CHAPTER 6: CONTINUOUS PROBABILITY DISTRIBUTIONS

Uniform, Normal, and Exponential Models

I. Continuous Probability Basics

A continuous random variable can take on any value in an interval. Because there are infinite possible values, we use a **Probability Density Function (PDF)**, denoted as $f(x)$.

- The total area under the PDF curve is always **1**.
- Probability is defined as the area under the curve between two points a and b.

II. The Uniform Distribution

This is the simplest distribution, used when the probability is "evenly spread" over an interval from a to b. The "curve" is actually a flat rectangular line.

$$f(x) = \frac{1}{b-a} \quad \text{for } a \leq x \leq b$$

Calculation: To find $P(x_1 < x < x_2)$, simply calculate the area of the rectangle:

$$\text{Area} = \text{Base} \times \text{Height} = (x_2 - x_1) \times \frac{1}{b-a}$$

III. The Normal Distribution

The most important distribution in all of statistics. It is symmetrical and bell-shaped, defined by its mean (μ) and standard deviation (σ).

1. Characteristics

- The highest point is at the mean, which is also the median and mode.
- The curve extends to infinity in both directions but never touches the x-axis.

2. The Standard Normal (Z)

To find probabilities, we convert any normal variable x into a **Standard Normal Variable (z)** with $\mu = 0$ and $\sigma = 1$:

$$z = \frac{x - \mu}{\sigma}$$

We then use the **Z-Table** to find the cumulative probability $P(Z \leq z)$.

3. Using the Z-Table

- **Left-tail:** $P(X < a) \rightarrow$ Look up z directly.
- **Right-tail:** $P(X > a) \rightarrow 1 - P(Z < z)$.
- **Between:** $P(a < X < b) \rightarrow P(Z < z_b) - P(Z < z_a)$.

IV. The Exponential Distribution

Often used to model the **time or distance between arrivals** (e.g., time between customers entering a bank).

$$f(x) = \frac{1}{\mu} e^{-x/\mu}$$

Cumulative Probability Function:

$$P(X \leq x) = 1 - e^{-x/\mu}$$

Where μ is the mean time between events.

V. Teacher's Strategy: Visualizing the Area

Students often struggle with Z-tables because they forget which "side" they are looking at.

- **Always sketch the curve.** Shade the area you are looking for.
- If your z value is positive, the area to the left must be greater than 0.50.
- If you are looking for a "between" probability, you are essentially subtracting a small tail from a larger tail.

VI. Step-by-Step Example

Problem: Monthly utility bills are normally distributed with $\mu = 100$ and $\sigma = 20$. What is the probability a bill is *more* than $130?

Logic: 1. **Standardize:** $z = (130 - 100)/20 = 30/20 = 1.5$. 2. **Table Look-up:** Find $z = 1.50$ in the Z-table. The area to the left is 0.9332. 3. **Adjust for Tail:** Since we want "more than," we need the area to the *right*. 4. **Result:** $1 - 0.9332 = \mathbf{0.0668}$ (or 6.68%).

VII. Practice Set

1. For a uniform distribution from $a = 10$ to $b = 50$, what is the height of the density function? What is $P(x < 20)$?

2. A process has a mean of 500 and std dev of 50. Find the z-score for $x = 425$.

3. The time between arrivals at a car wash follows an exponential distribution with a mean of 10 minutes ($\mu = 10$). What is the probability the next car arrives within 5 minutes?

4. If $P(Z < z) = 0.9750$, what is the value of z? (Work backwards using the table).

VIII. Answer Key

1. **Height:** $1/(50-10) = \mathbf{0.025}$. **Prob:** $(20-10) \times 0.025 = \mathbf{0.25}$. 2. $z = (425 - 500)/50 = -75/50 = \mathbf{-1.5}$. 3. $P(X \leq 5) = 1 - e^{-5/10} = 1 - e^{-0.5} \approx \mathbf{0.3935}$. 4. $z = \mathbf{1.96}$.