Introduction au traitement du signal

Amphi 4 - Echantillonnage

MINES ParisTech, Tronc commun 1A

Echantillonnage

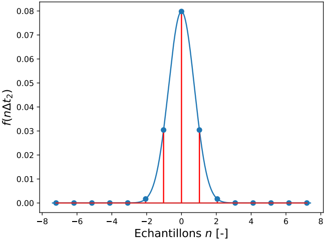
Signal continu

Fréquence d'échantillonnage :
44.1kHz

Echantillonnage régulier

Pour $f \in L^2$ et un pas de temps Δt, on considère les *échantillons*

$$x[n] = f(n\Delta t)$$



- Peut-on retrouver f à partir de x ?
- Dans le cas contraire, quelle information a-t-on perdu ?

Formule de Poisson

Lemme (Formule de Poisson)

Soit $f \in \mathcal{L}^2$. Pour tout $\Delta t > 0$, on a l'égalité

$$\sum_{n=-\infty}^{+\infty} f(n\Delta t) = \frac{1}{\Delta t} \sum_{n=-\infty}^{+\infty} \mathcal{F}f\left(\frac{2\pi n}{\Delta t}\right)$$

Lien entre les spectres des signaux

Reconstituer f à partir de $x \Longleftrightarrow$ Reconstituer $\mathcal{F}f$ à partir de $X(\Omega)$

Proposition

Soit $x[n] = f(n\Delta t)$. *On a*

$$X(\Omega) = \frac{1}{\Delta t} \sum_{n=-\infty}^{+\infty} \mathcal{F}f\left(\frac{\Omega - 2\pi n}{\Delta t}\right)$$

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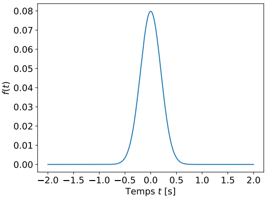
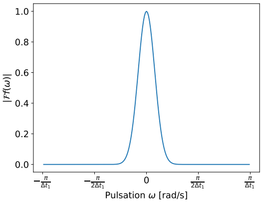
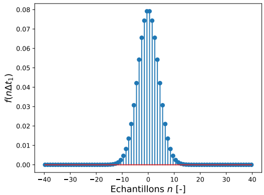


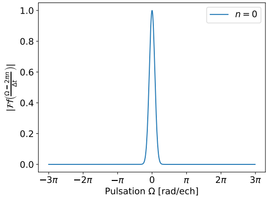
Signal continu $f(t)$



Spectre continu $\mathcal{F}f(\omega)$

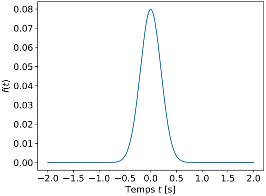


Signal discret $x[n] = f(n\Delta t_1)$



Terme $n = 0$: $\mathcal{F}f\left(\frac{\Omega}{\Delta t_1}\right)$

$$X(\Omega) = \frac{1}{\Delta t} \sum_{n=-\infty}^{+\infty} \mathcal{F}f\left(\frac{\Omega - 2\pi n}{\Delta t}\right)$$



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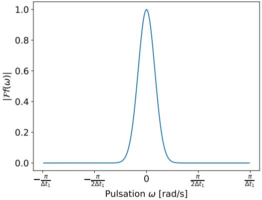

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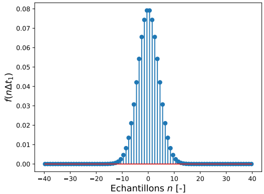

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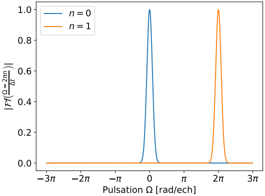

Terme $n = 1$: $\mathcal{F}f\left(\frac{\Omega - 2\pi n}{\Delta t_1}\right)$

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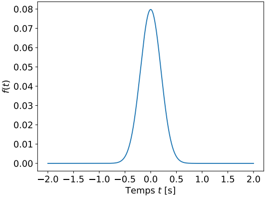

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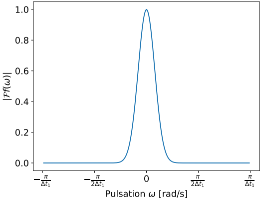

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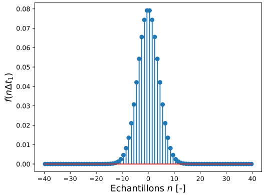

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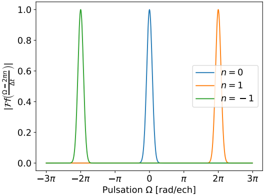
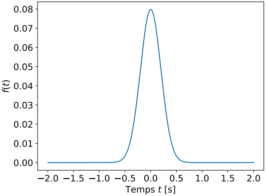


Terme $n = -1$: $\mathcal{F}f\left(\frac{\Omega + 2\pi n}{\Delta t_1}\right)$

$$X(\Omega) = \frac{1}{\Delta t} \sum_{n=-\infty}^{+\infty} \mathcal{F}f\left(\frac{\Omega - 2\pi n}{\Delta t}\right)$$



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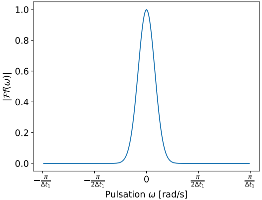

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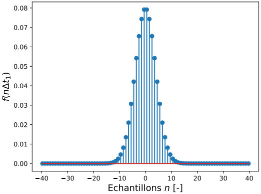

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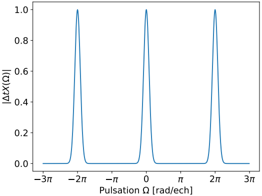
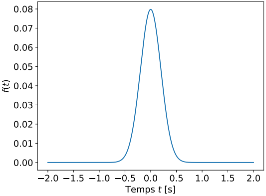


Spectre discret $\Delta t_1 X(\Omega)$

$$X(\Omega) = \frac{1}{\Delta t} \sum_{n=-\infty}^{+\infty} \mathcal{F}f\left(\frac{\Omega - 2\pi n}{\Delta t}\right)$$



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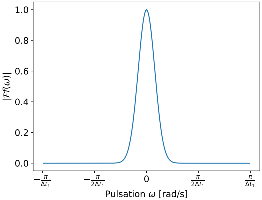

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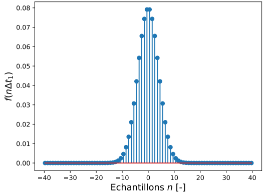

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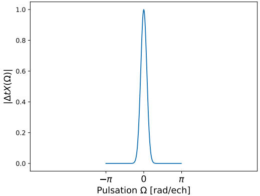
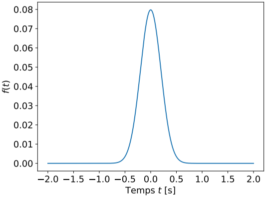


$\mathbb{1}_{[-\pi,\pi]}(\Omega)X(\Omega)$

$$X(\Omega) = \frac{1}{\Delta t} \sum_{n=-\infty}^{+\infty} \mathcal{F}f\left(\frac{\Omega - 2\pi n}{\Delta t}\right)$$



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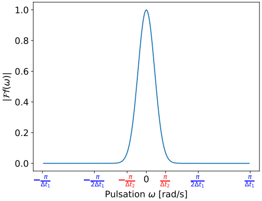

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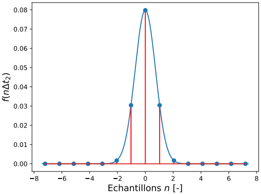
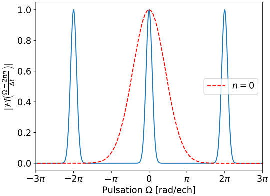


Signal discret $x[n] = f(n\Delta t_2)$



Terme $n = 0$: $\mathcal{F}f\left(\frac{\Omega}{\Delta t_2}\right)$

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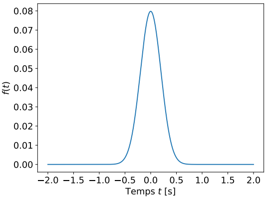

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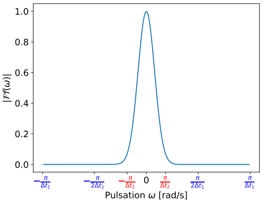

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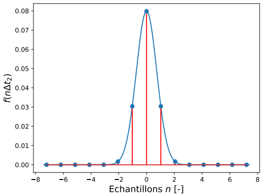

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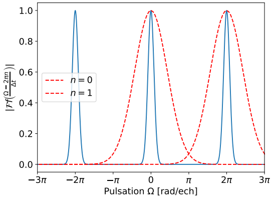

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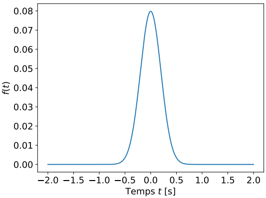

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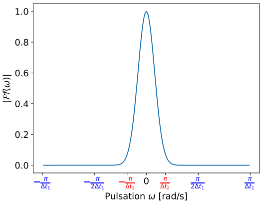

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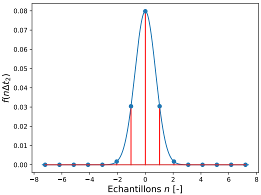

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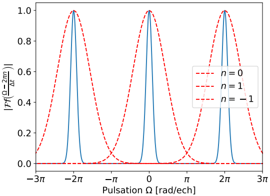

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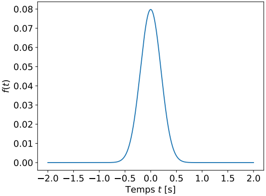

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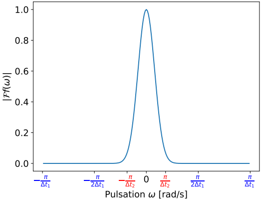

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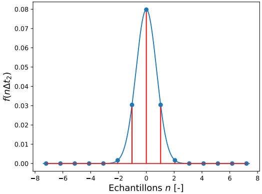

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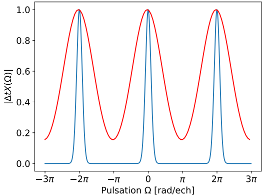

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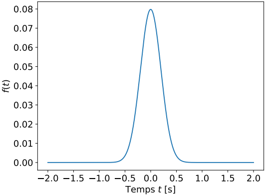

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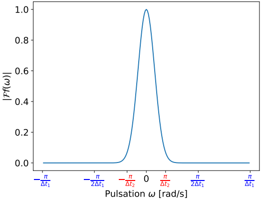

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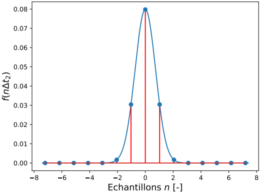

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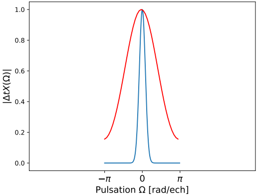

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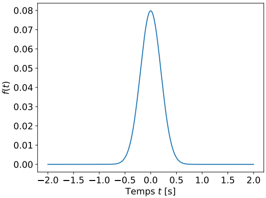

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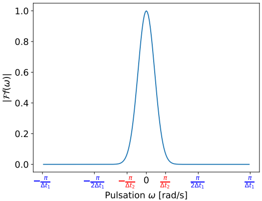

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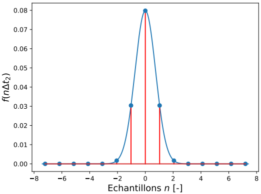

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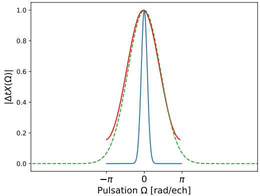

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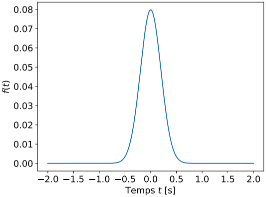

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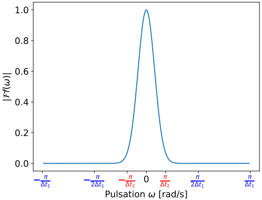

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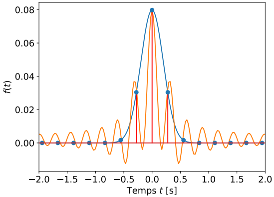
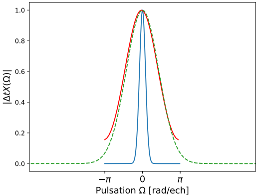


Signal reconstitué



$\mathbb{1}_{[-\pi,\pi]}(\Omega)X(\Omega)$

Le cas favorable

Théorème (Shannon)

Soit $f \in L^2$. On peut reconstituer f à partir de ses échantillons à un pas Δt par la formule

$$f(t) = \sum_{n=-\infty}^{+\infty} f(n\Delta t) \operatorname{sinc}\left(\frac{\pi}{\Delta t}(t - n\Delta t)\right)$$

ssi le spectre de f est nul en dehors de $\left[\frac{-\pi}{\Delta t}, \frac{\pi}{\Delta t}\right]$. En d'autres termes, la fréquence d'échantillonnage ν_{ech} doit satisfaire

$$\nu_{ech} > 2\nu_{max}$$

où ν_{max} est la fréquence maximale "contenue" dans f

Repliement spectral

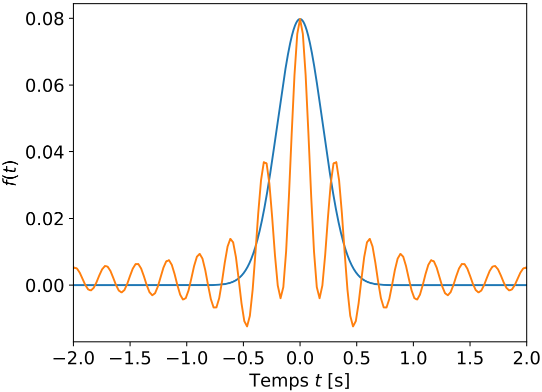


Illustration (Source : Jesse Mason – Youtube)

Repliement spectral

Source : https ://svi.nl/

Illustration (Source : Jesse Mason – Youtube)

En pratique

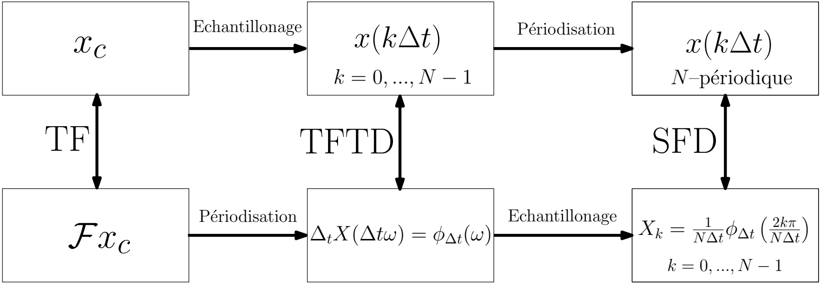
- Calcul de la TFD

$$\tilde{X}_k = X\left(\frac{2k\pi}{N}\right) = \frac{1}{\Delta t}\sum_{n=-\infty}^{+\infty} \mathcal{F}f\left(\frac{\frac{2k\pi}{N} - 2\pi n}{\Delta t}\right)$$
$$= \mathcal{F}f\left(\frac{2k\pi}{N\Delta t}\right) + \text{repliement spectral}$$

Proposition

f est à support compact $\Longleftrightarrow$ $\hat{f}$ est à support infini.

Echantillonnage et périodisation



Autres dualités : contraction / dilatation, translation / rotation

Au-delà de l'échantillonnage régulier

Caméra à 1 pixel

R. G. Baraniuk, Compressive Sensing [Lecture Notes], Signal Processing Magazine, July 2007.

Sous-échantillonnage

Proposition

Soit x un signal discret et $\check{x}$ le signal défini par

$$\check{x}(n) = x(Dn)$$

Alors la TFTD de $\check{x}$ est donnée par

$$\check{X}(\omega) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(\frac{\omega + 2k\pi}{D}\right)$$

Sur-échantillonnage

Proposition

Soit x un signal discret et y le signal défini par

$$y(n) = \begin{cases} x(p) & \text{si } n = pl \\ 0 & \text{sinon} \end{cases} \tag{1}$$

La TFTD de y est donnée par

$$Y(\omega) = X(l\omega). \tag{2}$$