The Set Properties Generating Geometry and Physics

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Abstract. Volume and the Minkowski distances/Lp norms (e.g., Euclidean distance) are derived from a set and limit-based foundation without referencing the primitives and relations of geometry. Sequencing a set of n number of distance intervals/dimensions in all n-at-a-time permutations via a strict linear order of successor/predecessor relations is a cyclic set limiting n to at most 3. Therefore, all other interval lengths have different types that can only be related to a distance interval length via unit-factoring, conversion ratios. The ratios and geometry proofs provide simpler derivations and new insights into the spacetime, gravity, charge force, and Einstein-Planck equations and their corresponding constants. All proofs are verified in Coq.

Contents

1. Introduction

Mathematical (real) analysis can construct differential calculus from a set and limit-based foundation without the need to reference the primitives and relations of Euclidean geometry, like side, angle, slope, etc. But the Riemann and Lebesgue integrals and measure theory (for example, Hilbert spaces and the Lebesgue measure) use the Euclidean volume equation as a definition. The inner product, vector norm, and metric space use Euclidean distance and its properties as definitions [Gol76] [Rud76]. Engineering and most physics assume only 3 dimensions of space. Here, a countable set and limit-based foundation generates: distance, volume, a limit of 3 dimensions. and some well-known physics equations and constants.

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Intuition might suggest deriving the n-dimensional Euclidean volume (n-volume) equation, $v = \prod_{i=1}^{n} s_i$, from the Cartesian product of the cardinals (number of members), $|x_i|$, of n number of countable sets, $x_i \in \{x_1, \cdots, x_n\} : v_c = \prod_{i=1}^{n} |x_i|$. But if each set, x_i, is a countable set of intervals, then assuming the function of an n-tuple of interval lengths is an n-volume is circular logic. The derivation in this article takes care to avoid that circular logic. The derivation of the Euclidean volume equation from $v_c = \prod_{i=1}^{n} |x_i|$ lays a set and limit-based foundation for deriving the L_p norms (Minkowski distances), for example Euclidean distance.

Every Euclidean and non-Euclidean n-volume, v, has a corresponding cuboid n-volume, $v = d^n$, which is an instance of a cuboid countable volume, $v_c = \prod_{i=1}^{n} |x|$. And an n-volume can only be the sum of n-volumes, where each summed n-volume corresponds to a cuboid Euclidean n-volume: $v_c = \sum_{i=1}^{m} |x_i|^n \Rightarrow v = \sum_{i=1}^{m} d_i^n \Rightarrow d = (\sum_{i=1}^{m} d_i^n)^{1/n}$, which are the Minkowski distances that have the properties of a metric space. Therefore, every distance measure that is an inverse function of an n-volume has a corresponding Minkowski distance.

Proving that a set of n number of independent domain intervals/dimensions of the same type sequenced in all n-at-a-time permutations (orders) via a strict linear order of successor/predecessor relations is a cyclic set limited to $n \leq 3$, implies that all additional interval lengths have different types that can only be related to geometric interval lengths via unit-factoring, conversion ratios. The ratios combined with the Minkowski distance and volume proofs provide simpler derivations of the spacetime, Newton's gravity, Coulomb's charge force, and Einstein-Planck equations and exposes the ratios that generate the gravity, charge, and Planck constants.

All the proofs in this article are trivial. But to ensure confidence, all the proofs have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

Derivatives and anti-derivative integrals divide the domain intervals and range interval such that $\forall\ \Delta x_i \subset [x_a, x_b]\ \exists\ \Delta f_i \subset [f(x_a), f(x_b)]$. The size the infinitesimals, Δx_i and Δf_i, are proportionate the lengths of $[x_a, x_b]$ and $[f(x_a), f(x_b)]$, which makes it difficult for differential equations and integrals to directly express the Cartesian mappings between the p_x number of size κ infinitesimals in one interval and the p_y number of the *same* size κ infinitesimals in a different-sized interval. Further, using integrals and measures that define Euclidean volume (for example, Riemann and Lebesgue) to derive Euclidean volume would be circular logic.

Therefore, a different tool is used here. A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals (infinitesimals), where each infinitesimal has the *same* size, κ. The ruler is both an inner and outer measure of an interval.

Definition 2.1. Ruler measure, M: $\forall\ [a, b] \subset \mathbb{R},\ \ s = b - a\ \ \wedge\ \ \kappa > 0\ \ \wedge\ \ (p = floor(s/\kappa)\ \ \vee\ \ p = ceiling(s/\kappa))\ \ \wedge\ \ M = \sum_{i=1}^{p} \kappa = p\kappa.$

Theorem 2.2. *Ruler convergence:* $M = \lim_{\kappa \to 0} p\kappa = s.$

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : y \leq x, y \in \mathbb{Z}, x \in \mathbb{R}\})$:

$$\forall \kappa > 0, p = floor(s/\kappa) \wedge 0 \leq |floor(s/\kappa) - s/\kappa| < 1 \Rightarrow |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by κ:

$$\forall \kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\begin{aligned} &\forall \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ &\Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \delta = \epsilon \quad := \quad M = \lim_{\kappa \to 0} p\kappa = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}}, 3.14_{\kappa=10^{-2}}, 3.141_{\kappa=10^{-3}}, ..., \pi_{\lim_{\kappa \to 0}}$.

LEMMA 2.3. $\forall\, n \geq 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa$.

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\begin{aligned} &\forall \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta \\ &\Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa \to 0} \kappa^n = 0. \end{aligned} \tag{2.5}$$

$$\lim_{\kappa \to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa \to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa. \tag{2.6}$$

$\square$

3. Euclidean Volume

DEFINITION 3.1. Countable volume, v_c, is the number of Cartesian product mappings (n-tuples) between the members of n number of disjoint, countable domain sets:

$$\exists\, n, v_c \in \mathbb{N}, \quad x_i, x_j \in \{x_1, \cdots, x_n\}, \quad x_i \bigcap x_{j \neq i} = \emptyset : \quad v_c = \textstyle\prod_{i=1}^n |x_i|. \tag{3.1}$$

THEOREM 3.2. *Euclidean volume, $v = \prod_{i=1}^n s_i$, is the countable volume case, $v_c = \prod_{i=1}^n |x_i|$, of countable sets of same-sized, size κ, infinitesimals in each domain interval, $[a_i, b_i]$, and the range interval, $[v_a, v_b]$.*

$$v_c = \textstyle\prod_{i=1}^n |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^n s_i, \;\; v = v_a - v_b, \;\; s_i = b_i - a_i. \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing $|x_i|$ number of size c subintervals and apply ruler convergence (2.2):

$$\forall\, i\, n \in \mathbb{N}, \;\; i \in [1, n], \; \kappa > 0 \;\wedge\; floor(s_i/\kappa) = |x_i| \;\Rightarrow\; s_i = \lim_{\kappa \to 0}(|x_i| \cdot \kappa). \tag{3.3}$$

$$s_i = \lim_{\kappa \to 0}(|x_i| \cdot \kappa) \quad \Leftrightarrow \quad \textstyle\prod_{i=1}^n s_i = \prod_{i=1}^n \lim_{\kappa \to 0}(|x_i| \cdot \kappa). \tag{3.4}$$

$$\textstyle\prod_{i=1}^n s_i = \prod_{i=1}^n \lim_{\kappa \to 0}(|x_i| \cdot \kappa) \quad \Leftrightarrow \quad \prod_{i=1}^n s_i = \lim_{\kappa \to 0}(\prod_{i=1}^n |x_i|) \cdot \kappa^n. \tag{3.5}$$

Apply lemma 2.3 to equation 3.5:

$$\begin{aligned} &\textstyle\prod_{i=1}^n s_i = \lim_{\kappa \to 0}(\prod_{i=1}^n |x_i|) \cdot \kappa^n \quad \wedge \quad \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa \\ &\Rightarrow \quad \textstyle\prod_{i=1}^n s_i = \lim_{\kappa \to 0}(\prod_{i=1}^n |x_i|) \cdot \kappa. \end{aligned} \tag{3.6}$$

Apply the ruler (2.1) and ruler convergence (2.2) to v:

$$\exists\, v \in \mathbb{R} :\; v_c = floor(v/\kappa) \quad \Leftrightarrow \quad v = \lim_{\kappa\to 0} v_c \cdot \kappa. \tag{3.7}$$

Multiply both sides of the countable volume equation 3.1 by κ:

$$v_c = \prod_{i=1}^{n} |x_i| \quad \Leftrightarrow \quad v_c \cdot \kappa = (\prod_{i=1}^{n} |x_i|) \cdot \kappa \tag{3.8}$$

Apply the ruler (2.1) and ruler convergence (2.2) to equation 3.8:

$$v_c \cdot \kappa = (\prod_{i=1}^{n} |x_i|) \cdot \kappa \quad \Leftrightarrow \quad \lim_{\kappa\to 0} v_c \cdot \kappa = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|) \cdot \kappa. \tag{3.9}$$

Combine equations 3.7, 3.9, and 3.6:

$$\begin{aligned} v = \lim_{\kappa\to 0} v_c \cdot \kappa \quad \wedge \quad \lim_{\kappa\to 0} v_c \cdot \kappa = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|) \cdot \kappa \quad \wedge \\ \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|) \cdot \kappa = \prod_{i=1}^{n} s_i \quad \Leftrightarrow \quad v = \prod_{i=1}^{n} s_i. \end{aligned} \tag{3.10}$$ □

4. Distance

4.1. Countable cuboid n-volume.

DEFINITION 4.1. The countable cuboid volume, d_c^n, is the sum of m number of sets of countable cuboid volumes.

$$\forall\, n \in \mathbb{N}, \quad d_c \in \{0, \mathbb{N}\} \quad \exists\, m \in \mathbb{N}, \quad x_1, \cdots, x_m \in X, \quad \bigcap_{i=1}^{m} x_i = \emptyset :$$
$$d_c^n = \sum_{i=1}^{m} |x_i|^n.$$

4.2. Minkowski distance (L_p norm).

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

THEOREM 4.2. *The Minkowski distances (L_p norms) are derived from the sum of countable cuboid n-volumes* (4.1).

$$d_c^n = \sum_{i=1}^{m} |x_i|^n \quad \Rightarrow \quad \exists\, d, s_1, \cdots, s_m \in \mathbb{R} : \quad d = (\sum_{i=1}^{m} s_i^n)^{1/n}.$$

PROOF. Apply the ruler (2.1):

$$\exists\, d, s_1, \cdots, s_m \in \mathbb{R} :\; d_c = floor(d/\kappa) \quad \wedge \quad |x_i| = floor(s_i/\kappa). \tag{4.1}$$

Apply the ruler convergence (2.2):

$$d_c^n = \sum_{i=1}^{m} |x_i|^n \;\Rightarrow\; d^n = \lim_{\kappa\to 0}(d_c\kappa)^n = \lim_{\kappa\to 0} \sum_{i=1}^{m}(|x_i|\kappa)^n = \sum_{i=1}^{m} s_i^n. \tag{4.2}$$

$$d^n = \sum_{i=1}^{m} s_i^n \quad \Leftrightarrow \quad d = (\sum_{i=1}^{m} s_i^n)^{1/n}. \tag{4.3}$$ □

4.3. Distance inequality. Proving that all Minkowski distances (L_p norms) satisfy the metric space triangle inequality requires another inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N}, \;\; v_a, v_b \geq 0 :\;\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand the n-volume, $(v_a^{1/n} + v_b^{1/n})^n$, using the binomial expansion:

$$\begin{aligned} \forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq v_a + v_b + \\ \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n. \end{aligned} \tag{4.4}$$

Take the n^{th} root of both sides of the inequality:

$$\forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \tag{4.5}$$ □

4.4. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m,n \in \mathbb{N},\;\; a_i, b_i \geq 0 :\;\; (\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$\forall\, m,n \in \mathbb{N},\;\; v_a, v_b \geq 0 : \quad v_a = \sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge \\ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad ((\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} = \\ (\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \qquad \square \tag{4.6}$$

4.5. Metric Space. All Minkowski distances (L_p norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,w) \leq d(u,v) + d(v,w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.7}$$

Apply the distance inequality (4.3) to the inequality 4.7:

$$(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \;\; \wedge \;\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ \Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ \Rightarrow \quad d(u,w) = (u^p + w^p)^{1/p} \leq \\ (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u,v) + d(v,w). \qquad \square \tag{4.8}$$

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,v) = d(v,u).$

PROOF. By the commutative law of addition:

$$\forall\, p \,:\, p \geq 1, \quad d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ \Rightarrow \quad d(u,v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v,u). \qquad \square \tag{4.9}$$

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\; [a_2, b_2], \quad u = b_1 - a_1,\; v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\; v \geq 0. \tag{4.10}$$

$$p \geq 1,\;\; u, v \geq 0 \quad \Rightarrow \quad d(u,v) = (u^p + v^p)^{1/p} \geq 0. \tag{4.11}$$ $\square$

THEOREM 4.8. *Identity of Indiscernibles:* $d(u,u) = 0.$

PROOF. From the non-negativity property (4.7):

$$d(u,w) \geq 0 \quad \wedge \quad d(u,v) \geq 0 \quad \wedge \quad d(v,w) \geq 0 \\ \Rightarrow \quad \exists\, d(u,w) = d(u,v) = d(v,w) = 0. \tag{4.12}$$

$$d(u,w) = d(v,w) = 0 \quad \Rightarrow \quad u = v. \tag{4.13}$$

$$d(u,v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u,u) = 0. \tag{4.14}$$

□

5. Applications to physics

5.1. At most 3 dimensions of physical space. The following two proofs are in the physics section because limiting the domain intervals/dimensions to a strict linearly ordered set that can be sequenced in all n-at-a-time orders, is an additional restriction on volume and distance used to explain why physical space is limited to a cyclic set of 3 dimensions.

DEFINITION 5.1. Strict linearly ordered set:

$$\forall\, i\, n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i\ \in \{x_1, \ldots, x_n\}, \\ \textit{successor}\ x_i = x_{i+1}\ \ \wedge\ \ \textit{predecessor}\ x_{i+1} = x_i.$$

DEFINITION 5.2. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i\, j\, n \in \mathbb{N},\ \ \forall\, x_i\, x_j \in \{x_1, \ldots, x_n\},\ \ \textit{successor}\ x_i = x_j \ \ \Leftrightarrow\ \ \textit{predecessor}\ x_j = x_i.$$

THEOREM 5.3. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n\ \ \wedge\ \ j = 1\ \ \Rightarrow\ \ \textit{successor}\ x_n = x_1\ \ \wedge\ \ \textit{predecessor}\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (5.2):

$$i = n\ \ \wedge\ \ j = 1\ \ \wedge\ \ \textit{successor}\ x_i = x_j\ \ \Rightarrow\ \ \textit{successor}\ x_n = x_1. \tag{5.1}$$

Applying the definition of the symmetry property (5.2) to conclusion 5.1:

$$\textit{successor}\ x_i = x_j\ \ \Rightarrow\ \ \textit{predecessor}\ x_j = x_i\ \ \Rightarrow\ \ \textit{predecessor}\ x_1 = x_n. \tag{5.2}$$

□

THEOREM 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (5.3).

DEFINITION 5.5. Successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize). \tag{5.3}$$

DEFINITION 5.6. Predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1). \tag{5.4}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:
(5.5)
$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize).$$

Prove that every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \tag{5.6}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \tag{5.7}$$
$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1). \tag{5.8}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.9}$$
$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \tag{5.10}$$
$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1). \tag{5.11}$$
$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize). \tag{5.12}$$
$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.13}$$
$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \tag{5.14}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\forall\ setsize > 3: \quad \neg Successor(1, 3, setsize > 3) \\ \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.15}$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$\forall\ setsize > 3: \quad \neg Predecessor(1, 3, setsize > 3) \\ \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \tag{5.16}$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$\forall\ setsize > 3: \quad \neg Adjacent(1, 3, setsize > 3) \\ \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \tag{5.17}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

5.2. Spacetime geometry. From the Minkowski distance proof (4.2), two linearly independent interval lengths, r_1 and r_2, defines a 2-space having corresponding cuboid 2-volumes that sum to a 2-volume: $r^2 = r_1^2 + r_2^2$. And from the 3D proof (5.4), if r, r_2, and r_2 are 3-dimensional distances, then a higher dimension interval length, t, must have a different type that is related to the distances via unit-factoring, conversion ratios:

$$(5.18)\quad r^2 = r_1^2 + r_2^2 \quad \wedge \quad \exists\, r_c, t_c, c \in \mathbb{R} : \; (r_c/t_c) = c = r/t \quad \Rightarrow \quad (ct)^2 = r_1^2 + r_2^2.$$

$$(5.19)\quad (ct)^2 = r_1^2 + r_2^2 \quad \wedge \quad \exists\, r_v, t_v, v \in \mathbb{R} : \; (r_v/t_v) = v = r_2/t \\ \Rightarrow \quad r_1 = \sqrt{(ct)^2 - (vt)^2} = ct\sqrt{1-(v/c)^2}.$$

Local distance, r_1, contracts relative to r as $v \to c$:

$$(5.20)\qquad r_1 = ct\sqrt{1-(v/c)^2} \quad \wedge \quad ct = r \quad \Rightarrow \quad r_1 = r\sqrt{1-(v/c)^2}.$$

Local time, t_1, dilates (gets smaller) relative to t as $v \to c$:

$$(5.21)\qquad r_1 = ct\sqrt{1-(v/c)^2} \quad \wedge \quad t_1 = r_1/c \quad \Rightarrow \quad t_1 = t\sqrt{1-(v/c)^2}.$$

Using the (-+++) form of Minkowski's flat spacetime [**Bru17**], the size of the event separation interval, r_1^2, is:

$$(5.22)\quad r_1^2 = -r_2^2 + r^2 \;\wedge\; \exists\, r_c, t_c, c \in \mathbb{R} : \; (r_c/t_c)t = ct = r_2 \;\wedge\; r^2 = x^2 + y^2 + z^2 \\ \Rightarrow \quad r_1^2 = -(ct)^2 + x^2 + y^2 + z^2.$$

5.3. Newton's gravity force equation. m_1 and m_2, are the sizes of two independent masses, where each size κ component of a mass exerts a force on each size κ component of the other mass. If p_1 and p_2 are the number of size κ components in each mass, then the total force, F, is equal to the total number of forces, $p_1 \cdot p_2$, and proportionate to the size, κ, of each component. Applying the ruler (2.1) and volume proof (3.2), where the force, F, is defined as the rest mass, m_0, times acceleration, a:

$$(5.23)\quad p_1 = floor(m_1/\kappa) \quad \wedge \quad p_2 = floor(m_2/\kappa) \quad \wedge \quad F := m_0 a \propto (p_1 \cdot p_2)\kappa \\ \Rightarrow \quad F := m_0 a \propto \lim_{\kappa\to 0}(p_1 \cdot p_2)\kappa = \lim_{\kappa\to 0}(p_1 \cdot p_2)\kappa^2 = \lim_{\kappa\to 0} p_1\kappa \cdot p_2\kappa = m_1 m_2.$$

$$(5.24)\quad F := m_0 a := m_0 r/t^2 \propto m_1 m_2 \;\wedge\; m_0 = m_1 \;\Rightarrow\; r \propto m_2 \;\Rightarrow \\ \exists\, m_G, r_G \in \mathbb{R} : \; r = (r_G/m_G)m_2,$$

where: r is Euclidean distance, t is time, and r_G/m_G is a unit-factoring ratio.

$$(5.25)\quad m_0 = m_1 \quad \wedge \quad r = (m_G/r_G)m_2 \quad \wedge \quad F = m_0 r/t^2 \\ \Rightarrow \quad F = m_0 r/t^2 = (r_G/m_G)m_1 m_2/t^2.$$

Using the local frame of reference, $r = ct\sqrt{1-(v/c)^2}$, and $v = 0$:

$$(5.26)\quad r = ct \quad \wedge \quad F = (r_G/m_G)m_1 m_2/t^2 \quad \Rightarrow \\ F = ((r_G/m_G)c^2 m_1 m_2/r^2 = G m_1 m_2/r^2,$$

where the constant, $G = (r_G/m_G)c^2$, has the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$. And where $|v| > 0$, $F = (r_G/m_G)(c^2 - v^2)m_1 m_2/r^2$.

5.4. Coulomb's charge force. q_1 and q_2, are the sizes of two independent charges, where each size κ component of a charge exerts a force on each size κ component of the other charge. If p_1 and p_2 are the number of size κ components in each charge, then the total force, F, is equal to the total number of forces, $p_1 \cdot p_2$, and proportionate to the size, κ, of each component. Applying the ruler (2.1) and volume proof (3.2), where the force, F, is defined as the rest mass, m_0, times acceleration, a:

$$(5.27)\quad p_1 = floor(q_1/\kappa) \quad \wedge \quad p_2 = floor(q_2/\kappa) \quad \wedge \quad F \propto (p_1 \cdot p_2)\kappa$$
$$\Rightarrow \quad F := m_0 a \propto \lim_{\kappa \to 0}(p_1 \cdot p_2)\kappa = \lim_{\kappa \to 0}(p_1 \cdot p_2)\kappa^2 = \lim_{\kappa \to 0} p_1 v/\kappa \cdot p_2 \kappa = q_1 q_2.$$

$$(5.28)\quad F := m_0 a := m_0 r/t^2 \propto q_1 q_2 \quad \wedge$$
$$m_0 = (m_G/r_G)(r_C/q_C)q_1 = (m_G r_C/q_C r_G)q_1 \quad \Rightarrow \quad r \propto q_2$$
$$\Rightarrow \quad \exists\, q_C, r_C \in \mathbb{R} : \; r = (r_C/q_C)q_2,$$

where: r is Euclidean distance, t is time, m_G/q_C and r_C/q_C are unit-factoring ratios.

$$(5.29)\quad m_0 = (m_G r_C/q_C r_G)q_1 \quad \wedge \quad r = (r_C/q_C)q_2 \quad \wedge \quad F = m_0 r/t^2$$
$$\Rightarrow \quad F = m_0 r/t^2 = (m_G/r_G)(r_C/q_C)^2 q_1 q_2/t^2.$$

Using the local frame of reference, $r = ct\sqrt{1-(v/c)^2}$, and $v = 0$:

$$(5.30)\quad r = ct = (r_c/t_c)t \quad \wedge \quad a_G = r_c/t_c^2 \quad \wedge \quad F = (m_G/r_G)(r_C/q_C)^2 q_1 q_2/t^2$$
$$\Rightarrow \quad F = ((m_G/r_G)(r_C/q_C)^2(r_c/t_c)^2)q_1 q_2/r^2 =$$
$$(m_G a_G)(r_c/r_G)(r_C/q_C)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2,$$

where the predicted charge constant, $k_e = (m_G a_G)(r_c/r_G)(r_C/q_C)^2$, has the SI units: $N \cdot m^2 \cdot c^{-2}$. And where $|v| > 0$, $F = ((m_G/r_G)(r_C/q_C)^2)(c^2 - v^2)q_1 q_2/r^2$.

5.5. Einstein-Planck and energy-charge equations: Combining the ratio (constant first derivative) equations: $m = (m_p/r_p)r$ and $r/t = r_c/t_c = c \Rightarrow m(ct)^2 = (m_p/r_p)r \cdot r^2$. Dividing both sides by t^2: $E = mc^2 = (m_p/r_p)r \cdot (r/t)^2 = (m_p/r_p)r \cdot (r_c/t_c)^2 = (m_p r_c/r_p t_c)c \cdot r = (m_p r_c c) \cdot (r/(r_p t_c)) = h \cdot f$, which is the Einstein-Planck equation, where the Planck constant is, $h = m_p r_c c$, and $f = r/(r_p t_c)$ is the frequency in cycles per second. $h/c = m_p r_c = k_W$, where r_c is the work displacement (Compton wavelength) of a particle with the rest mass, m_p.

Likewise, for charge, $r = (r_C/q_C)q = (r_p/m_p)m \Rightarrow m = (m_p/r_p)(r_C/q_C)q \Rightarrow E = mc^2 = (m_p/r_p)(r_C/q_C)qc^2 = (m_p r_c c) \cdot (r_C q/r_p q_C t_c) = h \cdot f'$.

6. Insights and implications

(1) The Euclidean volume proof (3.2) shows that "flat" space is where the ratio of a range infinitesimal size to a neighbor range infinitesimal size is 1 and the distance infinitesimals are all the same size, for example: $\mathrm{d}x =, \mathrm{d}y =, \mathrm{d}z = \kappa$, independent of the corresponding domain interval sizes ($[0,x]$, $[0,y]$, $[0,z]$) and infinitesimals of other types (for example, $\mathrm{d}t$, $\mathrm{d}m$ and $\mathrm{d}q$) are proportionate to κ, independent of their corresponding interval sizes ($[0,t]$, $[0,m]$, $[0,q]$). Whereas, the infinitesimal sizes in

derivative-based measures of curvature (for example, the Laplacian and Riemann curvature tensors) vary with their corresponding interval sizes. The ratio of range intervals as a measure of curvature maps directly to adjacent geodesics in general relativity and might help to simplify and calculate solutions to Einstein's field equations.

(2) It was shown that every distance measure that is a bijective, inverse function of an n-volume has a corresponding Minkowski distance (4.2), which have the properties of metric space (4.5). Therefore, if the definition of a complete metric space allows functions that cannot be reduced to a Minkowski distance, then the definition of a complete metric space is not a sufficient filter to obtain only "geometric" distance measures within an n-volume.

(3) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line (equation), where it is assumed that the straight line length is the smallest distance [**Joy98**]. And proofs that a straight line (equation) is the smallest distance have equated the straight line to the Euclidean distance.

All geometric distance measures are functions that can be reduced to a function of the sum of cuboid n-volumes, which are the Minkowski distances (4.2), $d = (\sum_{i=1}^{m} s_i^n)^{1/n}$. If m represents the number of domain intervals, one interval from each dimension, then $1 \leq n \leq m$. And $m = 2 \Rightarrow 1 \leq n \leq 2$, which constrains all Minkowski distances to a range from Manhattan distance (the largest distance) to Euclidean distance (the smallest distance) in Euclidean (flat) 2-space.

(4) Hilbert spaces allow fractional dimensions (fractals), which is the case of intersecting domain sets and requires generalizing the countable volume definition (3.1), $v_c = \prod_{i=1}^{n} |x_i|$, to: $v_c = \prod_{i=1}^{n}(|x_i| - |x_i \bigcap(\bigcup_{j=1, i\neq j}^{n} x_j)|)$.

(5) Compare the distance sum inequality (4.4),

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n},$$

used to prove that all Minkowski distances satisfy the metric space triangle inequality property (4.5), to Minkowski's sum inequality:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n)^n)^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

Note the exponent difference in the left side of the two inequalities:

$$(\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \quad vs. \quad (\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n}.$$

Minkowski's sum inequality proof assumes: convexity and the L_p space inequalities (for example, Hölder's inequality or Mahler's inequality) or the triangle inequality. In contrast, the distance (sum) inequality is a more fundamental inequality that does not require the assumptions of the Minkowski sum inequality.

(6) From the 3D proof (5.4), more than 3 dimensions of intervals must have a different type with lengths that can only be related to a geometric length, r, via constant, unit-factoring, conversion ratios (both direct and inverse proportionate ratios). The direct proportion ratios for time, mass, and charge are: $r = (r_c/t_c)t = ct = (r_G/m_G)m = (r_C/q_C)q$. An inverse proportion ratio is the mass-displacement ratio: $mr = (m_p r_c) = k_W$. The

speed of light, c, the gravity constant, G, the charge constant, k_e, and the Planck constant, h were all derived from these constant ratios.

(7) Here, the spacetime equations (5.2) were derived much more simply from the properties of the set-based derivations of Minkowski distance (4.2) and 3D space proofs (5.4). And the equations were derived without using the Lorentz transformations and Einstein's postulates that other derivations have required.

(8) The derivations in this article show that the spacetime, gravity force, charge force, and Einstein-Planck equations all depend on time being proportionate to distance: $r = (r_c/t_c)t = ct$. For example, from the derivation of Newton's gravity equation (5.26), where $v = 0$: $G = (r_G/m_G)c^2$. Likewise, from the derivation of Coulomb's charge force equation (5.30) the constant, where $v = 0$: $k_e = (m_G/r_G)(r_C/q_C)^2c^2$. And from the derivation of the Planck constant (5.5), $h = (m_p r_c)c$.

(9) The derivation of the Planck-Einstein equation (5.5), $h = (m_p r_c)c = k_W c$, shows that the Planck constant, h, is the product of two fundamental constants, the work constant, $h/c = m_p r_c = k_W \approx 2.2102190943 \cdot 10^{-42}\ kg\ m$, and the speed of light constant, c.

The mass-displacement equation, $k_W = m_p r_c$, simplifies physics equations that contain both c and h. The equation says that, for a given amount of energy, a larger mass will have a smaller displacement, which makes those simplified equations more intuitive.

(10) Applying the ruler (2.1) and volume proof (3.2) to the Cartesian product of same-sized, infinitesimal mass forces and charge forces to derive Newton's gravity force (5.3) and Coulomb's charge force (5.4) equations provide some firsts and some insights into physics:
   (a) These are the first derivations to not assume the inverse square law or Gauss's divergence theorem.
   (b) These are the first derivations to show that the definition of force, $F := m_0 a$, containing acceleration, $a = r/t^2$, where r is a distance that is proportionate to time, t, generates the inverse square law.
   (c) Using Occam's razor, those versions of constants like: charge, vacuum magnetic permeability, etc. that contain the value 4π might be incorrect because those constants are based on the less parsimonious assumption that the inverse square law is due to Gauss's flux divergence on a sphere having the surface area, $4\pi r^2$.
   (d) The derived relativistic gravity and charge force equations are: $F = (r_G/m_G)(c^2 - v^2)m_1 m_2/r^2$ (5.26) and
   $F = ((m_G/r_G)(r_C/q_C)^2)(c^2 - v^2)q_1 q_2/r^2$ (5.30).
   Note that $v \to c \Rightarrow F \to 0$. That is, gravity and charge have less effect on a (charged) mass with a higher momentum.

(11) A state is represented by a constant value. And a constant value, by definition, cannot vary with distance and time interval lengths. Therefore, the spin states of two quantum entangled electrons and the polarization states of two quantum entangled photons are independent of the amount of distance and time between the entangled particles.

(12) It was proved that sequencing through a set, having a strict linear order via the successor/predecessor relations in all n-at-time permutations, is a

cyclic set with at most 3 members (5.4).

(a) Sequencing through cyclic set of 3 members starting at each set member is the basis of the permutation (Levi-Civita) symbol, ϵ_{ijk}, where: $\epsilon_{ijk} = 1$ when sequenced in successor order $((i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\})$, $\epsilon_{ijk} = -1$ when sequenced in predecessor order $((i, j, k) \in \{(3, 2, 1), (2, 1, 3), (1, 3, 2)\})$, and otherwise $\epsilon_{ijk} = 0$. The ϵ_{ijk} values are the components of the Levi-Civita pseudo-tensor.

(b) Using Occam's razor, a strict linear order and symmetry is the most parsimonious explanation for observing only a cyclic set of 3 dimensions of physical distance and volume.

(c) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (5.4), each member being an ordered and symmetric set of 3 dimensions (three balls).

(d) Each of 3 ordered and symmetric dimensions of space can have only 3 sequentially ordered and symmetric state values. The 3 vector orientations (-1, 0, 1) per dimension makes physical space chiral (allows mirror images, clockwise/counter-clockwise rotations, and the right-hand rule that limits the vector cross product to 3 dimensions).

(e) Each of the 3 ordered and symmetric dimensions of space could correspond to an unordered collection (bag) of discrete state values. Bags are non-deterministic. For example, every time an unordered binary state is "pulled" from a bag, there is a 50 percent chance of getting one of the binary values.

(13) It was shown that some fundamental geometry (volume and the Minkowski distances/L_p norms) and physics (gravity force and charge force) are derived from the combinatorial (Cartesian product) mappings between the same-sized, size κ, infinitesimals of real-valued domain intervals. The proofs and derivations in this article show that the ruler (2.1) is a tool to directly express and solve such combinatorial relations that occur often in geometry, probability, physics, etc. that might be difficult to directly express with differential equations and integrals.

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