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Review Article

The $\mathcal{Z}$-Transform Method and Delta Type Fractional Difference Operators

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The Caputo-, Riemann-Liouville-, and Grünwald-Letnikov-type difference initial value problems for linear fractional-order systems are discussed. We take under our consideration the possible solutions via the classical $\mathcal{Z}$-transform method. We stress the formula for the image of the discrete Mittag-Leffler matrix function in the $\mathcal{Z}$-transform. We also prove forms of images in the $\mathcal{Z}$-transform of the expressed fractional difference summation and operators. Additionally, the stability problem of the considered systems is studied.

1. Introduction

Fractional operators are generalizations of the corresponding operators of integer order, as well in continuous as in discrete case. The particular interpretation of the fractional derivative was presented, for example, in [1] or [2]. Fractional differences used in models of control systems could be understand as (1) an approximation of continuous operators (see [3]) and (2) a possibility of involving some memory to difference systems, that is, systems in which the current state depends on the full history of systems' states. Systems with fractional derivatives and differences are widely discussed in many papers. However, different authors could use various definitions of operators. Some comparisons between three basic types of fractional difference operators were studied in [4] and in [5] for multistep case. There are many papers, in which the authors usually involve in discrete case the Grünwald-Letnikov-type fractional operator, see, for example, [2, 6–14] for case $h = 1$ and also for general case $h > 0$. However, the exact formulas for solutions are not used to considered linear initial value problems. The authors mainly use the recurrence method for stating their results. Using the particular type of definitions of fractional difference operators it was possible to state the exact formulas for solutions to initial value problems with the Riemann-Liouville- and the Caputo-type fractional difference operators. The clue point is that solutions to initial value problems with the Grünwald-Letnikov-type operator have the same values as those for systems with the Riemann-Liouville-type operator.

Basic properties of fractional sums and operators were developed firstly in [15] and continued by Atici and Eloe [16, 17], Baleanu and Abdeljawad [18, 19]. Another concept of the fractional sum/difference was introduced in [20–22].

Here we attempt to review methods of solutions for fractional difference systems. By comparing both types of solutions we produce the formula for the $\mathcal{Z}$-transform of particular type of the discrete Mittag-Leffler matrix function. There are few papers where various kinds of discrete Mittag-Leffler functions are discussed. The important paper is [23], where discrete type and q-discrete analogues of Mittag-Leffler function are presented. Their relations to fractional differences of the particular case are investigated. The author considers also applications of these functions to numerical analysis and integrable systems. However, the formulas of the considered functions mainly depend on the class of the type of calculus. For example in [19] the authors consider discrete Mittag-Leffler functions connected with nabla Caputo fractional linear difference systems and use there the nabla discrete Laplace transform. In [24] the authors consider a linear nabla (q, h)-difference equations of noninteger order. In [25] the sequential discrete fractional equations with some particular convention of the fractional difference are solved using the kind of fractional nabla discrete Mittag-Leffler functions.

Table 1: Basic $\mathcal{Z}$-transform formulas used in the text.

$y := y(n)$	$\mathcal{Z}[y](z)$
$\widetilde{\varphi}_\alpha(n) := \binom{n+\alpha-1}{n} = (-1)^n \binom{-\alpha}{n}$	$\left(\frac{z}{z-1}\right)^\alpha$
$({}_a\Delta_h^{-\alpha}x)(a+\alpha h+nh)$	$\left(\frac{hz}{z-1}\right)^\alpha X(z)$
$\varphi^*_{k,\alpha}(n)$	$\frac{1}{z^k}\left(\frac{z}{z-1}\right)^{k\alpha+1}$
$({}_a\Delta_{h,*}^{\alpha}x)(a+(1-\alpha)h+nh)$	$z\left(\frac{hz}{z-1}\right)^{-\alpha} X(z) - zh^{-\alpha}\left(\frac{z}{z-1}\right)^{1-\alpha} x(a)$
$E_{(\alpha,\beta)}(\lambda,n) = \sum_{k=0}^{\infty}\lambda^k \widetilde{\varphi}_{k\alpha+\beta}(n-k)$	$\left(\frac{z}{z-1}\right)^\beta \left(1-\frac{\lambda}{z}\left(\frac{z}{z-1}\right)^\alpha\right)^{-1}$
$\varphi_{k,\alpha}(n)$	$\frac{1}{z^k}\left(\frac{z}{z-1}\right)^{(k+1)\alpha}$
$({}_a\Delta_h^{\alpha}x)(a+(1-\alpha)h+nh)$	$z\left(\frac{hz}{z-1}\right)^{-\alpha} X(z) - zh^{-\alpha}x(a)$
$({}_0\widetilde{\Delta}_h^{\alpha}y)(nh+h)$	$z\left(\frac{hz}{z-1}\right)^{-\alpha} X(z) - zh^{-\alpha}x(a)$

There is the possible use of the generalized Laplace transformation on time scales, for that see, for example, [26]. Such method was used in [16] and [27] under the method called the R-transform and the Laplace transform, respectively, for the situation where on the right side of the system there are functions that depend only on time. Additionally, in the dissertation [28] the possible use of the Laplace transform for semilinear systems is mentioned. More classical solution of the problem is also presented in [29], where the authors use the $\mathcal{Z}$-transform to give the conditions for stability of systems with the Caputo-type operator. The conclusions were derived based on the image in the complex domain. The main advantage of the use of the $\mathcal{Z}$-transform is to introduce the natural language for discrete systems; it means to work with sequences instead of discrete functions defined on various domains.

The stability property is the main issue in dynamical systems. As the most of descriptions of behaviour of processes are digital due to computing tools used at computers, it is important to know some conditions for stability of discrete-time systems. The results are similar for different operators. In [30, 31] the study of the stability problem for discrete fractional systems with the nabla Riemann-Liouville-type difference operator is presented. The authors stress that the $\mathcal{Z}$-transform can be used as a very effective method for stability analysis of systems. In engineering problems more often systems are used with the Grünwald-Letnikov-type fractional difference. One of the attempts to the stability of fractional difference systems is the notion of practical stability, see [6, 7]. Since the linear systems are defined by some matrices, the conditions connected with eigenvalues of these matrices are presented, for instance, in [11, 13, 14].

Our paper is rather motivated by the idea of presenting images via $\mathcal{Z}$-transform of fractional difference operators and consequences into stability property. We receive results for stability of linear systems with three operators and with steps $h > 0$. In the case of the Grünwald-Letnikov-type fractional operator our results are similar to those in [11, 13, 14]. However, steps in the proof of stability condition are based on [31].

The paper is organized as follows. In Section 2, we present the preliminary material needed for further reading. Solutions to the considered linear initial value problems with the Caputo-type fractional difference and the image of the unknown vector function are constructed in Section 3. As the simple consequence of considering two types of solution: one from the classical approach and the second from the $\mathcal{Z}$-transform we can write the images of the one-parameter Mittag-Leffler functions. Sections 4 and 5 deal with images of the unknown vector functions for linear initial value problems with the Riemann-Liouville- and the Grünwald-Letnikov-type fractional differences, respectively. Some comparisons for the $\mathcal{Z}$-transforms of Caputo-type and Riemann-Liouville-type operators are presented in Section 6. In Section 7 we study the stability of the considered systems. Section 8 provides the brief conclusions. Additionally, Table 1 presents basic $\mathcal{Z}$-transform formulas used in the text.

2. Preliminaries

Now, we recall the necessary definitions and technical propositions that are used in the sequel therein the paper. Let $h > 0$, $a \in \mathbb{R}$ and define $(h\mathbb{N})_a := \{a, a+h, a+2h, \ldots\}$ and $\sigma(t) := t+h$ for any $t \in (h\mathbb{N})_a$. Hence $t = a + nh$ for $n \in \mathbb{N}_0$. For a function $x : (h\mathbb{N})_a \to \mathbb{R}$ the *forward h-difference operator* is defined as $(\Delta_h x)(t) := (x(\sigma(t)) - x(t))/h$, where $t \in (h\mathbb{N})_a$, see [21, 22]. For an arbitrary $\alpha \in \mathbb{R}$ and $t \in \mathbb{R}$ such that $t/h \in \mathbb{R} \setminus \{-1, -2, \ldots\}$, the *h-factorial function* is defined by $t_h^{(\alpha)} := h^\alpha(\Gamma((t/h)+1)/\Gamma((t/h)+1-\alpha))$, where Γ is the Euler gamma function, see [21, 22]. We use the convention that division at a pole yields zero. For $h = 1$ we write shortly $t^{(\alpha)}$ instead of $t_1^{(\alpha)}$ and Δ instead of Δ_1.

Definition 1 (see [22]). For a function $x : (h\mathbb{N})_a \to \mathbb{R}$ the *fractional h-sum of order* $\alpha > 0$ is given by $({}_a\Delta_h^{-\alpha} x)(t) := (1/\Gamma(\alpha)) \sum_{s=0}^{n} (t - \sigma(a + sh))_h^{(\alpha-1)} x(a + sh)h$, where $t = a + (\alpha + n)h$, $n \in \mathbb{N}_0$ and additionally, $({}_a\Delta_h^{0} x)(t) := x(t)$.

For $a = 0$ we write shortly $\Delta_h^{-\alpha}$ instead of ${}_0\Delta_h^{-\alpha}$. Note that ${}_a\Delta_h^{-\alpha} x : (h\mathbb{N})_{a+\alpha h} \to \mathbb{R}$. From Definition 1 it follows that for $t = a + (\alpha + n)h$

$$\begin{aligned}({}_a\Delta_h^{-\alpha} x)(t) &= h^{\alpha} \sum_{s=0}^{n} \frac{\Gamma(\alpha + n - s)}{\Gamma(\alpha)\Gamma(n - s + 1)} x(a + sh) \\ &= h^{\alpha} \sum_{s=0}^{n} \binom{n - s + \alpha - 1}{n - s} \overline{x}(s),\end{aligned} \tag{1}$$

where $\overline{x}(s) := x(a+sh)$. Observe that formula (1) has the form of the convolution of sequences. Let us call for the moment the family of binomial functions defined on $\mathbb{Z}$, parameterized by $\mu \in \mathbb{R}$, and given by values: $\widetilde{\varphi}_\mu(n) = \binom{n+\mu-1}{n}$ for $n \in \mathbb{N}_0$ and $\widetilde{\varphi}_\mu(n) = 0$ for $n < 0$. Then using

$$\widetilde{\varphi}_\alpha(n) := \binom{n + \alpha - 1}{n} = (-1)^n \binom{-\alpha}{n}, \tag{2}$$

we can write formula (1) in the form of the convolution of $\widetilde{\varphi}_\alpha$ and $\overline{x}$. In fact

$$({}_a\Delta_h^{-\alpha} x)(t) = h^{\alpha} (\widetilde{\varphi}_\alpha * \overline{x})(n), \tag{3}$$

where "$*$" denotes a convolution operator; that is, $(\widetilde{\varphi}_\alpha * \overline{x})(n) := \sum_{s=0}^{n} \binom{n-s+\alpha-1}{n-s} \overline{x}(s)$.

We recall here that the $\mathcal{Z}$-transform of a sequence $\{y(n)\}_{n\in\mathbb{N}_0}$ is a complex function given by $Y(z) := \mathcal{Z}[y](z) = \sum_{k=0}^{\infty} y(k)z^{-k}$, where $z \in \mathbb{C}$ is a complex number for which this series converges absolutely.

Note that $\mathcal{Z}[\widetilde{\varphi}_\alpha](z) = \sum_{k=0}^{\infty} \binom{k+\alpha-1}{k} z^{-k} = \sum_{k=0}^{\infty} (-1)^k \binom{-\alpha}{k} z^{-k} = (z/(z-1))^{\alpha}$. In the stability analysis it is important to prove to which classes of sequences $\widetilde{\varphi}_\alpha$ belongs. Let $\{y(n)\}_{n\in\mathbb{N}_0}$ be a sequence and let $\|y\|_1 := \sum_{s=0}^{\infty} |y(s)|$ and $\|y\|_\infty := \sup_{n\in\mathbb{N}_0} |y(n)|$ be the norms defined in the space of sequences. Moreover, let $\ell^p(\mathbb{N}_0)$, $p \in \{1, \infty\}$, be the spaces of all sequences satisfying $\|y\|_p < \infty$.

Lemma 2. *For $\alpha < 0$ and such that $\alpha \notin \mathbb{Z}_-$ the sequence $\widetilde{\varphi}_\alpha$ belongs to $\ell^1(\mathbb{N}_0)$; that is, $\sum_{s=0}^{\infty} |\widetilde{\varphi}_\alpha(s)| = \|\widetilde{\varphi}_\alpha\|_1 < \infty$.*

Proof. Let $\beta := -\alpha$, $\beta > 0$, and $[\beta] = k$ be the integer part of β; that is, $k - 1 < \beta \le k$ and $k \in \mathbb{N}_0$. Then for $s \in \mathbb{N}_0$ such that $0 \le s \le k$ we have $\binom{\beta}{s} > 0$ and for $s > k$ we get $(-1)^s \binom{\beta}{s} > 0$. Therefore

$$\begin{aligned}\|\widetilde{\varphi}_\alpha\|_1 &= \sum_{s=0}^{\infty} |\widetilde{\varphi}_\alpha(s)| = \sum_{s=0}^{k} \binom{\beta}{s} + \sum_{s=k+1}^{\infty} (-1)^s \binom{\beta}{s} \\ &= \sum_{s=0}^{k} \binom{\beta}{s} + \sum_{s=0}^{\infty} (-1)^s \binom{\beta}{s} - \sum_{s=0}^{k} (-1)^s \binom{\beta}{s} \\ &= 2 \sum_{s=0}^{[k/2]} \binom{\beta}{2s + 1} < \infty.\end{aligned} \tag{4}$$

Hence $\widetilde{\varphi}_\alpha \in \ell^1(\mathbb{N}_0)$. □

Proposition 3. *For $\alpha \in (0, 1)$ one has $\widetilde{\varphi}_\alpha \in \ell^\infty(\mathbb{N}_0)$.*

Proof. Note that $\widetilde{\varphi}_\alpha(0) = 1$ and

$$\begin{aligned}|\widetilde{\varphi}_\alpha(n)| &= \left| \frac{(-\alpha)(-\alpha - 1) \cdot \ldots \cdot (-\alpha - (n - 1))}{n!} \right| \\ &\le \frac{1 \cdot 2 \cdot \ldots \cdot n}{n!} = 1,\end{aligned} \tag{5}$$

for $n \ge 1$. Hence $\|\widetilde{\varphi}_\alpha\|_\infty = 1$ and consequently, $\widetilde{\varphi}_\alpha \in \ell^\infty(\mathbb{N}_0)$; that is, $\|\widetilde{\varphi}_\alpha\|_\infty = \sup_{n\in\mathbb{N}_0} |\widetilde{\varphi}_\alpha(n)| < \infty$. □

Proposition 4. *For $t = a + \alpha h + nh \in (h\mathbb{Z})_{a+\alpha h}$ let one denote $y(n) := ({}_a\Delta_h^{-\alpha} x)(t)$ and $\overline{x}(n) := x(a + nh)$. Then*

$$\mathcal{Z}[y](z) = \left(\frac{hz}{z - 1} \right)^{\alpha} X(z), \tag{6}$$

where $X(z) := \mathcal{Z}[\overline{x}](z)$.

Proof. In fact as we have from (3) that $y(n) = ({}_a\Delta_h^{-\alpha} x)(t) = h^{\alpha} (\widetilde{\varphi}_\alpha * \overline{x})(n)$, then $\mathcal{Z}[y](z) = h^{\alpha} \mathcal{Z}[\widetilde{\varphi}_\alpha](z) X(z)$. Moreover, $\mathcal{Z}[\widetilde{\varphi}_\alpha](z) = (z/(z - 1))^{\alpha}$, then we easily see equality (6). □

For $h = 1$, (6) can be shortly written as $\mathcal{Z}[{}_a\Delta^{-\alpha} x](z) = (z/(z-1))^{\alpha} X(z)$, where $({}_a\Delta^{-\alpha} x)(a + \alpha + n) =: y(n)$ is treated as a sequence.

Two fractional h-sums can be composed as follows.

Proposition 5 (see [32]). *Let x be a real-valued function defined on $(h\mathbb{N})_a$, where $a, h > 0$. For $\alpha, \beta > 0$ the following equalities hold: $({}_{a+\beta h}\Delta_h^{-\alpha} ({}_a\Delta_h^{-\beta} x))(t) = ({}_a\Delta_h^{-(\alpha+\beta)} x)(t) = ({}_{a+\alpha h}\Delta_h^{-\beta} ({}_a\Delta_h^{-\alpha} x))(t)$, where $t \in (h\mathbb{N})_{a+(\alpha+\beta)h}$.*

In [22] the authors prove the following lemma that gives transition between fractional summation operators for any $h > 0$ and $h = 1$.

Lemma 6. *Let $x : (h\mathbb{N})_a \to \mathbb{R}$ and $\alpha > 0$. Then, $({}_a\Delta_h^{-\alpha} x)(t) = h^{\alpha} ({}_{a/h}\Delta_1^{-\alpha} \widetilde{x})(t/h)$, where $t \in (h\mathbb{N})_{a+\alpha h}$ and $\widetilde{x}(s) = x(sh)$.*

In our consideration the crucial role is played by the power rule formula, see [21]: $({}_a\Delta_h^{-\alpha} \psi)(t) = (\Gamma(\mu + 1)/\Gamma(\mu + \alpha + 1))(t - a + \mu h)_h^{(\mu+\alpha)}$, where $\psi(r) = (r - a + \mu h)_h^{(\mu)}$, $r \in (h\mathbb{N})_a$, $t \in (h\mathbb{N})_{a+\alpha h}$. Then for $\psi \equiv 1$ we have $\mu = 0$. Nextly, taking $t = nh + a + \alpha h$, $n \in \mathbb{N}_0$, we get $({}_a\Delta_h^{-\alpha} 1)(t) = (1/\Gamma(\alpha + 1))(t - a)_h^{(\alpha)} = (\Gamma(n + \alpha + 1)/\Gamma(\alpha + 1)\Gamma(n + 1))h^{\alpha} = \binom{n+\alpha}{n} h^{\alpha}$. Particularly, we have $\mathcal{Z}[{}_a\Delta_h^{-\alpha} 1](z) = h^{\alpha}/(1 - (1/z))^{\alpha+1} = h^{\alpha}(z/(z - 1))^{\alpha+1}$.

Now, we adapt the above formulas to multivariable fractional-order linear case and define Mittag-Leffler matrix functions. In [19] the discrete Mittag-Leffler function is introduced in the following way.

Definition 7. Let $\alpha, \beta, z \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$. The *discrete Mittag-Leffler two-parameter function* is defined as

$$\mathscr{E}_{(\alpha,\beta)}(\lambda, z) := \sum_{k=0}^{\infty} \lambda^k \frac{(z + (k-1)(\alpha - 1))^{(k\alpha)} (z + k(\alpha - 1))^{(\beta - 1)}}{\Gamma(\alpha k + \beta)}. \quad (7)$$

We introduce the *two-parameter Mittag-Leffler function* defined in different manner and show that both definitions give the same values. Moreover, we use the function for the matrix case and prove by two ways its image in the $\mathscr{Z}$-transform. Let us define the following function:

$$E_{(\alpha,\beta)}(\lambda, n) := \sum_{k=0}^{\infty} \lambda^k \widetilde{\varphi}_{k\alpha+\beta}(n - k). \quad (8)$$

In fact, in the paper we use two of them, namely, $E_{(\alpha,\alpha)}(\lambda, n) = \sum_{k=0}^{\infty} \lambda^k \widetilde{\varphi}_{k\alpha+\alpha}(n - k)$ and $E_{(\alpha)}(\lambda, n) := E_{(\alpha,1)}(\lambda, n) = \sum_{k=0}^{\infty} \lambda^k \widetilde{\varphi}_{k\alpha+1}(n - k)$. For $\alpha = 1$ we have that $E_{(1,1)}(\lambda, n) = E_{(1)}(\lambda, n) = (1 + \lambda)^n$.

In our consideration an important tool is the image of $E_{(\alpha,\beta)}(\lambda, \cdot)$ with respect to the $\mathscr{Z}$-transform.

Proposition 8. *Let $E_{(\alpha,\beta)}(\lambda, \cdot)$ be defined by (8). Then*

$$\mathscr{Z}\left[E_{(\alpha,\beta)}(\lambda, \cdot)\right](z) = \left(\frac{z}{z-1}\right)^{\beta} \left(1 - \frac{\lambda}{z}\left(\frac{z}{z-1}\right)^{\alpha}\right)^{-1}, \quad (9)$$

where $|z| > 1$ *and* $|z - 1|^{\alpha} |z|^{1-\alpha} > |\lambda|$.

Proof. By basic calculations we have

$$\begin{aligned}
\mathscr{Z}\left[E_{(\alpha,\beta)}(\lambda, \cdot)\right](z) &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \lambda^k \binom{n - k + k\alpha + \beta - 1}{n - k} z^{-n} \\
&= \sum_{k=0}^{\infty} \lambda^k z^{-k} \sum_{s=0}^{\infty} \binom{s + k\alpha + \beta - 1}{s} z^{-s} \\
&= \sum_{k=0}^{\infty} \left(\frac{\lambda}{z}\right)^k \sum_{s=0}^{\infty} (-1)^s \binom{-k\alpha - \beta}{s} z^{-s} \\
&= \left(\frac{z}{z-1}\right)^{\beta} \sum_{k=0}^{\infty} \left(\frac{\lambda}{z}\right)^k \left(\frac{z}{z-1}\right)^{k\alpha} \\
&= \left(\frac{z}{z-1}\right)^{\beta} \left(1 - \frac{\lambda}{z}\left(\frac{z}{z-1}\right)^{\alpha}\right)^{-1},
\end{aligned} \quad (10)$$

where the summation exists for $|z| > 1$ and $|z - 1|^{\alpha} |z|^{1-\alpha} > |\lambda|$. □

Proposition 9. *Let $\alpha \in (0, 1]$ and $\beta < \alpha + 1$. Let R be the set of all roots of the following equation:*

$$(z - 1)^{\alpha} = \lambda z^{\alpha - 1}. \quad (11)$$

If all elements from R are strictly inside the unit circle, then $\lim_{n\to\infty} E_{(\alpha,\beta)}(\lambda, n) = 0$.

Proof. If all roots of (11) are strictly inside the unit circle, then using theorem of final value for the $\mathscr{Z}$-transform, we easily get the assertion. □

By Proposition 9 we get that for some order α there is a "good" λ such that all elements from R are inside the unit circle.

Corollary 10. *Let $\lambda \in \mathbb{R}$. All elements from R (Proposition 9) are inside the unit circle if and only if $-2^{\alpha} < \lambda < 0$.*

3. Caputo-Type Operator

In this section we recall the definition of the Caputo-type operator and consider initial value problems of fractional-order systems with this operator. We discuss the problem of solvability of fractional-order systems defined by difference equations with the Caputo-type operator. In [33] versions of solutions of scalar fractional-order difference equation are given.

Nextly, we define the family of functions $\varphi^*_{k,\alpha} : \mathbb{Z} \to \mathbb{R}$, parameterized by $k \in \mathbb{N}_0$ and by $\alpha \in (0, 1]$ with the following values:

$$\varphi^*_{k,\alpha}(n) := \begin{cases} \dbinom{n - k + k\alpha}{n - k}, & \text{for } n \in \mathbb{N}_k, \\ 0, & \text{for } n < k. \end{cases} \quad (12)$$

Proposition 11. *Let $\varphi^*_{k,\alpha}$ be the function defined by (12). Then*

$$\mathscr{Z}\left[\varphi^*_{k,\alpha}\right](z) = \frac{1}{z^k}\left(\frac{z}{z-1}\right)^{k\alpha+1}, \quad (13)$$

for z such that $|z| > 1$.

Proof. This needs the following simple calculations:

$$\begin{aligned}
&\mathscr{Z}\left[\varphi^*_{k,\alpha}\right](z) \\
&= \sum_{n=0}^{\infty} \varphi^*_{k,\alpha}(n) z^{-n} = \sum_{n=k}^{\infty} \binom{n - k + k\alpha}{n - k} z^{-n} \\
&= z^{-k} \sum_{s=0}^{\infty} \binom{s + k\alpha}{s} z^{-s} = z^{-k} \sum_{s=0}^{\infty} (-1)^s \binom{-k\alpha - 1}{s} z^{-s} \\
&= z^{-k}\left(1 - \frac{1}{z}\right)^{-k\alpha - 1} = \frac{1}{z^k}\left(\frac{z}{z-1}\right)^{k\alpha+1},
\end{aligned} \quad (14)$$

where $|z^{-1}| < 1$ to ensure the existence of the summation. □

Note that, for any $\alpha \in (0, 1]$, $k, n \in \mathbb{N}_0$, the following relations hold:

(1) $\varphi^*_{0,\alpha}(n) = 1$, $\varphi^*_{k,\alpha}(k) = 1$, and $\varphi^*_{1,\alpha}(1) = 1$;

(2) $\varphi^*_{k,\alpha}(n) = \Gamma(n - k + k\alpha + 1)/\Gamma(k\alpha + 1)\Gamma(n - k + 1)$ and as the division by pole gives zero, the formula works also for $n < k$;

(3) $\varphi^*_{k,\alpha}(n) = (1/\Gamma(k\alpha+1))(n-k+k\alpha)^{(k\alpha)} = \binom{n-k+k\alpha}{k\alpha}$;

(4) $\mathcal{Z}[\varphi^*_{1,\alpha}](z) = (1/z)(z/(z-1))^{\alpha+1} = (1/(z-1))(z/(z-1))^{\alpha}$;

(5) $\mathcal{Z}[\varphi^*_{k,\alpha}](z) = (1/z^k)\cdot(z/(z-1))^{k\alpha+1} = (1/(z-1)^k)\cdot(z/(z-1))^{k(\alpha-1)+1}$ for $k \in \mathbb{N}_0$;

(6) For $\alpha = 1$ we have $\varphi^*_{k,1}(n) = \binom{n}{k}$ and $\Delta^{-1}\binom{n-1}{k} = \binom{n}{k+1}$.

According to item (6) in the properties of functions $\varphi^*_{k,\alpha}$ for $\alpha = 1$, we put some comment. Note that $\mathcal{Z}\left[\Delta^{-1}\binom{n}{k}\right](z) = (z/(z-1))\cdot\mathcal{Z}\left[\binom{n}{k}\right](z) = (z/(z-1))\cdot(1/z^k)\cdot(z/(z-1))^{k+1} = z^2/(z-1)^{k+2}$ and $\mathcal{Z}\left[\binom{n}{k+1}\right](z) = (1/z^{k+1})\cdot(z/(z-1))^{k+2} = z/(z-1)^{k+2}$. Therefore, $\mathcal{Z}\left[\Delta^{-1}\binom{n}{k}\right](z) \neq \mathcal{Z}\left[\binom{n}{k+1}\right](z)$, but taking $\mathcal{Z}\left[\Delta^{-1}\binom{n-1}{k}\right](z) = (z/(z-1))\cdot\mathcal{Z}\left[\binom{n-1}{k}\right](z) = (z/(z-1))\cdot(1/z^{k+1})\cdot(z/(z-1))^{k+1} = z/(z-1)^{k+2}$ so $\mathcal{Z}\left[\Delta^{-1}\binom{n-1}{k}\right](z) = \mathcal{Z}\left[\binom{n}{k+1}\right](z)$.

Taking particular value of $z = n+\alpha-1 \in \mathbb{Z}_\nu$ and considering the family of functions $\varphi^*_{k,\alpha}$ the particular case of the Mittag-Lefler function from Definition 7 can be rewritten as $\mathcal{E}_{(\alpha,1)}(A, n+\nu) = \sum_{k=0}^{\infty} A^k \varphi^*_{k,\alpha}(n) = E_{(\alpha)}(A,n)$, where $\nu = \alpha-1$. Note that in fact $E_{(\alpha)}(A,n) = \sum_{k=0}^{n} A^k\varphi^*_{k,\alpha}(n)$, so the right hand side is finite and consequently; values $E_{(\alpha)}(A,n)$ always exist. For $\alpha = 1$ there is the delta exponential function: $E_{(1)}(A,n) = (I+A)^n$.

The properties of two-indexed functions $\varphi^*_{k,s}$ were given in [34, Proposition 2.5] and generalized for n-indexed functions $\varphi^*_{k,\alpha}$ in [34] and in [35] with multiorder (α). In the paper therein, we use functions $\varphi^*_{k,\alpha}$ with step $h = 1$ and one order. The next proposition is the particular case for the multi-indexed functions.

Proposition 12 (see [35]). *Let $\alpha \in (0,1]$ and $\nu = \alpha - 1$. Then for $n \in \mathbb{N}_0$ one has*

$$\left(\Delta^{-\alpha}\varphi^*_{k,\alpha}\right)(n+\nu) = \varphi^*_{k+1,\alpha}(n). \tag{15}$$

The next step is to present Caputo-type h-difference operator.

Definition 13 (see [32]). Let $\alpha \in (0,1]$. The *Caputo-type h-difference operator* ${}_a\Delta^{\alpha}_{h,*}$ of order α for a function $x : (h\mathbb{N})_a \to \mathbb{R}$ is defined by

$$\left({}_a\Delta^{\alpha}_{h,*}x\right)(t) := \left({}_a\Delta_h^{-(1-\alpha)}\left(\Delta_h x\right)\right)(t), \tag{16}$$

where $t \in (h\mathbb{N})_{a+(1-\alpha)h}$.

Note that ${}_a\Delta^{\alpha}_{h,*}x : (h\mathbb{N})_{a+(1-\alpha)h} \to \mathbb{R}$. Moreover, for $\alpha = 1$, the Caputo-type h-difference operator takes the form: $({}_a\Delta^{1}_{h,*}x)(t) = ({}_a\Delta^{0}_{h}(\Delta_h x))(t) = (\Delta_h x)(t)$, where $t \in (h\mathbb{N})_a$. Observe that using function $\widetilde{\varphi}_{1-\alpha}$ we can write the Caputo-type difference in the following way:

$$\left({}_a\Delta^{\alpha}_{h,*}x\right)(t) = h^{-\alpha}\left(\widetilde{\varphi}_{1-\alpha} * \Delta\overline{x}\right)(n), \tag{17}$$

where $t = a + (1-\alpha)h + nh$ and $\overline{x}(n) = x(a+nh)$.

For the Caputo-type fractional difference operator there exists the inverse operator that is the tool in recurrence and direct solving fractional difference equations.

Proposition 14 (see [32]). *Let $\alpha \in (0,1]$, $h > 0$, $a = (\alpha-1)h$, and x be a real-valued function defined on $(h\mathbb{N})_a$. The following formula holds $(\Delta_h^{-\alpha}({}_a\Delta^{\alpha}_{h,*}x))(t) = x(t) - x(a)$, $t \in (h\mathbb{N})_{\alpha h}$.*

Similarly as in Lemma 6 there exists the transition formula for the Caputo-type operator between the cases for any $h > 0$ and $h = 1$.

Lemma 15 (see [21]). *Let $x : (h\mathbb{N})_a \to \mathbb{R}$ and $\alpha > 0$. Then, $({}_a\Delta^{\alpha}_{h,*}x)(t) = h^{-\alpha}({}_{a/h}\Delta^{\alpha}_{1,*}\widetilde{x})(t/h)$, where $t \in (h\mathbb{N})_{a+\alpha h}$ and $\widetilde{x}(s) = x(sh)$.*

From this moment for the case $h = 1$ we write ${}_{a/h}\Delta^{\alpha}_{*} := {}_{a/h}\Delta^{\alpha}_{1,*}$ and similarly, number 1 is omitted for two other operators and fractional summations. Then the result from Proposition 14 can be stated as

$$\left(\Delta^{-\alpha}\left({}_{a/h}\Delta^{\alpha}_{*}\widetilde{x}\right)\right)(n+\alpha) = \widetilde{x}(n+\alpha) - x(a), \tag{18}$$

where $n \in \mathbb{N}_0$, $\widetilde{x}(s) = x(sh)$ and $a = (\alpha-1)h$.

Proposition 16. *For $a \in \mathbb{R}$, $\alpha \in (0,1]$ let one define $y(n) := ({}_a\Delta^{\alpha}_{h,*}x)(t)$, where $t \in (h\mathbb{N})_{a+(1-\alpha)h}$ and $t = a + (1-\alpha)h + nh$. Then*

$$\mathcal{Z}[y](z) = h^{-\alpha}\left(\frac{z}{z-1}\right)^{1-\alpha}\left((z-1)X(z) - zx(a)\right), \tag{19}$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ and $\overline{x}(n) := x(a+nh)$.

Proof. The equality follows from $({}_a\Delta^{\alpha}_{h,*}x)(t) = h^{-\alpha}(\widetilde{\varphi}_{1-\alpha} * \Delta\overline{x})(n)$. Then using the formula for $\mathcal{Z}[\Delta\overline{x}](z) = (z-1)X(z) - z\overline{x}(0) = (z-1)X(z) - zx(a)$ after simple calculations we get (19). □

Observe that for $\alpha = 1$ we have $\mathcal{Z}[y](z) = (1/h)((z-1)X(z) - z\overline{x}(0))$, which agrees with the transform of difference Δ_h of $\overline{x}$. Using the $\mathcal{Z}$-transform of the Caputo-type operator for $\varphi^*_{k,\alpha}$, we can calculate that for $k \in \mathbb{N}_1$ the following relation holds:

$$\left({}_0\Delta^{\alpha}_{*}\varphi^*_{k,\alpha}\right)(n+1-\alpha) = \varphi^*_{k-1,\alpha}(n), \tag{20}$$

while for $k = 0$ it becomes zero; that is, $({}_0\Delta^{\alpha}_{*}\varphi^*_{0,\alpha})(n+1-\alpha) = 0$.

Firstly, let us consider systems with Caputo-type operator for $h > 0$, given by the following form:

$$\left({}_a\Delta^{\alpha}_{h,*}x\right)(nh) = Ax(nh+a), \quad n \in \mathbb{N}_0, \tag{21}$$

where $x : (h\mathbb{Z})_a \to \mathbb{R}^p$, $\alpha \in (0,1]$, $a = (\alpha-1)h$, and A is $p\times p$ real matrix. Let us consider the following initial condition: $x(a) = x_0 \in \mathbb{R}^p$ for (21). System (21) with $x(a) = x_0$ can be rewritten in the form

$$\left({}_\nu\Delta^{\alpha}_{*}\widetilde{x}\right)(n) = \widetilde{A}\widetilde{x}(n+\nu), \quad n \in \mathbb{N}_0, \tag{22a}$$

$$\widetilde{x}(\nu) = x_0 \in \mathbb{R}^p, \tag{22b}$$

where $\nu = \alpha - 1$, $\widetilde{x}(n+\nu) = x(nh+a)$, and $\widetilde{A} = h^{\alpha}A$. We also use the notation $\overline{x}(n) := \widetilde{x}(n+\nu) = x(nh+a)$, what helps to avoid the problem with domains and allows to see the variable vector as a sequence.

Proposition 17 (see [4]). *Let $\alpha \in (0,1]$ and $a = (\alpha-1)h$. The linear initial value problem*

$$\left({}_{a}\Delta^{\alpha}_{h,*}\, x\right)(t) = Ax(t+a), \quad t \in (h\mathbb{N})_0 \tag{23a}$$

$$x(a) = x_0, \quad x_0 \in \mathbb{R}^p \tag{23b}$$

has the unique solution given by the formula

$$\begin{aligned} x(t) &= \mathscr{E}_{(\alpha)}\left(Ah^{\alpha}, \frac{t}{h}\right)x_0 = E_{(\alpha)}\left(Ah^{\alpha}, n\right)x_0 \\ &= \sum_{k=0}^{\infty} A^k h^{k\alpha} \varphi^{*}_{k,\alpha}(n)\, x_0, \end{aligned} \tag{24}$$

where $t \in (h\mathbb{N})_a$ and $t/h = n + \nu = n + \alpha - 1 \in \mathbb{N}_{\alpha-1}$.

The next proposition presents the proof of the $\mathscr{Z}$-transform of the one-parameter Mittag-Leffler matrix function, where the idea of transforming a linear equation is used.

Proposition 18. *Let $\alpha \in (0,1]$ and $M(n) := E_{(\alpha)}(Ah^{\alpha}, n) = \sum_{k=0}^{\infty} A^k h^{k\alpha}\varphi^{*}_{k,\alpha}(n)$. Then the $\mathscr{Z}$-transform of the one-parameter Mittag-Leffler function $E_{(\alpha)}(Ah^{\alpha}, \cdot)$ is given by*

$$\mathscr{Z}[M](z) = \frac{z}{z-1}\left(I - \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha} Ah^{\alpha}\right)^{-1}. \tag{25}$$

Proof. Let us consider problem (22a)-(22b) and take the $\mathscr{Z}$-transform on both sides of (22a). Then for $h = 1$ from (19), we get $(z/(z-1))^{1-\alpha}((z-1)X(z) - zx(a)) = Ah^{\alpha}X(z)$, where $X(z) = \mathscr{Z}[\overline{x}](z)$ and $\overline{x}(n) := x(a+nh)$. And after calculations we see that $X(z) = (z/(z-1))(I - (1/z)(z/(z-1))^{\alpha}Ah^{\alpha})^{-1}x_0$; hence we get $x(t) = \overline{x}(n) = \mathscr{Z}^{-1}[X(z)](n) = \mathscr{Z}^{-1}[\mathscr{Z}[M](z)](n)x_0$, that is, the solution of the considered initial value problem. And then taking initial conditions as vectors from standard basis from $\mathbb{R}^p$ we get the fundamental matrix M, whose transform form should agree with the given. □

Let us consider the Caputo difference initial value problem for the linear system that contains only one equation with $A = \lambda h^{-\alpha}$, that is, $({}_{a}\Delta^{\alpha}_{h,*}\, x)(t) = \lambda h^{-\alpha}x(t+a)$, $x(a) = 1$. Then $E_{(\alpha)}(\lambda, n) = \mathscr{Z}^{-1}[(z/(z-1))(1-(\lambda/z)(z/(z-1))^{\alpha})^{-1}](n) = \mathscr{Z}^{-1}[\sum_{k=0}^{\infty}(\lambda^k/z^k)(z/(z-1))^{k\alpha+1}](n)$ is the scalar one-parameter Mittag-Leffler function. Observe that the $\mathscr{Z}$-transform of this scalar Mittag-Leffler function can be obtained from Proposition 18 by substituting $I = 1$ and $Ah^{\alpha} = \lambda$ into (25).

4. Riemann-Liouville-Type Operator

In this section we recall the definition of the Riemann-Liouville-type operator and consider initial value problems of fractional-order systems with this operator. Similarly as in the case the Caputo-type operator we discuss the problem of solvability of fractional-order systems of difference equations. Family of functions $\varphi^{*}_{k,\alpha}$ is useful for solving systems with Caputo-type operator. We also formulate the similar family of functions that are used in solutions of systems with Riemann-Liouville-type operator. Let us define the family of functions $\varphi_{k,\alpha} : \mathbb{Z} \to \mathbb{R}$ parameterized by $k \in \mathbb{N}_0$ and by $\alpha \in (0,1]$ with the following values:

$$\varphi_{k,\alpha}(n) := \begin{cases} \dbinom{n-k+k\alpha+\alpha-1}{n-k}, & \text{for } n \in \mathbb{N}_k, \\ 0, & \text{for } n < k. \end{cases} \tag{26}$$

Proposition 19. *Let $\varphi_{k,\alpha}$ be the function defined by (26). Then*

$$\mathscr{Z}[\varphi_{k,\alpha}](z) = \frac{1}{z^k}\left(\frac{z}{z-1}\right)^{k\alpha+\alpha}, \tag{27}$$

for z such that $|z| > 1$.

Proof. By simple calculations we have

$$\begin{aligned} \mathscr{Z}[\varphi_{k,\alpha}](z) &= \sum_{n=0}^{\infty}\varphi_{k,\alpha}(n)\, z^{-n} \\ &= \sum_{n=k}^{\infty}\binom{n-k+(k+1)\alpha-1}{n-k} z^{-n} \\ &= z^{-k}\sum_{s=0}^{\infty}\binom{s+(k+1)\alpha-1}{s} z^{-s} \\ &= z^{-k}\sum_{s=0}^{\infty}(-1)^s\binom{-(k+1)\alpha}{s} z^{-s} \\ &= z^{-k}\left(1-\frac{1}{z}\right)^{-k\alpha-\alpha} = \frac{1}{z^k}\left(\frac{z}{z-1}\right)^{(k+1)\alpha}, \end{aligned} \tag{28}$$

where the summation exists for $|z^{-1}| < 1$. □

Note that for any $\alpha \in (0,1]$ and $n, k \in \mathbb{N}_0$ the following relations hold:

(1) $\varphi_{0,\alpha}(0) = 1$, $\varphi_{k,\alpha}(k) = 1$, and $\varphi_{1,\alpha}(1) = 1$.

(2) $\varphi_{k,\alpha}(n) \geq 0$ and $\varphi^{*}_{k,\alpha}(n) \geq 0$.

(3) $\varphi_{k,\alpha}(n) = (n-(k+1)+(k+1)\alpha)^{((k+1)\alpha-1)}/\Gamma((k+1)\alpha)$.

(4) $\mathscr{Z}[\varphi_{0,\alpha}](z) = (z/(z-1))^{\alpha}$, $\mathscr{Z}[\varphi_{1,\alpha}](z) = (1/z)(z/(z-1))^{2\alpha}$.

(5) $\mathscr{Z}[\varphi_{k,\alpha}](z) = (1/z^k)(z/(z-1))^{(k+1)\alpha}$ for $k \in \mathbb{N}_0$.

(6) For $\alpha = 1$, we have $\varphi_{k,1}(n) = \varphi^{*}_{k,1}(n) = \binom{n}{k}$.

Taking particular value of $z = n + \alpha - 1 \in \mathbb{Z}_{\nu}$ and considering the family of functions $\varphi_{k,\alpha}$ the particular case of Mittag-Leffler function from Definition 7 can be written as $\mathscr{E}_{(\alpha,\alpha)}(A, n+\nu) = \sum_{k=0}^{\infty} A^k\varphi_{k,\alpha}(n) = E_{(\alpha,\alpha)}(A, n)$, where $\nu = \alpha - 1$. Note that in fact $E_{(\alpha,\alpha)}(A,n) = \sum_{k=0}^{n} A^k\varphi_{k,\alpha}(n)$, so the right hand side is finite and consequently; values $E_{(\alpha,\alpha)}(A,n)$ always exist. For $\alpha = 1$ there is a delta exponential function: $E_{(1,1)}(A,n) = (I + A)^n$.

For the family of functions $\varphi_{k,\alpha}$ we have similar behaviour for fractional summations as for $\varphi^*_{k,\alpha}$ so we can state the following proposition.

Proposition 20. *Let $\alpha \in (0,1]$ and $\nu = \alpha - 1$. Then for $n \in \mathbb{N}_0$, one has*

$$\left(\Delta^{-\alpha}\varphi_{k,\alpha}\right)(n+\nu) = \varphi_{k+1,\alpha}(n). \tag{29}$$

Proof. In the proof we use the fact that $\mathcal{Z}[(\Delta^{-\alpha}\varphi_{k,\alpha})(n+\nu)](z) = (1/z)(z/(z-1))^{\alpha} \cdot \mathcal{Z}[\varphi_{k,\alpha}](z)$. Then using item 5 from properties of functions $\varphi_{k,\alpha}$, we see that $\mathcal{Z}[(\Delta^{-\alpha}\varphi_{k,\alpha})(n+\nu)](z) = (1/z^{k+1})(z/(z-1))^{(k+2)\alpha} = \mathcal{Z}[\varphi_{k+1,\alpha}](z)$. Hence taking the inverse transform we get the assertion. □

The next presented operator is called the Riemann-Liouville-type fractional h-difference operator. The definition of the operator can be found, for example, in [16] (for $h = 1$) or in [21, 22] (for any $h > 0$).

Definition 21. Let $\alpha \in (0,1]$. The Riemann-Liouville-type fractional h-difference operator ${}_a\Delta^{\alpha}_h$ of order α for a function $x : (h\mathbb{N})_a \to \mathbb{R}$ is defined by

$$\left({}_a\Delta^{\alpha}_h x\right)(t) := \left(\Delta_h\left({}_a\Delta^{-(1-\alpha)}_h x\right)\right)(t), \tag{30}$$

where $t \in (h\mathbb{N})_{a+(1-\alpha)h}$.

For $\alpha \in (0,1]$ one can get

$$\begin{aligned}\left({}_a\Delta^{\alpha}_{h,*} x\right)(t) &= \left({}_a\Delta^{\alpha}_h x\right)(t) - \frac{x(a)\cdot(t-a)^{(-\alpha)}_h}{\Gamma(1-\alpha)} \\ &= \left({}_a\Delta^{\alpha}_h x\right)(t) - \frac{x(a)}{h^{\alpha}}\begin{pmatrix}\frac{t-a}{h} \\ -\alpha\end{pmatrix},\end{aligned} \tag{31}$$

where $t \in (h\mathbb{N})_a$. Moreover for $\alpha = 1$ one has $({}_a\Delta^1_{h,*} x)(t) = ({}_a\Delta^1_h x)(t) = (\Delta_h x)(t)$.

The next propositions give useful identities of transforming fractional difference equations into fractional summations.

Proposition 22 (see [33]). *Let $\alpha \in (0,1]$, $h > 0$, $a = (\alpha-1)h$, and x be a real-valued function defined on $(h\mathbb{N})_a$. The following formula holds $({}_0\Delta^{-\alpha}_h({}_a\Delta^{\alpha}_h x))(t) = x(t) - x(a)\cdot(h^{1-\alpha}/\Gamma(\alpha))t^{(\alpha-1)}_h = x(t) - x(a)\cdot\binom{t/h}{\alpha-1}$, $t \in (h\mathbb{N})_{\alpha h}$.*

In [21] the authors show that similarly as in Lemmas 6 and 15 there exists the transition formula for the Riemann-Liouville-type operator between the cases for any $h > 0$ and $h = 1$.

Lemma 23. *Let $x : (h\mathbb{N})_a \to \mathbb{R}$ and $\alpha > 0$. Then $({}_a\Delta^{\alpha}_h x)(t) = h^{-\alpha}({}_{a/h}\Delta^{\alpha}_1 \tilde{x})(t/h)$, where $t \in (h\mathbb{N})_{a+(1-\alpha)h}$ and $\tilde{x}(s) = x(sh)$.*

For the case $h = 1$, we write ${}_{a/h}\Delta^{\alpha} := {}_{a/h}\Delta^{\alpha}_1$. Then the result from Proposition 22 can be stated as

$$\left(\Delta^{-\alpha}\left({}_{a/h}\Delta^{\alpha}\tilde{x}\right)\right)(n+\alpha) = \tilde{x}(n+\alpha) - x(a)\cdot\binom{n+\alpha}{n+1}, \tag{32}$$

where $n \in \mathbb{N}$, $\tilde{x}(s) = x(sh)$ and with $a = (\alpha-1)h$.

Proposition 24. *For $a \in \mathbb{R}$, $\alpha \in (0,1]$, let one define $y(n) := ({}_a\Delta^{\alpha}_h x)(t)$, where $t \in (h\mathbb{N})_{a+(1-\alpha)h}$ and $t = a + (1-\alpha)h + nh$. Then*

$$\mathcal{Z}[y](z) = z\left(\frac{hz}{z-1}\right)^{-\alpha} X(z) - zh^{-\alpha}x(a), \tag{33}$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ and $\overline{x}(n) := x(a+nh)$.

Proof. Let $h = 1$. Then the assertion for $h > 0$ comes from Lemma 23. Let $f(n) = ({}_{\nu}\Delta^{-(1-\alpha)} x)(n)$. Then $\mathcal{Z}[y](z) = \mathcal{Z}[\Delta f][z] = (z-1)F(z) - zf(0)$, where $F(z) = \mathcal{Z}[f](z) = (z/(z-1))^{1-\alpha}X(z)$ and $f(0) = x(a) = \overline{x}(0)$. Then we easily see equality (33). □

For $\alpha = 1$ we have the same as for the Caputo-type operator $\mathcal{Z}[y](z) = (1/h)((z-1)X(z) - z\overline{x}(0))$, which also agrees with the transform of difference Δ_h of $\overline{x}$.

Using the $\mathcal{Z}$-transform of the Riemann-Liouville-type operator for $k \in \mathbb{N}_1$ we can calculate that

$$\left({}_0\Delta^{\alpha}\varphi_{k,\alpha}\right)(n+1-\alpha) = \varphi_{k-1,\alpha}(n), \tag{34}$$

while for $k = 0$ we get $\varphi_{0,\alpha}(n) = \binom{n+\alpha-1}{n} = \tilde{\varphi}_{\alpha}(n)$ and consequently, $({}_0\Delta^{\alpha}\varphi_{0,\alpha})(n+1-\alpha) = ({}_0\Delta^{\alpha}\tilde{\varphi}_{\alpha})(n+1-\alpha) = 0$.

Now, let us consider systems with the Riemann-Liouville-type operator for $h > 0$, given by the following form:

$$\left({}_a\Delta^{\alpha}_h x\right)(nh) = Ax(nh+a), \quad n \in \mathbb{N}_0, \tag{35}$$

where $x : (h\mathbb{Z})_a \to \mathbb{R}^p$, $\alpha \in (0,1]$, $a = (\alpha-1)h$, and A is $p\times p$ real matrix and with initial condition $x(a) = x_0 \in \mathbb{R}^p$. System (35) with $x(a) = x_0$ can be rewritten as follows:

$$\left({}_{\nu}\Delta^{\alpha}\tilde{x}\right)(n) = \widetilde{A}\tilde{x}(n+\nu), \quad n \in \mathbb{N}_0, \tag{36a}$$

$$\tilde{x}(\nu) = x_0 \in \mathbb{R}^p, \tag{36b}$$

where $\nu = \alpha - 1$ and $\tilde{x}(n+\nu) = x(nh+a)$, $\widetilde{A} = h^{\alpha}A$. We also use the notation $\overline{x}(n) := \tilde{x}(n+\nu) = x(nh+a)$, what helps to avoid the problem with domains and allows to see the variable vector as a sequence.

Proposition 25 (see [4]). *Let $\alpha \in (0,1]$ and $a = (\alpha-1)h$. The linear initial value problem,*

$$\left({}_a\Delta^{\alpha}_h x\right)(t) = Ax(t+a), \quad t \in (h\mathbb{N})_0, \tag{37a}$$

$$x(a) = x_0, \quad x_0 \in \mathbb{R}^p, \tag{37b}$$

has the unique solution given by the formula

$$x(t) = \mathscr{E}_{(\alpha,\alpha)}\left(Ah^{\alpha}, \frac{t}{h}\right)x_0 = E_{(\alpha,\alpha)}\left(Ah^{\alpha}, n\right)x_0 = \sum_{k=0}^{\infty} A^k h^{k\alpha} \varphi_{k,\alpha}(n)\, x_0, \tag{38}$$

where $t \in (h\mathbb{N})_0$ *and* $t/h = n + \nu = n + \alpha - 1 \in \mathbb{N}_{\alpha-1}$.

Proposition 26. *Let* $\alpha \in (0,1]$ *and* $M(n) := E_{(\alpha,\alpha)}(Ah^{\alpha}, n)$. *Then the* $\mathscr{Z}$*-transform of the Mittag-Leffler function* $E_{(\alpha,\alpha)}(Ah^{\alpha}, \cdot)$ *is given by*

$$\mathscr{Z}[M](z) = \left(\frac{z}{z-1}\right)^{\alpha}\left(I - \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha} Ah^{\alpha}\right)^{-1}. \tag{39}$$

Proof. Let us consider problem (36a)-(36b) and take the $\mathscr{Z}$-transform on both sides of (36a). Then we get from (33) for $h = 1$, $z(z/(z-1))^{-\alpha}X(z) = zx(a) + \widetilde{A}X(z)$, where $X(z) = \mathscr{Z}[\overline{x}](z)$ and $\overline{x}(n) := x(a+nh) = \widetilde{x}(\nu+n)$. And after calculations we see that $X(z) = (z/(z-1))^{\alpha}(I - (1/z)(z/(z-1))^{\alpha}Ah^{\alpha})^{-1}x_0$; hence $x(t) = \overline{x}(n) = \mathscr{Z}^{-1}[X(z)](n) = \mathscr{Z}^{-1}[\mathscr{Z}[M](z)](n)\cdot x_0$ is the solution of the considered initial value problem. And then taking initial conditions as vectors from standard basis from $\mathbb{R}^p$ we get the fundamental matrix M, whose transform $\mathscr{Z}[M]$ is given by (39). □

Let $x : (h\mathbb{N})_a \to \mathbb{R}$. Then for $A = \lambda h^{-\alpha}$ we have $({}_a\Delta_h^{\alpha} x)(t) = \lambda h^{-\alpha}x(t+a)$, $x(a) = 1$. Taking into account the image of the scalar Mittag-Leffler function in the $\mathscr{Z}$-transform we get $E_{(\alpha,\alpha)}(\lambda, n) = \mathscr{Z}^{-1}[(z/(z-1))^{\alpha}(1 - (1/z)(z/(z-1))^{\alpha}\lambda)^{-1}](n)$, where $I = 1$ and $Ah^{\alpha} = \lambda$ are substituted into (39). Hence using the power series expansion the scalar Mittag-Leffler function $E_{(\alpha,\alpha)}(\lambda,\cdot)$ can be written as $E_{(\alpha,\alpha)}(\lambda, n) = \mathscr{Z}^{-1}[\sum_{k=0}^{\infty}((\lambda^k/z^k)(z/(z-1))^{k\alpha+\alpha})](n)$.

5. Grünwald-Letnikov-Type Operator

The third type of the operator, which we take under our consideration, is the Grünwald-Letnikov-type fractional h-difference operator, see, for example, [2, 6–14] for cases $h = 1$ and also for general case $h > 0$.

Definition 27. Let $\alpha \in \mathbb{R}$. The *Grünwald-Letnikov-type h-difference operator* ${}_a\widetilde{\Delta}_h^{\alpha}$ *of order* α *for a function* $x : (h\mathbb{N})_a \to \mathbb{R}$ is defined by

$$\left({}_a\widetilde{\Delta}_h^{\alpha} x\right)(t) := \sum_{s=0}^{(t-a)/h} a_s^{(\alpha)} x(t - sh), \tag{40}$$

where

$$a_s^{(\alpha)} = (-1)^s \binom{\alpha}{s}\left(\frac{1}{h^{\alpha}}\right) \quad \text{with} \quad \binom{\alpha}{s} = \begin{cases} 1, & \text{for } s = 0, \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-s+1)}{s!}, & \text{for } s \in \mathbb{N}. \end{cases} \tag{41}$$

Proposition 28. *For* $a \in \mathbb{R}$, $\alpha \in (0,1]$, *let one define* $y(n) := ({}_a\widetilde{\Delta}_h^{\alpha} x)(t)$, *where* $t \in (h\mathbb{N})_a$ *and* $t = a + nh$, $n \in \mathbb{N}_0$. *Then*

$$\mathscr{Z}[y](z) = \left(\frac{hz}{z-1}\right)^{-\alpha} X(z), \tag{42}$$

where $X(z) = \mathscr{Z}[\overline{x}](z)$ *and* $\overline{x}(n) := x(a+nh)$.

Proof. Note that $\mathscr{Z}[y](z) = \mathscr{Z}[{}_a\widetilde{\Delta}_h^{\alpha}\overline{x}](z) = \mathscr{Z}[a_s^{(\alpha)} * \overline{x}][z] = \mathscr{Z}[a_s^{(\alpha)}]\cdot X(z) = (hz/(z-1))^{-\alpha}X(z)$. □

The following comparison has been proven in [4].

Proposition 29. *Let* $a = (\alpha-1)h$. *Then* $\nabla_h({}_a\Delta_h^{-(1-\alpha)} x)(nh) = ({}_0\widetilde{\Delta}_h^{\alpha} y)(nh)$, *where* $y(nh) := x(nh+a)$, *for* $n \in \mathbb{N}_0$, *or* $x(t) = y(t-a)$ *for* $t \in (h\mathbb{N})_a$ *and* $(\nabla_h x)(nh) = (x(nh) - x(nh-h))/h$.

Proposition 30. *Let* $a = (\alpha-1)h$. *Then* $({}_0\widetilde{\Delta}_h^{\alpha} y)(t+h) = ({}_a\Delta_h^{\alpha} x)(t)$, *where* $x(t) = y(t-a)$ *for* $t \in (h\mathbb{N})_a$.

Proof. Let $f(n) := ({}_0\widetilde{\Delta}_h^{\alpha} y)(nh+h)$. Then by Proposition 28 we get $\mathscr{Z}[f](z) = z(hz/(z-1))^{-\alpha}X(z) - z({}_0\widetilde{\Delta}_h^{\alpha} y)(0) = z(hz/(z-1))^{-\alpha}X(z) - zh^{-\alpha}y(0) = z(hz/(z-1))^{-\alpha}\cdot X(z) - zzh^{-\alpha}x(a) = \mathscr{Z}[g]$, where $X(z) = \mathscr{Z}[\overline{x}](z)$, $\overline{x}(n) = y(nh) = x(a+nh)$ and $g(n) := ({}_a\Delta_h^{\alpha} x)(t)$, $t \in (h\mathbb{N})_{a+(1-\alpha)h}$. Hence taking the inverse transform we get the assertion. □

From Proposition 30 we have the possibility of stating exact formulas for solutions of initial value problems with the Grünwald-Letnikov-type difference operator by the comparison with parallel problems with the Riemann-Liouville-type operator. And we can use the same formula for the $\mathscr{Z}$-transform of $f(n) := ({}_0\widetilde{\Delta}_h^{\alpha} y)(nh+h)$ as for the Riemann-Liouville-type operator. Then many things could be easily done for systems with the Grünwald-Letnikov-type difference operator.

The next proposition is proved in [36] for the case $h = 1$.

Proposition 31. *The linear initial value problem,*

$$\left({}_0\widetilde{\Delta}^{\alpha} y\right)(t+h) = Ay(t), \quad t \in (h\mathbb{N})_0, \qquad y(0) = y_0, \quad y_0 \in \mathbb{R}^p, \tag{43}$$

has the unique solution given by the formula $y(t) = \Phi(t)y_0$ *for any* $t \in (h\mathbb{N})_0$ *and* $n \times n$ *dimensional state transition matrices* $\Phi(t)$ *are determined by the recurrence formula* $\Phi(t+h) = (Ah^{\alpha} + I\alpha)\Phi(t)$ *with* $\Phi(0) = I$.

Taking into account Propositions 25 and 30 we can state the following result.

Proposition 32. *The linear initial value problem,*

$$\left({}_0\widetilde{\Delta}_h^{\alpha} y\right)(t+h) = Ay(t), \quad t \in (h\mathbb{N})_0, \qquad y(0) = y_0, \quad y_0 \in \mathbb{R}^p, \tag{44}$$

has the unique solution given by the formula

$$y(t) = E_{(\alpha,\alpha)}\left(Ah^{\alpha}, \frac{t}{h}\right)y_0, \tag{45}$$

where $t \in (h\mathbb{N})_0$.

As a simple conclusion we have that

$$\Phi(nh) = \mathcal{Z}^{-1}\left[\left(\frac{z}{z-1}\right)^{\alpha}\left(I - \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha} Ah^{\alpha}\right)^{-1}\right](n). \tag{46}$$

6. Comparisons and Examples

Here we present some comparisons for the $\mathcal{Z}$-transforms of Caputo- and Riemann-Liouville-type operators as the third one used in systems with the left shift gives the same solutions as the Riemann-Liouville-type operator. Then we present the example with mixed operators in two equations. In this chapter we also state the formula in the complex domain of the solution to the initial value problem for semilinear systems like for control systems.

Proposition 33. *Let* $y(n) := (({}_a\Delta_h^{\alpha} - {}_a\Delta_{h,*}^{\alpha})x)(t)$, *where* $t = a + (1-\alpha)h + nh$. *Then*

$$\mathcal{Z}[y](z) = h^{-\alpha} z\left(\left(\frac{z}{z-1}\right)^{1-\alpha} - 1\right)x(a). \tag{47}$$

Proof. It is enough to compare Propositions 16 and 24. We can also equivalently take under consideration (31) in the real domain and observe that $\mathcal{Z}[y](z) = h^{-\alpha}\mathcal{Z}\left[\binom{n-\alpha+1}{-\alpha}\right](z)x(a) = h^{-\alpha}\left(z\sum_{s=0}^{\infty}(-1)^s\binom{\alpha-1}{s}z^{-s} - z\right)x(a) = h^{-\alpha}z((z/(z-1))^{1-\alpha} - 1)x(a)$. □

Comparing the $\mathcal{Z}$-transform of two types of the Mittag-Leffler functions we get the following relation.

Proposition 34. *Let* $\alpha \in (0,1]$, $n \in \mathbb{N}_0$. *Then*

$$E_{(\alpha,\alpha)}(A,n) = \left(E_{(\alpha)}(A,\cdot) * \widetilde{\varphi}_{\alpha-1}\right)(n). \tag{48}$$

Proof. In fact if we take the $\mathcal{Z}$-transform of the right side, then we have

$$\begin{aligned}
&\mathcal{Z}\left[E_{(\alpha)}(A,\cdot) * \widetilde{\varphi}_{\alpha-1}\right](z) \\
&= \mathcal{Z}\left[E_{(\alpha)}(A,\cdot)\right](z)\,\mathcal{Z}\left[\widetilde{\varphi}_{\alpha-1}\right](z) \\
&= \frac{z}{z-1}\left(I - \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha} Ah^{\alpha}\right)^{-1}\left(\frac{z}{z-1}\right)^{\alpha-1} \\
&= \left(\frac{z}{z-1}\right)^{\alpha}\left(I - \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha} Ah^{\alpha}\right)^{-1}.
\end{aligned} \tag{49}$$

Using formula (39) we get $\mathcal{Z}[E_{(\alpha)}(A,\cdot) * \widetilde{\varphi}_{\alpha-1}](z) = \mathcal{Z}[E_{(\alpha,\alpha)}(A,\cdot)](z)$ and consequently, we reach the assertion. □

In the following example we consider control systems with the Caputo- and Riemann-Liouville-type operators. We get the same transfer function for initial condition $x(a) = 0$.

Example 35. Let $\alpha \in (0,1]$, $h = 1$, and $a = \alpha - 1$. Let us consider linear initial value problems of the following forms:

(a) for the Caputo-type operator:

$$\begin{aligned}
\left({}_a\Delta_*^{\alpha} x\right)(n) &= Ax(a+n) + Bu(n), \\
y(n) &= Cx(a+n) + Du(n),
\end{aligned} \tag{50}$$

(b) for the Riemann-Liouville-type operator:

$$\begin{aligned}
\left({}_a\Delta^{\alpha} x\right)(n) &= Ax(a+n) + Bu(n), \\
y(n) &= Cx(a+n) + Du(n)
\end{aligned} \tag{51}$$

with initial condition $x(a) = 0$, and constant matrices $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times m}$, $C \in \mathbb{R}^{q\times p}$, $D \in \mathbb{R}^{q\times m}$.

We solve the problems using the $\mathcal{Z}$-transform. Since $x(a) = 0$, then applying the $\mathcal{Z}$-transform to the first equations in both systems, we get the following equation: $(z/(z-1))^{1-\alpha}(z-1)X(z) = AX(z) + BU(z)$. Then $X(z) = (1/(z-1)) \cdot (z/(z-1))^{\alpha-1} \cdot (I - (1/(z-1))(z/(z-1))^{\alpha-1}A)^{-1}BU(z) = (1/z) \cdot (z/(z-1))^{\alpha} \cdot (I - (1/z)(z/(z-1))^{\alpha}A)^{-1}BU(z)$, where $X = \mathcal{Z}[\overline{x}]$, $\overline{x}(n) = x(a+n)$, and $U = \mathcal{Z}[u]$. Consequently, for the output of the considered systems in the complex domain, we have the following formula: $Y(Z) = (1/z)(z/(z-1))^{\alpha}C(I - (1/z)(z/(z-1))^{\alpha}A)^{-1}BU(z) + DU(z)$, where $Y = \mathcal{Z}[y]$. Then the solution for the output in the real domain is as follows: $y(n) = C(f*u)(n) + Du(n)$, where $f(n) = E_{(\alpha,\alpha)}(A, n-1)B$. We can easily check the value at the beginning that $y(0) = Du(0)$, as $E_{(\alpha,\alpha)}(-1) = 0$.

7. Towards Stability Analysis

Let us introduce the stability notions for fractional difference systems involving the Caputo-, Riemann-Liouville-, and Grünwald-Letnikov-type operators.

Firstly, let us recall that the constant vector x_{eq} is an *equilibrium point* of fractional difference system (21) if and only if $({}_a\Delta_*^{\alpha} x_{\mathrm{eq}})(t) = f(t, x_{\mathrm{eq}}) = 0$ (and $({}_a\Delta^{\alpha_i} x_{\mathrm{eq}})(t) = f(t, x_{\mathrm{eq}})$ in the case of the Riemann-Liouville difference systems of the form (35)) for all $t \in (h\mathbb{N})_0$.

Assume that the trivial solution $x \equiv 0$ is an equilibrium point of fractional difference system (21) (or (35)). Note that there is no loss of generality in doing so because any equilibrium point can be shifted to the origin via a change of variables.

Definition 36. The equilibrium point $x_{\mathrm{eq}} = 0$ of (21) (or (35)) is said to be

(a) *stable* if, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|x(a)\| < \delta$ implies $\|x(a+kh)\| < \epsilon$, for all $k \in \mathbb{N}_0$,

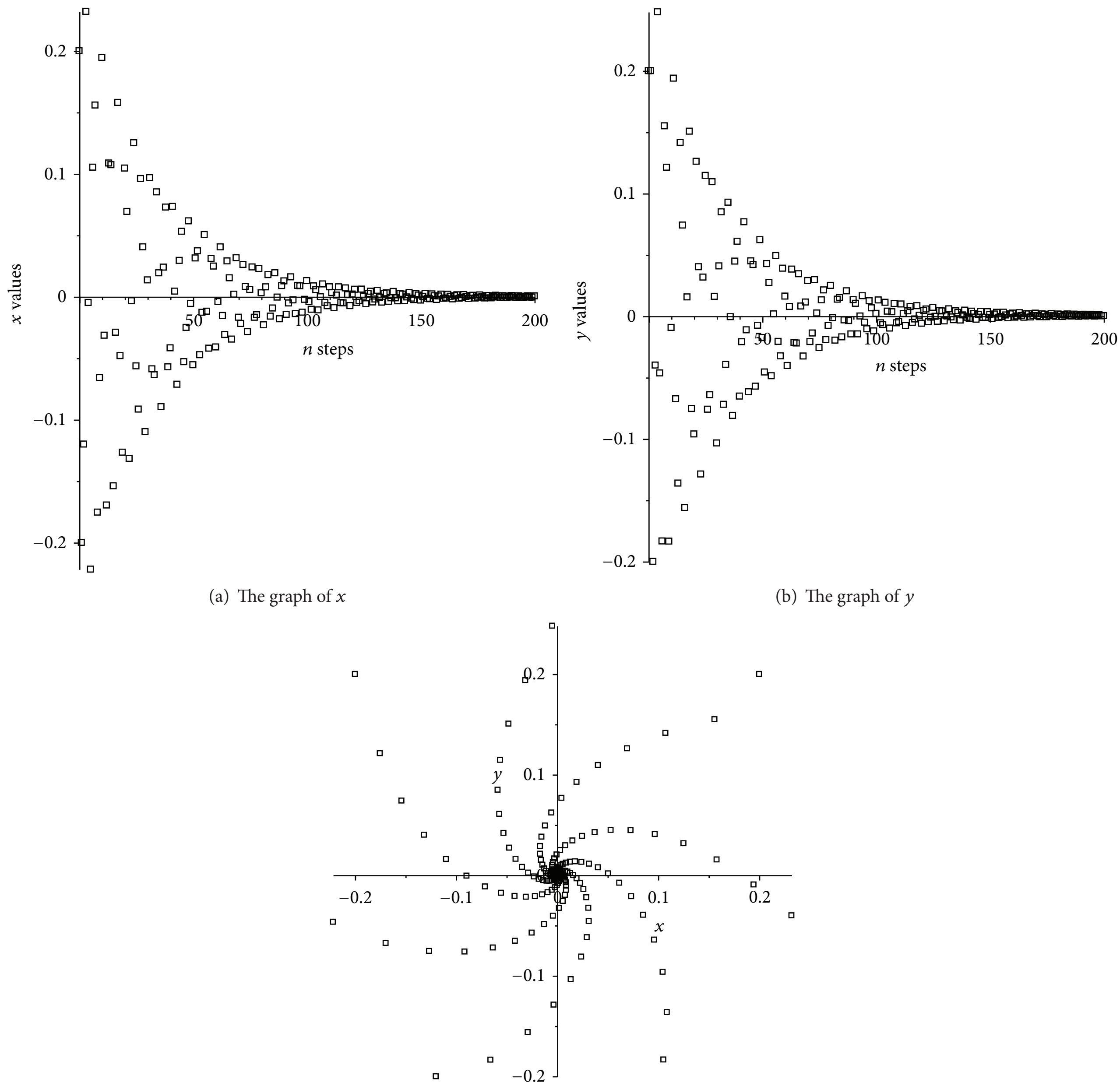
(b) *attractive* if there exists $\delta > 0$ such that $\|x(a)\| < \delta$ implies

$$\lim_{k\to\infty} x(a+kh) = 0, \tag{52}$$

(iv) *asymptotically stable* if it is stable and attractive.

The fractional difference systems (21), (35) are called *stable/asymptotically stable* if their equilibrium point $x_{\mathrm{eq}} = 0$ is stable/asymptotically stable.

Example 37. Let us consider systems (23a)-(23b) or (37a)-(37b) but with $A = 0$. Then for system (23a)-(23b) we have that $x(a+nh) = \varphi_{0,\alpha}^{*}(n)x_0 = x_0$, while for system (37a)-(37b)



(a) The graph of x

(b) The graph of y

(c) Phase trajectory

Figure 1: The solution of the Caputo-type fractional-order difference initial value problem for linear system (21) with the initial condition $x(0) = 0.2$, $y(0) = 0.2$, where the matrix A is given in Example 40 and $\alpha = 0.8$, $h = 0.5$. In this case the roots of (53) are strictly inside the unit circle.

the solution satisfies the following $\lim_{n\to\infty} x(a + nh) = \lim_{n\to\infty} \varphi_{0,\alpha}(n)x_0 = 0$. Hence, then systems with the Caputo-type operator are stable and systems with the Riemann-Liouville operator are attractive hence asymptotically stable.

Proposition 38. *Let R be the set of all roots of the equation*

$$\det\left(I - \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha} Ah^{\alpha}\right) = 0. \quad (53)$$

Then the following items are satisfied.

(a) *If all elements from R are strictly inside the unit circle, then systems (21) and (35) are asymptotically stable.*

(b) *If there is $z \in R$ such that $|z| > 1$, both systems (21) and (35) are not stable.*

Proof. The main tool in the proof is Proposition 9 and the method of decomposition of the matrix of systems into Jordan's blocks. Some basic tool is similar to those used in [30]. However in the proof we do not use the same tools as in the cited paper. Only the beginning could be read as similar. The $\mathscr{Z}$-transform of $\overline{x}(\cdot)$ is $X(z) = (z/(z-1))^{\beta}(I - (1/z)(z/(z-1))^{\alpha}Ah^{\alpha})^{-1}x_0 = P^{-1}(z/(z-1))^{\beta}(I - (1/z)(z/(z-1))^{\alpha}Jh^{\alpha})^{-1}Px_0$, where $\beta = 1$ for systems with the Caputo-type operator and $\beta = \alpha$ for systems with the Riemann-Liouville-type operator. Moreover the condition from Proposition 9 that $\beta < \alpha + 1$ is satisfied. Let $A = PJP^{-1}$, where P is invertible and $J = \operatorname{diag}(J_1, \dots, J_s)$ and J_l are Jordan's blocks of order $r_l, l = 1, \dots, s$. Let $k_i \in \mathbb{N}_1$ be algebraic multiplicities of eigenvalues $\lambda_i \in \operatorname{Spec}(A)$; let $p_i \in \mathbb{N}_1$ be their geometric multiplicities, $i = 1, \dots, m$. The matrix $w^{-1}(J) = (I - (1/z)(z/(z-1))^{\alpha} J h^{\alpha})^{-1}$ is a diagonal block matrix. The number of blocks corresponding to λ_i is p_i and their form is given by the upper triangular matrix

$$\begin{pmatrix} w^{-1}(\lambda_i) & w^{-2}(\lambda_i) & \dots & w^{-r_q}(\lambda_i) \\ 0 & w^{-1}(\lambda_i) & \dots & w^{-r_q+1}(\lambda_i) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w^{-1}(\lambda_i) \end{pmatrix}, \quad (54)$$

where $q = 1, \dots, p_i$ and r_q is the size of the block. Because $\det(w(A)) = \det(w(J)) = \prod_{i=1}^{m} w^{k_i}(\lambda_i)$ and $X(z)$ is formed by a linear combination of convolutions of $\widetilde{\varphi}_{\beta}$ with the rational functions from (54), R is the set of all poles of $(z-1)X(z)$ as $\alpha - \beta + 1 > 0$.

(a) If all elements from R are strictly inside the unit circle, then $\lim_{n \to \infty} \overline{x}(n) = \lim_{z \to 1}(z-1)X(z) = 0$.

(b) If there is an element from R that $|z| > 1$, then $\limsup_{n \to \infty} |x(a+n)| = \infty$ and systems (21) and (35) are not stable. □

Now we can easily derive some basic corollary.

Corollary 39. *If there is* $\lambda \in \operatorname{Spec}(A)$, *such that* $|\lambda| > (2/h)^{\alpha}$, *then system (21) is not asymptotically stable. Or equivalently, if system (21) is asymptotically stable, then* $|\lambda| \le (2/h)^{\alpha}$ *for all* $\lambda \in \operatorname{Spec}(A)$.

Example 40. Let us consider the family of planar systems with order $\alpha \in (0, 1]$ described by the matrix $A = \left[\begin{smallmatrix} -2^{\alpha} & -2^{\alpha} \\ 2^{\alpha} & -2^{\alpha} \end{smallmatrix}\right]$. Observe that $\det A = 2^{2\alpha+1} \neq 0$ so the considered planar systems have the unique equilibrium point $x_e = 0$. Since condition (53) does not depend on the type of an operator, we take into account systems (21) with the Caputo-type difference operator. In our case we get for (i) $\alpha = 0.8$, $h = 0.5$ that elements from R are inside the unit circle and the system is stable, trajectories are presented at Figure 1 but for (ii) $\alpha = 0.5$, $h = 0.5$ we get that the system is unstable, see Figure 2. In this case we can have stability for smaller step h. For the given matrix A and α, the stability of the systems depends also on the length of the step. It is easy to see that $\lambda_1 = -2^{\alpha} - 2^{\alpha} i$, $\lambda_2 = -2^{\alpha} + 2^{\alpha} i \in \operatorname{Spec}(A)$ and $|\lambda_j| = 2^{\alpha+0.5}$, $j = 1, 2$. Then for $h = 0.5$ we have $(2/h)^{\alpha} = 2^{2\alpha}$ and consequently, if $h = 0.5$ and $\alpha < 0.5$, then by Corollary 39 system (21) is not asymptotically stable. Additionally, one can see that for $\alpha = 0.5$ and $h = 0.5$ we get $|\lambda_j| = (2/h)^{\alpha} = 2$ and the equilibrium point is not stable. Therefore, the condition $|\lambda| \le (2/h)^{\alpha}$ given for eigenvalues of the matrix describing the system is not sufficient for asymptotic stability of the system.

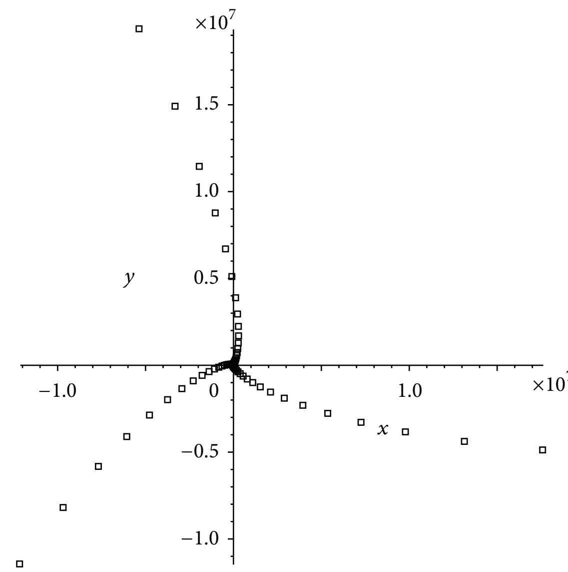


Figure 2: The solution of the Caputo-type fractional-order difference initial value problem for linear system (21) with the initial condition $x(0) = 0.2$, $y(0) = 0.2$, where the matrix A is given in Example 40 and $\alpha = 0.5$, $h = 0.5$. In this case the roots of (53) are outside the unit circle.

8. Conclusions

The Caputo-, Riemann-Liouville-, and Grünwald-Letnikov-type fractional-order difference initial value problems for linear systems are discussed. We stress the formula for the $\mathcal{Z}$-transform of the discrete Mittag-Leffler matrix function. Additionally, the stability problem of the considered systems is studied. The next natural step is to consider the semilinear control systems and develop the controllability property via the complex domain.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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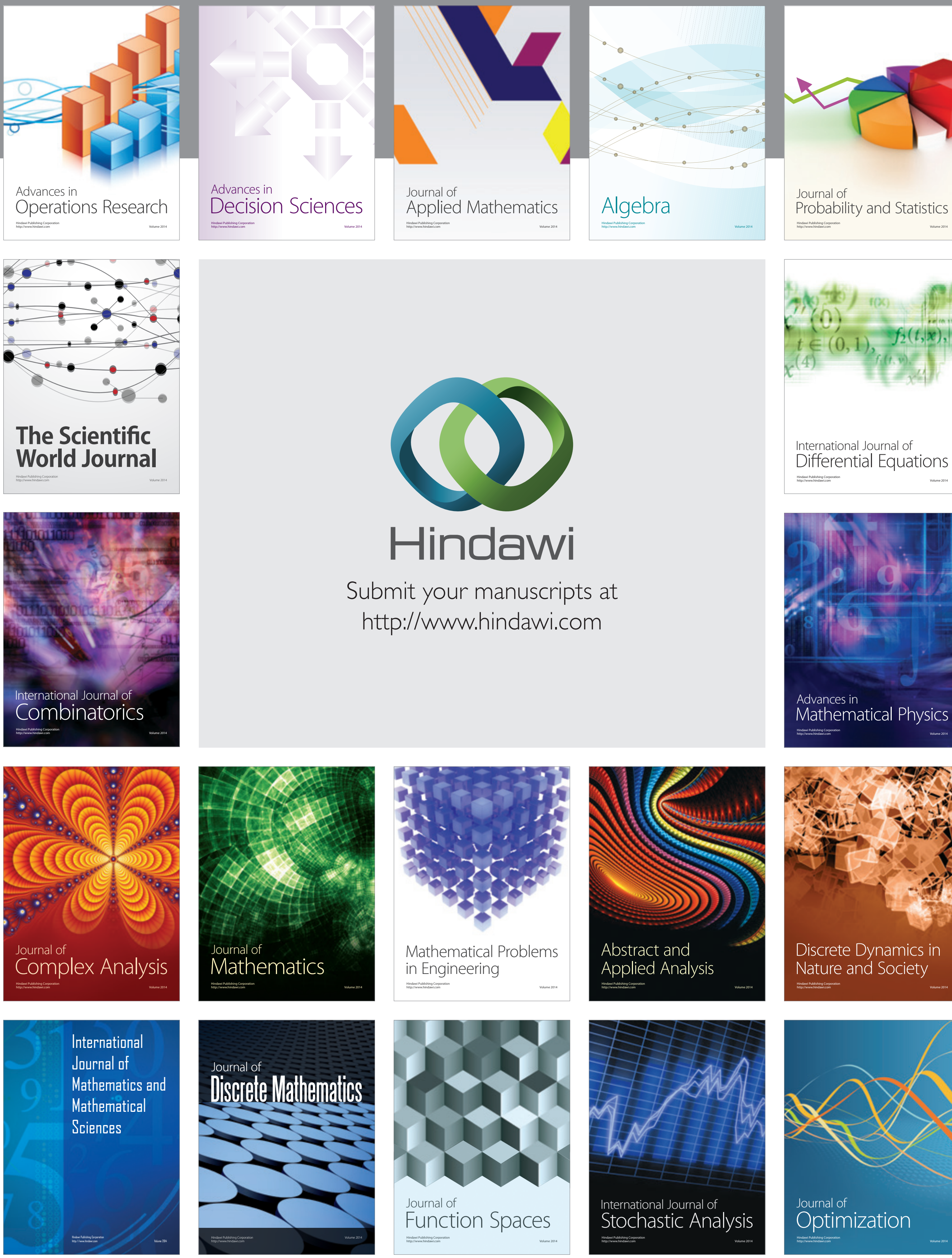
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