Name: ______________________________ PUID ______________________________

STAT 350 Worksheet #4

Independence Property: The independence property states that if two events are known to be **independent** then the occurrence of one event does not affect the probability of the other. Similarly, two events are **dependent** if the occurrence of one event changes the probability of the other event occurring.

For two non-empty events $\boldsymbol{A}$ and $\boldsymbol{B}$ belonging to the same sample space $\boldsymbol{\Omega}$,

- If $\boldsymbol{A}$ is independent of $\boldsymbol{B}$, then it implies that:
  - $\boldsymbol{P(A|B) = P(A)}$
  - $\boldsymbol{P(B|A) = P(B)}$
  - The **special multiplication rule** also holds $\boldsymbol{P(A \cap B) = P(A)P(B)}$**.**
  - All complementary events are also independent.
    - Example: Suppose $\boldsymbol{A}$ is independent of $\boldsymbol{B}$

$$\boldsymbol{P(A'|B) = 1 - P(A|B)}$$
$$\boldsymbol{= 1 - P(A)}$$
$$\boldsymbol{= P(A')}$$

Therefore, $\boldsymbol{A'}$ is also independent of $\boldsymbol{B}$.

- If $\boldsymbol{A}$ and $\boldsymbol{B}$ are dependent, then
  - $\boldsymbol{P(A|B) \neq P(A)}$ and $\boldsymbol{P(B|A) \neq P(B)}$
  - Cannot use special multiplication rule! Must resort to **general multiplication rule** $\boldsymbol{P(A \cap B) = P(B|A)P(A) = P(A|B)P(A)}$ or other methods.

Always check what properties the sets you are working with have before trying to calculate any probabilities. Do not make up your own rules!

1. Let $\boldsymbol{A, B,}$ and $\boldsymbol{C}$ be three events belonging to the same sample space $\boldsymbol{\Omega}$ with the following probabilities:

$\boldsymbol{P(A) = 0.6}$	$\boldsymbol{P(A \cap B) = 0.6}$	$\boldsymbol{P(A \cap B \cap C) = 0.12}$
$\boldsymbol{P(B) = 0.7}$	$\boldsymbol{P(A \cap C) = 0.12}$	
$\boldsymbol{P(C) = 0.2}$	$\boldsymbol{P(B \cap C) = 0.14}$	

   a. Are the three events **mutually independent** (see lecture slides)? Either mathematically show they are mutually independent or provide a counter example to show they are not.

   b. Compute $\boldsymbol{P(A' \cap B' \cap C')}$ by combining De'Morgan's Law, Conplement Rule, and Inclusion Exclusion Principle.

   c. If you determined the sets are mutually **independent** recalculate $\boldsymbol{P(A' \cap B' \cap C')}$ utilizing the property of **mutually independence**. If you determine they were not **mutually independent,** still calculate $\boldsymbol{P(A' \cap B' \cap C')}$ as if they were and compare with your answer in **part b**.

2. Let $\boldsymbol{A}, \boldsymbol{B}$, and $\boldsymbol{C}$ be three events belonging to the same sample space $\boldsymbol{\Omega}$ with the following probabilities:

$\boldsymbol{P(A) = 0.6}$ $\qquad$ $\boldsymbol{P(A \cap B) = 0.42}$ $\qquad$ $\boldsymbol{P(A \cap B \cap C) = 0.084}$

$\boldsymbol{P(B) = 0.7}$ $\qquad$ $\boldsymbol{P(A \cap C) = 0.12}$

$\boldsymbol{P(C) = 0.2}$ $\qquad$ $\boldsymbol{P(B \cap C) = 0.14}$

   a. Are the three events **mutually independent** (see lecture slides)? Either mathematically show they are mutually independent or provide a counter example to show they are not.

   b. Compute $\boldsymbol{P(A' \cap B' \cap C')}$ by combining De'Morgan's Law, Complement Rule, and Inclusion Exclusion Principle.

   c. If you determined the sets are mutually **independent** recalculate $\boldsymbol{P(A' \cap B' \cap C')}$ utilizing the property of **mutually independence**. If you determine they were not **mutually independent,** still calculate $\boldsymbol{P(A' \cap B' \cap C')}$ as if they were and compare with your answer in **part b**.

It is important to understand the difference between **independent events** and **mutually exclusive (disjoint) events**—these concepts may sound similar but are fundamentally different!

- **Independent events** have no influence on each other. The outcome of one event does not affect the probability of the other event occurring. For example, flipping a coin and rolling a die are independent events because knowing the coin's result doesn't tell you anything about the die roll's outcome. Independent events can both happen together. In fact, their probability of occurring together is given by $\boldsymbol{P(A \cap B) = P(A)P(B)}$.
- **Mutually exclusive events**, on the other hand, cannot occur at the same time. If one event happens, the other is guaranteed not to happen. For instance, rolling a 6 and rolling an odd number on a single die are mutually exclusive because a single die roll cannot satisfy both conditions at once. Mutually exclusive events can **never** happen together. If events A and B are mutually exclusive, $\boldsymbol{P(A \cap B) = 0}$.

3. In a distant galaxy, treasure hunters search for magical chests containing two gems: one from the Red Nebula and one from the Blue Comet. Each gem is assigned an integer value from 1 to 100, chosen randomly with equally likely outcomes.

Consider these events for a given chest:

- **Event R:** The Red Nebula gem's value is a prime number. *Recall a prime number is a positive integer greater than 1 that has exactly two distinct positive divisors: 1 and itself. In other words, a number p is prime if it cannot be divided evenly by any positive integer other than 1 and p.*
- **Event B:** The Blue Comet gem's value is a perfect square. A perfect square is a number that can be expressed as the square of an integer. In other words, if $\boldsymbol{n}$ is a perfect square, there exists some integer $\boldsymbol{k}$ such that: $\boldsymbol{n = k^2}$.
- **Event T:** The total value (Red + Blue) is greater than 120.
- **Event U:** The total value (Red + Blue) is less than or equal to 120.

   a. Are the events $\boldsymbol{R}$ and $\boldsymbol{B}$ **independent** or **mutually exclusive**? Justify your answer.

   b. Are the events $\boldsymbol{T}$ and $\boldsymbol{U}$ **independent** or **mutually exclusive**? Justify your answer.

A **random variable (r.v.)**, is a **function** that maps is a function that maps each outcome $\boldsymbol{\omega}$ in a sample space $\boldsymbol{\Omega}$ to a unique numerical value. That is, for any outcome $\boldsymbol{\omega} \in \boldsymbol{\Omega}$, the random variable produces a value $\boldsymbol{X}(\boldsymbol{\omega})$. Despite its name, a random variable is not truly a "variable" in the traditional sense, nor is it inherently random. Instead, it is a **deterministic** function that maps **random outcomes** to **numerical values**. By translating random outcomes into numbers, random variables provide a framework to analyze and understand random processes systematically.

Probabilities are assigned to events through the inverse mapping of the random variable. The probability that a random variable $\boldsymbol{X}$ takes on a specific value $\boldsymbol{x}$, denoted as $\boldsymbol{P}(\boldsymbol{X} = \boldsymbol{x})$ corresponds to the probability of the set of outcomes $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ for which $\boldsymbol{X}(\boldsymbol{\omega}) = \boldsymbol{x}$. Similarly, for an interval, $\boldsymbol{P}(\boldsymbol{a} \leq \boldsymbol{X} \leq \boldsymbol{b})$ is the probability of the set of outcomes for which $\boldsymbol{a} \leq \boldsymbol{X}(\boldsymbol{\omega}) \leq \boldsymbol{b}$.

4. In a secured vault, a sealed treasure chest is known to contain exactly three coins. Each coin is selected independently from a set of two denominations with the following values and probabilities:
   - **Gold Coin**: **100 units** (**probability 0.2**)
   - **Silver Coin**: **20 units** (**probability 0.8**)

Let $\boldsymbol{X}$ be the random variable representing the total monetary value of the three coins drawn from the chest. Although X deterministically maps each outcome (i.e., a specific sequence of three coins) to a numerical total, the probability associated with any total value is determined by the inverse mapping from that value back to the outcomes in the sample space.

a. What are the possible values that $\boldsymbol{X}$ can take on with positive probability? This collection of values is called the **support** of the **random variable**. Note that these values arise from the different combinations of gold and silver coins and correspond to the invers mapping $\boldsymbol{X}^{-1}(\{\boldsymbol{x}\})$.

$\mathbf{Supp}(\boldsymbol{X}) = \{\boldsymbol{x} \in \mathbb{R} | \boldsymbol{P}(\boldsymbol{X} = \boldsymbol{x}) > \mathbf{0}\} = \{ \qquad \}$

b. For each value of $\boldsymbol{x}$ in the support of $\boldsymbol{X}$, list the specific outcomes (coin sequences) in the sample space that map to $\boldsymbol{x}$ (i.e., describe the inverse image $\boldsymbol{X}^{-1}(\{\boldsymbol{x}\})$). Explain how these outcomes collectively determine the probability $\boldsymbol{P}(\boldsymbol{X} = \boldsymbol{x})$.

c. Calculate the following probability $\boldsymbol{P}(\boldsymbol{X} = \mathbf{300})$ by summing the probabilities of all outcomes for which $\boldsymbol{X}(\boldsymbol{\omega}) = \mathbf{300}$. Hint: Each coin is selected independently.

For the rest of the semester, we will not use functional notation $\boldsymbol{X}(\boldsymbol{\omega})$ or explicitly refer to the inverse mapping $\boldsymbol{X}^{-1}(\{\boldsymbol{x}\})$, as these details are cumbersome and unnecessary. However, it is important to understand that a random variable assigns a numerical value to each outcome (the thing that is random) in the sample space.

A **probability distribution** for a random variable specifies the probabilities associated with all its potential values. When a random variable is discrete, we refer to the probability distribution as a **probability mass function (pmf)**.
The support of a random variable is defined as
Symbolic Representation: $\boldsymbol{p}_{\boldsymbol{X}}(\boldsymbol{x}) = \boldsymbol{P}(\boldsymbol{X} = \boldsymbol{x})$
The **support** of a **discrete random variable** (r.v.), is the set of all possible values that have a **strictly positive probability** with respect to the probability mass function.

$$\mathbf{Supp}(\boldsymbol{X}) = \{\boldsymbol{x} \in \mathbb{R} \mid \boldsymbol{p}_{\boldsymbol{X}}(\boldsymbol{x}) > \mathbf{0}\}$$

For probability mass functions to be valid it must satisfy the following axioms.

1. **Probabilities must be between 0 and 1 inclusive:** $\mathbf{0} \leq p_X(x) \leq \mathbf{1}$ for all $x \in \mathbb{R}$.
2. **The probabilities must sum to 1**: $\sum_{x \in \text{Supp}(X)} p_X(x)$

5. The table below defines a possible probability mass function.

x	-3	-2	-1	0	1	2	3
$p_X(x)$	k	k	$2k$	$8k$	$2k$	k	k

   a. Find a value $\boldsymbol{k}$ such that makes it a valid **pmf**.

   b. Determine the probability that $\boldsymbol{X}$ takes on non-negative values.

   c. Determine the following probability $P(\{X = -1\} \cup \{X = -2\} \cup \{X = 1\} \cup \{X = 2\})$.

   d. Given that $\boldsymbol{X}$ takes on non-negative values determine the probability that it takes strictly positive values.

The probability of two discrete random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$ is defined jointly and called the **joint probability mass function**.

Symbolic Representation: $p_{X,Y}(x, y) = P(\{X = x\} \cap \{Y = y\})$

6. The following is a joint probability mass function for two random variables $\boldsymbol{X}$ and $\boldsymbol{Y}$.

x, y	1	2	3	4
1	$p_{X,Y}(1,1) = \frac{1}{144}$	$p_{X,Y}(1,2) = \frac{1}{72}$	$p_{X,Y}(1,3) = \frac{1}{288}$	$p_{X,Y}(1,4) = \frac{41}{288}$
2	$p_{X,Y}(2,1) = \frac{1}{144}$	$p_{X,Y}(2,2) = \frac{93}{144}$	$p_{X,Y}(2,3) = \frac{1}{144}$	$p_{X,Y}(2,4) = \frac{1}{144}$
3	$p_{X,Y}(3,1) = \frac{1}{144}$	$p_{X,Y}(3,2) = \frac{1}{72}$	$p_{X,Y}(3,3) = \frac{1}{288}$	$p_{X,Y}(3,4) = \frac{41}{288}$

   a. Determine the marginal distribution for $\boldsymbol{X}$.

   b. Determine the marginal distribution for $\boldsymbol{Y}$.

   c. If a random trial is performed find the probability that $\boldsymbol{Y}$ takes an even value given that $\boldsymbol{X}$ takes an odd value.