Adaptive Feedback Control of Fractional Order Discrete State-Space Systems

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Abstract

The paper is devoted to the application of Fractional Calculus concepts to modeling, identification and control of discrete-time systems. Fractional Difference Equations (FΔE) models are presented and their use in identification, state estimation and control context is discussed. The Fractional Difference state-space model is proposed for that purpose. For such a model stability conditions are given. A Fractional Kalman Filter (FKF) for this model is recalled. The proposed state-space model and Fractional Order Difference Equation are used in an identification procedure which produces very accurate results. Finally, the state-space model is used in closed-loop state feedback control form together with FKF as a state estimator. The latter is also given in an adaptive form together with FKF and a modification of Recursive Least Squares (RLS) algorithm as a parameters identification procedure. All the algorithms presented were tested in simulations and the example results are given in the paper.

1 INTRODUCTION

The theory of fractional calculus has a long and prominent history. In fact, one may trace it back to the very origins of differential calculus itself. Such mathematicians as Riemman, Euler or Liouville made some serious research along the lines of non-integer order derivatives and integrals. However, its complexity prevented it from practical applications until only very recently. In the last decade, the results of works on the theory of chaos revealed some relations with fractional derivatives and integrals. This in turn renewed the interest in the field. Some fundamental facts of the fractional calculus theory and its properties may be found in e.g., [8, 12, 13, 7]. As far as the applications of the fractional calculus are concerned there is a large volume of research on viscoelasticity/damping, see e.g., [15, 1] and chaos/fractals, see e.g., [6, 2]. Also other areas of science and technology have started to pay more attention to these concepts and it may be noted that fractional calculus is being adopted in the fields of signal processing, system modeling and identification, and control to name just the few. What is most interesting from our point of view is the application of fractional calculus in the last two areas. Several researchers on automatic control have proposed control algorithms both in frequency [11] and time [5] domains based on the concepts of fractional calculus. This work is still in a fairly early stage and still a lot remains to be done. This paper presents an approach to adaptive control basing on fractional difference equation models of discrete-time dynamic system. Some preliminary results are presented in [14]. Section 2 of the paper brings some fundamental results on discrete fractional calculus, shows the basic discrete-time state-space equations model, presents stability condition and also introduces the Fractional Kalman Filter (FKF) used in feedback controller in subsequent sections. In Section 3 an identification procedure for fractional discrete-time model is given and the simulation example proving its validity is presented. The procedure is a modification of Recursive Least Squares (RLS) recast in a fractional systems settings. Finally, in Section 4 a simple state-feedback controller for the fractional discrete-time system is proposed together with its adaptive versions using either only Fractional Kalman Filter as a state estimator or FKF coupled with a modified RLS identification algorithm presented in Section 3.

2 FRACTIONAL CALCULUS AND DYNAMIC MODELS

Let us start our exposition with some basics of the fractional calculus used throughout the paper. In this paper, as a definition of fractional discrete derivative, a following Grünwald-Letnikov definition [8, 12] will be used.

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Definition 1

$$\Delta^n x_k = \frac{1}{h^n}\sum_{j=0}^{k}(-1)^j \left(\begin{array}{c} n \\ j \end{array}\right) x_{k-j} \tag{1}$$

where, $n \in \mathbb{R}$, is a fractional degree, $\mathbb{R}$, is the set of real numbers, and h is a sampling time (this later will be equal to 1), $k \in \mathbb{N}$ is a number of sample for which the derivative is calculated. ■

The term $\left(\begin{array}{c} n \\ j \end{array}\right)$ in Definition 1 can be obtained from the following relation:

$$\left(\begin{array}{c} n \\ j \end{array}\right) = \begin{cases} 1 & \text{for} \quad j = 0 \\ \frac{n(n-1)\ldots(n-j+1)}{j!} & \text{for} \quad j > 0 \end{cases} \tag{2}$$

According to this definition it is possible to obtain the discrete equivalent of the derivative (when n is positive), of the integral (when n is negative) or when n equal to 0 original function.

Let us assume a traditional (integer order) discrete space system,

$$x_{k+1} = Ax_k + Bu_k \tag{3}$$
$$y_k = Cx_k \tag{4}$$

This equation (3) could be rewritten as follows,

$$\Delta^1 x_{k+1} = A_d x_k + Bu_k \tag{5}$$

where $\Delta^1 x_k$ is a first order difference for x_k and $A_d = A - I$ where I is the identity matrix.

The value of the space vector for time instance $k+1$ can be obtained from the first order difference definition

$$\Delta^1 x_{k+1} = x_{k+1} - x_k.$$

what gives the following relation,

$$x_{k+1} = \Delta^1 x_{k+1} + x_k.$$

This line of reasoning can in principle be generalized for difference of any, even non-integer, order. This can be thought of as a discrete version of a derivative of non-integer order, i.e., non-integer order finite difference just like standard finite difference is used as a discrete version of a standard derivative (see e.g., [16]).

Following similar calculations to those presented in integer case one can come to the following in general case when n is a real number (not necessarily an integer)

$$\Delta^n x_{k+1} = A_d x_k + Bu_k \tag{6}$$
$$x_{k+1} = \Delta^n x_{k+1} - \sum_{j=1}^{k+1}(-1)^j \left(\begin{array}{c} n \\ j \end{array}\right) x_{k-j+1} \tag{7}$$
$$y_k = Cx_k \tag{8}$$

The values of fractional order differences are obtained according to (6), from this values the next state vector is calculated using relation (7). Output equation is given by (8).

When system equations do not have the same degree, then the discrete state-space model has a somewhat different form. It can be presented as follows:

$$\Delta^\Upsilon x_{k+1} = A_d x_k + Bu_k \tag{9}$$

In this case the next state is evaluated, according to (7) as follows

$$x_{k+1} = \Delta^\Upsilon x_{k+1} - \sum_{j=1}^{k+1}(-1)^j \Upsilon_j x_{k-j+1}. \tag{10}$$

where:

$$\Upsilon_k = \text{diag}\begin{bmatrix} \left(\begin{array}{c} n_1 \\ k \end{array}\right) \\ \vdots \\ \left(\begin{array}{c} n_N \\ k \end{array}\right) \end{bmatrix}, \quad \Delta^\Upsilon x_{k+1} = \begin{bmatrix} \Delta^{n_1} x_{1,k+1} \\ \vdots \\ \Delta^{n_N} x_{N,k+1} \end{bmatrix}$$

and $n_1, \ldots, n_N$ are degrees of system equations and N is a number of these equations. The output equation is the same as in the previous case, i.e., equation (8)

This system can be rewritten as an infinite dimensional system

$$\begin{bmatrix} x_{k+1} \\ x_k \\ x_{k-1} \\ \vdots \end{bmatrix} = \mathbb{A}\begin{bmatrix} x_k \\ x_{k-1} \\ x_{k-2} \\ \vdots \end{bmatrix} + \mathbb{B}u_k \tag{11}$$

$$y_k = \mathbb{C}\begin{bmatrix} x_k \\ x_{k-1} \\ x_{k-2} \\ \vdots \end{bmatrix} \tag{12}$$

where

$$\mathbb{A} = \begin{bmatrix} (A_d + \Upsilon_1) & -(-1)^2\Upsilon_2 & -(-1)^3\Upsilon_3 & \ldots \\ I & 0 & 0 & \ldots \\ 0 & I & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

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$$\mathbb{B} = \begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \mathbb{C} = \begin{bmatrix} C & 0 & 0 & \dots \end{bmatrix}$$

2.1 Stability

The stability condition for such a redefined system is analogical to the traditional integer order state-space systems [4].

Theorem 1 *The system defined by equations (11) and (12) is asymptotically stable iff*

$$|\mathbb{A}| < 1$$

where $|.|$ denotes matrix norm defined as a $\max|\lambda_i|$ where λ_i is an $i-th$ eigenvalue of the matrix $\mathbb{A}$.

□

The number of factors Υ_j in matrix $\mathbb{A}$ in real application has to be reduced. This reduction may cause the decrease of accuracy of the stability determination. Especially when the system is close to the stability margin.

2.2 Fractional Order Difference Equation

In order to identify the system parameters the difference equation describing input-output dynamic relation is more convenient than the state-space representation. Let the system equations (8) and (9) be rewritten using $\mathcal{Z}$ transform with zero initial conditions ($x_j = 0$ for $j \leq 0$) as follows

$$\begin{aligned} z\Delta^{\Upsilon}(z)X(z) &= A_d X(z) + BU(z) \\ Y(z) &= CX(z) \end{aligned}$$

This immediately gives the relation

$$\frac{Y(z)}{U(z)} = C(I(z\Delta^{\Upsilon}(z)) - A_d)^{-1}B$$

where Δ^{Υ} is also a polynomial of z.
This provide to the relation

$$\begin{aligned} G(z) &= \frac{Y(z)}{U(z)} \\ &= \frac{b_{N-1}z^{N-1}\Delta^{n^*_{N-1}}(z) + \dots + b_1 z\Delta^{n^*_1}(z) + b_0}{z^N\Delta^{n^*_N}(z) + \dots + a_1 z\Delta^{n^*_1}(z) + a_0} \end{aligned}$$

where

$$n^*_N = n_1 + n_2 + \dots + n_N \tag{13}$$

Then, the Fractional Difference Equation is as follows:

$$\begin{aligned} \Delta^{n^*_N} y_k + a_{N-1}\Delta^{n^*_{N-1}} y_{k-1} + \dots + a_0 y_{k-N} = \\ b_{N-1}\Delta^{n^*_{N-1}} u_{k-1} + \dots + b_0 u_{k-N} \end{aligned} \tag{14}$$

Where factors a_k and b_k for $k = 0 \dots N-1$ are for example entries of the system matrices in the following canonical form:

$$A_d = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & -a_{N-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \dots & b_{N-1} \end{bmatrix}$$

2.3 Fractional Kalman Filter (FKF)

Using (9,10 and 8) the stochastic discrete state-space model could be introduced as follows

$$\begin{aligned} \Delta^{\Upsilon} x_{k+1} &= A_d x_k + Bu_k + \omega_k \\ x_{k+1} &= \Delta^{\Upsilon} x_{k+1} - \sum_{j=1}^{k+1} (-1)^j \Upsilon_j x_{k-j+1} \\ y_k &= Cx_k + \nu_k \end{aligned}$$

For such a system defined in state-space equations form we may introduce a simplified Kalman Filter (called Fractional Kalman Filter) by the following set of equations (see [14] for details):

$$\begin{aligned} \Delta^{\Upsilon} \tilde{x}_{k+1} &= A_d \hat{x}_k + Bu_k \\ \tilde{x}_{k+1} &= \Delta^{\Upsilon} \tilde{x}_{k+1} - \sum_{j=1}^{k+1} (-1)^j \Upsilon_j \hat{x}_{k-j+1} \\ \tilde{P}_k &= (A_d + \Upsilon_1) P_{k-1} (A_d + \Upsilon_1)^T + \\ & \quad + Q_{k-1} + \sum_{j=2}^{k} \Upsilon_j P_{k-j} \Upsilon_j^T \\ K_k &= \tilde{P}_k C^T (C\tilde{P}_k C^T + R_k)^{-1} \\ \hat{x}_k &= \tilde{x}_k + K_k(y_k - C\tilde{x}_k) \\ P_k &= (I - K_k C)\tilde{P}_k \end{aligned}$$

where: R_k is a symmetric covariance matrix of output noise ν_k and Q_k is a symmetric covariance matrix of system noise ω_k. Both of the noise signals are assumed to be independent noises with zero expected value. The $\tilde{P}_k$ is a prediction of estimation error symmetric covariance matrix, and P_k is an estimation error symmetric covariance matrix.

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3 FRACTIONAL SYSTEMS IDENTIFICATION

Fractional difference equation model of a dynamic system presented in Section 2.2 can form a basis for a control law. However, in order to construct any control law it is essential to know the parameters of the model.Using difference equation defined in previous section it is possible to determine parameters estimation process in the following way. This reasoning is in principle a version of RLS approach to parametric identification.

$$\varphi_k = [\ -\Delta^{n^*_{N-1}} y_{k-1} \quad \dots \quad -y_{k-N} \qquad (15)$$

$$\Delta^{n^*_{N-1}} u_{k-1} \quad \dots \quad u_{k-N}\ ] \qquad (16)$$

$$\theta^T = [\ a_{N-1} \quad \dots \quad a_0 \quad b_{N-1} \quad \dots \quad b_0\ ] \qquad (17)$$

$$Y_k = [\ \Delta^{n^*_N} y_k s\ ] \qquad (18)$$

The parameters could be obtained in by solving the equation (usually overdetermined)

$$\begin{bmatrix} Y_k \\ Y_{k-1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \varphi_k \\ \varphi_{k-1} \\ \vdots \end{bmatrix} \theta \qquad (19)$$

The use of the approach is demonstrated in the following Section.

3.1 Identification Example

Let us assume the following fractional order state-space system:

$$A_d = \begin{bmatrix} 0 & 1 \\ -0.35 & -0.1 \end{bmatrix}, \quad N = \begin{bmatrix} 0.8 \\ 0.7 \end{bmatrix} \qquad (20)$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [\ 0.3 \quad 0.4\ ] \qquad (21)$$

$$\mathrm{E}[\omega_k \omega_k^T] = 0.03\mathbb{I}, \quad \mathrm{E}[\nu_k \nu_k^T] = 0.02 \qquad (22)$$

Noises ω_k and ν_k are independent, Gaussian and zero mean.

In the identification process the deterministic model and difference equation were used. This can affect in the final results accuracy. In addition calculating of the fractional order difference could be very sensitive to noise and also decrease the identification accuracy.

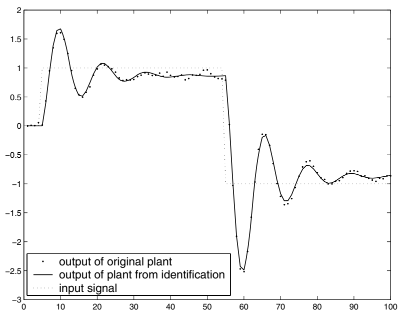

Figure 1. Step response of the system with known and identified parameters.

Results of identification are

$$\begin{aligned} a_1 &= 0.1102 & a_0 = 0.3475 \\ b_1 &= 0.3990 & b_0 = 0.2969 \end{aligned}$$

and, as it is could be seen, the accuracy despite of the noise is quite high.

The step response of the system for identified parameters compared with the exact step response of the system with known parameters is given in Figure 1.

4 STATE FEEDBACK CONTROL

State feedback configuration under consideration is depicted in Fig. 2.

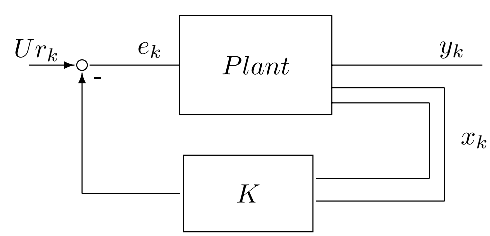

Figure 2. State feedback controller scheme

Where the plant is given in the form (9) and (8) For the control signal given by the following expression

$$e_k = Ur_k - Kx_k \qquad (23)$$

where $K = [k_0, k_1, \dots, k_N]$ is a state feedback controller matrix, the system matrix is given by the equation

$$A_d^* = A_d - BK \qquad (24)$$

For system matrices in canonical form presented in Section 2.2 the control defined by (23) leads to the very well

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known in traditional state-feedback theory direct link between plant, controller and closed loop system parameters

$$
\begin{aligned}
a_0^* &= a_0 + k_0 \\
a_1^* &= a_1 + k_1 \\
\vdots & \quad \vdots \\
a_{N-1}^* &= a_{N-1} + k_{N-1}
\end{aligned}
$$

5 STATE FEEDBACK WITH FKF AND ADAPTIVE STATE FEEDBACK CONTROLLERS

In order to use the state feedback controller presented in previous Section to the real life problems it may be essential to assume that both the state and the output of a fraction state space model are corrupted by noise. Moreover it is also reasonable to assume that the parameters of the model are not entirely known. Thus, we come to the solution composed of a combination of state feedback controller with FKF in the case of unknown/corrupted state and the combination of state feedback controller with both FKF and parameters identification. The building blocks for these combinations were discussed earlier. Now, let us present the examples of the results of their use.

5.1 State Feedback Controller with FKF

The control plant given by equations (20) and (21). The reference system is given by following system matrix

$$A_d = \begin{bmatrix} 0 & 1 \\ -0.2 & -0.4 \end{bmatrix} \tag{25}$$

In this case the controller matrix obtained using (25) is equal to $K = [-0.15, 0.3]$.

The output noise ν_k is increased to $\mathrm{E}[\nu_k \nu_k^T] = 0.1$ in order to better shows the FKF ability.

The FKF is used to estimate state vector which we assume as non-accessible directly from the plant. Starting value of error covariance matrix is set to $P_0 = 100I$ and value of initial state vector is $x_0 = [0, 0]^T$. Performance index matrices are $Q_k = 0.03\mathbb{I}$ and $R_k = 0.1$ respectively.

Simulation results of use of state feedback controller with FKF are given and compare to the reference (non noise) plant in Fig. 3. The estimated state variables are also compared with original.

As may be noticed the response of the system with known and un-corrupted state and the system with state estimated with FKF match each other quite accurately.

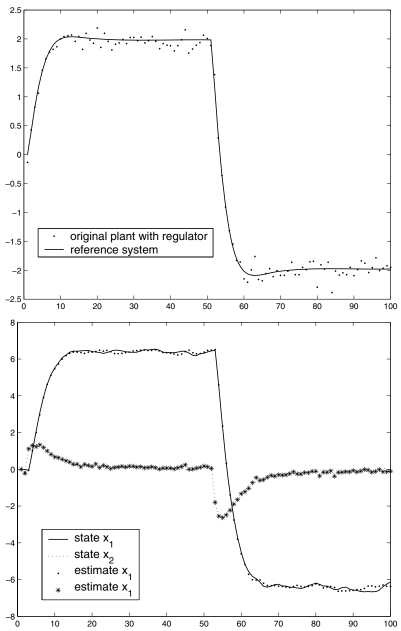

Figure 3. Results of using state feedback controller with Fractional Kalman Filter, outputs and states

5.2 Example of Adaptive State Feedback Controller

In this case we assume that not only the state vector but also the parameters of the fractional state space model are not known. Therefore, the parameters of the system were estimated by recursive least square (RLS) algorithm. Those parameters in turn are used by the Fractional Kalman Filter to estimate state vector of the system. Estimated state vector is used to determine the control signal according to the control law obtained from estimated parameters.

Initial conditions for estimated state vector and parameters vector were set to zero. Starting values of covariance estimate error matrix in FKF and covariance identification error in RLS algorithm were equal to $100\mathbb{I}$.

The parameters obtained from identification by RLS algorithm have the following values:

$$
\begin{aligned}
a_1 &= 0.1656 \qquad a_0 = 0.3529 \\
b_1 &= 0.3717 \qquad b_0 = 0.3120
\end{aligned}
$$

We may see the identification process in Fig. 4. Simulation results of control (system response) for this case may be observed in Fig. 5. The response of the adaptive system is close to the reference system. The estimated state variables are also compared to the original one. At the beginning of the control process the quality of control is slightly worse

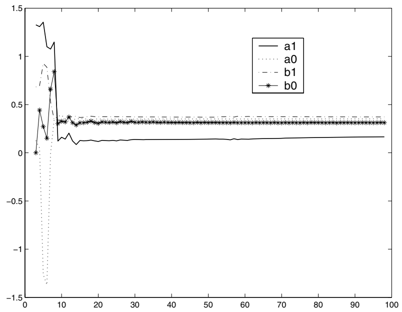

Figure 4. Parameters estimated in control process

than in previous case, but we must take into account the adaptation of the system.

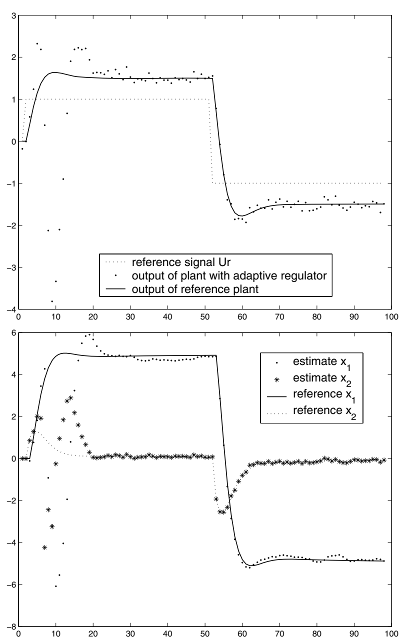

Figure 5. Results of using adaptive state feedback controller with Fractional Kalman Filter and RLS identification

6 CONCLUSIONS

A detailed discussion of use of the tools of fractional calculus in the control context has been presented. The fractional calculus turned out to be quite useful in several control domains. The fractional dynamic models have been introduced both in the form of FΔE and state-space. The stability condition for the models has been given. For parameters identification and state vector estimation purposes the fractional version of the well known KF and RLS algorithms were described. The Fractional Kalman Filter is a useful tool for optimal estimation of unknown or noise corrupted state of the system while a modified RLS algorithm turned out to be a good approach to parameters identification for fractional systems models. The state feedback pole placement control were presented. The adaptive versions of the state feedback control were also derived and tested. The simulation examples given suggest that also this more sophisticated approaches work in the case of fractional systems.

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