# Some Set Properties Underlying Geometry and Physics

George. M. Van Treeck

Abstract. Volume and the Minkowski distances/p-norms (e.g., Manhattan and Euclidean distance) are derived from a set and limit-based foundation without referencing the primitives and relations of geometry. Sequencing a strict linearly ordered set in all n-at-a-time orders via successor/predecessor relations is a cyclic set of at most 3 members. Therefore, all other domain interval lengths have different types from a cyclic set of 3 distance domain interval lengths. Constant ratios between different types of interval lengths and the set proofs provide simpler derivations of the: spacetime, Newton's gravity, Coulomb's charge force, Planck-Einstein, quantum-relativity gravity equations, and corresponding constants. All proofs are verified in Coq.

Contents



## 1. Introduction

Mathematical analysis can construct differential calculus from a set and limit-based foundation without referencing the primitives and relations of Euclidean geometry, like straight line, angle, shape, etc., which provides a more rigorous foundation to calculus. But volume in the Riemann integral, Lebesgue integral, measure theory, and distance in the vector magnitude and metric space criteria are definitions motivated by Euclidean geometry. [Gol76] [Rud76] Here, volume and distance is derived from a set and limit-based foundation. The derivations provide a better definition of definition of distance than metric space.

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A well-known set-based motivation of Euclidean volume is the cardinal, $v_c$, of a set of Cartesian product n-tuples: $v_c = \prod_{i=1}^{n} |x_i|$, where $|x_i|$ is the cardinal of the countable, disjoint set, $x_i$. But, where each $x_i$ is a set of subintervals of an interval, $[a_i, b_i] \subset \mathbb{R}$, and $s_i = b_i - a_i$, there have been no proofs that $v_c = \prod_{i=1}^{n} |x_i| \Rightarrow v = \prod_{i=1}^{n} s_i$, hence, Euclidean volume being defined rather than derived. In this article, it will be proved that: $v_c = \prod_{i=1}^{n} |x_i| \Rightarrow v = \prod_{i=1}^{n} s_i$.

$v_c = \prod_{i=1}^{n} |x_i| = f(|x_1|, \cdots, |x_n|, n)$ and $v = \prod_{i=1}^{n} s_i = f(s_1, \cdots, s_n, n) \Rightarrow f$ is a single-valued function. If $f$ is a bijective function, then $f^{-1}$ is also single-valued, where $f^{-1}(v_c, n) = |x_1| = \cdots = |x_n|$ and $f^{-1}(v, n) = s_1 = \cdots = s_n$. Therefore, $d \in \{s_1, \cdots, s_n\}$ and $s_1 = \cdots = s_n \Rightarrow v = \prod_{i=1}^{n} s_i = \prod_{i=1}^{n} d = d^n$. Where $f$ is bijective, it will be proved that $v_c = \sum_{j=1}^{m} v_{c_i} \Rightarrow d^n = \sum_{i=1}^{m} d_i^n$. $d$ is the $\rho$-norm (Minkowski distance), which will be proved to imply the metric space properties.

Sequencing the sets, $\{|x_1|, \cdots, |x_n|\}$ and $\{s_1, \cdots, s_n\}$, from $i = 1$ to $n$, is a strict linear (total) order, where a total order is defined in terms of successor and predecessor functions. Sequencing a set via successor and predecessor functions in all n-at-a-time orders, requires a "symmetry" constraint, where every set member is either a successor or predecessor to every other set member. A strict linearly ordered and symmetric set will be proved to be a cyclic set, where $n \leq 3$.

Therefore, if $\{x_1, x_2, x_3\}$ is a strict linearly ordered and symmetric set of 3 "distance" dimensions, then another dimension, $x_4$, must have a different type (is a member of different set). Calculus divides domain intervals into the same number of subintervals, where each subinterval of a distance domain interval maps to a proportionate-sized subinterval of some other type of domain interval, which is expressed by a unit-factoring ratio, for example, *meters/second*.

Simpler and shorter derivations of the: spacetime, Newton's gravity, Coulomb's charge force, Planck-Einstein, quantum-relativity gravity equations, and their corresponding constants are provided using some constant ratios combined with the results of the volume, distance, and 3D proofs. The ratios and proofs also simplify Einstein's field (general relativity) equations.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq23**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

## 2. Ruler measure and convergence

A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, $p$, of size $\kappa$ subintervals. The ruler is both an inner and outer measure of an interval.

Definition 2.1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\ [a, b] \subset \mathbb{R}$, $s = b - a \ \wedge\ 0 < \kappa \leq 1 \ \wedge\ (p = floor(s/\kappa) \ \vee\ p = ceiling(s/\kappa))$.

Theorem 2.2. *Ruler convergence:* $M = \lim_{\kappa \to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\ \kappa > 0,\ p = floor(s/\kappa)\ \wedge\ 0 \leq |floor(s/\kappa) - s/\kappa| < 1 \ \Rightarrow\ |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by $\kappa$:

$$\forall\ \kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$(2.3)\quad \forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta$$
$$\Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \epsilon \quad := \quad M = \lim_{\kappa\to 0} p\kappa = s. \quad \square$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, ..., \pi_{\lim_{\kappa\to 0}}$.

LEMMA 2.3. $\forall\, n \geq 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa$.

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$(2.4)\quad n \geq 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|.$$

$$(2.5)\quad \forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta$$
$$\Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa\to 0} \kappa^n = 0.$$

$$(2.6)\qquad \lim_{\kappa\to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa\to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa. \qquad \square$$

## 3. Volume

DEFINITION 3.1. A countable n-volume is the number of ordered combinations (n-tuples), $v_c$, of the members of $n$ number of disjoint, countable domain sets, $x_i$:

$$(3.1)\quad \exists\, n \in \mathbb{N},\ v_c \in \{0,\ \mathbb{N}\},\ \ x_i \in \{x_1, \cdots, x_n\} : \ \bigcap_{i=1}^n x_i = \emptyset \ \wedge \ v_c = \prod_{i=1}^n |x_i|.$$

THEOREM 3.2. *Euclidean volume,*

$$(3.2)\quad \forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\ \ [v_a, v_b] \subset \mathbb{R},\ \ s_i = b_i - a_i,\ \ v = v_b - v_a :$$
$$v_c = \prod_{i=1}^n |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^n s_i.$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.

$$(3.3)\quad v_c = \prod_{i=1}^n |x_i| \ \Leftrightarrow \ v_c\kappa = (\prod_{i=1}^n |x_i|)\kappa \ \Leftrightarrow \ \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa.$$

Apply the ruler (2.1) and ruler convergence (2.2) to equation 3.3:

$$(3.4)\quad \exists\, v, \kappa \in \mathbb{R} : \ v_c = floor(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa.$$

Apply lemma 2.3 to equation 3.4:

$$(3.5)\qquad v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa^n = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa).$$

Apply the ruler (2.1) and ruler convergence (2.2) to $s_i$:

$$(3.6)\qquad \exists\, s_i, \kappa \in \mathbb{R} : \ floor(s_i/\kappa) = |x_i| \quad \Rightarrow \quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i.$$

$$(3.7)\qquad v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa) \ \wedge \ \lim_{\kappa\to 0}(|x_i|\kappa) = s_i \ \Rightarrow \ v = \prod_{i=1}^n s_i \qquad \square$$

THEOREM 3.3. *Sum of volumes:*

$$(3.8)\quad \forall\, x_{i,j} \in \{x_{i_1}, \cdots, x_{i_m}\} = x_i : \ v_c = \prod_{i=1}^n |x_i| \quad \wedge \quad v_{c_j} = \prod_{i=1}^n |x_{i,j}| \quad \wedge$$
$$v_c = \sum_{j=1}^m v_{c_j} \quad \Rightarrow \quad \exists\, s_i, s_{i,j} \in \mathbb{R} : \ \prod_{i=1}^n s_i = \sum_{j=1}^m (\prod_{i=1}^n s_{i,j}).$$

The formal proof, "sum_of_volumes," is in the Coq file, euclidrelations.v.

PROOF. From the Euclidean volume theorem (3.2):

$$v_c = \prod_{i=1}^n |x_i| \;\Rightarrow\; v = \prod_{i=1}^n s_i \;\wedge\; v_{c_j} = \prod_{i=1}^n |x_{i,j}| \;\Rightarrow\; v_j = \prod_{i=1}^n s_{i,j}. \tag{3.9}$$

Apply the ruler (2.1) and ruler convergence (2.2):

$$\exists\, v, v_j, \kappa \in R: \quad v_c = floor(v/\kappa) \quad \wedge \quad v_{c_j} = floor(v_i/\kappa) \\ \Rightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa \quad \wedge \quad v_i = \lim_{\kappa\to 0} v_{c_j}\kappa. \tag{3.10}$$

$$v_c = \sum_{j=1}^m v_{c_j} \quad \Leftrightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\sum_{j=1}^m v_{c_j})\kappa. \tag{3.11}$$

Apply lemma 2.3 to equation 3.11:

$$\lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa \;\wedge\; v = \lim_{\kappa\to 0}(\sum_{j=1}^m v_{c_j})\kappa \;\wedge\; v_i = \lim_{\kappa\to 0} v_{c_j}\kappa \\ \Rightarrow \quad v = \lim_{\kappa\to 0}(\sum_{j=1}^m v_{c_j})\kappa^n = \lim_{\kappa\to 0}\sum_{j=1}^m (v_{c_j}\kappa) = \sum_{j=1}^m v_j. \tag{3.12}$$

$$v = \prod_{i=1}^n s_i \quad \wedge \quad v_j = \prod_{i=1}^n s_{i,j} \quad \wedge \quad v = \sum_{j=1}^m v_j \\ \Rightarrow \quad \prod_{i=1}^n s_i = \sum_{j=1}^m \prod_{i=1}^n s_{i,j}. \quad \square \tag{3.13}$$

## 4. Distance

DEFINITION 4.1. Distance, $d:\ d = f^{-1}(v,n) \Leftrightarrow v = f(d,n)$, where $\forall [a_i,b_i] \in \{[a_1,b_1],\cdots [a_n,b_n]\}$, $[v_a,v_b] \subset \mathbb{R}$, $s_i = b_i - a_i$, $v = v_b - v_a \;:\; v = \prod_{i=1}^n s_i = f(s_1,\cdots,s_n,n) = f(d,n) = \prod_{i=1}^n d := d^n$.

### 4.1. Minkowski distance ($\rho$-norm).

THEOREM 4.2. *Minkowski distance ($\rho$-norm):*

$$v_c = \prod_{i=1}^n |x_i| = \sum_{j=1}^m (\prod_{i=1}^n |x_{i,j}|) = \sum_{j=1}^m v_{c_i} \quad \Rightarrow \quad d^n = \sum_{i=1}^m d_i^n.$$

*The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.*

PROOF. From the sum of volumes proof (3.3), where all subintervals of all intervals are the same size, $\kappa$:

$$\prod_{i=1}^n |x_i| = \sum_{j=1}^m (\prod_{i=1}^n |x_{i,j}|) \quad \Rightarrow \quad \prod_{i=1}^n s_i = \sum_{j=1}^m (\prod_{i=1}^n s_{i,j}) \tag{4.1}$$

$$v = \prod_{i=1}^n s_i = \prod_{i=1}^n d = d^n \quad \wedge \quad v_i = \prod_{i=1}^n s_{i,j} = \prod_{i=1}^n d_i = d_i^n \quad \wedge \\ \prod_{i=1}^n s_i = \sum_{j=1}^m (\prod_{i=1}^n s_{i,j}) \;\Rightarrow\; d^n = \prod_{i=1}^n s_i = \sum_{j=1}^m (\prod_{i=1}^n s_{i,j}) = \sum_{i=1}^m d_i^n. \quad \square \tag{4.2}$$

**4.2. Distance inequality.** The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N},\;\; v_a, v_b \geq 0: \;\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$\forall\, v_a, v_b \geq 0: \quad v_a + v_b \leq v_a + v_b + \\ \sum{}_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum{}_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n. \tag{4.3}$$

Take the $n^{th}$ of both sides of the inequality 4.3:

$$\forall\, v_a, v_b \geq 0, n \in \mathbb{N}: v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad \square \tag{4.4}$$

**4.3. Distance sum inequality.** The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m,n \in \mathbb{N},\;\; a_i, b_i \geq 0 :\;\; (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$\begin{aligned}(4.5)\quad & \forall\, m,n \in \mathbb{N},\;\; v_a, v_b \geq 0 :\quad v_a = \textstyle\sum_{i=1}^{m} a_i^n \quad\wedge\quad v_b = \sum_{i=1}^{m} b_i^n \quad\wedge \\ & (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad\Rightarrow\quad ((\textstyle\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} = \\ & (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \quad \square\end{aligned}$$

**4.4. Metric Space.** All Minkowski distances ($\rho$-norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad\Rightarrow\quad d(u,w) \leq d(u,v) + d(v,w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(4.6)\qquad (u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}.$$

Apply the distance inequality (4.3) to the inequality 4.6:

$$\begin{aligned}(4.7)\quad & (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \;\wedge\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ & \wedge \quad v_a = u^p + v^p \quad\wedge\quad v_b = v^p + w^p \\ \Rightarrow\quad & (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ & \Rightarrow\quad d(u,w) = (u^p + w^p)^{1/p} \leq \\ & (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u,v) + d(v,w). \quad \square\end{aligned}$$

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad\Rightarrow\quad d(u,v) = d(v,u).$

PROOF. By the commutative law of addition:

$$\begin{aligned}(4.8)\quad & \forall\, p : p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ & \Rightarrow\quad d(u,v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v,u). \quad \square\end{aligned}$$

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad\Rightarrow\quad d(u,w) \geq 0.$

PROOF. By definition, the length of an interval is always $\geq 0$:

$$(4.9)\qquad \forall\, [a_1, b_1],\; [a_2, b_2], \quad u = b_1 - a_1,\; v = b_2 - a_2, \quad\Rightarrow\quad u \geq 0,\; v \geq 0.$$

$$(4.10)\qquad p \geq 1,\;\; u, v \geq 0 \quad\Rightarrow\quad d(u,v) = (u^p + v^p)^{1/p} \geq 0. \qquad \square$$

THEOREM 4.8. *Identity of Indiscernibles:* $d(u,u) = 0.$

PROOF. From the non-negativity property (4.7):

$$d(u,w) \geq 0 \quad \wedge \quad d(u,v) \geq 0 \quad \wedge \quad d(v,w) \geq 0 \quad \Rightarrow \quad \exists\, d(u,w) = d(u,v) = d(v,w) = 0. \tag{4.11}$$

$$d(u,w) = d(v,w) = 0 \quad \Rightarrow \quad u = v. \tag{4.12}$$

$$d(u,v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u,u) = 0. \tag{4.13}$$

□

### 4.5. The properties limiting a set to at most 3 members.

DEFINITION 4.9. Totally ordered set:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i \;\in \{x_1, \ldots, x_n\}, \quad \mathit{successor}\; x_i = x_{i+1} \;\wedge\; \mathit{predecessor}\; x_{i+1} = x_i.$$

DEFINITION 4.10. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i,\; j,\; n \in \mathbb{N},\; \forall\, x_i,\; x_j \in \{x_1, \ldots, x_n\},\; \mathit{successor}\; x_i = x_j \;\Leftrightarrow\; \mathit{predecessor}\; x_j = x_i.$$

THEOREM 4.11. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n \;\wedge\; j = 1 \;\Rightarrow\; \mathit{successor}\; x_n = x_1 \;\wedge\; \mathit{predecessor}\; x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (4.9) assigns a unique label to each set member and assigns unique successors and predecessors for all set members except for the successor of $x_n$ and the predecessor of $x_1$. Therefore, the only member that can be a successor of $x_n$, without creating a contradiction, is $x_1$. And the only member that can be a predecessor of $x_1$, without creating a contradiction, is $x_n$. Applying the symmetry property (4.10):

$$i = n \;\wedge\; j = 1 \;\wedge\; \mathit{successor}\; x_i = x_j \;\Rightarrow\; \mathit{successor}\; x_n = x_1. \tag{4.14}$$

Applying the definition of the symmetry property (4.10) to conclusion 4.14:

$$\mathit{successor}\; x_i = x_j \;\Rightarrow\; \mathit{predecessor}\; x_j = x_i \;\Rightarrow\; \mathit{predecessor}\; x_1 = x_n. \tag{4.15}$$

□

THEOREM 4.12. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (4.11).

DEFINITION 4.13. (Cyclic) Successor of $m$ is $n$:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (n = m+1 \leq \mathit{setsize}). \tag{4.16}$$

DEFINITION 4.14. (Cyclic) Predecessor of $m$ is $n$:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (n = m - 1 \geq 1). \tag{4.17}$$

DEFINITION 4.15. Adjacent: member $m$ is sequentially adjacent to member $n$ if the successor of $m$ is $n$ or the predecessor of $m$ is $n$. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{4.18}$$

Every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (m = setsize \wedge n = 1). \tag{4.19}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (n = m + 1 \leq setsize). \tag{4.20}$$

$$Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (n = setsize \wedge m = 1). \tag{4.21}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m + 1 \leq setsize). \tag{4.22}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m - 1 \geq 1). \tag{4.23}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \wedge m = 1). \tag{4.24}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \wedge n = setsize). \tag{4.25}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m + 1 \leq setsize). \tag{4.26}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m - 1 \geq 1). \tag{4.27}$$

Member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not ($\neg$) a successor of member 1 for all $setsize > 3$:

$$\begin{aligned} &\neg Successor(1, 3, setsize > 3) \\ &\qquad \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \end{aligned} \tag{4.28}$$

Member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not ($\neg$) a predecessor of member 1 for all $setsize > 3$:

$$\begin{aligned} &\neg Predecessor(1, 3, setsize > 3) \\ &\qquad \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \end{aligned} \tag{4.29}$$

For all $setsize > 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not symmetric):

$$\begin{aligned} &\neg Adjacent(1, 3, setsize > 3) \\ &\qquad \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \end{aligned} \tag{4.30}$$ □

## 5. Applications to physics

From the 3D proof (4.12), dividing a set of domain intervals into the same number of subintervals, a 3-dimensional distance subinterval length, $r$, maps to proportionately sized subinterval lengths of other types, $t$, $m$, and $q$, where:

$$r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q. \tag{5.1}$$

**5.1. Spacetime equations.** From the volume proof (3.2), two disjoint distance intervals, $[0, r]$ and $[0, r']$, define a 2-volume. From the Minkowski distance proof (4.2), the distance interval lengths, $r$ and $r'$, are inverse functions of two 2-volumes, $v$ and $v'$ having the sizes, $v = r^2$ and $v' = r'^2$. And if $r$ and $r'$ are independent, then $\exists\ r_{Total} : r_{Total}^2 = r^2 + r'^2$. And if dependent, then either $\exists\ r_\nu : r^2 = r'^2 + r_\nu^2$ or $r'^2 = r^2 + r_\nu^2$. The same spacetime equations result from any case – only the notation differs. For traditional notation, the $r^2 = r'^2 + r_\nu^2$ case is chosen. Combined with the 3D proof (4.12):

$$\exists\, \mu,\ \nu \in \mathbb{R} :\ r = \mu t \quad \wedge \quad r_\nu = \nu t \quad \wedge \quad \exists\, r_\nu \in \mathbb{R} :\ r^2 = {r'}^2 + r_\nu^2$$
$$\Rightarrow \quad (\mu t)^2 = {r'}^2 + (\nu t)^2 \quad \Rightarrow \quad r' = \sqrt{(\mu t)^2 - (\nu t)^2} \ = \ \mu t\sqrt{1-(\nu/\mu)^2}. \tag{5.2}$$

Local (proper) distance, $r'$, contracts relative to coordinate distance, $r$, as $\nu \to \mu$:

$$r' = \mu t\sqrt{1-(\nu/\mu)^2} \quad \wedge \quad \mu t = r \quad \Rightarrow \quad r' = r\sqrt{1-(\nu/\mu)^2}. \tag{5.3}$$

From equation 5.2, coordinate length, $t$, dilates relative to local length, $t'$, as $\nu \to \mu$:

$$\mu t = r'/\sqrt{1-(\nu/\mu)^2} \quad \wedge \quad r' = \mu t' \quad \Rightarrow \quad t = t'/\sqrt{1-(\nu/\mu)^2}. \tag{5.4}$$

One form of the flat Minkowski spacetime event interval is:

$$\mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_\nu^2 \quad \wedge \quad \mathrm{d}r_\nu^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \quad \wedge \quad \mathrm{d}(\mu t) = \mathrm{d}r$$
$$\Rightarrow \quad \mathrm{d}r'^2 = \mathrm{d}(\mu t)^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2 - \mathrm{d}x_3^2. \tag{5.5}$$

**5.2. Newton's gravity force and the constant, G.** From equation 5.1:

$$\forall\, m_1, m_2, m, r \in \mathbb{R} :\ m_1 m_2 = m^2 \quad \wedge \quad m = (m_c/r_c)r$$
$$\Rightarrow \quad m_1 m_2 = ((m_c/r_c)r)^2 \quad \Rightarrow \quad (r_c/m_c)^2 m_1 m_2/r^2 = 1. \tag{5.6}$$

$$r = (r_c/t_c)t = ct \ \Rightarrow \ mr = (m_c/r_c)(ct)^2 \ \Rightarrow \ ((r_c/m_c)/c^2)mr/t^2 = 1. \tag{5.7}$$

$$((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/m_c)^2 m_1 m_2/r^2 = 1$$
$$\Rightarrow \quad F := mr/t^2 = ((r_c/m_c)c^2)m_1 m_2/r^2 = G m_1 m_2/r^2, \tag{5.8}$$

where Newton's constant, $G = (r_c/m_c)c^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$.

**5.3. Coulomb's charge force and constant, $k_e$.** From equation 5.1:

$$\forall\, q_1, q_2, q, r \in \mathbb{R} :\ q_1 q_2 = q^2 \quad \wedge \quad q = (q_c/r_c)r$$
$$\Rightarrow \quad q_1 q_2 = ((q_c/r_c)r)^2 \quad \Rightarrow \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1. \tag{5.9}$$

$$r = (r_c/t_c)t = ct \ \Rightarrow \ mr = (m_c/r_c)(ct)^2 \ \Rightarrow \ ((r_c/m_c)/c^2)mr/t^2 = 1. \tag{5.10}$$

$$((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1$$
$$\Rightarrow \quad F := mr/t^2 = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2. \tag{5.11}$$

$$r_c/t_c = c \quad \wedge \quad F = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2$$
$$\Rightarrow \quad F = (m_c(r_c/t_c^2))(r_c/q_c)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2, \tag{5.12}$$

where Coulomb's constant, $k_e = (m_c(r_c/t_c^2))(r_c/q_c)^2$, conforms to the SI units: $N \cdot m^2 \cdot C^{-2}$.

**5.4. 3 fundamental constants.** $c_t$, $c_m$, and $c_q$.

$$c_t = r_c/t_c \approx 2.99792458 \cdot 10^8 m\ s^{-1}. \tag{5.13}$$

$$G = (r_c/m_c)c_t^2 \quad \Rightarrow \quad c_m = r_c/m_c \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}. \tag{5.14}$$

$$k_e = ((m_c/r_c)c_t^2)(r_c/q_c)^2 \Rightarrow \quad c_q = r_c/q_c \approx 8.6175172023 \cdot 10^{-18} m\ C^{-1}. \tag{5.15}$$

**5.5. Principle of conservation.** An amount of distance corresponds to an inversely proportionate amount of another type of measure. The ratios $c_t/c_m$ and $c_t/c_q$ yields 3 conservation constants, $k_t$, $k_m$, and $k_q$ that are the basis of particle-wave behavior:

$$c_t/c_m = (m_c/r_c)(r_c/t_c) = (m_c r_c)/(t_c r_c) = k_m/k_t. \tag{5.16}$$

$$c_t/c_q = (q_c/r_c)(r_c/t_c) = (q_c r_c)/(t_c r_c) = k_q/k_t. \tag{5.17}$$

**5.6. Planck-Einstein equation:** Applying both the relative measure ratios 5.1 and the conservation ratios 5.5:

$$m(ct)^2 = mr^2 \ \wedge \ m = m_c r_c/r = k_m/r \quad \Rightarrow \quad m(ct)^2 = (k_m/r)r^2 = k_m r. \tag{5.18}$$

$$\begin{aligned} & m(ct)^2 = k_m r \quad \wedge \quad r/t = r_c/t_c = c \\ & \qquad \Rightarrow \quad E := mc^2 = k_m r/t^2 = (k_m(r/t))\,(1/t) = (k_m c)(1/t) = hf, \end{aligned} \tag{5.19}$$

where the Planck constant $h = k_m c$ and the frequency $f = 1/t$.

$$k_m = m_c r_c = h/c \approx 2.21022 \cdot 10^{-42}\ kg\ m. \tag{5.20}$$

$$k_t = t_c r_c = k_m/(c_t/c_m) \approx 5.47493 \cdot 10^{-78}\ s\ m. \tag{5.21}$$

$$k_q = q_c r_c = (c_t/c_q)k_t \approx 1.90466 \cdot 10^{-52}\ C\ m. \tag{5.22}$$

**5.7. Quantum-special relativity extensions to Newton's gravity force.** The total mass of a particle is $m = \sqrt{m_0^2 + m_{ke}^2}$, where $m_0$ is the rest mass and $m_{ke}$ is the kinetic energy-equivalent mass (energy imparted by photons and gravitons). Applying both the relative measure ratios 5.1 and the conservation ratios 5.5:

$$\begin{aligned} & m_0 = (m_c/r_c)r \quad \wedge \quad m_{ke} = m_c r_c/r \quad \wedge \quad m = \sqrt{m_0^2 + m_{ke}^2} \\ & \qquad \Rightarrow \quad m = \sqrt{((m_c/r_c)r)^2 + ((m_c r_c)/r)^2}. \end{aligned} \tag{5.23}$$

$$\begin{aligned} & \exists\, m : \ m_1 m_2 = m^2 = ((m_c/r_c)r)^2 + ((m_c r_c)/r)^2 \\ & \qquad \Rightarrow \quad m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2) = 1. \end{aligned} \tag{5.24}$$

Deriving the gravity force from Einstein's field equations requires the assumption that velocity in the local frame of reference is zero. Newton's gravity force in the local frame of reference comes from the spacetime equation, 5.2:

$$r' = \sqrt{(ct)^2 - (vt)^2} \quad \Rightarrow \quad m_0 r' = (m_c/r_c)((ct)^2 - (vt)^2). \tag{5.25}$$

$$m_0 r' = (m_c/r_c)((ct)^2 - (vt)^2) \quad \Rightarrow \quad ((r_c/m_c)/(c^2 - v^2))m_0 r'/t^2 = 1. \tag{5.26}$$

$$\begin{aligned} & ((r_c/m_c)/(c^2 - v^2))m_0 r'/t^2 = 1 \\ & \qquad \wedge \quad m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2) = 1 \\ & \Rightarrow \quad F := m_0 r'/t^2 = ((m_c/r_c)(c^2 - v^2))m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2). \end{aligned} \tag{5.27}$$

**5.8. Quantum-special relativity extensions to Coulomb's charge force.** Applying $m = (m_c/q_c)q$ to the quantum-relativistic gravity equation (5.7):

$$F = (m_c/q_c)^2(m_c/r_c)(c^2 - v^2)q_1 q_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2). \tag{5.28}$$

## 6. Insights and implications

(1) Volume and distance derived from the same abstract, countable set of n-tuples provides a unifying and more rigorous set and limit-based foundation under integration, measure theory, the vector magnitude, and the metric space axioms without relying on the geometric primitives and relations in Euclidean geometry [**Joy98**], axiomatic geometry [**Lee10**], and vector analysis [**Wey52**].

(2) The definition of a complete metric space is **not** sufficient to model the geometric notion of distance because it ignores the intimate (bijective) relation between distance and volume. A more sufficient and useful definition of a distance measure is a function that can be reduced to a Minkowski distance: $d : d^n = \sum_{i=1}^m d_i^n$. Or a distance measure is a function that reduces to a Gaussian-like distance: $d^n = \sum_{i=1}^m \alpha_i d_i^n$, where each $\alpha_i$ is a function returning a scalar value $\geq 0$.

For example, vector spaces and Riemann manifolds are inner product spaces, where the distance in an infinitesimal region around every point is Euclidean ($n = 2$-Minkowski or a flat $n = 2$-Gaussian distance). Another example are the $\alpha_i$ curvature functions, which are the diagonal components of the metric tensor in Einstein's field equations [**Ein15**].

(3) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [**Joy98**]. And proofs that the straight line length is the smallest distance equate the straight line length to Euclidean distance.

Without using the notion of a straight line: Euclidean volume was derived from a set of n-tuples (3.2). And all distance measures derived from an Euclidean 2-volume (area) are Minkowski distances (4.2), where $n \in \{1, 2\}$. $n = 1$ is the Manhattan (largest) distance case, $d = \sum_{i=1}^m s_i$. $n = 2$ is the Euclidean (smallest) distance case, $d = (\sum_{i=1}^m s_i^2)^{1/2}$. Hilbert spaces allow non-integer values of $n$ (fractals). In that case, $1 \leq n \leq 2$ and $d$ decreases monotonically as $n \to 2$.

(4) The left side of the distance sum inequality (4.4),

$$(\sum_{i=1}^m (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}, \tag{6.1}$$

differs from the left side of Minkowski's sum inequality [**Min53**]:

$$(\sum_{i=1}^m (a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}. \tag{6.2}$$

The two inequalities are only the same where $n = 1$. The distance sum inequality is a more fundamental inequality because its proof does not require convexity and Hölder's inequality that are required to prove the Minkowski sum inequality. And the distance sum inequality is derived from the definitions of volume and distance, which makes it more directly related to geometry.

(5) The derivations of the spacetime equations, in this article (5.1), differ from other derivations:

   (a) The derivations, here, do not rely on the Lorentz transformations or Einsteins' postulates [**Ein15**]. The derivations do not even require the notion of light.

   (b) The derivations, here, rely only on the Euclidean volume proof (3.2), the Minkowski distances proof (4.1), and the 3D proof (4.12), which provides the insight that the properties of physical space creates a maximum speed and the spacetime equations.
   (c) The same derivations are also valid for spacemass and spacecharge.
(6) The flat spacetime interval equation was derived from a 2-dimensional equation (5.5). which is generalized to: $\mathrm{d}r'^2 = \alpha_1 \mathrm{d}(\mu t)^2 - \mathrm{d}r_\nu^2$, where $\mathrm{d}r_\nu^2 = \alpha_2 \mathrm{d}x_1^2 + \alpha_3 \mathrm{d}x_2^2 + \alpha_4 \mathrm{d}x_3^2$. The $4 \times 4$ metric tensor, $g_{\mu,\nu} = diag(\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4)$, in Einstein's field equations, can be simplified to the $2 \times 2$ metric tensor, $g_{i,j} = diag(\alpha_1, -1)$. The $2 \times 2$ metric tensor allows using a 2-dimensional Gaussian curvature, which is much simpler to calculate than the 4-dimensional Ricci curvature. And the $2 \times 2$ tensors reduce the number of independent equations to solve. The solutions to the $2 \times 2$ tensor equations places a constraint on the ranges of $\mathrm{d}r_\nu$'s 3 subcomponents.
(7) The gravity (5.8), charge (5.12), and Planck (5.19) constants were all derived from more fundamental constants, $(r_c/t_c) = c_t$, $(r_c/m_c) = c_m$, $(r_c/q_c) = c_q$, and $m_c r_c = k_m$. And all depend on the speed of light constant, $c_t$: For example, $G = c_m c_t^2$, $k_e = (c_q^2/c_m)c_t^2$, and $h = k_m c_t$.
(8) Algebraic manipulation of Coulomb's constant, $k_e = (r_c/q_c)^2((m_c/r_c)c^2)$ $= (m_c(r_c/t_c^2))(r_c/q_c)^2$, contains the term, $r_c/t_c^2$, which suggests a maximum acceleration constant for particles with mass or charge.
(9) Applying the ratios to derive Newton's gravity force (5.2) and Coulomb's charge force (5.3) equations provide:
   (a) Derivations that do not assume the inverse square law or Gauss's flux divergence theorem. **Note:** the components of the Ricci and metric tensors in Einstein's field equations have the units, $1/\text{distance}^2$ [**Wey52**], which is an assumption of the inverse square law.
   (b) The first derivations to show that the inverse square law and the property of force as mass times acceleration are the result of the conversion ratios, $r = (r_c/t_c)t = (r_c/m_c)m$.
(10) The quantum-special relativity extension to Newton's gravity equation (5.26) makes empirically verifiable predictions.
   (a) In Newton's gravity force, Gauss's gravity law, and Einstein's field equations, the force, $F \to \infty$ as the distance, $r \to 0$. But, the quantum component of the quantum-special relativity equation causes $F \to 0$ as $r \to 0$. Where the distance between two point-like particles is less than approximately $10^{-6}$ $m$, the gravity force should be measurably smaller than at $10^{-4}$ $m$, which implies larger black hole radii. The quantum effect, $F \to 0$ as $r \to 0$, helps explain the quantum phenomena, radioactivity and tunneling.
   The quantum effect can be approximated in Einstein's field equations by adding a second metric tensor, where the tensor components have the units, "*distance*$^2$."
   (b) Further, Newton's gravity constant, $G$, Gauss's gravity constant, $4\pi G$, and Einstein's gravity constant, $k = 8\pi G/c^4$, [**Wey52**], are only valid where the local velocity, $v = 0$. The special relativity component replaces $G$ with "$((m_c/r_c)(c^2 - v^2)) = c_m(c_t^2 - v^2)$".

For example, in the local (proper) frame of reference, an observer on a star orbiting the fringe of a galaxy at relativistic speeds will measure a slower gravitational acceleration than predicted by a constant G. And, in the coordinate frame of reference, for example, an observer on earth looking at the same star orbiting that distant galaxy would measure a faster gravitational acceleration than predicted by G.

(11) There is no constant ratio mapping a discrete value to a continuously varying value. Therefore, the discrete spin states of two quantum entangled particles and the polarization states of two quantum entangled photons are independent of continuously varying distance and time interval lengths.

(12) The inner product space in vector spaces, Riemann manifolds, etc. assumes any number of possible dimensions. For example, the Gram-Schmidt process is a method to find an orthogonal vector for any $n$-dimensional vector [**Coh21**]. None of those disciplines have exposed the properties that can limit a geometry to 3 dimensions.

But the set-based, first-order logic proof that a strict linearly ordered and symmetric set is a cyclic set of at most 3 members (4.12) is the simplest and most logically rigorous explanation for observing only 3 dimensions of physical space: simpler and more rigorous than parallel dimensions that cannot be detected and simpler and more rigorous than extra dimensions rolled up into infinitesimal balls that are too small to detect.

(a) Higher order dimensions could be strictly ordered but not symmetric and, thus, not sequentially reachable from the 3 cyclic dimensions.
(b) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (4.12), each member being an ordered and symmetric set of 3 dimensions (three 3-dimensional balls).
(c) Each of 3 ordered and symmetric dimensions of space can have at most 3 sequentially ordered and symmetric state values, for example, an ordered and symmetric set of 3 vector orientations, $\{-1, 0, 1\}$, per dimension of space and at most 3 spin states per plane, etc.
If the states are not sequentially ordered (a bag of states), then a selected state is undetermined until observed (like Schrodinger's cat being both alive and dead until the box is opened).

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George Van Treeck, 668 Westline Dr., Alameda, CA 94501