VECTOR SPACES AS UNIONS OF PROPER SUBSPACES

APOORVA KHARE

ABSTRACT. In this note, we find a sharp bound for the minimal number (or in general, indexing set) of subspaces of a fixed (finite) codimension needed to cover any vector space V over any field. If V is a finite set, this is related to the problem of partitioning V into subspaces.

1. THE MAIN THEOREM

Consider the following well-known problem in linear algebra (which is used, for example, to produce vectors not on root hyperplanes in Lie theory):

No vector space over an infinite field is a finite union of proper subspaces.

The question that we answer in this short note, is:

Given any vector space V over a ground field $\mathbb{F}$, and $k \in \mathbb{N}$, what is the smallest number (or in general, indexing set) of proper subspaces of codimension k, whose union is V?

To state our main result, we need some definitions.

Definition 1.1.

(1) Compare two sets I, J as follows: $J > I$ if there is no one-to-one map $f : J \to I$. Otherwise $J \leq I$.

(2) Given a vector space V over a field $\mathbb{F}$, define $\mathbb{P}(V)$ to be the set of lines in V; thus, $\mathbb{P}(V)$ is in bijection with $(V \setminus \{0\})/\mathbb{F}^{\times}$.

This paper is devoted to proving the following theorem.

Theorem 1.2. *Suppose V is a vector space over a field $\mathbb{F}$, and I is an indexing set. Also fix $1 \leq k < \dim_{\mathbb{F}} V$, $k \in \mathbb{N}$. Then V is a union of "I-many" proper subspaces of codimension at least k, if and only if $I \geq \nu(\mathbb{F}, V, k)$, where*

$$\nu(\mathbb{F}, V, k) := \begin{cases} \lceil |\mathbb{P}(V)|/|\mathbb{P}(V/\mathbb{F}^k)| \rceil, & \text{if } |V| < \infty; \\ \mathbb{N}, & \text{if } |\mathbb{F}| = \dim_{\mathbb{F}} V = \infty; \\ \mathbb{F}^k \coprod \{\infty\}, & \text{otherwise.} \end{cases}$$

(We will see in the proof, that the countable cover in the case of $\dim_{\mathbb{F}} V = |\mathbb{F}| = \infty$ is different in spirit from the constructions in the other cases.) We now mention a few examples and variants, before proving the theorem in general.

(1) No finite-dimensional vector space over $\mathbb{R}$ (hence, also over $\mathbb{C}$) is a union of countably many proper subspaces. Here is a simple measure-theoretic proof by S. Chebolu: suppose $V = \bigcup_{n>0} V_n$, with $V_n \subsetneq V \ \forall n \in \mathbb{N}$. Let μ be the Lebesgue measure on V; recall that μ is countably subadditive. We now get a contradiction:

$$\mu(V) = \mu \left(\bigcup_n V_n \right) \leq \sum_{n \in \mathbb{N}} \mu(V_n) = 0,$$

since each V_n has measure zero, being a proper subspace.

(2) On the other hand, suppose V is a finite-dimensional vector space over a finite field $\mathbb{F} = \mathbb{F}_q$ (with q elements); how many proper subspaces would cover it? (Hence, $k = 1$ here.) The answer is the same for all V; we mention a proof (by R. Walia; see [7]) for the simplest example of $V_2 = \mathbb{F}_q^2$.

Lemma 1.3. V_2 *is a union of $q+1$ lines (but not q lines).*

Proof. Consider the lines spanned by $(1, \alpha)$ (for each $\alpha \in \mathbb{F}_q$) and $(0, 1)$. These are $q+1$ lines, and each pair of lines has only the origin in common (since two points determine a line). Since each line has q points, the union of all these lines has size $1 + (q+1)(q-1) = q^2$ (where the "1" counts the origin). This counting argument also shows that a smaller number of lines can not cover all of V_2. $\square$

Remark 1.4. Thus, we should really think of $q+1$ as $\mathbb{F} \coprod \{\infty\} = \mathbb{P}(\mathbb{F}^2)$.

(3) Note that the proof of Lemma 1.3 shows that the $q+1$ lines actually provide a *partition* of the finite vector space V_2 - namely, a set of subspaces that are pairwise disjoint except for the origin, and cover all of V.

The theory of partitions of finite vector spaces has been extensively studied - see, for instance, [1, 2, 3, 4, 5]. We remark that this theory of partitions keeps track of the dimensions of the subspaces involved. Moreover, it has applications in error-correcting codes and combinatorial designs - see [3, §1] for more references.

(4) There is a school of thought that considers vector spaces over "$\mathbb{F}_1$ (the field with one element)", to morally be defined - and more precisely, they are finite sets. The way to get results using this philosophy, is to work the analogous results out for finite fields $\mathbb{F}_q$, and take $q \to 1^+$ (though it is a non-rigorous procedure, given that there usually is more than one generalization to $\mathbb{F}_q$).

As for our two problems, the results are clear: a set of size > 1 (which is analogous to $\dim_{\mathbb{F}_q}(V) > 1$) is a union of two proper subsets - where $2 = 1+1 = q+1$ - but not of one proper subset. The analogue for codimension k subspaces, is: how many subsets $W \subset V$ with $|V \setminus W| \geq k$, does it take to cover V?

The answer to this question is 2 if V is infinite, and if $|V| = n$, then the answer is $\left\lceil \dfrac{n}{n-k} \right\rceil$. Note that this is exactly the statement of Theorem 1.2 for finite vector spaces V in both cases, because $\mathbb{P}(V)/\mathbb{P}(V/\mathbb{F}_q^k) = (q^n - 1)/(q^{n-k} - 1)$, and for $0 < k < n$,

$$\lim_{q \to 1^+} \left\lceil \frac{q^n - 1}{q^{n-k} - 1} \right\rceil = \left\lceil \frac{n}{n-k} \right\rceil.$$

(5) The next variant involves modules over a finite-dimensional $\mathbb{F}$-algebra A, when $\mathbb{F}$ is infinite. It generalizes Theorem 1.2 when $k = 1$.

Proposition 1.5. *Suppose $\dim_{\mathbb{F}} A < \infty = |\mathbb{F}|$. Now if $M = \bigoplus_{i \in I} Am_i$ is any direct sum of cyclic A-modules, then M is a union of "J-many" proper submodules if and only if it M is not cyclic and $J \geq \nu(\mathbb{F}, M, 1)$.*

In other words, M is a union of "$\mathbb{F}$-many" proper submodules if I is finite (and not a singleton), and a countable union if I is infinite, since $\mathbb{F}$ and $\mathbb{F} \coprod \{\infty\}$ are in bijection if $\mathbb{F}$ is infinite.

Proof. If M is cyclic, the result is clear, since some submodule must contain the generator. So now assume that M is not cyclic; note that each cyclic A-module is a quotient of A, hence finite-dimensional. So if I is finite, then $\dim_{\mathbb{F}} M < \infty$, and every proper submodule is a subspace of codimension between 1 and $\dim_{\mathbb{F}} M$. By Theorem 1.2, we then need at least "$\mathbb{F}$-many" proper submodules to cover M.

On the other hand, $|M| = |\mathbb{F}|$, and for each $m \in M$, we have the submodule Am containing it. The result follows (for finite I) if we can show that Am is a proper submodule for all $m \in M$. But if $Am = M$ for some $m = \sum_{i \in I} a_i m_i$, then we can find $b_j \in A$ such that $b_j m = m_j \ \forall j \in I$, whence $b_j a_i = \delta_{ij} \ \forall i, j$. This is a contradiction if $|I| > 1$, since it implies that every $a_i \in A^\times$ is a unit, hence annihilated only by $0 \in A$.

On the other hand, if I is infinite, then for any sequence of subsets

$$\emptyset = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I, \qquad \bigcup_{n \in \mathbb{N}} I_n = I, \qquad I_n \neq I \ \forall n,$$

define the proper A-submodule $M_n := \bigoplus_{i \in I_n} Am_i$. Then $M = \bigcup_{n \in \mathbb{N}} M_n$ yields a countable cover by proper A-submodules. Evidently, finitely many proper submodules cannot cover M (again by Theorem 1.2), since $\mathbb{F}$ is infinite, and each submodule is a subspace as well. $\square$

(6) The last variant that we mention, is the following question, which generalizes Theorem 1.2 for finite $\mathbb{F}$:

Given a finitely generated abelian group G, how many proper subgroups are needed to cover it?

For instance, if $G = (\mathbb{Z}/5\mathbb{Z}) \oplus (\mathbb{Z}/25\mathbb{Z}) \oplus (\mathbb{Z}/12\mathbb{Z})$, then G has a quotient: $G \to (\mathbb{Z}/5\mathbb{Z})^2 = \mathbb{F}_5^2 \to 0$. Now to cover G by proper subgroups, we can cover $\mathbb{F}_5^2$ by six lines (by Theorem 1.2), and lift them to a cover of G. Moreover, it can be shown that G cannot be covered by five or fewer proper subgroups.

Thus, the question for abelian groups seems to be related to the question for fields. We explore this connection in a later work [6].

2. Proof for infinite fields

In this section, we show Theorem 1.2 for infinite fields (and some other cases too). Define *projective k-space* $\mathbb{F}P^k := \mathbb{P}(\mathbb{F}^{k+1})$.

Remark 2.1.

(1) In what follows, we freely interchange the use of (cardinal) numbers and sets while comparing them by inequalities. For instance, $I \geq A/B$ and $I \geq n$ mean, respectively, that $I \times B \geq A$ and $I \geq \{1, 2, \ldots, n\}$. Similarly, $\dim_{\mathbb{F}} V$ may denote any basis of V - or merely its cardinality.

We also write $\cong$ below, for bijections between sets (in other contexts and later sections, $\cong$ may also denote bijections of $\mathbb{F}$-vector spaces).

(2) $\mathbb{F}P^k$ is parametrized by the following lines:

$$(1, \alpha_1, \ldots, \alpha_k); \ (0, 1, \alpha_2, \ldots, \alpha_k); \ \ldots; \ (0, 0, \ldots, 0, 1),$$

where all α_i are in $\mathbb{F}$. If $\mathbb{F}$ is infinite, then this is in bijection with each of the following sets: $\mathbb{F}, \mathbb{F}^k, \mathbb{F} \coprod \{\infty\}, \mathbb{F}^k \coprod \{\infty\}$.

We now show a series of results, that prove the theorem when $\mathbb{F}$ is infinite.

Lemma 2.2. *($\mathbb{F}, V, k, I$ as above.) If $I \geq \mathbb{F}P^k$, then V is a union of "I-many" proper subspaces of codimension at least k, if and only if $\dim_{\mathbb{F}} V > k$.*

Proof. The result is trivial if $\dim_{\mathbb{F}} V \leq k$, and if not, then we start by fixing any $\mathbb{F}$-basis B of V. Fix $v_0, v_1, \ldots, v_k \in B$, and call the complement B'. Now define, for each $1 \leq i \leq k$ and each $x = (0, \ldots, 0, 1, \alpha_i, \alpha_{i+1}, \ldots, \alpha_k) \in \mathbb{F}P^k$, the codimension k-subspace V_x of V, spanned by B' and $v_{i-1} + \sum_{j=i}^{k} \alpha_j v_j$.

We claim that $V = \bigcup_{x \in \mathbb{F}P^k} V_x$. Indeed, any $v \in V$ is of the form $v' + \sum_{j=0}^{k} \beta_j v_j$, with $\beta_j \in \mathbb{F} \ \forall j$, and v' in the span of B'. Now if β_i is the first nonzero coefficient, then $v \in V_x$, where $x = (0, \ldots, 0, 1, \beta_i^{-1}\beta_{i+1}, \ldots, \beta_i^{-1}\beta_k)$, with the 1 in the ith coordinate. $\square$

Proposition 2.3. *Suppose $I < \mathbb{F}^k \coprod \{\infty\}$. If I or $\dim_{\mathbb{F}} V$ is finite, then V cannot be written as a union of "I-many" subspaces of codimension $\geq k$.*

Proof. This proof is long - and hence divided into steps.

(1) The first step is to show it for $k = 1$. Suppose we are given V and $\{V_i : i \in I\}$. Suppose the result fails and we do have $V = \bigcup_{i \in I} V_i$. We then seek a contradiction.

   (a) We first find a subcollection $\{V_i : i \in I' \subset I\}$ of subspaces that cover V, such that no V_i is in the union of the rest.

   If I is finite, this is easy: either the condition holds, or there is some V_i that is contained in the union of the others; now remove it and proceed by induction on $|I|$.

   The case of finite-dimensional V is from [8]. We need to use induction on $d = \dim_{\mathbb{F}} V$ to prove the result. It clearly holds if $V = \mathbb{F}^1$; now suppose that it holds for all $d < \dim_{\mathbb{F}} V$. We first reduce our collection $\{V_i : i \in I\}$ to a subcollection indexed by $I' \subset I$, say, as follows:
   Every chain of proper subspaces of V is finite (since $\dim_{\mathbb{F}} V < \infty$), whence its upper bound is in the chain (note that this fails if $|I| = \dim_{\mathbb{F}} V = \infty$). So for every chain of subspaces, remove all of them except the upper bound.
   We are left with $\{V_i : i \in I'\}$, where if $i \neq j$ in I', then $V_j \not\subseteq V_i$, or $V_i \cap V_j \subsetneq V_j$. Now use the induction hypothesis: no V_j is a union of "I-many" (hence "I'-many") proper subspaces. So

   $$V_j \supsetneq \bigcup_{i \in I', i \neq j} (V_j \cap V_i) = V_j \bigcap \bigcup_{i \in I', i \neq j} V_i,$$

   whence no V_j is contained in the union of the others, as desired.

   (b) Having found such a subcollection, we now obtain the desired contradiction:
   For all $i \in I'$, choose $v_i \in V_i$ such that $v_i \notin V_j \, \forall i \neq j$. There are at least two such, so choose $v_1 = v_{i_1}, v_2 = v_{i_2}$, with $i_1 \neq i_2$ in I'. Now consider $S := \{v_1 + \alpha v_2 : \alpha \in \mathbb{F}\} \coprod \{v_2\}$. Since $V = \bigcup_{i \in I'} V_i$, for each vector $v \in S$, choose some i such that $v \in V_i$. This defines a function $f : \mathbb{F} \coprod \{\infty\} \to I'$, and this is not injective by assumption. Thus some two elements of S are in the same V_i, and we can solve this system of linear equations to infer that both v_1 and v_2 are in V_i. Hence $i_1 = i = i_2$, a contradiction.

(2) We now show the result for general k. We have two cases. If $\mathbb{F}$ is infinite, then we are done by the previous part and the final part of Remark 2.1. The other case is when $\mathbb{F}$ is finite - say $\mathbb{F} = \mathbb{F}_q$ - whence I is finite. In this case, take any set of subspaces $V_1, \ldots, V_i$

of codimension $\geq k$, with $i = |I| < q^k + 1$; we are to show that $\bigcup_j V_j \subsetneq V$.

We now reduce the situation to that of a finite-dimensional quotient V' of V as follows. First, we may increase each V_i to a codimension k subspace. Next,

$$\dim_{\mathbb{F}} (V_1/(V_1 \cap V_2)) = \dim_{\mathbb{F}} ((V_1 + V_2)/V_2) \leq \dim_{\mathbb{F}}(V/V_2) < \infty, \quad (2.4)$$

whence $\dim_{\mathbb{F}} V/(V_1 \cap V_2) \leq \dim_{\mathbb{F}}(V/V_1) + \dim_{\mathbb{F}}(V/V_2) < \infty$. Now proceed inductively to show that $V_0 := \bigcap_{j=1}^i V_j$ has finite codimension in V; more precisely, $\dim_{\mathbb{F}}(V/V_0)$ is bounded above by $\sum_{j=1}^i \dim_{\mathbb{F}}(V/V_j)$.

Thus, we quotient by V_0, and end up with codimension-k subspaces V_j' covering the finite-dimensional quotient $V' = V/V_0$. Now if $\dim_{\mathbb{F}} V' = n$, then we are covering $q^n - 1$ nonzero vectors in V' by proper subspaces V_j', each with at most $q^{n-k} - 1$ nonzero vectors. Thus the number of subspaces needed, is at least $\geq \frac{q^n - 1}{q^{n-k} - 1} > q^k$, as claimed.

$\square$

The following result concludes the proof for infinite fields, by the last part of Remark 2.1.

Lemma 2.5. *If $\mathbb{F}$ and $\dim_{\mathbb{F}} V$ are both infinite, then V is a countable union of proper subspaces.*

Proof. (As for Proposition 1.5.) Fix any (infinite) basis B of V, and a sequence of proper subsets $\emptyset = B_0 \subset B_1 \subset \ldots$ of B, whose union is B. Now define V_n to be the span of B_n for all n. Then the V_n's provide a cover of V by proper subspaces, each of infinite codimension in V. $\square$

3. Proof for finite fields

We now complete the proof. In what follows, we will crucially use some well-known results on partitions of finite vector spaces. These are found in [2, Lemmas 2,4], though the first part below was known even before [1]).

Lemma 3.1. *Suppose V is an n-dimensional vector space over the finite field $\mathbb{F} = \mathbb{F}_q$ (for some $q, n \in \mathbb{N}$), and we also fix $d \in \mathbb{N}$.*

  (1) *V can be partitioned using only d-dimensional subspaces, if and only if $d | n$. (The number of such subspaces is $(q^n - 1)/(q^d - 1)$.)*
  (2) *Let $1 < d < n/2$. Then V can be partitioned into one $(n - d)$-dimensional subspace, and q^{n-d} subspaces of dimension d.*

We now show most of the main result, for finite fields.

Proposition 3.2. *Suppose V is a finite set. Then V is covered by "I-many" subspaces of codimension at least k, if and only if $I \geq \mathbb{P}(V)/(\mathbb{P}(V/\mathbb{F}^k))$.*

Proof. If V is finite, then so are $\mathbb{F}$ and $\dim_{\mathbb{F}} V$. We may also assume that the subspaces that cover V are of codimension exactly equal to k. Now suppose V is covered by "I-many" such subspaces, and $\dim_{\mathbb{F}} V = n \in \mathbb{N}$. Then we need to cover $q^n - 1$ nonzero vectors by proper subspaces, each with $q^{n-k} - 1$ nonzero vectors, whence

$$I \geq \frac{q^n - 1}{q^{n-k} - 1} = \frac{\mathbb{P}(V)}{\mathbb{P}(V/\mathbb{F}^k)},$$

as required.

We now show the converse: if $\dim_{\mathbb{F}} V = n$, and $(n-k)|n$, then we are done by the first part of Lemma 3.1, since there exists a partition. In the other case, we illustrate the proof via an example that can easily be made rigorous. We first fix $\mathbb{F} = \mathbb{F}_q$; now suppose $n = 41$ and $k = 29$. We must, then, find $\lceil (q^{41} - 1)/(q^{12} - 1) \rceil = q^{29} + q^{17} + q^5 + 1$ subspaces of codimension 29, that cover $\mathbb{F}^{41}$.

Now set $d = 12$ and apply the second part of Lemma 3.1; thus,

$$\mathbb{F}^{41} = \mathbb{F}^{29} \coprod (\mathbb{F}^{12})^{\coprod q^{29}}.$$

In other words, we have q^{29} 12-dimensional subspaces, and one extra subspace of dimension 29. Now apply the same result again (with $d = 12$ and replacing $n = 41$ by 29) to get

$$\mathbb{F}^{41} = \mathbb{F}^{17} \coprod (\mathbb{F}^{12})^{\coprod q^{17}} \coprod (\mathbb{F}^{12})^{\coprod q^{29}}.$$

(For a general n, k, apply the result repeatedly with $d = n - k$ and n replaced by $n - d, n - 2d, \dots$, until there remains one subspace of dimension between d and $2d$, and "almost disjoint" subspaces of codimension k.)

To conclude the proof, it suffices to cover $V_1 = \mathbb{F}_q^{17}$ with $q^5 + 1$ subspaces of dimension 12. To do this, fix some 7-dimensional subspace V_0 of V_1, and consider $V_1/V_0 \cong \mathbb{F}_q^{10}$. By the first part of Lemma 3.1, this has a partition into $(q^5 + 1)$ 5-dimensional subspaces. Lift this partition to V_1; this provides the desired (remaining) $q^5 + 1$ subspaces of codimension 29 in $\mathbb{F}^{41}$. $\square$

The last part of the main result can now be shown, using this result.

Proof of Theorem 1.2. The above results show the theorem except in the case when $\mathbb{F}$ is finite, but $\dim_{\mathbb{F}} V$ is not. In this case, by Proposition 2.3, we only need to show that V can be covered by $q^k + 1$ subspaces of codimension k. To see this, quotient V by a codimension $2k$ subspace V_0; now by Proposition 3.2, V/V_0 can be covered by

$$\frac{\mathbb{P}(V/V_0)}{\mathbb{P}((V/V_0)/\mathbb{F}^k)} = \frac{(q^{2k} - 1)/(q - 1)}{(q^k - 1)/(q - 1)} = q^k + 1$$

subspaces of codimension k. Lift these to V for the desired cover. $\square$

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E-mail address: apoorva@math.ucr.edu

Dept. of Mathematics, Univ. of California at Riverside, Riverside, CA 92521