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From Davidenko Method to Zhang Dynamics for Nonlinear Equation Systems Solving

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***Abstract*—The solving of nonlinear equation systems (e.g., complex transcendental dispersion equation systems in waveguide systems) is a fundamental topic in science and engineering. Davidenko method has been used by electromagnetism researchers to solve time-invariant nonlinear equation systems (e.g., the aforementioned transcendental dispersion equation systems). Meanwhile, Zhang dynamics (ZD), which is a special class of neural dynamics, has been substantiated as an effective and accurate method for solving nonlinear equation systems, particularly time-varying nonlinear equation systems. In this paper, Davidenko method is compared with ZD in terms of efficiency and accuracy in solving time-invariant and time-varying nonlinear equation systems. Results reveal that ZD is a more competent approach than Davidenko method. Moreover, discrete-time ZD models, corresponding block diagrams, and circuit schematics are presented to facilitate the convenient implementation of ZD by researchers and engineers for solving time-invariant and time-varying nonlinear equation systems online. The theoretical analysis and results on Davidenko method, ZD, and discrete-time ZD models are also discussed in relation to solving time-varying nonlinear equation systems.**

***Index Terms*—Comparison, Davidenko method, time-invariant nonlinear equation systems, time-varying nonlinear equation systems, Zhang dynamics (ZD).**

I. Introduction

SOLVING nonlinear equation systems is a fundamental topic in science and engineering [1]–[13]. Chan and Chen [2] presented an ion-sensitive field-effect transistor readout circuit that incorporates a novel nonlinear temperature compensation method, and they needed to solve for the design parameters in nonlinear equation systems. Brambilla *et al.* [4] specified that the harmonic balance method has been extensively adopted to compute the steady-state behavior of mildly nonlinear circuits in the frequency domain, which commonly employs Newton's method to solve the nonlinear equation system that models the circuit to facilitate simulation in the frequency domain. Kekatpure *et al.* [7] asserted that the allowable electromagnetic modes in microphotonic structures are determined by solving Maxwell's equations for the given geometry. These researchers explained that the final step in the procedure is the application of boundary conditions at the interfaces, which yields the dispersion equation systems that must be solved to obtain these modes. These dispersion equation systems are generally transcendental except in a few simple cases [7]–[9]. Transcendental equation systems represent a type of nonlinear equation systems, and their solutions cannot be expressed in terms of elementary mathematical functions. Moreover, many electromagnetic problems are typically described by an intricate eigenvalue problem, and the solution to such problems determines the value of an unknown parameter (e.g., propagation coefficient or resonance frequency). The problem can often be transformed into a nonlinear equation system with respect to this parameter [11]. Furthermore, complex transcendental expressions are generally obtained in many electromagnetic problems, such as microstrip antennas in multilayered dielectric structures. The complex poles or singularities of such expressions must be identified and extracted to accelerate the convergence of the adopted numerical algorithm [9]. The kinematic control of manipulators also involves solving nonlinear equation systems [10]. Therefore, effective and accurate methods for solving nonlinear equation systems are significant.

Prior to discussing the methods for solving nonlinear equation systems, such equation systems must be described. Nonlinear equation systems can be classified into two classes in terms of the relationship with time t, namely, time-invariant and time-varying nonlinear equation systems. The former is expressed in the form of $f(x) = 0$, and the latter can be written as $f(x, t) = 0$. Time-invariant nonlinear equation systems are generally preferred to time-varying ones when establishing mathematical models for scientific or engineering problems given the lack of effective methods, as well as for simplification. However, time-invariant models may be less accurate in solving time-varying problems.

Time-invariant nonlinear equation systems can be solved with numerous methods. In particular, Newton's and Muller's methods are often used to solve such systems; however, these techniques strongly rely on initial values [9], [11], [14]. These values should be close to the solutions of the equation systems. Without appropriate initial values, both methods may fail to

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solve such equation systems [8], [15], [16]. Moreover, solutions are typically difficult to estimate for researchers and engineers. Thus, Davidenko method (or Davidenko's method) was proposed as an improvement of Newton's method. Davidenko method relaxes the restriction on the choice of initial values and can converge to the solutions [9]. Therefore, this approach has been used by electromagnetic researchers to solve time-invariant nonlinear equation systems [8], [9], [12]. Meanwhile, Zhang dynamics (ZD) is a special class of neural dynamics and it is depicted in the form of a dynamical system with selectable activation functions [17]–[21]. ZD is considered an effective and accurate method for solving both time-invariant and time-varying nonlinear equation systems [17]–[22]. This class is generalized from the Zhang neural network (ZNN) [23]–[26]. ZNN originated from the research on the Hopfield neural network [23], [27]. Furthermore, the neural dynamics (e.g., ZD) approach is regarded as a powerful alternative for solving online problems because of its potential suitability for very large scale integration (VLSI) implementation, high-speed processing, and parallel-distributed properties [28], [29]. In addition, other methods such as dynamical Newton-like method for solving nonlinear equation systems can be seen from [30]–[32].

This paper mainly compares Davidenko method with ZD in terms of efficiency and accuracy in solving nonlinear equation systems. The results indicate the superiority of the latter. The remainder of this paper is thus organized into six sections. Following the Introduction, Sections II and III compare Davidenko method and ZD in terms of efficiency and accuracy in solving time-invariant and time-varying nonlinear equation systems, respectively. Section IV presents the discrete-time ZD models, which can be implemented by VLSI. Section V introduces the theoretical analysis of the convergence and accuracy of the methods and the models for solving time-varying nonlinear equation systems. Section VI presents an engineering application of ZD to the online kinematic control of a physical robot manipulator. Finally, Section VII concludes the study with final remarks. The main contributions of this paper are described as follows.

1) Davidenko method and ZD are compared as two effective methods for solving nonlinear equation systems. The results indicate the superiority of the latter.
2) A conjecture of Davidenko method is proposed for solving time-varying nonlinear equation systems. Moreover, the results of a corresponding numerical experiment are presented and discussed.
3) The discrete-time implementations of ZD for solving nonlinear equation systems are specified. These implementations may assist researchers and engineers in solving such systems online.
4) The theoretical analysis and results of Davidenko method, ZD, and discrete-time ZD models are elucidated in relation to solving time-varying nonlinear equation systems.
5) The engineering application of ZD to the online kinematic control of a two-link physical robot manipulator via online solution of a time-varying nonlinear equation system substantiates the physical realizability, accuracy and online computation ability of ZD.

II. Time-Invariant Nonlinear Equation Systems Solving

In this section, the problem is first described, which involves solving general time-invariant nonlinear equation systems. Then, Davidenko method and ZD are explained and compared for solving such systems.

A. Problem Description

The time-invariant nonlinear equation system that must be solved with n unknowns are formulated as follows:

$$\begin{cases} f_1(x_1, x_2, \ldots, x_n) = 0 \\ f_2(x_1, x_2, \ldots, x_n) = 0 \\ \qquad \vdots \qquad\qquad \vdots \\ f_n(x_1, x_2, \ldots, x_n) = 0 \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}$ is the kth unknown variable, with $k = 1, 2, \ldots, n$, and $f_k(x_1, x_2, \ldots, x_n)$ is smooth and differentiable. System (1) can also be described in the vector form

$$\mathbf{f}(\mathbf{x}) = \mathbf{0} \tag{2}$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}$; $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})]^{\mathrm{T}}$; and superscript T denotes the transpose of a vector or matrix. A minimum of one theoretical solution $\mathbf{x}^*$ is assumed to exist in (2). For convenient comparison, the two methods are presented beginning with the simplest time-invariant nonlinear equation system indicated below. This equation system involves only one unknown variable [i.e., $n = 1$ case of nonlinear equation system (1)]

$$f(x) = 0 \tag{3}$$

where x is the unknown variable.

B. Davidenko Method

The basic concept of Davidenko method is the transformation of Newton's method from a system of n nonlinear equations with n unknowns into a set of n first-order ordinary differential equations (ODEs) in a dummy variable t [8]. The formula of Newton's method, which is commonly adopted for solving nonlinear equation system (3) is written as follows [8]:

$$x_{i+1} = x_i - \frac{f(x_i)}{\mathrm{d}f(x_i)/\mathrm{d}x} \tag{4}$$

where i denotes the iteration index, and

$$x_{i+1} = x_i + \Delta x_i \tag{5}$$

where Δx_i is a correction term for x_i. The substitution of (5) into (4) yields the following equation [8]:

$$\frac{\mathrm{d}f(x_i)}{\mathrm{d}x}\Delta x_i = -f(x_i). \tag{6}$$

Newton's method relies strongly on the initial value denoted by either x_0 or $x(0)$. The iteration fails to converge to the solution if an improper initial value is set. This condition

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reflects a significant disadvantage of Newton's method [15]. Reference [8] points out that reducing the size of the correction term (i.e., $|\Delta x_i|$) by incorporating factor ϵ into (6) can eliminate this disadvantage, thereby producing the following equation:

$$\frac{\mathrm{d}f(x_i)}{\mathrm{d}x}\Delta x_i = -\epsilon f(x_i) \tag{7}$$

where $0 < \epsilon < 1$. Equation (7) can then be rewritten as

$$\frac{\Delta x_i}{\epsilon} = -\frac{f(x_i)}{\mathrm{d}f(x_i)/\mathrm{d}x}.$$

When $\epsilon \to 0$, $\Delta x_i \to 0$ is obtained in view of (7). Therefore, Δx_i can be replaced with a differential term (i.e., $\mathrm{d}x$) [8]. Correspondingly, $f(x_i)$ and ϵ are rewritten as $f(x)$ and $\mathrm{d}t$, respectively. Therefore, the following continuous differential equation was obtained in [8]:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{f(x)}{\mathrm{d}f(x)/\mathrm{d}x}$$

where t is now a scalar-valued independent variable that is introduced merely to construct the algorithm. This variable has no basis in the original problem (3). The following equation (i.e., Davidenko method for solving 1-D nonlinear equation systems) is thus obtained with $\dot{x} = \mathrm{d}x/\mathrm{d}t$ and $f'(x) = \mathrm{d}f(x)/\mathrm{d}x$:

$$\dot{x} = -\frac{f(x)}{f'(x)}. \tag{8}$$

When the Jacobian matrix of $\mathbf{f}(\mathbf{x})$ is nonsingular and depicted as follows:

$$J(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

Equation (8) can be extended to the n-dimensional case as follows:

$$\dot{\mathbf{x}} = -J^{-1}(\mathbf{x})\mathbf{f}(\mathbf{x}) \tag{9}$$

with $\dot{\mathbf{x}} = \mathrm{d}\mathbf{x}/\mathrm{d}t$ [8]. Equation (9) is the general form of Davidenko method and can be considered as a system of n ODEs. These equations can be integrated depending on an initial condition $\mathbf{x}(0)$ and can be implemented via (analog) computers or circuits [8]. Given random initial values, $\mathbf{x}$ can converge to a solution of (2) [8]. Davidenko method is regarded as an effective method for solving time-invariant nonlinear equation systems (e.g., complicated transcendental equation systems with complex roots) [8], [9], [12]. Reference [8] provides theoretical proof for the effectiveness of the method and points out that it can facilitate an exponential convergence rate.

C. Zhang Dynamics

The basic idea behind ZD involves forcing error functions to converge to zero. Specifically, an error function for solving (3) is defined as follows:

$$e(t) = f(x) - 0$$

which is monitored and controlled at zero. Note that t denotes the computational time instant, and $e(t)$ is the computational error at the time instant. When ZD forces $e(t)$ to be zero, x changes with t and maintains this condition for theoretical solution x^*. The time derivative of $e(t)$ [i.e., $\mathrm{d}e(t)/\mathrm{d}t$] is selected according to the following ZD design formula to obtain an exponential or significantly accelerated convergence rate:

$$\frac{\mathrm{d}e(t)}{\mathrm{d}t} = -\gamma\phi(e(t)). \tag{10}$$

Equation (10) can be rewritten as follows:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}t} = -\gamma\phi(f(x)) \tag{11}$$

where design parameter $\gamma > 0$ is used to scale the convergence rate [17], [18]. In circuit implementations, γ normally corresponds to the reciprocal of a capacitance parameter [23]. If possible, the value of γ must be large to obtain a rapid convergence rate. Besides, $\phi(\cdot) : \mathbb{R} \to \mathbb{R}$ is the activation function that should be monotonically increasing and odd. Such functions are illustrated in [18]. If the following linear activation function is adopted:

$$\phi(u) = u$$

then the convergence rate of ZD (10) is exponential for any random initial value $e(0)$ [25], [26]. Following (11), one can derive

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = -\gamma\phi(f(x)).$$

Given time $t \in [0, \infty)$, $f'(x) = \mathrm{d}f(x)/\mathrm{d}x$ and $\dot{x} = \mathrm{d}x/\mathrm{d}t$, under the assumption of $f'(x) \neq 0$, the following first-order ODE (i.e., the formula of ZD for solving 1-D time-invariant nonlinear equation systems) is obtained:

$$\dot{x} = -\frac{\gamma\phi(f(x))}{f'(x)}. \tag{12}$$

The convergence rate of ZD is exponential when the linear activation function is used for solving time-invariant nonlinear equation systems with random initial value $x(0)$ [14]. Moreover, the convergence rate of ZD when either the power-sum activation function or the power-sigmoid activation function is used is superior to that observed when the linear activation function is used in terms of certain error ranges (locally or globally) [18]. Equation (12) can also be readily simulated through MATLAB ODE routines (e.g., ODE45 or ODE15s) with a random initial value $x(0)$. Consequently, $x(t)$ converges quickly to the solution.

Another error function is defined as follows for solving the general time-invariant nonlinear equation system depicted in (2):

$$\mathbf{e}(t) = \mathbf{f}(\mathbf{x}) - \mathbf{0}$$

i.e., $\mathbf{e}(t) = \mathbf{f}(\mathbf{x})$. By extending (11) to its vector form, the following equation is obtained:

$$\frac{\mathrm{d}\mathbf{f}(\mathbf{x})}{\mathrm{d}t} = -\gamma\phi(\mathbf{f}(\mathbf{x})) \tag{13}$$

whose expanded form yields the following equation:

$$J(\mathbf{x})\dot{\mathbf{x}} = -\gamma\phi(\mathbf{f}(\mathbf{x})).$$

With $J(\mathbf{x})$ being nonsingular, the ZD formula for solving n-dimensional time-invariant nonlinear equation systems is expressed as follows:

$$\dot{\mathbf{x}} = -\gamma J^{-1}(\mathbf{x})\phi(\mathbf{f}(\mathbf{x})). \tag{14}$$

Various specific examples have substantiated the effectiveness and accuracy of ZD for solving time-invariant nonlinear equation systems [14], [17], [18].

D. Theoretical Comparison

In this section, different aspects of Davidenko method and ZD are compared in relation to solving time-invariant nonlinear equation systems.

On the one hand, the two approaches are relatively correlated with each other, and they share favorable characteristics. Davidenko method can be viewed as a special case of ZD for solving time-invariant nonlinear equation systems. That is, the ZD formula (14) for solving n-dimensional time-invariant nonlinear equation systems reduces to (9) when $\gamma = 1$ and when the linear activation function is used. Meanwhile, ZD formula (12) is reduced to the formula of Davidenko method (8) for solving 1-D time-invariant nonlinear equation systems. As mentioned previously, both Davidenko method and ZD can converge to the theoretical solutions given random initial values.

On the other hand, Davidenko method and ZD differ intrinsically from each other. The basic concept of Davidenko method is the transformation of Newton's method from a system of n nonlinear equations with n unknowns into a set of n first-order ODEs in a dummy variable t. In this approach, $\mathrm{d}x/\mathrm{d}t$ is derived from an improved formula for Newton's method on the assumption that $\epsilon \to 0$ (i.e., Davidenko *et al.* [8] replaced $\Delta x/\epsilon$ with $\mathrm{d}x/\mathrm{d}t$). By contrast, the basic concept of ZD involves forcing the error function to take a value of zero. To this end, Zhang *et al.* [19] proposed the ZD design formula to solve time-invariant nonlinear equation systems in scalar form (10) and vector form (13). Expanding (10) and (13) can yield the expressions of $\mathrm{d}x/\mathrm{d}t$ and $\mathrm{d}\mathbf{x}/\mathrm{d}t$, respectively. Besides, the value of parameter γ determines the convergence rate of ZD using some specific activation function. The convergence rate can be improved by increasing the value of γ. This increase can be realized in numerical experiments, simulations, and/or hardware implementations (e.g., VLSI). The convergence of ZD can also be expedited for solving time-invariant nonlinear equation systems when other suitable activation functions are used. Note that the disadvantage of the Davidenko method for solving time-invariant nonlinear equation systems lies in the fact that its convergence cannot be accelerated.

In brief, Davidenko method can be viewed as a special case of ZD for solving time-invariant nonlinear equation systems. Therefore, the effectiveness of ZD is substantiated by the use of Davidenko method in addressing electromagnetic problems through solving time-invariant nonlinear equation systems. ZD possesses all of the favorable characteristics of Davidenko method; however, ZD is considered superior to Davidenko method because of its adjustable (accelerated) convergence rate in solving time-invariant nonlinear equation systems.

III. Time-Varying Nonlinear Equation Systems Solving

In this section, the solving of time-varying nonlinear equation systems is investigated. Moreover, Davidenko method and ZD are compared further in the context of solving time-varying nonlinear equation systems on the basis of a specific example. The results of the comparison verifies the superiority of ZD.

A. Problem Description

The time-varying nonlinear equation system that must be solved with n unknowns is generally formulated as follows:

$$\begin{cases} f_1(x_1(t), x_2(t), \ldots, x_n(t), t) = 0 \\ f_2(x_1(t), x_2(t), \ldots, x_n(t), t) = 0 \\ \quad\vdots \qquad\qquad \vdots \\ f_n(x_1(t), x_2(t), \ldots, x_n(t), t) = 0 \end{cases} \tag{15}$$

where $x_k(t) \in \mathbb{R}$ is the kth unknown variable, with $k = 1, 2, \ldots, n$, and $f_k(x_1(t), x_2(t), \ldots, x_n(t), t)$ is smooth and differentiable. System (15) can also be described as follows:

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{0} \tag{16}$$

with $\mathbf{x} = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}}$. A minimum of one theoretical solution $\mathbf{x}^*$ is assumed to exist for the problem at any time instant t. Given only one unknown variable (i.e., $n = 1$), the simplest case of the problem is

$$f(x, t) = 0. \tag{17}$$

B. Time-Varying Davidenko Method

Davidenko method (an improvement on Newton's method) has conventionally been used as a method for solving time-invariant nonlinear equation systems. Newton's method is based on the fixed-point theorem, which also concentrates on solving time-invariant equation systems [15]. Nonetheless, almost no theoretical bases have been established regarding the feasibility of adopting this approach for solving time-varying nonlinear equation systems. This lack may be the reason why Davidenko *et al.* [8] and other researchers did not derive the theoretical formula of Davidenko method for solving time-varying nonlinear equation systems from the original Newton's method. The formula of Davidenko method is conjectured in this paper as follows:

$$\dot{\mathbf{x}} = -J^{-1}(\mathbf{x}, t)\mathbf{f}(\mathbf{x}, t) \tag{18}$$

for solving the problem based on its commonly used formula (9) when the Jacobian matrix of $\mathbf{f}(\mathbf{x}, t)$ denoted by $J(\mathbf{x}, t)$ is nonsingular.

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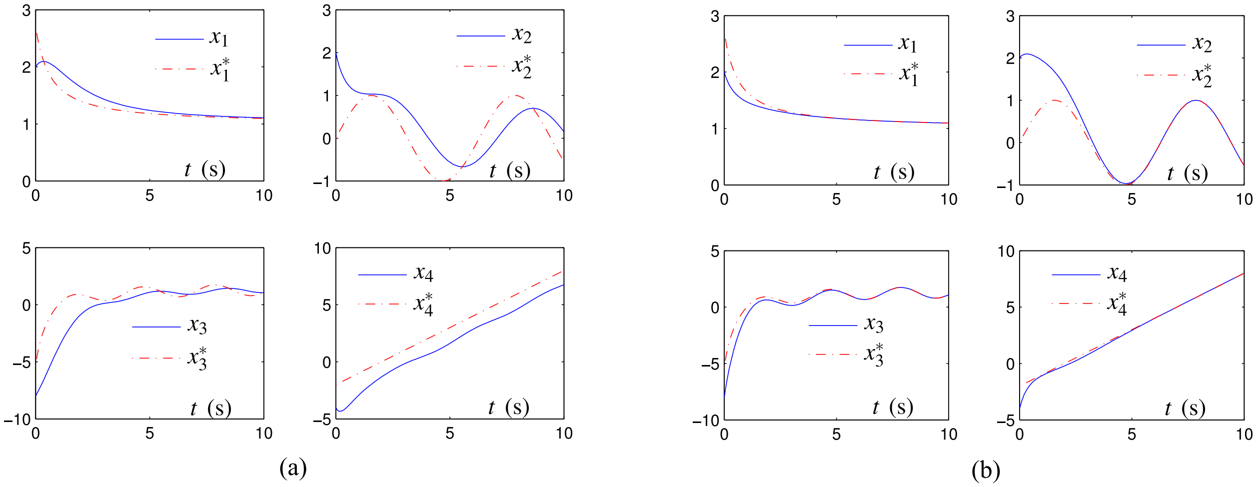


Fig. 1. Performance comparison between Davidenko method (18) and ZD (23) for solving the time-varying nonlinear equation system depicted in (24), where x_k^* denotes the kth element of the theoretical solution and x_k denotes the kth element of the solution by the adopted method with $k = 1$–4. (a) Solution by Davidenko method. (b) Solution by ZD.

The corresponding theoretical analysis for the conjecture is presented in Section V.

C. Zhang Dynamics

ZD can be used to solve various types of time-varying problems because of its sensitivity to problem changes (i.e., ZD sufficiently includes the derivative information of the problem with respect to time t). An error function for solving (17) via ZD is defined as follows:

$$e(t) = f(x,t) - 0 \tag{19}$$

i.e., $e(t) = f(x,t)$. The ZD design formula is considered

$$\frac{\mathrm{d}e(t)}{\mathrm{d}t} = -\gamma\phi(e(t)) \tag{20}$$

and the definitions of $\phi(\cdot)$ and γ are given in Section II-C. ZD is a systematic and unified method; therefore, the effects of the selected $\phi(\cdot)$ and γ on the performance of ZD in terms of solving time-varying nonlinear equation systems are similar to the aforementioned effects. Substituting (19) into (20) yields the following equation:

$$\frac{\mathrm{d}f(x,t)}{\mathrm{d}t} = -\gamma\phi(f(x,t))$$

whose expanded form generates the following expression:

$$\frac{\partial f(x,t)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f(x,t)}{\partial t} = -\gamma\phi(f(x,t)).$$

Under the assumption of $\partial f(x,t)/\partial x \neq 0$, the formula of ZD for solving 1-D nonlinear equation system (17) is obtained as follows:

$$\dot{x} = -\frac{\gamma\phi(f(x,t)) + \dfrac{\partial f(x,t)}{\partial t}}{\dfrac{\partial f(x,t)}{\partial x}}.$$

Another error function for solving the n-dimensional time-varying nonlinear equation system (16) is defined as $\mathbf{e}(t) = \mathbf{f}(\mathbf{x},t) - \mathbf{0}$, i.e.,

$$\mathbf{e}(t) = \mathbf{f}(\mathbf{x},t). \tag{21}$$

By extending (20), the vector form of the ZD formula is obtained again

$$\frac{\mathrm{d}\mathbf{e}(t)}{\mathrm{d}t} = -\gamma\phi(\mathbf{e}(t)). \tag{22}$$

The substitution of (21) into (22) yields the following

$$\frac{\mathrm{d}\mathbf{f}(\mathbf{x},t)}{\mathrm{d}t} = -\gamma\phi(\mathbf{f}(\mathbf{x},t))$$
$$J(\mathbf{x},t)\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + \frac{\partial\mathbf{f}(\mathbf{x},t)}{\partial t} = -\gamma\phi(\mathbf{f}(\mathbf{x},t)).$$

Given that $J(\mathbf{x},t)$ is nonsingular, the formula of ZD for solving n-dimensional time-varying nonlinear equation systems is

$$\dot{\mathbf{x}} = -J^{-1}(\mathbf{x},t)\left(\gamma\phi(\mathbf{f}(\mathbf{x},t)) + \frac{\partial\mathbf{f}(\mathbf{x},t)}{\partial t}\right). \tag{23}$$

The convergence analysis regarding the use of ZD in solving time-varying nonlinear equation systems is elucidated in Section V. Various examples in [20]–[22] have confirmed the effectiveness and accuracy of ZD given such equation systems.

D. Numerical Comparison

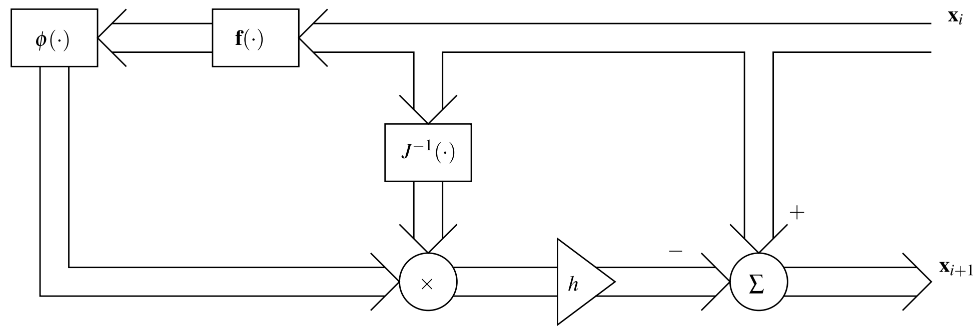
In this section, Davidenko method and ZD are compared through a specific example [21] in relation to solving time-varying nonlinear equation systems.

The specific time-varying nonlinear equation system to be solved are formulated as

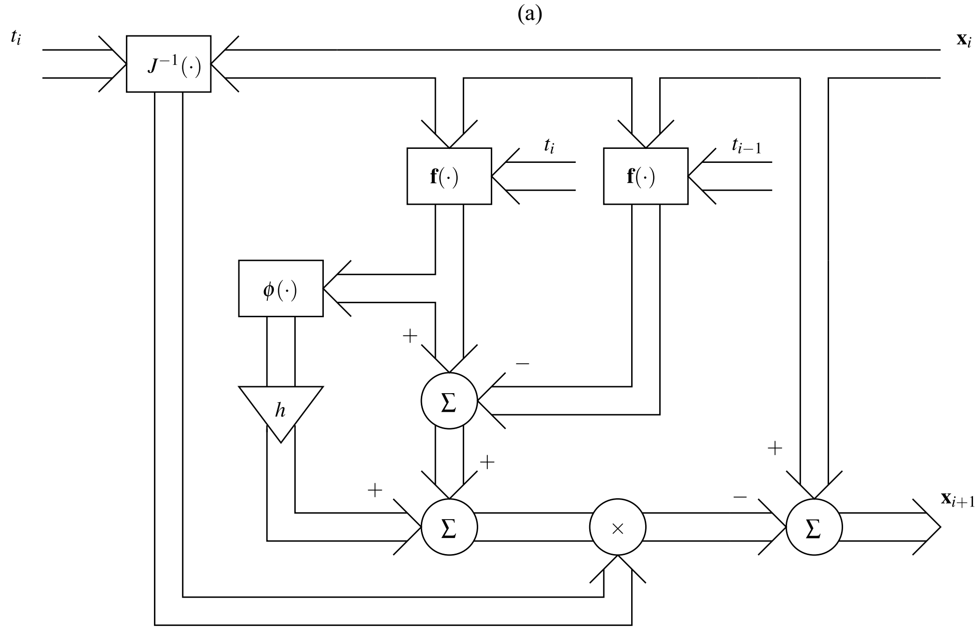
$$\begin{cases} \ln x_1(t) - 1/(t+1) = 0 \\ x_1(t)x_2(t) - \exp(1/(t+1))\sin(t) = 0 \\ x_1^2(t) - \sin(t)x_2(t) + x_3(t) - 2 = 0 \\ x_1^2(t) + x_2^2(t) + x_3(t) + x_4^2(t) - t = 0 \end{cases} \tag{24}$$

whose theoretical solution is given as follows for the convenient solution comparison:

$$\mathbf{x}^*(t) = \begin{bmatrix} x_1^*(t) \\ x_2^*(t) \\ x_3^*(t) \\ x_4^*(t) \end{bmatrix} = \begin{bmatrix} \exp(1/(t+1)) \\ \sin(t) \\ 2 - \exp(2/(t+1)) + \sin^2(t) \\ t-2 \end{bmatrix}.$$



(a)



(b)

Fig. 2. Block diagrams of discrete-time ZD models for solving time-invariant and time-varying nonlinear equations. Block diagram of discrete-time. (a) ZD model (26) for solving time-invariant nonlinear equation systems and (b) ZD model (27) for solving time-varying nonlinear equation systems.

In the numerical comparison, MATLAB routine ODE45 is adopted under the same setting for the two methods, which thus guarantees that the comparison is fair [33]. The performance of the time-varying Davidenko method with respect to solving (24) is presented in Fig. 1(a). The initial value is $\mathbf{x}(0) = [2, 2, -8, -4]^{\mathrm{T}}$. Correspondingly, Fig. 1(b) presents the performance of ZD in terms of solving (24) when the linear activation function is used, $\gamma = 1$, and the initial value is maintained. Fig. 1 verifies that ZD (23) is superior to the time-varying Davidenko method (18) in terms of solving time-varying nonlinear equation systems. Specifically, the convergence performance and accuracy of the former are better than those of the latter. Nonetheless, the performance of the time-varying Davidenko method in solving (24) is consistent with the corresponding theoretical analysis shown in Section V. The solution error $|x_k - x_k^*|$ is bounded, and the values of the errors are evidently related to the time-varying rate of the problem. Fig. 1(a) illustrates that the accuracy of the time-varying Davidenko method is better for x_1^* than that for the accelerated time-varying x_4^*.

IV. Discrete-Time Implementations of ZD

For the potential hardware implementation of ZD based on digital circuits or digital computers, discrete-time ZD models for solving online nonlinear equation systems are presented by adopting Euler difference rules with the corresponding block diagrams and circuit schematics shown.

A. Models Derivation

Equation (14) is discretized by adopting Euler forward difference rule with term $\mathbf{O}(\tau)$ neglected to obtain the discrete-time ZD model for solving time-invariant nonlinear equation systems. Consequently, the following expression is acquired [17]:

$$\frac{\mathbf{x}_{i+1} - \mathbf{x}_i}{\tau} = -\gamma J^{-1}(\mathbf{x}_i)\phi(\mathbf{f}(\mathbf{x}_i)) \tag{25}$$

where τ denotes the step size of fictitious time, and i denotes the iteration index with $i = 0, 1, 2, \ldots$ From (25), the discrete-time ZD model for solving time-invariant nonlinear equation systems is obtained as follows:

$$\mathbf{x}_{i+1} = \mathbf{x}_i - hJ^{-1}(\mathbf{x}_i)\phi(\mathbf{f}(\mathbf{x}_i)) \tag{26}$$

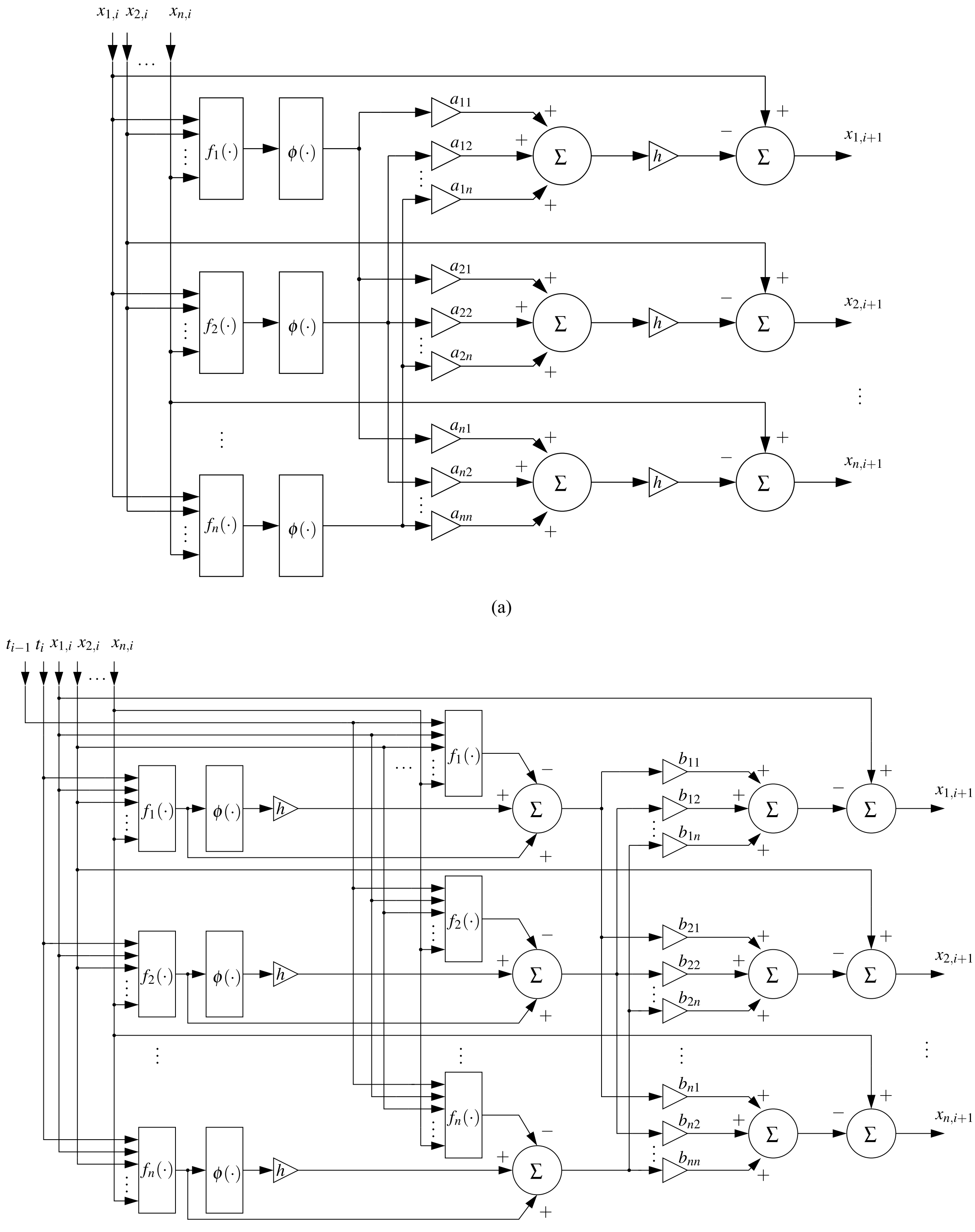


Fig. 3. Circuit schematics that realize discrete-time ZD models for solving time-invariant and time-varying nonlinear equation systems. Circuit schematic that realizes discrete-time. (a) ZD model (26) for solving time-invariant nonlinear equation systems, where (26) is rewritten as $\mathbf{x}_{i+1} = \mathbf{x}_i - hA\phi(\mathbf{f}(\mathbf{x}_i))$ with $A = J^{-1}(\mathbf{x}_i)$ and a_{lj} denotes the entry at position (l, j) in matrix A and (b) ZD model (27) for solving time-varying nonlinear equation systems, where (27) is rewritten as $\mathbf{x}_{i+1} = \mathbf{x}_i - B(h\phi(\mathbf{f}(\mathbf{x}_i, t_i)) + \mathbf{f}(\mathbf{x}_i, t_i) - \mathbf{f}(\mathbf{x}_i, t_{i-1}))$ with $B = J^{-1}(\mathbf{x}_i, t_i)$ and b_{lj} denotes the entry at position (l, j) in matrix B.

with $h = \gamma\tau$. When the linear activation function is adopted and $h = 1$, this time-invariant model for solving (16) can be rewritten as $\mathbf{x}_{i+1} = \mathbf{x}_i - J^{-1}(\mathbf{x}_i, t_i)\mathbf{f}(\mathbf{x}_i, t_i)$. In view of (26), different values of h and $\phi(\cdot)$ result in variations in the performances of the discrete-time ZD model for solving time-invariant nonlinear equation systems. When the linear

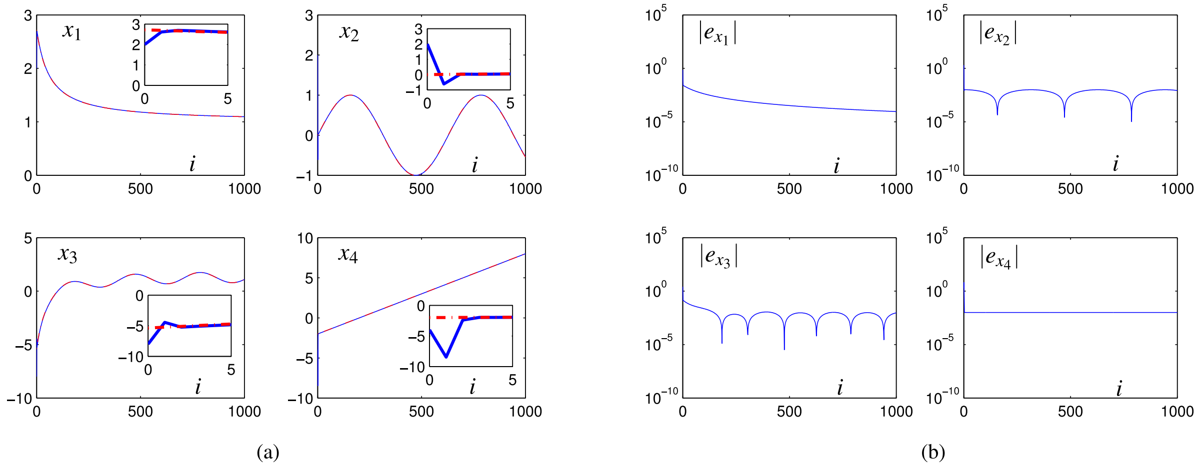


Fig. 4. Performance of time-invariant ZD model (26) for solving the time-varying nonlinear equation system depicted in (24) given $h = 1$ and $\tau = 0.01$ s; in (a), the red dash-dotted curves represent the theoretical solution and the blue solid curves represent the solution generated by the time-invariant ZD model. (a) Solution by time-invariant ZD model (26). (b) Solution errors.

activation function is selected and $h = \tau$, the discrete-time ZD model (26) is also the discrete-time model of Davidenko method for solving time-invariant nonlinear equation systems. The block diagram of (26) is shown in Fig. 2(a). The circuit schematic that reflects the model is illustrated in Fig. 3(a) to guide the hardware implementation of the discrete-time ZD model specifically for solving time-invariant nonlinear equation systems. Reference [19] presents some of the numerical experiment results that substantiate the efficacy of (26) for solving time-invariant nonlinear equation systems; thus, those findings are omitted in the current paper.

The corresponding discrete-time ZD model for solving time-varying nonlinear equation systems can be obtained by discretizing (23). By applying the Euler forward-difference rule to this equation and disregarding the term $\mathbf{O}(\tau)$, the following equation is obtained:

$$\frac{\mathbf{x}_{i+1} - \mathbf{x}_i}{\tau} = -J^{-1}(\mathbf{x}_i, t_i)\left(\gamma\phi(\mathbf{f}(\mathbf{x}_i, t_i)) + \frac{\partial \mathbf{f}(\mathbf{x}_i, t_i)}{\partial t_i}\right)$$

where the definitions of i and τ are as before and $t_i = i\tau$. On the basis of the Euler backward-difference rule, $\partial \mathbf{f}(\mathbf{x}_i, t_i)/\partial t_i$ is replaced with $(\mathbf{f}(\mathbf{x}_i, t_i) - \mathbf{f}(\mathbf{x}_i, t_{i-1}))/\tau$. The following equation is then generated:

$$\frac{\mathbf{x}_{i+1} - \mathbf{x}_i}{\tau} = -J^{-1}(\mathbf{x}_i, t_i)\left(\gamma\phi(\mathbf{f}(\mathbf{x}_i, t_i)) + \frac{\mathbf{f}(\mathbf{x}_i, t_i) - \mathbf{f}(\mathbf{x}_i, t_{i-1})}{\tau}\right).$$

Given the aforementioned definition of h, the discrete-time ZD model for solving time-varying nonlinear equation systems is obtained. The model is formulated as follows:

$$\mathbf{x}_{i+1} = \mathbf{x}_i - J^{-1}(\mathbf{x}_i, t_i)(h\phi(\mathbf{f}(\mathbf{x}_i, t_i)) + \mathbf{f}(\mathbf{x}_i, t_i) - \mathbf{f}(\mathbf{x}_i, t_{i-1})). \tag{27}$$

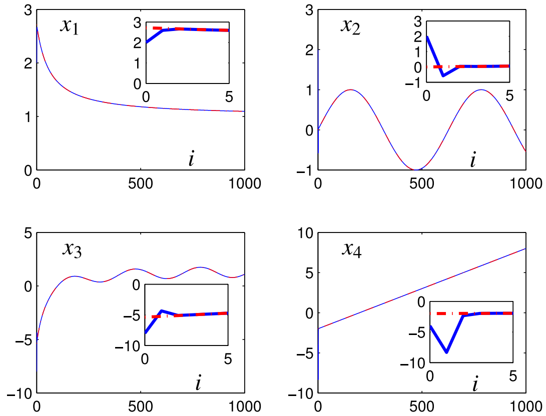
The corresponding block diagram of discrete-time ZD model (27) for solving time-varying nonlinear equation systems is displayed in Fig. 2(b). As noted previously, different values of h and $\phi(\cdot)$ may vary the performances of the discrete-time ZD model (27) in solving time-varying nonlinear equation systems. The circuit schematic that realizes the model is displayed in Fig. 3(b) to guide for the potential hardware implementation of the discrete-time ZD model for solving time-varying nonlinear equation systems.

B. Numerical Experiments

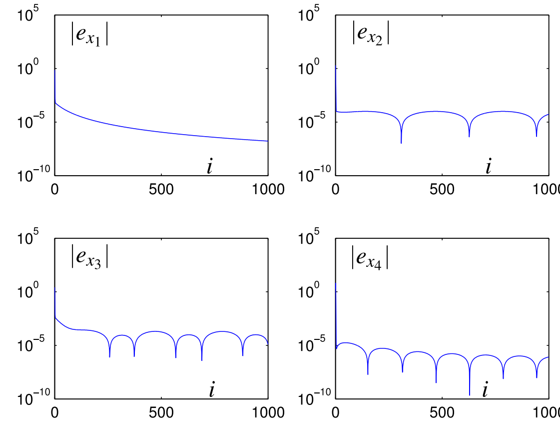
The results of the numerical experiments regarding the specific system (24) of time-varying nonlinear equations given the two discrete-time models are presented and discussed in this section to substantiate and emphasize the effectiveness of the time-varying discrete-time ZD model in solving time-varying nonlinear equation systems.

The results of (26) and (27) in relation to solving the same specific time-varying nonlinear equation system (24) are depicted in Figs. 4 and 5, respectively. The initial value is $\mathbf{x}(0) = [2, 2, -8, -4]^{\mathrm{T}}$; moreover, $h = 1$, $\tau = 0.01$ s and the linear activation function is used. When $i > 300$, the solution errors (i.e., $|e_{x_k}| = |x_k - x_k^*|$ with $k = 1, 2, 3, 4$) of the time-varying ZD (27) for solving (24) are maintained under approximately 10^{-4}. These errors are significantly smaller than those of the time-invariant discrete-time ZD model (26), i.e., about 10^{-2}. This case affirms that the time-varying discrete-time ZD model is considerably more accurate (i.e., roughly 100 times) and more effective for solving time-varying nonlinear equation systems than the time-invariant discrete-time ZD model.

In summary, this section introduces two discrete-time ZD models for solving both time-invariant and time-varying nonlinear equation systems. The corresponding block diagrams and circuit schematics are also presented to guide the implementations of hardware for solving such problems online with the advantages of the neural dynamics (e.g., parallel processing). The results of the numerical experiments emphasize and substantiate the accuracy, effectiveness, and superiority of the time-varying discrete-time ZD model.



(a)



(b)

Fig. 5. Performance of time-varying ZD model (27) for solving the time-varying nonlinear equation system depicted in (24) given $h = 1$ and $\tau = 0.01$ s; in (a), the red dash-dotted curves represent the theoretical solution and the blue solid curves represent the solution generated by the time-varying ZD model. (a) Solution by time-varying ZD model (27). (b) Solution errors.

V. Theoretical Analysis and Results

This section presents the theoretical analysis and results about Davidenko method and ZD for solving time-varying nonlinear equation systems, showing the advantages of ZD over Davidenko method in terms of accuracy. The theoretical analysis of ZD in relation to solving time-invariant nonlinear equation systems can be readily conducted in a similar manner; therefore, the description of this process is omitted.

A. Continuous-Time Model Analysis

The following theorem is drawn to support the conjecture of the time-varying Davidenko method (18).

Theorem 1: Consider time-varying nonlinear equation system (16) with nonsingular Jacobian matrix $J(\mathbf{x}, t)$, whose partial derivative with respect to time t is uniformly bounded as $\|\partial \mathbf{f}(\mathbf{x}, t)/\partial t\|_2 \le \delta$, $\forall t \in [0, \infty)$, and $\exists 0 \le \delta < \infty$. Starting from any initial state $\boldsymbol{x}(0)$, the steady-state residual error of the time-varying Davidenko method (18) for solving (16) is upper-bounded as

$$\lim_{t\to\infty} \|\boldsymbol{f}(\boldsymbol{x}, t)\|_2 \le \delta.$$

Proof: Let a Lyapunov function candidate be defined as $V(t) = \|\mathbf{f}(\mathbf{x}, t)\|_2^2/2$. $V(t)$ is evidently positive-definite because $V(t) = \mathbf{f}^{\mathrm{T}}(\mathbf{x}, t)\mathbf{f}(\mathbf{x}, t)/2 > 0$ for $\mathbf{f}(\mathbf{x}, t) \ne \mathbf{0} \in \mathbb{R}^n$ and $V(t) = 0$ for $\mathbf{f}(\mathbf{x}, t) = \mathbf{0} \in \mathbb{R}^n$ only. Consequently, the following equation is obtained:

$$\dot{V}(t) = \mathbf{f}^{\mathrm{T}}(\mathbf{x}, t)\frac{\mathrm{d}\mathbf{f}(\mathbf{x}, t)}{\mathrm{d}t} = \mathbf{f}^{\mathrm{T}}(\mathbf{x}, t)\left(J(\mathbf{x}, t)\dot{\mathbf{x}} + \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial t}\right).$$

Substituting (18) into the equation above generates the following equation:

$$\begin{aligned}\dot{V}(t) &= \mathbf{f}^{\mathrm{T}}(\mathbf{x}, t)\left(J(\mathbf{x}, t)\left(-J^{-1}(\mathbf{x}, t)\mathbf{f}(\mathbf{x}, t)\right) + \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial t}\right)\\ &= -\mathbf{f}^{\mathrm{T}}(\mathbf{x}, t)\mathbf{f}(\mathbf{x}, t) + \mathbf{f}^{\mathrm{T}}(\mathbf{x}, t)\frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial t}.\end{aligned}$$

Moreover

$$\mathbf{f}^{\mathrm{T}}(\mathbf{x}, t)\frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial t} \le \|\mathbf{f}^{\mathrm{T}}(\mathbf{x}, t)\|_2 \cdot \left\|\frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial t}\right\|_2 \le \delta\|\mathbf{f}^{\mathrm{T}}(\mathbf{x}, t)\|_2.$$

Therefore

$$\begin{aligned}\dot{V}(t) &\le -\mathbf{f}^{\mathrm{T}}(\mathbf{x}, t)\mathbf{f}(\mathbf{x}, t) + \delta\left\|\mathbf{f}^{\mathrm{T}}(\mathbf{x}, t)\right\|_2\\ &= -\|\mathbf{f}(\mathbf{x}, t)\|_2^2 + \delta\|\mathbf{f}(\mathbf{x}, t)\|_2\\ &= -\|\mathbf{f}(\mathbf{x}, t)\|_2(\|\mathbf{f}(\mathbf{x}, t)\|_2 - \delta).\end{aligned}$$

During the solution process, the equation above falls into one of the following three situations: 1) $\|\mathbf{f}(\mathbf{x}, t)\|_2 - \delta > 0$; 2) $\|\mathbf{f}(\mathbf{x}, t)\|_2 - \delta = 0$; and 3) $\|\mathbf{f}(\mathbf{x}, t)\|_2 - \delta < 0$. The corresponding analysis is presented as follows.

1) If $\|\mathbf{f}(\mathbf{x}, t)\|_2 - \delta > 0$ is satisfied, then $\dot{V}(t) < 0$. This situation implies that $\mathbf{f}(\mathbf{x}, t)$ approaches $\mathbf{0} \in \mathbb{R}^n$.
2) If $\|\mathbf{f}(\mathbf{x}, t)\|_2 - \delta = 0$ is satisfied (i.e., $\|\mathbf{f}(\mathbf{x}, t)\|_2 = \delta$, a so-called ball surface), then $\dot{V}(t) \le 0$. This situation implies that $\mathbf{f}(\mathbf{x}, t)$ either approaches $\mathbf{0} \in \mathbb{R}^n$ or remains on the ball surface. In other words, $\|\mathbf{f}(\mathbf{x}, t)\|_2$ remains within the ball of radius δ.
3) If $\|\mathbf{f}(\mathbf{x}, t)\|_2 - \delta < 0$ is satisfied, i.e., $\|\mathbf{f}(\mathbf{x}, t)\|_2 < \delta$, then $\dot{V}(t)$ is less than a positive scalar value (containing $\dot{V}(t) \le 0$ and $\dot{V}(t) > 0$). The worst case is $\dot{V}(t) > 0$, in which the values of $V(t)$ and $\|\mathbf{f}(\mathbf{x}, t)\|_2$ increase. A certain time instant must exist in which $\|\mathbf{f}(\mathbf{x}, t)\|_2 - \delta = 0$. This instance falls into the second situation, i.e., $\dot{V}(t) \le 0$, and the worst is $\|\mathbf{f}(\mathbf{x}, t)\|_2 = \delta$.

In view of the above situations, the residual error $\|\mathbf{f}(\mathbf{x}, t)\|_2$ of Davidenko method for solving the time-varying nonlinear equation system is upper-bounded by δ; i.e., in mathematics, $\lim_{t\to\infty} \|\mathbf{f}(\mathbf{x}, t)\|_2 \le \delta$. The proof is thus completed. ■

Correspondingly, the following theorems are established regarding the adoption of ZD for solving time-varying nonlinear equation systems.

Theorem 2: Consider time-varying nonlinear equation system (16) with nonsingular Jacobian matrix $J(\boldsymbol{x}, t)$. If the monotonically increasing odd activation function $\phi(\cdot)$ is adopted for ZD (23), then $\|\mathbf{f}(\mathbf{x}, t)\|_2$ is large-scale asymptotically convergent to zero.

Proof: Consider the Lyapunov function candidate $L(t) = \|\mathbf{f}(\mathbf{x}, t)\|_2^2/2$. By following the same derivation procedure as in the proof of Theorem 1, it can be concluded that $L(t)$ is positive-definite and

$$\dot{L}(t) = \mathbf{f}^{\mathrm{T}}(\mathbf{x}, t)\left(J(\mathbf{x}, t)\dot{\mathbf{x}} + \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial t}\right).$$

Substituting (23) into the equation above yields

$$\dot{L}(t) = \mathbf{f}^{\mathrm{T}}(\mathbf{x}, t)\left(J(\mathbf{x}, t)\left(-J^{-1}(\mathbf{x}, t)\left(\gamma\phi(\mathbf{f}(\mathbf{x}, t)) + \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial t}\right)\right) + \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial t}\right)$$

from which the following equation can promptly be acquired:

$$\dot{L}(t) = -\gamma \mathbf{f}^{\mathrm{T}}(\mathbf{x}, t)\phi(\mathbf{f}(\mathbf{x}, t)) = -\gamma \sum_{k=1}^{n} f_k(\mathbf{x}, t)\phi(f_k(\mathbf{x}, t)).$$

As $\phi(\cdot)$ is monotonically increasing and odd, one can derive $f_k(\mathbf{x}, t)\phi(f_k(\mathbf{x}, t)) > 0$ for $f_k(\mathbf{x}, t) \neq 0$ and $f_k(\mathbf{x}, t)\phi(f_k(\mathbf{x}, t)) = 0$ for $f_k(\mathbf{x}, t) = 0$ only.

Therefore, $\dot{L}(t) < 0$ for $\|\mathbf{f}(\mathbf{x}, t)\|_2 \neq 0$ and $\dot{L}(t) = 0$ for $\|\mathbf{f}(\mathbf{x}, t)\|_2 = 0$. In other words, the negative-definiteness of $\dot{L}(t)$ is guaranteed. Besides, if $\|\mathbf{f}(\mathbf{x}, t)\|_2 \to \infty$, the Lyapunov candidate $L(t)$ tends to be infinite. Thus, by Lyapunov theory [25], [26], [34]–[36], $\|\mathbf{f}(\mathbf{x}, t)\|_2$ is large-scale asymptotically convergent to zero. Besides, $\mathbf{x}(t)$ converges to theoretical solution $\mathbf{x}^*(t)$ from any initial state $\mathbf{x}(0)$ when the residual error $\mathbf{f}(\mathbf{x}, t)$ converges to $\mathbf{0} \in \mathbb{R}^n$. The proof is thus completed. ■

Theorem 3: Consider time-varying nonlinear equation system (16) with nonsingular Jacobian matrix $J(\mathbf{x}, t)$. If the linear activation function is adopted for ZD (23) starting from any initial $\mathbf{f}(\mathbf{x}(0), 0)$, then $\mathbf{f}(\mathbf{x}, t)$ is exponentially convergent to $\mathbf{0} \in \mathbb{R}^n$ with the convergence rate being γ.

Proof: With nonsingular $J(\mathbf{x}, t)$, one can readily derive $\mathrm{d}\mathbf{f}(\mathbf{x}, t)/\mathrm{d}t = -\gamma\phi(\mathbf{f}(\mathbf{x}, t))$ from the formula of ZD (23) for solving the time-varying nonlinear equation system. When the linear activation is adopted, $\mathrm{d}\mathbf{f}(\mathbf{x}, t)/\mathrm{d}t = -\gamma \mathbf{f}(\mathbf{x}, t)$. Beginning from any initial $\mathbf{f}(\mathbf{x}(0), 0)$, the following equation is obtained:

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}(0), 0)\exp(-\gamma t).$$

In other words, $\mathbf{f}(\mathbf{x}, t)$ exponentially converges to $\mathbf{0} \in \mathbb{R}^n$ with the convergence rate being γ. The proof is thus completed. ■

In view of Theorem 3, a large value of λ leads to accelerated convergence of ZD. In terms of the circuit implementations, the value of parameter γ should be selected properly according to the constraints of the circuit implementations (e.g., the constraints of circuit components). Since γ corresponds to the reciprocal of a capacitance parameter in the potential circuit implementations, the value of γ may have a finite upper bound, being as large as that of order 10^{12}, for instance [34]. Therefore, the convergence rate cannot be arbitrarily large, and an optimal convergence rate exists. Normally, with the constraints of the circuit implementations satisfied, the value of parameter γ should be set as large as possible to achieve the optimal convergence rate. The convergence of ZD with the application of the power-sigmoid activation function is superior to that observed when the linear activation function is adopted. This conclusion can be readily proven in a similar manner [25]; therefore, this proof is omitted in this paper due to the space limitation.

B. Discrete-Time Model Analysis

The following theorems are established in terms of the efficiency of using the discrete-time ZD models for solving time-varying nonlinear equation systems. Three definitions [37] are numerated in the Appendix as a basis for such theorems to complete the theory.

Theorem 4: Consider time-varying nonlinear equation system (16) with nonsingular Jacobian matrix $J(\mathbf{x}, t)$, whose partial derivative with respect to time t is uniformly bounded as $\|\partial\mathbf{f}(\mathbf{x}, t)/\partial t\|_2 \leq \delta$, $\forall t \in [0, \infty)$, and $\exists 0 \leq \delta < \infty$. Residual error $\|\mathbf{f}(\mathbf{x}_{i+1}, t_{i+1})\|_2$ is $O(\tau)$ for the time-invariant discrete-time ZD model (26) that addresses (16) when the linear activation function is adopted and $h = 1$.

Proof: When the linear activation function is adopted and $h = 1$, the time-invariant discrete-time ZD model (26) for solving (16) can be rewritten as $\mathbf{x}_{i+1} = \mathbf{x}_i - J^{-1}(\mathbf{x}_i, t_i)\mathbf{f}(\mathbf{x}_i, t_i)$. With $\Delta\mathbf{x}_i = \mathbf{x}_{i+1} - \mathbf{x}_i = -J^{-1}(\mathbf{x}_i, t_i)\mathbf{f}(\mathbf{x}_i, t_i)$, adopting the Taylor expansion formula at $(\mathbf{x}_i, t_i)$ leads to

$$\begin{aligned}
&\|\mathbf{f}(\mathbf{x}_{i+1}, t_{i+1})\|_2 \\
&= \left\|\mathbf{f}(\mathbf{x}_i, t_i) + J(\mathbf{x}_i, t_i)\Delta\mathbf{x}_i + \frac{\partial \mathbf{f}(\mathbf{x}_i, t_i)}{\partial t}\tau + \mathbf{O}(\tau^2)\right\|_2 \\
&= \Bigg\|\mathbf{f}(\mathbf{x}_i, t_i) + J(\mathbf{x}_i, t_i)(-J^{-1}(\mathbf{x}_i, t_i)\mathbf{f}(\mathbf{x}_i, t_i)) + \frac{\partial \mathbf{f}(\mathbf{x}_i, t_i)}{\partial t}\tau \\
&\quad + \mathbf{O}(\tau^2)\Bigg\|_2 \\
&= \left\|\mathbf{0} + \frac{\partial \mathbf{f}(\mathbf{x}_i, t_i)}{\partial t}\tau + \mathbf{O}(\tau^2)\right\|_2 = O(\tau)
\end{aligned}$$

where, being of the same order of magnitude, $\mathbf{O}(\|\Delta\mathbf{x}_i\|_2^2)$ is absorbed into $\mathbf{O}(\tau^2)$ [38]. The proof is therefore completed. ■

Theorem 5: Time-varying discrete-time ZD model (27) is 0-stable.

Proof: According to Definition 1 in the Appendix, (27) is an one-step model and its characteristic polynomial can be determined with $P(q) = q - 1 = 0$, which has only one root on the unit disk, i.e., $q = 1$. Therefore, the time-varying discrete-time ZD model (27) is 0-stable. The proof is thus completed. ■

Theorem 6: Consider time-varying nonlinear equation system (16) with nonsingular Jacobian matrix $J(\mathbf{x}, t)$. Time-varying discrete-time ZD model (27) is consistent and convergent with the order of truncation error being $\mathbf{O}(\tau^2)$.

Proof: The expressions $\dot{\mathbf{x}}(t_i) = (\mathbf{x}_{i+1} - \mathbf{x}_i)/\tau + \mathbf{O}(\tau)$ and $\partial\mathbf{f}(\mathbf{x}_i, t_i)/\partial t = (\mathbf{f}(\mathbf{x}_i, t_i) - \mathbf{f}(\mathbf{x}_i, t_{i-1}))/\tau + \mathbf{O}(\tau)$ are precise. With the above terms $\mathbf{O}(\tau)$ considered in the discretization procedure of (23), the following equation is established:

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \tau\mathbf{O}(\tau) - J^{-1}(\mathbf{x}_i, t_i)(h\phi(\mathbf{f}(\mathbf{x}_i, t_i)) + \mathbf{f}(\mathbf{x}_i, t_i) - \mathbf{f}(\mathbf{x}_i, t_{i-1}) + \tau\mathbf{O}(\tau))$$

which then yields

$$\mathbf{x}_{i+1} = \mathbf{x}_i - J^{-1}(\mathbf{x}_i, t_i)(h\phi(\mathbf{f}(\mathbf{x}_i, t_i)) + \mathbf{f}(\mathbf{x}_i, t_i) - \mathbf{f}(\mathbf{x}_i, t_{i-1})) + \mathbf{O}(\tau^2).$$

Eliminating $\mathbf{O}(\tau^2)$ from the equation above yields the formula that is identical to the time-varying discrete-time ZD

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

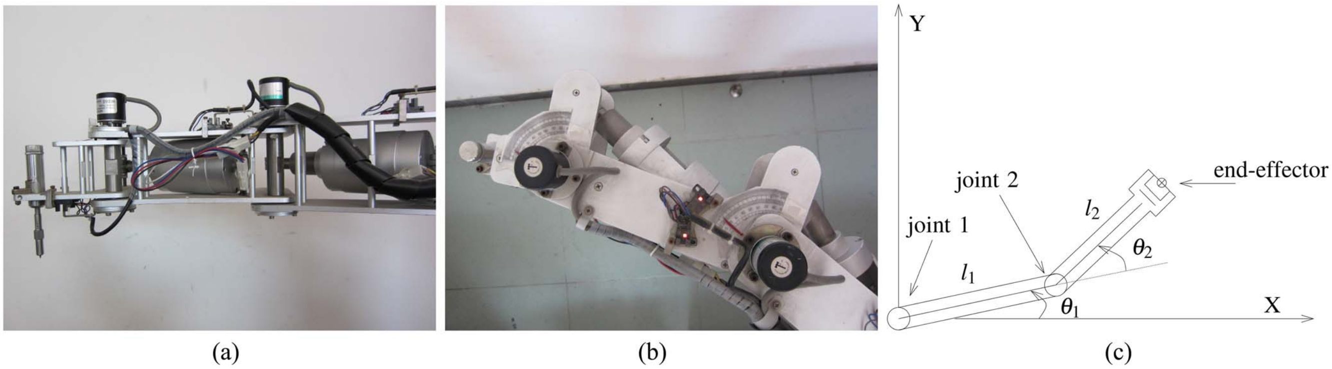


Fig. 6. Two different views and geometry of the robot manipulator. (a) Lateral view of the robot manipulator. (b) Vertical view of the robot manipulator. (c) Geometry of the robot manipulator.

model (27), and thus the truncation error of (27) is $\mathbf{O}(\tau^2)$. Hence, model (27) is consistent of order 2 according to Definition 2 in the Appendix. So, the model is both 0-stable and consistent as supported by Theorem 5. Therefore, the time-varying discrete-time ZD model (27) is consistent and convergent with the truncation error being $\mathbf{O}(\tau^2)$ on the basis of Definition 3 in the Appendix. The proof is thus completed. ■

Theorem 7: Consider time-varying nonlinear equation system (16) with nonsingular Jacobian matrix $J(\boldsymbol{x}, t)$. Steady-state residual error $\lim_{i\to\infty}\|\boldsymbol{f}(\boldsymbol{x}_{i+1}, t_{i+1})\|_2$ of the time-varying discrete-time ZD model (27) for solving (16) is $O(\tau^2)$.

Proof: In view of Definition 2, Theorems 2, 5, and 6, the following result is obtained:

$$\mathbf{x}^*_{i+1} = \mathbf{x}_{i+1} + \mathbf{O}(\tau^2)$$

in which the value of i is adequately large. By applying the Taylor expansion formula at $\mathbf{x}^*_{i+1}$, the following equation is obtained:

$$\begin{aligned}\|\mathbf{f}(\mathbf{x}_{i+1}, t_{i+1})\|_2 &= \left\|\mathbf{f}\left(\mathbf{x}^*_{i+1} - \mathbf{O}\left(\tau^2\right), t_{i+1}\right)\right\|_2 \\ &= \left\|\mathbf{f}\left(\mathbf{x}^*_{i+1}, t_{i+1}\right) + \frac{\partial \mathbf{f}\left(\mathbf{x}^*_{i+1}, t_{i+1}\right)}{\partial \mathbf{x}}\mathbf{O}\left(\tau^2\right) + \mathbf{O}\left(\tau^4\right)\right\|_2 \\ &= \left\|\mathbf{0} + J\left(\mathbf{x}^*_{i+1}, t_{i+1}\right)\mathbf{O}\left(\tau^2\right) + \mathbf{O}\left(\tau^4\right)\right\|_2 \\ &= O\left(\tau^2\right)\end{aligned}$$

which completes the proof. ■

The theoretical analysis and results presented in this section regarding residual error $\|\mathbf{f}(\mathbf{x}, t)\|_2$ are universal for both the unique-solution case and the multisolution case. In the multisolution case, state $\mathbf{x}(t)$ converges to theoretical solution $\mathbf{x}^*(t)$ which is closest to the initial state $\mathbf{x}(0)$ through ZD. For convenient comparison and presentation, only the specific example with a unique solution is presented in this paper.

VI. Application to Robot Manipulator

This section presents the application of ZD to the online kinematic control of a two-link physical robot manipulator (as shown in Fig. 6) via the online solution of a time-varying nonlinear equation system. Specifically, Fig. 6(a) and (b) presents the robot manipulator used in this application, which is controlled via a host computer. The computer is a personal digital computer equipped with a Pentium E5300 2.6-GHz CPU, 4-GB DDR3 memory, and a Windows XP Professional operating system. The computer sends commands and signals to the robot motion-control module (e.g., a control card of peripheral component interconnect) to control the motion of the robot manipulator. The geometry of the robot manipulator is shown in Fig. 6(c) with the physical lengths of the links being $l_1 = 0.185$ m and $l_2 = 0.174$ m. In view of Fig. 6(c), the forward-kinematic equation of the robot manipulator can be derived as follows:

$$\mathbf{r} = \begin{bmatrix} r_{\mathrm{X}} \\ r_{\mathrm{Y}} \end{bmatrix} = \begin{bmatrix} l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) \\ l_1\sin(\theta_1) + l_2\sin(\theta_1+\theta_2) \end{bmatrix} = \Phi(\theta)$$

where $\theta = [\theta_1, \theta_2]^{\mathrm{T}} \in \mathbb{R}^2$ denotes the joint-angle vector, $\mathbf{r} \in \mathbb{R}^2$ denotes the end-effector position vector, and $\Phi(\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the nonlinear and differentiable function of the forward kinematics. The inverse kinematic problem of robot manipulators is to solve for joint-angle $\theta(t)$ corresponding to the desired end-effector path $\mathbf{r}_{\mathrm{d}}(t)$ at any time instant $t \in [0, T]$ with T denoting the task duration. Specifically, the following time-varying nonlinear equation system needs to be solved online:

$$\mathbf{f}(\theta, t) = \Phi(\theta) - \mathbf{r}_{\mathrm{d}}(t) = \mathbf{0}. \tag{28}$$

In the application, the control period of the robot manipulator is 0.05 s, with the task duration being $T = 8$ s and the initial configuration being $\theta(0) = [5\pi/36, 17\pi/180]^{\mathrm{T}}$ rad. The desired end-effector path is set as $\mathbf{r}_{\mathrm{d}}(t) = [r_{\mathrm{X}}(0) + 0.003t, r_{\mathrm{Y}}(0) + 0.006t]^{\mathrm{T}}$, where $r_{\mathrm{X}}(0)$ and $r_{\mathrm{Y}}(0)$ denote, respectively, the X- and Y-axis components of the initial position vector $\Phi(\theta(0))$. To solving for $\theta(t)$ online from (28), time-varying discrete-time ZD model (27) equipped with the linear activation function is adopted with $h = 1$ and $\tau = 0.05$ s. The snapshots for the task execution of the robot manipulator synthesized by ZD is shown in Fig. 7. The final result of the task execution and its measurement are shown in Fig. 8, and the line segment generated by the end-effector is about 5.45 cm. Compared with the desired

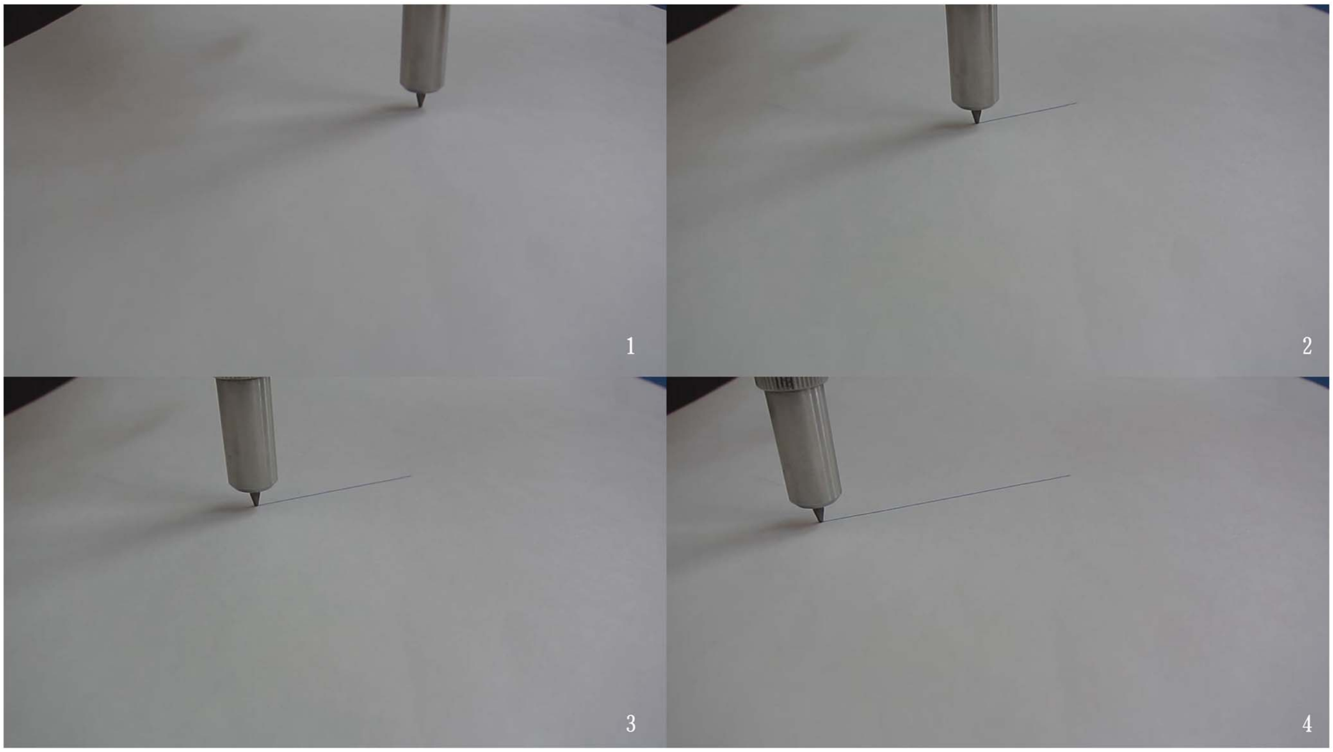


Fig. 7. Snapshots for the task execution of the robot manipulator when its end-effector tracks the desired line-segment path.

TABLE I
COMPARISON ON FORMULAS OF DAVIDENKO METHOD AND ZD FOR SOLVING NONLINEAR EQUATION SYSTEMS

Nonlinear Equation System	$\mathbf{f}(\mathbf{x}) = \mathbf{0}$	$\mathbf{f}(\mathbf{x},t) = \mathbf{0}$
Davidenko Method	$\dot{\mathbf{x}} = -J^{-1}(\mathbf{x})\mathbf{f}(\mathbf{x})$	$\dot{\mathbf{x}} = -J^{-1}(\mathbf{x},t)\mathbf{f}(\mathbf{x},t)$
Continuous-Time ZD	$\dot{\mathbf{x}} = -\gamma J^{-1}(\mathbf{x})\phi(\mathbf{f}(\mathbf{x}))$	$\dot{\mathbf{x}} = -J^{-1}(\mathbf{x},t)(\gamma\phi(\mathbf{f}(\mathbf{x},t)) + \partial\mathbf{f}(\mathbf{x},t)/\partial t)$
Discrete-Time ZD	$\mathbf{x}_{i+1} = \mathbf{x}_i - hJ^{-1}(\mathbf{x}_i)\phi(\mathbf{f}(\mathbf{x}_i))$	$\mathbf{x}_{i+1} = \mathbf{x}_i - J^{-1}(\mathbf{x}_i,t_i)(h\phi(\mathbf{f}(\mathbf{x}_i,t_i)) + \mathbf{f}(\mathbf{x}_i,t_i) - \mathbf{f}(\mathbf{x}_i,t_{i-1}))$

* Note: If the linear activation function is adopted, the discrete-time ZD model with $h = \tau$ for solving $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ is identical to the discrete-time model of Davidenko method for solving $\mathbf{f}(\mathbf{x}) = \mathbf{0}$.

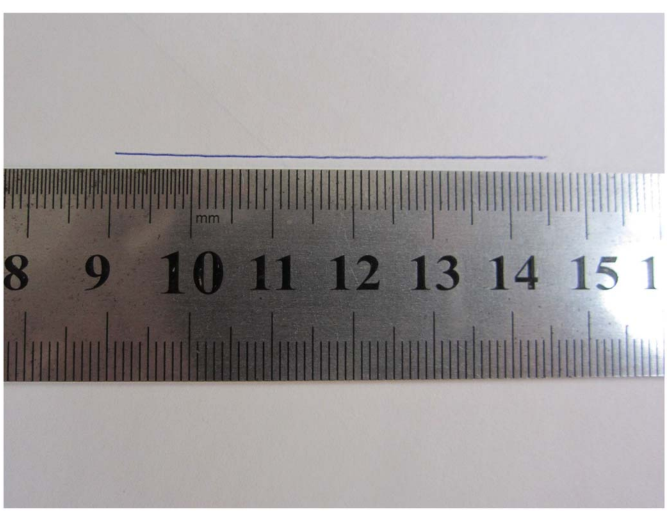


Fig. 8. Measurement at the end of the task execution.

length (i.e., about 5.37 cm), the error is only 0.08 cm (i.e., 8×10^{-4} m), which substantiates the physical realizability, accuracy, and online computation ability of ZD.

VII. CONCLUSION

In this paper, Davidenko method has first been compared with ZD in terms of solving time-invariant nonlinear equation systems. The findings have revealed that Davidenko method can be a special case of ZD, and the latter is superior to the former. A conjecture (i.e., time-varying Davidenko method) has been proposed to further compare the two methods for solving time-varying nonlinear equation systems. In addition to the theoretical analysis and theorems, the results of the corresponding numerical experiments have substantiated that ZD is more accurate than the time-varying Davidenko method and displays a significantly faster convergence rate. The application of ZD to the online kinematic control of the physical robot manipulator has further illustrated the physical realizability, accuracy, and online computation ability of ZD.

Table I is further given, which compares the formulas of Davidenko method and ZD for solving nonlinear equation systems. This paper may thus guide the software development or hardware implementations for solving nonlinear equation systems online. In summary, ZD is a powerful method for solving nonlinear equation systems, especially time-varying nonlinear equation systems. Time-varying problems show a good research prospect because most practical problems are essentially time-varying. Thus, ZD is expected to motivate emerging topics in system engineering and science because of its high efficiency in solving time-varying problems.

APPENDIX

In this part, three definitions [37] are presented as a basis for the proof of Theorems 5 and 6.

Definition 1: An N step method/model $\sum_{r=0}^{N}\alpha_r\chi_{\kappa+r} = \tau\sum_{r=0}^{N}\beta_r g_{\kappa+r}$ can be checked for 0-stability by determining the roots of the characteristic polynomial $P_N(q) = \sum_{r=0}^{N}\alpha_r q^r$. If the roots of $P_N(q)$ are such that $|q| \leq 1$ and any root on the unit circle (i.e., $|q| = 1$) is simple, then the N step method is 0-stable. In addition, the 0-stability is sometimes called Dahlquist stability or root stability.

Definition 2: An N step method is said to be consistent of order p if its truncation error is $O(\tau^p)$ with $p > 0$ for the smooth exact solution.

Definition 3: An N step method is convergent, i.e., $\chi_{[(t-t_0)/\tau]} \to \chi^*(t)$, for all $t \in [t_0, t_f]$, as $\tau \to 0$, if and only if the method is consistent and 0-stable, where t_0 denotes the first computational time instant and t_f denotes the final one. In other words, consistency plus 0-stability leads to convergence. In particular, a 0-stable and consistent method converges with the order of its truncation error.

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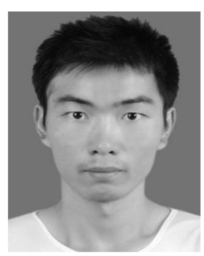

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