Chapter 4
Riemann–Liouville q-Fractional Calculi

Abstract In this chapter we investigate q-analogues of the classical fractional calculi. We study the q-Riemann–Liouville fractional integral operator introduced by Al-Salam (Proc. Am. Math. Soc. 17, 616–621, 1966; Proc. Edinb. Math. Soc. 2(15), 135–140, 1966/1967) and by Agarwal (Proc. Camb. Phil. Soc. 66, 365–370, 1969). We give rigorous proofs of existence of the fractional q-integral and q-derivative. Therefore we establish a q-analogue of Abel's integral equation and its solutions.

4.1 Classical Fractional Calculi

In this section, we introduce some classical fractional calculi which we will consider their q-analogues. We also state a set of relations and properties of these fractional calculi for which we investigate their q-counterparts. First we denote by $\mathscr{A}C^{(n)}[a,b]$, $n \in \mathbb{N}$, to the set of functions f which have continuous derivatives up to order $n-1$ on $[a,b]$ with $f^{(n-1)} \in \mathscr{A}C[a,b]$, i.e. absolutely continuous on $[a,b]$. Functions of $\mathscr{A}C[a,b]$ can be characterized via the following Taylor's formula taken from [175].

Lemma A. The set $\mathscr{A}C^{(n)}[a,b]$ consists of those, and only those, functions f, which are represented in the form

$$f(x) = \sum_{k=0}^{n-1} c_k (x-a)^k + \frac{1}{n-1!} \int_a^x (x-t)^{n-1} \phi(t)\, dt,$$

where $\phi \in L_1(a,b)$ and the c_k's are arbitrary constants. Moreover, $\phi(x) = f^{(n)}(x)$ a.e., and $c_k = \dfrac{f^{(k)}(a)}{k!}$, $k = 0, 1, \ldots, n-1$.

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The first fractional integral operator we define in this section is that of Riemann and Liouville. It is connected to Abel's integral equation

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\phi(t)\,dt = f(x),\ x>a,\ \alpha>0,\quad f\in L_1(a,b), \tag{4.1}$$

cf. [269, P. 32], see also [213,227,262]. The following theorem, cf. [118,269], solves Abel's equation concretely.

Theorem 4.1. *The Abel's integral equation* (4.1) *with* $0<\alpha<1$ *has a unique solution in* $L_1(a,b)$ *if and only if the function* $f_{1-\alpha}$ *defined by*

$$f_{1-\alpha}(x) := \frac{1}{\Gamma(1-\alpha)}\int_a^x (x-t)^{-\alpha} f(t)\,dt$$

is absolutely continuous on $[a,b]$ *and* $f_{1-\alpha}(a)=0$. *If these later conditions are fulfilled, then the unique solution* ϕ *is given explicitly by*

$$\phi(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{-\alpha} f(t)\,dt = \frac{d}{dx} f_{1-\alpha}(x),\ a.e. \tag{4.2}$$

If $f\in \mathscr{A}C[a,b]$, then $f_{1-\alpha}\in\mathscr{A}C[a,b]$ and (4.2) becomes

$$\phi(x) = \frac{1}{\Gamma(1-\alpha)}\Big[\frac{f(a)}{(x-a)^{\alpha}} + \int_a^x \frac{f'(s)}{(x-s)^{\alpha}}\,ds\Big].$$

The following Cauchy formulae can be considered as the nth primitive of a function $f\in L_1(a,b)$.

$$\int_a^x\int_a^{x_{n-1}}\ldots\int_a^{x_1} f(t)\,dt\,dx_1\ldots dx_{n-1} = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\,dt, \tag{4.3}$$

$$\int_x^b\int_{x_{n-1}}^b\ldots\int_{x_1}^b f(t)\,dt\,dx_1\ldots dx_{n-1} = \frac{1}{(n-1)!}\int_x^b (t-x)^{n-1} f(t)\,dt, \tag{4.4}$$

$n\in\mathbb{N}$. Since the right hand sides of (4.3) and (4.4) exist also for non integer values of n, the Riemann–Liouville fractional integral can be considered as an extension of (4.3) and (4.4) when we replace n by $\alpha\in\mathbb{R}^+$ and $n-1!$ by $\Gamma(\alpha)$. Indeed, for $\alpha\in(0,\infty)$ and $f\in L_1(a,b)$, the fractional Riemann–Liouville integral is defined to be

$$\begin{aligned} I_{a+}^{\alpha} f(x) &:= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt,\\ I_{b-}^{\alpha} f(x) &:= \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt, \end{aligned} \tag{4.5}$$

$x \in (a, b)$ with respect to $x = a$ and $x = b$, respectively. In some literature, they are called left-sided and right-sided Riemann–Liouville fractional integrals, respectively. It is known, see e.g [211, 227, 235, 269], that if $f \in L_1(a, b)$, then both of $I^{\alpha}_{a+} f$ and $I^{\alpha}_{b-} f$ exist a.e. and they are $L_1(a, b)$-functions. Moreover, for $f \in L_1(a, b)$, we have

$$\lim_{\alpha \to 0+} I^{\alpha}_{a+} f(x) = \lim_{\alpha \to 0+} I^{\alpha}_{b-} f(x) = f(x) \quad \text{a.e.} \tag{4.6}$$

We now pass to the definition of the fractional derivative of arbitrary order. The existence of the fractional derivative is connected to the solvability of the Abel's integral equation (4.1). For $f \in L_1(a, b)$, the left and right sided Riemann–Liouville fractional derivatives of order α, $\alpha \in \mathbb{R}^+$, are defined formally by

$$D^{\alpha}_{a+} f(x) := D^k I^{k-\alpha}_{a+} f(x) = \frac{1}{\Gamma(k-\alpha)} \frac{d^k}{dx^k} \int_a^x (x-t)^{k-\alpha-1} f(t)\, dt, \tag{4.7}$$

and

$$\begin{aligned} D^{\alpha}_{b-} f(x) &:= (-1)^k D^k I^{k-\alpha}_{b-} f(x) \\ &= \frac{(-1)^k}{\Gamma(k-\alpha)} \frac{d^k}{dx^k} \int_x^b (t-x)^{k-\alpha-1} f(t)\, dt, \end{aligned} \tag{4.8}$$

$x \in (a, b)$, $k = \lceil \alpha \rceil$, respectively, $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α. The fractional derivatives $D^{\alpha}_{a+} f$, $D^{\alpha}_{b-} f$ exist if $f \in L_1(a, b)$ and $I^{k-\alpha}_{a+} f$, $I^{k-\alpha}_{b-} f$ are of the class $\mathscr{AC}^{(k)}[a, b]$, see [269]. For example, the Riemann–Liouville fractional derivatives of x^c, $c \in \mathbb{R}$, is given by

$$D^{\alpha}_{a+} x^c = \frac{\Gamma(c+1)}{\Gamma(c-\alpha+1)} x^{c-\alpha} \quad (\alpha \geq 0). \tag{4.9}$$

Of course if $\alpha < 0$, (4.9) gives the Riemann–Liouville fractional integral value of x^c.

Now we state some properties of the Riemann–Liouville fractional calculus for which we derive their q-analogues. We confine ourselves to the case of the left-sided Riemann–Liouville fractional calculus since we shall study its basic analogue. Let α, $\beta \in \mathbb{R}^+$. If $f \in L_1(a, b)$, then the semigroup property

$$I^{\beta}_{a+} I^{\alpha}_{a+} f(x) = I^{\alpha}_{a+} I^{\beta}_{a+} f(x) = I^{\alpha+\beta}_{a+} f(x), \tag{4.10}$$

holds for almost all $x \in [a, b]$. If $f(x)$ satisfies the conditions

$$f \in L_1(a, b) \text{ and } I^{k-\alpha}_{a+} f \in \mathscr{AC}^{(k)}[a, b], \quad k = \lceil \alpha \rceil,$$

then

$$\begin{cases} D_{a+}^{\alpha-j} f \in L_1(a,b), & j = 0,1,\dots,k, \\ \\ D_{a+}^{\alpha-j} f \in AC^{(j)}[a,b], & j = 1,2,\dots,k-1. \end{cases} \tag{4.11}$$

Moreover, for functions satisfying the appropriate mentioned conditions

$$D_{a+}^{\alpha} I_{a+}^{\alpha} f(x) = f(x), \quad \text{a.e.} \tag{4.12}$$

$$D_{a+}^{\alpha} I_{a+}^{\beta} f(x) = I_{a+}^{\beta-\alpha} f(x), \quad \text{a.e. if } \beta \geqslant \alpha \geqslant 0, \tag{4.13}$$

$$D_{a+}^{\alpha} I_{a+}^{\beta} f(x) = D_{a+}^{\alpha-\beta} f(x), \quad \text{a.e. if } \alpha > \beta \geqslant 0. \tag{4.14}$$

In addition,

$$I_{a+}^{\alpha} D_{a+}^{\alpha} f(x) = f(x) - \sum_{j=1}^{k} D_{a+}^{\alpha-j} f(a^+) \frac{(x-a)^{\alpha-j}}{\Gamma(1+\alpha-j)}. \tag{4.15}$$

If $f \in L_1(a,b)$ and $D_{a+}^{-(n-\beta)} f \in \mathscr{A}C^{(n)}[a,b]$, $n = \lceil \beta \rceil$, then the equality

$$I_{a+}^{\alpha} D_{a+}^{\beta} f(x) = D_{a+}^{\beta-\alpha} f(x) - \sum_{j=1}^{n} D_{a+}^{\beta-j} f(a^+) \frac{(x-a)^{\alpha-j}}{\Gamma(1+\alpha-j)} \tag{4.16}$$

holds almost everywhere in (a,b) for any $\alpha > 0$. Moreover, if

$$0 \leqslant k-1 \leqslant \alpha < k, \ \alpha + \beta < k, \ \text{and } D_{a+}^{-(n-\alpha)} f \in \mathscr{A}C^{(n)}[a,b],$$

then

$$D_{a+}^{\alpha} D_{a+}^{\beta} f(x) = D_{a+}^{\alpha+\beta} f(x) - \sum_{j=1}^{n} D_{a+}^{\beta-j} f(a^+) \frac{(x-a)^{-\alpha-j}}{\Gamma(1-\alpha-j)}, \tag{4.17}$$

holds almost everywhere in (a,b). Finally,

$$I_{a+}^{\alpha} D_{a+}^{\alpha} f(x) = f(x) \ \text{ and } \ I_{a+}^{\alpha} D_{a+}^{\beta} f(x) = D_{a+}^{\beta-\alpha} f(x),$$

whenever $f \in I_{a+}^{\alpha}(L_1)$ and $f \in I_{a+}^{\beta}(L_1)$, respectively, cf. [269]. Here, by $I_{a+}^{\alpha}(L_1)$ we denote the set of functions f represented by the left sided fractional integral of order α ($\alpha > 0$) of a summable function, i.e. $f = I_{a+}^{\alpha}\phi$, $\phi \in L_1(a,b)$. For proofs of these and other properties, see e.g. [85, 235, 269].

The other types of fractional derivatives we discuss here are those of Grünwald–Letnikov and Caputo. It is known that the nth derivative of a function f defined on (a,b) is given by

$$f^{(n)}(x) = \lim_{h \to 0} \frac{\Delta_h^n f(x)}{h^n}, \tag{4.18}$$

where

$$\Delta_h^n f(x) = \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(x - rh).$$

Starting from this formula, Grünwald [120] and Letnikov [180] developed an approach to fractional differentiation for which the formal definition of the fractional derivative $\mathbf{D}_{a+}^{\alpha} f(x)$ is the limit

$$\mathbf{D}_{a+}^{\alpha} f(x) := \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - jh), \ \alpha \in \mathbb{R}^+. \tag{4.19}$$

Grünwald in [120] gave a formal proof while Letnikov in [180] gave a rigorous proof of the fact that $\mathbf{D}^{-\alpha}$ coincides with the Riemann–Liouville fractional integral operator when $\alpha > 0$ and f is continuous on $[a, b]$, cf. [178]. The following theorem gives the conditions on which the Grünwald–Letnikov fractional derivative is nothing but the Riemann–Liouville fractional derivative.

Theorem 4.2 ([235]). *Assume that $f \in \mathscr{AC}^{(n)}[a, b]$. Then for every α, $0 < \alpha < n$, the Riemann–Liouville fractional derivative $D_{a+}^{\alpha} f(t)$ exists and coincides with the Grünwald–Letnikov derivative $\mathbf{D}_{a+}^{\alpha} f(t)$, i.e. if*

$$0 \leqslant m - 1 \leqslant \alpha \leqslant m \leqslant n$$

then for $a < t < b$ the following holds:

$$\begin{aligned} \mathbf{D}_{a+}^{\alpha} f(t) = D_{a+}^{\alpha} f(t) &= \sum_{j=0}^{m-1} \frac{f^{(j)}(a^+)(t-a)^{j-\alpha}}{j!} \\ &+ \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-1-\alpha} f^{(m)}(\tau)\, d\tau. \end{aligned} \tag{4.20}$$

It is known that in applied problems, we require definitions of fractional problems which allow the involvement of initial conditions with physical meanings, i.e. conditions like

$$f^{(j)}(a^+) = b_j, \quad j = 1, 2, \ldots, k, \tag{4.21}$$

where the b_j's and $k \in \mathbb{N}$ are given constants. The Riemann–Liouville approach leads to initial conditions containing the limit values of the Riemann–Liouville fractional derivatives at the lower end point $t = a$, for example

$$\lim_{t \to a+} D_{a+}^{\alpha - j} f(t) = c_j, \quad j = 1, 2, \ldots, k, \tag{4.22}$$

where the c_j's and $k \in \mathbb{N}$ are given constants and α is not an integer. It is frequently stated that the physical meaning of initial conditions of the form (4.22) is unclear or even non existent, see e.g. [269, P. 78]. The requirement for physical interpretation of such initial conditions was most clearly formulated recently by Diethelm et al. [83]. In [134], Heymans and Podlubny show that initial conditions of the form (4.22) for Riemann–Liouville fractional differential equations may have physical meanings, and that the corresponding quantities can be obtained from measurements. However, the problem of interpretation of initial conditions of the type (4.22) remains open. In [69, 70], Caputo gave a solution for this problem when he defined the fractional derivative of order α, ${}_a^C D_t^\alpha f(t)$, by the formula

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, \tag{4.23} $$

$n-1 < \alpha \leqslant n$, $n \in \mathbb{N}$. The Caputo approach leads to initial conditions of the form (4.21) which are physically accepted. Recently, several initial value problems based on the Caputo fractional operator have been studied, see e.g. [90, 91, 233, 236]. The relationship between the Caputo fractional derivative and the Riemann–Liouville fractional operator is given by

$$ {}_a^C D_t^\alpha f(t) = I_{a+}^{n-\alpha} f^{(n)}(t). $$

If the function $f(t)$ has $n+1$ continuous derivatives in $[a, b]$, then cf. [235],

$$ \lim_{\alpha \to n} {}_a^C D_t^\alpha f(t) = f^{(n)}(t) \quad \text{for } t \in [a, b]. $$

Another advantage of the Caputo fractional derivative is that ${}_a^C D_t^\alpha c = 0$, where c is a constant, while the Riemann–Liouville fractional derivative of a constant c is

$$ D_{a+}^\alpha c = D^n I_{a+}^{n-\alpha} c = \frac{c}{\Gamma(n-\alpha+1)} D^n (x-a)^{n-\alpha} = \frac{c}{\Gamma(1-\alpha)} (x-a)^{-\alpha}, $$

where $n = \lceil \alpha \rceil$. Since we will establish a method for using the q-Laplace transform defined by Hahn in [123] to solve fractional q-difference equations, we state some results concerning the Laplace transform before closing this section.

Let $f \in L_1(0, b)$. If $I_{a+}^{n-\alpha} f \in \mathscr{A}C^{(n)}[0, b]$ for any $b > 0$ such that $I_{a+}^{n-\alpha} f(x)$ is of exponential order as $x \to \infty$. That is, there exist nonnegative constants K, T and $p_0 \geqslant 0$, such that

$$ |I_{a+}^{n-\alpha} f(x)| \leqslant K e^{p_0 x} \quad (x > T)\,. $$

Then the Laplace transform of the Riemann–Liouville fractional derivative $D_{a+}^\alpha f(x)$ exists, cf. [291], and satisfies the following identity

$$\begin{aligned}\mathscr{L}(D^{\alpha}_{a+}f)(p) &= \int_0^{\infty} e^{-pt} D^{\alpha}_{a+} f(t)\, dt \\ &= p^{\alpha} F(p) - \sum_{k=0}^{n-1} p^k D^{\alpha}_{a+} f(0^+) \quad (n = \lceil \alpha \rceil)\,,\end{aligned}$$

for all p that satisfies $\mathrm{Re}(p) > p_0$, cf. [269, 291].

Similarly, the Laplace transform of the Caputo derivative ${}_a^C D_t^{\alpha} f(x)$ exists if $f(x) \in \mathscr{A}C^{(n)}[0,b]$ for every $b > 0$ and $f(x)$ is of exponential order as $x \to \infty$. In this case we can find nonnegative real number p_0 such that the Laplace transform of the Caputo derivative satisfies the identity

$$\begin{aligned}\mathscr{L}({}_0^C D_t^{\alpha} f)(p) &= \int_0^{\infty} e^{-pt} {}_0^C D_t^{\alpha} f(t)\, dt \\ &= p^{\alpha} F(p) - \sum_{k=0}^{n-1} p^{\alpha-k-1} f^{(k)}(0),\end{aligned}$$

for all p satisfying $\mathrm{Re}\, p > \mathrm{Re}(p_0)$. The Laplace transform method is usually used for solving applied problems. So, it is clear that the Laplace transform of the Caputo derivative allows utilization of initial values of classical integer-order derivative with known physical interpretations, while the Laplace transform of the Riemann–Liouville fractional derivative allows utilization of initial conditions of the form

$$D^{\alpha-j} f(t)|_{t=0} = b_j \qquad (j = 1, 2, \ldots, \lceil \alpha \rceil)\,,$$

which may form problems without known physical interpretations.

4.2 Recent History of Fractional q-Calculus

Over the last decades, fractional calculus became an area of intense research and developments. There are many surveys for the history of the fractional calculus, see the survey of Butzer and Westphal [68], the survey of Machado et al. [197] reported some of the major documents and events in the area of fractional calculus that took place since 1974 up to 2010. They also introduced a poster [196] illustrates the major contribution during the period 1966–2010. To the best of our knowledge, the recent developments in the theory of fractional q-calculus is not well reported. Therefore, we aim to cover in brief in this section the recent developments in the theory of fractional q-calculus. A q-analogue of the Riemann–Liouville fractional integral operator is introduced in [19] by Al-Salam through

$$I_q^{\alpha} f(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \left(qt/x; q\right)_{\alpha-1} f(t)\, d_q t, \tag{4.24}$$

$\alpha \notin \{-1,-2,\ldots\}$. Al-Salam defined this q-analogue as an extension of the following q-Cauchy formula who introduced in[18]

$$\begin{aligned} I^n_{q,a} f(x) &:= \int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_1} f(t)\, d_q t\, d_q x_1 \ldots d_q x_{n-1} \\ &= \frac{x^{n-1}}{\Gamma_q(n)} \int_a^x \big(qt/x;q\big)_{n-1} f(t)\, d_q t. \end{aligned} \tag{4.25}$$

For the convenience of the reader, we shall use the notation $I^n_q f(x)$ instead of $I^n_{q,0} f(x)$. This basic analogue of Riemann–Liouville fractional integral is also given independently by Agarwal, [17], who defined the q-fractional derivative to be

$$D^\alpha_q f(x) := I^{-\alpha}_q f(x) = \frac{x^{-\alpha-1}}{\Gamma_q(-\alpha)} \int_0^x \big(qt/x;q\big)_{-\alpha-1} f(t)\, d_q t.$$

Using the series representation of the q-integration (1.19), identity (4.24) reduces to

$$I^\alpha_q f(x) = x^\alpha (1-q)^\alpha \sum_{n=0}^\infty q^n \frac{(q^\alpha;q)_n}{(q;q)_n} f(xq^n). \tag{4.26}$$

Al-Salam, [19], defined a fractional q-integral operator $K^{-\alpha}_q$ by

$$\begin{aligned} K^{-\alpha}_q \phi(x) &:= \frac{q^{-\frac{1}{2}\alpha(\alpha-1)}}{\Gamma_q(\alpha)} \int_x^\infty t^{\alpha-1} \big(x/t;q\big)_{\alpha-1} \phi(tq^{1-\alpha})\, d_q t, \\ K^0_q \phi(x) &:= \phi(x), \end{aligned} \tag{4.27}$$

where $\alpha \neq -1,-2,\ldots$ as a generalization of the q-Cauchy formula

$$\begin{aligned} K^{-n}_q \phi(x) &= \int_x^\infty \int_{x_{n-1}}^\infty \cdots \int_{x_1}^\infty \phi(t)\, d_q t\, d_q x_1 \ldots d_q x_{n-1} \\ &= \frac{q^{-\frac{1}{2}n(n-1)}}{\Gamma_q(n)} \int_x^\infty t^{n-1} \, (x/t;q)_{n-1}\, \phi(tq^{1-n})\, d_q t, \end{aligned}$$

which he introduced in [18] for a positive integer n. The fractional q-integral operator $K^{-\alpha}_q$ is a q-analogue of Liouville fractional integral, K^α_-, defined by

$$K^\alpha_- \phi(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{\phi(t)}{(t-x)^{1-\alpha}}\, dt \quad (x > 0;\ \mathrm{Re}(\alpha) > 0). \tag{4.28}$$

Al-Salam formally proved the following semigroup identity, cf. [19],

$$K^\alpha_q K^\beta_q \phi(x) = K^{\alpha+\beta}_q \phi(x),$$

for all α and β. It is remarked by Ismail, cf. [143, P. 553] that

$$\int_0^\infty f(x) K_q^{-\alpha} g(x)\, d_q x = \int_0^\infty g(xq^{-\alpha}) I_q^\alpha f(x)\, d_q x. \tag{4.29}$$

There is a slight error on (4.29) as we shall see in Theorem 5.14. Al-Salam and Agarwal introduced their q-analogue with only zero as a lower point of the q-integration. Recently, Rajović et al. [254, 255] allowed the lower point of the q-integration in (4.24) to be nonzero and introduced the fractional definition:

Definition 4.2.1. The fractional q-integral is

$$I_{q,c}^\alpha f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_c^x (qt/x;q)_{\alpha-1} f(t)\, d_q t,$$

and the fractional q-derivative is

$$D_{q,c}^\alpha f(x) = D_q^{\lceil \alpha \rceil} I_{q,c}^{\alpha - \lceil \alpha \rceil} f(x).$$

Rajvoicć et al. [255] proved the following properties

Proposition 4.3. *Let $\alpha, \beta \in \mathbb{R}^+$. The q-fractional integration has the following semigroup property*

$$I_{q,c}^\alpha I_{q,c}^\beta f(x) = I_{q,c}^{\alpha+\beta} f(x).$$

Proposition 4.4. *For $\alpha \in \mathbb{R}^+$*

$$D_{q,c}^\alpha I_{q,c}^\alpha f(x) = f(x).$$

Al-Salam [19] and Agarwal [17] defined a two parameter q-fractional operator by

$$K_q^{\eta,\alpha} \phi(x) := \frac{q^{-\eta} x^\eta}{\Gamma_q(\alpha)} \int_x^\infty \left(x/t;q\right)_{\alpha-1} t^{-\eta-1} \phi(tq^{1-\alpha})\, d_q t, \tag{4.30}$$

$\alpha \neq -1, -2, \ldots$. This is a q-analogue of the Erdélyi and Sneddon fractional operator, cf. [93, 96],

$$K^{\eta,\alpha} \phi(x) = \frac{x^\eta}{\Gamma(\alpha)} \int_x^\infty \left(t - x\right)^{\alpha-1} t^{-\eta-1} \phi(t)\, dt. \tag{4.31}$$

By means of (1.20), (4.30) can be written as

$$K_q^{\eta,\alpha} \phi(x) = (1-q)^\alpha \sum_{k=0}^\infty q^{k\eta} \frac{(q^\alpha;q)_k}{(q;q)_k} \phi(xq^{-\alpha-k}).$$

Al-Salam derived formally the following semigroup identity for η, α and β in $\mathbb{R}^+$.

$$K_q^{\eta,\alpha} K_q^{\eta+\alpha,\beta} f(x) = K_q^{\eta,\alpha+\beta} f(x).$$

Al-Salam did not consider the problem of the existence of $K_q^{-\alpha}$, $K_q^{\eta,\alpha}\phi$. In [112], Galue generalizes the Erdélyi–Kober fractional q-integral operator of arbitrary order α introduced by Agarwal in [17] as follows

$$I_q^{\eta,\mu,\beta} f(x) = \frac{\beta x^{-\beta(\eta)}}{\Gamma_q(\eta)} \int_0^x \left(qt^\beta/x^\beta; q\right)_{\mu-1} t^{\beta(\eta+1)-1} f(t)\, d_q t,$$

$$\mathrm{Re}(\beta) > 0,\ \mathrm{Re}(\mu) > 0,\ \eta \in \mathbb{C}.$$

Then, he investigated various rules of composition for the above-defined operator.

Recently, Purohit and Yadav in [250] introduced two new fractional q-integral operators by

$$\begin{aligned} I_q^{\alpha,\beta,\eta} f(x) = {} & \frac{x^{-\beta-1} q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)} \\ & \times \int_0^x (qt/x;q)_{\alpha-1} \varepsilon^{-\frac{q^{\alpha+1}t}{x}} \left[{}_2\phi_1\left(q^{\alpha+\beta}, q^{-\eta}; q^\alpha; q, q\right)\right] \\ & \times f(t)\, d_q t, \end{aligned} \tag{4.32}$$

$$\begin{aligned} K_q^{\alpha,\beta,\eta} f(x) = {} & \frac{q^{-\eta(\alpha+\beta)-\frac{\alpha(\alpha+1)}{2}-2\beta}}{\Gamma_q(\alpha)} \\ & \times \int_x^\infty (x/t;q)_{\alpha-1} t^{-\beta-1} \varepsilon^{-\frac{q^{\alpha+1}t}{x}} \left[{}_2\phi_1\left(q^{\alpha+\beta}, q^{-\eta}; q^\alpha; q, q\right)\right] \\ & \times f(tq^{1-\alpha})\, d_q t \end{aligned} \tag{4.33}$$

where $\alpha > 0$, β is a real number, η is a nonnegative real number, and ε is the q-translation operator defined in (1.15). The series representations of the operators in (4.32) and (4.33) are

$$I_q^{\alpha,\beta,\eta} f(x) = x^{-\beta} q^{-\eta(\alpha+\beta)} (1-q)^\alpha \times$$
$$\sum_{n=0}^{\eta} q^n \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n} \sum_{k=0}^{\infty} q^k \frac{(q^{\alpha+n};q)_k}{(q;q)_k} f(xq^k),$$

and

$$K_q^{\alpha,\beta,\eta} = x^{-\beta} q^{-\eta(\alpha+\beta)-\frac{\alpha(\alpha+1)}{2}-\beta}(1-q)^{\alpha}\times$$
$$\sum_{n=0}^{\eta} q^n \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n} \sum_{k=0}^{\infty} q^{k\beta} \frac{(q^{\alpha+n};q)_k}{(q;q)_k} f(xq^{-\alpha-k}).$$

It is straightforward to see that Purohit and Yadav integral operators can be regarded as extensions of Riemann–Liouville, Weyl and Kober fractional q-integral operators with the following fractional relations

$$\begin{aligned} I_q^{\alpha,0,\eta} &= I_q^{\alpha}, \\ K_q^{\alpha,0,\eta} &= q^{-\alpha(\alpha+1)/2} K_q^{\eta,\alpha}, \\ K_q^{\alpha,-\alpha,\eta} &= K_q^{\alpha}. \end{aligned}$$

Purohit and Yadav derived the fractional q-integration by parts formula:

$$\int_0^{\infty} f(x) K_q^{\alpha,\beta,\gamma} g(x)\, d_q x = q^{-\alpha(\alpha+1)/2-\beta} \int_0^{\infty} g(xq^{-\alpha}) I_q^{\alpha,\beta,\gamma} f(x)\, d_q x, \quad (4.34)$$

for $\alpha > 0$, β is real, and η is a nonnegative integer, provided that both of the q-integrals in (4.34) exist. The operators (4.32) and (4.33) are q-extensions of the operators

$$I_x^{\alpha,\beta,\eta} f = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha, 1-\frac{t}{x}\right) f(t)\, dt, \quad (4.35)$$

$$J_x^{\alpha,\beta,\eta} f = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (x-t)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\eta; \alpha, 1-\frac{t}{x}\right) f(t)\, dt,$$

where ${}_2F_1$ is the Gaussian hypergeometric functions defined by

$${}_2F_1(a,b;c,x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}.$$

4.2.1 The Time Scale Fractional q-Calculus

The time scale fractional calculus was introduced by Hilger and Aulbad in [46, 135] then developed by Bohner and Petrson in [59–61]. It merges the theory of difference equations and the theory of differential equations into one theory. A time scale means a closed subset of the real line. A merger of the Riemann–Liouville fractional q-calculus and Riemann–Liouville fractional q-calculus on time scale was initiated in the paper of Atici and Eloe in [44, 2007]. They use the time scale calculus notations to develop Hahn q-Laplace transform operator, ${}_qL_s$, on the time scale $\mathbb{T} := \{q^n : n \in \mathbb{N}_0\} \cup \{0\}$ and define a fractional q-difference equation on $\mathbb{T}$ and finally apply the q-transform method to find solutions. A development for the Caputo fractional q-derivative in the time scale fractional calculus can be found in [2].

4.3 q-Abel Integral Equation

In this section, we define a q-analogue of Abel's integral equation. So we need the q-analogues of the classical classes $L_1(a,b)$ and $\mathscr{A}C[a,b]$ introduced in Sect. 1.2. In Sect. 2.2, we introduced a q-analogue of the class of integrable functions. Now, we introduce a q-analogue of $\mathscr{A}C[0,a]$ and we denote it by $\mathscr{A}C_q[0,a]$.

Definition 4.3.1. A function f defined on $[0,a]$ is called q-absolutely continuous if f is q-regular at zero, and there exists $K > 0$ such that

$$\sum_{j=0}^{\infty} \left| f(tq^j) - f(tq^{j+1}) \right| \leq K \quad \text{for all } t \in (qa, a]. \tag{4.36}$$

If (4.36) holds then it can be extended throughout $(0,a]$. To see this, it suffices to investigate the case when $x \in (0,a]$. Indeed, if $x \in (0,a]$ then there exists $t \in (qa,a]$ and $k \in \mathbb{N}$ such that $x = tq^k$. Then

$$\begin{aligned} \sum_{j=0}^{\infty} \left| f(xq^j) - f(xq^{j+1}) \right| &= \sum_{j=k}^{\infty} \left| f(tq^j) - f(tq^{j+1}) \right| \\ &\leqslant \sum_{j=0}^{\infty} \left| f(tq^j) - f(tq^{j+1}) \right| < \infty. \end{aligned}$$

We shall use $\mathscr{A}C_q[0,a]$ to denote the class of q-absolutely continuous functions on $[0,a]$.

Theorem 4.5. *Let f be a function defined on $[0,a]$. Then, the function $f \in \mathscr{A}C_q[0,a]$ if and only if there exists a constant c and a function ϕ in $\mathscr{L}_q^1[0,a]$ such that*

$$f \in \mathscr{A}C_q[0,a] \Longleftrightarrow f(x) = c + \int_0^x \phi(u)\, d_q u \quad \textit{for all } x \in [0,a]. \tag{4.37}$$

Moreover, the constant c and the function ϕ are uniquely determined via $c = f(0)$ and

$$\phi(x) = D_q f(x) \quad \textit{for all } x \in (0,a].$$

Proof. Assume that $f \in \mathscr{A}C_q[0,a]$. Consequently, from the fundamental theorem of q-calculus, Theorem 1.10, we obtain

$$f(x) = f(0) + \int_0^x D_q f(t)\, d_q t, \quad x \in [0,a],$$

proving necessity. Assume that f is given by (4.37) for $\phi \in \mathscr{L}_q^1[0,a]$. Then for $x \in [0,a]$, we have

$$\lim_{n\to\infty} f(xq^n) = \lim_{n\to\infty} \left(c + x(1-q)\sum_{k=n}^{\infty} q^k \phi(xq^k) \right) = c.$$

Therefore, the function f is q-regular at zero. Applying Theorem 1.10 again, we obtain $D_q f(x) = \phi(x)$, $x \in (0,a]$. Hence, $D_q f(x) \in \mathscr{L}_q^1[0,a]$. This directly leads to (4.36), proving sufficiency. The uniqueness of c and ϕ holds by construction. □

Definition 4.3.2. Let $\mathscr{A}C_q^{(n)}[0,a]$, $n \in \mathbb{N}$, be the space of all functions f defined on $[0,a]$ such that f, $D_q f$, …, $D_q^{n-1} f$ are q-regular at zero and $D_q^{n-1} f(x) \in \mathscr{A}C_q[0,a]$.

When $n = 1$, we simply write $\mathscr{A}C_q[0,a]$ for $\mathscr{A}C_q^{(1)}[0,a]$.

Theorem 4.6. *A function $f : [0,a] \to \mathbb{C}$ lies in $\mathscr{A}C_q^{(n)}[0,a]$ if and only if there exists a function $\phi \in \mathscr{L}_q^1[0,a]$ such that*

$$f(x) = \sum_{k=0}^{n-1} c_k x^k + \frac{x^{n-1}}{\Gamma_q(n)} \int_0^x \big(qt/x;q\big)_{n-1} \phi(t)\, d_q t, \tag{4.38}$$

where

$$x \in (0,a], \quad n \in \mathbb{N}, \quad c_k = \frac{D_q^k f(0)}{\Gamma_q(k+1)},$$

and $\phi(x) = D_q^n f(x)$ for all $x \neq 0$.

Proof. First, we prove sufficiency. Assume that f has the form (4.38). Then, the q-derivative of order j for $x \neq 0$ is

$$D_q^j f(x) = \sum_{k=j}^{n-1} c_k (1-q^k)(1-q^{k-1})\dots(1-q^{k-j+1})(1-q)^{-j} x^{k-j}$$
$$+ \frac{x^{n-j-1}}{\Gamma_q(n-j)} \int_0^x (qt/x;q)_{n-j-1} \phi(t)\, d_q t,$$

where $j = 0, 1, \ldots, n-1$. We can prove by induction that

$$D_q^j f(0) = c_j \frac{(q;q)_j}{(1-q)^j} \quad (j = 0, 1, \ldots, n-1).$$

Since

$$\left| \int_0^{xq^m} (qt/xq^m; q)_{n-j-1} \phi(t)\, d_q t \right| = \left| \sum_{r=0}^{\infty} xq^{m+r}(1-q)(q^{r+1}; q)_{n-j-1} \phi(xq^{m+r}) \right|$$
$$\leqslant \sum_{r=m}^{\infty} xq^r (1-q) |\phi(xq^r)|$$

and $\phi \in \mathscr{L}_q^1[0,a]$, then

$$\lim_{m\to\infty} D_q^j f(xq^m) = c_j \frac{(q;q)_j}{(1-q)^j} = D_q^j f(0) \quad (j = 0, 1, \ldots, n-1)\,.$$

Hence, the functions $D_q^j f$, $j = 0, 1, \ldots, n-1$, are q-regular at zero. Clearly,

$$D_q^n f(x) = \phi(x) \in \mathscr{L}_q^1[0,a],$$

proving sufficiency. As for necessity, we use mathematical induction. The case $n = 1$ is proved in Theorem 4.5. Now, assume that (4.38) holds at $n = m$. Let $f(x) \in \mathscr{AC}_q^{(m+1)}[0,a]$. That is, $f, \ldots, D_q^m f$ are q-regular at zero and $D_q^m f \in \mathscr{AC}_q[0,a]$. Hence, $D_q f \in \mathscr{AC}_q^{(m)}[0,a]$. Then, from the induction hypothesis we have

$$D_q f(x) = \sum_{j=0}^{m-1} c_j x^j + \frac{x^{m-1}}{\Gamma_q(m)} \int_0^x \big(qt/x;q\big)_{m-1} \phi(t)\, d_q t, \tag{4.39}$$

where $x \in (0,a]$,

$$c_j = \frac{D_q^j D_q f(0)}{\Gamma_q(j+1)} = \frac{D_q^{j+1} f(0)}{\Gamma_q(j+1)} \quad \text{and} \quad \phi(x) = D_q^{m+1} f(x)$$

for all $x \neq 0$. Now, replacing x by t and t with u in (4.39); integrating from 0 to x and applying Lemma 1.20 yield

$$f(x) = \sum_{k=0}^{m} \frac{D_q^k f(0)}{\Gamma_q(k+1)} x^k + \frac{x^m}{\Gamma_q(m+1)} \int_0^x \big(qt/x;q\big)_m D_q^{m+1} f(t)\, d_q t,$$

which completes the proof. □

As in the case of the Riemann–Liouville fractional derivative, the existence of the fractional q-Riemann–Liouville derivative holds only for a more restrictive class of functions. For this reason, we study a q-analogue of Abel's integral equation.

Theorem 4.7. *The q-Abel integral equation*

$$\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x \big(qt/x;q\big)_{\alpha-1}\,\phi(t)\,d_qt = f(x) \quad (0<\alpha<1,\ x\in(0,a]) \tag{4.40}$$

has a unique solution $\phi(x)\in\mathscr{L}_q^1[0,a]$ if and only if

$$I_q^{1-\alpha}f(x)\in\mathscr{A}C_q[0,a] \quad \textit{and} \quad I_q^{1-\alpha}f(0)=0. \tag{4.41}$$

Moreover, the unique solution $\phi(x)$ is given by

$$\phi(x) = D_{q,x}I_q^{1-\alpha}f(x). \tag{4.42}$$

Proof. First of all, we prove that the q-Abel integral equation cannot have more than one $\mathscr{L}_q^1[0,a]$-solution. Assume that (4.40) has a solution $\phi\in\mathscr{L}_q^1[0,a]$. Hence, replacing x with t and t with u in (4.40), multiplying both sides of the equation by $x^{-\alpha}\big(qt/x;q\big)_{-\alpha}$, and then integrating from 0 to x, we obtain

$$\begin{aligned}\frac{x^{-\alpha}}{\Gamma_q(\alpha)}\int_0^x \big(qt/x;q\big)_{-\alpha}\int_0^t t^{\alpha-1}\big(qu/t;q\big)_{\alpha-1}\phi(u)\,d_qu\,d_qt \\ = x^{-\alpha}\int_0^x \big(qt/x;q\big)_{-\alpha}f(t)\,d_qt.\end{aligned}$$

But from (1.66),

$$\begin{aligned}\frac{x^{-\alpha}}{\Gamma_q(\alpha)}\int_0^x \big(qt/x;q\big)_{-\alpha}\int_0^t t^{\alpha-1}\big(qu/t;q\big)_{\alpha-1}\phi(u)\,d_qu\,d_qt \\ = \frac{B_q(1-\alpha,\alpha)}{\Gamma_q(\alpha)}\int_0^x \phi(t)\,d_qt.\end{aligned}$$

Consequently,

$$I_q\phi(x) = I_q^{1-\alpha}f(x). \tag{4.43}$$

Therefore, any $\mathscr{L}_q^1[0,a]$-solution of (4.40) must satisfy the relation (4.43). Suppose that ϕ and ψ are two $\mathscr{L}_q^1[0,a]$-solutions of (4.40). From relation (4.43) and the linearity of q-integration, we obtain

$$I_q(\phi-\psi)(x) = \int_0^x (\phi(t)-\psi(t))\,d_qt = 0 \quad \text{for all } x\in(0,a].$$

Hence, from the fundamental theorem of q-calculus, Theorem 1.10, we obtain $\phi \equiv \psi$ on $(0, a]$. That is, we have at most one solution in $\mathscr{L}_q^1[0, a]$. Now we prove necessity. Assume that (4.40) has a solution $\phi \in \mathscr{L}_q^1[0, a]$. Then, ϕ satisfies (4.43). Applying $D_q = D_{q,x}$ to (4.43), we obtain (4.42). By Theorem 4.5 and (4.42), $I_q^{1-\alpha} f \in \mathscr{A}C_q[0, a]$ and $I_q^{1-\alpha} f(0) = 0$.

Conversely, assume that f satisfies (4.41) and $\phi(x) := D_{q,x} I_q^{1-\alpha} f(x)$. Then $\phi \in \mathscr{L}_q^1(0, a)$. We prove that ϕ is a solution of (4.40). Indeed, let g be the function defined by

$$g(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \big(qt/x; q\big)_{\alpha-1} \phi(t)\, d_q t \quad \text{for all} \quad x \in (0, a]. \tag{4.44}$$

It suffices to show that $g(x) = f(x)$, $x \in (0, a]$. From the necessity part, ϕ should have the form (4.42) with g instead of f. Thus,

$$\begin{aligned} \phi(x) &= D_{q,x} I_q^{1-\alpha} f(x) = D_{q,x} I_q^{1-\alpha} g(x) \\ &= D_{q,x} \frac{x^{-\alpha}}{\Gamma_q(1-\alpha)} \int_0^x \big(qt/x; q\big)_{-\alpha} g(t)\, d_q t, \end{aligned}$$

for all $x \in (0, a]$. The linearity of the q-integration implies

$$D_{q,x}\big(I_q^{1-\alpha}\big(f - g\big)(x)\big) = 0 \quad \text{for all} \quad x \in (0, a]. \tag{4.45}$$

Therefore, the function

$$c(x) := I_q^{1-\alpha}\big(f - g\big)(x) \quad \text{for all} \ \ x \in (0, a], \tag{4.46}$$

is a q-periodic function, i.e. $c(x) = c(qx)$. The function $I_q^{1-\alpha} g(x) \in \mathscr{A}C_q[0, a]$ because it satisfies (4.43) with $g(x)$ on the right hand side. Moreover, $I_q^{1-\alpha} g(0) = 0$. Thus,

$$I_q^{1-\alpha}\big(f - g\big)(x) \in \mathscr{A}C_q[0, a].$$

Accordingly, $I_q^{1-\alpha}\big(f - g\big)$ is q-regular at zero with $I_q^{1-\alpha}\big(f - g\big)(0) = 0$. Consequently, $c(x) \equiv 0$. That is,

$$\frac{x^{-\alpha}}{\Gamma_q(1-\alpha)} \int_0^x \big(qt/x; q\big)_{-\alpha} (f(t) - g(t))\, d_q t = 0 \ \text{ for all } \ x \in (0, a].$$

The previous equation is in the form (4.40). By the uniqueness of its solutions, we obtain $f(x) = g(x)$, for all $x \in (0, a]$. □

Notice that in the previous proof, we have used the fact that the only q-regular q-periodic functions are the constants. Indeed, if c has these two properties, then

$$c(x) = c(qx) = \ldots c(q^n x) \longrightarrow c(0) \quad \text{as } n \longrightarrow \infty.$$

We would like to mention also that another proof of sufficiency could be derived by direct computations using a technique similar to that applied in proving necessity. The following two examples confirm the validity of the theorem.

Example 4.3.1. Consider the q-Abel integral equation

$$\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x \big(qt/x;q\big)_{\alpha-1}\,\phi(t)\,d_qt = x \quad (0<\alpha<1,\ x\in(0,a]). \tag{4.47}$$

An easy computation gives that

$$I_q^{1-\alpha}f(x) = \frac{x^{2-\alpha}}{\Gamma_q(3-\alpha)}.$$

Then, $I_q^{1-\alpha}f(0) = 0$ and $I_q^{1-\alpha}f(x) \in \mathscr{A}C_q[0,a]$. Consequently, (4.47) has a unique solution given by

$$\phi(x) = \frac{x^{1-\alpha}}{\Gamma_q(2-\alpha)}.$$

Substitute with $\phi(t)$ on the left hand side of (4.47) and make the substitution $t/x = u$, we obtain

$$\begin{aligned}\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x \big(qt/x;q\big)_{\alpha-1}\,\phi(t)\,d_qt &= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)\Gamma_q(1-\alpha)}\int_0^x (qt/x;q)_{\alpha-1}t^{1-\alpha}\,d_qt\\ &= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)\Gamma_q(2-\alpha)}\int_0^1 (qu;q)_{\alpha-1}u^{1-\alpha}\,d_qu\\ &= \frac{x}{\Gamma_q(\alpha)\Gamma_q(2-\alpha)}B_q(\alpha,2-\alpha) = x.\end{aligned}$$

Example 4.3.2. Let $N\in\mathbb{N}_0$. Consider the q-Abel integral equation

$$\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x \big(qt/x;q\big)_{\alpha-1}\,\phi(t)\,d_qt = (x;q)_N, \tag{4.48}$$

$$0<\alpha<1, \quad \text{and} \quad x\in(0,a].$$

Since

$$f(x) := (x;q)_N = \sum_{j=0}^{N}(-1)^j \begin{bmatrix} N \\ j\end{bmatrix}_q q^{j(j-1/2)}x^j,$$

then

$$I_q^{1-\alpha}f(x) = \Gamma_q(N+1)\sum_{j=0}^{N}(-1)^j\frac{q^{j(j-1)/2}}{\Gamma_q(N-j+1)\Gamma_q(j-\alpha+2)}x^{j+1-\alpha}.$$

Hence, $I_q^{1-\alpha} f(0) = 0$ and it is a q-absolutely continuous function. Therefore, (4.48) has a unique solution give by

$$\phi(x) = \Gamma_q(N+1) \sum_{j=0}^{N} (-1)^j \frac{q^{j(j-1)/2}}{\Gamma_q(N-j+1)\Gamma_q(j-\alpha+1)} x^{j-\alpha}.$$

Substituting with $\phi(t)$ into the left hand side of (4.48), we obtain

$$\begin{aligned}
&\frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}\, \phi(t)\, d_qt \\
&= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \sum_{j=0}^{N} (-1)^j \frac{q^{j(j-1)}/2}{\Gamma_q(N-j+1)\Gamma_q(j-\alpha+1)} \int_0^x (qt/x;q)_{\alpha-1} t^{j-\alpha}\, d_qt \\
&= \frac{\Gamma_q(N+1)}{\Gamma_q(\alpha)} \sum_{j=0}^{N} (-1)^j \frac{q^{j(j-1)}/2}{\Gamma_q(N-j+1)\Gamma_q(j-\alpha+1)} B_q(\alpha, j-\alpha+1) \\
&= \sum_{j=0}^{N} (-1)^j \begin{bmatrix} N \\ j \end{bmatrix}_q q^{j(j-1)/2} x^j = f(x).
\end{aligned}$$

As in Theorem 4.1, the solution ϕ of q-Abel's equation (4.40) should be the fractional q-derivative of order α of f. Starting from this observation, we define the basic fractional derivative as follows:

Definition 4.3.3. Let f be a function defined on $[0,a]$. For $\alpha > 0$ the fractional q-derivative of order α is defined to be

$$D_q^\alpha f(x) := \phi(x) = D_q^k I_q^{k-\alpha} f(x) \quad (k = \lceil \alpha \rceil), \tag{4.49}$$

provided that

$$f(x) \in \mathscr{L}_q^1[0,a] \quad \text{and} \quad I_q^{k-\alpha} f(x) \in \mathscr{A}C_q^{(k)}[0,a]. \tag{4.50}$$

For example,

$$D_q^\alpha x^{\beta-1} = \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\alpha)} x^{\beta-\alpha-1} \quad (\beta > 0;\ \alpha \in \mathbb{R}). \tag{4.51}$$

Theorem 4.7 guarantees that $D_q^\alpha f(x) \in \mathscr{L}_q^1[0,a]$, provided (4.50) is fulfilled.

The following lemma gives the relationship between the fractional q-derivative of order α and $I_q^{-\alpha}$, $\alpha > 0$.

Lemma 4.8. *Let* $f \in \mathscr{L}_q^1[0,a]$ *and* $\alpha > 0$, $k = \lceil \alpha \rceil$, *be such that* $I_q^{k-\alpha} f \in \mathscr{A}C_q^{(k)}[0,a]$. *Then*

$$D_q^\alpha f(x) = I_q^{-\alpha} f(x) = \frac{x^{-\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \big(qt/x;q\big)_{-\alpha-1} f(t)\, d_qt \;\; \textit{for} \;\; x \in (0,a].$$

Proof. From (4.49), we have

$$\begin{aligned} D_q^\alpha f(x) &= D_q^k I_q^{k-\alpha} f(x) \\ &= D_q^k \left(\frac{x^{k-\alpha-1}}{\Gamma_q(k-\alpha)} \int_0^x \big(qt/x;q\big)_{k-\alpha-1} f(t)\, d_qt \right). \end{aligned} \tag{4.52}$$

Set

$$h(t,x) := \frac{x^{k-\alpha-1}}{\Gamma_q(k-\alpha)} \big(qt/x;q\big)_{k-\alpha-1} f(t).$$

Hence,

$$h(xq^r, xq^j) = 0 \quad (r = 0,1,\ldots,j-1; \quad j = 1,2,\ldots,k)\,.$$

Applying Lemma 1.12 to (4.52), we obtain

$$\begin{aligned} D_q^\alpha f(x) &= \int_0^x D_{q,x}^k \frac{x^{k-\alpha-1}}{\Gamma_q(k-\alpha)} \big(qt/x;q\big)_{k-\alpha-1} f(t)\, d_qt \\ &= \frac{x^{-\alpha-1}}{\Gamma_q(-\alpha)} \int_0^x \big(qt/x;q\big)_{-\alpha-1} f(t)\, d_qt = I_q^{-\alpha} f(x) \end{aligned}$$

□

4.4 Some Properties of q-Riemann–Liouville Fractional Integral Operator

In this section, we derive basic properties of the Riemann–Liouville fractional q-integral operator in certain function spaces.

Lemma 4.9. *Let* $\alpha \in \mathbb{R}^+$ *and* $f : (0,a] \to \mathbb{C}$ *be a function. If* $f \in \mathscr{L}_q^1[0,a]$ *then* $I_q^\alpha f \in \mathscr{L}_q^1[0,a]$ *and*

$$\left\| I_q^\alpha f \right\|_1 \le \frac{a^\alpha}{\Gamma_q(\alpha+1)} \, \|f\|_1 \,. \tag{4.53}$$

Proof. Let f be a function defined on $(0,a]$ such that $f \in \mathscr{L}_q^1[0,a]$. Then from (4.24) we get for $x \in (qa,a]$

$$
\begin{aligned}
\int_0^x |I_q^\alpha f(t)|\, d_q t &= \frac{1}{\Gamma_q(\alpha)} \int_0^x t^{\alpha-1} \left| \int_0^t \left(qu/t;q\right)_{\alpha-1} f(u)\, d_q u \right| d_q t \\
&\leqslant \frac{1}{\Gamma_q(\alpha)} \int_0^x t^{\alpha-1} \int_0^t \left(qu/t;q\right)_{\alpha-1} |f(u)|\, d_q u\, d_q t \\
&= \frac{x^{\alpha+1}(1-q)^2}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} q^{n\alpha} \sum_{m=0}^{\infty} q^{n+m} \left(q^{m+1};q\right)_{\alpha-1} |f(xq^{n+m})| \qquad (4.54) \\
&= \frac{x^{\alpha+1}(1-q)^2}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} q^{n\alpha} \sum_{k=n}^{\infty} q^{k} \left(q^{k-n+1};q\right)_{\alpha-1} |f(xq^{k})| \\
&= \frac{x^{\alpha+1}(1-q)}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} q^{k} |f(xq^{k})| \sum_{n=0}^{k} (1-q) q^{n\alpha} \left(q^{k-n+1};q\right)_{\alpha-1}.
\end{aligned}
$$

But from (1.61) we obtain

$$
\sum_{n=0}^{k} (1-q) q^{n\alpha} \left(q^{k-n+1};q\right)_{\alpha-1} = B_q(1,\alpha) \left(q^{k+1};q\right)_{\alpha}.
$$

Since $(q^{k+1};q)_\alpha \leq 1$ for all $k \in \mathbb{N}_0$, then

$$
\begin{aligned}
\int_0^x |I_q^\alpha f(t)|\, d_q t &\leqslant B_q(1,\alpha) x^\alpha \sum_{k=0}^{\infty} q^k (1-q) |f(xq^k)| \\
&= \frac{x^\alpha}{\Gamma_q(\alpha)} B_q(1,\alpha) \int_0^x |f(t)|\, d_q t \qquad (4.55) \\
&\leqslant \frac{a^\alpha}{\Gamma_q(\alpha+1)} \|f\|_1 .
\end{aligned}
$$

Hence $I_q^\alpha f \in \mathscr{L}_q^1[0,a]$ and (4.53) is proved. ◻

We shall also need the following lemma:

Lemma 4.10. *Let $\gamma \in \mathbb{R}$, $\gamma < 1$, and let $g \in C_\gamma[0,a]$. Then*

(1) $I_q^\alpha g \in C_\gamma[0,a]$ and

$$
\left\| I_q^\alpha g \right\|_{C_\gamma} \leq a^\alpha \frac{\Gamma_q(1-\gamma)}{\Gamma_q(1+\alpha-\gamma)} \|g\|_{C_\gamma} . \qquad (4.56)
$$

(2) If we additionally assume that $\gamma \leq \alpha$, then $I_q^\alpha g \in C[0,a]$,

Proof. To prove that $I_q^\alpha g \in C_\gamma[0,a]$, it is equivalent to prove $x^\gamma I_q^\alpha g \in C[0,a]$. Since $x^\gamma g(x) \in C[0,a]$, there exits $M > 0$ such that

$$M := \max_{0\le x\le a} |x^\gamma g(x)|\,.$$

Hence,

$$\begin{aligned}\left|x^\gamma I_q^\alpha g(x)\right| &\le \frac{M}{\Gamma_q(\alpha)} x^{\gamma+\alpha-1}\int_0^x (qt/x;q)_{\alpha-1} t^{-\gamma}\, d_q t\\ &= \frac{Mx^\alpha}{\gamma_q(\alpha)}\int_0^1 (q\xi;q)_{\alpha-1}\xi^{-\gamma}\, d_q\xi = \frac{Mx^\alpha}{\Gamma_q(\alpha)} B_q(\alpha, 1-\gamma).\end{aligned}$$

Hence $\lim_{x\to 0} x^\gamma I_q^\alpha g(x) = 0$ and we can assume the continuity of the function $x^\gamma I_q^\alpha g$ at zero by assuming that its value at zero is zero. Now we prove the continuity of the function $x^\gamma I_q^\alpha g(x)$ at any $x_0 \ne 0$. Since $x^\gamma g(x) \in C[0,a]$, it is uniformly continuous on $[0,a]$. Hence, $\forall \epsilon > 0$ there exists $\delta > 0$ such that for all $x,\ y \in [0,a]$

$$|x-y| < \delta \longrightarrow |x^\gamma g(x) - y^\gamma g(y)| < \epsilon.$$

Assume that $|x - x_0| < \delta$ and we may assume that $x_0 - \delta > 0$. Consequently,

$$0 \ne (x_0 - \delta, x_0 + \delta).$$

Now

$$\begin{aligned}|x - x_0| < \delta &\longrightarrow |xq^k - x_0 q^k| < \delta\\ &\longrightarrow \left|x(q^k)^\gamma g(xq^k) - (x_0q^k)^\gamma g(x_0q^k)\right| < \epsilon,\end{aligned}$$

for all $k \in \mathbb{N}_0$. Consequently,

$$\sum_{k=0}^\infty q^{k(1-\gamma)}\frac{(q^\alpha;q)_k}{(q;q)_k}\left|x(q^k)^\gamma g(xq^k) - (x_0q^k)^\gamma g(x_0q^k)\right| < \frac{(q^{\alpha+1-\gamma};q)_\infty}{(q^{1-\gamma};q)_\infty}\epsilon.$$

That is,

$$\left|x^{\gamma-\alpha} I_q^\alpha g(x) - x_0^{\gamma-\alpha} I_q^\alpha g(x_0)\right| < \frac{(q^{\alpha+1-\gamma};q)_\infty}{(q^{1-\gamma};q)_\infty}\epsilon.$$

Hence, $x^{\gamma-\alpha} I_q^\alpha g(x) \in C[0,a]$. Consequently,

$$x^\gamma I_q^\alpha g(x) = x^\alpha x^{\gamma-\alpha} I_q^\alpha g(x) \in C[0,a].$$

That is, $I_q^\alpha g \in C_\gamma[0,a]$.

Now we prove Inequality (4.56). Indeed,

$$\begin{aligned}\left\|I_q^\alpha g\right\|_{C_\gamma} &= \max_{0\le x\le a}\left|x^\gamma I_q^\alpha g(x)\right|\\ &= \max_{0\le x\le a}\left|\frac{x^{\gamma+\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} g(t)\,d_qt\right|\\ &\le \|g\|_{C_\gamma}\max_{0\le x\le a}\left|\frac{x^{\gamma+\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} t^{-\gamma}\,d_qt\right|. \end{aligned} \tag{4.57}$$

Since $0 < \gamma < 1$, the q-integration on the most right side of (4.57) converges. Moreover, it can be calculated by making the substitution $t = x\xi$. This gives

$$\int_0^x (qt/x;q)_{\alpha-1} t^{-\gamma}\,d_qt = x^{1-\gamma}\int_0^1 (q\xi;q)_{\alpha-1}\xi^{-\gamma}\,d_q\xi = x^{-\gamma+1}B_q(\alpha,1-\gamma)\,. \tag{4.58}$$

Substituting (4.58) into (4.57), we get the required inequality. The proof of (2) follows by using (1) and noting that

$$I_q^\alpha g(x) = x^{\alpha-\gamma} I_q^\gamma g(x) \quad \text{for all } x\in[0,a],$$

and the product of continuous functions is continuous. □

Lemma 4.11. *Let* $\alpha > 0$, $\gamma < 1$ *and* $g\in C_\gamma[0,a]$. *Then*

(i) If $\gamma < \alpha$, *then* $lim_{x\to 0^+} I_q^\alpha g(x) = 0$.
(ii) If $\alpha = \gamma$, *then*

$$\lim_{x\to 0+} x^\gamma g(x) = c \quad \textit{if and only if} \quad \lim_{x\to 0+} I_q^\alpha g(x) = c\Gamma_q(1-\alpha). \tag{4.59}$$

Proof. Since

$$I_q^\alpha g(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} t^{-\gamma}\left(t^\gamma g(t)-c\right)\,d_qt + c\frac{x^{\alpha-\gamma}}{\Gamma_q(\alpha)}B_q(\alpha,1-\gamma)$$

and

$$\left|\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} t^{-\gamma}\left(t^\gamma g(t)-c\right)\,d_qt\right| \le \frac{B_q(\alpha,1-\gamma)}{\Gamma_q(\alpha)}x^{\alpha-\gamma}\max_{0\le t\le x}|t^\gamma g(t)-c|,$$

we obtain from the continuity of the function $t^\gamma g$ at zero that

$$\lim_{x\to 0+}\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} t^{-\gamma}\left(f(t)-f(0)\right)\,d_qt = 0.$$

Hence

$$\lim_{x\to 0^+} I_q^\alpha g(x) = c \lim_{x\to 0^+} \frac{x^{\alpha-\gamma}}{\Gamma_q(\alpha)} B_q(\alpha, 1-\gamma) = \begin{cases} 0, & \gamma < \alpha, \\ c\Gamma_q(1-\alpha), & \gamma = \alpha. \end{cases}$$

Now we prove the sufficient condition in (4.59). Therefore, we assume that

$$\lim_{x\to 0^+} I_q^\alpha g(x) = d\Gamma_q(1-\alpha),$$

where d is a constant. Since $\Gamma_q(1-\alpha)$ can be represented as

$$(1-q)^\alpha \sum_{k=0}^{\infty} q^{(1-\alpha)k} \frac{(q^\alpha;q)_k}{(q;q)_k},$$

then

$$I_q^\alpha g(x) - c\Gamma_q(1-\alpha) = (1-q)^\alpha \sum_{k=0}^{\infty} q^{k(1-\alpha)} \frac{(q^\alpha;q)_k}{(q;q)_k} \left((xq^k)^\alpha g(xq^k) - d\right). \quad (4.60)$$

Since $\lim_{x\to 0^+} x^\alpha g(x) = c$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that

$$\left|(xq^k)^\alpha g(xq^k) - d\right| \leq M \quad \text{for all} \quad x \in (-\delta, \delta).$$

Consequently,

$$q^{k(1-\alpha)} \frac{(q^\alpha;q)_k}{(q;q)_k} \left|(xq^k)^\alpha g(xq^k) - d\right| \leq C q^{k(1-\alpha)}, \; C := M \frac{(1-q)^\alpha}{(q^\alpha;q)_\infty}.$$

Hence, the series on the right hand side of (4.60) is uniformly convergent to zero on $(-\delta, \delta)$, and we can calculate the limit as $x \to 0^+$ on (4.60) term by term. This gives $c - d = 0$. That is, $d = c$, completing the proof of (ii) and the lemma. □

In the following two lemmas, we study the limits $\lim_{q\to 1^-} I_q f(x)$ and $\lim\limits_{\alpha\to 0^+} I_q^\alpha f(x)$.

Lemma 4.12. *Let $\alpha > 0$ and f be a function defined on $[0, a]$, $a > 0$. If f is Riemann integrable on $[0, x]$, then*

$$\lim_{q\to 1^-} I_q^\alpha f(x) = I^\alpha f(x) \quad (x \in (0, a]; \; \alpha > 0).$$

Proof. Let $x \in (0, a]$ be fixed. From (1.59) and (1.12), we obtain for $0 < t < x$

$$\begin{aligned}\lim_{q\to 1^-} x^{\alpha-1}\big(qt/x;q\big)_{\alpha-1} &= \lim_{q\to 1^-} x^{\alpha-1}\frac{\big(qt/x;q\big)_\infty}{\big(q^\alpha t/x;q\big)_\infty}\\ &= x^{\alpha-1}\left(1-\frac{t}{x}\right)^{\alpha-1} = (x-t)^{\alpha-1}\end{aligned}$$

uniformly for $0 < t < x$. Hence, given $\epsilon > 0$ there exists $\delta > 0$ such that for all $t \in \{q^k x,\ k \in \mathbb{N}\}$, we have

$$\left|x^{\alpha-1}\big(qt/x;q\big)_{\alpha-1} - (x-t)^{\alpha-1}\right| \leqslant \epsilon/2$$

whenever $0 < 1-q < \delta$. That is,

$$\left|I_q^\alpha f(x) - \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\, d_qt\right| \leqslant \frac{\epsilon}{2\Gamma_q(\alpha)}\int_0^x |f(t)|\, d_qt.$$

Since

$$\begin{aligned}\left|I_q^\alpha f(x) - I^\alpha f(x)\right| &\leqslant \left|I_q^\alpha f(x) - \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\, d_qt\right|\\ &\quad + \left|\frac{1}{\Gamma_q(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\, d_qt - I_\alpha f(x)\right|,\end{aligned}$$

then the lemma follows if f is Riemann integrable on $[0, x]$. □

Lemma 4.13. *Let $\nu > 0$ and let $f \in H_\nu[0, a]$. Then for $\alpha > 0$*

$$I_q^\alpha f(x) = \frac{x^\alpha}{\Gamma_q(\alpha+1)} f(0) + F(x),$$

where

$$F(x) = O(x^{\alpha+\nu}).$$

Proof.

$$\begin{aligned}I_q^\alpha f(x) &= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} f(t)\, d_qt\\ &= \frac{x^\alpha}{\Gamma_q(\alpha+1)} f(0) + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}(f(t)-f(0))\, d_qt.\end{aligned}$$

Set

$$F(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1}(f(t) - f(0))\, d_q t.$$

Since $f \in H_\nu[0,a]$, there exists $c > 0$ such that

$$|f(t) - f(0)| < ct^\nu \quad \text{for all} \quad t \in [0,a].$$

Therefore,

$$|F(x)| \leq \frac{cx^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} t^\nu\, d_q t = cx^{\alpha+\nu} \frac{\Gamma_q(\nu+1)}{\Gamma_q(\nu+\alpha+1)},$$

proving the lemma. □

Lemma 4.14. *Let* $f \in \mathscr{L}_q^1[0,a]$. *Then*

$$\lim_{\alpha\to 0+} I_q^\alpha f(x) = f(x) \quad \textit{for all} \quad x \in (0,a].$$

Proof. Let $x \in (0,a]$ be fixed. From (4.26), we have

$$\lim_{\alpha\to 0+} I_q^\alpha f(x) = \lim_{\alpha\to 0+} x^\alpha (1-q)^\alpha \sum_{n=0}^\infty q^n \frac{(q^\alpha;q)_n}{(q;q)_n} f(xq^n). \tag{4.61}$$

Since

$$q^n \frac{(q^\alpha;q)_n}{(q;q)_n} |f(xq^n)| \leqslant q^n |f(xq^n)| \quad (n \in \mathbb{N}_0,\ \alpha \in (0,1]),$$

from Weierstrass M-test, the series in (4.61) is uniformly convergent for $\alpha \in (0,1)$. Thus,

$$\lim_{\alpha\to 0+} \sum_{n=0}^\infty q^n \frac{(q^\alpha;q)_n}{(q;q)_n} f(xq^n) = f(x).$$

Hence,

$$\begin{aligned}\lim_{\alpha\to 0+} I_q^\alpha f(x) &= \lim_{\alpha\to 0+} x^\alpha (1-q)^\alpha \lim_{\alpha\to 0+} \sum_{n=0}^\infty q^n \frac{(q^\alpha;q)_n}{(q;q)_n} f(xq^n) \\ &= f(x).\end{aligned}$$

□

From now on, we use the notation $I_q^0 f(x) = f(x)$.

4.5 q-Riemann–Liouville Fractional Calculus

In this section, we investigate the main properties of the Riemann–Liouville fractional q-integral and q-derivative. The property

$$I_q^{\beta} I_q^{\alpha} f(x) = I_q^{\alpha} I_q^{\beta} f(x) = I_q^{\alpha+\beta} f(x) \quad (\alpha;\ \beta \geqslant 0) \tag{4.62}$$

is a q-analogue of the semigroup property (4.10). It is established by Agarwal in [17]. For the convenience of the reader, we prove (4.62) in the following lemma.

Lemma 4.15. *If $f \in \mathscr{L}_q^1[0,a]$ then the semi group property (4.62) holds.*

Proof. Using (4.26) we obtain

$$I_q^{\alpha}(I_q^{\beta} f(x)) = x^{\alpha+\beta}(1-q)^{\alpha+\beta} \sum_{k=0}^{\infty} q^{k(1+\beta)} \frac{(q^{\alpha};q)_k}{(q;q)_k} \sum_{m=0}^{\infty} q^m \frac{(q^{\beta};q)_m}{(q;q)_m} f(xq^{n+m}).$$

Making the substitution $n = k + m$ we obtain

$$I_q^{\alpha}(I_q^{\beta} f(x)) = x^{\alpha+\beta}(1-q)^{\alpha+\beta} \sum_{k=0}^{\infty} q^{k(1+\beta)} \frac{(q^{\alpha};q)_k}{(q;q)_k} \sum_{n=k}^{\infty} q^{n-k} \frac{(q^{\beta};q)_{n-k}}{(q;q)_{n-k}} f(xq^{n}). \tag{4.63}$$

Since $f \in \mathscr{L}_q^1[0,a]$, the double series (4.63) is absolutely convergent. Then, we can interchange the order of summations to obtain

$$I_q^{\alpha}(I_q^{\beta} f(x)) = x^{\alpha+\beta}(1-q)^{\alpha+\beta} \sum_{n=0}^{\infty} q^n f(xq^n) \sum_{k=0}^{n} q^{k\beta} \frac{(q^{\alpha};q)_k}{(q;q)_k} \frac{(q^{\beta};q)_{n-k}}{(q;q)_{n-k}}.$$

It is straight forward to see that

$$\frac{(q^{\beta};q)_{n-k}}{(q;q)_{n-k}} = \frac{(q^{-n};q)_k}{(q^{1-n-\beta};q)_k} q^{(1-\beta)k}.$$

Consequently,

$$I_q^{\alpha}(I_q^{\beta} f(x)) = x^{\alpha+\beta}(1-q)^{\alpha+\beta} \sum_{n=0}^{\infty} q^n f(xq^n) \frac{(q^{\beta};q)_n}{(q;q)_n} {}_2\phi_1(q^{-n}, q^{\alpha}; q^{1-n-\beta}; q, q). \tag{4.64}$$

Applying (1.11) we obtain

$${}_2\phi_1(q^{-n}, q^{\alpha}; q^{1-n-\beta}; q, q) = \frac{(q^{1-n-\beta-\alpha};q)_n}{(q^{1-n-\beta};q)_n} q^{n\alpha} = \frac{(q^{\alpha+\beta};q)_n}{(q^{\beta};q)_n}. \tag{4.65}$$

Substituting (4.65) into (4.64), we obtain the series representation of $I_q^{\alpha+\beta} f(x)$ and (4.62) follows. □

Similar to property (4.12), we have the following:

Lemma 4.16. *If* $f \in \mathscr{L}_q^1[0,a]$ *then*

$$D_q^\alpha I_q^\alpha f(x) = f(x) \quad (\alpha > 0;\ x \in (0,a])\,. \tag{4.66}$$

Proof. It is obvious that if $\alpha = n, n \in \mathbb{N}$, then $D_q^n I_q^n f(x) = f(x)$. For a non integer and positive α, $n-1 < \alpha < n,\ \ n \in \mathbb{N}$, we apply the semi group identity (4.62), to obtain

$$\begin{aligned} D_q^\alpha I_q^\alpha f(x) &= D_q^n I_q^{n-\alpha} I_q^\alpha f(x) \\ &= D_q^n I_q^n f(x) = f(x), \end{aligned}$$

for all $x \in (0,a]$. □

The converse of (4.66), which is a q-analogue of (4.15), is the following:

Lemma 4.17. *Let* $\alpha \in \mathbb{R}^+$ *and* $n := \lceil \alpha \rceil$. *If*

$$f \in \mathscr{L}_q^1[0,a] \quad \textit{such that} \quad I_q^{n-\alpha} f \in \mathscr{A}C_q^{(n)}[0,a],$$

then

$$I_q^\alpha D_q^\alpha f(x) = f(x) - \sum_{j=1}^{n} D_q^{\alpha-j} f(0^+) \frac{x^{\alpha-j}}{\Gamma_q(\alpha-j+1)},\ x \in (0,a]. \tag{4.67}$$

Proof. Set

$$h(t,x) := x^\alpha \, (qt/x;q)_\alpha \, D_q^\alpha f(t), \quad x \in (0,a], \quad 0 < t \leqslant x.$$

Hence, $h(x,qx) = 0$ for all $x \in (0,a]$. Accordingly, applying Lemma 1.12 yields

$$\begin{aligned} I_q^\alpha D_q^\alpha f(x) &= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x \left(qt/x;q\right)_{\alpha-1} D_q^\alpha f(t)\, d_q t \\ &= D_{q,x}\left(\frac{x^\alpha}{\Gamma_q(\alpha+1)} \int_0^x \left(qt/x;q\right)_\alpha D_q^\alpha f(t)\, d_q t\right) \\ &= D_{q,x}\left(\frac{x^\alpha}{\Gamma_q(\alpha+1)} \int_0^x \left(qt/x;q\right)_\alpha D_q^n I_q^{n-\alpha} f(t)\, d_q t\right) \end{aligned} \tag{4.68}$$

Now, applying the q-integration by parts (1.28) n times on the last q-integral of (4.68), we obtain

$$\begin{aligned}&\tfrac{x^\alpha}{\Gamma_q(\alpha+1)}\int_0^x \left(qt/x;q\right)_\alpha D_q^\alpha f(t)\,d_qt\\ &\quad=-\textstyle\sum_{j=1}^n D_{q,x}^{\alpha-j}f(0^+)\tfrac{x^{\alpha-j+1}}{\Gamma_q(\alpha-j+2)}+\tfrac{x^{\alpha-n}}{\Gamma_q(\alpha-n+1)}\int_0^x \left(qt/x;q\right)_{\alpha-n} I_q^{n-\alpha}f(t)\,d_qt\\ &\quad=-\textstyle\sum_{j=1}^n D_{q,x}^{\alpha-j}f(0^+)\tfrac{x^{\alpha-j+1}}{\Gamma_q(\alpha-j+2)}+I_q^{\alpha-n+1}I_q^{n-\alpha}f(x).\end{aligned}\tag{4.69}$$

Applying the semigroup identity (4.62) on (4.69) yields

$$\begin{aligned}&\frac{x^\alpha}{\Gamma_q(\alpha+1)}\int_0^x \left(qt/x;q\right)_\alpha D_q^\alpha f(t)\,d_qt=\\ &\qquad I_qf(x)-\sum_{j=1}^n D_q^{\alpha-j}f(0^+)\frac{x^{\alpha-j+1}}{\Gamma_q(\alpha-j+2)}.\end{aligned}\tag{4.70}$$

Computing the q-derivative of both sides of (4.70) for $x\in(0,a]$ and combining the result with (4.68) we end with (4.67). $\square$

From (4.67), the equality

$$I_q^\alpha D_q^\alpha f(x)=f(x)\quad\text{for all}\quad x\in(0,a]$$

holds if and only if

$$D_q^{\alpha-j}f(0^+)=0\quad(j=1,2,\ldots,n)\,.$$

In the following four lemmas, we discuss the results of combining q-fractional integral and derivative of not necessarily equal orders. These results are q-analogues of relations (4.13), (4.14), (4.16), (4.17), and (4.11).

Lemma 4.18. *If $f\in\mathscr{L}_q^1[0,a]$ then*

$$D_q^\alpha I_q^\beta f(x)=I_q^{\beta-\alpha}f(x)\quad(\beta\geqslant\alpha\geqslant0;\ x\in(0,a])\,.\tag{4.71}$$

If in addition $D_q^{\alpha-\beta}f(x)$ exists in $(0,a]$ then

$$D_q^\alpha I_q^\beta f(x)=D_q^{\alpha-\beta}f(x)\quad(\alpha>\beta\geqslant0)\,.\tag{4.72}$$

Proof. First, assume that $\beta\geqslant\alpha$. Then, $\beta=\alpha+(\beta-\alpha)$ and from (4.62) we obtain

$$D_q^\alpha I_q^\beta f(x)=D_q^\alpha I_q^\alpha I_q^{\beta-\alpha}f(x)=I_q^{\beta-\alpha}f(x).$$

Now let $\beta < \alpha$, $m := \lceil \alpha \rceil$ and $n := \lceil \alpha - \beta \rceil$. Then $n \leqslant m$. Using (4.49) and (4.62), we get

$$\begin{aligned} D_q^\alpha I_q^\beta f(x) &= D_q^m I_q^{m-\alpha} I_q^\beta f(x) = D_q^m I_q^{m-n} I_q^{n-\alpha+\beta} f(x) \\ &= D_q^n I_q^{n-\alpha+\beta} f(x) = D_q^{\alpha-\beta} f(x). \end{aligned}$$

□

Lemma 4.19. *Let $f \in \mathscr{L}_q^1[0,a]$ such that $I_q^{n-\beta} f \in \mathscr{A}C_q^{(n)}[0,a]$, where $\beta > 0$ and $n := \lceil \beta \rceil$. Then for any $\alpha \geqslant 0$*

$$I_q^\alpha D_q^\beta f(x) = D_q^{-\alpha+\beta} f(x) - \sum_{j=1}^{n} D_q^{\beta-j} f(0^+) \frac{x^{\alpha-j}}{\Gamma_q(\alpha - j + 1)} \quad \textit{for } x \in (0,a]. \tag{4.73}$$

Proof. Since $\frac{1}{\Gamma_q(z)}$ has zeros at the negative integers, identity (4.73) holds for any $\beta > 0$ when $\alpha = 0$. Therefore, we assume that $\alpha > 0$. We distinguish between two cases. First, if $\alpha \geqslant \beta$ then from (4.62) and (4.67) we obtain

$$\begin{aligned} I_q^\alpha D_q^\beta f(x) &= I_q^{\alpha-\beta} \left(I_q^\beta D_q^\beta \right) f(x) \\ &= I_q^{\alpha-\beta} \left(f(x) - \sum_{j=1}^{n} D_q^{\beta-j} f(0^+) \frac{x^{\beta-j}}{\Gamma_q(\beta - j + 1)} \right) \\ &= I_q^{\alpha-\beta} f(x) - \sum_{j=1}^{n} D_q^{\beta-j} f(0^+) \frac{x^{\alpha-j}}{\Gamma_q(\alpha - j + 1)}, \end{aligned}$$

for all $x \in (0,a]$. If $\beta > \alpha$ then from (4.71) and (4.67), we get

$$\begin{aligned} I_q^\alpha D_q^\beta f(x) &= D_q^{\beta-\alpha} \left(I_q^\beta D_q^\beta f(x) \right) \\ &= D_q^{\beta-\alpha} \left(f(x) - \sum_{j=1}^{n} D_q^{\beta-j} f(0^+) \frac{x^{\beta-j}}{\Gamma_q(\beta - j + 1)} \right) \\ &= D_q^{\beta-\alpha} f(x) - \sum_{j=1}^{n} D_q^{\beta-j} f(0^+) \frac{x^{\alpha-j}}{\Gamma_q(\alpha - j + 1)}, \end{aligned}$$

for all $x \in (0,a]$. □

Lemma 4.20. *Let $\beta > 0$ and $n := \lceil \beta \rceil$. Assume that*

$$f \in \mathscr{L}_q^1[0,a] \quad \textit{and} \quad I_q^{n-\beta} f(x) \in \mathscr{A}C_q^{(n)}[0,a].$$

Then

$$D_q^{\alpha} D_q^{\beta} f(x) = D_q^{\alpha+\beta} f(x) - \sum_{j=1}^{n} D_q^{\beta-j} f(0^+) \frac{x^{-\alpha-j}}{\Gamma_q(-\alpha-j+1)}$$

for all $x \in (0, a]$*, provided that* $D_q^{\alpha+\beta} f(x)$ *exists for any* $\alpha > 0$.

Proof. Let $x \in (0, a]$. From (4.67) and (4.72), we conclude that

$$\begin{aligned} D_q^{\alpha} D_q^{\beta} f(x) &= D_q^{\alpha+\beta} I_q^{\beta} D_q^{\beta} f(x) \\ &= D_q^{\alpha+\beta} \left[f(x) - \sum_{j=1}^{n} D_q^{\beta-j} f(0^+) \frac{x^{\beta-j}}{\Gamma_q(\beta-j+1)} \right] \\ &= D_q^{\alpha+\beta} f(x) - \sum_{j=1}^{n} D_q^{\beta-j} f(0^+) \frac{x^{-\alpha-j}}{\Gamma_q(-\alpha-j+1)}. \end{aligned}$$

□

Remark 4.5.1. Similar to (4.20), if $\alpha,\ \beta > 0$, $I_q^{k-\alpha} \in \mathscr{A}C_q^{(k)}[0, a]$ and $D_q^{\alpha+\beta} f(x)$ exists, where $k := \lceil \alpha \rceil$ then

$$D_q^{\beta} D_q^{\alpha} f(x) = D_q^{\beta+\alpha} f(x) - \sum_{i=1}^{k} D_q^{\alpha-i} f(0^+) \frac{x^{-\beta-i}}{\Gamma_q(-\beta-i+1)},$$

for all $x \in (0, a]$. If

$$D_q^{\beta-j} f(0^+) = 0 \quad (j = 1, 2, \ldots, n)$$

and

$$D_q^{\alpha-j} f(0^+) = 0 \quad (j = 1, 2, \ldots, k)\,,$$

then we have the simplified relation

$$D_q^{\alpha} D_q^{\beta} = D_q^{\beta} D_q^{\alpha} = D_q^{\alpha+\beta}.$$

Lemma 4.21. *Let f be such that*

$$f \in \mathscr{L}_q^1(0, a) \quad \textit{and} \quad I_q^{k-\alpha} f \in \mathscr{A}C_q^{(k)}[0, a], \tag{4.74}$$

where $\alpha > 0$ *and* $k := \lceil \alpha \rceil$. *Then*

$$D_q^{\alpha-j} f(x) \in \mathscr{L}_q^1[0, a] \quad (j = 0, 1, 2, \ldots, k-1)\,. \tag{4.75}$$

Moreover,

$$D_q^{\alpha-j} f(x) \in \mathscr{A}C_q^{(j)}[0,a] \quad (j = 1,2,\ldots,k-1). \tag{4.76}$$

Proof. The proof of (4.75) follows immediately from (4.74) and Lemma 4.9. As for (4.76), we apply Lemma 4.18 to obtain

$$\begin{aligned} D_q^{\alpha-j} f(x) &= D_q^{k-j} I_q^{k-j-(\alpha-j)} f(x) \\ &= D_q^{k-j} I_q^{k-\alpha} f(x) \in \mathscr{A}C_q^{(j)}[0,a] \quad \text{for} \quad j = 1,2,\ldots,k-1. \end{aligned}$$

□

Similar to property (4.20), we have the following representation of the q-fractional derivative.

Lemma 4.22. *Let $\alpha > 0$, $k := \lceil \alpha \rceil$. If $f \in \mathscr{A}C_q^{(k)}[0,a]$ then*

$$D_q^\alpha f(x) = \sum_{j=0}^{k-1} \frac{D_q^j f(0^+)}{\Gamma_q(-\alpha+j+1)} x^{-\alpha+j} + \frac{x^{k-\alpha-1}}{\Gamma_q(k-\alpha)} \int_0^x \left(qt/x;q\right)_{k-\alpha-1} D_q^k f(t)\, d_qt, \tag{4.77}$$

for all $x \in (0,a]$. Furthermore, $D_q^\alpha f(0^+) = 0$ if and only if

$$D_q^j f(0^+) = 0 \quad (j = 0,1,\ldots,k-1).$$

Proof. Since $f \in \mathscr{A}C_q^{(k)}[0,a]$, then from Theorem 4.6 we obtain

$$f(x) = \sum_{j=0}^{k-1} \frac{D_q^j f(0^+)}{\Gamma_q(j+1)} x^j + \frac{x^{k-1}}{\Gamma_q(k)} \int_0^x \left(qt/x;q\right)_{k-1} D_q^k f(t)\, d_qt$$

for all $x \in (0,a]$. Consequently,

$$f(x) = \sum_{j=0}^{k-1} \frac{D_q^j f(0^+)}{\Gamma_q(j+1)} x^j + I_q^k D_q^k f(x) \quad \text{for all } x \in (0,a]. \tag{4.78}$$

Applying the operator D_q^α to both sides of (4.78) and using (4.71) yield

$$\begin{aligned} D_q^\alpha f(x) &= \sum_{j=0}^{k-1} \frac{D_q^j f(0^+)}{\Gamma_q(j-\alpha+1)} x^{j-\alpha} + D_q^\alpha I_q^k D_q^k f(x) \\ &= \sum_{j=0}^{k-1} \frac{D_q^j f(0^+)}{\Gamma_q(j-\alpha+1)} x^{j-\alpha} + I_q^{k-\alpha} D_q^k f(x) \end{aligned}$$

$$= \sum_{j=0}^{k-1} \frac{D_q^j f(0^+)}{\Gamma_q(j-\alpha+1)} x^{j-\alpha} + \frac{x^{k-\alpha-1}}{\Gamma_q(k-\alpha)} \int_0^x \left(qt/x;q\right)_{k-\alpha-1} D_q^k f(t)\, d_q t,$$

which is (4.77). The rest of the proof arises from (4.77). □

4.6 A Complex Variable Approach for Riemann–Liouville Fractional q-Derivative

In this section, we represent Jackson q-derivative of order n and Riemann–Liouville fractional q-derivative of order α in terms of complex integration. The complex approach for Jackson q-derivative is already known. See for example [148, 149].

Theorem 4.23. *Let Γ be a simple closed positively oriented contour such that the zero point lies inside Γ. If $f(\xi)$ is analytic in some simply connected domain D containing Γ and z is any nonzero point lies inside Γ then*

$$D_q^n f(z) = \frac{\Gamma_q(n+1)}{2\pi i} \int_\Gamma \frac{f(\xi)}{(\xi-z)(\xi-qz)\ldots(\xi-q^n z)}\, d\xi. \tag{4.79}$$

Proof. Since $0, z$ lie inside Γ, the set of points $\left\{zq^k,\ k\in\mathbb{N}_0\right\}$ lies inside Γ. Hence, from the Cauchy's integral formula, see for example [268, P. 143], we obtain

$$f(q^k z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{\xi - q^k z}\, d\xi \quad (k\in\mathbb{N}_0). \tag{4.80}$$

Substituting with the value of $f(q^k z)$ from (4.80) into (1.25), we obtain

$$D_q^n f(z) = (-1)^n z^{-n}(1-q)^{-n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-nk+\frac{k(k+1)}{2}} \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{\xi - q^k z}\, d\xi$$

$$= \frac{(-1)^n z^{-n}(1-q)^{-n}}{2\pi i} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-nk+\frac{k(k+1)}{2}} \int_\Gamma \frac{f(\xi)}{\xi} \sum_{r=0}^{\infty} \left(\frac{q^k z}{\xi}\right)^r d\xi,$$

where we expand $1/(1-(q^k z/\xi))$ and use that $|z| < |\xi|$. Hence, by changing the order of summations and applying the formula

$$\sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_q q^{\binom{k}{2}} a^k = (a;q)_m,$$

with $m = n$ and $a = q^{-n+r+1}$, we obtain

$$\begin{aligned} D_q^n f(z) &= \frac{(-1)^n z^{-n}(1-q)^{-n}}{2\pi i} \sum_{r=0}^{\infty} z^r \int_{\Gamma} \frac{f(\xi)}{\xi^{r+1}} (q^{-n+r+1};q)_n \, d\xi \\ &= \frac{(-1)^n z^{-n}(1-q)^{-n}}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{\xi} \left[\sum_{r=0}^{\infty} (q^{-n+r+1};q)_n \left(\frac{z}{\xi}\right)^r \right] d\xi. \end{aligned}$$

But

$$\sum_{r=0}^{\infty} (q^{-n+r+1};q)_n \left(\frac{z}{\xi}\right)^r = \sum_{r=n}^{\infty} (q^{-n+r+1};q)_n \left(\frac{z}{\xi}\right)^r. \tag{4.81}$$

Making the substitution $k = r - n$, we obtain

$$\sum_{r=0}^{\infty} (q^{-n+r+1};q)_n \left(\frac{z}{\xi}\right)^r = \sum_{k=0}^{\infty} (q^{k+1};q)_n \left(\frac{z}{\xi}\right)^{k+n}.$$

Since

$$(q^{k+1};q)_n = \frac{(q;q)_{n+k}}{(q;q)_k} = (q;q)_n \frac{(q^{n+1};q)_k}{(q;q)_k},$$

from the q-binomial theorem we obtain

$$\begin{aligned} \sum_{r=0}^{\infty} (q^{-n+r+1};q)_n \left(\frac{z}{\xi}\right)^r &= (q;q)_n \left(\frac{z}{\xi}\right)^n \frac{(q^{n+1}z/\xi;q)_\infty}{(z/\xi;q)_\infty} \\ &= \left(\frac{z}{\xi}\right)^n \frac{(q;q)_n}{(z/\xi;q)_{n+1}}. \end{aligned} \tag{4.82}$$

Substituting (4.81) into (4.82), we obtain (4.79). □

Observe that the denominator of the integrand of (4.79) can be replaced by $\xi^{n+1}(z/\xi;q)_{n+1}$. If we replace n by a non integer α, the function $\xi^{\alpha+1}$ no longer has a pole at $\xi = 0$ but a branch point. So, for $n \in \mathbb{N}$ we define Γ_n to be a simple closed positively oriented contour like the one in Fig. 4.1. We assume that the inner contour is a circle of radius ϵ_n and we denote it by c_{ϵ_n}. We choose ϵ_n so that $\epsilon_n \neq |z|q^k$ for all $k \in \mathbb{N}$. This leads to the following theorem

Theorem 4.24. *Let $R > 0$ and $h(\xi)$ be an analytic function in $|z| < R$. Let G be a branch domain of the logarithmic function and $f(\xi) = \xi^a h(\xi)$, $a > -1$, $\xi \in \Omega := D_R \cap G$. Then for each $z \in \Omega$*

$$D_q^\alpha f(z) = \lim_{n\to\infty} \int_{\Gamma_n} \frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}} \, d\xi,$$

where $\Gamma_n \subset \Omega$ is the contour in Fig. 4.1.

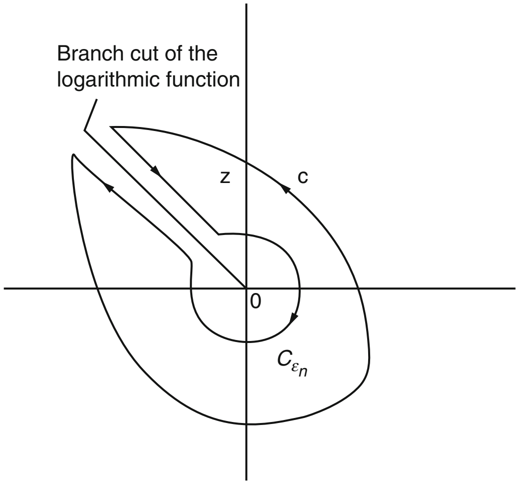


Fig. 4.1 Contour integration Γ_n

Proof. Since z is an interior point of Γ_n, and the set of points $\{zq^k,\ k = 0, 1, \ldots, n\}$ lies inside Γ_n, from the Cauchy's residue theorem we obtain

$$\int_{\Gamma_n} \frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}\,d\xi = 2\pi i \sum_{k=0}^{n} Res(f;a_k);\ a_k := zq^k,$$

$$\begin{aligned}\operatorname{Res}(f;a_k) &= \lim_{\xi\to a_k}(\xi - a_k)\frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}\\ &= q^k\, z^{-\alpha}\,\frac{(q^{-\alpha};q)_k(q^{\alpha+1};q)_\infty}{(q;q)_k(q;q)_\infty}.\end{aligned}$$

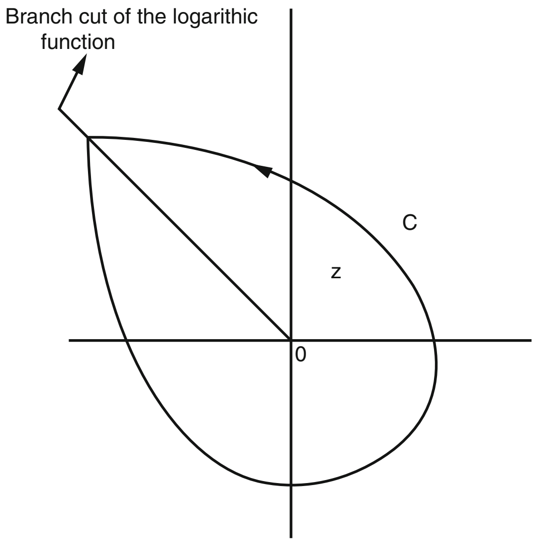
Hence,

$$\frac{\Gamma_q(\alpha+1)}{2\pi i}\int_{\Gamma_n}\frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}\,d\xi = (1-q)^{-\alpha}\sum_{k=0}^{n} q^k\,\frac{(q^{-\alpha};q)_k}{(q;q)_k}\,f(zq^k). \quad (4.83)$$

Since $f \in \mathscr{L}_q^1(G)$, the convergence of the finite sums in the previous equation as $n \to \infty$ is confirmed. Therefore, from (4.26) by letting $n \to \infty$ in (4.83) we get $D_q^\alpha f(z)$. □

In the following lemma, we need to integrate the function $f(\xi)/\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}$ on the branch cut of ξ^α or equivalently the logarithmic function. We apply the celebrated technique of integrating functions with branch cuts along their branch cuts. See for example [268]. Briefly, the value of the function on each side of the cut being determined by continuity from that side.

Fig. 4.2 Contour integration C



Lemma 4.25. *Let f, Ω, and α be as in Theorem 4.24. Then*

$$D_q^{\alpha} f(z) = \int_C \frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}\, d\xi,$$

where C is a positively oriented contour in Fig. 4.2.

Proof. From Fig. 4.3, it is obvious that

$$\int_{\Gamma_n} \frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}\, d\xi = \left[\sum_{i=1}^{2}\int_{R_i} + \int_{C_{\epsilon_n}} + \int_C\right] \frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}\, d\xi.$$

Since the function $\dfrac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}$ is analytic in the region bounded by R_i, $i = 1, 2$, its integration is zero on R_i, $i = 1, 2$. Hence, using

$$\left|\int_{C_{\epsilon_n}} \frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}\, d\xi}\right| \leq M(\epsilon_n; f) \max_{|\xi|=\epsilon_n} \left|\frac{1}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}\right| 2\pi\, \epsilon_n \to 0$$

as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \int_{\Gamma_n} \frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}\, d\xi} = \int_C \frac{f(\xi)}{\xi^{\alpha+1}(z/\xi;q)_{\alpha+1}}\, d\xi,$$

and the lemma follows. □

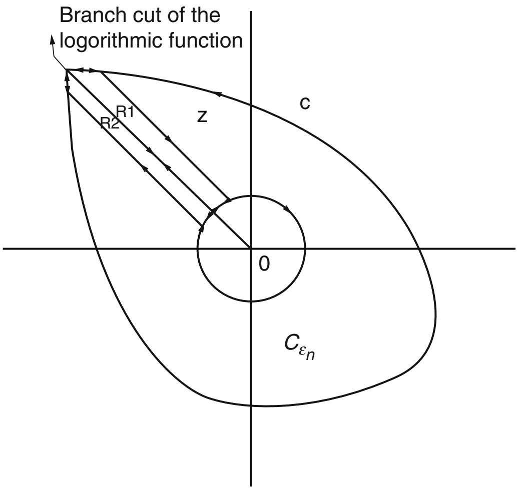


Fig. 4.3 The contour Γ_n can be represented as the sum of the contours C_{ϵ_n}, C and the contours R_1, R_2

Theorem 4.26. *Let $\alpha \in \mathbb{R}$ and G be an open simply connected domain of the complex plane containing the origin. Let f_n, $n \in \mathbb{N}$, be a family of complex functions defined on G, except may be at the origin. If for $z \in G \setminus \{0\}$*

(1) $D_q^\alpha f_n$ exists for each $z \in G$,
(2) $\sum_{n,m} q^m |f_n(zq^m)| < \infty$,

then

$$D_q^\alpha \sum_{n=0}^{\infty} f_n(z) = \sum_{n=0}^{\infty} D_q^\alpha f_n(z).$$

Proof. Set $F(z) := \sum_{n=0}^{\infty} f_n(z)$. For $z \in G$

$$\begin{aligned} D_q^\alpha F(z) &= z^{-\alpha}(1-q)^{-\alpha} \sum_{m=0}^{\infty} q^m \frac{(q^{-\alpha};q)_m}{(q;q)_m} F(zq^m) \\ &= z^{-\alpha}(1-q)^{-\alpha} \sum_{m=0}^{\infty} q^m \frac{(q^{-\alpha};q)_m}{(q;q)_m} \sum_{n=0}^{\infty} f_n(zq^m). \end{aligned}$$

But from the absolute convergence of the series $\sum_{n,m} q^m f_n(zq^m)$, we can interchange the order of summations to obtain

$$\begin{aligned} D_q^\alpha F(z) &= \sum_{n=0}^{\infty} z^{-\alpha}(1-q)^{-\alpha} \sum_{m=0}^{\infty} q^m \frac{(q^{-\alpha};q)_m}{(q;q)_m} f_n(zq^m) \\ &= \sum_{n=0}^{\infty} D_q^\alpha f_n(z). \end{aligned}$$

$\square$

4.7 The Law of Exponents

We have previously mentioned that Agarwal in [17] proved the semigroup property (4.62), which is

$$I_q^{\alpha}\left(I_q^{\beta} f(x)\right) = I_q^{\alpha+\beta} f(x) = I_q^{\beta}\left(I_q^{\alpha} f(x)\right) \quad (\alpha,\, \beta > 0).$$

Sometimes this rule is called the law of exponents. This rule may not be generalized when α and / or β are negative real numbers. As an example, take $\alpha = -1/2$ and $\beta = -3/2$ and $f(x) = x^{1/2}$. Then,

$$I_q^{-1/2} f(x) = D_q^{1/2} f(x) = \Gamma_q(3/2)$$
$$I_q^{-3/2} f(x) = D_q^{3/2} f(x) = 0$$

Therefore,

$$D_q^{3/2} D_q^{1/2} f(x) = \frac{\Gamma_q(3/2)}{\Gamma_q(-1/2)} x^{-3/2},$$

while

$$D_q^{1/2} D_q^{3/2} f(x) = 0.$$

That is,

$$D_q^{3/2} D_q^{1/2} x^{1/2} \neq D_q^{1/2} D_q^{3/2} x^{1/2}.$$

In the following theorem, we state precise conditions under which the law of exponents holds for arbitrary fractional operator.

Theorem 4.27. *Let α_1, α_2 and β be positive real numbers and let G be an open, simply connected domain of the complex plane containing the origin. If $f(z) = z^{\beta-1}\sum_{n=0}^{\infty} a_n z^n$ is defined on G, except may be at the origin then*

$$D_q^{\alpha_1} D_q^{\alpha_2} f(z) = D_q^{\alpha_2} D_q^{\alpha_1} f(z),$$

for all $z \in G \setminus \{0\}$ if

(a) $\alpha_1 < \beta$ and α_2 is arbitrary.
(b) $\alpha_1 \geq \beta$, α_2 is arbitrary, and $a_k = 0$ for $k = 0, 1, \ldots, m-1$, where m is the smallest integer greater than or equal to α_1.

Proof. The proof is similar to the proof of [213, Theorem 3] and is omitted. □

4.8 Miscellaneous Examples

It is known that

$$D_{0+}^{1/2}\exp(x) = \frac{1}{\sqrt{\pi\, x}} + \exp(x)\,\mathrm{erf}(\sqrt{x}), \tag{4.84}$$

$$I_{0+}^{1/2}\exp(x) = \exp(x)\,\mathrm{erf}(\sqrt{x}). \tag{4.85}$$

The following theorem gives q-analogues of (4.84) and (4.85).

Theorem 4.28. *For $|x| < \frac{1}{1-q}$, we have*

$$D_q^{1/2} e_q(x(1-q)) = \frac{1}{\sqrt{x}\,\Gamma_q(1/2)} + e_q(x(1-q))\,\mathrm{Erf}(\sqrt{x};\sqrt{q}), \tag{4.86}$$

$$I_q^{1/2} e_q(x(1-q)) = e_q(x(1-q))\,\mathrm{Erf}(\sqrt{x};\sqrt{q}). \tag{4.87}$$

For $x \in \mathbb{C}$, we have

$$D_q^{1/2} E_q(x(1-q)/\sqrt{q}) = \frac{1}{\sqrt{x}\,\Gamma_q(1/2)} + \frac{E_q(x(1-q))\,\mathrm{erf}(\sqrt{x};\sqrt{q})}{\sqrt{q}\,K_q(1/2)}, \tag{4.88}$$

$$I_q^{1/2} E_q(x\sqrt{q}(1-q)) = \frac{1}{K_q(1/2)} E_q(x(1-q))\,\mathrm{erf}(\sqrt{x};\sqrt{q}). \tag{4.89}$$

Proof. First, we prove the identities in (4.86) and (4.87). From the series representation of the function e_q, we obtain

$$\begin{aligned} D_q^{1/2} e_q(x(1-q)) &= \sum_{n=0}^{\infty} \frac{x^{n-1/2}}{\Gamma_q(n+1/2)} \\ &= \frac{1}{\Gamma_q(1/2)\sqrt{x}} + \sum_{n=0}^{\infty} \frac{x^{n+1/2}}{\Gamma_q(n+3/2)}. \end{aligned} \tag{4.90}$$

On the other hand,

$$\begin{aligned} &\mathrm{Erf}(\sqrt{x};\sqrt{q})e_q(x(1-q)) \\ &= \frac{(1-q)}{\Gamma_q(1/2)} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \frac{x^{n+1/2}}{\Gamma_q(n+1)(1-q^{n+1/2})} \times \sum_{n=0}^{\infty} \frac{x^n}{\Gamma_q(n+1)} \\ &= \frac{(1-q)}{\Gamma_q(1/2)} \sum_{n=0}^{\infty} \frac{x^{n+1/2}}{\Gamma_q(n+1)} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} \frac{q^k}{1-q^{k+1/2}}. \end{aligned}$$

Since

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} \frac{q^k}{1-q^{k+1/2}} = \frac{1}{1-q^{1/2}} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{1/2};q)_k}{(q^{3/2};q)_k (q;q)_k} q^{(n+1)k}$$
$$= \frac{1}{1-q^{1/2}} {}_2\phi_1\left(q^{1/2}, q^{-n}; q^{3/2}; q, q^{n+1}\right)$$
$$= \frac{1}{1-q^{1/2}} \frac{(q;q)_n}{(q^{3/2};q)_n},$$

where we used (1.11). Hence,

$$\begin{aligned} \mathrm{Erf}(\sqrt{x};\sqrt{q}) e_q(x(1-q)) &= \frac{1-q}{1-q^{1/2}\Gamma_q(1/2)} \sum_{n=0}^{\infty} \frac{(1-q)^n x^{n+1/2}}{(q^{3/2};q)_n} \\ &= \sum_{n=0}^{\infty} \frac{x^{n+1/2}}{\Gamma_q(n+3/2)}. \end{aligned} \tag{4.91}$$

Combining (4.90) and (4.91) yield (4.86). The proof of (4.87) follows from (4.91) and the fact that

$$I_q^{1/2} e_q(x(1-q)) = \sum_{n=0}^{\infty} \frac{x^{n+1/2}}{\Gamma_q(n+3/2)}.$$

□

As for the identities in (4.88) and (4.89), one can verify that

$$D_q^{1/2} E_q(x(1-q)/\sqrt{q}) = \frac{1}{\sqrt{x}\Gamma_q(1/2)} + q^{-1/2} \sum_{k=0}^{\infty} q^{k^2/2} \frac{x^{k+\frac{1}{2}}}{\Gamma_q(k+3/2)}, \tag{4.92}$$

while

$$E_q(x(1-q)) \,\mathrm{erf}(\sqrt{x};\sqrt{q}) = K_q(1/2) \sum_{k=0}^{\infty} q^{k^2/2} \frac{x^{k+\frac{1}{2}}}{\Gamma_q(k+3/2)}. \tag{4.93}$$

Substituting (4.93) into (4.92), we obtain (4.88). The proof of (4.89) follows from (4.93) and

$$I_q^{1/2} E_q(x\sqrt{q}(1-q)) = \sum_{k=0}^{\infty} q^{k^2/2} \frac{x^{k+\frac{1}{2}}}{\Gamma_q(k+3/2)}.$$

□

Corollary 4.29.

$$
\begin{aligned}
D_q^{1/2}\left(e_q(x(1-q))\,\mathrm{Erf}(\sqrt{x};\sqrt{q})\right) &= e_q(x(1-q)),\\
I_q^{1/2}\left(e_q(x(1-q))\,\mathrm{Erf}(\sqrt{x};\sqrt{q})\right) &= \left(e_q(x(1-q))-1\right),\\
D_q^{1/2}\left(E_q(x(1-q))\,\mathrm{erf}(\sqrt{x};\sqrt{q})\right) &= K_q(1/2)E_q(\sqrt{q}(1-q)x),\\
I_q^{1/2}\left(E_q(x(1-q))\,\mathrm{erf}(\sqrt{x};\sqrt{q})\right) &= \sqrt{q}K_q(1/2)\left(E_q(x(1-q)/\sqrt{q})-1\right).
\end{aligned}
$$

The following result is a q-analogue for the one proved in [178, P. 261].

Theorem 4.30. *For* $|x| < \frac{1}{1-q}$,

$$\gamma_q(a,x) = \Gamma_q(a)e_q(-x(1-q))I_q^a E_q(q^a x(1-q)), \tag{4.94}$$

$$\Gamma_q(a,x) = \Gamma_q(a)E_q(q^{-1}x(1-q))I_q^a e_q(q^{-1}x(1-q)). \tag{4.95}$$

Proof. First, we prove the first identity. A direct computation shows

$$I_q^a E_q(q^a x(1-q)) = \sum_{k=0}^{\infty} q^{ak} q^{\binom{k}{2}} \frac{x^{k+a}}{\Gamma_q(k+a+1)}.$$

Hence,

$$
\begin{aligned}
&e_q(-x(1-q))I_q^a E_q(q^a x(1-q))\\
&= \frac{1}{\Gamma_q(a+1)}\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+a}}{\Gamma_q(n+1)}{}_2\phi_1(q^{-n},q;q^{a+1};q,q^{n+a}).
\end{aligned}
$$

Using (1.11) we obtain

$$e_q(-x(1-q))I_q^a E_q(q^a x(1-q)) = \frac{\gamma_q(a,x)}{\Gamma_q(a)},$$

and (4.94) follows. The proof of (4.95) follows from

$$
\begin{aligned}
&\Gamma_q(a+1)E_q(q^{-1}x(1-q))I_q^a e_q(q^{-1}x(1-q))\\
&= \sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}} \frac{x^{n+a}}{\Gamma_q(n+1)}{}_2\phi_1(q^{-n},q;q^{a+1};q,q).
\end{aligned}
$$

Then using (1.10), we obtain (4.95). □

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