

## 1.2 The Gauss map and the Weingarten map.

In order to state the general formula, we make the following definitions: Let  $Y$  be an (immersed) oriented hypersurface. At each  $x \in Y$  there is a unique (positive) unit normal vector, and hence a well defined **Gauss map**

$$\nu : Y \rightarrow S^{n-1}$$

assigning to each point  $x \in Y$  its unit normal vector,  $\nu(x)$ . Here  $S^{n-1}$  denotes the unit sphere, the set of all unit vectors in  $\mathbf{R}^n$ .

The normal vector,  $\nu(x)$  is orthogonal to the tangent space to  $Y$  at  $x$ . We will denote this tangent space by  $TY_x$ . For our present purposes, we can regard  $TY_x$  as a subspace of  $\mathbf{R}^n$ : If  $t \mapsto \gamma(t)$  is a differentiable curve lying on the hypersurface  $Y$ , (this means that  $\gamma(t) \in Y$  for all  $t$ ) and if  $\gamma(0) = x$ , then  $\gamma'(0)$  belongs to the tangent space  $TY_x$ . Conversely, given any vector  $v \in TY_x$ , we can always find a differentiable curve  $\gamma$  with  $\gamma(0) = x$ ,  $\gamma'(0) = v$ . So a good way to think of a tangent vector to  $Y$  at  $x$  is as an “infinitesimal curve” on  $Y$  passing through  $x$ .

### Examples:

1. Suppose that  $Y$  is a portion of an  $(n-1)$  dimensional linear or affine subspace sitting in  $\mathbf{R}^n$ . For example suppose that  $Y = \mathbf{R}^{n-1}$  consisting of those points in  $\mathbf{R}^n$  whose last coordinate vanishes. Then the tangent space to  $Y$  at every point is just this same subspace, and hence the normal vector is a constant. The Gauss map is thus a constant, mapping all of  $Y$  onto a single point in  $S^{n-1}$ .
2. Suppose that  $Y$  is the sphere of radius  $R$  (say centered at the origin). The Gauss map carries every point of  $Y$  into the corresponding (parallel) point of  $S^{n-1}$ . In other words, it is multiplication by  $1/R$ :

$$\nu(y) = \frac{1}{R}y.$$

3. Suppose that  $Y$  is a right circular cylinder in  $\mathbf{R}^3$  whose base is the circle of radius  $r$  in the  $x^1, x^2$  plane. Then the Gauss map sends  $Y$  onto the equator of the unit sphere,  $S^2$ , sending a point  $x$  into  $(1/r)\pi(x)$  where  $\pi : \mathbf{R}^3 \rightarrow \mathbf{R}^2$  is projection onto the  $x^1, x^2$  plane.

Another good way to think of the tangent space is in terms of a **local parameterization** which means that we are given a map  $X : M \mapsto \mathbf{R}^n$  where  $M$  is some open subset of  $\mathbf{R}^{n-1}$  and such that  $X(M)$  is some neighborhood of  $x$  in  $Y$ . Let  $y^1, \dots, y^{n-1}$  be the standard coordinates on  $\mathbf{R}^{n-1}$ . Part of the requirement that goes into the definition of parameterization is that the map  $X$  be **regular**, in the sense that its Jacobian matrix

$$dX := \left( \frac{\partial X}{\partial y^1}, \dots, \frac{\partial X}{\partial y^{n-1}} \right)$$