1 Задача 4.1

Если некоторое выпуклое трехмерное тело спроектировать на плоскость, характеризуемую нормалью n, то площадь получившейся проекции будет равна S(n). Выразите среднее по направлениям нормали $< S(\vec{n}) >_{\vec{n}}$ через интегральные характеристики тела (например, такие, как объем, площадь поверхности, ее средняя кривизна, наибольшее или наименьшее сечение, и т.п.)

Решение(неоконченное)

$S(\vec{n}) = \int_D \vec{n} d\vec{S} = \int_{D^*} \vec{n^*} d\vec{S} = S(\vec{n^*})$, где $\vec{n^*} = -\vec{n}$,

а D и D^* – противоположные части поверхности тела

$2S(\vec{n}) = S(\vec{n}) + S(-\vec{n}) = \oint \vec{n} d\vec{S} = \int_V div(\vec{n}) dV$

$\vec{n} =< \cos\theta,\ \sin\phi\cos\theta,\ -\cos\phi\sin\theta >$

$\vec{\nabla} =< \frac{\partial}{\partial r},\ \frac{1}{r}\frac{\partial}{\partial\theta},\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} >$

$\vec{\nabla}\cdot\vec{n} = \frac{1}{r}\frac{\partial}{\partial\theta}\sin\phi\cos\theta - \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\cos\phi\sin\theta = \frac{\sin\phi}{r}(\cos\theta + 1)$

$\int_V div(\vec{n}) dV = \sin\phi(\cos\theta + 1)\iiint r\sin\tilde{\theta} dr d\tilde{\phi} d\tilde{\theta} = \frac{\sin\phi(\cos\theta+1)}{2}\iint r^2(\tilde{\theta},\tilde{\phi})\sin\tilde{\theta} d\tilde{\phi} d\tilde{\theta}$

$I_{2n-1} = \iint r^2(\tilde{\theta},\tilde{\phi})\sin^{2n-1}\tilde{\theta}\, d\tilde{\phi}\, d\tilde{\theta} = \iint r^2(\tilde{\theta},\tilde{\phi})\sin^{2n+1}\tilde{\theta}\, d\tilde{\phi}\, d\tilde{\theta} +$

$+2\iint \frac{1}{2}\tilde{r}^2(\tilde{\theta},\tilde{\phi})\sin^{2n-1}\tilde{\theta}\, d\tilde{\phi}\, d\tilde{\theta} = I_{2n+1} + 2\int \sin^{2n-1}\tilde{\theta}\, S(\tilde{\theta})\, d\tilde{\theta}$, где $S(\tilde{\theta})$ - сечение

на уровне θ

$I_{2n+1} = I_{2n-1} - 2\int \sin^{2n-1}\tilde{\theta}\, S(\tilde{\theta})\, d\tilde{\theta} = I_{2n-3} - 2\int(\sin^{2n-1}\tilde{\theta} + \sin^{2n-3}\tilde{\theta})\, S(\tilde{\theta})\, d\tilde{\theta}$

$I_{2n+1} = I_1 - 2\int(\sin^{2n-1}\tilde{\theta} + \sin^{2n-3}\tilde{\theta} + \cdots + \sin^1\tilde{\theta}) S(\tilde{\theta}) d\tilde{\theta} = I_1 - 2\int \sin\tilde{\theta}\frac{1-\sin^{2n}\tilde{\theta}}{1-\sin^2\tilde{\theta}} S(\tilde{\theta}) d\tilde{\theta}$

$\lim\limits_{n\to\infty} I_{2n+1} = I_1 - 2\int \sin\tilde{\theta}\lim\limits_{n\to\infty}\frac{1-\sin^{2n}\tilde{\theta}}{1-\sin^2\tilde{\theta}}\, S(\tilde{\theta})\, d\tilde{\theta}$

$0 = I_1 - 2\int \frac{\sin\tilde{\theta}}{\cos^2\tilde{\theta}}\, S(\tilde{\theta})\, d\tilde{\theta} \Rightarrow I_1 = 2\int \frac{\sin\tilde{\theta}}{\cos^2\tilde{\theta}}\, S(\tilde{\theta})\, d\tilde{\theta} = 2\int S(\tilde{\theta})\, d\sec\tilde{\theta}$

$I_1 = 2[S(\tilde{\theta})\sec\tilde{\theta}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int \sec\tilde{\theta}\, dS(\tilde{\theta})] = 2[0 - \int \sec\tilde{\theta}\, dS(\tilde{\theta})] = -2\int \sec\tilde{\theta}\, dS(\tilde{\theta})$

$< S(\vec{n}) >_{\vec{n}} = \frac{1}{4\pi}\int S(\vec{n})\, d\Omega = \frac{1}{4\pi}\iint S(\vec{n})\, d\phi\, d\theta =$

$= \frac{1}{4\pi}\cdot\frac{1}{2}\iint \frac{|\sin\phi|(\cos\theta+1)}{2}\, d\phi\, d\theta \cdot 2\int S(\tilde{\theta})\, d\sec\tilde{\theta} = \frac{2\cdot 4\cdot(\pi+2)}{4\pi\cdot 2\cdot 2} 2\int S(\tilde{\theta})\, d\sec\tilde{\theta}$

$< S(\vec{n}) >_{\vec{n}} = \frac{\pi+2}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} S(\tilde{\theta})\, d\sec\tilde{\theta}$