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Modeling of discrete-time fractional-order state space systems using the balanced truncation method

Rafał Stanisławski*, Marek Rydel, Krzysztof J. Latawiec

Department of Electrical, Control and Computer Engineering, Opole University of Technology, ul. Prószkowska 76, 45-758 Opole, Opole, Poland

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Abstract

This paper presents a new approach to approximation of linear time-invariant (LTI) discrete-time fractional-order state space SISO systems by means of the SVD-originated balanced truncation (BT) method applied to an FIR-based representation of the fractional-order system. This specific representation of the system enables to introduce simple, analytical formulas for determination of the Cholesky factorizations of the controllability and observability Gramians, which contributes to significant improvement of the computational efficiency of the BT method. As a model reduction result for the fractional-order systems we obtain a low-order rational (integer-order) state space system. Simulation experiments show a high efficiency of the introduced methodology both in terms of the approximation accuracy of the model and low time complexity of the approximation algorithm.

Keywords: Fractional-order system; Discrete-time system; Model order reduction; Balanced truncation method.

1. Introduction

Fractional-order dynamic systems have attracted a considerable research interest. It is the specific properties of fractional-order models that make them more adequate in modeling of selected (not only) industrial systems. However, the discrete-time Grünwald–Letnikov fractional-

* Corresponding author.
E-mail addresses: r.stanislawski@po.opole.pl (R. Stanisławski), m.rydel@po.opole.pl (M. Rydel), k.latawiec@po.opole.pl (K.J. Latawiec).

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order difference (FD), to be considered here, may lead to computational explosion. Therefore, a number of various concepts have been developed to approximate, or model discrete-time fractional difference systems. Those concepts are mainly based on two types of applications, where 1) approximators are used to model the discrete-time fractional-order difference involved in a fractional-order system and 2) the whole fractional-order system is modeled by a rational, integer-order approximator.

In the first case, the solutions have led to e.g. least-squares (LS) fit of an impulse/step response of a discrete-time integer-order IIR filter [3,5,29]. On the other hand, a fit of a FIR filter to FD has been analyzed in the frequency and time domains [11,12,25], also in terms of time-varying filters [23]. An alternative approach has been the employment of an approximating filter incorporating discrete-time Laguerre functions [24]. In this case, Laguerre-based equivalents to the fractional-order difference, called Laguerre-based difference, have been proposed [24]. The approximation of the fractional-order difference is obtained as a finite-length implementation of the Laguerre-based difference, called finite Laguerre-based difference. Another approach to use the Laguerre filters has been proposed in [14].

In the second case, there are a number of methods for approximation of fractional order systems by integer, high order approximators [9,20,21]. One of the conceptually simplest, but computationally involving, approximation methods for fractional-order systems is the involvement of both approximation of the fractional-order difference and system performance in a high-order state space model, by use of the so-called 'expanded' state equation [6,15]. In this case, the fractional-order difference is calculated using the finite-length implementation of the Grünwald–Letnikov difference. In [22] it has been shown that the expanded state equation model can be effectively reduced to a low-order approximator using the balanced truncation (BT) method, which however suffers from computational burden. A similar application of model order reduction methods for fractional-order continuous-time systems under the Oustaloup approximation of the fractional-order derivative has been presented in [13].

This paper presents a new method for approximation of discrete-time fractional-order state space systems using the balanced truncation applied to a specific, FIR-based representation of a fractional-order system. The paper is organized as follows. Having introduced the approximation problem for discrete-time fractional-order systems in Section 1, a representation method for discrete-time fractional-order state space systems is presented in Section 2. Section 3 recalls the model order reduction problem for discrete-time state space systems via the Balanced Truncation method. The main result in terms of a simple, analytical solution method for the Gramian factorizations used in the BT algorithm is presented in Section 4. A simulation example of Section 5 confirms the effectiveness of the introduced methodology both in terms of high modeling accuracy and low time complexity of the approximation algorithm. Conclusions of Section 6 complete the paper.

2. System representation

Consider a commensurate fractional-order discrete-time state space single input single output (SISO) system

$$\begin{aligned}\Delta^{\alpha} x(t+1) &= Ax(t) + Bu(t), \quad x_0 \\ y(t) &= Cx(t) + Du(t)\end{aligned} \tag{1}$$

where $t = 0, 1, \ldots,$ $x(t) \in \Re^n$ is the state vector, $u(t)$ and $y(t)$ are input and output signals, respectively, $A \in \Re^{n\times n}$, $B \in \Re^{n\times 1}$, $C \in \Re^{1\times n}$ and $D \in \Re$.

Remark 1. It is well known that for state-space systems to be controlled there is $D = 0$. We still continue with $D \neq 0$ in order to retain the generality of the state-space model, being valid for the state-space controller description as well.

Fractional order difference $\Delta^\alpha x(t+1)$ can be represented by the well known Grünwald–Letnikov fractional-order difference (FD) [15,23]

$$\Delta^\alpha x(t+1) = \sum_{j=0}^{t+1} (-1)^j \binom{\alpha}{j} x(t-j+1) \qquad t = 0, 1, \ldots \tag{2}$$

with

$$\binom{\alpha}{j} = \begin{cases} 1 & j = 0 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & j > 0 \end{cases}$$

The main problem encountered in implementation of the fractional-order difference is that the sum is calculated from 0 to $t+1$, so each incoming sample increases a complication of Eq. (2), leading to computational explosion for $t \to \infty$. The same problem occurs when implementing e.g. Tustin [28] and Al-Alaoui-based schemes [1], as well as some other methods [24]. Clearly, finite-length expansions must be used when implementing fractional-order differences. Specifically, the finite fractional difference (FFD) can be used as an approximator to FD [15,19,23]

$$\Delta^\alpha x(t+1) \cong \sum_{j=0}^{\underline{L}} (-1)^j \binom{\alpha}{j} x(t-j+1) \qquad t = 0, 1, \ldots \tag{3}$$

with $x(l) = 0 \; \forall \; l < 0$ and the implementation length $\underline{L}$ assuming 'sufficiently' high values.

In order to set up the FD approximation problem in the state-space model reduction framework we recall the 'expanded' rational (integer-order) state space model of the fractional-order system of Eq. (1) [6,15]

$$\begin{aligned} \tilde{x}(t+1) &= \tilde{A}\tilde{x}(t) + \tilde{B}u(t), \\ \tilde{y}(t) &= \tilde{C}\tilde{x}(t) + Du(t) \end{aligned} \tag{4}$$

obtained by combination of Eqs. (1) and (3). Unfortunately, the dimensions of the modeling problem are very high, that is $\tilde{A} \in \Re^{L\times L}$, $\tilde{B} \in \Re^{L\times 1}$, $\tilde{C} \in \Re^{1\times L}$ with $L = n\underline{L}$. Even though we have developed a nice model reduction scheme for the problem [22], the computational effort was rather high.

In this paper we introduce a new, computationally sound 'L-expanded' (integer-order) state space model of Eq. (1) in form of Eq. (4). Additionally, we offer a new *analytical* method for computation of the Cholesky factorizations of the controllability and obervability Gramians, thus making our new model reduction approach for fractional-order state space systems very effective.

3. Model order reduction for discrete-time state space systems

Consider a discrete-time state space dynamical system of integer-order described by Eq. (4). Then the model order reduction aims to approximate this system by a model of a lower order ($k \ll L$)

$$\begin{aligned}\tilde{x}_r(t+1) &= \tilde{A}_r\tilde{x}_r(t)+\tilde{B}_r u(t)\\ \tilde{y}_r(t) &= \tilde{C}_r\tilde{x}_r(t)+Du(t)\end{aligned} \tag{5}$$

with $\tilde{A}_r \in \Re^{k\times k}$, $\tilde{B}_r \in \Re^{k\times 1}$ and $\tilde{C}_r \in \Re^{1\times k}$.

The reduction of a model is not a unique operation and there are several techniques for model order reduction. Among the reduction methods, great attention has been given to the polynomial approximations [7], Krylov-based methods [2,4,8] and SVD-based methods [2,10,16,17]. One of the most popular algorithms is the Balanced Truncation (BT) method [2,16,31] which is an SVD-based method. BT is well-known to yield a good system approximation, which additionally allows to '*a priori*' assess an error bound for the reduced model [2,10,16], however it requires a high computational effort.

The BT method relies on the concept of Balanced Realization and determination of a dominant part of the model. The states that are difficult to reach, i.e. require a large amount of energy to reach, and are difficult to observe, i.e. yield small amounts of observation energy, correspond to low eigenvalues of the controllability and observability Gramians. On this basis a reduced-order model can be obtained by eliminating states which are difficult to reach and are simultaneously difficult to observe by 'cutting off' the low-reach and low-energy submatrices in the state space model.

The controllability and observability Gramians, W_c and W_o respectively, for discrete–time dynamical systems are usually obtained as the solutions of two Lyapunov equations

$$\tilde{A}W_c\tilde{A}^T - W_c + \tilde{B}\tilde{B}^T = 0 \tag{6}$$

$$\tilde{A}^T W_o\tilde{A} - W_o + \tilde{C}^T\tilde{C} = 0 \tag{7}$$

where $\tilde{A} \in \Re^{L\times L}$, $\tilde{B} \in \Re^{L\times 1}$ and $\tilde{C} \in \Re^{1\times L}$ are the system matrices of Eq. (4).

In order to balance the model, the linear state transformation $x \to Tx$ has to be determined. As a result, the controllability and observability Gramians of the balanced model are diagonal matrices, with decreasing Hankel singular values at the main diagonal

$$TW_cT^T = \left(T^T\right)^{-1}W_oT^{-1} = \Sigma = diag(\sigma_i) \tag{8}$$

where σ_i, $i = 1, \ldots, L$, are called the Hankel singular values of the system.

The balanced model ($\overline{A} = T\tilde{A}T^{-1}$, $\overline{B} = T\tilde{B}$, $\overline{C} = \tilde{C}T^{-1}$) can be easily partitioned to the 'dominant' (order k) and 'weak' (order $L-k$) subsystems as

$$\begin{aligned}\begin{bmatrix}\overline{x}_1(t+1)\\ \overline{x}_2(t+1)\end{bmatrix} &= \begin{bmatrix}\overline{A}_{11} & \overline{A}_{12}\\ \overline{A}_{21} & \overline{A}_{22}\end{bmatrix}\begin{bmatrix}\overline{x}_1(t)\\ \overline{x}_2(t)\end{bmatrix} + \begin{bmatrix}\overline{B}_1\\ \overline{B}_2\end{bmatrix}u(t)\\ \tilde{y} &= \begin{bmatrix}\overline{C}_1 & \overline{C}_2\end{bmatrix}\begin{bmatrix}\overline{x}_1(t)\\ \overline{x}_2(t)\end{bmatrix} + D\,u(t)\end{aligned} \tag{9}$$

The reduced model is obtained by elimination of the 'weak' subsystem from the balanced model of Eq. (9)

$$\tilde{A}_r = \overline{A}_{11} \quad \tilde{B}_r = \overline{B}_1 \quad \tilde{C}_r = \overline{C}_1 \tag{10}$$

The transformation matrix is not unique and there are several algorithms to determine the matrix T, which in theory give identical results, however in practice lead to different numerical properties [2,27]. Balancing the whole high-order complex model may turn out to

Table 1
(a) square root and (b) balancing-free square root algorithms

1a/1b) Compute a Cholesky factorization of controllability and observability Gramians S and R such that:

$$W_c = S^T S \quad W_o = R^T R$$

2a/2b) Compute the SVD decomposition:

$$SR^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

where $\Sigma_1 = diag(\sigma_1, \ldots, \sigma_k)$, $\Sigma_2 = diag(\sigma_{k+1}, \ldots, \sigma_L)$,
$\sigma_1 \geq \ldots \geq \sigma_k > \sigma_{k+1} \geq \ldots \geq \sigma_L > 0$

3a) Create transformation matrix and its right inverse:

$$T = \Sigma_1^{-\frac{1}{2}} U_1^T R \quad T^{\#} = S^T V_1 \Sigma_1^{-\frac{1}{2}}$$

3b) Compute the QR decompositions:

$$S^T U_1 = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} X \quad R^T V_1 = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} Y$$

4b) Compute the SVD decompositions:

$$Z_1^T W_1 = U_E \Sigma_E V_E^T$$

5b) Create transformation matrix and its right inverse:

$$T = \Sigma_E^{-\frac{1}{2}} U_E^T Z_1^T \quad T^{\#} = W_1 V_E \Sigma_E^{-\frac{1}{2}}$$

be numerically ill-conditioned, especially for large systems due to the usually rapid decay of the Hankel singular values σ_i. For complex models, the transformation matrix often has properties similar to a singular matrix and it is important to avoid formulas involving matrix inverses in the algorithm. Therefore, robust numerical algorithms calculate simultaneously the rectangular transformation matrix $T \in \Re^{k \times L}$ and its (nonunique) right inverse $T^{\#} \in \Re^{L \times k}$ such that $TT^{\#} = I$. This operation allows to balance and reduce the high-order model at the same time. In this case the reduced model is finally obtained as

$$\tilde{A}_r = T\tilde{A}T^{\#} \quad \tilde{B}_r = T\tilde{B} \quad \tilde{C}_r = \tilde{C}T^{\#} \tag{11}$$

One of the simplest methods for determining the transformation matrix is the square root (SR) algorithm [2] presented in steps 1a to 3a in Table 1. Another, more numerically robust and accurate method, but more computationally complicated is the balancing-free square root (BFSR) algorithm [2,26,27], see steps 1b to 5b in Table 1. Note that both SR and BFSR algorithms require obtaining the Cholesky factorizations for both controllability and observability Gramians. Square-root solvers for discrete-time Lyapunov equations are widely available [18], however the execution could be very time consuming or even fail due to the RAM shortages for equations of very high order. It is worth emphasizing that selection of the Cholesky factorization for the Gramians is the most time-consuming operation for the presented algorithms. For these reasons the SVD-based methods are not usually used for models with a number of state variables $L > 10^4$. For models of higher complexity there can be employed algorithms that compute approximate solutions $W_c \approx \hat{W}_c = \hat{S}^T \hat{S}$ and $W_o \approx \hat{W}_o = \hat{R}^T \hat{R}$, e.g. the alternating directions implicit (ADI) iteration methods [17,32]. Our contribution is substantial in that we eliminate the necessity of either time-consuming solution of the Lyapunov equations or using the approximate solutions to W_c and W_o. Our analytical solution offers essential savings in computational complexity of the model reduction process.

4. Main result

As mentioned above, selection of the Cholesky factorizations for the controllability and observability Gramians, S and R, respectively, has a crucial role in system approximation by means of the SVD-based order reduction methods. Here we present the main results of the paper, which is a new, simple method for 1) FIR-based representation of fractional-order system (1) as an integer-order state space system (4) and 2) *analytical* selection of S and R. We will call our combination of the FIR representation and the BT algorithm as the FIR-BT methodology.

Lemma 1. *Consider a discrete-time fractional-order state space system as in Eq. (1) with the Grünwald–Letnikov fractional difference (FD) as in Eq. (2). For the FFD approximator to FD, described by Eq. (3), the system of Eq. (1) can be modeled by Eq. (4), with* $\tilde{A} \in \Re^{L\times L}$, $\tilde{B} \in \Re^{L\times 1}$ *and* $\tilde{C} \in \Re^{1\times L}$ *calculated as*

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} \tag{12}$$

$$\tilde{B} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T \tag{13}$$

$$\tilde{C} = C\begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_L \end{bmatrix} \tag{14}$$

with the elements $\phi_i \in \Re^{n\times 1}$, $i = 1, \dots, L$, *calculated recursively*

$$\phi_i = \begin{cases} B & i = 1 \\ A_\alpha \phi_{i-1} - \sum_{j=2}^{i-1} (-1)^j \binom{\alpha}{j} \phi_{i-j} & i = 2, \dots, L \end{cases} \tag{15}$$

where $A_\alpha = A + I\alpha$ *and the matrices* A, B, C *and* D *as in Eq. (1).*

Proof. It is well known that a (stable) discrete-time LTI SISO system can be described by the Laurent expansion as

$$y(t) = \sum_{i=0}^{\infty} \underline{\phi}_i q^{-i} u(t) \tag{16}$$

where $\underline{\phi}_i$, $i = 1, 2, \dots,$ denotes the infinite impulse response (IIR) of the system. The finite impulse response (FIR) is now

$$\tilde{y}(t) = \sum_{i=0}^{L} \underline{\phi}_i q^{-i} u(t) \tag{17}$$

where L is the number of FIR components. On the other hand, combining Eqs. (2) and (3) we can model the state equation of the fractional-order system as

$$x(t+1) = A_\alpha x(t) + Bu(t) - \sum_{j=2}^{L} (-1)^j \binom{\alpha}{j} x(t-j+1) \tag{18}$$

with $A_\alpha = A + I\alpha$. Taking into account the state equation of Eq. (18) and the output equation as in Eq. (1), the finite impulse response parameters $\underline{\phi}_i$, $i = 0, 1, \ldots, L$, of the system are as follows

$$\underline{\phi}_i = \begin{cases} D & i = 0 \\ C\phi_i & i = 1, \ldots, L \end{cases} \tag{19}$$

where ϕ_i, $i = 1, \ldots, L$, can be calculated in a recurrent way as follows

$$\begin{aligned}
\phi_1 &= B \\
\phi_2 &= A_\alpha \phi_1 \\
\phi_3 &= A_\alpha \phi_2 - \binom{\alpha}{2}\phi_1 \\
&\vdots \\
\phi_i &= A_\alpha \phi_{i-1} - \sum_{j=2}^{i-1} (-1)^j \binom{\alpha}{j} \phi_{i-j} \\
&\vdots \\
\phi_L &= A_\alpha \phi_{L-1} - \sum_{j=2}^{L-1} (-1)^j \binom{\alpha}{j} \phi_{L-j}
\end{aligned}$$

It is well known that the FIR model as in Eq. (17) can be presented in the state space form of Eq. (4), with $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ as in (12), (13) and (14), respectively, see e.g. [30]. □

Lemma 1 leads to formulation of a simple analytical method for selection of the Cholesky factorizations of the controlability and observability Gramians for the underlying system.

Theorem 1. *Consider a discrete-time fractional-order state space system as in Eq. (1) described by the integer-order state space model introduced in Lemma 1. Then the controllability and observability Gramian factorizations $S \in \Re^{L\times L}$ and $R \in \Re^{L\times L}$, respectively, can be selected in an analytical way as*

$$S = I_{L\times L} \tag{20}$$

$$R = \begin{bmatrix} C\phi_L & 0 & 0 & \ldots & 0 \\ C\phi_{L-1} & C\phi_L & 0 & \ldots & 0 \\ C\phi_{L-2} & C\phi_{L-1} & C\phi_L & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C\phi_1 & C\phi_2 & C\phi_3 & \ldots & C\phi_L \end{bmatrix} \tag{21}$$

where ϕ_i, $i = 1, \ldots, L$, are as in Eq. (15) and C as in Eq. (1).

Proof. The controllability Gramian for the state space system introduced in Lemma 1 is calculated by solving the discrete-time Lyapunov Eq. (6). Therefore, the coefficients of the matrix W_c are solving the set of equations

$$\begin{bmatrix} w_{11}^c & w_{12}^c & w_{13}^c & \dots & w_{1L}^c \\ w_{21}^c & w_{22}^c - w_{11}^c & w_{23}^c - w_{12}^c & \dots & w_{2L}^c - w_{1(L-1)}^c \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{L1}^c & w_{L2}^c - w_{(L-1)1}^c & w_{L3}^c - w_{(L-1)2}^c & \dots & w_{LL}^c - w_{(L-1)(L-1)}^c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{22}$$

Finally, taking into account that the solution of Eq. (22) is $w_{ij}^c = 1$ for $i = j$, and $w_{ij}^c = 0$ for $j \neq i$, $i, j = 1, \dots, L$, the controllability Gramian W_c and its Cholesky factorization S are identical and equal to the identity matrix

$$W_c = S = I \tag{23}$$

The observability Gramian is obtained by solving the discrete-time Lyapunov Eq. (7). Therefore, the coefficients of the matrix W_o are solving the set of equations

$$\begin{bmatrix} w_{11}^o - w_{22}^o & w_{12}^o - w_{23}^o & \dots & w_{1(L-1)}^o - w_{2L}^o & w_{1L}^o \\ w_{21}^o - w_{32}^o & w_{22}^o - w_{33}^o & \dots & w_{2(L-1)}^o - w_{3L}^o & w_{2L}^o \\ w_{31}^o - w_{42}^o & w_{32}^o - w_{43}^o & \dots & w_{3(L-1)}^o - w_{4L}^o & w_{3L}^o \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{(L-1)1}^o - w_{L2}^o & w_{(L-1)2}^o - w_{L3}^o & \dots & w_{(L-1)(L-1)}^o - w_{LL}^o & w_{(L-1)L}^o \\ w_{L1}^o & w_{L2}^o & \dots & w_{L(L-1)}^o & w_{LL}^o \end{bmatrix}$$
$$= \begin{bmatrix} \underline{\phi}_1^2 & \underline{\phi}_1\underline{\phi}_2 & \dots & \underline{\phi}_1\underline{\phi}_{L-1} & \underline{\phi}_1\underline{\phi}_L \\ \underline{\phi}_2\underline{\phi}_1 & \underline{\phi}_2^2 & \dots & \underline{\phi}_2\underline{\phi}_{L-1} & \underline{\phi}_2\underline{\phi}_L \\ \underline{\phi}_3\underline{\phi}_1 & \underline{\phi}_3\underline{\phi}_2 & \dots & \underline{\phi}_2\underline{\phi}_{L-1} & \underline{\phi}_3\underline{\phi}_L \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{\phi}_L\underline{\phi}_1 & \underline{\phi}_L\underline{\phi}_2 & \dots & \underline{\phi}_L\underline{\phi}_{L-1} & \underline{\phi}_L^2 \end{bmatrix} \tag{24}$$

Finally, by solving Eq. (24) the coefficients w_{ij}^o, $i, j = 1, \dots, L$, form the observability Gramian W_o as follows

$$W_o = \begin{bmatrix} \sum_{j=1}^{L} \underline{\phi}_j^2 & \sum_{j=1}^{L-1} \underline{\phi}_j\underline{\phi}_{j+1} & \sum_{j=1}^{L-2} \underline{\phi}_j\underline{\phi}_{j+2} & \dots & \underline{\phi}_1\underline{\phi}_L \\ \sum_{j=2}^{L} \underline{\phi}_j\underline{\phi}_{j-1} & \sum_{j=2}^{L} \underline{\phi}_j^2 & \sum_{j=2}^{L-1} \underline{\phi}_j\underline{\phi}_{j+1} & \dots & \underline{\phi}_2\underline{\phi}_L \\ \sum_{j=3}^{L} \underline{\phi}_j\underline{\phi}_{j-2} & \sum_{j=3}^{L} \underline{\phi}_j\underline{\phi}_{j-1} & \sum_{j=3}^{L} \underline{\phi}_j^2 & \dots & \underline{\phi}_3\underline{\phi}_L \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{\phi}_L\underline{\phi}_1 & \underline{\phi}_L\underline{\phi}_2 & \underline{\phi}_L\underline{\phi}_3 & \dots & \underline{\phi}_L^2 \end{bmatrix} \tag{25}$$

Now, it is easy to show that the Cholesky factorization R of the matrix W_o as in Eq. (25) can be presented in form of Eq. (21). This completes the proof. □

The system introduced in Lemma 1 is equivalent to the fractional-order system (1) for finite time $t = 1, \dots, L$, only, that is for the FFD. In a general case, for $t > L$ the state space system presented in Lemma 1 is an integer-order approximation of the fractional-order system, with an upper bound for the FIR approximation error as derived in [30].

The results of Theorem 1, in terms of the system matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and the Cholesky factorization of the controllability and observability Gramians, can be used in the SR and

BFSR algorithms for effective determination of the transformation matrix T and its right inverse $T^{\#}$ and finally to obtain a low-order, 'regular' (integer-order) state space model which approximates the discrete-time fractional-order system of Eq. (1).

5. Simulation example

Consider the discrete-time fractional-order state space system as in Eq. (1) of order $\alpha = 0.9$ with

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccc|c} 2.37 & -4.3849 & 2.602023 & -0.5886251 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ \hline 1 & -1.8 & 0.9 & 0 & 0 \end{array}\right]$$

Using the Gramian factorizations S (Eq. (20)) and R (Eq. (21)) presented in Theorem 1 and applying the SR-based BT methodology (see Table 1), the system described by the FIR approximation as in Lemma 1 with $L = 10^4$ is reduced to two models of orders $k = 6$ and $k = 8$. An example model of order $k = 6$ is presented below

$$\left[\begin{array}{c|c} \tilde{A}_r & \tilde{B}_r \\ \hline \tilde{C}_r & D \end{array}\right] = \left[\begin{array}{cccccc|c} 0.9795 & -8.696e{-3} & -1.248e{-2} & -9.271e{-3} & 2.355e{-2} & 2.464e{-4} & -1.133 \\ -8.696e{-3} & 0.9752 & -4.934e{-2} & -0.1034 & 0.1038 & 3.029e{-3} & -0.2478 \\ -1.248e{-2} & -4.934e{-2} & 0.8823 & -0.3676 & 0.16172 & 6.393e{-3} & -0.3482 \\ 9.271e{-3} & 0.1034 & 0.3676 & 0.8656 & 0.2041 & -3.656e{-3} & 0.2409 \\ -2.355e{-2} & -0.1038 & -0.1617 & 0.2041 & 0.4216 & 5.8522{-2} & -0.6362 \\ 2.464e{-4} & 3.029e{-3} & 6.393e{-3} & 3.656e{-3} & -5.852e{-2} & 0.9967 & 5.861e{-3} \\ \hline -1.133 & -0.2478 & -0.3482 & -0.2409 & 0.6362 & 5.861e{-3} & 0 \end{array}\right]$$

Frequency responses for the FIR approximation and the two FIR-BT models for $k = 6$ and $k = 8$ are presented in Fig. 1. Also, Table 2 contains both frequency and time domain approximation errors for the analyzed models in terms of:

- DCE - steady state approximation error,
- MSE_ω - mean square approximation error for the frequency characteristics in the frequency range $\omega \in [10^{-4}, \pi]$,
- MSE_t - mean square approximation error for the step response in the discrete-time range $t \in [0, 2000]$.

The results of Fig. 1 and Table 2 show that our FIR-BT model gives a very good approximation accuracy both in the frequency and time domains. Fig. 1 demonstrates that the frequency characteristics of all the analyzed models are hardly distinguishable from the actual system characteristic. Plots of the approximation errors for particular models presented in Fig. 1 illustrate that the performance of the FIR-BT model with $k = 8$ is, unsurprisingly, visibly better than for $k = 6$. Moreover, it can be seen th3at an approximation error for this model in the low frequency range is a result of the finite-length FIR approximation, whereas the BT reduction method affects the model performance in the medium/high frequency range.

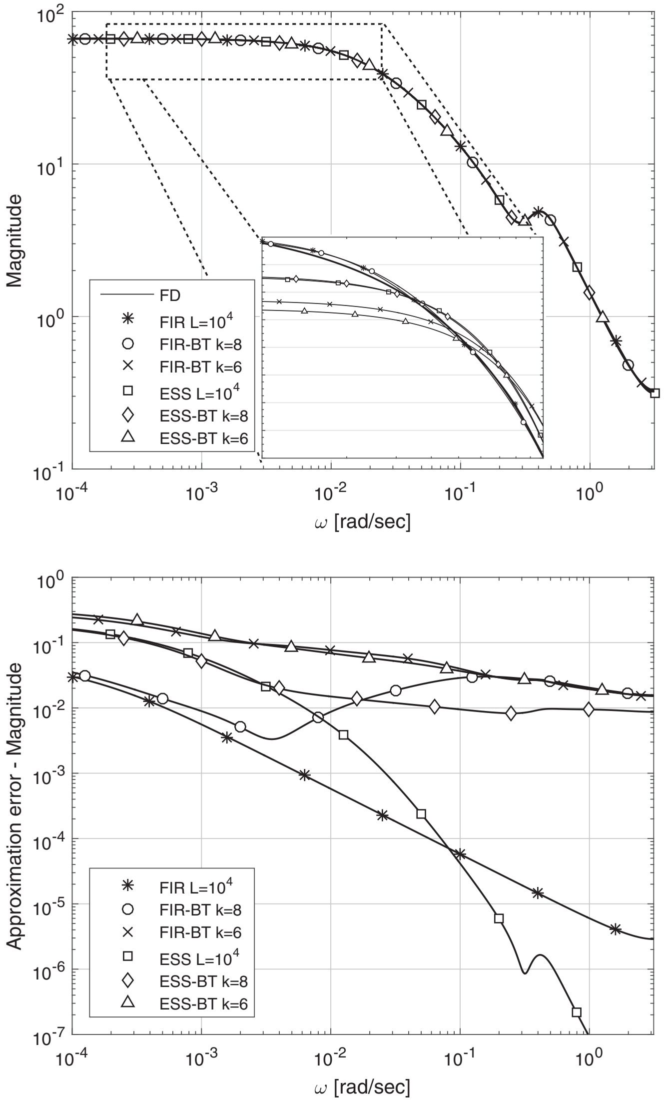

Fig. 1. Frequency responses and approximation errors for the reduced models.

The mean square approximation error for the step response shows that the model of order $k = 8$ gives better results than the FIR model, so in this specific case the BT reduction does not affect the time performances of the model. Also, the BT-based model of order $k = 6$ shows a good performance in the time domain.

The presented methodology is compared with another method for approximation of a fractional-order system by use of the BT reduction method applied to the expanded state space (ESS) representation of fractional order system (ESS-BT) (see [22]). Results presented in Fig. 1 and Table 2 show that the FIR-BT method is more effective than the ESS-BT and

Table 2
Approximation errors

	DCE	MSE_ω	MSE_t
FIR model ($L = 10^4$)	0.0632	7.1427e−05	1.7027e−03
FIR-BT model ($L = 10^4$, $k = 8$)	0.0699	3.9447e−04	1.2891e−03
FIR-BT model ($L = 10^4$, $k = 6$)	0.3025	1.0988e−02	2.8774e−02
ESS model ($J = 2500$)	0.2164	3.2287e−03	8.0224e−03
ESS-BT model ($J = 2500$, $k = 8$)	0.2113	2.9938e−03	7.2517e−03
ESS-BT model ($J = 2500$, $k = 6$)	0.3340	1.3065e−02	3.9676e−02

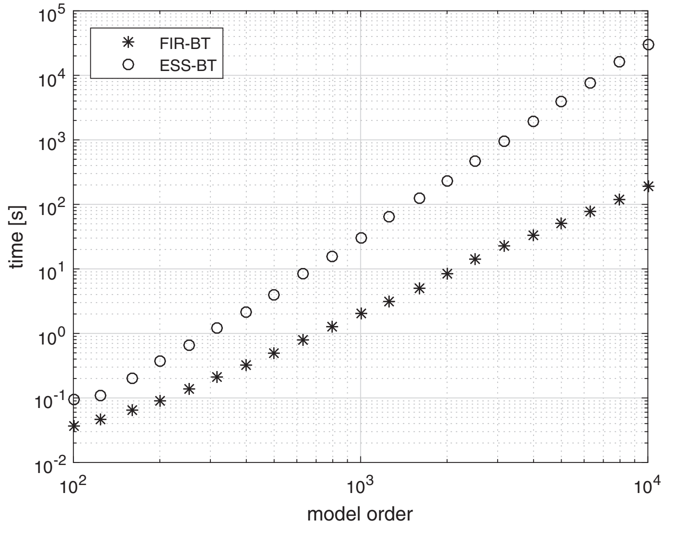

Fig. 2. Time complexity as a function of model order.

provides better performance both in frequency and time domains. Table 2 shows the values of the approximation errors for the obtained models.

It is worth mentioning that the main computational burden of the ESS-BT method is generated by determination of the Gramian factorizations S and R, which are directly given by Theorem 1 for the the FIR-BT method. Therefore, the reduction using the FIR-BT method is some 100 times faster than for the ESS-BT algorithm in the considered example. The time complexities for the approximations via the introduced FIR-BT methodology vs. the ESS-BT approach for various model orders are presented in Fig. 2. It can be seen that the time complexity for the FIR-BT method is a class of the quadratic- instead of the cubic-growth for the ESS-BT algorithm. This fact illustrates the computational superiority of the introduced method, especially for models of very high orders ($nL > 10^4$).

6. Conclusion

This paper has presented a new approximation method for modeling of discrete-time fractional-order state space systems by 'regular' (integer-order) state-space models. The introduced method is based on the SVD order reduction method applied to a specific representation of the state space system. Since controllability and observability Gramian factorizations are given by simple analytical formulas presented in this paper, the proposed method is computationally more efficient and faster than other model order reduction methods for approximation of fractional-order systems. A simulation example shows that a simple, low-order model obtained by using the introduced methodology can be very effective in approximation of fractional-order systems both in the time and frequency domains over a wide frequency range.

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