THEORY OF PROBABILITY AND ITS APPLICATIONS

Volume XXV Number 4

1980

ON THE CONVERGENCE RATE IN THE CENTRAL LIMIT THEOREM FOR WEAKLY DEPENDENT RANDOM VARIABLES

A. N. TIKHOMIROV

(*Translated by B. Seckler*)

1. Introduction

Let $X_1, X_2, \cdots$ be a narrow-sense stationary sequence of random variables, which are assumed to have zero means and finite variances. Put $\sigma_n^2 = \mathbf{E}(\sum_{j=1}^n X_j)^2$ and form the sum

$$S_n = \frac{1}{\sigma_n} \sum_{j=1}^n X_j.$$

Many publications have been devoted to investigating conditions under which this sum is asymptotically normal, i.e., when

$$F_n(Z) = \mathbf{P}\{S_n < Z\} \to \Phi(Z)$$

as $n \to \infty$, where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2}\, dy$$

(for instance, see [3], Chapter 18). Set

$$\Delta_n = \sup_z |F_n(z) - \Phi(z)|.$$

This paper investigates the rate with which Δ_n tends to zero when n goes to infinity.

This problem has been studied in detail for independent summands (for example, see [5], Chapter 5). Some very interesting results were obtained for dependent variables X_j by V. A. Statulyavichus in [6] and C. Stein in [10].

To a considerable extent, this paper was written under the influence of the article of Stein just mentioned. However, in contrast to Stein, we make use of the apparatus of characteristic functions, whereas Stein in general questioned the advantage of using this apparatus when investigating the rate of decay for weakly dependent variables.

In what follows, we shall assume that the sequence $\{X_j\}_{j=1}^{\infty}$ satisfies one of the following weak dependency conditions:

(a) strong mixing (s.m.): as $n\to\infty$,

$$\alpha(n)=\sup|\mathbf{P}(AB)-\mathbf{P}(A)\mathbf{P}(B)|\to 0,$$

where the supremum is over all $A\in\mathfrak{M}_{-\infty}^{k}$ and $B\in\mathfrak{M}_{k+n}^{\infty}$ ($\mathfrak{M}_a^b$ denotes the σ-algebra generated by the random variables X_j when $j\in[a,b]$);

(b) complete regularity: as $n\to\infty$,

$$\rho(n)=\sup\frac{|\mathbf{E}\xi\eta-\mathbf{E}\xi\mathbf{E}\eta|}{\sqrt{\mathbf{E}(\xi-\mathbf{E}\xi)^2\mathbf{E}(\eta-\mathbf{E}\eta)^2}}\to 0,$$

where the supremum is over all random variables ξ and η which are measurable with respect to the σ-algebras $\mathfrak{M}_{-\infty}^{k}$ and $\mathfrak{M}_{k+n}^{\infty}$, respectively, and have finite second moments;

(c) m-dependence: any two vectors of the form $(X_{a-p}, X_{a-p+1},\cdots, X_{a-1}, X_a)$ and $(X_b, X_{b+1},\cdots, X_{b+q})$ are independent for $b-a>m$.

We are interested in the rate of decay of Δ_n as a function of the restrictions imposed on the coefficients $\alpha(n)$, $\rho(n)$ and moments of the random variables X_j.

When $\alpha(n)$ decreases like a power, we obtain the following result.

Theorem 1. *Let there exist for the sequence $X_1, X_2,\cdots$ constants $K>0$ and $\beta>1$ such that the inequalities*

$$\alpha(n)\leqq Kn^{-\beta(2+\delta)(1+\delta)/\delta^2}$$

and

$$\mathbf{E}|X_1|^{2+\delta}<\infty$$

hold for all $n\geqq 1$ and some δ, $0<\delta\leqq 1$. Then

$$\sigma^2=\mathbf{E}X_1^2+2\sum_{k=2}^{\infty}\mathbf{E}X_1X_k<\infty,$$

and if $\sigma^2>0$, then there is a contant A_1 depending just on K, β and δ such that

$$\Delta_n\leqq A_1 n^{-(\delta/2)(\beta-1)/(\beta+1)}.$$

When $\alpha(n)$ decreases exponentially, the bound for Δ_n is optimal up to a logarithmic factor.

Theorem 2. *Suppose that positive constants K and β exist such that the inequality*

$$\alpha(n)\leqq K e^{-\beta n}$$

holds for all n and

$$\mathbf{E}|X_1|^{2+\delta}<\infty$$

for some δ, $0<\delta\leqq 1$. Then there is an A_2 depending just on K, β and δ such that

$$\Delta_n\leqq A_2 n^{-\delta/2}\log^{1+\delta} n.$$

If instead of requiring that the s.m. coefficient $\alpha(n)$ decrease exponentially, one requires that the maximal correlation coefficient $\rho(n)$ decrease exponentially, then the power on the logarithm in the bound for Δ_n may be lowered.

Theorem 3. *If constants K and β exist such that the inequality*

$$\rho(n) \leqq K e^{-\beta n}$$

holds for all $n \geqq 1$ and

$$\mathbf{E}|X_1|^{2+\delta} < \infty$$

for some δ, $0 < \delta \leqq 1$, then there is an A_3 depending on K, β and δ such that

$$\Delta_n \leqq A_3 n^{-\delta/2} \log^{1+\delta/2} n.$$

The power of the logarithm in Theorem 2 may also be lowered for a s.m. condition but under more stringent restrictions on the moments of the summands.

Theorem 4. *Suppose that the conditions of Theorem 2 hold and that*

$$\mathbf{E}|X_1|^{4+\gamma} < \infty$$

for some positive γ. Then there are A_4 and A_5 depending just on K, β and γ such that, first,

$$\Delta_n \leqq A_4 n^{-1/2} \log n,$$

and second,

$$|F_n(z) - \Phi(z)| \leqq \frac{A_5 \log^3 n}{\sqrt{n}(1+|z|)^4}.$$

And finally, for m-dependent variables, we can succeed in getting rid of the logarithm in the estimate for Δ_n and obtain the next result.

Theorem 5. *Let $X_1, X_2, \cdots$ be a narrow-sense stationary sequence of m-dependent random variables such that*

$$\mathbf{E}|X_1|^3 < \infty.$$

Then absolute constants C_1 and C_2 exist such that

$$\Delta_n \leqq C_1 \frac{b_m^2 E^{1/3}|X_1|^3}{\sigma^3 \sqrt{n}} + C_2 \frac{m b_m E^{1/3}|X_1|^3 \log n}{\sigma^2 n}$$

where $b_m = \max_{1 \leqq p \leqq m+1} \mathbf{E}^{1/3}|\sum_{\nu=1}^{p} X_\nu|^3$.

The method for proving Theorems 1–5 is presented in Section 2 of the paper. The idea behind the method was described in the author's article [7], in which weakened versions of Theorem 2 and 4 were proved.

Section 3 proves Theorem 5 in detail. The proofs of Theorems 1–4 essentially differ little from the proof of Theorem 5 and their detailed presentation would be too tedious. Therefore, Section 4 will be confined to giving the plan of proof of Theorems 1–4 with an indication of the main differences from the proof of Theorem 5.

Section 5 describes examples of stationary sequences satisfying a s.m. condition for which lower bounds are obtained for Δ_n. The resultant bounds show what rate of decay can be attained for Δ_n under various relationships between the rate of decay of the s.m. coefficient $\alpha(n)$ and the number of moments of the summands X_j. In particular, it turns out that an estimate of order $n^{-1/2}$ cannot be obtained for Δ_n if the s.m. coefficient $\alpha(n)$ decreases only as a power and X_j has at most three moments. The results of Section 5 were stated earlier in the author's paper [8].

The author wishes to express his deepest thanks to I. A. Ibragimov to whom he is indebted for the publication of this article.

2. Method of Proof

First of all, let us establish some notation. The symbol B with or without a subscript will denote a bounded quantity and the symbol $\theta(t)$ (also with or without a subscript) will denote a function such that $|\theta(t)| \leqq 1$. Further, for all $j = 1, 2, \cdots, n$ and $\nu = 1, 2, \cdots, k$, put

$$S_j^{(\nu)} = \frac{1}{\sigma_n} \sum_{|j-l|>\nu m} X_l, \qquad S_j^{(0)} = S_n.$$

In the definition $S_j^{(\nu)}$, the number m is an integer (in general depending on n): its choice is determined by the nature of the dependency of the sequence $X_1, X_2, \cdots$ and the rate of decay of the dependency.

Let $f_n(t)$ be the characteristic function of the sum S_n. Consider the relation

$$f_n'(t) = \frac{i}{\sigma_n} \sum_{j=1}^n \mathbf{E} X_j e^{itS_j^{(0)}}.$$

Adding and subtracting the quantities $\mathbf{E} X_j e^{itS_j^{(1)}}$ from the right-hand side of this relation, we arrive at

$$f_n'(t) = \frac{i}{\sigma_n} \left[\sum_{j=1}^n \mathbf{E} X_j (e^{it(S_j^{(0)} - S_j^{(1)})} - 1) e^{itS_j^{(1)}} + \sum_{j=1}^n \mathbf{E} X_j e^{itS_j^{(1)}} \right],$$

and this process may be continued to obtain the relation

$$\begin{aligned} f_n'(t) = \frac{i}{\sigma_n} \Bigg\{ & \sum_{j=1}^n \mathbf{E} X_j e^{itS_j^{(1)}} + \sum_{j=1}^n \mathbf{E} X_j (e^{it(S_j^{(0)} - S_j^{(1)})} - 1) e^{itS_j^{(2)}} \\ & + \sum_{j=1}^n \sum_{r=3}^k \mathbf{E} X_j \prod_{l=1}^{r-1} (e^{it(S_j^{(l-1)} - S_j^{(l)})} - 1) e^{itS_j^{(r)}} \\ & + \sum_{j=1}^n \mathbf{E} X_j \prod_{l=1}^k (e^{it(S_j^{(l-1)} - S_j^{(l)})} - 1) e^{itS_j^{(k)}} \Bigg\}. \end{aligned} \tag{2.1}$$

Observe that, for all $r = 1, 2, \cdots, k$, the random variables of the form $X_j \prod_{l=1}^{r-1} (e^{it(S_j^{(l-1)} - S_j^{(l)})} - 1)$ and $e^{itS_j^{(r)}}$ are weakly dependent since the first variable is measurable with respect to the σ-algebra generated by the random variables X_p for $|p-j| \leqq (r-1)m$ and the second one is measurable with respect to the σ-algebra generated by the random variables X_p for $|p-j| > rm$ (the "distance" between these σ-algebras is greater than m). Therefore, by choosing m appropriately, the expectation of the product of these variables may be approximated by the product of the corresponding expectations. Now, if K is chosen "not very large", then $\mathbf{E}\, e^{itS_j^{(r)}}$ will be close to $f_n(t)$ for $r=1, 2, \cdots, k$. On the other hand, by increasing k one can make the last term in relation (2.1) arbitrarily small in the region $|t| \leqq \gamma_n \sqrt{n}$. The quantity γ_n may go to zero at a rate determined by the rate of decay of the dependency of the sequence $\{X_j\}_{j=1}^{\infty}$ (γ_n will behave for example like $O(1/m)$). In addition, the relation

$$\mathbf{E}X_j \prod_{l=1}^{r-1} (\exp\{it(S_j^{(l-1)} - S_j^{(l)})\} - 1) = \theta_{jr}(t) B_1 \left(\frac{|t| B_2}{\gamma_n \sqrt{n}}\right)^{r-1}$$

holds for all $r=2, \cdots, k$ and, for $r=2$,

$$\frac{i}{\sigma_n} \sum_{j=1}^{n} \mathbf{E}X_j (\exp\{it(S_j^{(0)} - S_j^{(1)})\} - 1) = -t + \frac{\theta(t) t^2 Bm}{\sqrt{n}}.$$

By virtue of what was said above, relation (2.1) may be reduced to the form

$$f_n'(t) = -t f_n(t) + \theta_1(t) \frac{t^2 Bm}{\sqrt{n}} f_n(t) + \theta_2(t) B \frac{t^2 m}{n} \tag{2.2}$$

in the region $|t| \leqq \gamma_n \sqrt{n}$. Integrating it in this region, we obtain

$$f_n(t) = \exp\left\{-\frac{t^2}{2} + \theta(t) \frac{|t|^3 Bm}{\sqrt{n}}\right\} + \theta(t) \frac{B|t|m}{n},$$

which by use of Esseen's inequality (for example, see [3], p. 27) yields in turn the required bounds for Δ_n.

3. Proof of Theorem 5

We shall now carry out in detail the method described above as applied to Theorem 5 with a narrow-sense stationary sequence of m-dependent random variables $X_1, X_2, \cdots$ which have finite third moments.

Let us prove a number of lemmas which we shall use in deriving a relation of the form (2.2).

Lemma 3.1. *The inequality*

$$\mathbf{E}\left| X_j \prod_{l=1}^{r-1} (e^{it(S_j^{(l-1)} - S_j^{(l)})} - 1) \right| \leqq \left(\frac{|t| b_m \sqrt{2}}{\sigma_n}\right)^{r-1} \mathbf{E}^{1/3} |X_1|^3$$

holds for all $j = 1, 2, \cdots, n$ and all $r = 1, 2, \cdots, k$.

PROOF. For $j = 1, 2, \cdots, n$ and for $r = 1, 2, \cdots, k$, put

$$\xi_j^{(r)} = e^{it(S_j^{(r-1)} - S_j^{(r)})} - 1.$$

(This notation will be used throughout the entire paper.)

The $\xi_j^{(r)}$ are measurable with respect to the σ-algebra generated by the random variables X_p for $(r-1)m < |p - j| \leqq pm$ and hence, for any j, the sequence of variables $\{\xi_j^{(r)}\}_{r=1}^k$ is a sequence of 1-dependent variables. This implies, in particular, that the $\xi_j^{(r)}$ are independent when the subscript r takes on either all even values or all odd values.

Apply Hölder's inequality to $\mathbf{E}|X_j \prod_{l=1}^{r-1} \xi_j^{(l)}|$ to obtain

$$\mathbf{E}\left|X_j \prod_{l=1}^{r-1} \xi_j^{(l)}\right| \leqq E^{1/2}\left|X_j \prod_l{}' \xi_j^{(l)}\right|^2 E^{1/2}\left|\prod_l{}'' \xi_j^{(l)}\right|^2$$

($\prod'$ in this means that the product is over all even indices l and $\prod''$ that it is over all odd indices.) By virtue of the 1-dependence of the variables $\xi_j^{(l)}$, we have

$$\mathbf{E}\left|X_j \prod_{l=1}^{r-1} \xi_j^{(l)}\right| \leqq \mathbf{E}^{1/2}|X_1|^2 \prod_{l=1}^{r-1} E^{1/2}|\xi_j^{(l)}|^2. \tag{3.1}$$

The following inequality holds for all $x \in R^1$ and all $k \geqq 0$:

$$\left|e^{ix} - \sum_{\nu=0}^{k} \frac{(ix)^\nu}{\nu!}\right| \leqq \frac{|x|^{k+1}}{(k+1)!}. \tag{3.2}$$

By inequality (3.2),

$$\mathbf{E}|\xi_j^{(l)}|^2 \leqq t^2 \mathbf{E}|S_j^{(l-1)} - S_j^{(l)}|^2. \tag{3.3}$$

It is not hard to show that

$$\mathbf{E}|S_j^{(l-1)} - S_j^{(l)}|^2 \leqq \frac{2b_m^2}{\sigma_n^2}. \tag{3.4}$$

Inequalities (3.1), (3.3) and (3.4) yield the required estimate. The lemma is proved.

Lemma 3.2. *The inequality*

$$\left|\sum_{j=1}^{n} \mathbf{E}X_j \prod_{l=1}^{r-1} \xi_j^{(l)} e^{itS_j^{(l)}} - \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\left(X_j \prod_{l=1}^{r-1} \xi_j^{(l)}\right) \sum_{p=1}^{n} \mathbf{E}\, e^{itS_p^{(r)}}\right|$$
$$\leqq B(r-1)mE^{1/3}|X_1|^3\left(\frac{|t|b_m\sqrt{2}}{\sigma_n}\right)^{r-1}$$

holds for all $r = 2, \cdots, k$.

PROOF. First of all, observe that the variables X_j and $\xi_j^{(l)}$, $l = 1, 2, \cdots, r-1$, are independent of the random variable $S_j^{(r)}$ and so

$$\mathbf{E}X_j \prod_{l=1}^{r-1} \xi_j^{(l)} e^{itS_j^{(r)}} = \mathbf{E}X_j \prod_{l=1}^{r-1} \xi_j^{(l)} \mathbf{E}\, e^{itS_j^{(r)}}. \tag{3.5}$$

Then, by the stationary property of the sequence, we have the equality

$$\mathbf{E}X_j \prod_{l=1}^{r-1} \xi_j^{(l)} = \mathbf{E}X_{j_0} \prod_{l=1}^{r-1} \xi_{j_0}^{(l)} \tag{3.6}$$

for all j: $j \in [(r-1)m, n-(r-1)m]$, where $j_0 = (r-1)m+1$. Relations (3.5) and

(3.6) imply that

$$\sum_{j=1}^{n} \mathbf{E}X_j \prod_{l=1}^{r-1} \xi_j^{(l)} e^{itS_j^{(r)}} = \left(\sum_{j=1}^{j_0-1} + \sum_{j=n-j_0+1}^{n}\right)\mathbf{E}X_j \prod_{l=1}^{r-1} \xi_j^{(l)} \mathbf{E}\, e^{itS_j^{(r)}}$$
$$+\mathbf{E}\left(X_{j_0} \prod_{l=1}^{r-1} \xi_{j_0}^{(l)}\right)\left(\sum_{j=j_0}^{n-j_0} \mathbf{E}\, e^{itS_j^{(r)}}\right).$$

We may continue to transform this last relation as follows:

$$\sum_{j=1}^{n} \mathbf{E}X_j \prod_{l=1}^{r-1} \xi_j^{(l)} e^{itS_j^{(r)}} = \mathbf{E}X_{j_0} \prod_{l=1}^{r-1} \xi_{j_0}^{(l)} \sum_{j=1}^{n} \mathbf{E}\, e^{itS_j^{(r)}}$$
$$(3.7) \qquad +\left(\sum_{j=1}^{j_0-1} + \sum_{j=n-j_0+1}^{n}\right)\mathbf{E}X_i \prod_{l=1}^{r-1} \xi_j^{(l)} \mathbf{E}\, e^{itS_j^{(r)}}$$
$$-\left(\sum_{j=1}^{j_0-1} + \sum_{j=n-j_0+1}^{n}\right)\mathbf{E}X_{j_0} \prod_{l=1}^{r-1} \xi_{j_0}^{(l)} \mathbf{E}\, e^{itS_j^{(r)}}$$

In addition, by relation (3.6), we have

$$(3.8) \qquad \sum_{j=1}^{n} \mathbf{E}X_j \prod_{l=1}^{r-1} \xi_j^{(l)} = (n-2(r-1)m)\mathbf{E}X_{j_0} \prod_{l=1}^{r-1} \xi_{j_0}^{(l)}$$
$$+\left(\sum_{j=1}^{j_0-1} + \sum_{j=n-j_0+1}^{n}\right)\mathbf{E}X_j \prod_{l=1}^{r-1} \xi_j^{(l)}.$$

From relations (3.7) and (3.8), it follows that

$$\sum_{j=1}^{n} \mathbf{E}X_j \prod_{l=1}^{r-1} \xi_j^{(l)} e^{itS_j^{(r)}} - \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}X_j \prod_{l=1}^{r-1} \xi_j^{(l)} \sum_{p=1}^{n} \mathbf{E}\, e^{itS_p^{(r)}}$$
$$(3.9) \qquad = \left(\sum_{j=1}^{j_0-1} + \sum_{j=n-j_0+1}^{n}\right)\mathbf{E}X_j \prod_{l=1}^{r-1} \xi_j^{(l)}\left(\mathbf{E}\, e^{itS_j^{(r)}} - \frac{1}{n}\sum_{p=1}^{n} \mathbf{E}\, e^{itS_p^{(r)}}\right)$$
$$-\left(\sum_{j=1}^{j_0-1} + \sum_{j=n-j_0+1}^{n}\right)\mathbf{E}X_{j_0} \prod_{l=1}^{r-1} \xi_{j_0}^{(l)}\left(\mathbf{E}\, e^{itS_j^{(r)}} - \frac{1}{n}\sum_{p=1}^{n} \mathbf{E}\, e^{itS_p^{(r)}}\right).$$

Relation (3.9) and Lemma 3.1 readily lead to the required estimate. The lemma is proved.

Lemma 3.3. *The inequality*

$$\left|\frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\, e^{itS_j^{(r)}} - f_n(t)\right| \leqq B_1 \sqrt{\frac{m}{n}}\frac{|t|}{\sigma_n} r b_m + B_2 \frac{|t|^2 r b_m^2}{\sigma_n^2}|f_n(t)|$$

holds for all $r = 1, 2, \cdots, k$.

PROOF. Consider the relation

$$(3.10) \quad f_n(t) - \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\, e^{itS_j^{(r)}} = \mathbf{E}\left[\frac{1}{n}\sum_{j=1}^{n}\left(1-\exp\left\{-\frac{it}{\sigma_n}\sum_{|j-p|\leqq rm} X_p\right\}\right)\right] e^{itS_j^{(0)}}.$$

Put

$$\eta_j^{(r)} = 1 - \exp\left\{-\frac{it}{\sigma_n} \sum_{|j-p|\leqq rm} X_p\right\}$$

and rewrite (3.10) in the form

$$(3.11) \qquad f_n(t) - \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\, e^{itS_j^{(r)}} = \mathbf{E}\, e^{itS_j^{(0)}} \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\eta_j^{(r)} + \mathbf{E}\left[\frac{1}{n}\sum_{j=1}^{n} (\eta_j^{(r)} - \mathbf{E}\eta_j^{(r)})\right] e^{itS_j^{(0)}}.$$

Applying Hölder's inequality to the last term on the right-hand side of (3.11), we obtain

$$(3.12) \qquad \left|\mathbf{E}\left[\frac{1}{n}\sum_{j=1}^{n} (\eta_j^{(r)} - \mathbf{E}\eta_j^{(r)})\right] e^{itS_j^{(0)}}\right| \leqq \frac{1}{n}\mathbf{E}^{1/2}\left|\sum_{j=1}^{n} (\eta_j^{(r)} - \mathbf{E}\eta_j^{(r)})\right|^2.$$

We now estimate the expression on the right-hand side of (3.12). Note that

$$\mathbf{E}\left|\sum_{j=1}^{n} (\eta_j^{(r)} - \mathbf{E}\eta_j^{(r)})\right|^2 = \sum_{j=1}^{n}\sum_{p=1}^{n} \operatorname{cov}(\eta_j^{(r)}, \eta_p^{(r)})$$

(here $\operatorname{cov}(\xi, \eta) = \mathbf{E}(\xi - \mathbf{E}\xi)(\overline{\eta - \mathbf{E}\eta})$). It is not hard to see that if $|j-p| > 3rm$, then the variables $\eta_j^{(r)}$ and $\eta_p^{(r)}$ are independent and hence

$$\operatorname{cov}(\eta_j^{(r)}, \eta_p^{(r)}) = 0.$$

Thus,

$$(3.13) \qquad \mathbf{E}\left|\sum_{j=1}^{n} (\eta_j^{(r)} - \mathbf{E}\eta_j^{(r)})\right|^2 = \sum_{j=1}^{n}\sum_{|p-j|\leqq 3rm} \operatorname{cov}(\eta_j^{(r)}, \eta_p^{(r)}).$$

By Hölder's inequality, we have

$$|\operatorname{cov}(\eta_j^{(r)}, \eta_p^{(r)})| \leqq \mathbf{E}^{1/2}|\eta_j^{(r)}|^2\mathbf{E}^{1/2}|\eta_p^{(r)}|^2.$$

Recalling inequalities (3.3) and (3.4) (the variables $\eta_j^{(r)}$ essentially do not differ from the $\xi_j^{(r)}$ considered earlier), we find that

$$(3.14) \qquad |\operatorname{cov}(\eta_j^{(r)}, \eta_p^{(r)})| \leqq \frac{|t|^2}{\sigma_n^2} Bb_m^2 r.$$

From (3.13) and (3.14), it follows that

$$(3.15) \qquad \mathbf{E}\left|\sum_{j=1}^{n} (\eta_j^{(r)} - \mathbf{E}\eta_j^{(r)})\right|^2 \leqq Bnr^2 m \frac{|t|^2}{\sigma_n^2} b_m^2.$$

The conclusion of the lemma now follows from inequalities (3.13)–(3.15) and equality (3.11). The lemma is proved.

Lemma 3.4. *The following inequality holds*:

$$\sum_{j=1}^{n} \frac{i}{\sigma_n}\mathbf{E}X_j(\exp\{it(S_j^{(0)} - S_j^{(1)})\} - 1) = -t + \theta(t)\frac{t^2}{\sigma_n^3} Bnb_m^2\mathbf{E}^{1/3}|X_1|^3.$$

PROOF. We apply inequality (3.2) obtaining

$$\frac{i}{\sigma_n}\sum_{j=1}^n \mathbf{E}X_j\xi_j^{(1)} = -\frac{t}{\sigma_n}\sum_{j=1}^n \mathbf{E}X_j(S_j^{(0)}-S_j^{(1)}) + \frac{\theta(t)t^2}{\sigma_n}\sum_{j=1}^n \mathbf{E}|X_j(S_j^{(0)}-S_j^{(1)})^2|. \tag{3.16}$$

By virtue of the m-dependence,

$$\mathbf{E}X_jS_j^{(1)}=0$$

for all $j=1, 2, \cdots, n$. Hence,

$$\sum_{j=1}^n \mathbf{E}X_j(S_j^{(0)}-S_j^{(1)})=\sigma_n^2, \tag{3.17}$$

and

$$\mathbf{E}|X_j(S_j^{(0)}-S_j^{(1)})^2| \leqq B\frac{b_m^2}{\sigma_n^2}\mathbf{E}^{1/3}|X_1|^3. \tag{3.18}$$

The required result follows from relations (3.16)–(3.18), and the lemma is proved.

Relation (2.1) can now be transformed by the use of Lemmas 3.1–3.4. First of all, note that by the m-dependence, for all $j=1, 2, \cdots, n$, we have the equality

$$\mathbf{E}X_j \exp\{itS_j^{(1)}\}=0,$$

and therefore (2.1) may be rewritten in the form

$$\begin{aligned} f_n'(t) = {} & \sum_{r=2}^k \frac{i}{\sigma_n}\sum_{j=1}^n \mathbf{E}X_j \prod_{l=1}^{r-1}\xi_j^{(l)}\frac{1}{n}\sum_{p=1}^n \mathbf{E}\, e^{itS_j^{(r)}} \\ & + \theta_1(t)\sum_{r=2}^k \frac{1}{\sigma_n} Brm\left(\frac{|t|b_m\sqrt{2}}{\sigma_n}\right)^{r-1}\mathbf{E}^{1/3}|X_1|^3 \\ & + \theta_2(t)\frac{n}{\sigma_n}\left(\frac{|t|b_m\sqrt{2}}{\sigma_n}\right)^k \mathbf{E}^{1/3}|X_1|^3. \end{aligned} \tag{3.19}$$

(We have applied Lemma 3.1 to the last term, and Lemma 3.2 to the first, on the right-hand side of (2.1).)

We now apply Lemmas 3.1 and 3.3 to the first sum in (3.19). We obtain

$$\begin{aligned} f_n'(t) = {} & \sum_{r=2}^k \frac{i}{\sigma_n}\sum_{j=1}^n \mathbf{E}X_j \prod_{l=1}^{r-1}\xi_j^{(l)} f_n(t) + \theta_1(t)\frac{n}{\sigma_n}\left(\frac{|t|b_m\sqrt{2}}{\sigma_n}\right)^k \mathbf{E}^{1/3}|X_1|^3 \\ & + \theta_2(t)\sum_{r=2}^k B_1\frac{n}{\sigma_n} r\left(\frac{|t|b_m\sqrt{2}}{\sigma_n}\right)^{r-1}\left[\sqrt{\frac{m}{n}}\frac{|t|b_m}{\sigma_n} + \frac{t^2b_m^2}{\sigma_n^2}|f_n(t)|\right]\mathbf{E}^{1/3}|X_1|^3 \\ & + \theta_3(t)\sum_{r=2}^k B_2\frac{rm}{\sigma_n}\left(\frac{|t|b_m\sqrt{2}}{\sigma_n}\right)^{r-1}\mathbf{E}^{1/3}|X_1|^3. \end{aligned} \tag{3.20}$$

For $r=2$, we use Lemma 3.4 on the right-hand side of (3.20), obtaining

$$\begin{aligned} f_n'(t) &= -tf_n(t) + \frac{\theta_1(t)B_1 n b_m^2 \mathbf{E}^{1/2}|X_1|^3}{\sigma_n^3} t^2 f_n(t) \\ &\quad + \theta_2(t) \sum_{r=3}^{k} B_2 \frac{n}{\sigma_n} r\left(\frac{|t|b_m\sqrt{2}}{\sigma_n}\right)^{r+1} \mathbf{E}^{1/3}|X_1|^3 f_n(t) \\ &\quad + \theta_3(t) \sum_{r=3}^{k} \frac{n}{\sigma_n}\left(\frac{|t|b_m\sqrt{2}}{\sigma_n}\right)^{r-1} f_n(t)\mathbf{E}^{1/3}|X_1|^3 \\ &\quad + \theta_4(t)\frac{n}{\sigma_n}\left(\frac{|t|b_m\sqrt{2}}{\sigma_n}\right)^{k} \mathbf{E}^{1/3}|X_1|^3 \\ &\quad + \theta_5(t) \sum_{r=2}^{k} B_3 \frac{m}{\sigma_n}\left(\frac{|t|b_m\sqrt{2}}{\sigma_n}\right)^{r-1} r\mathbf{E}^{1/3}|X_1|^3 \\ &\quad + \theta_6(t) \sum_{r=2}^{k} B_4 \frac{\sqrt{mn}}{\sigma_n} r\left(\frac{|t|b_m\sqrt{2}}{\sigma_n}\right)^{r} \mathbf{E}^{1/3}|X_1|^3. \end{aligned} \tag{3.21}$$

If we put

$$T_1 = \frac{\sigma_n}{2b_m\sqrt{2}}$$

and consider (3.21) in the region $|t| \leqq T_1$, it will assume the form

$$f_n'(t) = -tf_n(t) + \theta_1(t)c_n^{(1)}\frac{t^2}{\sigma_n} f_n(t) + \theta_2(t)c_n^{(2)}\frac{|t|}{\sigma_n^2} + \theta_3(t)c_n^{(3)}\frac{t^2}{\sigma_n^2}, \tag{3.22}$$

where

$$c_n^{(1)} = \frac{B_1 n b_m^2 \mathbf{E}^{1/3}|X_1|^3}{\sigma_n^2}, \qquad c_n^{(2)} = B_2 m b_m \mathbf{E}^{1/3}|X_1|^3,$$

$$c_n^{(3)} = \frac{b_m^2 n \mathbf{E}^{1/3}|X_1|^3}{2^k \sigma_n} + B_3 \frac{\sqrt{nm}\, b_m^2 \mathbf{E}^{1/3}|X_1|^3}{\sigma_n}.$$

Integrating (3.22) in this region, we find that

$$\begin{aligned} f_n(t) &= \exp\left\{-\frac{t^2}{2} + \frac{c_n^{(1)}}{\sigma_n}\int_0^t \theta_1(u)u^2\,du\right\} \\ &\quad \times\left[1 + \int_0^t \left(\frac{c_n^{(2)}\theta_2(u)u}{\sigma_n^2} + \frac{c_n^{(3)}\theta_3(u)u^2}{\sigma_n^2}\right)\right. \\ &\quad \left.\times \exp\left\{\frac{u^2}{2} - \frac{c_n^{(1)}}{\sigma_n}\int_0^u \theta_1(z)z^2\,dz\right\} du\right. \end{aligned} \tag{3.23}$$

for all t such that $|t| \leqq T_1$.

Consider the function

$$b(u,t) = -\frac{t^2}{2} + \frac{u^2}{2} + \frac{c_n^{(1)}}{\sigma_n}\int_u^t \theta_1(z)z^2\,dz.$$

Clearly,

$$\left|\int_u^t \theta_1(z)z^2\,dz\right| \leqq \int_{|u|}^{|t|} z^2\,dz \leqq |t|(t^2-u^2).$$

If we now put

$$T_2=\frac{\sigma_n}{4c_n^{(1)}},$$

and $T_0=\min(T_1, T_2)$, then we obtain the inequality

$$\operatorname{Re} b(u,t)\leqq -\frac{t^2}{4}+\frac{u^2}{4} \tag{3.24}$$

for all t such that $|t|\leqq T_0$.

In view of this last inequality, (3.23) may be rewritten in this region in the form

$$\begin{aligned} f_n(t)&=\exp\left\{-\frac{t^2}{2}+\theta_1(t)\frac{c_n^{(1)}|t|^3}{\sigma_n}\right\}\\ &\quad+e^{-t^2/4}\int_0^t\left(c_n^{(2)}\frac{\theta_2(u)u}{\sigma_n^2}+c_n^{(3)}\frac{\theta_3(u)u^2}{\sigma_n^2}\right)e^{u^2/4}\,du. \end{aligned} \tag{3.25}$$

From (3.25) and the relation

$$\left|\int_0^t u^r e^{u^2/4}\,du\right|=\theta(t)B\,e^{t^2/4}\min(|t|^{r+1}, |t|^{r-1})$$

we obtain the relation

$$\begin{aligned} f_n(t)&=\exp\left\{-\frac{t^2}{2}+\frac{c_n^{(1)}\theta_1(t)t^3}{\sigma_n}\right\}\\ &\quad+\frac{c_n^{(2)}\theta_2(t)B}{\sigma_n^2}\min\{1, |t|^2\}+\frac{c_n^{(3)}\theta_s(t)|t|}{\sigma_n^2} \end{aligned}$$

in the region $|t|\leqq T_0$. The following inequality therefore holds in this region:

$$|f_n(t)-e^{-t^2/2}|\leqq B\left\{\frac{|t|^3c_n^{(1)}}{\sigma_n}e^{-t^2/4}+\frac{c_n^{(2)}\min\{1, |t|^2\}}{\sigma_n^2}+\frac{c_n^{(3)}|t|}{\sigma_n^2}\right\}.$$

To complete the proof, we merely have to apply Esseen's inequality and in so doing choose the number k occurring in the definition of $c_n^{(3)}$ to satisfy $2^k\geqq\sqrt{n}$.

4. On the Proofs of Theorems 1–4

As was mentioned above, the proofs of Theorems 1–4 differ from the proof of Theorem 5 in a purely technical way. In the derivation of inequalities analogous to those obtained in Lemmas 3.1–3.4, we have to estimate the covariance of random variables which are measurable with respect to "sufficiently remote" σ-algebras. This makes it possible to state the following proposition.

Proposition 1. *Suppose that the random variable ξ is measurable with respect to the σ-algebra $\mathfrak{M}_{-\infty}^k$ and the random variable η is measurable with respect to*

$\mathfrak{M}_{k+n}^{\infty}$. *Then the following inequalities hold*:

(i) *if* $\mathbf{E}\xi^2 < \infty$ *and* $\mathbf{E}\eta^2 < \infty$, *then*

$$|\mathbf{E}\xi\eta - \mathbf{E}\xi\mathbf{E}\eta| \leqq \rho(n)\mathbf{E}^{1/2}\xi^2\mathbf{E}^{1/2}\eta^2;$$

(ii) *if* $|\xi| \leqq C_1$, $|\eta| \leqq C_2$ *a.s., then*

$$|\mathbf{E}\xi\eta - \mathbf{E}\xi\mathbf{E}\eta| \leqq 16C_1C_2\alpha(n);$$

(iii) *if* $|\xi| \leqq C$ *a.s. and* $\mathbf{E}|\xi|^p < \infty$ *for some* $p > 1$, *then*

$$|\mathbf{E}\xi\eta - \mathbf{E}\xi\mathbf{E}\eta| \leqq B[\alpha(n)]^{1-1/p}\mathbf{E}^{1/p}|\xi|^p;$$

(iv) *if* $\mathbf{E}|\xi|^p < \infty$ *and* $\mathbf{E}|\eta|^q < \infty$ *for* $p > 1$ *and* $q > 1$ *such that* $1/p + 1/q < 1$, *then*

$$|\mathbf{E}\xi\eta - \mathbf{E}\xi\mathbf{E}\eta| \leqq B[\alpha(n)]^{1-1/p-1/q}\mathbf{E}^{1/p}|\xi|^p\mathbf{E}^{1/q}|\eta|^q.$$

The first of these inequalities is a trivial consequence of the definition of $\rho(n)$. The proofs of inequalities (ii)–(iv) may be found in [3], p. 306 and in [1].

In what follows, we shall assume that $\mathbf{E}|X_1|^3 < \infty$, and only sequences satisfying a s.m. condition will be considered (the changes in the proof for the case of sequences satisfying the complete regularity condition present no difficulties).

Let us state the lemmas analogous to Lemmas 3.1–3.4 whose results will be needed in going from relation (2.1) to an equation of the form (2.2) for sequences satisfying the s.m. condition.

Lemma 4.1. *The inequality*

$$\mathbf{E}\left|X_j \prod_{l=1}^{r-1} \xi_j^{(l)}\right| \leqq \left(\frac{|t|b_m 2}{\sigma_n}\right)^{2(r-1)} \mathbf{E}^{1/3}|X_1|^3 + B\mathbf{E}^{1/3}|X_1|^3[\alpha(m)]^{1/3}\left[r^{1/3}\left(\frac{|t|b_m 4}{\sigma_n}\right)^{r-1} + r2^{2(r-1)}[\alpha(m)]^{1/3}\right]$$

holds for all $j = 1, 2, \cdots, n$ *and* $r = 1, 2, \cdots k$.

Lemma 4.2. *The inequality*

$$\left|\sum_{j=1}^{n} \mathbf{E}X_j \prod_{l=1}^{r-1} \xi_j^{(l)}\, e^{itS_j^{(r)}} - \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}X_j \prod_{l=1}^{r-1} \xi_j^{(l)} \sum_{j=1}^{n} \mathbf{E}\, e^{itS_j^{(r)}}\right|$$
$$\leqq B\mathbf{E}^{1/3}|X_1|^3\left[rm\left(\frac{4|t|b_m}{\sigma_n}\right)^r + m[\alpha(m)]^{1/3}r^2\left(\frac{4|t|b_m}{\sigma_n}\right)^{r/2} + [\alpha(m)]^{2/3}2^r(mr^2 2^r + n)\right]$$

holds for all $r = 1, 2, \cdots, k$.

Lemma 4.3. *The inequality*

$$\left|f_n(t) - \frac{1}{n}\sum_{j=1}^{n} \mathbf{E}\, e^{itS_j^{(r)}}\right| \leqq B_1 r^2\left(\frac{|t|b_m}{\sigma_n}\right)^2 |f_n(t)| + \frac{B_2|t|b_m}{\sigma_n\sqrt{n}} r^{3/2}\sqrt{m}$$

holds for all $r = 1, 2, \cdots, k$.

Lemma 4.4. *The following inequality holds*:

$$\sum_{j=1}^{n} \frac{i}{\sigma_n} \mathbf{E} X_j (e^{it(S_j^{(0)} - S_j^{(1)})} - 1)$$

$$= -t + \theta_1(t) \frac{t^2 B n b_m^2 \mathbf{E}^{1/3} |X_1|^3}{\sigma_n^3} + \theta_2(t) B \frac{n^2}{\sigma_n^2} |t| [\alpha(m)]^{1/3} \mathbf{E}^{2/3} |X_1|^3$$

The proofs of Lemmas 4.1–4.4 are similar to those of Lemmas 3.1–3.4. In Section 3 we could replace the expectation of the product of variables which are measurable with respect to sufficiently remote σ-algebras by the product of their expectations because of the m-dependency; however, here we need to estimate in addition the difference between the expectation of the product of the "remote" variables and the product of their expectations.

We point out one further fact that causes additional technical difficulties. Namely, if the random variable ξ is measurable with respect to the σ-algebra generated by the random variables X_j when $j \in [a, b]$, and η is measurable with respect to the σ-algebra generated by the variables X_j when $j \in (-\infty, a-m) \cup (b+m, \infty)$, then for a sequence of m-dependent variables, the random variables ξ and η will be independent (hence, $\operatorname{cov}(\xi, \eta) = 0$). But for sequences satisfying the strong mixing condition or complete regularity condition, we are unable to estimate $\operatorname{cov}(\xi, \eta)$ directly in terms of $\alpha(n)$ or $\rho(n)$. This difficulty has to be circumvented by introducing random variables which are measurable with respect to the σ-algebras $\mathfrak{M}_{-\infty}^{a-m}$ and $\mathfrak{M}_{b+m}^{\infty}$.

We shall give only the proof of Lemma 4.1, which is apparently more difficult technically than the proofs of Lemmas 4.2–4.4.

First of all, introduce the variables

$$\tilde{\xi}_j^{(l)} = \exp\left\{\frac{it}{\sigma_n} \sum_{p=j+(l-1)m}^{j+lm} X_p\right\} - 1$$

and

$$\hat{\xi}_j^{(l)} = \exp\left\{\frac{it}{\sigma_n} \sum_{p=j-lm}^{j-(l-1)m} X_p\right\} - 1$$

(if $j \leqq lm$, then the summation is to begin with $p = 1$; if $j > n - lm$, then the summation goes up to $p = n$; but if $j < (l-1)m$ or $j > n-(l-1)m$, then the corresponding sum is regarded as equal to 0).

It is not hard to see that

$$|\xi_j^{(l)}| \leqq |\tilde{\xi}_j^{(l)}| + |\hat{\xi}_j^{(l)}|$$

for all $j = 1, 2, \cdots, n$ and $l = 1, 2, \cdots, k$. By virtue of this inequality,

$$\mathbf{E}\left|X_j \prod_{l=1}^{r-1} \xi_j^{(l)}\right| \leqq \sum_{p=0}^{r-1} \sum \mathbf{E}\left|X_j \prod_{\nu=1}^{p} \tilde{\xi}_j^{(l_\nu)} \prod_{\mu=p+1}^{r-1} \hat{\xi}_j^{(l_\mu)}\right| \tag{4.1}$$

(the second summation in this is over all sets of indices $1 \leqq l_1 < l_2 < \cdots < l_p \leqq r-1$ and $1 \leqq l_{p+1} < l_{p+2} < \cdots < l_{r-1} \leqq r-1$ such that $l_\nu \neq l_\mu$ when $\nu \neq \mu$).

Applying Hölder's inequality to the summands on the right-hand side of (4.1), we obtain

$$\mathbf{E}\left|X_j \prod_{\nu=1}^{p} \tilde{\xi}_j^{(l_\nu)} \prod_{\mu=p+1}^{r-1} \hat{\xi}_j^{(l_\mu)}\right|$$
$$\leqq \mathbf{E}^{2/3}\left|X_j \prod_\nu{}' \tilde{\xi}_j^{(l_\nu)} \prod_\mu{}' \hat{\xi}_j^{(l_\mu)}\right|^{3/2} \mathbf{E}^{1/3}\left|\prod_\nu{}'' \tilde{\xi}_j^{(l_\nu)} \prod_\mu{}'' \hat{\xi}_j^{(l_\mu)}\right|^3,$$

where $\prod'$ denotes a product over all even indices and $\prod''$ a product over all odd ones.

Observe now that the variables $\tilde{\xi}_j^{(l_\nu)}$ and $\hat{\xi}_j^{(l_\mu)}$ are measurable with respect to the σ-algebras $\mathfrak{M}_{j-l_\nu m}^{j-(l_\nu-1)m}$ and $\mathfrak{M}_{j+(l_\mu-1)m}^{j+l_\mu m}$, respectively.

For even or odd indices ν or μ, the corresponding σ-algebras are "separated" by at least m variables X_q. Also, it is not hard to see that $\max(|\hat{\xi}_j^{(l)}|, |\tilde{\xi}_j^{(l)}|) \leqq 2$. Applying (ii) of Proposition 1 consecutively, we obtain

$$\mathbf{E}\left|X_j \prod_\nu{}' \tilde{\xi}_j^{(l_\nu)} \prod_\mu \hat{\xi}_j^{(l_\mu)}\right|^{3/2} \leqq \mathbf{E}^{1/2}|X_1|^3 \prod_\nu{}' \mathbf{E}|\tilde{\xi}_j^{(l_\nu)}|^{3/2} \prod_\mu{}' \mathbf{E}|\hat{\xi}_j^{(l_\mu)}|^{3/2} \tag{4.2}$$
$$+B_1 r 2^{(r-1)}[\alpha(m)]^{1/2}\mathbf{E}^{1/2}|X_1|^3$$

and

$$\mathbf{E}\left|\prod_\nu{}'' \tilde{\xi}_j^{(l_\nu)} \prod_\mu{}'' \hat{\xi}_j^{(l_\mu)}\right|^3 \leqq Br2^{(r-1)}[\alpha(m)] + \prod_\nu{}'' \mathbf{E}|\xi_j^{(l_\nu)}|^3 \prod_\mu{}'' \mathbf{E}|\hat{\xi}_j^{(l_\mu)}|^3. \tag{4.3}$$

Noting that

$$\max(\mathbf{E}|\tilde{\xi}_j^{(l_\nu)}|^3, \mathbf{E}|\hat{\xi}_j^{(l_\mu)}|^3) \leqq \left(\frac{|t|b_m}{\sigma_n}\right)^3,$$

we obtain from (4.2) and (4.3) the inequality

$$\mathbf{E}\left|X_j \prod_\nu \tilde{\xi}_j^{(l_\nu)} \prod_\mu \hat{\xi}_j^{(l_\mu)}\right| \leqq \left(\frac{|t|b_m}{\sigma_n}\right)^{r-1} \mathbf{E}^{1/3}|X_1|^3$$
$$+B_1[\alpha(m)]^{1/3}\left(\frac{2|t|b_m}{\sigma_n}\right)^{(r-1)/2} r^{1/3}\mathbf{E}^{1/3}|X_1|^3 \tag{4.4}$$
$$+B_2[\alpha(m)]^{2/3} r 2^{(r-1)}\mathbf{E}^{1/3}|X_1|^3.$$

To complete the proof of the lemma, merely observe that the right-hand side of (4.4) does not depend on the indices l_ν and l_μ and that there are at most 2^{r-1} summands in (4.1). The lemma is proved.

Now put $T_0 = \sigma_n/(32b_m)$, and choose k so that

$$k^2 16^k [\alpha(m)]^{1/3} \leqq 1. \tag{4.5}$$

Applying the inequalities obtained in Lemmas 4.1–4.4, we can derive the following relation in the region $|t| \leqq T_0$ similarly to what was done in Section 3:

$$f_n'(t) = -tf_n(t) + \theta_1(t) \frac{B_1 n E^{1/3}|X_1|^3}{\sigma_n} \left(\frac{|t| b_m}{\sigma_n}\right)^2 f_n(t)$$

$$+ \theta_2(t) B_2 \left(\frac{|t| b_m}{\sigma_n}\right)^2 \sqrt{m} \frac{\sqrt{n}}{\sigma_n} \mathbf{E}^{1/3}|X_1|^3 + \theta_3(t) B_3 \frac{|t| b_m \mathbf{E}^{1/3}|X_1|^3 m}{\sigma_n^2} \tag{4.6}$$

$$+ \theta_4(t) B_4 [\alpha(m)]^{1/3} \frac{n}{\sigma_n} \mathbf{E}^{1/3}|X_1|^3 + \theta_5(t) \frac{n}{\sigma_n} E^{1/3}|X_1|^3 \left(\frac{|t| b_m}{\sigma_n}\right)^k.$$

When $\alpha(n)$ decreases exponentially, we can choose m so that

$$[\alpha(m)]^{1/3} \leqq Cn^{-2}$$

($m \leqq A_0 \log n$, where the constant A_0 is chosen so that besides condition (4.5), k satisfies the condition

$$\left(\frac{|t| b_m}{\sigma_n}\right)^{k-2} \leqq \frac{c}{n}$$

for all t such that $|t| \leqq T_0$).

With these choices of m and k and from the fact that $b_m = O(m)$ and $\sigma_n = O(\sqrt{n})$, relation (4.6) leads to the inequality

$$|f_n(t) - e^{-t^2/2}| \leqq B_1 \frac{|t|^3 \log^2 n}{\sqrt{n}} e^{-t^2/4} + B_2 \frac{\log^2 n}{n} |t| + B_3 \frac{\log n}{n} \min\{1, |t|\} \tag{4.7}$$

which holds in the region $|t| \leqq \gamma\sqrt{n}/\log^2 n$ (γ is a positive constant not depending on n).

Inequality (4.7) obviously implies the assertion in Theorem 2. We can choose for example, in the conditions of Theorem 1

$$m = O\left(\frac{1}{n^{2(\beta+1)}}\right), \qquad n \to \infty,$$

and k, as before, from condition (4.5).

5. A Lower Bound for the Convergence Rate for a Class of Stationary Sequences

We form a Markov chain whose states are all integers. Let $\alpha_0 = 1/2$ and $0 < \alpha_k < 1$ for all $k \geqq 1$. Define transition probabilities as follows: for all $k \geqq 0$, let

$$p_{kk+1} = p_{-k-k-1} - \alpha_k$$

and, for all $k \geqq 1$,

$$p_{k0} = p_{-k0} = 1 - \alpha_k.$$

It is not hard to see that the transition probabilities so prescribed define a chain whose states all comprise a single ergodic class of period 1.

Further, let $\beta_0 = \beta_1 = 1$, and, for $k \geqq 2$,

$$\beta_k = \prod_{\nu=1}^{k-1} \alpha_\nu. \tag{5.0}$$

Let $f_{ij}^{(k)}$ denote the probability of getting from state i into state j for the first time at step k. In this notation, for all $k \geqq 1$,

$$f_{00}^{(k)} = \beta_{k-1} - \beta_k. \tag{5.1}$$

The relation (5.1) implies that the chain will be recurrent if and only if $\lim_{k\to\infty} \beta_k = 0$. In addition, if the series of the β_k is convergent, then all states of the chain will be positive and the stationary distribution will be of the form

$$\pi_j = \pi_{-j} = \tfrac{1}{2}\beta_{j-1}\pi_0, \tag{5.2}$$

where $j \geqq 1$.

A more detailed description of the structure of such a chain may be found in Chung's book [9], pp. 107–109.

From the way the chain is formed and equalities (5.0)–(5.2), it follows that the chain is defined by merely giving the probabilities $f_{00}^{(k)}$ for all $k \geqq 1$. Put

$$\begin{aligned} f_{00}^{(1)} &= 0, \\ f_{00}^{(k)} &= \frac{A}{k^{(2+\delta)(p+1)+1}} \quad (k \geqq 2), \end{aligned} \tag{5.3}$$

where $0 < \delta < 1$ and p is positive. The constant A is chosen to fit the condition

$$\sum_{k=1}^{\infty} f_{00}^{(k)} = 1,$$

We shall assume that the initial distribution is taken to be the stationary distribution defined by equality (5.1). Define a function f on the state space of the chain as follows:

$$f(j) = \operatorname{sgn}\{j\}|j|^p. \tag{5.4}$$

Let ξ_k denote the state of the chain at time k and put $X_k = f(\xi_k)$. The sequence $X_1, X_2, \cdots$ formed in this way is a narrow-sense stationary sequence of random variables.

In [2], Yu. A. Davydov studied the behavior of s.m. coefficients for such sequences. In particular, his results imply in this case that a s.m. coefficient behaves as follows:

$$\alpha(n) \asymp n^{-(2+\delta)p-1-\delta} \tag{5.5}$$

($a_n \asymp b_n$ denotes the fact that there exist positive constants C_1 and C_2 such that $C_1 b_n \leqq a_n \leqq C_2 b_n$ for all n).

From (5.1)–(5.4), it follows that, for all $\gamma < \delta + (1+\delta)p$,

$$\mathbf{E}|X_1|^{2+\gamma} < \infty \tag{5.6}$$

and

$$\mathbf{E}|X_1|^{2+\delta+(1+\delta)/p} = \infty. \tag{5.7}$$

On the other hand, Δ_n satisfies the estimate

$$\Delta_n \geqq Cn^{-\delta/2} \log^{-2-\delta} n. \tag{5.8}$$

To prove this last inequality, we make use of Doeblin's representation of a sum of random variables connected in a Markov chain in the form of a random number of independent summands. A detailed description of this method may be found in Chung's book [9], Part 1, Section 14.

Introduce the following random variables: τ_ν, the time that state 0 is reached for the νth time, $\nu \geqq 1$; $\rho_\nu = \tau_{\nu+1} - \tau_\nu$, the νth recurrence time at state 0. It is known that the variables τ_1 and ρ_ν, $\nu \geqq 1$, are independent, that moreover the ρ_ν are identically distributed and that

$$\mathbf{P}\{\rho_\nu = k\} = f_{00}^{(k)}. \tag{5.9}$$

For all $\nu \geqq 1$, let

$$Y_\nu = \sum_{k=\tau_\nu+1}^{\tau_{\nu+1}} X_k.$$

Define an integer-valued random variable $l(n)$ by the inequalities

$$\tau_{l(n)} \leqq n < \tau_{l(n)+1};$$

$l(n)$ is the number of times state 0 is reached in n steps. Further, let

$$Y' = \sum_{k=1}^{\tau_1} X_k \quad \text{and} \quad Y'' = \sum_{k=\tau_{l(n)}+1}^{n} X_k.$$

In this notation, the following relation holds:

$$\tilde{S}_n = \sum_{k=1}^{n} X_k = Y' + \sum_{\nu=1}^{l(n)-1} Y_\nu + Y''. \tag{5.10}$$

The random variables Y_ν are independent and identically distributed and, for all $k \geqq 1$,

$$\mathbf{P}\left\{ Y_1 = \sum_{j=1}^{k} j^p \right\} = \mathbf{P}\left\{ Y_1 = -\sum_{j=1}^{k} j^p \right\} = \tfrac{1}{2} f_{00}^{(k)}. \tag{5.11}$$

From (5.3) and (5.11), it follows that, for any δ' such that $0 < \delta' < \delta$,

$$\mathbf{E}|Y_1|^{2+\delta'} < \infty \tag{5.12}$$

and

$$\mathbf{E}|Y_1|^{2+\delta} = \infty. \tag{5.13}$$

Defining the normalizing factor in the following way:

$$D = \pi_0 \mathbf{E} Y_1^2,$$

we put

$$S_n = \frac{1}{\sqrt{Dn}} \sum_{k=1}^{n} X_k,$$

and

$$\Delta_n = \sup_z |\mathbf{P}\{S_n < z\} - \Phi(z)|.$$

To estimate Δ_n, observe that

$$\Delta_n \geqq \mathbf{P}\{S_n > B \log n\} - (1 - \Phi(B \log n)), \tag{5.14}$$

in which, for any $q > 1$,

$$1 - \Phi(B \log n) = o(n^{-q}) \quad \text{as } n \to \infty. \tag{5.15}$$

We let

$$n' = [n\pi_0 - \sqrt{n} \log n] \quad \text{and} \quad n'' = [n\pi_0 + \sqrt{n} \log n],$$

and extend the inequality (5.14) using (5.15):

$$\begin{aligned} \Delta_n \geqq \mathbf{P}&\left\{ \sum_{\nu=1}^{n'} Y_\nu > \frac{b\sqrt{Dn} \log n}{2} \right\} \\ &- \mathbf{P}\left\{ Y' + \sum_{\nu=n'+1}^{l(n)-1} Y_\nu + Y'' > \frac{B\sqrt{Dn} \log n}{2} \right\} - o(n^{-q}) \quad \text{as } n \to \infty. \end{aligned} \tag{5.16}$$

To estimate the subtrahend on the right-hand side of (5.16), we need an estimate for the probability of the difference between $l(n)$ and its expectation. Using the relationship between the events

$$\{l(n) \in [k_1, k_2]\} = \{\tau_{k_1} \leqq n, \tau_{k_2} > n\} = \left\{ \tau_1 + \sum_{\nu=1}^{k_1-1} \rho_\nu \leqq n, \tau_1 + \sum_{\nu=1}^{k_2} \rho_\nu > n \right\}$$

and the independence of the random variables τ_1 and ρ_ν, we can obtain the estimate

$$\mathbf{P}\{l(n) \in [n', n'']\} = 1 - o(n^{-\delta''/2}) \quad \text{as } n \to \infty, \tag{5.17}$$

where δ'' is independent of n and satisfies the condition $\delta < \delta''$.

It is not hard to see that

$$\begin{aligned} \mathbf{P}&\left\{ Y' + \sum_{\nu=n'}^{l(n)-1} Y_\nu + Y'' > \frac{B\sqrt{Dn} \log n}{2} \right\} \\ &\leqq \mathbf{P}\left\{ Y' > \frac{B\sqrt{Dn} \log n}{6} \right\} + \mathbf{P}\left\{ Y'' > \frac{B\sqrt{Dn} \log n}{6} \right\} \\ &\quad + \mathbf{P}\left\{ \sum_{\nu=n'}^{(n)-1} Y_\nu > \frac{B\sqrt{Dn} \log n}{6} \right\}. \end{aligned} \tag{5.18}$$

Relation (5.17) now leads to the inequality

$$\begin{aligned} \mathbf{P}&\left\{ \sum_{\nu=n'}^{l(n)-1} Y_\nu > \frac{B\sqrt{Dn} \log n}{6} \right\} \\ &\leqq \mathbf{P}\left\{ \max_{n'<k\leqq n''} \sum_{\nu=n'}^{k} Y_\nu > \frac{B\sqrt{Dn} \log n}{6} \right\} + o(n^{-\delta''/2}) \quad \text{as } n \to \infty. \end{aligned} \tag{5.19}$$

Applying Kolmogorov's inequality, we find that

$$\mathbf{P}\left\{\max_{n'<k\leqq n''}\sum_{\nu=n'}^{k} Y_\nu > \frac{B\sqrt{Dn}\log n}{6}\right\} \leqq \frac{\mathbf{E}(\sum_{\nu=1}^{n''-n'} Y_\nu)^2}{B_1 n \log^2 n} = o(n^{-1/2}),\ n\to\infty. \tag{5.20}$$

By studying the distributions of the random variables Y' and Y'', we can prove for any $\delta'<\delta$ that

$$\mathbf{E}|Y'|^{1+\delta'}<\infty \quad \text{and} \quad \mathbf{E}|Y''|^{1+\delta'}<\infty.$$

This implies that as $n\to\infty$

$$\mathbf{P}\left\{Y' > \frac{B\sqrt{Dn}\log n}{6}\right\} = o(n^{-1/2}), \tag{5.21}$$

$$\mathbf{P}\left\{Y'' > \frac{B\sqrt{Dn\log n}}{6}\right\} = o(n^{-1/2}). \tag{5.22}$$

We now estimate the probability

$$\mathbf{P}\left\{\sum_{\nu=1}^{n'} Y_\nu > \frac{B\sqrt{Dn}\log n}{6}\right\}.$$

Note that, by (5.11),

$$\mathbf{P}\{|Y_1|>x\} = \sum_k \frac{A}{k^{(2+\delta)(p+1)+1}},$$

in which the summation is over all k such that $\sum_{j=1}^k j^p > x$. From this, it is possible to deduce that there is a constant C_0 such that

$$\mathbf{P}\{|Y_1|>x\} \sim \frac{C_0}{x^{2+\delta}}. \tag{5.23}$$

Since Y_1 has a symmetric distribution and (5.23) is satisfied, the results of A. V. Nagaev's paper [4] imply the relation ($n\to\infty$)

$$\mathbf{P}\left\{\sum_{\nu=1}^{n'} Y_\nu > x\right\} = n'\mathbf{P}\{Y_1>x\}(1+o(1))$$

for all x such that $x/\log x > \sqrt{nD}$. Putting $x = B_1\sqrt{Dn}\log n$, we find for all sufficiently large n that

$$\begin{aligned}\mathbf{P}\left\{\sum_{\nu=1}^{n'} Y_\nu > B_1\sqrt{Dn}\log n\right\} &\geqq \frac{n}{2}\mathbf{P}\left\{Y_1 > B_1\sqrt{nD}\log n\right\} \\ &\geqq B_2 n^{-\delta/2}\log^{-2-\delta}.\end{aligned} \tag{5.24}$$

From relations (5.14)–(5.22) and (5.24), it follows for all sufficiently large n that

$$\Delta_n \geqq B_3 n^{-\delta/2}\log^{-2-\delta} n,$$

as required.

Received by the editors
January 18, 1979

REFERENCES

[1] YU. A. DAVYDOV, *Convergence of distributions generated by stationary stochastic processes*, Theory Prob. Appl., 13 (1968), pp. 691–696.

[2] YU. A. DAVYDOV, *Mixing conditions for Markov chains*, Theory Prob. Appl., 18 (1973) pp. 312–328.

[3] I. A. IBRAGIMOV AND YU. V. LINNIK, *Independent and Stationary Sequences of Random Variables*, Wolters-Noordhoff, Groningen, 1971.

[4] A. V. NAGAEV, *Limit theorems considering large deviations with Cramér's condition violated*, Izv. Akad. Nauk Uzbekh. SSR, Ser. Fiz.-Mat., 2 (1969), pp. 17–22. (In Russian.)

[5] V. V. PETROV, *Sums of Independent Random Variables*, Springer-Verlag, New York, 1975.

[6] V. A. STATULYAVICHUS, *On limit theorems for dependent random variables*, Second Vilnius Conf. on Prob. Theory and Math. Statistics, Abstracts of Reports, Vol. 3, Vilnius, 1977, pp. 212–215. (In Russian.)

[7] A. N. TIKHOMIROV, *On the convergence rate in the central limit theorem for weakly dependent random variables*, Vestnik Leningrad. Gos. Univ., Ser. Mat., Mekh., Astron., 7 (1976), pp. 158–159. (In Russian.)

[8] A. N. TIKHOMIROV, *The convergence rate in the central limit theorem for Markov chains*, Theory Prob. Appl., 20 (1975), pp. 676–677.

[9] K. L. CHUNG, *Markov Chains with Stationary Transition Probabilities*, Springer-Verlag, Berlin, 2nd ed., 1967.

[10] C. STEIN, *A bound for the error in the normal approximation to the distribution of a sum of dependent random variables*, Proc. Sixth Berkeley Symp. Math. Statist. and Prob., II, 1972, pp. 583–602.