Name: ____________________________ PUID ____________________________

STAT 350 Worksheet #3

Conditional probability allows us to understand how the likelihood of one event changes when we have additional information about another event. It does not assume one event causes another. Instead, it refines our understanding of probabilities by narrowing down the possibilities based on what we know.
For example:

- If you know it is cloudy, the likelihood of rain may increase because clouds are often associated with rain.
- If you know the card drawn from a deck is a spade, it impacts the probability that the card is a face card.

For example, imagine you are selecting a gummy bear from one of several jars. The probability of selecting a red gummy depends on which jar you chose. If you know the jar you picked, you can narrow down the probability of getting a red gummy. Conditional probability helps us formalize this reasoning.

We write the conditional probability of drawing a red gummy from the **i^th^ jar** as:

$$\boldsymbol{P}(\textbf{Red Gummy}|\textbf{Jar}_{\boldsymbol{i}})$$

Here, the vertical bar | means **"given that."** It indicates that we are finding the probability of the event drawing a **Red Gummy**, given that the jar selected is $\textbf{Jar}_{\boldsymbol{i}}$.

1. To understand the concept of conditional probability, consider three jars filled with gummy bears:
   - $\textbf{Jar}_\mathbf{1}$ contains **30 red**, **10 green**, and **10 blue** gummies.
   - $\textbf{Jar}_\mathbf{2}$ contains **20 red** and **40 green** gummies.
   - $\textbf{Jar}_\mathbf{3}$ contains **35 yellow** gummies

   Answer the following questions using formal probability statements and clearly show your work:

   a. Suppose you were handed $\textbf{Jar}_\mathbf{3}$ and you randomly sample a single gummy bear. Compute the probability that the gummy bear you sampled would be **red**.

   b. Suppose instead you were handed $\textbf{Jar}_\mathbf{2}$ and you randomly sample a single gummy bear. Compute the probability that the gummy bear you sampled would be **red**.

   c. Suppose you were handed $\textbf{Jar}_\mathbf{1}$ and you randomly sample a single gummy bear. Compute the probability that the gummy bear you sampled would be either **red** or **blue**.

   d. Instead suppose you were not allowed to see the contents of the selected jar but are aware of the distribution of colors in each jar. You randomly sample **one gummy**, and it is **yellow**. Determine the probability that the gummy came from each of the three jars ($\textbf{Jar}_\mathbf{1}$, $\textbf{Jar}_\mathbf{2}$, or $\textbf{Jar}_\mathbf{3}$).

A tree diagram is a visual representation of the general multiplication rule. It illustrates all possible outcomes of a sequence of events, where each branch of the tree represents a possible event, and probabilities are assigned to these branches. By following the branches of the tree, we can calculate the probabilities of different outcomes by multiplying the conditional probabilities along the paths.
Tree diagrams also rely on the principles of **mutual exclusivity** and **exhaustiveness**:

- **Mutual exclusivity** ensures that each complete path through the tree represents a distinct and non-overlapping outcome.
- **Exhaustiveness** ensures that all possible outcomes are represented exactly once in the tree, so the probabilities of the mutually exclusive outcomes add up to 1.

Together, these two properties ensure that the tree diagram forms a **partition** of the sample space, dividing it into distinct and complete subsets of outcomes.

2. Considering the same jars of gummy bears, suppose you randomly select one jar, with each jar being **"equally likely"** to be chosen**:**

$$\boldsymbol{P}(\mathbf{Jar_1}) = \boldsymbol{P}(\mathbf{Jar_2}) = \boldsymbol{P}(\mathbf{Jar_3}) = \frac{\mathbf{1}}{\mathbf{3}}$$

Next you **sample two gummies** from the selected jar **without replacement**.

a. **Create a tree diagram** to represent all possible outcomes of sampling two gummies. The tree diagram should convey all probabilities clearly, showing:
   - Unconditional probabilities (e.g., the probability of selecting each jar).
   - Conditional probabilities (e.g., the probabilities of drawing specific gummy colors on the first and second draws, accounting for the reduced contents of the jar after the first draw).
   - Intersection probabilities (e.g., the probabilities at the end of each path, representing the probability of a specific sequence of events occurring).

b. Logically explain why your tree diagram satisfies both **mutual exclusivity** (each complete path represents a distinct, non-overlapping outcome) and **exhaustiveness** (all possible outcomes of the experiment are included in the diagram, and their probabilities sum to 1).

c. Using your tree diagram, compute the following probabilities (try to maintain mathematical formalism):

   i. The probability of drawing two yellow gummies.

   ii. The probability of drawing two blue gummies.

   iii. The probability of drawing two green gummies given that you know the samples came from $\mathbf{Jar_1}$.

   iv. The probability of drawing two green gummies given that you know the samples came from $\mathbf{Jar_2}$.

   v. Use the **law of total probability** to determine the probability of sampling two green gummies and relate it to the paths of the tree diagram.

Bayes' Theorem provides a way to compute **conditional probabilities** when directly calculating them is difficult. In particular, it allows us to express a probability in terms of its **reverse conditional**, which is often easier to determine. It helps answer questions like:

"Given an observed outcome, how should we determine the probability of an event that may have led to it?"

Beyond this basic use, Bayes' Theorem also provides a framework for updating probabilities when additional evidence is observed, refining our estimates step by step.

"Given an initial observation, how should we revise our probabilities after receiving additional evidence?"

The first application focuses on **computing a conditional probability** when its direct computation is complex, while the second extends this concept to **sequential updating**, where each new observation refines our understanding of the underlying probabilities.

3. To understand the concept of Bayes formula, consider the same three jars filled with gummy bears:
   - $\mathbf{Jar_1}$ contains **30 red**, **10 green**, and **10 blue** gummies.
   - $\mathbf{Jar_2}$ contains **20 red** and **40 green** gummies.
   - $\mathbf{Jar_3}$ contains **35 yellow** gummies

Suppose you randomly select one jar, with each jar being **"equally likely"** to be chosen. Next, you draw one gummy bear without looking at the contents of the jar and observe that it is **red**.

a. Compute the probability that the red gummy bear came from each jar ($\mathbf{Jar_1}$, $\mathbf{Jar_2}$, or $\mathbf{Jar_3}$).

b. Now assume you draw a **second gummy bear** from the same jar, and it is also red. Compute the updated probabilities that the jar you sampled from was from $\mathbf{Jar_1}$, $\mathbf{Jar_2}$, or $\mathbf{Jar_3}$. **Your tree diagram may be helpful in answering this question.**