# topology

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## 1 metric spaces

**Definition 1.1.** Let $(X, d)$ be a metric space. A *curve* in $X$ is a continuous mapping $\gamma : [0, 1] \to X$.

**Definition 1.2.** Let $\gamma$ be a curve in $(X, d)$. Let $P[0, 1]$ denote the set of partitions of $[0, 1]$. For each $\mathcal{P} \in P[0, 1]$, with components $p_0 < \cdots < p_n$, define

$$\ell_{\mathcal{P}}(\gamma) = \sum_{i=1}^{n} d(\gamma(p_{i-1}), \gamma(p_i)).$$

The *length* of $\gamma$ is

$$\ell(\gamma) = \sup_{\mathcal{P}} \ell_{\mathcal{P}}(\gamma).$$

**Proposition 1.3.**

$$\lim_{|\mathcal{P}| \to 0} \ell_{\mathcal{P}}(\gamma) = \ell(\gamma).$$

*Proof.* Fix $\epsilon > 0$. We need to find $\delta > 0$ such that $\ell(\gamma) - \epsilon < \ell_{\mathcal{Q}}(\gamma) \leq \ell(\gamma)$. By the definition of the supremum, there exists $\mathcal{P} \in P[0, 1]$ such that

$$\ell(\gamma) - \epsilon < \ell_{\mathcal{P}}(\gamma) \leq \ell(\sigma).$$

Let $\epsilon' > 0$ be arbitrary. Since $\sigma$ is continuous on a compact space, it is uniformly continuous (REF), hence there exists $\eta > 0$ such that $|t - t'| < \eta$ implies $d(\gamma(t), \gamma(t')) < \epsilon'$ for all $t, t' \in [0, 1]$. Let $\delta = \min(\eta, |\mathcal{P}|)$.

Suppose $\mathcal{Q} \in P[0, 1]$ is such that $|\mathcal{Q}| < \delta$. Say it is given by $q_0 < \cdots < q_m$. Let $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$. For each $i = 1, \ldots, n$, there exists a finite set $\{r_{i,k}\}_{k=0}^{K_i} \subset \mathcal{R}$ partitioning $[p_{i-1}, p_i]$. Then

$$\begin{aligned}
\ell_{\mathcal{R}}(\gamma)) &= \sum_{i=1}^{n} \sum_{k=1}^{K_i} d(\gamma(r_{i,k-1}, \gamma(i, r_k))) \\
&\leq \sum_{i=1}^{n} d(\gamma(\max_{q_j \leq r_{i,0}} q_j), \gamma(r_{i,0})) + \sum_{k=1}^{K_i} d(\gamma(r_{i,k-1}), \gamma(r_{i,k})) + d(\gamma(r_{i,K_i}), \gamma(\min_{q_j \geq r_{i,K_i}} q_j)) \\
&\leq \sum_{i=1}^{n} d(\gamma(\max_{q_j \leq r_{i,0}} q_j), \gamma(\min_{q_j \geq r_{i,K_i}} q_j))
\end{aligned}$$

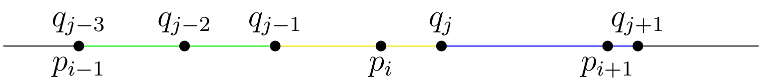


Figure 1: This is an example of what $\mathcal{P} \cup \mathcal{Q}$ could look like. The green line extends from $q_{j-3}$ to $q_j$ and represents the domain of one summand, and the blue line extends from $q_{j-1}$ to $q_{j+1}$ and represents the next summand. The yellow line from $q_{j-1}$ to $q_j$ represents the overlap; this region is "double counted".

$$\leq \ell_{\mathcal{Q}}(\gamma) + 2n\epsilon'.$$

Figure 1 demonstrates our strategy here.

Now

$$\ell(\gamma) - \epsilon < \ell_{\mathcal{P}}(\gamma) \leq \ell_{\mathcal{R}}(\gamma) \leq \ell_{\mathcal{Q}}(\gamma) + 2n\epsilon',$$
$$\ell_{\mathcal{R}}(\gamma) - 2n\epsilon' \leq \ell_{\mathcal{Q}}(\gamma).$$

But our choice of $\epsilon'$ was arbitrary; if we pick, say,

$$\epsilon' = \frac{1}{2n} \cdot \frac{\ell_{\mathcal{P}}(\gamma) - (\ell(\gamma) - \epsilon)}{4}$$

then we can choose $\delta$ in such a way that that, whenever $|\mathcal{Q}| < \delta$, we have

$$\ell(\gamma) - \epsilon < \ell_{\mathcal{P}}(\gamma) - 2n\epsilon' \leq \ell_{\mathcal{Q}}(\gamma) < \ell(\gamma)$$

as desired. □

**Proposition 1.4.** Let $s(t) = \ell(\gamma|_{[0,t]})$ for $0 \leq t \leq 1$. Then $s$ is continuous.

*Proof.* Let $\{t_n\}_1^\infty$ be a convergent sequence in $[0,1]$ with limit $t$. Then

$$\begin{aligned}\lim_{n\to\infty} s(t_n) &= \lim_{n\to\infty} \ell(\gamma|_{[0,t]}) \\ &= \lim_{n\to\infty} \lim_{|\mathcal{P}_n|\to 0, \mathcal{P}_n \in P[0,t_n]} \ell_{\mathcal{P}_n}(\gamma) \\ &= \lim_{|\mathcal{P}|\to 0, \mathcal{P}\in P[0,t]} \ell_{\mathcal{P}}(t) = \ell(\sigma|_{[0,t]}) \\ &= s(t).\end{aligned}$$

□