The Countable Set Mappings Generating Geometry

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ABSTRACT. Countable distance is a function of the number of domain-to-range set mappings. Countable volume is a function of the number of range-to-range (distance) set mappings. The countable distance and volume mappings between sets of size c subintervals of domain and range intervals generate the properties of metric space, the Lp norms (for example, Manhattan and Euclidean distance), and the volume equation as c goes to 0. The volume proof is used to derive Coulomb's charge force and Newton's gravity force equations without using other laws of physics or Gauss's divergence theorem. A symmetry constraint on a strict totally ordered set limits the set to at most 3 members, for example, 3 dimensions of space. All proofs are verified in Coq.

CONTENTS

1. Introduction

The definitions of metric space, Euclidean distance, and Euclidean area/volume in mathematical analysis [**Gol76**] [**Rud76**] are motivated by Euclidean geometry [**Joy98**] rather than derived from an abstract set and limit-based foundation. An abstract set and limit-based foundation exposes: the constraint between countable domain and range sets that makes a space flat or curved; the countable domain-to-range set mapping that makes Euclidean distance the smallest possible distance between two distinct points in flat space; the set operation and constraint generating the properties of metric space; etc.

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As an easy, low-detail introduction, each disjoint, countable domain set, x_i, has a corresponding range (distance) set, y_i, where abstract, countable distance is defined as the cardinal of the union of the range sets: $d_c = |\bigcup_{i=1}^{n} y_i|$. As the intersection of the range sets increases, more domain set members can map to a single range set member. Therefore, the cardinal of the union range set, $d_c = |\bigcup_{i=1}^{n} y_i|$, is a function of the number of domain-to-range set mappings.

And because area/volume is always calculated as a function of distances, abstract, countable area/volume is a function of the number of range-to-range (distance) set mappings, which, by the rule of product, the largest possible number of mappings is the Cartesian product of the number of members in each disjoint, range (distance) set: $v_c = | \times_{i=1}^{n} y_i|$.

The number of mappings between sets of same-sized, size c, subintervals of domain and range intervals generate the properties of metric space, the absolute value metric, all L_p norms (Minkowski distances, for example, Manhattan and Euclidean distance), and the volume equation as $c \to 0$. Some applications to physics are shown.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq15**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: https://github.com/treeck/RASRGeometry.

2. Ruler measure and convergence

In this article, geometric relations are derived from the number of correspondences (mappings) between the set of size c subintervals in each interval (as $c \to 0$). A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals, where each subinterval has the *same* size, c. The ruler makes it easy to derive geometric relations from the number of possible mappings between the p_x number of subintervals in one interval and the p_y number of subintervals in another interval.

Definition 2.1. Ruler measure, M: $\forall\, c,\ s \in \mathbb{R},\ [a, b] \subset \mathbb{R},\ s = b-a \wedge c > 0 \wedge (p = floor(s/c) \ \vee\ p = ceiling(s/c)) \ \wedge\ M = \sum_{i=1}^{p} c = pc.$

Theorem 2.2. *Ruler convergence:* $M = \lim_{c\to 0} pc = s.$

The proof is trivial but is included here for completeness. The theorem, "limit_c_0_M_eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

Proof. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y :\ y \le x,\ y \in \mathbb{Z},\ x \in \mathbb{R}\})$:

$$\forall\, c > 0,\ p = floor(s/c)\ \wedge\ 0 \le |floor(s/c) - s/c| < 1\ \Rightarrow\ 0 \le |p - s/c| < 1. \tag{2.1}$$

Multiply all sides of inequality 2.1 by c:

$$\forall\, c > 0, \quad 0 \le |p - s/c| < 1 \quad \Rightarrow \quad 0 \le |pc - s| < |c|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \delta\ :\ |pc - s| < |c| = |c - 0| < \delta \\ &\qquad \Rightarrow\quad \forall\, \epsilon = \delta :\ |c - 0| < \delta\ \wedge\ |pc - s| < \epsilon\ :=\ M = \lim_{c\to 0} pc = s. \quad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}},\ 3.14_{c=10^{-2}},\ 3.141_{c=10^{-3}}, ..., \pi_{\lim_{c\to 0}}.$

3. Distance

Notation conventions: Vertical bars around a set, $|\{\cdots\}|$, or list, $|[\cdots]|$, indicates the cardinal (the number of members in the set or list).

3.1. Countable distance. Distance in one dimension is independent of distance in every other other dimension. Therefore, each disjoint domain set, x_i, has its own independent range (distance) set, y_i. The countable distance spanning the disjoint domain sets is the cardinal, d_c, of the union range (distance) set.

It will be shown in the next subsections that the constraint, $|x_i| = |y_i|$, generates Manhattan and Euclidean distance at the boundaries (generates flat space/rectilinear distances). Generalizing distance and volume beyond flat space is shown in the last section of this article.

DEFINITION 3.1. Countable distance, d_c, in flat space:

$$d_c = |\bigcup_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i|.$$

3.2. Union-Sum Inequality. The inequality, $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$, is used often in this article. The proof is trivial but is included here for completeness.

The proof follows from the associative law of addition where the sum of set sizes is equal to the size of all the set members appended into a list and the commutative law of addition that allows sorting that list into a list of unique members (the *union* set) and a list of duplicates. For example, $y_1 = \{a, b, c\}$ and $y_2 = \{c, d, e\} \Rightarrow \bigcup_{i=1}^{2} |y_i| = |\{a, b, c, d, e\}| = 5 < \sum_{i=1}^{2} |y_i| = |[a, b, c, c, d, e]| = 6$.

LEMMA 3.2. *Union-Sum Inequality:* $|\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|$.

PROOF. A formal proof, union_sum_inequality, using sorting into a set of unique members (*union* set) and a list of duplicates, is in the file euclidrelations.v.

$$\sum_{i=1}^{n} |y_i| = |append_{i=1}^{n} y_i| = |sort(append_{i=1}^{n} y_i)| \\ = |\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i|. \tag{3.1}$$

$$|\bigcup_{i=1}^{n} y_i| + |duplicates_{i=1}^{n} y_i| = \sum_{i=1}^{n} |y_i| \quad \wedge \quad |duplicates_{i=1}^{n} y_i| \geq 0 \\ \Rightarrow \quad |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i|. \quad \square \tag{3.2}$$

3.3. Countable distance range. From the countable distance definition (3.1), $d_c = |\bigcup_{i=1}^{n} y_i|$, as the amount of intersection increases, more domain set members can map to a single range set member. Therefore, the countable distance, d_c, is a function of the total number of domain-to-range set member mappings.

Each domain set, x_i has its own independent range set, y_i. From the countable distance constraint (3.1), where $|x_i| = |y_i| = p_i$, the countable distance, d_c, ranges from a function of the sum of 1-1 correspondence mappings, $d_c = \mathrm{f}(\sum_{i=1}^{n}(1 \cdot |y_i|)) = \mathrm{f}(\sum_{i=1}^{n} p_i)$, to a function of the sum of all-to-each (Cartesian product) mappings, $d_c = \mathrm{f}(\sum_{i=1}^{n}(|x_i| \cdot |y_i|)) = \mathrm{f}(\sum_{i=1}^{n} p_i^2)$.

Applying the ruler (2.1) and ruler convergence theorem (2.2) to the smallest and largest total number of domain-to-range set mapping cases converges to the real-valued Manhattan and Euclidean distance relations.

3.4. Manhattan distance.

THEOREM 3.3. *Manhattan (largest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$d = \sum_{i=1}^{n} s_i, \quad d = d_m - d_0, \quad s_i = b_i - a_i.$$

The formal proof, "taxicab_distance," is in the Coq file, euclidrelations.v.

PROOF.
From the countable distance definition (3.1) and the union-sum inequality (3.2), the largest possible countable distance, d_c, is the equality case:

$$(3.3)\quad d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \quad \Rightarrow \quad d_c \leq \sum_{i=1}^{n} p_i$$
$$\Rightarrow \quad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i.$$

Multiply both sides of equation 3.3 by c and take the limit:

$$(3.4)\quad d_c = \sum_{i=1}^{n} p_i \Rightarrow d_c \cdot c = \sum_{i=1}^{n} (p_i \cdot c) \Rightarrow \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c).$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to the definition of d:

$$(3.5)\qquad d = d_m - d_0 \quad \Rightarrow \quad \exists\, c\, d :\ floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c.$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$(3.6)\quad s_i = b_i - a_i \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i.$$

Combine equations 3.5, 3.4, 3.6:

$$(3.7)\quad d = \lim_{c\to 0} d_c \cdot c \quad \wedge \quad \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) \quad \wedge$$
$$\lim_{c\to 0} (p_i \cdot c) = s_i \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c = \sum_{i=1}^{n} \lim_{c\to 0} (p_i \cdot c) = \sum_{i=1}^{n} s_i. \quad \square$$

3.5. Euclidean distance.

THEOREM 3.4. *Euclidean (smallest) distance, d, is the size of the range interval, $[d_0, d_m]$, corresponding to a set of disjoint domain intervals,*
$\{[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]\}$, *where:*

$$d^2 = \sum_{i=1}^{n} s_i^2, \quad d = d_m - d_0, \quad s_i = b_i - a_i.$$

The formal proof, "Euclidean_distance," is in the Coq file, euclidrelations.v.

PROOF.
Apply the rule of product to the largest number of domain-to-range set mappings, where all p_i number of range set members, y_i, map to each of the p_i number of members in the domain set, x_i, which, by the rule of product, is the Cartesian product, $|y_i| \cdot |x_i|$:

$$(3.8)\qquad |x_i| = |y_i| = p_i \quad \Rightarrow \quad \sum_{i=1}^{n} |y_i| \cdot |x_i| = \sum_{i=1}^{n} p_i^2.$$

From the countable distance definition (3.1) and the union-sum inequality (3.2), choose the equality case:

$$(3.9)\quad d_c = |\bigcup_{i=1}^{n} y_i| \leq \sum_{i=1}^{n} |y_i| \quad \wedge \quad |x_i| = |y_i| = p_i \quad \Rightarrow \quad d_c \leq \sum_{i=1}^{n} p_i$$
$$\Rightarrow \quad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i.$$

Square both sides of equation 3.9 $(x = y \Leftrightarrow f(x) = f(y))$:

$$(3.10)\qquad \exists\, p_i,\ d_c :\ d_c = \sum_{i=1}^{n} p_i \quad \Leftrightarrow \quad \exists\, p_i,\ d_c :\ d_c^2 = (\sum_{i=1}^{n} p_i)^2.$$

Apply the square of sum inequality, $(\sum_{i=1}^{n} p_i)^2 \geq \sum_{i=1}^{n} p_i^2$, to equation 3.10 and select the smallest area (the equality) case:

$$d_c^2 = (\textstyle\sum_{i=1}^{n} p_i)^2 = \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j$$
$$= \textstyle\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n} p_i \sum_{j=1, j\neq i}^{n} p_j \geq \sum_{i=1}^{n} p_i^2 \quad \Rightarrow \quad \exists\, p_i : d_c^2 = \sum_{i=1}^{n} p_i^2. \tag{3.11}$$

Multiply both sides of equation 3.11 by c^2, simplify, and take the limit.

$$d_c^2 = \textstyle\sum_{i=1}^{n} p_i^2 \;\Rightarrow\; d_c^2 \cdot c^2 = \sum_{i=1}^{n} p_i^2 \cdot c^2 \;\Leftrightarrow\; (d_c \cdot c)^2 = \sum_{i=1}^{n} (p_i \cdot c)^2$$
$$\Rightarrow \quad \lim_{c\to 0}(d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2. \tag{3.12}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) and square both sides:

$$\exists\, c\, d \in \mathbb{R} : \; floor(d/c) = d_c \quad \Rightarrow \quad d = \lim_{c\to 0} d_c \cdot c \quad \Rightarrow \quad d^2 = \lim_{c\to 0} (d_c \cdot c)^2. \tag{3.13}$$

Apply the ruler (2.1) and ruler convergence theorem (2.2) to each domain interval:

$$s_i = b_i - a_i \quad \wedge \quad floor(s_i/c) = |x_i| = |y_i| = p_i \quad \Rightarrow \quad \lim_{c\to 0} p_i \cdot c = s_i. \tag{3.14}$$

Combine equations 3.13, 3.12, 3.14:

$$d^2 = \lim_{c\to 0}(d_c \cdot c)^2 \quad \wedge \quad \lim_{c\to 0}(d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 \quad \wedge$$
$$\lim_{c\to 0}(p_i \cdot c) = s_i \quad \Rightarrow \quad d^2 = \lim_{c\to 0}(d_c \cdot c)^2 = \textstyle\sum_{i=1}^{n} \lim_{c\to 0}(p_i \cdot c)^2 = \sum_{i=1}^{n} s_i^2. \quad \square \tag{3.15}$$

3.6. Metric Space. All distances, $d(u, w)$, satisfying the countable distance definition (3.1), where the ruler is applied, generates the properties of metric space. The formal proofs: triangle_inequality, non_negativity, identity_of_ indiscernibles, and symmetry are in the Coq file, euclidrelations.v.

THEOREM 3.5. *Triangle Inequality:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \leq d(u, v) + d(v, w)$.

PROOF. Apply the ruler measure (2.1), the countable distance condition (3.1), union-sum inequality (3.2), and then ruler convergence (2.2).

$$\begin{aligned}
&\forall\, c > 0,\; d(u, w),\; d(u, v),\; d(v, w) \;: \\
&\qquad |y_1| = floor(d(u, v)/c) \;\wedge\; |y_2| = floor(d(v, w)/c) \;\wedge \\
&\qquad d_c = floor(d(u, w)/c) \;\wedge\; d_c = |y_1 \cup y_2| \leq |y_1| + |y_2| \\
&\qquad \Rightarrow floor(d(u, w)/c) \leq floor(d(u, v)/c) + floor(d(v, w)/c) \\
&\qquad \Rightarrow floor(d(u, w)/c) \cdot c \leq floor(d(u, v)/c) \cdot c + floor(d(v, w)/c) \cdot c \\
&\Rightarrow \lim_{c\to 0} floor(d(u, w)/c) \cdot c \leq \lim_{c\to 0} floor(d(u, v)/c) \cdot c + \lim_{c\to 0} floor(d(v, w)/c) \cdot c \\
&\qquad \Rightarrow d(u, w) \leq d(u, v) + d(v, w). \quad \square
\end{aligned} \tag{3.16}$$

THEOREM 3.6. *Non-negativity:* $d_c = |y_1 \cup y_2| \Rightarrow d(u, w) \geq 0$.

PROOF. By definition, a set always has a size (cardinal) ≥ 0:

$$\begin{aligned}
&\forall\, c > 0,\; d(u, w) \;: \quad floor(d(u, w)/c) = d_c \quad \wedge \quad d_c = |y_1 \cup y_2| \geq 0 \\
&\qquad \Rightarrow \quad floor(d(u, w)/c) = d_c \geq 0 \quad \Rightarrow \quad d(u, w) = \lim_{c\to 0} d_c \cdot c \geq 0. \quad \square
\end{aligned} \tag{3.17}$$

THEOREM 3.7. *Identity of Indiscernibles:* $d(w, w) = 0$.

PROOF. Apply the triangle inequality property (3.5):

$$\forall\, d(u, v) = d(v, w) = 0 \;\wedge\; d(u, w) \leq d(u, v) + d(v, w) \;\Rightarrow\; d(u, w) \leq 0. \tag{3.18}$$

Combine the non-negativity property (3.6) and the previous inequality (3.18):

$$d(u,w) \geq 0 \ \wedge \ d(u,w) \leq 0 \ \Leftrightarrow \ 0 \leq d(u,w) \leq 0 \ \Rightarrow \ d(u,w) = 0. \tag{3.19}$$

Combine the result of step 3.19 and the condition, $d(u,v) = 0$, in step 3.18.

$$d(u,w) = 0 \ \wedge \ d(u,v) = 0 \ \Rightarrow \ w = v. \tag{3.20}$$

Combine the condition, $d(v,w) = 0$, in step 3.18 and the result of step 3.20.

$$d(v,w) = 0 \ \wedge \ w = v \ \Rightarrow \ d(w,w) = 0. \tag{3.21}$$

□

THEOREM 3.8. *Symmetry:* $|x_i| = |y_i| \Rightarrow d(u,v) = d(v,u)$.

PROOF.
The range of countable distances (3.3) is a function of domain-to-range set members, under the constraint, $|x_i| = |y_i|$, is:

$$|x_i| = |y_i| = p_i \quad \Rightarrow \quad d_c = \mathrm{f}(\textstyle\sum_{i=1}^{n} |x_i| \cdot |y_i|^q) = \mathrm{f}(\sum_{i=1}^{n} p_i^{1+q}), \ q \in \{0, \ 1\}. \tag{3.22}$$

Applying the ruler to real-valued domain and range intervals, where s_i is the size of domain interval, $[a_i, b_i]$, generates the range of distances from Manhattan distance (3.3), $d(x,y) = \mathrm{f}(\sum_{i=1}^{n} s_i^1)$, to Euclidean distance (3.4), $d(x,y) = \mathrm{f}(\sum_{i=1}^{n} s_i^2)$. Generalizing:

$$\forall\, p \ : \ p \geq 0, \quad d(x,y) = \mathrm{f}(\textstyle\sum_{i=1}^{n} s_i^p). \tag{3.23}$$

From the previous Manhattan and Euclidean distance proofs, distance is a function of domain interval sizes, s_i. For all $x, y \in \mathbb{R}$, there are two domain interval cases:

Case #1: $n = 1$ domain interval: In this case, $d(x,y)$ is the distance from domain value x to domain value y in $\mathbb{R}$, which make x and y the boundary values of a domain interval having size, $s : s = |x - y|$. And applying equation 3.23, yields the absolute value metric:

$$d(x,y) = \mathrm{f}(s^p) = \mathrm{f}(|x-y|^p) \Rightarrow d(u,v) = \mathrm{f}(|u-v|^p) = \mathrm{f}(|v-u|^p) = d(v,u). \tag{3.24}$$

Case #2: $n = 2$ domain intervals: In this case, $d(x,y)$ is the distance, where x and y are the sizes of two domain intervals, $[a_1, b_1|$ and $[a_2, b_2]$: $s_1 = x = |a_1 - b_1|$ and $s_2 = y = |a_2 - b_2|$.

$$\begin{aligned} d(x,y) &= \mathrm{f}(s_1^p + s_2^p) = \mathrm{f}(x^p + y^p) \\ &\Rightarrow \quad d(u,v) = \mathrm{f}(u^p + v^p) = \mathrm{f}(v^p + u^p) = d(v,u). \end{aligned} \tag{3.25}$$

□

4. Euclidean Volume

$\mathbb{R}^n$, the Lebesgue measure, Riemann integral, and Lebesgue integral define (assume) area/volume to be the product of domain interval lengths. The goal here is to derive the area/volume equation from an abstract, set-based definition of volume without assuming the product of interval lengths.

Area/volume is always calculated as a function of distances. Therefore, an abstract, countable area/volume is a function of the number of range-to-range (distance) set mappings, which, by the rule of product, the largest number of mappings is the Cartesian product of the number of members in each disjoint, range set:

DEFINITION 4.1. Euclidean (largest possible) Countable Volume , v_c, is the cardinal of the set of n-tuples of members from countable, disjoint, range sets:

$$v_c = |\times_{i=1}^{n} y_i| : \quad \textstyle\bigcap_{i=1}^{n} x_i = \bigcap_{i=1}^{n} y_i = \emptyset \quad \wedge \quad |x_i| = |y_i|.$$

THEOREM 4.2. *Euclidean volume, v, is length of the range interval, $[v_0, v_m]$, equal to product of domain interval lengths, $\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$:*

$$v = \prod_{i=1}^n s_i, \ \ v = v_m - v_0, \ \ s_i = b_i - a_i.$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.
Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i, containing p_i number of subintervals.

$$\forall\, i\, n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad floor(s_i/c) = p_i = |x_i| = |y_i|. \tag{4.1}$$

Apply the ruler convergence theorem (2.2) to equation 4.1:

$$floor(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c\to 0}(p_i \cdot c) = s_i. \tag{4.2}$$

Apply the associative law of multiplication to derive the countable volume (4.1) in terms of p_i:

$$v_c = |\times_{i=1}^n y_i| = \prod_{i=1}^n |y_i| \quad \wedge \quad |y_i| = p_i \quad \Rightarrow \quad v_c = \prod_{i=1}^n p_i. \tag{4.3}$$

Multiply both sides of equation 4.3 by c^n:

$$v_c \cdot c^n = (\prod_{i=1}^n p_i) \cdot c^n = \prod_{i=1}^n (p_i \cdot c). \tag{4.4}$$

$$\forall\, n, v_c\ \in \mathbb{N}\, \exists\, p \in \mathbb{R}\ :\ p^n = v_c \ \Rightarrow\ v_c \cdot c^n = p^n \cdot c^n = (p \cdot c)^n = \prod_{i=1}^n (p_i \cdot c). \tag{4.5}$$

Apply the ruler (2.1) and ruler convergence (2.2) to the range interval, $[v_0, v_m]$ (where $v = v_m - v_0$), and then combine with equations 4.5 and 4.2:

$$floor(v/c^n) = p^n \Rightarrow v = \lim_{c\to 0}(p \cdot c)^n = \prod_{i=1}^n \lim_{c\to 0}(p_i \cdot c) = \prod_{i=1}^n s_i. \tag{4.6}$$ □

5. Applications to physics

5.1. Coulomb's charge force. q_1 and q_2, are the sizes of two independent charge intervals, where each infinitesimal size c subinterval of a charge interval exerts a quantum force, $m_C a_C$, on each size c subinterval of the other charge interval. The total force, F, is proportionate to the total number of forces, which is the Cartesian product of the infinitesimal size c components multiplied times the quantum charge force, $m_C a_C$. Applying the ruler, $p_1 = floor(q_1/c)$ and $p_2 = floor(q_2/c)$, and the Cartesian product, $p_1 \times p_2$, of size c components yields:

$$F \propto m_C a_C (\lim_{c\to 0} p_1 c \cdot \lim_{c\to 0} p_2 c) = m_C a_C \int_0^{q_2} \int_0^{q_1} \mathrm{d}^2 c = m_C a_C (q_1 q_2). \tag{5.1}$$

For every size, q, of a charge interval, another interval having size, r, can be defined, where $r : \ q = (q_C/r_C) r$ and q_C/r_C is a unit-factoring conversion ratio.

$$\forall\, q_1, q_2 \geq 0\ \exists\, q \in \mathbb{R}\ :\ q^2 = q_1 q_2\ \wedge\ (q_C/r_C) r = q\ \Rightarrow\ ((q_C/r_C) r)^2 = q_1 q_2. \tag{5.2}$$

$$\begin{aligned} &((q_C/r_C) r)^2 = q_1 q_2 \ \wedge\ F \propto m_C a_C (q_1 q_2) \\ &\qquad \Rightarrow \quad F \propto m_C a_C ((q_C/r_C) r)^2 = m_C a_C (q_1 q_2) \\ &\qquad\qquad \Rightarrow \quad F = m_C a_C = (m_C a_C r_C^2/q_C^2) q_1 q_2/r^2 = k_c q_1 q_2/r^2. \end{aligned} \tag{5.3}$$

where $k_C = m_C a_C r_C^2/q_C^2$ corresponds to the SI units: Nm^2C^{-2}. And multiplying both sides of equation 5.3 by xy:

$$\forall\ x,\ y \geq 1,\ m_0 = x \cdot m_C,\ a = y \cdot a_C,\ d = r/(x \cdot y) \Rightarrow F = m_0 a = k_c q_1 q_2/d^2. \tag{5.4}$$

5.2. Newton's gravity force equation. m_1 and m_2, are the sizes of two independent mass intervals, where each infinitesimal size c subinterval of a mass interval exerts a quantum force, $m_G a_G$, on each size c subinterval of the other mass interval. The total force, F, is proportionate to the total number of forces, which is the Cartesian product of the size c components) multiplied times the quantum gravity force, $m_G a_G$. Applying the ruler, $p_1 = floor(m_1/c)$ and $p_2 = floor(m_2/c)$, and the Cartesian product, $p_1 \times p_2$, of size c components yields:

$$F \propto m_G a_G(\lim_{c\to 0} p_1 c \cdot \lim_{c\to 0} p_2 c) = m_G a_G \int_0^{m_2}\int_0^{m_1} \mathrm{d}^2 c = m_G a_G(m_1 m_2). \tag{5.5}$$

For every size, m, of a mass interval, another interval having size, r, can be defined, where $r: \ m = (m_G/r_G)r$ and m_G/r_G is a unit-factoring conversion ratio.

$$\begin{aligned} \forall\, m_1, m_2 \geq 0\, \exists\, m \in \mathbb{R} : \quad m^2 = m_1 m_2 \quad \wedge \quad (m_G/r_G)r = m \\ \Rightarrow \quad ((m_G/r_G)r)^2 = m_1 m_2. \end{aligned} \tag{5.6}$$

$$\begin{aligned} ((m_G/r_G)r)^2 = m_1 m_2 \ \wedge \ F \propto m_G a_G(m_1 m_2) \\ \Rightarrow \quad F \propto m_G a_G((m_G/r_G)r)^2 = m_G a_G(m_1 m_2) \\ \Rightarrow \quad F = m_G a_G = (a_G r_G^2/m_G) m_1 m_2/r^2. \end{aligned} \tag{5.7}$$

$$\begin{aligned} \exists\, t_G \in \mathbb{R}: \ r_G/t_G^2 = a_G \quad \wedge \quad F = (a_G r_G^2/m_G) m_1 m_2/r^2 \\ \Rightarrow \quad F = (r_G^3/m_G t_G^2) m_1 m_2/r^2 = G m_1 m_2/r^2, \end{aligned} \tag{5.8}$$

where $G = r_G^3/m_G t_G^2$ corresponds to the SI units: $m^3 kg^{-1} s^{-2}$. And multiplying both sides of equation 5.8 by xy:

$$\forall\, x,\ y \geq 1,\, m_0 = x \cdot m_G,\, a = y \cdot a_G,\, d = r/(x \cdot y) \Rightarrow F = m_0 a = G m_1 m_2/d^2. \tag{5.9}$$

5.3. Spacetime equations. For any Euclidean distance interval having size, r, a interval having size, t, can be defined, where $r = (r_c/t_c)t = ct$, and $r_c/t_c = c$ is a unit-factoring conversion ratio.

Applying the ruler to two intervals, $[0, d_1]$ and $[0, d_2]$, in two inertial (independent, non-accelerating) frames of reference, the distance (and time) spanning the two domain intervals converges to a range of distances (and times) from Manhattan (3.3) to Euclidean distance (3.4).

$$\begin{aligned} r^2 = d_1^2 + d_2^2 \quad \wedge \quad r = (r_c/t_c)t = ct \quad \Rightarrow \quad (ct)^2 = d_1^2 + d_2^2 \\ \Leftrightarrow \quad d_1^2 = (ct)^2 - (x^2 + y^2 + z^2), \end{aligned} \tag{5.10}$$

where $d_2^2 = x^2 + y^2 + z^2$, which is one form of Minkowski's well-known flat spacetime interval equation [**Bru17**]. And, the time dilation and length contraction equations also follow directly by dividing both sides of $(ct)^2 = d_1^2 + d_2^2$ by t^2 and using $v = d/t$.

5.4. 3 dimensional balls. Countable distance, $d_c = |\bigcup_{i=1}^n y_i|$, (3.1), countable volume, $d_c = |\times_{i=1}^n y_i|$, (4.1), Manhattan distance (3.3), Euclidean distance (3.4), and volume (4.2) requires that a set of intervals/dimensions can be assigned a *strict total order* (i = 1 to n). And the commutative properties of union, addition, and multiplication allow sequencing through each interval (dimension) in every possible order. **Note** that "jumping" from member 1 to member m of a set requires calculating an offset that is an implicit traversal of successor/predecessor relations. Therefore, "strict" sequencing (no jumping over other members) via the successor and predecessor relations of a strict totally ordered set in every possible order requires each set member to be sequentially adjacent (either a successor or predecessor) to every other member, herein referred to as a symmetric geometry.

It will now be proved that the constraint (coexistence) of symmetry on a sequentially ordered set defines a cyclic set that contains at most 3 members, in this case, 3 dimensions of ordered and symmetric distance and volume. If there are higher dimension of space, then the cyclic property prevents sequencing from the 3 lower, cyclic set of dimensions to any higher dimensions.

Definition 5.1. Ordered geometry:

$$\forall\, i\; n \in \mathbb{N},\;\; i \in [1, n-1],\;\; \forall\, x_i\; \in \{x_1, \ldots, x_n\},$$
$$\mathit{successor}\; x_i = x_{i+1} \;\;\wedge\;\; \mathit{predecessor}\; x_{i+1} = x_i.$$

Definition 5.2. Symmetric geometry (every set member is sequentially adjacent to every other member):

$$\forall\, i\; j\; n \in \mathbb{N},\;\; \forall\, x_i\; x_j \in \{x_1, \ldots, x_n\},\;\; \mathit{successor}\; x_i = x_j \;\;\Leftrightarrow\;\; \mathit{predecessor}\; x_j = x_i.$$

Theorem 5.3. *An ordered and symmetric set is a cyclic set.*

$$i = n \;\;\wedge\;\; j = 1 \;\;\Rightarrow\;\; \mathit{successor}\; x_n = x_1 \;\;\wedge\;\; \mathit{predecessor}\; x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

Proof. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. From the properties of a symmetric geometry (5.2):

$$i = n \;\;\wedge\;\; j = 1 \;\;\wedge\;\; \mathit{successor}\; x_i = x_j \;\;\Rightarrow\;\; \mathit{successor}\; x_n = x_1. \tag{5.11}$$

Applying the definition of a symmetric geometry (5.2) to conclusion 5.11:

$$\mathit{successor}\; x_i = x_j \;\;\Rightarrow\;\; \mathit{predecessor}\; x_j = x_i \;\;\Rightarrow\;\; \mathit{predecessor}\; x_1 = x_n. \tag{5.12}$$ □

Theorem 5.4. *An ordered and symmetric set is limited to at most 3 members.*

The lemmas and formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first-order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

Proof.
It was proved that an ordered and symmetric set is a cyclic set (5.3). In other words, the successors and predecessors of an ordered and symmetric set are cyclic:

Definition 5.5. Cyclic successor of m is n:

$$\mathit{Successor}(m, n, \mathit{setsize}) \leftarrow (m = \mathit{setsize} \wedge n = 1) \vee (n = m + 1 \leq \mathit{setsize}). \tag{5.13}$$

Definition 5.6. Cyclic predecessor of m is n:

$$\mathit{Predecessor}(m, n, \mathit{setsize}) \leftarrow (m = 1 \wedge n = \mathit{setsize}) \vee (n = m - q \geq 1). \tag{5.14}$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the cyclic successor of m is n or the cyclic predecessor of m is n. Notionally:
(5.15)
$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize).$$

Prove that every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \tag{5.16}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \tag{5.17}$$
$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1). \tag{5.18}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.19}$$
$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - q \geq 1). \tag{5.20}$$
$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1). \tag{5.21}$$
$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize). \tag{5.22}$$
$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize). \tag{5.23}$$
$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - q \geq 1). \tag{5.24}$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not ($\neg$) adjacent:

$$\begin{aligned} \forall\ setsize > 3: \quad & \neg Successor(1, 3, setsize > 3) \\ & \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \end{aligned} \tag{5.25}$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\ setsize > 3: \quad & \neg Predecessor(1, 3, setsize > 3) \\ & \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \end{aligned} \tag{5.26}$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$\begin{aligned} \forall\ setsize > 3: \quad & \neg Adjacent(1, 3, setsize > 3) \\ & \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \end{aligned} \tag{5.27}$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

(1) In flat space, where $|x_i| = |y_i| = p_i$, the largest possible number of domain-to-range set mappings (the largest intersection, where the countable distance $d_c :\ d_c = |\bigcup_{i=1}^{n} y_i|$) is where each domain set, x_i, member maps to every member of the corresponding range set, y_i, which, by the rule of product, is the Cartesian product of the number of domain and range set members: $d_c = \mathrm{f}(\sum_{i=1}^{n} |x_i| \cdot |y_i|) = \mathrm{f}(\sum_{i=1}^{n} p_i^2)$. The largest possible

number of domain-to-range set mappings is the set-based reason that Euclidean distance (3.4) is the smallest possible distance between two distinct points (in flat space).

(2) Generalizing the countable distance and volume constraint, $|x_i| = |y_i|$, to $|x_i| = |y_i|^q$, $q \geq 0$, generates all the L^p norms (Minkowski distances), $||L||_p = (\sum_{i=1}^{n} s_i^p)^{1/p}$. For example, using the same proof pattern as for Euclidean distance (3.4): $|y_i| = p_i \Rightarrow |x_i| = p_i^q \Rightarrow \sum_{i=1}^{n} |x_i| \cdot |y_i| = \sum_{i=1}^{n} p_i^q \cdot p_i = \sum_{i=1}^{n} p_i^{q+1} \leq d_c^{q+1} \ldots d = (\sum_{i=1}^{n} s_i^{q+1})^{1/(q+1)}$.

(3) $-1 \leq q < 0 \Rightarrow 0 \leq p < 1$ and generates L^p norms that do not satisfy the metric space triangle inequality property.

(4) The curvature of a space around a point is typically measured in terms of second order differential equations, e.g., the Laplacian. A set-based measure of the amount of curvature is how far q deviates from the value, 1, in the countable distance and volume constraint, $|x_i| = |y_i|^q$.

(5) Manhattan distance and rectangular Euclidean volume are both cases of disjoint range sets, $\bigcap_{i=1}^{n} y_i = \emptyset$, in flat space (where $|x_i| = |y_i|$):

$$d_c = |\bigcup_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset.$$

$$v_c = |\times_{i=1}^{n} y_i| : \quad \bigcap_{i=1}^{n} x_i = \emptyset \quad \wedge \quad |x_i| = |y_i| \quad \wedge \quad \bigcap_{i=1}^{n} y_i = \emptyset.$$

(6) There are combinatorial relationships between countable sets of subintervals of intervals in statistics, probability, physics, etc., where the ruler is an applicable tool. For example, applying the ruler (2.1) and ruler convergence (2.2) to the Cartesian product of same-sized, infinitesimal charge forces and mass forces allowed deriving Coulomb's charge force (5.1) and Newton's gravity force (5.5) equations in a few steps each, without using other laws of physics or Gauss's divergence theorem.

(7) **The Proportionate Interval Principle:** The derivations of the charge force, gravity force, and spacetime equations shows that all Euclidean distance intervals having a size, r, have proportionately sized intervals of other types: $r = (r_C/q_C)q = (r_G/m_G)m = (r_c/t_c)t = ct$, where the conversion ratios are for unit-factor analysis.

   (a) Some versions of the charge constant, vacuum magnetic permeability constant, fine structure constant, etc. contain the value 4π because the creators assumed flux divergence on the surface of a sphere, $4\pi r^2$. Using Occam's razor, the mapping of rectangular geometric area to rectangular charge and mass areas provides more parsimonious derivations of the inverse square law, charge, and gravity force equations than flux divergence. Therefore, those versions of the constants containing the value 4π might be incorrect.

   (b) $(r_G/m_G)m \cdot ct = r^2 \Rightarrow m = (m_G/r_G c)r^2/t$. For a constant mass, m, as the distance, r, to the mass decreases, then time, t, must also decrease (time slows down), which agrees with relativity theory and observation. Also, $(r_G/m_G)m \cdot (ct)^2 = r^3 \Rightarrow E = mc^2 = (m_G/r_G)r^3/t^2 = (m_G/r_G)rv^2$. Likewise, for charge, $q = (q_C/r_C c)r^2/t$. And $E = qc^2 = (q_C/r_C)r^3/t^2 = (q_C/r_C)rv^2$.

   (c) If there are quantum values of charge, q_C, and mass, m_G, then there are quantum distances (wavelengths), r_C and r_G, where the charge and gravity forces do not exist (are not defined) at smaller distances.

(d) A countable set of values has measure 0. Therefore, there is no proportion relationship of a countable set of values to distance. Therefore, a countable set of state value changes with respect to time are independent of distance. For example, the change in the spin values of two quantum entangled electrons and the change in polarization of two quantum entangled photons are independent of the distance between the entangled particles.

(8) Any higher dimensions of space not being sequentially reachable from the lower 3 dimensions because the lower 3 dimensions are a cyclic set is a more parsimonious explanation of not seeing any higher dimensions than the higher dimensions being rolled into infinitesimal balls, which requires an additional explanation of what causes the higher dimensions to be rolled up and additional equations describing the rollups.

(9) If there are higher dimensions of ordered and symmetric space, then there is a set of three members, each member being an ordered and symmetric set of 3 dimensions (three boxes), yielding a total of 9 ordered and symmetric dimensions of space.

(10) Each ordered and symmetric ball can have at most 3 ordered and symmetric dimensions of discrete states of the same type, for example, a set of 3 binary values, 1 and -1, indicating vector orientation.

(11) Each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values of the same type per ball, for example, spin values: -1, 0, 1 per orthogonal plane in the ball.

(12) Each of the 3 possible ordered and symmetric dimensions of discrete physical states could contain an unordered collection (bag) of discrete state values. Bags (of states) are non-deterministic. For example, every time an unordered binary state is measured, there is a 50 percent chance of having one of the binary values.

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