The Set Mappings Generating Geometry and Physics

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Abstract. The Euclidean volume equation is derived from a set and limit-based foundation. Distance as a function of volume is used for simple derivations of the Minkowski distances (for example, Manhattan and Euclidean distance) and the properties of properties metric space. The Euclidean volume proof provides simpler and more rigorous derivations of Newton's gravity force and Coulomb's charge force equations (without using the inverse square law or Gauss's divergence theorem). The derivations of the gravity and charge forces exposes a ratio (constant first derivative) principle that allows simpler derivations of the spacetime equations and some general relativity equations. A symmetry property can limit geometric distance and volume to 3 dimensions per ball and a limit of 3 3-dimensional balls. All proofs are verified in Coq.

Contents

1. Introduction

Metric space, Euclidean distance, and Euclidean area/volume are opaque definitions in mathematical analysis [**Gol76**] [**Rud76**] motivated by Euclidean geometry [**Joy98**]. Cartesian geometry motivates the idea of volume and distance in terms of Cartesian product mappings (coordinates/n-tuples). But the set of n-tuples is countable. Therefore, if each n-tuple corresponds to a real value (point), a countable infinity of points sizes has measure 0. And if each n-tuple corresponds to an infinitesimal volume or distance, then volume and distance are defined as the sums of volumes and distances, which is circular logic.

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In this article, countable volume and countable distance are defined as the the number of n-tuples. Proving that the Euclidean volume equation is an instance of the countable volume definition provides insight into what makes Euclidean space "flat" and has applications to physics. Proving that all distance functions in Euclidean (flat) space are the L_p norms/Minkowski distances (for example, Manhattan and Euclidean distance) shows that all "geometric" distance equations are inverse functions of volume and also explains why Euclidean distance the smallest distance in Euclidean space. These are insights into geometry that measure theory and point-set topology have not provided.

All the proofs in this article are trivial. But to ensure confidence, all the proofs have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

In order to compute areas and volumes, integrals divide all intervals into the *same* number subintervals (infinitesimals), where the size of the infinitesimals in each interval can *vary*, which makes it difficult for integrals to directly express the number of mappings between the p_x number of size c infinitesimals in one interval and the p_y number of size c infinitesimals in another interval.

Therefore, a different tool is used here. A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals (infinitesimals), where each infinitesimal has the *same* size, c. The ruler is both an inner and outer (floor and ceiling) measure of an interval.

Definition 2.1. Ruler measure, M: $\forall\, [a, b] \subset \mathbb{R},\ \ s = b - a \ \wedge\ c > 0 \ \wedge$ $(p = \mathit{floor}(s/c) \ \vee\ p = \mathit{ceiling}(s/c)) \ \wedge\ M = \sum_{i=1}^{p} c = pc.$

Theorem 2.2. *Ruler convergence:* $M = \lim_{c \to 0} pc = s.$

The formal proof, "limit_c_0_M_eq_exact_size," in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $\mathit{floor}(x) = \mathit{max}(\{y :\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\ p = \mathit{floor}(s/c) \ \wedge\ 0 \le |\mathit{floor}(s/c) - s/c| < 1 \ \Rightarrow\ 0 \le |p - s/c| < 1. \tag{2.1}$$

Multiply all sides of inequality 2.1 by c:

$$\forall\, c > 0, \quad 0 \le |p - s/c| < 1 \quad \Rightarrow \quad 0 \le |pc - s| < |c|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \epsilon, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \quad \wedge \quad \epsilon = \delta \\ &\qquad \Rightarrow \quad |c - 0| < \delta \quad \wedge \quad |pc - s| < \epsilon \quad := \quad M = \lim_{c \to 0} pc = s. \end{aligned} \tag{2.3}$$ □

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = \mathit{floor}(s/c) \Rightarrow\ p \cdot c = 3.1_{c=10^{-1}},\ 3.14_{c=10^{-2}},\ 3.141_{c=10^{-3}}, ..., \pi_{\lim_{c\to 0}}$.

Lemma 2.3. $\forall\, n \ge 1,\ 0 < c^n \le c \quad \Rightarrow \quad \lim_{c \to 0} c^n = \lim_{c \to 0} c.$

Proof. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$\forall\, n \ge 1 \quad \wedge \quad c:\ 0 < c^n \le c \quad \Rightarrow \quad |c - c^n| < |c|. \tag{2.4}$$

$$|c - c^n| < |c| \quad \Rightarrow \quad |c - c^n| < |c - 0|. \tag{2.5}$$

$$\forall\, \epsilon, \delta : \ |c - c^n| < |c - 0| < \delta \quad \wedge \quad \delta = \epsilon$$
$$\Rightarrow \quad |c - 0| < \delta \quad \wedge \quad |c - c^n| < \epsilon \quad := \quad \lim_{c \to 0} c^n = 0. \tag{2.6}$$

$$\lim_{c \to 0} c^n = 0 \quad \wedge \quad \lim_{c \to 0} c = 0 \quad \Rightarrow \quad \lim_{c \to 0} c^n = \lim_{c \to 0} c. \tag{2.7}$$ □

3. Euclidean Volume

DEFINITION 3.1. Countable volume, v_c is the number of Cartesian product mappings (n-tuples) between the members of n number of disjoint, countable domain sets:

$$\exists\, n, v_c \in \mathbb{N}, \quad x_1, \cdots, x_n : \quad v_c = \prod_{i=1}^n |x_i|, \quad \bigcap_{i=1}^n x_i = \emptyset$$

THEOREM 3.2. *Euclidean volume, v, defined as length of the range interval, $[v_a, v_b]$, that is equal to product of domain interval lengths, $\{[a_1, b_1], \ldots, [a_n, b_n]\}$, is an instance of the countable volume (3.1):*

$$v_c = \prod_{i=1}^n |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^n s_i, \ \ v = v_a - v_b, \ \ s_i = b_i - a_i.$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of size c subintervals and apply ruler convergence (2.2):

$$\forall\, i\ n \in \mathbb{N}, \ \ i \in [1, n], \ c > 0 \ \ \wedge \ \ floor(s_i/c) = |x_i| \ \ \Rightarrow \ \ s_i = \lim_{c \to 0}(|x_i| \cdot c). \tag{3.1}$$

Decompose the right-hand side of the volume equation into the right-hand side of the countable n-volume definition (3.1).

$$s_i = \lim_{c \to 0}(|x_i| \cdot c) \quad \Leftrightarrow \quad \prod_{i=1}^n s_i = \lim_{c \to 0} \prod_{i=1}^n (|x_i| \cdot c). \tag{3.2}$$

$$\prod_{i=1}^n s_i = \lim_{c \to 0} \prod_{i=1}^n (|x_i| \cdot c) \quad \Leftrightarrow \quad \prod_{i=1}^n s_i = \lim_{c \to 0}(\prod_{i=1}^n |x_i|) \cdot c^n. \tag{3.3}$$

$$v_c = \prod_{i=1}^n |x_i| \quad \Leftrightarrow \quad \lim_{c \to 0} v_c \cdot c^n = \lim_{c \to 0}(\prod_{i=1}^n |x_i|) \cdot c^n = \prod_{i=1}^n s_i. \tag{3.4}$$

Decompose the left-hand side of the volume equation into the left-hand side of the countable n-volume definition (3.1).

$$\exists\, v \in \mathbb{R} : \ v_c = floor(v/c) \quad \Leftrightarrow \quad v = \lim_{c \to 0} v_c \cdot c. \tag{3.5}$$

Apply lemma 2.3 to equation 3.5:

$$v = \lim_{c \to 0} v_c \cdot c \quad \wedge \quad \lim_{c \to 0} c^n = \lim_{c \to 0} c \quad \Leftrightarrow \quad v = \lim_{c \to 0} v_c \cdot c^n. \tag{3.6}$$

Combine equations 3.6 and 3.4:

$$v = \lim_{c \to 0} v_c \cdot c^n \quad \wedge \quad \lim_{c \to 0} v_c \cdot c^n = \prod_{i=1}^n s_i \quad \Leftrightarrow \quad v = \prod_{i=1}^n s_i. \tag{3.7}$$ □

4. Distance

4.1. Countable distance. Only like types can be added together. For example, a scalar can only be the sum of scalars and a vector can only be the sum of vectors. Likewise, an n-volume can only be the sum of n-volumes. d_c^n is an integer (countable) n-volume, where d_c is an integer (countable) n-distance. Therefore, the countable n-distance, d_c, is a function of the summed countable n-volumes.

Definition 4.1. The countable n-volume, d_c^n, is the sum of m number of disjoint, countable n-volumes.

$$\forall\, n \in \mathbb{N}, \quad d_c \in \{0, \mathbb{N}\} : \quad \Rightarrow \quad \exists\, m \in \mathbb{N}, \quad x_1, \cdots, x_m \in X : \quad d_c^n = \textstyle\sum_{i=1}^m |x_i|^n.$$

4.2. Minkowski distance (L_p norm).
The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

Theorem 4.2. *Minkowski distance (L_p norm) is an instance of the sum of disjoint, countable n-volumes* (4.1).

$$d_c^n = \textstyle\sum_{i=1}^m |x_i|^n \quad \Rightarrow \quad \exists\, d, s_1, \cdots, s_m \in \mathbb{R} : \quad d = (\textstyle\sum_{i=1}^m s_i^n)^{1/n}.$$

Proof. Apply the ruler (2.1) and ruler convergence (2.2):

$$\exists\, d, s_1, \cdots, s_m \in \mathbb{R} : \; d_c = floor(d/c) \quad \wedge \quad |x_i| = floor(s_i/c) \quad \wedge \tag{4.1}$$
$$d_c^n = \textstyle\sum_{i=1}^m |x_i|^n \quad \Rightarrow \quad d^n = \lim_{c\to 0}(d_c \cdot c)^n = \lim_{c\to 0} \sum_{i=1}^m (|x_i| \cdot c)^n = \sum_{i=1}^m s_i^n.$$

$$d^n = \textstyle\sum_{i=1}^m s_i^n \quad \Leftrightarrow \quad d = (\textstyle\sum_{i=1}^m s_i^n)^{1/n}. \tag{4.2}$$ □

4.3. Distance inequality. Proving that the Minkowski distance (L_p norm) satisfies the metric space triangle inequality requires another inequality. And for completeness, that inequality must be proved. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

Theorem 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N}, \;\; v_a, v_b \geq 0 : \;\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

Proof. Expand the n-volume, $(v_a^{1/n} + v_b^{1/n})^n$, using the binomial expansion:

$$\forall\, v_a, v_b \geq 0 : \quad v_a + v_b \leq (v_a + v_b + \tag{4.3}$$
$$\textstyle\sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k}) = (v_a^{1/n} + v_b^{1/n})^n.$$

Take the n^{th} root of both sides of the inequality:

$$\forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \tag{4.4}$$ □

4.4. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

Theorem 4.4. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N}, \;\; a_i, b_i \geq 0 : \;\; (\textstyle\sum_{i=1}^m (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^m a_i^n)^{1/n} + (\sum_{i=1}^m b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$\begin{aligned}(4.5)\quad \forall\, m,n \in \mathbb{N},\ v_a,v_b \geq 0: \quad & v_a = \textstyle\sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge \\ (v_a+v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad & \Rightarrow \quad ((\textstyle\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} = \\ & (\textstyle\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \quad \square\end{aligned}$$

4.5. Metric Space. All Minkowski distances (L_p norms) have the properties of metric space.

The formal proofs: symmetry, triangle_inequality, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Symmetry:* $d(s_1,s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,v) = d(v,u)$.

PROOF. By the commutative law of addition:

$$\begin{aligned}(4.6)\quad \forall\, p : p \geq 1, \quad & d(s_1,s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ & \Rightarrow \quad d(u,v) = (u^p+v^p)^{1/p} = (v^p+u^p)^{1/p} = d(v,u). \quad \square\end{aligned}$$

THEOREM 4.6. *Triangle Inequality:*
$d(s_1,s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,w) \leq d(u,v) + d(v,w)$.

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(4.7)\qquad (u^p+w^p)^{1/p} \leq ((u^p+w^p) + 2v^p)^{1/p} = ((u^p+v^p) + (v^p+w^p))^{1/p}.$$

Apply the distance inequality (4.3) to the inequality 4.7:

$$\begin{aligned}(4.8)\quad & (u^p+w^p)^{1/p} \leq ((u^p+v^p)+(v^p+w^p))^{1/p} \ \wedge\ (v_a+v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ & \qquad \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ \Rightarrow \quad & (u^p+w^p)^{1/p} \leq ((u^p+v^p)+(v^p+w^p))^{1/p} \leq (u^p+v^p)^{1/p} + (v^p+w^p)^{1/p} \\ & \qquad \Rightarrow \quad d(u,w) = (u^p+w^p)^{1/p} \leq \\ & \qquad (u^p+v^p)^{1/p} + (v^p+w^p)^{1/p} = d(u,v) + d(v,w). \quad \square\end{aligned}$$

THEOREM 4.7. *Non-negativity:* $d(s_1,s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u,w) \geq 0$.

PROOF. By definition, the length of an interval is always ≥ 0:

$$(4.9)\qquad \forall\, [a_1,b_1],\ [a_2,b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0.$$

$$(4.10)\qquad p \geq 1,\ u,v \geq 0 \quad \Rightarrow \quad d(u,v) = (u^p+v^p)^{1/p} \geq 0. \qquad \square$$

THEOREM 4.8. *Identity of Indiscernibles:* $d(u,u) = 0$.

PROOF. From the non-negativity property (4.7):

$$\begin{aligned}(4.11)\quad & d(u,w) \geq 0 \quad \wedge \quad d(u,v) \geq 0 \quad \wedge \quad d(v,w) \geq 0 \\ & \qquad \Rightarrow \quad \exists\, d(u,w) = d(u,v) = d(v,w) = 0.\end{aligned}$$

$$(4.12)\qquad d(u,w) = d(v,w) = 0 \quad \Rightarrow \quad u = v.$$

$$(4.13)\qquad d(u,v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u,u) = 0. \qquad \square$$

5. Applications to physics

5.1. Newton's gravity force equation. m_1 and m_2, are the sizes of two independent mass intervals, where each size c component of a mass interval exerts a force on each size c component of the other mass interval. If p_1 and p_2 are the number of size c components in each mass interval, then the total force, F, is equal to the total number of forces, $p_1 \cdot p_2$, and proportionate to the size, c, of each component. Applying the ruler (2.1) and volume proof (3.2):

$$\begin{aligned}(5.1)\quad & p_1 = floor(m_1/c) \quad \wedge \quad p_2 = floor(m_2/c) \quad \wedge \quad F := m_0 a \propto (p_1 \cdot p_2)c \\ & \Rightarrow \quad F := m_0 a \propto \lim_{c\to 0}(p_1 \cdot p_2)c = \lim_{c\to 0}(p_1 \cdot p_2)c^n = \lim_{c\to 0} p_1 c \cdot p_2 c = m_1 m_2,\end{aligned}$$

where the force, F, is defined as the rest mass, m_0, times acceleration, a.

$$\begin{aligned}(5.2)\quad & F := m_0 a = m_0 r/t^2 \propto m_1 m_2 \ \wedge \ m_0 = m_1 \ \Rightarrow \ r \propto m_2 \ \Rightarrow \\ & \exists\, m_G, r_c \in \mathbb{R}: \ r = (\mathrm{d}r/\mathrm{d}m)m_2 = (r_c/m_G)m_2,\end{aligned}$$

where: r is Euclidean distance, t is time, and r_c/m_G is a unit-factoring proportion ratio.

$$\begin{aligned}(5.3)\quad & m_0 = m_1 \quad \wedge \quad r = (m_G/r_c)m_2 \quad \wedge \quad F = m_0 r/t^2 \\ & \Rightarrow \quad F = m_0 r/t^2 = (r_c/m_G)m_1 m_2/t^2.\end{aligned}$$

From the definition of force, $F := m_0 a$:

$$(5.4)\quad \textstyle\int_0^t a\mathrm{d}t = r/t \ \Rightarrow \ \exists\, t_c, r_c \in \mathbb{R}: \ r/t = (\mathrm{d}r/\mathrm{d}t) = r_c/t_c \ \Rightarrow \ t = (t_c/r_c)r.$$

$$\begin{aligned}(5.5)\quad & t = (t_c/r_c)r \quad \wedge \quad F = (r_c/m_G)m_1 m_2/t^2 \quad \Rightarrow \\ & F = (r_c/m_G)(r_c^2/t_c^2)m_1 m_2/r^2 = (r_c^3/m_G t_c^2)m_1 m_2/r^2 = G m_1 m_2/r^2,\end{aligned}$$

where the gravitational constant, $G = r_c^3/m_G t_c^2$, has the SI units: $m^3 kg^{-1} s^{-2}$.

5.2. Coulomb's charge force. q_1 and q_2, are the sizes of two independent charge intervals, where each size c component of a charge interval exerts a force on each size c component of the other charge interval. If p_1 and p_2 are the number of size c components in each charge interval, then the total force, F, is equal to the total number of forces, $p_1 \cdot p_2$, and proportionate to the size, c, of each component. Applying the ruler (2.1) and volume proof (3.2):

$$\begin{aligned}(5.6)\quad & p_1 = floor(q_1/c) \quad \wedge \quad p_2 = floor(q_2/c) \quad \wedge \quad F \propto (p_1 \cdot p_2)c \\ & \Rightarrow \quad F := m_0 a \propto \lim_{c\to 0}(p_1 \cdot p_2)c = \lim_{c\to 0}(p_1 \cdot p_2)c^n = \lim_{c\to 0} p_1 c \cdot p_2 c = q_1 q_2,\end{aligned}$$

where the force, F, is defined as the rest mass, m_0, times acceleration, a.

$$\begin{aligned}(5.7)\quad & F := m_0 a = m_0 r/t^2 \propto q_1 q_2 \quad \wedge \\ & m_0 = (\mathrm{d}m/\mathrm{d}q)q_1 = (m_G/q_C)q_1 \quad \Rightarrow \quad r \propto q_2 \\ & \Rightarrow \quad \exists\, q_C, r_c \in \mathbb{R}: \ r = (\mathrm{d}r/\mathrm{d}q)q_2 = (r_c/q_C)q_2,\end{aligned}$$

where: r is Euclidean distance, t is time, m_G/q_C and r_c/q_C are unit-factoring proportion ratios.

$$(5.8)\quad m_0 = (m_G/q_C)q_1 \quad \wedge \quad r = (q_C/r_c)q_2 \quad \wedge \quad F = m_0 r/t^2$$
$$\Rightarrow \quad F = m_0 r/t^2 = (m_G/q_C)(r_c/q_C)q_1 q_2/t^2 = (m_G r_c/q_C^2)q_1 q_2/t^2.$$

From the definition of force, $F := m_0 a$:

$$(5.9)\quad \textstyle\int_0^t a\mathrm{d}t = r/t \;\Rightarrow\; \exists\, t_c, r_c \in \mathbb{R} : \; r/t = (\mathrm{d}r/\mathrm{d}t) = r_c/t_c \;\Rightarrow\; t = (t_c/r_c)r.$$

$$(5.10)\quad t = (t_c/r_c)r \quad \wedge \quad a_G = r_c/t_c^2 \quad \wedge \quad F = (m_G r_c/q_C^2)q_1 q_2/t^2 \quad \Rightarrow$$
$$F = (r_c^2/t_c^2)(m_G r_c/q_C^2)q_1 q_2/r^2 = ((m_G a_G)r_c^2/q_C^2)q_1 q_2/r^2 = k_c q_1 q_2/r^2,$$

where the charge constant, $k_C = (m_G a_G)r_c^2/q_C^2$, has the SI units: Nm^2C^{-2}.

5.3. Spacetime equations. As shown in the derivations of Newton's gravity force (5.1) and Coulomb's charge force (5.2) equations: $r = (r_c/t_c)t = ct$, where r is the Euclidean distance and $r_c/t_c = c$ is a unit-factoring proportion ratio. And, the smallest distance (and time) spanning the two inertial (independent, non-accelerating) frames of reference, $[0, r_1]$ and $[0, r_2]$, is the Euclidean distance, r.

$$(5.11)\qquad r = ct \quad \Rightarrow \quad (ct)^2 = r_1^2 + r_2^2 \quad \Leftrightarrow \quad r_1^2 = (ct)^2 - (x^2 + y^2 + z^2),$$

where $r_2^2 = x^2 + y^2 + z^2$, which is one form of Minkowski's flat spacetime interval equation [**Bru17**]. And the length contraction and time dilation equations also follow directly from $(ct)^2 = r_1^2 + r_2^2$, where $v = r_1/t$:

$$(5.12)\quad r_2^2 = (ct)^2 - r_1^2 \;\wedge\; L = r_2 \;\Rightarrow\; L^2 = c^2t^2 - r_1^2 \;\Rightarrow\; L = ct\sqrt{1 - (v/c)^2}.$$

$$(5.13)\qquad L = ct\sqrt{1 - (v/c)^2} \quad \wedge \quad L_0 = ct \quad \Rightarrow \quad L = L_0\sqrt{1 - (v/c)^2}.$$

$$(5.14)\qquad L = ct\sqrt{1 - (v/c)^2} \;\wedge\; t' = L/c \;\Rightarrow\; t' = t\sqrt{1 - (v/c)^2}.$$

5.4. Some general relativity equations: Combining the ratio (constant first derivative) equations into partial differential equations: $r = (r_c/m_G)m = ct \Rightarrow (r_c/m_G)m \cdot ct = r^2 \Rightarrow m = (m_G/r_c c)r^2/t = (m_G/r_c c)rv$. For a constant mass, m, a decrease in the distance, r, between two mass centers causes a decrease in time, t, (time slows down). v is the relativistic orbital velocity at distance, r. $(r_c/m_G)m \cdot (ct)^2 = r^3 \Rightarrow E = mc^2 = (m_G/r_c)r^3/t^2$. And $(ct)^2 = r^2 \Rightarrow c^2 = v^2 \Rightarrow (r_c/m_G)mv^2 = c^2 r \Rightarrow KE = mv^2/2 = (m_G c^2/2r_c)r$.

$c = r_c/t_c \approx 3 \cdot 10^8 ms^{-1}$ and $G = r_c^3/m_G t_c^2 = (r_c/m_G)(r_c/t_c)^2 \approx 6.7 \cdot 10^{-11} m^3 kg^{-1} s^{-2} \Rightarrow r_c/m_G \approx (6.7 \cdot 10^{-11} m^3 kg^{-1} s^{-2}/(3 \cdot 10^8 m\ s^{-1})^2 \approx 7.4 \cdot 10^{-28} m\ kg^{-1}$, which can be used to quantify the constants in the previously derived equations. For example, $m = (m_G/r_c c)rv \approx (1/((7.4 \cdot 10^{-28} m\ kg^{-1})(3 \cdot 10^8 m\ s^{-1})))rv \approx (4.5 \cdot 10^{18} kg\ s\ m^{-2})rv$.

Likewise, for charge, $r = (r_c/q_C)q = ct \Rightarrow q = (q_C/r_c c)r^2/t = (q_C/r_c c)rv$, $E = qc^2 = (q_C/r_c)r^3/t^2$, and $KE = qv^2/2 = (q_C c^2/2r_c)r$. And if the ratio of an electron's mass to charge is m_G/q_C, then $m_G/q_C \approx 9.1 \cdot 10^{-31} kg/1.6 \cdot 10^{-19} C \approx 5.7 \cdot 10^{-12} kgC^{-1}$. And using Coulomb's constant in ratio form: $k_C = (r_c/t_c)^2(m_G r_c/q_C^2) \approx 9 \cdot 10^9 Nm^2C^{-2} \approx (3 \cdot 10^8 m\ s^{-1})^2(5.7 \cdot 10^{-12} kg\ C^{-1})(r_c/q_c) \Rightarrow r_c/q_C \approx 1.7 \cdot 10^5 m\ C^{-1}$. Therefore, $q = (q_C/r_c c)rv \approx (1/((1.7 \cdot 10^5 m\ C^{-1})(3 \cdot 10^8 m\ s^{-1})))rv \approx (1.9 \cdot 10^{-13} C\ s\ m^{-2})rv$.

5.5. 3 dimensional balls. Countable volume, $v_c = \prod_{i=1}^{n} |x_i|$, Euclidean volume, $v = \prod_{i=1}^{n} s_i$, and all Minkowski distances, $d = (\sum_{i=1}^{n} s_i^n)^{1/n}$, require that a set of domain intervals/dimensions can be assigned a *total order*. A total order is defined in terms of successor and predecessor relations, where, in this case, the successor and predecessor relations are specified by the integers $i = 1$ to n that map to a set of domain intervals/dimensions.

But the commutative properties of union, multiplication, and addition allow sequencing through each interval (dimension) in every possible order. And "jumping" (indexing) over set members to another member requires calculating an offset, which is implicitly sequencing via the successor and predecessor relations.

Therefore, sequencing directly via the successor and predecessor relations from one set member to every other member requires each set member to be sequentially adjacent (either a successor or predecessor) to every other member, herein referred to as a symmetry constraint. It will now be proved that coexistence of the symmetry constraint on a sequentially ordered set defines a cyclic set that contains at most 3 members, in this case, 3 dimensions per ball and 3 3-dimensional balls.

Definition 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i \;\in \{x_1, \ldots, x_n\},$$
$$\textit{successor}\; x_i = x_{i+1} \;\;\wedge\;\; \textit{predecessor}\; x_{i+1} = x_i.$$

Definition 5.2. Symmetry Constraint (every set member is sequentially adjacent to every other member):

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; \textit{successor}\; x_i = x_j \;\;\Leftrightarrow\;\; \textit{predecessor}\; x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$i = n \;\;\wedge\;\; j = 1 \;\;\Rightarrow\;\; \textit{successor}\; x_n = x_1 \;\;\wedge\;\; \textit{predecessor}\; x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

Proof. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry constraint (5.2):

$$i = n \;\;\wedge\;\; j = 1 \;\;\wedge\;\; \textit{successor}\; x_i = x_j \;\;\Rightarrow\;\; \textit{successor}\; x_n = x_1. \tag{5.15}$$

Applying the definition of the symmetry constraint (5.2) to conclusion 5.15:

$$\textit{successor}\; x_i = x_j \;\;\Rightarrow\;\; \textit{predecessor}\; x_j = x_i \;\;\Rightarrow\;\; \textit{predecessor}\; x_1 = x_n. \tag{5.16}$$ □

Theorem 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

Proof.
It was proved that an ordered and symmetric set is a cyclic set (5.3).

DEFINITION 5.5. Successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize). \tag{5.17}$$

DEFINITION 5.6. Predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1). \tag{5.18}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{5.19}$$

Prove that every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \tag{5.20}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \tag{5.21}$$

$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1). \tag{5.22}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.23}$$

$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \tag{5.24}$$

$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1). \tag{5.25}$$

$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize). \tag{5.26}$$

$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.27}$$

$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \tag{5.28}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Successor(1, 3, setsize > 3) \\ & \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \end{aligned} \tag{5.29}$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Predecessor(1, 3, setsize > 3) \\ & \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \end{aligned} \tag{5.30}$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\ setsize > 3 : \quad & \neg Adjacent(1, 3, setsize > 3) \\ & \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \end{aligned} \tag{5.31}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

(1) It was proved that Euclidean area/volume is a "flat" space, where each n-tuple corresponds to an infinitesimal having the same size (3.2).

(2) The same-sized Euclidean volume infinitesimals constrains a distance measure within each Euclidean volume infinitesimal to the same size. Therefore, each distance infinitesimal in an Euclidean area/volume is an inverse function of each Euclidean area/volume infinitesimal. And it was proved that all distances that are inverse functions of Euclidean area/volume are the L_p norm/Minkowski distances (4.3).

(3) All "geometric" distances, d, that are inverse functions of both Euclidean and non-Euclidean volumes, v, are of the form:
$v = (g(n)d)^{f(n)} = \sum_{i=1}^{m} (g(n)s_i)^{f(n)} \quad \Leftrightarrow$
$d = ((\sum_{i=1}^{m} (g(n)s_i)^{f(n)})^{f^{-1}(n)})/g(n)$, where m is the number of dimensions and $1 \leq n \leq m$. An example using the gamma function as an instance of the generalized distance equation, where:
$g(n) = \Gamma(1/n), \quad f(n) = n, \quad m = n = 2 \quad \Rightarrow$
$s = \sum_{i=1}^{2} (\Gamma(1/2)d)^2 = 2\pi d^2$, which is the surface area, s, of a sphere having the diameter, d. And the distance, d, is an inverse function of the surface area, s: $d = \sqrt{s}/2\pi$.

(4) If the definition of a complete metric space allows equations that are not instances of the above generalized distance equation, then the definition of a complete metric space is not a sufficient filter for geometric distances.

(5) The volume-based inequality, $v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n$, generates the distance inequality, $(v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}$ (4.3), which was used to help derive the metric space triangle inequality (4.6) and derive the distance sum inequality (4.4). Compare the distance sum inequality (4.4):

$$(\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

to the Minkowski's sum inequality:

$$(\sum_{i=1}^{m} (a_i^n + b_i^n)^n)^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

Note the difference in the left side of the two inequalities:

$$(\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \quad vs. \quad (\sum_{i=1}^{m} (a_i^n + b_i^n)^{\mathbf{n}})^{1/n}.$$

The derivation of the distance sum inequality is much simpler and shorter than the derivation of Minkowski's sum inequality. Unlike Minkowski's sum inequality proof, the distance (sum) inequality proofs do not depend on: convexity, L_p space inequalities (for example, Hölder's inequality or Mahler's inequality), or the triangle inequality, which indicates that the distance (sum) inequality is a more fundamental inequality than the other distance-related inequalities.

(6) Proofs that Euclidean distance is the smallest distance between two distinct points have equated Euclidean distance to a straight line (equation), where it is assumed that the straight line length is the smallest distance [**Joy98**]. And proofs that a straight line (equation) is the smallest distance have equated the straight line to the Euclidean distance. There have been no set and limit-based explanations of why the Euclidean distance/straight line length is the smallest distance.

The Cartesian product mappings between same-sized countable sets, $|x_i|^n$, generate both Euclidean volume (3.2) and the Minkowski distances (4.2). In the Minkowski equation, $d = (\sum_{i=1}^{m} s_i^n)^{1/n}$, if m represents the number of domain intervals/countable sets (one from each dimension), then $1 \le n \le m$. And $m = 2 \Rightarrow 1 \le n \le 2$, which constrains all flat 2-space Minkowski distances to a range from Manhattan distance (the largest distance) to Euclidean distance (the smallest distance).

(7) Applying the ruler (2.1) and volume proof (3.2) to the Cartesian product of same-sized, infinitesimal mass forces and charge forces to derive Newton's gravity force (5.1) and Coulomb's charge force (5.2) equations provide several firsts and some insights into physics:
   (a) These are the first deductive derivations of the gravity and charge forces. All other derivations have been empirical and use Newton's induction, which is not fully provable, for example, assumes the inverse square law, which is based on empirical observation.
   (b) These are the first derivations to not use the inverse square law or Gauss's divergence theorem.
   (c) These are the first derivations to show that the definition of force, $F := m_0 a$, containing acceleration, a, generates the inverse square law: $\int_0^t a\mathrm{d}t = r/t \;\Rightarrow\; \exists\, t_c, r_c \in \mathbb{R} : r/t = r_c/t_c \;\Rightarrow\; t = (t_c/r_c)r$. Using the same derivation steps as for Coulomb's charge force: $F := m_0 a = m_0 r/t^2 = (r_c/t_c)^2(m_x r_c/x_x^2)x_1 x_2/r^2 = k_x x_1 x_2/r^2$.
   (d) Using Occam's razor, those versions of constants like: charge, vacuum magnetic permeability, fine structure, etc. that contain the value 4π might be incorrect because those constants are based on the less parsimonious assumption that the inverse square law is due to Gauss's flux divergence on a sphere having the surface area, $4\pi r^2$.
   (e) These are the first derivations to show that the gravity force, charge force, spacetime, and general relativity all depend on time being proportionate to distance: $r = (r_c/t_c)t = ct$.
   (f) The derivations of the gravity and charge force equations expose a ratio (constant first derivative) principle. Combining the constant first derivatives (ratios) into partial differential equations allows simple algebraic derivations of some general relativity equations (5.4) without the need for integrating second derivative (spacetime curvature) tensors.
   (g) A state is represented by a constant value. And a constant value, by definition, cannot vary with distance and time interval lengths. Therefore, the spin states of two quantum entangled electrons and the polarization states of two quantum entangled photons are independent of the amount of distance and time between the entangled particles.

(8) It was proved that a totally ordered set with a symmetry constraint is a cyclic set with at most 3 members (5.3). And the definitions of distance and volume both require a total order and symmetry, which provides several insights:
   (a) Using Occam's razor, a cyclic set of at most 3 members is the most parsimonious explanation of only observing 3 dimensions of geometric

distance and volume.

(b) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (5.4), each member being an ordered and symmetric set of 3 dimensions (three balls).

(c) Each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values of the same type, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values of the same type per 3-dimensional ball, for example, vector orientation values: -1, 0, 1 per orthogonal direction in the ball.

(d) Each of the 3 possible ordered and symmetric dimensions of discrete physical states could contain an unordered collection (bag) of discrete state values. Bags are non-deterministic. For example, every time an unordered binary state is "pulled" from a bag, there is a 50 percent chance of getting one of the binary values.

(9) Functions that are a bijection (1-1 correspondence) between the elements of real-valued intervals is a primary tool in mathematics. But, in this article, it was shown that some fundamental geometry (volume and the Minkowski distances/L_p norms) and physics (gravity force and charge force) are derived from the combinatorial mappings between the elements of real-valued intervals. And only 3 dimensions of geometric space is also due to combinatorics.

(10) The proofs and derivations in this article show that the ruler (2.1) is a tool to directly express some combinatorial relations in geometry, probability, physics, etc. that are difficult to directly express with differential equations and integrals.

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