

The Set Mappings Generating Geometry and Physics

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ABSTRACT. The Euclidean volume equation is derived from a set and limit-based foundation. Distance as a function of volume is used for simple derivations of the Minkowski distances (for example, Manhattan and Euclidean distance), the Minkowski inequality, and the properties of properties metric space. The Euclidean volume proof provides simpler and more rigorous derivations of Newton's gravity force and Coulomb's charge force equations (without using the inverse square law or Gauss's divergence theorem). The derivations of the gravity and charge force equations exposes a ratio (constant first derivative) principle that allows simpler derivations of the spacetime equations and some general relativity equations. A symmetry property can limit geometric distance and volume to 3 dimensions per ball and a limit of 3 3-dimensional balls. All proofs are verified in Coq.

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1. Introduction

Metric space, Euclidean distance, and area/volume are opaque definitions in mathematical analysis [Gol76] [Rud76] motivated by Euclidean geometry [Joy98]. Deriving those definitions from a set and limit-based foundation, without relying on any of the primitives and relations of Euclidean geometry, explains aspects of geometry and physics that opaque definitions cannot provide.

In this article, the Euclidean area/volume equation is derived from a set of n -tuples without assuming the product of interval lengths, without summing infinitesimal volumes, without multiplying scalars times a unit area/volume, and without using the notions of geometry, like unit area/volume, line, angle, rectangle, etc. Distance as a function of volume is used for simple derivations of the Minkowski distances (for example, Manhattan and Euclidean distance), the Minkowski inequality, and the properties of properties metric space without relying on any of the primitives and relations of Euclidean geometry.

A symmetry constraint on the mapping between a set of integers and a set of domain intervals/dimensions (a totally ordered set) can limit geometric distance and volume to 3 dimensions per ball and a limit of 3 3-dimensional balls.

The Euclidean volume proof is used to provide simpler and more rigorous derivations of Newton's gravity force and Coulomb's charge force equations (without using the inverse square law or Gauss's divergence theorem). The derivations of the gravity and charge forces expose a ratio (constant first derivative) principle that generates the spacetime equations and some general relativity equations.

All the proofs in this article are trivial. But, to ensure confidence in the correctness, all the proofs in this article have been verified using the Coq proof verification system [Coq15]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: <https://github.com/treeck/RASRGeometry>.

2. Ruler measure and convergence

Note: In order to compute areas and volumes, integrals divide all intervals into the *same* number subintervals (infinitesimals), where the size of the infinitesimals in each interval can *vary*, which makes it difficult for integrals to directly express the number of mappings between the p_x number of size c infinitesimals in one interval and the p_y number of size c infinitesimals in another interval.

In contrast to the integral, a ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p , of whole subintervals (infinitesimals), where each infinitesimal has the *same* size, c .

DEFINITION 2.1. Ruler measure, M : $\forall c, s \in \mathbb{R}, [a, b] \subset \mathbb{R}, s = b - a \wedge c > 0 \wedge (p = \text{floor}(s/c) \vee p = \text{ceiling}(s/c)) \wedge M = \sum_{i=1}^p c = pc$.

THEOREM 2.2. *Ruler convergence:* $M = \lim_{c \rightarrow 0} pc = s$.

The proof is trivial but is included here for completeness. The theorem, "limit_c_0_M.eq_exact_size," and formal proof is in the Coq file, euclidrelations.v.

PROOF. (epsilon-delta proof)

By definition of the floor function, $\text{floor}(x) = \max(\{y : y \leq x, y \in \mathbb{Z}, x \in \mathbb{R}\})$:

$$(2.1) \quad \forall c > 0, p = \text{floor}(s/c) \wedge 0 \leq |\text{floor}(s/c) - s/c| < 1 \Rightarrow 0 \leq |p - s/c| < 1.$$

Multiply all sides of inequality 2.1 by c :

$$(2.2) \quad \forall c > 0, \quad 0 \leq |p - s/c| < 1 \Rightarrow 0 \leq |pc - s| < |c|.$$

$$(2.3) \quad \forall \delta : |pc - s| < |c| = |c - 0| < \delta \\ \Rightarrow \quad \forall \epsilon = \delta : |c - 0| < \delta \wedge |pc - s| < \epsilon := M = \lim_{c \rightarrow 0} pc = s. \quad \square$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = \text{floor}(s/c) \Rightarrow p \cdot c = 3.1_{c=10^{-1}}, 3.14_{c=10^{-2}}, 3.141_{c=10^{-3}}, \dots, \pi \lim_{c \rightarrow 0}$.

3. Euclidean Volume

DEFINITION 3.1. Countable Volume:

$$v_c = |\times_{i=1}^n x_i| = \prod_{i=1}^n |x_i|, \quad \bigcap_{i=1}^n x_i = \emptyset$$

LEMMA 3.2. $\forall c > 0, \lim_{c \rightarrow 0} c^n = \lim_{c \rightarrow 0} c$.

PROOF.

$$(3.1) \quad q > 1 \quad \wedge \quad n > 1 \quad \Rightarrow \quad q^n > q > 1 \quad \Rightarrow \quad 0 < 1/q^n < 1/q$$

$$(3.2) \quad 0 < 1/q^n < 1/q \quad \wedge \quad c = 1/q \quad \Rightarrow \quad 0 < c^n < c.$$

$$(3.3) \quad 0 < c^n < c \quad \Rightarrow \quad 0 < |c - c^n| < |c| = |c - 0|.$$

$$(3.4) \quad 0 < |c - c^n| < |c - 0| \quad \Rightarrow \quad \forall \delta : |c - c^n| < |c - 0| < \delta \\ \Rightarrow \quad \forall \epsilon = \delta : |c - 0| < \delta \quad \wedge \quad |c - c^n| < \epsilon := \lim_{c \rightarrow 0} c^n = 0.$$

$$(3.5) \quad \lim_{c \rightarrow 0} c^n = 0 \quad \wedge \quad \lim_{c \rightarrow 0} c = 0 \quad \Rightarrow \quad \lim_{c \rightarrow 0} c^n = \lim_{c \rightarrow 0} c. \quad \square$$

THEOREM 3.3. *Euclidean volume, v , is length of the range interval, $[v_u, v_w]$, which is equal to product of domain interval lengths, $\{[a_1, b_1], [a_2, b_2], \dots, [a_n, b_n]\}$:*

$$v = \prod_{i=1}^n s_i, \quad v = v_w - v_u, \quad s_i = b_i - a_i.$$

The formal proof, “Euclidean_volume,” is in the Coq file, euclidrelations.v.

PROOF.

Use the ruler (2.1) to partition each of the domain intervals, $[a_i, b_i]$, into a set, x_i , containing p_i number of subintervals.

$$(3.6) \quad \forall i \ n \in \mathbb{N}, \quad i \in [1, n], \quad c > 0 \quad \wedge \quad \text{floor}(s_i/c) = p_i = |x_i|.$$

Apply the ruler convergence theorem (2.2) to equation 3.6:

$$(3.7) \quad \text{floor}(s_i/c) = p_i \quad \Rightarrow \quad \lim_{c \rightarrow 0} (p_i \cdot c) = s_i.$$

$$(3.8) \quad v_c = \prod_{i=1}^n |x_i| \quad \wedge \quad |x_i| = p_i \quad \Rightarrow \quad v_c = \prod_{i=1}^n p_i \\ \Rightarrow \quad \lim_{c \rightarrow 0} v_c \cdot c = \lim_{c \rightarrow 0} (\prod_{i=1}^n p_i) \cdot c.$$

Note: that multiplying both sides of $v_c = \prod_{i=1}^n p_i$ by an infinitesimal volume, c^n , in the previous step is assuming the volume equation and would make the logic circular. Instead both sides are multiplied by c and, next, apply lemma 3.2 to equation 3.8:

$$(3.9) \quad \lim_{c \rightarrow 0} c^n = \lim_{c \rightarrow 0} c \quad \wedge \quad \lim_{c \rightarrow 0} v_c \cdot c = \lim_{c \rightarrow 0} (\prod_{i=1}^n p_i) \cdot c \\ \Rightarrow \quad \lim_{c \rightarrow 0} (v_c \cdot c) = \lim_{c \rightarrow 0} (\prod_{i=1}^n p_i) \cdot c^n = \lim_{c \rightarrow 0} \prod_{i=1}^n (p_i \cdot c).$$

By ruler convergence (2.2):

$$(3.10) \quad \exists v \in \mathbb{R} : v_c = \text{floor}(v/c) \quad \Rightarrow \quad v = \lim_{c \rightarrow 0} (v_c \cdot c).$$

Combine equation 3.10 with equation 3.9:

$$(3.11) \quad v = \lim_{c \rightarrow 0} (v_c \cdot c) \quad \wedge \quad \lim_{c \rightarrow 0} (v_c \cdot c) = \lim_{c \rightarrow 0} \prod_{i=1}^n (p_i \cdot c) \\ \Rightarrow \quad v = \lim_{c \rightarrow 0} \prod_{i=1}^n (p_i \cdot c).$$

Combine equation 3.7 and equation 3.11:

$$(3.12) \quad \lim_{c \rightarrow 0} (p_i \cdot c) = s_i \quad \wedge \quad v = \lim_{c \rightarrow 0} \prod_{i=1}^n (p_i \cdot c) \quad \Rightarrow \quad v = \prod_{i=1}^n s_i. \quad \square$$

4. Distance

4.1. n-distance.

DEFINITION 4.1. n-distance, d :

$$v = \prod_{i=1}^n d = d^n \quad \Leftrightarrow \quad d = v^{1/n}.$$

4.2. Minkowski distance. Only like types can be added together. For example, only scalars can be added to a scalar and only vectors can be added to a vector. Likewise, an n-dimensional volume (an n-volume) can only be the sum of n-volumes.

THEOREM 4.2. *Minkowski distance: All distances that are a function of volume are Minkowski distances.*

$$v = \prod_{i=1}^n d = d^n \quad \Rightarrow \quad d = (\sum_{i=1}^m s_i^n)^{1/n}$$

The formal proof, “Minkowski_distance,” is in the Coq file, euclidrelations.v.

PROOF.

$$(4.1) \quad \forall v, v_1, \dots, v_m : v = \sum_{i=1}^m v_i \quad \wedge \quad v = d^n \quad \Rightarrow \quad d^n = v = \sum_{i=1}^m v_i.$$

An n-volume can only be the sum of n-volumes:

$$(4.2) \quad v = \sum_{i=1}^m v_i \quad \wedge \quad \exists s_i \in \mathbb{R} : s_i^n = v_i \\ \Rightarrow \quad v = d^n = \sum_{i=1}^m s_i^n \quad \Leftrightarrow \quad d = (\sum_{i=1}^m s_i^n)^{1/n}. \quad \square$$

4.3. Countable distance. Applying the ruler to Minkowski distances yields a countable distance, d_c .

DEFINITION 4.3. Countable distance, d_c :

$$d = (\sum_{i=1}^m s_i^n)^{1/n} \quad \wedge \quad d_c = \text{floor}(d/c) \quad \wedge \quad |x_i| = \text{floor}(s_i/c) \\ \Rightarrow \quad d_c^n = \sum_{i=1}^m |x_i|^n.$$

LEMMA 4.4. *Countable distance generates Minkowski distance*

$$d_c^n = \sum_{i=1}^m |x_i|^n \quad \Rightarrow \quad d = (\sum_{i=1}^m s_i^n)^{1/n}$$

PROOF. Apply the ruler (2.1) and ruler convergence (2.2):

$$(4.3) \quad \exists d, s_1, \dots, s_m \in \mathbb{R} : d_c = \text{floor}(d/c) \quad \wedge \quad |x_i| = \text{floor}(s_i/c) \quad \wedge \\ d_c^n = \sum_{i=1}^m |x_i|^n \quad \Rightarrow \quad d^n = \lim_{c \rightarrow 0} (d_c \cdot c)^n = \lim_{c \rightarrow 0} \sum_{i=1}^m (|x_i| \cdot c)^n = \sum_{i=1}^m s_i^n. \quad \square$$

4.4. Minkowski inequality. $(\sum_{i=1}^m a_i + b_i)^{1/n} \leq \sum_{i=1}^m a_i^{1/n} + \sum_{i=1}^m b_i^{1/n}$ is used in this article for the proof that the metric space triangle inequality is generated by the Minkowski distance. But all published proofs of the Minkowski inequality assume the triangle inequality, convexity, or Hölder's inequality which precludes the proofs being used to derive the triangle inequality. Therefore, a new proof of the Minkowski inequality, for all $n \in \mathbb{R}$, that does not assume the triangle inequality, convexity, or Hölder's inequality is presented here.

THEOREM 4.5. *Minkowski inequality*

$$\forall m, n \in \mathbb{N}, a_1, \dots, a_m, b_1, \dots, b_m \geq 0 : \sum_{i=1}^m (a_i + b_i)^{1/n} \leq \sum_{i=1}^m a_i^{1/n} + \sum_{i=1}^m b_i^{1/n}.$$

PROOF.

$$(4.4) \quad \forall v_a, v_b, v_x, v_y \geq 0, n \in \mathbb{N} : v_x = (v_a + v_b) \wedge v_y = (v_a^{1/n} + v_b^{1/n})^n \\ \Rightarrow v_x/v_y = (v_a + v_b)/(v_a^{1/n} + v_b^{1/n})^n.$$

Expand the denominator using the binomial expansion:

$$(4.5) \quad v_x/v_y = (v_a + v_b)/(v_1^{1/n} + v_2^{1/n})^n = \\ (v_a + v_b)/(v_a + v_b + \sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k) + \sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k}.$$

$$(4.6) \quad (v_a + v_b)/(v_a + v_b + \\ \sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k) + \sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} \wedge \\ \sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k) + \sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} \geq 0 \\ \Rightarrow (v_a + v_b)/(v_a + v_b + \\ \sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k) + \sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} \leq 1 \\ \Rightarrow v_a + v_b \leq (v_a + v_b + \\ \sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k) + \sum_{i=1}^n \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = \\ (v_a^{1/n} + v_b^{1/n})^n.$$

$$(4.7) \quad v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

$$(4.8) \quad (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \wedge v_a = a_i \wedge v_b = b_i \\ \Rightarrow (a_i + b_i)^{1/n} \leq a_i^{1/n} + b_i^{1/n} \\ \Rightarrow \sum_{i=1}^m (a_i + b_i)^{1/n} \leq \sum_{i=1}^m a_i^{1/n} + \sum_{i=1}^m b_i^{1/n} \quad \square$$

4.5. Metric Space. The properties of metric space have been motivated by Euclidean distance. But, all Minkowski distances generate the properties of metric space. The formal proofs: symmetry, triangle_inequality, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.6. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} \Rightarrow d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$(4.9) \quad \forall p : 1 \leq p \leq 2, \quad d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \quad \square$$

THEOREM 4.7. *Triangle Inequality:* $d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} \Rightarrow d(u, w) \leq d(u, v) + d(v, w)$.

PROOF. $\forall p \geq 1, \quad k > 0, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(4.10) \quad (u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}.$$

Using the Minkowski's inequality (4.5):

$$(4.11) \quad (a + b)^{1/p} \leq a^{1/p} + b^{1/p} \\ \Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p}. \quad \square$$

THEOREM 4.8. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} \Rightarrow d(u, w) \geq 0$.

PROOF. By definition, the length of an interval is always ≥ 0 :

$$(4.12) \quad \forall [a_1, b_1], [a_2, b_2], \quad s_1 = b_1 - a_1, \quad s_2 = b_2 - a_2, \quad \Rightarrow \quad s_1 \geq 0, \quad s_2 \geq 0.$$

$$(4.13) \quad s_1 \geq 0, \quad s_2 \geq 0 \quad \Rightarrow \quad d(s_1, s_2) = (\sum_{i=1}^2 s_i^p)^{1/p} \geq 0. \quad \square$$

THEOREM 4.9. *Identity of Indiscernibles:* $d(w, w) = 0$.

PROOF. Apply the triangle inequality property (4.7):

$$(4.14) \quad \forall d(u, v) = d(v, w) = 0 \wedge d(u, w) \leq d(u, v) + d(v, w) \Rightarrow d(u, w) \leq 0.$$

Combine the non-negativity property (4.8) and the previous inequality (4.14):

$$(4.15) \quad d(u, w) \geq 0 \wedge d(u, w) \leq 0 \Leftrightarrow 0 \leq d(u, w) \leq 0 \Rightarrow d(u, w) = 0.$$

Combine the result of step 4.15 and the condition, $d(u, v) = 0$, in step 4.14.

$$(4.16) \quad d(u, w) = 0 \wedge d(u, v) = 0 \Rightarrow w = v.$$

Combine the condition, $d(v, w) = 0$, in step 4.14 and the result of step 4.16.

$$(4.17) \quad d(v, w) = 0 \wedge w = v \Rightarrow d(w, w) = 0. \quad \square$$

5. Applications to physics

5.1. Newton's gravity force equation. m_1 and m_2 , are the sizes of two independent mass intervals, where each size c component of a mass interval exerts a force on each size c component of the other mass interval. If p_1 and p_2 are the number of size c components in each mass interval, then the total force, F , is equal to the total number of forces, which is proportionate to the Cartesian product, $p_1 \cdot p_2$, and proportionate to the size, c , of each component. Applying the volume proof (3.3) (and lemma 3.2):

$$(5.1) \quad p_1 = \text{floor}(m_1/c) \wedge p_2 = \text{floor}(m_2/c) \wedge F := m_0 a \propto (p_1 p_2) c \\ \Rightarrow F := m_0 a \propto \lim_{c \rightarrow 0} (p_1 p_2) c = \lim_{c \rightarrow 0} (p_1 p_2) c^2 = \lim_{c \rightarrow 0} (p_1 c \cdot p_2 c) = m_1 m_2,$$

where the force, F , is defined as the rest mass, m_0 , times acceleration, a .

Note that integrals have no means of directly specifying the p_1 and p_2 of size c infinitesimals. Therefore, it is difficult to use integrals to rigorously derive: $\lim_{c \rightarrow 0}(p_1 p_2)c = m_1 m_2$.

$$(5.2) \quad F := m_0 a = m_0 r / t^2 \propto m_1 m_2 \quad \wedge \quad m_0 = m_1 \Rightarrow r \propto m_1 \Rightarrow \\ \exists m_G, r_c \in \mathbb{R} : r = (dr/dm)m_2 = (r_c/m_G)m_2,$$

where: r is Euclidean distance, t is time, and r_c/m_G is a unit-factoring proportion ratio.

$$(5.3) \quad m_0 = m_1 \quad \wedge \quad r = (m_G/r_c)m_2 \quad \wedge \quad F = m_0 r / t^2 \\ \Rightarrow \quad F = m_0 r / t^2 = (r_c/m_G)m_1 m_2 / t^2.$$

From equation the definition of force, $F := m_0 a$:

$$(5.4) \quad \int_0^t a dt = r/t \Rightarrow \exists t_c, r_c \in \mathbb{R} : t/r = (dt/dr) = t_c/r_c \Rightarrow t = (t_c/r_c)r.$$

$$(5.5) \quad t = (t_c/r_c)r \quad \wedge \quad F = (r_c/m_G)m_1 m_2 / t^2 \Rightarrow \\ F = (r_c/m_G)(r_c^2/t_c^2)m_1 m_2 / r^2 = (r_c^3/m_G t_c^2)m_1 m_2 / r^2 = G m_1 m_2 / r^2,$$

where the gravitational constant, $G = r_c^3/m_G t_c^2$, has the SI units: $m^3 kg^{-1} s^{-2}$.

5.2. Coulomb's charge force. q_1 and q_2 , are the sizes of two independent charge intervals, where each size c component of a charge interval exerts a force on each size c component of the other charge interval. If p_1 and p_2 are the number of size c components in each charge interval, then the total force, F , is equal to the total number of forces, which is proportionate to the Cartesian product, $p_1 \cdot p_2$, and proportionate to the size, c , of each component. Applying the volume proof (3.3) (and lemma 3.2):

$$(5.6) \quad p_1 = \text{floor}(q_1/c) \quad \wedge \quad p_2 = \text{floor}(q_2/c) \quad \wedge \quad F \propto (p_1 p_2)c \\ \Rightarrow \quad F := m_0 a \propto \lim_{c \rightarrow 0}(p_1 p_2)c = \lim_{c \rightarrow 0}(p_1 p_2)c^2 = \lim_{c \rightarrow 0}(p_1 c \cdot p_2 c) = q_1 q_2,$$

where the force, F , is defined as the rest mass, m_0 , times acceleration, a .

$$(5.7) \quad F := m_0 a = m_0 r / t^2 \propto q_1 q_2 \quad \wedge \\ m_0 = (dm/dq)q_1 = (m_G/q_C)q_1 \Rightarrow r \propto q_1 \\ \Rightarrow \exists q_C, r_c \in \mathbb{R} : r = (dr/dq)q_2 = (r_c/q_C)q_2,$$

where: r is Euclidean distance, t is time, m_G/q_C and r_c/q_C are unit-factoring proportion ratios.

$$(5.8) \quad m_0 = (m_G/q_C)q_1 \quad \wedge \quad r = (q_C/r_c)q_2 \quad \wedge \quad F = m_0 r / t^2 \\ \Rightarrow \quad F = m_0 r / t^2 = (m_G/q_C)(r_c/q_C)q_1 q_2 / t^2 = (m_G r_c / q_C^2)q_1 q_2 / t^2.$$

From equation the definition of force, $F := m_0 a$:

$$(5.9) \quad \int_0^t a dt = r/t \Rightarrow \exists t_c, r_c \in \mathbb{R} : t/r = (dt/dr) = t_c/r_c \Rightarrow t = (t_c/r_c)r.$$

$$(5.10) \quad t = (t_c/r_c)r \quad \wedge \quad a_G = r_c/t_c^2 \quad \wedge \quad F = (m_G r_c / q_C^2)q_1 q_2 / t^2 \Rightarrow \\ F = (r_c^2/t_c^2)(m_G r_c / q_C^2)q_1 q_2 / r^2 = ((m_G a_G) r_c^2 / q_C^2)q_1 q_2 / r^2 = k_C q_1 q_2 / r^2,$$

where the charge constant, $k_C = (m_G a_G) r_c^2 / q_C^2$, has the SI units: $N m^2 C^{-2}$.

5.3. Spacetime equations. As shown in the derivations of Newton's gravity force (5.1) and Coulomb's charge force (5.2) equations: $r = (r_c/t_c)t = ct$, where $r_c/t_c = c$ is a unit-factoring proportion ratio. And, the smallest distance (and time) spanning the two inertial (independent, non-accelerating) frames of reference, $[0, r_1]$ and $[0, r_2]$, is the Euclidean distance, r .

$$(5.11) \quad r = ct \Rightarrow (ct)^2 = r_1^2 + r_2^2 \Leftrightarrow r_1^2 = (ct)^2 - (x^2 + y^2 + z^2),$$

where $r_2^2 = x^2 + y^2 + z^2$, which is one form of Minkowski's flat spacetime interval equation [Bru17]. And the length contraction and time dilation equations also follow directly from $(ct)^2 = r_1^2 + r_2^2$, where $v = r_1/t$:

$$(5.12) \quad r_2^2 = (ct)^2 - r_1^2 \wedge L = r_2 \Rightarrow L^2 = c^2 t^2 - v^2 \Rightarrow L = ct\sqrt{1 - (v/c)^2}.$$

$$(5.13) \quad L = ct\sqrt{1 - (v/c)^2} \wedge L_0 = ct \Rightarrow L = L_0\sqrt{1 - (v/c)^2}.$$

$$(5.14) \quad L = ct\sqrt{1 - (v/c)^2} \wedge t' = L/c \Rightarrow t' = t\sqrt{1 - (v/c)^2}.$$

5.4. Some general relativity equations: Combining the ratio (constant first derivative) equations into partial differential equations: $r = (r_c/m_G)m = ct \Rightarrow (r_c/m_G)m \cdot ct = r^2 \Rightarrow m = (m_G/r_c)r^2/t = (m_G/r_c)rv$. For a constant mass, m , a decrease in the distance, r , between two mass centers causes a decrease in time, t , (time slows down). v is the relativistic orbital velocity at distance, r . $(r_c/m_G)m \cdot (ct)^2 = r^3 \Rightarrow E = mc^2 = (m_G/r_c)r^3/t^2$. And $(ct)^2 = r^2 \Rightarrow c^2 = v^2 \Rightarrow (r_c/m_G)mv^2 = c^2 r \Rightarrow KE = mv^2/2 = (m_G c^2/2r_c)r$.

Given that $c = r_c/t_c \approx 3 \cdot 10^8 \text{ms}^{-1}$ and $G = r_c^3/m_G t_c^2 = (r_c/m_G)(r_c/t_c)^2 \approx 6.7 \cdot 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2} \Rightarrow r_c/m_G \approx (6.7 \cdot 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2})/(3 \cdot 10^8 \text{m s}^{-1})^2 \approx 7.4 \cdot 10^{-28} \text{m kg}^{-1}$, which can be used to quantify the constants in the previously derived equations. For example, $m = (m_G/r_c c)rv \approx (1/((7.4 \cdot 10^{-28} \text{m kg}^{-1})(3 \cdot 10^8 \text{m s}^{-1})))rv \approx (4.5 \cdot 10^{18} \text{kg s m}^{-2})rv$.

Likewise, for charge, $r = (r_c/q_C)q = ct \Rightarrow q = (q_C/r_c c)r^2/t = (q_C/r_c c)rv$, $E = qc^2 = (q_C/r_c)r^3/t^2$, and $KE = qv^2/2 = (q_C c^2/2r_c)r$. And if the ratio of an electron's mass to charge is m_G/q_C , then $m_G/q_C \approx 9.1 \cdot 10^{-31} \text{kg}/1.6 \cdot 10^{-19} \text{C} \approx 5.7 \cdot 10^{-12} \text{kg C}^{-1}$. And using Coulomb's constant in ratio form: $k_C = (r_c/t_c)^2(m_G r_c/q_C^2) \approx 9 \cdot 10^9 \text{Nm}^2 \text{C}^{-2} \approx (3 \cdot 10^8 \text{m s}^{-1})^2(5.7 \cdot 10^{-12} \text{kg C}^{-1})(r_c/q_C) \Rightarrow r_c/q_C \approx 1.7 \cdot 10^5 \text{m C}^{-1}$. Therefore, $q = (q_C/r_c c)rv \approx (1/((1.7 \cdot 10^5 \text{m C}^{-1})(3 \cdot 10^8 \text{m s}^{-1})))rv \approx (1.9 \cdot 10^{-13} \text{C s m}^{-2})rv$.

5.5. 3 dimensional balls. Countable volume, $v_c = \prod_{i=1}^n |x_i|$, Euclidean volume, $v = \prod_{i=1}^n s_i$, and all Minkowski distances, $d = (\sum_{i=1}^n s_i^2)^{1/2}$, require that a set of domain intervals/dimensions can be assigned a *total order*. A total order is defined in terms of successor and predecessor relations, where, in this case, the successor and predecessor relations are specified by the integers $i = 1$ to n that map to a set of domain intervals/dimensions.

But the commutative properties of union, multiplication, and addition allow sequencing through each interval (dimension) in every possible order. And "jumping" (indexing) over set members to another member requires calculating an offset, which is implicitly sequencing via the successor and predecessor relations.

Therefore, sequencing directly via the successor and predecessor relations from one set member to every other member requires each set member to be sequentially adjacent (either a successor or predecessor) to every other member, herein referred

to as a symmetry constraint. It will now be proved that coexistence of the symmetry constraint on a sequentially ordered set defines a cyclic set that contains at most 3 members, in this case, 3 dimensions per ball and 3 3-dimensional balls.

DEFINITION 5.1. Ordered geometry:

$$\forall i \ n \in \mathbb{N}, \ i \in [1, n-1], \ \forall x_i \in \{x_1, \dots, x_n\}, \\ \text{successor } x_i = x_{i+1} \ \wedge \ \text{predecessor } x_{i+1} = x_i.$$

DEFINITION 5.2. Symmetry Constraint (every set member is sequentially adjacent to every other member):

$$\forall i \ j \ n \in \mathbb{N}, \ \forall x_i \ x_j \in \{x_1, \dots, x_n\}, \ \text{successor } x_i = x_j \Leftrightarrow \text{predecessor } x_j = x_i.$$

THEOREM 5.3. An ordered and symmetric set is a cyclic set.

$$i = n \ \wedge \ j = 1 \ \Rightarrow \ \text{successor } x_n = x_1 \ \wedge \ \text{predecessor } x_1 = x_n.$$

The formal proof, “ordered_symmetric_is_cyclic,” is in the Coq file, `threed.v`.

PROOF. A total order (5.1) defines unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1 . Therefore, the only member that can be a successor of x_n , without creating a contradiction, is x_1 . And the only member that can be a predecessor of x_1 , without creating a contradiction, is x_n . Applying the symmetry constraint (5.2):

$$(5.15) \quad i = n \ \wedge \ j = 1 \ \wedge \ \text{successor } x_i = x_j \ \Rightarrow \ \text{successor } x_n = x_1.$$

Applying the definition of the symmetry constraint (5.2) to conclusion 5.15:

$$(5.16) \quad \text{successor } x_i = x_j \ \Rightarrow \ \text{predecessor } x_j = x_i \ \Rightarrow \ \text{predecessor } x_1 = x_n. \quad \square$$

THEOREM 5.4. An ordered and symmetric set is limited to at most 3 members.

The lemmas and formal proofs in the Coq file `threed.v` are:

Lemmas: `adj111`, `adj122`, `adj212`, `adj123`, `adj133`, `adj233`, `adj213`, `adj313`, `adj323`, and `not_all_mutually_adjacent_gt_3`.

The following proof uses Horn clauses (a subset of first order logic) that uses unification and resolution. Horn clauses make it clear which facts satisfy a proof goal.

PROOF.

It was proved that an ordered and symmetric set is a cyclic set (5.3).

DEFINITION 5.5. Successor of m is n :

$$(5.17) \quad \text{Successor}(m, n, \text{setsize}) \leftarrow (m = \text{setsize} \wedge n = 1) \vee (n = m + 1 \leq \text{setsize}).$$

DEFINITION 5.6. Predecessor of m is n :

$$(5.18) \quad \text{Predecessor}(m, n, \text{setsize}) \leftarrow (m = 1 \wedge n = \text{setsize}) \vee (n = m - q \geq 1).$$

DEFINITION 5.7. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n . Notionally:

$$(5.19) \quad \text{Adjacent}(m, n, \text{setsize}) \leftarrow \text{Successor}(m, n, \text{setsize}) \vee \text{Predecessor}(m, n, \text{setsize}).$$

Prove that every member is adjacent to every other member, where $setsize \in \{1, 2, 3\}$:

$$(5.20) \quad Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1).$$

$$(5.21) \quad Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize).$$

$$(5.22) \quad Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1).$$

$$(5.23) \quad Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize).$$

$$(5.24) \quad Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - q \geq 1).$$

$$(5.25) \quad Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1).$$

$$(5.26) \quad Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize).$$

$$(5.27) \quad Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize).$$

$$(5.28) \quad Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - q \geq 1).$$

Must prove that for all $setsize > 3$, there exist non-adjacent members. For example, the first and third members are not (\neg) adjacent:

$$(5.29) \quad \forall setsize > 3 : \quad \neg Successor(1, 3, setsize > 3) \\ \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize).$$

That is, member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not a successor of member 1 for all $setsize > 3$.

$$(5.30) \quad \forall setsize > 3 : \quad \neg Predecessor(1, 3, setsize > 3) \\ \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3).$$

That is, member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not a predecessor of member 1 for all $setsize > 3$.

$$(5.31) \quad \forall setsize > 3 : \quad \neg Adjacent(1, 3, setsize > 3) \\ \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square$$

That is, for all $setsize > 3$, some elements are not sequentially adjacent to every other element (not symmetric).

6. Insights and implications

- (1) It was shown that all distances that are a function of an n-volume are Minkowski distances (4.2). And the Minkowski distances have the properties defining metric space (4.5). Therefore, the criteria of a distance measure being a function equivalent to a Minkowski distance (or all functions derived from an n-volume) would filter out functions that metric space would allow.
- (2) A metric in the form $d(x, y)$ is usually interpreted as the distance between two points, x and y . The derivation of the properties of metric space from the Minkowski distances indicates a more correct interpretation of $d(x, y)$ is the distance spanning two domain sets (intervals) having the sizes, x and y .
- (3) The Minkowski inequality proof in this article (4.5) is the first proof to not assume the triangle inequality, convexity, or Hölder's inequality.

- (4) The ratio of two related volumes (4.5), $v_x/v_y = (v_a + v_b)/(v_a^{1/n} + v_b^{1/n})^n$, generates: the Minkowski inequality, the triangle inequality, and convexity.
- (5) Mathematical analysis (for example, point-set topology) has not been able to provide a set and limit-based explanations of why Euclidean distance is the smallest distance between two distinct points.

The bidirectional implication between countable distance (4.4), d_c , and Minkowski distance (4.2) exposes the countable domain-to-domain set mappings that generate distance. Each countable n-volume, $|x_i|^n$, is the number of mappings between countable countable domain sets, where each domain set, x_i , has the same cardinal, $|x_1| = \dots = |x_n|$.

Flat space is where each size c domain set member maps to at least one size c member of another domain set and maps at most once to other domain set members. Therefore, the countable distance, d_c in flat space ranges from the sum of bijective mappings (the sum of 1-1 correspondences), which converges to Manhattan distance (by lemma 4.4), for example, $d = a + b + c$, to the sum of each size c domain set member mapping to each size c member of another domain set once (the sum of the Cartesian product mappings), which converges to Euclidean distance (by lemma 4.4): $d = (a^2 + b^2 + c^2)^{1/2}$.

$\forall a, b, c > 0, 1 \leq p < 2 : (a^2 + b^2 + c^2)^{1/2} < (a^p + b^p + c^p)^{1/p}$, which shows that Euclidean distance is the smallest distance in flat space.

- (6) As shown in the derivations of Euclidean volume, Newton's gravity force, and Coulomb's charge force, the ruler (2.1) plus ruler convergence (2.2) is a tool to directly express some counting relations in geometry, probability, physics, etc. that is difficult to directly express with integrals.
- (7) Applying the volume proof (3.3) to the Cartesian product of same-sized, infinitesimal mass forces and charge forces to derive Newton's gravity force (5.1) and Coulomb's charge force (5.2) equations provide several firsts and some insights into physics.
 - (a) These are the first deductive derivations of the gravity and charge forces. All other derivations have been empirical and use Newton's induction, which is not fully provable, for example, assumes the inverse square law based on empirical observation.
 - (b) These are the first derivations to not use the inverse square law or Gauss's divergence theorem.
 - (c) These are the first derivations to show that the definition of force, $F := m_0 a$, containing acceleration, $a : \int_0^t a dt = r/t \Rightarrow \exists t_c, r_c \in \mathbb{R} : t/r = t_c/r_c \Rightarrow r = (r_c/t_c)t$, generates the inverse square law: $F := m_0 a = m_0 r/t^2 = (r_c/t_c)^2 (m_x r_c/x_x^2) x_1 x_2 / r^2 = k_x x_1 x_2 / r^2$.
 - (d) Using Occam's razor, those versions of constants like: charge, vacuum magnetic permeability, fine structure, etc. that contain the value 4π are probably incorrect because those constants are based on the less parsimonious assumption that the inverse square law is due to Gauss's flux divergence on a sphere having the surface area, $4\pi r^2$.
 - (e) These are the first derivations to show that time is proportionate to distance: $r = (r_c/t_c)t = ct$, which is used to derive the spacetime equations (5.3) without the notion of the speed of light.

- (f) The derivations show for the first time how gravity force, charge force, spacetime, and general relativity all depend on time being proportionate to distance.
 - (g) Combining the constant first derivatives (ratios) into partial differential equations allows simple derivations of some general relativity equations (5.4) without the need for integrating second derivative (spacetime curvature) tensors.
 - (h) A state is represented by a constant value. Therefore, a state value does not vary with distance and time interval lengths. For example, the spin values of two quantum entangled electrons and the polarization of two quantum entangled photons are independent of the amount of distance and time between the entangled particles.
- (8) It was proved that a totally ordered set with a symmetry constraint is a cyclic set with at most 3 members (5.3). And the definitions of distance and volume both require a total order and symmetry, which provides several insights:
- (a) Using Occam's razor, a cyclic set of at most 3 members is the most parsimonious explanation of only observing 3 dimensions of geometric distance and volume.
 - (b) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (5.4), each member being an ordered and symmetric set of 3 dimensions (three balls), yielding a total of at most 9 ordered and symmetric dimensions of geometric space.
 - (c) Each dimension of discrete physical states can have at most 3 ordered and symmetric discrete state values of the same type, which allows $3 \cdot 3 \cdot 3 = 27$ possible combinations of discrete values of the same type per 3-dimensional ball, for example, vector orientation values: -1, 0, 1 per orthogonal direction in the ball.
 - (d) Each of the 3 possible ordered and symmetric dimensions of discrete physical states could contain an unordered collection (bag) of discrete state values. Bags are non-deterministic. For example, every time an unordered binary state is "pulled" from a bag, there is a 50 percent chance of getting one of the binary values.

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