Positive Linear Systems Consisting of n Subsystems With Different Fractional Orders

Tadeusz Kaczorek

***Abstract*—A new class of positive linear systems consisting of n subsystems with different fractional orders is introduced. Solution to the set of matrix linear differential equations with different fractional orders is derived. Necessary and sufficient conditions for the positivity of the fractional systems are established. It is shown that the linear electrical circuits composed of resistors, supercondensators, coils, and voltage (current) sources are positive systems with different fractional orders.**

***Index Terms*—Electrical circuit, fractional order, linear, positive, solution.**

I. Introduction

A DYNAMICAL system is called positive if and only if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in [4], [9], [10], and [21]. Variety of models having positive linear behavior can be found in engineering, management sciences, economics, social sciences, biology, medicine, etc. Mathematical fundamentals of the fractional calculus are given in the monographs [25]–[27], [29] and the fractional differential equations and their applications have been addressed in [20], [22], and [23]. The positive fractional linear systems have been introduced in [4], [13], [15], and [16]. The stability analysis of standard and positive linear and nonlinear systems has been addressed in [5], [6], [12], [25], and [31]. The stability of fractional linear 1-D discrete-time and continuous-time systems has been investigated in the papers [1], [8], [28], and [30] and of 2-D fractional positive linear systems in [14]. The notion of practical stability of positive fractional discrete-time 1-D and 2-D linear systems has been introduced in [11] and [19].The mathematical modeling of fractional reaction-diffusion systems with different order time derivatives has been investigated in [3] and the numerical simulation of the fractional order control systems in [2]. The reachability of fractional positive continuous-time linear systems has been addressed in [17] and of positive fractional 2-D linear systems in [18]. Some recent interesting results in fractional systems theory and its applications can be found in [7], [23], [24], and [32].

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In this paper a new class of positive linear systems consisting of n subsystems with different fractional orders will be introduced. Using the Laplace transform method the solution to the set of matrix linear different fractional orders will be derived and necessary and sufficient conditions for the positivity of the fractional systems will be established.

The paper is organized as follows. In Section II the set of n matrix linear differential equations with different fractional orders is introduced and its solutions is derived using Laplace transform method. Necessary and sufficient conditions for the positivity of the fractional linear systems are established in Section III. In Section IV it is shown that the linear electrical circuits are positive systems with different fractional orders. Concluding remarks are given in Section V.

The following notation will be used in this paper. The set of real $n \times m$ matrices will be denoted by $\Re^{n\times m}$ and the set of $n \times m$ real matrices with nonnegative entries will be denoted by $\Re_+^{n\times m}$ ($\Re_+^n = \Re_+^{n\times 1}$). The $n \times n$ identity matrix will be denoted by I_n.

II. Linear Differential Equations With Different Fractional Orders and Their Solutions

In this paper the following Caputo definition of the fractional derivative will be used:

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^\infty \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \tag{2.1}$$

where $n-1 < \alpha < n,\ n \in N\{1,2,\ldots\}$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \mathrm{Re}(x) > 0 \tag{2.2}$$

is the gamma Euler function and

$$f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n}. \tag{2.3}$$

It is well known [8], [13] that the Laplace transform $(\mathcal{L})$ of (1) is given by the formula

$$\begin{aligned} L\left[\frac{d^\alpha f(t)}{dt^\alpha}\right] &= \int_0^\infty \frac{d^\alpha f(t)}{dt^\alpha} e^{-st} dt \\ &= s^\alpha F(s) - \sum_{k=1}^n s^{\alpha-k} f^{(k-1)}(0^+), \\ n-1 < \alpha < n,&\ n \in N \end{aligned} \tag{2.4}$$

where $F(s) = L[f(t)]$ and $n-1 < \alpha < n,\ n \in N$.

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Consider a fractional matrix linear system described by the equation

$$\begin{bmatrix} \frac{d^{\alpha_1}x_1}{dt^{\alpha_1}} \\ \vdots \\ \frac{d^{\alpha_n}x_n}{dt^{\alpha_n}} \end{bmatrix} = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \dots & \vdots \\ A_{n1} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} u,$$
$$p_k - 1 < \alpha_k < p_k;\ p \in N = \{1, 2, \dots\},\ k = 1, \dots, n \tag{2.5}$$

where $x_k \in \Re_k^{\bar{n}}$ $k = 1, \dots, n$ are the state vectors, $A_{kj} \in \Re^{\bar{n}_k \times \bar{n}_j}$, $B_k \in \Re^{\bar{n}_k \times m}$; $k, j = 1, \dots, n$ and $u \in \Re^m$ is the input vector.

Initial conditions for (2.5) have the form

$$x_k^{(j)}(0) = x_{k0}^{(j)} \in \Re^{\bar{n}_k} \quad k = 1, \dots, n;\ j = 0, 1, \dots, p_k - 1. \tag{2.6}$$

Theorem 2.1: The solution of the (2.5) for $p_k - 1 < \alpha_k < p_k,\ k = 1, \dots, n$ with initial conditions (2.6) has the form

$$x(t) = \int_0^t [\Phi_1(t-\tau)B_{10} + \dots + \Phi_n(t-\tau)B_{n0}]\, u(\tau)d\tau + \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \begin{bmatrix} \sum\limits_{j_1=1}^{p_1} \frac{t^{k_1\alpha_1 + \dots + k_n\alpha_n + j_1 - 1}}{\Gamma(k_1\alpha_1 + \dots + k_n\alpha_n + j_1)} x_{10}^{(j_1-1)} \\ \vdots \\ \sum\limits_{j_n=1}^{p_n} \frac{t^{k_1\alpha_1 + \dots + k_n\alpha_n + j_n - 1}}{\Gamma(k_1\alpha_1 + \dots + k_n\alpha_n + j_n)} x_{n0}^{(j_n-1)} \end{bmatrix} \tag{2.7}$$

where terms are defined as shown in (2.8a)–(2.8c) at the bottom of the page.

Proof: Using the Laplace transforms

$$X_k(s) = \mathcal{L}[x_k(t)],\ k = 1, \dots, nU(s) = \mathcal{L}[u(t)] \tag{2.9}$$

and (2.4) we may write the (2.5) for $p_k - 1 < \alpha_k < p_k;\ p_k \in N,\ k = 1, \dots, n$ in the form

$$\begin{bmatrix} I_{\bar{n}_1}s^{\alpha_1} - A_{11} & -A_{12} & \dots & -A_{1n-1} & -A_{1n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ -A_{n1} & -A_{n2} & \dots & -A_{nn-1} & I_{\bar{n}_n}s^{\alpha_n} - A_{nn} \end{bmatrix} \times \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} U(s) + \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{\alpha_1 - j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{\alpha_n - j_n} x_{n0}^{(j_n-1)} \end{bmatrix}. \tag{2.10}$$

From (2.10) we have

$$\begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} = \begin{bmatrix} I_{\bar{n}_1}s^{\alpha_1} - A_{11} & -A_{12} & \dots & -A_{1n-1} & -A_{1n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ -A_{n1} & -A_{n2} & \dots & -A_{nn-1} & I_{\bar{n}_n}s^{\alpha_n} - A_{nn} \end{bmatrix}^{-1} \times \left\{ \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} U(s) + \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{\alpha_1 - j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{\alpha_n - j_n} x_{n0}^{(j_n-1)} \end{bmatrix} \right\}. \tag{2.11}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \Re^N,\ N = \bar{n}_1 + \dots + \bar{n}_n,\ x_0 = \begin{bmatrix} x_{10} \\ \vdots \\ x_{n0} \end{bmatrix},$$
$$B_{10} = \begin{bmatrix} B_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ B_{n0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B_n \end{bmatrix} \tag{2.8a}$$

$$\Phi_1(t) = \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \frac{t^{(k_1+1)\alpha_1 + k_2\alpha_2 + \dots + k_n\alpha_n - 1}}{\Gamma[(k_1+1)\alpha_1 + k_2\alpha_2 + \dots + k_n\alpha_n]}$$
$$\vdots$$
$$\Phi_n(t) = \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \frac{t^{k_1\alpha_1 + \dots + k_{n-1}\alpha_{n-1} + (k_n+1)\alpha_n - 1}}{\Gamma[k_1\alpha_1 + \dots + k_{n-1}\alpha_{n-1} + (k_n+1)\alpha_n]} \tag{2.8b}$$

$$T_{k_1 \dots k_n} = \begin{cases} I_N & \text{for } k_1 = \dots = k_n = 0 \\ \begin{bmatrix} A_{11} & \dots & A_{1n} \\ 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 0 \end{bmatrix} & \text{for } k_1 = 1, k_2 = \dots = k_n = 0 \\ \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 0 \\ A_{n1} & \dots & A_{nn} \end{bmatrix} & \text{for } k_1 = \dots = k_{n-1} = 0, k_n = 1 \\ T_{10\dots0}T_{01\dots1} + \dots + T_{0\dots01}T_{1\dots10} & \text{for } k_1 = \dots = k_n = 1 \\ \vdots & \\ T_{10\dots0}T_{k_1-1,k_2,\dots,k_n} + \dots + T_{0\dots01}T_{k_1,\dots,k_{n-1},k_n-1} & \text{for } k_1 + \dots + k_n > 0 \\ 0 & \text{for at least one } k_i < 0,\ i = 1, \dots, n. \end{cases} \tag{2.8c}$$

Comparing the coefficients at the same powers of $s^{-\alpha_k}$, it is easy to verify that

$$\begin{bmatrix} I_{\bar{n}_1} - A_{11}s^{-\alpha_1} & \dots & -A_{1n}s^{-\alpha_1} \\ \vdots & \dots & \vdots \\ -A_{n1}s^{-\alpha_n} & \dots & I_{\bar{n}_n} - A_{nn}s^{-\alpha_n} \end{bmatrix} \times \left[\sum_{k_1=0}^{\infty}\dots\sum_{k_n=0}^{\infty} T_{k_1\dots k_n} s^{-(k_1\alpha_1+k_2\alpha_2+\dots+k_n\alpha_n)}\right] = I_N \tag{2.12}$$

where the matrices $T_{k_1\dots k_n}$ are defined by (2.8c).

Using (2.12) we obtain (2.13), at the bottom of the page.

Substitution of (2.13) into (2.11) yields (2.14), at the bottom of the page.

Applying the inverse Laplace transform (L^{-1}) and the convolution theorem to (2.14) we obtain (2.15), at the bottom of the next page, since $L^{-1}\left[1/s^{\alpha+1}\right] = t^{\alpha}/\Gamma(\alpha+1)$. □

In this particular case, if $0 < \alpha_k < 1,\ k = 1,\dots,n\ (p_1 = \dots = p_n = 1)$, then

$$\sum_{k_1=0}^{\infty}\dots\sum_{k_n=0}^{\infty} T_{k_1\dots k_n} \begin{bmatrix} \sum_{j_1=1}^{p_1} \frac{t^{k_1\alpha_1+\dots+k_n\alpha_n+j_1-1}}{\Gamma(k_1\alpha_1+\dots+k_n\alpha_n+j_1)} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} \frac{t^{k_1\alpha_1+\dots+k_n\alpha_n+j_n-1}}{\Gamma(k_1\alpha_1+\dots+k_n\alpha_n+j_n)} x_{n0}^{(j_n-1)} \end{bmatrix} = \Phi_0(t)x_0 \tag{2.16}$$

where

$$\Phi_0(t) = \sum_{k_1=0}^{\infty}\dots\sum_{k_n=0}^{\infty} T_{k_1\dots k_n} \frac{t^{k_1\alpha_1+\dots+k_n\alpha_n}}{\Gamma(k_1\alpha_1+\dots+k_n\alpha_n+1)}. \tag{2.17}$$

1) Conclusion 2.1: The solution of the equation

$$\begin{bmatrix} \frac{d^{\alpha_1}x_1}{dt^{\alpha_1}} \\ \frac{d^{\alpha_2}x_2}{dt^{\alpha_2}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{2.18}$$

$$\begin{aligned} &\begin{bmatrix} I_{\bar{n}_1}s^{\alpha_1} - A_{11} & -A_{12} & \dots & -A_{1n-1} & -A_{1n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ -A_{n1} & -A_{n2} & \dots & -A_{nn-1} & I_{\bar{n}_n}s^{\alpha_n} - A_{nn} \end{bmatrix}^{-1} \\ &= \left\{ \begin{bmatrix} I_{\bar{n}_1}s^{\alpha_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & I_{\bar{n}_n}s^{\alpha_n} \end{bmatrix} \begin{bmatrix} I_{\bar{n}_1} - A_{11}s^{-\alpha_1} & \dots & -A_{1n}s^{-\alpha_1} \\ \vdots & \dots & \vdots \\ -A_{n1}s^{-\alpha_n} & \dots & I_{\bar{n}_n} - A_{nn}s^{-\alpha_n} \end{bmatrix} \right\}^{-1} \\ &= \begin{bmatrix} I_{\bar{n}_1} - A_{11}s^{-\alpha_1} & \dots & -A_{1n}s^{-\alpha_1} \\ \vdots & \dots & \vdots \\ -A_{n1}s^{-\alpha_n} & \dots & I_{\bar{n}_n} - A_{nn}s^{-\alpha_n} \end{bmatrix}^{-1} \begin{bmatrix} I_{\bar{n}_1}s^{-\alpha_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & I_{\bar{n}_n}s^{-\alpha_n} \end{bmatrix} \\ &= \sum_{k_1=0}^{\infty}\dots\sum_{k_n=0}^{\infty} T_{k_1\dots k_n} s^{-(k_1\alpha_1+k_2\alpha_2+\dots+k_n\alpha_n)} \begin{bmatrix} I_{\bar{n}_1}s^{-\alpha_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & I_{\bar{n}_n}s^{-\alpha_n} \end{bmatrix}. \end{aligned} \tag{2.13}$$

$$\begin{aligned} \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} &= \sum_{k_1=0}^{\infty}\dots\sum_{k_n=0}^{\infty} T_{k_1\dots k_n} s^{-(k_1\alpha_1+k_2\alpha_2+\dots+k_n\alpha_n)} \begin{bmatrix} I_{\bar{n}_1}s^{-\alpha_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & I_{\bar{n}_n}s^{-\alpha_n} \end{bmatrix} \\ &\quad \times \left\{ \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} U(s) + \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{\alpha_1-j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{\alpha_n-j_n} x_{10}^{(j_n-1)} \end{bmatrix} \right\} \\ &= \sum_{k_1=0}^{\infty}\dots\sum_{k_n=0}^{\infty} T_{k_1\dots k_n} \left\{ \begin{aligned} &\begin{bmatrix} B_{10}s^{-[(k_1+1)\alpha_1+k_2\alpha_2+\dots+k_n\alpha_n]} + \dots + \\ + \dots + B_{n0}s^{-[k_1\alpha_1+\dots+k_{n-1}\alpha_{n-1}+(k_n+1)\alpha_n]} \end{bmatrix} U(s) \\ &+ s^{-(k_1\alpha_1+\dots+k_n\alpha_n)} \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{-j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{-j_n} x_{n0}^{(j_n-1)} \end{bmatrix} \end{aligned} \right\}. \end{aligned} \tag{2.14}$$

$$\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = L^{-1}\begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix}$$
$$= L^{-1}\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} T_{k_1\dots k_n}\left\{\begin{array}{c}\begin{bmatrix} B_{10}s^{-[(k_1+1)\alpha_1+k_2\alpha_2+\cdots+k_n\alpha_n]}+\cdots+ \\ +\cdots+B_{n0}s^{-[k_1\alpha_1+\cdots+k_{n-1}\alpha_{n-1}+(k_n+1)\alpha_n]}\end{bmatrix}U(s) \\ +s^{-(k_1\alpha_1+\cdots+k_n\alpha_n)}\begin{bmatrix}\sum_{j_1=1}^{p_1}s^{-j_1}x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n}s^{-j_n}x_{n0}^{(j_n-1)}\end{bmatrix}\end{array}\right\}$$
$$= \int_0^t [\Phi_1(t-\tau)B_{10}+\cdots+\Phi_n(t-\tau)B_{n0}]\,u(\tau)d\tau$$
$$+\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} T_{k_1\dots k_n}\begin{bmatrix}\sum_{j_1=1}^{p_1}\frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n+j_1-1}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+j_1)}x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n}\frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n+j_n-1}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+j_n)}x_{n0}^{(j_n-1)}\end{bmatrix} \quad (2.15)$$

for $0 < \alpha_k < 1$ $k = 1, 2$ with initial condition

$$x_k(0) = x_{k0},\ k = 1, 2 \quad (2.19)$$

has the form

$$x(t) = \Phi_0(t)x_0 + \int_0^t [\Phi_1(t-\tau)B_{10} + \Phi_2(t-\tau)B_{20}]\,u(\tau)d\tau \quad (2.20)$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix},$$
$$B_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, B_{20} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad (2.21)$$
$$\Phi_0(t) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} T_{k_1k_2}\frac{t^{k_1\alpha_1+k_2\alpha_2}}{\Gamma(k_1\alpha_1+k_2\alpha_2+1)} \quad (2.22a)$$
$$\Phi_1(t) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} T_{k_1k_2}\frac{t^{(k_1+1)\alpha_1+k_2\alpha_2-1}}{\Gamma[(k_1+1)\alpha_1+k_2\alpha_2]} \quad (2.22b)$$
$$\Phi_2(t) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} T_{k_1k_2}\frac{t^{k_1\alpha_1+(k_2+1)\alpha_2-1}}{\Gamma[k_1\alpha_1+(k_2+1)\alpha_2]} \quad (2.22c)$$
$$T_{k_1k_2} = \begin{cases} \begin{bmatrix} I_{\bar n_1} & 0 \\ 0 & I_{\bar n_2} \end{bmatrix} & \text{for } k_1 = k_2 = 0 \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} & \text{for } k_1 = 1, k_2 = 0 \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} & \text{for } k_1 = 0, k_2 = 1 \\ T_{10}T_{k_1-1,k_2} + T_{01}T_{k_1,k_2-1} & \text{for } k_1 + k_2 > 0 \\ 0 & \text{for } k_1 < 0 \\ & \text{or/and } k_2 < 0. \end{cases} \quad (2.23)$$

III. Positive Fractional Systems

Definition 3.1: The fractional system (2.5) is called positive if $x_k(t) \in \Re_+^{\bar n_k},\ k = 1, \dots, n,\ t \geq 0$ for any initial conditions $x_{k0} \in \Re_+^{\bar n_k},\ k = 1, \dots, n$, and all input vectors $u \in \Re_+^m,\ t \geq 0$.

Let M_n be the set of $n \times n$ Metzler matrices, i.e., real matrices with nonnegative off-diagonal entries.

Theorem 3.1: The fractional system (2.5) for $p_k - 1 < \alpha_k < p_k; p_k \in N,\ k = 1, \dots, n$ is positive if and only if

$$A = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \dots & \vdots \\ A_{n1} & \dots & A_{nn} \end{bmatrix} \in M_N \quad (3.1a)$$
$$\begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} \in \Re_+^{N\times m}. \quad (3.1b)$$

Proof: To simplify the notation the proof will be given for $n = 2$. First we shall show that

$$\Phi_k(t) \in \Re_+^{\bar n\times\bar n}\bar n = \bar n_1 + \bar n_2 \quad \text{for } k = 0, 1, 2 \text{ and } t \geq 0 \quad (3.2)$$

only if (3.1a) holds.

From the expansions (2.21) we have

$$\Phi_0(t) = \begin{bmatrix} I_{\bar n_1} & 0 \\ 0 & I_{\bar n_2} \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)} + \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} + \cdots \quad (3.3a)$$
$$\Phi_1(t) = \begin{bmatrix} I_{\bar n_1} & 0 \\ 0 & I_{\bar n_2} \end{bmatrix}\frac{t^{\alpha_1-1}}{\Gamma(\alpha_1)} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}\frac{t^{2\alpha_1-1}}{\Gamma(2\alpha_1)} + \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}\frac{t^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} + \cdots \quad (3.3b)$$
$$\Phi_2(t) = \begin{bmatrix} I_{\bar n_1} & 0 \\ 0 & I_{\bar n_2} \end{bmatrix}\frac{t^{\alpha_2-1}}{\Gamma(\alpha_2)} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}\frac{t^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} + \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}\frac{t^{2\alpha_2-1}}{\Gamma(2\alpha_2)} + \cdots \quad (3.3c)$$

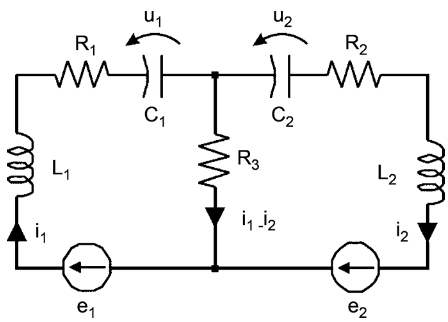

Fig. 1. Electrical circuit.

From (3.3) it follows that $\Phi_k(t) \in \Re_+^{\bar{n}\times\bar{n}},\ k = 0,1,2$ for small value of $t > 0$ only if the condition (3.1a) is satisfied.

In a similar way as in [8] and [10] it can be shown that if (3.1) holds, then

$$\Phi_0(t) \in \Re_+^{\bar{n}\times\bar{n}} t \geq 0 \tag{3.4}$$

and

$$\Phi_1(t)B_{10} + \Phi_2(t)B_{01} \in \Re_+^{\bar{n}\times\bar{n}} t \geq 0. \tag{3.5}$$

In this case from (2.20) we have $x(t) \in \Re_+^{\bar{n}},\ t \geq 0$ since by definition $x_0 \in \Re_+^{\bar{n}}$ and $u(t) \in \Re_+^m,\ t \geq 0$. The remaining part of the proof is similar to [8] and [10]. □

IV. Fractional Linear Electrical Circuits

Consider linear electrical circuits composed of resistors, supercondensators (ultracapacitors), coils, and voltage (current) sources. As the state variables (the components of the state vector x) the voltage across the supercondensators and the currents in the coils are usually chosen. It is well-known [2], [16] that the current $i(t)$ in supercondensator with its voltage $u_C(t)$ is related by the formula

$$i_C(t) = C\frac{d^\alpha u_C(t)}{dt^\alpha} \text{ for } 0 < \alpha < 1 \tag{4.1}$$

where C is the capacity of the supercondensator.

Similarly, the voltage $u_L(t)$ on the coil with its current $i_L(t)$ is related by the formula

$$u_L(t) = L\frac{d^\beta i_L(t)}{dt^\beta} \text{ for } 0 < \beta < 1 \tag{4.2}$$

where L is the inductance of the coil.

Using the relations (4.1), (4.2), and Kirchhoff's laws, we may write for the fractional linear circuits the following state equation:

$$\begin{bmatrix} \frac{d^\alpha x_C}{dt^\alpha} \\ \frac{d^\beta x_L}{dt^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_C \\ x_L \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e \tag{4.3}$$

where the components of $x_C \in \Re^{n_1}$ are voltages across the supercondensators, the components of $x_L \in \Re^{n_2}$ are currents in coils and the components of $e \in \Re^m$ are the voltages of the circuit.

1) *Example 4.1:* Consider the linear electrical circuit shown on Fig. 1 with known resistances R_1, R_2, R_3, capacitances C_1, C_2, inductances L_1, L_2, and sources voltages e_1, e_2.

Using relations (4.1), (4.2), and Kirchhoff's laws, we may write for the circuit the following equations:

$$\begin{aligned} i_1 &= C_1\frac{d^{\alpha_1}u_1}{dt^{\alpha_1}},\ i_2 = C_2\frac{d^{\alpha_2}u_2}{dt^{\alpha_2}}, \\ e_1 &= (R_1+R_3)i_1 + L_1\frac{d^{\beta_1}i_1}{dt^{\beta_1}} + u_1 - R_3 i_2, \\ e_2 &= (R_2+R_3)i_2 + L_2\frac{d^{\beta_2}i_2}{dt^{\beta_2}} + u_2 - R_3 i_1. \end{aligned} \tag{4.4}$$

Equation (4.4) can be written in the form

$$\begin{bmatrix} \frac{d^{\alpha_1}u_1}{dt^{\alpha_1}} \\ \frac{d^{\alpha_2}u_2}{dt^{\alpha_2}} \\ \frac{d^{\beta_1}i_1}{dt^{\beta_1}} \\ \frac{d^{\beta_2}i_2}{dt^{\beta_2}} \end{bmatrix} = A\begin{bmatrix} u_1 \\ u_2 \\ i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{4.5}$$

where

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \\ -\frac{1}{L_1} & 0 & -\frac{R_1+R_3}{L_1} & \frac{R_3}{L_1} \\ 0 & -\frac{1}{L_2} & \frac{R_3}{L_2} & -\frac{R_2+R_3}{L_2} \end{bmatrix}, \\ B &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \end{aligned} \tag{4.6}$$

From (4.6) it follows that the fractional electrical circuit is not positive since the matrix A has some negative off-diagonal entries.

If the fractional linear circuit is not positive but the matrix B has nonnegative entries (see for example the circuits on Fig. 1) then using the state-feedback

$$e = K\begin{bmatrix} x_C \\ x_L \end{bmatrix} \tag{4.7}$$

we may usually choose the gain matrix $K \in \Re^{m\times n}$ $(n = n_1 + n_2)$ so that the closed-loop system matrix [obtained by substitution of (4.7) into (4.3)]

$$A_c = A + BK \tag{4.8}$$

is a Metzler matrix.

Theorem 4.1: Let A be not a Metzler matrix but $B \in \Re_+^{n\times m}$. Then there exists a gain matrix K such that the closed-loop system matrix $A_c \in M_n$ if and only if

$$\operatorname{rank}[B, A_c - A] = \operatorname{rank}B. \tag{4.9}$$

Proof: By the Kronecker-Cappely theorem, the equation

$$BK = A_c - A \tag{4.10}$$

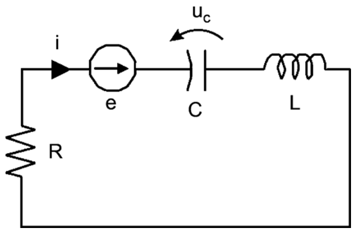

Fig. 2. Simple electrical circuit.

has a solution K for any given B and $A_c - A$ if and only if the conditions (4.9) is satisfied. $\square$

2) Example 4.2 (Continuation of Example 4.1): Let

$$A_c = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \\ \frac{a_1}{L_1} & 0 & -\frac{R_1+R_3}{L_1} & \frac{a_3}{L_1} \\ 0 & \frac{a_2}{L_2} & \frac{a_4}{L_2} & -\frac{R_2+R_3}{L_2} \end{bmatrix} \quad \text{for } a_k \geq 0 \; k = 1,2,3,4. \tag{4.11}$$

In this case the condition (4.9) is satisfied since

$$\begin{aligned} &\operatorname{rank}[B, A_c - A] \\ &= \operatorname{rank} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{L_1} & 0 & \frac{a_1+1}{L_1} & 0 & 0 & \frac{a_3-R_3}{L_1} \\ 0 & \frac{1}{L_2} & 0 & \frac{a_2+1}{L_2} & \frac{a_4-R_3}{L_2} & 0 \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} = 2. \end{aligned} \tag{4.12}$$

Equation (4.10) has the form

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} \begin{bmatrix} \mathrm{k}_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a_1+1}{L_1} & 0 & 0 & \frac{a_3-R_3}{L_1} \\ 0 & \frac{a_2+1}{L_2} & \frac{a_4-R_3}{L_2} & 0 \end{bmatrix} \tag{4.13}$$

and its solution is

$$K = \begin{bmatrix} \mathrm{k}_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} = \begin{bmatrix} a_1+1 & 0 & 0 & a_3 - R_3 \\ 0 & a_2+1 & a_4 - R_3 & 0 \end{bmatrix}. \tag{4.14}$$

The matrix (4.14) has nonnegative entries if $a_k \geq 0$ for $k = 1,2,3,4$.

On the following two examples of fractional linear circuits we shall show that it is not always possible to choose the gain matrix K so that the following two conditions are satisfied:

1) the closed-loop system matrix $A_c \in M_n$;
2) the closed-loop system is asymptotically stable.

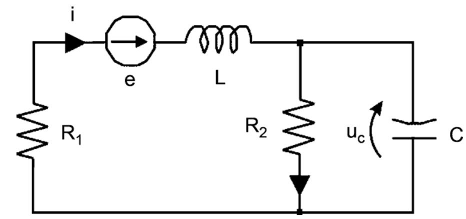

Fig. 3. Electrical circuit.

3) Example 4.3: Consider the fractional linear circuit shown on Fig. 2 with given resistance R, capacitance C, inductance L, and source of voltage e.

Using (4.1), (4.2), and the second Kirchhoff's law, we obtain for the circuit the state equation

$$\begin{bmatrix} \frac{d^\alpha u_C}{dt^\alpha} \\ \frac{d^\beta i}{dt^\beta} \end{bmatrix} = A \begin{bmatrix} u_C \\ i \end{bmatrix} + Be \quad 0 < \alpha < 1; 0 < \beta < 1 \tag{4.15}$$

where

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}. \tag{4.16}$$

From (4.16) it follows that A is not a Metzler matrix but $B \in \Re_+^2$. It is easy to see that the condition (4.9) is satisfied for

$$A_c = \begin{bmatrix} 0 & \frac{1}{C} \\ \frac{a}{L} & \frac{b-R}{L} \end{bmatrix} \tag{4.17}$$

and from (4.10) we obtain

$$\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} [\, k_1 \quad k_2 \,] = \begin{bmatrix} 0 & 0 \\ \frac{a+1}{L} & \frac{b}{L} \end{bmatrix} \tag{4.18}$$

and

$$K = [\, k_1 \quad k_2 \,] = [\, a+1 \quad b \,]. \tag{4.19}$$

Note that the characteristic polynomial of the matrix (4.17)

$$\begin{aligned} &\det \begin{bmatrix} I_{n_1} s^\alpha - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} s^\beta - A_{22} \end{bmatrix} \\ &= \begin{vmatrix} s^\alpha & -\frac{1}{C} \\ -\frac{a}{L} & s^\beta + \frac{R-b}{L} \end{vmatrix} \\ &= s^{\alpha+\beta} + \frac{R-b}{L} s^\alpha - \frac{a}{LC} \end{aligned} \tag{4.20}$$

has one negative coefficient and the closed-loop circuit is unstable for $a \geq 0$ and any b.

4) Example 4.4: Consider the fractional linear system shown on Fig. 3 with given resistances R_1, R_2, capacitance C, inductance L, and source of voltage e.

Using the relations (4.1), (4.2), and the second Kirchhoff's law, we obtain for the circuit the state equation

$$\begin{bmatrix} \frac{d^\alpha u_C}{dt^\alpha} \\ \frac{d^\beta i}{dt^\beta} \end{bmatrix} = A \begin{bmatrix} u_C \\ i \end{bmatrix} + Be \tag{4.21}$$

where

$$A = \begin{bmatrix} -\frac{1}{R_2 C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_1}{L} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}. \tag{4.22}$$

The matrix A is not a Metzler matrix but $B \in \Re_+^2$. It is easy to check that the condition (4.9) is satisfied for

$$A_c = \begin{bmatrix} -\frac{1}{R_2C} & \frac{1}{C} \\ \frac{a}{L} & \frac{b-R_1}{L} \end{bmatrix}, \ a, \ b \geq 0 \tag{4.23}$$

and from (4.10) we obtain

$$\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} [\, k_1 \quad k_2 \,] = \begin{bmatrix} 0 & 0 \\ \frac{a+1}{L} & \frac{b}{L} \end{bmatrix} \tag{4.24}$$

and

$$K = [\, k_1 \quad k_2 \,] = [\, a+1 \quad b \,]. \tag{4.25}$$

In this case the characteristic polynomial of the matrix (4.23) has the form

$$\begin{aligned} p(s) &= \begin{vmatrix} s^\alpha + \frac{1}{R_2C} & -\frac{1}{C} \\ -\frac{a}{L} & s^\beta + \frac{R_1-b}{L} \end{vmatrix} \\ &= s^{\alpha+\beta} + \frac{R_1-b}{L}s^\alpha + \frac{1}{R_2C}s^\beta \\ &\quad + \frac{R_1 - aR_2 - b}{R_2CL} \end{aligned} \tag{4.26}$$

and it is possible to choose the values of parameters a, b so that the closed-loop system is asymptotically stable [14].

The following question arises. What are the necessary and sufficient conditions under which there exists a gain matrix K such that the closed-loop system is positive and asymptotically stable? This problem will be the topic of a subsequent paper.

V. Concluding Remarks

A new class of positive linear systems consisting of n subsystems with different fractional orders has been introduced. Solution to the set of matrix linear differential equations with different fractional orders has been derived using Laplace transform method (Theorem 2.1). It has been shown that the fractional linear systems are positive if and only if the system matrix A is a Metzler matrix and the matrix B has nonnegative entries (Theorem 3.1). It has been also shown that linear electrical circuits are positive systems with different fractional orders. If the systems matrix A is not Metzler matrix but the matrix B has nonnegative entries then there exists a gain matrix of the state-feedbacks such that the closed-loop system matrix is a Metzler matrix if and only if the condition (4.9) is satisfied (Theorem 4.1). The consideration have been illustrated by examples of linear electrical circuits. An open problem has been formulated.

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