A Geometric Construction for the Evaluation of Mean Curvature

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February 26, 2019

Abstract

We give a relationship that yields an effective geometric way of evaluating mean curvature of surfaces. The approach is reminiscent of the Gausss contour based evaluation of intrinsic curvature. The presented formula may have a number of potential applications including estimating the normal vector and mean curvature on triangulated surfaces. Given how brief is its derivation, it is truly surprising that this formula does not appear in the existing literature on differential geometry at least according to the authors search. We hope to learn about a reference containing this result.

1 Overview

1.1 Theoretical Framework

The presented formula admits a number of straightforward generalizations including to higher dimensions and submanifolds of co-dimension other than one. Here, we discuss the formula for a two dimensional surface embedded in a three dimensional Euclidean space.

Consider a regular patch P within a smooth surface S. Supposed that the area of P is A and that its smooth contour boundary is Γ. Let $\mathbf{N}$ be the normal to the surface and $\mathbf{n}$ be the exterior normal to Γ within S. Let B_{α}^{α} denote the mean curvature of S consistent with the direction of N [1], [2]. Then

$$\int_P \mathbf{N} B_{\alpha}^{\alpha} dS = \int_{\Gamma} \mathbf{n} d\Gamma \tag{1}$$

In particular, for a patch shrinking to a point we have,

$$\mathbf{N} B_{\alpha}^{\alpha} = \lim \frac{1}{A} \int_{\Gamma} \mathbf{n} d\Gamma \tag{2}$$

The right hand side of equation (2) is of coordinate free form and yields a pure geometric interpretation of mean curvature.

One immediate corrolary of equation (1) is that the integral $\int_{\Gamma} \mathbf{n} d\Gamma$ vanishes for any contour in the interior of a minimal surface.

The derivation is quite brief and we cannot help but wonder whether equation (1) could have really been overlooked. The quantity $\mathbf{N} B_{\alpha}^{\alpha}$ equals the surface divergence of the surface covariant basis $\mathbf{S}_{\alpha}$:

$$\mathbf{N} B_{\alpha}^{\alpha} = \nabla^{\alpha} \mathbf{S}_{\alpha} \tag{3}$$

By Gauss' theorem on the patch P, we have

$$\int_P \nabla^{\alpha} \mathbf{S}_{\alpha} dS = \int_{\Gamma} n^{\alpha} \mathbf{S}_{\alpha} d\Gamma \tag{4}$$

Recognizing that $n^{\alpha} \mathbf{S}_{\alpha} = \mathbf{n}$, we arrive at formula (1).

Formula (1) suggests a natural way of carrying over the concepts of the normal vector and mean curvature to triangulated surfaces. We note in advace that the proposed calculation yields an object analogous to the product $\mathbf{N} B_{\alpha}^{\alpha}$ is a more fundamental object than either $\mathbf{N}$ or B_{α}^{α} and more often than not B_{α}^{α} arises precisely in the combination $\mathbf{N} B_{\alpha}^{\alpha}$.

1.2 Geometric Construction

Suppose that on a triangulated surface, triangles T_i, typically six in number with areas A_i meet at the node O. Let $\mathbf{n}_i$ be unit vectors in the plane of T_i perpendicular to the triangle edges opposite of O and let $\mathbf{n}_i$ point away from O. Then the vector quanity

$$\mathbf{B} = \frac{\sum a_i \mathbf{n}_i}{\sum A_i} \tag{5}$$

may be taken as the definition of vector mean curvature for triangulated surface.

The quantity $\mathbf{B}$ has a number of properties in common with its continuous analogue $\mathbf{N}B_\alpha^\alpha$ including:

1. $\mathbf{B} = 0$ for flat surfaces
2. More generally, $\mathbf{B}$ is the gradient of the total area with respect to the location of the vertices. Therefore, $\mathbf{B} = 0$ for minimal triangulated surfaces.

In conclusion, the central relationships (1) and (2) are analogous to Gauss' geometric construction for intrinsic curvature and may lead to effective computational applications.

1.3 Correction

While the above technique is readily implemented in a numerical framework for triangulated surfaces, a minor correction has to be introduced to account for the fact even for a very fine meshes, smooth, curved surfaces are approximated by flat faces. As such, consider a point O on the triangulated surface, surrounded by triangles of arbitrary orientations.

Consider a sphere that has been triangulated and consider only the triangulation around one point. Let the origin be at the center of the circle, and let R be the radius of this circle. Define point O to be the point we wish to find the curvature at, and let $R\hat{o}$ be its position. Let points P_i be the points which O shares edges with, and let them have positions $R\hat{p}_i$. Because all points lie on the sphere, all these vectors have magnitude R. Note that the real number R, unit vector $\hat{o}$, and unit vectors $\hat{p}_i$ for all i perfectly parametrize the problem for the case of an arbitrary triangulation. The question is then: what is the curvature about point $R\hat{o}$?

1.4 Solution

We know that the curvature $\vec{B}$ for a triangulation is:

$$\vec{B} = \frac{\sum_i a_i \cdot \hat{n}_i}{\sum_i A_i} \tag{6}$$

Where A_i is the area of the ith triangle, a_i is the exterior edge length of the ith triangle, and $\hat{n}_i$ is the exterior normal of the ith triangle. We can combine $a_i \cdot \hat{n}_i$ into the exterior normal vector $\vec{n}_i$ that points in the direction of the exterior normal and has a magnitude given by the length of the exterior edge, a_i. We can calculate this by taking the cross product of the vector defining the exterior edge of the ith triangle, $R(\hat{p}_{i+1} - \hat{p}_i)$, and the unit surface normal of the ith triangle, $\hat{N}_i$.

Consider the ith triangle, which will be defined by the points $R\hat{o}$, $R\hat{p}_i$, and $R\hat{p}_{i+1}$. Consider the cross product of the edges of this triangle: $R(\hat{p}_i - \hat{o}) \times R(\hat{p}_{i+1} - \hat{o}) = R^2(\hat{p}_i - \hat{o}) \times (\hat{p}_{i+1} - \hat{o})$. Define the surface normal vector the following way for reasons of convenience:

$$\vec{N}_i = (\hat{p}_i - \hat{o}) \times (\hat{p}_{i+1} - \hat{o}) \tag{7}$$

This has a non-unitary magnitude and points in the direction of the unit surface normal. Then, since the area of a triangle is half the magnitude of the cross product of the edges, $A_i = \frac{1}{2}R^2\|\vec{N}_i\|$.

The exterior normal vector $\vec{n}_i = a_i\hat{n}_i$ can be found by taking the cross product of the exterior edge vector with the unit surface normal: $\vec{n}_i = R(\hat{p}_{i+1} - \hat{p}_i) \times \frac{\vec{N}_i}{\|\vec{N}_i\|}$. Then:

$$
\begin{aligned}
\vec{B} &= \frac{\sum_i \vec{n}_i}{\sum_i A_i} \\
&= \frac{\sum_i R(\hat{p}_{i+1} - \hat{p}_i) \times \frac{\vec{N}_i}{\|\vec{N}_i\|}}{\sum_i \frac{1}{2} R^2 \|\vec{N}_i\|} \\
&= \frac{2}{R} \left(\frac{\sum_i (\hat{p}_{i+1} - \hat{p}_i) \times \frac{\vec{N}_i}{\|\vec{N}_i\|}}{\sum_i \|\vec{N}_i\|} \right)
\end{aligned} \tag{8}
$$

2 A Triangulation of Congruent Isosceles Triangles

2.1 Problem

Now consider the particular case where the triangulation around a particular point O on the surface of a sphere (at position vector $R\hat{o}$) is composed of congruent isosceles triangles so that the boundary is a regular polygon. This is important in the commonly occurring case where the triangulation algorithm tries to create triangles that are as equilateral as possible (e.g., in a Delaunay triangulation). What is the curvature at point O both in general and as the triangulated mesh gets finer?

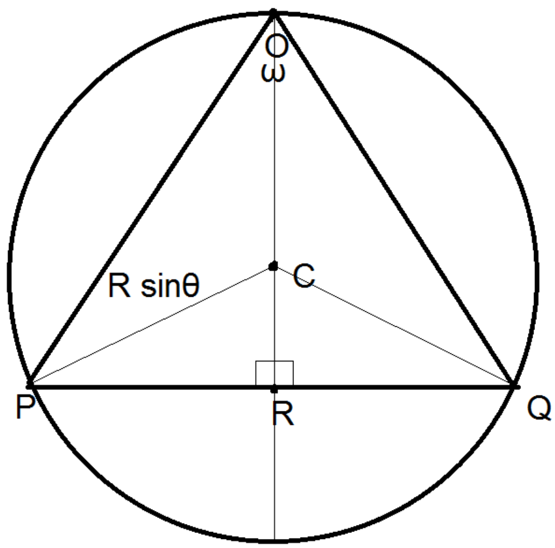
2.2 Solution

We are interested in the curvature of point O on the surface of a triangulated sphere of radius R. Suppose this point is surrounded by congruent isosceles triangles whose surface normals make an angle θ with the position vector for point O. As this mesh gets finer and finer, θ will get smaller, but we will take this limit later and for now simply call this angle θ. Also suppose that there are n faces incident at point O. As we will see, these three variables, θ, n, and R, will be enough to completely parametrize this problem.

Because all of the triangles are congruent, we can take the ratio of the component of the exterior normal pointing in the $-\hat{o}$ direction to the area for only one triangle and that will be equal to the curvature of point O. Therefore: what is the exterior normal and the area of one of these incident triangles?

First, let us see what we can derive from knowing that the surface normals of the triangles make an angle θ with $\hat{o}$. These triangles also have the point O at one of their corners. Consider all the planes with a surface normal that makes an angle θ with $\hat{o}$ and intersect with the sphere on the point O. All the incident triangles must lie on these planes. These planes intersect with the sphere to form a circle of radius $R\sin\theta$. All the corners of the triangle must lie on this circle because this circle is defined by the constraints that its points lie both on the plane and on the surface of the sphere, which are the same constraints that define the corners of the triangle. Because all the corners lie on the circle, the center of this circle must be the circumcenter of the triangle. We have thus determined that each triangle is circumscribed by a circle of radius $R\sin\theta$.

Let us define a new variable ω whose value we will find later, and say that for each of the triangles, the corner at point O has angle ω. The other two corners, therefore, have angle $\frac{\pi}{2} - \frac{\omega}{2}$. Let us call the other two corners of the triangle P and Q. Recall that this triangle is circumscribed in a circle of radius $R\sin\theta$. Call the center of this circle point C. Call the midpoint of PQ point R. We will try to find the base (length of PQ) and the height (length of OR) of this triangle. From that, we can find the area. Then, after finding the tilt of this triangle from its normal vector, we can find the exterior normal, and then calculate the curvature at point O.



Consider the triangle PCR. Because the angle POQ measures ω, the angle PCQ must measure 2ω, and the angle PCR must measure ω. Because PC is the radius of the circle, $R\sin\theta$, length PR must measure $(R\sin\theta)(\sin\omega)$. Then, because R bisects PQ, length PQ must measure $2R\sin\theta\sin\omega$.

Next, consider the triangle POR. We will find the height of the triangle, length OR. We know that length PR measures $R\sin\theta\sin\omega$, so length OR must be this length times the tangent of angle OPR, $\frac{\pi}{2}-\frac{\omega}{2}$.

Now, consider the tilt of the plane containing this triangle, because the exterior normal will be pointing in this direction. By symmetry, only the component of the exterior normal in the $-\hat{o}$ direction will contribute towards the curvature of point O. Consider the same geometry described but on a sphere so that points O, P, and Q are on the surface of the sphere. Call the center of the sphere point S. Then, the angle OSC must measure θ because SC is normal to the plane of the triangle. Also, CR is perpendicular to both the surface normal of the triangle and the exterior edge of the triangle PQ, so it must point in the same direction as the exterior normal. The projection of the unit vector in this direction onto the $-\hat{o}$ direction can be derived to be $\sin\theta$, so to find the curvature, we must multiply the length of the exterior edge with $\sin\theta$ and divide by the area.

Call half the base of the triangle, length PR, $\frac{1}{2}b = R\sin\theta\sin\omega$, and the height of the triangle, length OR, $h = \frac{1}{2}b\cdot\tan\left(\frac{\pi}{2}-\frac{\omega}{2}\right)$. Then, the curvature at point O is:

$$
\begin{aligned}
B &= \frac{l \cdot \sin\theta}{\frac{1}{2} l \cdot h} \\
&= \frac{2\sin\theta}{h} \\
&= \frac{2}{R} \cdot \frac{\sin\theta}{\sin\theta \sin\omega \cdot \tan\left(\frac{\pi}{2} - \frac{\omega}{2}\right)} \\
&= \frac{2}{R} \cdot \frac{\tan\frac{\omega}{2}}{\sin\omega} \\
&= \frac{2}{R} \cdot \frac{\sin\frac{\omega}{2} / \cos\frac{\omega}{2}}{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}} \\
&= \frac{2}{R} \cdot \frac{1}{2\cos^2\frac{\omega}{2}} \\
&= \frac{2}{R} \cdot \frac{1}{2\left(\sqrt{\frac{1+\cos\omega}{2}}\right)^2} \\
&= \frac{2}{R} \cdot \frac{1}{1+\cos\omega}
\end{aligned}
\tag{9}
$$

For fine meshes in which n triangles are incident at point O, $\cos\omega$ approaches $\cos\frac{2\pi}{n}$, which is the value for a planar regular polygon. The most common value of n in such a triangulation is 6. In this case, $\cos\omega = \frac{1}{2}$, and:

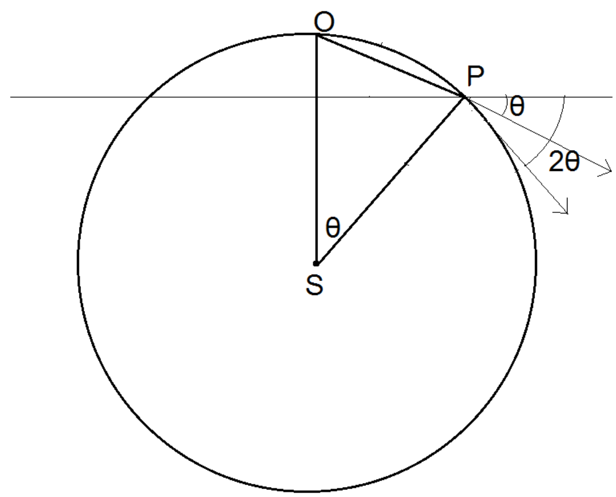
$$
B = \frac{4}{3} \cdot \frac{1}{R} \tag{10}
$$

Which is indeed the value that has been found in simulations.

A more intuitive explanation of why the correction factor is needed is that it corrects for two things:

1. There are only a finite number of triangles incident at a given point. If we were to approximate the continuous case, not only would the mesh have to get finer and finer, but the number of incident triangles at a point would have to approach infinity. This would mean ω approaches 0, which implies B approaches $1/R$. However, this is still off by a factor of 2.

2. The exterior normal vectors in the triangulated case do not approach the exterior normal vectors in the continuous case even as the mesh gets finer and finer and the number of incident triangles approaches infinity. For a sphere, the angle between the exterior normal vector and the tangent vector at point O is always half what it should be. This is demonstrated in the figure below. With a small-angle approximation for θ, this leads to the curvature being half of what it should be. Accounting for this source of error, B approaches $2/R$.



The next two sections relax the assumption of congruent isosceles triangles and refine the correction factor until it is nearly exact.

3 A Triangulation of Non-Congruent Isosceles Triangles

3.1 Problem

Now we keep the assumption that the triangles surrounding O are isosceles but relax the assumption that they are congruent. What correction can we make so that the curvature comes out to be $2/R$?

3.2 Solution

Suppose that the angle of the normal with $\hat{o}$ for the ith triangle is θ_i and the angle of the ith triangle at point O is ω_i. By the same arguments from the previous section, this triangle must be circumscribed by a circle of radius $R\sin\theta$. Since this triangle is isosceles, by the same line of reasoning as in the previous section:

1. Its base must be $l_i = 2R\sin\theta_i\sin\omega_i$.
2. Its height must be:

$$\begin{aligned} h_i &= \frac{1}{2}l_i\tan\left(\frac{\pi}{2}-\frac{\omega_i}{2}\right) \\ &= R\sin\theta_i\sin\omega\cot\left(\frac{\omega_i}{2}\right) \\ &= 2R\sin\theta_i\sin\left(\frac{\omega_i}{2}\right)\cos\left(\frac{\omega_i}{2}\right)\cot\left(\frac{\omega_i}{2}\right) \\ &= 2R\sin\theta_i\cos^2\left(\frac{\omega_i}{2}\right) \\ &= 2R\sin\theta_i\left(\frac{1+\cos\omega_i}{2}\right) \\ &= R\sin\theta_i(1+\cos\omega_i) \end{aligned} \tag{11}$$

3. Its area must therefore be:

$$
\begin{aligned}
A_i &= \frac{1}{2} l_i h_i \\
&= \frac{1}{2}(2R\sin\theta_i \sin\omega_i)(R\sin\theta_i(1+\cos\omega_i)) \\
&= R^2 \sin^2\theta_i \sin\omega_i(1+\cos\omega_i)
\end{aligned} \tag{12}
$$

4. The contribution of exterior normal vector from the ith triangle must be $n_i = l_i \sin\theta_i$ because this is its projection onto onto $\hat{o}$ (the analytical normal). Therefore:

$$
\begin{aligned}
n_i &= l_i \sin\theta_i \\
&= 2R\sin^2\theta_i \sin\omega_i
\end{aligned} \tag{13}
$$

A critical assumption was made here. The assumption is that the calculated curvature vector will point in the direction of $\hat{o}$, the analytical normal. Previously, symmetry arguments were used to show that in the case of congruent triangles, this is exactly correct, but this is not the case here. So far simulations show that the curvature vector lies very close to the analytical normal, and gets closer with increasing triangulation, but it is usually not exactly the same. However, the correction factor we find has been shown to work well on real triangulations.

Putting all this together, the curvature must be:

$$
\begin{aligned}
B &= \frac{\sum_i n_i}{\sum_i A_i} \\
&= \frac{\sum_i 2R\sin^2\theta_i \sin\omega_i}{\sum_i R^2 \sin^2\theta_i \sin\omega_i(1+\cos\omega_i)} \\
&= \frac{2}{R} \cdot \frac{\sum_i \sin^2\theta_i \sin\omega_i}{\sum_i \sin^2\theta_i \sin\omega_i(1+\cos\omega_i)}
\end{aligned} \tag{14}
$$

Because this is a sphere, we know that the curvature should be $\frac{2}{R}$. Therefore, we should apply some correction so that the fraction above comes out to be 1. We can either multiply the exterior length of each triangle by a factor or the area by a factor. This factor should not depend on θ, only the planar properties of the triangle. Furthermore, if we multiply the length of each triangle by a correction factor, when the exterior normals are added up, they might not point in the direction we want them to. Therefore, the solution here is to multiply each area by a correction factor that is dependent only on a triangle's planar geometry.

Suppose we multiplied the area of triangle i by the factor $Q_i = \frac{1}{1+\cos\omega_i}$. Then we would get:

$$
\begin{aligned}
B &= \frac{\sum_i n_i}{\sum_i A_i Q_i} \\
&= \frac{\sum_i 2R\sin^2\theta_i \sin\omega_i}{\sum_i R^2 \sin^2\theta_i \sin\omega_i(1+\cos\omega_i)\frac{1}{1+\cos\omega_i}} \\
&= \frac{2}{R} \cdot \frac{\sum_i \sin^2\theta_i \sin\omega_i}{\sum_i \sin^2\theta_i \sin\omega_i} \\
&= \frac{2}{R}
\end{aligned} \tag{15}
$$

Which is exactly the answer we want to get. Therefore, the solution is to multiply the area of triangle i by $Q_i = \frac{1}{1+\cos\omega_i}$ before adding them up.

4 A Triangulation of Arbitrary Triangles

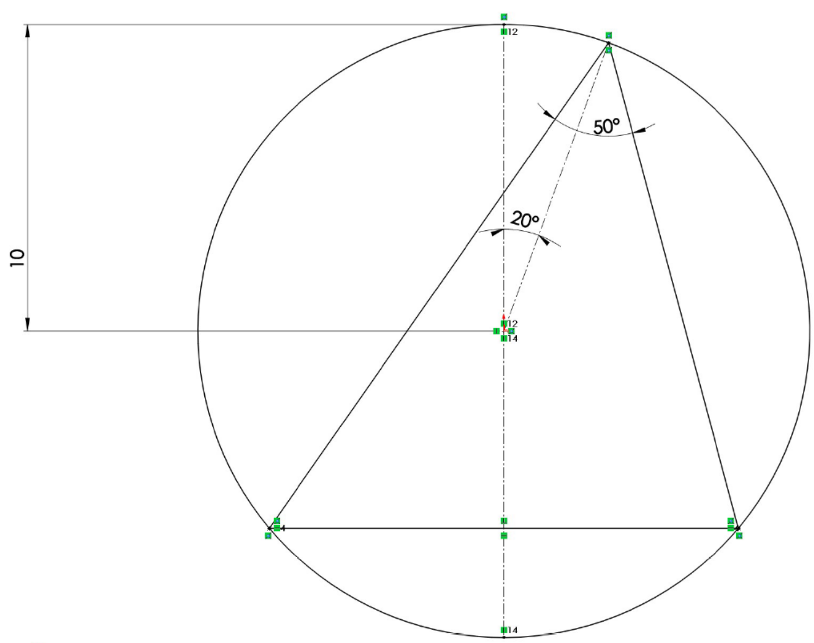
4.1 Problem

We now return to the first question that was posed but with a more concrete understanding. Suppose we relax all assumptions and suppose point O is surrounded by triangles of arbitrary shape and size. What is the correction factor Q_i to apply to the area so that the curvature comes out to be what it is in the continuous realm, $\frac{2}{R}$?

4.2 Solution

We must first choose how to parametrize each triangle. Previously, when it was known that each triangle was isosceles, we chose to use the two angles θ_i and ω_i to completely parametrize a triangle on a sphere of known radius. θ_i gave us the radius of the circle in which the triangle was circumscribed, $R\sin\theta_i$, and the projection ratio of the exterior normal onto the analytical normal at O, $\sin\theta_i$; ω_i gave us planar angles of the isosceles triangle as well as the value of Q_i; and θ_i and ω_i together gave us side lengths and the area of the triangle.

Now that we can no longer assume the triangle is isosceles, we must choose how to parametrize any arbitrary triangle. We can do so by using the angles θ_i and ω_i in addition to an angle which we will call ϕ_i. ϕ_i will be defined in the following way: draw a line from the circumcenter of the triangle so that it intersects with the exterior edge (the edge opposite to point O) at a right angle. Also draw a line from the circumcenter to point O. Then, we will define angle ϕ_i as the angle between the two lines. In the example triangle below, for example, ω_i is 50 degrees, ϕ_i is 20 degrees, and the radius of the circumcircle, $R\sin\theta_i$, is 10.



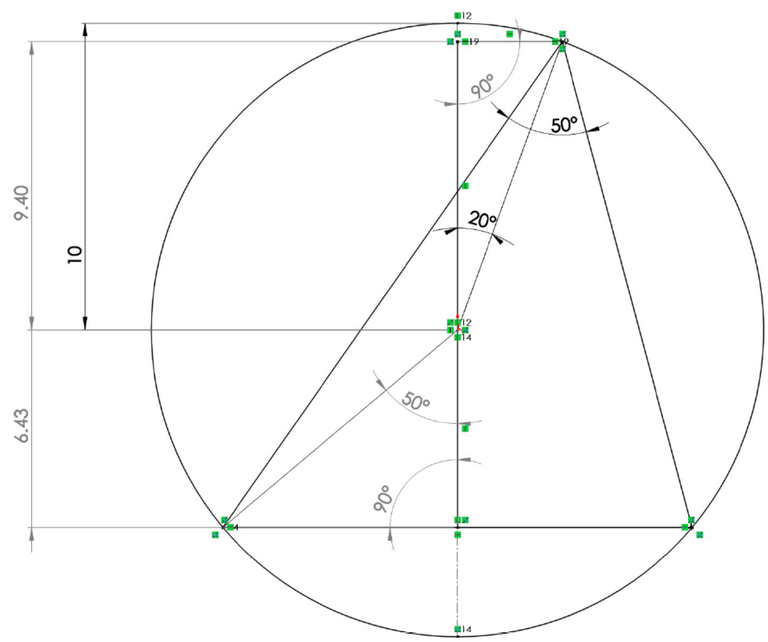
It turns out that ϕ_i is also equal to the difference between the two angles of the triangle other than ω_i. For an isosceles triangle, this angle will be 0, and otherwise it will be some other value. Therefore, ϕ_i is a measure of the skewness of a triangle.

From this parametrization, we can easily compute the base l_i and height h_i of the triangle, which allows us to compute the area A_i and with some additional information gives us the exterior normal projection n_i:

1. The radius of the circumcircle is $R\sin\theta_i$
2. The length of base l_i depends only on the radius of the circle circle and ω_i, so it is the same as in the case for an isosceles triangle: $2R\sin\theta_i\sin\omega_i$

3. The height h_i can be broken up into two pieces: (a) the length of the line that starts at the circumcenter and ends at its perpendicular intersection with the exterior edge, and (b) the length of the line that starts at the circumcenter and ends at the vertical projection of point O onto the line. In the example above, it is the vertical dashed line broken into two pieces at the circumcenter.
   (a) The portion between the circumcenter and the exterior edge is equal to $R \sin\theta_i \cos\omega_i$. To show this, draw a triangle connecting one of the points on the exterior edge, the center of the circumscribed circle, and the intersection of the height line with the exterior edge. Then, the angle of the triangle at the circumcenter is ω_i, it is a right triangle, and its hypotenuse is $R \sin\theta_i$. The length of the adjacent side is therefore $R \sin\theta_i \cos\omega_i$.
   (b) The portion between the circumcenter and the projection of point O onto the height line is equal to $R \sin\theta_i \cos\phi_i$. To show this, construct another triangle between the center of the circumcircle, point O, and its projection onto the height line. This is again a right triangle with a hypotenuse of $R \sin\theta_i$, and its angle at the circumcenter is ϕ_i, by definition. The length of the adjacent side is therefore $R \sin\theta_i \cos\phi_i$.

   Summing these two segments together, we get a total height of $h_i = R \sin\theta_i(\cos\omega_i + \cos\phi_i)$. This calculation is shown for the example triangle below.



4. The projection of the exterior normal onto the analytical normal is no longer just $\sin\theta_i$ but rather $\sin\theta_i \cos\phi_i$. I have not come up with a very convincing proof of this yet but this is a spherical coordinate transformation.
5. The above facts imply $A_i = \frac{1}{2} l_i h_i = R^2 \sin^2\theta_i \cos\omega_i(\cos\omega_i + \cos\phi_i)$ and $n_i = 2R \sin^2\theta_i \cos\omega_i \cos\phi_i$

The curvature will then be:

$$
\begin{aligned}
B &= \frac{\sum_i n_i}{\sum_i A_i} \\
&= \frac{\sum_i 2R \sin^2\theta_i \cos\omega_i \cos\phi_i}{R^2 \sin^2\theta_i \cos\omega_i(\cos\omega_i + \cos\phi_i)} \\
&= \frac{2}{R} \cdot \frac{\sum_i \sin^2\theta_i \cos\omega_i \cos\phi_i}{\sin^2\theta_i \cos\omega_i(\cos\omega_i + \cos\phi_i)}
\end{aligned} \tag{16}
$$

The appropriate correction factor on the area, therefore, is:

$$Q_i = \frac{\cos\phi_i}{\cos\omega_i + \cos\phi_i} \tag{17}$$

The only assumption made for this correction factor to hold is that the B vector points in the direction of the normal vector, which becomes more accurate with finer meshes. Notice also that Q_i becomes the $\frac{1}{1+\cos\omega_i}$ in the case where $\phi_i = 0$, which happens for isosceles triangles. This correction factor has been found to hold very accurately in simulations, even for randomized triangulations.