Some Set Properties Underlying Geometry and Physics

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Abstract

Euclidean volume and some distance equations are proved to be instances of abstract sets of countable, ordered combinations (n-tuples). A permutation property is proved to limit a set of unordered elements to a cyclic list of at most 3 elements, e.g., 3 distance elements (height, width, and depth) and 3 non-distance elements (time, mass, and charge). The proofs indicate each 3-dimensional distance unit interval length corresponds to unit interval lengths of time, mass, and charge. Derivations of well-known gravity, charge, electromagnetic, relativity, and quantum physics equations from unit ratios are shorter, simpler, more rigorous. The unit ratios also allow simple quantum extensions to classical and relativity equations. The proofs are verified in Rocq.

[1]

CONTENTS

Keywords: mathematical physics, gravity, charge, electromagnetism, relativity, quantum physics.

INTRODUCTION

Derivations of the gravity, charge, electromagnetic, relativity, and quantum physics equations, in this article, are shorter and simpler because the equations and constants are derived from "first principles," a small set of math proof-motivated unit ratios. And the derivations are more rigorous because of fewer assumptions that are not math-proof motivated.

Using the unit ratios to add quantum extensions to Schwarzschild's time dilation and black hole metric[2][3], Newton's gravity force[4], and Coulomb's charge force[5], create equations predicting that where the quantum effects are *measurable*, the constants; gravity, G, charge, k_e, and Einstein's $\kappa = 8\pi G/c^4$[6][7] do *not* exist. And the quantum-extended equations also predict that as distance between two point masses and between two point charges go to zero, the forces do *not* go to infinity.

In contrast, common practice has been to derive empirical laws from other empirical laws rather than deriving from first principles. Examples of deriving empirical laws from other empirical laws are: 1) deriving Newton's law of gravity from Kepler's laws [4], or from Gauss's law [5], or from Einstein's general relativity equations[6][7], and 2) Maxwell deriving the electromagnetic wave equation from several empirical laws[5].

Some assumptions without math proof motivations in relativity theory are: 1) assuming 3 dimensions of space and 1 dimension of time and 2) assuming that the laws of physics are the same at local Euclidean-like coordinate point[8][7]. Integration, topology, measure theory, Hilbert spaces, linear algebra, (pseudo-)Riemann geometry, etc. have no restriction on the number of dimensions [9][10][11][7][12]. And the uncountable density of reals places no restrictions on the relative sizes of domain intervals[9][10].

If there were a proof that sets with certain properties are limited to at most 3 elements and the set of physical space dimensions had those same properties, then the assumption would have a math proof motivation. And if unit ratios are are derived from the properties of Euclidean space, then there is a math proof motivation that the unit ratios and all laws derived from the unit ratios are the same at each local Euclidean-like coordinate point.

Therefore, the first step is to provide analytic (set, sequence, limit, combinatorial) proofs that can motivate some common physics assumptions and expose some simple ratios from which many physics equations can be derived. Each informal proof, in this article, has a corresponding formal proof, which has been verified using using the Rocq proof verification system [13], a verification tool used by mathematicians world-wide. The formal proofs are in the Rocq files, "euclidrelations.v" and "threed.v," which are included as ancillary files.

Let $|x_i|$ be the cardinal of (number of elements in) the countable set, $x_i \in \{x_1, \cdots, x_n\}$. And let the countable range value, v_c be the number of ordered combinations (n-tuples) of $x_1, \cdots, x_n$. The Euclidean volume equation will be proved to be an instance of v_c, where each n-tuple corresponds to a value, $\kappa \in \mathbb{R}$, (each κ the same size) and as $|x_i| \to \infty$, $\kappa \to 0$:

$$\forall\, v_c, |x_i| \in \{0,\ \mathbb{N}\},\ \ x_i \in \{x_1, \cdots, x_n\}, v_c = \textstyle\prod_{i=1}^{n} |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^{n} s_i,\ \ s_i, v \in \mathbb{R}. \quad (1)$$

For all $n > 1$, there are many cases where different domain values, $|x_1|, \cdots, |x_n|$, multiplied yield the same range value, v_c. Inferring a domain value, d_c, from v_c, requires an inverse (bijective) function, $d_c = f_n^{-1}(v_c)$ and $v_c = f_n(d_c)$. The simplest bijective case for $n = 1$ is: $v_c = |x_1| = d_c$. The simplest bijective case for all n that includes the $n = 1$ case is:

$$\exists\, d_c, v_c, |x_i| \in \{0, \mathbb{N}\} :\ v_c = \textstyle\prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n. \quad (2)$$

A set of n-tuples being the union of disjoint subsets of n-tuples implies that the domain value, d_c, is also the inverse function of the sum of n-tuples, where it will be proved that:

$$d_c^n = v_c = \sum_{i=1}^m v_{c_i} = \sum_{i=1}^m (\prod_{j=1}^n |x_{i,j}|) \quad \Rightarrow \quad d^n = \sum_{i=1}^m (\prod_{j=1}^n s_{i,j}). \tag{3}$$

Where each $s_{i,j}$ is $\pm$-signed, the $n = 2$ case is the dot product.

Where each v_{c_i} also is the bijective function, $v_{c_i} = d_{c_i}^n$:

$$d_c^n = \sum_{i=1}^m d_{c_i}^n \quad \Rightarrow \quad d^n = \sum_{i=1}^m d_i^n. \tag{4}$$

$|d|$ is the p-norm (Minkowski distance) [14], which will be proved to imply the metric space properties [10]. The $n = 2$ case is the Euclidean distance.

Many sets contain unordered elements, where there is no intrinsic property making a set element the first, second, $\cdots$, or last (n-th), for example, the set, {height, width, depth} $\equiv$ {depth, width, height}, and the set, {time, mass, charge} $\equiv$ {mass, charge, time}. A set of unordered elements can be placed in a "list," where the list is sequenced one-by-one in a total order via successor and predecessor functions, $successor(x_i) = x_{i+1}$ and $predecessor(x_i) = x_{i-1}$, thereby allowing calculation of the number of n-tuples, volumes, and distances, for example, $v = \prod_{i=1}^n s_i$ and $v = \prod_{i=n}^1 s_i$.

But the union, intersection, multiplication, and addition operations defining the total number of n-tuples and the corresponding volume and distance equations are commutative. The commutative property allows calculating the *same* totally ordered list of n number of elements in all of the $n!$ possible unique sequences (permutations).

Only a cyclic list, where $successor(x_n) = x_1$ and $predecessor(x_1) = x_n$, allows sequencing 1 through n (and n through 1), and also allows each element to be the first element of some of those $n!$ permutations. For example, where $n = 3$, the volume cyclic successor sequences are: $v = s_1 \times s_2 \times s_3$. $v = s_2 \times s_3 \times s_1$, $v = s_3 \times s_1 \times s_2$, and the cyclic predecessor sequences are: $v = s_1 \times s_3 \times s_2$. $v = s_2 \times s_1 \times s_3$, $v = s_3 \times s_2 \times s_1$.

The only cyclic lists that allows all $n!$ permutations using the cyclic successor and predecessor functions are where each list element is sequentially adjacent (either an *immediate* cyclic successor or an *immediate* cyclic predecessor) to every other element, referred to as an immediate symmetric cyclic list (ISCL). An ISCL will be proved to contain $n \leq 3$ elements.

Application of the ISCL and properties of Euclidean geometry to physics:

1. **ISCL:** $\{r_1, r_2, r_3\}$ is an ISCL of 3 "distance" dimensions and $\{t\ (time), m\ (mass), q\ (charge)\}$ is the ISCL of 3 "non-distance" dimensions, each dimension $\subseteq \mathbb{R}$. The total number of dimensions with the common property, $\subseteq \mathbb{R}$, is 6: $r_1,\ r_2,\ r_3,\ t,\ m,\ q$.

2. **Corresponding units:** The Euclidean volume and distance equations are derived from abstract sets of n-tuples, where each n-tuple has a corresponding value, $\kappa \in \mathbb{R}$ (each κ the same size). For countably infinite sets of n-tuples, x and y, each κ_x, has a corresponding κ_y. Therefore, each unit interval length, r_p, of $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$ corresponds to unit interval lengths: t_p of time, t; m_p of mass, m; and q_p of charge, q.

Some might inquire why the relationship of the time, mass, and charge units are with respect to a unit of $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$ instead of units of $\{r_1,\ r_2,\ r_3\}$ individually. The dimensions, $\{r_1,\ r_2,\ r_3\}$, can be rotated and permuted in every way causing changes of the values of $\{r_1,\ r_2,\ r_3\}$, while the Euclidean distance, r, between two points remains the constant, salient value.

A consequence of the 1-1 correspondence of the units are the direct proportion ratios: $r = (r_p/t_p)t$, $r = (r_p/m_p)m$, and $r = (r_p/q_p)q$, letting: $c_t = r_p/t_p$, $c_m = r_p/m_p$, and $c_q = r_p/q_p$. For example, the constant, c_t, corresponds to the speed of light, c.

Algebraic manipulation of the 3 direct proportion ratios yields 3 inverse proportion ratios, $rt = t_p r_p$, $rm = m_p r_p$. and $rq = q_p r_p$, letting: $k_t = t_p r_p$, $k_m = m_p r_p$, and $k_q = q_p r_p$. Examples of constants containing k_m are: reduced Planck, $\hbar = k_m c_t$, charge, $k_e = k_m c_t^2/q_p^2 = 4\pi/\varepsilon_0$, and permeability, $\mu_0 = 4\pi k_m/q_p^2$, where q_p is the reduced Planck charge unit.

RULER MEASURE AND CONVERGENCE

Definition .1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\ s, \kappa \in \mathbb{R}, \quad 0 < |\kappa| \leq 1$, $(p = floor(s/\kappa) \quad \vee \quad p = ceiling(s/\kappa))$.

Theorem .2. *Ruler convergence:* $M = \lim_{\kappa \to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

Proof. (epsilon-delta proof)
$floor(x)$ is the integer part of x. Therefore:

$$floor(x) = max(\{y :\ y \leq x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\}) \quad \Rightarrow \quad |floor(x) - x| < 1. \tag{5}$$

$$|floor(s/\kappa) - s/\kappa| < 1 \quad \wedge \quad p = floor(s/\kappa) \quad \Rightarrow \quad |p - s/\kappa| < 1. \tag{6}$$

Multiply both sides of inequality 6 by $|\kappa|$:

$$|p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{7}$$

$$\forall\, \epsilon = p \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < p \quad \Rightarrow \quad |\kappa - 0| < p \ \wedge \ |p\kappa - s| < \epsilon$$
$$:= \quad M = \lim_{\kappa \to 0} p\kappa = s. \quad \square \tag{8}$$

The following is an example of ruler convergence where, $s = \pi \ \Rightarrow \ p = floor(s/\kappa) \Rightarrow p\kappa = 3.1_{\kappa=10^{-1}},\ 3.14_{\kappa=10^{-2}},\ 3.141_{\kappa=10^{-3}}, ..., \ \pi_{\lim_{\kappa\to 0}}$.

Lemma .3. $\forall\, n \geq 1,\ 0 < |\kappa| < 1 : \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa \to 0} \kappa$.

Proof. The formal proof , "lim_c_to_n_eq_lim_c," is in the Rocq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa \leq 1 \quad \Rightarrow \quad 0 < \kappa^n \leq \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| \leq |\kappa| = |\kappa - 0|. \tag{9}$$

$$\forall\, \epsilon = p \quad \wedge \quad |\kappa - \kappa^n| \leq |\kappa - 0| < p \quad \Rightarrow \quad |\kappa - 0| < p \quad \wedge \quad |\kappa - \kappa^n| < p = \epsilon$$
$$:= \quad \lim_{\kappa\to 0} \kappa^n = 0. \tag{10}$$

$$\lim_{\kappa\to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa \to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa. \tag{11}$$

$\square$

VOLUME

Euclidean volume

Theorem .4. *Euclidean volume,*

$$\forall\, v_c, d_c, |x_i| \in \{0,\ \mathbb{N}\},\ \ x_i \in \{x_1, \cdots, x_n\},\ v_c = \textstyle\prod_{i=1}^n |x_i| \ \Rightarrow \ v = \prod_{i=1}^n s_i, \quad s_i, v \in \mathbb{R}. \tag{12}$$

The formal proof, "Euclidean_volume," is in the Rocq file, euclidrelations.v.

Proof.

$$v_c = \textstyle\prod_{i=1}^n |x_i| \quad \Leftrightarrow \quad v_c\kappa = (\prod_{i=1}^n |x_i|)\kappa \quad \Leftrightarrow \quad \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{13}$$

Apply the ruler (.1) and ruler convergence (.2) to equation 13:

$$\exists\, v, \kappa \in \mathbb{R} : v_c = floor(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa \quad \wedge \quad \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\textstyle\prod_{i=1}^n |x_i|)\kappa$$
$$\Rightarrow \quad v = \lim_{\kappa\to 0}(\textstyle\prod_{i=1}^n |x_i|)\kappa. \quad (14)$$

Apply lemma .3 to equation 14:

$$v = \lim_{\kappa\to 0}(\textstyle\prod_{i=1}^n |x_i|)\kappa \quad \wedge \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa$$
$$\Rightarrow \quad v = \lim_{\kappa\to 0}(\textstyle\prod_{i=1}^n |x_i|)\kappa^n = \lim_{\kappa\to 0}(\prod_{i=1}^n(|x_i|\kappa)). \quad (15)$$

Apply the ruler (.1) and ruler convergence (.2) to s_i:

$$\exists\, s_i, \kappa \in \mathbb{R} : \ floor(s_i/\kappa) = |x_i| \quad \Rightarrow \quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i. \quad (16)$$

$$v = \lim_{\kappa\to 0}(\textstyle\prod_{i=1}^n |x_i|\kappa) \quad \wedge \quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i \quad \Leftrightarrow \quad v = \prod_{i=1}^n s_i \quad (17)$$

□

Lemma .5. *The number of n-tuples, v_c, is the sum of the number of n-tuples, v_{c_i}, in each subset of n-tuples, implies a volume is the sum of volumes,*

$$v_c = \textstyle\sum_{i=1}^m v_{c_i} \quad \Rightarrow \quad v = \sum_{i=1}^m v_i, \quad v, v_i \in \mathbb{R}.$$

The formal proof, "sum_of _volumes," is in the Rocq file, euclidrelations.v.

Proof. From the condition of this theorem:

$$v_c = \textstyle\sum_{i=1}^m v_{c_i} \quad \Leftrightarrow \quad \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}\sum_{j=1}^m(v_{c_i}\kappa). \quad (18)$$

Apply lemma .3 to equation 18:

$$\lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\textstyle\sum_{j=1}^m v_{c_i})\kappa \quad \wedge \quad \lim_{\kappa\to 0}\kappa^n = \lim_{\kappa\to 0}\kappa$$
$$\Leftrightarrow \ \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}\textstyle\sum_{j=1}^m(v_{c_i}\kappa). \quad (19)$$

Apply the ruler (.1) and ruler convergence theorem (.2) to equation 19:

$$\exists\, v, v_i : \ v = \lim_{\kappa\to 0} v_c\kappa \quad \wedge \quad \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}\textstyle\sum_{j=1}^m(v_{c_i}\kappa)$$
$$\Rightarrow \quad v == \lim_{\kappa\to 0}\textstyle\sum_{j=1}^m(v_{c_i}\kappa). \quad (20)$$

Apply the ruler (.1) and ruler convergence theorem (.2) to equation 20:

$$v == \lim_{\kappa\to 0}\textstyle\sum_{j=1}^m(v_{c_i}\kappa) \quad \wedge \quad \exists\, v_i, v_{c_i} : \ v_i = \lim_{\kappa\to 0} v_{c_i}\kappa \quad \Rightarrow \quad v = \sum_{j=1}^m v_i. \quad (21)$$

□

DISTANCE

Definition .6. Bijective, countable domain value, d_c:

$$\exists\, v_c, d_c, |x_i| \in \{0,\ \mathbb{N}\},\ \ x_i \in \{x_1, \cdots, x_n\}, \quad v_c = \textstyle\prod_{i=1}^{n} |x_i| = \prod_{i=1}^{n} d_c = d_c^n. \tag{22}$$

Vector inner product

Theorem .7. *Sum of volumes distance:*

$$d_c^n = v_c = \textstyle\sum_{i=1}^{m} v_{c_i} \quad \Rightarrow \quad d^n = \sum_{i=1}^{m} (\prod_{j=1}^{n} s_{i_j}).$$

The formal proof, "sum_of _volumes _distance," is in the Rocq file, euclidrelations.v.

Proof. From the sum of volumes lemma .5 and the Euclidean volume theorem .4:

$$\begin{aligned} d_c^n = \textstyle\sum_{i=1}^{m} v_{c_i} \quad \Rightarrow \quad d^n = \sum_{i=1}^{m} v_i \quad \wedge \quad v_i = \prod_{j=1}^{n} s_{i_j} & \\ \Rightarrow \quad d^n = \textstyle\sum_{i=1}^{m} v_i = \sum_{i=1}^{m} (\prod_{j=1}^{n} s_{i_j}). \quad \square & \end{aligned} \tag{23}$$

Note: In the volume proof (.4), where κ is negative, volume is negative. From the lemma (.3), $\forall\ n \geq 1,\ 0 < |\kappa| < 1 : \lim_{\kappa \to 0} \kappa^n = \lim_{\kappa \to 0} \kappa,\ \exists\ \kappa_i \in \{\kappa_1, \cdots, \kappa_n\}$. Therefore, κ_i can have negative values, where $\prod_{i=1}^{n} \kappa_i = \kappa^n$. And where $\kappa_i < 0$, the corresponding $s_{i,j} < 0$. Therefore, the $n = 2$ case of the sum of volumes distance is the dot product.

Minkowski distance (p-norm)

Theorem .8. *Minkowski distance (p-norm):*

$$d_c^n = v_c = \textstyle\sum_{i=1}^{m} v_{c_i} = \sum_{i=1}^{m} d_{c_i}^n \quad \Leftrightarrow \quad d^n = \sum_{i=1}^{m} d_i^n.$$

The formal proof, "Minkowski_distance," is in the Rocq file, euclidrelations.v.

Proof. From the sum of volumes distance theorem .7 and the Euclidean volume theorem .4:

$$\begin{aligned} d_c^n = v_c = \textstyle\sum_{i=1}^{m} v_{c_i} \quad \Rightarrow \quad d^n = v = \sum_{i=1}^{m} v_i \quad \wedge \quad v_i = \prod_{j=1}^{n} d_i = d_i^n & \\ \Rightarrow \quad d^n = \textstyle\sum_{i=1}^{m} d_i^n \quad \square & \end{aligned} \tag{24}$$

Theorem .9. *Distance triangle inequality*

$$\forall\, n \in \mathbb{N},\;\; v_a, v_b \geq 0: \;\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

The formal proof, distance_inequality, is in the Rocq file, euclidrelations.v.

Proof. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$\forall\, v_a, v_b \geq 0: \quad v_a + v_b \leq v_a + v_b + \textstyle\sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^{k} + \\ \textstyle\sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{k} (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n. \quad (25)$$

Take the n^{th} root of both sides of the inequality 25:

$$\forall\, v_a, v_b \geq 0,\;\; n \in \mathbb{N}: \quad v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \quad \Rightarrow \quad (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \quad (26)$$

□

Theorem .10. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N},\;\; a_i, b_i \geq 0: (\textstyle\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \leq (\textstyle\sum_{i=1}^{m} a_i^n)^{1/n} + (\textstyle\sum_{i=1}^{m} b_i^n)^{1/n}.$$

The formal proof, distance_sum_inequality, is in the Rocq file, euclidrelations.v.

Proof. Apply the distance triangle inequality (.9):

$$\forall\, m, n \in \mathbb{N},\;\; v_a, v_b \geq 0: \;\; v_a = \textstyle\sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \textstyle\sum_{i=1}^{m} b_i^n \quad \wedge \quad (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ \Rightarrow \quad ((\textstyle\sum_{i=1}^{m} a_i^n) + (\textstyle\sum_{i=1}^{m} b_i^n))^{1/n} = (\textstyle\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \leq \\ (\textstyle\sum_{i=1}^{m} a_i^n)^{1/n} + (\textstyle\sum_{i=1}^{m} b_i^n)^{1/n}. \quad \square \quad (27)$$

Metric Space

All Minkowski distances (p-norms) imply the metric space properties. The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Rocq file, euclidrelations.v.

Theorem .11. *Triangle Inequality:*

$$d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p}, \quad p \geq 1, \quad u = s_1, \quad w = s_2, \quad v = w/k \\ \Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w).$$

Proof. $\forall\, p \geq 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{28}$$

Apply the distance triangle inequality (.9) to the inequality 28:

$$\begin{aligned} &(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \quad \wedge \quad (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ &\qquad\qquad \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ &\quad \Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ &\Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \quad \square \end{aligned} \tag{29}$$

Theorem .12. *Symmetry:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p}, \quad p \geq 1, \quad u = s_1, \quad v = s_2 \quad \Rightarrow \quad d(u, v) = d(v, u).$

Proof. By the commutative law of addition:

$$\begin{aligned} &\forall\, p \,:\, p \geq 1, \quad d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ &\qquad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \quad \square \end{aligned} \tag{30}$$

Theorem .13. *Non-negativity:*

$$d(s_1, s_2) = (\textstyle\sum_{i=1}^{2} s_i^p)^{1/p}, \quad p \geq 1, \quad u = s_1, \quad w = s_2, \quad v = w/k \quad \Rightarrow \quad d(u, w) \geq 0.$$

Proof. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0. \tag{31}$$

$$p \geq 1,\ u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \tag{32}$$

□

Theorem .14. *Identity of Indiscernibles: $d(u, u) = 0$.*

Proof. From the non-negativity property (.13):

$$\begin{aligned} &d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0 \\ &\qquad \Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0. \end{aligned} \tag{33}$$

$$d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v. \tag{34}$$

$$d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \tag{35}$$

□

Properties limiting a set to at most 3 elements

The following definitions and proof use first order logic. A Horn clause-like expression is used, here, to make the proof easier to read. By convention, the proof goal is on the left side and supporting facts are on the right side of the implication sign ($\leftarrow$). The formal proofs in the Rocq file threed.v are: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.

Definition .15. Immediate Cyclic Successor of m is n:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\}:$$
$$Successor(m, n, setsize) \quad \leftarrow \quad (m = setsize \wedge n = 1) \quad \vee \quad (n = m + 1 \leq setsize). \quad (36)$$

Definition .16. Immediate Cyclic Predecessor of m is n:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\}:$$
$$Predecessor(m, n, setsize) \quad \leftarrow \quad (m = 1 \wedge n = setsize) \quad \vee \quad (n = m - 1 \geq 1). \quad (37)$$

Definition .17. Adjacent: element m is sequentially adjacent to element n if the immediate cyclic successor of m is n or the immediate cyclic predecessor of m is n. Notionally:

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\}:$$
$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \;\vee\; Predecessor(m, n, setsize). \quad (38)$$

Definition .18. Immediate Symmetric (every set element is sequentially adjacent to every other element):

$$\forall\, x_m, x_n \in \{x_1, \cdots, x_{setsize}\}: \quad Adjacent(m, n, setsize). \quad (39)$$

Theorem .19. *An immediate symmetric cyclic list (ISCL) is limited to at most 3 elements.*

Proof.

Every element is adjacent to every other element, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1,1,1) \leftarrow Successor(1,1,1) \leftarrow (m = setsize \wedge n = 1). \tag{40}$$

$$Adjacent(1,2,2) \leftarrow Successor(1,2,2) \leftarrow (n = m + 1 \leq setsize). \tag{41}$$

$$Adjacent(1,2,3) \leftarrow Successor(1,2,3) \leftarrow (n = m + 1 \leq setsize). \tag{42}$$

$$Adjacent(2,1,3) \leftarrow Predecessor(2,1,3) \leftarrow (n = m - 1 \geq 1). \tag{43}$$

$$Adjacent(3,1,3) \leftarrow Successor(3,1,3) \leftarrow (n = setsize \wedge m = 1). \tag{44}$$

$$Adjacent(1,3,3) \leftarrow Predecessor(1,3,3) \leftarrow (m = 1 \wedge n = setsize). \tag{45}$$

$$Adjacent(2,3,3) \leftarrow Successor(2,3,3) \leftarrow (n = m + 1 \leq setsize). \tag{46}$$

$$Adjacent(3,2,3) \leftarrow Predecessor(3,2,3) \leftarrow (n = m - 1 \geq 1). \tag{47}$$

Element 2 is the only immediate successor of element 1 for all $setsize \geq 3$, which implies element 3 is not ($\neg$) an immediate successor of element 1 for all $setsize \geq 3$:

$$\neg Successor(1,3,setsize \geq 3) \leftarrow Successor(1,2,setsize \geq 3) \leftarrow (n = m + 1 \leq setsize). \tag{48}$$

Element $n = setsize > 3$ is the only immediate predecessor of element 1, which implies element 3 is not ($\neg$) an immediate predecessor of element 1 for all $setsize > 3$:

$$\neg Predecessor(1,3,setsize \geq 3) \leftarrow Predecessor(1,setsize,setsize > 3) \leftarrow$$
$$(m = 1 \wedge n = setsize > 3). \tag{49}$$

For all $setsize > 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not immediate symmetric):

$$\neg Adjacent(1,3,setsize > 3) \leftarrow$$
$$\neg Successor(1,3,setsize > 3) \quad \wedge \quad \neg Predecessor(1,3,setsize > 3). \quad \square \tag{50}$$

The Symmetric goal matches the Adjacent goals 40 and fails for all “setsize” greater than three.

APPLICATIONS TO PHYSICS

Derivation of direct and inverse proportion unit ratios

Application of the ISCL (.19) and properties of Euclidean geometry (.4) to physics:

1. **ISCL:** $\{r_1, r_2, r_3\}$ is an ISCL of 3 "distance" dimensions and $\{t\ (time), m\ (mass), q\ (charge)\}$ is the ISCL of 3 "non-distance" dimensions, each dimension $\subseteq \mathbb{R}$. The total number of dimensions with the common property, $\subseteq \mathbb{R}$, is 6: $r_1,\ r_2,\ r_3,\ t,\ m,\ q$.

2. **Corresponding units:** The Euclidean volume and distance equations are derived from abstract sets of n-tuples, where each n-tuple has a corresponding value, $\kappa \in \mathbb{R}$ (each κ the same size). For countably infinite sets of n-tuples, x and y, each κ_x, has a corresponding κ_y. Therefore, each unit interval length, r_p, of $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$ corresponds to unit interval lengths: t_p of time, t; m_p of mass, m; and q_p of charge, q.

A corresponding number of units can be expressed as, $r/r_p = \tau/\tau_p$, where $\tau \in \{t, m, q\}$. Multiplying both sides by r_p yields 3 direct proportion ratios:

$$\forall\, r_p, \tau_p, r, \tau \subseteq \mathbb{R},\ r_p, \tau_p\ \textbf{constants},\ \tau \in \{t, m, q\} :\ r/r_p = \tau/\tau_p \quad \Rightarrow \quad r = (r_p/\tau_p)/\tau$$
$$\Rightarrow \quad r = (r_p/t_p)t \quad \wedge \quad r = (r_p/m_p)m \quad \wedge \quad r = (r_p/q_p)q. \quad (51)$$

And for conciseness, let:

$$r_p/t_p = c_t \quad \wedge \quad r_p/m_p = c_m \quad \wedge \quad r_p/q_p = c_q. \quad (52)$$

Algebraic manipulation of the 3 direct proportion ratios yields 3 inverse proportion ratios:

$$r/t = r_p/t_p \quad \wedge \quad r/m = r_p/m_p \quad \Rightarrow \quad (r/t)/(r/m) = (r_p/t_p)/(r_p/m_p)$$
$$\Rightarrow \quad (mr)/(tr) = (m_p r_p)/(t_p r_p) \quad \Rightarrow \quad \exists\, r, t, m \in \mathbb{R} :\ mr = m_p r_p, \quad tr = t_p r_p. \quad (53)$$

$$r/t = r_p/t_p \quad \wedge \quad r/q = r_p/q_p \quad \Rightarrow \quad (r/t)/(r/q) = (r_p/t_p)/(r_p/q_p)$$
$$\Rightarrow \quad (qr)/(tr) = (q_p r_p)/(t_p r_p) \quad \Rightarrow \quad \exists\, r, t, q \in \mathbb{R} :\ qr = q_p r_p, \quad tr = t_p r_p. \quad (54)$$

And for conciseness, let:

$$t_p r_p = k_t \quad \wedge \quad m_p r_p = k_m \quad \wedge \quad q_p r_p = k_q. \quad (55)$$

Derivation of G, and the Newton, Gauss, and Poisson gravity laws

From equations 52, linear acceleration, r/t^2, can be related to mass, m, and distance, r:

$$r = c_m m\ \wedge\ r = c_t t\ \Rightarrow\ r/(c_t t)^2 = c_m m/r^2\ \Rightarrow\ r/t^2 = (c_m c_t^2)m/r^2 = Gm/r^2, \quad (56)$$

where $G = c_m c_t^2$, conforms to the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$ [4].

Newton's law follows from multiplying both sides of equation 56 by m:

$$r/t^2 = Gm/r^2 \;\Leftrightarrow\; F := mr/t^2 = Gm^2/r^2. \tag{57}$$

$$F = Gm^2/r^2 \;\wedge\; \forall\, m \in \mathbb{R}:\; \exists\, m_1, m_2 \in \mathbb{R}:\; m_1 m_2 = m^2 \quad\Rightarrow\quad F = Gm_1 m_2/r^2. \tag{58}$$

A new interpretation of Gauss's and Poisson's gravity laws is presented here. Equation 56 relates linear acceleration, r/t^2, to mass and distance. Gauss's gravity field, $\mathbf{g}$, and Poisson's gravity field, $-\nabla\Phi(\tilde{\mathbf{r}}, t)$ is the angular (orbital) acceleration, $2\pi r/t^2$. Applying equation 56:

$$r/t^2 = Gm/|\tilde{\mathbf{r}}|^2 \quad\wedge\quad \mathbf{g} = -\nabla\Phi(\tilde{\mathbf{r}}, t) = 2\pi r/t^2 \quad\Rightarrow\quad \mathbf{g} = -\nabla\Phi(\tilde{\mathbf{r}}, t) = 2\pi Gm/|\tilde{\mathbf{r}}|^2 \tag{59}$$

$$\mathbf{g} = -\nabla\Phi(\tilde{\mathbf{r}}, t) = 2\pi Gm/|\tilde{\mathbf{r}}|^2 \quad\Rightarrow\quad \nabla\cdot\mathbf{g} = \nabla^2\Phi(\tilde{\mathbf{r}}, t) = -4\pi Gm/|\tilde{\mathbf{r}}|^3 = -4\pi G\rho, \tag{60}$$

where $\rho = m/|\tilde{\mathbf{r}}|^3$ is the mass density:

Derivation of the special relativity equations

$$\forall\, \tau \in \{t, m, q\},\; r^2 = r'^2 + r_v^2,\; \exists\, \mu, \nu:\; r = \mu\tau \quad\wedge\quad r_v = \nu\tau \quad\Rightarrow\quad (\mu\tau)^2 = r'^2 + (\nu\tau)^2$$
$$\Rightarrow\quad r' = \sqrt{(\mu\tau)^2 - (\nu\tau)^2} \;=\; \mu\tau\sqrt{1 - (\nu/\mu)^2}. \tag{61}$$

Local frame distance, r', contracts relative to a distant observer frame distance, r, as $\nu \to \mu$:

$$r' = \mu\tau\sqrt{1 - (\nu/\mu)^2} \quad\wedge\quad \mu\tau = r \quad\Rightarrow\quad r' = r\sqrt{1 - (\nu/\mu)^2}. \tag{62}$$

A distant observer frame type, τ, dilates relative to the local observer frame type, τ', as $\nu \to \mu$:

$$\mu\tau = r'/\sqrt{1 - (\nu/\mu)^2} \quad\wedge\quad r' = \mu\tau \quad\Rightarrow\quad \tau = \tau'/\sqrt{1 - (\nu/\mu)^2}. \tag{63}$$

Where τ is type, time, the space-like flat Minkowski spacetime event interval is:

$$\mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_v^2 \quad\wedge\quad \mathrm{d}r_v^2 = \mathrm{d}r_1^2 + \mathrm{d}r_2^2 + \mathrm{d}r_3^2 \quad\wedge\quad \mathrm{d}(\mu\tau) = \mathrm{d}r$$
$$\Rightarrow\quad \mathrm{d}r'^2 = \mathrm{d}(\mu\tau)^2 - \mathrm{d}r_1^2 - \mathrm{d}r_2^2 - \mathrm{d}r_3^2, \quad \text{or in 6 dimensions}:$$
$$\mathrm{d}r'^2 = \mathrm{d}(c_t t)^2 + \mathrm{d}(c_m m)^2 + \mathrm{d}(c_q q)^2 - \mathrm{d}r_1^2 - \mathrm{d}r_2^2 - \mathrm{d}r_3^2. \tag{64}$$

Derivation of Schwarzschild's gravitational time dilation and black hole metric

From equations 62 and 51:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-(1)(v^2/c^2)} \wedge c_m m/r = 1 \Rightarrow \sqrt{1-(v^2/c^2)} = \sqrt{1-c_m m v^2/rc^2}. \quad (65)$$

Where v_{escape} is the escape velocity:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-c_m m v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv_{escape}^2$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2c_m m v_{escape}^2/rc^2}. \quad (66)$$

For a photon, the escape velocity, $v_{escape} = c$.

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-2c_m m v_{escape}^2/rc^2}. \wedge v_{escape} = c$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2c_m mc^2/rc^2}. \quad (67)$$

Combining equation 67 with the derivation of G (58):

$$c_m c^2 = G \wedge \sqrt{1-(v^2/c^2)} = \sqrt{1-2c_m mc^2/rc^2} \Rightarrow \sqrt{1-(v^2/c^2)} = \sqrt{1-2Gm/rc^2}. \quad (68)$$

Combining equation 68 with equation 63 yields Schwarzschild's gravitational time dilation:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-2Gm/rc^2} \wedge t' = t\sqrt{1-(v^2/c^2)} \Rightarrow t' = t\sqrt{1-2Gm/rc^2}. \quad (69)$$

From equations 62 and 68, where Schwarzschild's defined the black hole event horizon radius as $\alpha := 2Gm/c^2$ [2] [3]:

$$r' = r\sqrt{1-(v/c)^2} = r\sqrt{1-2Gm/rc^2} \quad \wedge \quad \alpha := 2Gm/c^2 \quad \Rightarrow \quad r' = r\sqrt{1-\alpha/r}. \quad (70)$$

Applying equation 70 to the time-like spacetime interval equation 64:

$$r' = r\sqrt{1-\alpha/r} \quad \wedge \quad \mathrm{d}s^2 = \mathrm{d}r'^2 - \mathrm{d}r^2$$
$$\Rightarrow \quad \mathrm{d}s^2 = (\sqrt{1-\alpha/r}\mathrm{d}r)^2 - (\mathrm{d}r'/\sqrt{1-\alpha/r})^2 = (1-\alpha/r)\mathrm{d}r^2 - (1-\alpha/r)^{-1}\mathrm{d}r'^2. \quad (71)$$

$$\mathrm{d}s^2 = (1-\alpha/r)\mathrm{d}r^2 - (1-\alpha/r)^{-1}\mathrm{d}r'^2 \quad \wedge \quad \mathrm{d}r = \mathrm{d}(ct) \quad \wedge \quad c = 1 \quad \wedge \quad \lim_{\mathrm{d}s\to 0} \mathrm{d}r' = \mathrm{d}r$$
$$\Rightarrow \quad \lim_{\mathrm{d}s\to 0} \mathrm{d}s^2 = (1-\alpha/r)\mathrm{d}t^2 - (1-\alpha/r)^{-1}\mathrm{d}r^2. \quad (72)$$

Using spherical coordinates to translate from 2D to 4D yields the $+ - --$ form of Schwarzschild's black hole metric [2] [3]:

$$\begin{aligned}
&\mathrm{d}s^2 = (1-\alpha/r)\mathrm{d}t^2 - (1-\alpha/r)^{-1}\mathrm{d}r^2 \\
&\qquad\Rightarrow\quad \mathrm{d}s^2 = (1-\alpha/r)\mathrm{d}t^2 - (1-\alpha/r)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + sin^2\theta\mathrm{d}\phi^2) \\
&\qquad\quad\Rightarrow\quad g_{\mu,\nu} = diag[(1-\alpha/r),\ -(1-\alpha/r)^{-1},\ -r^2(\mathrm{d}\theta^2),\ -r^2(sin^2\theta\mathrm{d}\phi^2)]. \quad (73)
\end{aligned}$$

Simple method to find general relativity solutions

Einstein's field equation is:

$$G_{\mu,\nu} = \mathbf{R} + \frac{1}{2}Rg_{\mu,\nu} = \kappa T_{\mu,\nu}, \tag{74}$$

where: $G_{\mu,\nu}$ is Einstein's tensor, $\mathbf{R}$ is the Ricci curvatature, R is the scalar curvature, $g_{\mu,\nu}$ is the metric tensor, $\kappa = 8\pi G/c^4$, and $T_{\mu,\nu}$ is the stress-energy tensor [8][7].

The goal of the field equation is to determine the geodesic path and acceleration of a particle caused by the distribution of mass and energy as specified in the stress-energy tensor, $T_{\mu,\nu}$. This requires solving the equation for the metric tensor, $g_{\mu,\nu}$. But the metric tensor has a complex nonlinear relation to the Ricci and scalar curvature, which makes it complicated to determine the metric. Often, there are no exact solutions.

In this article, the laws of physics are the independent variables. The infinitesimal space near every coordinate point on a pseudo-Riemann surface is Euclidean, where the physics laws derived from the ratios and special relativity equations (61) determine the metric, $g_{\mu,\nu}$, which, in turn, determines the changes in the distribution of mass and energy, $T_{\mu,\nu}$. For example, when two black holes approach each other, the laws of physics determine the metric, which, in turn, determines what happens next to the black holes. This leads to the following steps to solve for the metric tensor, $g_{\mu,\nu}$, independent of Einstein's field equation.

Step 1) Use the ratios and relativity equations to define functions returning scalar values for each component of the metric, $g_{\nu,\mu}$, in Einstein's field equations [8][7]: An example is the previous Schwarzschild black hole metric (65).

Step 2) The 4D flat spacetime interval equation is an instance of the 2D equation (64), $\mathrm{d}r'^2 = \mathrm{d}(ct)^2 - \mathrm{d}r_v^2$. As shown in equation 73, derive the metric first as a 2D metric and then expanded to a 4D metric.

(Optional) The 2D metric tensor allows using the much simpler 2D Ricci curvature and scalar curvature. And the 2D tensors reduce the number of independent equations to solve, which can next be used to set constraints on the solutions in the 4D tensors.

Step 3) One simple method to translate from 2D to 4D is to use spherical coordinates, where r and t remain unchanged and two added dimensions are the angles, ϕ, and θ. For example, the 2D Schwarzschild metric was translated to 4D using this method in equation 73. The spherical coordinates can then be translated to other types of coordinates.

Derivation of k_e, Coulomb charge law, ε_0, E, and Gauss electric field law

From equations 52, linear acceleration, r/t^2, can be related charge, q, and distance, r:

$$r = c_q q \ \wedge \ r = c_t t \ \Rightarrow \ r/(c_t t)^2 = c_q q/r^2 \ \Rightarrow \ r/t^2 = (c_q c_t^2)q/r^2, \tag{75}$$

$$r = c_m m = c_q q \quad \Rightarrow \quad m = (c_q/c_m)q. \tag{76}$$

Combining equations 75 and 76 yields Coulombs's law:

$$r/t^2 = (c_q c_t^2)q/r^2 \ \wedge \ m = (c_q/c_m)q \ \Rightarrow \ F := mr/t^2 = (c_q^2 c_t^2/c_m)q^2/r^2 = k_e q^2/r^2. \tag{77}$$

where $k_e = c_q^2 c_t^2/c_m$, conforms to the SI units: $kg \cdot m^3 \cdot s^{-2} \cdot C^{-2} = N \cdot m^2 \cdot C^{-2}$ [5].

$$\forall\, q \in \mathbb{R}\, \exists\, q_1, q_2 \in \mathbb{R}: \ q_1 q_2 = q^2 \quad \wedge \quad F = k_e q^2/r^2 \quad \Rightarrow \quad F = k_e q_1 q_2/r^2. \tag{78}$$

A new interpretation of Gauss's electric field law is presented here. Equation 75 relates the linear acceleration, r/t^2, to charge and distance. Gauss's electric field, **E**, relates angular (orbital) acceleration, $2\pi r/t^2$ to charge and distance. Applying equation 77:

$$F_C = mr/t^2 = k_e q^2/r^2 \quad \Rightarrow \quad \exists\, F_E \in \mathbb{R}: \ F_E = 2\pi(mr/t^2) = 2\pi k_e q^2/r^2 \tag{79}$$

$$F_E = 2\pi k_e q^2/r^2 = q(2\pi(k_e q/r^2)) \quad \wedge \quad \exists\, E \in R: \ E = 2\pi k_e q/r^2 \quad \Rightarrow \quad F_E = qE. \tag{80}$$

The electric field, $E := 2\pi k_e q/r^2$, conforms to the SI units $kg \cdot m \cdot s^{-2} \cdot C^{-1} = N \cdot C^{-1}$.

$$\mathbf{E} = 2\pi k_e q/|\vec{\mathbf{r}}|^2 \quad \Rightarrow \quad \nabla \cdot \mathbf{E} = -4\pi k_e q/|\vec{\mathbf{r}}|^3. \tag{81}$$

$$\nabla \cdot \mathbf{E} = -4\pi k_e q/|\vec{\mathbf{r}}|^3 \quad \wedge \quad \varepsilon_0 := 1/4\pi k_e \quad \wedge \quad \rho = q/|\vec{\mathbf{r}}|^3 \quad \Rightarrow \quad \nabla \cdot \mathbf{E} = -\rho/\varepsilon_0, \tag{82}$$

which is Gauss's electric field law [5] in differential form.

Derivation of B, μ_0, and Lorentz's law

Applying the distance contraction equation, 62, to equation 80, where r' is the distant observer frame of reference and r is local (rest) frame of reference:

$$r' = r/\sqrt{1 - v^2/c^2} \quad \wedge \quad F' = 2\pi k_e q^2/r'^2 \quad \Rightarrow \quad F = 2\pi k_e q^2(1 - v^2/c^2)/r^2. \tag{83}$$

From equation 80:

$$E = 2\pi k_e q/r^2 \ \wedge \ F = 2\pi k_e q^2(1 - v^2/c^2)/r^2 \ \Rightarrow \ F = q(E - (((2\pi k_e/c^2)q/r^2))v^2). \tag{84}$$

$$F = q(E - (((2\pi k_e/c^2)q/r^2))v^2) \ \wedge \ \exists\, B :\ B = (2\pi k_e/c^2)vq/r^2 \ \Rightarrow \ F = q(E - Bv). \tag{85}$$

Translating equation 85 to vector form:

$$F = q(E - Bv) \quad \Rightarrow \quad \mathbf{F} = q(\mathbf{E} - \mathbf{B} \times \vec{\mathbf{v}}). \tag{86}$$

$$\mathbf{B} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \mathbf{B})) \quad \wedge \quad \mathbf{F} = q(\mathbf{E} - \mathbf{B} \times \vec{\mathbf{v}}) \quad \Rightarrow \quad \mathbf{F} = q(\mathbf{E} + \vec{\mathbf{v}} \times \mathbf{B}), \tag{87}$$

which is Lorentz law in the distant observer frame of reference, where the magnetic field, $B = (2\pi k_e/c^2)vq/r^2$, conforms to the base SI units: $kg \cdot s^{-1} \cdot C^{-1} \ = \ kg \cdot s^{-2} \cdot A^{-1} \ = \ T$.

Based on the assumption (having no math proof-motivation) of a magnetic dipole, the magnetic field divergence outward, $\nabla \cdot \mathbf{B}$, at a point, P, normal to a surface enclosing a charge equals the field divergence inward, $\nabla \cdot \mathbf{B}$:

$$\mathbf{B} = (2\pi k_e/c^2)vq/|\vec{\mathbf{r}}|^2 \quad \Rightarrow \quad \nabla \cdot \mathbf{B} = -(4\pi k_e/c^2)q/|\vec{\mathbf{r}}|^3 + (4\pi k_e/c^2)q/|\vec{\mathbf{r}}|^3 = 0. \tag{88}$$

$$\nabla \cdot \mathbf{B} = -(4\pi k_e/c^2)q/|\vec{\mathbf{r}}|^3 + (4\pi k_e/c^2)q/|\vec{\mathbf{r}}|^3 = 0 \quad \Rightarrow \quad \mu_0 = (4\pi k_e/c^2), \tag{89}$$

where $\mu_0 = 4\pi k_e/c^2$ conforms to the SI units $kg \cdot m \cdot C^{-2} = kg \cdot m \cdot s^{-2} A^{-2}$.

Note: In the last section of this article, it will be shown that the CODATA definition, $\mu_0 = 2h\alpha/cq_e^2$ [15], and $\mu_0 = 4\pi k_e/c^2$ both reduce to the same equation, $\mu_0 = 4\pi k_m/q_p^2$.

Derivation of Maxwell-Faraday's law

From the magnetic field equation 86, where the electric and magnetic fields are propagating at the speed, $v = c$:

$$B = (2\pi k_e/c^2)qv/r^2 \quad \wedge \quad v = c \quad \wedge \quad r = ct \quad \Rightarrow \quad B = (2\pi k_e/c^3)q/t^2. \tag{90}$$

$$B = (2\pi k_e/c^3)q/t^2 \quad \Rightarrow \quad \partial B/\partial t = -(4\pi k_e/c^3)q/t^3. \tag{91}$$

$$\partial B/\partial t = -(4\pi k_e/c^3)q/t^3 \quad \wedge \quad r = ct \quad \Rightarrow \quad \partial B/\partial t = -4\pi k_e q/r^3. \tag{92}$$

Faraday's law is the case, where the electric field, $\mathbf{E}$, is a single radial vector, $\mathbf{E_{\tilde{r}_1}}$ acting on some point, P. Therefore, at point, P, $\mathbf{E_{\tilde{r}_2}} = 0$ and $\mathbf{E_{\tilde{r}_3}} = 0$. From the Lorentz equation 87, the magnetic field, $\mathbf{B}$ at point P, is perpendicular to $\mathbf{E_{\tilde{r}_1}}$ (aligned on $\tilde{\mathbf{r}}_2$). For a constant force, the Lorentz equation makes the electric field, $\mathbf{E_{\tilde{r}_1}}$, a function of $\mathbf{B_{\tilde{r}_2}}$. Combining with equation 81:

$$\mathbf{E_{\tilde{r}_1}} = 2\pi k_e q/|\tilde{\mathbf{r}}_2|^2 = f(\mathbf{B_{\tilde{r}_2}}) \quad \wedge \quad \mathbf{E_{\tilde{r}_2}} = 0 \quad \wedge \quad \mathbf{E_{\tilde{r}_3}} = 0 \quad \Rightarrow \quad \nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial \tilde{\mathbf{r}}_1^2} & \frac{\partial}{\partial \tilde{\mathbf{r}}_2^2} & \frac{\partial}{\partial \tilde{\mathbf{r}}_3^2} \\ \mathbf{E_{\tilde{r}_1}} & 0 & 0 \end{vmatrix}$$
$$= (\frac{\partial \mathbf{E}_{r_3}}{\partial \tilde{\mathbf{r}}_2^2} - \frac{\partial \mathbf{E}_{r_2}}{\partial \tilde{\mathbf{r}}_3^2})\hat{\mathbf{i}} + (\frac{\partial \mathbf{E}_{r_1}}{\partial \tilde{\mathbf{r}}_3^2} - \frac{\partial \mathbf{E}_{r_3}}{\partial \tilde{\mathbf{r}}_1^2})\hat{\mathbf{j}} + (\frac{\partial \mathbf{E}_{r_2}}{\partial \tilde{\mathbf{r}}_1^2} - \frac{\partial \mathbf{E}_{r_1}}{\partial \tilde{\mathbf{r}}_2^2})\hat{\mathbf{k}}$$
$$= 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + (0 - \frac{\partial 2\pi k_e q/|\tilde{\mathbf{r}}_2|^2}{\partial |\tilde{\mathbf{r}}_2|^2})\hat{\mathbf{k}} \quad = \quad 4\pi k_e q/|\vec{\mathbf{r}}_2|^3\hat{\mathbf{k}} \quad \equiv \quad \nabla \times \mathbf{E} = 4\pi k_e q/|\vec{\mathbf{r}}|^3. \tag{93}$$

Combining equations 93 and 92 yields Maxwell-Faraday's law [5]:

$$\nabla \times \mathbf{E} = 4\pi k_e q/|\vec{\mathbf{r}}|^3 \quad \wedge \quad \partial \mathbf{B}/\partial t = -4\pi k_e q/|\vec{\mathbf{r}}|^3 \quad \Rightarrow \quad \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t. \tag{94}$$

Derivation of the fine structure constant, α

The ratio of two subtypes of force implies ratios of the form:

$$\alpha_\tau = \frac{F_{\tau_1}}{F_{\tau_2}} = \frac{K\tau_1^2/r^2}{K\tau_2^2/r^2} = \frac{\tau_1^2}{\tau_2^2}. \tag{95}$$

For example, where q_e is the elementary (electron) charge ($1.60217663 \cdot 10^{-19}\ C$), and q_p is reduced Planck charge unit, and the fine structure electron coupling constant has the empirical value $7.2973525705 \cdot 10^{-3}$:

$$\alpha_q = q_e^2/q_p^2 \approx 7.2973525705 \cdot 10^{-3}. \tag{96}$$

The gravitational fine structure constant is: $\alpha_G = m_e^2/m_p^2$.

Derivation of $\hbar$, h, and the Einstein-Planck relation

[5][16] Applying both the direct proportion ratio (107), and inverse proportion ratio (53):

$$m = k_m/r \quad \wedge \quad r = ct \quad \Rightarrow \quad m(ct)^2 = (k_m/r)r^2 = k_m r. \tag{97}$$

$$m(ct)^2 = k_m r \quad \Rightarrow \quad E := mc^2 = k_m r/t^2. \tag{98}$$

$$E = mc^2 = k_m r/t^2 \;\; \wedge \;\; r/t = c \;\; \Rightarrow \;\; E = mc^2 = (k_m c/t. \tag{99}$$

The next 4 steps show that $k_m c = \hbar$, the *reduced* Planck constant. From the fine structure equationx 96:

$$\alpha = q_e^2/q_p^2 \quad \wedge \quad q_e = 1.60217663 \cdot 10^{-19}\, C \quad \wedge \quad \alpha \approx 7.2973525705 \cdot 10^{-3}$$
$$\Rightarrow \quad q_p = \sqrt{q_e^2/\alpha} \approx 1.87554604 \cdot 10^{-18}\, C. \tag{100}$$

From equations 89 and 100:

$$\mu_0 = 4\pi k_e/c_t^2 = 4\pi c_q^2/c_m = 4\pi(r_p/q_p)^2/(r_p/m_p) = 4\pi m_p r_p/q_p^2 = 4\pi k_m/q_p^2. \tag{101}$$

$$\mu_0 = 4\pi k_m/q_p^2 \;\; \wedge \;\; q_p \approx 1.87554604 \cdot 10^{-18}\, C \;\; \wedge \;\; \mu_0 \approx 1.25663706127(20) \cdot 10^{-6}\, kg\ m\ C^{-2}$$
$$\Rightarrow \quad k_m = \mu_0 q_p^2/4\pi \approx 3.51767294 \cdot 10^{-43}\, kg\ m. \tag{102}$$

$$k_m \approx 3.51767294 \cdot 10^{-43}\, kg\ m \quad \wedge \quad c = 2.99792458 \cdot 10^{8}\, m\ s^{-1}$$
$$\Rightarrow \quad k_m c \approx 1.054571817 \cdot 10^{-34}\, kg\ m^2\ s^{-1} = \hbar. \tag{103}$$

Combining equations 99 and 103:

$$E = mc^2 = k_m c/t\ kg\ m^2\ s^{-2} \;\; \wedge \;\; k_m c = \hbar\ kg\ m^2\ s^{-1} \;\; \Rightarrow \;\; E = \hbar/t\ kg\ m^2\ s^{-2}. \tag{104}$$

$$E = \hbar/t\ kg\ m^2\ s^{-2} \quad \wedge \quad \omega = (2\pi/t)\ radian\ s^{-1} \quad \Rightarrow$$
$$E_\omega = \hbar(2\pi/t) = \hbar\omega\ kg\ m^2\ radian\ s^{-2}. \tag{105}$$

$$E_\omega = \hbar\omega\ kg\ m^2\ radian\ s^{-2} \quad \wedge \quad f = (1/t)\ cycle\ s^{-1} = (2\pi/t)\ radian\ s^{-1} = \omega \quad \Rightarrow$$
$$E_f = \hbar\omega(2\pi/2\pi) = 2\pi\hbar f = hf\ kg\ m^2\ cycle\ s^{-2}. \tag{106}$$

Derivation of the unit ratio and reduced Planck unit values

The direct proportion ratios are:

$$c_t = c \approx 2.99792458 \cdot 10^8 \ m \ s^{-1}. \tag{107}$$

$$G = c_m c_t^2 \quad \wedge \quad G \approx 6.67430(15) \cdot 10^{-11} m^3/kg/s^2 \quad \Rightarrow$$

$$c_m = G/c_t^2 \approx 7.426162 \cdot 10^{-28} m \ kg^{-1}. \tag{108}$$

$$k_e = c_q^2 c_t^2 / c_m \quad \wedge \quad k_e \approx 8.9875517923(13) \cdot 10^9 Nm^2/C^2 \quad \Rightarrow$$

$$c_q = \sqrt{k_e c_m / c_t^2} \approx 8.6175182 \cdot 10^{-18} m \ C^{-1}. \tag{109}$$

From equation 103, the inverse proportion ratios are:

$$\hbar \approx 1.054571783 \cdot 10^{-34} \ kg \ m^2/s \quad \wedge \quad c_t = 2.99792458 \cdot 10^8 \ m/s$$

$$\Rightarrow \quad k_m = \hbar / c_t \approx 3.51767283 \cdot 10^{-43} \ kg \ m. \tag{110}$$

$$r_p = \sqrt{r_p^2} = \sqrt{k_m c_m} \approx 1.616255 \cdot 10^{-35} \ m. \tag{111}$$

$$k_t = r_p / c_t \approx 8.71363 \cdot 10^{-79} \ s \ m. \tag{112}$$

$$k_q = r_p / c_q \approx 3.031361 \cdot 10^{-53} \ C \ m. \tag{113}$$

From the fine structure equation 100, the reduced Planck units are:

$$q_p = \sqrt{q_e^2/\alpha} \approx 1.87554604 \cdot 10^{-18} \ C. \tag{114}$$

$$r_p = q_p c_q \approx 1.616255 \cdot 10^{-35} \ m. \tag{115}$$

$$t_p = r_p / c_t \approx 5.391246 \cdot 10^{-44} \ s. \tag{116}$$

$$m_p = r_p / c_m \approx 2.176434 \cdot 10^{-8} \ kg. \tag{117}$$

Derivation of the Compton wavelength, λ

[5][16] From equations 55 and 100:

$$\lambda = 2\pi r = 2\pi k_m / m = 2\pi k_m c / mc \quad \wedge \quad h = 2\pi\hbar = 2\pi k_m c \quad \Rightarrow \quad \lambda = h/mc. \tag{118}$$

Derivation of Schrödenger's position-space wave equation

Start with the previously derived "base" Einstein-Planck relation 104 and multiply the kinetic energy component by mc/mc:

$$mc^2 = \hbar/t \quad \Rightarrow \quad \exists\, V(r,t):\ \hbar/t = \hbar/2t + V(r,t). \tag{119}$$

$$\hbar/t = h/2t + V(r,t) \quad \Rightarrow \quad \hbar/t = \hbar mc/2mct + V(r,t). \tag{120}$$

And from the distance-to-time (speed of light) ratio (107):

$$\hbar/t = \hbar mc/2mct + V(r,t) \quad \wedge \quad r = ct \quad \Rightarrow \quad \hbar/t = \hbar mc^2/2mcr + V(r,t). \tag{121}$$

$$\hbar/t = \hbar mc^2/2mcr + V(r,t) \ \wedge\ \hbar/t = mc^2 \quad \Rightarrow \quad \hbar/t = \hbar^2/2mcrt + V(r,t). \tag{122}$$

$$\hbar/t = \hbar^2/2mcrt + V(r,t) \quad \wedge \quad r = ct \quad \Rightarrow \quad \hbar/t = \hbar^2/2mr^2 + V(r,t). \tag{123}$$

Multiply both sides of equation 123 by a probability density function, $\Psi(r,t)$.

$$\hbar/t = \hbar^2/2mr^2 + V(r,t) \quad \Rightarrow \quad (\hbar/t)\Psi(r,t) = (\hbar^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t). \tag{124}$$

$$\begin{aligned}&(\hbar/t)\Psi(r,t) = ((\hbar)^2/2mr^2)\Psi(r,t) + V(r,t)\Psi(r,t) \quad \wedge \\ &\qquad \forall\ \Psi(r,t):\ \partial^2\Psi(r,t)/\partial r^2 = (-1/r^2)\Psi(r,t) \quad \wedge \quad \partial\Psi(r,t)/\partial t = (i\ 1/t)\Psi(r,t) \\ &\qquad\qquad \Rightarrow \quad i\hbar\partial\Psi(r,t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t),\end{aligned} \tag{125}$$

which is the one-dimensional position-space Schrödenger's equation [17][16].

$$\begin{aligned}&i\hbar\partial\Psi(r,t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(r,t)/\partial r^2 + V(r,t)\Psi(r,t) \quad \wedge \quad ||\vec{\mathbf{r}}|| = r \\ &\qquad \Rightarrow \quad \exists\ \vec{\mathbf{r}}:\ i\hbar\partial\Psi(\vec{\mathbf{r}},t)/\partial t = -(\hbar^2/2m)\partial^2\Psi(\vec{\mathbf{r}},t)/\partial\vec{\mathbf{r}}^2 + V(\vec{\mathbf{r}},t)\Psi(\vec{\mathbf{r}},t),\end{aligned} \tag{126}$$

which is the 3-dimensional position-space Schrödenger's equation [17] [16].

Derivation of Dirac's wave equation

Start with the previously derived "base" Einstein-Planck relation 104:

$$mc^2 = \hbar/t \quad \Rightarrow \quad \exists\, V(r,t):\ mc^2/2 + V(r,t) = \hbar/t \quad \Rightarrow \quad 2\hbar/t - 2V(r,t) = mc^2. \tag{127}$$

$$\begin{aligned}&\forall\, V(r,t):\ V(r,t) = \hbar/t \quad \wedge \quad r = ct \quad \wedge \quad 2\hbar/t - 2V(r,t) = mc^2 \\ &\qquad\qquad \Rightarrow \quad 2\hbar/t - i2\hbar c/r = mc^2.\end{aligned} \tag{128}$$

Use the ratios, $r = c_q q$, and, $r = ct$. to multiply each term on the left side of equation 128 by 1:

$$qc_q/r = qc_q/ct = 1 \wedge 2\hbar/t - i2\hbar c/r = mc^2 \Rightarrow 2\hbar(qc_q/c)/t^2 - i\hbar((qc_q/c)/r^2)c = mc^2. \quad (129)$$

Applying a quantum amplitude equation in complex form to equation **??**:

$$A_0 = 2(c_q/c)((1/t) - i(1/r)) \quad \wedge \quad 2\hbar(qc_q/c)/t^2 - i\hbar((qc_q/c)/r^2)c = mc^2$$
$$\Rightarrow \hbar\partial(-qA_0)/\partial t - i\hbar(\partial(-qA_0)/\partial r)c = mc^2. \quad (130)$$

Translating equation 130 to moving (rest frame) coordinates via the Lorentz factor, $\gamma_0 = 1/\sqrt{1-(v/c)^2}$:

$$\hbar\partial(-qA_0)/\partial t - i\hbar(\partial(-qA_0)/\partial r)c = mc^2$$
$$\Rightarrow \quad \gamma_0\hbar\partial(-qA_0)/\partial t - \gamma_0 i\hbar(\partial(-qA_0)/\partial r)c = mc^2. \quad (131)$$

Translating equation 131 to vector form:

$$\gamma_0\hbar(\partial(-qA_0)/\partial t) - \gamma_0 i\hbar(\partial(-qA_0)/\partial r)c = mc^2 \quad \wedge$$
$$||\vec{\mathbf{r}}|| = r \quad \wedge \quad ||\vec{\mathbf{A}}|| = A_0 \quad \wedge \quad ||\vec{\gamma}|| = \gamma_0$$
$$\Leftrightarrow \quad \exists\, \vec{\mathbf{r}}, \vec{\mathbf{A}}, \vec{\gamma} : \gamma_0\hbar(\partial(-qA_0)/\partial t) - \vec{\gamma}\cdot i\hbar(\partial(-q\vec{\mathbf{A}})/\partial r)c = mc^2. \quad (132)$$

Multiplying both sides of equation 132 by the spinor, Ψ, yields:Dirac's equation [18]:

$$\gamma_0\hbar(\partial(-qA_0)/\partial t) - \vec{\gamma}\cdot i\hbar(\partial(-q\vec{\mathbf{A}})/\partial r)c = mc^2 \quad \Rightarrow$$
$$\gamma_0\hbar(\partial(-qA_0)/\partial t)\Psi - \vec{\gamma}\cdot i\hbar(\partial(-q\vec{\mathbf{A}})/\partial r)c\Psi = mc^2\Psi \quad (133)$$

Total of a type

Applying both the direct (107) and inverse proportion ratios (110) to the Euclidean distance:

$$r = \sqrt{r_1^2 + r_2^2}, \quad r_1 = c_\tau \tau, \quad r_2 = k_\tau/\tau, \quad \tau \in \{t, m, q\}$$
$$\Rightarrow \quad r = \sqrt{(c_\tau\tau)^2 + (k_\tau/\tau)^2} \quad \Leftrightarrow \quad \tau = \sqrt{(r/c_\tau)^2 + (k_\tau/r)^2}. \quad (134)$$

Quantum extension to general relativity

The simplest way to demonstrate how to add quantum physics to general relativity is by extending Schwarzschild's gravitational time dilation equation and black hole metric. Apply the total of a type equation 134 to the derivation of Schwarzschild's time dilation and metric (65):

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-(v^2/c^2)(r/r)} \quad \wedge \quad r = \sqrt{(c_m m)^2 + (k_m/m)^2} = Q_m$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-Q_m v^2/rc^2}. \quad (135)$$

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-Q_m v^2/rc^2} \quad \wedge \quad KE = mv^2/2 = mv^2_{escape}$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2Q_m v^2_{escape}/rc^2}. \quad (136)$$

For a photon, the escape velocity, $v_{escape} = c$.

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-2Q_m v^2_{escape}/rc^2}. \wedge v_{escape} = c$$
$$\Rightarrow \quad \sqrt{1-(v^2/c^2)} = \sqrt{1-2Q_m c^2/rc^2} = \sqrt{1-2Q_m/r}. \quad (137)$$

Combining equation 137 with equation 63 yields Schwarzschild's gravitational time dilation with a quantum mass effect:

$$\sqrt{1-(v^2/c^2)} = \sqrt{1-2Q_m/r} \quad \wedge \quad t' = t\sqrt{1-(v^2/c^2)} \quad \Rightarrow \quad t' = t\sqrt{1-2Q_m/r}. \quad (138)$$

Schwarzschild defined the black hole event horizon radius, $\alpha := 2Gm/c^2$. The radius with the quantum extension is $\alpha := 2Q_m$. At this point the exact same equations 70 through 73 yield what looks like the same Schwarzschild black hole metric.

Quantum extension to Newton's gravity force

The quantum mass effect is easier to understand in the context Newton's gravity equation than in general relativity, because the metric equations and solutions in the EFEs are much more complex. From equations 134 and 51:

$$m/\sqrt{(r/c_m)^2 + (k_m/r)^2} = 1 \quad \wedge \quad r^2/(ct)^2 = 1 \quad \Rightarrow \quad r^2/(ct)^2 = m/\sqrt{(r/c_m)^2 + (k_m/r)^2}$$
$$\Rightarrow \quad r^2/t^2 = c^2 m/\sqrt{(r/c_m)^2 + (k_m/r)^2}. \quad (139)$$

$$r^2/t^2 = c^2m/\sqrt{(r/c_m)^2 + (k_m/r)^2} \quad \Rightarrow \quad (m/r)(r^2/t^2) = (m/r)(c^2m/\sqrt{(r/c_m)^2 + (k_m/r)^2})$$
$$\Rightarrow \quad F := mr/t^2 = c^2m^2/\sqrt{(r^4/c_m^2) + k_m^2}. \quad (140)$$

$$F = c^2m^2/\sqrt{(r^4/c_m^2) + k_m^2} \quad \wedge \quad \forall\, m \in \mathbb{R},\, \exists\, m_1, m_2 \in \mathbb{R} :\ m_1m_2 = m^2$$
$$\Rightarrow \quad F = c^2m_1m_2/\sqrt{(r^4/c_m^2) + k_m^2}. \quad (141)$$

Quantum extension to Coulomb's charge force

$$q^2/((r/c_q)^2 + (k_q/r)^2) = 1 \quad \wedge \quad r^2/(ct)^2 = 1 \quad \Rightarrow \quad r^2/(ct)^2 = q^2/((r/c_q)^2 + (k_q/r)^2)$$
$$\Rightarrow \quad r^2/t^2 = c^2q^2/((r/c_q)^2 + (k_q/r)^2). \quad (142)$$

$$(1/r)(r^2/t^2) = (1/r)(c^2q^2/((r/c_q)^2 + (k_q/r)^2)) \quad \Rightarrow \quad r/t^2 = c^2q^2/(r^3/c_q^2 + k_q^2/r). \quad (143)$$

$$\forall\, q \in \mathbb{R} :\ \exists\, q_1, q_2 \in \mathbb{R} :\ q_1q_2 = q^2 \quad \wedge \quad r/t^2 = c^2q^2/(r^3/c_q^2 + k_q^2/r) \quad \Rightarrow$$
$$\exists\, q_1, q_2 \in \mathbb{R} :\ r/t^2 = c^2q_1q_2/(r^3/c_q^2 + k_q^2/r). \quad (144)$$

$$r/t^2 = c^2q_1q_2/(r^3/c_q^2 + k_q^2/r) \quad \wedge \quad m = r/c_m \quad \Rightarrow \quad F := mr/t^2 = (c^2/c_m)q_1q_2/(r^2/c_q^2 + k_q^2/r^2). \quad (145)$$

INSIGHTS AND IMPLICATIONS

1. The ruler measure (.1) and convergence theorem (.2) were shown to be useful tools for proving that a countable sets of n-tuples imply a corresponding real-valued equation.

2. The notion of distance was derived as the inverse function of the sum of Euclidean volumes (.7), which suggests defining a distance measures as the inverse function of the sum of subset Euclidean and non-Euclidean volumes,

   $$\forall\, n \in \mathbb{N}, \quad f_n, g \in C^2, \quad d = f_n^{-1}(\textstyle\sum_{i=1}^m f_n(v_i)) = f_n^{-1}(\sum_{i=1}^m g(s_{i,1}, \cdots, s_{i,n})) : \quad (146)$$

   (a) shows the intimate relation between distance and volume that definitions, like inner product space and metric space, ignore [9][10];

   (b) is a more simple and concise definition of a distance measure, having the metric space properties [9][10].

3. The left side of the distance sum inequality (.10),

$$(\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}, \tag{147}$$

differs from the left side of Minkowski's sum inequality [14]:

$$(\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \tag{148}$$

   (a) The two inequalities are only the same where $n = 1$.

   (b) The distance sum inequality (.10) is a more fundamental inequality because the proof does not require the convexity and Hölder's inequality assumptions required to prove the Minkowski sum inequality [14].

   (c) $\forall\ n > 1,\ v > 0$: The distance sum inequality term, $v_i^n = a_i^n + b_i^n$: $d = v^{1/n} = (\sum_{i=1}^{m} v_i^n)^{1/n}$, is the Minkowski distance, which makes it directly related to geometry. But the Minkowski sum inequality term, $d = v^{1/n} = (\sum_{i=1}^{m}((v_i^n)^{\mathbf{n}}))^{1/n}$ $= (\sum_{i=1}^{m} v_i^{\mathbf{n^2}})^{1/n}$, is *not* a Minkowski distance.

   (d) The distance sum inequality might be applicable to machine learning.

4. *Combinatorics.* The number of n-tuples, $v_c = \prod_{i=1}^{n} |x_i|$, was proven to imply: the Euclidean volume equation (.4). And the notion of distance was derived as the inverse function of the sum of the number of n-tuples (.7) (which includes the inner product) and the Minkowski distance equation (.8), where the Minkowsk distance case of $n = 1$ is the Manhattan distance and the case of $n = 2$ is the Euclidean distance equation. These equations were derived without relying on the geometric primitives and relations in Euclidean geometry [19][20], axiomatic geometry [21][22][23] [24][25], trigonometry [26] [27] calculus [28][26][29], and vector analysis [7].

5. *Combinatorics.* The commutative property of the operations defining volume and distance equations limits a domain set to $n \leq 3$ elements (.19). Other elements must have different types (are elements of different sets).

   (a) For example, the vector inner product space, in physical space, can only be extended beyond 3 dimensions if and only if the other dimensions have non-distance types, for example, dimensions of time, mass, and charge.

(b) If there are more than 6 dimensions, $\{r_1, r_2, r_3, t, m, q\}$, then the dimensions are not $\subseteq \mathbb{R}$. For example, a "dimension" might a countable set of states.

(c) If each type of quantum state is an ISCL, then there are at most 3 states of the same type: 3 orientations per dimension of space (+1,0-1), 3 quark color charges, {red, green, blue}, 3 quark anti-color charges, and so on.

6. A discrete (point) valued state has measure 0 (zero-length interval). The ratio of a time or distance interval length to zero is undefined, which is the reason quantum entangled (discrete) state values exist independent of (non-deterministic with respect to) distance (position), time, mass, and charge. Therefore, a state value in space and time is undetermined until observed, like Schrödenger's poisoned cat being both alive and dead until the box is opened [17].

7. For each unit interval length, r_p, of a 3-dimensional distance, $r subseteq \mathbb{R}$, there are unit interval lengths of other types of dimensions, forming unit ratios (107): $c_t = r_p/t_p$, $c_m = r_p/m_p$, $c_q = r_p/q_p \Leftrightarrow$ the inverse proportion ratios (53): $k_t = r_p t_p$, $k_m = r_p m_p$, $k_t = r_p q_p$, where r_p, t_p, m_p, and q_p are the reduced Planck units (114).

8. An empirical law *describes* the relations between the variables in an equation. Deriving empirical laws from the ratios *explains* the reasons for relations between the variables in an equation. Further, all the derivations of the physics equations from the ratios are much shorter and simpler than other derivations, which shows that the ratios are an important tool for physicists and engineers.

9. As shown in the subsection deriving the Schwarzschild's gravitational time dilation and black hole metric (65) [2][3] using ratios illustrates a simple way of finding solutions to Einstein's field equations.

10. c_t, c_m, and c_q are the *maximum* ratios: $r = \sqrt{r_1^2 + r_2^2 + r_3^2} \Rightarrow r_1,\ r_2,\ r_3 \leq r \Rightarrow \forall\, \tau \in \{t, m, q\} : r_1/\tau,\ r_2/\tau,\ r_3/\tau \leq r/\tau = r_p/\tau_p = c_\tau$. For example, the speeds of gravity and light waves are limited to the speed, c_t.

11. c_t is a component of the constants: $G = c_m c_t^2$, $k_e = c_q^2 c_t^2/c_m$, $\varepsilon_0 = 1/4\pi k_e = 1/4\pi(c_q^2 c_t^2/c_m)$, and $\hbar = k_m c_t$. The only constant, derived in this article, that does not contain c_t is vacuum permeability: $\mu_0 = 4\pi k_e/c_t^2 = 4\pi c_q^2/c_m$.

12. The fine structure electron coupling constant, α was derived from the ratio of two subtypes of charge force that reduces to the unit-less ratio $\alpha = q_e^2/q_p^2 \approx 0.0072973526$ (96), which is the empirical CODATA value [15].

    (a) The CODATA electron coupling version of the fine structure constant, α is defined as: $\alpha = q_e^2/4\pi\varepsilon_0\hbar c) = q_e^2/2\varepsilon_0 hc$ [15]. The following steps show that the CODATA definition reduces to the ratio-derived equation:

$$\varepsilon_0 := 1/4\pi k_e = 1/(4\pi(c_q^2 c_t^2/c_m) \quad \wedge \quad \hbar = k_m c_t \quad \wedge \quad h = 2\pi\hbar$$
$$\Rightarrow \quad \varepsilon_0 hc = 2\pi k_m c_t^2/(4\pi(c_q^2/c_m)c_t^2) = k_m/(2(c_q^2/c_m))$$
$$= m_p r_p/(2((r_p/q_p)^2/(r_p/m_p))) = q_p^2/2. \quad (149)$$

$$\alpha = q_e^2/2\varepsilon_0 hc \quad \wedge \quad \varepsilon_0 hc = q_p^2/2 \quad \Rightarrow \quad \alpha = q_e^2/2(q_p^2/2) = q_e^2/q_p^2. \quad (150)$$

    (b) The fine structure electron coupling constant is the ratio of electron static charge force to the propagating quantum charge (photon/electromagnetic) wave force, caused by a moving charged particle.

    (c) The gravitational fine structure electron coupling constant, α_G, can be expressed as the ratio of two subtypes of Newton's gravity force, where m_e is the rest electron mass and m_p is the reduced Planck mass unit: $\alpha_G = m_e^2/m_p^2$.

    The currently used gravitational fine structure constant, $\alpha_G = Gm_e^2/\hbar c = (c_m c_t^2)m_e^2/(k_m c_t)c_t = c_m m_e^2/k_m = (r_p/m_p)m_e^2/m_p r_p = m_e^2/m_p^2$.

13. The following will show that the NIST/CODATA standard definition of $\mu_0 = 2h\alpha/cq_e^2$ [15] and the derivation of $\mu_0 = 4\pi k_e/c_t^2 = 1/\varepsilon c^2$, in this article, are equivalent. Using the ratios to reduce the CODATA definition:

$$h = 2\pi k_m c_t, \;\; \alpha = q_e^q/q_p^2 \;\; \Rightarrow: \;\; \mu_0 = 2h\alpha/cq_e^2 = 4\pi k_m c_t(q_e^2/q_p^2)/c_t q_e^2 = 4\pi k_m/q_p^2. \quad (151)$$

    Using the ratios to reduce the definition of μ_0 derived in this article:

$$\mu_0 = 4\pi k_e/c_t^2 = 4\pi c_q^2/c_m = 4\pi(r_p/q_p)^2/(r_p/m_p) = 4\pi m_p r_p/q_p^2 = 4\pi k_m/q_p^2. \quad (152)$$

    Both equations reducing to the same base equation shows that the old definition, $\mu_0 = 4\pi k_e/c_t^2 = 1/\varepsilon c^2$, which differs slightly in value from the NIST/CODATA definition, $\mu_0 = 2h\alpha/cq_e^2$, is due to differences in measurement methods.

14. Empirical and hypothesized laws of physics use an *opaque* constant to make the units in an equation balance. The opacity has led to the *incorrect* assumptions of those constants being fundamental (atomic) constants.

    In this article, some opaque constants are derived directly from (composed of) the ratios: gravity, $G = c_m c_t^2$ (58), charge, $k_e = c_q^2 c_t^2 / c_m$ (77), and reduced Planck $\hbar = k_m c_t$ (100). $\varepsilon_0 = 1/4\pi k_e = 1/4\pi c_m/((c_q^2/c_m)c_t^2)$ (82) and $\mu_0 = 4\pi k_e/c_t^2 = 4\pi c_q^2/c_m$ (88).

    And the quantum extensions to: Schwarzschild's gravitational time dilation (137) Newton's gravity force (141), and Coulomb's charge force show, that where the quantum effects become measurable, the constants G, k_e, ε_0, and μ_0 no longer exist (are no longer valid).

    Therefore, G, k_e, ε_0, and μ_0 are *not* fundamental constants.

15. Constants that use notions of temperature and pressure for example, the Boltzmann constant [15], are probably not possible to define solely in terms of Planck units.

16. The derivations of: $\nabla \cdot \mathbf{g} = -4\pi G\rho$ from $\mathbf{g} = 2\pi Gm/|\tilde{\mathbf{r}}|^2$ (59), and $\nabla \cdot \mathbf{E} = -\rho/\varepsilon_0$ from $\mathbf{E} = 2\pi k_e q/|\tilde{\mathbf{r}}|^2$ (82), show that the use of mass and charge density, ρ show that:

    (a) Mass and charge density as motivations are unnecessary complexities. Equation 56 used to derive Newton's law and equation 75 used to derive Coulomb's law relate linear acceleration, r/t^2 to mass and charge. It is small step to recognize Gauss's $\mathbf{g}$, Poisson's $-\nabla\Phi(\tilde{\mathbf{r}}, t)$, and Gauss's $\mathbf{E}$ relate the angular (orbital) acceleration, $2\pi r/t^2$, to mass and charge, where differentiation yields the gravity field and electric field laws.

        The derivations are much simpler and shorter using angular (orbital) acceleration, $2\pi r/t^2$. The orbit length, $2\pi r$, corresponds to the Gauss's line integral length of a curve circumscribing a sphere containing the mass or charge.

    (b) Mass and charge density obfuscates the pattern, $\nabla \cdot f(x, r) = -4\pi k_x x/|\tilde{\mathbf{r}}|^3$, being derived from the inverse square pattern, $f(x, r) = 2\pi k_x x/|\tilde{\mathbf{r}}|^2$. The constant, G (derived from the inverse square relation (56)), is used in Einstein's constant, $\kappa = 8\pi G/c^4$, and the use of energy density in the stress energy tensor, $T_{\mu,\nu}$, in Einstein's field equations [7] also obfuscate the inverse square pattern.

17. The derivation Lorentz's law here (86) differs from other relativistic derivations, in that equations for the magnetic field, $\mathbf{B}$, and magnetic permeability, μ_0, are derived in the process. The equation, $\mathbf{B} = (2\pi k_e/c^2)vq/|\vec{\mathbf{r}}|^2$, allows a simple derivation of the Maxwell-Faraday law (94).

18. Einstein's relativity equations: 1) assume the Lorentz transformations, 2) assume the laws of physics are same at each coordinate point, 3) assume the notion of light, and 4) assume that the speed of light is the same at each coordinate point [8][7].

    The derivations, in this article, were made without those assumptions (does not even require the notion of light). Assuming Euclidean space near each coordinate point creates unit ratios, where all equations (laws) derived from the unit ratios must be the same at each coordinate point.

19. Using the quantum units, r_p and t_p: r_p/t_p^2, suggests a maximum linear acceleration for masses. And $2\pi r_p/t_p^2$ suggests a maximum orbital or rotational acceleration.

20. The Einstein-Planck relations, $E_\omega = \hbar\omega\ kg\ m^2\ radian\ s^{-2}$ (105) and $E_f = hf\ kg\ m^2$ $cycle\ s^{-2}$ (106), were both derived from the base Einstein-Planck relation, $E = \hbar/t\ kg\ m^2\ s^{-2}$ (104). And the Schrödenger's wave equation (126) [17] and Dirac's wave equation (133) [18] were both derived from the base Einstein-Planck relation.

21. The quantum extensions to: Schwarzschild's gravitational time dilation (137), black hole metric (73), Newton's gravity force (141), and Coulomb's charge force (144) make quantifiable predictions:

    (a) For gravity, $\lim_{r\to 0} F = c^2 m_1 m_2/k_m$, and for charge, $\lim_{r\to 0} F = 0$. Finite maximum gravity and charge forces: 1) eliminates the problem of forces going to infinity as $r \to 0$, 2) allows radioactivity without the need for a weak force, 3) finite sloped energy well walls explains quantum tunneling, and 4) gravity is stronger at very large distances than predicted by classical and relativity equations.

    (b) The quantum-extended relativity and classic equations reduce to the non-extended relativity and classic equations and constants, where the distance between masses and charges is sufficiently large or the masses and charges sufficiently large that the quantum effects are not measurable. *Note* that G, k_e, ε_0,

μ_0, and κ (Einstein's constant, which contains G) do not exist (are not valid), where the quantum effects becomes measurable.

(c) The covariant tensor components, in Einstein's field equations, that had the units $1/distance^2$, will now have the more complex units, $1/\sqrt{(distance^4/c_\tau^2) + k_\tau^2}$, $\tau \in \{t, m\}$.

(d) $1/\sqrt{(distance^4/c_\tau^2) + k_\tau^2}$ implies that as distance $\to 0$, spacetime curvature peaks at the Planck unit distance, r_p (114). The ratio of quantum units, m_p/r_p^3, might indicate a maximum mass density. A finite force (spacetime curvature) and finite mass density would imply that black holes have sizes > 0 (are not singularities). Black hole evaporation might be possible. If there was a "big bang," then it might not have originated from a singularity.

[2] K. Schwarzschild, Über das gravitationsfeld eines massenpunktes nach der einsteinschen theorie, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften **7**, 198 (1916), https://archive.org/stream/sitzungsberichte1916deutsch #page/188/mode/2up.

[3] S. Antoci and A. Loinger, An english translation of k. schwarzschild's 1916 article: On the gravitational field of a mass point according to einstein's theory, ARXIV.org (1999), https://arxiv.org/abs/physics/9905030.

[4] I. Newton, *The Mathematical Principles of Natural Philosophy* (Patristic Publishing, 2019) reprint.

[5] R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics* (California Institute of Technology, 2010) https://www.feynmanlectures.caltech.edu/.

[6] A. Einstein, Die grundlage der allgemeinen relativitätstheorie (the foundation of the general theory of relativity), Annalen der Physik (1916).

[7] H. Weyl, *Space-Time-Matter* (Dover Publications Inc, 1952).

[8] A. Einstein, *Relativity, The Special and General Theory* (Princeton University Press, 2015).

[9] R. R. Goldberg, *Methods of Real Analysis* (John Wiley and Sons, 1976).

[10] W. Rudin, *Principles of Mathematical Analysis* (McGraw Hill Education, 1976).

[11] W. Conradie and V. Goranko, *Logic and Discrete Mathematics* (Wiley, 2015).

[12] X. P. N. Guigui, N. Miolane, Introduction to riemannian geometry and geometric statistics: from basic theory to implementation with geomstats, Foundations and Trends in Machine Learning **16**, 182 (2023), https://inria.hal.science/hal-03766900v2, hal-00644459.

[13] Rocq, Rocq proof assistant (2025), https://rocq-prover.org/docs.

[14] H. Minkowski, *Geometrie der Zahlen* (Chelsea, 1953) reprint.

[15] CODATA, Physical constant values (2022), https://physics.nist.gov/cuu/Constants/index.html.

[16] M. Jain, *Quantum Mechanics: A Textbook for Undergraduates* (PHI Learning Private Limited, New Delhi, India, 2011).

[17] E. Schrödenger, *Collected Papers On Wave Mechanics* (Minkowski Institute Press, 2020).

[18] P. Dirac, *The Principles of Quantum Mechanics* (Snowball Publishing, 1957).

[19] D. E. Joyce, Euclid's elements (1998), http://aleph0.clarku.edu/ djoyce/java/elements/elements .html.

[20] E. S. Loomis, *The Pythagorean Proposition* (NCTM, 1968).

[21] G. D. Birkhoff, A set of postulates for plane geometry (based on scale and protractors), Annals of Mathematics **33** (1932).

[22] D. Hilbert, *The Foundations of Geometry (2cd Ed)* (Chicago: Open Court, 1980) http://www.gutenberg.org/ebooks/17384.

[23] J. M. Lee, *Axiomatic Geometry* (American Mathematical Society, 2010).

[24] A. Tarski and S. Givant, Tarski's system of geometry, The Bulletin of Symbolic Logic **5**, 175 (1999).

[25] O. Veblen, A system of axioms for geometry, Trans. Amer. Math. Soc **5**, 343 (1904), https://doi.org/10.1090/S0002-9947-1904-1500678-X.

[26] A. Bogomolny, *Pythagorean Theorem* (Interactive Mathematics Miscellany and Puzzles, 2010) http://www.cut-the-knot.org/pythagoras/ CalculusProofCorrectedVersion.shtml.

[27] J. Zimba, On the possibility of trigonometric proofs of the pythagorean theorem, Forum Geometricorum **9**, 275 (2009).

[28] B. C. Berndt, Ramanujan-100 years old (fashioned) or 100 years new (fangled)?, The Mathematical Intelligencer **10** (1988).

[29] M. Staring, The pythagorean proposition a proof by means of calculus, Mathematics Magazine **69**, 45 (1996).