The Set Properties Generating Geometry and Physics

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Abstract. Volume and the Minkowski distances/Lp norms (e.g., Manhattan and Euclidean distance) are derived from a set and limit-based foundation without referencing the primitives and relations of geometry. Sequencing a strict linearly ordered set in all n-at-a-time permutations via successor/predecessor relations is a cyclic set of at most 3 members. Therefore, all other interval lengths have different types from a cyclic set of 3 distance interval lengths. Unit-factoring ratios between different types of interval lengths and the set proofs provide simpler derivations of the spacetime, Lorentz, Newton's gravity, Coulomb's charge force, Planck-Einstein, quantum-relativity gravity equations and corresponding constants. All proofs are verified in Coq.

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1. Introduction

Mathematical (real) analysis can construct differential calculus from a set and limit-based foundation without the need to reference the primitives and relations of Euclidean geometry, like straight line, angle, shape, etc., providing a more rigorous foundation to calculus. But volume in the Riemann integral, Lebesgue integral, and measure theory and distance in the vector norm and metric space axioms are all definitions motivated by Euclidean geometry. [Gol76] [Rud76] Here, volume and distance are motivated and derived from a set and limit-based foundation.

A well-known set-based motivation of Euclidean volume is the number of members (cardinal), v_c, of an abstract, countable set of Cartesian product n-tuples:

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$v_c = \prod_{i=1}^{n} |x_i|$, where $|x_i|$ is the cardinal of the countable, disjoint set, x_i. Where each x_i is a set of size κ partitions of $[a_i, b_i] \subset \mathbb{R}$ and $s_i = b_i - a_i$, it will be proved that Euclidean volume, $v = \prod_{i=1}^{n} s_i$, is an instance of $v_c = \prod_{i=1}^{n} |x_i|$.

Traditionally, distance measures (metric spaces) are defined as functions satisfying a set of Euclidean geometry-motivated axioms. Instead, here, all distance measures are defined as functions that are inverse (bijective) functions of the countable set-based n-volume, $v_c = \prod_{i=1}^{n} |x_i|$. And, therefore, all functions that are bijective functions of an Euclidean n-volume are also distance measures.

Further, all n-volumes and corresponding distance measures can be partitioned into the sum of m number of sub-n-volumes and sub-distance measures. All distance measures that are the sum of sub-distance measures are metric spaces.

For example, $\exists\, d, d_i \in \mathbb{R} : \; v = d^n = \sum_{i=1}^{m} d_i^n = \sum_{i=1}^{m} v_i$. d and d_i are distance measures (bijective functions of v and v_i). And d is the L_p norm (Minkowski distance), which will be proved to imply the properties of a metric space.

In the prior equations, sequencing a set, from $i = 1$ to n, is a strict linear (total) order that set theory defines in terms of successor and predecessor functions. But sequencing a strict linear order in all n-at-a-time orders requires an additional "symmetry" constraint, where every set member is either a successor or predecessor to every other set member, which will be proved to be a cyclic set, where $n \leq 3$.

Therefore, where $\{x, y, z\}$ is a strict linearly ordered and symmetric set of 3 "distance" dimensions, then a fourth dimension, t, must have a different type (is a member of different set). A grid within an Euclidean-like volume around a coordinate maps some amount on the x axis to a proportionate amount on the t axis expressed as a constant, unit-factoring, conversion ratio (linear transformation), for example, *meters/second*.

Some ratios combined with the results of the volume and distance proofs provide simpler derivations of the spacetime, Lorentz, Newton's gravity, Coulomb's charge force, Planck-Einstein, MOND-like quantum-relativistic gravity equations, and corresponding constants. Impacts on Einstein's field equations are also discussed.

All the proofs in this article have been verified using using the Coq proof verification system [**Coq23**]. The formal proofs are in the Coq files, "euclidrelations.v" and "threed.v," at: `https://github.com/treeck/RASRGeometry`.

2. Ruler measure and convergence

Derivatives and integrals use a 1-1 correspondence between the infinitesimals of each interval, where the size of the infinitesimals in each interval are proportionate to the size of the containing interval, which precludes using derivatives and integrals to directly express many-to-many (Cartesian product) mappings between same-sized, size κ, infinitesimals in different-sized intervals. Further, using tools that define Euclidean volume and distance precludes using those tools to derive Euclidean volume and distance.

Therefore, a different tool is used here. A ruler (measuring stick) measures the size of each interval *approximately* as the sum of the nearest integer number, p, of whole subintervals (infinitesimals), where each infinitesimal has the *same* size, κ, across all intervals. The ruler is both an inner and outer measure of an interval.

Definition 2.1. Ruler measure, $M = \sum_{i=1}^{p} \kappa = p\kappa$, where $\forall\, [a, b] \subset \mathbb{R}$, $s = b - a \;\wedge\; \kappa > 0 \;\wedge\; (p = floor(s/\kappa) \;\vee\; p = ceiling(s/\kappa))$.

THEOREM 2.2. *Ruler convergence:* $M = \lim_{\kappa\to 0} p\kappa = s$.

The formal proof, "limit_c_0_M_eq_exact_size," is in the file, euclidrelations.v.

PROOF. (epsilon-delta proof)
By definition of the floor function, $floor(x) = max(\{y : \ y \leq x, \ y \in \mathbb{Z}, \ x \in \mathbb{R}\})$:

$$\forall\, \kappa > 0, \ p = floor(s/\kappa) \ \wedge \ 0 \leq |floor(s/\kappa) - s/\kappa| < 1 \ \Rightarrow \ |p - s/\kappa| < 1. \tag{2.1}$$

Multiply both sides of inequality 2.1 by κ:

$$\forall\, \kappa > 0, \quad |p - s/\kappa| < 1 \quad \Rightarrow \quad |p\kappa - s| < |\kappa| = |\kappa - 0|. \tag{2.2}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |p\kappa - s| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |p\kappa - s| < \delta = \epsilon \quad := \quad M = \lim_{\kappa\to 0} p\kappa = s. \qquad \square \end{aligned} \tag{2.3}$$

The following is an example of ruler convergence for the interval, $[0, \pi]$: $s = \pi - 0$, and $p = floor(s/\kappa) \Rightarrow p \cdot \kappa = 3.1_{\kappa=10^{-1}}, \ 3.14_{\kappa=10^{-2}}, \ 3.141_{\kappa=10^{-3}}, ..., \pi_{\lim_{\kappa\to 0}}$.

LEMMA 2.3. $\forall\, n \geq 1, \quad 0 < \kappa < 1 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa$.

PROOF. The formal proof , "lim_c_to_n_eq_lim_c," is in the Coq file, euclidrelations.v.

$$n \geq 1 \quad \wedge \quad 0 < \kappa < 1 \quad \Rightarrow \quad 0 < \kappa^n < \kappa \quad \Rightarrow \quad |\kappa - \kappa^n| < |\kappa| = |\kappa - 0|. \tag{2.4}$$

$$\begin{aligned} &\forall\, \epsilon = \delta \quad \wedge \quad |\kappa - \kappa^n| < |\kappa - 0| < \delta \\ &\qquad \Rightarrow \quad |\kappa - 0| < \delta \quad \wedge \quad |\kappa - \kappa^n| < \delta = \epsilon \quad := \quad \lim_{\kappa\to 0} \kappa^n = 0. \end{aligned} \tag{2.5}$$

$$\lim_{\kappa\to 0} \kappa^n = 0 \quad \wedge \quad \lim_{\kappa\to 0} \kappa = 0 \quad \Rightarrow \quad \lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa. \qquad \square \tag{2.6}$$

3. Volume

DEFINITION 3.1. An n-volume is the number of ordered combinations (n-tuples), v_c, of the members of n number of disjoint, countable domain sets, x_i:

$$\exists\, n \in \mathbb{N}, \ v_c, \in \{0, \ \mathbb{N}\}, \ x_i \in \{x_1, \cdots, x_n\}, \ \bigcap_{i=1}^n x_i = \emptyset : \ \ v_c = \prod_{i=1}^n |x_i|. \tag{3.1}$$

THEOREM 3.2. *Euclidean volume,*

$$\begin{aligned} &\forall [a_i, b_i] \in \{[a_1, b_1], \cdots [a_n, b_n]\}, \ [v_a, v_b] \subset \mathbb{R}, \ s_i = b_i - a_i, \ v = v_b - v_a, \\ &\qquad v_c = \prod_{i=1}^n |x_i| \quad \Rightarrow \quad v = \prod_{i=1}^n s_i. \end{aligned} \tag{3.2}$$

The formal proof, "Euclidean_volume," is in the Coq file, euclidrelations.v.

PROOF.

$$v_c = \prod_{i=1}^n |x_i| \ \Leftrightarrow \ v_c\kappa = (\prod_{i=1}^n |x_i|)\kappa \ \Leftrightarrow \ \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{3.3}$$

Apply the ruler (2.1) and ruler convergence (2.2) to v:

$$\exists\, v, \kappa \in \mathbb{R} : \ v_c = floor(v/\kappa) \quad \Rightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa. \tag{3.4}$$

Apply lemma 2.3 to equation 3.4:

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|)\kappa^n = \lim_{\kappa\to 0}(\prod_{i=1}^n |x_i|\kappa). \tag{3.5}$$

Apply the ruler (2.1) and ruler convergence (2.2) to s_i:

$$\exists\, s_i, \kappa \in \mathbb{R} : \ floor(s_i/\kappa) = |x_i| \quad \Rightarrow \quad \lim_{\kappa\to 0}(|x_i|\kappa) = s_i. \tag{3.6}$$

$$v = \lim_{\kappa\to 0}(\prod_{i=1}^{n} |x_i|\kappa) \;\wedge\; \lim_{\kappa\to 0}(|x_i|\kappa) = s_i \;\Rightarrow\; v = \prod_{i=1}^{n} s_i \tag{3.7}$$ □

THEOREM 3.3. *Sum of volumes:*

$$\forall\, x_{i,j} \in \{x_{i_1}, \cdots, x_{i_m}\} = x_i : \; v_c = \prod_{i=1}^{n} |x_i| \quad \wedge \quad v_{c_j} = \prod_{i=1}^{n} |x_{i,j}| \quad \wedge \\ v_c = \sum_{j=1}^{m} v_{c_j} \quad \Rightarrow \quad \exists\, s_i, s_{i,j} \in \mathbb{R} : \; \prod_{i=1}^{n} s_i = \sum_{j=1}^{m}(\prod_{i=1}^{n} s_{i,j}). \tag{3.8}$$

The formal proof, "sum_of_volumes," is in the Coq file, euclidrelations.v.

PROOF. From the Euclidean volume theorem (3.2):

$$v_c = \prod_{i=1}^{n} |x_i| \;\Rightarrow\; v = \prod_{i=1}^{n} s_i \;\wedge\; v_{c_j} = \prod_{i=1}^{n} |x_{i,j}| \;\Rightarrow\; v_j = \prod_{i=1}^{n} s_{i,j}. \tag{3.9}$$

Apply the ruler (2.1) and ruler convergence (2.2):

$$\exists\, v, v_j, \kappa \in R : \quad v_c = floor(v/\kappa) \quad \wedge \quad v_{c_j} = floor(v_i/\kappa) \\ \Rightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa \quad \wedge \quad v_i = \lim_{\kappa\to 0} v_{c_j}\kappa. \tag{3.10}$$

$$v_c = \sum_{j=1}^{m} v_{c_j} \quad \Leftrightarrow \quad v = \lim_{\kappa\to 0} v_c\kappa = \lim_{\kappa\to 0}(\sum_{j=1}^{m} v_{c_j})\kappa. \tag{3.11}$$

Apply lemma 2.3 to equation 3.11:

$$\lim_{\kappa\to 0} \kappa^n = \lim_{\kappa\to 0} \kappa \;\wedge\; v = \lim_{\kappa\to 0}(\sum_{j=1}^{m} v_{c_j})\kappa \;\wedge\; v_i = \lim_{\kappa\to 0} v_{c_j}\kappa \\ \Rightarrow \quad v = \lim_{\kappa\to 0}(\sum_{j=1}^{m} v_{c_j})\kappa^n = \lim_{\kappa\to 0} \sum_{j=1}^{m}(v_{c_j}\kappa) = \sum_{j=1}^{m} v_j. \tag{3.12}$$

$$v = \prod_{i=1}^{n} s_i \quad \wedge \quad v_j = \prod_{i=1}^{n} s_{i,j} \quad \wedge \quad v = \sum_{j=1}^{m} v_j \\ \Rightarrow \quad \prod_{i=1}^{n} s_i = \sum_{j=1}^{m} \prod_{i=1}^{n} s_{i,j}. \tag{3.13}$$ □

4. Distance

4.1. Minkowski distance (L_p norm).

THEOREM 4.1. *Minkowski distance (L_p norm):*

$$\prod_{i=1}^{n} |x_i| = \sum_{j=1}^{m}(\prod_{i=1}^{n} |x_{i,j}|) \quad \Rightarrow \quad \exists\, d, d_i \in \mathbb{R} : \quad d^n = \sum_{i=1}^{m} d_i^n.$$

The formal proof, "Minkowski_distance," is in the Coq file, euclidrelations.v.

PROOF. From the sum of volumes proof (3.3):

$$\prod_{i=1}^{n} |x_i| = \sum_{j=1}^{m}(\prod_{i=1}^{n} |x_{i,j}|) \quad \Rightarrow \quad \prod_{i=1}^{n} s_i = \sum_{j=1}^{m}(\prod_{i=1}^{n} s_{i,j}) \tag{4.1}$$

$$\exists\, d, d_i, s_i, s_{i,j} \in \mathbb{R} : \; s_1 = \cdots = s_n = d \quad \wedge \quad s_{i_1} = \cdots = s_{i_m} = d_i \quad \wedge \\ \prod_{i=1}^{n} s_i = \sum_{j=1}^{m}(\prod_{i=1}^{n} s_{i,j}) \;\Rightarrow\; d^n = \prod_{i=1}^{n} s_i = \sum_{j=1}^{m}(\prod_{i=1}^{n} s_{i,j}) = \sum_{i=1}^{m} d_i^n. \tag{4.2}$$ □

4.2. Distance inequality. Proving that all Minkowski distances (L_p norms) satisfy the metric space triangle inequality requires another inequality. The formal proof, distance_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.2. *Distance inequality*

$$\forall\, n \in \mathbb{N}, \;\; v_a, v_b \geq 0 : \;\; (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}.$$

PROOF. Expand $(v_a^{1/n} + v_b^{1/n})^n$ using the binomial expansion:

$$\forall\, v_a, v_b \geq 0: \quad v_a + v_b \leq v_a + v_b + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^{n-k} (v_b^{1/n})^k + \sum_{i=1}^{n} \binom{n}{k} (v_a^{1/n})^k (v_b^{1/n})^{n-k} = (v_a^{1/n} + v_b^{1/n})^n. \tag{4.3}$$

Take the n^{th} of both sides of the inequality 4.3:

$$\forall\, v_a, v_b \geq 0, n \in \mathbb{N} : v_a + v_b \leq (v_a^{1/n} + v_b^{1/n})^n \Rightarrow (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n}. \tag{4.4}$$ □

4.3. Distance sum inequality. The formal proof, distance_sum_inequality, is in the Coq file, euclidrelations.v.

THEOREM 4.3. *Distance sum inequality*

$$\forall\, m, n \in \mathbb{N},\ \ a_i, b_i \geq 0 :\ \ (\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}.$$

PROOF. Apply the distance inequality (4.2):

$$\begin{aligned} &\forall\, m, n \in \mathbb{N},\ \ v_a, v_b \geq 0 : \quad v_a = \sum_{i=1}^{m} a_i^n \quad \wedge \quad v_b = \sum_{i=1}^{m} b_i^n \quad \wedge \\ &(v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \quad \Rightarrow \quad ((\sum_{i=1}^{m} a_i^n) + (\sum_{i=1}^{m} b_i^n))^{1/n} = \\ &\qquad (\sum_{i=1}^{m} (a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}. \end{aligned} \tag{4.5}$$ □

4.4. Metric Space. All Minkowski distances (L_p norms) have the properties of metric space.

The formal proofs: triangle_inequality, symmetry, non_negativity, and identity_of_indiscernibles are in the Coq file, euclidrelations.v.

THEOREM 4.4. *Triangle Inequality:*
$d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \leq d(u, v) + d(v, w).$

PROOF. $\forall\, p \geq 1, \quad k > 1, \quad u = s_1, \quad w = s_2, \quad v = w/k$:

$$(u^p + w^p)^{1/p} \leq ((u^p + w^p) + 2v^p)^{1/p} = ((u^p + v^p) + (v^p + w^p))^{1/p}. \tag{4.6}$$

Apply the distance inequality (4.2) to the inequality 4.6:

$$\begin{aligned} &(u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \ \wedge \ (v_a + v_b)^{1/n} \leq v_a^{1/n} + v_b^{1/n} \\ &\qquad \wedge \quad v_a = u^p + v^p \quad \wedge \quad v_b = v^p + w^p \\ &\Rightarrow \quad (u^p + w^p)^{1/p} \leq ((u^p + v^p) + (v^p + w^p))^{1/p} \leq (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} \\ &\qquad \Rightarrow \quad d(u, w) = (u^p + w^p)^{1/p} \leq \\ &\qquad\qquad (u^p + v^p)^{1/p} + (v^p + w^p)^{1/p} = d(u, v) + d(v, w). \end{aligned} \tag{4.7}$$ □

THEOREM 4.5. *Symmetry:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, v) = d(v, u).$

PROOF. By the commutative law of addition:

$$\begin{aligned} &\forall\, p\ :\ p \geq 1, \quad d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} = (s_1^p + s_2^p)^{1/p} \\ &\qquad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} = (v^p + u^p)^{1/p} = d(v, u). \end{aligned} \tag{4.8}$$ □

THEOREM 4.6. *Non-negativity:* $d(s_1, s_2) = (\sum_{i=1}^{2} s_i^p)^{1/p} \quad \Rightarrow \quad d(u, w) \geq 0.$

PROOF. By definition, the length of an interval is always ≥ 0:

$$\forall\, [a_1, b_1],\ [a_2, b_2], \quad u = b_1 - a_1,\ v = b_2 - a_2, \quad \Rightarrow \quad u \geq 0,\ v \geq 0. \tag{4.9}$$

$$p \geq 1,\ \ u, v \geq 0 \quad \Rightarrow \quad d(u, v) = (u^p + v^p)^{1/p} \geq 0. \tag{4.10}$$ □

THEOREM 4.7. *Identity of Indiscernibles: $d(u, u) = 0$.*

PROOF. From the non-negativity property (4.6):

$$\begin{aligned} d(u, w) \geq 0 \quad \wedge \quad d(u, v) \geq 0 \quad \wedge \quad d(v, w) \geq 0 \\ \Rightarrow \quad \exists\, d(u, w) = d(u, v) = d(v, w) = 0. \end{aligned} \tag{4.11}$$

$$d(u, w) = d(v, w) = 0 \quad \Rightarrow \quad u = v. \tag{4.12}$$

$$d(u, v) = 0 \quad \wedge \quad u = v \quad \Rightarrow \quad d(u, u) = 0. \tag{4.13}$$ □

4.5. The properties limiting a set to at most 3 members.

DEFINITION 4.8. Totally ordered set:

$$\begin{aligned} &\forall\, i\ n \in \mathbb{N},\ \ i \in [1, n-1],\ \ \forall\, x_i\ \in \{x_1, \ldots, x_n\}, \\ &\qquad\qquad\qquad\qquad \textit{successor } x_i = x_{i+1} \ \wedge\ \textit{predecessor } x_{i+1} = x_i. \end{aligned}$$

DEFINITION 4.9. Symmetry (every set member is sequentially adjacent to every other member):

$$\forall\, i,\ j,\ n \in \mathbb{N},\ \forall\, x_i,\ x_j \in \{x_1, \ldots, x_n\},\ \textit{successor } x_i = x_j \ \Leftrightarrow\ \textit{predecessor } x_j = x_i.$$

THEOREM 4.10. *A strict linearly ordered and symmetric set is a cyclic set.*

$$i = n \ \wedge\ j = 1 \ \Rightarrow\ \textit{successor } x_n = x_1 \ \wedge\ \textit{predecessor } x_1 = x_n.$$

The formal proof, "ordered_symmetric_is_cyclic," is in the Coq file, threed.v.

PROOF. A total order (4.8) assigns a unique label to each set member and assigns unique successors and predecessors for all set members except for the successor of x_n and the predecessor of x_1. Therefore, the only member that can be a successor of x_n, without creating a contradiction, is x_1. And the only member that can be a predecessor of x_1, without creating a contradiction, is x_n. Applying the symmetry property (4.9):

$$i = n \ \wedge\ j = 1 \ \wedge\ \textit{successor } x_i = x_j \ \Rightarrow\ \textit{successor } x_n = x_1. \tag{4.14}$$

Applying the definition of the symmetry property (4.9) to conclusion 4.14:

$$\textit{successor } x_i = x_j \ \Rightarrow\ \textit{predecessor } x_j = x_i \ \Rightarrow\ \textit{predecessor } x_1 = x_n. \tag{4.15}$$ □

THEOREM 4.11. *An ordered and symmetric set is limited to at most 3 members.*

The formal proofs in the Coq file threed.v are:

`Lemmas: adj111, adj122, adj212, adj123, adj133, adj233, adj213, adj313, adj323, and not_all_mutually_adjacent_gt_3.`

The following proof uses Horn clauses (a subset of first order logic), which makes it clear which facts satisfy a proof goal.

PROOF.
It was proved that an ordered and symmetric set is a cyclic set (4.10).

DEFINITION 4.12. (Cyclic) Successor of m is n:

$$Successor(m, n, setsize) \leftarrow (m = setsize \wedge n = 1) \vee (n = m + 1 \leq setsize). \tag{4.16}$$

DEFINITION 4.13. (Cyclic) Predecessor of m is n:

$$Predecessor(m, n, setsize) \leftarrow (m = 1 \wedge n = setsize) \vee (n = m - 1 \geq 1). \tag{4.17}$$

DEFINITION 4.14. Adjacent: member m is sequentially adjacent to member n if the successor of m is n or the predecessor of m is n. Notionally:

$$Adjacent(m, n, setsize) \leftarrow Successor(m, n, setsize) \vee Predecessor(m, n, setsize). \tag{4.18}$$

Prove that every member is adjacent to every other member, where $setsize \in \{1,\ 2,\ 3\}$:

$$Adjacent(1, 1, 1) \leftarrow Successor(1, 1, 1) \leftarrow (m = setsize \wedge n = 1). \tag{4.19}$$

$$Adjacent(1, 2, 2) \leftarrow Successor(1, 2, 2) \leftarrow (n = m + 1 \leq setsize). \tag{4.20}$$
$$Adjacent(2, 1, 2) \leftarrow Successor(2, 1, 2) \leftarrow (n = setsize \wedge m = 1). \tag{4.21}$$

$$Adjacent(1, 2, 3) \leftarrow Successor(1, 2, 3) \leftarrow (n = m + 1 \leq setsize). \tag{4.22}$$
$$Adjacent(2, 1, 3) \leftarrow Predecessor(2, 1, 3) \leftarrow (n = m - 1 \geq 1). \tag{4.23}$$
$$Adjacent(3, 1, 3) \leftarrow Successor(3, 1, 3) \leftarrow (n = setsize \wedge m = 1). \tag{4.24}$$
$$Adjacent(1, 3, 3) \leftarrow Predecessor(1, 3, 3) \leftarrow (m = 1 \wedge n = setsize). \tag{4.25}$$
$$Adjacent(2, 3, 3) \leftarrow Successor(2, 3, 3) \leftarrow (n = m + 1 \leq setsize). \tag{4.26}$$
$$Adjacent(3, 2, 3) \leftarrow Predecessor(3, 2, 3) \leftarrow (n = m - 1 \geq 1). \tag{4.27}$$

Member 2 is the only successor of member 1 for all $setsize > 3$, which implies member 3 is not ($\neg$) a successor of member 1 for all $setsize > 3$:

$$\begin{aligned} &\neg Successor(1, 3, setsize > 3) \\ &\qquad \leftarrow Successor(1, 2, setsize > 3) \leftarrow (n = m + 1 \leq setsize). \end{aligned} \tag{4.28}$$

Member $n = setsize > 3$ is the only predecessor of member 1, which implies member 3 is not ($\neg$) a predecessor of member 1 for all $setsize > 3$:

$$\begin{aligned} &\neg Predecessor(1, 3, setsize > 3) \\ &\qquad \leftarrow Predecessor(1, setsize, setsize > 3) \leftarrow (m = 1 \wedge n = setsize > 3). \end{aligned} \tag{4.29}$$

For all $setsize > 3$, some elements are not ($\neg$) sequentially adjacent to every other element (not symmetric):

$$\begin{aligned} &\neg Adjacent(1, 3, setsize > 3) \\ &\qquad \leftarrow \neg Successor(1, 3, setsize > 3) \wedge \neg Predecessor(1, 3, setsize > 3). \quad \square \end{aligned} \tag{4.30}$$

5. Applications to physics

From the 3D proof (4.11), the interval lengths: t (time), m (mass), and q (charge) have different types (are from different sets) from a 3-dimensional distance interval length, r.

5.1. Axiom of geometric measures. An Euclidean-like volume around a coordinate (point) maps an amount on the r axis to proportionate amounts on the t, m, and q axes, which are expressed as constant, unit-factoring, conversion ratios:

$$r = (r_c/t_c)t = (r_c/m_G)m = (r_c/q_C)q. \tag{5.1}$$

5.2. Axiom of conservation. A change in one type of measure must cause an inversely proportionate change in another type of measure:

$$r = (t_c r_c)/t = (m_G r_c)/m = (q_C r_c)/q. \tag{5.2}$$

5.3. Spacetime and Lorentz equations. From the Euclidean volume proof (3.2), two disjoint intervals, $[0, r]$ and $[0, r']$, defines an Euclidean 2-volume. From the Minkowski distance proof (4.1), $\forall\, r \geq r' \,\exists\, r_v \in \mathbb{R} :\, r^2 = {r'}^2 + r_v^2$. And from the 3D proof (4.11), let u and v be constant, unit-factoring conversion ratios from some type, t, to distances r and r_v.

$$\begin{aligned} &\forall\, r \geq r' \,\exists\, r_v \in \mathbb{R} :\, r^2 = r'^2 + r_v^2 \quad \wedge \quad \exists\, u,\, v \in \mathbb{R} :\, r = ut \quad \wedge \quad r_v = vt \\ &\quad \Rightarrow \quad (ut)^2 = {r'}^2 + (vt)^2 \quad \Rightarrow \quad r' = \sqrt{(ut)^2 - (vt)^2} \;=\; ut\sqrt{1 - (v/u)^2}. \end{aligned} \tag{5.3}$$

Local (proper) distance, r', contracts relative to coordinate distance, r, as $v \to u$:

$$r' = ut\sqrt{1 - (v/u)^2} \quad \wedge \quad ut = r \quad \Rightarrow \quad r' = r\sqrt{1 - (v/u)^2}. \tag{5.4}$$

From equation 5.3, coordinate length, t, dilates relative to local length, t', as $v \to u$:

$$ut = r'/\sqrt{1 - (v/u)^2} \quad \wedge \quad r' = ut' \quad \Rightarrow \quad t = t'/\sqrt{1 - (v/u)^2}. \tag{5.5}$$

Using $r^2 = r'^2 + r_v^2$ from equation 5.3, where r_v is a 3-dimensional distance, one form of the flat Minkowski's spacetime event interval is:

$$\begin{aligned} &\mathrm{d}r^2 = \mathrm{d}r'^2 + \mathrm{d}r_v^2 \quad \wedge \quad \mathrm{d}r_v^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 \quad \wedge \quad \mathrm{d}(ut) = \mathrm{d}r \\ &\qquad\qquad \Rightarrow \quad \mathrm{d}r'^2 = \mathrm{d}(ut)^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2 - \mathrm{d}x_3^2. \end{aligned} \tag{5.6}$$

The Lorentz transformations follow from equation 5.4 and the Galilean transformation, r = r' + vt:

$$r' = r/\sqrt{1 - (v/u)^2} \quad \wedge \quad r = r' + vt \quad \Rightarrow \quad r' = (r - vt)/\sqrt{1 - (v/u)^2}. \tag{5.7}$$

$$\begin{aligned} &r' = (r - vt)/\sqrt{1 - (v/u)^2} \quad \wedge \quad r = ut \quad \wedge \quad r' = ut' \\ &\quad \Rightarrow \quad t' = (t - (vt/u))/\sqrt{1 - (v/u)^2} = (t - (vr/u^2))/\sqrt{1 - (v/u)^2}. \end{aligned} \tag{5.8}$$

5.4. Newton's gravity force and the constant, G. From equation 5.1:

$$\begin{aligned} &\forall\, m_1, m_2, m, r \in \mathbb{R} :\, m_1 m_2 = m^2 \quad \wedge \quad m = (m_G/r_c)r \\ &\quad \Rightarrow \quad m_1 m_2 = m^2 = ((m_G/r_c)r)^2 \quad \Rightarrow \quad (r_c/m_G)^2 m_1 m_2/r^2 = 1. \end{aligned} \tag{5.9}$$

$$r = r_c/t_c = ct \quad \wedge \quad mr = ((m_G/r_c)r)(ct) \quad \Rightarrow \quad mr = (m_G/r_c)(ct)^2. \tag{5.10}$$

$$mr = (m_G/r_c)(ct)^2 \quad \Rightarrow \quad ((r_c/m_G)/c^2)mr/t^2 = 1. \tag{5.11}$$

$$\begin{aligned} &((r_c/m_G)/c^2)mr/t^2 = 1 \quad \wedge \quad (r_c/m_G)^2 m_1 m_2/r^2 = 1 \\ &\quad \Rightarrow \quad F := mr/t^2 = ((r_c/m_G)c^2)m_1 m_2/r^2 = G m_1 m_2/r^2, \end{aligned} \tag{5.12}$$

where Newton's constant, $G = (r_c/m_G)c^2$, has the SI units: $m^3 \cdot kg^{-1} \cdot s^{-2}$.

5.5. Coulomb's charge force and constant. From equation 5.1:

$$r = (r_c/m_G)m = (r_c/q_C)q_1 \Rightarrow m = (m_G/q_C)q_1. \tag{5.13}$$

Substituting equations 5.13 and 5.1 into equation 5.12:

$$m = (m_G/q_C)q \quad \wedge \quad r_c/t_c = c \quad \wedge \quad F = ((r_c/m_G)c^2)m_1m_2/r^2 \\ \Rightarrow \quad F = (m_G/q_C)(r_c/q_C)(r_c/t_c)^2 q_1q_2/r^2. \tag{5.14}$$

$$a_G = r_c/t_c^2 \quad \wedge \quad F = (m_G/q_C)(r_c/q_C)(r_c/t_c)^2 q_1q_2/r^2 \\ \Rightarrow \quad F = (m_G a_G)(r_c/q_C)^2 q_1q_2/r^2 = k_e q_1q_2/r^2, \tag{5.15}$$

where Coulomb's constant, $k_e = (m_G a_G)(r_c/q_C)^2$, has the SI units: $N \cdot m^2 \cdot C^{-2}$.

5.6. Planck-Einstein equation: Applying both the relative measure ratios 5.1 and the conservation ratios 5.2:

$$m(ct)^2 = mr^2 \quad \wedge \quad mr = m_G r_c = k_m. \quad \Rightarrow \quad m(ct)^2 = k_m r. \tag{5.16}$$

$$m(ct)^2 = k_m r \quad \wedge \quad r_c/t_c = r/t = c \\ \Rightarrow \quad E = mc^2 = k_m r/t^2 = (k_m(r/t))\,(1/t) = (k_m c)(1/t) = hf, \tag{5.17}$$

where the Planck constant $h = k_m c$ and the frequency $f = 1/t$.

$$h = k_m c \quad \Rightarrow \quad k_m \approx 2.2102190943 \cdot 10^{-42}\ kg\ m. \tag{5.18}$$

5.7. MOND-like quantum-relativistic gravity. The total mass of a particle is $m = \sqrt{m_0^2 + m_{ke}^2}$, where m_0 is the rest mass and m_{ke} is the kinetic energy-equivalent mass. Applying both the relative measure ratios 5.1 and the conservation ratios 5.2:

$$m_0 = (m_G/r_c)r \quad \wedge \quad m_{ke} = m_G r_c/r \quad \wedge \quad m = \sqrt{m_0^2 + m_{ke}^2} \\ \Rightarrow \quad m = \sqrt{((m_G/r_c)r)^2 + ((m_G r_c)/r)^2}. \tag{5.19}$$

Applying equation 5.19 to equation 5.9:

$$\exists\, m : \ m_1m_2 = m^2 = ((m_G/r_c)r)^2 + ((m_G r_c)/r)^2 \\ \Rightarrow \quad m_1m_2/(((m_G/r_c)r)^2 + ((m_G r_c)/r)^2) = 1. \tag{5.20}$$

From equation 5.3, if r is the proper distance, then $r = \sqrt{(ct)^2 - (vt)^2}$:

$$r = \sqrt{(ct)^2 - (vt)^2} \quad \Rightarrow \quad m_0 r = (m_G/r_c)((ct)^2 - (vt)^2). \tag{5.21}$$

$$m_0 r = (m_G/r_c)((ct)^2 - (vt)^2) \quad \Rightarrow \quad ((r_c/m_G)/(c^2 - v^2))m_0 r/t^2 = 1. \tag{5.22}$$

$$((r_c/m_G)/(c^2 - v^2))m_0 r/t^2 = 1 \\ \wedge \quad m_1m_2/(((m_G/r_c)r)^2 + ((m_G r_c)/r)^2) = 1 \\ \Rightarrow \quad F := m_0 r/t^2 = ((m_G/r_c)(c^2 - v^2))m_1m_2/(((m_G/r_c)r)^2 + ((m_G r_c)/r)^2). \tag{5.23}$$

5.8. Quantum-relativistic charge. Applying $m = (m_G/q_C)q$ to the quantum-relativistic gravity equation (5.7):

$$F = (r_c/m_G)(c^2 - v^2)(m_G/q_C)^2 q_1q_2/(((m_G/r_c)r)^2 + ((m_G r_c)/r)^2). \tag{5.24}$$

6. Insights and implications

(1) Proving that volume and the distance are instances of the same abstract, set-based definition of a countable n-volume provides a unifying set and limit-based foundation under volume and distance without using the geometric primitives and relations required in Euclidean geometry [Joy98], axiomatic geometry [Lee10], and vector analysis [Wey52].

(2) The Gaussian distance, $d = (\sum_{i=1}^{m} a_i s_i^n)^{1/n}$, where a_i is a function returning a scalar value, inherits the properties of a metric space from being a bijective function of volumes (4.4). Note that the a_i function values correspond to the $g(i,i)$ components in the metric tensor, $g(\mu,\nu)$, used in Einstein's field (general relativity) equations [Wey52].

(3) The interval length, $s = b - a$, in the ruler measure (2.1) can be replaced with a $\pm$-signed integer length, where $s = (b-a \Leftarrow a = \omega : -(b-a) \Leftarrow b = \omega)$ and where ω is the local origin value. The $\pm$-signed interval lengths extends the set and limit-based volume and distance foundation, in this article, to include vectors.

(4) Euclid's proof that Euclidean distance is the smallest distance between two distinct points equate Euclidean distance to a straight line, where it is assumed that the straight line length is the smallest distance [Joy98]. And proofs that a straight line is the smallest distance equate the straight line to the Euclidean distance.

Using the calculus of variations for a shortest distance proof would result in circular logic due to the Euclidean assumptions in the definition of the Riemann and Lebesgue integrals.

It was proved that all "flat" distances are Minkowski distances (4.1). In a flat Euclidean 2-volume (flat area), the Minkowski distances, range from $1 \leq n \leq 2$, where $n = 1$ is the Manhattan (largest) distance case, $d = \sum_{i=1}^{m} s_i$, and $n = 2$ is the Euclidean (smallest) distance case, $d = (\sum_{i=1}^{m} s_i^2)^{1/2}$.

(5) Compare the distance sum inequality (4.3),

$$(\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n}, \tag{6.1}$$

used to prove that all Minkowski distances satisfy the metric space triangle inequality property (4.4), to Minkowski's sum inequality:

$$(\sum_{i=1}^{m}(a_i^n + b_i^n)^n)^{1/n} \leq (\sum_{i=1}^{m} a_i^n)^{1/n} + (\sum_{i=1}^{m} b_i^n)^{1/n} \tag{6.2}$$

[Min53]. Note the difference in the left side of each equation:

$$\forall\, n > 1,\ 0 < a_i^n, b_i^n < 1 :\ (\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \ > \ (\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n}. \tag{6.3}$$

$$\forall\, n > 1,\ a_i^n, b_i^n \geq 1 :\ (\sum_{i=1}^{m}(a_i^n + b_i^n))^{1/n} \ < \ (\sum_{i=1}^{m}(a_i^n + b_i^n)^{\mathbf{n}})^{1/n}. \tag{6.4}$$

The distance sum inequality is a more fundamental inequality because its proof does not require the convexity and various inequality theorems required to prove the Minkowski sum inequality. And the distance sum inequality is derived from the definitions of volume and distance, which makes it more directly related to geometry.

(6) From the 3D proof (4.11), more intervals than the 3 dimensions of distance intervals must have different types with lengths that are related to a 3-dimensional distance interval length, r, via constant, unit-factoring,

conversion ratios, both direct (5.1) and inverse (5.2) proportion ratios. In SI units:

$c_t = r_c/t_c \approx 2.99792458 \cdot 10^8 m\ s^{-1}$.
$c_m = r_c/m_G \approx 7.4261602691 \cdot 10^{-28} m\ kg^{-1}$.
$c_q = r_c/q_C \approx 8.6175172023 \cdot 10^{-18} m\ C^{-1}$.
$k_m = m_G r_c \approx 2.2102190943 \cdot 10^{-42}\ kg\ m$.
$k_q = q_C r_c \approx$ *To-Be-Determined* $C\ m$.
$k_t = t_c r_c \approx$ *To-Be-Determined* $s\ m$.

(7) The derivations in this article show that the spacetime (5.3), gravity force (5.10), charge force (5.15), and Planck (5.17) constants all depend on time being proportionate to distance: $r = (r_c/t_c)t = ct$. For example, $G = (r_c/m_G)c^2$, $k_e = (m_G/q_C)(r_c/q_C)c^2$, and $h = (m_G r_c)c = k_m c$.

(8) The ratios make all distance, wavelength, time, frequency, mass, charge, etc. sizes relative to each other. There are no absolute sizes > 0. That might imply there is no Planck-like (quantum) distance, time, mass, charge, etc.

(9) The derivations of the spacetime equations and Lorentz transformations, here (5.3), differ from other derivations.
   (a) The derivations, here, are much shorter and simpler.
   (b) The derivations of the spacetime equations, here, do not rely on the Lorentz transformations or Einsteins' postulates [**Ein15**]. The derivations do not even require the notion of light.
   (c) The derivations, here, rely only on geometry: the Euclidean volume proof (3.2), the Minkowski distances proof (4.1), and the 3D proof (4.11), which provides the insight that the geometry of physical space creates: 1) a maximum speed, c; 2) the spacetime equations; and 3) the Lorentz transformations.
   (d) The distance-to-mass ratio, $r = (r_c/m_G)m$, and distance-to-charge ratio, $r = (r_c/q_C)q$, can replace the distance-to-time ratio, $r = (r_c/t_c)t = ct$, in the spacetime derivations, to derive corresponding spacemass and spacecharge equations.

(10) Applying the ratios to derive Newton's gravity force (5.4) and Coulomb's charge force (5.5) equations provide:
   (a) Derivations that do not assume the inverse square law or Gauss's flux divergence theorem. **Note:** the components of the Ricci and metric tensors in Einstein's field equations have the units, 1/distance^2 [**Wey52**], which is an assumption of the inverse square law.
   (b) The first derivations to show that the inverse square law and the property of force as mass times acceleration are the result of the conversion ratios, $r = (r_c/t_c)t = (r_c/m_G)m$.

The quantum-relativistic extension to Newton's gravity equation (5.22) makes empirically verifiable predictions.
   (a) In Newton's gravity force, Gauss's gravity law, and Einstein's field (general relativity) equations, the force, $F \to \infty$ as the distance, $r \to 0$. But, in the quantum-relativistic extension to Newton's gravity equation, $F \to 0$ as $r \to 0$. Where the distance between particles is less than approximately $10^{-4}\ m$, the gravity force should start to measurably decrease, which implies larger black hole radii and maybe

allows black hole evaporation.

(b) Further, the quantum-relativistic gravity equation indicates that Newton's gravity constant, G, Gauss's constant, $4\pi G$, and Einstein's gravity constant, $k = 8\pi G/c^4$, [**Wey52**], are only valid where the local velocity, $v = 0$. Deviation from G as $v \to c$ should be measurable.

(c) Adapting the quantum-relativistic extension to Newton's gravity equation (5.22) to Einsteins field (general relativity) equations requires replacing G in the constant k with "$((m_G/r_c)(c^2 - v^2))$", and adding terms to Einstein's metric tensor, where the derivatives of the spacial components have the form "$1/(((m_G/r_c)r)^2 + ((m_G r_c)/r)^2)$".

(11) There is no constant ratio converting discrete state values to a continuously varying interval lengths. Therefore, the spin states of two quantum entangled particles and the polarization states of two quantum entangled photons are independent of varying distance and time interval lengths.

(12) Linear algebra, vector analysis, differential geometry, etc. assume any number of possible dimensions. For example, the Gram-Schmidt process is a method to find an orthogonal vector for any n-dimensional vector [**Coh21**]. None of those disciplines have exposed the properties that can limit a geometry to 3 dimensions. But the proof that a strict linearly ordered and symmetric set is a cyclic set of at most 3 members (4.11) is the simplest explanation for observing only 3 dimensions of physical space.

(a) If there are higher dimensions of ordered and symmetric geometric space, then there is a set of at most three members (4.11), each member being an ordered and symmetric set of 3 dimensions (three 3-dimensional balls).

(b) Each of 3 ordered and symmetric dimensions of space can have at most 3 sequentially ordered and symmetric state values, for example, an ordered and symmetric set of 3 vector orientations, $\{-1, 0, 1\}$, per dimension of space and at most 3 spin states per dimension, etc.

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