Quantum Optimization Tools

An overview lecture
27 Jan 2023

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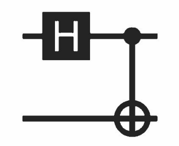
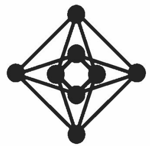
Several types of hardware for quantum and quantum-inspired algorithms

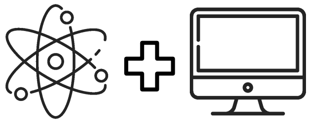


Universal quantum computers

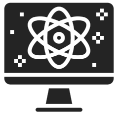
- Trapped Ions
- Cold atoms
- Superconductors
- Integrated photonics



Quantum annealers



Hybrid quantum-classical computers



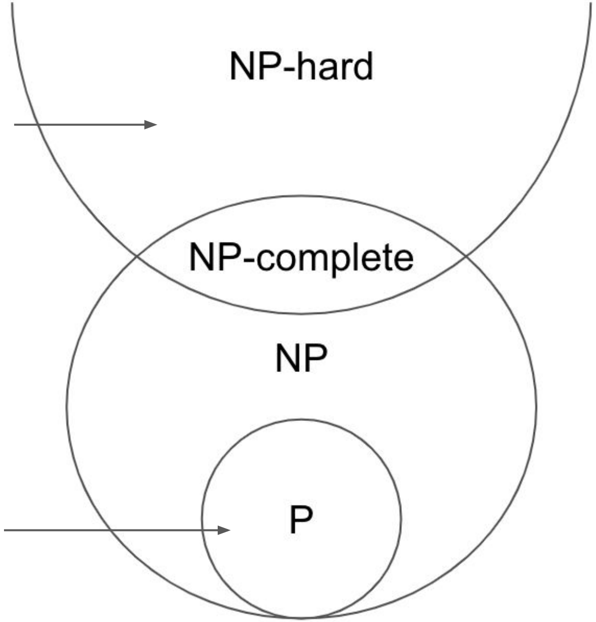
Quantum inspired computing

Software
- CPU
- GPU

Hardware+Software
- FPGA
- ASIC

Some problems are easy some problems are hard

- Traveling salesman problem
- Max-CUT
- Max-SAT
- **Ising Glass problem**
- **Quadratic unconstrained binary optimization**
- etc



Matrix multiplication
Finding inverse matrix
Array sorting

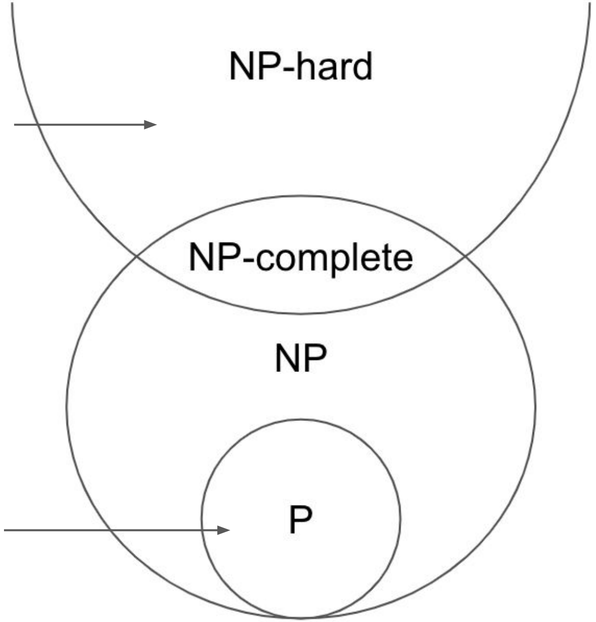
Some problems are easy some problems are hard

1M$ question

P vs NP

- Traveling salesman problem
- Max-CUT
- Max-SAT
- **Ising Glass problem**
- **Quadratic unconstrained binary optimization**
- etc

Matrix multiplication
Finding inverse matrix
Array sorting

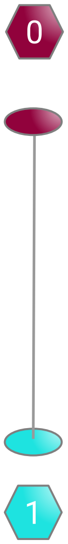


5-min course on quantum mechanics

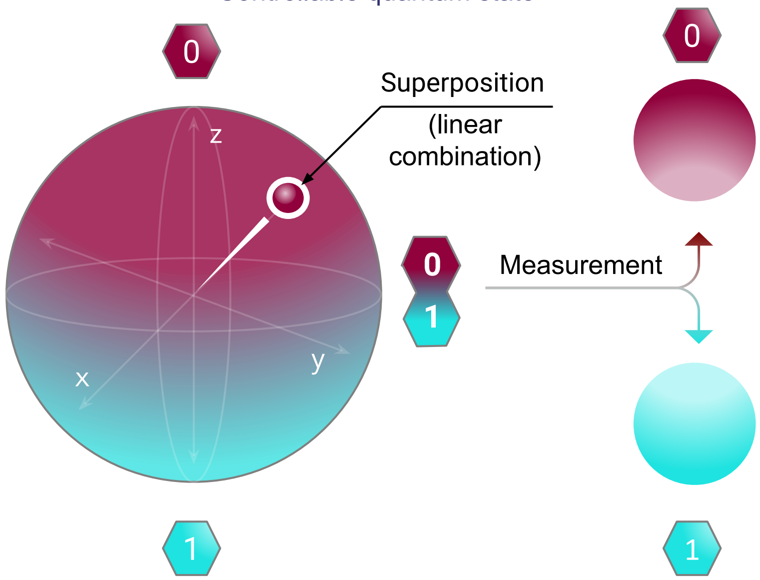
Quantum case of bit

Quantum bit = qubit

Classical **bit**
(binary system)



Quantum **bit**
Controllable quantum state



Math representation of state

1. Quantum states are vectors in Hilbert states
   a. States are normalized to 1

2. Evolution of state is described by unitary operator

$$0 \leftrightarrow |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad 1 \leftrightarrow |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\alpha, \beta \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1$$

Math representation of many qubits

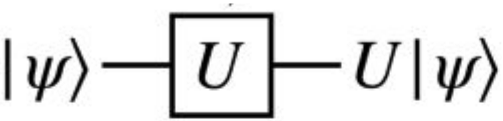
$$\begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a \begin{bmatrix} c \\ d \end{bmatrix} \\ b \begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$$

$$101 \leftrightarrow |101\rangle \equiv |1\rangle \otimes |0\rangle | \otimes 1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{array}{cl} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} & \begin{array}{l} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \end{array}$$

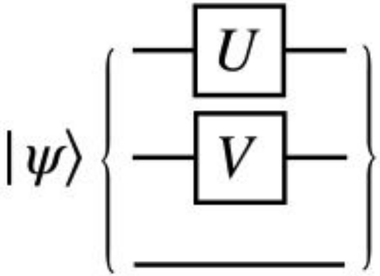
$\text{bin}('101') = 5$

Quantum gates

$$|\psi\rangle = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} \qquad U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad U|\psi\rangle = \begin{bmatrix} Ac_0 + Bc_1 \\ Cc_0 + Dc_1 \end{bmatrix}$$



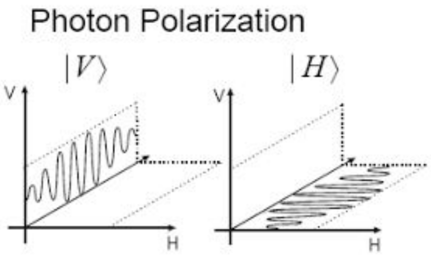
Representation of several gates as a matrix



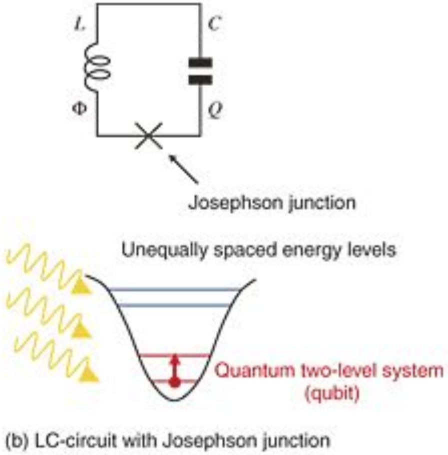
$$(U \otimes V \otimes \mathbf{I})\,|\psi\rangle$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a\begin{bmatrix} e & f \\ g & h \end{bmatrix} & b\begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ c\begin{bmatrix} e & f \\ g & h \end{bmatrix} & d\begin{bmatrix} e & f \\ g & h \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{bmatrix}$$

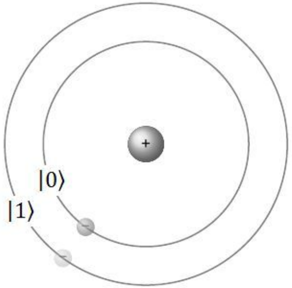
How to make qubit at ~~home~~ lab?



Photons



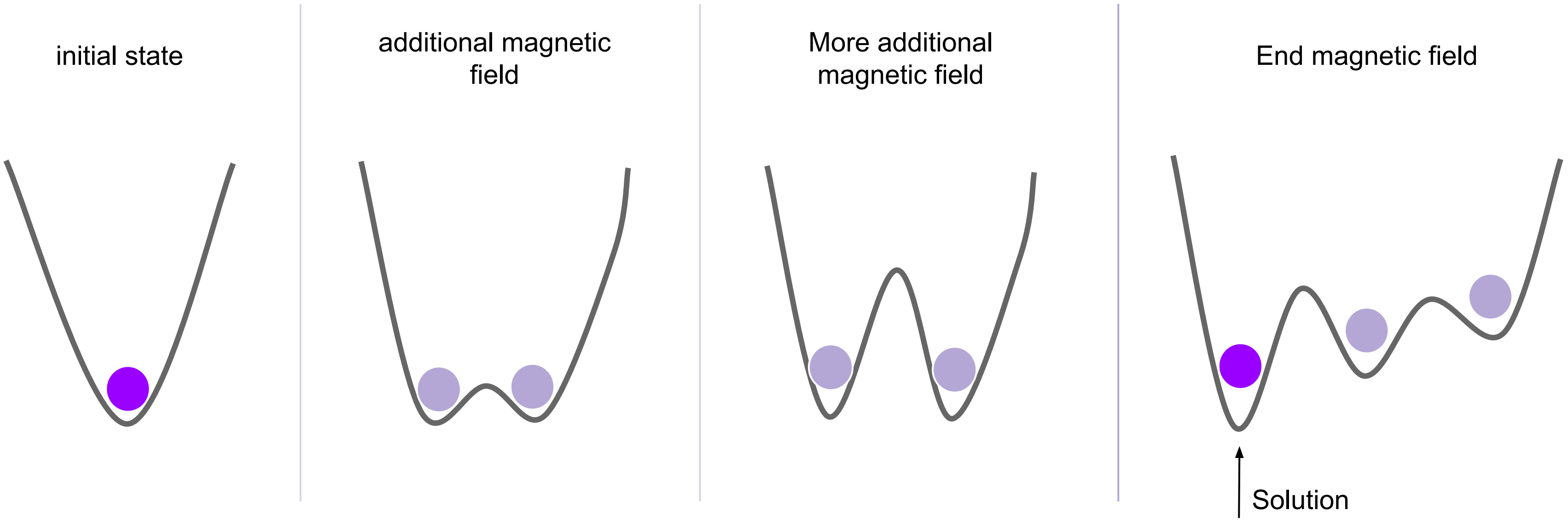
Superconductors



Atoms

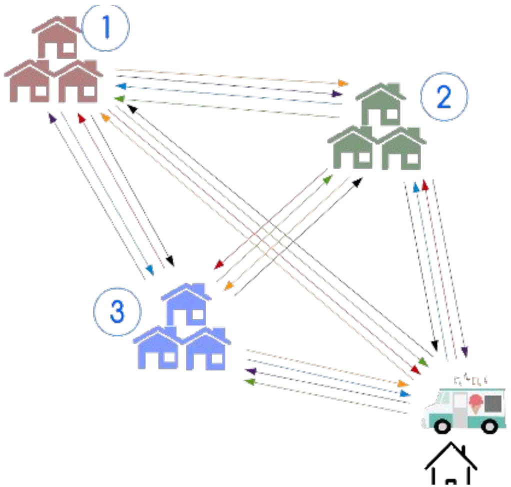
ADIABATIC QUANTUM COMPUTING

Adiabatic quantum computing



Ising model and other NP-hard problems

NP-complete problems are equivalent
And we could map some NP-hard problems to another NP-hard problems



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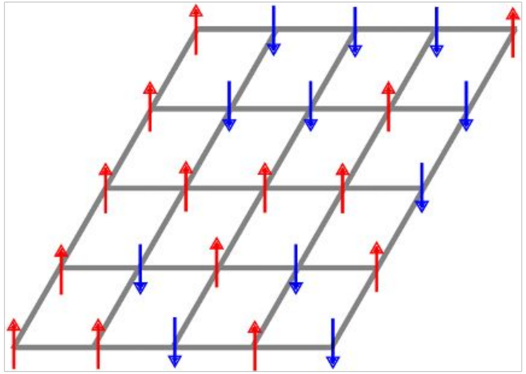
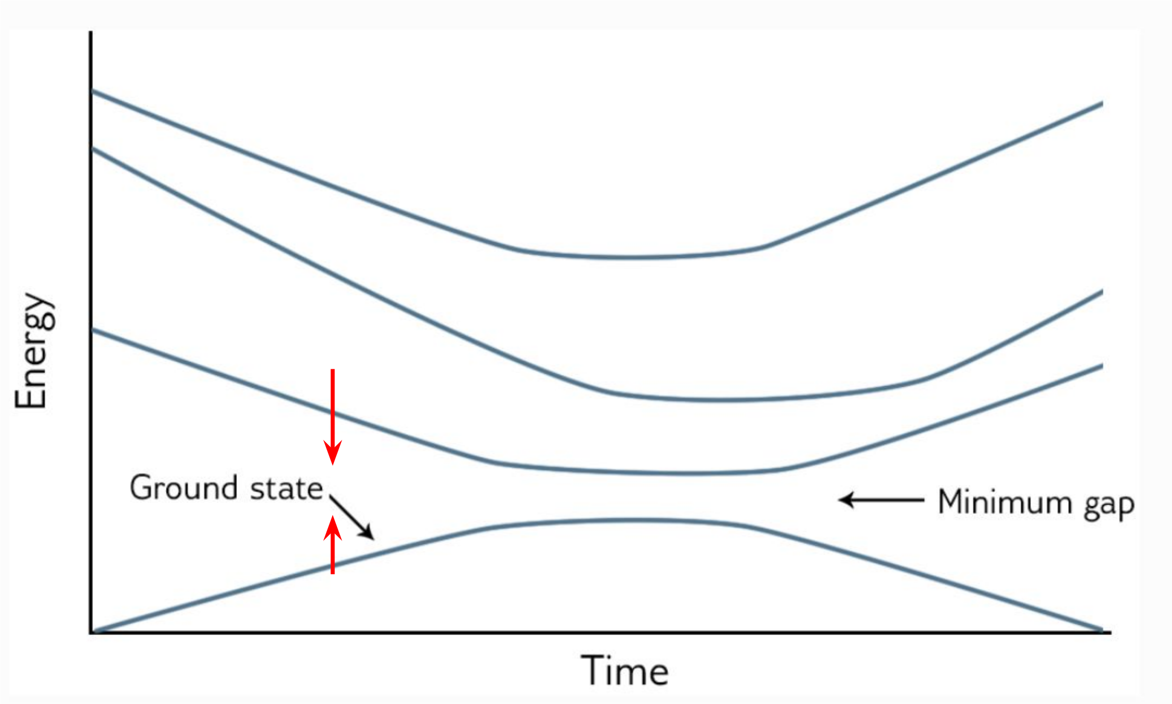
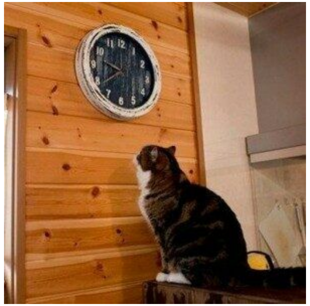


Some problems in adiabatic computing

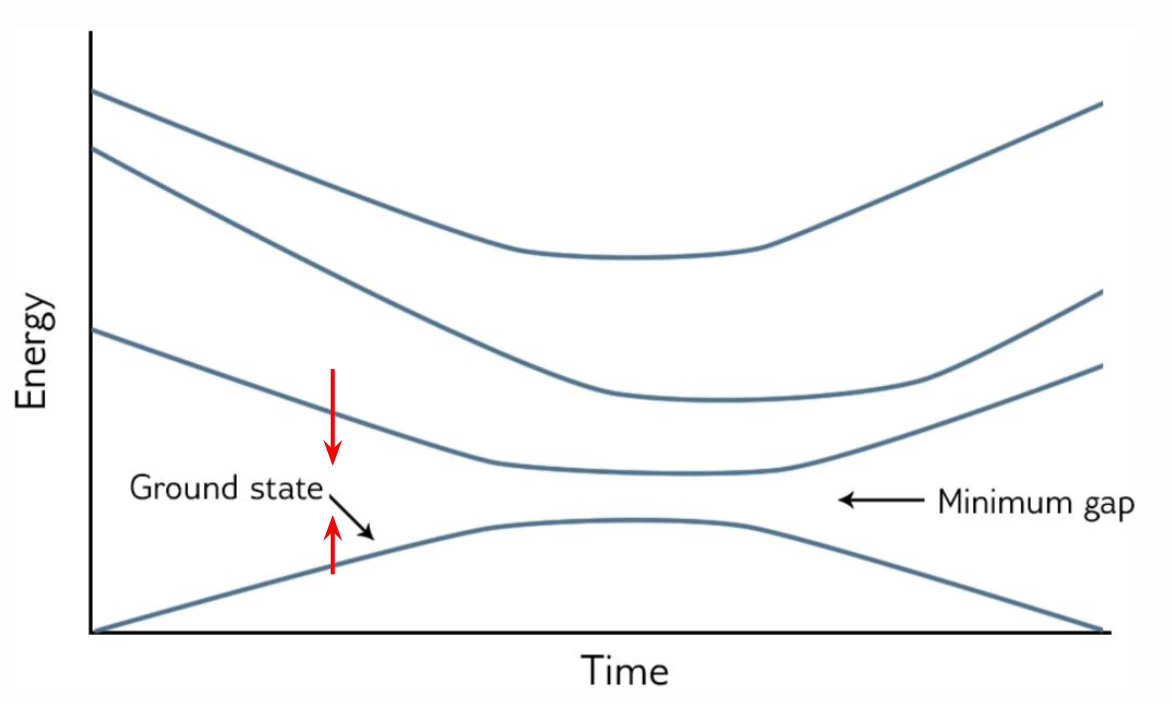


source: docs.dwavesys.com

Some problems in adiabatic computing



waiting for adiabatic quantum evolution



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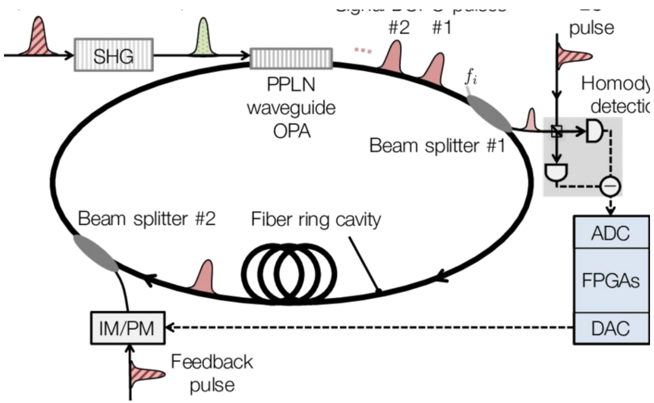
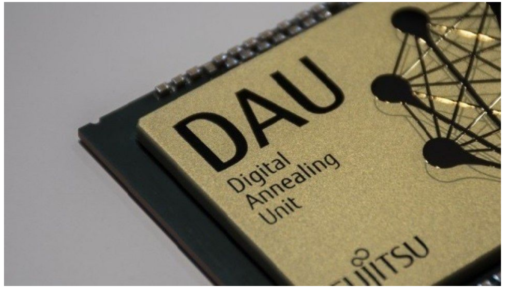
Several types of hardware for quantum and quantum-inspired algorithms



Coherent Ising Machine
[10.1088/2058-9565/aa8190](https://doi.org/10.1088/2058-9565/aa8190)



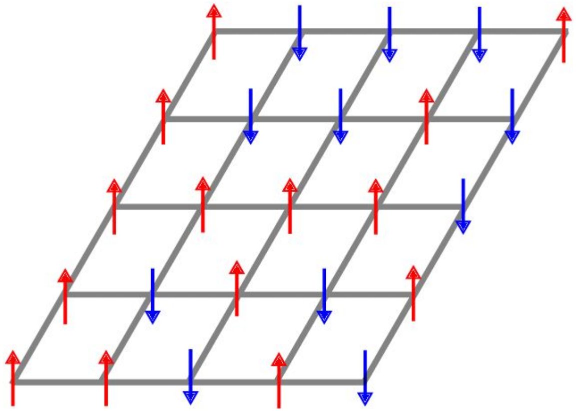
Digital annealer
https://www.fujitsu.com/global/services/business-services/digital-annealer/

Ising model

The Ising model is a mathematical model of statistical physics designed to describe the magnetization of a material.

$$H(s) = -\sum_{\langle ij \rangle} J_{ij} s_i s_j - \sum_j h_j s_j$$

$$s_i = \pm 1$$



It is necessary to find a configuration s in which the system is in the state with the lowest energy.

In general, the problem is NP-hard.

Ising model

The final Hamiltonian is constructed from 2 types of terms:

1. The objective function of the optimization problem (Hp)
2. Restrictions on the scope of values and definitions of the objective function (Hc)

$$H = H_P + H_C$$

Classic problem

$$\max_x \sum_i p_i x_i$$

$$\text{s.t.} \sum_i w_i x_i \leq W$$

QUBO model

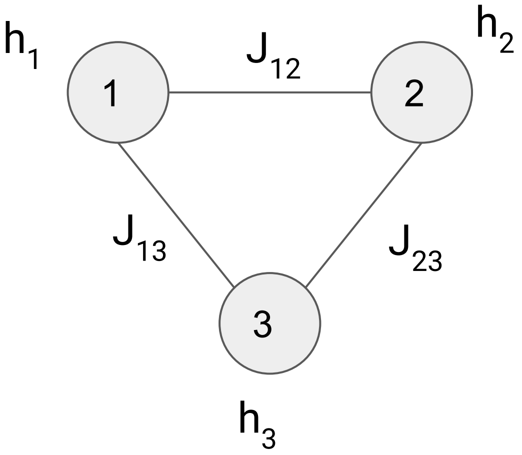
$$-\sum_i p_i x_i$$

$$+ \left(\sum_{d=0}^{D-1} d y_d - \sum_{i=0}^{n-1} w_i x_i \right)^2 + \left(\sum_{d=0}^{D-1} y_d - 1 \right)^2$$

Graph interpretation

Sometimes it is useful to represent the Ising model in the form of a graph, where the weight of an edge between a vertex reflects the interaction between the spins, and the weight of an individual vertex corresponds to the effect of an external field on the spin.

$$H(s) = -\sum_{\langle ij \rangle} J_{ij} s_i s_j - \sum_j h_j s_j$$



QUBO problem

$$f_Q : \mathbb{B}^n \rightarrow \mathbb{R}$$

$$f_Q(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} x_i x_j \qquad q_{ij} \in \mathbb{R} \quad \forall i, j$$

$$x = \underset{x \in \mathbb{B}^n}{\arg\min} \, f_Q(x)$$

Trivial cases

$$x = (0, 0, ..., 0), \\ \text{if} \quad q_{ij} \geq 0 \quad \forall i, j$$

$$x = (1, 1, ..., 1), \\ \text{if} \quad q_{ij} < 0 \quad \forall i, j$$

$$q_{ij} = 0, \quad \forall i \neq j \\ x_i = 1, \quad \text{if} \quad q_{ii} < 0 \\ x_i = 0 \quad \text{if} \quad q_{ii} > 0$$

QUBO = ISING

The mapping of the Ising model to QUBO is done by replacing variables

$$s \rightarrow 2x - 1$$

$$H(s) = -\sum_{\langle ij \rangle} J_{ij} s_i s_j - \sum_j h_j s_j$$

$$f_Q(x) = \sum_{i=1}^{n} \sum_{j=1}^{i} q_{ij} x_i x_j + C$$

$$q_{ij} = \begin{cases} -4J_{ij} & \text{if } i \neq j \\ \sum_{\langle ki \rangle} 2J_{ki} + \sum_{\langle il \rangle} 2J_{il} + h_i & \text{if } i = j \end{cases}$$

$$C = -\sum_{\langle ij \rangle} J_{ij} - \sum_j h_j$$

Useful articles

A.Lukas. Ising formulations of many NP problems

Y.Yamamoto et al. Coherent Ising machines – optical neural networks operating at the quantum limit

M.Aramon. Physics-Inspired Optimization for Quadratic Unconstrained Problems Using a Digital Annealer

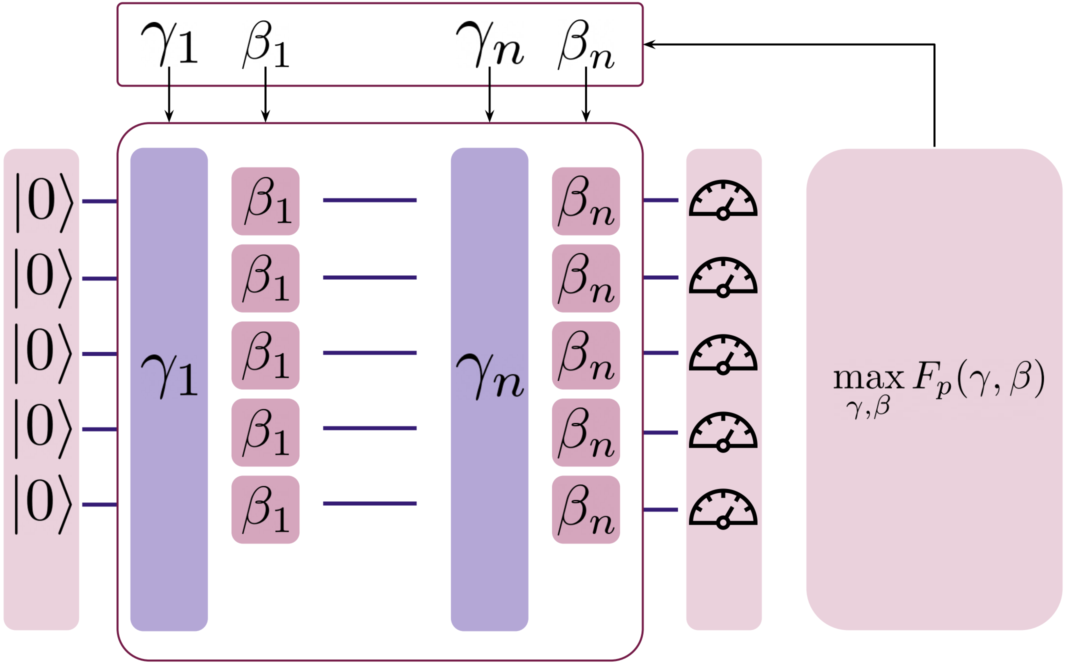
Suzuki-Trotter decomposition

$$e^{A+B} = \lim_{n\to\infty} (e^{A/n} e^{B/n})^n$$

$$e^{-iH(t)} = e^{-i((1-\frac{t}{T})H_0 + \frac{t}{T}H_P)} = \lim_{n\to\infty} (e^{-i(1-\frac{t}{T})H_0/n} e^{-i\frac{t}{T}H_P/n})^n$$

$$e^{-iH(t)} = (e^{-i(1-\frac{t}{rT})H_0} e^{-i\frac{t}{rT}H_P})^r + O(t^2/r)$$

QAOA
Quantum approximate optimization algorithm

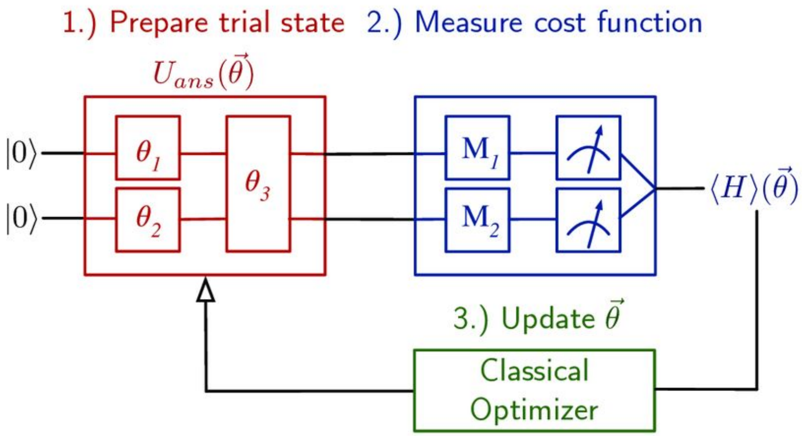


$$|\psi(\vec{\gamma}, \vec{\beta})\rangle = U_B(\beta_p)U_C(\gamma_p) \dots U_B(\beta_1)U_C(\gamma_1) \mid + \rangle$$

$$\langle\psi(\vec{\gamma}, \vec{\beta})\,|H_P|\psi(\vec{\gamma}, \vec{\beta})\rangle$$

Source: E. Farhi, J. Goldstone, S. Gutmann. A Quantum Approximate Optimization Algorithm. arXiv:1411.4028

General type of variational quantum algorithms



DOI: 10.3390/e23060657