

# SOME SET PROPERTIES UNDERLYING GEOMETRY AND PHYSICS

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ABSTRACT. Euclidean volume and the Minkowski distances, are derived from sets of ordered combinations (n-tuples). A combinatorial property of distance can limit distance to a set to 3 dimensions. Other dimensions have different types (not members of the distance set) with constant ratios of a distance unit interval length to unit interval lengths of time, mass, and charge. The proofs and ratios are used to: 1) derive well-known gravity, charge, electromagnetic equations, special relativity, Schwarzschild time dilation and metric equations, and quantum physics equations; 2) derive the gravity, charge, vacuum permittivity, permeability, Planck, and fine structure constants; 3) add quantum extensions to gravity and charge equations. All the proofs are verified in Rocq.

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## 1. INTRODUCTION

Many physics equations either assume Euclidean space (for example, the gravity, charge, electromagnetism, and Schrödinger's equations) [8][17], or assume that the space near each local coordinate point is Euclidean (Riemann and pseudo-Riemann spaces, special and general relativity) [7][16][25]. Although Euclidean geometry permeates math and physics, mathematical analysis defines Euclidean volume as the product of interval sizes  $\subset \mathbb{R}^n$  and defines the Euclidean distance equation [9][19], Both definitions are motivated by Cartesian and Euclidean geometry rather than deriving the equations from a set and limit-based foundation.

Proving that sets of ordered combinations (n-tuples) imply the Euclidean volume equation and some distance equations, including the Euclidean distance equation,

provides a foundation for: 1) a rigorous proof that can limit distance to 3 dimensions (other dimensions are members of different sets), 2) short and simple derivations of what have been empirical equations and constants, 3) simpler and shorter derivations of some well-known relativity and quantum physics equations, and 4) adding quantum extensions to classical and relativity equations.

Proving that sets of n-tuples imply the Euclidean volume and distance equations, without circular logic, requires deriving the equations without contamination of the volume and distance notions of interval, straight line, side, angle, etc. A “ruler” measure, which does not require the notions of interval, etc., will be used to prove that sets of n-tuples imply the Euclidean volume equation and some distance equations, including Euclidean distance.

The proofs in this article have been verified using the Rocq proof verification system [18]. The formal proofs are in the Rocq files, “euclidrelations.v” and “threed.v,” at: <https://github.com/treeck/RASRGGeometry>.

Let  $|x_i|$  be the cardinal of (number of elements in) in the countable set,  $x_i$ . And let  $v_c$  be the integer number of ordered combinations (n-tuples) of the members of  $x_1, \dots, x_n$ . The ruler measure will be used to prove the set of n-tuples imply the Euclidean volume equation:

$$\forall v_c, d_c, |x_i| \in \{0, \mathbb{N}\}, x_i \in \{x_1, \dots, x_n\},$$

$$v_c = \prod_{i=1}^n |x_i| \Rightarrow v = \prod_{i=1}^n s_i, \quad s_i, v \in \mathbb{R}. \quad (1.1)$$

For all  $n > 1$ , there are an infinite number of combinations of domain values,  $s_1, \dots, s_n$ , that multiplied yield the same range value,  $v$ . Inferring a domain value,  $d$ , from  $v$ , requires an inverse (bijective) function,  $d = f_n^{-1}(v)$  and  $v = f_n(d)$ . The simplest bijective case, for all  $n$ , extends the  $n = 1$  case,  $v_c = |x_1| = d_c^1$ :

$$\exists d_c, v_c, |x_i| \in \{0, \mathbb{N}\} : v_c = \prod_{i=1}^n |x_i| = \prod_{i=1}^n d_c = d_c^n. \quad (1.2)$$

A set of n-tuples being the union of subsets of n-tuples implies that distance is also the inverse function of the sum of volumes:

$$d_c^n = v_c = \sum_{i=1}^m v_{c_i} = \sum_{i=1}^m (\prod_{j=1}^n |x_{i,j}|) \Rightarrow d^n = \sum_{i=1}^m (\prod_{j=1}^n s_{i,j}). \quad (1.3)$$

Where each  $s_{i,j}$  is  $\pm$  signed, the  $n = 2$  case is the inner product.

Where each  $v_{c_i}$  is also the bijective function,  $v_{c_i} = d_{c_i}^n$ , it will proved that:

$$d_c^n = \sum_{i=1}^m d_{c_i}^n \Rightarrow d^n = \sum_{i=1}^m d_i^n. \quad (1.4)$$

$|d|$  is the  $p$ -norm (Minkowski distance) [15], which will be proved to imply the metric space properties [19]. The  $n = 2$  case is the Euclidean distance.

Volume and distance are derived from sets of ordered combinations (n-tuples). Volume and distance have another combinatorial (permutation) property.

Calculating volume and distance, both n-tuples and real-valued equations, requires multiplication and addition. The commutative properties of multiplication and addition requires being able to sequence (multiply and add) a set of  $n$  number of values in any one of  $n!$  permutations.

Reliably re-sequencing a set of values in the same order requires assigning a sequential order to the values. Further, the *only* sequential order that allows starting with any value and sequence in a repeatable order, is a cyclic order.

Reliably re-sequencing of a cyclic set in all  $n!$  permutations, is a symmetry, where every set member is either an *immediate* cyclic successor or an *immediate* cyclic predecessor to every other set member, which is, herein, referred to as an

“immediate symmetric” cyclic set (ISCS). First-order logic will be used to prove an ISCS has  $n \leq 3$  members.

Application to physics uses the following 3 hypotheses:

- (1) **ISCS:** Physical distance is an ISCS of 3 dimensions,  $\{r_1, r_2, r_3\}$ , and  $\{t$  (*time*),  $m$  (*mass*),  $q$  (*charge*) $\}$  is the ISCS of “non-distance” dimensions, each dimension  $\subseteq \mathbb{R}$ . Physical space is 6-dimensional:  $r_1$ - $r_2$ - $r_3$ - $t$ - $m$ - $q$ .
- (2) **Cartesian:** Each local coordinate point is the origin of a Cartesian grid (the space near each local coordinate point is Euclidean), where for each Cartesian axis unit interval length,  $r_p$ , of distance, there is a constant Cartesian axis unit interval length:  $t_p$  of time;  $m_p$  of mass; and  $q_p$  of charge, such that:  $r = (r_p/t_p)t = (r_p/m_p)m = (r_p/q_p)q$ , where  $r_p/t_p = c_t$ ,  $r_p/m_p = c_m$ , and  $r_p/q_p = c_q$ .
- (3) **Maximum ratios** The Cartesian axis unit ratios,  $c_t$ ,  $c_m$ , and  $c_q$  are the largest ratios. For example, the speed of light is limited to  $c_t$ .

A consequence of these hypotheses is that all equations derived from combining the constant ratios are the same equations at each local coordinate point, which is the reason the laws of physics are same at each local coordinate point.

The Newton’s gravity [17] and Coulomb’s charge force [8] equations will be derived from the ratios, where:

$$F = (c_m c_t^2) m_1 m_2 / r^2 = G m_1 m_2 / r^2 \quad \text{and} \quad (1.5)$$

$$F = (c_q^2 c_t / c_m) q_1 q_2 / r^2 = k_e q_1 q_2 / r^2. \quad (1.6)$$

The ratio,  $c_m$ , is calculated from the empirical values of  $G$ , and  $c_t$  (the speed of light) in Newton’s equation. Next, the ratio,  $c_q$ , is calculated from the values of  $k_e$ ,  $c_t$ , and  $c_m$  in Coulomb’s equation. The derivations from the ratios will show that  $G$ ,  $k_e$ ,  $\epsilon_0$ ,  $\mu_0$ , and  $\hbar$  are **not** fundamental (atomic) constants.

Algebraic manipulation of the 3 direct proportion ratios yields 3 inverse proportion ratios,  $r = t_p r_p / t = m_p r_p / m = q_p r_p / q$ , where  $k_t = t_p r_p$ ,  $k_m = m_p r_p$ , and  $k_q = q_p r_p$ . The combination of the direct and inverse proportion ratios are used to derive the Planck relation and the reduced Planck constant,  $\hbar = k_m c_t$ . The values of  $k_t$ ,  $k_m$ , and  $k_q$  are calculated from the values of  $\hbar$ ,  $c_t$ ,  $c_m$ , and  $c_q$ .

$r_p$ ,  $t_p$ ,  $m_p$ , and  $q_p$  are the Planck units, which are calculated from  $G$ ,  $c_t$ , and  $\hbar$ . The fine structure electron coupling constant,  $\alpha$ , is derived, in this article, as the ratio of two forces that reduces to the ratio of subtypes:  $\alpha = q_e^2 / q_p^2$ , which is much simpler and more elucidating than the standard equation,  $\alpha = q_e^2 / 4\pi\epsilon_0 \hbar c$  [5].

The proofs and the 3 direct proportion ratios are used to provide simple derivations of: the gravitational constant,  $G$ , the Newton, Gauss, and Poisson gravity equations, Coulomb’s charge force and charge constant,  $k_e$ , the special relativity equations, the Schwarzschild time dilation and black hole metric equations (pointing to a simplified method of finding solutions to Einstein’s general relativity equations), the Gauss, Lorentz, and Faraday electromagnetic equations, the vacuum permittivity,  $\epsilon_0$ , and vacuum permeability,  $\mu_0$ , constants.

The ratios and Planck relation are used to derive the Compton wavelength, the position-space Schrödinger, and the Dirac wave equations. And, finally, the inverse proportion ratios are also used to add quantum extensions to some general relativity and classical physics equations.

## 2. RULER MEASURE AND CONVERGENCE

**Definition 2.1.** Ruler measure,  $M = \sum_{i=1}^p \kappa = p\kappa$ , where  $\forall s, \kappa \in \mathbb{R}$ ,  $0 < \kappa \leq 1$ ,  $(p = \text{floor}(s/\kappa) \quad \vee \quad p = \text{ceiling}(s/\kappa))$ .

**Theorem 2.2.** *Ruler convergence:*  $M = \lim_{\kappa \rightarrow 0} p\kappa = s$ .

The formal proof, “limit\_c.0\_M.eq\_exact\_size,” is in the file, euclidrelations.v.

*Proof.* (epsilon-delta proof)

By definition of the floor function,  $\text{floor}(x) = \max(\{y : y \leq x, y \in \mathbb{Z}, x \in \mathbb{R}\})$ :

$$p = \text{floor}(s/\kappa) \quad \wedge \quad 0 \leq |\text{floor}(s/\kappa) - s/\kappa| < 1 \quad \Rightarrow \quad |p - s/\kappa| < 1. \quad (2.1)$$

Multiply both sides of inequality 2.1 by  $\kappa$ :

$$\forall 0 < \kappa \leq 1, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \quad (2.2)$$

$$\begin{aligned} \forall \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \delta = \epsilon \quad := \quad M = \lim_{\kappa \rightarrow 0} p\kappa = s. \quad \square \end{aligned} \quad (2.3)$$

The following is an example of ruler convergence for the interval,  $[0, \pi]$ :  $s = \pi - 0$ , and  $p = \text{floor}(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}}, 3.14_{\kappa=10^{-2}}, 3.141_{\kappa=10^{-3}}, \dots, \pi_{\lim_{\kappa \rightarrow 0}}$ .

**Lemma 2.3.**  $\forall n \geq 1, \quad 0 < \kappa \leq 1 \quad \Rightarrow \quad \lim_{\kappa \rightarrow 0} \kappa^n = \lim_{\kappa \rightarrow 0} \kappa$ .

*Proof.* The formal proof, “lim\_c.to\_n.eq\_lim\_c,” is in the Rocq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa \leq 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \quad (2.4)$$

$$\begin{aligned} \forall \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta \\ \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa \rightarrow 0} \kappa^n = 0. \end{aligned} \quad (2.5)$$

$$\lim_{\kappa \rightarrow 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa \rightarrow 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa \rightarrow 0} \kappa^n = \lim_{\kappa \rightarrow 0} \kappa. \quad (2.6)$$

□

## 3. VOLUME

**Theorem 3.1.** *Euclidean volume,*

$$\begin{aligned} \forall v_c, d_c, |x_i| \in \{0, \mathbb{N}\}, \quad x_i \in \{x_1, \dots, x_n\}, \\ v_c = \prod_{i=1}^n |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^n s_i, \quad s_i, v \in \mathbb{R}. \end{aligned} \quad (3.1)$$

The formal proof, “Euclidean\_volume,” is in the Rocq file, euclidrelations.v.

*Proof.*

$$v_c = \prod_{i=1}^n |x_i| \Leftrightarrow v_c \kappa = (\prod_{i=1}^n |x_i|) \kappa \Leftrightarrow \lim_{\kappa \rightarrow 0} v_c \kappa = \lim_{\kappa \rightarrow 0} (\prod_{i=1}^n |x_i|) \kappa. \quad (3.2)$$

Apply the ruler (2.1) and ruler convergence (2.2) to equation 3.2:

$$\begin{aligned} \exists v, \kappa \in \mathbb{R} : v_c = \text{floor}(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa \rightarrow 0} v_c \kappa \quad \wedge \\ \lim_{\kappa \rightarrow 0} v_c \kappa = \lim_{\kappa \rightarrow 0} (\prod_{i=1}^n |x_i|) \kappa \quad \Rightarrow \quad v = \lim_{\kappa \rightarrow 0} (\prod_{i=1}^n |x_i|) \kappa. \end{aligned} \quad (3.3)$$

Apply lemma 2.3 to equation 3.3:

$$\begin{aligned} v = \lim_{\kappa \rightarrow 0} (\prod_{i=1}^n |x_i|) \kappa \quad \wedge \quad \lim_{\kappa \rightarrow 0} \kappa^n = \lim_{\kappa \rightarrow 0} \kappa \\ \Rightarrow \quad v = \lim_{\kappa \rightarrow 0} (\prod_{i=1}^n |x_i|) \kappa^n = \lim_{\kappa \rightarrow 0} (\prod_{i=1}^n |x_i| \kappa). \end{aligned} \quad (3.4)$$

Apply the ruler (2.1) and ruler convergence (2.2) to  $s_i$ :

$$\exists s_i, \kappa \in \mathbb{R} : \text{floor}(s_i/\kappa) = |x_i| \Rightarrow \lim_{\kappa \rightarrow 0} (|x_i|\kappa) = s_i. \quad (3.5)$$

$$v = \lim_{\kappa \rightarrow 0} (\prod_{i=1}^n |x_i|\kappa) \quad \wedge \quad \lim_{\kappa \rightarrow 0} (|x_i|\kappa) = s_i \quad \Leftrightarrow \quad v = \prod_{i=1}^n s_i \quad (3.6)$$

□

#### 4. DISTANCE

**Definition 4.1.** Countable distance,

$$\forall v_c, d_c, |x_i| \in \{0, \mathbb{N}\}, x_i \in \{x_1, \dots, x_n\}, v_c = \prod_{i=1}^n |x_i| = \prod_{i=1}^n d_c = d_c^n. \quad (4.1)$$

**Lemma 4.2.** A volume is the sum of volumes,

$$v_c = \sum_{i=1}^m v_{c_i} \Rightarrow v = \sum_{i=1}^m v_i, \quad v, v_i \in \mathbb{R}.$$

The formal proof, “sum\_of\_volumes,” is in the Rocq file, euclidrelations.v.

*Proof.* From the condition of this theorem:

$$v_c = \sum_{i=1}^m v_{c_i} \Leftrightarrow \lim_{\kappa \rightarrow 0} v_c \kappa = \lim_{\kappa \rightarrow 0} \sum_{j=1}^m (v_{c_i} \kappa). \quad (4.2)$$

Apply lemma 2.3 to equation 4.2:

$$\begin{aligned} \lim_{\kappa \rightarrow 0} v_c \kappa &= \lim_{\kappa \rightarrow 0} (\sum_{j=1}^m v_{c_i}) \kappa \quad \wedge \quad \lim_{\kappa \rightarrow 0} \kappa^n = \lim_{\kappa \rightarrow 0} \kappa \\ &\Leftrightarrow \lim_{\kappa \rightarrow 0} v_c \kappa = \lim_{\kappa \rightarrow 0} \sum_{j=1}^m (v_{c_i} \kappa). \end{aligned} \quad (4.3)$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to equation 4.3:

$$\begin{aligned} \exists v, v_i : v &= \lim_{\kappa \rightarrow 0} v_c \kappa \quad \wedge \quad \exists v_i, v_{c_i} : v_i = \lim_{\kappa \rightarrow 0} v_{c_i} \kappa \\ \wedge \quad \lim_{\kappa \rightarrow 0} (d_c \kappa)^n &= \lim_{\kappa \rightarrow 0} \sum_{j=1}^m (v_{c_i} \kappa) \Rightarrow v = \sum_{i=1}^m v_i^n. \quad \square \end{aligned} \quad (4.4)$$

#### 4.1. Sum of volumes distance.

**Theorem 4.3.** Sum of volumes distance:

$$d_c^n = \sum_{i=1}^m v_{c_i} \Rightarrow d^n = \sum_{i=1}^m (\prod_{j=1}^n s_{i_j}).$$

The formal proof, “sum\_of\_volumes\_distance,” is in the Rocq file, euclidrelations.v.

*Proof.* From lemma 4.2 and the Euclidean volume theorem 3.1:

$$\begin{aligned} d_c^n = \sum_{i=1}^m v_{c_i} &\Rightarrow d^n = \sum_{i=1}^m v_i, \quad \wedge \quad v_i = \prod_{j=1}^n s_{i_j} \\ &\Rightarrow d^n = \sum_{i=1}^m (\prod_{j=1}^n s_{i_j}). \quad \square \end{aligned} \quad (4.5)$$

#### 4.2. Minkowski distance (p-norm).

**Theorem 4.4.** Minkowski distance (p-norm):

$$d_c^n = v_c = \sum_{i=1}^m v_{c_i} = \sum_{i=1}^m d_{c_i}^n \Leftrightarrow d^n = \sum_{i=1}^m d_i^n.$$

The formal proof, “Minkowski\_distance,” is in the Rocq file, euclidrelations.v.

*Proof.* From lemma 4.2 and the Euclidean volume theorem 3.1:

$$\begin{aligned} d_c^n = v_c = \sum_{i=1}^m v_{c_i} &\Rightarrow d^n = v = \sum_{i=1}^m v_i \quad \wedge \quad v_i = \prod_{j=1}^n d_i = d_i^n \\ &\Rightarrow d^n = \sum_{i=1}^m d_i^n \quad \square \end{aligned} \quad (4.6)$$

**4.3. Distance inequality.** The formal proof, `distance_inequality`, is in the Rocq file, `euclidrelations.v`.

**Theorem 4.5.** *Distance inequality*

$$\forall n \in \mathbb{N}, v_a, v_b \geq 0 : (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

*Proof.* Expand  $(v_a^{1/n} + v_b^{1/n})^n$  using the binomial expansion:

$$\forall v_a, v_b \geq 0 : v_a + v_b \leq v_a + v_b +$$

$$\sum_{i=1}^n \binom{n}{i} (v_a^{1/n})^{n-i} (v_b^{1/n})^i + \sum_{i=1}^n \binom{n}{i} (v_a^{1/n})^i (v_b^{1/n})^{n-i} = (v_a^{1/n} + v_b^{1/n})^n. \quad (4.7)$$

Take the  $n^{th}$  root of both sides of the inequality 4.7:

$$\forall v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad (4.8) \quad \square$$

**4.4. Distance sum inequality.** The formal proof, `distance_sum_inequality`, is in the Rocq file, `euclidrelations.v`.

**Theorem 4.6.** *Distance sum inequality*

$$\forall m, n \in \mathbb{N}, a_i, b_i \geq 0 : (\sum_{i=1}^m (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}.$$

*Proof.* Apply the distance inequality (4.5):

$$\begin{aligned} \forall m, n \in \mathbb{N}, v_a, v_b \geq 0 : v_a &= \sum_{i=1}^m a_i^n \quad \wedge \quad v_b = \sum_{i=1}^m b_i^n \quad \wedge \\ (v_a + v_b)^{1/n} &\leq v_a^{1/n} + v_b^{1/n} \Rightarrow ((\sum_{i=1}^m a_i^n) + (\sum_{i=1}^m b_i^n))^{1/n} = \\ &(\sum_{i=1}^m (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}. \quad \square \end{aligned} \quad (4.9)$$

**4.5. Metric Space.** All Minkowski distances ( $p$ -norms) imply the metric space properties. The formal proofs: `triangle_inequality`, `symmetry`, `non_negativity`, and `identity_of_indiscernibles` are in the Rocq file, `euclidrelations.v`.

**Theorem 4.7.** *Triangle Inequality:*

$$d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} \Rightarrow d(u, w) \leq d(u, v) + d(v, w).$$

*Proof.*  $\forall p \geq 1, k > 1, u = s_1, w = s_2, v = w/k:$

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \quad (4.10)$$

Apply the distance inequality (4.5) to the inequality 4.10:

$$\begin{aligned} (u^p + w^p)^{1/p} &\leq ((u^p + v^p) + (v^p + w^p))^{1/p} \quad \wedge \quad (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ &\quad \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ \Rightarrow (u^p + w^p)^{1/p} &\leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ &\Rightarrow d(u, w) = (u^p + w^p)^{1/p} \leq \\ &(u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \quad \square \end{aligned} \quad (4.11)$$

**Theorem 4.8.** *Symmetry:*  $d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} \Rightarrow d(u, v) = d(v, u).$

*Proof.* By the commutative law of addition:

$$\begin{aligned} \forall p : p \geq 1, d(s_1, s_2) &= (\sum_{i=1}^2 s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ &\Rightarrow d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \quad \square \end{aligned} \quad (4.12)$$

**Theorem 4.9.** *Non-negativity:*  $d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} \Rightarrow d(u, w) \geq 0$ .

*Proof.* By definition, the length of an interval is always  $\geq 0$ :

$$\forall [a_1, b_1], [a_2, b_2], \quad u = b_1 - a_1, v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0, v \geq 0. \quad (4.13)$$

$$p \geq 1, \quad u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \quad (4.14)$$

□

**Theorem 4.10.** *Identity of Indiscernibles:*  $d(u, u) = 0$ .

*Proof.* From the non-negativity property (4.9):

$$\begin{aligned} d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0 \\ \Rightarrow \quad \exists d(u, w) = d(u, v) = d(v, w) = 0. \end{aligned} \quad (4.15)$$

$$d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v. \quad (4.16)$$

$$d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \quad (4.17)$$

□

**4.6. Properties limiting a set to at most 3 members.** The following definitions and proof use first order logic. A Horn clause-like expression is used, here, to make the proof easier to read. By convention, the proof goal is on the left side and supporting facts are on the right side of the implication sign ( $\leftarrow$ ). The formal proofs in the Rocq file `threed.v` are:

**Lemmas:** `adj111`, `adj122`, `adj212`, `adj123`, `adj133`, `adj233`, `adj213`, `adj313`, `adj323`, and `not_all_mutually_adjacent_gt_3`.

**Definition 4.11.** Immediate Cyclic Successor of  $m$  is  $n$ :

$$\begin{aligned} \forall x_m, x_n \in \{x_1, \dots, x_{\text{setsize}}\} : \\ \text{Successor}(m, n, \text{setsize}) \leftarrow (m = \text{setsize} \wedge n = 1) \vee (n = m + 1 \leq \text{setsize}). \end{aligned} \quad (4.18)$$

**Definition 4.12.** Immediate Cyclic Predecessor of  $m$  is  $n$ :

$$\begin{aligned} \forall x_m, x_n \in \{x_1, \dots, x_{\text{setsize}}\} : \\ \text{Predecessor}(m, n, \text{setsize}) \leftarrow (m = 1 \wedge n = \text{setsize}) \vee (n = m - 1 \geq 1). \end{aligned} \quad (4.19)$$

**Definition 4.13.** Adjacent: Member  $m$  is sequentially adjacent to member  $n$  if the immediate cyclic successor of  $m$  is  $n$  or the immediate cyclic predecessor of  $m$  is  $n$ . Notionally:

$$\begin{aligned} \forall x_m, x_n \in \{x_1, \dots, x_{\text{setsize}}\} : \\ \text{Adjacent}(m, n, \text{setsize}) \leftarrow \text{Successor}(m, n, \text{setsize}) \vee \text{Predecessor}(m, n, \text{setsize}). \end{aligned} \quad (4.20)$$

**Definition 4.14.** Immediate Symmetric (every set member is sequentially adjacent to every other member):

$$\forall x_m, x_n \in \{x_1, \dots, x_{\text{setsize}}\} : \quad \text{Adjacent}(m, n, \text{setsize}). \quad (4.21)$$

**Theorem 4.15.** *An immediate symmetric cyclic set is limited to at most 3 members.*

*Proof.*

Every member is adjacent to every other member, where  $setsize \in \{1, 2, 3\}$ :

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \quad (4.22)$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \quad (4.23)$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1). \quad (4.24)$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize). \quad (4.25)$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \quad (4.26)$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1). \quad (4.27)$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize). \quad (4.28)$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize). \quad (4.29)$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \quad (4.30)$$

Member 2 is the only immediate successor of member 1 for all  $setsize \geq 3$ , which implies member 3 is not ( $\neg$ ) an immediate successor of member 1 for all  $setsize \geq 3$ :

$$\neg Successor(1, 3, setsize \geq 3) \\ \leftarrow Successor(1, 2, setsize \geq 3) \leftarrow (n = m + 1 \leq setsize). \quad (4.31)$$

Member  $n = setsize > 3$  is the only immediate predecessor of member 1, which implies member 3 is not ( $\neg$ ) an immediate predecessor of member 1 for all  $setsize > 3$ :

$$\neg Predecessor(1, 3, setsize \geq 3) \\ \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \quad (4.32)$$

For all  $setsize > 3$ , some elements are not ( $\neg$ ) sequentially adjacent to every other element (not immediate symmetric):

$$\neg Adjacent(1, 3, setsize > 3) \\ \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \quad (4.33)$$

The Symmetric goal matches Adjacent goals 4.22 and fails for all “setsize” greater than three.

## 5. APPLICATIONS TO PHYSICS

Where distance is an immediate symmetric cyclic set (ISCS) of dimensions, the 3D proof (4.15) requires more dimensions to have non-distance types (members of other sets). Let  $\tau = \{t \text{ (time)}, m \text{ (mass)}, q \text{ (charge)}\}$  be the ISCS of type “non-distance” dimensions, where for each Cartesian axis unit length,  $r_p$ , of distance,  $r$ , there are Cartesian axis unit lengths:  $t_p$  of time,  $t$ ;  $m_p$  of mass,  $m$ ; and  $q_p$  of charge,  $q$ , such that:

$$r = (r_p/t_p)t = (r_p/m_p)m = (r_p/q_p)q, \quad (5.1)$$

where  $c_t$ ,  $c_m$ , and  $c_q$  are the maximum ratios:

$$c_t = r_p/t_p, \quad c_m = r_p/m_p, \quad c = c_t = r_p/q_p. \quad (5.2)$$



**5.1. Derivation of the constant,  $G$ , and the gravity laws of Newton, Gauss, and Poisson.** From equation 5.2:

$$\begin{aligned} r = c_m m \quad \wedge \quad r = c_t t \quad \Rightarrow \quad r/(c_t t)^2 = c_m m/r^2 \\ \Rightarrow \quad r/t^2 = (c_m c_t^2) m/r^2 = Gm/r^2, \end{aligned} \quad (5.3)$$

where  $G = c_m c_t^2$ , conforms to the SI units:  $m^3 \cdot kg^{-1} \cdot s^{-2}$  [17].

Newton's law follows from multiplying both sides of equation 5.3 by  $m$ :

$$r/t^2 = Gm/r^2 \Leftrightarrow F := mr/t^2 = Gm^2/r^2. \quad (5.4)$$

$$F = Gm^2/r^2 \wedge \forall m \in \mathbb{R} : \exists m_1, m_2 \in \mathbb{R} : m_1 m_2 = m^2 \Rightarrow F = Gm_1 m_2/r^2. \quad (5.5)$$

Equation 5.3 relates linear acceleration,  $r/t^2$ , to mass and distance. Gauss's gravity field,  $\mathbf{g}$ , and Poisson's gravity field,  $-\nabla\Phi(r, t)$ , relates orbital acceleration,  $2\pi r/t^2$ , to mass and distance. Multiplying both sides of equation 5.3 by  $2\pi$  and differentiating yields Gauss's and Poisson's laws [8]:

$$\mathbf{g} = -\nabla\Phi(\vec{r}, t) = 2\pi r/t^2 = 2\pi Gm/r^2 \Rightarrow \nabla \cdot \mathbf{g} = \nabla^2\Phi(\vec{r}, t) = -4\pi Gm/r^3. \quad (5.6)$$

$$\nabla \cdot \mathbf{g} = \nabla^2\Phi(\vec{r}, t) = -4\pi Gm/r^3 \wedge \rho = m/r^3 \Rightarrow \nabla \cdot \mathbf{g} = \nabla^2\Phi(\vec{r}, t) = -4\pi G\rho. \quad (5.7)$$

**5.2. Derivation of Coulomb's charge constant,  $k_e$ , and charge force.** [8]

From equation 5.2:

$$r = c_q q \Rightarrow r^2 = c_q^2 q^2 \Rightarrow c_q^2 q^2/r^2 = 1. \quad (5.8)$$

$$r = c_m m = c_t t \Rightarrow mr = ((1/c_m)r)(c_t t) = (c_t^2/c_m)t^2 \Rightarrow (c_m/c_t^2)mr/t^2 = 1. \quad (5.9)$$

$$\begin{aligned} c_q^2 q^2/r^2 = 1 \quad \wedge \quad (c_m/c_t^2)mr/t^2 = 1 \\ \Rightarrow \quad F := mr/t^2 = (c_q^2 c_t^2/c_m)q^2/r^2 = k_e q^2/r^2. \end{aligned} \quad (5.10)$$

$k_e = c_q^2 c_t^2/c_m$ , conforms to the SI units:  $kg \cdot m^3 \cdot s^{-2} \cdot C^{-2} = N \cdot m^2 \cdot C^{-2}$  [8].

$$\exists q_1, q_2 \in \mathbb{R} : q_1 q_2 = q^2 \quad \wedge \quad F = k_e q^2/r^2 \Rightarrow F = k_e q_1 q_2/r^2. \quad (5.11)$$

**5.3. 3 fundamental direct proportion ratios.**  $c_t$ ,  $c_m$ , and  $c_q$ :

$$c_t = r_p/t_p \approx 2.99792458 \cdot 10^8 m \, s^{-1}. \quad (5.12)$$

$$G = c_m c_t^2 = c_m c_t^2 \Rightarrow c_m = r_p/m_p \approx 7.4261602691 \cdot 10^{-28} m \, kg^{-1}. \quad (5.13)$$

$$k_e = c_q^2 c_t^2/c_m \Rightarrow c_q = r_p/q_p \approx 8.6175172023 \cdot 10^{-18} m \, C^{-1}. \quad (5.14)$$

**5.4. 3 fundamental inverse proportion ratios.**  $k_t$ ,  $k_m$ , and  $k_q$ :

$$\begin{aligned} r/t = r_p/t_p, \quad r/m = r_p/m_p \Rightarrow (r/t)/(r/m) = (r_p/t_p)/(r_p/m_p) \Rightarrow \\ (mr)/(tr) = (m_p r_p)/(t_p r_p) \Rightarrow mr = m_p r_p = k_m, \quad tr = t_p r_p = k_t. \end{aligned} \quad (5.15)$$

$$\begin{aligned} r/t = r_p/t_p, \quad r/q = r_p/q_p \Rightarrow (r/t)/(r/q) = (r_p/t_p)/(r_p/q_p) \Rightarrow \\ (qr)/(tr) = (q_p r_p)/(t_p r_p) \Rightarrow qr = q_p r_p = k_q, \quad tr = t_p r_p = k_t. \end{aligned} \quad (5.16)$$

**5.5. Planck relation and constant,  $\hbar$ .** [11] Applying both the direct proportion ratio (5.12), and inverse proportion ratio (5.15):

$$r = ct \quad \wedge \quad m = k_m/r \quad \Rightarrow \quad m(ct)^2 = (k_m/r)r^2 = k_m r. \quad (5.17)$$

$$\begin{aligned} m(ct)^2 &= k_m r \quad \wedge \quad r/t = c \\ \Rightarrow \quad E := mc^2 &= k_m r/t^2 = (k_m c)(1/t) = \hbar\omega = \hbar\omega(2\pi/2\pi) = hf, \end{aligned} \quad (5.18)$$

where the reduced Planck constant,  $\hbar = k_m c$ , angular frequency,  $\omega = 1/t$ , the full Planck constant,  $h = 2\pi\hbar$ , and the cycles per second frequency (Hertz),  $f = 1/2\pi t$ .

$$k_m = m_p r_p = \hbar/c \approx 3.5176729162 \cdot 10^{-43} \text{ kg } m. \quad (5.19)$$

$$k_t = t_p r_p = k_m c_m / c_t \approx 8.7136291599 \cdot 10^{-79} \text{ s } m. \quad (5.20)$$

$$k_q = q_p r_p = k_t c_t / c_q \approx 3.0313607071 \cdot 10^{-52} \text{ C } m. \quad (5.21)$$

**5.6. 4 quantum (Planck) units.** Length ( $r_p$ ), time ( $t_p$ ), mass ( $m_p$ ), charge ( $q_p$ ):

$$r_p = \sqrt{r_p^2} = \sqrt{c_t k_t} = \sqrt{c_m k_m} = \sqrt{c_q k_q} \approx 1.6162550244 \cdot 10^{-35} \text{ m}. \quad (5.22)$$

$$t_p = r_p / c_t \approx 5.3912464472 \cdot 10^{-44} \text{ s}. \quad (5.23)$$

$$m_p = r_p / c_m \approx 2.176434343 \cdot 10^{-8} \text{ kg}. \quad (5.24)$$

$$q_p = r_p / c_q \approx 1.875546038 \cdot 10^{-18} \text{ C}. \quad (5.25)$$

**5.7. Subtype ratios.** The ratios of two subtypes of force implies ratios of the form:  $\alpha_\tau = \frac{F_{\tau_1}}{F_{\tau_2}} = \frac{K\tau_1^2/r^2}{K\tau_2^2/r^2} = \frac{\tau_1^2}{\tau_2^2}$ . For example, where  $q_e$  is the elementary (electron) charge ( $1.60217663 \cdot 10^{-19} \text{ C}$ ), and  $q_p$  is Planck charge unit, the fine structure electron coupling constant is:

$$\alpha_q = q_e^2 / q_p^2 \approx 0.0072973526. \quad (5.26)$$

**5.8. Space-time-mass-charge.** Let  $r$  be an Euclidean distance. Then by the Minkowski distance theorem (4.4),  $r^2 = \sum_{i=1}^m r_i^2$ . Let,  $r' = r_1$  and  $r_v^2 = (\sum_{i=2}^m r_i^2)$ . From the 3D theorem (4.15) and Cartesian hypothesis (2):

$$\begin{aligned} \forall \tau \in \{t, m, q\}, \quad r^2 &= r'^2 + r_v^2, \quad \exists \mu, \nu : r = \mu\tau \quad \wedge \quad r_v = \nu\tau \\ \Rightarrow \quad (\mu\tau)^2 &= r'^2 + (\nu\tau)^2 \quad \Rightarrow \quad r' = \sqrt{(\mu\tau)^2 - (\nu\tau)^2} = \mu\tau\sqrt{1 - (\nu/\mu)^2}. \end{aligned} \quad (5.27)$$

Rest frame distance,  $r'$ , contracts relative to stationary frame distance,  $r$ , as  $\nu \rightarrow \mu$ :

$$r' = \mu\tau\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad \mu\tau = r \quad \Rightarrow \quad r' = r\sqrt{1 - (\nu/\mu)^2}. \quad (5.28)$$

Stationary frame type,  $\tau$ , dilates relative to the rest frame type,  $\tau'$ , as  $\nu \rightarrow \mu$ :

$$\mu\tau = r' / \sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad r' = \mu\tau' \quad \Rightarrow \quad \tau = \tau' / \sqrt{1 - (\nu/\mu)^2}. \quad (5.29)$$

Where  $\tau$  is type, time, the space-like flat Minkowski spacetime event interval is:

$$\begin{aligned} dr^2 &= dr'^2 + dr_v^2 \quad \wedge \quad dr_v^2 = dr_1^2 + dr_2^2 + dr_3^2 \quad \wedge \quad d(\mu\tau) = dr \\ &\Rightarrow \quad dr'^2 = d(\mu\tau)^2 - dr_1^2 - dr_2^2 - dr_3^2. \end{aligned} \quad (5.30)$$

**5.9. Derivation of Schwarzschild's gravitational time dilation and black hole metric.** [21] [1] From equations 5.28 and 5.1:

$$\begin{aligned} \sqrt{1 - (v^2/c^2)} &= \sqrt{1 - (v^2/c^2)} \quad \wedge \quad c_m m/r = 1 \\ &\Rightarrow \sqrt{1 - (v^2/c^2)} = \sqrt{1 - (c_m m)v^2/rc^2}. \end{aligned} \quad (5.31)$$

Where  $v_{\text{escape}}$  is the escape velocity:

$$\begin{aligned} \sqrt{1 - (v^2/c^2)} &= \sqrt{1 - (c_m m)v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv_{\text{escape}}^2 \\ &\Rightarrow \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2c_m mv_{\text{escape}}^2/rc^2}. \end{aligned} \quad (5.32)$$

$$\sqrt{1 - (v^2/c^2)} = \lim_{v_{\text{escape}} \rightarrow c} \sqrt{1 - 2c_m mv_{\text{escape}}^2/rc^2} = \sqrt{1 - 2c_m mc^2/rc^2}. \quad (5.33)$$

Combining equation 5.33 with the derivation of  $G$  (5.5):

$$\begin{aligned} c_m c^2 = G \quad \wedge \quad \sqrt{1 - (v^2/c^2)} &= \sqrt{1 - 2c_m mc^2/rc^2} \\ &\Rightarrow \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2Gm/rc^2}. \end{aligned} \quad (5.34)$$

Combining equation 5.34 with equation 5.29 yields Schwarzschild's gravitational time dilation [21] [1]:

$$\begin{aligned} \sqrt{1 - (v^2/c^2)} &= \sqrt{1 - 2Gm/rc^2} \quad \wedge \quad t' = t\sqrt{1 - (v^2/c^2)} \\ &\Rightarrow t' = t\sqrt{1 - 2Gm/rc^2}. \end{aligned} \quad (5.35)$$

Schwarzschild defined the black hole event horizon radius,  $r_s := 2Gm/c^2$ . From equations 5.28 and 5.34:

$$\begin{aligned} r' &= r\sqrt{1 - (v/c)^2} \quad \wedge \quad \sqrt{1 - (v/c)^2} = \sqrt{1 - 2Gm/rc^2} \quad \wedge \quad r_s := 2Gm/c^2 \\ &\Rightarrow r' = r\sqrt{1 - 2Gm/rc^2} = r\sqrt{1 - r_s/r}. \end{aligned} \quad (5.36)$$

Applying equation 5.36 to the time-like spacetime interval equation 5.30:

$$\begin{aligned} r' &= r\sqrt{1 - r_s/r} \quad \wedge \quad ds^2 = dr'^2 - dt^2 \quad \Rightarrow \\ ds^2 &= (\sqrt{1 - r_s/r}dr)^2 - (dr'/\sqrt{1 - r_s/r})^2 = (1 - r_s/r)dr^2 - (1 - r_s/r)^{-1}dr'^2. \end{aligned} \quad (5.37)$$

General relativity does not have a special frame of reference, so let  $r' = r$ .

$$\begin{aligned} ds^2 &= (1 - r_s/r)dr^2 - (1 - r_s/r)^{-1}dr^2 \quad \wedge \quad dr = d(ct) \quad \wedge \quad c = 1 \\ &\Rightarrow ds^2 = (1 - r_s/r)dt^2 - (1 - r_s/r)^{-1}dr^2. \end{aligned} \quad (5.38)$$

Using spherical coordinates to translate from 2D to 4D yields Schwarzschild's black hole metric [21] [1]:

$$\begin{aligned} ds^2 &= (1 - r_s/r)dt^2 - (1 - r_s/r)^{-1}dr^2 = f(r, t) \\ &\Rightarrow ds^2 = (1 - r_s/r)dt^2 - (1 - r_s/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) = f(r, t, \theta, \phi) \\ &\Rightarrow g_{\mu,\nu} = \text{diag}[1 - r_s/r, (1 - r_s/r)^{-1}, r^2(d\theta^2), r^2(\sin^2\theta d\phi^2)]. \end{aligned} \quad (5.39)$$

**5.10. Simple derivations of solutions to Einstein's general relativity (field equation).** Step 1) Use the ratios to define functions returning scalar values for each component of the metric,  $g_{\nu,\mu}$ , in Einstein's field equations [7] [25]: All functions derived from the ratios and special relativity are valid metrics, for example, the previous Schwarzschild black hole metric derivation using the ratios (5.9).

Step 2) Express the Einstein field equation as 2D tensors: As shown in equation 5.39, the Schwarzschild metric was first derived as a 2D metric and then expanded to a 4D metric. Further, the 4D flat spacetime interval equation (5.30) is an instance of the 2D equation,  $dr'^2 = d(ct)^2 - dr_v^2$ .

The 2D metric tensor allows using the much simpler 2D Ricci curvature and scalar curvature. And the 2D tensors reduce the number of independent equations to solve, which can be used to set constraints on the solutions in the 4D tensors.

Step 3) One simple method to translate from 2D to 4D is to use spherical coordinates, where  $r$  and  $t$  remain unchanged and two added dimensions are the angles,  $\phi$ , and  $\theta$ . For example, the 2D Schwarzschild metric was translated to 4D using this method in equation 5.39.

**5.11. Relativistic Lorentz law, and vacuum permeability,  $\mu_0$ .** Coulomb's charge force equation 5.10 relates linear acceleration,  $r/t^2$ , to charge and distance. Gauss's electric field,  $\mathbf{E}$ , relates orbital (or rotational) acceleration,  $2\pi r/t^2$  to charge and distance:

$$F_C = mr/t^2 = k_e q^2/r^2 \quad \Rightarrow \quad \exists F_E \in \mathbb{R} : F_E = m(2\pi r/t^2) = 2\pi k_e q^2/r^2. \quad (5.40)$$

Applying the distance contraction equation 5.28 to equation 5.40, where  $r$  is the stationary frame of reference and  $r'$  is moving particle (rest) frame of reference:

$$r = r'/\sqrt{1 - v^2/c^2} \quad \wedge \quad F = 2\pi k_e q^2/r^2 \quad \Rightarrow \quad F = 2\pi k_e q^2(1 - v^2/c^2)/r'^2. \quad (5.41)$$

$$E := 2\pi k_e q/r'^2 \quad \Rightarrow \quad F = q(E - v^2(2\pi k_e/c^2)q/r'^2). \quad (5.42)$$

$$B := (2\pi k_e/c^2)vq/r'^2 \quad \Rightarrow \quad F = q(E - vB). \quad (5.43)$$

$$F = q(E - vB) \quad \Rightarrow \quad \mathbf{F} = q(\mathbf{E} - \vec{v} \times \mathbf{B}), \quad (5.44)$$

which is Lorentz law in the rest frame of reference. And

$$\mathbf{F} = q(\mathbf{E} + \vec{v} \times \mathbf{B}), \quad (5.45)$$

is Lorentz law in the stationary frame of reference. The direction of rotation (curl) depends on your frame of reference.

The electric field,  $E := 2\pi k_e q/r'^2$ , conforms to the SI units  $kg \cdot m \cdot s^{-2} \cdot C^{-1} = N \cdot C^{-1}$  and the magnetic field,  $B = (2\pi k_e/c^2)vq/r'^2$ , conforms to the base SI units:  $kg \cdot s^{-1} \cdot C^{-1} = kg \cdot s^{-2} \cdot A^{-1} = T$ .

$$B := (2\pi k_e/c^2)vq/r'^2 \quad \wedge \quad B := \mu_0 H \quad \wedge \quad \mu_0 := 4\pi k_e/c^2 \quad \Rightarrow \quad H = vq/2r'^2, \quad (5.46)$$

where  $\mu_0 = 4\pi k_e/c^2$  conforms to the SI units  $kg \cdot m \cdot C^{-2} = kg \cdot m \cdot s^{-2} A^{-2}$  and  $H = vq/2r'^2$  conforms to the SI units  $C \cdot s^{-1} \cdot m^{-1} = A \cdot m^{-1}$ .

**5.12. Vacuum permittivity,  $\varepsilon_0$ , and Gauss's law for electric fields.** From equation 5.42:

$$E = 2\pi k_e q/r^2 \quad \Leftrightarrow \quad \mathbf{E} = 2\pi k_e q/\mathbf{r}^2 \quad \Rightarrow \quad \nabla \cdot \mathbf{E} = -4\pi k_e q/\mathbf{r}^3. \quad (5.47)$$

$$\nabla \cdot \mathbf{E} = -4\pi k_e q/\mathbf{r}^3 \quad \wedge \quad \varepsilon_0 := 1/4\pi k_e \quad \wedge \quad \rho = q/\mathbf{r}^3 \quad \Rightarrow \quad \nabla \cdot \mathbf{E} = -\rho/\varepsilon_0, \quad (5.48)$$

which is Gauss's electric field law [8].

**5.13. Derivation of Faraday's law.** From the magnetic field equation 5.43, where the velocity of light,  $v = c$ :

$$B = (2\pi k_e/c^2)qv/r^2 \quad \wedge \quad v = c \quad \wedge \quad r = ct \quad \Rightarrow \quad B = (2\pi k_e/c^3)q/t^2. \quad (5.49)$$

$$B = (2\pi k_e/c^3)q/t^2 \quad \Rightarrow \quad \partial B/\partial t = -(4\pi k_e/c^3)q/t^3. \quad (5.50)$$

$$\partial B/\partial t = -(4\pi k_e/c^3)q/t^3 \quad \wedge \quad r = ct \quad \Rightarrow \quad \partial B/\partial t = -4\pi k_e q/r^3. \quad (5.51)$$

From equation 5.47:

$$\mathbf{E} = 2\pi k_e q/\mathbf{r}^2 \quad \Rightarrow \quad \nabla \times \mathbf{E} = 4\pi k_e q/\mathbf{r}^3. \quad (5.52)$$

Combining equations 5.52 and 5.51 yields Faraday's law [8]:

$$\nabla \times \mathbf{E} = 4\pi k_e q/\mathbf{r}^3 \quad \wedge \quad \partial \mathbf{B}/\partial t = -4\pi k_e q/\mathbf{r}^3 \quad \Rightarrow \quad \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t. \quad (5.53)$$

**5.14. Compton wavelength,  $\lambda$ .** [11] From equations 5.15 and 5.18:

$$\begin{aligned} r = k_m/m \quad \wedge \quad h = 2\pi k_m c \\ \Rightarrow \quad \lambda = 2\pi r = 2\pi k_m/m = (2\pi k_m/m)(c/c) = h/mc. \end{aligned} \quad (5.54)$$

**5.15. Schrödinger's position-space equation.** Start with the previously derived Planck relation 5.18 and multiply the kinetic energy component by  $mc/mc$ :

$$\begin{aligned} mc^2 = \hbar\omega = \hbar/t \quad \Rightarrow \quad \exists V(r, t) : \hbar/t = \hbar/2t + V(r, t) \\ \Rightarrow \quad \hbar/t = \hbar mc/2mct + V(r, t). \end{aligned} \quad (5.55)$$

And from the distance-to-time (speed of light) ratio (5.12):

$$\hbar/t = \hbar mc/2mct + V(r, t) \quad \wedge \quad r = ct \quad \Rightarrow \quad \hbar/t = \hbar mc^2/2mcr + V(r, t). \quad (5.56)$$

$$\hbar/t = \hbar mc^2/2mcr + V(r, t) \quad \wedge \quad \hbar/t = mc^2 \quad \Rightarrow \quad \hbar/t = \hbar^2/2mcrt + V(r, t). \quad (5.57)$$

$$\hbar/t = \hbar^2/2mcrt + V(r, t) \quad \wedge \quad r = ct \quad \Rightarrow \quad \hbar/t = \hbar^2/2mr^2 + V(r, t). \quad (5.58)$$

Multiply both sides of equation 5.58 by a function,  $\Psi(r, t)$ .

$$\hbar/t = \hbar^2/2mr^2 + V(r, t) \Rightarrow (\hbar/t)\Psi(r, t) = (\hbar^2/2mr^2)\Psi(r, t) + V(r, t)\Psi(r, t). \quad (5.59)$$

$$\begin{aligned} (\hbar/t)\Psi(r, t) &= (\hbar^2/2mr^2)\Psi(r, t) + V(r, t)\Psi(r, t) \quad \wedge \\ \forall \Psi(r, t) : \partial^2 \Psi(r, t)/\partial r^2 &= (-1/r^2)\Psi(r, t) \quad \wedge \quad \partial \Psi(r, t)/\partial t = (i/t)\Psi(r, t) \\ \Rightarrow \quad i\hbar \partial \Psi(r, t)/\partial t &= -(\hbar^2/2m)\partial^2 \Psi(r, t)/\partial r^2 + V(r, t)\Psi(r, t), \end{aligned} \quad (5.60)$$

which is the one-dimensional position-space Schrödinger's equation [20] [11].

$$\begin{aligned} i\hbar \partial \Psi(r, t)/\partial t &= -(\hbar^2/2m)\partial^2 \Psi(r, t)/\partial r^2 + V(r, t)\Psi(r, t) \quad \wedge \quad ||\mathbf{r}'|| = r \\ \Rightarrow \quad \exists \mathbf{r}' : i\hbar \partial \Psi(\mathbf{r}', t)/\partial t &= -(\hbar^2/2m)\partial^2 \Psi(\mathbf{r}', t)/\partial \mathbf{r}'^2 + V(\mathbf{r}', t)\Psi(\mathbf{r}', t), \end{aligned} \quad (5.61)$$

which is the 3-dimensional position-space Schrödinger's equation [20] [11].

**5.16. Dirac's wave equation.** Using the derived Planck relation 5.18:

$$\begin{aligned} mc^2 = \hbar/t \quad \Rightarrow \quad \exists V(r, t) : mc^2/2 + V(r, t) = \hbar/t \\ \Rightarrow \quad 2\hbar/t - 2V(r, t) = mc^2. \end{aligned} \quad (5.62)$$

$$\begin{aligned} \forall V(r, t) : V(r, t) = i\hbar/t \quad \wedge \quad r = ct \quad \wedge \quad 2\hbar/t - 2V(r, t) = mc^2 \\ \Rightarrow \quad 2\hbar/t - i2\hbar c/r = mc^2. \end{aligned} \quad (5.63)$$

Use the charge ratio,  $r = c_q q$ , and time ratio,  $r = ct$ . to multiply each term on the left side of equation 5.63 by 1:

$$\begin{aligned} qc_q/r = qc_q/ct = 1 \quad \wedge \quad 2\hbar/t - i2\hbar c/r = mc^2 \\ \Rightarrow \quad 2\hbar(qc_q/c)/t^2 - i2\hbar((qc_q/c)/r^2)c = mc^2. \end{aligned} \quad (5.64)$$

Applying a quantum amplitude equation in complex form to equation 5.65:

$$\begin{aligned} A_0 = (c_q/c)((1/t) - i(1/r)) \quad \wedge \quad 2\hbar(qc_q/c)/t^2 - i2\hbar((qc_q/c)/r^2)c = mc^2 \\ \Rightarrow \quad 2\hbar\partial(-qA_0)/\partial t - i2\hbar(\partial(-qA_0)/\partial r)c = mc^2. \end{aligned} \quad (5.65)$$

Translating equation 5.65 to moving (rest frame) coordinates via the Lorentz factor,  $\gamma_0 = 1/\sqrt{1 - (v/c)^2}$ :

$$\begin{aligned} 2\hbar\partial(-qA_0)/\partial t - i\hbar h(\partial(-qA_0)/\partial r)c = mc^2 \\ \Rightarrow \quad \gamma_0 2\hbar\partial(-qA_0)/\partial t - \gamma_0 i2\hbar(\partial(-qA_0)/\partial r)c = mc^2. \end{aligned} \quad (5.66)$$

Multiplying both sides of equation 5.66 by  $\Psi(r, t)$ :

$$\begin{aligned} \gamma_0 2\hbar\partial(-qA_0)/\partial t - \gamma_0 i2\hbar(\partial(-qA_0)/\partial r)c = mc^2 \\ \Rightarrow \quad \gamma_0 2\hbar(\partial(-qA_0)/\partial t)\Psi(r, t) - \gamma_0 i2\hbar(\partial(-qA_0)/\partial r)c\Psi(r, t) = mc^2\Psi(r, t). \end{aligned} \quad (5.67)$$

Applying the vectors to equation 5.67:

$$\begin{aligned} \gamma_0 2\hbar(\partial(-qA_0)/\partial t)\Psi(r, t) - \gamma_0 i2\hbar(\partial(-qA_0)/\partial r)c\Psi(r, t) = mc^2\Psi(r, t) \quad \wedge \\ ||\vec{r}|| = r \quad \wedge \quad ||\vec{A}|| = A_0 \quad \wedge \quad ||\vec{\gamma}|| = \gamma_0 \quad \wedge \quad \Leftrightarrow \quad \exists \vec{r}, \vec{A}, \vec{\gamma} : \\ \gamma_0 2\hbar(\partial(-qA_0)/\partial t)\Psi(r, t) - \vec{\gamma} \cdot i2\hbar(\partial(-q\vec{A})/\partial r)c\Psi(\vec{r}, t) = mc^2\Psi(\vec{r}, t). \end{aligned} \quad (5.68)$$

Adding a  $\frac{1}{2}$  spin to equation 5.65 yields Dirac's wave equation [6] [11]:

$$\begin{aligned} \gamma_0 2\hbar(\partial(-qA_0)/\partial t)\Psi(r, t) - \vec{\gamma} \cdot i2\hbar(\partial(-q\vec{A})/\partial r)c\Psi(\vec{r}, t) = mc^2\Psi(\vec{r}, t) \\ \wedge \quad A_0 = \frac{1}{2}(c_q/c)((1/t) - i(1/r)) \\ \Rightarrow \quad \gamma_0 \hbar(\partial(-qA_0)/\partial t)\Psi(r, t) - \vec{\gamma} \cdot i\hbar(\partial(-q\vec{A})/\partial r)c\Psi(\vec{r}, t) = mc^2\Psi(\vec{r}, t). \end{aligned} \quad (5.69)$$

**5.17. Total mass.** The total mass of a particle is  $m = \sqrt{m_0^2 + m_{ke}^2}$ , where  $m_0$  is the rest mass and  $m_{ke}$  is the kinetic energy-equivalent mass. Applying both the direct (5.12) and inverse proportion ratios (5.15):

$$\begin{aligned} m_0 = r/c_m \quad \wedge \quad m_{ke} = k_m/r \quad \wedge \quad m = \sqrt{m_0^2 + m_{ke}^2} \\ \Rightarrow \quad m = \sqrt{(r/c_m)^2 + (k_m/r)^2}. \end{aligned} \quad (5.70)$$

**5.18. Quantum extension to general relativity.** The simplest way to demonstrate how to add quantum physics to general relativity is by extending Schwarzschild's time dilation equation and black hole metric (5.9). Start by changing equation 5.31 in the Schwarzschild derivation:

$$\begin{aligned}\sqrt{1 - (v^2/c^2)} &= \sqrt{1 - (v^2/c^2)(r/r)} \quad \wedge \quad r = \sqrt{(c_m m)^2 + (k_m/m)^2} = Q_m \\ &\Rightarrow \quad \sqrt{1 - (v^2/c^2)} = \sqrt{1 - Q_m v^2/rc^2}.\end{aligned}\quad (5.71)$$

$$\begin{aligned}\sqrt{1 - (v^2/c^2)} &= \sqrt{1 - Q_m v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv_{escape}^2 \\ &\Rightarrow \quad \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2Q_m v_{escape}^2/rc^2}.\end{aligned}\quad (5.72)$$

$$\begin{aligned}\sqrt{1 - (v^2/c^2)} &= \lim_{v_{escape} \rightarrow c} \sqrt{1 - 2Q_m v_{escape}^2/rc^2} \\ &\Rightarrow \quad \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2Q_m c^2/rc^2} = \sqrt{1 - 2Q_m/r}.\end{aligned}\quad (5.73)$$

Combining equation 5.73 with equation 5.29 yields Schwarzschild's gravitational time dilation with a quantum mass effect:

$$\begin{aligned}\sqrt{1 - (v^2/c^2)} &= \sqrt{1 - 2Q_m/r} \quad \wedge \quad t' = t\sqrt{1 - (v^2/c^2)} \\ &\Rightarrow \quad t' = t\sqrt{1 - 2Q_m/r}.\end{aligned}\quad (5.74)$$

Schwarzschild defined the black hole event horizon radius,  $r_s := 2Gm/c^2$ . The radius with the quantum extension is  $r_s := 2Q_m$ . At this point the exact same equations 5.36 through 5.39 yield what looks like the same Schwarzschild black hole metric.

**5.19. Quantum extension to Newton's gravity force.** The quantum mass effect is easier to understand in the context Newton's gravity equation than in general relativity, because the metric equations and solutions in the EFEs are much more complex. From equations 5.75 and 5.1:

$$\begin{aligned}m/\sqrt{(r/c_m)^2 + (k_m/r)^2} &= 1 \quad \wedge \quad r^2/(ct)^2 = 1 \\ &\Rightarrow \quad r^2/(ct)^2 = m/\sqrt{(r/c_m)^2 + (k_m/r)^2} \\ &\Rightarrow \quad r^2/t^2 = c^2 m/\sqrt{(r/c_m)^2 + (k_m/r)^2}.\end{aligned}\quad (5.75)$$

$$\begin{aligned}r^2/t^2 &= c^2 m/\sqrt{(r/c_m)^2 + (k_m/r)^2} \\ &\Rightarrow \quad (m/r)(r^2/t^2) = (m/r)(c^2 m/\sqrt{(r/c_m)^2 + (k_m/r)^2}) \\ &\Rightarrow \quad F := mr/t^2 = c^2 m^2/\sqrt{(r^4/c_m^2) + k_m^2}.\end{aligned}\quad (5.76)$$

$$\begin{aligned}F &= c^2 m^2/\sqrt{(r^4/c_m^2) + k_m^2} \quad \wedge \quad \forall m \in \mathbb{R}, \exists m_1, m_2 \in \mathbb{R} : m_1 m_2 = m^2 \\ &\Rightarrow \quad F = c^2 m_1 m_2/\sqrt{(r^4/c_m^2) + k_m^2}.\end{aligned}\quad (5.77)$$

## 5.20. Quantum extension to Coulomb's force.

$$\begin{aligned}
 q^2/((r/c_q)^2 + (k_q/r)^2) &= 1 \quad \wedge \quad r^2/(ct)^2 = 1 \\
 \Rightarrow r^2/(ct)^2 &= q^2/((r/c_q)^2 + (k_q/r)^2) \\
 \Rightarrow r^2/t^2 &= c^2 q^2/((r/c_q)^2 + (k_q/r)^2). \quad (5.78)
 \end{aligned}$$

$$\begin{aligned}
 (1/r)(r^2/t^2) &= (1/r)(c^2 q^2/((r/c_q)^2 + (k_q/r)^2)) \\
 \Rightarrow r/t^2 &= c^2 q^2/(r^3/c_q^2 + k_q^2/r). \quad (5.79)
 \end{aligned}$$

$$\begin{aligned}
 \forall q \in \mathbb{R} : \exists q_1, q_2 \in \mathbb{R} : q_1 q_2 = q^2 \quad \wedge \quad r/t^2 = c^2 q^2/(r^3/c_q^2 + k_q^2/r) \\
 \Rightarrow \exists q_1, q_2 \in \mathbb{R} : r^2/t^2 = c^2 q_1 q_2/(r^3/c_q^2 + k_q^2/r). \quad (5.80)
 \end{aligned}$$

$$\begin{aligned}
 r^2/t^2 = c^2 q_1 q_2/(r^3/c_q^2 + k_q^2/r) \quad \wedge \quad m = r/c_m \\
 \Rightarrow F := mr/t^2 = (c^2/c_m) q_1 q_2/(r^2/c_q^2 + k_q^2/r^2). \quad (5.81)
 \end{aligned}$$

## 6. INSIGHTS AND IMPLICATIONS

- (1) The ruler measure (2.1) and convergence theorem (2.2) were shown to be useful tools for proving that a countable sets of n-tuples imply a corresponding real-valued equation.
- (2) Defining all Euclidean and non-Euclidean distance measures as the inverse function of the sum of subset volumes:

$$\forall n, d : d = f_n^{-1}(v) = f_n^{-1}(\sum_{i=1}^m v_i) : \quad (6.1)$$

- (a) shows the intimate relation between distance and volume that definitions, like inner product space and metric space, ignore [25] [9] [19];
- (b) is a more simple and concise definition of a distance measure that includes the properties used in the definitions of inner product space and metric space [25] [9] [19].
- (3) The left side of the distance sum inequality (4.6),

$$(\sum_{i=1}^m (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}, \quad (6.2)$$

differs from the left side of Minkowski's sum inequality [15]:

$$(\sum_{i=1}^m (a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}. \quad (6.3)$$

- (a) The two inequalities are only the same where  $n = 1$ .
- (b) The distance sum inequality (4.6) is a more fundamental inequality because the proof does not require the convexity and Hölder's inequality assumptions required to prove the Minkowski sum inequality [15].
- (c) The distance sum inequality term,  $\forall n > 1, v_i^n = a_i^n + b_i^n : d = v^{1/n} = (\sum_{i=1}^m v_i^n)^{1/n}$ , is the Minkowski distance, which makes it directly related to geometry. But the Minkowski sum inequality term,  $\forall n > 1, v > 0 : d = v^{1/n} = (\sum_{i=1}^m ((v_i^n)^{\mathbf{n}}))^{1/n} = (\sum_{i=1}^m v_i^{\mathbf{n}^2})^{1/n}$ , is *not* a Minkowski distance.
- (d) The distance sum inequality might be applicable to machine learning.



- (4) **Combinatorics**, the set of ordered combinations of countable, disjoint sets ( $n$ -tuples),  $v_c = \prod_{i=1}^n |x_i|$ , was proven to imply: the Euclidean volume equation (3.1), the sum of volumes equation (4.3) (which includes the inner product), and the Minkowski distance equation (4.4) (which includes the Manhattan and Euclidean distance equations), without relying on the geometric primitives and relations in Euclidean geometry [12] [14], axiomatic geometry [3] [10] [13] [23] [24], trigonometry [4] [26] calculus [2] [4] [22], and vector analysis [25].
- (5) **Combinatorics**, repeatable sequencing through an ordered set of  $n$  number of members to yield all  $n!$  permutations of its members (without jumping around) was proved to be an immediate symmetric cyclic set (ISCS) having  $n \leq 3$  members (4.15). Higher dimensions must have different types (members of different sets).
- (a) For example, the vector inner product space can only be extended beyond 3 dimensions if and only if the higher dimensions have non-distance types, for example, time.
  - (b) As shown in the special relativity section (5.8), there is 6-dimensional space-time-mass-charge.
  - (c) If each type of quantum state is an ISCS, there are at most 3 states of the same type: 3 orientations per dimension of space, 3 quark color charges, {red, green, blue}, 3 quark anti-color charges, and so on.
  - (d) If the states are not ordered (a bag of states), then a state value is undetermined (or superimposed) until observed (like Schrödinger's poisoned cat being both alive and dead until the box is opened [20]).
  - (e) A discrete (point) value has measure 0 (zero-length interval size). The ratio of a time or distance interval length to zero is undefined, which is the reason quantum entangled (discrete) state values exist independent of time and distance.
- (6) For each Cartesian axis unit,  $r_p$ , of a 3-dimensional distance interval having a length,  $r$ , there are Cartesian axis units of other types of intervals forming unit ratios (5.3):  $c_t = r_p/t_p$ ,  $c_m = r_p/m_p$ ,  $c_q = r_p/q_p \Leftrightarrow$  the inverse proportion ratios (5.4):  $k_t = r_p t_p$ ,  $k_m = r_p m_p$ ,  $k_t = r_p q_p$ , where  $r_p$ ,  $t_p$ ,  $m_p$ , and  $q_p$  are the Planck units.
- (7) Empirical laws *describe* relations. Deriving the laws from the ratios *explains* the relations. Further, all the derivations of the physics equations from the ratios were much shorter and simpler than other derivations, which shows that the ratios are an important tool for physicists and engineers.
- (8) As shown in subsection 5.10, the derivation of the Schwarzschild's time dilation and black hole metric (5.9) [21] [1] using ratios exposed a way of simplifying the finding of solutions to Einstein's field equations.
- (9) The speed of light ratio,  $c_t$ , is a component of the constants:  $G = c_m c_t^2$ ,  $k_e = c_q^2 c_t^2 / c_m$ ,  $\varepsilon_0 = 1/4\pi k_e = 1/4\pi (c_q^2 c_t^2 / c_m)$ ,  $\hbar = k_m c_t$ .
- The only constant derived, in this article, that does not contain  $c_t$  is vacuum permeability:  $\mu_0 = 4\pi k_e / c_t^2 = 4\pi c_q^2 / c_m$ .
- (10) Using the quantum (Planck) units,  $r_p$  and  $t_p$ :  $r_p/t_p^2 \approx 5.5607262989 \cdot 10^{51} \text{ m s}^{-2}$ , which suggests a maximum acceleration for masses. And  $2\pi r_p/t_p^2$  would be maximum orbital or rotational acceleration.

- (11) The simplification of  $\mu_0$  into the quantum units shows two interesting relationships:

$$\begin{aligned}\mu_0 &= 4\pi \frac{k_e}{c_t^2} = 4\pi \frac{c_q^2}{c_m} = 4\pi \frac{(r_p/q_p)^2}{r_p/m_p} = 4\pi \frac{m_p r_p}{q_p^2} = 4\pi \frac{k_m}{q_p^2} \\ &\approx 4\pi \frac{3.5176729162 \cdot 10^{-43}}{3.5176729162 \cdot 10^{-35}} = 4\pi \cdot 10^{-7} \text{ kg m C}^{-2} = 4\pi \cdot 10^{-7} \text{ H m}^{-1}. \quad (6.4)\end{aligned}$$

- (a) The first time  $k_m = m_p r_p$  appears is in the derivation of the Planck relation and Planck constant,  $\hbar = k_m c$  (5.5), the second time in the Compton wavelength,  $r = k_m/m$  (5.14). And now,  $k_m$  appears as a components of  $k_e$ ,  $\varepsilon_0$ , and  $\mu_0$ .
- (b) It is an open question why  $\frac{k_m}{q_p^2} = 1.0 \cdot 10^{-7}$  exactly.
- (12) The fine structure constant,  $\alpha$  was derived from the ratio of two forces of two subtypes which reduces to ratio of the square of the subtypes (5.7).
- (a) The CODATA electron coupling version of the fine structure constant,  $\alpha$  is defined as:  $\alpha = q_e^2/4\pi\varepsilon_0\hbar c = q_e^2/2\varepsilon_0\hbar c$  [5].
- (i) The derivation of  $\alpha$ , in this article (5.7), is much simpler because it is the ratio of two subtypes of charge: elementary (electron) charge,  $q_e^2$  and Planck charge,  $q_p^2$ :  $\alpha = q_e^2/q_p^2 \approx 0.0072973526$ , which is the empirical CODATA value [5].
- (ii) The following steps show that the CODATA definition reduces to the ratio-derived equation:

$$\begin{aligned}\varepsilon_0 &:= 1/4\pi k_e = 1/(4\pi(c_q^2 c_t^2/c_m)) \quad \wedge \quad \hbar = k_m c_t \quad \wedge \quad h = 2\pi\hbar \\ &\Rightarrow \quad \varepsilon_0 \hbar c = 2\pi k_m c_t^2 / (4\pi(c_q^2/c_m)c_t^2) = k_m / (2(c_q^2/c_m)) \\ &= m_p r_p / (2((r_p/q_p)^2/(r_p/m_p))) = q_p^2/2. \quad (6.5)\end{aligned}$$

$$\alpha = q_e^2/2\varepsilon_0\hbar c \quad \wedge \quad \varepsilon_0 \hbar c = q_p^2/2 \quad \Rightarrow \quad \alpha = q_e^2/2(q_p^2/2) = q_e^2/q_p^2. \quad (6.6)$$

- (b) Other fine structure constants can also be expressed more simply as the ratios of two subtypes of fields, for example, an electron gravity coupling constant can be expressed as the ratio of the rest electron mass to a Planck mass unit:  $\alpha_{G_m} = m_e^2/m_p^2$ .
- (13) Empirical and hypothesized laws of physics use an *opaque* constant,  $K$ , that is defined to make an equation, where the units balance,  $g = Kf(r, t, \dots)$ . The opacity has led to the *incorrect* assumptions of those constants,  $K$ , being fundamental (atomic) constants.

In this article, some opaque constants are derived directly from (composed of) the ratios: gravity,  $G = c_m c_t^2$  (5.5), charge,  $k_e = c_q^2 c_t^2/c_m$  (5.10), and Planck  $h = k_m c_t$  (5.18).  $\varepsilon_0 = 1/4\pi k_e = 1/4\pi c_m / ((c_q^2/c_m)c_t^2)$  (5.48) and  $\mu_0 = 4\pi k_e/c_t^2 = 4\pi c_q^2/c_m$  (5.45).

And the quantum extensions to: Schwarzschild's time dilation (5.73) Newton's gravity force (5.77), and Coulomb's charge force show, that where the quantum effects become measurable, the constants  $G$ ,  $k_e$ ,  $\varepsilon_0$ , and  $\mu_0$  no longer exist (are no longer valid).

Therefore,  $G$ ,  $k_e$ ,  $\varepsilon_0$ ,  $\mu_0$ , and  $h$  are **not** fundamental (atomic) constants.

- (14) The derivations of:  $\nabla \cdot \mathbf{g} = -4\pi G\rho$  from  $\mathbf{g} = 2\pi Gm/r^2$  (5.6),  $\nabla \cdot \mathbf{E} = -\rho/\varepsilon_0$  from  $\mathbf{E} = 2\pi k_e q/r^2$  (5.48), and  $\partial \mathbf{B}/\partial t = -\mu_0 \rho$  from  $\mathbf{B} = 2\pi k_e q/r^2$

(5.51), show that the use of mass and charge density,  $\rho$ , are unnecessary complications that obfuscates the pattern,  $\nabla \cdot f(x, y, r) = -2k_{x,y}y/r^3$ , being derived from the inverse square pattern,  $f(x, y, r) = k_{x,y}y/r^2$ . And the energy density in the stress-energy tensor,  $T_{\mu,\nu}$ , in Einstein's field equations [25] also obfuscates the inverse square assumption.

- (15) Einstein's relativity equations assume the Lorentz transformations, that the laws of physics are same at each coordinate point, and that the speed of light is the same at each coordinate point [7] [25]. The derivations, in this article, were made without those assumptions (does even require the notion of light). Assuming Cartesian coordinates at each coordinate point, creates unit ratios, where all equations (laws) derived from the unit ratios must be the same at each coordinate point. Deriving numeric values for the ratios assumes that the ratio,  $c_t$ , is the maximum speed.
- (16) The derivation of the magnetic field from special relativity (5.42) shows that magnetic field,  $\mathbf{B}$ , is the spacetime bend (curl) of the electric field,  $\mathbf{E}$  due to relativistic velocities. The magnetic force is a torque caused by spacetime bending of the radial charge force.

Charged particles have an orientation that rotates, where a half-spin is a  $\pi$  radians rotation of the orientation (dipole moment). Positive and negative charges with orientations in the same direction have opposite spins (angular momentum).

- (17) The quantum extensions to: Schwarzschild's time dilation (5.73) black hole metric (5.39), Newton's gravity force (5.77), and Coulomb's charge force (5.80) make quantifiable predictions:
- (a) The gravitation and charge forces peak at finite amounts as  $r \rightarrow 0$ : for gravity,  $\lim_{r \rightarrow 0} F = c^2 m_1 m_2 / k_m$ , and for charge,  $\lim_{r \rightarrow 0} F = 0$ . Finite maximum gravity and charge forces: 1) allows radioactivity, finite sloped energy well walls; and 2) eliminates the problem of forces going to infinity as  $r \rightarrow 0$ , which might eliminate the need to hypothesize the existence of a weak force and strong force.
  - (b) The quantum-extended Schwarzschild time dilation and metric, gravity, and charge equations reduce to the classic equations, where the distance between masses and charges is sufficiently large or the masses and charges sufficiently large that the quantum effect is not measurable. **Note** that  $G$ ,  $k_e$ ,  $\varepsilon_0$ ,  $\mu_0$ , and  $\kappa$  (Einstein's constant, which contains  $G$ ) do not exist (are not valid), where the quantum effects becomes measurable.
  - (c) And the covariant tensor components, in Einstein's field equations, that had the units  $1/\text{distance}^2$ , will now have the more complex units,  $1/\sqrt{(\text{distance}^4/c_m^2) + k_m^2}$ .
  - (d)  $1/\sqrt{(\text{distance}^4/c_m^2) + k_m^2}$  implies that as distance  $\rightarrow 0$ , spacetime curvature peaks at a finite amount, which might imply that black holes have sizes  $> 0$  (might not be singularities). Black hole evaporation might be possible. If there was a "big bang," then it might not have originated from a singularity.

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