Some Set Properties Underlying Geometry and Physics

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Abstract. Euclidean volume and the Minkowski distances (Manhattan, Euclidean, etc. distances) are derived from a set and limit-based foundation without referencing the primitives and relations of geometry. Sequencing a strict linearly ordered set in all n-at-a-time orders via successor/predecessor relations is proved to be a cyclic set of at most 3 members. A cyclic set of 3 distance domain interval lengths are related to other types of domain interval lengths by unit-factoring ratios. The ratios are used to provide simple and short derivations of: the gravity (G), charge (k_e), and Planck (h) constants, the spacetime, Schwarzchild's gravitational time dilation, Einstein's gravitational lens, Planck relation, Compton wavelength, de Broglie wavelength, and quantum-gravity equations. All the proofs are verified in Coq.

Contents

1. Introduction

Mathematical analysis can construct differential calculus from a set and limit-based foundation without referencing the primitives and relations of Euclidean geometry, like straight line, angle, etc., which provides a more rigorous foundation and deeper understanding of geometry and physics. But Euclidean volume in the Riemann integral, Lebesgue integral, measure theory, and distance in the vector magnitude and metric space criteria are definitions motivated by Euclidean geometry [Gol76] [Rud76] rather than derived from a set and limit-based foundation.

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An intuitive, set-based motivation of Euclidean volume is the number, v_c, of ordered combinations (n-tuples): $v_c = \prod_{i=1}^{n} |x_i|$, where $|x_i|$ is the cardinal of the countable, disjoint set, x_i. But, some well-known analysis textbooks do not provide a proof that, $\lim_{\kappa \to 0} v_c \cdot \kappa = \lim_{\kappa \to 0}(\prod_{i=1}^{n} |x_i|) \cdot \kappa \Rightarrow v = \prod_{i=1}^{n} s_i$, where each set, x_i, is a set of size κ subintervals of each interval, $[a_i, b_i] \subset \mathbb{R}$, and where $s_i = b_i - a_i$. [**Gol76**] [**Rud76**]. In this article, that proof is provided.

$v_c = \prod_{i=1}^{n} |x_i| = f(|x_1|, \cdots, |x_n|, n)$. If f is a bijective function, then $\exists\ d_c : d_c = f^{-1}(v_c, n)$ and $v_c = f(d_c, n) = f(|x_1|, \cdots, |x_n|, n)$. If f is also isomorphic, then $\forall\ |x_i|,\ d_c = |x_1| = \cdots = |x_n|$ and $v_c = \prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n$.

Where $v_c = f(|x_1|, \cdots, |x_n|, n)$ is a bijective and isomorphic function, it will be proved that $\lim_{\kappa \to 0} v_c \cdot \kappa = \lim_{\kappa \to 0}(\sum_{j=1}^{m} v_{c_i}) \cdot \kappa \Rightarrow d^n = \sum_{i=1}^{m} d_i^n$. d is the ρ-norm (Minkowski distance) [**Min53**], which will be proved to imply the metric space properties [**Rud76**].

Sequencing the domain sets, $x_1, \cdots, x_n$, from $i = 1$ to n, is a strict linear (total) order, where a total order is defined in terms of successor and predecessor relations [**CG15**]. Sequencing a set, via successor and predecessor relations, in all n-at-a-time orders, requires a "symmetry" constraint, where every set member is either a successor or predecessor to every other set member. A strict linearly ordered and symmetric set will be proved to be a cyclic set, where $n \leq 3$.

Therefore, if $\{s_1, s_2, s_3\}$ is a strict linearly ordered and symmetric set of 3 "distance" dimensions, then another dimension, s_4, must have a different type (is a member of different set). In an orthogonal Cartesian grid around a point, there are constant, unit-factoring ratios between a unit of a 3-dimensional distance, r, and units of other orthogonal dimensions: $r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q = \cdots$.

The ratio constants are used for simple and short derivations of: the gravity (G), charge (k_e), and Planck (h) constants. And the ratio constants are also used for simpler and shorter derivations of: spacetime, Schwarzchild gravitational time dilation, Einstein's gravitational lens, Planck relation, Compton wavelength, de Broglie wavelength, quantum-relativity gravity, and quantum-relativity charge equations.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq23**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of size κ subintervals. The ruler is both an inner and outer measure of an interval.

Definition 2.1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\ [a, b] \subset \mathbb{R}$, $s = b - a\ \wedge\ 0 < \kappa \leq 1\ \wedge\ (p = floor(s/\kappa)\ \vee\ p = ceiling(s/\kappa))$.

Theorem 2.2. *Ruler convergence:* $M = \lim_{\kappa \to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\ \kappa > 0,\ p = floor(s/\kappa)\ \wedge\ 0 \leq |floor(s/\kappa) - s/\kappa| < 1\ \Rightarrow\ |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by κ:

$$\forall\, \kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \epsilon \quad := \quad M = \lim_{\kappa \to 0} p\kappa = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, \ldots, \pi_{\lim_{\kappa\to 0}}$.

Lemma 2.3. $\forall\, n \geq 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa \to 0} \kappa.$

Proof. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa\to 0} \kappa^n = 0. \end{aligned} \tag{2.5}$$

$$\lim_{\kappa\to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa\to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa. \tag{2.6}$$

$\square$

3. Volume

Definition 3.1. A countable n-volume is the number of ordered combinations (n-tuples), v_c, of the members of n number of disjoint, countable domain sets, x_i:

$$\exists\, n \in \mathbb{N},\ v_c \in \{0,\ \mathbb{N}\},\ \ x_i \in \{x_1, \cdots, x_n\} : \ \bigcap_{i=1}^n x_i = \emptyset \ \wedge \ v_c = \prod_{i=1}^n |x_i|. \tag{3.1}$$

Theorem 3.2. *Euclidean volume,*

$$\begin{aligned} &\forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\},\ \ [v_a, v_b] \subset \mathbb{R},\ \ s_i = b_i - a_i,\ \ v = v_b - v_a : \\ &\qquad\qquad v_c = \prod_{i=1}^n |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^n s_i. \end{aligned} \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

Proof.

$$v_c = \prod_{i=1}^n |x_i| \ \Leftrightarrow \ v_c\kappa = (\prod_{i=1}^n |x_i|)\kappa \ \Leftrightarrow \ \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{3.3}$$

Apply the ruler (2.1) and ruler convergence (2.2) to equation 3.3:

$$\exists\, v, \kappa \in \mathbb{R} : \ v_c = floor(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{3.4}$$

Apply lemma 2.3 to equation 3.4:

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa^n = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa). \tag{3.5}$$

Apply the ruler (2.1) and ruler convergence (2.2) to s_i:

$$\exists\, s_i, \kappa \in \mathbb{R} : \ floor(s_i/\kappa) = |x_i| \quad \Rightarrow \quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i. \tag{3.6}$$

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa) \ \wedge \ \lim_{\kappa\to 0}(|x_i|\kappa) = s_i \ \Rightarrow \ v = \prod_{i=1}^n s_i \tag{3.7}$$

$\square$

4. Distance

Definition 4.1. Countable distance, $d_c = f(v_c, n) = f(|x_1|, \cdots, |x_n|, n) = \prod_{i=1}^n |x_i|$ is bijective and isomorphic: $v_c = \prod_{i=1}^n |x_i| = \prod_{i=1}^n d_c = d_c^n$.

4.1. Minkowski distance (ρ-norm).

THEOREM 4.2. *Minkowski distance (ρ-norm):*

$$v_c = \textstyle\sum_{j=1}^m v_{c_i} \quad \Rightarrow \quad \exists\, d, d_i \in \mathbb{R} : \; d^n = \textstyle\sum_{i=1}^m d_i^n.$$

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

PROOF. Apply the countable distance definition (4.1) to the assumption:

$$v_c = \textstyle\prod_{i=1}^n |x_i| = \prod_{i=1}^n d_c = d_c^n \quad \wedge \quad v_{c_i} = \prod_{j=1}^n |x_{i_j}| = \prod_{i=1}^n d_{c_i} = d_{c_i}^n \quad \wedge \quad v_c = \sum_{j=1}^m v_{c_i} \quad \Rightarrow \quad d_c^n = \sum_{j=1}^m d_{c_i}^n. \tag{4.1}$$

Multiply both sides of equation 4.1 by κ and take the limit:

$$d_c^n = \textstyle\sum_{j=1}^m d_{c_i}^n \quad \Leftrightarrow \quad \lim_{\kappa\to 0} d_c^n \kappa = \lim_{\kappa\to 0} \sum_{j=1}^m d_{c_i}^n \kappa. \tag{4.2}$$

Apply lemma 2.3 to equation 4.1:

$$\lim_{\kappa\to 0} d_c^n \kappa = \lim_{\kappa\to 0} \textstyle\sum_{j=1}^m d_{c_i}^n \kappa \quad \wedge \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa$$
$$\Leftrightarrow \lim_{\kappa\to 0} d_c^n \kappa^n = \lim_{\kappa\to 0} \textstyle\sum_{j=1}^m d_{c_i}^n \kappa^n \Leftrightarrow \lim_{\kappa\to 0} (d_c\kappa)^n = \lim_{\kappa\to 0} \sum_{j=1}^m (d_{c_i}\kappa)^n. \tag{4.3}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to equation 4.3:

$$\exists\, d, d_i : \; d_c = floor(d/\kappa), \; d = \lim_{\kappa\to 0} d_c\kappa \quad \wedge \quad d_{c_i} = floor(d_i/\kappa), \; d_i = \lim_{\kappa\to 0} d_{c_i}\kappa \quad \Rightarrow$$
$$d^n = \lim_{\kappa\to 0} (d_c\kappa)^n = \lim_{\kappa\to 0} \textstyle\sum_{j=1}^m (d_{c_i}\kappa)^n = \sum_{j=1}^m d_i^n. \tag{4.4}$$

□

4.2. Distance inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance inequality*

$$\forall\, n \in \mathbb{N}, \;\; v_a, v_b \ge 0 : \;\; (v_a + v_b)^{1/n} \le v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$\forall\, v_a, v_b \ge 0 : \quad v_a + v_b \le v_a + v_b + \textstyle\sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n. \tag{4.5}$$

Take the n^{th} of both sides of the inequality 4.5:

$$\forall\, v_a, v_b \ge 0, n \in \mathbb{N} : v_a + v_b \le (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \le v_a^{1/n} + v_b^{1/n}. \tag{4.6}$$

□

4.3. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.4. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N}, \;\; a_i, b_i \ge 0 : \;\; \left(\textstyle\sum_{i=1}^m (a_i^n + b_i^n)\right)^{1/n} \le \left(\sum_{i=1}^m a_i^n\right)^{1/n} + \left(\sum_{i=1}^m b_i^n\right)^{1/n}.$$

PROOF. Apply the distance inequality (4.3):

$$\forall\, m, n \in \mathbb{N}, \;\; v_a, v_b \ge 0 : \quad v_a = \textstyle\sum_{i=1}^m a_i^n \quad \wedge \quad v_b = \sum_{i=1}^m b_i^n \quad \wedge$$
$$(v_a + v_b)^{1/n} \le v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad \left(\left(\textstyle\sum_{i=1}^m a_i^n\right) + \left(\sum_{i=1}^m b_i^n\right)\right)^{1/n} =$$
$$\left(\textstyle\sum_{i=1}^m (a_i^n + b_i^n)\right)^{1/n} \le \left(\sum_{i=1}^m a_i^n\right)^{1/n} + \left(\sum_{i=1}^m b_i^n\right)^{1/n}. \tag{4.7}$$

□

4.4. Metric Space. All Minkowski distances (ρ-norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.5. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \le d(u, v) + d(v, w)$.

PROOF. $\forall\, p \ge 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \le ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.8}$$

Apply the distance inequality (4.3) to the inequality 4.8:

$$\begin{aligned}(u^p + w^p)^{1/p} \le ((u^p + v^p) + (v^p + w^p))^{1/p} \quad \wedge \quad (v_a + v_b)^{1/n} \le v_a^{1/n} + v_b^{1/n} \\ \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ \Rightarrow \quad (u^p + w^p)^{1/p} \le ((u^p + v^p) + (v^p + w^p))^{1/p} \le (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ \Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \le \\ (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \quad \square\end{aligned} \tag{4.9}$$

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u)$.

PROOF. By the commutative law of addition:

$$\begin{aligned}\forall\, p \,:\, p \ge 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \quad \square\end{aligned} \tag{4.10}$$

THEOREM 4.7. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \ge 0$.

PROOF. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \ge 0,\ v \ge 0. \tag{4.11}$$

$$p \ge 1,\ \ u, v \ge 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \ge 0. \tag{4.12}$$

$\square$

THEOREM 4.8. *Identity of Indiscernibles:* $d(u, u) = 0$.

PROOF. From the non-negativity property (4.7):

$$\begin{aligned}d(u, w) \ge 0 \quad \wedge \quad d(u, v) \ge 0 \quad \wedge \quad d(v, w) \ge 0 \\ \Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0.\end{aligned} \tag{4.13}$$

$$d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v. \tag{4.14}$$

$$d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \tag{4.15}$$

$\square$

4.5. Set properties limiting a set to at most 3 members.

DEFINITION 4.9. Totally ordered set:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i\;\; \in \{x_1, \ldots, x_n\},$$
$$successor\ x_i = x_{i+1} \;\;\wedge\;\; predecessor\ x_{i+1} = x_i.$$

DEFINITION 4.10. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i,\; j,\; n \in \mathbb{N},\; \forall\, x_i,\; x_j \in \{x_1, \ldots, x_n\},\; successor\ x_i = x_j \;\Leftrightarrow\; predecessor\ x_j = x_i.$$

THEOREM 4.11. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n \;\;\wedge\;\; j = 1 \;\;\Rightarrow\;\; successor\ x_n = x_1 \;\;\wedge\;\; predecessor\ x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (4.9) assigns a unique label to each set member and assigns unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (4.10):

$$i = n \;\;\wedge\;\; j = 1 \;\;\wedge\;\; successor\ x_i = x_j \;\;\Rightarrow\;\; successor\ x_n = x_1. \tag{4.16}$$

Applying the definition of the symmetry property (4.10) to conclusion 4.16:

$$successor\ x_i = x_j \;\;\Rightarrow\;\; predecessor\ x_j = x_i \;\;\Rightarrow\;\; predecessor\ x_1 = x_n. \tag{4.17}$$ □

THEOREM 4.12. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (4.11).

DEFINITION 4.13. (Cyclic) Successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize). \tag{4.18}$$

DEFINITION 4.14. (Cyclic) Predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1). \tag{4.19}$$

DEFINITION 4.15. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{4.20}$$

Every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$(4.21)\qquad Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (m = setsize \wedge n = 1).$$

$$(4.22)\qquad Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (n = m+1 \leq setsize).$$

$$(4.23)\qquad Adjacent(2,1,2) \leftarrow Successor(2,1,2) \leftarrow (n = setsize \wedge m = 1).$$

$$(4.24)\qquad Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m+1 \leq setsize).$$

$$(4.25)\qquad Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m-1 \geq 1).$$

$$(4.26)\qquad Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \wedge m = 1).$$

$$(4.27)\qquad Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \wedge n = setsize).$$

$$(4.28)\qquad Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m+1 \leq setsize).$$

$$(4.29)\qquad Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m-1 \geq 1).$$

Member 2 is the only successor of member 1 for all $setsize \geq 3$, which implies member 3 is not ($\neg$) a successor of member 1 for all $setsize \geq 3$:

$$(4.30)\quad \neg Successor(1,3,setsize \geq 3) \\ \leftarrow Successor(1,2,setsize \geq 3) \leftarrow (n = m+1 \leq setsize).$$

Member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not ($\neg$) a predecessor of member 1 for all $setsize > 3$:

$$(4.31)\quad \neg Predecessor(1,3,setsize \geq 3) \\ \leftarrow Predecessor(1,setsize,setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).$$

For all $setsize \geq 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not symmetric):

$$(4.32)\quad \neg Adjacent(1,3,setsize > 3) \\ \leftarrow \neg Successor(1,3,setsize > 3) \wedge \neg Predecessor(1,3,setsize > 3). \quad \square$$

5. Applications to physics

From the volume proof (3.2), two disjoint 3D distance intervals, $[0, r_1]$ and $[0, r_2]$, define a 2-volume. From the Minkowski distance proof (4.2), $\exists\ r : r^2 = r_1^2 + r_2^2$. And from the 3D proof (4.12), for some non-distance type, $\tau\ :\ \tau \in \{t\ (time),\ m\ (mass),\ q\ (charge), \cdots\}$, there exist unit-factoring ratios, $\mu,\ \nu_1,\ \nu_2$:

$$(5.1)\quad \forall\, r, r_1, r_2 :\ r^2 = r_1^2 + r_2^2 \quad \wedge \quad r = \mu\tau \quad \wedge \quad r_1 = \nu_1\tau \quad \wedge \quad r_2 = \nu_2\tau \\ \Rightarrow \quad (\mu\tau)^2 = (\nu_1\tau)^2 + (\nu_2\tau)^2 \quad \Rightarrow \quad \mu \geq \nu_1 \quad \wedge \quad \mu \geq \nu_2.$$

For a constant r, μ is a constant, maximum-possible, unit-factoring ratio, where:

$$(5.2)\ \ \mu \in \{r_c/t_c,\ r_c/m_c,\ r_c/q_c, \cdots\} :\ r = (r_c/t_c)t = (r_c/m_c)m = (r_c/q_c)q = \cdots .$$

5.1. Space-time-mass-charge equations. Form equation 5.1:

$$(5.3)\quad \forall\, r, r', r_\nu, \mu, \nu :\ r^2 = r'^2 + r_\nu^2 \quad \wedge \quad r = \mu\tau \quad \wedge \quad r_\nu = \nu\tau \\ \Rightarrow \quad r' = \sqrt{(\mu\tau)^2 - (\nu\tau)^2} \ = \ \mu\tau\sqrt{1 - (\nu/\mu)^2}.$$

Rest distance, r', contracts relative to coordinate distance, r, as $\nu \to \mu$:

$$(5.4)\qquad r' = \mu\tau\sqrt{1-(\nu/\mu)^2} \quad \wedge \quad \mu\tau = r \quad \Rightarrow \quad r' = r\sqrt{1-(\nu/\mu)^2}.$$

Coordinate length, τ, dilates relative to the rest length, τ', as $\nu \to \mu$:

$$\mu\tau = r'/\sqrt{1-(\nu/\mu)^2} \quad \wedge \quad r' = \mu\tau' \quad \Rightarrow \quad \tau = \tau'/\sqrt{1-(\nu/\mu)^2}. \tag{5.5}$$

Where τ is time, the space-like flat Minkowski spacetime event interval is:

$$\mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_\nu^2 \quad \wedge \quad \mathrm{d}r_\nu^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \quad \wedge \quad \mathrm{d}(\mu\tau) = \mathrm{d}r \\ \Rightarrow \quad \mathrm{d}r'^2 = \mathrm{d}(\mu\tau)^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2 - \mathrm{d}x_3^2. \tag{5.6}$$

5.2. Newton's gravity force and the constant, G. From equation 5.2:

$$\forall\, m_1, m_2, m, r \in \mathbb{R}:\ m_1 m_2 = m^2 \quad \wedge \quad r = (r_c/m_c)m \\ \Rightarrow \quad m_1 m_2 = ((m_c/r_c)r)^2 \quad \Rightarrow \quad (r_c/m_c)^2 m_1 m_2/r^2 = 1. \tag{5.7}$$

$$r = (r_c/t_c)t = ct \ \Rightarrow\ mr = (m_c/r_c)(ct)^2 \ \Rightarrow\ ((r_c/m_c)/c^2)mr/t^2 = 1, \tag{5.8}$$

$$((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/m_c)^2 m_1 m_2/r^2 = 1 \\ \Rightarrow \quad F := mr/t^2 = ((r_c/m_c)c^2)m_1 m_2/r^2 = G m_1 m_2/r^2, \tag{5.9}$$

where Newton's constant, $G = (r_c/m_c)c^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$.

Note the relationship between $E = mc^2$ and G:

$$rc^2 = (r_c/m_c)mc^2 = Gm. \tag{5.10}$$

5.3. Coulomb's charge force and constant, k_e. From equation 5.2:

$$\forall\, q_1, q_2, q, r \in \mathbb{R}:\ q_1 q_2 = q^2 \quad \wedge \quad r = (r_c/q_c)q \\ \Rightarrow \quad q_1 q_2 = ((q_c/r_c)r)^2 \quad \Rightarrow \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1. \tag{5.11}$$

$$r = (r_c/t_c)t = ct \ \Rightarrow\ mr = (m_c/r_c)(ct)^2 \ \Rightarrow\ ((r_c/m_c)/c^2)mr/t^2 = 1. \tag{5.12}$$

$$((r_c/m_c)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/q_c)^2 q_1 q_2/r^2 = 1 \\ \Rightarrow \quad F := mr/t^2 = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2. \tag{5.13}$$

$$r_c/t_c = c \quad \wedge \quad F = ((m_c/r_c)c^2)(r_c/q_c)^2 q_1 q_2/r^2 \\ \Rightarrow \quad F = (m_c(r_c/t_c^2))(r_c/q_c)^2 q_1 q_2/r^2 = k_e q_1 q_2/r^2, \tag{5.14}$$

where Coulomb's constant, $k_e = (m_c(r_c/t_c^2))(r_c/q_c)^2$, conforms to the SI units: $N \cdot m^2 \cdot C^{-2}$.

5.4. 3 fundamental direct proportion ratios. c_t, c_m, and c_q:

$$c_t = r_c/t_c \approx 2.99792458 \cdot 10^8 m\ s^{-1}. \tag{5.15}$$

$$G = (r_c/m_c)c_t^2 = c_m c_t^2 \quad \Rightarrow \quad c_m = r_c/m_c \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}. \tag{5.16}$$

$$k_e = (c_t^2/c_m)(r_c/q_c)^2 \quad \Rightarrow \quad c_q = r_c/q_c \approx 8.6175172023 \cdot 10^{-18} m\ C^{-1}. \tag{5.17}$$

5.5. 3 fundamental inverse proportion ratios. c_t, c_m, and c_q (5.4) $\Leftrightarrow$ k_t, k_m, and k_q:

$$(5.18)\quad r/t = r_c/t_c, \quad r/m = r_c/m_c \quad \Leftrightarrow \quad (r/t)/(r/m) = (r_c/t_c)/(r_c/m_c) \quad \Leftrightarrow \quad (mr)/(tr) = (m_c r_c)/(t_c r_c) \quad \Leftrightarrow \quad mr = m_c r_c = k_m, \quad tr = t_c r_c = k_t.$$

$$(5.19)\quad r/t = r_c/t_c, \quad r/q = r_c/q_c \quad \Leftrightarrow \quad (r/t)/(r/q) = (r_c/t_c)/(r_c/q_c) \quad \Leftrightarrow \quad (qr)/(tr) = (q_c r_c)/(t_c r_c) \quad \Leftrightarrow \quad qr = q_c r_c = k_q, \quad tr = t_c r_c = k_t.$$

$$(5.20)\quad k_m = m_c r_c = h/c \approx 2.21022 \cdot 10^{-42}\ kg\ m.$$

$$(5.21)\quad k_t = t_c r_c = k_m/(c_t/c_m) \approx 5.47494 \cdot 10^{-78}\ s\ m.$$

$$(5.22)\quad k_q = q_c r_c = (c_t/c_q)k_t \approx 1.90466 \cdot 10^{-52}\ C\ m.$$

5.6. Schwarzchild's point-mass gravitational time dilation. From equations 5.5, 5.1, 5.2, and 5.10:

$$(5.23)\quad t' = t\sqrt{1-(v^2/c^2)(r/r)} \quad \wedge \quad r = (r_c/m_c)m \quad \Rightarrow \quad t' = t\sqrt{1-((r_c/m_c)m)v^2/rc^2}.$$

$$(5.24)\quad t' = t\sqrt{1-(r_c/m_c)mv^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv^2_{orbital} \quad \Rightarrow \quad t' = t\sqrt{1-2(r_c/m_c)mv^2_{orbital}/rc^2}.$$

$$(5.25)\quad t' = \lim_{v_{orbital}\to c} t\sqrt{1-2(r_c/m_c)mv^2_{orbital}/rc^2} = t\sqrt{1-2(r_c/m_c)mc^2/rc^2} = t\sqrt{1-2Gm/rc^2}.$$

5.7. Einstein's gravitational lens. An incremental deflection, ds, of light traveling in a straight line is proportionate to the lens mass, m and inverse square proportionate to the distance from the lens mass, r: d$s \propto m/r^2$. And from the energy-gravity equation 5.24:

$$(5.26)\quad \mathrm{d}s \propto m/r^2 \ \wedge\ \lim_{v_{orbital}\to c} 2(r_c/m_c)mv^2/rc^2 = (2G/c^2)m/r \quad \Rightarrow \quad \exists\, \mathrm{d}s, \mathrm{d}r :\ \mathrm{d}s = ((2G/c^2)m/r^2)\mathrm{d}r \quad \Rightarrow \quad s = \textstyle\int \mathrm{d}s = \int_{-r}^{r}((2G/c^2)m/r^2)\mathrm{d}r = 4Gm/rc^2.$$

5.8. Planck relation and constant, h. [Jai11] Applying both the direct proportion (5.15), $r/t = r_c/t_c = c$, and inverse proportion (5.18), $mr = m_c r_c = k_m$, ratios:

$$(5.27)\quad m(ct)^2 = mr^2 \ \wedge\ m = m_c r_c/r = k_m/r \quad \Rightarrow \quad m(ct)^2 = (k_m/r)r^2 = k_m r.$$

$$(5.28)\quad m(ct)^2 = k_m r \quad \wedge \quad r/t = r_c/t_c = c \quad \Rightarrow \quad E := mc^2 = k_m r/t^2 = (k_m(r/t))\,(1/t) = (k_m c)(1/t) = hf,$$

where the Planck constant, $h = k_m c$, and the frequency, $f = 1/t$.

5.9. Compton wavelength, r. [Jai11] From equations 5.18 and 5.28:

$$(5.29)\quad mr = k_m \quad \Rightarrow \quad r = k_m/m = (k_m/m)(c/c) = h/mc.$$

5.10. de Broglie wavelength, r. [Jai11] From equations 5.3 and 5.29:

$$v = r'/t = c\sqrt{1-(v'/c)^2} \quad \wedge \quad r = h/mc \quad \Rightarrow \quad r = (h/mv)\sqrt{1-(v'/c)^2}, \tag{5.30}$$

where $r_\nu/t = v'$ is the rest frame of reference velocity and $r'/t = v$ is the velocity observed from the stationary frame of reference.

5.11. Total mass. The total mass of a particle is $m = \sqrt{m_0^2 + m_{ke}^2}$, where m_0 is the rest mass and m_{ke} is the kinetic energy-equivalent mass. Applying both the direct (5.15) and inverse proportion ratios (5.18):

$$\begin{aligned} m_0 = (m_c/r_c)r \quad \wedge \quad m_{ke} = m_c r_c/r \quad \wedge \quad m = \sqrt{m_0^2 + m_{ke}^2} \\ \Rightarrow \quad m = \sqrt{((m_c/r_c)r)^2 + ((m_c r_c)/r)^2}. \end{aligned} \tag{5.31}$$

The effect is easier to understand in terms of Newtonian gravity than in general relativity.

5.12. Quantum-special relativity extensions to Newton's gravity force. Inserting the quantum effect into Einstein's field equations and finding solutions would be difficult. Applying the quantum effect to Newton's gravity equation is much simpler.

$$\begin{aligned} \exists\, m : m_1 m_2 = m^2 = ((m_c/r_c)r)^2 + ((m_c r_c)/r)^2 \\ \Rightarrow \quad m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2) = 1. \end{aligned} \tag{5.32}$$

Applying the spacetime equation 5.4 to equation 5.9:

$$r' = r\sqrt{1-(v/c)^2} \quad \wedge \quad Gmr/t^2 = 1 \quad \Rightarrow \quad G\sqrt{1-(v/c)^2})mr'/t^2 = 1, \tag{5.33}$$

where r'/t^2 is the acceleration observed from a stationary frame of reference.

$$\begin{aligned} G\sqrt{1-(v/c)^2})mr'/t^2 = 1 \quad \wedge \quad m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2) = 1 \\ \Rightarrow \quad F := mr'/t^2 = (G\sqrt{1-(v/c)^2})m_1 m_2/(((m_c/r_c)r)^2 + ((m_c r_c)/r)^2). \end{aligned} \tag{5.34}$$

5.13. Quantum-special relativity extensions to Coulomb's charge force.

$$F = (k_e\sqrt{1-(v/c)^2})q_1 q_2/(((q_c/r_c)r)^2 + ((q_c r_c)/r)^2). \tag{5.35}$$

6. Insights and implications

(1) Deriving volume and distance from the same abstract, countable set of n-tuples provides a single, unifying set and limit-based foundation under Euclidean geometry without relying on the geometric primitives and relations in Euclidean geometry [Joy98], axiomatic geometry [Lee10], and vector analysis [Wey52].

(2) The definition of a complete metric space [Rud76] ignores the intimate relation between distance and volume. A more sufficient definition is: a distance measure is the inverse (bijective) function of volume.

(3) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [Joy98]. And analytic proofs that the straight line length is the smallest distance equate the straight line length to Euclidean distance.

Without using the notion of a straight line: Euclidean volume was derived from a set of n-tuples (3.2). And all distance measures (bijective, isomorphic functions of n-volumes) derived from Euclidean 2-volumes (areas) are Minkowski distances (4.2), where $n \in \{1,2\}$: $n=1$ is the Manhattan (largest) distance case, $d = \sum_{i=1}^{m} s_i$. $n = 2$ is the Euclidean (smallest) distance case, $d = (\sum_{i=1}^{m} s_i^2)^{1/2}$. For the case, $n \in \mathbb{R}$, $1 \leq n \leq 2$: d decreases monotonically as n goes from 1 to 2.

(4) The left side of the distance sum inequality (4.4),

$$(\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}, \tag{6.1}$$

differs from the left side of Minkowski's sum inequality [**Min53**]:

$$(\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \tag{6.2}$$

The two inequalities are only the same where $n = 1$. The distance sum inequality is a more fundamental inequality because its proof does not require the convexity and Hölder's inequality assumptions required to prove the Minkowski sum inequality. And the distance sum inequality is derived from volume and distance, which makes it directly related to geometry, which should make it useful in search and machine learning.

(5) The derivations of the spacetime equations, in this article (5.1), differ from other derivations:
   (a) The derivations, here, do not rely on the Lorentz transformations or Einsteins' postulates [**Ein15**]. The derivations do not even require the notion of light.
   (b) The derivations, here, rely only on the Euclidean volume proof (3.2), the Minkowski distances proof (4.1), and the 3D proof (4.12), which provides the insight that the properties of physical space creates a maximum speed and the spacetime equations. For example, from the direct proportion equations 5.1 and 5.2, $\mu = r_c/t_c$ is always the maximum ratio (the speed of light).
   (c) The same derivations are also valid for spacemass and spacecharge.
   (d) **Both** special and general relativity result from equation 5.1, where the ratio of any 3-dimensional distance, r, to other types is always a constant in all inertial (special relativity) frames and all accelerating (general relativity) frames. For example, the special relativity time dilation equation 5.5 was derived from constant distance-to-time ratios and combined with a constant distance-to-mass ratio, $r = (r_c/m_c)m$, (5.4) yields the Schwarzchild's gravitational time dilation equation (5.25).

(6) A simplification to Einstein's field equations, $\mathbf{G}_{\mu,\nu} - g_{\mu,\nu} = k\mathbf{T}_{\mu,\nu}$, [**Wey52**] comes from the flat spacetime interval equation being derived from a 2-dimensional equation (5.6), which is generalized to: $\mathrm{d}r'^2 = \alpha_0 \mathrm{d}(\mu\tau)^2 - \mathrm{d}r_\nu^2$, where $\mathrm{d}r_\nu^2 = \alpha_1 \mathrm{d}x_1^2 + \alpha_2 \mathrm{d}x_2^2 + \alpha_3 \mathrm{d}x_3^2$. Therefore, the 4×4 metric tensor, $g_{\mu,\nu} = diag(\alpha_0, -\alpha_1, -\alpha_2, -\alpha_3)$ can be simplified to a 2×2 metric tensor, $g_{\mu,\nu} = diag(\alpha_0, -1)$. The 2×2 metric tensor allows using a 2-dimensional Gaussian curvature in the Einstein tensor, $\mathbf{G}_{\mu,\nu}$, which is much simpler to calculate than the 4-dimensional Ricci and scalar curvature. And the 2×2 tensors reduce the number of independent equations to solve.

(7) The direct proportion ratios, $r_c/t_c = c_t$, $r_c/m_c = c_m$, $(r_c/q_c) = c_q \Leftrightarrow$ the inverse proportion ratios, $t_c r_c = k_t$, $m_c r_c = k_m$, and $q_c r_c = k_q$ (5.5) are the properties of a symmetry group.
   (a) The inverse square law for gravity (5.7) and charge (5.11) is a result of the direct proportion ratios.
   (b) The combination of direct and inverse proportion ratios create the particle-wave equations: Planck relation (5.8), Compton wavelength (5.29), and de Broglie wavelength (5.30).
   (c) The gravity, G (5.9), charge k_e (5.14), and Planck h (5.28) constants were all derived from the constant proportion ratios. Therefore, G, k_e, and h are **not** "fundamental" constants.
   (d) G, k_e, and h all depend on the speed of light ratio, c_t: $G = c_m c_t^2$, $k_e = (c_q^2/c_m)c_t^2$, and $h = k_m c_t$.
   (e) $k_e = (c_q^2/c_m)c_t^2 = ((m_c/r_c)(r_c/t_c)^2)c_q^2 = (m_c(r_c/t_c^2))c_q^2$, where the term, r_c/t_c^2, suggests a maximum acceleration constant.
   (f) The ratios used to derive k_e (5.14) do not contain the value, 4π, which indicates the current "standard" definitions of k_e in terms of vacuum permitivity, ε_0, and permeability, μ_0, where $k_e = 1/4\pi\varepsilon_0$ and $k_e = \mu_0 c^2/4\pi$, are **not** logically derived in orthogonal Cartesian coordinates. Likewise, the logic of the reduced Planck constant, $\hbar = h/2\pi$, in orthogonal coordinates, might need to be reconsidered.

(8) The quantum-special relativity extensions to Newton's gravity force (5.34) and Coulomb's charge force (5.35) make quantifiable predictions.
   (a) $\lim_{r\to 0} F = 0$. The classical-quantum boundaries are where the gravity and charge forces between 2 point-like particles peak: $r/c_m = k_m/r$ and $r/c_q = k_q/r$, is $r \approx 4.05135 \cdot 10^{-35}\ m$ for **both** gravity and charge ($r = \sqrt{c_m k_m} = \sqrt{c_q k_q} = \sqrt{r_c^2}$).
   If the quantum effects exists, then:
      (i) Black holes have measurable sizes > 0 – no singularities.
      (ii) The finite gravity-charge well allows radioactivity and quantum tunneling.
      (iii) As the kinetic energy (temperature) decreases, more particles will stay within their gravity-charge well distance, $r << 4.05135 \cdot 10^{-35}\ m$, allowing superconductivity and Bose-Einstein condensates.
      (iv) Gravitational time dilation near an object with mass peaks at $r \approx 4.05135 \cdot 10^{-35}\ m$.
   (b) The term, r'/t^2, in equation 5.34 is the acceleration observed from the stationary frame. From earth, as a stationary frame, equation 5.34 predicts the planet Mercury's orbiting velocity (angular acceleration) around the sun would appear to be slightly slower than predicted by Newton's gravity equation 5.9. The slower orbiting velocity is due to the Lorentz factor, $\sqrt{1-(v/c)^2}$, which was also used to derive Schwarzchild's gravitational time dilation (5.25). Therefore, the slower relativistic velocity and gravitational time dilation are directly related to each other.

(9) A constant ratio cannot map a constant value to continuously varying values. Therefore, the discrete spin states of two quantum entangled particles and the polarization states of two quantum entangled photons are independent of continuously varying distance and time interval lengths.

(10) The set-based, first-order logic proof that a strict linearly ordered and symmetric set is a cyclic set of at most 3 members (4.12) is the simplest and most logically rigorous explanation for observing only 3 dimensions of physical space, which is less contrived and more rigorous than: parallel dimensions that cannot be detected or extra dimensions rolled up into infinitesimal balls that are too small to detect.
   (a) Higher order dimensions must have different types (members of different sets), for example, dimensions of time, mass, and charge.
   (b) Each of 3 ordered and symmetric dimensions of space can have at most 3 sequentially ordered and symmetric state values, for example, an ordered and symmetric set of 3 vector orientations, $\{-1, 0, 1\}$, per dimension of space and at most 3 spin states per plane, etc.
   If the states are not sequentially ordered (a bag of states), then a state value is undetermined until observed (like Schrodinger's cat being both alive and dead until the box is opened). That is, there is no "axiom of choice" that allows selecting a particular state.

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