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THE MODELLING AND ANALYSIS OF FRACTIONAL-ORDER CONTROL SYSTEMS IN FREQUENCY DOMAIN

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Abstract

This paper deals with fractional-order controlled systems and fractional-order controllers in the frequency domain. The mathematical description by fractional transfer functions and properties of these systems are presented. The new ways for modelling of fractional-order systems are illustrated with a numerical example and obtained results are discussed in conclusion.

Key words: fractional-order controller, fractional-order system, fractional calculus, stability.

1. INTRODUCTION

Fractional calculus is a generalization of integration and derivation to non-integer order fundamental operator ${}_aD_t^\alpha$, where a and t are the limits of the operation. The two definitions used for the general fractional differintegral are the Grünwald definition and the Riemann-Liouville definition [4].

The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with Leibniz and L'Hospital in 1695. Fractional calculus was used for modelling of physical systems, but we can find only few works dealing with the application of this mathematical tool in control theory (e.g. [1, 6, 5, 8]).

The aim of this paper is to show, how by using the fractional calculus, we can obtain a more general structure for the classical PID controller, for controlled systems with memory and hereditary behaviour, and how to model fractional-order systems in the frequency domain. On the other hand, we can analyse the fractional-order systems and specify the conditions of stability in Bode's and Nyquist's frequency response.

2. FRACTIONAL-ORDER CONTROL CIRCUIT

We will be studying the control system shown in Fig.1, where $G_c(j\omega)$ is the controller transfer function and $G_s(j\omega)$ is the controlled system transfer function.

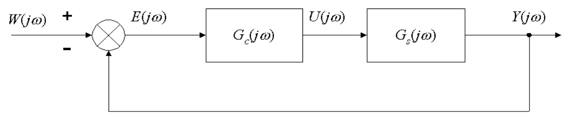

Figure 1: Feed - back control loop

2.1 Fractional-order controlled system

The fractional controlled system will be represented with fractional model with frequency transfer function given by the following expression:

$$G_s(j\omega) = \frac{Y(j\omega)}{U(j\omega)} = \frac{b_m(j\omega)^{\alpha_m} + \ldots + b_1(j\omega)^{\alpha_1} + b_0(j\omega)^{\alpha_0}}{a_n(j\omega)^{\beta_n} + \ldots + a_1(j\omega)^{\beta_1} + a_0(j\omega)^{\beta_0}} = \frac{\sum_{k=0}^{m} b_k(j\omega)^{\alpha_k}}{\sum_{k=0}^{n} a_k(j\omega)^{\beta_k}}, \tag{1}$$

where β_k, α_k $(k = 0, 1, 2, \ldots, n)$ are generally real numbers, $\beta_n > \ldots > \beta_1 > \beta_0$, $\alpha_m > \ldots > \alpha_1 > \alpha_0$ and a_k, b_k $(k = 0, 1, \ldots, n)$ are arbitrary constants.

Identification methods [3, 5, 9, 10] for determination of the coefficients a_k, b_k and α_k, β_k $(k = 0, 1, \ldots, n)$ were developed, based on minimisation of the difference between the calculated and experimentally measured values. The optimisation consists in minimising the square norm of the difference between the measured frequency response $F(\omega)$ and the frequency response of the model $G_s(j\omega)$. The quadratic criterion is given by the expression:

$$Q = \sum_{m=0}^{M} W^2(\omega_m)|F(\omega_m) - G_s(j\omega_m)|^2, \tag{2}$$

where $W(\omega_m)$ is a weighting function and M is the number of measured values of frequencies.

2.2 Fractional order controller

The fractional $PI^\lambda D^\delta$ controller will be represented by frequency transfer function given in the following expression:

$$G_c(j\omega) = \frac{U(j\omega)}{E(j\omega)} = K + \frac{T_i}{(j\omega)^\lambda} + T_d(j\omega)^\delta, \tag{3}$$

where λ and δ are arbitrary real numbers $(\lambda, \delta \geq 0)$, K is the proportional constant, T_i is the integration constant and T_d is the derivation constant.

Taking $\lambda = 1$ and $\delta = 1$, we obtain a classic PID controller. If $\lambda = 0$ and/or $T_i = 0$, we obtain a PD^δ controller, etc. All these types of controllers are particular cases of the $PI^\lambda D^\delta$ controller, which is more flexible and gives an opportunity to better adjust the dynamical properties of the fractional-order control system.

The $PI^\lambda D^\delta$ controller with complex zeros and poles located anywhere in the left-hand s-plane may be rewritten as

$$G_c(j\omega) = C\frac{((j\omega)/\omega_n)^{\delta+\lambda} + (2\xi(j\omega)^\lambda)/\omega_n + 1}{(j\omega)^\lambda}, \tag{4}$$

where C is a gain, ξ is the dimensionless damping ratio and ω_n is the natural frequency. Normally, we choose $0.9 > \xi > 0.7$. When $\xi = 1$, the condition is called critical damping [2].

The tuning of $PI^\lambda D^\delta$ controller parameters is determined according to the given requirements. These requirements are, for example, the stability measure, the accuracy of the regulation process, dynamical properties etc. One of the methods being developed is the method (modification of roots locus method) of dominant roots [6], based on the given stability measure and the damping measure of the control circuit. Another possible way to obtain the controller parameters is using the tuning formula, based on gain and phase margins specifications.

3. FREQUENCY CHARACTERISTICS

The graphical interpretation of the frequency transfer function for different values of the angular frequency ω in the range $\omega \in \langle 0, \infty \rangle$ is called the frequency characteristic.

The practical meaning of the frequency characteristics is in the determination of the stability of control circuits from the behavior of the frequency characteristic. Stability is an asymptotic qualitative criterion of the quality of the control circuit and is the primary and necessary condition for correct functioning of every control circuit. It is difficult to evaluate the stability of a fractional-order control circuit by examining its characteristic equation either by finding its dominant roots or by algebraic methods. The methods given are suitable for integer-order systems. In our case we use Bode's and Nyquist's frequency characteristic to investigate the stability [7].

For the frequency transfer of the open control circuit $G_o(j\omega)$ with respect to (1) and (3), we have

$$G_o(j\omega) = \frac{Y(j\omega)}{W(j\omega)} = \left(\frac{T_i + K(j\omega)^{\lambda} + T_d(j\omega)^{\delta+\lambda}}{(j\omega)^{\lambda}} \right) \frac{\sum_{k=0}^{m} b_k (j\omega)^{\alpha_k}}{\sum_{k=0}^{n} a_k (j\omega)^{\beta_k}}. \tag{5}$$

It is known from the investigation theory of stability of regulation circuits that the system is stable if the roots of the characteristic equation are negative or have negative real parts if they are complex conjugate. This means they are located left of the imaginary axis of the complex s-plane of the roots. In the case of frequency methods of evaluating the stability we transform the plane of the roots s into the complex plane $G_o(j\omega)$ while the transformation is realised according to the transfer function of the open circuit. During the transformation all the roots of the characteristic equation are mapped from the s-plane into the critical point $(-1, i0)$ in the plane $G_o(j\omega)$. The mapping of the s-plane into plane $G_o(j\omega)$ is conformal, that is, the directions and locations of points in the s-plane are preserved also in the plane $G_o(j\omega)$.

In the Nyquist frequency characteristic to find out the stability of the control circuit, it is necessary to investigate the behavior of the curve $G_o(j\omega)$ for $\omega \in \langle 0, \infty \rangle$ relative to the critical point $(-1; i0)$. Based on the above it follows that also the fractional-order control circuit is stable, if the frequency characteristic passes on the right side of the critical point, when going in the direction of increasing values ω. If the frequency characteristic passes on the left of the critical point, the circuit is non-stable. The passing of the frequency characteristic through the critical point means the circuit is on the border of stability.

4. ILLUSTRATIVE EXAMPLE

We give in this section an example of modelling the stable dynamical system by using fractional calculus in frequency domain. The fractional-order control system consists of the real controlled system with the frequency transfer function

$$G_s(j\omega) = \frac{1}{0.8(j\omega)^{2.2} + 0.5(j\omega)^{0.9} + 1} \tag{6}$$

and the fractional-order PD^{δ} controller, designed on the stability measure $S_t = 2.0$ and damping measure $\xi = 0.4$, with the frequency transfer function in the form:

$$G_c(j\omega) = 50.0 + 5.326(j\omega)^{1.286}. \tag{7}$$

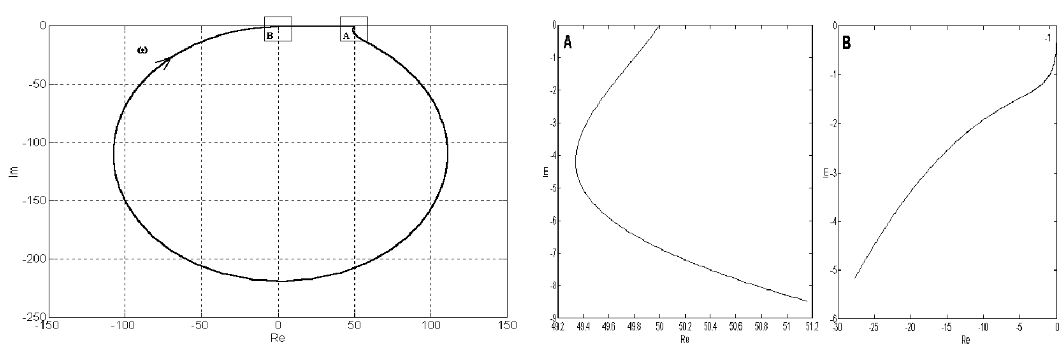

Figure 2: Nyquist plot (and details) for (6) and (7)

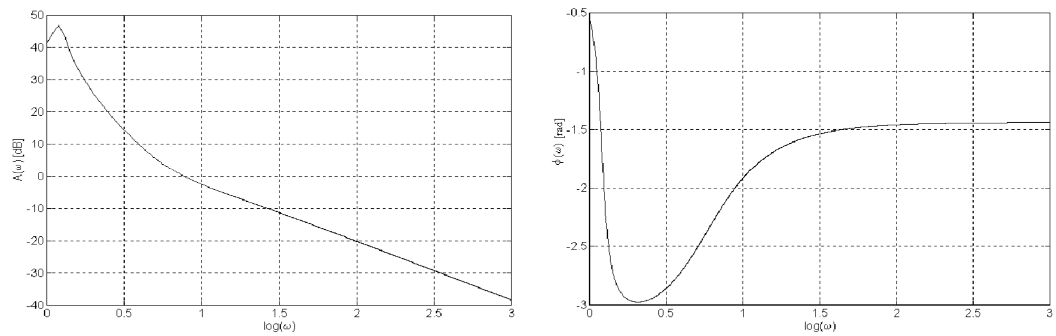

Figure 3: Bode plots for (6) and (7)

In Fig. 2 and Fig. 3 are showed the frequency characteristics of the fractional-order control system consists of the fractional-order controlled object (6) and the fractional-order controller (7).

5. CONCLUSION

The above methods make it possible to model and analyse fractional-order control systems in the frequency domain. The stability of a fractional-order control system can be investigated via the behavior of the frequency plot (see Fig. 2 and Fig. 3). These properties of the fractional-order systems can be used in the controller parameters design. The results of this and previous works also show that fractional-order controllers are robust. This can even lead to qualitatively different dynamical phenomena in control circuits.

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