Hilbert spaces

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1 adjoints

Proposition 1.1. If an operator $T : \mathcal{H} \to \mathcal{H}$ has an adjoint then it is bounded.

Proof. T is bounded if and only if it is continuous. By the closed graph theorem, it suffices to show $\Gamma(T)$ is closed. In particular, we must show that if $x_n \to x$ and $Tx_n \to z$, then $Tx = z$. So suppose $x_n \to x$ and $Tx_n \to y$. Then, for any y,

$$\langle z, y \rangle = \langle Tx_n, y \rangle = \langle x_n, T^* y \rangle.$$

Taking limits,

$$\langle z, y \rangle = \langle x, T^* y \rangle.$$

But also $\langle x, T^* y \rangle = \langle Tx, y \rangle$, so

$$\langle z, y \rangle = \langle Tx, y \rangle.$$

Since this is true for all $y \in Y$, it must be that $Tx = z$ as desired. $\square$