# Modeling and identification of fractional first-order systems with Laguerre-Grünwald-Letnikov fractional-order differences

RafałStanisł awski, Krzysztof J. Latawiec, MarianŁukaniszyn,
Wojciech Czuczwara, and Ryszard Kopka
Department of Electrical, Control and Computer Engineering,
Opole University of Technology, ul. Prószkowska 76, 45-758 Opole, Poland
{r.stanislawski, k.latawiec, m.lukaniszyn, w.czuczwara, r.kopka}@po.opole.pl

***Abstract*—This paper introduces a method for modeling and identification of a simple dynamical system described by fractional-order differential equation. The Grünwald-Letnikov fractional-order derivative is approximated by a discrete-time Laguerre-based model, giving rise to a new discrete-time integer-order equation modeling the considered system. An application example involves a supercapacitor charging circuit. High accuracy of parametric identification for the circuit model, under moderate computational effort, is achieved on a real-life experimental data.**

## I. INTRODUCTION

Dynamical systems modeled by noninteger-order differential equations, to be called fractional-order or fractional systems, have been given considerable research effort both by the academia and practicians focused on modeling, identification and control [18]. One of the main problems in modeling and identification of fractional systems is the necessity of effective approximation, or modeling, of a noninteger-order derivative (or fractional derivative), with the effectiveness meaning both high approximation accuracy and possibly low computational burden. Up to date, two main approaches to the problems in modeling of a fractional derivative have been proposed, the first one relying on approximation of the operator 'equivalent' $s^\alpha$ of the derivative, possibly supported with frequency analysis of the element $(j\omega)^\alpha$ [10], [16], [17] and the second one involving a discrete-time representation of the fractional derivative that is fractional difference [1], [9], [11], [12], also possibly supplemented with frequency analysis of the model. In this paper, the second approach is advocated, mainly because of the fact that final implementation of a model of the fractional derivative is anyway performed in a digital way.

It has been shown that, under mild conditions, discretized fractional derivatives of both Riemann-Liouville and Grünwald-Letnikov eventually lead to the same fractional difference, called the Grunwald-Letnikov (GL) difference [8]. This paper is aimed at modeling and identification of simple fractional systems using an efficient model of the GL difference based on discrete-time Laguerre filters.

## II. CLASS OF FRACTIONAL-ORDER SYSTEMS

A general class of linear fractional-order systems can be described by the transfer function

$$G(s^\alpha) = \frac{b_m(s^\alpha)^m + b_{m-1}(s^\alpha)^{m-1} + \cdots + b_1 s^\alpha + b_0}{a_n(s^\alpha)^n + a_{n-1}(s^\alpha)^{n-1} + \cdots + a_1 s^\alpha + b_0} \quad (1)$$

where $\alpha \in (0, 2)$ is the fractional order. In this paper, we are interested in the simplest case of $m \le n = 1$, in which the transfer function is written as

$$G(s^\alpha) = \frac{b_1 s^\alpha + b_0}{a_1 s^\alpha + a_0} \quad (2)$$

We call the underlying system as the "fractional first-order system", meaning that e.g. the first-order inertia includes the noninteger-order derivative. In other words, for $n = 1$ we talk about the first-order system and it is fractional for $0 < \alpha < 2$. Well, the simultaneous use of the terms "fractional-order" and "first-order" might be confusing, but still some other counter-candidates to our "fractional first-order system" could be welcome here. Anyway, a discussion on terminology for fractional dynamical systems is desirable.

We proceed with another version of the model (2), introducing the gain $K$ which will support our identification accuracy considerations.

$$G(s^\alpha) = K\frac{\tau_1 s^\alpha + 1}{\tau_2 s^\alpha + 1} \quad (3)$$

Note that this model is often specialized to the Cole-Cole model [2], [3], [5] and used e.g. in diagnostics of insulation systems in oil transformers and in medical diagnostics.

Write a fractional differential equation of the system

$$\tau_2 D^\alpha y(t) + y(t) = K\left[\tau_1 D^\alpha u(t) + u(t)\right] \quad (4)$$

where $u(t)$ and $y(t)$ are the system input and output, respectively, and $D^\alpha u(t)$ and $D^\alpha y(t)$ denote their respective GL fractional-order derivatives. The fractional-order derivatives in Eqn. (4) can be approximated by the GL fractional-order differences

$$D^\alpha x(t) \cong \frac{\Delta^\alpha x(k+1)}{T^\alpha} \quad (5)$$

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where $T$ is the sampling period for a discrete-time implementation of the derivatives and $x(t)$ stands for either $u(t)$ or $y(t)$. The rationale for using the forward shift operator will be explained later on.

## III. Fractional-order differences

### A. GL fractional-order difference

The GL Fractional-order Difference (FD) used in this paper is of form [9]

$$\begin{aligned}\Delta^{\alpha}_{FD}x(k) &= \sum_{j=0}^{k} P_j(\alpha)x(k)q^{-j} \\ &= x(k) + \sum_{j=1}^{k} P_j(\alpha)x(k)q^{-j} \quad k=0,1,.. \end{aligned} \tag{6}$$

where $q^{-1}$ is the backward shift operator and

$$P_j(\alpha) = (-1)^j C_j(\alpha) \tag{7}$$

with

$$C_j(\alpha) = \binom{\alpha}{j} = \begin{cases} 1 & j=0 \\ \frac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} & j>0 \end{cases} \tag{8}$$

The fractional difference (6) can be alternatively given as

$$\begin{aligned}\Delta^{\alpha}_{FD}x(k) &= x(k) + \sum_{j=1}^{\infty} P_j(\alpha)x(k-j) \\ &= x(k) + X_{FD}(k) \quad k=0,1,... \end{aligned} \tag{9}$$

where $x(l)=0$ for all $l<0$.

Implementation of the difference FD is only feasible in case of finite, or 'truncated' summation in Eqn. (9), that is in form of Finite Fractional-order Difference (FFD), with the filter $X_{FFD}(k)$ substituted for the filter $X_{FD}(k)$ [12]

$$\begin{aligned}\Delta^{\alpha}_{FFD}x(k) &= x(k) + \sum_{j=1}^{J} P_j(\alpha)x(k-j) \\ &= x(k) + X_{FFD}(k) \quad k=0,1,... \end{aligned} \tag{10}$$

where $J=\min(k,\overline{J})$ and $\overline{J}$ is the upper bound for $j$.

### B. Laguerre-based modeling

*1) OBF modeling of integer-order systems:* A stable linear discrete-time system having infinite impulse response can be desribed as

$$G(z) = \sum_{j=1}^{\infty} g_j z^{-j} \tag{11}$$

where $g_j$, $j=1,2,...$, is the impulse response. The system (11) can be modeled with the Laurent expansion [4], [18]

$$G(z) = \sum_{j=1}^{\infty} c_j L_j(z^{-1}) \tag{12}$$

including a series $L_j(z^{-1})$ of Orthonormal Basis Functions (OBF) and weighting parameters $c_j$, $j=1,2,...$, describing the dynamics of the model. Various OBFs can be used in practice [4], [18]. However, the most popular are simple Laguerre and Kautz functions, with their pole(s) modeling the 'dominant' dynamics. In case of the Laguerre functions/filters to be used hereinafter

$$L_j(z^{-1}) = \frac{\underline{k}z^{-1}}{1-pz^{-1}} \left[\frac{-p+z^{-1}}{1-pz^{-1}}\right]^{j-1} \quad j=1,2,... \tag{13}$$

a single lowpass 1st-order filter ($j=1$) is cascaded with allpass filters ($j>1$), with $p<1$ being the single dominant pole and $\underline{k}=\sqrt{1-p^2}$.

*2) Laguerre-based fractional difference:* Consider the fractional difference FD defined as in Eqns. (6) or (9). The two expansions can be interpreted as 'a sort of' infinite impulse response, being a special case of OBF. Now, we can generalize the fractional difference to include an OBF-based (fractional) difference. In particular, we are interested in the simple Laguerre-based Difference (LD) defined as [11], [15], [16]

$$\begin{aligned}\Delta^{\alpha}_{LD}x(k) &= x(k) + \sum_{j=1}^{\infty} c_j L_j(q^{-1})x(k) \\ &= x(k) + X_{LD}(k) \quad k=0,1,... \end{aligned} \tag{14}$$

with $x(l)=0 \ \forall \ l<0$.

In Ref. [11], conditions have been derived for which the filters FD and LD are equivalent, that is $X_{FD}(k)=X_{LD}(k)$. Even though the two fractional differences can be equivalent, they comprise an infinite number of elements, which may ultimately lead to computational explosion. Therefore, finite implementations of those differences are applicable, to mention the FFD for the former one. Finite implementation of the LD, that is Finite Laguerre-based Difference (FLD) can be more effective than FFD [15], [16]. However, it is not until the powerful combination of FFD and FLD has been offered that both the modeling accuracy and computational effectiveness have been achieved [11], [15], [16]. This combination, called Finite (combined) Fractional/Laguerre-based Difference (FFLD), has been found superior to many other discrete-time models of the GL fractional derivative [16].

The efficient FFLD approximator to FD/LD will be used hereafter

$$\begin{aligned}\Delta^{\alpha}_{FFLD}x(k) &= x(k) + \sum_{i=1}^{J} P_i(\alpha)x(k)q^{-i} \\ &\quad + \sum_{j=1}^{M} c_j L_j(q^{-1})q^{-\overline{J}}x(k) \\ &= x(k) + X_{FFLD}(k) \quad k=0,1,... \end{aligned} \tag{15}$$

where $M$ is a number of Laguerre filters used in the model.

The finite number of the Laguerre filters used in FFLD results in modeling errors. A detailed analysis of those errors has been given in Refs. [11], [15], with the culmination in Ref. [16] showing the real power of FFLD.

*Remark 1:* Note that the pole $p=p(\alpha)$ and parameters $c_j=c_j(\alpha)$, $j=1,...,M$, can be calculated in an analytical way [11], [15].

## IV. Application: Simple fractional system

Fig. 1 shows a diagram of a fractional circuit to charge the supercapacitor $C_\alpha$ through the resistor $R$. The internal resistance of the supercapacitor is $R_s$.

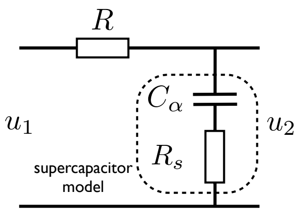


Fig. 1. Supercapacitor charging cicrcuit.

A fractional differential equation of the supercapacitor charging circuit is as in Eqn. (4), with $u(t) = u_1(t)$, $y(t) = u_2(t)$, $\tau_1 = R_s C_\alpha$, $\tau_2 = (R + R_s)C_\alpha$ and $K = 1$; compare [2], [3]. In this equation, we still admit the gain $K$ different from unity as the gain estimate $\hat{K}$ being as close to 1 as possible will be an additional indicator of the accuracy of the modeling/identification method. Now, we consider the following fractional differential equation of the circuit

$$\tau_2 D^\alpha u_2(t) + u_2(t) = K\left[\tau_1 D^\alpha u_1(t) + u_1(t)\right] \qquad (16)$$

The fractional derivatives in Eqn. (16) could be approximated as in Eqn. (5), using the FFLD discrete-time implementation. However, approximating the two derivatives would lead to computational burden in modeling and parameter estimation for the simple system. To essentially simplify the computations let us get back to the transfer function (3) which can be easily manipulated to give

$$G(s^\alpha) = K\left[c + \frac{1-c}{\tau_2 s^\alpha + 1}\right] = G_1(s^\alpha) + G_2(s^\alpha) \qquad (17)$$

where $c = \tau_1/\tau_2$, with $K$ still left to be estimated in order to verify whether its estimate $\hat{K}$ is close to unity.

Now, the transform of the output voltage is $U_2(s^\alpha) = BU_1(s) + G_2(s^\alpha)U_1(s) = U_{21}(s^\alpha) + U_{22}(s^\alpha)$, where $B = Kc$, with only a single derivative being necessary to be approximated as in Eqn. (5) for the second component $u_{22}(t)$ of the output voltage. The following fractional differential equation for that output component is now considered instead of Eqn. (16)

$$\tau_2 D^\alpha u_{22}(t) + u_{22}(t) = K(1-c)u_1(t) \qquad (18)$$

Using the FFLD approximator for the signal component $u_{22}(t)$ we have

$$\Delta^\alpha u_{22}(k+1) = u_{22}(k+1) + X_{FFLD}(k+1) \qquad (19)$$

$$X_{FFLD}(k+1) = \sum_{i=1}^{J} P_i(\alpha)x(k-i+1) + \sum_{j=1}^{M} c_j L_j(q^{-1})x(k-\overline{J}+1) \qquad (20)$$

with $x(k) = u_{22}(k)$. Equations (5) and (18) to (20) lead to the following discrete-time model of the fractional differential equation (18)

$$u_{22}(k+1) + X_{FFLD}(k+1) = -au_{22}(k) + bu_1(k) \qquad (21)$$

where $a = a(\alpha) = T^\alpha/\tau_2$ and $b = aK(1-c)$.

After some manipulations, including lowering the time indices by one at both sides of Eqn. (21), we obtain

$$u_{22}(k) = -Au_{22}(k-1) + bu_{22}(k-1) - \sum_{i=2}^{J} P_i(\alpha)u_{22}(k-i) - \sum_{j=1}^{M} c_j L_j(q^{-1})u_{22}(k-\overline{J}) \qquad (22)$$

where $A = a + P_1(\alpha)$.

A final equation for $u_2(k)$ can be written as

$$u_2(k) = Bu_1(k) + u_{22}(k) \qquad (23)$$

Alternatively, the model equation (23) can be rewritten as

$$U_2(k) = -Au_{22}(k-1) + Bu_1(k) + bu_1(k-1) \qquad (24)$$

where

$$U_2(k) = u_2(k) + \sum_{i=2}^{J} P_i(\alpha)u_{22}(k\text{-}i) + \sum_{j=1}^{M} c_j L_j(q^{-1})u_{22}(k\text{-}\overline{J}) \qquad (25)$$

*Remark 2:* It is essential that the approximation (5) employs the forward (rather than backward) difference in a way used in fractional-order state space models [6], [7], [12], [13], [14]. This results in the occurrence of the unity delay in the model (22), which corresponds to the connection of the non-delay continuous-time system with the zero-order hold. Possible lack of the (unity) delay in case of using the backward difference would contrast with the physical interpretation of a discretized non-delay continuous-time system, in particular subject to its control.

**Case 1:** fractional order $\alpha$ and resistance $R_s$ are known.

The model equation (24) can be rewritten in the familiar, linear regression form

$$U_2(k) = \varphi^T(k)\Theta \qquad (26)$$

where $\varphi^T(k) = [-u_{22}(k-1),\ u_1(k),\ u_1(k-1)]$, $\Theta^T = [A,\ B,\ b]$. The unknown parameter vector $\Theta$ can be analytically estimated by means of the familiar least squares (LS) method to obtain the estimators $\hat{A}$, $\hat{B}$, $\hat{b}$ and finally, taking into account that $a = T^\alpha/\tau_2$, $c = \tau_1/\tau_2$, $b = aK(1-c)$ and $B = Kc$, the estimators for the unknown circuit parameters $\hat{\tau}_1$, $\hat{\tau}_2$, $\hat{K}$ as follows

$$\hat{\tau}_2 = \frac{T^\alpha}{\hat{a}}, \quad \hat{K} = \frac{\hat{B} + \hat{b}}{\hat{a}}, \quad \hat{\tau}_1 = \frac{\hat{B}}{\hat{K}}\hat{\tau}_2 \qquad (27)$$

where $\hat{a} = \hat{A} - P_1(\alpha)$.

*Remark 3:* In our circuit case, the gain $K$ should be close to unity and could not be estimated. However, we purposefully estimate the gain, just to verify if $\hat{K}$ is close to unity, this being an additional indicator of the accuracy of the estimation method, in addition to the MSE.

**Case 2:** order $\alpha$ and resistance $R_s$ are unknown.

Estimation of the unknown parameters $A$, $B$, $b$ and $\alpha$ is performed in a numerical way by minimization of e.g. the LS loss function

$$\min_{A,B,b,\alpha} \sum_{i=1}^{N} \left[u_2(k) - \hat{u}_2(k)\right]^2 \tag{28}$$

where $u_2(k)$ is the measured circuit output and $\hat{u}_2(k)$ is the estimated output of the circuit model (23), comprising the parameter estimators $\hat{A}$, $\hat{B}$, $\hat{b}$ and $\hat{\alpha}$. Parameter estimators for $\tau_1$, $\tau_2$ and $K$ are as in Eqns. (27), with $\hat{\alpha}$ substituted for $\alpha$. Since the Matlab's `fminsearch` minimization algorithm may be stuck in local minima, we effectively use the genetic algorithm (GA).

## V. Experimental verification

The usefullness of the introduced modeling approach for fractional first-order systems has been confirmed in a number of simulations. Here, we verify the method on real-life data measured from the BUC 0.47F supercapacitor charging circuit (supplied by the step input 3.5V through the buffer circuit), with $R = 200\Omega$ and $R_s = 6.5\Omega$. Measurements of the step response $u_2(k)$ at $T = 0.1s$ are plotted in Fig. 2 together with the output estimate $\hat{u}_2(k)$ for $\overline{J} = 15$, $M = 25$ and $N = 7000$. The two summation bounds in FFLD are relatively low, thus making the computation load quite low. The unknown circuit parameters are estimated to be $\hat{\alpha} = 0.725$, $\hat{\tau}_1 = 0.3527$, $\hat{\tau}_2 = 1.205$ and $\hat{K} = 1.049$. The gain estimate is close to unity, which supports our claim on high accuracy of the method, in addition to the low MSE equal to $5.8e^{-4}$.

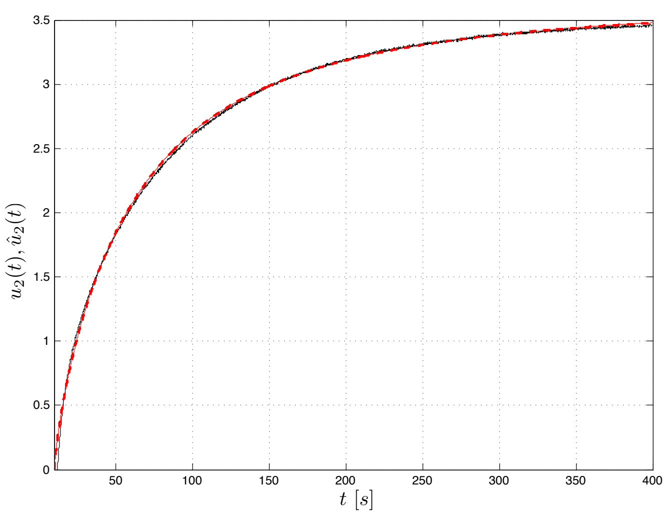


Fig. 2. Measured (black) vs. estimated (red) output voltages.

## VI. Conclusion

The paper has introduced an efficient method for modeling and identification of a simple fractional-order system, called here a fractional first-order system. An effective modeling tool is the discrete-time Laguerre-based approximator of the fractional derivative, developed by the authors and called the FFLD. The high accuracy of modeling and identification has been confirmed on a real-life data collected from a supercapacitor charging circuit. Future research is directed to extension of the method to more complex fractional systems.

## References


[1] Y. Q. Chen and K. L. Vinagre, "A new IIR-type digital fractional order differentiator," *Signal Processing*, vol. 83, no. 11, pp. 2359–2365, 2003.

[2] T. J. Freeborn, B. Maundy, and A. S. Elwakil, "Measurement of supercapacitor fractional-order model parameters from voltage excited step response," *IEEE J. Emerging Sel. Topics on Circuits Systems*, vol. 3, no. 3, pp. 367–376, 2013.

[3] ——, "Fractional-order models of supercapacitors, batteries and fuel cells: a survey," *Materials for Renewable and Sustainable Energy*, vol. 4, no. 9, 2015.

[4] P. S. C. Heuberger, P. M. J. Van den Hof, and B. Wahlberg, *Modeling and identification with rational orthogonal basis functions*. Springer-Verlag, London, UK, 2005.

[5] Y. P. Kalmykov, W. T. Coffey, D. S. F. Crothers, and S. V. Titov, "Microscopic models for dielectric relaxation in disordered systems," *Physical Review E*, vol. 70, no. 041103, 2004.

[6] K. J. Latawiec, R. Stanisławski, M. Gałek, and M.Ł ukaniszyn, "Modeling and identification of a fractional-order discrete-time SISO Laguerre-Wiener system," in *Proceedings of the 19th International Conference on Methods and Models in Automation and Robotics, Miedzyzdroje, Poland*, 2014, pp. 165–168.

[7] C. Monje, Y. Chen, B. Vinagre, D. Xue, and V. Feliu, *Fractional-order Systems and Controls: Fundamentals and Applications*. Series on Advances in Industrial Control, Springer, London, UK, 2010.

[8] P. Ostalczyk, "Equivalent descriptions of a discrete-time fractional-order linear system and its stability domains," *International Journal of Applied Mathematics and Computer Science*, vol. 22, no. 3, pp. 533–538, 2012.

[9] I. Petráš and B. Vinagre, "Practical application of digital fractional-order controller to temperature control," *Acta Montanistica Slovaca*, vol. 7, no. 2, pp. 131–137, 2002.

[10] D. Sierociuk and A. Dzieliński, "New method of fractional order integrator analog modeling for orders 0.5 and 0.25," in *Proceedings of the 16th International Conference on Methods and Models in Automation and Robotics, Miedzyzdroje, Poland*, 2011, pp. 137–141.

[11] R. Stanisławski, "New Laguerre filter approximators to the Grünwald-Letnikov fractional difference," *Mathematical Problems in Engineering*, vol. 2012, pp. 1–21, 2012, Article ID: 732917.

[12] R. Stanisławski and K. J. Latawiec, "Normalized finite fractional differences – the computational and accuracy breakthroughs," *International Journal of Applied Mathematics and Computer Science*, vol. 22, no. 4, pp. 907–919, 2012.

[13] ——, "Stability analysis for discrete-time fractional-order LTI state-space systems. Part I: New necessary and sufficient conditions for asymptotic stability," *Bulletin of the Polish Academy of Sciences, Technical Sciences*, vol. 61, no. 2, pp. 353–361, 2013.

[14] ——, "Stability analysis for discrete-time fractional-order LTI state-space systems. Part II: New stability criterion for FD-based systems," *Bulletin of the Polish Academy of Sciences, Technical Sciences*, vol. 61, no. 2, pp. 362–370, 2013.

[15] R. Stanisławski, K. J. Latawiec, and M.Ł ukaniszyn, "The steady-state error issue in Laguerre-based modeling of the GL fractional difference." in *Proceedings of the IEEE International Symposium on Industrial Electronics (ISIE), Taipei, Taiwan*, 2013, pp. 1–6.

[16] ——, "A comparative analysis of Laguerre-based approximators to the Grünwald-Letnikov fractional-order difference," *Mathematical Problems in Engineering*, vol. 2015, pp. 1–10, 2015, Article ID: 512104.

[17] C. Tseng, S. Pei, and S. Hsia, "Computation of fractional derivatives using Fourier transform and digital FIR diferentiator," *Signal Processing*, vol. 80, no. 1, pp. 151–159, 2000.

[18] P. Van den Hof, P. Heuberger, and J. Bokor, "System identification with generalized orthonormal basis functions," *Automatica*, vol. 31, no. 12, pp. 1821–1834, 1995.