$$S=\left(\frac{1}{2}\right)^3\left(1+\frac{1}{2}\right)+\left(\frac{1\cdot 3}{2\cdot 4}\right)^3\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+...=\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^3 H_{2n}=$$

$$=\sum_{n=1}^{\infty}\left[\frac{(1/2)_n}{(1)_n}\right]^3 H_{2n}$$

$$H_{2n}=\sum_{n=1}^{2n}\frac{1}{k}=\sum_{n=1}^{n}\frac{1}{2k}+\sum_{n=1}^{n}\frac{1}{2k-1}=\frac{1}{2}\sum_{n=1}^{n}\frac{1}{k}+\frac{1}{k-\frac{1}{2}}$$

$$\partial_c\Big|_{c=\frac{1}{2}}\frac{(c)_n}{(\frac{3}{2}-c)_n}=\frac{(c)_n}{(\frac{3}{2}-c)_n}\left(\frac{1}{c}+...+\frac{1}{c+n-1}+\frac{1}{\frac{3}{2}-c}+...+\frac{1}{\frac{1}{2}-c+n}\right)\Big|_{c=\frac{1}{2}}=$$

$$=\frac{(1/2)_n}{(1)_n}\left(\frac{1}{\frac{1}{2}}+...+\frac{1}{n-\frac{1}{2}}+\frac{1}{1}+...+\frac{1}{n}\right)=\frac{(1/2)_n}{(1)_n}\,2H_{2n}$$

$$S=\frac{1}{2}\partial_c\Big|_{c=\frac{1}{2}}\sum_{n=1}^{\infty}\frac{(1/2)_n^2(c)_n}{(1)_n^2(3/2-c)_n}=\frac{1}{2}\partial_c\Big|_{c=\frac{1}{2}}{}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},c\\ 1,\frac{3}{2}-c\end{matrix};1\right)$$

$${}_3F_2\left(\begin{matrix}a,b,c\\ 1+a-b,1+a-c\end{matrix};1\right)=\frac{\Gamma(1+\frac{a}{2})\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)\Gamma(1+a-b-c)}$$

$$a=b=\frac{1}{2}$$

$$S=\frac{1}{2}{}_3F_2\left(\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1\end{matrix};1\right)\Big(-\psi(1)-\psi(1/4)+\psi(3/4)+\psi(1/2)\Big)=\frac{\pi}{2\Gamma^4\left(\frac{3}{4}\right)}(\pi-2\ln 2)$$