulomb's charge force, and Einstein-Planck equations and exposes the ratios composing their corresponding constants. All proofs are verified in Coq.

CONTENTS

1. Introduction

Mathematical (real) analysis can construct differential calculus from a set and limit-based foundation without the need to reference the primitives of Euclidean geometry, like straight line, angle, slope, etc. But the Riemann and Lebesgue integrals and measure theory (for example, Hilbert spaces and the Lebesgue measure) define Euclidean volume as the product of interval lengths. And the vector norm and metric space use Euclidean distance and its properties as definitions [**Gol76**] [**Rud76**]. Here, volume and distance are derived from a simple set and limit-based foundation without the hand-waving references to side, angle, triangle, rectangle, etc. for justification.

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A Cartesian-based definition of volume is the cardinal, v_c, of a set of n-tuples, $v_c = \prod_{i=1}^{n} |x_i|$, where $|x_i|$ is the cardinal of the countable set, x_i. Deriving Euclidean volume, $v = \prod_{i=1}^{n} s_i$, from $v_c = \prod_{i=1}^{n} |x_i|$, where $|x_i|$ is the number of partitions of the interval, $[a_i, b_i] \subset \mathbb{R}$, and $s_i = b_i - a_i$, requires avoiding the circular logic of summing volumes, where each n-tuple corresponds to a range interval having a size equal to the product of the partition sizes. One generalization to include non-Euclidean volumes is: $v_c = f_c(\prod_{i=1}^{n} |x_i|)$, where $v = f(\prod_{i=1}^{n} s_i)$.

And an n-volume is the sum of disjoint n-volumes. Therefore, all volumes are of the form: $v = f(\prod_{i=1}^{n} s_i) = \sum_{i=1}^{m} v_i = \sum_{i=1}^{m} f_i(\prod_{j=1}^{n} s_{i,j})$. $\exists\, d, d_i \in \mathbb{R} :\ d^n = \prod_{i=1}^{n} s_i$ and $d_i = \prod_{j=1}^{n} s_{i,j} \Rightarrow \exists\ f\ :\ d = (f^{-1}(v))^{1/n} = (f^{-1}(\sum_{i=1}^{m} f_i(d_i^n)))^{1/n}$, where the quasi-metric, d, is herein defined to be a "geometric" distance. Proving that Euclidean volume and the L_p norms (Minkowski distances), $d = (\sum_{i=1}^{m} d_i^n)^{1/n}$, are derived from the same definition of countable n-volume provides a set and limit-based foundation under both volume and distance.

It is commonly assumed that the dimensions of physical space can be sequenced in any n-at-a-time order. But *deterministic* sequencing of a set from 1 to n (for example, a set of n number domain intervals or dimensions) implies a strict linear order that set theory defines in terms of successor and predecessor functions. Sequencing a strict linear order in all n-at-a-time orders (permutations) requires the "symmetry" property of every set member being sequentially adjacent (either a successor or predecessor) to every other set member. It will be proved that a "symmetric" linearly ordered set is a cyclic set, where n $\leq$ 3.

Therefore, an interval length that is not in a cyclic set of 3 "distance" interval lengths has a different type (member of a different set) that can be related to a 3-dimensional distance via unit-factoring, conversion ratios (linear transformations). The ratios combined with the geometry proofs provide simpler derivations of the Lorentz, spacetime, Newton's gravity, Coulomb's charge force, and Einstein-Planck equations and exposes the ratios that compose the gravity, charge, and Planck constants. Impacts on Einstein's field (general relativity) equations are also shown.

All the proofs in this article are trivial. But to ensure confidence, all the proofs have been verified using using the Coq proof verification system [**Coq23**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

Derivatives and integrals use a 1-1 correspondence between the infinitesimals of each interval, where the size of the infinitesimals in each interval are proportionate to the size of the interval, which precludes using derivatives and integrals to directly express many-to-one, one-to-many, and many-to-many (Cartesian product) mappings between same-sized, size κ, infinitesimals in different-sized intervals. Further, using tools that define Euclidean volume and distance precludes using those tools to derive Euclidean volume and distance.

Therefore, a different tool is used here. A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals (infinitesimals), where each infinitesimal has the *same* size, κ. The ruler is both an inner and outer measure of an interval.

Definition 2.1. Ruler measure, M: $\forall\, [a,b] \subset \mathbb{R},\ \ s = b - a\ \ \wedge\ \ \kappa > 0\ \ \wedge$ $(p = floor(s/\kappa)\ \ \vee\ \ p = ceiling(s/\kappa))\ \ \wedge\ \ M = \sum_{i=1}^{p} \kappa = p\kappa$.

THEOREM 2.2. *Ruler convergence:* $M = \lim_{\kappa \to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : y \leq x, y \in \mathbb{Z}, x \in \mathbb{R}\})$:

$$\forall \kappa > 0, p = floor(s/\kappa) \wedge 0 \leq |floor(s/\kappa) - s/\kappa| < 1 \Rightarrow |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by κ:

$$\forall \kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\begin{aligned} &\forall \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ &\Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \delta = \epsilon \quad := \quad M = \lim_{\kappa \to 0} p\kappa = s. \end{aligned} \tag{2.3}$$ $\square$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, ..., \pi_{\lim_{\kappa \to 0}}$.

LEMMA 2.3. $\forall n \geq 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa$.

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\begin{aligned} &\forall \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta \\ &\Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa \to 0} \kappa^n = 0. \end{aligned} \tag{2.5}$$

$$\lim_{\kappa \to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa \to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa. \tag{2.6}$$ $\square$

3. Volume

DEFINITION 3.1. Countable volume size, v_c, is the number of ordered combinations (n-tuples) of the members of n number of disjoint, countable domain sets, x_i:

$$\exists n \in \mathbb{N}, v_c, \in \{0, \mathbb{N}\}, x_i \in \{x_1, \cdots, x_n\}, \bigcap_{i=1}^n x_i = \emptyset : \quad v_c = f_c(\prod_{i=1}^n |x_i|). \tag{3.1}$$

THEOREM 3.2. *Euclidean volume size,* $v = \prod_{i=1}^n s_i$, *is the equality case of countable volume size,* $v_c = \prod_{i=1}^n |x_i|$, *where each countable set,* x_i, *is the set of partitions of an interval,* $[a_i, b_i] \subset \mathbb{R}$.

$$\begin{aligned} &\forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\}, \ [v_a, v_b] \subset \mathbb{R}, \ s_i = b_i - a_i, \ v = v_a - v_b, \\ &\qquad v_c = \prod_{i=1}^n |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^n s_i. \end{aligned} \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing $|x_i|$ number of size κ partitions and apply ruler convergence (2.2):

$$\forall\ i\ n \in \mathbb{N}, \ i \in [1, n], \ \kappa > 0 \ \wedge \ floor(s_i/\kappa) = |x_i| \ \Rightarrow \ s_i = \lim_{\kappa \to 0}(|x_i| \cdot \kappa). \tag{3.3}$$

$$s_i = \lim_{\kappa \to 0}(|x_i| \cdot \kappa) \quad \Leftrightarrow \quad \prod_{i=1}^n s_i = \prod_{i=1}^n \lim_{\kappa \to 0}(|x_i| \cdot \kappa). \tag{3.4}$$

$$\prod_{i=1}^n s_i = \prod_{i=1}^n \lim_{\kappa \to 0}(|x_i| \cdot \kappa) \quad \Leftrightarrow \quad \prod_{i=1}^n s_i = \lim_{\kappa \to 0}(\prod_{i=1}^n |x_i|) \cdot \kappa^n. \tag{3.5}$$

Apply lemma 2.3 to equation 3.5:

$$\prod_{i=1}^{n} s_i = \lim_{\kappa\to 0}(\prod_{i=1}^{n}|x_i|)\cdot\kappa^n \quad \wedge \quad \lim_{\kappa\to 0}\kappa^n = \lim_{\kappa\to 0}\kappa \quad \Rightarrow \quad \prod_{i=1}^{n} s_i = \lim_{\kappa\to 0}(\prod_{i=1}^{n}|x_i|)\cdot\kappa. \tag{3.6}$$

Apply the ruler (2.1) and ruler convergence (2.2) to v:

$$\exists\, v\in\mathbb{R}:\ v_c = floor(v/\kappa) \quad \Leftrightarrow \quad v = \lim_{\kappa\to 0} v_c\cdot\kappa. \tag{3.7}$$

Multiply both sides of the countable volume equation 3.1 by κ:

$$v_c = \prod_{i=1}^{n}|x_i| \quad \Leftrightarrow \quad v_c\cdot\kappa = (\prod_{i=1}^{n}|x_i|)\cdot\kappa \tag{3.8}$$

$$v_c\cdot\kappa = (\prod_{i=1}^{n}|x_i|)\cdot\kappa \quad \Leftrightarrow \quad \lim_{\kappa\to 0} v_c\cdot\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^{n}|x_i|)\cdot\kappa. \tag{3.9}$$

Combine equations 3.7, 3.9, and 3.6:

$$v = \lim_{\kappa\to 0} v_c\cdot\kappa \quad \wedge \quad \lim_{\kappa\to 0} v_c\cdot\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^{n}|x_i|)\cdot\kappa \quad \wedge \quad \lim_{\kappa\to 0}(\prod_{i=1}^{n}|x_i|)\cdot\kappa = \prod_{i=1}^{n} s_i \quad \Rightarrow \quad v = \prod_{i=1}^{n} s_i. \tag{3.10}$$ □

4. Distance

4.1. Geometric distance.

THEOREM 4.1. *Geometric distance, d:*
$d = (f^{-1}(\sum_{i=1}^{m} f_i(d_i^n)))^{1/n} = (f^{-1}(\sum_{i=1}^{m} f_i(d_i^n)))^{1/n}$.

PROOF. From the Euclidean volume proof (3.2):

$$\forall\, v_E = \prod_{i=1}^{n} s_i \ \Rightarrow\ \exists\, v, f:\ v = f(\prod_{i=1}^{n} s_i) = \sum_{i=1}^{m} v_i = \sum_{i=1}^{m} f_i(\prod_{j=1}^{n} s_{i,j}). \tag{4.1}$$

$$\exists\, d, d_i\in\mathbb{R}:\ d^n = \prod_{i=1}^{n} s_i \ \wedge\ v_i^n = \prod_{j=1}^{n} s_{i,j} \ \Rightarrow\ v = f(d^n) = \sum_{i=1}^{m} f_i(d_i^n). \tag{4.2}$$

$$\Rightarrow \quad \exists\, f:\ d = (f^{-1}(v))^{1/n} = (f^{-1}(\sum_{i=1}^{m} f_i(d_i^n)))^{1/n}. \tag{4.3}$$ □

4.2. Countable cuboid n-volume size.

DEFINITION 4.2. The countable cuboid volume size, d_c^n, is the sum of m number of disjoint countable cuboid volume sizes.

$$\forall\, n\in\mathbb{N},\quad d_c\in\{0,\mathbb{N}\}\quad \exists\, m\in\mathbb{N},\quad x_i\in\{x_1,\cdots,x_m\},\quad \bigcap_{i=1}^{m} x_i = \emptyset:\quad d_c^n = \sum_{i=1}^{m}|x_i|^n.$$

4.3. Minkowski distance (L_p norm).

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

THEOREM 4.3. *The Minkowski distances (L_p norms) are derived from the sum of countable cuboid n-volume sizes* (4.2).

$$d_c^n = \sum_{i=1}^{m}|x_i|^n \quad \Rightarrow \quad \exists\, d, s_1,\cdots,s_m\in\mathbb{R}:\quad d = (\sum_{i=1}^{m} s_i^n)^{1/n}.$$

PROOF. Apply the ruler (2.1):

$$\exists\, d, s_1,\cdots,s_m\in\mathbb{R}:\ d_c = floor(d/\kappa) \quad \wedge \quad |x_i| = floor(s_i/\kappa). \tag{4.4}$$

Apply the ruler convergence (2.2):

$$|x_i| = floor(s_i/\kappa) \quad \Rightarrow \quad s_i = \lim_{\kappa\to 0}|x_i|\cdot\kappa. \tag{4.5}$$

$$d_c^n = \sum_{i=1}^{m}|x_i|^n \quad \Rightarrow \quad d^n = \lim_{\kappa\to 0}(d_c\cdot\kappa)^n = \lim_{\kappa\to 0}(\sum_{i=1}^{m}(|x_i|^n)\cdot\kappa. \tag{4.6}$$

Apply lemma 2.3 to equation 4.6 and substitute equation 4.5:

$$(4.7)\quad d^n = \lim_{\kappa\to 0}(\textstyle\sum_{i=1}^m(|x_i|^n)\cdot\kappa \quad\wedge\quad \lim_{\kappa\to 0}\kappa^n = \lim_{\kappa\to 0}\kappa$$
$$\Rightarrow\quad d^n = \lim_{\kappa\to 0}\textstyle\sum_{i=1}^m(|x_i|^n)\cdot\kappa^n = \lim_{\kappa\to 0}\sum_{i=1}^m(|x_i|\cdot\kappa)^n.$$

Apply equation 4.5 to equation 4.7:

$$(4.8)\quad d^n = \lim_{\kappa\to 0}\textstyle\sum_{i=1}^m(|x_i|\cdot\kappa)^n \quad\wedge\quad s_i = \lim_{\kappa\to 0}|x_i|\cdot\kappa \quad\Rightarrow\quad d^n = \sum_{i=1}^m s_i^n.$$

$$(4.9)\qquad d^n = \textstyle\sum_{i=1}^m s_i^n \quad\Leftrightarrow\quad d = (\sum_{i=1}^m s_i^n)^{1/n}. \qquad\square$$

4.4. Distance inequality. Proving that all Minkowski distances (L_p norms) satisfy the metric space triangle inequality requires another inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance inequality*

$$\forall\, n\in\mathbb{N},\ v_a, v_b \geq 0:\ (v_a+v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand the n-volume, $(v_a^{1/n}+v_b^{1/n})^n$, using the binomial expansion:

$$(4.10)\quad \forall\, v_a, v_b \geq 0:\quad v_a+v_b \leq v_a+v_b+$$
$$\textstyle\sum_{i=1}^{n}\binom{n}{k}(v_a^{1/n})^{n-k}(v_b^{1/n})^k + \sum_{i=1}^{n}\binom{n}{k}(v_a^{1/n})^k(v_b^{1/n})^{n-k} = (v_a^{1/n}+v_b^{1/n})^n.$$

Take the n^{th} root of both sides of the inequality:

$$(4.11)\qquad \square$$
$$\forall\, v_a, v_b \geq 0,\ n\in\mathbb{N}:\ v_a+v_b \leq (v_a^{1/n}+v_b^{1/n})^n \quad\Rightarrow\quad (v_a+v_b)^{1/n} \leq v_a^{1/n}+v_b^{1/n}.$$

4.5. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.5. *Distance sum inequality*

$$\forall\, m,n\in\mathbb{N},\ a_i, b_i \geq 0:\ (\textstyle\sum_{i=1}^m(a_i^n+b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.4):

$$(4.12)\quad \forall\, m,n\in\mathbb{N},\ v_a, v_b \geq 0:\quad v_a = \textstyle\sum_{i=1}^m a_i^n \quad\wedge\quad v_b = \sum_{i=1}^m b_i^n \quad\wedge$$
$$(v_a+v_b)^{1/n} \leq v_a^{1/n}+v_b^{1/n} \quad\Rightarrow\quad ((\textstyle\sum_{i=1}^m a_i^n)+(\sum_{i=1}^m b_i^n))^{1/n} =$$
$$(\textstyle\sum_{i=1}^m(a_i^n+b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}. \quad\square$$

4.6. Metric Space. All Minkowski distances (L_p norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.6. *Triangle Inequality:*
$d(s_1,s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad\Rightarrow\quad d(u,w) \leq d(u,v)+d(v,w).$

PROOF. $\forall\, p\geq 1,\quad k>1,\quad u=s_1,\quad w=s_2,\quad v=w/k$:

$$(4.13)\qquad (u^p+w^p)^{1/p} \leq ((u^p+w^p)+2v^p)^{1/p} = ((u^p+v^p)+(v^p+w^p))^{1/p}.$$

Apply the distance inequality (4.4) to the inequality 4.13:

$$(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \;\wedge\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}$$
$$\wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p$$
$$\Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p}$$
$$\Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq$$
$$(u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \quad \square \tag{4.14}$$

THEOREM 4.7. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$\forall\, p : p \geq 1, \quad d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p}$$
$$\Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \quad \square \tag{4.15}$$

THEOREM 4.8. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0. \tag{4.16}$$

$$p \geq 1,\ u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \tag{4.17}$$

$\square$

THEOREM 4.9. *Identity of Indiscernibles:* $d(u, u) = 0.$

PROOF. From the non-negativity property (4.8):

$$d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0$$
$$\Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0. \tag{4.18}$$

$$d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v. \tag{4.19}$$

$$d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \tag{4.20}$$

$\square$

4.7. The properties limiting a set to at most 3 members.

DEFINITION 4.10. Totally ordered set:

$$\forall\, i\ n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i\ \in \{x_1, \ldots, x_n\},$$
$$\textit{successor}\ x_i = x_{i+1} \ \wedge\ \textit{predecessor}\ x_{i+1} = x_i.$$

DEFINITION 4.11. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i\ j\ n \in \mathbb{N},\ \ \forall\, x_i\ x_j \in \{x_1, \ldots, x_n\},\ \ \textit{successor}\ x_i = x_j \ \Leftrightarrow\ \textit{predecessor}\ x_j = x_i.$$

THEOREM 4.12. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n \ \wedge\ j = 1 \ \Rightarrow\ \textit{successor}\ x_n = x_1 \ \wedge\ \textit{predecessor}\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

Proof. A total order (4.10) assigns a unique label to each set member and assigns unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (4.11):

$$i = n \ \wedge \ j = 1 \ \wedge \ \mathit{successor}\ x_i = x_j \ \Rightarrow \ \mathit{successor}\ x_n = x_1. \tag{4.21}$$

Applying the definition of the symmetry property (4.11) to conclusion 4.21:

$$\mathit{successor}\ x_i = x_j \ \Rightarrow \ \mathit{predecessor}\ x_j = x_i \ \Rightarrow \ \mathit{predecessor}\ x_1 = x_n. \tag{4.22}$$ □

Theorem 4.13. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

Proof.
It was proved that an ordered and symmetric set is a cyclic set (4.12).

Definition 4.14. (Cyclic) Successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (n = m + 1 \leq \mathit{setsize}). \tag{4.23}$$

Definition 4.15. (Cyclic) Predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (n = m - 1 \geq 1). \tag{4.24}$$

Definition 4.16. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$\mathit{Adjacent}(m, n, \mathit{setsize}) \leftarrow \mathit{Successor}(m, n, \mathit{setsize}) \vee \mathit{Predecessor}(m, n, \mathit{setsize}). \tag{4.25}$$

Prove that every member is adjacent to every other member, where $\mathit{setsize} \in \{1,\ 2,\ 3\}$:

$$\mathit{Adjacent}(1, 1, 1) \leftarrow \mathit{Successor}(1, 1, 1) \leftarrow (m = \mathit{setsize} \wedge n = 1). \tag{4.26}$$

$$\mathit{Adjacent}(1, 2, 2) \leftarrow \mathit{Successor}(1, 2, 2) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{4.27}$$

$$\mathit{Adjacent}(2, 1, 2) \leftarrow \mathit{Successor}(2, 1, 2) \leftarrow (n = \mathit{setsize} \wedge m = 1). \tag{4.28}$$

$$\mathit{Adjacent}(1, 2, 3) \leftarrow \mathit{Successor}(1, 2, 3) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{4.29}$$

$$\mathit{Adjacent}(2, 1, 3) \leftarrow \mathit{Predecessor}(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \tag{4.30}$$

$$\mathit{Adjacent}(3, 1, 3) \leftarrow \mathit{Successor}(3, 1, 3) \leftarrow (n = \mathit{setsize} \wedge m = 1). \tag{4.31}$$

$$\mathit{Adjacent}(1, 3, 3) \leftarrow \mathit{Predecessor}(1, 3, 3) \leftarrow (m = 1 \wedge n = \mathit{setsize}). \tag{4.32}$$

$$\mathit{Adjacent}(2, 3, 3) \leftarrow \mathit{Successor}(2, 3, 3) \leftarrow (n = m + 1 \leq \mathit{setsize}). \tag{4.33}$$

$$\mathit{Adjacent}(3, 2, 3) \leftarrow \mathit{Predecessor}(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \tag{4.34}$$

Member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not ($\neg$) a successor of member 1 for all $setsize > 3$:

$$\neg Successor(1, 3, setsize > 3) \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \tag{4.35}$$

Member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not ($\neg$) a predecessor of member 1 for all $setsize > 3$:

$$\neg Predecessor(1, 3, setsize > 3) \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \tag{4.36}$$

For all $setsize > 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not symmetric):

$$\neg Adjacent(1, 3, setsize > 3) \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \tag{4.37}$$

5. Applications to physics

From the 3D proof (4.13), the interval lengths: t (time), m (mass), and q (charge) have different types (are from different sets) from a 3-dimensional interval length, r, that can be related via constant, unit-factoring, conversion ratios:

$$r = (r_c/t_c)t = ct = (r_c/m_G)m = (r_c/q_C)q, \tag{5.1}$$

5.1. Spacetime and Lorentz equations. From the Euclidean volume proof (3.2), two independent (disjoint) intervals, $[0, r]$ and $[0, r']$, defines an Euclidean 2-space. From the Minkowski distance proof (4.3), the interval lengths, r and r', are inverse functions of 2 cuboid 2-volumes, which sum to a cuboid 2-volume:

$$r_v^2 = r^2 + r'^2 \quad \wedge \quad \exists\, r_c,\ t_c,\ c,\ v \in \mathbb{R} :\ r/t = r_c/t_c = c \quad \wedge \quad r_v/t = v \quad \Rightarrow \quad r' = \sqrt{(ct)^2 - (vt)^2} \ = \ ct\sqrt{1 - (v/c)^2}. \tag{5.2}$$

Local (proper) distance, r', contracts relative to coordinate distance, r, as $v \to c$:

$$r' = ct\sqrt{1 - (v/c)^2} \quad \wedge \quad ct = r \quad \Rightarrow \quad r' = r\sqrt{1 - (v/c)^2}. \tag{5.3}$$

The Lorentz transformations follow from equation 5.3 and Galilean transformation:

$$r' = r/\sqrt{1 - (v/c)^2} \quad \wedge \quad r = r' + vt \quad \Rightarrow \quad r' = (r - vt)/\sqrt{1 - (v/c)^2}. \tag{5.4}$$

$$r' = (r - vt)/\sqrt{1 - (v/c)^2} \quad \wedge \quad r = ct \quad \wedge \quad r' = ct' \quad \Rightarrow \quad t' = (t - (vt/c))/\sqrt{1 - (v/c)^2} = (t - (vr/c^2))/\sqrt{1 - (v/c)^2}. \tag{5.5}$$

From equation 5.2, coordinate time, t, dilates relative to local time, t, as $v \to c$:

$$ct = r'/\sqrt{1 - (v/c)^2} \quad \wedge \quad r' = ct' \quad \Rightarrow \quad t = t'/\sqrt{1 - (v/c)^2}. \tag{5.6}$$

Using $r_v^2 = r^2 + r'^2$ from equation 5.2, where r' is a 3-dimensional distance, the "$-{+}{+}{+}$" form of Minkowski's spacetime event interval [**Ein15**] is:

$$\mathrm{d}r_v^2 = \mathrm{d}r^2 + \mathrm{d}r'^2 \quad \wedge \quad \mathrm{d}r'^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \quad \wedge \quad \mathrm{d}(ct) = \mathrm{d}r_v \quad \Rightarrow \quad \mathrm{d}r^2 = -\mathrm{d}(ct)^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2. \tag{5.7}$$

5.2. Einstein's field (general relativity) equations. From the "geometric" distance theorem (4.1), the spacetime interval equation (5.7) generalizes to: $f(ds^2) = \sum_{i=0}^{3} f_i(dx_i^2)$. Einstein used Gaussian (affine) geometry, where $f_i = g_{i,i}$: $ds^2 = \sum_{i=0}^{3} g_{i,i}(dx_i^2)$ and where the metric tensor in Einstein's field equations (EFEs) is $g_{\mu,\nu} = diag(g_{0,0}, g_{1,1}, g_{2,2}, g_{3,3})$ [**Ein15**] [**Wey52**].

The Gaussian curvature of a surface is limited to, at most, 3-space. Einstein created a Gaussian-like curvature, $G_{\mu,\nu}$, induced by $g_{\mu,\nu}$, in 4-space via the Ricci tensor, $\mathbf{R}_{\mu,\nu}$, and it's scalar curvature, R, where $G_{\mu,\nu} = \mathbf{R}_{\mu,\nu} - (1/2)Rg_{\mu,\nu}$. And the sum of two tensors is a tensor, $T_{\mu,\nu} : G_{\mu,\nu} + g_{\mu,\nu} = kT_{\mu,\nu}$, where Einstein defined the conversion factor $k = 8\pi G/c^4$ [**Wey52**]. Newton's gravitational constant, G, was defined rather than derived from more basic constants.

5.3. Newton's gravity force equation. From equation 5.1:

$$\forall\, m_1 m_2 = m^2 = (m_G/r_c)^2 r^2 \quad \Rightarrow \quad (r_c/m_G)^2 m_1 m_2/r^2 = 1. \tag{5.8}$$

From equation 5.4, the proper distance, $r = ct\sqrt{1-(v/c)^2}$, and where $v = 0$:

$$mr = ((m_G/r_c)r)(ct) = (m_G/r_c)(ct)^2 \quad \Rightarrow \quad ((r_c/m_G)/c^2)mr/t^2 = 1. \tag{5.9}$$

$$\begin{aligned}((r_c/m_G)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/m_G)^2 m_1 m_2/r^2 = 1 \\ \Rightarrow \quad F := mr/t^2 = ((r_c/m_G)c^2)m_1 m_2/r^2 = Gm_1 m_2/r^2,\end{aligned} \tag{5.10}$$

where the constant, $G = (r_c/m_G)c^2$, has the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$. And where $|v| > 0$, $F = (r_c/m_G)(c^2 - v^2)m_1 m_2/r^2$.

5.4. Coulomb's charge force. From equation 5.1:

$$r = (r_c/m_G)m = (r_c/q_C)q_1 \Rightarrow m = (m_G/q_C)q_1. \tag{5.11}$$

Substituting equations 5.11 and 5.1 into equation 5.10:

$$\begin{aligned}m = (m_G/q_C)q_1 \quad \wedge \quad r_c/t_c = c \quad \wedge \quad F = ((r_c/m_G)c^2)m_1 m_2/r^2 \\ \Rightarrow \quad F = (m_G/q_C)(r_c/q_C)(r_c/t_c)^2 q_1 q_2/r^2.\end{aligned} \tag{5.12}$$

$$\begin{aligned}a_G = r_c/t_c^2 \quad \wedge \quad F = (m_G/q_C)(r_c/q_C)(r_c/t_c)^2 q_1 q_2/r^2 \\ \Rightarrow \quad F = (m_G a_G)(r_c/q_C)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2,\end{aligned} \tag{5.13}$$

where the charge constant, $k_e = (m_G a_G)(r_c/q_C)^2$, has the SI units: $N \cdot m^2 \cdot C^{-2}$. And where $|v| > 0$, $F = (m_G/q_C)(r_c/q_C)(c^2 - v^2)q_1 q_2/r^2$.

5.5. Work and Einstein-Planck equations: From the ratios 5.1:

$$\begin{aligned}m = (m_G/r_c)r \quad \wedge \quad r = (r_c/t_c)t = ct \\ \Rightarrow mr = (m_G/r_c)r^2 = (m_G/r_c)(r_c/t_c)^2 t^2 = (t/t_c)^2 m_G r_c = k_W \\ \approx 2.2102190943 \cdot 10^{-42}\ kg\ m,\end{aligned} \tag{5.14}$$

where r is the displacement (Compton wavelength) of the mass, m. Dividing both sides of, $m(ct)^2 = mr^2 = k_W r$, by t^2 yields the Einstein-Planck equation:

$$m(ct)^2 = k_W r \Rightarrow E = mc^2 = k_W r/t^2 = k_W(r/t)(1/t) = (k_W c)(1/t) = hf, \tag{5.15}$$

where the Planck constant $h = k_W c$ and the frequency $f = 1/t$.

6. Insights and implications

(1) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [**Joy98**]. And proofs that a straight line is the smallest distance equate the straight line to the Euclidean distance.

The calculus of variations cannot be used to prove that the smallest distance is the Euclidean distance in Euclidean space because the integrals make Euclidean assumptions, which would result in circular logic.

In Euclidean space, if m represents the number of dimensions, then $m = 2 \Rightarrow 1 \leq n \leq 2$, which constrains the Minkowski distances (4.3) to a range from Manhattan distance (the largest distance, $d = (\sum_{i=1}^{2} s_i^1)^{1/1}$, to Euclidean distance (the smallest distance, $d = (\sum_{i=1}^{2} s_i^2)^{1/2}$).

(2) Hilbert spaces allow fractional dimensions (fractals), which is the case of intersecting distance sets and requires generalizing the countable cuboid distance definition (4.2) from: $d_c = (\sum_{i=1}^{m} |x_i|^n)^{1/n}$ to:

$d_c = (\sum_{i=1}^{m} |x_i|^n - |x_i \bigcap (\bigcup_{j=1, i \neq j}^{n} x_j)|)^{1/n}$.

Distance measures are used in shortest path and least cost path search algorithms with various applications, including machine learning. For example, intersecting domain sets allow neural networks to generalize a response across the intersecting domain sets, x_i and x_j.

(3) Compare the distance sum inequality (4.5),

$$(\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n},$$

used to prove that all Minkowski distances satisfy the metric space triangle inequality property (4.6), to Minkowski's sum inequality:

$$(\sum_{i=1}^{m} (a_i^n + b_i^n)^n)^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

Note the exponent difference in the left side of the two inequalities:

$$(\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \quad vs. \quad (\sum_{i=1}^{m} (a_i^n + b_i^n)^{\mathbf{n}})^{1/n}.$$

The proof of Minkowski's sum inequality assumes convexity (the triangle inequality) and the L_p space inequalities (for example, H́older's inequality or Mahler's inequality). In contrast, the distance (sum) inequality is a more fundamental inequality because its proof does not require the assumptions required for proving the Minkowski sum inequality.

(4) From the 3D proof (4.13), more intervals than the 3 dimensions of distance intervals must have different types with lengths that are related to a 3-dimensional distance interval length, r, via constant, unit-factoring, conversion ratios (both direct and inverse proportion ratios). The direct proportion ratios for time, mass, and charge are: $r = (r_c/t_c)t = ct = (r_c/m_G)m = (r_c/q_C)q$. An inverse proportion ratio is the work ratio: $m_p r_c = k_W$. In SI units:

$c_l = r_c/t_c \approx 2.99792458 \cdot 10^8 m\ s^{-1}$.
$c_m = r_c/m_G \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}$.
$c_q = r_c/q_C \approx 8.6175172023 \cdot 10^{-18} m\ C^{-1}$.
$k_W = (t/t_c)^2 m_G r_c = m_p r_c \approx 2.2102190943 \cdot 10^{-42}\ kg\ m$.

(5) The gravity, charge, and Planck constants are not fundamental constants because they are all derived from more basic direct proportion ratios.
(6) The derivations in this article show that the spacetime, gravity force, charge force, and Einstein-Planck equations all depend on time being proportionate to distance: $r = (r_c/t_c)t = ct$. For example, from the derivation of Newton's gravity equation (5.9), where $v = 0$: $G = (r_c/m_G)c^2$. Likewise, from the derivation of Coulomb's charge force equation (5.13) the constant, where $v = 0$: $k_e = (m_G/q_C)(r_c/q_C)c^2$. And from the derivation of the Planck constant (5.15): $h = (m_p r_c)c = k_W c$.
(7) The derivation of the speed of light, work, and Einstein-Planck equations shows that all wave, frequency, displacement, and particle sizes are due to the relations between conversion ratios. Just as all inertial frame of references are relative, all sizes are relative (no absolute sizes > 0).
(8) The derivations of the spacetime equations and Lorentz transformations, here (5.1), differ from all other derivations and provide insights that the other derivations cannot provide.
  (a) The derivations, here, are much shorter and simpler.
  (b) The derivations of the spacetime equations, here, do not rely on the Lorentz transformations or Einsteins' postulates [**Ein15**]. The derivations do not even require the notion of light.
  (c) The derivations, here, rely only on geometry: the Euclidean volume proof (3.2), the Minkowski distances proof (4.3), and the 3D proof (4.13), which provides the insight that the geometry of physical space creates: 1) a maximum speed, c; 2) the spacetime equations; and 3) the Lorentz transformations.
(9) Applying the ratios to derive Newton's gravity force (5.3) and Coulomb's charge force (5.4) equations provide some firsts and some new insights into physics:
  (a) These are the first derivations of Newton's gravity and Coulomb's charge force equations to not assume the inverse square law or Gauss's flux divergence theorem. Note: the components of Ricci and metric tensors in Einstein's field (general relativity) equations have the units, $1/\text{distance}^2$, which is an assumption of the inverse square law.
  (b) These are the first derivations to show that the inverse square law, the property of force as mass times acceleration are the result of the conversion ratios, $r = (r_c/t_c)t = (r_c/m_G)m$.
  (c) Using Occam's razor, those versions of constants like: Gauss' gravity and charge constants, vacuum magnetic permeability, etc. that contain the value 4π might be incorrect because those constants are based on the less parsimonious assumption that the inverse square law is due to Gauss's flux divergence on a sphere having the surface area, $4\pi r^2$.
  (d) The derivations predict that values G and k_e are constants, in a local frame of reference, only where the local velocity is zero. The derived relativistic gravity and charge force equations are:
  $F = (r_c/m_G)(c^2 - v^2)m_1 m_2/r^2$ (5.9) and
  $F = (m_G/q_C)(r_c/q_C)(c^2 - v^2)q_1 q_2/r^2$ (5.13).

Note: the common use of the Lorentz factor, $\gamma = 1/\sqrt{1 - v^2/c^2}$, in Newton's equation, $F = Gm_1m_2/r^2\gamma$, is not derivable.

(i) Therefore, Einstein's gravity constant, $k = 8\pi G/c^4$ [**Wey52**], is only valid when the local velocity is 0. Otherwise, $k = 8\pi(r_G/m_G)(c^2 - v^2)/c^4$. As $v \to c \Rightarrow F \to 0$, implies a universe expanding faster than predicted by a constant k and also predicts an accelerating expansion.

(ii) From the spacetime equation 5.2, r_v is a constant, where $r^2 = r_v^2 - r'^2 \Rightarrow \lim_{r\to 0} r^2 = \lim_{r\to 0} r_v^2 - r'^2 = \lim_{r\to 0}(r_v/t)^2 - (r'/t)^2 = \lim_{r\to 0} c^2 - v^2 = 0$. Therefore, the inverse square law and flux divergence are approximations that only holds where $r >> 0$. The inverse square law not holding at very small distances reduces the mass density of black holes (increases the radius).

(10) There is no unit-factoring ratio converting a constant state value to continuously varying distance, time, mass, and charge values. Therefore, the spin states of two quantum entangled particles and the polarization states of two quantum entangled photons are independent of the amount of distance between the particles and independent of time (instantaneous).

(11) It was proved that sequencing through a set, having a strict linear order via the successor/predecessor relations in all n-at-a-time permutations, is a cyclic set with n $\leq$ 3 (4.13), which is why there are only 3 dimensions of physical space.

(a) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (4.13), each member being an ordered and symmetric set of 3 dimensions (three 3-dimensional balls).

(b) Each of 3 ordered and symmetric dimensions of space can have at most 3 sequentially ordered and symmetric state values, for example, an ordered and symmetric set of 3 vector orientations, $\{-1, 0, 1\}$, per dimension of space.

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