Numerical Test for Stability Evaluation of Discrete-Time Circuits and Systems

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***Abstract*—In this paper, a new numerical test for stability evaluation of discrete-time circuits and systems is presented. It is based on modern root-finding techniques at the complex plane employing the Delaunay triangulation and the Cauchy's Argument Principle. The method evaluates if a system is stable and returns possible values and multiplicities of unstable zeros of the characteristic equation. For state-space discrete-time models, the developed test evaluates complex function related to the characteristic equation on the complex plane, so it does not require computation of state-matrix eigenvalues. The developed method is general because it allows to analyze systems whose the characteristic equation is not only a polynomial. Therefore, the verification of the proposed method is presented in benchmarks for both integer- and fractional-order systems.**

***Index Terms*—Discrete-time systems, Circuit stability, Stability analysis, Digital signal processing, Control engineering.**

I. Introduction

STABILITY evaluation represents a fundamental problem in circuit and system engineering, control engineering and signal processing. It is well known that a discrete-time linear time-invariant (LTI) system is (asymptotically) stable if and only if all zeros of the characteristic equation are within a unit circle at the complex z-plane [1], [2]. This condition is independent of the system modeling, i.e., zeros can be evaluated either for (i) the denominator of the transfer function or (ii) the characteristic equation of the state-space model. In both cases, determination of zeros requires searching for roots of a polynomial. A direct searching for roots of a polynomial employs computation of eigenvalues of the companion matrix, which requires matrix computations. Therefore, numerous tests were developed which evaluate stability of discrete-time systems based on polynomial coefficients only, e.g., refer to Schur-Cohn and Jury tests [3]. However, such tests return only information if a considered system is stable and do not return values of unstable poles when the system is unstable. Hence, those approaches do not allow to evaluate how far from the stability margin the considered system is. Furthermore, standard approaches (i.e., Schur-Cohn and Jury tests) do not allow to execute optimization/compensation/tuning procedures leading to shifting of unstable zeros into the unit circle at the z-plane.

Therefore, we have developed a numerical test for stability evaluation based on modern root-finding techniques at the complex plane employing the Delaunay triangulation and the Cauchy's Argument Principle [4], [5]. It allows an exploration of various systems by analyzing characteristic equation ($f(z) = 0$), also including those containing singular points and branch cuts. In the proposed method, the area outside the unit circle ($|z| = 1$) is transformed into its interior ($z = w^{-1}$, $F(w) = f(z^{-1}) = 0$) and singular points are removed if possible. Then, the obtained function is sampled within the unit circle using the Delaunay triangulation and sign changes are detected for its real and imaginary parts. If real and imaginary parts of the function change the sign for sample points belonging to the same triangle, then it may contain a zero of the $F(w)$ function. The existence of the function zero and its multiplicity is finally verified with the use of the Cauchy's Argument Principle.

The developed numerical test is general because it allows to analyze systems whose the characteristic equation is not only a polynomial. Hence, it is valid for discrete-time systems of integer order as well as for fractional-order systems. It returns values of unstable zeros of the characteristic equation with their multiplicity if the system is unstable. It allows to evaluate how far from the stability margin the considered system is. Hence, the developed stability test can be applied to the system optimization/compensation/tuning for its stabilization. Since the developed test evaluates complex function related to the characteristic equation on the complex plane, it does not require computation of eigenvalues.

II. Stability of Discrete-Time Circuits and Systems

In this section, we present fundamental definitions and stability conditions for discrete-time systems that are considered in the remainder of this paper.

A. Integer Order Systems

Let us consider discrete-time LTI system of integer order defined by the following state-space equations:

$$\begin{aligned} \mathbf{x}(n+1) &= \mathbf{A}\mathbf{x}(n) + \mathbf{B}\mathbf{u}(n) \\ \mathbf{y}(n) &= \mathbf{C}\mathbf{x}(n) + \mathbf{D}\mathbf{u}(n) \end{aligned} \tag{1}$$

where $\mathbf{x}$ is the state vector ($\mathbf{x} \in \mathbb{R}^L$), $\mathbf{y}$ is the output vector ($\mathbf{y} \in \mathbb{R}^Q$), $\mathbf{u}$ is the input vector ($\mathbf{u} \in \mathbb{R}^P$), $\mathbf{A}$ is the state matrix, $\mathbf{B}$ is the input matrix, $\mathbf{C}$ is the output matrix and $\mathbf{D}$ is the direct transmission matrix [1]. The characteristic equation of the system (1) is given by

$$f(z) = \det(z\mathbf{I} - \mathbf{A}) = 0 \tag{2}$$

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where $\mathbf{I}$ denotes the identity matrix and z denotes the $\mathcal{Z}$-transform variable. For LTI systems, the function $f(z)$ is a polynomial of the z variable

$$f(z) = \prod_{i=1}^{L}(z - p_i). \tag{3}$$

For systems defined by the transfer function in the z-domain, the $f(z)$ function represents its denominator.

B. Fractional Order Systems

For discrete-time fractional-order LTI system, the state-space equations take the following form [6], [7]:

$$\begin{aligned} \Delta^{\alpha}\mathbf{x}(n+1) &= \mathbf{A}_f\mathbf{x}(n) + \mathbf{B}\mathbf{u}(n) \\ \mathbf{y}(n) &= \mathbf{C}\mathbf{x}(n) + \mathbf{D}\mathbf{u}(n) \end{aligned} \tag{4}$$

where $\mathbf{A}_f = \mathbf{A} - \mathbf{I}$ and $\alpha \in (0, 2)$ is the fractional order. The fractional difference is defined by the following equations:

$$\Delta^{\alpha}x(n) = \sum_{j=0}^{n}(-1)^j\binom{\alpha}{j}q^{-j}x(n) \tag{5}$$

where q^{-1} is the backward shift operator. For such a system, the characteristic equation takes the form

$$f(z) = \det[z(1 - z^{-1})^{\alpha}\mathbf{I} - \mathbf{A}_f] = 0. \tag{6}$$

In both cases (2), (6), the system is stable if and only if all zeros of $f(z)$ are inside the unit circle at the complex z-plane [1], [2]. It means that the system is unstable if there exists at least a single zero of $f(z)$ outside the unit circle. To consider the region outside the unit circle, the transformation of variables is applied

$$z = w^{-1}. \tag{7}$$

To avoid numerical issues resulting from the singularity in zero, the factorization of the characteristic equation is applied if possible. Then, the following equation is considered within the unit circle:

$$F(w) = w^L f(w^{-1}) = 0. \tag{8}$$

III. Numerical Test for Stability Evaluation

Let us consider the system which is stable if and only if (8) does not have any zeros inside the unit circle. The goal of the algorithm is to verify if any such a zero exists and locate it by employing the Delaunay triangulation of the unit circle at the complex plane [4], [5]. The accuracy, for which the zeros are located, is set by the parameter Δr.

The algorithm can be summarized as follows:

1) Initialize the algorithm with a set of initial vertices, $V = \{v_1, v_2, ..., v_n\}$, that forms a convex hull enveloping the unit circle in the complex plane.
2) Apply the Delaunay triangulation to the set V. The result is a triangular mesh that covers the unit circle and a set $T = \{t_1, t_2, ..., t_n\}$, where each element of T consists of three vertices from V that forms a triangle.
3) For each triangle t in T and each of its edges, if the length of the considered edge is greater than or equal Δr, calculate the complex value of the function (8) at its ends. If either the real or imaginary part has a different sign at the ends of the edge, then add a new vertice v to the set V that is placed at the midpoint of this edge.
4) If at least one vertice has been added in the previous step, then proceed to step 2.
5) For every triangle t in T, evaluate if any of the triangle's edge length is lower than or equal Δr. For such triangles, apply the Cauchy's Argument Principle by numerically integrating the phase change of function (8) along the triangle edges.
6) If the integral is equal to $2\pi q$, where q is any positive integer, then there is a zero located within the triangle with multiplicity equal to q. In such a case, the system is unstable. The approximated location of the zero is in the geometric center of the triangle.

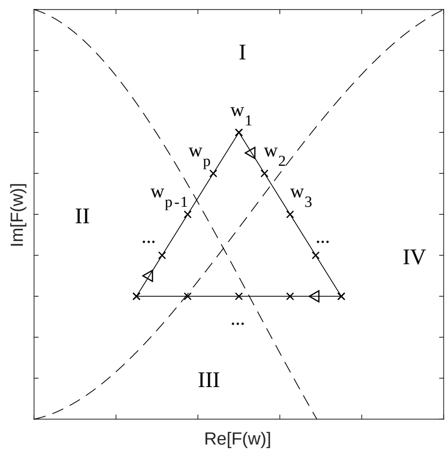


Fig. 1: Contour integral over the contour C that is a circumference of a triangle t in the complex w-plane. Dashed lines represent either $Re[F(w)] = 0$ and $Im[F(w)] = 0$. I-IV represent the quadrants of $F(w)$. Zero is located inside the triangle area at the point where the dashed lines cross.

If any zero of the characteristic equation (8) is located inside the unit circle, then it must be located inside one of the triangles. If the zero is located within a given triangle, then it is necessary that the function values in its close vicinity will fall into different quadrants of the complex-valued plane, refer to Fig. 1. The sign of the real and imaginary parts of the complex function $F(w)$ is used to determine the quadrant it belongs to:

I - $Re[F(w)] > 0, Im[F(w)] > 0$
II - $Re[F(w)] < 0, Im[F(w)] > 0$
III - $Re[F(w)] < 0, Im[F(w)] < 0$
IV - $Re[F(w)] > 0, Im[F(w)] < 0$.

The algorithm utilizes this fact and evaluates if any edge crosses quadrants. If two vertices of the same edge differ in

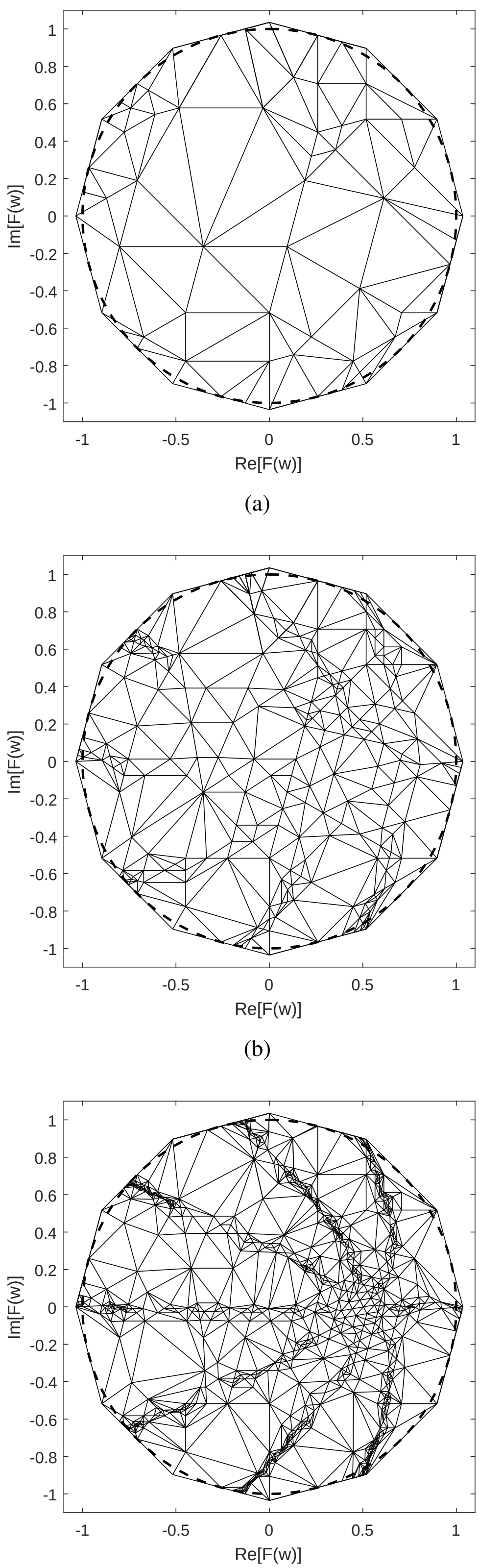


Fig. 2: Algorithm convergence for the system with characteristic equation's zeros: $(0.5, 0.5, 1.25 - 0.5j, 1.25 + 0.5j)$, precision $\Delta r = 0.001$. Dashed line represents the unit circle. Plots (a)-(c) for 3, 5 and 7 iterations. After 17 iterations, the double zero is located at $0.4997 + 0.0009j$.

sign for either the real or imaginary part, a new point is added to the set V in the midpoint of the edge. As the triangular mesh spans the entire region of the unit circle, new vertices will be added near the location of a zero.

After evaluating edges, if any new vertice has been added to V, then the Delaunay triangulation is executed again and a new set of triangles T is computed. With each iteration, the algorithm converges to the solution by decreasing the mesh size near zeros. This is repeated until the required accuracy Δr of locating zeros is achieved.

In the next step, the existence of zeros is verified in the candidate triangles. The Delaunay triangulation ensures that triangles do not overlap and therefore each zero (or multiple zeros) must be located inside a single triangle. Only the triangles with edge length lower than Δr are considered as these are obtained in the last step of the convergent algorithm. To evaluate if a zero of the characteristic equation (8) is located inside a triangle, the Cauchy's Argument Principle is used. This requires the calculation of a contour integral

$$q = \frac{1}{2\pi j} \oint_C \frac{F'(w)}{F(w)} \, dw \tag{9}$$

where C is the circumference of the triangle.

As stated before, each triangle forms a closed contour over the region where zero may be located. If (8) only has zeros, e.g., because all singularities are removed, the value of q can be either:

- zero - for regular point which is not a function zero,
- positive integer - for a function zero of order q.

By using the transformation (7) of the unstable region of the complex z-plane into the unit circle on w-plane, the system is unstable for any positive q. Note that the accuracy of locating zeros is dictated by the parameter Δr. Hence, if more then a single zero is located within a triangle, then algorithm treats it as a multiple zero.

A. Algorithm Convergence

To locate zeros of the characteristic equation, a perfect mesh of an arbitrarily-small size could be used that uniformly covers the area of the complex plane. The accuracy parameter Δr controls the resolution of the algorithm. The higher the accuracy required, the greater the computational overhead and it is impractical to perform calculations for all polygons that constitute such a perfect mesh. Hence, to address this issue, the algorithm uses an initial mesh with limited number of vertices. The mesh is constructed using the Delaunay triangulation. The triangular mesh is evaluated and, as additional vertices are added, the solution converges to zeros. In this case, the convergence is exhibited by fine meshing and decreasing triangle edge lengths around zeros. The behavior of the algorithm is to some extent similar to the gradient algorithm for location of a convex function extreme value.

Fig. 2 demonstrates the algorithm convergence. The characteristic equation of the system has four zeros: $[0.5, 0.5, 1.25 - 0.5j, 1.25 + 0.5j]$. The precision Δr of the algorithm is set to 0.001 and the total number of iterations of the Delaunay triangulation is 17.

B. Cauchy's Argument Principle

In order to verify if a zero is present inside a triangle, the Cauchy's Argument Principle is used and the integral (9) is computed. This integral can also be represented as

$$\frac{1}{2\pi j}\oint_C \frac{F'(w)}{F(w)}\,dw = \frac{1}{2\pi}\sum_{p=1}^{P} Arg\left(\frac{F(w_{p+1})}{F(w_p)}\right) \tag{10}$$

if C is divided into P sections $(w_p,\, w_{p+1})$ with $(w_{p+1} = w_1)$ [4], [5]. Hence, to compute the above integral, it is sufficient to numerically integrate the phase change of the function $F(w)$ along the C contour (Fig. 1).

C. Parallelization

The proposed algorithm is fully parallelizable on modern computing architectures since all function evaluations are completely independent tasks.

IV. Numerical Results

The code is developed in Matlab [8] environment which returns information if the system is stable with (i) value of the first-found unstable zero of (8) and its multiplicity or (ii) values of all zeros of (8) in the unit circle with their multiplicities. The first mode of operation (i) allows to reduce time of computations in comparison to (ii) when the system has many unstable zeros.

A. Integer Order Systems

In our investigations, the following benchmarking tests are considered.

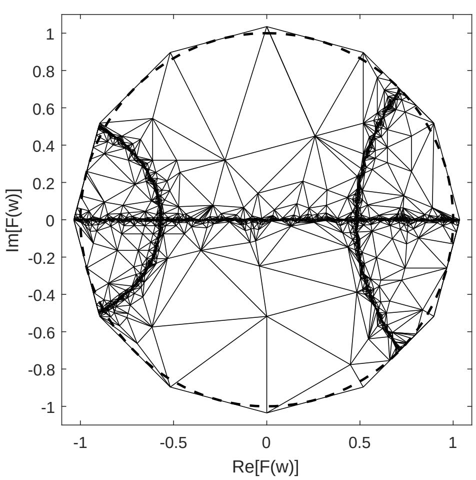
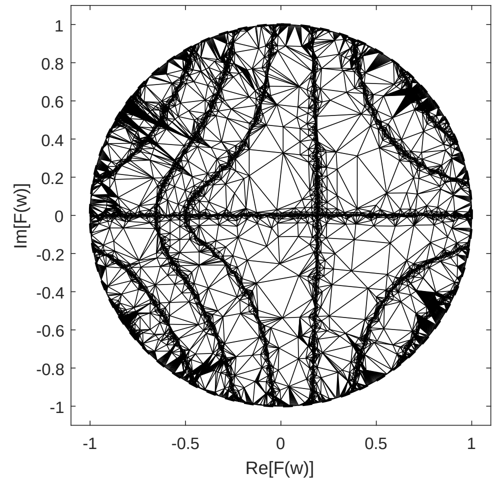
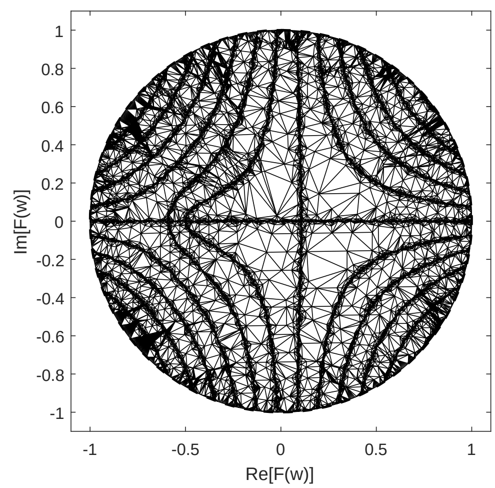


Fig. 3: Convergence of the algorithm for a system described by the characteristic equation (12). Dashed line represents the unit circle. Precision $\Delta r = 0.001$, number of iterations equal to 20, zero located at $-0.5711 - 0.0002j$.



(a)



(b)

Fig. 4: Algorithm convergence ($\Delta r = 0.001$) for the system (13) with unstable zero at $w_0 = -0.5$. (a) $L = 50$ and (b) $L = 100$.

1) Stability evaluation for low-order system: Let us consider the system with the transfer function given by [9].

$$T(z) = \frac{N(z)}{D(z)} = \frac{2z-1}{z^4 + 1.1z^3 - 0.8z^2 + 0.1z - 0.9}. \tag{11}$$

For the system to be stable, all zeros of the characteristic equation must be located inside the unit circle. As one of $D(z)$ zeros is at -1.7549, this system is unstable. By applying (7) to the denominator of (11), one obtains

$$\begin{aligned} D(w^{-1}) &= w^{-4}(-0.9w^4 + 0.1w^3 - 0.8w^2 + 1.1w + 1) \\ &= w^{-4}F(w). \end{aligned} \tag{12}$$

Since the outer region of the complex plane is mapped into the unit circle, the stability condition is that no zero of (12) should be inside the unit circle. Fig. 3 shows the convergence of the proposed algorithm for (12). One of the zeros of $F(w)$

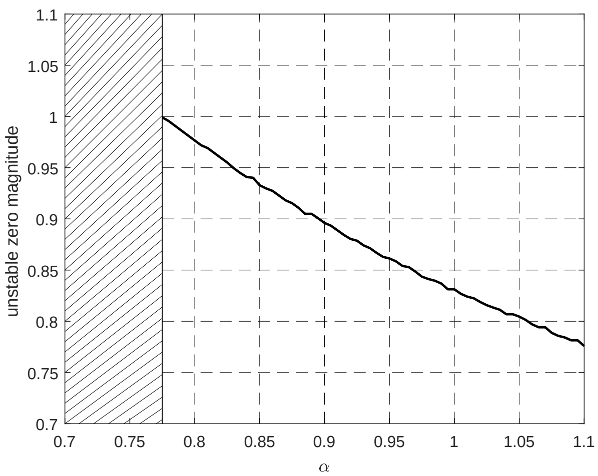


Fig. 5: Tuning unstable-zero location by α parameter for system (14). Bold line is the distance of the unstable zero to origin of complex w-plane, dashed area is the system-stability range of α. For transition point of $F(w)$ zero into the stability range: $w_0 = 0.5944 - 0.8030j$ and $\alpha = 0.7749$.

is at $-0.5711 - 0.0002j$ and therefore the system is unstable. In order to validate the result, Matlab $roots$ function is used with (12) and finds the unstable zero at -0.5698, which is consistent with our result.

2) Stability evaluation for high-order system: The proposed numerical technique is well suited for cases when many zeros of the characteristic equation are stable and a few are unstable. Let us consider the system whose characteristic equation is given by

$$f(z) = (z-2)\prod_{i=1}^{L}[z - (-1)^i 0.4(1 + \frac{i}{L})]. \tag{13}$$

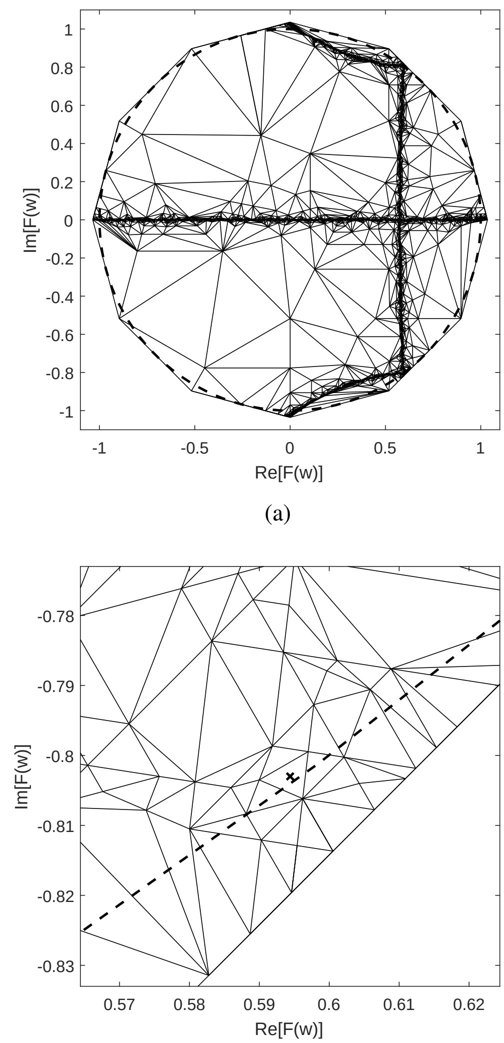
A single zero of this characteristic equation is located in the unstable region of the complex z-plane, i.e. $z = 2$. Our algorithm first, applies the transformation (7) and then proceeds to locate the zero inside the unit circle. With this approach, only the region of the unit circle is analyzed.

Tab. I lists the results and the computation times of the algorithm for varying number of stable zeros. Clearly, $roots$ function in Matlab is much faster than the developed method for all tested L values. However, the proposed method is able to evaluate stability for characteristic equations with $F(w)$ containing singularities and branch cuts [4], [5].

Additionally with the higher order of the characteristic equation, it may become necessary to increase the accuracy of the algorithm Δr. The algorithm converges by analyzing the sign changes of the real and imaginary parts of the function. With increased number of zeros, the number of quadrant transitions of the function value inside the unit circle may also be increased, e.g., refer to Fig. 4.

B. Fractional Order Systems

1) Benchmark 1: Let us consider fractional-order system with [6]



(a)

(b)

Fig. 6: Algorithm convergence for (14) with system near stability range of α parameter ($\alpha = 0.775$, $\Delta r = 0.001$). (a) the Delaunay triangulation over the unit circle, (b) the close-up view of the triangle with one unstable zero of the conjugate zero pair, i.e., $w_0 = 0.5944 - 0.8030j$.

TABLE I: Algorithm results and computation times for high-order system with varying number of stable zeros L. $iter$ is the number of Delaunay triangulation iterations, $time$ is the computation time of the developed algorithm and $roots\ time$ is computation time when applying Matlab $roots$ function. Precision $\Delta r = 0.001$. Zero is located at $w_0 = -0.5$.

L	$iter$	$time$ (s)	w_0	$roots\ time$ (s)
50	18	18.7798	-0.50227-0.0006511j	0.0029
100	18	28.9823	-0.50227-0.0006511j	0.0084
150	17	45.6513	-0.50227-0.0006511j	0.0188
200	19	77.8948	-0.49919-0.0013154j	0.0332
250	18	106.7658	-0.49945-0.0008754j	0.0572
300	18	157.2958	-0.50285+0.0003389j	0.0814
350	17	183.9027	-0.50285+0.0003389j	0.1188
400	16	203.5486	-0.50285+0.0003389j	0.1587
450	17	250.7676	-0.50285+0.0003389j	0.2128
500	18	294.7975	-0.50227-0.0006511j	0.2656

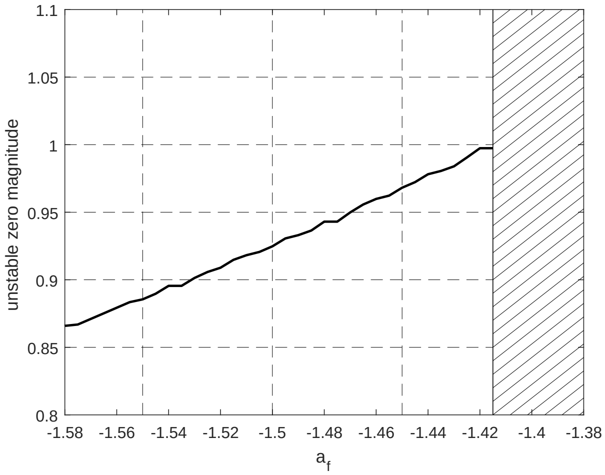


Fig. 7: Tuning unstable-zero location by a_f parameter in (15). Bold line is the distance of the unstable zero to origin of complex w-plane, dashed area is the system-stability range of a_f. For transition point of $F(w)$ zero into the stability range: $w_0 = -0.9974 - 0.0015j$ and $a_f = -1.415$.

$$\mathbf{A}_f = \begin{bmatrix} 0.6 & -1.45 \\ 1 & -1 \end{bmatrix}. \tag{14}$$

The characteristic equation of this system takes the form (6) and the system is stable only when all zeros of $F(w)$ are outside the unit circle. Using the developed method, the system stabilization can be performed, refer to Fig. 5. Starting from $\alpha = 1.1$, one can decrease α value which results in increase of unstable zero magnitude at the w-plane. Continuing this tuning of the α parameter, one can obtain the stable system for $\alpha < 0.7749$. The mesh for which the system approaches the stability limit is presented in Fig. 6. As seen, the mesh becomes denser for conjugate zero points at the unit circle.

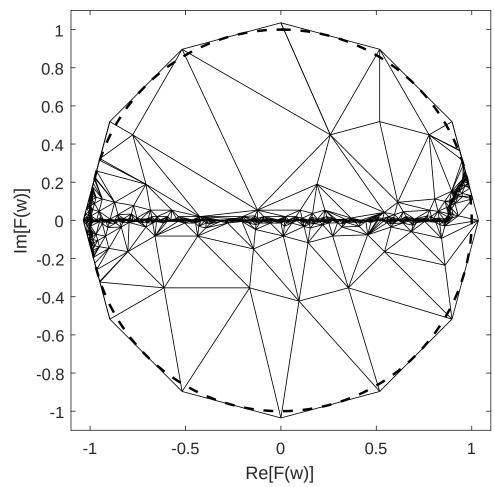
2) Benchmark 2: Let us consider fractional-order scalar system with [10]

$$\mathbf{A}_f = \begin{bmatrix} a_f \end{bmatrix} \tag{15}$$

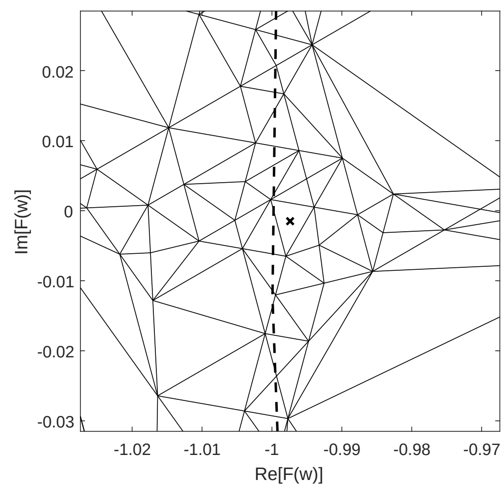
where a_f is the parameter and $\alpha = 0.5$. The characteristic equation of this system takes the form (6). Using the developed method, the stable system can be designed by tuning value of the a_f parameter, refer to Fig. 7. Starting from $a_f = -1.6$, one can increase a_f value which results in increase of unstable zero magnitude at the w-plane. Continuing tuning of the a_f parameter, one can obtain the stable system for $a_f > -1.415$. The mesh for which system approaches the stability limit is presented in Fig. 8. As seen, the mesh becomes denser for the zero point at the unit circle.

3) Benchmark 3: Let us consider fractional-order system with [10]:

$$\mathbf{A}_f = \begin{bmatrix} -1 & 0 & 0.1 & 0 \\ 0 & -1 & -0.01 & 0 \\ 0.02 & 0 & -0.8 & -0.03 \\ 0.77 & 0.05 & -0.9 & -1 \end{bmatrix}. \tag{16}$$



(a)



(b)

Fig. 8: Algorithm convergence for (15) with system near stability range of a_f ($a_f = -1.415$, $\Delta r = 0.001$). (a) the Delaunay triangulation over the unit circle, (b) the close-up view of the triangle with one unstable zero, i.e., $w_0 = -0.9974 - 0.0015j$.

The characteristic equation of this system takes the form (6) and the system is stable only when all zeros of $F(w)$ are outside the unit circle. By modifying the α parameter, the system stabilization can be performed, refer to Fig. 9. Starting from $\alpha = 0.12$, one can increase α value so as to increase the magnitude of the unstable zero at the w-plane. Continuing this tuning of the α parameter, one can obtain the stable system for $\alpha > 0.184$. The mesh for which the system approaches the stability limit is presented in Fig. 10. As seen, the mesh becomes denser for the zero point at the unit circle.

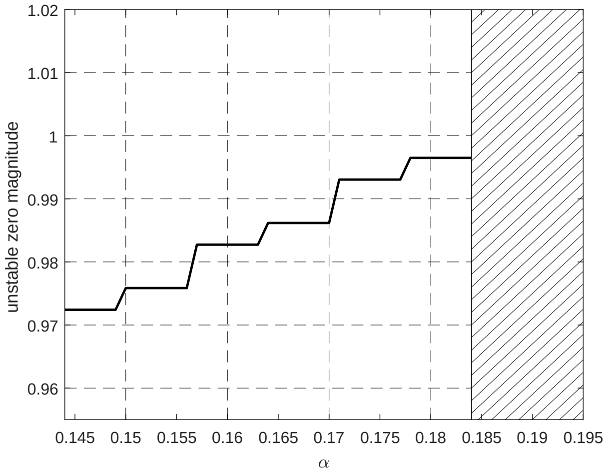


Fig. 9: Tuning unstable-zero location by α parameter for system (16). Bold line is the distance of the unstable zero to origin of complex w-plane, dashed area is the system-stability range of α. For transition point of $F(w)$ zero into the stability range: $w_0 = -0.9974 - 0.0015j$ and $\alpha = 0.184$.

V. Conclusion

The flexible and general numerical test for stability evaluation of discrete-time circuits and systems is developed. It employs modern root-finding techniques at the complex plane employing the Delaunay triangulation and the Cauchy's Argument Principle. The algorithm can be applied to systems with characteristic equation not being a polynomial. Hence, it is applicable to discrete-time systems of integer order as well as to fractional-order systems. It returns values of unstable zeros of the characteristic equation with their multiplicity if the system is unstable. Numerical tests demonstrate its value for the stabilization of discrete-time systems.

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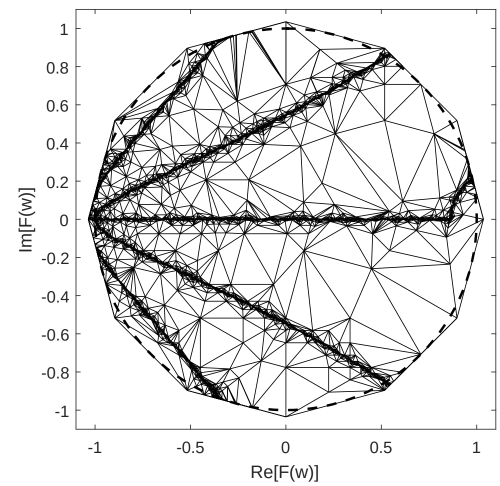
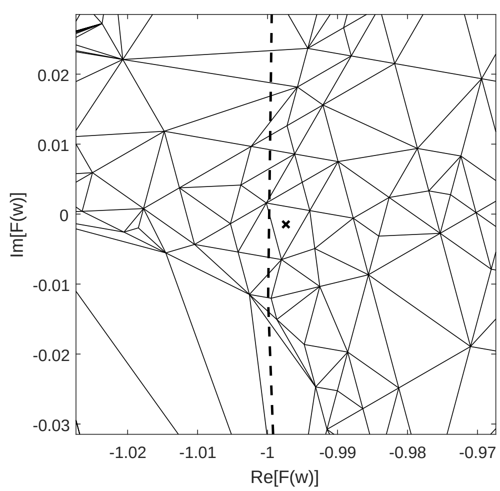


(a)



(b)

Fig. 10: Algorithm convergence for (16) with system near stability range of α ($\alpha = 0.184$, $\Delta r = 0.001$). (a) the Delaunay triangulation over the unit circle, (b) the close-up view of the triangle with one unstable zero, i.e., $w_0 = -0.9974 - 0.0015j$.