Name: ____________________________ PUID ____________________________

STAT 350 Worksheet #2

In set theory, we often use symbols and notation to describe collections of objects, called sets. Here are some key symbols and their meanings:

- $\in$: Means **“is an element of.”** For example, $\boldsymbol{x} \in \boldsymbol{E}$ means $\boldsymbol{x}$ belongs to the set $\boldsymbol{E}$.
- $\mathbb{Z}$**:** Is a special symbol to denote a very specific set, the set of integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$.
- $\mathbb{N}$**:** Is a special symbol to denote a very specific set, the set of natural numbers$\{1, 2, 3, \dots\}$.
- $\cap$**:** This is referred to as a binary operator that represents the **intersection** of two sets. In English, we often associate the concept of **intersection** with the word **'and,'** as it includes only those elements that satisfy the conditions of being in both sets simultaneously.
- $\cup$**:** This is referred to as a binary operator that represents the **union** of two sets. In English, we often associate the concept of **union** with the word **'or,'** as it includes all elements that satisfy the condition of being in either one of the sets or in both.

Set-builder notation is a concise way to define a set by specifying the properties that its elements must satisfy. For example, consider the set $\boldsymbol{E} = \{\boldsymbol{x} \in \mathbb{Z}^{+} |\ \boldsymbol{x}\ \textbf{is even and}\ \boldsymbol{x} \leq \mathbf{10}\}$**.** This define $\boldsymbol{E}$ as the set of all positive integers x that are both **even** and **at most 10**.

1. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be sets defined as follows: $\boldsymbol{A} = \{\boldsymbol{x} \in \mathbb{Z} | -\mathbf{5} \leq \boldsymbol{x} \leq \mathbf{5}\}$ and $\boldsymbol{B} = \{\boldsymbol{x} \in \mathbb{N} | \boldsymbol{x}\ \textbf{is even and}\ \boldsymbol{x} \leq \mathbf{10}\}$.
   Further consider the sets $\boldsymbol{C} = \boldsymbol{A} \cap \boldsymbol{B}$ and $\boldsymbol{D} = \boldsymbol{A} \cup \boldsymbol{B}$.
   a. Write out the expanded set of elements of $\boldsymbol{A}$, then separately write out the expanded set of elements for$\boldsymbol{B}$.

   b. Determine the elements contained in $\boldsymbol{C}$, and separately determine the elements contained in $\boldsymbol{D}$.

   c. Using **set-builder notation,** express the set $\boldsymbol{C}$.

   d. Formally describe the set $\boldsymbol{D}$ in terms of the sets $\boldsymbol{A}$ and $\boldsymbol{B}$, combining English and the **'element of'** ($\in$) symbol.

Probability is a function $\boldsymbol{P}(\cdot)$ that takes a set (or event) $\boldsymbol{E}$ as input and outputs a real number p in the interval $[\mathbf{0},\mathbf{1}]$.

Axioms

1. For an event $\boldsymbol{E}$, $\mathbf{0} \leq \boldsymbol{P}(\boldsymbol{E}) \leq \mathbf{1}$. (I should never see a probability answer given that is negative or greater than 1.)
2. $\boldsymbol{P}(\boldsymbol{\Omega}) = \mathbf{1}$, where $\boldsymbol{\Omega}$ is a special symbol denoting the entire sample space.
3. For any event $\boldsymbol{E}$, we have that $\boldsymbol{P}(\boldsymbol{E}) = \sum_{\boldsymbol{\omega} \in \boldsymbol{E}} \boldsymbol{P}(\boldsymbol{\omega})$. In other words, we add up the probabilities of all the simple events in $\boldsymbol{E}$ to obtain the probability of the event.
4. It follows that $\boldsymbol{P}(\emptyset) = \mathbf{0}$, where $\emptyset$ is a special symbol denoting the **empty set** which is the set of no outcomes.

2. Using these axioms, answer the following questions:
   a. What does it mean for $\boldsymbol{P}$ to be a function that operates on sets rather than directly on elements of the sample space or numerical values? Why must the input to $\boldsymbol{P}(\cdot)$ always be a set?

   b. Explain why the following statement is not a valid probability expression: $\boldsymbol{P}(\boldsymbol{A}) \cap \boldsymbol{P}(\boldsymbol{B}) \cap \boldsymbol{P}(\boldsymbol{C})$.

   c. If $\boldsymbol{A} \subset \boldsymbol{B}$, use **axiom 3** to justify why $\boldsymbol{P}(\boldsymbol{A}) < \boldsymbol{P}(\boldsymbol{B})$.

   d. The complement of a set $\boldsymbol{E}$, denoted $\boldsymbol{E}'$, is defined as $\boldsymbol{E}' = \{\boldsymbol{\omega} \in \boldsymbol{\Omega} | \boldsymbol{\omega} \notin \boldsymbol{E}\}$. Using **axiom 2** and **axiom 3** to derive the complement rule $\boldsymbol{P}(\boldsymbol{E}') = \mathbf{1} - \boldsymbol{P}(\boldsymbol{E})$.

3. Let $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$ be two events of a sample space $\boldsymbol{\Omega}$, with known probabilities:

$$\boldsymbol{P}(\boldsymbol{E_1}) = \mathbf{0.3} \qquad \boldsymbol{P}(\boldsymbol{E_2}) = \mathbf{0.6} \qquad \boldsymbol{P}(\boldsymbol{E_1} \cup \boldsymbol{E_2}) = \mathbf{0.75}$$

Calculate the following probabilities. Make sure to write out the probability statements explicitly before performing any calculations and include all intermediate steps.

Why Formality and Intermediate Steps Matter:
Writing probability statements explicitly and showing intermediate steps ensures:

1. **Clarity:** Identifies the correct rules and logic to apply.
2. **Accuracy:** Reduces errors, especially in multi-step calculations or when overlapping probabilities are involved.
3. **Preparation for Complexity:** Builds habits needed for tackling more advanced probability and mathematics problems.
4. **Communication Skills:** Clear steps and reasoning improve the ability to explain and justify your work.

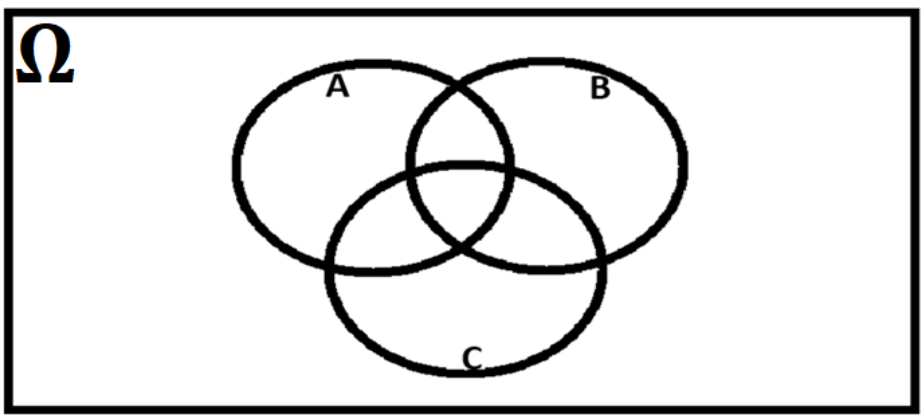
a) Calculate the probability that both $\boldsymbol{E_1}'$ and $\boldsymbol{E_2}'$ occur simultaneously.

b) Calculate the probability that both $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$ occur simultaneously.

c) Calculate the probability that both E_1' and E_2 occur simultaneously.

4. A festival raffle has a total of N tickets, divided into the following categories of winners:
   - $|A| = 40$: Tickets that win electronics.
   - $|B| = 30$: Tickets that win gift cards.
   - $|C| = 20$: Tickets that win home appliances.
   - $|A \cap B| = 10$: Tickets that win both electronics and gift cards.
   - $|A \cap C| = 5$: Tickets that win both electronics and home appliances.
   - $|B \cap C| = 3$: Tickets that win both gift cards and home appliances.
   - $|A \cap B \cap C| = 2$: Tickets that win in all three categories.
   - The remaining **432 tickets** do not win any prizes.

Fill in the Venn Diagram Below to help you solve the problems.



a. **Determine N:** Utilizing the **inclusion-exclusion principle** and **additional knowledge**, calculate the total number of tickets N. One form of the inclusion-exclusion principle for three sets states:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

b. **Probabilities:**
After determining N, calculate the following probabilities:
   I. The probability of randomly selecting a ticket that wins in **exactly one category**.

   II. The probability of randomly selecting a ticket that wins in **at least two categories**.

   III. The probability of randomly selecting a ticket that wins in **exactly two categories**.

   IV. The probability of randomly selecting a ticket that does **not win electronics** and **does not win any gift cards**.