Dedicated to Professor M.P. Kaźmierkowski
on the occasion of his 70th birthday

Necessary and sufficient conditions for stability of fractional discrete-time linear state-space systems

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Abstract. In the paper the problems of practical stability and asymptotic stability of fractional discrete-time linear systems are addressed. Necessary and sufficient conditions for practical stability and for asymptotic stability are established. The conditions are given in terms of eigenvalues of the state matrix of the system. In particular, it is shown that (similarly as in the case of fractional continuous-time linear systems) in the complex plane exists such a region, that location of all eigenvalues of the state matrix in this region is necessary and sufficient for asymptotic stability. The parametric description of boundary of this region is given. Moreover, it is shown that Schur stability of the state matrix (all eigenvalues have absolute values less than 1) is not necessary nor sufficient for asymptotic stability of the fractional discrete-time system. The considerations are illustrated by numerical examples.

Key words: linear system, discrete-time, fractional, practical stability, asymptotic stability.

1. Introduction

Dynamical systems described by fractional order differential or difference equations have been investigated in several areas such as viscoelasticity, electrochemistry, diffusion processes, automatic control, power electronic, etc. (see [1–10], for example, and references therein).

The problem of stability of linear fractional order systems has recently had considerable attention. In the case of continuous-time linear systems exist analytic, LMI and frequency domain conditions for asymptotic stability (see [11–18], for example). In the case of fractional discrete-time linear systems existing conditions are mainly sufficient. This follows from the fact that asymptotic stability of the system is equivalent to asymptotic stability of the corresponding discrete-time linear system of natural order with an infinite number of delays. In practical problems only the bounded number of delays (called the length of practical implementation) can be considered. In this case the corresponding discrete-time linear system of natural order has a finite number of delays and it is called the practical realization of the fractional order system. Asymptotic stability of this system is called the practical stability of the fractional system.

The sufficient conditions for practical stability with a given length of practical implementation for fractional order discrete-time systems are derived in [19–21]. Simple necessary and sufficient analytic conditions for practical stability of scalar fractional discrete-time linear systems and simple necessary and sufficient analytic conditions for practical stability and for asymptotic stability of fractional discrete-time linear systems with diagonal state matrix are established in [22] and [23], respectively.

Recently, the equivalent descriptions of discrete-time linear systems of fractional order and its stability domains were considered in [24].

Simple analytic necessary and sufficient conditions for practical stability and for asymptotic stability of positive fractional discrete-time linear systems are given in [25–27] for 1D systems and in [27, 28] for 2D systems.

The aim of the paper is to give the necessary and sufficient conditions for practical stability and for asymptotic stability of fractional discrete-time linear systems described by the state-space model in the general case. In particular, we show that in the complex plane exists such a region, that location of all eigenvalues of the state matrix in this region is necessary and sufficient for asymptotic stability of the system.

The paper is organized as follows. Formulation of the problem is given in Sec. 2. Necessary and sufficient conditions for practical stability and for asymptotic stability are established in Sec. 3. Illustrative examples are given in Sec. 4 and concluding remarks in Sec. 5.

2. Problem formulation

Consider the discrete-time linear system of fractional order α described by the homogeneous state equation

$$\Delta^\alpha x_{i+1} = A x_i, \qquad \alpha \in (0, 1), \qquad (1)$$

with the initial condition x_0, where $x_i \in \Re^n$, $A \in \Re^{n \times n}$,

$$\Delta^\alpha x_i = \sum_{k=0}^{i} (-1)^k \binom{\alpha}{k} x_{i-k} \qquad (2)$$

is the fractional difference of order $\alpha \in (0, 1)$ of the discrete-time function x_i [26–28] and

$$\binom{\alpha}{k} = \frac{\alpha!}{k!(\alpha - k)!}. \qquad (3)$$

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Using (2) we may write Eq. (1) in the form

$$x_{i+1} = A_0 x_i + \sum_{k=1}^{i} A_k x_{i-k}, \qquad (4)$$

where

$$A_0 = A + I\alpha, \qquad A_k = I c_k(\alpha), \qquad k = 1, 2, ..., \quad (5)$$

I is the $n \times n$ identity matrix and

$$c_k(\alpha) = (-1)^k \begin{pmatrix} \alpha \\ k+1 \end{pmatrix}, \qquad k = 1, 2, ... \quad (6)$$

The state Eq. (4), also (1), can be obtained by the use of Grunwald-Letnikov approximation of the fractional order derivative [3].

The coefficients (6) can be computed by the following simple algorithm suitable for computer programming [25]

$$c_{k+1}(\alpha) = c_k(\alpha)\frac{k+1-\alpha}{k+2}, \qquad k = 1, 2, ... \quad (7)$$

with $c_1(\alpha) = 0.5\alpha(1 - \alpha)$.

Equation (4) describes the discrete-time linear system with increasing number of delays.

From (7) it follows that $c_k(\alpha) > 0$ for all $\alpha \in (0, 1)$ and $k = 1, 2, ...$. Moreover, coefficients $c_k(\alpha)$ strongly decrease for increasing k. Therefore, in practical problems it is assumed that k is bounded by some natural number L. This number is called the length of practical implementation. In this case the Eq. (4) can be written in the form

$$x_{i+1} = \begin{cases} A_0 x_i + \sum_{k=1}^{i} A_k x_{i-k} & \text{for } i = 0, 1, \cdots, L \\ A_0 x_i + \sum_{k=1}^{L} A_k x_{i-k} & \text{for } i = L+1, L+2, \cdots \end{cases}$$
$$(8)$$

with the initial condition $x_0 \in \Re^n$. The equation (8) describes a linear discrete-time system with L delays in state.

The time-delay system (8) is called the practical realization of the fractional system (1). From [26] we have the following definitions.

Definition 1. The fractional system (1) is called practically stable if the system (8) is asymptotically stable.

Definition 2. The fractional system (1) is called asymptotically stable if the system (8) is asymptotically stable for $L \to \infty$.

From Definition 1 and theory of asymptotic stability of discrete-time linear system we have the following theorem.

Theorem 1. The fractional system (1) with given length L of practical implementation is practically stable if and only if

$$w(z) \neq 0, \qquad |z| \geq 1, \quad (9)$$

where

$$w(z) = \det\{Iz - A_0 - \sum_{k=1}^{L} A_k z^{-k}\}. \quad (10)$$

The characteristic equation of the system for $z \neq 0$ can be written in the form

$$\det\{Iz^{L+1} - A_0 z^L - \sum_{k=1}^{L} A_k z^{L-k}\} = 0. \quad (11)$$

To check practical stability of the fractional system (1) we can apply the existing methods to the asymptotic stability analysis of the discrete-time systems (8) with delays. However, these methods may be inconvenient with respect to a high degree of Eq. (11) for a large length of practical implementation.

The problem of practical stability of the fractional system (1) has been considered in [19–21] for standard systems and in [25–28] for positive systems. In [25] it has been shown that practical stability and asymptotic stability of the positive fractional system (1) are equivalent to asymptotic stability of the corresponding natural order positive discrete-time system without delays of the same size as the system (1). Recall that the system (1) is positive if and only if all entries of the matrix $A_0 = A + I\alpha$ are non-negative [25–27].

In the case of standard systems, the problem of checking of the asymptotic stability is more complicated and less advanced. This follows from the fact that asymptotic stability of the fractional discrete-time system (1) is equivalent to asymptotic stability of the system (8) with $L \to \infty$.

Simple necessary and sufficient analytic conditions for practical stability of scalar fractional discrete-time system (1) (A is a real number) and simple necessary and sufficient analytic conditions for practical stability and for asymptotic stability of the system (1) with diagonal matrix A were established in [22] and [23], respectively.

Recently, the method for evaluation of the stability domain in the space of parameters of the discrete-time linear system (1) was proposed in [24].

The aim of the paper is to give the new necessary and sufficient conditions for practical stability and for asymptotic stability of the fractional system (1).

3. Solution of the problem

3.1. Practical stability. Substitution (5) in (10) gives

$$w(z) = \det\{(z - \sum_{k=1}^{L} c_k(\alpha)z^{-k})I - A_0\}. \quad (12)$$

The characteristic equation $w(z) = 0$ can be written in the form

$$\prod_{i=1}^{n} w_i(z) = 0, \quad (13)$$

where

$$w_i(z) = z - \sum_{k=1}^{L} c_k(\alpha)z^{-k} - \lambda_i(A_0) \quad (14)$$

and $\lambda_i(A_0)$ denotes i-th eigenvalue of $A_0 = A + I\alpha$ ($i = 1, 2, ..., n$).

From the above and Theorem 1 it follows that the fractional system (1) with given length L of practical implementation is practically stable if and only if all roots of all equations

$w_i(z) = 0$ $(i = 1, 2, ..., n)$ are stable, i.e. have absolute values less than 1. Therefore, first we consider the problem of stability of roots of the equation

$$z - \sum_{k=1}^{L} c_k(\alpha) z^{-k} - \rho = 0 \qquad (15)$$

in dependence of $\rho = \lambda_i(A_0)$.

To the stability analysis we apply the D-decomposition method. For state of the art of this method see [29].

According to the D-decomposition method, the boundary of stability region in the complex ρ-plane is given by the formula

$$\rho(\omega) = e^{j\omega} - \sum_{k=1}^{L} c_k(\alpha) e^{-jk\omega}, \qquad \omega \in [0, 2\pi], \qquad (16)$$

which is obtained by substitution $z = \exp(j\omega)$, $\omega \in [0, 2\pi]$ (boundary of the unit circle in the complex z-plane) in equation (15).

The closed curve (16) divides the complex ρ-plane into two regions. One of them is a bounded region consisting the origin of this plane (see Fig. 1). This region will be denoted by $S(\alpha, L)$.
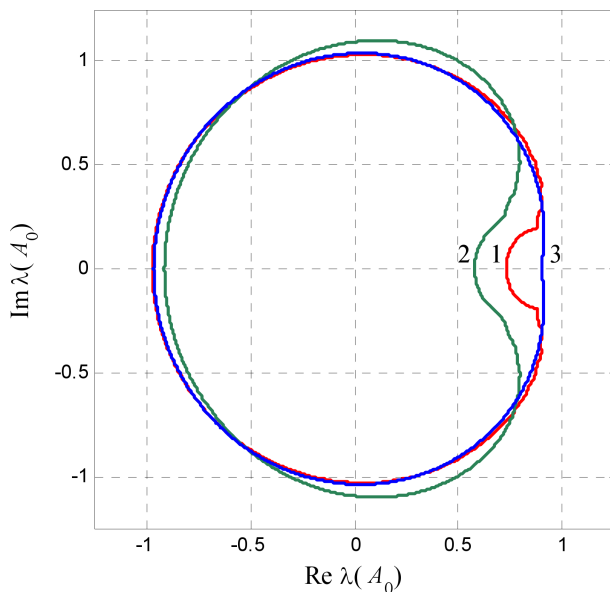

Fig. 1. Regions $S(\alpha, L)$ for $L = 50$ and $\alpha = 0.1$ (boundary 1), $\alpha = 0.5$ (boundary 2) and $\alpha = 0.9$ (boundary 3)

It is easy to check that (16) for $\alpha = 0$ and for $\alpha = 1$ describes the unit circle, i.e. a circle with centre at point (0,0) and radius 1.

From the left hand side of (11) for $z = 0$ one has $\det(-A_L) = \det(-Ic_L(\alpha)) \neq 0$. This means that $z = 0$ is not a root of the characteristic equation (11) and also the equation (15).

Lemma 1. All roots of Eq. (15) are stable if and only if $\rho \in S(\alpha, L)$.

Proof. According to the D-decomposition method, for proof that $S(\alpha, L)$ is a stability region in the complex ρ-plane it is

sufficient to show that all roots of (15) are stable for at least one point in $S(\alpha, L)$. We choose $\rho = 0$.

Equation (15) for $\rho = 0$ can be written in the form

$$z^{L+1} - \sum_{k=1}^{L} c_k(\alpha) z^{L-k} = 0. \qquad (17)$$

In [30] it has been shown that if

$$1 > |a_{n-1}| + \cdots + |a_1| + |a_0|, \qquad (18)$$

then all roots of the polynomial

$$z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \qquad (19)$$

have absolute values less than 1.

The condition (19) for the left hand side of (17) has the form

$$1 > \sum_{k=1}^{L} c_k(\alpha). \qquad (20)$$

Since $\sum_{k=1}^{\infty} c_k(\alpha) = 1 - \alpha$ [27], we have

$$1 - \sum_{k=1}^{L} c_k(\alpha) > 1 - \sum_{k=1}^{\infty} c_k(\alpha) = 1 - 1 + \alpha = \alpha > 0. \quad (21)$$

Hence, the condition (20) holds. This means that the equation (15) for $\rho = 0$ has $L+1$ roots which satisfy the condition $|z_r| < 1$ $(r = 1, 2, ..., L+1)$. Because $\rho = 0 \in S(\alpha, L)$, $S(\alpha, L)$ is the stability region for all roots of (15) and the proof is completed.

Taking into consideration all functions (14), from (13) and Lemma 1 we obtain the following theorem.

Theorem 2. The fractional system (1) with the given length L of practical implementation is practically stable if and only if all eigenvalues $\lambda_i(A_0)$ $(i = 1, 2, ..., n)$ are located in the stability region $S(\alpha, L)$, i.e. $\lambda_i(A_0) \in S(\alpha, L)$ for all $i = 1, 2, ..., n$.

For $\omega = 0$ and $\omega = 2\pi$ from (16) one obtains

$$\rho(0) = 1 - \sum_{k=1}^{L} c_k(\alpha), \qquad (22)$$

$$\rho(\pi) = -1 - \sum_{k=1}^{L} c_k(\alpha)(-1)^k. \qquad (23)$$

Lemma 2. If all eigenvalues $\lambda_i(A_0)$ are real, then the fractional system (1) with given length L of practical implementation is practically stable if and only if

$$\rho(\pi) < \lambda_i(A_0) < \rho(0), \qquad i = 1, 2, ..., n. \qquad (24)$$

Proof. It is easy to see that $\rho(\pi) < \rho(0)$ and the interval $(\rho(\pi), \rho(0))$ of the real axis lies in the stability region $S(\alpha, L)$ for any fixed $\alpha \in (0, 1)$. The proof directly follows from Theorem 2.

Lemma 2 also follows from [23].

Practical stability regions $S(\alpha, L)$ for $L = 50$ and a few values of fractional order $\alpha \in (0, 1)$ are shown in Fig. 1 on the plane of eigenvalues of $A_0 = A + I\alpha$. The closed curve (16) crosses real axis in points $\rho(\pi)$ and $\rho(0)$. From

(22), (23) and (6) it follows that $\rho(\pi) = -1$, $\rho(0) = 1$ for $\alpha = 0$ and for $\alpha = 1$. In these cases (16) describes a circle with centre at point (0,0) and radius 1. It is easy to check that $\rho(\pi) > -1$ and $\rho(0) < 1$ for all $\alpha \in (0, 1)$. This means that if α grows from 0 to 1, then point $\rho(\pi)$ in complex plane beginning moves in right and then in left, but point $\rho(0)$ moves in opposite direction, i.e. beginning in left and then in right.

Figure 2 shows the regions $S(\alpha, L)$ for $\alpha = 0.1$ and a few values of the length L of practical implementation. From Fig. 2 we can see that (for fixed α) the greater value of L results in the lowest value of $\rho(0)$. The values of $\rho(\pi)$ are nearly the same for different values of L.
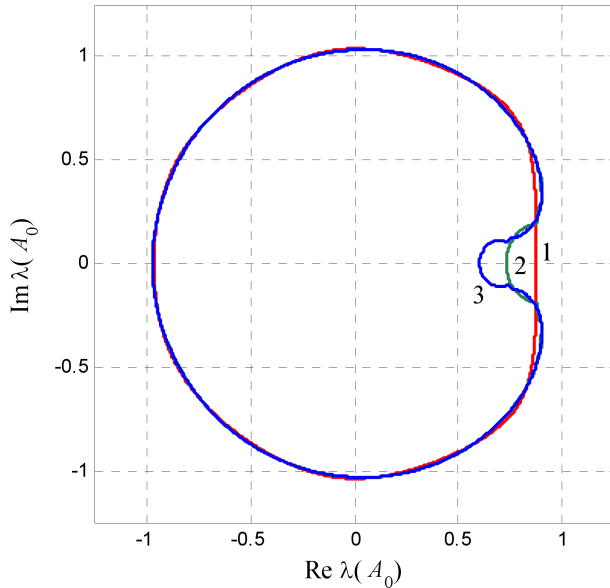

Fig. 2. Regions $S(\alpha, L)$ for $\alpha = 0.1$ and $L = 5$ (boundary 1), $L = 50$ (boundary 2) and $L = 500$ (boundary 3)

From Figs. 1, 2 and Lemma 2 it follows that for any fixed $\alpha \in (0, 1)$ there exists a circle $D_1 = D_1(\rho_0, r_1)$ with the centre

$$\rho_0 = 0.5(\rho(0) + \rho(\pi)) = -\sum_{k=2,4,...}^{L} c_k(\alpha) \qquad (25)$$

and radius

$$r_1 = 0.5(\rho(0) - \rho(\pi)) = 1 - \sum_{k=1,3,...}^{L} c_k(\alpha), \qquad (26)$$

which entirely lies in the stability region $S(\alpha, L)$.

Another circle located entirely in the stability region $S(\alpha, L)$, denoted by $D_2 = D_2(0, r_2)$, has centre at origin of the complex plane and radius $r_2 = \rho(0)$, where $\rho(0)$ is defined by (22).

From (6) and (3) it follows that

$$-\sum_{k=1}^{L} c_k(\alpha) = \sum_{k=1}^{L} (-1)^{k+1} \begin{pmatrix} \alpha \\ k+1 \end{pmatrix}$$
$$= \sum_{k=2}^{L+1} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix}. \qquad (27)$$

Using the equality [31] ($\Gamma(\cdot)$ is the Euler gamma function)

$$\sum_{k=0}^{L+1} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} = \frac{\Gamma(L + 2 - \alpha)}{\Gamma(1 - \alpha)\Gamma(L + 2)}, \qquad (28)$$

one obtains

$$\sum_{k=2}^{L+1} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} = \frac{\Gamma(L + 2 - \alpha)}{\Gamma(1 - \alpha)\Gamma(L + 2)} - 1 + \alpha. \qquad (29)$$

Substitution (27) and (29) into (22) gives

$$\rho(0) = 1 - \sum_{k=1}^{L} c_k(\alpha) = \frac{\Gamma(L + 2 - \alpha)}{\Gamma(1 - \alpha)\Gamma(L + 2)} + \alpha. \qquad (30)$$

This means that radius $r_2 = \rho(0)$ of circle $D_2 = D_2(0, r_2)$ can be computed from (30).

From the above we have the following sufficient conditions for practical stability.

Theorem 3. The fractional system (1) with given length L of practical implementation is practically stable if at least one of the following conditions is satisfied:

1. all eigenvalues of $A_0 = A + I\alpha$ are located in circle $D_1 = D_1(\rho_0, r_1)$
2. all eigenvalues of A_0 are located in circle $D_2 = D_2(0, r_2)$, i.e.

$$|\lambda_i(A_0)| < \frac{\Gamma(L + 2 - \alpha)}{\Gamma(1 - \alpha)\Gamma(L + 2)} + \alpha, \quad i = 1, 2, ..., n. \quad (31)$$

The sufficient condition similar to (31) has been given in [19]. This condition has the form

$$|\lambda_i(A_0)| < \frac{\Gamma(L + 1 - \alpha)}{\Gamma(1 - \alpha)\Gamma(L + 1)} + \alpha, \quad i = 1, 2, ..., n. \quad (32)$$
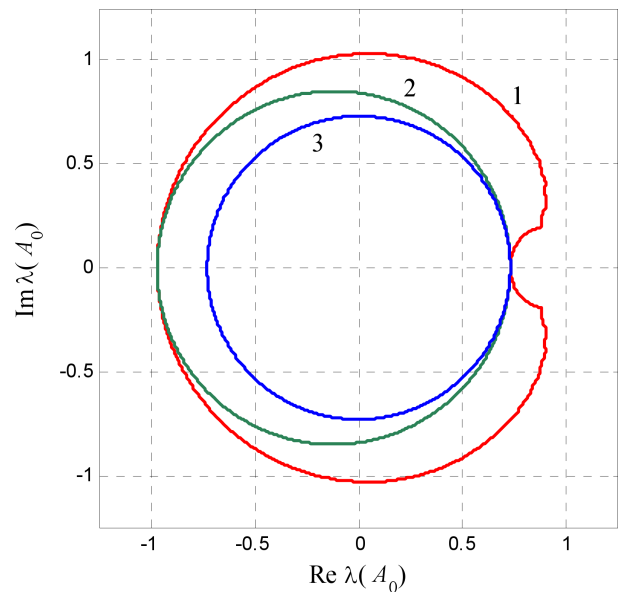

Fig. 3. Practical stability region $S(\alpha, L)$ for $\alpha = 0.1$, $L = 50$ (boundary 1) and circles D_1 (boundary 2) and D_2 (boundary 3)

Region $S(\alpha, L)$ with $\alpha = 0.1$, $L = 50$ and circles D_1 and D_2 are shown in Fig. 3. From (25), (26) and (30) for $\alpha = 0.1$ and $L = 50$ we have:

- $\rho_0 = -0.1207$, $r_1 = 0.8517$ for circle $D_1 = D_1(\rho_0, r_1)$
- $r_2 = 0.7310$ for circle $D_2 = D_2(0, r_2)$.

3.2. Asymptotic stability. First we prove the following important lemma.

Lemma 3. The following equality is true

$$\sum_{k=1}^{\infty} c_k(\alpha) z^{-k} = z - \alpha - (z-1)^{\alpha} z^{1-\alpha}. \quad (33)$$

Proof. From the Newton's generalized binomial formula (α is a real number)

$$(a+b)^{\alpha} = \sum_{j=0}^{\infty} \binom{\alpha}{j} a^{\alpha-j} b^j \quad (34)$$

for $a = z$ and $b = -1$ we obtain

$$(z-1)^{\alpha} = \sum_{j=0}^{\infty} \binom{\alpha}{j} z^{\alpha-j} (-1)^j$$
$$= z^{\alpha} + \sum_{j=1}^{\infty} \binom{\alpha}{j} z^{\alpha-j} (-1)^j. \quad (35)$$

By multiplying both sides of (35) by $z^{1-\alpha}$ we have

$$(z-1)^{\alpha} z^{1-\alpha} = z + \sum_{j=1}^{\infty} \binom{\alpha}{j} z^{1-j} (-1)^j$$
$$= z - \sum_{j=1}^{\infty} \binom{\alpha}{j} z^{1-j} (-1)^{j-1}. \quad (36)$$

If we let $1 - j = -k$, then

$$(z-1)^{\alpha} z^{1-\alpha} = z - \sum_{k=0}^{\infty} \binom{\alpha}{k+1} z^{-k} (-1)^k$$
$$= z - \alpha - \sum_{k=1}^{\infty} \binom{\alpha}{k+1} z^{-k} (-1)^k. \quad (37)$$

Finally, using (6) from (37) we obtain (33). This completes the proof.

From Definition 2 it follows that in the case of asymptotic stability we must consider the characteristic function (12) of the system (8) with $L \to \infty$, of the form

$$w(z) = \det\left\{\left(z - \sum_{k=1}^{\infty} c_k(\alpha) z^{-k}\right) I - (A + I\alpha)\right\}. \quad (38)$$

Using (33) in equation $w(z) = 0$ one obtains

$$\det\{(z-1)^{\alpha} z^{1-\alpha} I - A)\} = 0. \quad (39)$$

The characteristic equation (39) can be written in the form (13), where

$$w_i(z) = (z-1)^{\alpha} z^{1-\alpha} - \lambda_i(A) \quad (40)$$

and $\lambda_i(A)$ denotes i-th eigenvalue of A ($i = 1, 2, ..., n$).

From the above, Definition 2 and Theorem 1 for $L \to \infty$ it follows that the fractional system (1) is asymptotically stable if and only if all roots of all equations $w_i(z) = 0$ ($i = 1, 2, ..., n$) are stable, where $w_i(z)$ is given by (40).

We consider the stability problem of roots of the equation

$$(z-1)^{\alpha} z^{1-\alpha} - \eta = 0 \quad (41)$$

in dependence of $\eta = \lambda_i(A)$.

Substituting $z = \exp(j\omega)$, $\omega \in [0, 2\pi]$, in equation (41) we obtain the parametric description of boundary of the stability region in the complex η-plane

$$\eta(\omega) = (e^{j\omega} - 1)^{\alpha} (e^{j\omega})^{1-\alpha}, \quad \omega \in [0, 2\pi]. \quad (42)$$

It is easy to check that (42) for $\alpha = 0$ and $\alpha = 1$ describes the circle with centre at point (0,0) and radius 1 and the circle with centre at point (-1,0) and radius 1, respectively.

The closed curve (42) divide the complex η-plane into two regions, one bounded and one unbounded (see Fig. 4). Denote by $S(\alpha)$ the bounded region.
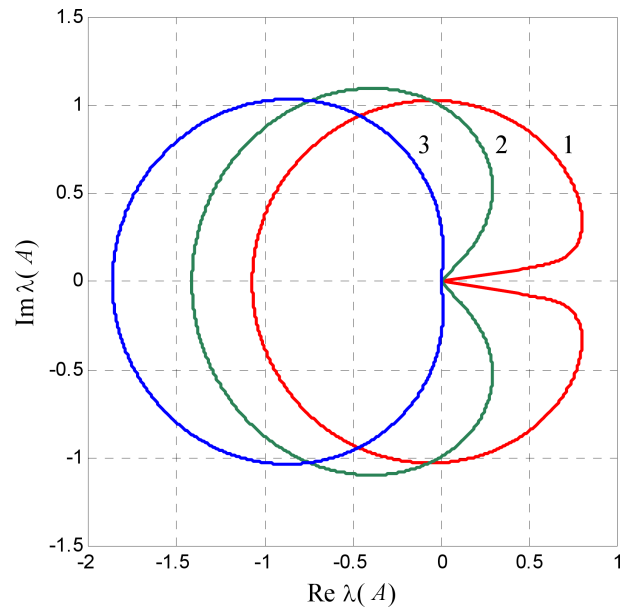

Fig. 4. Regions $S(\alpha)$ for $\alpha = 0.1$ (boundary 1), $\alpha = 0.5$ (boundary 2) and $\alpha = 0.9$ (boundary 3)

Lemma 4. If η is real, then all roots of (41) are stable if and only if

$$-2^{\alpha} < \eta < 0. \quad (43)$$

Proof. In [23] it has been shown that if A is a real number then the system (1) is asymptotically stable if and only if $-2^{\alpha} < A < 0$. This means that the condition (43) is necessary and sufficient for asymptotic stability of roots of (41) with real η and the proof is completed.

Lemma 5. All roots of (41) are stable if and only if $\eta \in S(\alpha)$.

Proof. It is sufficient to show that in $S(\alpha)$ exists at least one point such that all roots of (41) are stable.

From (42) for $\omega = 0$ and $\omega = \pi$ we have $\eta(0) = 0$ and $\eta(\pi) = -2^{\alpha}$. This means that the interval $(-2^{\alpha}, 0)$ of the

real axis lies in the stability region $S(\alpha)$. This interval, according to Lemma 4, is the stability region for all roots of (41) with real η. Hence, in $S(\alpha)$ exist such points η for which all roots of (41) are stable. This completes the proof.

Asymptotic stability regions $S(\alpha)$ for $\alpha = 0.1$, $\alpha = 0.5$ and $\alpha = 0.9$ are shown in Fig. 4.

Taking into considerations (13) and (40), from Lemma 5 we obtain the following theorem.

Theorem 4. The fractional system (1) is asymptotically stable if and only if all eigenvalues of the matrix A are located in the stability region $S(\alpha)$ with boundary (42).

Similar result has been recently obtained in [24].

From Theorem 4 and Lemma 4 we have the following lemmas.

Lemma 6. If all eigenvalues $\lambda_i(A)$ are real, then the fractional system (1) is asymptotically stable if and only if

$$-2^\alpha < \lambda_i(A) < 0, \quad i = 1, 2, ..., n. \qquad (44)$$

Lemma 7. If all eigenvalues $\lambda_i(A)$ of A are real, then the fractional system (1) is asymptotically stable for $\alpha \in (\alpha_{\min}, 1)$, where $\alpha_{\min} = \max\{\alpha_i, \ i = 1, 2, .., n\}$ and

- $\alpha_i = 0$ if $-1 < \lambda_i(A) < 0$,
- $\alpha_i = \log_2(-\lambda_i(A))$ if $-2 < \lambda_i(A) < -1$, where $\log_2 a$ is the base 2 logarithm of a. Moreover, if $\lambda_i(A) = -1$, then the condition (44) holds for $0 \le \alpha_i < 1$.

From Fig. 4 and (44) it follows that for any fixed $\alpha \in (0, 1)$ there exists a circle $D_3 = D_3(\eta_0, r_3)$ with the centre $\eta_0 = -2^{\alpha-1}$ and radius $r_3 = 2^{\alpha-1}$ which lies in the region $S(\alpha)$. If, for example, $\alpha = 0.3$ then $\eta_0 = -2^{-0.7} = -0.6156$ and $r_3 = 0.6156$.

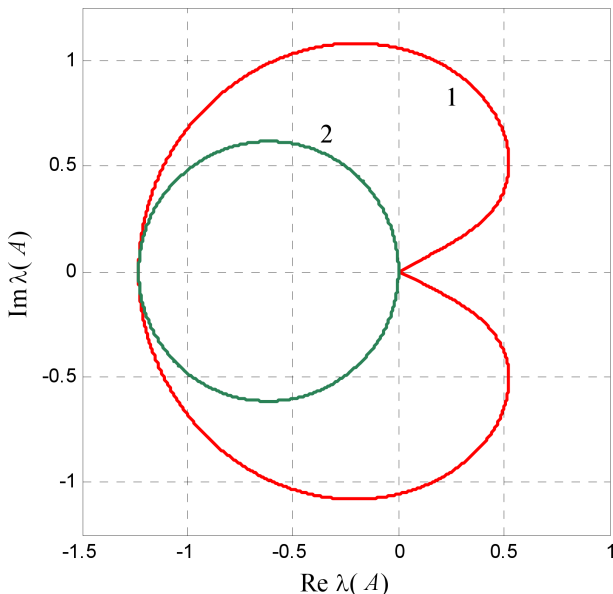The region $S(\alpha)$ with $\alpha = 0.3$ and circle D_3 are shown in Fig. 5.

Fig. 5. Region $S(\alpha)$ for $\alpha = 0.3$ (boundary 1) and circle D_3 (boundary 2)

From the above we have the following sufficient condition for asymptotic stability.

Lemma 8. The fractional system (1) is asymptotically stable if all eigenvalues of A are located in circle $D_3 = D_3(\eta_0, r_3)$.

From Theorem 4 and Figs. 4 and 5 we have the following important remarks.

Remark 1. If the state matrix A has at least one real positive eigenvalue, then the fractional system (1) is not asymptotically stable for all $\alpha \in (0, 1)$.

Remark 2. Schur stability of the state matrix A (all eigenvalues have absolute values lees than 1) is not necessary nor sufficient for asymptotic stability of the fractional system (1).

Given in this section conditions for asymptotic stability of the system (1) can be formulated in terms of eigenvalues of the matrix $A_0 = A + I\alpha$. In particular, Theorem 4 can be formulated in equivalent form as follows.

Theorem 4a. The fractional system (1) is asymptotically stable if and only if all eigenvalues of the matrix $A_0 = A + I\alpha$ are located in the stability region $S_0(\alpha)$ with boundary

$$\eta_0(\omega) = \alpha + (e^{j\omega} - 1)^\alpha (e^{j\omega})^{1-\alpha}, \quad \omega \in [0, 2\pi]. \quad (45)$$

4. Illustrative examples

Example 1. Check practical stability with length $L = 50$ of practical implementation of the fractional system (1) with $\alpha = 0.1$ and the state matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.5 & -0.03 & 0.9 & 0.06 \\ 0.3 & 0 & 0 & -1 \\ 0.09 & 0.04 & 0.08 & 0.02 \end{bmatrix}. \qquad (46)$$

Matrix $A_0 = A + I\alpha$ has the following eigenvalues:

$$\lambda_{1,2} = -0.1654 \pm j0.7715;$$
$$\lambda_{3,4} = 0.3604 \pm j0.3463. \qquad (47)$$

Computing from (25) and (26) centre and radius of the circle $D_1 = D_1(\rho_0, r_1)$ one obtains $\rho_0 = -0.1207$ and $r_1 = 0.8517$.

Circle $D_2 = D_2(0, r_2)$ has radius $r_2 = 0.7310$. Radius $r_2 = \rho(0)$ can be computed from (22) or (30).

Practical stability region $S(\alpha, L)$ for $\alpha = 0.1$, $L = 50$, eigenvalues (47) and circles D_1 and D_2 are shown in Fig. 6.

From Fig. 6 it follows that eigenvalues (47) lie in the practical stability region (the necessary and sufficient condition of Theorem 2 holds) and also in circle D_1 (the sufficient condition 1 of Theorem 3 holds). Hence, the system is practically stable.

It is easy to see that not all eigenvalues (47) are located in circle D_2, or equivalently, the sufficient condition (31) is not satisfied because

$$\max |\lambda_i(A_0)| = 0.7890 > r_2 = 0.7310.$$

This means that the sufficient condition 2. of Theorem 3 is not satisfied.
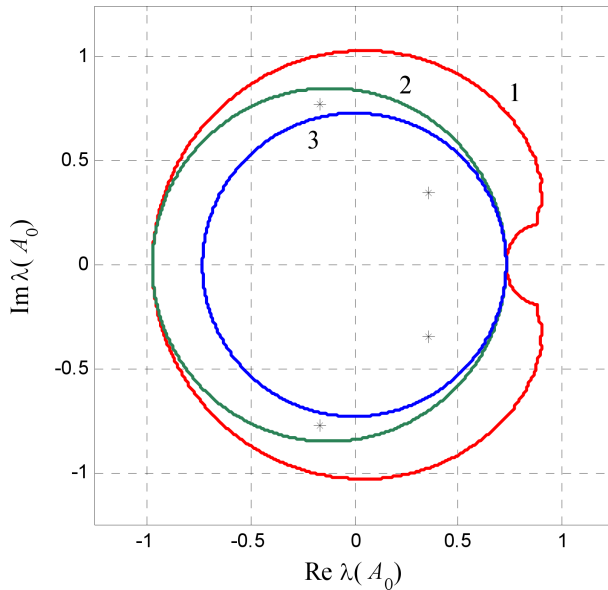
Fig. 6. Region $S(\alpha, L)$ (boundary 1), eigenvalues (47) (*), circles D_1 (boundary 2) and D_2 (boundary 3)

Example 2. Check asymptotic stability of the fractional system (1) with $\alpha = 0.1$ and the state matrix

$$A = \begin{bmatrix} -1 & 0 & 0.1 & 0 \\ 0 & -1 & -0.01 & 0 \\ 0.02 & 0 & -0.8 & -0.03 \\ 0.77 & 0.05 & -0.9 & -1 \end{bmatrix}. \qquad (48)$$

Matrix A has the following real eigenvalues: $\lambda_1 = -0.7249$; $\lambda_2 = -1.1363$; $\lambda_3 = -0.9388$; $\lambda_4 = -1$.

From Lemma 7 we have that the fractional system (1) with the matrix (48) is asymptotically stable for $\alpha \in (\alpha_{\min}, \; 1)$, where $\alpha_{\min} = \log_2(1.1363) = 0.1843$. Hence, the system is not asymptotically stable for $\alpha = 0.1$.

Note, that the matrix (48) is not Schur stable ($|\lambda_2| = 1.1363 > 1$) but the system (1), (48) is asymptotically stable for any $\alpha \in (0.1843, \; 1)$.

5. Concluding remarks

The problems of practical stability and asymptotic stability of the discrete-time linear system (1) of the fractional order $\alpha \in (0, \; 1)$ have been addressed. Necessary and sufficient conditions for practical stability (Theorem 2, Lemma 2) and for asymptotic stability (Theorem 4, Lemmas 6 and 7) have been established. The conditions are given in terms of eigenvalues of the state matrix A (or the matrix $A + I\alpha$).

In particular, it has been shown that (similarly as for fractional continuous-time linear systems [11]) in the complex plane there exists such a region, that location of all eigenvalues of the state matrix A in this region is necessary and sufficient for asymptotic stability. The parametric description of a boundary of this region has the form (42). This important result has been recently presented in [24] and more recently in [32]. It should be noted that the result of Theorem 4 and similar results of [24] and [32] have been obtained inde-

pendently, with different proofs and more or less in the same time.

Moreover, it has been shown that Schur stability of the state matrix A (all eigenvalues have absolute values lees than 1) is not necessary nor sufficient for asymptotic stability of the fractional system (1).

REFERENCES

[1] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[2] C.A. Monje, Y.Q. Chen, B.M. Vinagre, D. Xue, and V. Feliu-Batlle, *Fractional-order Systems and Controls Fundamentals and Applications*, Springer, London, 2010.

[3] M.M. Meerschaert and C. Tadjeran, "Finite difference approximations for fractional advection-dispersion flow equations", *J. Computational and Applied Mathematics* 172 (1), 65–77 (2004).

[4] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations,* Elsevier, Amsterdam, 2006.

[5] L. Debnath, "Recent applications of fractional calculus to science and engineering", *Int. J. Mathematics and Mathematical Sciences* 54, 3413–3442 (2003).

[6] A. Dzieliński, D. Sierociuk, and G. Sarwas, "Some applications of fractional order calculus", *Bull. Pol. Ac.: Tech.* 58 (4), 583–592 (2010).

[7] P. Lino and G. Maione, "Loop-shaping and easy tuning of fractional-order proportional integral controllers for position servo systems", *Asian J. Control* 15 (3), 796–805 (2012).

[8] Y. Luo and Y-Q. Chen, *Fractional Order Motion Controls*, John Wiley & Sons Ltd, Chichester, 2013.

[9] Y-H. Chang, C-I. Wu, H-W. Lin, H-C. Chen, and C-W. Chang, "Fractional order integral sliding-mode flux observer for direct field-oriented induction machines", *Int. J. Innovative Computing, Information and Control* 8 (7A), 4851–4198, (2012).

[10] R. Melicio, V.M.F. Mendes, and J.P.S. Catalao, "Power converter topologies and fractional-order controllers: Wind energy applications", *Proc. Int. Conf. on Power Electronics, Electrical Drives and Motion* 1, 1334–1338 (2010).

[11] D. Matignon, "Stability result on fractional differential equations with applications to control processing", *Proc. IMACS-SMC* 2, 963–968 (1996).

[12] I. Petras, "Stability of fractional-order systems with rational orders: a survey", *Fractional Calculus & Applied Analysis. Int. J. Theory and Applications* 12 (3), 269–298 (2009).

[13] M.S. Tavazoei and M. Haeri, "Note on the stability of fractional order systems", *Mathematics and Computers in Simulation* 79 (5), 1566–1576 (2009).

[14] M. Busłowicz, "Stability of linear continuous-time fractional order systems with delays of the retarded type", *Bull. Pol. Ac.: Tech.* 56 (4), 319–324 (2008).

[15] M. Busłowicz, "Stability analysis of linear continuous-time fractional systems of commensurate order", *J. Automation, Mobile Robotics and Intelligent Systems* 3 (1), 16–21 (2009).

[16] M. Busłowicz, "Stability of state-space models of linear continuous-time fractional order systems", *Acta Mechanica et Automatica* 5 (2), 15–22 (2011).

[17] M. Busłowicz, "Stability analysis of continuous-time linear systems consisting of n subsystems with different fractional orders", *Bull. Pol. Ac.: Tech.* 60 (2), 279–284 (2012).

[18] J. Sabatier, M. Moze, and C. Farges, "LMI stability conditions for fractional order systems", *Computers and Mathematics with Applications* 59 (5), 1594–1609 (2010).

[19] A. Dzieliński and D. Sierociuk, "Stability of discrete fractional order state-space systems", *J. Vibration and Control* 14, 1543–1556 (2008).

[20] S. Guermah, S. Djennoune, and M. Bettayeb, "A new approach for stability analysis of linear discrete-time fractional-order systems", in *New Trends in Nanotechnology and Fractional Calculus Applications*, ed. D Baleanu, pp. 151-162, Springer, Berlin, 2010.

[21] R. Stanisławski, W.P. Hunek, and K.J. Latawiec, "Finite approximations of a discrete-time fractional derivative", *Proc. 16th Int. Conf. Methods and Models in Automation and Robotics* 1, 142–145 (2011).

[22] M. Busłowicz, "Practical stability of scalar discrete-time linear systems of fractional order", in *Automation of Discrete Processes: Theory and Applications*, eds. A. Świerniak and J. Krystek, vol. 1, pp. 31–40, Printing House of Silesian University of Technology, Gliwice, 2012, (in Polish).

[23] M. Busłowicz, "Simple analytic conditions for stability of fractional discrete-time linear systems with diagonal state matrix", *Bull. Pol. Ac.: Tech.* 60 (4), 809–814 (2012).

[24] P. Ostalczyk, "Equivalent descriptions of a discrete-time fractional-order linear system and its stability domains", *Int. J. Applied Mathematics and Computer Science* 22 (3), 533–538 (2012).

[25] M. Busłowicz and T. Kaczorek, "Simple conditions for practical stability of linear positive fractional discrete-time linear systems", *Int. J. Applied Mathematics and Computer Science* 19 (2), 263–269 (2009).

[26] T. Kaczorek, "Practical stability of positive fractional discrete-time systems", *Bull. Pol. Ac.: Tech.* 56 (4), 313–317 (2008).

[27] T. Kaczorek, *Selected Problems of Fractional Systems Theory*, Springer, Berlin, 2011.

[28] T. Kaczorek, "Practical stability and asymptotic stability of positive fractional 2D linear systems", *Asian J. Control* 12 (2), 200–207 (2010).

[29] E.N. Gryazina, B.T. Polyak, and A.A. Tremba, "*D*-decomposition technique state-of-the-art", *Automation and Remote Control* 69 (12), 1991-2026 (2008).

[30] J. Ackermann, *Sampled-data Control Systems*, Springer, Berlin, 1985.

[31] R. Hilfer, *Application of Fractional Calculus in Physics*, World Scientific, Singapore 2000.

[32] R. Stanisławski and K. J. Latawiec, "Stability analysis for discrete-time fractional-order LTI state-space systems. Part I: New necessary and sufficient conditions for asymptotic stability", *Bull. Pol. Ac.: Tech.*, 61 (2), 353–361 (2013).