Task 1

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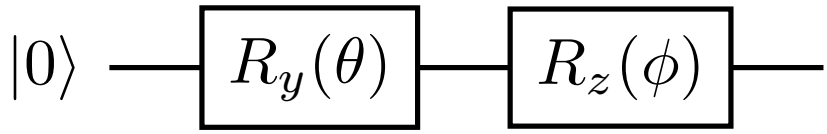
1 Part 1

A variational quantum circuit that is able to generate the most general 1-qubit state, starting from the initial state $|0\rangle$ is the following:



Proof:
A general state on the Bloch sphere can be parameterized by two angles, $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$, as:

$$|\theta, \phi\rangle = \begin{bmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{bmatrix}$$

The matrix representation of our circuit is:

$$\begin{aligned} R_z(\phi)R_y(\theta) &= \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix} \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta/2)e^{-i\phi/2} & -\sin(\theta/2)e^{i\phi/2} \\ \sin(\theta/2)e^{-i\phi/2} & \cos(\theta/2)e^{i\phi/2} \end{bmatrix} \end{aligned}$$

Our initial state is $|0\rangle$. The resulting state after the circuit is:

$$\begin{aligned} R_z(\phi)R_y(\theta)|0\rangle &= \begin{bmatrix} \cos(\theta/2)e^{-i\phi/2} & -\sin(\theta/2)e^{i\phi/2} \\ \sin(\theta/2)e^{-i\phi/2} & \cos(\theta/2)e^{i\phi/2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= e^{-i\phi/2} \begin{bmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{bmatrix} = |\theta, \phi\rangle \end{aligned}$$

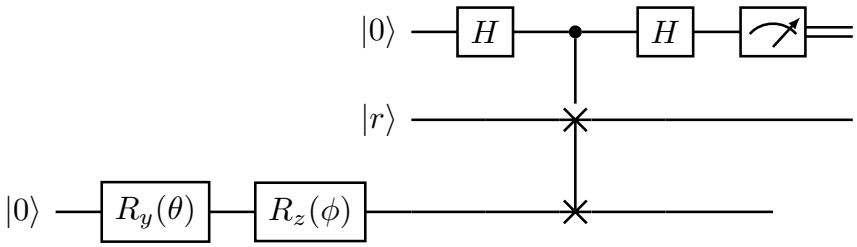
which is equivalent to $|\theta, \phi\rangle$ modulo a global phase. ■

2 Part 2

Let $|r\rangle$ be a random 1-qubit state. From above, we know that it can be written as $|r\rangle = |\theta_0, \phi_0\rangle$ for some angles θ_0, ϕ_0. Since we don't know the angles ahead of time, we can't find the random state by just tuning our gates to $R_z(\phi_0)$ and $R_y(\theta_0)$. Instead, we can use the SWAP test and machine learning to discover the angles. The principle behind this approach is minimizing a cost function in terms of the overlap between the random state $|r\rangle$ and the controlled state $|\theta, \phi\rangle$

$$\text{cost} \propto 1 - |\langle r|\theta, \phi\rangle|^2$$

We can implement this cost function using the SWAP test as in the following circuit:



Proof:
The wavefunction of the system immediately before measurement is:

$$\frac{1}{2}|0\rangle\left(|\theta, \phi\rangle|r\rangle + |r\rangle|\theta, \phi\rangle\right) + \frac{1}{2}|1\rangle\left(|\theta, \phi\rangle|r\rangle - |r\rangle|\theta, \phi\rangle\right)$$

Let M be the value measured of the ancilla qubit, in the z - basis, $\{|0\rangle, |1\rangle\}$, returning the value 0 or 1 respectively. Suppose we run the circuit n times, so that we obtain a set of measurements $\{M_i\}_{i=1}^n$. Then in the limit $n \to \infty$, we can compute the overlap:

$$\lim_{n\to\infty}\left(1 - \frac{2}{n}\sum_{i=1}^{n} M_i\right) = |\langle r|\theta, \phi\rangle|^2$$

The extension to the cost function is straightforward ■.
We then run this circuit many times ($n \gg 1$) so that we can approach the limiting value as mentioned in the proof. Once we have the resulting

measurements, the optimal angles θ_0, ϕ_0 are the solution to the minimization problem:

$$(\theta_0, \phi_0) = \arg\min_{(\theta,\phi)} \frac{2}{n} \sum_{i=1}^{n} M_i$$

We can solve this via an appropriate minimization algorithm.

2.1 Implementation

We can implement the circuit and the optimization using the Pennylane Python library:

```
import pennylane as qml
from pennylane import numpy as np
```

For the purposes of illustration, in this implementation we let the random state be $|r\rangle = |\theta_0 = \pi/8, \phi_0 = \pi/4\rangle$. In reality, in order to be actually random, we would not know this ahead of time.

```
t_0 = 0.125*np.pi # ~ 0.3927
p_0 = 0.25*np.pi  # ~ 0.7854
```

Now we are ready to implement our circuit. We use `QubitStateVector` to define the random state:

```
num_shots = 1000
dev = qml.device('default.qubit', wires=3, shots=num_shots)

@qml.qnode(dev)
def var_swap(params):
  # random state preparation
  qml.QubitStateVector([
      np.cos(t_0/2.),
      np.sqrt(0.5)*(1+1j)*np.sin(t_0/2.)
  ], wires=1)
  # actual circuit follows:
  qml.RY(params[0], wires=2)
  qml.RZ(params[1], wires=2)
  qml.Hadamard(wires=0)
  qml.CSWAP(wires=[0,1,2])
  qml.Hadamard(wires=0)
  return qml.sample(qml.PauliZ(0))
```

Now we set up the optimizer. Define the cost function:

```
def cost(params):
    return 1-(1./num_shots)*np.sum(var_swap(params))
```

Define the initial parameters:

```
params = np.array([0.,0.])
```

Now we define the optimization algorithm. Because our cost function has a stochastic output (via the call to `qml.sample`), we are limited to using *gradient-free* methods of optimization. In this case we can use Rotosolve via the `RotosolveOptimizer`.[1]

```
opt = qml.RotosolveOptimizer()
```

So with that we are ready to run the optimization:

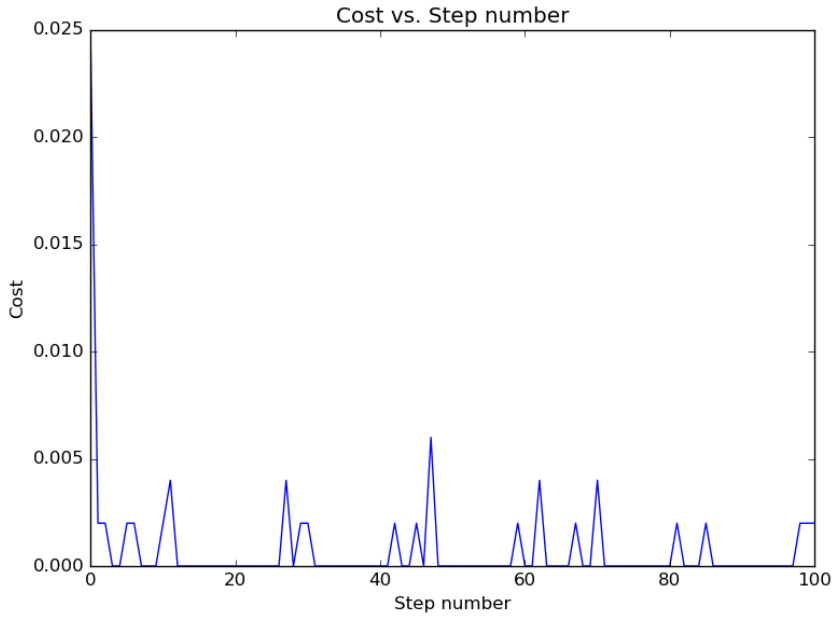
```
steps = 100
best_cost = [cost(params)]
best_params = params
for i in range(steps):
  params = opt.step(cost, params)
  new_cost = cost(params)
  if(new_cost < best_cost[-1]):
    best_params = params
  best_cost.append(new_cost)

print(best_params)
```

```
[0.40703541 0.93070817]
```

The optimization gives $\theta =$ `0.40703540572409325` and $\phi =$ `0.9307081740331506`, which differs from the actual best parameters by:
$\theta - \pi/8 = 0.0143, \phi - \pi/4 = 0.145$

[1] Pennylane also has an API for the RotoselectOptimizer, which not only can find the best θ, ϕ minimizing the cost function, but can also find the best rotation gates for us! However, we already know the best set of rotation gates – they are the gates we used to prepare the "random" state.

We visually check the progression of our optimizer with a plot of cost-vs-steps:



3 Part 3

In this part we extend the analysis from the first two parts to perform a SWAP test on a N - qubit product state.
Now the random state $|r\rangle$ has each qubit in either the $|0\rangle$ or $|1\rangle$ state:

$$|r\rangle = \bigotimes_{i=1}^{N} |k_i\rangle, \qquad k_i \in \{0, 1\}$$

Since we know that each qubit in the product state is limited to either $|0\rangle$ or $|1\rangle$, we can change our approach slightly to simplify things. We drop the parametric rotation gates from the circuit in Part 2, and instead determine each qubit in the random state via a grid search. That is, we perform a multi-qubit SWAP test by comparing the random state $|r\rangle$ to the test state $|t\rangle$, where $|t\rangle = |t_1 t_2 \cdots t_N\rangle$, $t_i \in \{0, 1\}$. Our estimate for the random state $|r\rangle$ will be the state $|t^*\rangle$ that maximizes the overlap $|\langle r|t^*\rangle|^2$ Equivalently, we can seek the minimizer of the function $1 - |\langle r|t\rangle|^2$:

$$|t^*\rangle = \arg\min_{t_1 t_2 \cdots t_N} (1 - |\langle r|t_1 t_2 \cdots t_N\rangle|^2)$$

We will find the state $|t^*\rangle$ by simple grid search.

3.1 Implementation

Let's say the number of qubits is 4 to start, and our random state is $|r\rangle = |0011\rangle$. Since we require an ancilla qubit for every swap test, that means our circuit will require 12 qubits total.

```
import pennylane as qml
from pennylane import numpy as np

n_qubits = 4 * 3

num_shots = 1000
dev4 = qml.device('default.qubit', wires=n_qubits, shots=num_shots)

def init_random_state(state):
  qml.BasisState(state, wires=[1,4,7,10])

def swap(init_wire, theta):
  qml.RY(theta, wires=init_wire+2)
  qml.Hadamard(wires=init_wire)
  qml.CSWAP(wires=[init_wire, init_wire+1, init_wire+2])
  qml.Hadamard(wires=init_wire)

@qml.qnode(dev4)
def n_swap(params):
  init_random_state(np.array([0,0,1,1]))
  swap(0, params[0])
  swap(3, params[1])
  swap(6, params[2])
  swap(9, params[3])
  return [qml.sample(qml.PauliZ(i)) for i in range(0,11,3)]
```

Our cost function is the similar to the last section:

```
def cost(params):
  results = n_swap(params)
  return [1-(1./num_shots)*np.sum(results[i]) for i in range(4)]
```

And we run the simulation:

```
param_set = [np.array([np.pi*i,np.pi*j,np.pi*k,np.pi*l])
             for i in range(0,2)
             for j in range(0,2)
             for k in range(0,2)
             for l in range(0,2)]
cost_set = []
for param in param_set:
  cost_set.append(cost(param))
```

Finally we find the optimal value by grid search:

```
best_cost = 1
best_params = []
for r in range(len(cost_set)):
  run_cost = np.sum(cost_set[r])
  if run_cost < best_cost:
    best_cost = run_cost
    best_params = param_set[r]

print(best_params, best_cost)
```

```
[0.         0.         3.14159265 3.14159265] 0.0
```

Which gives the results as expected.

4 References

Wikipedia - SWAP test
Quantum Fingerprinting
Learning the quantum algorithm for state overlap