Driving Test 2008

Tuesday, 18 March 9am – 11am

Using the account details given below, attempt all 3 tasks. Write one C++ file per task. As your working directory you may choose `C:\temp\OOP_your_name`*. Make sure to write in the answers to each question on the* ANSWER SHEET*. Once you have finished, save the 3 files* `TaskN.cpp`, `N=1...3`, *to the directory* `C:\temp\SUBMIT_your_name` *and contact the invigilator.* **Do not log off!** *Hand in your answer sheet and present your submission directory. The invigilator will then reconnect the network cable so that you can email your solutions to* `rn@ic.ac.uk` *and to yourself. You are then free to leave.*

```
Username  :  *****
Password  :  *****
Domain    :  MA215-xx (or MA410-xx)   (this computer)
```

1. *Bubble sort* [**30 marks**]
   Read in a list of N `double` numbers from the file `data.txt` and then sort the list with the help of bubble sort (see below). You may assume that the list is not longer than $10,000$ elements. Output the sorted list to the file `sorted.txt` and state on screen how many swaps have taken place during the sorting process.

   Bubble sort:

   - repeat
   - for $i = 1, \ldots, N-1$: if $x_i > x_{i+1}$ then swap $x_i \leftrightarrow x_{i+1}$.
   - until no neighbours have been swapped

   [*Note: 75% of the marks for this task can be obtained by asking the user for the number N. Full marks for detecting N automatically from the given data file.*]

2. *Monte Carlo Integration* [**50 marks**]

   - *Pseudo random number generation*
     For this task, you will need only the following univariate random number generation algorithms. Here we assume that you are able to generate from a uniform $\sim \mathrm{U}[0,1]$.
     Box-Müller: Generate from $\mathrm{N}(0,1)$. Set $(X,Y) := \sqrt{-2\log U_2}\,(\cos(2\pi U_1), \sin(2\pi U_1))$, where $U_1, U_2 \sim \mathrm{U}[0,1]$, $U_2 \neq 0$. Then $X, Y \sim \mathrm{N}(0,1)$.
     [*Hint: Use* `pi=4*atan(1.0);` *or the macro* `M_PI` *for π.*]
   - *Monte Carlo integration*
     Suppose we wish to estimate the value of an integral $\boxed{\omega = \int h(x)\,\mathrm{d}x}$ which is analytically intractable. On writing the integral as $\omega = \int \phi(x)\, f(x)\,\mathrm{d}x$, where f is a probability density function (pdf) and $\phi := \frac{h}{f}$, the Monte Carlo estimator of the integral ω is defined as
     $$\hat{\omega} := \frac{1}{n}\sum_{i=1}^{n} \phi(X_i) = \frac{1}{n}\sum_{i=1}^{n} \frac{h(X_i)}{f(X_i)},$$
     where $X_i \sim f(\cdot)$ are independent pseudo random numbers. A very common example for the integral $\omega = \int_a^b h(x)\ \mathrm{d}x = \int h(x)\,\mathbb{I}_{[a,b]}\ \mathrm{d}x$ is to take a uniform distribution $f \sim \mathrm{U}[a,b]$ so that the above estimator reduces to $\hat{\omega} := \frac{b-a}{n}\sum\limits_{i=1}^{n} h(X_i)$.

Create an abstract base class `RV` that will hold information for univariate random variables. Define the pure `virtual` member functions `generate()`, `pdf(const double &)` and `cdf(const double &)`. These functions should return a pseudo random number of type `double` from the given distribution and, where this is possible, evaluate its pdf/cdf at the given argument, respectively. From the base class derive classes for some common random univariate variables and implement the above functions. You should include at least the following distributions: uniform $\mathrm{U}[a,b]$ and normal $\mathrm{N}(\mu,\sigma^2)$.

Then define a function (or a class) `MC` that performs a Monte Carlo integration for a given function and a given random distribution.

You could test your code with the integrals $\int_{\frac{1}{4}}^{1} 1 \, \mathrm{d}x = \frac{3}{4}$ and $\int_0^\infty e^{-x} \, \mathrm{d}x = 1$.

Hint: In order to be able to apply the MC code to any integral and to any distribution, you should always assume the integrand to be defined on the whole of $\mathbb{R}$. *E.g.* $\int_0^\pi \sin(x) \, \mathrm{d}x = \int_{-\infty}^{\infty} \sin(x) \, \mathbb{I}_{[0,\pi]} \, \mathrm{d}x$.

NOTE: Marks for the task below will only be awarded, if all the previous tasks have been completed to a satisfactory level.

3. *Templates* [**20 marks**]

   Adapt your code from 2. so that now you use templates. I.e. create an abstract base class template `RV<T>` that will hold information for different kinds of random variables. Later on, `T` could be a `double`, an `int` or a `point` in 2d. Define the pure `virtual` member functions `generate()`, `pdf(const T &)` and `cdf(const T &)`. These functions should return a pseudo random number of type `T` from the given distribution and, where possible, evaluate its pdf/cdf at the given argument, respectively.

   Observe that task 2. reduces to the class `RV<double>`.

   Then define a function (or a class) template `MC<T>` that performs a Monte Carlo integration for a given function and a given random distribution. Finally, provide classes `uniform2d` and `normal2d` that inherit from `RV<point>`, where `point` can be based on the class from Exercise 4.1 (*Shapes's areas*) or on the class `complex`, and that generate from a bivariate uniform and normal $\mathrm{N}(\underline{m},\Sigma)$, respectively. Here you may assume that $\Sigma \in \mathbb{R}^{2\times 2}$ is diagonal.

   You could test your code with the integrals $\int_{\frac{1}{4}}^{1} \int_{-4}^{4} 1 \, \mathrm{d}x \, \mathrm{d}y = 6$ and $\int_0^\infty \int_0^1 e^{-x} \, y \, \mathrm{d}y \, \mathrm{d}x = \frac{1}{2}$.

Name		CID	

Answer Sheet A

1. How many swaps take place when sorting the following lists?

List	Number of swaps
3 2 5 4 7 6 9 8 1 9.5 4.5	
10 9 8 7 6 5 4 3 2 1	

2. Test your code for the integral $\int_0^\pi \sin(x)\ \mathrm{d}x$. For $n = 1000$, and a test sample of $\{\hat{\omega}_j\}_{j=1}^{100}$, display the sample averages and sample variances in a table for the following distributions: $\mathrm{U}[0,\pi]$, $\mathrm{N}(0,1)$ and $\mathrm{N}(\frac{\pi}{2},1)$. Display all values to 4 decimal digits.

Distribution	Sample Mean	Sample Variance
$\mathrm{U}[0,\pi]$		
$\mathrm{N}(0,1)$		
$\mathrm{N}(\frac{\pi}{2},1)$		

3. Use your classes to approximately compute the integral

$$\int_{-1}^{1}\int_{-3}^{3} e^{-x^2+x\,y}\ \cos(\pi\,x)\ \sin(\pi\,y)\ \mathrm{d}x\ \mathrm{d}y$$

Display your results for $\hat{\omega}$ in the table below. Display all values to 4 decimal digits.

Distribution	$n = 100$	$n = 1000$	$n = 10000$