SOME SET PROPERTIES UNDERLYING GEOMETRY AND PHYSICS

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Abstract. Volume and distance equations are proved to be instances of ordered sets of combinations (n-tuples). The combinatorial properties can limit volume and distance to a set of 3 dimensions. More dimensions have different types (are members of other sets), with ratios of a distance unit to units of time, mass, and charge. The proofs and ratios are used to: 1) derive well-known gravity, charge, electromagnetic equations, special and general relativity equations, and quantum physics equations; 2) derive the gravity, charge, vacuum permittivity, vacuum permeability, Planck, and fine structure constants; 3) add quantum extensions to gravity and charge equations. All the proofs are verified in Rocq.

Contents

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1. Introduction

The Riemann integral, Lebesgue integral, and Lebesgue measure define Euclidean volume as the product of interval sizes $\subset \mathbb{R}^n$ [8] [15]. And Euclidean distance, vector magnitude, and the many "spaces": vector, inner product, metric, Hausdorff, Cauchy, etc. are definitions, in analysis [18] [8] [15].

Justification of the definitions uses finger-pointing to geometry because the definitions only *describe* aspects of volume and distance. Deriving volume and distance equations from an abstract set and limit-based foundation exposes the principles that *explain*, cause, the volume and distance equations, which provides some tools for mathematical analysis, physics, and engineering.

All the proofs in this article have been verified using using the Rocq proof verification system [14]. The formal proofs are in the Rocq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

Using integral and differential calculus and σ-algebras (for example, the Lebesgue measure) to prove the volume and distance theorems in this article would result in circular logic due to inherent Euclidean assumptions. Therefore, a "ruler" measure of intervals, $[a, b] \subset \mathbb{R}$, will be used to prove the theorems.

Where $|x_i|$ is the cardinal of (number of elements in) a countable set, x_i, the countable number of ordered combinations (n-tuples) is v_c. The ruler measure will be used to prove the Euclidean volume relation:

$$\begin{aligned} &\forall\, x_i \in \{x_1, \cdots, x_n\} = X, \quad \textstyle\bigcap_{x_i \in X} x_i = \emptyset: \quad v_c = \prod_{i=1}^{n} |x_i| \\ &\qquad\qquad \Leftrightarrow \quad v = \textstyle\prod_{i=1}^{n} s_i, \quad s_i = b_i - a_i, \quad [a_i, b_i] \subset \mathbb{R}. \end{aligned} \tag{1.1}$$

For all $n > 1$, there are an infinite number of combinations of domain values, $s_i \cdots s_n$, that multiply to the same range value, v. The volume function, $v = \prod_{i=1}^{n} s_i$ is only bijective where, $\exists\, d: \ v = f(d)$ and $d = f^{-1}(v) = f^{-1}(\prod_{i=1}^{n} s_i)$.

The simplest bijective case of countable n-volumes, where $n \geq 1$, is:

$$\exists\, d_c, v_c, |x_i| \in \{0, \mathbb{N}\}: \ v_c = \textstyle\prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n. \tag{1.2}$$

The ruler measure will be used to prove that:

$$d_c^n = \textstyle\sum_{i=1}^{m} v_{c_i} = \sum_{i=1}^{m} (\prod_{j=1}^{n} |x_{i,j}|) \ \Leftrightarrow \ d^n = \sum_{i=1}^{m} v_i = \sum_{i=1}^{m} (\prod_{j=1}^{n} s_{i,j}). \tag{1.3}$$

The $n = 2$ case is the basis of the inner product. Where each v_{c_i} is also a bijective function, $v_{c_i} = d_{c_i}^n$, the ruler measure will be used to prove that:

$$d_c^n = \textstyle\sum_{i=1}^{m} d_{c_i}^n \quad \Leftrightarrow \quad d^n = \sum_{i=1}^{m} d_i^n. \tag{1.4}$$

$|d|$ is the p-norm (Minkowski distance) [12], which will be proved to imply the metric space properties [15]. The $n = 2$ case is, obviously, the Euclidean distance.

Volume and distance are derived from sets of ordered combinations (n-tuples). Volume and distance have another combinatorial (permutation) property.

Calculating volume requires multiplying a sequentially ordered set of domain values. And calculating distance requires summing a sequentially ordered set of values. The commutative properties of multiplication and addition allows sequencing an ordered set in all $n!$ permutations.

Reliably re-sequencing a set of domain values in the same order requires assigning a sequential order to the values. Further, the *only* sequential order, where you can start with any value and sequence in a repeatable order, is a cyclic order.

The second sequenced member must be either the *immediate* cyclic successor or *immediate* cyclic predecessor. And repeatable, sequencing of a cyclic set in all $n!$ permutations, is a symmetry, where every set member is either an *immediate* cyclic successor or an *immediate* cyclic predecessor to every other set member is, herein, referred to as an "immediate symmetric" cyclic set (ISCS). An ISCS will be proved to have $n \leq 3$ members.

An ISCS of 3 "distance" dimensions requires more dimensions to have non-distance types (be members of other sets). Let $\tau = \{t\ (time), m\ (mass), q\ (charge)\}$ be an ISCS of "non-distance" dimensions $\subseteq \mathbb{R}$, where for each subinterval (unit) length, r_c, of distance interval length, r, there are unit lengths: t_c of time interval length, t; m_c of mass interval length, m; and q_c of charge interval length, q, such that: $r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q$.

The proofs and the 3 direct proportion ratios are used to provide simple derivations of: the Newton, Gauss, and Poisson gravity equations, the special relativity equations, the Schwarzschild time dilation and black hole metric equations (pointing to a simplified method of finding solutions to Einstein's general relativity equations), Coulomb's charge force, Gauss, Lorentz, and Faraday electromagnetic equations [and the constants: gravity (G), charge (k_e), vacuum permittivity (ε_0), and vacuum permeability (μ_0)].

Next, algebraic manipulation of the 3 direct proportion ratios yields 3 inverse proportion ratios, $r = t_c r_c/t = m_c r_c/m = q_c r_c/q$. The Planck units and fine structure constant, α, are derived from ratios of subtypes: $\alpha = q_e^2/q_p^2$ where q_e is the elementary (electron) charge and q_p: $q_c^2/q_p^2 = 2\pi$, is the Planck charge unit.

The combination of the direct and inverse proportion ratios are used to derive the Planck relation, the Planck constant, $h = (m_c r_c)(r_c/t_c)$, the Compton, position-space Schrödenger, and Dirac wave equations. The inverse proportion ratios are also used to add quantum extensions to some general relativity and classical physics equations. Finally, the derivations expose some incorrect assumptions commonly used in physics today.

2. Ruler measure and convergence

A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of size κ subintervals. The ruler is both an inner and outer measure of an interval.

Definition 2.1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\ [a, b] \subset \mathbb{R}$, $s = b - a \ \wedge\ 0 < \kappa \leq 1 \ \wedge\ (p = floor(s/\kappa) \ \vee\ p = ceiling(s/\kappa))$.

Theorem 2.2. *Ruler convergence:* $M = \lim_{\kappa\to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, 0 < \kappa \leq 1, p = floor(s/\kappa) \wedge 0 \leq |floor(s/\kappa) - s/\kappa| < 1 \Rightarrow |p - s/\kappa| < 1. \quad (2.1)$$

Multiply both sides of inequality 2.1 by κ:

$$\forall\, 0 < \kappa \leq 1, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \qquad (2.2)$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \delta = \epsilon \quad := \quad M = \lim_{\kappa\to 0} p\kappa = s. \quad \Box \end{aligned} \qquad (2.3)$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, ..., \pi_{\lim_{\kappa\to 0}}$.

Lemma 2.3. $\forall\, n \geq 1, \quad 0 < \kappa \leq 1 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa$.

Proof. The formal proof , "lim_c_to_n_eq_lim_c," is in the Rocq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa \leq 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \quad (2.4)$$

$$\forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta$$
$$\Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa\to 0} \kappa^n = 0. \quad (2.5)$$

$$\lim_{\kappa\to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa\to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa. \quad (2.6)$$

□

3. Volume

Definition 3.1. A countable n-volume is the number of ordered combinations (n-tuples), v_c, of the members of n number of disjoint, countable domain sets, x_i:

$$x_i \in \{x_1, \cdots, x_n\} = X,\ |x_i| \in \{0, \mathbb{N}\} :\ \bigcap_{x_i \in X} x_i = \emptyset \ \wedge\ v_c = \prod_{i=1}^{n} |x_i|. \quad (3.1)$$

Theorem 3.2. *Euclidean volume,*

$$\forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\ [v_a, v_b] \subset \mathbb{R},\ s_i = b_i - a_i,\ v = v_b - v_a :$$
$$v_c = \prod_{i=1}^{n} |x_i| \quad \Leftrightarrow \quad v = \prod_{i=1}^{n} s_i. \quad (3.2)$$

The formal proof, "Euclidean_volume," is in the Rocq file, euclidrelations.v.

Proof.

$$v_c = \prod_{i=1}^{n} |x_i| \ \Leftrightarrow\ v_c\kappa = (\prod_{i=1}^{n} |x_i|)\kappa \ \Leftrightarrow\ \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|)\kappa. \quad (3.3)$$

Apply the ruler (2.1) and ruler convergence (2.2) to equation 3.3:

$$\exists\, v, \kappa \in \mathbb{R} :\ v_c = floor(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa \quad \wedge$$
$$\lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|)\kappa \quad \Rightarrow \quad v = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|)\kappa. \quad (3.4)$$

Apply lemma 2.3 to equation 3.4:

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|)\kappa \quad \wedge \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa$$
$$\Rightarrow \quad v = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|)\kappa^n = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|\kappa). \quad (3.5)$$

Apply the ruler (2.1) and ruler convergence (2.2) to s_i:

$$\exists\, s_i, \kappa \in \mathbb{R} :\ floor(s_i/\kappa) = |x_i| \quad \Rightarrow \quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i. \quad (3.6)$$

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|\kappa) \quad \wedge \quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i$$
$$\Leftrightarrow \ v = \lim_{\kappa\to 0}(|x_i|\kappa) = \prod_{i=1}^{n} s_i \quad \square \quad (3.7)$$

4. Distance

Definition 4.1. Countable distance,

$$\exists\, n \in \mathbb{N},\ v_c, d_c \in \{0,\ \mathbb{N}\},\ x_i \in \{x_1, \cdots, x_n\} = X :\ \bigcap_{x_i \in X} x_i = \emptyset \quad \wedge$$
$$d_c = |x_1| = \cdots = |x_n| \quad \wedge \quad v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n. \quad (4.1)$$

Lemma 4.2. *A volume is the sum of volumes,*

$$v_c = d_c^n = \sum_{i=1}^{m} v_{c_i} \quad \Leftrightarrow \quad v = \sum_{i=1}^{m} v_i, \quad v, v_i \in \mathbb{R}.$$

The formal proof, "sum_of _volumes," is in the Rocq file, euclidrelations.v.

Proof. From the condition of this theorem:

$$v_c = \sum_{i=1}^m v_{c_i} \quad \Leftrightarrow \quad \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0} \sum_{j=1}^m (v_{c_i}\kappa). \tag{4.2}$$

Apply lemma 2.3 to equation 4.2:

$$\begin{aligned} &\lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\textstyle\sum_{j=1}^m v_{c_i})\kappa \quad \wedge \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0}\kappa \\ &\Leftrightarrow \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}\textstyle\sum_{j=1}^m (v_{c_i})\kappa^n \Leftrightarrow \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}\textstyle\sum_{j=1}^m (v_{c_i}\kappa). \end{aligned} \tag{4.3}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to equation 4.3:

$$\begin{aligned} &\exists\, v, v_i:\ v = floor(d/\kappa),\ v = \lim_{\kappa\to 0} v_c\kappa \\ \wedge\quad & v_{c_i} = floor(v_i/\kappa),\ v_i = \lim_{\kappa\to 0} v_{c_i}\kappa \quad \wedge \quad \lim_{\kappa\to 0}(d_c\kappa)^n = \lim_{\kappa\to 0}\textstyle\sum_{j=1}^m (v_{c_i}\kappa) \\ &\Leftrightarrow \quad v = \lim_{\kappa\to 0}(d_c\kappa)^n = \lim_{\kappa\to 0}\textstyle\sum_{j=1}^m (v_{c_i}\kappa) = \sum_{j=1}^m v_i^n. \quad \square \end{aligned} \tag{4.4}$$

4.1. **Sum of volumes distance.**

Theorem 4.3. *Sum of volumes distance:*

$$v_c = d_c^n = \sum_{i=1}^m v_{c_i} \quad \Leftrightarrow \quad d^n = \sum_{i=1}^m (\prod_{j=1}^n s_{i_j}).$$

The formal proof, "sum_of_volumes_distance," is in the Rocq file, euclidrelations.v.

Proof. From lemma 4.2 and the Euclidean volume theorem 3.2:

$$\begin{aligned} v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad &\Leftrightarrow \quad d^n = \textstyle\sum_{i=1}^m (\prod_{j=1}^n v_i) \quad \wedge \quad v_i = \prod_{j=1}^n s_{i_j} \\ v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad &\Leftrightarrow \quad d^n = \textstyle\sum_{i=1}^m (\prod_{j=1}^n s_{i_j}). \quad \square \end{aligned} \tag{4.5}$$

4.2. **Minkowski distance (p-norm).**

Theorem 4.4. *Minkowski distance (p-norm):*

$$v_c = d_c^n = \sum_{i=1}^m v_{c_i} = \sum_{i=1}^m d_{c_i}^n \quad \Leftrightarrow \quad d^n = \sum_{i=1}^m d_i^n.$$

The formal proof, "Minkowski_distance," is in the Rocq file, euclidrelations.v.

Proof. From lemma 4.2 and the Euclidean volume theorem 3.2:

$$\begin{aligned} v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad &\Leftrightarrow \quad d^n = \textstyle\sum_{i=1}^m v_i \quad \wedge \quad v_i = \prod_{j=1}^n d_i = d_i^n \\ v_c = d_c^n = \textstyle\sum_{i=1}^m v_{c_i} \quad &\Leftrightarrow \quad d^n = \textstyle\sum_{i=1}^m d^n. \quad \square \end{aligned} \tag{4.6}$$

4.3. **Distance inequality.** The formal proof, distance_inequality, is in the Rocq file, euclidrelations.v.

Theorem 4.5. *Distance inequality*

$$\forall\, n \in \mathbb{N},\ \ v_a, v_b \geq 0:\ \ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

Proof. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$\begin{aligned} &\forall\, v_a, v_b \geq 0: \quad v_a + v_b \leq v_a + v_b + \\ &\textstyle\sum_{i=1}^n \binom{n}{k}(v_a^{1/n})^{n-k}(v_b^{1/n})^k + \sum_{i=1}^n \binom{n}{k}(v_a^{1/n})^k(v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n. \end{aligned} \tag{4.7}$$

Take the n^{th} root of both sides of the inequality 4.7:

$$\forall\, v_a, v_b \geq 0,\ n \in \mathbb{N}:\ v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \ \Rightarrow\ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \tag{4.8}$$

$\square$

4.4. **Distance sum inequality.** The formal proof, distance_sum_inequality, is in the Rocq file, euclidrelations.v.

Theorem 4.6. *Distance sum inequality*

$$\forall\, m,n \in \mathbb{N},\;\; a_i, b_i \geq 0:\;\; (\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

Proof. Apply the distance inequality (4.5):

$$\forall\, m,n \in \mathbb{N},\;\; v_a, v_b \geq 0:\quad v_a = \sum_{i=1}^{m} a_i^n \quad\wedge\quad v_b = \sum_{i=1}^{m} b_i^n \quad\wedge$$
$$(v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad\Rightarrow\quad ((\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} =$$
$$(\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \quad \square \tag{4.9}$$

4.5. **Metric Space.** All Minkowski distances (p-norms) imply the metric space properties. The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Rocq file, euclidrelations.v.

Theorem 4.7. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad\Rightarrow\quad d(u,w) \leq d(u,v) + d(v,w).$

Proof. $\forall\, p \geq 1,\quad k > 1,\quad u = s_1,\quad w = s_2,\quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.10}$$

Apply the distance inequality (4.5) to the inequality 4.10:

$$(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \;\;\wedge\;\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}$$
$$\wedge \quad v_a = u^p + v^p \quad\wedge\quad v_b = v^p + w^p$$
$$\Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p}$$
$$\Rightarrow \quad d(u,w) = (u^p + w^p)^{1/p} \leq$$
$$(u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u,v) + d(v,w). \quad \square \tag{4.11}$$

Theorem 4.8. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad\Rightarrow\quad d(u,v) = d(v,u).$

Proof. By the commutative law of addition:

$$\forall\, p \,:\, p \geq 1,\quad d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p}$$
$$\Rightarrow \quad d(u,v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v,u). \quad \square \tag{4.12}$$

Theorem 4.9. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad\Rightarrow\quad d(u,w) \geq 0.$

Proof. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\; [a_2, b_2],\quad u = b_1 - a_1,\; v = b_2 - a_2,\quad\Rightarrow\quad u \geq 0,\; v \geq 0. \tag{4.13}$$
$$p \geq 1,\;\; u, v \geq 0 \quad\Rightarrow\quad d(u,v) = (u^p + v^p)^{1/p} \geq 0. \tag{4.14}$$

$\square$

Theorem 4.10. *Identity of Indiscernibles:* $d(u,u) = 0.$

Proof. From the non-negativity property (4.9):

$$d(u,w) \geq 0 \quad\wedge\quad d(u,v) \geq 0 \quad\wedge\quad d(v,w) \geq 0$$
$$\Rightarrow \quad \exists\, d(u,w) = d(u,v) = d(v,w) = 0. \tag{4.15}$$
$$d(u,w) = d(v,w) = 0 \quad\Rightarrow\quad u = v. \tag{4.16}$$

$$d(u,v)=0 \quad \wedge \quad u=v \quad \Rightarrow \quad d(u,u)=0. \tag{4.17}$$

□

4.6. Set properties limiting a set to at most 3 members. The following definitions and proof use first order logic. A Horn clause-like expression is used, here, to make the proof easier to read. By convention, the proof goal is on the left side and supporting facts are on the right side of the implication sign ($\leftarrow$). The formal proofs in the Rocq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

Definition 4.11. Immediate Cyclic Successor of m is n:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\}: \\ Successor(m,n,setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m+1 \leq setsize). \tag{4.18}$$

Definition 4.12. Immediate Cyclic Predecessor of m is n:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\}: \\ Predecessor(m,n,setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m-1 \geq 1). \tag{4.19}$$

Definition 4.13. Adjacent: Member m is sequentially adjacent to member n if the immediate cyclic successor of m is n or the immediate cyclic predecessor of m is n. Notionally:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\}: \\ Adjacent(m,n,setsize) \leftarrow Successor(m,n,setsize) \vee Predecessor(m,n,setsize). \tag{4.20}$$

Definition 4.14. Immediate Symmetric (every set member is sequentially adjacent to every other member):

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\}: \quad Adjacent(m,n,setsize). \tag{4.21}$$

Theorem 4.15. *An immediate symmetric cyclic set is limited to at most 3 members.*

Proof.
Every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (m = setsize \wedge n = 1). \tag{4.22}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (n = m+1 \leq setsize). \tag{4.23}$$

$$Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (n = setsize \wedge m = 1). \tag{4.24}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m+1 \leq setsize). \tag{4.25}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m-1 \geq 1). \tag{4.26}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \wedge m = 1). \tag{4.27}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \wedge n = setsize). \tag{4.28}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m+1 \leq setsize). \tag{4.29}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m-1 \geq 1). \tag{4.30}$$

Member 2 is the only immediate successor of member 1 for all $setsize \geq 3$, which implies member 3 is not ($\neg$) an immediate successor of member 1 for all $setsize \geq 3$:

$$\neg Successor(1, 3, setsize \geq 3) \\ \leftarrow Successor(1, 2, setsize \geq 3) \leftarrow (n = m + 1 \leq setsize). \quad (4.31)$$

Member $n = setsize > 3$ is the only immediate predecessor of member 1, which implies member 3 is not ($\neg$) an immediate predecessor of member 1 for all $setsize >$ 3:

$$\neg Predecessor(1, 3, setsize \geq 3) \\ \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \quad (4.32)$$

For all $setsize > 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not immediate symmetric):

$$\neg Adjacent(1, 3, setsize > 3) \\ \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \quad (4.33)$$

The Symmetric goal matches Adjacent goals 4.22 and fails for all "setsize" greater than three.

5. Applications to physics

Where distance is an immediate cyclic set of dimensions, the 3D proof (4.15) requires more dimensions to have non-distance types. Let $\tau = \{t\ (time), m\ (mass), q\ (charge)\}$ be an immediate cyclic set of type "non-distance" dimensions, where for each subinterval (unit) length, r_c, of distance interval length, r, there are unit lengths: t_c of time interval length, t; m_c of mass interval length, m; and q_c of charge interval length, q, such that:

$$r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q. \quad (5.1)$$

5.1. Derivation of the constant, G, and the gravity laws of Newton, Gauss, and Poisson.

From equation 5.1:

$$r = (r_c/m_c)m \quad \wedge \quad r = (r_c/t_c)t = ct \quad \Rightarrow \quad r/(ct)^2 = (r_c/m_c)m/r^2 \\ \Rightarrow \quad r/t^2 = ((r_c/m_c)c^2)m/r^2 = Gm/r^2, \quad (5.2)$$

where the constant, $G = (r_c/m_c)c^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$ [13].

Newton's law follows from multiplying both sides of equation 5.2 by m:

$$r/t^2 = Gm/r^2 \;\Leftrightarrow\; F := mr/t^2 = Gm^2/r^2. \quad (5.3)$$

$$F = Gm^2/r^2 \;\wedge\; \forall\, m \in \mathbb{R} : \exists\, m_1, m_2 \in \mathbb{R} : m_1 m_2 = m^2 \;\Rightarrow\; F = Gm_1 m_2/r^2. \quad (5.4)$$

From equation 5.2, Gauss's gravity field, $\mathbf{g}$ and Poisson's gravity field, $\nabla\Phi(r, t)$:

$$\mathbf{g} = -\nabla\Phi(\vec{r}, t) := r/t^2 = Gm/r^2 \\ \Rightarrow \quad \nabla \cdot \mathbf{g} = \nabla^2\Phi(\vec{r}, t) = -2Gm/r^3 = (-2Gm/r^3)(2\pi/2\pi) \quad \wedge \quad \rho = m/2\pi r^3 \\ \Rightarrow \quad \nabla \cdot \mathbf{g} = \nabla^2\Phi(\vec{r}, t) = -4\pi G\rho. \quad (5.5)$$

5.2. **Space-time-mass-charge.** Let r be an Euclidean distance. Then by the Minkowski distance theorem (4.4), $r^2 = \sum_{i=1}^{m} r_i^2$. Let, $r' = r_1$ and $r_v^2 = (\sum_{i=2}^{m} r_i^2)$. From equation 5.1, there are ratios μ and ν such that:

$$\forall\, \tau \in \{t, m, q\},\ r^2 = r'^2 + r_v^2,\ \exists\, \mu, \nu :\ r = \mu\tau \quad \wedge \quad r_v = \nu\tau$$
$$\Rightarrow \quad (\mu\tau)^2 = r'^2 + (\nu\tau)^2 \quad \Rightarrow \quad r' = \sqrt{(\mu\tau)^2 - (\nu\tau)^2} \ = \ \mu\tau\sqrt{1 - (\nu/\mu)^2}. \quad (5.6)$$

Rest frame distance, r', contracts relative to stationary frame distance, r, as $\nu \to \mu$:

$$r' = \mu\tau\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad \mu\tau = r \quad \Rightarrow \quad r' = r\sqrt{1 - (\nu/\mu)^2}. \quad (5.7)$$

Stationary frame type, τ, dilates relative to the rest frame type, τ', as $\nu \to \mu$:

$$\mu\tau = r'/\sqrt{1 - (\nu/\mu)^2} \quad \wedge \quad r' = \mu\tau' \quad \Rightarrow \quad \tau = \tau'/\sqrt{1 - (\nu/\mu)^2}. \quad (5.8)$$

Where τ is type, time, the space-like flat Minkowski spacetime event interval is:

$$\mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_v^2 \quad \wedge \quad \mathrm{d}r_v^2 = \mathrm{d}r_1^2 + \mathrm{d}r_2^2 + \mathrm{d}r_3^2 \quad \wedge \quad \mathrm{d}(\mu\tau) = \mathrm{d}r$$
$$\Rightarrow \quad \mathrm{d}r'^2 = \mathrm{d}(\mu\tau)^2 - \mathrm{d}r_1^2 - \mathrm{d}r_2^2 - \mathrm{d}r_3^2. \quad (5.9)$$

5.3. **Derivation of Schwarzschild's gravitational time dilation and black hole metric.** [17] [1] From equations 5.7 and 5.1:

$$\sqrt{1 - (v^2/c^2)} = \sqrt{1 - (v^2/c^2)(r/r)} \quad \wedge \quad r = (r_c/m_c)m$$
$$\Rightarrow \quad \sqrt{1 - (v^2/c^2)} = \sqrt{1 - ((r_c/m_c)m)v^2/rc^2}. \quad (5.10)$$

Where v_{escape} is the escape velocity:

$$\sqrt{1 - (v^2/c^2)} = \sqrt{1 - ((r_c/m_c)m)v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv_{escape}^2$$
$$\Rightarrow \quad \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2(r_c/m_c)mv_{escape}^2/rc^2}. \quad (5.11)$$

$$\sqrt{1 - (v^2/c^2)} = \lim_{v_{escape} \to c} \sqrt{1 - 2(r_c/m_c)mv_{escape}^2/rc^2}$$
$$= \sqrt{1 - 2(r_c/m_c)mc^2/rc^2}. \quad (5.12)$$

Combining equation 5.12 with the derivation of G (5.4):

$$(r_c/m_c)c^2 = G \quad \wedge \quad \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2(r_c/m_c)mc^2/rc^2}$$
$$\Rightarrow \quad \sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2Gm/rc^2}. \quad (5.13)$$

Combining equation 5.13 with equation 5.8 yields Schwarzschild's gravitational time dilation:

$$\sqrt{1 - (v^2/c^2)} = \sqrt{1 - 2Gm/rc^2} \quad \wedge \quad t' = t\sqrt{1 - (v^2/c^2)}$$
$$\Rightarrow \quad t' = t\sqrt{1 - 2Gm/rc^2}. \quad (5.14)$$

Schwarzschild defined the black hole event horizon radius, $r_s := 2Gm/c^2$. From equations 5.7 and 5.15:

$$r' = r\sqrt{1 - (v/c)^2} \quad \wedge \quad \sqrt{1 - (v/c)^2} = \sqrt{1 - 2Gm/rc^2}$$
$$\Rightarrow \quad r' = r\sqrt{1 - 2Gm/rc^2} = r\sqrt{1 - r_s/r}. \quad (5.15)$$

Using the time-like spacetime interval, where $\mathrm{d}s^2$ is negative:

$$r' = r\sqrt{1 - r_s/r} \quad \wedge \quad \mathrm{d}s^2 = \mathrm{d}r'^2 - \mathrm{d}r^2$$
$$\Rightarrow \quad \mathrm{d}s^2 = (\sqrt{1 - r_s/r}\mathrm{d}r')^2 - (\mathrm{d}r/\sqrt{1 - r_s/r})^2 = (1 - r_s/r)\mathrm{d}r'^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2. \tag{5.16}$$

$$\mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}r'^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2 \quad \wedge \quad \mathrm{d}r' = \mathrm{d}(ct) \quad \wedge \quad c = 1$$
$$\Rightarrow \quad \mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}t^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2. \tag{5.17}$$

Using spherical coordinates to translate from 2D to 4D yields Schwarzschild's black hole metric:

$$\mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}t^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2 = f(r, t)$$
$$\Rightarrow \quad \mathrm{d}s^2 = (1 - r_s/r)\mathrm{d}t^2 - (1 - r_s/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + sin^2\theta\mathrm{d}\phi^2) = f(r, t, \theta, \phi)$$
$$\Rightarrow \quad g_{\mu,\nu} = diag[1 - r_s/r),\ (1 - r_s/r)^{-1},\ r^2(\mathrm{d}\theta^2),\ r^2(sin^2\theta\mathrm{d}\phi^2)]. \tag{5.18}$$

5.4. **Simple derivations of solutions to Einstein's general relativity (field) equation.** Step 1) Use the ratios to define functions returning scalar values for each component of the metric, $g_{\nu,\mu}$, in Einstein's field equations [6] [18]: All functions derived from the ratios, where the units on each side of the equation balance, are valid metrics, for example, the previous Schwarzschild black hole metric derivation using the ratios (5.3).

Step 2) Express the EFE as 2D tensors: As shown in equation 5.18, the Schwarzschild metric was first derived as a 2D metric and then expanded to a 4D metric. Further, the 4D flat spacetime interval equation (5.9) is an instance of the 2D equation, $\mathrm{d}r'^2 = \mathrm{d}(ct)^2 - \mathrm{d}r_v^2$, where $\mathrm{d}r_v^2$ is the magnitude of a 3-dimensional vector.

The 2D metric tensor allows using the much simpler 2D Ricci curvature and scalar curvature. And the 2D tensors reduce the number of independent equations to solve.

Step 3) One simple method to translate from 2D to 4D is to use spherical coordinates, where r and t remain unchanged and two added dimensions are the angles, ϕ, and θ. For example, the 2D Schwarzschild metric was translated to 4D using this method in equation 5.18.

5.5. **Derivation of Coulomb's charge constant, k_e, and charge force.**

$$r = (r_c/q_c)q \quad \Rightarrow \quad r^2 = (r_c/q_c)^2q^2 \quad \Rightarrow \quad (r_c/q_c)^2q^2/r^2 = 1. \tag{5.19}$$

$$r = (r_c/t_c)t = ct \quad \wedge \quad r = (r_c/m_c)m = ct$$
$$\Rightarrow \quad mr = (m_c/r_c)rct = (m_c/r_c)(ct)^2 \quad \Rightarrow \quad ((r_c/m_c)/c^2)mr/t^2 = 1. \tag{5.20}$$

$$((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/q_c)^2q_1q_2/r^2 = 1$$
$$\Rightarrow \quad F := mr/t^2 = ((m_c/r_c)c^2)(r_c/q_c)^2q^2/r^2 = k_eq^2/r^2. \tag{5.21}$$

where Coulomb's constant, $k_e = ((m_c/r_c)c^2)(r_c/q_c)^2$, conforms to the base SI units: $kg \cdot m^3 \cdot s^{-2} \cdot C^{-2}$, which is equivalent to the charge SI units: $N \cdot m^2 \cdot C^{-2}$ [7].

$$\exists\ q_1, q_2 \in \mathbb{R} :\ q_1q_2 = q^2 \quad \wedge \quad F = k_eq^2/r^2 \quad \Rightarrow \quad F = k_eq_1q_2/r^2. \tag{5.22}$$

5.6. **Relativistic electromagnetism, Lorentz law, and vacuum permeability, μ_0.** Applying the distance contraction equation 5.7 to Coulomb's charge force equation 5.21, where r is the stationary frame of reference and r' is moving particle frame of reference:

$$r = r'/\sqrt{1-v^2/c^2} \quad \wedge \quad F = k_e q^2/r^2 \quad \Rightarrow \quad F = k_e q^2(1-v^2/c^2)/r'^2. \tag{5.23}$$

$$F = k_e q^2(1-v^2/c^2)/r'^2 \quad \wedge \quad E := k_e q/r'^2 \Rightarrow F = q(E - v^2(k_e/c^2)q/r'^2). \tag{5.24}$$

$$F = q(E - v^2(k_e/c^2)q/r'^2) \quad \wedge \quad B := (k_e/c^2)vq/r'^2 \quad \Rightarrow \quad F = q(E - vB). \tag{5.25}$$

$$F = q(E - vB) \quad \Rightarrow \quad \mathbf{F} = q(\mathbf{E} + \vec{\mathbf{v}} \times \mathbf{B}), \tag{5.26}$$

which is Lorentz's electromagnetic force law, where: the change to a plus sign on the right side of the equation indicates the right-hand curl (torque) rule, the electric field, $E := k_e q/r'^2$, conforms to the SI units $kg \cdot m \cdot s^{-2} \cdot C^{-1} = N \cdot C^{-1}$ and the magnetic field, $B = (k_e/c)q/r^2$, conforms to the base SI units: $kg \cdot s^{-1} \cdot C^{-1} = kg \cdot s^{-2} \cdot A^{-1} = T$.

$$B := (k_e/c^2)vq/r'^2 \quad \wedge \quad B = \mu_0 H \quad \Rightarrow \quad \mu_0 = k_e/c^2 \quad \wedge \quad H = vq/r'^2, \tag{5.27}$$

where $\mu_0 = k_e/c^2 = (r_c/q_c)^2/(r_c/m_c)$ conforms to the SI units $kg \cdot m \cdot C^{-2} = kg \cdot m \cdot s^{-2} A^{-2}$ and $H = vq/r'^2$ conforms to the SI units $m \cdot s^{-1} \cdot C = A \cdot m$.

5.7. **Vacuum permittivity, ε_0, and Gauss's law for electric fields.** From equation 5.24:

$$E = k_e q/r^2 \quad \Leftrightarrow \quad \mathbf{E} = k_e q/\vec{\mathbf{r}}^2. \tag{5.28}$$

$$\mathbf{E} = k_e q/\vec{\mathbf{r}}^2 \quad \Rightarrow \quad \nabla \cdot \mathbf{E} = -2k_e q/\vec{\mathbf{r}}^3 = 2k_e q/\vec{\mathbf{r}}^3. \tag{5.29}$$

$$\nabla \cdot \mathbf{E} = -2k_e q/\vec{\mathbf{r}}^3 \quad \wedge \quad \rho = q/\vec{\mathbf{r}}^3 \quad \Rightarrow \quad \nabla \cdot \mathbf{E} = -2k_e \rho. \tag{5.30}$$

$$\nabla \cdot \mathbf{E} = -2k_e \rho \quad \wedge \quad \varepsilon_0 := 2/k_e \Rightarrow \quad \nabla \cdot \mathbf{E} = -\rho/\varepsilon_0, \tag{5.31}$$

which is Gauss's electric field law [7].

5.8. **Derivation of Faraday's law.** From the magnetic field equation 5.25, where the magnetic field, B, velocity is propagating at the speed of light, $v = c$:

$$B = (k_e/c^2)qv/r^2 \quad \wedge \quad v = c \quad \wedge \quad r = ct \quad \Rightarrow \quad B = (k_e/c^3)q/t^2. \tag{5.32}$$

$$B = (k_e/c^3)q/t^2 \quad \Rightarrow \quad \partial B/\partial t = -(2k_e/c^3)q/t^3. \tag{5.33}$$

$$\partial B/\partial t = -(2k_e/c^3)q/t^3 \quad \wedge \quad r = ct \quad \Rightarrow \quad \partial B/\partial t = -2k_e q/r^3. \tag{5.34}$$

From equation 5.28:

$$\mathbf{E} = k_e q/\vec{\mathbf{r}}^2 \quad \Rightarrow \quad \nabla \times \mathbf{E} = 2k_e q/\vec{\mathbf{r}}^2 \tag{5.35}$$

Combining equations 5.35 and 5.34 yields Faraday's law [7]:

$$\nabla \times \mathbf{E} = 2k_e q/\vec{\mathbf{r}}^3 \quad \wedge \quad \partial \mathbf{B}/\partial t = -2k_e q/\vec{\mathbf{r}}^3 \quad \Rightarrow \quad \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t. \tag{5.36}$$

5.9. **3 fundamental direct proportion ratios.** c_t, c_m, and c_q:

$$c_t = r_c/t_c \approx 2.99792458 \cdot 10^8 m\ s^{-1}. \tag{5.37}$$

$$G = (r_c/m_c)c_t^2 = c_m c_t^2 \quad \Rightarrow \quad c_m = r_c/m_c \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}. \tag{5.38}$$

$$k_e = (c_t^2/c_m)c_q^2 \quad \Rightarrow \quad c_q = r_c/q_c \approx 8.6175172023 \cdot 10^{-18} m\ C^{-1}. \tag{5.39}$$

5.10. **3 fundamental inverse proportion ratios.** k_t, k_m, and k_q:

$$r/t = r_c/t_c, \quad r/m = r_c/m_c \quad \Rightarrow \quad (r/t)/(r/m) = (r_c/t_c)/(r_c/m_c) \quad \Rightarrow$$
$$(mr)/(tr) = (m_c r_c)/(t_c r_c) \quad \Rightarrow \quad mr = m_c r_c = k_m, \quad tr = t_c r_c = k_t. \tag{5.40}$$

$$r/t = r_c/t_c, \quad r/q = r_c/q_c \quad \Rightarrow \quad (r/t)/(r/q) = (r_c/t_c)/(r_c/q_c) \quad \Rightarrow$$
$$(qr)/(tr) = (q_c r_c)/(t_c r_c) \quad \Rightarrow \quad qr = q_c r_c = k_q, \quad tr = t_c r_c = k_t. \tag{5.41}$$

5.11. **Planck relation and constant, h.** [9] Applying both the direct proportion ratio (5.37), and inverse proportion ratio (5.40):

$$r = ct \quad \wedge \quad m = k_m/r \quad \Rightarrow \quad m(ct)^2 = (k_m/r)r^2 = k_m r. \tag{5.42}$$

$$m(ct)^2 = k_m r \quad \wedge \quad r/t = r_c/t_c = c$$
$$\Rightarrow \quad E := mc^2 = k_m r/t^2 = (k_m(r/t))\,(1/t) = (k_m c)(1/t) = hf = \hbar\omega, \tag{5.43}$$

where the Planck constant, $h = k_m c$, the cycles per second (Hertz) frequency, $f = 1/t$, the reduced Planck constant, $\hbar = h/2\pi$, and the angular frequency, $\omega = 2\pi f = 2\pi/t$.

$$k_m = m_c r_c = h/c \approx 2.2102190943 \cdot 10^{-42}\ kg\ m. \tag{5.44}$$
$$k_t = t_c r_c = k_m c_m/c_t \approx 5.4749346710 \cdot 10^{-78}\ s\ m. \tag{5.45}$$
$$k_q = q_c r_c = k_t c_t/c_q \approx 1.9046601056 \cdot 10^{-52}\ C\ m. \tag{5.46}$$

5.12. **Compton wavelength.** [9] From equations 5.40 and 5.43:

$$mr = k_m \quad \wedge \quad h = k_m c \quad \Rightarrow \quad r = k_m/m = (k_m/m)(c/c) = h/mc. \tag{5.47}$$

5.13. **4 quantum units.** Distance (r_c), time (t_c), mass (m_c), and charge (q_c):

$$r_c = \sqrt{r_c^2} = \sqrt{c_t k_t} = \sqrt{c_m k_m} = \sqrt{c_q k_q} \approx 4.0513505432 \cdot 10^{-35}\ m. \tag{5.48}$$
$$t_c = r_c/c_t \approx 1.3513850782 \cdot 10^{-43}\ s. \tag{5.49}$$
$$m_c = r_c/c_m \approx 5.4555118613 \cdot 10^{-8}\ kg. \tag{5.50}$$
$$q_c = r_c/c_q \approx 4.7012967286 \cdot 10^{-18}\ C. \tag{5.51}$$

5.14. **Subtype ratios.** $\frac{F_1}{F_2} = \frac{K\tau_1^2/r^2}{K\tau_2^2/r^2} = \frac{\tau_1^2}{\tau_2^2}$.

$$\text{Planck length}, r_p : r_c^2/r_p^2 = 2\pi \Rightarrow r_p = r_c/\sqrt{2\pi} \approx 1.6162550244 \cdot 10^{-35}\ m. \tag{5.52}$$
$$\text{Planck time}, \quad t_p : t_c^2/t_p^2 = 2\pi \quad \Rightarrow t_p = t_c/\sqrt{2\pi} \approx 5.3912464472 \cdot 10^{-44}\ s. \tag{5.53}$$
$$\text{Planck mass}, m_p : m_c^2/m_p^2 = 2\pi \Rightarrow m_p = m_c/\sqrt{2\pi} \approx 2.176434343 \cdot 10^{-8} kg. \tag{5.54}$$
$$\text{Planck charge}, q_p : q_c^2/q_p^2 = 2\pi \Rightarrow q_p = q_c/\sqrt{2\pi} \approx 1.875546038 \cdot 10^{-18}\ C. \tag{5.55}$$

Where q_e is the elementary (electron) charge ($1.60217663 \cdot 10^{-19}\ C$), the fine structure constant, α is:

$$\alpha q_c^2/q_e^2 = 2\pi \ \Rightarrow\ \alpha = 2\pi q_e^2/q_c^2 = q_e^2/(q_c/\sqrt{2\pi})^2 = q_e^2/q_p^2 \approx 0.0072973526. \tag{5.56}$$

5.15. **Schrödenger's position-space equation.** Start with the previously derived Planck relation 5.43 and multiply the kinetic energy component by mc/mc:

$$mc^2 = \hbar\omega = 2\pi\hbar/t \quad \Rightarrow \quad \exists\, V(r,t) :\; 2\pi\hbar/t = 2\pi\hbar/t/2t + V(r,t)$$
$$\Rightarrow \quad 2\pi\hbar/t = 2\pi\hbar mc/2mct + V(r,t). \quad (5.57)$$

And from the distance-to-time (speed of light) ratio (5.37):

$$2\pi\hbar/t = 2\pi\hbar mc/2mct + V(r,t) \quad \wedge \quad r = ct$$
$$\Rightarrow \quad 2\pi\hbar/t = 2\pi\hbar mc^2/2mcr + V(r,t). \quad (5.58)$$

$$2\pi\hbar/t = 2\pi\hbar mc^2/2mcr + V(r,t) \quad \wedge \quad 2\pi\hbar/t = mc^2$$
$$\Rightarrow \quad 2\pi\hbar/t = (2\pi)^2\hbar^2/2mcrt + V(r,t). \quad (5.59)$$

$$2\pi\hbar/t = 2\pi\hbar^2/2mcrt + V(r,t) \quad \wedge \quad r = ct$$
$$\Rightarrow \quad 2\pi\hbar/t = (2\pi)^2\hbar^2/2mr^2 + V(r,t). \quad (5.60)$$

Multiply both sides of equation 5.60 by a function, $\Psi(r,t)$.

$$2\pi\hbar/t = (2\pi)^2\hbar^2/2mr^2 + V(r,t)$$
$$\Rightarrow \quad (2\pi\hbar/t)\Psi(r,t) = ((2\pi)^2\hbar^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t). \quad (5.61)$$

$$(2\pi\hbar/t)\Psi(r,t) = ((2\pi)^2\hbar^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t) \quad \wedge$$
$$\forall\, \Psi(r,t) :\; \partial^2\Psi(r,t)/\partial r^2 = (-(2\pi)^2/r^2)\Psi(r,t) \quad \wedge \quad \partial\Psi(r,t)/\partial t = (i\, 2\pi/t)\Psi(r,t)$$
$$\Rightarrow \quad i\hbar\partial\Psi(r,t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t), \quad (5.62)$$

which is the one-dimensional position-space Schrödenger's equation [16] [9].

$$i\hbar\partial\Psi(r,t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t) \quad \wedge \quad ||\vec{\mathbf{r}}|| = r$$
$$\Rightarrow \quad \exists\, \vec{\mathbf{r}} :\; i\hbar\partial\Psi(\vec{\mathbf{r}},t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(\vec{\mathbf{r}},t)/\partial\vec{\mathbf{r}}^2 + V(\vec{\mathbf{r}},t)\Psi(\vec{\mathbf{r}},t), \quad (5.63)$$

which is the 3-dimensional position-space Schrödenger's equation [16] [9].

5.16. **Dirac's wave equation.** Using the derived Planck relation 5.43:

$$mc^2 = h/t \quad \Rightarrow \quad \exists\, V(r,t) :\; mc^2/2 + V(r,t) = h/t$$
$$\Rightarrow \quad 2h/t - 2V(r,t) = mc^2. \quad (5.64)$$

$$\forall\, V(r,t) :\; V(r,t) = ih/t \quad \wedge \quad r = ct \quad \wedge \quad 2h/t - 2V(r,t) = mc^2$$
$$\Rightarrow \quad 2h/t - i2hc/r = mc^2. \quad (5.65)$$

Use the charge ratio, c_q, and time ratio, $c_t = c$ to multiply each term on the left side of equation 5.65 by 1:

$$qc_q/r = qc_q/ct = 1 \quad \wedge \quad 2h/t - i2hc/r = mc^2$$
$$\Rightarrow \quad 2h(-qc_q/c)/t^2 - i2h((-qc_q/c)/r^2)c = mc^2. \quad (5.66)$$

where a negative sign is added to q to indicate an attractive force between an electron and a nucleus.

Applying a quantum amplitude equation in complex form to equation 5.67, where τ is a unit time and ρ is a unit wavelength:

$$A_0 = (c_q/c)((1/t)\tau - i(1/r)\rho) \ \wedge \ 2h(-qc_q/c)/t^2 - i2h((-qc_q/c)/r^2)c = mc^2$$
$$\Rightarrow \quad 2h\partial(-qA_0)/\partial t - i2h(\partial(-qA_0)/\partial r)c = mc^2. \quad (5.67)$$

Translating equation 5.67 to moving coordinates via the Lorentz factor, $\gamma_0 = 1/\sqrt{1-(v/c)^2}$:

$$2h\partial(-qA_0)/\partial t - i2h(\partial(-qA_0)/\partial r)c = mc^2$$
$$\Rightarrow \quad \gamma_0 2h\partial(-qA_0)/\partial t - \gamma_0 i2h(\partial(-qA_0)/\partial r)c = mc^2. \quad (5.68)$$

Multiplying both sides of equation 5.68 by $\Psi(r,t)$:

$$\gamma_0 2h\partial(-qA_0)/\partial t - \gamma_0 i2h(\partial(-qA_0)/\partial r)c = mc^2$$
$$\Rightarrow \quad \gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \gamma_0 i2h(\partial(-qA_0)/\partial r)c\Psi(r,t) = mc^2\Psi(r,t). \quad (5.69)$$

Applying the vectors to equation 5.69:

$$\gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \gamma_0 i2h(\partial(-qA_0)/\partial r)c\Psi(r,t) = mc^2\Psi(r,t) \ \wedge$$
$$||\vec{\mathbf{r}}|| = r \quad \wedge \quad ||\vec{\mathbf{A}}|| = A_0 \quad \wedge \quad ||\vec{\gamma}|| = \gamma_0 \quad \wedge \quad \Leftrightarrow \quad \exists\, \vec{\mathbf{r}}, \vec{\mathbf{A}}, \vec{\gamma} :$$
$$\gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \vec{\gamma}\cdot i2h(\partial(-q\vec{\mathbf{A}})/\partial r)c\Psi(\vec{\mathbf{r}},t) = mc^2\Psi(\vec{\mathbf{r}},t). \quad (5.70)$$

Adding a $\frac{1}{2}$ angular rotation (spin-$\frac{1}{2}$) of π to equation 5.67 allows substituting the reduced Planck constant, $\hbar = h/2\pi$, into equation 5.70, which yields Dirac's wave equation [5] [9]:

$$\gamma_0 2h(\partial(-qA_0)/\partial t)\Psi(r,t) - \vec{\gamma}\cdot i2h(\partial(-q\vec{\mathbf{A}})/\partial r)c\Psi(\vec{\mathbf{r}},t) = mc^2\Psi(\vec{\mathbf{r}},t)$$
$$\wedge \ A_0 = \pi(c_q/c)((1/t)\tau - i(1/r)\rho)$$
$$\Rightarrow \quad \gamma_0\hbar(\partial(-qA_0)/\partial t)\Psi(r,t) - \vec{\gamma}\cdot i\hbar(\partial(-q\vec{\mathbf{A}})/\partial r)c\Psi(\vec{\mathbf{r}},t) = mc^2\Psi(\vec{\mathbf{r}},t). \quad (5.71)$$

5.17. **Total mass.** The total mass of a particle is $m = \sqrt{m_0^2 + m_{ke}^2}$, where m_0 is the rest mass and m_{ke} is the kinetic energy-equivalent mass. Applying both the direct (5.37) and inverse proportion ratios (5.40):

$$m_0 = r/(r_c/m_c) = r/c_m \quad \wedge \quad m_{ke} = (m_c r_c)/r = k_m/r \quad \wedge$$
$$m = \sqrt{m_0^2 + m_{ke}^2} \quad \Rightarrow \quad m = \sqrt{(r/c_m)^2 + (k_m/r)^2}. \quad (5.72)$$

5.18. **Quantum extension to general relativity.** The simplest way to demonstrate how to add quantum physics to general relativity is by extending Schwarzschild's time dilation equation and black hole metric (5.3). Start by changing equation 5.10 in the Schwarzschild derivation:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-(v^2/c^2)(r/r)} \ \wedge \ r = \sqrt{(c_m m)^2 + (k_m/m)^2} = Q_m$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1 - Q_m v^2/rc^2}. \quad (5.73)$$

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-Q_m v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv_{escape}^2$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2Q_m v_{escape}^2/rc^2}. \quad (5.74)$$

$$\sqrt{1-(v^2/c^2)} = \lim_{v_{escape}\to c} \sqrt{1-2Q_m v_{escape}^2/rc^2}$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2Q_m c^2/rc^2} = \sqrt{1-2Q_m/r}. \quad (5.75)$$

Combining equation 5.75 with equation 5.8 yields Schwarzschild's gravitational time dilation with a quantum mass effect:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-2Q_m/r} \quad \wedge \quad t' = t\sqrt{1-(v^2/c^2)}$$
$$\Rightarrow \quad t' = t\sqrt{1-2Q_m/r}. \quad (5.76)$$

Schwarzschild defined the black hole event horizon radius, $r_s := 2Gm/c^2$. The radius with the quantum extension is $r_s := 2Q_m$. At this point the exact same equations 5.15 through 5.18 yield what looks like the same Schwarzschild black hole metric.

5.19. **Quantum extension to Newton's gravity force.** The quantum mass effect is easier to understand in the context Newton's gravity equation than in general relativity, because the metric equations and solutions in the EFEs are much more complex. From equations 5.77 and 5.1:

$$m/\sqrt{(r/c_m)^2+(k_m/r)^2} = 1 \quad \wedge \quad r^2/(ct)^2 = 1$$
$$\Rightarrow \quad r^2/(ct)^2 = m/\sqrt{(r/c_m)^2+(k_m/r)^2}$$
$$\Rightarrow \quad r^2/t^2 = c^2 m/\sqrt{(r/c_m)^2+(k_m/r)^2}. \quad (5.77)$$

$$r^2/t^2 = c^2 m/\sqrt{(r/c_m)^2+(k_m/r)^2}$$
$$\Rightarrow \quad (m/r)(r^2/t^2) = (m/r)(c^2 m/\sqrt{(r/c_m)^2+(k_m/r)^2})$$
$$\Rightarrow \quad F := mr/t^2 = c^2m^2/\sqrt{(r^4/c_m^2)+k_m^2}. \quad (5.78)$$

$$F = c^2m^2/\sqrt{(r^4/c_m^2)+k_m^2} \quad \wedge \quad \forall\, m \in \mathbb{R},\ \exists\, m_1, m_2 \in \mathbb{R}:\ m_1m_2 = m^2$$
$$\Rightarrow \quad F = c^2m_1m_2/\sqrt{(r^4/c_m^2)+k_m^2}. \quad (5.79)$$

5.20. **Quantum extension to Coulomb's force.**

$$q^2/((r/c_q)^2+(k_q/r)^2) = 1 \quad \wedge \quad r^2/(ct)^2 = 1$$
$$\Rightarrow \quad r^2/(ct)^2 = q^2/((r/c_q)^2+(k_q/r)^2)$$
$$\Rightarrow \quad r^2/t^2 = c^2q^2/((r/c_q)^2+(k_q/r)^2). \quad (5.80)$$

$$(1/r)(r^2/t^2) = (1/r)(c^2q^2/((r/c_q)^2+(k_q/r)^2))$$
$$\Rightarrow \quad r/t^2 = c^2q^2/(r^3/c_q^2+k_q^2/r). \quad (5.81)$$

$$\forall\, q \in \mathbb{R}:\ \exists\, q_1, q_2 \in \mathbb{R}:\ q_1q_2 = q^2 \quad \wedge \quad r/t^2 = c^2q^2/(r^3/c_q^2+k_q^2/r)$$
$$\Rightarrow \quad \exists\, q_1, q_2 \in \mathbb{R}:\ r^2/t^2 = c^2q_1q_2/(r^3/c_q^2+k_q^2/r). \quad (5.82)$$

$$r^2/t^2 = c^2 q_1 q_2/(r^3/c_q^2 + k_q^2/r) \quad \wedge \quad m = r/c_m$$
$$\Rightarrow \quad F := mr/t^2 = (c^2/c_m) q_1 q_2/(r^2/c_q^2 + k_q^2/r^2). \quad (5.83)$$

6. Insights and implications

(1) The ruler measure (2.1) and convergence theorem (2.2) were shown to be useful tools for proving that a real-valued equation is the only instance of an abstract, countable set relation and that same set relation is the only instance of that same real-valued equation.

(2) Combinatorics, the ordered set of combinations of countable, disjoint sets (n-tuples), $v_c = \prod_{i=1}^{n} |x_i|$, was proven to imply: the Euclidean volume equation (3.2), the sum of volumes equation (4.3) (which includes the inner product), and the Minkowski distance equation (4.4) (which includes the Manhattan and Euclidean distance equations), without relying on the geometric primitives and relations in Euclidean geometry [10], axiomatic geometry [11], and vector analysis [18].

(3) Where the total n-volume is both the sum and subtraction of n-volumes, the $n = 2$ case is the vector inner product. The distributive and associate laws of multiplication and addition allow the $\pm$ signed volumes to represented as each domain interval length multiplied by a $\pm$-signed unit values:

$$\alpha_i, \beta_i \in \{-1, 1\}, \quad d^2 = \textstyle\sum_{i=1}^{m} (a_i \alpha_i)(b_i \beta_i) := \mathbf{a} \cdot \mathbf{b}. \quad (6.1)$$

(4) Defining all Euclidean and non-Euclidean distance measures as

$$\forall\, n,\ d: \quad f(d) = v = \textstyle\sum_{i=1}^{m} v_i, \quad d = f^{-1}(v) = f(\sum_{i=1}^{m} v_i)^{-1} : \quad (6.2)$$

   (a) shows the intimate relation between distance and volume that definitions, like inner product space and metric space, ignore [18] [8] [15];

   (b) is a more simple and concise definition of a distance measure that includes the properties used in the definitions of inner product space and metric space [18] [8] [15];

(5) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equates Euclidean distance to a straight line [10]. And analytic proofs sum infinitesimal Euclidean distances, $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$, where the Euler-Lagrange equation is used find the minimum solution, which is the straight line equation, $y = mx + b$ [2]. In both cases, it is assumed that the straight line is the smallest distance.

Without using the notion of a straight line: The Minkowski distance, $d = (\sum_{i=1}^{m} d_i^n)^{1/n}$, was derived without the notion of straight lines (4.4). And the proof that all Minkowski distances imply the triangle inequality (4.11) is also a proof that Euclidean distance is the shortest distance:

$$d(u, w) \le d(u, v) + d(v, w)$$
$$\Rightarrow \quad d(u, w) = (d_1^2 + d_2^2)^{1/2} \le (d_1^2)^{1/2} + (d_2^2)^{1/2} = d_1 + d_2. \quad (6.3)$$

(6) The left side of the distance sum inequality (4.6),

$$(\textstyle\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \le (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}, \quad (6.4)$$

differs from the left side of Minkowski's sum inequality [12]:

$$(\textstyle\sum_{i=1}^{m} (a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \le (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \quad (6.5)$$

(a) The two inequalities are only the same where $n = 1$.
(b) The distance sum inequality (4.6) is a more fundamental inequality because the proof does not require the convexity and Hölder's inequality assumptions required to prove the Minkowski sum inequality [12].
(c) The distance sum inequality term, $\forall\, n > 1,\ v_i^n = a_i^n + b_i^n$: $d = v^{1/n} = (\sum_{i=1}^{m} v_i^n)^{1/n}$, is the Minkowski distance, which makes it directly related to geometry. For example, the $m = 1$ case of the distance sum inequality was used to prove that all Minkowski distances imply the triangle inequality (4.7), where the $n = 2$ case of the triangle inequality is also a proof that the Euclidean distance is always the shortest distance in an Euclidean plane.
But the Minkowski sum inequality term, $\forall\, n > 1,\ v > 0$: $d = v^{1/n} = (\sum_{i=1}^{m} ((v_i^n)^{\mathbf{n}}))^{1/n} = (\sum_{i=1}^{m} v_i^{\mathbf{n^2}})^{1/n}$, is not a Minkowski distance.

(7) Combinatorics, repeatable sequencing through an ordered set to yield all $n!$ permutations of its members (without jumping around) was proved to be a cyclic set having $n \leq 3$ members (4.15). Higher dimensions must have different types (members of different sets).
(a) For example, the vector inner product space can only be extended beyond 3 dimensions if and only if the higher dimensions have non-distance types, for example, time.
(b) Order and symmetry probably limit the number of non-distance types $\subset \mathbb{R}$ to 3, for example: time, mass, and charge. As shown in the special relativity section (5.2), there is 6-dimensional space-time-mass-charge.
(c) Each of 3 immediate symmetric cyclic dimensions of space can have at most 3 immediate symmetric cyclic state values of the same type, for example, an immediate symmetric cyclic set of 3 orientations, $\{-1, 0, 1\}$, 3 quark color charges, {red, green, blue}, 3 quark anti-color charges, and so on.
(d) If the states are not ordered (a bag of states), then a state value is undetermined (or superimposed) until observed (like Schrödenger's poisoned cat being both alive and dead until the box is opened [16]). For a bag of states, there is **no** "axiom of choice" [4], an axiom often used in math proofs that allows selecting a particular set element (in this case, selecting a particular state).
(e) A discrete (point) value has measure 0 (zero-length interval size). The ratio of a time or distance interval length to zero is undefined, which is the reason quantum entangled state values exist independent of time and distance.

(8) For each unit, r_c, of a 3-dimensional distance interval having a length, r, there are units of other types of intervals forming unit ratios (5.9): $c_t = r_c/t_c$, $c_m = r_c/m_c$, $c_q = r_c/q_c \Leftrightarrow$ the inverse proportion ratios (5.10): $k_t = r_c t_c$, $k_m = r_c m_c$, $k_t = r_c q_c$. All physics equations at measurable distances, times, masses, and charges are derived from these ratios, which can be stated as the following 3 principals:
(a) **The Cartesian principle:** The direct proportion ratios state that for each unit of one type there is a bijective map to a proportionate-sized unit of another type.

(b) **The conservation principle:** The inverse proportion ratios state that a change in size of one type requires an inverse proportionate change in another type. "You don't get something for nothing."

(c) **The covariance principle:** Where the volume near every frame of reference coordinate point is Euclidean, the ratios are the same. Therefore, all ratio-derived physics equations are the same at every coordinate frame of reference.

(9) The derivations of empirical and hypothesized laws of physics from the ratios expose some **incorrect assumptions** currently used in physics:

(a) Empirical and hypothesized laws of physics use an *opaque* constant, K, that is defined to make an equation, where the units balance, $g = Kf(r, t, \cdots)$. The opacity has led to the *incorrect* assumptions of those constants being fundamental constants.
In this article, some opaque constants are derived directly from (composed of) the ratios: gravity, $G = c_m c_t^2$ (5.4), charge, $k_e = (c_q^2/c_m)c_t^2$ (5.21), and Planck $h = k_m c_t$ (5.43). $\varepsilon_0 = 2/k_e = 2/c_m/((c_q^2/c_m)c_t^2)$ (5.31) and $\mu_0 = k_e/c_t^2 = c_q^2/c_m$ (5.27).
And the quantum extensions to: Schwarzschild's time dilation (5.75) Newton's gravity force (5.79), and Coulomb's charge force show, that where the quantum effects become measurable, the constants G, k_e, ε_0, and μ_0 no longer exist (are no longer valid).
Therefore, G, k_e, ε_0, μ_0, and h are **not** fundamental constants.

(b) In quantum physics, the reduced Planck constant, $\hbar = h/2\pi$, is commonly used instead of the full Planck constant, h, because it simplifies finding solutions to some wave equations. For example, the Schrödenger (5.15) [16] and Dirac wave equations (5.16) [5] *appear* to use the reduced Planck constant, $\hbar$, with the cycles per second (Hertz) frequency, $1/t$, rather than the angular frequency, $2\pi/t$.
But Schrödenger, Dirac, and Planck hypothesized the Planck relation and constants without deriving them. In this article, the relation and constants were derived from the ratios yielding: $E = mc^2 = (k_m c)(1/t) = h(1/t)$ – **not** $E = mc^2 = \hbar(1/t)$. The use of $\hbar$ is only valid where $E = hf(2\pi/2\pi) = \hbar(2\pi/t)$.
It is an arithmetic error to use the reduced Planck constant, $\hbar$, with the cycles per second (Hertz) frequency, $1/t$. All valid equations using $\hbar$ must also use the angular frequency, $\omega = 2\pi/t$.
For example, the Schrödenger (5.15) and Dirac wave equations (5.16), were derived in this article, using the angular frequency, $2\pi/t$. As shown in the derivations, the Schrödenger equation is only valid if the distribution function, Ψ, contains 2π terms and the Dirac wave equation contains π in the amplitude, A_0.

(c) The derivations of: $\nabla \cdot \mathbf{g} = -4\pi G\rho$ from $\nabla \cdot \mathbf{g} = -2Gm/r^3$ (5.5), $\nabla \cdot \mathbf{E} = -\rho/\varepsilon_0$ (5.30) from $\nabla \cdot \mathbf{E} = -2k_e q/r^3$ (5.29), and $\partial \mathbf{B}/\partial t = -\mu_0 \rho$ from $\partial \mathbf{B}/\partial t = -2k_e q/r^3$ (5.34), show that the use of mass and charge density, ρ, is an unnecessary complication that obfuscates the pattern, $\partial f(x, y, r)/\partial r = -2k_{x,y} y/r^3$, and the inverse square pattern, $f(x, y, r) = k_{x,y} y/r^2$. Likewise, the $4\pi G$ in $\kappa = 2(4\pi G)/c^4$ and the energy density in the stress-energy tensor, $T_{\mu,\nu}$, in Einstein's field equations [18] also obfuscates the inverse square assumption.

(d) The derivation of the magnetic field from special relativity (5.25) shows that magnetic field, $\mathbf{B}$, is the spacetime bend (curl) of the electric field, $\mathbf{E}$. The magnetic force is a pseudo (fictitious) force that is the spacetime bending (torque) on a charge moving (translating and rotating) at relativistic velocities.

(10) Using the ratios instead of empirical constants in equations would show the shared principles underlying the different laws of physics. For example, the speed of light ratio, c_t, is a component of the constants: $G = c_m c_t^2$, $k_e = (c_q^2/c_m)c_t^2$, $\varepsilon_0 = 2/k_e = 2/((c_q^2/c_m)c_t^2)$, $h = k_m c_t$.

(11) Using the ratios instead of the opaque constants can sometimes simplify equations. For example: the Compton wavelength equation, $r = h/mc$, simplified to $r = k_m/m$; and the fine structure constant, $\alpha = q_e^2/2\varepsilon_0 hc$ simplified to $\alpha = q_e^2/q_p^2$.

(12) Empirical laws *describe* relations. Deriving the laws from the ratios *explains* the relations. Further, the derivations of the gravity, charge, relativity, electromagnetic, and quantum physics equations from the ratios were much shorter and simpler than other derivations, which shows that the ratios are an important new tool for physicists and engineers.

(13) As shown in subsection 5.4, the derivation of the Schwarzschild's time dilation and black hole metric (5.3) [17] [1] using ratios exposed a way of simplifying the finding of solutions to Einstein's field equations .

(14) Using the quantum units, r_c and t_c: $r_c/t_c^2 \approx 2.2184088232 \cdot 10^{51}\ m\ s^{-2}$, which suggests a maximum acceleration for masses.

(15) The simplification of μ_0 into the quantum units shows two interesting relationships:

$$\mu_0 = \frac{k_e}{c_t^2} = \frac{c_q^2}{c_m} = \frac{(r_c/q_c)^2}{r_c/m_c} = \frac{m_c r_c}{q_c^2} = \frac{k_m}{q_c^2}$$
$$\approx \frac{2.2102190930 \cdot 10^{-42}}{2.2102190930 \cdot 10^{-35}} = 1.0 \cdot 10^{-7}\ kg\ m\ C^{-2} = 1.0 \cdot 10^{-7}\ H\ m^{-1}. \quad (6.6)$$

(a) The first time $k_m = m_c r_c$ appears is in the derivation of the Planck relation and Planck constant, $h = k_m c$ (5.11), the second time in the Compton wavelength, $r = k_m/m$ (5.12). And now, k_m appears as a components of k_e and μ_0.

(b) It is an open question why $\frac{c_q^2}{c_m} = \frac{(r_c/q_c)^2}{r_c/m_c} = \frac{k_m}{q_c^2} = 1.0 \cdot 10^{-7}$ exactly.

(16) Two subtypes are related via the ratios of the same super-type (5.14).

(a) For example, the quantum charge and reduced Planck charge units are related via the ratio: $q_c^2/q_p^2 = 2\pi \Rightarrow q_p = q_c/\sqrt{2\pi}$.

(b) **Note** that if the value of k_m was calculated using the reduced Planck constant, $k_m = \hbar/c$, instead of using $k_m = h/c$ (5.44), then the quantum units would equal the Planck units: $r_c = r_p$, $t_c = t_p$, $m_c = m_p$, and $q_c = q_p$, and some other equations would appear more elegant. But that would create logical contradictions.

Specifically, the Planck relation was derived as, $mc^2 = (k_m c)(1/t)$ (5.11), **not** $mc^2 = (k_m c)(\mathbf{2}\pi/t)$. And using $k_m c = \hbar$, would create other problems, for example, it would make it impossible to derive the Compton wavelength without "magic happens here," where 2π suddenly appears in the derivation.

(c) The CODATA electron coupling version of the fine structure constant, α is defined as: $\alpha = q_e^2/4\pi\varepsilon_0\hbar c) = q_e^2/2\varepsilon_0 hc$ [3].

(i) The derivation of α, in this article (5.14), is much simpler because it is the ratio of two subtypes of charge: elementary (electron) charge, q_e^2 and Planck charge, q_p^2: $\alpha = q_e^2/q_p^2 \approx 0.0072973526$, which is the empirical CODATA value [3].

(ii) The following steps show that the CODATA definition reduces to the ratio-derived equation (using the archaic definition of vacuum permittivity, ε_0, in terms of Coulomb's constant, k_e):

$$\begin{aligned}\varepsilon_0 := 1/4\pi k_e = 1/(4\pi(c_q^2/c_m)c_t^2) \quad \wedge \quad h = k_m c_t \\ \Rightarrow \quad \varepsilon_0 hc = k_m c_t^2/(4\pi(c_q^2/c_m)c_t^2) = k_m/(4\pi(c_q^2/c_m)) \\ = m_c r_c/(4\pi((r_c/q_c)^2/(r_c/m_c))) = q_c^2/4\pi = q_p^2/2.\end{aligned} \tag{6.7}$$

$$\alpha = q_e^2/2\varepsilon_0 hc \quad \wedge \quad \varepsilon_0 hc = q_p^2/2 \quad \Rightarrow \quad \alpha = q_e^2/2(q_p^2/2) = q_e^2/q_p^2. \tag{6.8}$$

(iii) As shown above, CODATA defines the fine structure constant in terms of a relationship to the Planck constant and an **incorrect** definition of ε_0, hence, the ratio containing the reduced Planck unit, q_p: $\alpha = q_e^2/q_p^2$.
The quantum unit, q_c, appears naturally in the derivation of Coulomb's constant, k_e. However, this classical definition of $\varepsilon_0 = 1/4\pi k_e$ differs from the one derived naturally in the derivation of Gauss' law (5.31), where $\varepsilon_0 = 2/k_e$. Therefore, a better definition to describe particle interaction with a charge (electromagnetic) wave is: $\alpha = q_e^2/q_c^2$, where the current CODATA value would be divided by 2π.

(iv) Other fine structure constants can also be expressed more simply as the ratios of two subtypes of fields, for example, an electron gravity coupling constant can be expressed as the ratio of the rest electron mass to a quantum mass unit: $\alpha_{G_m} = m_e^2/m_p^2$ (or more correctly, $\alpha_{G_m} = m_e^2/m_c^2$).

(17) The derivation of the Schrödenger (5.15) and Dirac wave equations (5.16), in this article, differs from other derivations: The derivations, here, are much simpler: no spinors, no Dirac delta, no L_2-integrable assumptions, no distribution assumptions. And the derivations are more rigorous because the energy-momentum term, $h^2/2m$, was derived, in this article, from the Planck relation (5.60), where the Planck relation was also rigorously derived from the ratios (5.11). Other derivations assume (define) the Planck relation and the energy momentum relation as: $(\mathbf{p}\cdot\mathbf{p})/2m = \hbar^2/2m$ [16] [7], which, as previously discussed, is only valid where there are compensating 2π terms for angular frequency in the the equations.

(18) The quantum extensions to: Schwarzschild's time dilation (5.75) black hole metric (5.18), Newton's gravity force (5.79), and Coulomb's charge force (5.82) make quantifiable predictions:

(a) The gravitation and charge forces peak at finite amounts as $r \to 0$: for gravity, $\lim_{r\to 0} F = c^2 m_1 m_2/k_m$, and for charge, $\lim_{r\to 0} F = 0$.

Finite maximum gravity and charge forces: 1) allows radioactivity, finite sloped energy well walls, and possibly black hole evaporation; 2) eliminates the problem of forces going to infinity as $r \to 0$.

(b) The quantum-extended Schwarzschild time dilation and metric, gravity, and charge equations reduce to the classic equations, where the distance between masses and charges is sufficiently large or the masses and charges sufficiently large that the quantum effect is not measurable. **Note** that G, k_e, ε_0, μ_0, and κ (Einstein's constant, which contains G) are not valid (do not exist), where the quantum effects becomes measurable.

(c) And the covariant tensor components, in Einstein's field equations, that had the units $1/distance^2$, will now have the more complex units, $1/\sqrt{(distance^4/c_m^2) + k_m^2}$.

(d) $1/\sqrt{(distance^4/c_m^2) + k_m^2}$ implies that as distance $\to 0$, spacetime curvature peaks at a finite amount, which predicts that black holes might have sizes > 0 (might not be singularities). If there was a "big bang," then it might not have originated from a singularity.

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