Efficient Complex Root Finding Algorithm for Microwave and Optical Propagation Problems

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***Abstract*—Article relates to the use of innovative root finding algorithm (on a complex plane) to study propagation properties of microwave and optical waveguides. Problems of this type occur not only in the analysis of lossy structures, but also in the study of complex and leaky modes (radiation phenomena). The proposed algorithm is simple to implement and can be applied for functions with singularities and branch cuts in the complex plane (frequently occurring in these types of issues).**

***Index Terms*—root finding algorithm, propagation, waveguides**

I. Introduction

Many different types of propagation problems boil down to finding a zero of some specific function. In most cases the root represents a propagation coefficient. Then the corresponding field distribution, energy, losses and any other parameter can be evaluated directly. In some problems the function can be expressed in real terms only, however, in general such assumption is not sufficient - function must be complex. This situation occurs in the analysis of lossy structures, but also in the study of specific types of waves in lossless guides: complex modes (in shielded structures) and leaky modes in open waveguides.

There are many algorithms related to this problem (Newton's method [1], Muller's [2], Simplex [3]). Unfortunately, these are mostly local methods, which can improve an accuracy of the solution, with a pre-knowledge of its location. The most popular global methods are effective only for polynomial functions or such functions which can be approximated to the ones in a certain area [4], [5]. In most practical problems such approximation is not possible, as the investigated function have many singularities and branch cuts in the complex plane (often in a close vicinity of the root). All these reasons effectively prevent the use of the aforementioned methods in many propagation problems.

The technique proposed in this paper is the global iterative algorithm based on Delaunay triangulation [6] (and it is free from the restrictions listed above). Despite the high efficiency its implementation is relatively simple and easy to parallelize. The suggested algorithm consists of three stages. In the first stage the function is discretized on a self-adaptive triangular grid and a set of candidate points is determined. The set contains all points where signs of the real and imaginary part of the function changes simultaneously (roots, singularities and all points along branch cuts). In the second step the resulting set must be verified and all points which are not roots are rejected. The last step is to improve the accuracy of the evaluated zeros. At this point any local algorithm can be applied.

II. Algorithm Description

As mentioned, the algorithm [6] can be divided into three stages. However, (depending on the desired accuracy) the final step can be skipped.

Let us assume that $f(z)$ is a complex function defined on the domain $\Omega \subset \mathbb{C}$.

A. Preliminary Estimation

In the first stage the function $f(z)$ is discretized and approximated on a self-adaptive triangular grid. A parameter Δr denotes the resolution of the preliminary estimation - the distance below which zeros will not be distinguishable. Then a set of candidate points is determined (all points where signs of the real and imaginary part of the function changes simultaneously - roots, singularities and all points along branch cuts). The exact steps of this stage are described below:

1) In the first step, considered region Ω is covered with a initial mesh which nodes are collected in the following set $V = \{v_1, v_2, ..., v_N\}$. Usually, the four vertices of a rectangular domain are sufficient to initiate the procedure.
2) By applying Delaunay triangulation to the points collected in the set V, they become the vertices of triangles, completely covering the domain Ω.
3) The function is evaluated for all points of V and its values are stored in the second set $F = \{f_1, f_2, ..., f_N\}$.
4) All the edges of a length greater than the assumed accuracy Δr are analyzed. If the signs of the real or imaginary parts of the function are different at the vertices, then an extra point (located at the midpoint of the considered edge) is added to V.
5) If in the previous step at least one new point has been added to V, the procedure is repeated from step (2).
6) The real and imaginary parts of the function can be approximated separately on each triangle and two curves (consisting of line segments) representing zeros of the

real and imaginary parts of the function can be constructed: $C_R = \{z \in \Omega : Re(f(z)) = 0\}$ and $C_I = \{z \in \Omega : Im(f(z)) = 0\}$.
7) All the points where two curves C_R and C_I cross (signs of the real and imaginary part of the function changes simultaneously) are collected in set $S = \{s_1, s_2, ..., s_M\}$.

In step (3) the function is only evaluated at new points of V. Such remark is especially important for more complex function (then this part of the algorithm can be most time consuming).

From the description above procedure it is clear that the points in the set S are determined with an accuracy of Δr. Thus, the zeros located closer to each other (in a distance smaller than Δr) are treated as multiple roots. It should be emphasized that the set S can contain roots as well as singularities or points located at the branch cuts, and their elimination is a necessary next step.

B. Verification

An unambiguous identification of a root from the set of candidate point is not a trivial issue (in the case of complex function). To this end Cauchy's Argument Principle is applied for each point s_m. According to this principle, the following integral must be evaluated

$$q = \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz. \quad (1)$$

In general q is a sum of all zeros counted with their multiplicities, minus the sum of all poles counted with their multiplicities. Due to the contour C is a circle of radius Δr centered at s_m and the candidate point represents only one root or singularity, the parameter q can have only fixed value:

- a positive integer - root of order q,
- a negative integer - singularity of order $-q$,
- zero - regular point.

Unfortunately, the process does not guarantee that all the roots are found. However, the risk of root missing can be reduced by the integration (1) over the whole considered area Ω. Moreover, the risk can be further reduce by extending integration to higher moments of function $f(z)$ [7], [8] (the second moment eliminates the problem for single pair of root-pole).

C. Final Refinement (optional)

In order to improve the accuracy of the roots any known local algorithm can be applied, for instance Newton's [1], Muller's [2] or the Simplex method [3]. In general, all these techniques can be combined to obtain the best solution in a fewest number of iterations. Moreover, this stage can be simply parallelized, due to independence of the process for each root. In this publication the results are obtained with the use of Muller's algorithm with the greatest possible precision specified by the machine $\varepsilon = 2.22 \cdot 10^{-16}$.

III. Numerical tests

In order to demonstrate the validity of the method two guiding structures are analyzed. In the first one the lossy media is introduced, which implicates the complex values of propagation coefficients. In the second one, the complex values of propagation coefficients represents the radiation from the structure.

A. Partially loaded waveguide

The first structure is a partially loaded waveguide $WR-90$. The dimensions of the waveguide are $a = 22.86$ mm, $b = 10.16$ mm and it is loaded at full hight on a distance $d = 10$ mm (see - Fig. 1). The permittivity of the media is equal $\varepsilon_r = 5 - 2i$.

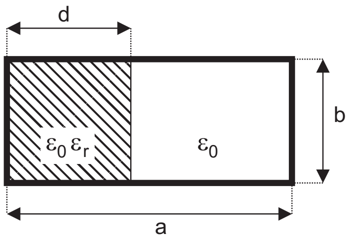

Fig. 1. Considered optical fiber

The propagation coefficient (due to lossy media its value must be complex) can be obtained from the following relation [9]:

$$f(z) = \kappa_2 \tan(\kappa_1 d) + \kappa_1 \tan(\kappa_2 (a - d)), \quad (2)$$

where z represents a normalized propagation coefficient. The coefficients κ_1 and κ_2 are defined as follows $\kappa_1 = k_0\sqrt{z^2 + \varepsilon_r}$, $\kappa_2 = k_0\sqrt{z^2 + 1}$, where $k_0 = 2\pi f / c$. The analysis is carried out for frequency $f = 4$ GHz.

The region analyzed is $\Omega = \{z \in \mathbb{C} : -2 < \Re(z) < 2 \wedge -2 < \Im(z) < 2\}$ whereas $\Delta r = 0.1$. The preliminary estimation (in 13 iterations and 807 evaluations of the function - see Fig. 2) results in 10 candidate points which are listed in Table I.

TABLE I
THE PRELIMINARY ESTIMATION AND VERIFICATION FOR THE PARTIALLY LOADED WAVEGUIDE PROBLEM

m	s_m	q	verification
1	$-0.55 - 1.78i$	-1	singularity
2	$-0.01 + 1.83i$	0.05	cut
3	$-0.49 - 0.72i$	1	root
4	$-1.60 - 0.00i$	-1	singularity
5	$0.49 + 0.72i$	1	root
6	$0.55 + 1.78i$	-1	singularity
7	$1.60 + 0.00i$	-1	singularity
8	$0.01 - 1.83i$	0.05	cut
9	$0.00 - 1.00i$	0.44	cut
10	$0.00 + 1.00i$	0.44	cut

The final results (only refined roots $\overline{s_m}$, singularities and points at the branch cuts are rejected) are collected in Table II. To confirm the validity of the analysis the function is evaluated in all points $\overline{s_m}$ and in their close neighborhood $\overline{s_m} + \delta_m$.

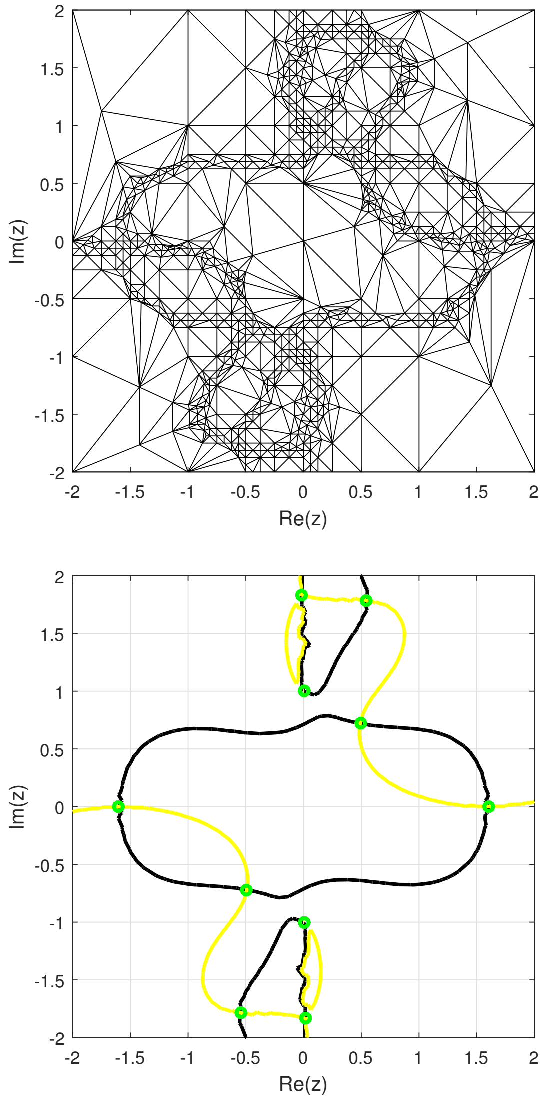

Fig. 2. The final triangulation and the real-zero (black) and imaginary-zero (yellow) curves for the partially loaded waveguide problem

TABLE II
THE FINAL RESULTS FOR THE PARTIALLY LOADED WAVEGUIDE PROBLEM ($\delta_m = \overline{s_m} \cdot 10^{-10}$)

m	$\overline{s_m}$	$\lvert f(\overline{s_m})\rvert$	$\lvert f(\overline{s_m} + \delta_m)\rvert$
3	$-0.48996 - 0.72038i$	$7.15 \cdot 10^{-15}$	$1.34 \cdot 10^{-8}$
5	$0.48996 + 0.72038i$	$7.15 \cdot 10^{-15}$	$1.34 \cdot 10^{-8}$

B. Optical Fiber

The second example concerns radiation processes (leaky waves) in optical fiber of radius $R = 0.5$ μm. The refractive index of the core is equal $n_2 = 2.9$, and the cladding, $n_1 = 1.55$ (see - Fig. 3).

From continuity of the fields at the boundary of these two media a set of homogeneous equations can be derived [10]. To obtain the solution (correct propagation coefficient) the

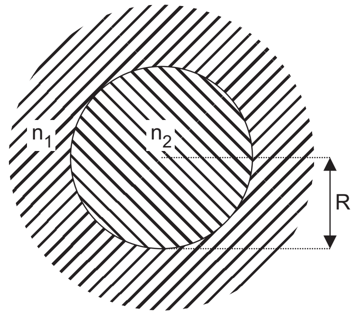

Fig. 3. Considered optical fiber

following determinant function must be zero:

$$f(z) = \begin{vmatrix} H & 0 & -J & 0 \\ 0 & n_1 H & 0 & -n_2 J \\ \frac{zmH}{R\kappa_1^2} & \frac{k_0 n_1 H'}{\kappa_1} & -\frac{zmJ}{R\kappa_2^2} & -\frac{k_0 n_2 J'}{\kappa_2} \\ \frac{k_0 n_1^2 H'}{\kappa_1} & -\frac{zmH}{\kappa_1^2} & -\frac{k_0 n_2^2 J'}{\kappa_2} & \frac{zmJ}{\kappa_2^2} \end{vmatrix}, \quad (3)$$

where z represents a normalized propagation coefficient, $J = J_m(\kappa_2 R)$ is a Bessel function of the first kind and $H = H_m^{(2)}(\kappa_1 R)$ is a Hankel function of the second kind. The coefficients κ_1 and κ_2 are defined as follows $\kappa_1 = k_0\sqrt{z^2 + n_1^2}$, $\kappa_2 = k_0\sqrt{z^2 + n_2^2}$, where $k_0 = 2\pi f/c$. The numerical tests are carried out for angular variation $m = 1$ and frequency $f = 50$ THz.

The region analyzed is $\Omega = \{z \in \mathbb{C} : -2 < \Re(z) < 2 \wedge -2 < \Im(z) < 2\}$ whereas $\Delta r = 0.1$. For this structure, the first stage - preliminary estimation requires 15 iterations (1109 evaluations of the function - see Fig. 4) and results in 6 candidate points which are listed in Table III.

TABLE III
THE PRELIMINARY ESTIMATION AND VERIFICATION FOR THE OPTICAL FIBER PROBLEM

m	s_m	q	verification
1	$1.32 + 0.76i$	1	root
2	$-0.02 + 1.59i$	-3	singularity
3	$-0.95 + 1.66i$	1	root
4	$-1.32 - 0.76i$	1	root
5	$0.02 - 1.59i$	-3	singularity
6	$0.95 - 1.66i$	1	root

The final results are collected in Table IV. The same results can be obtained directly from finite difference method (then roots s_m are represented by eigenvalues of the corresponding matrix operator) [11].

TABLE IV
THE FINAL RESULTS FOR THE OPTICAL FIBER PROBLEM($\delta_m = \overline{s_m} \cdot 10^{-10}$)

m	$\overline{s_m}$	$\lvert f(\overline{s_m})\rvert$	$\lvert f(\overline{s_m} + \delta_m)\rvert$
1	$1.31797 + 0.75858i$	$2.72 \cdot 10^{-16}$	$1.27 \cdot 10^{-10}$
3	$-0.94641 + 1.65406i$	$3.26 \cdot 10^{-16}$	$2.28 \cdot 10^{-10}$
4	$-1.31797 - 0.75858i$	$2.72 \cdot 10^{-16}$	$1.27 \cdot 10^{-10}$
6	$0.94641 - 1.65406i$	$3.26 \cdot 10^{-16}$	$2.28 \cdot 10^{-10}$

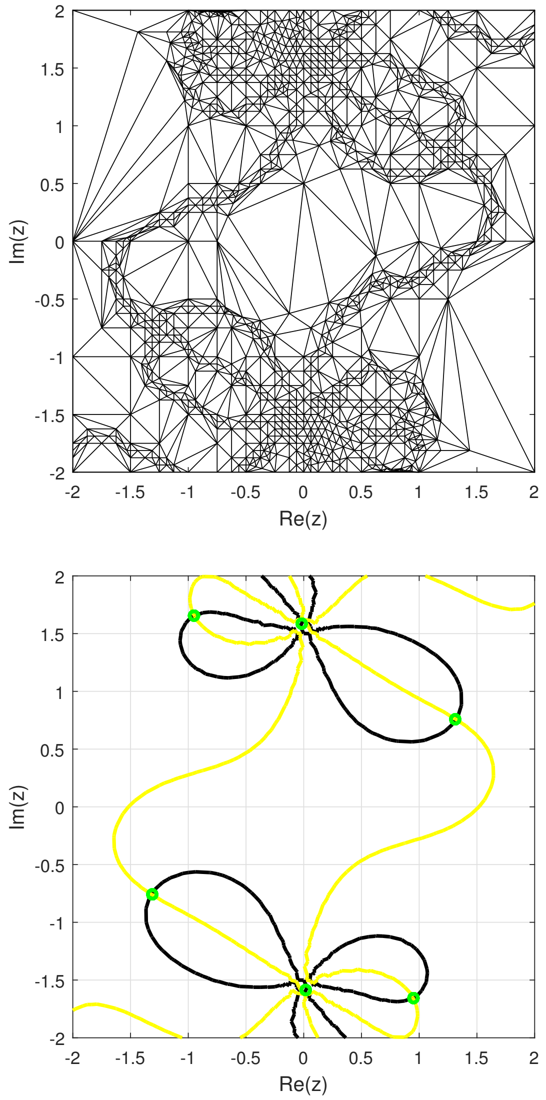

Fig. 4. The final triangulation and the real-zero (black) and imaginary-zero (yellow) curves for the optical fiber problem

IV. Conclusion

The numerical tests suggest that the algorithm can be successfully applied in the analysis of properties of the microwave and optical guides. Its implementation is simple and can be used in regions with singularities and branch cuts. Moreover, the algorithm is flexible and can be simply parallelized, which significantly improves its effectiveness for complex cases. The results also allow to suppose that the algorithm can be used to a much broader class of microwave and optical systems.

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