Sheaf Theory

Tom Lovering

September 24, 2010

Abstract

In this essay we develop the basic idea of a sheaf, look at some simple examples and explore areas of mathematics which become more transparent and easier to think about in light of this new concept. Though we attempt to avoid being too dependent on category theory and homological algebra, a reliance on the basic language of the subject is inevitable when we start discussing sheaf cohomology (but by omitting proofs when they become too technical we hope it is still accessible).

1 Introduction and Definitions

Mathematics is a curious activity. Most of us probably see it as some giant blob of knowledge and understanding, much of which is not yet understood and maybe never will be. When we study mathematics we focus on a certain 'area' together with some collection of associated theorems, ideas and examples. In the course of our study it often becomes necessary to restrict our study to how these ideas apply in a more specialised area. Sometimes they might apply in an interesting way, sometimes in a trivial way, but they will always apply in some way. While doing mathematics, we might notice that two ideas which seemed disparate on a general scale exhibit very similar behaviour when applied to more specific examples, and eventually be led to deduce that their overarching essence is in fact the same. Alternatively, we might identify similar ideas in several different areas, and after some work find that we can indeed 'glue them together' to get a more general theory that encompasses them all. In fancy terminology, mathematics is *a bit like a sheaf.*

In this essay we will attempt one such 'gluing job', by developing the theory of mathematical sheaves as a precise formulation of really general situations where there is a large and potentially vastly intractable 'blob' of data which we can (at least in theory) reconstruct uniquely by looking at how it behaves locally and piecing it together. From Hindu theology to the subject of history (or, for that matter, any academic subject), this vague idea occurs throughout human experience and it is therefore entirely unsurprising that it also has a starring role in much of human mathematics.

Let us consider a slightly more precise example. A common activity for schoolchildren in the UK around Easter time (a festival whose bizarre secular incarnation is even more confused than that of Christmas) is the decoration of hard-boiled eggs, often as part of a judged competition with high chocolatey stakes. This process involves taking a *topological space* (an egg) and attaching some *data* (paint) to it. Judging this competition presents a subtle challenge however. Eggs have the unfortunate feature of being 'round', and the judge's eyes are only capable of seeing about half of the egg from any given angle. The judge will therefore have to pick up the egg and turn it round to inspect the complete design. This involves making an assumption: *given enough angles of viewing, it is possible to mentally reconstruct a unique 'complete design.'* In particular we are assuming that there is such a thing as a complete design, even though no person can fully perceive it at a given time (even some clever arrangement of mirrors will only really be giving you some number of different angles simultaneously, which is not the same as being able to see the entire surface of the egg at once). This philosophical abstraction, which our brains seem to automatically take care of in the physical world, is more mind-boggling in mathematics, where we are far more careful about our assumptions. Thankfully, once we have defined a sheaf, we can stop worrying and proceed with the far more important task of awarding chocolate.

Before we go on to make a formal definition it will help to have one more very important example in our heads, hopefully strongly suggested by the previous paragraph. Very often in geometry one is dealing with a *real manifold*. This is another example of a topological space, but this time rather than studying paintings of manifolds, mathematicians tend to study *smooth functions* from the manifold to real vector spaces of a fixed dimension. Such functions might not be explicitly definable simultaneously at every point of the manifold (given there is likely to be no single co-ordinate system for the whole manifold), but we would like to be able to take the wide view and say that such functions still exist even if we can only work with their local properties.

So, keeping these ideas in our head, it will help to frame our definition in two stages, the first of which expresses the basic concept of wanting to attach data to a topological space and the second of which is a 'niceness' condition and goes on to make explicit our need for data at different points to be 'compatible' and 'gluable'.

Definition 1.1. *A* ***presheaf*** $\mathcal{F}$ *of abelian groups on a topological space* X *is an object associating with every open set* $U \subset X$ *an abelian group* $\mathcal{F}(U)$ *and with every inclusion* $U \subset V$ *of two open sets in* X *a group homomorphism* $\rho_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$. *Furthermore,* ρ_{UU} *must be the identity mapping and if* $U \subset V \subset W$ *then* $\rho_{WU} = \rho_{VU} \circ \rho_{WV}$.

The group $\mathcal{F}(U)$ *is called the group of* ***sections*** *over* U*, and the morphism* ρ_{VU} *is called the* ***restriction map*** *of* $\mathcal{F}(V)$ *onto* $\mathcal{F}(U)$.

This might seem mind-boggling at first, but what is going on is very simple if we think about our manifold example, for simplicity in the case of maps to $\mathbb{R}$, $\mathcal{F}(U)$ is simply the ring (hence abelian group) of smooth functions $f : U \to \mathbb{R}$, and the restriction maps are just restrictions of smooth functions to a smaller domain. In other words a more conventional notation for $\rho_{UV}(f)$ would be $f|_V$ - in fact, whenever there is no danger of ambiguity we use this notation instead.

One further remark worth making is that when in the definition I wrote 'abelian groups', I could have written any category whatesoever and the definition would still work. In reality, we will usually be working with sheaves of rings or sheaves of modules, and since both of these are abelian groups, the above definition will be sufficiently general. However, to make it explicit that every morphism in sight is actually a morphism of rings or modules it will often be appropriate to talk about sheaves of rings or modules instead.

Ok, so far so good. So now a couple more conditions to ensure sheaves are 'nicely-behaved'. In reality what they are saying is what we were talking about in the opening paragraphs about egg-painting: that if we have a load of sections on different small open sets with the property that on intersections any pair of sections 'agrees' (after being restricted), then there exists a corresponding section on the large union of these open sets, and such a section is unique. I.e. there exists a unique gluing whenever such a gluing is not trivially impossible. Let's make that totally precise now.

Definition 1.2. *A **sheaf** $\mathcal{F}$ is a presheaf that satisfies the following condition. Let $\{U_i\}_{i\in I}$ be any collection of open sets on X and $U = \bigcup_{i\in I} U_i$. Then:*

1. *(Existence of gluing) Whenever, for every $i \in I$, $s_i \in \mathcal{F}(U_i)$ and for every $i, j \in I$, $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ there exists a section $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$ for all $i \in I$.*

2. *(Uniqueness of gluing) Let $s, t \in \mathcal{F}(U)$ be such that $s|_{U_i} = t|_{U_i}$ for all $i \in I$. Then $s = t$.*

Hopefully after the introduction, that definition should seem fairly natural. So, let's test our intuition. It is indeed a quick check to verify that continuous, n-times differentiable or smooth functions on manifolds (real or complex) are all examples of sheaves. A couple of examples spell out the limits of this definition though.

Firstly, consider the presheaf of bounded functions from $\mathbb{R}$ to $\mathbb{R}$ (i.e. take $\mathbb{R}$ to be our topological space). Is this a sheaf? It turns out not! Indeed, if we consider the function $f(x) = x$, this is bounded on $(-N, N)$ for all N, but is not bounded on $\mathbb{R}$. It looks like in defining sheaves we have accidentally built in some fairly strong properties that cause this to fail.

Secondly, suppose we want to define a constant sheaf (one all of whose sections are 'the same'). This turns out to be impossible almost all of the time! To see why, suppose we have two disjoint open sets $U, V \subset X$. Then the existence and uniqueness of gluings induces an isomorphism between $\mathcal{F}(U\cup V)$ and $\mathcal{F}(U) \times \mathcal{F}(V)$ and hence unless the sheaf is trivial, $\mathcal{F}(U \cup V)$ is not isomorphic to $\mathcal{F}(U)$.

So sheaves seem fairly special. Perhaps it is important to remark in the other direction that they aren't *that* special however. I remember on first learning about sheaves imagining that they had something to do with analytic continuation on the complex plane. If the reader is having such thoughts, I will arrest them now by remarking that the extraordinary behaviour exhibited by complex analytic functions - where knowing how a function behaves in any open set, no matter how small, determines its behaviour on any other open set in the space where it can be defined - is a far stronger property than that possessed by general sheaves (where knowledge of local behaviour is required at every point and not just a single point).

It is also worth remarking that our example of a smooth manifold is actually in one sense a bit rubbish, because in this case it is a theorem that any section in $\mathcal{F}(U)$ can be extended to a section in $\mathcal{F}(X)$. Such a sheaf is called **flabby**, and one might argue that the lower sections are basically irrelevent: once you know what the sheaf is, all the necessary data can be obtained by looking at global sections. An example of a much 'skinnier' sheaf would be the sheaf of holomorphic functions on the Riemann sphere (the one-point compactification of $\mathbb{C}$). Here, by Liouville's theorem in complex function theory, the only global sections are the constant functions, but there is still a rich collection of holomorphic functions on small open sets. When we come to study the cohomology of sheaves in the final section, we will make these ideas more precise and see how the sheaf structure being less interesting in smooth manifolds limits the information we can extract about them from the topological theories of cohomology.

2 Morphisms, Localisation and Sheafification: An Overture to Category Theory

In this section we will define some useful ideas which will allow us to manipulate sheafs and presheaves with a little more flexibility. We will also learn that the language of category theory really is best for describing these objects, since everything we do in this section will be essentially trivial from a category-theoretical point of view.

2.1 Morphisms of sheaves

Firstly, having defined sheaves of abelian groups as generalisations of (fixed) abelian groups, it is natural to want to define morphisms between them in order to compare different sheaves. This aspiration becomes rather complex and unnatural unless the underlying topological space is the same for both the domain and codomain of the morphism. So how will we define a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{F}'$? Well, normal abelian groups can be viewed as sheaves on an indiscrete topology, and their morphisms consist of a single morphism of abelian groups. Now, in this more general situation, we have a whole collection of abelian groups, one for each open set, so it seems natural that for every open set U we will need a morphism of abelian groups $\phi_U : \mathcal{F}(U) \to \mathcal{F}'(U)$. Is this sufficient? Well, no, not quite. Sheaves also have extra structure in the form of restriction maps, and morphisms always have to pay due respect to structure. These considerations put together give rise to the following final definition.

Definition 2.1. *A **morphism of presheaves** of abelian groups on X, $\phi : \mathcal{F} \to \mathcal{F}'$, consists of a morphism $\phi_U : \mathcal{F}(U) \to \mathcal{F}'(U)$ of abelian groups for each open set U, and these morphisms must have the property that for any pair of open sets $U \subset V$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{F}'(V) \\ {\scriptstyle\rho_{VU}}\downarrow & & \downarrow{\scriptstyle\rho'_{VU}} \\ \mathcal{F}(U) & \xrightarrow[\phi_U]{} & \mathcal{F}'(U). \end{array}$$

A morphism of sheaves is defined in precisely the same way.

The reader who has done even a little category theory will not have failed to notice that this is the only possible natural definition of a morphism. Indeed, she will probably have observed that by making X into a category whose objects are open sets and whose arrows are inclusions, presheafs of abelian groups on X can be elegantly defined as contravariant functors from this category to the category of abelian groups. In other words, they are an arrow-reversing morphism of categories. Such morphisms of categories, functors, themselves admit morphisms called *natural transformations*, whose definition is of exactly the same form as that above.

Now we have morphisms, we have in particular a notion of isomorphism. Specifically, two sheaves $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic if there exist morphisms $\phi : \mathcal{F} \to \mathcal{F}', \phi' : \mathcal{F}' \to \mathcal{F}$ such that $\phi \circ \phi'$ and $\phi' \circ \phi$ are both the identity. That is to say, for every open set U in X, $\phi_U \circ \phi'_U$ and $\phi'_U \circ \phi_U$ are the identity as automorphisms of abelian groups. To category theorists this is obvious, but the point must be made that without defining the morphisms we could easily define a host of sheaves that are in essence exactly the same but have no way of knowing. As with any mathematical theory, the morphisms give the subject (some possibility of) structure and coherence. Sheaves can now be viewed as a category in their own right.

2.2 Local properties: stalks of a presheaf

The next concept we shall meet is one more specific to sheaf theory. We motivated the theory by describing it as a tool of keeping track of local data which connects together to form some global whole. However, the sheaf only gives us sections on open sets, and at some point we will want to have some concept of a section's 'local properties' near a given *point* x of the topological space. The construction is fairly natural, and works for all presheaves.

Definition 2.2. *Define an equivalence relation on the disjoint union $S = \bigcup_{U \ni x} \mathcal{F}(U)$ of all sections over open sets containing x as follows. We say that $u \in \mathcal{F}(U)$ and $v \in \mathcal{F}(V)$ are equivalent if there exists some open $W \subseteq U \cap V$ containing x such that $u|_W = v|_W$. We can then define the **stalk** $\mathcal{F}_x$ of $\mathcal{F}$ at x as the set of equivalence classes under this relation. The elements of a stalk are* *called **germs**, and there are natural projection maps $s \mapsto s_x$ taking sections to germs.*

It is easy to check that $\mathcal{F}_x$ inherits the abelian group structure from the sections (and will indeed inherit ring and module structure, or any other algebraic structure that involves only finite processes at the level of its axioms). Categorically, this is just a direct limit over the system whose objects are the $\{\mathcal{F}(U) : x \in U\}$ and whose arrows are the restriction maps between these objects.

So now we know what a presheaf looks like around any given point, which will be useful in lots of situations. While we are dealing with generalities it is now fairly natural to ask whether any kind of converse is possible: what is it possible for us to reconstruct if we are just given the stalks of a presheaf on a known topological space? As one might expect, it will turn out that we can recover sheaves, but that presheaves have lost information (lacking the nice local-to-global conditions).

2.3 From stalks to sheaves

We define the **plant** [1] $P\mathcal{F}$ generated by a presheaf $\mathcal{F}$ to be the set of all germs of $\mathcal{F}$ (i.e. the disjoint union of the stalks) together with the following topology. Fix a section $s \in \mathcal{F}(U)$, and write s_x to mean the germ of s at x. Then we take $\{s_x : x \in U\} \subseteq P\mathcal{F}$ to be open for all pairs (s, U), and let this be a basis for the topology on $P\mathcal{F}$. It should be remarked that this topology is often not Hausdorff.

It will also be useful to associate with a plant its projection map $\pi : P\mathcal{F} \to X$ which takes all germs $s_x \mapsto x$, really just keeping track of which germs are from which stalks. We shall now do a little actual maths, proving that plants have some nice properties.

Proposition 2.1. *Let $(P\mathcal{F}, \pi)$ be a plant. Then:*

1. *$\pi : P\mathcal{F} \to X$ is a local homeomorphism onto X.*
2. *The group operations are continuous. More specifically, if $P_2\mathcal{F}$ is the subspace of $(f_x, g_x) \in P\mathcal{F} \times P\mathcal{F}$ (pairs of elements in the same stalk). Then the map $P_2\mathcal{F} \to P\mathcal{F}$ defined by $(f_x, g_x) \mapsto f_x - g_x$ is continuous (and if we were working with presheaves over rings or modules, all relevent operations would be continuous).*

Proof. It is obvious that π is an open mapping and locally bijective. Furthermore, if $U \subseteq X$ is open then for every $s_x \in \pi^{-1}(U)$, let us suppose $s \in \mathcal{F}(V)$, $V \ni x$. Now define $t = s|_{U \cap V}$. Since t and s agree when restricted to $U \cap V$ they must give rise to the same germs on $U \cap V$, and hence $s_x \in \{t_x : x \in U \cap V\} \subset \pi^{-1}(U)$. In this way we can write $\pi^{-1}(U)$ as a union

[1] This terminology is nonstandard, for reasons that will become clear soon.

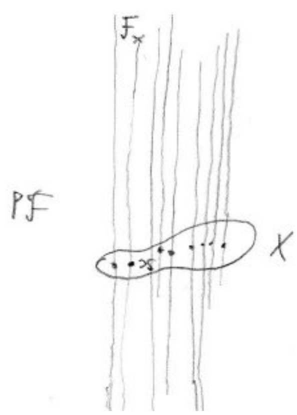
of open sets and so it is open, proving that π is continuous, and completing the proof of the first part.

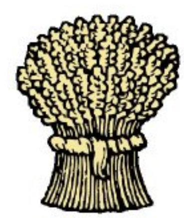
To check the group operations are continuous, we need to remark that $P\mathcal{F}$ is a quotient of the space S formed taking the disjoint union of sets $U \times \mathcal{F}(U)$ with $U \subset X$ open having the subspace topology and $\mathcal{F}(U)$ the discrete topology. It will help to define S_2 as the preimage of $P_2\mathcal{F}$ when the quotient is applied to the product space $S \times S$. A basis for the open sets in S is all sets of the form $V \times \{f\}$ where $V \subseteq U$ are open subsets of X and $f \in \mathcal{F}(U)$. If $\Delta : \langle (x, f), (x, g) \rangle \mapsto (x, f - g)$, then

$$\Delta^{-1}(V \times \{f\}) = \bigcup_{g \in \mathcal{F}(U)} \langle V \times \{f + g\}, V \times \{g\} \rangle \cap S_2.$$

This set is clearly open in S_2, hence the group operations are continuous in S and therefore, since projections onto quotient spaces are continuous open mappings, the group operations are also continuous in $P\mathcal{F}$. $\square$

Ok, so we now have a large topological space equipped with a locally homeomorphic projection map onto X, where the preimage of every point of X has the structure of an abelian group and where the groups 'vary continuously'. If we try to draw a picture of what this looks like, we might get something like the figure on the left below. The figure on the right is included to illustrate by way of comparison where the terminology for this essay might have originated: our 'plant' really looks like a (physical) sheaf with stalks all bound together by the underlying space X.





Looking at this picture, if we want to study the properties of $P\mathcal{F}$ it looks, especially given the group operations vary continuously, like taking ('almost-horizontal') *cross-sections* could be a good idea, and rather than always taking cross sections of the whole plant we might just want to look at small cross sections on small open sets U, at least to start with. How do we take a cross-section? We want a map $s : U \to P\mathcal{F}$ which selects one germ from each stalk of a point in U. In other words, we demand that $\pi \circ s = Id|_U$. Furthermore, of course we will insist that s is continuous. There are likely to be many different cross-sections, so we define $P\mathcal{F}(U)$ to be the set of all cross-sections of $P\mathcal{F}$ over U.

But this situation now feels very familiar. For each open set U of X we have this collection of objects $P\mathcal{F}(U)$ which inherit any algebraic structure pointwise from the stalks of $\mathcal{F}$, so in fact they will be abelian groups. Furthermore, since these objects are concrete functions, of course we have restsriction maps, so $P\mathcal{F}$ is itself a presheaf! However, we have much more, which we will gather in a theorem and forms the climax of this section.

Theorem 2.2. *The plant $P\mathcal{F}$ generated by a presheaf $\mathcal{F}$ over X gives rise, by way of its cross-sections and their restrictions, to a sheaf over X. Furthermore, the stalks of $P\mathcal{F}$ are identical to the stalks of $\mathcal{F}$, and if $\mathcal{F}$ is a sheaf then it is sheaf-isomorphic to $P\mathcal{F}$.*

Proof. First, we check that the two further sheaf axioms are satisfied by $P\mathcal{F}$. Let $U = \bigcup_{i \in I} U_i$ be a union of open sets in X, and $s_i : U_i \to P\mathcal{F}$ be sections on each set in the cover that agree on intersections. We need to check that the obvious gluing map ($s(x) := s_i(x)$ for any i such that $x \in U_i$) is continuous. But since $\pi \circ s = Id|_U$ (being the identity on each individual U_i), and π is an open mapping, this is immediate. It is also immediate that this is the unique gluing, because any other gluing would differ from this one at some point, and hence differ with one of the s_i at some point. Thus we have verified that $P\mathcal{F}$ does indeed have full sheaf structure.

Next we check that the stalks are fixed. To do this, we will need to prove that any two cross-sections agree on an open set. Note that for any open $U \subseteq X$ and any cross-section $s : U \to P\mathcal{F}$ the set $s(U)$ is open in $P\mathcal{F}$, since π is continuous. Let 0_U be the zero cross-section: whose image is the set of germs on U of the zero element of $\mathcal{F}(U)$. Then $0_U(U)$ is also open (just as s above), so for arbitrary s, $s(U) \cap 0_U(U)$ is open in $P\mathcal{F}$, and therefore, since π is an open mapping, $\{x \in U : s(x) = 0\}$ is open. Taking $s = f - g$ we get that for any two cross-sections $f, g \in P\mathcal{F}(U)$, the set on which $f = g$ is open.

But we are now almost done. The stalk of our new sheaf about a fixed point x is a set of equivalence classes of cross-sections where $f \approx g$ if they agree on some open set containing x. But putting this together with the above remarks, we can deduce that the equivalence classes are simply that $f \approx g$ if and only if $f(x) = g(x)$ (this is clearly a necessary condition, and our demonstration that the sets of agreement are open makes it also sufficient). But since $f(x)$ and $g(x)$ are both themselves germs of the presheaf $\mathcal{F}$, we have established a natural isomorphism $[f] \mapsto f(x)$ between $P\mathcal{F}_x$ and $\mathcal{F}_x$.

Finally, what happens if $\mathcal{F}$ was a sheaf to begin with? We need to match up sections of the original sheaf with cross-sections of its plant, so we'll try the first thing that comes into our heads. Fix U open and take $s \in \mathcal{F}(U)$ to the cross-section $P_U(s) : u \mapsto s_u$. We need to check that the maps $P_U : \mathcal{F}(U) \to P\mathcal{F}(U)$ are isomorphisms, and we need to check that they commute with restriction maps (recalling the definition of sheaf morphisms from section 2.1).

It is clear that P_U is a group homomorphism, being a pointwise projection onto the stalk which is a homomorphism. To prove injectivity, take s in the kernel, so $P_U(s)(u) = 0$ for all $u \in U$. I.e. for every $u \in U$ there exists an open set $U_u \ni u$ such that $s|_{U_u} = 0$. Hence by the uniqueness of gluings, $s = 0$. To prove surjectivity, we know that σ is an open mapping, so $\sigma(U) = \bigcup_{i \in I}\{s_{ix} : x \in U_i\}$ for some cover of U by disjoint open sets U_i and some sections $s_i \in \mathcal{F}(U_i)$. Since the U_i are disjoint there exists, by the existence of gluings, an $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$ for all i. It is clear then that $P_U(s) = \sigma$, proving that P_U is surjective and an isomorphism.

Finally, if $V \subseteq U$ then we need to check that the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{P_U} & P\mathcal{F}(U) \\ {\scriptstyle \rho_{UV}}\downarrow & & \downarrow{\scriptstyle |_V} \\ \mathcal{F}(V) & \xrightarrow[P_V]{} & P\mathcal{F}(V). \end{array}$$

But since everything is defined locally, this is fairly clear. After all, $(\rho_{UV}(s))_x = s_x$ for all $x \in V$, so the both directions simply give the map $V \to P\mathcal{F}(V)$ which takes $x \mapsto s_x$.

□

This theorem is rather interesting, in that it gives us a new characterisation of sheaves, and gives us a canonical way to make a sheaf out of a presheaf. The new characterisation can be read off as follows, and is that adopted for the whole of [4]. Note that we now drop the P and any mention of 'plants', moving between these two characterisations interchangably.

Corollary 2.3. *A sheaf of abelian groups on a topological space X can be written as a pair $(\mathcal{F}, \pi)$ where $\mathcal{F}$ is a topological space, $\pi : \mathcal{F} \to X$ is a local homeomorphism onto X, and, for every $x \in X$, $\pi^{-1}(x)$ is an abelian group whose group operations are continuous.*

Proof. This follows immediately from proposition 2.1 and theorem 2.2. □

Let us take a quick look at the first part of theorem 2.2 in more detail. What we have done over the course of this section is take a presheaf $\mathcal{F}$ and, by passing through its stalks, made a sheaf out of it. It is possible to show that in fact the sheaf we have made is in some sense the 'closest possible approximation' to $\mathcal{F}$ by a sheaf. More precisely it has the property that for any other sheaf $\mathcal{G}$, any morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ must be expressible as $\phi' \circ P$ where $P : \mathcal{F} \to P\mathcal{F}$ is the presheaf morphism discussed in the proof of theorem 2.2, and $\phi' : P\mathcal{F} \to \mathcal{G}$ is a morphism of sheaves. This property is by its own definition unique to a single isomorphism class of sheaves.

This is exactly analogous to many other constructions in algebra like those of free groups or algebraically closed fields. Our smug category theorist friend, though he would still have had to prove the theorems of this section, would probably have put money on it working. This is an example of a certain class of what are known as *adjunctions* - pairs of functors giving 'least upper bounds' and 'greatest lower bounds' in some sense. There is a functor from sheaves to presheaves called a *forgetful functor* which simply does nothing, but forgets that the sheaf you are working with has the nice gluing properties. What we have constructed, known as the *sheaving functor* is the *left adjoint* to the forgetful functor, taking presheaves to sheaves in the most general possible way they can in the sense of the previous paragraph. In particular, it should take presheaves which actually happen to be sheaves to themselves, as theorem 2.2 happily confirms.

So what have we learnt? In this chapter we developed some of the basics of the theory of sheaves: we defined their morphisms, then defined the fundamental notion of a stalk for managing local data, and learnt how it gave a valuable means for canonically constructing sheaves from presheaves. In this construction we gained a new more visual perspective on what a sheaf is - describing it as a large topological space, and we learnt that *for any reasonably compatible set of local data (defined at stalks), there exists a unique sheaf.* Now let us get to know a few of the more important sheaves and apply what we have learnt.

3 Exploring some sheaves

After the preceding section, we see that there are two (possibly three) obvious ways we might go about constructing a sheaf. We might just specify the sections directly and the check they satisfy the sheaf axioms. Alternatively we might describe what the stalks look like (or a suitable presheaf), build the corresponding plant, and check its cross-sections do give rise to a sheaf (which will require that the algebraic operations of the stalks act continuously in the plant topology). In each case, the objects we *don't* explicitly describe (either stalks or sections) are often quite complicated and interesting to study in their own right, being as they are automatically born out of the machinery of sheaf theory.

3.1 The constant sheaf and the localisation of nonlocal properties

We first try to construct sheaves corresponding to some of those objects which irritatingly were not quite sheaves in the first chapter. The most obvious (and a very important) example is the constant presheaf, which you will recall was the presheaf whose sections (or, if you like, its stalks) are all the same abelian group. It failed to be a sheaf most of the time because we saw that a section over a union of two disconnected open sets has to be the direct product of the sections over the two sets individually, by the gluing axiom. However, it's certainly a valid presheaf, so we can apply the procedure of the previous chapter to generate a sheaf from it. It is a useful exercise for the reader to do this themselves, referring to the previous chapter as necessary. Hopefully they obtain the following (which, should be noted, is largely a result of the definition of the topology on sheaves).

Definition 3.1. *The **constant sheaf** of an abelian group G on X, G_X, is the sheaf of locally constant functions $s : U \to G$. Its stalks are all equal to G, and its sections satisfy the isomorphism $G_X(U) \cong G^{conn(U)}$ whenever $conn(U)$, the number of connected components of U, can be defined.*

What happened here is interesting. We started with a global property: asking for globally constant functions, and we ended up with the corresponding local property. The same thing happens with other examples. A quite instructive but initially quite weird one might be what happens to the presheaf of polynomial functions $\mathbb{R} \to \mathbb{R}$. It is not a sheaf because, for example, we cannot glue together $(x \mapsto 0) : (-\infty, 0) \to \mathbb{R}$ and $(x \mapsto x^2) : (0, \infty) \to \mathbb{R}$ into a single polynomial function (if you like, because it would have to be nonzero, but has infinitely many roots). If we therefore apply the sheaving functor to this presheaf we end up with the **sheaf of locally polynomial functions**, where rather than having to be a polynomial, there just has to be an open set about any point on which you are a polynomial. This might sound a little artificial and ridiculous, but is very important to bear in mind in many areas where sheaves are used: for example in algebraic geometry, where sheaves of locally *rational* functions are of great importance.

This also concisely explains why our other examples of sheaves in the opening chapter were in fact sheaves. Things like continuous functions, n times differentiable functions and smooth functions already often have local properties. When we define a property like 'differentiable', we say something along the lines of "for every point x there exists an open ball about x such that...." It is therefore unsurprising that they form a sheaf - there is no more localisation of the property for the sheaving functor to do. If you like, the statement that smooth functions on a manifold form a sheaf could be expressed as

$$\text{smooth} \cong \text{locally smooth}.$$

So this gives us a nice picture of what the machinery of the sheaving functor is really doing in many examples, taking arbitrary properties and making them local with respect to the topology on X.

3.2 Sheaves of rings and locally ringed spaces

So far we have been looking only at sheaves of abelian groups. However, other types of sheaf are also important, and in section 3.2 every sheaf we meet will be a sheaf of rings (so its sections and stalks now have a continuous ring structure, and any morphisms of abelian groups are, we now insist, morphisms of rings). Since this is an essay being written by a Cambridge undergraduate, our rings will always be commutative with 1, though many of the results of this theory apply in more general situations.

The most obvious sheaves of rings have already been mentioned. Let us take, for example, the sheaf of continuous functions $(U \subseteq \mathbb{R}) \to \mathbb{R}$ where $\mathbb{R}$ has its usual topology. This is defined in terms of its sections, so we need to check these give a sheaf, but the discussion above about local and global properties, perhaps after a bit of rigorisation, should make this a straighforward process. It is natural to ask, having defined this very familiar concept in terms of sheaves, how well the topology and the stalks of this sheaf behave.

The topology is, perhaps surprisingly, not Hausdorff. To see why, let us consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x \geq 0. \end{cases}.$$

This function's germ at 0 is impossible to separate by open sets from the zero function's germ (since the germs of these two functions coincide for all $x < 0$), but they are definitely distinct germs ($0 \neq x$ for $x > 0$ arbitrarily small). Hence the topology is non-Hausdorff.

What are the stalks like? They definitely have the ring structure induced by the direct limit. However, they are slightly odd to think about. It certainly isn't true that there is a bijection between germs and global functions (as we saw in the above example where a function that was 0 for some period of time before going off to do other things had the same germ as the zero function for most of the time it was 0). One obvious thing that occurs to somebody thinking about these stalks C_x is that two functions which have different values at x definitely give rise to different germs. This thought gives rise to an obvious maximal ideal: the ideal of germs of functions which vanish at x. We know it is maximal because its quotient is clearly isomorphic to $\mathbb{R}$, which is a field (if you prefer, it is the kernel of the homomorphism $f_x \mapsto f(x)$, whose image is $\mathbb{R}$). In fact, a bit more thought shows this must be the unique maximal ideal. Indeed, any other maximal ideal must contain an element which is not 0 at x, and so by continuity this element corresponds to a function which is nonzero on a neighbourhood of x. But then we can invert it, and learn that the germ of 1 is in our ideal, so our ideal is actually the whole of C_x.

This property, that of having a unique maximal ideal, makes this ring seem rather 'small' and 'well-behaved' in some sense even though it is very much a newly defined entity. Such rings are called **local rings** and always arise in such situations where a ring of functions is expressed as a sheaf and one looks at the germs. This gives rise to the following very general definition.

Definition 3.2. *A **locally ringed space** is a pair $(X, \mathcal{O}_X)$, where X is a topological space, and $\mathcal{O}_X$ is a sheaf of rings on X whose stalks are local rings (each has a unique maximal ideal). $\mathcal{O}_X$ is called the **structure sheaf** of X.*

It is probably no exaggeration to say that this definition encapsulates one of the most important ideas in modern geometry, and cannot be underestimated. Almost all spaces geometers study can be defined in terms of locally-ringed spaces, and very often the idea of defining them in this manner has led to rich generalisations which have in turn led to a much deeper understanding of the great logical architecture supporting many of the structures with which we are familiar.

Given how important locally ringed spaces are, it is appropriate and useful for us to briefly detail how they can be compared with one another. Unfortunately, there is something of a technicality involved in that we will be comparing two sheaves with different base spaces, and therefore cannot directly apply the sheaf morphisms we defined already without combining it with a new idea called the *direct image functor*.

Definition 3.3. *If $f : X \to Y$ is a continuous map of topological spaces, and $\mathcal{F}$ is a sheaf over X, we define the* ***direct image*** *$f_*\mathcal{F}$ to be the sheaf on Y whose sections are $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ for every open set $U \subset Y$.*

Equipped with this gadget for changing the base space of sheaves we can now define morphisms of locally ringed spaces.

Definition 3.4. *Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be locally ringed spaces. Then a* ***morphism*** *between them is a pair (f, f') satisfying:*

1. *$f : X \to Y$ is a continuous map of topological spaces.*
2. *$f' : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of rings on Y.*
3. *For every $x \in X$, the localisation of f', $f'_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a local homomorphism (a ring homomorphism where the inverse image of a maximal ideal is maximal).*

In particular, we have the usual concept of isomorphism (two morphisms which compose both ways to yield the identity), and this turns out to be equivalent to the property that f must be a homeomorphism and f' a sheaf isomorphism. With these defined, we are ready to look at two major areas in modern geometry where these concepts have led to great advances.

3.3 Manifolds and schemes

I will now briefly describe two areas in which applying a sheaf-theoretic approach to a previously studied locally ringed space has led to rich and important new ideas and generalisations. We will look at differentiable manifolds, and how this concept admits generalisations, as well as how cotangent spaces can be defined abstractly within a locally ringed space. Then we will see how these ideas have been applied to algebraic geometry, sketching the ideas behind Alexander Grothendieck's revolutionary generalisation of zero sets of polynomials to abstract *schemes*. In fact, both of these concepts are, in light of our new sheaf-based formalism, closely related.

Firstly, let us take a look at n-dimensional differentiable manifolds with fresh eyes. Recall that they are topological spaces on which it is possible to do differentiation because they are locally homeomorphic to a subset of $\mathbb{R}^n$. In the language of locally-ringed spaces, we can proceed as follows. Let $(\mathbb{R}^n, \mathcal{O})$ be n-dimensional space equipped with the standard sheaf of rings of smooth functions (familiar from any undergraduate analysis course - note how the openness of the sets on which sections of a sheaf are defined becomes important). This is a locally ringed space, and defining a manifold is now easy with our new language.

Definition 3.5. *A smoothly differentiable n-dimensional* ***manifold*** *is any locally ringed space $(M, \mathcal{O}_M)$ which is locally isomorphic to a subset of $(\mathbb{R}^n, \mathcal{O})$.*

In other words, for any $x \in M$ there exists a neighbourhood U of x such that some map $f : U \to \mathbb{R}^n$ induces an isomorphism $(U, \mathcal{O}_M|_U) \cong (f(U), \mathcal{O}|_{f(U)})$ of locally ringed spaces. It is fairly straightforward to check that this definition is equivalent to the usual one with charts and atlases and so on. What is nice is that in this definition we really get to the heart of the matter of what a manifold is. It's not really some nonsense about gluing together a few co-ordinate systems - the property of interest is that it is locally isomorphic to euclidean space, at least for the purposes of doing differentiation.

There are also some practical uses. For example, we now have suggested to us a rather nice abstract definition of the cotangent space. Intuitively this space is the space of 'first order infinitesimals' around a function (or, in the language of the next section, the space of local 1-forms about a point). Equipped with the formalism of local rings, we can rigorously express infinitessimals as equivalence classes of functions which all head off in the same direction, to first order. Let $\mathcal{O}_x$ be the local ring at x, and M_x its maximal ideal. Since we care about infinitessimals, all our cotangents will be elements of M_x (i.e. they are zero, but might be going somewhere nonzero). However, being first order approximations, two elements of M_x will correspond to the same cotangent precisely whenever their difference is in $M_x^2 := \{fg : f, g \in M\}$. Hence we obtain a purely abstract definition of the cotangent space as M_x/M_x^2. It can be quickly checked that this is a vector space over the field $\mathcal{O}_x/M_x$, and indeed in smooth manifolds this coincides with the definition of cotangents as dual to the tangent space of derivation operations.

Above all, it must be becoming clear to the reader that many of these familiar geometrical concepts are actually much more general than one might expect. The previous paragraph allows us to define a cotangent space on any locally ringed space and operate on it in analogy with a differentiable manifold. More profoundly however, there was absolutely nothing special about the space $(\mathbb{R}^n, \mathcal{O})$ as our choice of model in the definition of a manifold. Indeed, replacing $\mathbb{R}$ with $\mathbb{C}$ and taking a sheaf of analytic functions, we get complex manifolds. A similar much more radical substitution into this sheaf-theoretic construction of a manifold allows us to define a *scheme*, an unbelievably deep and useful generalisation of the concept of *variety* (essentially, the set of points in a vector space which are the common zeroes of a fixed system of algebraic equations) from classical algebraic geometry.

Rather than a vector space with some differentiation happening, the underlying structure here is the *spectrum of a ring A* whose definition is rather involved but which we shall give briefly for the interested reader.

Firstly let $SpecA$ be the set of prime ideals of A. For any ideal a of A, let $V(a)$ be the set of prime ideals which contain a. Take the sets $V(a) \subset SpecA$ to be the closed sets generating a topology on $SpecA$ (it can be checked that this procedure works). We now have a nice topological space of prime ideals, whose topology is analogous to the Zariski topology in algebraic spaces.

Next, define $\mathcal{O}$ a sheaf of rings on $SpecA$ as follows. For $p \in SpecA$ let $A_p = A/p$, and we can now define $\mathcal{O}(U)$ to be the set of functions $s : U \to \coprod_{p \in U} A_p$ such that $s(p) \in A_p$ for all p and s is locally a quotient of elements of A. The local nature of this definition guarantees that this $\mathcal{O}$, which is manifestly a presheaf, is indeed a sheaf, and it generalises the sheaves of locally rational functions one would get on a more concrete algebraic space. Note that the stalks of this sheaf are just A_p but also, perhaps surprisingly, the ring of global sections $\mathcal{O}(SpecA)$ is canonically isomorphic to A.

Anyway, we have now made the spectrum of a ring into a locally ringed space $(SpecA, \mathcal{O})$, and can now define a scheme in exactly the same manner as our definition of a manifold.

Definition 3.6. *A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to the spectrum of a ring (which need not necessarily be the same ring in every neighbourhood).*

Of course, the schemes are simply a subcategory of the category of locally ringed spaces (so we have already defined scheme morphisms). It is well beyond the scope of this essay to go any further into the huge subject of algebraic geometry, but it is important to realise how absolutely vital the theory of sheaves is for defining locally ringed spaces and in turn for defining these schemes.

4 Sheaves in algebraic topology: steps towards unifying cohomology

In this final section we will start to explore one of the most important applications of sheaf theory - in providing a new cohomology whose definition makes it very easy to compare with the cornucopia of other cohomologies that have arisen in algebraic topology over its long and energetic history. Aware that some readers might not be totally happy with what a cohomology even is, "making something quite scary a lot scarier in order to make it less scary" might have been a more appropriate title. Though it will start quite gently, this chapter will be a narrative (omitting most of the quite technical proofs) of some actually very serious and involved 20th century mathematics, and will therefore of necessity be rather faster-paced than the rest of the essay. That said, we hope that even if a reader does not take away a completely clear picture of how they might go about proving these results, they will at least get a flavour of the power of the sheaf-theoretic approach in dealing with the very practical issue of comparing different cohomological theories.

4.1 Sheaves of modules and their complexes

In most of this chapter, our object of study will still be some locally ringed space $(X, \mathcal{O}_X)$. However, taking our sheaf formalism one level higher, we will normally be working with sheaves $\mathcal{F}$ (still over X) of $\mathcal{O}_X$-modules. Since we now have two sheaves in the mix, this will require a little clarification. We obviously define $\mathcal{F}(U)$ to be an $\mathcal{O}_X(U)$-module for every open U. However, in order to define the restriction maps we need to observe that for $V \subset U$, some $\mathcal{O}_X(V)$-modules can be naturally viewed as $\mathcal{O}_X(U)$-modules with scalar multiplication including a restriction map inside $\mathcal{O}_X$: $\lambda.x = \lambda|_V.x$. Thus our restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are (via this sidestep into a higher ring of scalars) $\mathcal{O}(U)$-module homomorphisms, and in any nested system of open sets when looking at restriction maps we must take the largest open set when identifying the ring of scalars to use.

We will let $\mathcal{A}$ denote the category of sheaves of $\mathcal{O}_X$-modules (it is a category because we know the morphisms - they are just sheaf morphisms but respecting the underlying module properties). Here the abstract nonsense begins. It is something that needs to be checked (and, in places, defined) that $\mathcal{A}$ is what is called an *abelian category.* We will define what an abelian category is, but refer the interested reader to [1] and [4] for verification that sheaves of modules are indeed an example of such a category.

Definition 4.1. *An* ***exact category*** *is one with a zero element where every morphism has a kernel and a cokernel and induces an isomorphism analogously to the first isomorphism theorem for groups. An* ***additive category*** *is one with a zero element and whose morphism sets $Hom(X, Y)$ are abelian groups such that compositions of morphisms are bilinear. It must also admit direct sums between pairs of objects (if X and Y are objects in the category, then so is $X \oplus Y$). An* ***abelian category*** *is exact and additive.*

If it scares you a bit, it isn't worth worrying too much about this definition - what it basically distils is the bare minimum of properties we will need in order to do most of the work in this section. One does not lose out too much by just thinking about elements of as nice modules (perhaps even forgetting that there are sheaves around if you are worried, in which case all of these facts are just basic facts about modules). The really important thing we can define in abelian categories is a complex, and a notion of its exactness or failure to be exact. Note that in most literature complexes can extend infinitely in both directions, but for our purposes the following definitions will suffice.

Definition 4.2. *A* ***complex*** *$A^\bullet$ is a sequence of objects A^i connected with morphisms d^i:*

$$0 \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \ldots$$

which have the property that for every i, $d^{i+1} \circ d^i = 0$. Where there is no danger of confusion, the superscripts are dropped and for example this relation can be written $d \circ d = 0$.

The complex is called ***exact*** *if $Ker(d^i) = Im(d^{i-1})$ for all i.*

A morphism of complexes $u : A^\bullet \to B^\bullet$ is just a sequence of morphisms

$u_k : A^k \to B^k$ such that the following diagram commutes.

$$\begin{array}{ccccccccc}
0 & \xrightarrow{d^{-1}} & A^0 & \xrightarrow{d^0} & A^1 & \xrightarrow{d^1} & A^2 & \xrightarrow{d^2} & \ldots \\
\downarrow & & \downarrow{\scriptstyle u_0} & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_2} & & \\
0 & \xrightarrow{d'^{-1}} & B^0 & \xrightarrow{d'^0} & B^1 & \xrightarrow{d'^1} & B^2 & \xrightarrow{d'^2} & \ldots
\end{array}$$

If a complex is not exact, we would like some measure of to what extent it is not exact. The condition $d^{i+1} \circ d^i = 0$ is the clue to how we might do this. This condition tells us that $\mathrm{Im}(d^i)$ is always a submodule of $\mathrm{Ker}(d^{i+1})$, which leads us to our abstract definition of the 'c' word.

Definition 4.3. *Let $A^\bullet$ be a complex in an abelian category. Define the kth* ***cohomology functor*** *H^k by*

$$H^k(A^\bullet) = Ker(d^k)/Im(d^{k-1}).$$

It is a functor (mapping of categories) because it also acts on morphisms u in the obvious way: $H^k(u)$ is the map u_k induces $H^k(A^\bullet) \to H^k(B^\bullet)$.

There is a technicality of great significance when working with sheaves. There are two possible definitions of the image of a sheaf morphism. It is possible for the 'obvious' image (the image of every individual morphism of groups of sections) to fail to be a sheaf and just be a presheaf, so instead one defines an image that remains inside the category of sheaves by an application of the sheafification functor. To avoid going into the technicalities (which are addressed in [8]), we just remark that this image respects the stalk structure, so *a sequence of sheaves is exact if and only if the induced sequences at every stalk are exact.* Critically, this can be translated as "if and only if it is *locally* exact."

Ok, great, so we have now defined complexes, exactness, and a precise way of measuring nonexactness. A reasonable question, at least to a reader unfamiliar with algebraic topology, is why any of this abstract nonsense is at all interesting. Well, it turns out that complexes like this turn up rather a lot in connection with topological spaces, and failures of exactness often correspond to topological properties like there being holes in your space, or different connected components. Having easy ways to compute these cohomologies in complicated spaces is therefore rather important. To put our new language into practice we will have a look at an actual sheaf of modules that is useful in fluid dynamics, and see some of these ideas in action.

4.2 Integration, differentiation, and the de-Rham complex

In the 'differential equations' course in the first term of one's education at Cambridge, the following apparently rather unremarkable observation is made, in order to facilitate a nice trick for solving certain differential equations.

Lemma 4.1. *Let p, q be smooth functions $\mathbb{R}^2 \to \mathbb{R}$. Then for there to exist a function f such that $df = p(x,y)dx + q(x,y)dy$ it must be the case that*

$$\frac{dq}{dx} = \frac{dp}{dy}.$$

Proof. Suppose such a function exists. Then by the chain rule $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ so, comparing coefficients, we have $p(x,y) = \frac{\partial f}{\partial x}$ and $q(x,y) = \frac{\partial f}{\partial y}$. Partially differentiating the first of these equations with respect to y and the second with respect to x, and using the symmetry of mixed partial derivatives, we get the formula required. $\square$

The quantity df, whatever it is, is called an 'exact differential' and then never mentioned again for the rest of the course. Even as a fresher in my first term, I had the natural curiosity to ponder whether there was a converse to this result. In other words, does the condition $\frac{dq}{dx} = \frac{dp}{dy}$ guarantee that $pdx + qdy$ is an exact differential? A clear answer was never really given, though in a course the following term on vector calculus we were effectively told that there is a converse which holds 'locally'. However, the following example might suggest the problem. Let $p(x,y) = \frac{-y}{x^2+y^2}, q(x,y) = \frac{x}{x^2+y^2}$, both smooth on $\mathbb{R}^2\backslash\{0\}$. Then

$$\frac{\partial p}{\partial y} = \frac{2y^2 - (x^2+y^2)}{(x^2+y^2)^2} = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{\partial q}{\partial x}.$$

However, if we think about what these functions are saying, the f satisfying $df = pdx + qdy$ would be a measure of angle about the origin. In other words, if we set $f(x,y) = \tan^{-1}(y/x)$ or something along these lines, we find that it (or a variant on it) does have this as its differential. But it is impossible to define an angle function smoothly on the whole of $\mathbb{R}^2\backslash\{0\}$, because once you go round the origin once you will have to jump back to where you started. So in this case there is no converse, or rather, the converse fails to happen to some extent, and this extent turns out to be completely measurable and inextricably bound up in the topology of $\mathbb{R}^2\backslash\{0\}$ and its mischievious hole. This also reveals why there *is* a local converse (provided points always have 'nice' neighbourhoods).

The discussion above is a bit startling taken out of context, in that we have really no rigorous definition of what these funny df, dx, dy symbols are. However, treating them as 'things we are allowed to integrate over' actually seems to work reasonably well. In fact, we will go with the following definition, which, though a little bit more rigorous, is still not quite perfect (but is much better for giving the reader a clear picture of what is happening).

Definition 4.4. *Let X be a smooth real n-dimensional manifold, and let $U \subset X$ be a co-ordinate patch with local co-ordinates $(x_1, x_2, ..., x_n)$. Then we let the* ***smooth 0-forms*** *Ω_X^0 be the sheaf of smooth functions on X and, for $p \in \{1, ..., n\}$ we define the* ***smooth p-forms*** *Ω_X^p to be the sheaf of Ω_X^0-modules with local basis $\{dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_p} : i_1, ..., i_p \in \{1, ..., n\}\}$ where for now the dx_i are just formal symbols respecting co-ordinate transformations and the $\wedge$ is an* *formal associative multilinear alternating product (so, for example, $dx_1 \wedge dx_2 = -dx_2 \wedge dx_1$, and $dx_1 \wedge dx_1 = 0$).*

Note that our definition is local in the sense that it is co-ordinate dependent. The sheaf Ω_X^p is constructed, like the egg in the opening paragraph, by first building all the p-forms you explicitly can with co-ordinates on small subsets, and then using the sheaf axioms to glue together any p-forms which agree on regions of overlap. This agreement is verified by means of the transition functions to get a change of co-ordinates on the overlap. This makes is therefore fairly messy and in some sense clearly the wrong definition, but it will suit our purposes quite nicely by retaining the direct link to integration theory without too much abstract nonsense in the way notationally. Note that really a p-form should just be thought as something to be integrated over a p-dimensional surface (with 0-forms, functions, being things you evaluate at points).

In our tale from the theory of differential equations, it was useful to have this thing called the 'chain rule'. However, given we are still in an abstract phase of development of the theory of p-forms, we can take it as a *definition* of an operation $d : \Omega_X^0 \to \Omega_X^1$.

Definition 4.5. *For f a 0-form, define its* ***exterior derivative***

$$d^0(f) = \sum_i \frac{\partial f}{\partial x_i} dx_i.$$

In general, we define the exterior derivatives of p-forms to be the unique sequence $(d^0, d^1, ..., d^{n-1}$, $d^i : \Omega_X^i \to \Omega_X^{i+1})$ of $\mathbb{R}$-linear maps which satisfy:

1. *d^0 is the map defined above - the dual of the gradient of a smooth function.*
2. *the modified Leibniz rule $d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge (d\omega_2)$ (where ω_1 is a p-form).*
3. *$d^{i+1} \circ d^i = 0$ for all i.*

Aha! And the reader who has remained conscious for the previous few pages will immediately recognise that this makes Ω_X into a *complex* $\Omega_X^\bullet$, and as proof that it is of significance, we give its name.

Definition 4.6. *The (sheaf-theoretic)* ***de Rham complex*** *is the complex*

$$0 \longrightarrow \Omega_X^0 \xrightarrow{d^0} \Omega_X^1 \xrightarrow{d^1} ... \xrightarrow{d^{n-1}} \Omega_X^n \longrightarrow 0\cdot$$

Hopefully an astute reader will now start to see why some of the definitions from the previous subsection might turn out to be useful. Let us just do a bit of computation to decipher some of our general definitions of exterior derivative. If one applies the axiomatic definition above, one deduces that in 2-dimensions we have

$$d(pdx + qdy) = (\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y})(dx \wedge dy).$$

In three dimensions one can easily check that d^0, d^1 and d^2 give analogues of, respectively, gradient, curl and divergence operators.

So, let us return to our problem above of finding a converse to lemma 4.1. We have just observed that the condition $\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = 0$ is equivalent to $d(pdx+qdy) = 0$. In other words, to the 1-form $\omega = pdx+qdy$ being in the kernel of the exterior derivative $d^1_X : \Omega^1_X(X) \to \Omega^2_X(X)$. So when do we have a converse to lemma 4.1? Well, simply when the kernel of d^1_X is precisely equal to the image of d^0_X. Recall that the latter lies strictly within the former, and using the language above this is exactly the same as saying **the first cohomology group is trivial**. What's more, even if there is no converse to lemma 4.1, the cohomology groups give us some measure of how far we are from having a converse.

And this kind of behaviour is useful for far more than exact solutions to differential equations in 2 dimensions. Upon it depends facts like the existence of a scalar or vector potential in vector calculus (such things exist provided appropriate cohomology groups are trivial in whatever 3-dimensional submanifold one wants to define the potential), and also it provides an indirect way of measuring the topology of the space. It turns out that if you take 2-dimensional space and remove k points, the first cohomology group (of the whole space) becomes a k-dimensional vector space (a basis for which is the set of forms corresponding to 'angle functions' about each point, directly generalising the example above). The cohomology group gets bigger in some sense, in direct proportion to the topology getting more complicated.

There is a subtlety floating around in that all these cohomologies are of the section over X, while the complex defined above is one of sheaves, so I will clarify things by making another definition.

Definition 4.7. *Let X be a real smooth n-dimensional manifold, with de-Rham complex $\Omega^\bullet_X$. We define the **global de-Rham complex** to be the complex $\Omega^\bullet_X(X)$ of global sections with differentials d^i_X (the portion of the sheaf morphism d^i mapping sections over X). Then we define the **de-Rham cohomology groups***

$$H^i_{DR}(X) = \frac{Ker(d^i_X : \Omega^i_X(X) \to \Omega^{i+1}_X(X))}{Im(d^{i-1}_X : \Omega^{i-1}_X(X) \to \Omega^i_X(X))}.$$

One final topological property locked in the de Rham complex will be of interest in the final part of our essay and is another example of where the language of sheaves is helpful. It is clear, from elementary calculus and the definition of the derivative at dimension 0, that $df = 0$ if and only if f is a locally constant function. In other words, the kernel of the map d^0 is precisely the sheaf of locally constant functions. Combining with the classical Poincaré lemma (that all closed p-forms are exact in a contractible space, and hence locally exact in a manifold) the following sequence is an exact sequence of sheaves (but not necessarily of presheaves!), where d^{-1} is simply an embedding into the kernel of d^0:

$$0 \longrightarrow \mathbb{R}_X \xrightarrow{d^{-1}} \Omega^0_X \xrightarrow{d^0} \Omega^1_X \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} \Omega^n_X \longrightarrow 0\cdot$$

A trivial consequence of this is that the zeroth de-Rham cohomology group measures (in its dimension as a real vector space) the number of connected components of the topological space X. A much deeper consequence is that in almost all smooth manifolds people work with, it binds the de-Rham cohomology inextricably to the sheaf $\mathbb{R}_X$ in such a profound way that we will eventually be able to compute the de-Rham cohomologies just by performing manipulations on the sheaf $\mathbb{R}_X$. This powerful idea, the subtle cohomology intrinsic to sheaves themselves, is what we now seek to achieve.

4.3 Cohomology of functors and the global sections functor

Our first stage towards developing a cohomology of sheaves will be to set up a general categorical construction called the 'derived functor'. Though this will mean doing things in much more generality than we care about, it will make it clearer exactly what we are doing without the clutter of sheaf notation everywhere. We will then apply this construction to the global sections functor $\Gamma : \mathcal{F} \mapsto \mathcal{F}(X)$ which takes a sheaf of rings (or modules) to its ring (or module) of global sections to obtain the sheaf cohomology. Most proofs in this section will be omitted, but [1],[4] and [5] all give expositions and in [5], which this author regards as the most down-to-earth exposition, all the results are proved.

Firstly, a few key definitions of category theory and homological algebra.

Definition 4.8. *A (covariant)* ***functor*** $F : \mathcal{A} \to \mathcal{B}$ *is a mapping of categories that takes objects*[2] *in $\mathcal{A}$ to objects in $\mathcal{B}$ and morphisms between objects in $\mathcal{A}$ to morphisms between the corresponding objects in $\mathcal{B}$. It must respect identity morphisms and composition of morphisms.*

If all the categories in sight are abelian, we say that F is ***exact*** *if whenever*

$$0 \to A \to B \to C \to 0$$

is a short exact sequence, then the following sequence (with morphisms also transformed under F) is also exact:

$$0 \to F(A) \to F(B) \to F(C) \to 0.$$

We say that F is ***left-exact*** *if, everything as above, we only have that*

$$0 \to F(A) \to F(B) \to F(C)$$

is exact. In other words, if it preserves (in some sense) injections but not necessarily surjections.

[2] If talk of objects and morphisms scares you it should not. An object is just something like a ring, and in all categories we will work with a morphism is just an appropriate map between two objects, in our case probably a ring homomorphism. Hence an example of a functor might be the so-called 'forgetful' functor from the category of groups from the category of sets, which takes a group to the set containing its elements, and the morphisms to the corresponding maps of sets - it just 'forgets' the group structure. Another functor in the other direction is the 'free group' functor which makes a free group out of any set, and a mapping of sets is mapped to a group homomorphism determined by the corresponding substitution of the generators.

Once the reader has finished scratching their head (which may take some time), they might guess where we are going next. Just as with the 1-forms in the differential equations example, here we will not be content to just say that a functor is left-exact (i.e. an exact functor which has *failed* to preserve surjections). We will be interested in some kind of measure of *to what extent* the functor is not exact. The obvious thing to try is just plugging in some exact sequences and measuring how non-surjective the maps $F(B) \to F(C)$ get. It turns out that such a process is just too unwieldy, and ends up with something which is a function of at least two objects in the category. We try something slightly more subtle, and though at first it looks hopeless, we are saved by the presence of a miraculous subcategory called the 'injectives.'

Regardless of our procedure, we will always have to pick some object to kick things off, let's call it A. We then construct a (possibly infinitely long - never hitting zero) exact complex

$$0 \to A \to A^0 \to A^1 \to ...$$

. If we apply F to this as it stands, we will get exactness at the first two stages, and then some other stuff will happen which is basically independent of A. Since we therefore don't care about these first two stages, it is usual to instead cut out the A and consider the composition of the two relevent morphisms as a single map, so we will measure the inexactness of the functor just by taking cohomology groups of the complex (preserving $F(A)$ as the 0th cohomology group, by left-exactness).

$$0 \to F(A^0) \to F(A^1) \to F(A^2) \to ...$$

Great! Great? Well, no, not really. This still gives us all sorts of answers, and by sticking extra objects in or moving things about we can get all sorts of different cohomologies in this way, even for A fixed. The procedure is tantalisingly close to being correct though. The main reason it fails is that once you know a functor doesn't tend to preserve a certain surjection, you can go sticking copies of that surjection all over your $A^\bullet$ complex, and immediately alter the higher cohomology. This seems hopeless... the only thing that could save us would be some kind of incredibly well-behaved subcategory of objects restricted to which F is in fact an exact functor, but one's gut (certainly my poorly trained gut) despairs that this might have to be different for different functors (or even starting objects!) and so making this idea work will be a nightmare. However, unbelievably, such a subcategory can at least be defined.

Definition 4.9. *We say that an object I is an* ***injective*** *if for any injection $\phi : A \to B$ if there is a map $\theta : A \to I$ it can be extended to $\theta' : B \to I$ in a way that respects the injection ($\theta' \circ \phi = \theta$).*

Of course, just because we can define it doesn't mean it exists. There might not actually be any injectives out there, or not enough to be useful. Fortunately, the following proposition (proved in [8]) tells us that in the categories we will be studying there are.

Proposition 4.2. *For any sheaf of rings $\mathcal{O}$, the category of $\mathcal{O}$-modules has enough injectives: the property that for any object A, there exists an injective I such that A can be embedded in I (i.e. $0 \to A \to I$ is exact).*

What makes injectives work so well is that their defining property allows you to compare two different complexes $0 \to A \to A^\bullet$ all of whose elements (after A) are injectives by starting with the identity morphism both ways between A and itself, and then extending to a direct comparison both ways of the first pair of injectives using the definition of an injective, and so on. These comparisons turn out to prove that whichever complex of injectives we use to make $0 \to A \to A^\bullet$ exact, the complex $0 \to F(A^\bullet)$ will always have the same cohomology groups!

Therefore we can construct a new set of functors called the *right-derived functors of* F which measure the extent to which F fails to be exact as a function of A, and we now know the procedure. Take any complex of injectives $A^\bullet$ such that $0 \to A \to A^0 \to A^1 \to ...$ is exact (this is possible because there are enough injectives). By the above discussion, the following is well-defined.

Definition 4.10. *Define the* ***right derived functors*** *by $R^nF(A) := H^n(F(A^\bullet))$.*

Now we have the right derived functors, we have a reasonable measure of how removing a factor of A from an otherwise perfectly well-behaved sequence (on which F is exact) causes it the exactness of the sequence to break up under F at the various stages down the line. The special injectives were vital in building such a 'well behaved sequence'. However, surely any other object whose surjections F preserves would have done just as good a job - such an object would itself have caused no damage to the exactness. This turns out to be the case, which makes the derived functors much easier to calculate. This is the F-dependent category of my previous nightmares, but with the help of our wonderful injectives to get us going, we now have a beautiful definition for it.

Definition 4.11. *An object B is F-**acyclic** if $R^nF(B)$ is trivial for all $n > 0$.*

As we predict above, the following theorem turns out to be true, and the proof is basically the same as the proof that a choice of complex of injectives does not affect the cohomology there.

Theorem 4.3. *Let A be any object, and $B^\bullet$ be a complex of F-acyclic objects such that*

$$0 \to A \to B^0 \to B^1 \to B^2 \to ...$$

is exact. Then

$$R^nF(A) = H^n(F(B^\bullet)).$$

Ok, so we've now done even more abstract nonsense which seemed to have little to do with sheaves. However, now we are going to apply it to the global sections functor. This is the functor that basically ignores all of the sheaf theory except the 'layer at the top' for both objects and morphisms. It is easy to check that this functor is left-exact, but that it may often fail to be actually exact. Recall our example of the sheaf of holomorphic functions on a compact Riemann surface having very few global sections relative to its diverse local structure. Phenomena like that are precisely what the derived functors will be able to give us some indications of - sheaves which, though locally exact, deviate from global exactness as the global sections which local sections expect to exist simply fail to because of nontrivial global structure.

With all the machinery above developed, we can go right ahead and define the sheaf cohomology.

Definition 4.12. *Let $\Gamma(X,-) : \mathcal{F} \mapsto \mathcal{F}(X), u \mapsto u_X$ be the global sections functor over X (from sheaves of modules to modules). Define the* ***sheaf cohomology*** *by*

$$H^n(X,\mathcal{F}) := R^n\Gamma(X,\mathcal{F}).$$

4.4 Carnival of the cohomologies

In this closing section, we take our shiny new cohomology of sheaves for a test-drive to see what it can do, uniting several different cohomology theories. Unless we say otherwise, all sheaves are sheaves of rings or modules, and hence there are always enough injectives so the previous discussion holds.

From the previous section, we can extract the following very important theorem.

Theorem 4.4. *Sheaf cohomology can be computed as follows. For $\mathcal{F}$ a sheaf, take any exact sequence (called a* ***resolution*** *of $\mathcal{F}$)*

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to ...$$

where each $\mathcal{F}^i$ is $\Gamma(X,-)$-acyclic. Then

$$H^n(X,\mathcal{F}) = H^n(\Gamma(X,\mathcal{F}^\bullet)).$$

This is handy because it will turn out that lots of types of sheaf, including those which arise when computing apparently disparate cohomologies, will be acyclic provided the topologies are not too wild. Some particularly important examples are:

- **Flabby** sheaves: where every section $s \in \mathcal{F}(U)$ can be extended to a global section.
- **(c)-Soft** sheaves: where every section over a (compact) *closed* set, defined using a limit of sections over open sets, can be extended to a global section.
- **Fine** sheaves: where every locally finite cover admits a partition of unity. I.e. there is a collection ϕ_α of sheaf endomorphisms, one for each open set U_α in the cover, such that $\phi_\alpha s_x = 0$ for all $x \notin U_\alpha$ but $\sum_\alpha \phi_\alpha = 1$.

We will now proceed to note some of the major isomorphisms in sheaf cohomology, without going into details of how to prove certain types of sheaf are soft, flabby, etc.

4.4.1 de Rham cohomology

We shall start with the example of cohomology we met earlier this chapter. Earlier I left you on a cliffhanger concerning the significance of the complex

$$0 \longrightarrow \mathbb{R}_X \xrightarrow{d^{-1}} \Omega^0_X \xrightarrow{d^0} \Omega^1_X \xrightarrow{d^1} \ldots \xrightarrow{d^{n-1}} \Omega^n_X \longrightarrow 0\cdot$$

Now I hope you can see where we are going. This sequence is, of course, a resolution of $\mathbb{R}_X$, and it turns out that the Ω^i_X are soft sheaves (see [4]), and smooth manifolds are sufficiently nice that any soft sheaf is Γ-acyclic, so we can plug this sequence into the above theorem and get the instant and astonishingly nontrivial isomorphism

$$H^n(X, \mathbb{R}_X) \cong H^n_{DR}(X).$$

What this tells us is that somehow taking our topological space X and quantifying how integrating over things in different ways can give different answers gives exactly the same data as taking our topological space X, and a constant sheaf $\mathbb{R}_X$ over it, and then measuring the extent to which global sections 'go missing' in exact sequences involving this sheaf.

4.4.2 Singular cohomology

Next we'll look at another type of cohomology on smooth manifolds people often come across - the (vector-space) dual of the standard homology of chains. We will find that it can be treated exactly as the de Rham complex above. This analysis is the author's own work, and he has found no such direct exposition in the literature (but almost certainly one exists somewhere).

Recall that the group $C_k(X)$ of k-chains on a manifold X is the free abelian group generated by the set of all k-paths, smooth maps $\gamma : [0,1]^k \to X$, where, to clarify, the 0-chains are $\mathbb{Z}$-linear combinations of points. There are **boundary** maps $\partial_k : C_k(X) \to C_{k-1}(X)$ which take a k-chain to its 'boundary', a $(k-1)$-chain. The boundary of a k-path is simply an alternating sum over the $(k-1)$-paths obtained by fixing one of the co-ordinates to be 0 or 1 and extending linearly gives the boundary of an arbitrary k-chain. This gives rise to a kind of complex, but going in the opposite direction from normal (hence 'homology' as opposed to '*co*homology').

However, we can construct a complex going the other way out of this by going to the vector space dual (the space of $\mathbb{Z}$-linear maps from $C_k(X)$ to $\mathbb{R}$). In other words, we set

$$C^k(X) := \mathrm{Hom}(C_k(X), \mathbb{R}). \quad d^k(f)(\gamma) := f(\partial_k \gamma).$$

One can check (by verifying that taking a boundary twice always gives 0) that this gives rise to a complex

$$0 \longrightarrow C^0(X) \xrightarrow{d^0} C^1(X) \xrightarrow{d^1} C^2(X) \xrightarrow{d^2} \ldots$$

And this in turn gives rise to the **singular cohomology groups**

$$H^k(X) = \mathrm{Ker}(d^k)/\mathrm{Im}(d^{k-1}).$$

We would like to compare this cohomology theory to a sheaf cohomology. To do this we will have to construct an exact complex of acyclic sheaves whose global sections give the complex above and whose first cohomology group is a nice sheaf (whose cohomology will be isomorphic to this one). That sounds like quite a lot to ask, but fortunately the obvious choice of sheaf does the job remarkably well.

Define $\mathcal{C}^k$ to be the *sheaf* of k-cocycles over X. This will have sections $\mathcal{C}^k(U) = \mathrm{Hom}(C_k(U), \mathbb{R})$ with obvious restriction maps, and happily (trivially) $\mathcal{C}^k(X) = C^k(X)$. Furthermore, just applying the above procedure to each open set of X, we construct a complex of sheaves

$$0 \longrightarrow \mathcal{C}^0 \xrightarrow{d^0} \mathcal{C}^1 \xrightarrow{d^1} \mathcal{C}^2 \xrightarrow{d^2} \ldots$$

A manifold is locally contractible (so each point has a neighbourhood with trivial homology), so this sequence is certainly an exact sequence of sheaves except at the first stage. Also, since these sheaves are clearly flabby (to get a global section f' extending $f \in \mathcal{C}^k(U)$ just set $f'(\gamma) = 0$ for any γ a k-path not contained in U, and $f'(\gamma) = f(\gamma)$ for all other γ, and then extend linearly) it will suffice to evaluate $\mathrm{Ker}(d^0)$. Well, if $d^0(f)$ being the zero map is equivalent to $f(\gamma(1)) - f(\gamma(0)) = f(\gamma(1) - \gamma(0)) = 0$ for any path γ. I.e. for any U, the sections of U in the kernel are those which are constant on the path connected components of U. That is to say (since all these sets are open, so path connectedness and connectedness agree)

$$\mathrm{Ker}(d^0) = \mathbb{R}_X.$$

In other words, the following sequence of sheaves is exact:

$$0 \longrightarrow \mathbb{R}_X \longrightarrow \mathcal{C}^0 \xrightarrow{d^0} \mathcal{C}^1 \xrightarrow{d^1} \mathcal{C}^2 \xrightarrow{d^2} \ldots$$

So just as with de-Rham cohomology, we apply our initial theorem to this sequence and deduce:

$$H^n(X, \mathbb{R}_X) \cong H^k(X).$$

Ah, so yet again, another apparently unrelated cohomology, this one relating to what happens to paths you draw on your manifold, turns out to actually just be the sheaf cohomology of $\mathbb{R}_X$. We can put this together with our result above to deduce one of the most important results in the topology of smooth manifolds.

Theorem 4.5 (De Rham's Theorem)**.** *For any smooth real finite-dimensional manifold X,*

$$H^n_{DR}(X) \cong H^n(X).$$

In other words, the map $[\omega] \mapsto [\gamma \mapsto \int_\gamma \omega]$ is a bijection of cohomology classes at every dimension.

4.4.3 Čech cohomology

Readers who have done some algebraic topology will be aware of the device of using simplicial complexes to calculate the homology of a space (or triangulating a closed 2-surface to compute its Euler characteristic). This is useful, because it is often easy to directly compute the homologies involved by just looking at your simplicial complex. One of the big limitations of the sheaf cohomology we have defined is that in abstract situations we really have no systematic way of computing it. Fortunately, there is an analogue of simplicial division which works most of the time, called the *Čech cohomology*, and is a valuable method for actually computing the sheaf cohomology in more complicated locally ringed spaces (like the schemes we met last chapter). Our main reference is [3].

Here is the general construction. We will work with $\mathcal{F}$ any sheaf of abelian groups over a topological space X. The Čech cohomology is taken with reference to an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X.

Definition 4.13. *A q-**simplex** σ is an ordered collection $(U_0, U_1, ..., U_q)$ of $q+1$ sets of $\mathcal{U}$ whose total intersection is nonempty. We call this intersection the **support** $|\sigma| := \bigcap_{i=1}^{q+1} U_i$ of σ. It will also be useful to have the notation $\sigma_k = (U_0, U_1, ..., U_{k-1}, U_{k+1}, ..., U_q)$ for the $(q-1)$-simplex obtained by removing U_k. Note that for each k, $|\sigma| \subseteq |\sigma_k|$.*

*Define a q-**cochain** ϕ to be a map which associates with every q-simplex σ some element of $\mathcal{F}(|\sigma|)$, and let $C^q(\mathcal{U}, \mathcal{F})$ be the abelian group of all q-cochains. We can define differentials $d^{q-1} : C^{q-1}(\mathcal{U}, \mathcal{F}) \to C^q(\mathcal{U}, \mathcal{F})$ by*

$$d^{q-1}(\phi)(\sigma) = \sum_{j=0}^{q} (-1)^j (\phi(\sigma_j))|_{|\sigma|}.$$

*One can check that these differentials, together with our groups of cochains, define a complex. It is called the **Čech complex**, and the corresponding cohomology is called the **Čech cohomology** with groups $\check{H}^q(\mathcal{U}, \mathcal{F})$.*

Ok, so quite an involved looking definition, but what is really happening? Well, a q-simplex has $q+1$ 'faces' σ_k, which are obtained by deleting one of the open sets and leaving the rest. These faces have supports which contain the support of the entire simplex, and therefore any element of $\mathcal{F}(|\sigma_k|)$ can be restricted (via the sheaf restriction maps) to an element of $\mathcal{F}(|\sigma|)$. Finally, the sum has to alternate in order for the differential to satisfy $d^2 = 0$, and in direct analogy with the standard boundary map from concrete simplicial complexes from elementary algebraic topology.

So we now have a totally different cohomology - it would be nice if we could compare it with our old friend the sheaf cohomology. The first step towards doing this is obviously to look at the kernel of the first boundary map (since to compare with sheaf theory we want to view our new complex as an acyclic resolution, as we did with the de Rham complex). A quick check reveals that the kernel is just $\mathcal{F}(X)$, which is rather nice.

The next step is rather technical, but hopefully the reader should now be happy enough with sheaves to be content with my just sketching it.[3] The abelian groups $C^q(\mathcal{U},\mathcal{F})$ need to themselves be made into a *sheaf* of abelian groups $\mathcal{C}^q(\mathcal{U},\mathcal{F})$ in the obvious way. Essentially, rather than maps ϕ having codomain $\mathcal{F}(|\sigma|)$ their codomain becomes the subsheaf $\mathcal{F}|_{|\sigma|}$. The differentials are essentially the same as those in the non-sheaf Čech complex, but acting at every level simultaneously.

Our check on the kernel of the first boundary map told us that the Čech complex, if exact, is a resolution of $\mathcal{F}$.

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{U},\mathcal{F}) \to \mathcal{C}^1(\mathcal{U},\mathcal{F}) \to \mathcal{C}^2(\mathcal{U},\mathcal{F}) \to ...$$

Aha, so now we are in business. In most (**but not all!**) situations it is the case that this resolution is actually exact and Γ-acyclic, and so we immediately obtain the isomorphism

$$\check{H}^q(\mathcal{U},\mathcal{F}) \cong H^q(X,\mathcal{F}).$$

Note that as a nice corollary, this tells us that (in such cases) the Čech cohomology is independent of the open cover $\mathcal{U}$ used. More details about some of the circumstances in which the above isomorphism holds are given in [3], which is available online. It certainly holds whenever $\mathcal{F}$ is flabby, for example, but it holds far more generally and is of great importance in many of the more abstract fields where the sheaf cohomology plays a key role in its own right and needs to be computed in certain examples.

Conclusion

In this essay we have gently introduced the reader to the idea of a sheaf in mathematics, and hopefully persuaded them that the language and formalism of sheaves, once fully understood, is very powerful and of interest in all sorts of areas, particularly geometry and topology. In particular, we saw how sheaves were important in defining abstract geometrical spaces, giving us a new angle on manifolds, and proving absolutely vital for defining schemes. We then took a whistle-stop tour through the basics of sheaf cohomology, a much deeper application of sheaf theory in providing a general mechanism (the absence of global sections in exact sequences of sheaves) which surprisingly gives rise to many of the major cohomological theories and sometimes shows naturally where theories are superficially dissimilar how they are related (in particular giving a 'high brow' proof of the de Rham theorem). The language of sheaves also allowed us to define the concept of Čech cohomology, providing an easy means of calculating sheaf cohomology in more abstract situations where the notion of cohomology is still useful but no concrete definition available.

[3] For a proper explanation see [3].

Bibliography

Below I detail books and articles of which I have used hard copies. However, I have also fleetingly made use of very many internet resources, but very few in sufficient detail or substance to have kept track of. I also have to thank Chris Elliott, Saul Glasman, James Cranch, Heather Macbeth and probably several others for the occasional conversation which has inspired my enthusiasm for the subject of sheaves and developed my understanding in a way that has been valuable in writing this essay.

[1] A. Dimca: Sheaves in Topology.

[2] R. Hartshorne: Algebraic Geometry.

[3] G. Chenevert: Sheaf Cohomology.

[4] G.E. Bredon: Sheaf Theory.

[5] S. Lang: Algebra (Revised 3rd Edition).

[6] F. Borceaux: Handbook of Categorical Algebra.

[7] http://wikipedia.org .

[8] B.R. Tennison: Sheaf Theory.