Notebooks of

Srinivasa Ramanujan

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VOLUME II

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PREFACE

The publication of this facsimile edition of the manuscripts of the late Srinivasa Ramanujan, in two volumes, has been made possible by the co-operative effort of three institutions : the University of Madras, the Tata Institute of Fundamental Research, and the Sir Dorabji Tata Trust. The suggestion to bring out this publication was made by Professor K. Chandrasekharan to Dr. H. J. Bhabha, Director of the Tata Institute of Fundamental Research, who got in touch with Dr. A. L. Mudaliar, Vice-Chancellor of the University of Madras, and Professor R. Choksi, Managing Trustee of the Sir Dorabji Tata Trust. The manuscripts were readily made available by the University, and the funds by the Trust. Professor Chandrasekharan has been in charge of the arrangements for the compilation, production, and distribution of the edition. Mr. S. Ramu and Mr. R. Subbu of the Commercial Printing Press have done the difficult work of reproducing the manuscripts.

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NOTEBOOK 2

2	2	144	110880
3	4	160	166320
4	6	168	221760
6	12	180	277200
8	24	192	332640
9	36	200	498960
10	48	216	554400
12	60	224	665280
16	120	240	720720
18	180	256	1081080
20	240	288	1441440
24	360	320	2162160
30	720	336	2882880
32	840	360	3603600
36	1260	384	4324320
40	1680	400	6486480
48	2520	432	7207200
60	5040	448	8648640
64	7560	480	10810800
72	10080	504	14414400
80	15120	512	17297280
84	20160	576	21621600
90	25200	600	32432400
96	27720	640	36756720
100	45360	672	43243200
108	50400	720	61261200
120	55440	768	73513440
128	83160	800	110270160

864	122522400	
896	147026880	
960	183783600	
1008	245044800	
1024	294053760	
1152	367567200	
1200	551350800	
1280	698377680	
1344	735134400	
1440	1102701600	
1536	1396755360	
1600	2095133040	
1680	2205403200	
1728	2327925600	
1792	2793510720	
1920	3491888400	
2016	4655851200	
2048	5587021440	
2304	6983776800	
2400	10475665200	
2688	13967553600	
2880	20951330400	
3072	27935107200	
3360	41902660800	
3456	48886437600	
3584	64250746560	
3600	73329656400	
3840	80313433200	
4032	97772875200	
4096	128501493120	57
4320	146659312800.	

If $p^3+q^3+r^3=s^3$

and $\begin{cases} m = (s+q)\sqrt{\dfrac{s-q}{r+p}} \text{ and} \\ n = (r-p)\sqrt{\dfrac{r+p}{s-q}} \end{cases}$

then

$$(pa^2+mab-rb^2)^3+(qa^2-nab+sb^2)^3$$
$$+(ra^2-mab-pb^2)^3=(sa^2-nab+qb^2)^3.$$

$$\frac{\chi^3(x)}{\chi(x^3)} = 1+3x\cdot\frac{\psi(-x^9)}{\psi(-x)}.$$

$$\frac{\chi^5(x)}{\chi(x^5)} = 1+5x\cdot\left\{\frac{\psi(-x^5)}{\psi(-x)}\right\}^2.$$

CHAPTER [illegible]

[illegible]agic squares can be constructed by combining two sets of letters so that the same letter may not appear in a row, a column or a corner. For example if we want to construct a square containing n rows and n columns, we should take two sets of n letters A, B, C, D, E &c and P, Q, R, S, T, &c, combine them as A+P, A+Q, A+R &c, B+P, B+Q, B+R &c, C+P, C+Q, C+R &c, &c &c and arrange them in such a way that any of these, say A, may not appear in the same row, column or corner.

Now we have algebraically constructed a square and the sum of the figures in the rows and columns will be equal if we give any value to the letters; yet the same figure may likely appear again owing to arithmetical values. This difficulty is removed from the following truths:

A+M differs from A+N as B+M from B+N, as C+M from C+N

Cor. 1. If A+P, A+Q, A+R &c are in A.P. then B+P, B+Q, B+R &c are also in A.P.

Cor. 2. If the values of A+P, A+Q, A+R, A+S &c are known, then from the value of B+R, those of B+Q, B+R, B+S &c are also known.

[illegible] we should not give separate values to A, B, C &c and to P, Q, R &c but we should give [illegible] to A+P, A+Q, &c.

Ex. 1. The values of $A+P$, $B+R$, $C+P$, $D+P$ are 8, 10, 11 and 14; and $C+R = 25$; find $A+R$, $B+R$ & $D+R$

$A+R = 22$; $B+R = 24$ & $D+R = 28$.

2. $A+P$, $A+Q$, $A+S$ are 5, 7 & 17 respectively and $B+Q = 23$ & $B+R = 26$; find $A+R$, $B+P$ & $B+S$

$A+R = 10$; $B+P = 21$ & $B+S = 33$.

2. To construct a square containing three rows

Let r be a row or a column, m middle r... or column and c a corner & x the mi[dd]le fig[ure]

(i) If r, m & c are different,

write $\frac{1}{3}(m_1 + m_2 + c_1 + c_2 - S)$ in the middle S being the whole Sum and supply the other figures.

Sol. $m_1 + m_2 + c_1 + c_2 = S + 3x$

$\therefore x = \frac{1}{3}(m_1 + m_2 + c_1 + c_2 - S)$

As this is the only relation existing between c, r & m we may supply the other figures as we choose.

(ii). If the columns and rows are equal and the corners different,

write $\frac{c_1 + c_2 - r}{3}$ in the middle.

Sol. By I 2(i) $x = \frac{1}{3}(m_1 + m_2 + c_1 + c_2 - S)$

But here $m_1 = m_2 = r$ & $S = 3r$

$\therefore x = \frac{c_1 + c_2 - r}{3}$

When the rows, columns and corners are all equal write $\frac{r}{3}$ in the middle

Sol $x = \dfrac{c_1 + c_2 - r}{3}$. But $c_1 = c_2 = r$; $\therefore x = \frac{r}{3}$.

Cor 1. The numbers in the two corners and in the middle row and column are in A.P.

Sol. Since the 2nd is one-third of the sum, the 1st & the 3rd are together twice the 2nd and consequently they are in A.P.

Ex. 1. construct a square when (i) $r = m = c = 15$

(ii) $r = m = c = 27$ and all no.s are odd

6	1	8
7	5	3
2	9	4

15	1	11
5	9	13
7	17	3

2. (i) $r = m = c = 36$ and all are even

(ii) $r = m = c = 63$ & all are multiples of 3.

14	4	18
16	12	8
6	20	10

24	9	30
27	21	15
12	33	18

N.B. The solution fails when the given sum is not a multiple of 3.

4. To construct a square for A+B+C+P+Q+R.

C+Q	A+P	B+R
A+R	B+Q	C+P
B+P	C+R	A+Q

	∧	∨	
∧	∨	×	∧
∨	×	∧	∨
×	∧	∨	

N.B. In order that the two corners may satis the given condition A, B, C must be in A and so also P, Q, R must be in A.P.

Ex 1. Neglecting only one corner construct a squ

(i) for 18

(ii) for 31 when all are odd.

10	2	7
4	6	9
5	11	3

14	5	12
7	11	13
10	15	6

2. (i) The corners are 16 & 19 and the rest 20.

(ii) The corners are 15 & 19, the columns 16, 17, 12 & the rows 6, 21 & 18.

10	2	8
4	5	11
6	13	1

1	2	3
8	9	4
7	6	5

To construct an oblong containing 3 rows and 4 columns.

$$A+C=2B+3D$$

A	$C+D$	$A+2D$	$C+3D$
$B+6D$	$B+4D$	$B+2D$	B
C	$A+D$	$C+2D$	$A+3D$

∨	∧	∨	∧
×	×	×	×
∧	∨	∧	∨

Ex. Construct an oblong (i) when the average is 8 (ii) when it is 15 and all numbers are odd.

1	13	3	15
11	9	7	5
12	2	14	4

1	25	5	29
21	17	13	9
23	3	27	7

6. To construct a square containing 4 rows and 4 columns.

i. When the corners, columns and rows are all different, arrange the middle four so that their sum may be equal to half the difference between the whole sum and the sum of the corners, the middle rows and the middle columns

ii. When the rows, columns & corners are equal

A B C D   P Q R S
D C B A   R S P Q

Add these two as $A+P$, $B+Q$ &c and fill up the other two rows. Or we may construct as the oblong in I 5.

A+P	D+S	C+Q	B+R
C+R	B+Q	A+S	D+P
B+S	C+P	D+R	A+Q
D+Q	A+R	B+P	C+S

$A+D = B+C$, $P+S = Q+R$

A+P	D+Q	D+R	A+S
B+S	C+R	C+Q	B+P
C+S	B+R	B+Q	C+P
D+P	A+Q	A+R	D+S

N.B. If $A+D = B+C$ & $P+R = Q+S$ the extreme middle four in the 1st sq. also satisfy the given condition

Ex. 1. Construct for 34 and 35.

1	14	11	8
12	7	2	13
6	9	16	3
15	4	5	10

1	14	15	4
8	11	10	5
12	7	6	9
13	2	3	16

1	15	11	8
12	7	2	14
6	9	17	3
16	4	5	10

2. Construct two different squares for 66.

1	30	27	8
28	7	2	29
6	25	32	3
31	4	5	26

9	22	19	16
20	15	10	21
14	17	24	11
23	12	13	18

3. Construct two different squares of 3 rows for 60.

28	1	31
23	20	17
9	39	12

25	3	32
27	20	13
8	37	15

If m is a multiple of n then a square of m rows can be formed of different squares of n rows.
Exception:– The central numbers in the squares of 3 rows are not different and consequently the square of 6 rows cannot be formed by the above method; however a regular square of 6 rows can be formed by making the corners of the three-rowed squares different.
If m is a multiple of 4 it may also be constructed as in I 6 (ii) second square.

1	62	59	8	9	54	51	16
60	7	2	61	52	15	10	53
6	57	64	3	14	49	56	11
63	4	5	58	55	12	13	50
17	46	43	24	25	38	35	32
44	23	18	45	36	31	26	37
22	41	48	19	30	33	40	27
47	20	21	42	39	28	29	34

1	58	59	4	5	62	63	8
16	55	54	13	12	51	50	9
24	47	46	21	20	43	42	17
25	34	35	28	29	38	39	32
33	26	27	36	37	30	31	40
48	23	22	45	44	19	18	41
56	15	14	53	52	11	10	49
57	2	3	60	61	6	7	64

8. To construct a square of odd rows & columns

A, B, C, D, E, F, G, H &c P, Q, R, S, T, U, V, W &c
A, B, C, D, E, F &c R, S, T, U, V, W, &c
A, B, C, D &c T, U, V, W &c
A, B &c V, W, &c

Thus arranging the letters and adding the two sets a square of any number of odd rows can be formed and we can find many ways of constructing a square and the peculiarities are common to all the odd squares.

A+P	E+R	D+T	C+Q	B+S
C+T	B+Q	A+S	E+P	D+R
E+S	D+P	C+R	B+T	A+Q
B+R	A+T	E+Q	D+S	C+P
D+Q	C+S	B+P	A+R	E+T

D+Q	E+S	A+P	B+R	C+T
E+R	A+T	B+Q	C+S	D+P
A+S	B+P	C+R	D+T	E+[illegible]
B+T	C+Q	D+S	E+P	A+R
C+P	D+R	E+T	A+Q	B+[illegible]

N.B. In the 2nd Square A+B+D+E must be equal

Ex. 1. Construct a 5 rowed Square for 65 and 66.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

1	24	20	12	9
15	7	4	22	18
25	16	13	10	2
8	5	23	19	11
17	14	6	3	26

2. Construct a seven rowed Square for 170 & 17[illegible]

1	26	45	15	41	12	30
38	9	34	5	23	42	19
27	46	16	35	13	31	2
10	28	6	24	43	20	39
47	17	36	14	32	3	21
29	7	25	44	14	40	11
18	37	8	33	4	22	48

1	49	41	33	25	17	9
18	10	2	43	42	34	26
35	27	19	11	3	44	36
45	37	29	28	20	12	4
13	5	46	38	30	22	21
23	15	14	6	47	39	31
40	32	24	16	8	7	48

$$\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\frac{1}{n+4}+\cdots\cdots+\frac{1}{2n}$$
$$=\frac{n}{2n+1}+\frac{1}{2^3-2}+\frac{1}{4^3-4}+\frac{1}{6^3-6}+\cdots+\frac{1}{(2n)^3-2n}$$

Sol. $\frac{1}{(2n)^3-2n}=\frac{1}{2}\cdot\frac{1}{2n-1}-\frac{1}{2n}+\frac{1}{2(2n+1)}$

$\therefore$ Right Side $=\frac{1}{2}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots\frac{1}{2n-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\cdots\frac{1}{2n}\right)$
$+\frac{1}{2}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n+1}\right)+\frac{n}{2n+1}-\frac{1}{2}$.

$=\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2n}\right)$
$=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}$
$=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}\right)$
$=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$.

Cor. $2\log_e 2=1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\frac{2}{6^3-6}+$ &c

Sol. By making n infinite in the above theorem we get $\log_e 2=\frac{1}{2}+\frac{1}{2^3-2}+\frac{1}{4^3-4}+$ &c

Ex. Shew that $\frac{n-1}{n+1}+\frac{n-2}{n+2}+\cdots\cdots+\frac{n-n}{n+n}$
$$=2n\left\{\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\cdots\cdots+\frac{1}{(2n-1)2n(2n+1)}\right\}-\frac{n}{2n+1}.$$

Sol. We have by II 1,
$\frac{1}{n+1}+\frac{1}{n+2}+\cdots\cdots+\frac{1}{2n}=\frac{n}{2n+1}+\frac{1}{1.2.3}+\frac{1}{3.4.5}+\cdots\cdots+\frac{1}{(2n-1)2n(2n+1)}$; multiplying both sides by $2n$ we have $\frac{2n}{n+1}+\frac{2n}{n+2}+\cdots+\frac{2n}{2n}=\frac{2n^2}{2n+1}+2n\left\{\frac{1}{1.2.3}+\frac{1}{3.4.5}+\text{ \&c to } n \text{ terms}\right\}$
Subtracting 1 from each term in the left and n from the right we get the result.

10.

2. $\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots\cdots+\frac{1}{3n+1}$

$=1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\frac{2}{9^3-9}+\cdots+\frac{2}{(3n)^3-3n}$.

Sol. As in II 1.

Cor. $\log_e 3=1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\frac{2}{9^3-9}+$ &c

Sol. R.S. $=\frac{1}{n+1}+\frac{1}{n+2}+\cdots\frac{1}{3n+1}$ when $n=\infty$

~~writing~~ $\frac{1}{dx}$ for n R.S. $=\frac{dx}{1+dx}+\frac{dx}{1+2dx}+\cdots$

$+\frac{dx}{3\cancel{+3dx}}=\int_0^3\frac{dx}{x}=\log_e 3.$

3. $\tan^{-1}\frac{1}{n+1}+\tan^{-1}\frac{1}{n+2}+\cdots\cdots+\tan^{-1}\frac{1}{3n+1}$

$=\tan^{-1}1+\tan^{-1}\frac{10}{5.8}+\tan^{-1}\frac{20}{14.35}+\cdots+\tan^{-1}\frac{18?}{(3n^2+2)(?}$

Sol. As in II 2.

4. $\left\{\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right\}+\left\{\frac{1}{2n+1}+\frac{1}{2n+3}+\cdots+\frac{1}{4n+1}\right.$

$=1+\frac{2}{4^3-4}+\frac{2}{8^3-8}+\cdots\cdots+\frac{2}{(4n)^3-4n}$.

$=\left(1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{1}{4n+1}\right)+\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{1}{2n}\right)$.

Sol. By proceeding as in II 1, R.S $=\Sigma\frac{1}{4n+1}-\frac{1}{2}\Sigma\frac{1}{2}$

$-\frac{1}{2}\Sigma\frac{1}{n}=\Sigma\frac{1}{4n+1}-\Sigma\frac{1}{n}-\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right.$

$=\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{4n+1}\right)-\left(\frac{1}{2n+2}+\frac{1}{2n+4}+\cdots+\frac{1}{4n}\right.$

$=\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots\frac{1}{2n}\right)+\left(\frac{1}{2n+1}+\frac{1}{2n+3}+\cdots+\frac{1}{4n+1}\right)$.

Again $\Sigma\frac{1}{4n+1}-\frac{1}{2}\Sigma\frac{1}{2n}-\frac{1}{2}\Sigma\frac{1}{n}=\Sigma\frac{1}{4n+1}-\Sigma\frac{1}{2n}$

$+\frac{1}{2}\Sigma\frac{1}{2n}-\frac{1}{2}\Sigma\frac{1}{n}=\left(1+\frac{1}{2}+\cdots\frac{1}{4n+1}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\cdots\right.$

$+\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{2n}\right)-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)$

$=\left(1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{1}{4n+1}\right)+\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{1}{2n}\right)$.

Cor. $\frac{3}{2}\log_e 2 = 1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \frac{2}{12^3-12} + \&c$

5. $\frac{2}{3}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{6n+1}\right)$

$= 1 + \frac{2}{6^3-6} + \frac{2}{12^3-12} + \frac{2}{18^3-18} + \cdots + \frac{2}{(6n)^3-6n}.$

Sol. By proceeding as in II 1. the Sum is

$\Sigma \frac{1}{6n+1} - \frac{1}{2}\Sigma\frac{1}{3n} - \frac{1}{3}\Sigma\frac{1}{2n} - \frac{1}{6}\Sigma\frac{1}{n} = L.S.$

Cor. $\frac{1}{2}\log_e 3 + \frac{1}{3}\log_e 4 = 1 + \frac{2}{6^3-6} + \frac{2}{12^3-12} + \frac{2}{18^3-18} + \&c$

N.B. $1 + \frac{2}{a^3-a} + \frac{2}{(2a)^3-2a} + \cdots + \frac{2}{(an)^3-(an)}$

cannot be expressed as in II 2. for all values of a except 2, 3, 4 and 6 though it can be summed up for all values of a when n becomes infinite. See chapter

Ex. 1. $\frac{1}{4}\log_e 2 = \frac{1}{2^3-2} + \frac{1}{6^3-6} + \frac{1}{10^3-10} + \&c$

2. $\log_e 2 = 1 - \frac{2}{2^3-2} + \frac{2}{4^3-4} - \frac{2}{6^3-6} + \&c$

3. $2\left\{1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \cdots + \frac{2}{(4n)^3-4n}\right\}$

$= \left\{1 + \frac{2}{2^3-2} + \frac{2}{4^3-4} + \cdots + \frac{2}{(4n)^3-4n}\right\} + \frac{1}{(4n+1)(4n+2)}$

$+ \frac{1}{2}\left\{1 + \frac{2}{2^3-2} + \frac{2}{4^3-4} + \cdots + \frac{2}{(2n)^3-2n}\right\}.$

4. $1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \cdots + \frac{2}{(4n)^3-4n}$

$= \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) + \left(\frac{1}{2n+1} + \frac{1}{2n+2} + \cdots + \frac{1}{4n+1}\right)$

5. $\frac{1}{3^3-3} + \frac{1}{9^3-9} + \frac{1}{15^3-15} + \&c = \frac{1}{4}\log_e 3 - \frac{1}{3}\log_e 2.$

6. $\frac{4}{3}\log_e 2 = 1 - \frac{2}{3^3-3} + \frac{2}{6^3-6} - \frac{2}{9^3-9} + \&c$

12.

7. $2\left\{1+\frac{2}{6^3-6}+\frac{2}{12^3-12}+\cdots+\frac{2}{(6n)^3-6n}\right\}$

$+\frac{1}{3}\left\{1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\cdots+\frac{2}{(2n)^3-2n}\right\}$

$=\left\{1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\cdots+\frac{2}{(3n)^3-3n}\right\}$

$+\left\{1+\frac{2}{2^3-2}+\frac{2}{4^3-4}+\cdots+\frac{2}{(6n)^3-6n}\right\}$

$+\frac{2}{(6n+1)(6n+2)(6n+3)}$,

8. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}+\cdots+\tan^{-1}\frac{1}{13}$

$=\frac{\pi}{2}+2\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{49}+\tan^{-1}\frac{3}{232}+\tan^{-1}\frac{4}{715}$

9. $2\left(\tan^{-1}\frac{1}{n+1}+\tan^{-1}\frac{1}{n+2}+\cdots+\tan^{-1}\frac{1}{2n+1}\right)$

$=\tan^{-1}\frac{n+1}{n}+\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{4}{137}+\tan^{-1}\frac{6}{667}$

$+\cdots+\tan^{-1}\frac{2n}{8n^4+2n^2+1}+$

$2\left\{\tan^{-1}\frac{1}{1\cdot 7}+\tan^{-1}\frac{1}{2\cdot 19}+\cdots+\tan^{-1}\frac{1}{n(4n^2+3)}\right\}$

10. $\tan^{-1}\frac{1}{n+1}+\tan^{-1}\frac{1}{n+2}+\cdots+\tan^{-1}\frac{1}{2n}$

$+\tan^{-1}\frac{1}{2n+1}+\tan^{-1}\frac{1}{2n+3}+\cdots+\tan^{-1}\frac{1}{4n+1}$

$=\frac{\pi}{4}+\tan^{-1}\frac{9}{53}+\tan^{-1}\frac{18}{599}+\cdots+\tan^{-1}\frac{9n}{32n^4+26n^2-1}$

$+\tan^{-1}\frac{4}{137}+\tan^{-1}\frac{8}{2081}+\cdots+\tan^{-1}\frac{2n}{128n^4+8n^2+1}$

6. If $A_n=3^n(n+\frac{1}{2})-\frac{1}{2}$, then

$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{A_n}$

$=n\left\{1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\frac{2}{9^3-9}+\cdots+\frac{2}{(3n)^3-3n}\right\}$

$$+(r-1)\left\{\frac{2}{(3A_0+3)^3-(3A_0+3)}+\frac{2}{(3A_0+6)^3-(3A_0+6)}+\cdots+\frac{2}{(3A_1)^3-3A_1}\right\}$$

$$+(r-2)\left\{\frac{2}{(3A_1+3)^3-(3A_1+3)}+\frac{2}{(3A_1+6)^3-(3A_1+6)}+\cdots+\frac{2}{(3A_2)^3-3A_2}\right\}$$

+ &c to r terms.

Sol. By II 2. we have

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}=1+\frac{2}{3^3-3}+\frac{2}{6^3-6}+\cdots+\frac{2}{(3n)^3-3n}$$

$$\frac{1}{3n+2}+\frac{1}{3n+3}+\cdots+\frac{1}{9n+4}=1+\frac{2}{3^3-3}+\cdots+\frac{2}{(9n+3)^3-(9n+3)}$$

$$\frac{1}{9n+5}+\frac{1}{9n+6}+\cdots+\frac{1}{27n+13}=1+\frac{2}{3^3-3}+\cdots+\frac{2}{(27n+12)^3-(27n+12)}$$

writing thus r times and then adding up all the terms we can get the result.

Cor. $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{\frac{3^n-1}{2}}$

$$=n+(n-1)\left(\frac{2}{3^3-3}\right)+(n-2)\left(\frac{2}{6^3-6}+\frac{2}{9^3-9}+\frac{2}{12^3-12}\right)$$

$$+(n-3)\left(\frac{2}{15^3-15}+\frac{2}{18^3-18}+\cdots+\frac{2}{39^3-39}\right)+\&c$$

to n terms.

N.B. The above theorems are very useful in finding $\Sigma\frac{1}{n}$. If a_1 & a_n are very great & a_1, a_2, a_3 &c are in A.P., then the approximate value of $\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}=\frac{2n}{a_1+a_n}$.

Ex. 1. $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{13}$

$$=3+\frac{1}{6}+\frac{1}{105}+\frac{1}{360}+\frac{1}{858}.$$

2. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{1000}=7\frac{1}{2}$ very nearly

7. $\tan^{-1}\frac{2}{(n+1)^2}+\tan^{-1}\frac{2}{(n+3)^2}+\tan^{-1}\frac{2}{(n+5)^2}+$ &c to r term

$=\tan^{-1}\frac{2r}{n^2+2nr+1}$.

Sol. $\tan^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{n+2}=\tan^{-1}\frac{2}{(n+1)^2}$

$\therefore$ L.S $=\tan^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{n+2r}=\tan^{-1}\frac{2r}{n^2+2nr+1}$.

Cor. $\tan^{-1}\frac{2}{(n+1)^2}+\tan^{-1}\frac{2}{(n+3)^2}+\tan^{-1}\frac{2}{(n+5)^2}+$ &c $=\tan^{-1}\frac{1}{n}$.

Ex. 1. $\tan^{-1}\frac{2}{(n+1)^2}+\tan^{-1}\frac{2}{(n+2)^2}+\tan^{-1}\frac{2}{(n+3)^2}+$ &c $=\tan^{-1}\frac{2n}{n^2}$

Sol. $\tan^{-1}\frac{2}{(n+1)^2}+\tan^{-1}\frac{2}{(n+3)^2}+$ &c $=\tan^{-1}\frac{1}{n}$

$\therefore\ \tan^{-1}\frac{2}{(n+2)^2}+\tan^{-1}\frac{2}{(n+4)^2}+$ &c $=\tan^{-1}\frac{1}{n+1}$.

$\therefore\ \tan^{-1}\frac{2}{(n+1)^2}+\tan^{-1}\frac{2}{(n+2)^2}+$ &c $=\tan^{-1}\frac{2n+1}{n^2+n-1}$.

N. B. If $n<\frac{\sqrt{5}-1}{2}$ add π to R.S.

2. $\tan^{-1}\frac{2}{(n+1)^2}-\tan^{-1}\frac{2}{(n+2)^2}+\tan^{-1}\frac{2}{(n+3)^2}-$ &c $=\tan^{-1}\frac{1}{n^2+n+1}$

3. $\tan^{-1}\frac{1}{2(n+1)^2}+\tan^{-1}\frac{1}{2(n+2)^2}+\tan^{-1}\frac{1}{2(n+3)^2}+$ &c $=\tan^{-1}\frac{1}{2n+1}$

4. $\frac{3\pi}{4}=\tan^{-1}\frac{2}{1^2}+\tan^{-1}\frac{2}{2^2}+\tan^{-1}\frac{2}{3^2}+$ &c

5. $\frac{\pi}{4}=\tan^{-1}\frac{1}{2.1^2}+\tan^{-1}\frac{1}{2.2^2}+$ &c $=\tan^{-1}\frac{2}{1^2}-\tan^{-1}\frac{2}{2^2}+$ &c

6. $\frac{\pi}{8}=\tan^{-1}\frac{1}{(1+\sqrt{2})^2}+\tan^{-1}\frac{1}{(1+2\sqrt{2})^2}+\tan^{-1}\frac{1}{(1+3\sqrt{2})^2}+$ &c

7. $\frac{\pi}{2}=\tan^{-1}\frac{8}{(1+\sqrt{5})^2}+\tan^{-1}\frac{8}{(3+\sqrt{5})^2}+\tan^{-1}\frac{8}{(5+\sqrt{5})^2}+$ &c

8. $\frac{\pi}{2}=\tan^{-1}\frac{2}{1^2}+\tan^{-1}\frac{2}{3^2}+\tan^{-1}\frac{2}{5^2}+$ &c

If $\alpha, \beta, \gamma, \delta$ &c are the roots of the equation $f(x)=0$, then $f(x) = f(0)\left(1-\frac{x}{\alpha}\right)\left(1-\frac{x}{\beta}\right)\left(1-\frac{x}{\gamma}\right)$ &c. Only if the test given at the end of the notebook is true.

$$f'(x) = f(x)\left(\frac{1}{x-\alpha}+\frac{1}{x-\beta}+\frac{1}{x-\gamma}+\&c\right)$$

2. $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}+\&c = -\frac{f'(0)}{f(0)}$.

9. i $\frac{\sin x}{x} = \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)$ &c

ii. $\cos x = \left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right)\left(1-\frac{4x^2}{5^2\pi^2}\right)$ &c

Sol. The roots of the equation $\frac{\sin x}{x}=0$ are $\pm\pi, \pm 2\pi, \pm 3\pi$ &c and those of $\cos x = 0$ are $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ &c. Applying the above theorem we get the result.

Cor. 1. $\frac{e^x-e^{-x}}{2x} = \left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{2^2\pi^2}\right)\left(1+\frac{x^2}{3^2\pi^2}\right)$ &c

2. $\frac{e^x+e^{-x}}{2} = \left(1+\frac{2^2x^2}{\pi^2}\right)\left(1+\frac{2^2x^2}{3^2\pi^2}\right)\left(1+\frac{2^2x^2}{5^2\pi^2}\right)$ &c

Sol. change x to xi in the above result.

3. $\cos\frac{x}{4}+\sin\frac{x}{4} = \left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{5\pi}\right)\left(1-\frac{x}{7\pi}\right)$ &c

4. $\frac{\sin(x+a)}{\sin a} = \left(1+\frac{x}{a}\right)\left(1-\frac{x}{\pi-a}\right)\left(1+\frac{x}{\pi+a}\right)\left(1-\frac{x}{2\pi-a}\right)$ &c

Ex. 1. $\frac{\cos(x+a)}{\cos a} = \left(1+\frac{x}{\frac{\pi}{2}+a}\right)\left(1-\frac{x}{\frac{\pi}{2}-a}\right)\left(1+\frac{x}{\frac{3\pi}{2}+a}\right)$ &c

2. $1+\frac{\sin x}{\sin a} = \left(1+\frac{x}{a}\right)\left(1+\frac{x}{\pi-a}\right)\left(1-\frac{x}{\pi+a}\right)\left(1-\frac{x}{2\pi-a}\right)$ &c.

N.B. If we know the value of $(1+a_1x)(1+a_2x)(1+a_3x)$ then it is possible to find $(1+a_1^n x^n)(1+a_2^n x^n)$ &c

10. $\cot x = \frac{1}{x} - \frac{1}{\pi - x} + \frac{1}{\pi + x} - \frac{1}{2\pi - x} + \frac{1}{2\pi + x} - \&c$

Sol. Equate the coeffts. of x in II.9 Cor 4.

Cor 1. $\tan x = \frac{1}{\frac{\pi}{2} - x} - \frac{1}{\frac{\pi}{2} + x} + \frac{1}{\frac{3\pi}{2} - x} - \frac{1}{\frac{3\pi}{2} + x} + \&c$

2. $\operatorname{cosec} x = \frac{1}{x} + \frac{1}{\pi - x} - \frac{1}{\pi + x} - \frac{1}{2\pi - x} + \&c$

3. $\sec x = \frac{1}{\frac{\pi}{2} - x} + \frac{1}{\frac{\pi}{2} + x} - \frac{1}{\frac{3\pi}{2} - x} - \frac{1}{\frac{3\pi}{2} + x} + \&c$

Sol. $\tan x = \cot\left(\frac{\pi}{2} - x\right)$; $\operatorname{cosec} x = \frac{1}{2}\left(\cot \frac{x}{2} + \tan \frac{x}{2}\right)$ and $\sec x = \operatorname{cosec}\left(\frac{\pi}{2} - x\right)$. Apply the above rules

11. $\tan^{-1}\frac{x}{a} - \tan^{-1}\frac{x}{\pi - a} + \tan^{-1}\frac{x}{\pi + a} - \tan^{-1}\frac{x}{2\pi - a} + \&c$

$= \tan^{-1}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \cot a\right)$

Sol. $L.S = \frac{1}{2i} \log_e \left\{ \frac{1 + \frac{ix}{a}}{1 - \frac{ix}{a}} \cdot \frac{1 - \frac{ix}{\pi - a}}{1 + \frac{ix}{\pi - a}} \&c \right\}$ Apply II.9 Cor 4.

Cor. 1. $\tan^{-1}\frac{x}{a} + \tan^{-1}\frac{x}{\pi - a} - \tan^{-1}\frac{x}{\pi + a} - \tan^{-1}\frac{x}{2\pi - a} + \&c$

$= \tan^{-1}\left(\frac{e^x - e^{-x}}{2} \operatorname{cosec} a\right)$

2. $\tan^{-1}\frac{x}{1} - \tan^{-1}\frac{x}{3} + \tan^{-1}\frac{x}{5} - \&c = \tan^{-1}\left\{\frac{e^{\frac{\pi x}{2}} - 1}{e^{\frac{\pi x}{2}} + 1}\right\}$

3. $\tan^{-1}\frac{x}{1} + \tan^{-1}\frac{x}{3} - \tan^{-1}\frac{x}{5} - \&c = \tan^{-1}\left(\frac{e^{\frac{\pi x}{2}} - 1}{\sqrt{2}\, e^{\frac{\pi x}{4}}}\right)$

Ex. 1. $\tan^{-1}\frac{x}{\frac{\pi}{2}-a} - \tan^{-1}\frac{x}{\frac{\pi}{2}+a} + \tan^{-1}\frac{x}{\frac{3\pi}{2}-a} - \&c$

$= \tan^{-1}(\tanh x \tan h)$

2. $\tan^{-1}\frac{x}{\frac{\pi}{2}-a} + \tan^{-1}\frac{x}{\frac{\pi}{2}+a} - \tan^{-1}\frac{x}{\frac{3\pi}{2}-a} - \&c$

$= \tan^{-1}\left(\frac{\sinh x}{\cos a}\right).$

3. $(1+\frac{1}{1^3})(1+\frac{1}{2^3})(1+\frac{1}{3^3})(1+\frac{1}{4^3})\;\&c = \frac{1}{\pi}\cosh(\pi\cos\frac{\pi}{6})$

Sol. $(1+\frac{1}{n^3}) = (1+\frac{1}{n})(1-\frac{1}{n}+\frac{1}{n^2})$

$= (1+\frac{1}{n})(1-\frac{1}{2n})^2\left\{1+\frac{3}{(2n-1)^2}\right\}$

$\therefore$ L.S $= (\frac{3}{2})^2\cdot\frac{2}{1}\cdot(\frac{3}{4})^2\frac{3}{2}(\frac{5}{6})^2\frac{4}{3}\;\&c \times (1+\frac{3}{1^2})(1+\frac{3}{3^2})(1+\frac{3}{5^2})$

$= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{5}{6}\;\&c\;(1+\frac{3}{1^2})(1+\frac{3}{3^2})(1+\frac{3}{5^2})\;\&c$

$= \frac{1}{\pi}\cosh\frac{\pi\sqrt{3}}{2}.$

4. $(1-\frac{1}{2^3})(1-\frac{1}{3^3})(1-\frac{1}{4^3})(1-\frac{1}{5^3})\;\&c = \frac{\cosh(\pi\cos\frac{\pi}{6})}{3\pi}$

Sol. $(1-\frac{1}{n^3}) = (1-\frac{1}{n})(1+\frac{1}{n}+\frac{1}{n^2})$

$= (1-\frac{1}{n})(1+\frac{1}{2n})^2\left\{1+\frac{3}{(2n+1)^2}\right\}$ & proceed as before

12. To find convergents to a root of the eq.^{n}

$1 = A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \&c$.

If $P_n = A_1P_{n-1} + A_2P_{n-2} + A_3P_{n-3} + \dots + A_{n-1}P_1$ and $P_1 = 1$, then $\frac{P_n}{P_{n+1}}$ approaches x when n becomes greater and greater.

18.

E.g. 1. $x+x^2=1$

$x = \frac{0}{1}, \frac{1}{1} \Big| \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}$, &c

2. $x+x^2+x^3=1$

$x = \frac{0}{1}, \frac{1}{1}, \frac{1}{2} \Big| \frac{2}{4}, \frac{4}{7}, \frac{7}{13}, \frac{13}{24}, \frac{24}{44}$, &c

3. $x+x^3=1$

$x = \frac{0}{1}, \frac{1}{1}, \frac{1}{1} \Big| \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{6}, \frac{6}{9}, \frac{9}{13}, \frac{13}{19}$ &c

4. $2x+x^2+x^3=1$

$x = \frac{0}{1}, \frac{1}{2}, \frac{2}{5} \Big| \frac{5}{13}, \frac{13}{33}, \frac{33}{84}, \frac{84}{214}$ &c

N.B. If $\frac{p}{q}$ & $\frac{r}{s}$ are two Consecutive convergents to x, then we may take $\frac{mp+nr}{mq+ns}$ in a suitable manner equivalent to x.

Ex. 1. Find convergents to $\log_e 2$.

Let $\log_e 2 = x$, then $e^x = 2$

$\therefore 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} +$ &c

$\therefore x = \frac{0}{1}, \frac{1}{1} \Big| \frac{1}{1\frac{1}{2}}, \frac{1\frac{1}{2}}{2\frac{1}{6}}, \frac{2\frac{1}{6}}{3\frac{1}{8}}, \frac{3\frac{1}{8}}{4\frac{61}{120}}$, &c

$= \frac{2}{3}, \frac{9}{13}, \frac{52}{75}, \frac{375}{541}$, &c.

2. If $e^{-x}=x$, shew that the convergents to x are $\frac{1}{2}, \frac{4}{7}, \frac{21}{37}, \frac{148}{261}$ &c.

Sol. $1 = 2x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} +$ &c.

CHAPTER III

1. If $P_0 + P_1 x + P_2 x^2 + P_3 x^3 + \&c = e^x (Q_0 + Q_1 x + Q_2 x^2 + \&c)$

then $P_0 f(0) + P_1 f'(0) + P_2 f''(0) + P_3 f'''(0) + P_4 f^{iv}(0) + \&c$

$= Q_0 f(1) + Q_1 f'(1) + Q_2 f''(1) + Q_3 f'''(1) + Q_4 f^{iv}(1) + \&c.$

Sol. The coeffts. of $f^n(0)$ in both sides are the same as those of x^n in both sides of the first equation which are equal.

Cor. 1. $P_0 + nP_1 x + n(n-1)P_2 x^2 + n(n-1)(n-2)P_3 x^3 + \&c$

$= Q_0 (1+x)^n + nQ_1 x(1+x)^{n-1} + n(n-1)Q_2 x^2 (1+x)^{n-2} + \&c$

Sol. Writing $(1+ax)^n$ for $f(x)$ in the above theorem we get $f(0) = 1$, $f'(0) = na$, $f''(0) = n(n-1)a^2$ &c and $f(1) = (1+a)^n$, $f'(1) = na(1+a)^{n-1}$, $f''(1) = n(n-1)a^2(1+a)^{n-2}$ &c

Cor 2. If $\phi(x) = e^x \psi(x)$, then

$$\phi(0) f(0) + \frac{\phi'(0) f'(0)}{\lfloor 1} + \frac{\phi''(0) f''(0)}{\lfloor 2} + \frac{\phi'''(0) f'''(0)}{\lfloor 3} + \&c$$

$$= \psi(0) f(1) + \frac{\psi'(0) f'(1)}{\lfloor 1} + \frac{\psi''(0) f''(1)}{\lfloor 2} + \frac{\psi'''(0) f'''(1)}{\lfloor 3} + \&c$$

Sol. Write $\frac{\phi^n(0)}{\lfloor n}$ for P_n & $\frac{\psi^n(0)}{\lfloor n}$ for Q_n in III 1.

2. $\frac{x}{n\lfloor 0} + \frac{x^2}{(n+1)\lfloor 1} + \frac{x^3}{(n+2)\lfloor 2} + \frac{x^4}{(n+3)\lfloor 4} + \&c$

$= e^x \left\{ \frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c \right\}$

Sol. 1. $L.S = \frac{1}{x^{n-1}} \left\{ \frac{x^n}{n\lfloor 0} + \frac{x^{n+1}}{(n+1)\lfloor 1} + \frac{x^{n+2}}{(n+2)\lfloor 2} + \&c \right\}$

$= \frac{1}{x^{n-1}} \int x^{n-1} e^x = e^x \left\{ \frac{x}{n} - \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} - \&c \right\}$

Sol. 2. Let $\phi(n) = \frac{x}{n\lfloor 0} + \frac{x^2}{(n+1)\lfloor 1} + \frac{x^3}{(n+2)\lfloor 2} + \&c$

Then $n\phi(n) = x + \frac{n}{n+1}\cdot\frac{x^2}{\underline{|1}} + \frac{n}{n+2}\cdot\frac{x^3}{\underline{|2}} + \&c$

and $x\phi(n+1) = \qquad \frac{x^2}{n+1} + \frac{x^3}{(n+2)\underline{|1}} + \&c$

$\therefore n\phi(n) + x\phi(n+1) = x + x^2 + \frac{x^3}{\underline{|2}} + \&c = xe^x$

$\therefore \phi(n) = e^x\frac{x}{n} - \frac{x}{n}\phi(n+1) = e^x\frac{x}{n} - e^x\frac{x^2}{n(n+1)} + \frac{x^2}{n(n+1)}\phi(n+2)$

&c &c.

Cor. 1. $\frac{f(0)}{n} + \frac{f'(0)}{(n+1)\underline{|1}} + \frac{f''(0)}{(n+2)\underline{|2}} + \&c$

$= \frac{f(1)}{n} - \frac{f'(1)}{n(n+1)} + \frac{f''(1)}{n(n+1)(n+2)} - \&c$

Cor. 2. $\frac{x}{\underline{|1}} + \left(1+\frac{1}{2}\right)\frac{x^2}{\underline{|2}} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^3}{\underline{|3}} + \&c$

$= e^x\left(\frac{x}{1\underline{|1}} - \frac{x^2}{2\underline{|2}} + \frac{x^3}{3\underline{|3}} - \&c\right)$

Sol. By III (1) we have $e^x\left\{\frac{x}{(n+1)\underline{|0}} - \frac{x^2}{(n+2)\underline{|1}} + \&c\right\}$

$= \frac{x}{(n+1)} + \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} + \&c$

Equating the coeff^{ts} of n we get the result.

3. If $\frac{1^n}{\underline{|0}}x + \frac{2^n}{\underline{|1}}x^2 + \frac{3^n}{\underline{|2}}x^3 + \&c = e^x f_n(x)$, then

$\frac{x}{n+1} - \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} - \&c$

$= \frac{f_0(x)}{n} - \frac{f_1(x)}{n^2} + \frac{f_2(x)}{n^3} - \frac{f_3(x)}{n^4} + \&c$

Sol. By III 2. we have $\frac{x}{(n+1)\underline{|0}} + \frac{x^2}{(n+2)\underline{|1}} + \frac{x^3}{(n+3)\underline{|2}} + \&c$

$= e^x\left\{\frac{x}{n+1} - \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} - \&c\right\}$

$= \frac{1}{n}\left(\frac{x}{\underline{|0}} + \frac{x^2}{\underline{|1}} + \frac{x^3}{\underline{|2}} + \&c\right) - \frac{1}{n^2}\left(\frac{x}{\underline{|0}} + \frac{2x^2}{\underline{|1}} + \frac{3x^3}{\underline{|2}} + \&c\right)$

$+ \frac{1}{n^3}\left(\frac{1^2 x}{\underline{|0}} + \frac{2^2 x^2}{\underline{|1}} + \frac{3^2 x^3}{\underline{|2}} + \&c\right) - \&c \ \&c$

$= e^x\left\{\frac{f_0(x)}{n} - \frac{f_1(x)}{n^2} + \frac{f_2(x)}{n^3} - \&c\right\}$.

4. $e^{x(e^a-1)} = 1 + \frac{a}{\lfloor\underline{1}} f_0(x) + \frac{a^2}{\lfloor\underline{2}} f_1(x) + \frac{a^3}{\lfloor\underline{3}} f_2(x) + \&c$

Sol. $e^{xe^a} = 1 + xe^a + \frac{x^2}{\lfloor\underline{2}} e^{2a} + \frac{x^3}{\lfloor\underline{3}} e^{3a} + \&c$. The coeff.^{t} of a^{n+1} is $\frac{1}{\lfloor\underline{n+1}} \left\{ \frac{1^n}{\lfloor\underline{0}} x + \frac{2^n}{\lfloor\underline{1}} x^2 + \frac{3^n}{\lfloor\underline{2}} x^3 + \&c \right\} = \frac{e^x}{\lfloor\underline{n+1}} f_n(x)$

$\therefore e^{xe^a} = e^x \left\{ 1 + \frac{a}{\lfloor\underline{1}} f_0(x) + \frac{a^2}{\lfloor\underline{2}} f_1(x) + \frac{a^3}{\lfloor\underline{3}} f_2(x) + \&c \right\}$

5. $f_n(x) = x \left\{ 1 + n f_0(x) + \frac{n(n-1)}{\lfloor\underline{2}} f_1(x) + \frac{n(n-1)(n-2)}{\lfloor\underline{3}} f_2(x) + \&c \right\}$

Sol. Differentiating both Sides in $\underline{\text{III}}$ 4 with respect to a, $xe^a e^{x(e^a-1)} = f_0(x) + \frac{a}{\lfloor\underline{1}} f_1(x) + \frac{a^2}{\lfloor\underline{2}} f_2(x) + \&c$
$= xe^a \left\{ 1 + \frac{a}{\lfloor\underline{1}} f_0(x) + \frac{a^2}{\lfloor\underline{2}} f_1(x) + \&c \right\}$. Equating the coeff.^{ts} of a^n we get the result.

Cor. The above result may be written thus

$$\begin{array}{ccccc} f_0(x), & f_1(x), & f_2(x), & f_3(x), & f_4(x) \\ & a_0 & b_0 & c_0 & d_0 \\ & & a_1 & b_1 & c_1 \\ & & & a_2 & b_2 \\ & & & & a_3 \end{array}\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\}$$

These are Successive diff.^{ces} a_n being equal to $x f_n(x)$.

6. If $f_n(x) = \phi_1(n) x + \phi_2(n) x^2 + \phi_3(n) x^3 + \ldots + \phi_{n+1}(n) x^{n+1}$

then $\frac{\phi_r(n)}{\lfloor\underline{0}} + \frac{\phi_{r-1}(n)}{\lfloor\underline{1}} + \frac{\phi_{r-2}(n)}{\lfloor\underline{2}} + \&c = \frac{r^n}{\lfloor\underline{r-1}}$.

Sol. $e^x f_n(x) = e^x \left\{ \phi_1(n) x + \phi_2(n) x^2 + \ldots\ldots + \phi_{n+1}(n) x^{n+1} \right\}$

But $e^x f_n(x) = \frac{1^n}{\lfloor\underline{0}} x + \frac{2^n}{\lfloor\underline{1}} x^2 + \frac{3^n}{\lfloor\underline{2}} x^3 + \&c$

Equating the coeff.^{ts} of x^r in both Sides we get the result.

7. $\phi_{r+1}(n) \lfloor\underline{r} = (r+1)^n - r \cdot r^n + \frac{r(r-1)}{\lfloor\underline{2}} (r-1)^n - \frac{r(r-1)(r-2)}{\lfloor\underline{3}} (r-2)^n$
$+ \frac{r(r-1)(r-2)(r-3)}{\lfloor\underline{4}} (r-3)^n - \&c$

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Sol. $f_n(x) = \phi_1(n)\, x + \phi_2(n)\, x^2 + \phi_3(n)\, x^3 + \&c$

$$= e^{-x}\left\{ \frac{1^n}{\lfloor 0} x + \frac{2^n}{\lfloor 1} x^2 + \frac{3^n}{\lfloor 2} x^3 + \&c \right\}$$

Equating the coeff^{ts} of x^{r+1} we can get the result

8. $\phi_r(n+1) = r\,\phi_r(n) + \phi_{r-1}(n)$.

Sol. $\phi_r(n+1) = \frac{1}{\lfloor r-1}\left\{ r^{n+1} - (r-1)(r-1)^{n+1} + \frac{(r-1)(r-2)}{\lfloor 2}(r-2)^{n+1} - \&c \right\}$

$\therefore \phi_r(n+1) - \phi_{r-1}(n) = \frac{1}{\lfloor r-1}\left\{ r.\, r^n - r(r-1)(r-1)^n + \frac{r(r-1)(r-2)}{\lfloor 2} \right.$

$$\left. \times (r-2)^n - \&c \right\} = r\,\phi_r(n).$$

Cor. The above theorem may be written thus

Write under each term the product of the coeff^{t}. and the index of x of that term together with the coeff^{t}. of the preceding one.

$f_0(x) = x$

$f_1(x) = x + x^2$

$f_2(x) = x + 3x^2 + x^3$

$f_3(x) = x + 7x^2 + 6x^3 + x^4$

$f_4(x) = x + 15x^2 + 25x^3 + 10x^4 + x^5$

$f_5(x) = x + 31x^2 + 90x^3 + 65x^4 + 15x^5 + x^6$

$f_6(x) = x + 63x^2 + 301x^3 + 350x^4 + 140x^5 + 21x^6 + x^7$

Ex. 1. If $\frac{a_1}{n+1} - \frac{a_2}{(n+1)(n+2)} + \frac{a_3}{(n+1)(n+2)(n+3)} - \&c$

$$= \frac{F(0)}{n} - \frac{F(1)}{n^2} + \frac{F(2)}{n^3} - \frac{F(3)}{n^4} + \&c$$

show that $F(n) = \phi_1(n)\, a_1 + \phi_2(n)\, a_2 + \phi_3(n)\, a_3 + \&c$

2. Shew that $\phi_{r+1}(n)$ is the coeff^{t}. of $\frac{x^n}{\lfloor n}$ in $\frac{e^x}{\lfloor r}(e^x - 1)^r$.

Sol. By III 7 we have $\phi_{r+1}(n)\lfloor r$

$$= (r+1)^n - \frac{r}{\lfloor 1} r^n + \frac{r(r-1)}{\lfloor 2}(r-1)^n - \&c$$

$= $ the coeff^{ts}. of $\frac{x^n}{\lfloor n}$ in $\left\{ e^{x(r+1)} - \frac{r}{\lfloor 1} e^{xr} + \frac{r(r-1)}{\lfloor 2} e^{x(r-1)} - \&c \right\}$

$=$ that of $\frac{x^n}{\lfloor n}$ in $e^x(e^x-1)^r$.

3. $\frac{d f_{n-1}(x)}{dx} = n f_{n-2}(x) + \frac{n(n-1)}{\lfloor 2} f_{n-3}(x) + \frac{n(n-1)(n-2)}{\lfloor 3} f_{n-4}(x) + \&c$

Sol. Differentiating both sides in III 4 with respect x and proceeding as in III 5 by differentiating the result with respect to a and equating the the coeff^{ts}. we can get the result.

4. $\int f_n(x)\, dx + \frac{1}{2} f_n(x) = \frac{f_{n+1}(x)}{n+1} + \frac{B_2}{\lfloor 2} n f_{n-1}(x)$

$- B_4 \frac{n(n-1)(n-2)}{\lfloor 4} f_{n-3}(x) + B_6 \frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 6} f_{n-5}(x) - \&c$

Sol. Integrating both sides in III 4. with respect to x we have $\frac{1}{e^a-1}\left\{1 + \frac{a}{\lfloor 1} f_0(x) + \frac{a^2}{\lfloor 2} f_1(x) + \frac{a^3}{\lfloor 3} f_2(x) + \&c\right\}$

$= \frac{1}{e^a-1} + x + \frac{a}{\lfloor 1}\int f_0(x)\,dx + \frac{a^2}{\lfloor 2}\int f_1(x)\,dx + \&c$

Equate the coeff^{ts}. of a^{n+1}

5. i If $\frac{1^n}{\lfloor 0} + \frac{2^n}{\lfloor 1} + \frac{3^n}{\lfloor 2} + \frac{4^n}{\lfloor 3} + \&c = e A_n$

shew that $A_0 = 1$, $A_1 = 2$, $A_2 = 5$, $A_3 = 15$, $A_4 = 52$, $A_5 = 203$, $A_6 = 877$, $A_7 = 4140$, $A_8 = 21147$ &c

ii. If $-\frac{1^n}{\lfloor 0} + \frac{2^n}{\lfloor 1} - \frac{3^n}{\lfloor 2} + \frac{4^n}{\lfloor 3} - \&c = \frac{A_n}{e}$ shew that

$A_0 = -1$, $A_1 = 0$, $A_2 = 1$, $A_3 = 1$, $A_4 = -2$, $A_5 = -9$, $A_6 = -9$, $A_7 = 50$, $A_8 = 267$ &c.

Sol $2 = 1+1$, $5 = 1+2.1+2$, $15 = 1+3.1+3.2+5$, $52 = 1+4.1+6.2+4.5+15$ &c Similarly for the 2nd also.

N.B. 1	2	5	15	52	203	877		-1	0	1	1	-2	-9	-9	50
	1	3	10	37	151	674			1	1	0	-3	-7	0	59
		2	7	27	114	523				0	-1	-3	-4	7	59
			5	20	87	409					-1	-2	-1	11	52
				15	67	322						-1	1	12	41
					52	255							2	11	29
						203								9	18
															9

6. Shew that

(i) $\frac{1^3}{\underline{|0}} + \frac{2^3}{\underline{|1}} + \frac{3^3}{\underline{|2}} + \frac{4^3}{\underline{|3}} + \&c = 3\left(\frac{1^2}{\underline{|0}} + \frac{2^2}{\underline{|1}} + \frac{3^2}{\underline{|2}} + \&c\right)$

(ii) $\frac{1^2(1^3+1)}{\underline{|0}} + \frac{2^2(2^3+1)}{\underline{|2}} + \frac{3^2(3^3+1)}{\underline{|3}} + \&c = 4\left(\frac{1^4}{\underline{|0}} + \frac{2^4}{\underline{|1}} + \frac{3^4}{\underline{|2}} + \&c\right)$

(iii) $\frac{1^3}{\underline{|0}} - \frac{2^3}{\underline{|1}} + \frac{3^3}{\underline{|2}} - \frac{4^3}{\underline{|3}} + \&c = \frac{1^2}{\underline{|0}} - \frac{2^2}{\underline{|1}} + \frac{3^2}{\underline{|2}} - \frac{4^2}{\underline{|3}} + \&c$

(iv) $\frac{1^6}{\underline{|0}} - \frac{2^6}{\underline{|1}} + \frac{3^6}{\underline{|2}} - \frac{4^6}{\underline{|3}} + \&c = \frac{1^5}{\underline{|0}} - \frac{2^5}{\underline{|1}} + \frac{3^5}{\underline{|2}} - \frac{4^5}{\underline{|3}} + \&c$

(v) $\frac{1^3(1^3+1)(1^2+1)}{\underline{|0}} - \frac{2^3(2^3+1)(2^2+1)}{\underline{|1}} + \frac{3^3(3^3+1)(3^2+1)}{\underline{|2}} - \&c$

$= 5\left(\frac{1^7}{\underline{|0}} - \frac{2^7}{\underline{|1}} + \frac{3^7}{\underline{|2}} - \frac{4^7}{\underline{|3}} + \&c\right)$

9. If $(a+b)^n \frac{x}{\underline{|0}} + (a+2b)^n \frac{x^2}{\underline{|1}} + (a+3b)^n \frac{x^3}{\underline{|2}} + \&c = e^x F_n(x)$,

then i. $\frac{F_0(x)}{n} - \frac{F_1(x)}{n^2} + \frac{F_2(x)}{n^3} - \&c$

$= \frac{x}{n+a+b} - \frac{b x^2}{(n+a+b)(n+a+2b)} + \frac{b^2 x^3}{(n+a+b)(n+a+2b)(n+a+3b)}$

$- \&c$

ii. $F_0(x) + y F_1(x) + \frac{y^2}{\underline{|2}} F_2(x) + \frac{y^3}{\underline{|3}} F_3(x) + \&c = x e^{y(a+b)} e^{x(e^{by}-1)}$

iii. $F_{n+1}(x) - (a+b) F_n(x) = bx\left\{F_n(x) + \frac{n}{\underline{|1}} F_{n-1}(x) b + \frac{n(n-1)}{\underline{|2}} F_{n-2}(x) b^2 + \&c\right\}$

iv. If $F_n(x) = \phi_1(n) x + \phi_2(n) x^2 + \phi_3(n) x^3 + \&c$, then

$\frac{\phi_n(n)}{\underline{|0}} + \frac{\phi_{n-1}(n)}{\underline{|1}} + \frac{\phi_{n-2}(n)}{\underline{|2}} + \&c = \frac{(a+nb)^n}{\underline{|n-1}}$.

N.B. If $F_{n+1}(x) - (a+b) F_n(x) = \psi_n(x)$, then

$$\begin{array}{ccccc}
\psi_0(x), & \psi_1(x), & \psi_2(x), & \psi_3(x), & \psi_4(x) \\
 & a_1 & b_1 & c_1 & d_1 \\
 & & a_2 & b_2 & c_2 \\
 & & & a_3 & b_3 \\
 & & & & a_4
\end{array}\left.\right\}$$

These are successive diffces, b times the previous term being subtracted from each term and a_n being equal to $b x F_n(x)$

V. $\phi_n(x)\lfloor{n-1} = (a+nb)^n - \frac{n-1}{\lfloor 1}(a+\overline{n-1}\,b)^n + \frac{(n-1)(n-2)}{\lfloor 2}(a+\overline{n-2}\,b)^n$
$- \frac{(n-1)(n-2)(n-3)}{\lfloor 3}(a+\overline{n-3}\,b)^n + \&c$

VI. $\phi_n(x+1) = (a+nb)\phi_n(x) + b\,\phi_{n-1}(x)$

N.B. Write under each term the product of $a+nb$, n being the index of x, and the coefft. of x of that term together with b times the coefft. of the preceding one

$F_0(x) = x$

$F_1(x) = (a+b)x + bx^2$

$F_2(x) = (a+b)^2x + b(2a+3b)x^2 + b^2x^3$

$F_3(x) = (a+b)^3x + b\{3(a+b)(a+2b)+b^2\}x^2 + 3b^2(a+2b)x^3 + b^3x^4$

$F_4(x) = (a+b)^4x + b\{2(a+b)(a+2b)+b^2\}(2a+3b)x^2 +$
$b^2\{6(a+2b)^2+b^2\}x^3 + 2b^3(2a+5b)x^4 + b^4x^5$

VII. $\phi_{n+1}(x)$ is the coefft. of $\frac{x^n}{\lfloor n}$ in $\frac{e^{x(a+b)}}{\lfloor n}(e^{bx}-1)^n$.

Ex. i. $\frac{1^3+1^2}{\lfloor 0} - \frac{3^3+3^2}{\lfloor 1} + \frac{5^3+5^2}{\lfloor 2} - \&c = 0$

ii. $\frac{1^4}{\lfloor 0} + \frac{2^4}{\lfloor 1} + \frac{3^4}{\lfloor 2} + \frac{4^4}{\lfloor 3} + \&c = 4\left(\frac{1^2}{\lfloor 0} + \frac{3^2}{\lfloor 1} + \frac{5^2}{\lfloor 2} + \&c\right)$

iii $\frac{1^7+1^6}{\lfloor 0} - \frac{2^7+2^6}{\lfloor 1} + \frac{3^7+3^6}{\lfloor 2} - \&c = \frac{1^4}{\lfloor 0} - \frac{3^4}{\lfloor 1} + \frac{5^4}{\lfloor 2} - \&c$

iv. $1^4 - \frac{3^4}{1\lfloor 1} + \frac{5^4}{2\lfloor 2} - \frac{7^4}{3\lfloor 3} + \&c = \left(1 - \frac{1}{1\lfloor 1} + \frac{1}{2\lfloor 2} - \frac{1}{3\lfloor 3} + \&c\right) - 4$

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10. $\phi(0) + \frac{x}{\lfloor 1}\phi(1) + \frac{x^2}{\lfloor 2}\phi(2) + \frac{x^3}{\lfloor 3}\phi(3) + \&c = e^x \phi_\infty(x)$.

where $\phi_n(x) = \phi_{n-1}(x) + \frac{a}{\lfloor 1 \lfloor n}\phi^n_{n-1}(x) + \frac{a^2}{\lfloor 2 (\lfloor n)^2}\phi^{2n}_{n-1}(x) +$

$\frac{a^3}{\lfloor 3 (\lfloor n)^3}\phi^{3n}_{n-1}(x) + \&c$ ad. inf. where $\phi_1(x) = \phi(x)$,

$\phi^n(x)$ is the nth diff.l coeff.t of $\phi(x)$ and a is ultimately made equal to x.

Sol. $\phi(0) + \frac{x}{\lfloor 1}\phi(1) + \frac{x^2}{\lfloor 2}\phi(2) + \frac{x^3}{\lfloor 3}\phi(3) + \&c$

$= e^x\left\{\cancel{\phi(x)} + \frac{\phi'(0)}{\lfloor 1}f_1(x) + \frac{\phi''(0)}{\lfloor 2}f_2(x) + \frac{\phi'''(0)}{\lfloor 3}f_3(x) + \&c\right\}$

$$= e^x\Big[\phi(x) + \frac{x}{2}\phi''(x) + \left\{\frac{x}{6}\phi'''(x) + \frac{x^2}{8}\phi^{IV}(x)\right\} +$$

$$\left\{\frac{x}{24}\phi^{IV}(x) + \frac{x^2}{12}\phi^{V}(x) + \frac{x^3}{48}\phi^{VI}(x)\right\} +$$

$$\left\{\frac{x}{120}\phi^{V}(x) + \frac{5}{144}x^2\phi^{VI}(x) + \frac{x^3}{48}\phi^{VII}(x) + \frac{x^4}{384}\phi^{VIII}(x)\right\}$$

$$+\left\{\frac{x}{720}\phi^{VI}(x) + \frac{x^2}{98}\phi^{VII}(x) + \frac{7x^3}{576}\phi^{VIII}(x) + \frac{x^4}{288}\phi^{IX}(x) + \frac{x^5}{3840}\phi^{X}(x)\right\} + \&c\ \&c\Big]$$

Collecting the last terms, the last but one terms &c we can get the result.

Cor. If x is great and $\phi''(x)$ can be neglected, then

$$e^{-x}\left\{\phi(0) + \frac{x}{\lfloor 1}\phi(1) + \frac{x^2}{\lfloor 2}\phi(2) + \&c\right\} = \phi(x) + \frac{x}{2}\phi''(x)$$

very nearly.

Sol. In the above solution neglecting the third and the other terms we get $\phi(0) + \frac{x}{\lfloor 1}\phi(1) + \frac{x^2}{\lfloor 2}\phi(2) + \&c$

$= e^x\left\{\phi(x) + \frac{x}{2}\phi''(x)\right\}$. . .

Ex. 1. Shew that $\log_e\left(\frac{x}{\lfloor 1}\sqrt{1} + \frac{x^2}{\lfloor 2}\sqrt{2} + \frac{x^3}{\lfloor 3}\sqrt{3} + \&c\right)$
$= x + \frac{1}{2}\log_e x - \frac{1}{8x} - \frac{1}{16x^2}$ very nearly.

2. $e^{-x}\left(\frac{x}{\lfloor 1}\log_e 2 + \frac{x^2}{\lfloor 2}\log_e 3 + \frac{x^3}{\lfloor 3}\log_e 4 + \&c\right)$
$= \log_e x + \frac{1}{2x} + \frac{1}{12x^2}$ very nearly.

3. $\log_e\left\{\phi(0) + \frac{100}{\lfloor 1}\phi(1) + \frac{100^2}{\lfloor 2}\phi(2) + \frac{100^3}{\lfloor 3}\phi(3) + \&c\right\}$
$= 100 + \log_e \frac{\phi(110) + \phi(90)}{2}$. nearly.

4. Shew that $\frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2}(1+\frac{1}{2}) + \frac{x^3}{\lfloor 3}(1+\frac{1}{2}+\frac{1}{3}) + \&c$
$= e^x(C + \log_e x)$ very nearly where C is the constant of the series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x}$.

11. If $e^{A_1 x + A_2 \frac{x^2}{2} + A_3 \frac{x^3}{3} + \&c}$
$= P_0 + P_1 x + P_2 x^2 + P_3 x^3 + \&c$

then $P_n = A_1 P_{n-1} + A_2 P_{n-2} + A_3 P_{n-3} + \&c$ to n terms
where $P_0 = 1$.

Sol:— Take logarithms of both sides, then differentiate them & equate the coeff^{ts}.

Cor. If $S_r = a_1^r + a_2^r + a_3^r + \cdots + a_n^r$ and P_r denotes the sum of the products of $a_1\, a_2,\, a_3 \ldots a_n$ taken r at a time, then
$rP_r = S_1 P_{r-1} - S_2 P_{r-2} + S_3 P_{r-3} - \&c$ where $P_0 = 1$.

Sol Apply the above theorem in $(1-a_1x)(1-a_2x)(1-a_3x)\&c$

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12. If $n^r + \frac{(n+1)^{r+1}}{a\lfloor 1} + \frac{(n+2)^{r+2}}{a^2 \lfloor 2} + \frac{(n+3)^{r+3}}{a^3 \lfloor 3} + \&c = F_r(n)$, then

$$F_{r+1}(n) = nF_r(n) + \frac{1}{a}F_{r+1}(n+1).$$

Sol. $F_{r+1}(n) = n^{r+1} + \frac{(n+1)^{r+2}}{a\lfloor 1} + \frac{(n+2)^{r+3}}{a^2 \lfloor 2} + \&c$

$= n\left\{n^r + \frac{(n+1)^{r+1}}{a\lfloor 1} + \frac{(n+2)^{r+2}}{a^2\lfloor 2} + \&c\right\}$

$+ \frac{1}{a}\left\{(n+1)^{r+1} + \frac{(n+2)^{r+2}}{a\lfloor 1} + \frac{(n+3)^{r+3}}{a^2\lfloor 2} + \&c\right\}$

$= nF_r(n) + \frac{1}{a}F_{r+1}(n+1)$.

We see from this identity that if we are able to find the sum for one value of r we can sum up the series for all values of r.

N.B. $F_r(n)$ is convergent when $a > e$ or $\not< e$ according as r is positive or not.

13. If $x = a\log_e x$, then $\frac{x^n}{n} = F_{-1}(n)$.

Sol. Let $f(n) = 1 + \frac{n}{a\lfloor 1} + \frac{n(n+2)}{a^2\lfloor 2} + \frac{n(n+3)^2}{a^3\lfloor 3} + \&c$

Multiplying $f(m)$ by $f(n)$ we get $f(m+n)$.

$\therefore f(n) = \{f(1)\}^n$. Let $f(1) = x$, then $x^n = f(n)$.

$\frac{f(n)-1}{n}$, when $n=0$, $= \frac{1}{a} + \frac{2}{a^2\lfloor 2} + \frac{3^2}{a^3\lfloor 3} + \&c$

$= \frac{1}{a}\left(1 + \frac{1}{a\lfloor 1} + \frac{3}{a^2\lfloor 2} + \&c\right) = \frac{1}{a}f(1) = \frac{x}{a}$.

i.e. $\frac{x^n - 1}{n}$, when $n=0$, $= \frac{x}{a}$ or $x = a\log_e x$.

N.B. The minimum value of $\frac{x}{\log_e x} = e$; if $a = e$, $f(n) = e^n$

$f(n)$ is convergent if $a > e$ & divergent if $a < e$.

Cor. $e^{x} = 1 + \frac{x}{e^{n}} + \frac{x(x+2n)}{e^{2n}\,\lfloor 2} + \frac{x(x+3n)^{2}}{e^{3n}\,\lfloor 3} + \&c$

14. If $aqx^{p} - x^{q} + 1 = 0$, then

$$x^{n} = 1 + \frac{n}{\lfloor 1}a + \frac{n(n+2p-q)}{\lfloor 2}a^{2} + \frac{n(n+3p-q)(n+3p-2q)}{\lfloor 3}a^{3}$$
$$+ \frac{n(n+4p-q)(n+4p-2q)(n+4p-3q)}{\lfloor 4}a^{4} + \&c$$

Sol. Similar to that of III 13.

Cor 1. $$\left(\frac{2}{1+\sqrt{1-4x}}\right)^{n} = 1 + nx + \frac{n(n+3)}{\lfloor 2}x^{2} + \frac{n(n+4)(n+5)}{\lfloor 3}x^{3}$$
$$+ \frac{n(n+5)(n+6)(n+7)}{\lfloor 4}x^{4} + \&c$$

Cor. 2. $$\left(x+\sqrt{1+x^{2}}\right)^{n} = 1 + \frac{n}{\lfloor 1}x + \frac{n^{2}}{\lfloor 2}x^{2} + \frac{n(n^{2}-1)}{\lfloor 3}x^{3}$$
$$+ \frac{n^{2}(n^{2}-4)}{\lfloor 4}x^{4} + \frac{n(n^{2}-1)(n^{2}-9)}{\lfloor 5}x^{5} + \&c$$

15. $$1 + \frac{1}{\lfloor 1}e^{-\left(1+\frac{x^{2}}{2}\right)} + \frac{3}{\lfloor 2}e^{-2\left(1+\frac{x^{2}}{2}\right)} + \frac{4^{2}}{\lfloor 3}e^{-3\left(1+\frac{x^{2}}{2}\right)}$$
$$+ \frac{5^{3}}{\lfloor 4}e^{-4\left(1+\frac{x^{2}}{2}\right)} + \&c$$
$$\doteq e^{1 - x + \frac{x^{2}}{3} - \frac{x^{3}}{36} - \cancel{\frac{x^{4}}{270}} - \&c} + \cancel{\frac{1}{1080}x^{4}}$$
correct
$$= e^{1+\frac{x^{2}}{2}}\left(1 - x + \frac{x^{2}}{3} - \frac{x^{3}}{36} - \cancel{\frac{x^{4}}{270}} - \&c\right) + \cancel{\frac{1}{1080}x^{4}}$$
correct

Sol. $e^{n} = 1 + \frac{1}{\lfloor 1}ne^{-n} + \frac{3}{\lfloor 2}n^{2}e^{-2n} + \&c$

$\therefore e^{e^{-n}} = 1 + \frac{1}{\lfloor 1}e^{-(n+e^{-n})} + \frac{3}{\lfloor 2}e^{-2(n+e^{-n})} + \&c$

Let $n + e^{-n} = 1 + \frac{x^{2}}{2}$. Solve the equation and find n.

N. B. This result is useful in finding the numerical value of $F_{a}(n)$ when a approaches e.

Ex. 1. $2^m = 1 + \frac{m}{2^n} + \frac{m(m+2n-1)}{2^{2n}\lfloor 2} + \frac{m(m+3n-1)(m+3n-2)}{2^{3n}\lfloor 3}$ + &c.

2. Find x^p when $\frac{(\log_e x)^m}{x^n} = a$. Sol. Let $x^n = y^m$.

3. Find x in terms of a in each of the following

i. $x^a = e^{\pm x}$ Sol. $a \log_e x = \pm x$.

ii. $x^a = a^{\pm x}$; Sol $a \log_e x = \pm x \log_e a$. $\therefore \frac{x}{\log_e x} = \pm \frac{a}{\log_e a}$

iii. $x = a e^{\pm x}$; Sol. let $x = \log_e y$, then $\log_e y = a\, y^{\pm 1}$

iv. $x = a^{\pm x}$; Sol. let $x \log_e a = \log_e y$, then $\log_e y = y^{\pm 1} \log_e a$.

v. $x^{\pm x} = a$; Sol let $x = \frac{1}{y}$, then $y = a^{\mp y}$

vi. $x e^{\pm x} = a$; Sol. let $x = \log_e y$, then $\log_e y = a\, y^{\pm 1}$

vii. $e^x \pm x = a$; Sol let $x = \log_e \log_e y$ then $e^a = y(\log_e y)^{\pm 1}$

viii $x \pm \log_e x = a$; Sol let $x = \log_e y$, then $e^a = y(\log_e y)^{\pm 1}$

4. Shew how to find the values of the following for numerical values of x.

i. $x^{x^{x^{\&c}}} = v$. then $x^v = v$.

ii. $x \pm e^{x \pm e^{x \pm \&c}} = v$ then $x \pm e^v = v$

iii. $\log_e \{x \log_e [x \log_e (x \ldots \text{ad inf.})]\}$

iv. $\pm \log_e \{x \pm \log_e [x \pm \log_e (x \pm \&c)]\}$

16. Writing $x^n \phi_n(x)$ for $F_n(x)$, we have

$$\phi_{n+1}(n) - \log_e x\, \phi_{n+1}(n+1) = n\, \phi_n(n).$$

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Case I If r is positive

Let $\phi_r(n) = \dfrac{\psi_1(r,n)}{(1-\log_e x)^{r+1}} + \dfrac{\psi_2(r,n)}{(1-\log_e x)^{r+2}} + \cdots + \dfrac{\psi_{r+1}(r,n)}{(1-\log_e x)^{2r+1}}$

Then $n\,\psi_t(r,n) + \psi_{t-1}(r+1,n+1) = \psi_t(r+1,n+1) + \psi_{t-1}(r+1,n)$.

Case II If r is negative, the terms in R.S continue as far as the term independent of $(1-\log_e x)$.

$\phi_{-1}(n) = \frac{1}{n}$.

$\phi_{-2}(n) = \dfrac{1-\log_e x}{n(n+1)} + \dfrac{1}{n^2(n+1)}$

$\phi_{-3}(n) = \dfrac{(1-\log_e x)^2}{n(n+1)(n+2)} + \dfrac{(3n+2)(1-\log_e x)}{n^2(n+1)^2(n+2)} + \dfrac{3n+2}{n^3(n+1)^2(n+2)}$

$\phi_0(n) = \dfrac{1}{1-\log_e x}$

$\phi_1(n) = \dfrac{n-1}{(1-\log_e x)^2} + \dfrac{1}{(1-\log_e x)^3}$

$\phi_2(n) = \dfrac{(n-1)(n-2)}{(1-\log_e x)^3} + \dfrac{(n-1)(n-2)\left(\frac{1}{n-2}+\frac{2}{n-1}\right)}{(1-\log_e x)^4} + \dfrac{1.3}{(1-\log_e x)^5}$

$\phi_3(n) = \dfrac{(n-1)(n-2)(n-3)}{(1-\log_e x)^4} + \dfrac{(n-1)(n-2)(n-3)\left(\frac{1}{n-3}+\frac{2}{n-2}+\frac{3}{n-1}\right)}{(1-\log_e x)^5}$

$\qquad + \dfrac{15n-35}{(1-\log_e x)^6} + \dfrac{1.3.5}{(1-\log_e x)^7}$,

Cor. 1. $e^x = (1-n)\left\{1 + \dfrac{x+n}{e^n \lfloor 1} + \dfrac{(x+2n)^2}{e^{2n}\lfloor 2} + \dfrac{(x+3n)^3}{e^{3n}\lfloor 3} + \&c\right\}$

2. $\psi_1(r,n) + \psi_2(r,n) + \psi_3(r,n) + \&c$ as far as the terms cease to continue in $\phi_r(n) = n^r$.

Sol. L.S $= \phi_r(n)$ when $x=1$ i.e $F_r(n)$ when $x=1$, i.e $F_r(n)$ when $a=\infty$ $= n^r$.

17. To expand x^m in ascending powers of h when $x^x = a^a e^h$

32. Let $\frac{x-a}{a} = \frac{A_1}{\underline{|1}} \cdot \frac{h}{a} - \frac{A_2}{\underline{|2}} \cdot \left(\frac{h}{a}\right)^2 + \frac{A_3}{\underline{|3}} \left(\frac{h}{a}\right)^3$ &c & $n = \frac{1}{1+\log_e a}$

then $A_r - n(r-2)A_{r-1} = n\left\{ r A_1 A_{r-1} + \frac{r(r-1)}{\underline{|2}} A_2 A_{r-2} + \frac{r(r-1)(r-2)}{\underline{|3}} A_3 A_{r-3} + \&c \right\}$ the last term being

$\frac{\underline{|r}}{\underline{|\frac{r-1}{2}}\,\underline{|\frac{r+1}{2}}} A_{\frac{r-1}{2}}$ or $\frac{\underline{|r}}{2\left(\underline{|\frac{r}{2}}\right)^2} A_{\frac{r}{2}}$ according as r is odd or even

Multiply the power and the coeff.t write under each term the sum of this product and $(r-1)$ times the coeff.t of the preceding term where r is the suffix of A.

$A_1 = n$

$A_2 = n^3$

$A_3 = 3n^5 + n^4$

$A_4 = 15n^7 + 10n^6 + 2n^5$

$A_5 = 105n^9 + 105n^8 + 40n^7 + 6n^6$

$A_6 = 945n^{11} + 1260n^{10} + 700n^9 + 196n^8 + 24n^7$

$A_7 = 10395n^{13} + 17325n^{12} + 12600n^{11} + 5068n^{10} + 1148n^9 + 120n^8$

N. B. For $\frac{a}{x}$ take $(r+1)$ times the coeff.t; for $\log\frac{x}{a}$ take r times the coeff.t and generally for $\left(\frac{x}{a}\right)^m$ take $(r-m)$ times the coeff.t.

Ex. 1. Shew that the sum of the coeff.ts of $A_r = (r-1)^{r-1}$

Sol. Put for a. Then $x^x = e^h$.

Let $x = \frac{1}{y}$, then $y^{\frac{1}{y}} = e^{-h}$ or $\frac{\log_e y}{y} = -h$.

$\therefore \frac{1}{y} = x = 1 + h - \frac{1^1}{\underline{|2}} h^2 + \frac{2^2}{\underline{|3}} h^3 - \frac{3^3}{\underline{|4}} h^4 + \&c$

$\therefore$ The sum of the coeff.ts of $A_r = (r-1)^{r-1}$

2. To expand x in ascending powers of h when

$$\sqrt[x]{x} = e^h \sqrt[a]{a}.$$

Sol. Let $x = \frac{1}{y}$. then $y^y = e^{-h}\left(\frac{1}{a}\right)^{\frac{1}{a}}$.

CHAPTER IV

Let $F_1(x) = e^x - 1$, $F_2(x) = e^{e^x-1} - 1$

$F_3(x) = e^{e^{e^x-1}-1}-1$ $\quad F_4(x) = e^{e^{e^{e^x-1}-1}-1}-1$ $\quad F_5(x) = e^{e^{e^{e^{e^x-1}-1}-1}-1}-1$ and so on.

Let us try to find the expansion of $F_n(x)$ in ascending powers of x and in ascending powers of n.

let $\underbrace{e^{e^{\cdot^{\cdot^{e^x-1}}}-1}-1}_{n\ e\text{'s}} = F_n(x) = x\phi_1(n) + x^2\phi_2(n) + x^3\phi_3(n) + \&c$

$= f_0(x) + n f_1(x) + n^2 f_2(x) + n^3 f_3(x) + \&c$

then $\log_e\left[1+\log_e\left\{1+\log_e(1+\cdots+\log_e\overline{1+x})\right\}\right]$ log^{ms} taken n times

$= F_{-n}(x) = x\phi_1(-n) + x^2\phi_2(-n) + x^3\phi_3(-n) + \&c$

$= f_0(x) - n f_1(x) + n^2 f_2(x) - n^3 f_3(x) + \&c$

Sol. we have $e^{F_{n-1}(x)}-1 = F_n(x)$; $\therefore F_{n-1}(x) = \log_e\{1+F_n(x)\}$

$\therefore F_0(x) = x$. $\therefore F_{-1}(x) = \log_e(1+x)$. $\therefore F_{-2}(x) = \log_e\{1+\log_e(1+x)\}$.

&c &c.

Cor. $F_0(x) = x$. and $f_0(x) = x$.

2. $\dfrac{dF_n(x)}{dx} \div \dfrac{dF_{n-1}(x)}{dx} = 1 + F_n(x)$.

Sol. $F_{n-1}(x) = \log_e\{1+F_n(x)\}$; differentiating both sides with respect to x we have $\dfrac{dF_n(x)}{dx} = \{1+F_n(x)\}\dfrac{dF_{n-1}(x)}{dx}$.

Cor. 1. $\dfrac{dF_n(x)}{dx} = \{1+F_1(x)\}\{1+F_2(x)\}\{1+F_3(x)\}\cdots\cdots\{1+F_n(x)\}$.

Sol. $F_n'(x) = \{1+F_n(x)\}F_{n-1}'(x) = \{1+F_n(x)\}\{1+F_{n-1}(x)\}F_{n-2}'(x) =$
$\{1+F_n(x)\}\{1+F_{n-1}(x)\}\{1+F_{n-2}(x)\}F_{n-3}'(x) = \&c\ \&c =$

$= \{1+F_n(x)\}\{1+F_{n-1}(x)\}\{1+F_{n-2}(x)\}\cdots\cdots\{1+F_1(x)\}\,F_0'(x)$.

But $F_0(x) = x$; $\therefore F_0'(x) = 1$.

Cor 2. $n\{\phi_n(x) - \phi_n(x-1)\} = (n-1)\,\phi_1(x)\,\phi_{n-1}(x-1) + (n-2)\,\phi_2(x)\,\phi_{n-2}(x-1)$
$+ (n-3)\,\phi_3(x)\,\phi_{n-3}(x-1) + \&c$

Sol. $F_{n-1}'(x)\{1+F_n(x)\} = F_n'(x)$. Here equate the coeff.ts of x^{x+1}.

3. $\dfrac{df_1(x)}{dx} = x - \tfrac{1}{2}f_1(x) + B_2 f_2(x) - B_4 f_4(x) + B_6 f_6(x) - \&c$

Sol. $F_n'(x) = \{1+F_1(x)\}\{1+F_2(x)\}\{1+F_3(x)\}\cdots\{1+F_n(x)\}$

$\therefore 1 + n\dfrac{df_1(x)}{dx} + \&c = e^{F_0(x) + F_1(x) + \cdots + F_{n-1}(x)}$.

$\therefore \log_e\left\{1 + n\dfrac{df_1(x)}{dx} + \&c\right\} = F_0(x) + F_1(x) + \&c$ to n terms

$= \psi(x) + \displaystyle\int_0^n F_n(x)\,dn - \tfrac{1}{2}F_n(x) + \frac{B_2}{\lfloor 3}\frac{dF_n(x)}{dn} - \frac{B_4}{\lfloor 4}\frac{d^3F_n(x)}{dn^3} + \&c$

where $\psi(x)$ is a function of x independent of n.

Equating the coeff.ts of n we get the result.

Cor. $\psi(x) = \displaystyle\int_0^x \frac{x - \dfrac{df_1(x)}{dx}}{f_1(x)}\,dx$

Sol. Since when $n=0$, $\log_e\left\{1 + n\dfrac{df_1(x)}{dx} + \&c\right\} = 0$

$\psi(x) = \dfrac{x}{2} - \dfrac{B_2}{2}f_1(x) + \dfrac{B_4}{4}f_3(x) - \dfrac{B_6}{6}f_5(x) + \&c$.

$\therefore \psi'(x) = \dfrac{1}{2} - \dfrac{B_2}{2}f_1'(x) + \dfrac{B_4}{4}f_3'(x) - \dfrac{B_6}{6}f_5'(x) + \&c$

$\therefore \psi'(x)f_1(x) = \tfrac{1}{2}f_1(x) - \dfrac{B_2}{2}f_1(x)f_1'(x) + \dfrac{B_4}{4}f_1(x)f_3'(x) - \&c$

$= \tfrac{1}{2}f_1(x) - B_2 f_2(x) + B_4 f_4(x) - \&c$ by IV 4.

$= x - f_1'(x)$; $\therefore \psi'(x) = \dfrac{x - f_1'(x)}{f_1(x)}$.

4. $\dfrac{dF_r(x)}{dr} = f_1(x)\dfrac{dF_r(x)}{dx}$ and hence $nf_n(x) = f_1(x)\dfrac{df_{n-1}(x)}{dx}$.

Sol. In IV 1 write $F_k(x)$ for x; then $F_{r+k}(x) = F_r\{F_k(x)\}$.

But $F_{r+k}(x) = F_k(x) + r\dfrac{dF_k(x)}{dk} + \dfrac{r^2}{\underline{|2}}\dfrac{d^2F_k(x)}{dk^2} + \&c$

and $F_r\{F_k(x)\} = F_k(x) + rf_1\{F_k(x)\} + r^2f_2\{F_k(x)\} + \&c$

Equating the coeff.ts of r we have $\dfrac{dF_k(x)}{dk} = f_1\{F_k(x)\}$

Let $F_k(x) = y$ and $F_k(y) = z$, then we have

$$\frac{dy}{dk} = f_1(y); \therefore \frac{dz}{dk} = f_1(y)\frac{dz}{dy}.$$

$\therefore \dfrac{dF_k(y)}{dk} = f_1(y)\dfrac{dF_k(y)}{dy}$. Equating the coeff.ts of k^{n+1} we have $nf_n(x) = f_1(x)f'_{n-1}(x)$.

Cor 1. If $f_n(x) = \left(\frac{x}{2}\right)^n\{\psi_1(n)x - \psi_2(n)x^2 + \psi_3(n)x^3 - \&c\}$

i. $n\psi_r(n) = n\psi_1(n-1)\psi_r(1) + (n+1)\psi_2(n-1)\psi_{r-1}(1) + (n+2)\psi_3(n-1)\psi_{r-2}(1) + (n+3)\psi_4(n-1)\psi_{r-3}(1) + \&c$

ii. $\phi_n(2r) = r^n\left\{\dfrac{\psi_1(n-1)}{r} - \dfrac{\psi_2(n-2)}{r^2} + \dfrac{\psi_3(n-3)}{r^3} - \&c\right\}$

Sol. $nf_n(x) = f_1(x)f'_{n-1}(x)$; here equate the coeff.ts of like powers of x. $\phi_n(r)$ is the coeff.t of x^n in $F_r(x)$ by I expansion. Again find the coeff.t of x^n by II expansion and equate the two results.

Cor 2. $(r+1)\psi_r(1) = \frac{1}{2}\psi_{r-1}(1) + \dfrac{B_2}{2}\psi_{r-2}(2) - \dfrac{B_4}{2^3}\psi_{r-4}(4) + \dfrac{B_6}{2^5}\psi_{r-6}(6) - \dfrac{B_8}{2^7}\psi_{r-8}(8) + \&c$

Sol. Equate the coeff^{ts} of x^n in IV 3.

5. $f_1(x) = (1+x) f_1\{\log_e(1+x)\}$.

Sol. In IV 1. write $\log_e(1+x)$ for x; then $F_{n-1}(x) =$
$\log_e(1+x) + n f_1\{\log_e(1+x)\} + n^2 f_2\{\log_e(1+x)\} + \&c$

$\therefore e^{F_{n-1}(x)} = (1+x)\, e^{n f_1\{\log_e(1+x)\} + n^2 f_2\{\log_e(1+x)\} + \&c}$

But $e^{F_{n-1}(x)} = 1 + F_n(x) = 1 + x + n f_1(x) + n^2 f_2(x) + \&c$

Equating the coeff^{ts} of n $f_1(x) = (1+x) f_1\{\log_e(1+x)\}$

6. i. The sum of the coeff^{ts} in $\phi_n(n)$ without the signs is $\frac{1}{n}$ and with signs $= \frac{1}{\lfloor n}$.

Sol. $F_1(x) = e^x - 1$ and $F_{-1}(x) = \log_e(1+x)$; equate the coeff^{ts}.

ii. $\psi_1(n) = 1$; $\psi_2(n-1) = \frac{n}{3}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right)$;

$\psi_3(n-2) = \frac{n(n-1)}{72}\left\{\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)^2 - \left(\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right) - \frac{1}{n} + \frac{1}{2}\right\}$

$\phi_1(2n) = 1$

$\phi_2(2n) = n$

$\phi_3(2n) = n^2 - \frac{n}{6}$

$\phi_4(2n) = n^3 - \frac{5n^2}{12} + \frac{n}{24}$

$\phi_5(2n) = n^4 - \frac{13}{18}n^3 + \frac{n^2}{6} - \frac{n}{90}$

$\phi_6(2n) = n^5 - \frac{77}{72}n^4 + \frac{89}{216}n^3 - \frac{91}{1440}n^2 + \frac{11n}{4320}$ $\left(n - \frac{1}{3}\right)$

$\phi_7(2n) = n^6 - \frac{29}{20}n^5 + \frac{175}{216}n^4 - \frac{149}{720}n^3 + \frac{96n^2}{4320} - \frac{n}{3360}$.

$\phi_1(2n) = n^0$; $\phi_2(2n) = n$; $\phi_3(2n) = n\left(n - \frac{1}{6}\right)$; $\phi_4(2n) = n\left(n - \frac{1}{6}\right)\left(n - \frac{1}{4}\right)$

$f_1(x) = \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{48} - \frac{x^5}{180} + \frac{11x^6}{8640} - \frac{x^7}{6720}$.

7. If $\frac{x}{1-nx} = y$ and $1-nx = z$, then
$$F_{2n}(x) = y + \frac{y^2}{6}\log_e z + \frac{y^3}{72}\left\{(\log_e z)^2 + (1-\log_e z)^2 - z\right\} + \&c$$
Sol. Apply IV 6 ii in IV 1.

Ex. 1. $f_1(x) f_1''(x) = f_1(x) - f_2(x) + 3B_2 f_3(x) - 5B_4 f_5(x) + \&c$

Sol. From IV 3 we have $f_1'(x) = x - \frac{1}{2} f_1(x) + B_2 f_2(x) - B_4 f_4(x) + B_6 f_6(x) - \&c$; differentiating both sides and multiplying the results by $f_1(x)$ we have
$$f_1(x) f_1''(x) = f_1(x) - \frac{1}{2} f_1(x) f_1'(x) + B_2 f_1(x) f_2'(x) - B_4 f_1(x) f_4'(x) + \&c$$
$$= f_1(x) - f_2(x) + 3B_2 f_3(x) - 5B_4 f_5(x) + \&c \text{ by IV 4.}$$

2. $$\frac{1}{2} F_n\left(\frac{1}{2n}\right) = \frac{1}{n} - \frac{\log 2}{3n^2} + \frac{\left(\frac{1}{2} + \log 2\right)^2}{9n^3} - \&c$$
Sol. Put $x = \frac{1}{2n}$ in IV 7.

8. i. $$\Sigma\frac{1}{1} + \Sigma\frac{1}{2} + \Sigma\frac{1}{3} + \cdots + \Sigma\frac{1}{x} = (x+1)\Sigma\frac{1}{x} - x.$$

ii. $$\left(\Sigma\frac{1}{1}\right)^2 + \left(\Sigma\frac{1}{2}\right)^2 + \left(\Sigma\frac{1}{3}\right)^2 + \cdots + \left(\Sigma\frac{1}{x}\right)^2 = (x+1)\left(\Sigma\frac{1}{x}\right)^2 - (2x+1)\Sigma\frac{1}{x} + 2x$$

iii. $$\left(\Sigma\frac{1}{1}\right)^3 + \left(\Sigma\frac{1}{2}\right)^3 + \left(\Sigma\frac{1}{3}\right)^3 + \cdots + \left(\Sigma\frac{1}{x}\right)^3 = (x+1)\left(\Sigma\frac{1}{x}\right)^3 - 3\left(x+\frac{1}{2}\right)\left(\Sigma\frac{1}{x}\right)^2$$
$$+ 3(2x+1)\Sigma\frac{1}{x} - 6x + \frac{1}{2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{x^2}\right).$$

9. If two functions of x be equal, then a general theorem can be formed by simply writing $\phi(n)$ instead of x^n in the original theorem.

Sol. Put $x = 1$ and multiply it by $f(0)$, then change x to x, x^2, x^3, x^4 &c and multiply $\frac{f'(0)}{\lfloor 1}, \frac{f''(0)}{\lfloor 2}, \frac{f'''(0)}{\lfloor 3}$ &c respectively and add up all the results. Then instead of x^n we have $f(x^n)$ for positive as

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well as negative values of n. Changing $f(x^n)$ to $\phi(n)$ we can get the result.

E.G. 1 We know that $\tan^{-1}x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}$. The theorem states a general theorem from this identity can be formed as follows:—

$$\frac{\phi(1)+\phi(-1)}{1} - \frac{\phi(3)+\phi(-3)}{3} + \frac{\phi(5)+\phi(-5)}{5} - \&c = \frac{\pi}{2}\phi(0)$$

Sol. $f(0)\left(\tan^{-1}1+\tan^{-1}1\right) = \frac{\pi}{2}f(0)$

$$\frac{f'(0)}{\lfloor 1}\left(\tan^{-1}x+\tan^{-1}\frac{1}{x}\right) = \frac{\pi}{2}\cdot\frac{f'(0)}{\lfloor 1}$$

$$\frac{f''(0)}{\lfloor 2}\left(\tan^{-1}x^2+\tan^{-1}\frac{1}{x^2}\right) = \frac{\pi}{2}\cdot\frac{f''(0)}{\lfloor 2}$$

&c &c &c &c

Adding up all the results we have

$$\frac{f(x)+f\left(\frac{1}{x}\right)}{1} - \frac{f(x^3)+f\left(\frac{1}{x^3}\right)}{3} + \frac{f(x^5)+f\left(\frac{1}{x^5}\right)}{5} - \&c$$

$= \frac{\pi}{2}f(1)$. Let $f(x^n) = \phi(n)$, then $f(1)=\phi(0)$

$$\frac{\phi(1)+\phi(-1)}{1} - \frac{\phi(3)+\phi(-3)}{3} + \&c = \frac{\pi}{2}\phi(0).$$

2. Similarly we can derive from $\frac{x}{1+x}+\frac{1}{x+1}=1$

$$\{\phi(1)+\phi(-1)\} - \{\phi(2)+\phi(-2)\} + \{\phi(3)+\phi(-3)\} - \&c = \phi(0)$$

N.B. 1. If $\phi(n)$ be substituted for x^n, $\phi'(0)$ must be substituted for $\log_e x$, $\phi''(0)$ for $(\log_e x)^2$ &c.

N.B. 2. If an infinite number of terms vanish it may assume the form $0\times\infty$ and have a definite value. This error in case of a function of x is a function of e^{-x} which rapidly decreases as x increases.

3. $\frac{\phi(1)-\phi(-1)}{1} - \frac{\phi(2)-\phi(-2)}{2} + \frac{\phi(3)-\phi(-3)}{3} - \&c = \phi'(0).$

Sol. $\log_e(1+x) - \log_e(1+\frac{1}{x}) = \log_e x$ Apply IV 9.

4. $\frac{\phi(1)}{1 \lfloor 1} - \frac{\phi(2)}{2 \lfloor 2} + \frac{\phi(3)}{3 \lfloor 3} - \&c = C\phi(0) + \phi'(0)$ nearly

where C is the constant of $\Sigma \frac{1}{x}$

Sol. change $\phi(n)$ to $\frac{\phi(n)}{\lfloor n}$ in the above result.

$\phi(-1), \phi(-2)$ &c vanish.

Cor. 1. $\frac{x}{1 \lfloor 1} - \frac{x^2}{2 \lfloor 2} + \frac{x^3}{3 \lfloor 3} - \&c = C + \log_e x$ nearly.

Here the error lies between $\frac{e^{-x}}{x}$ & $\frac{e^{-x}}{1+x}$.

2. If x becomes greater and greater

$\left(\frac{x}{\lfloor 1}\right)^n - \frac{1}{2}\cdot\left(\frac{x^2}{\lfloor 2}\right)^n + \frac{1}{3}\cdot\left(\frac{x^3}{\lfloor 3}\right)^n - \&c = n\left(\frac{x}{1 \lfloor 1} - \frac{x^2}{2 \lfloor 2} + \frac{x^3}{3 \lfloor 3} - \&c\right.$

10. $\phi(0) + \frac{n}{\lfloor 1}\phi(1) + \frac{n(n-1)}{\lfloor 2}\phi(2) + \&c$

$= \phi(n) + \frac{n}{\lfloor 1}\phi(n-1) + \frac{n(n-1)}{\lfloor 2}\phi(n-2) + \&c$

Sol. $1 + \frac{n}{\lfloor 1}x + \frac{n(n-1)}{\lfloor 2}x^2 + \&c = x^n + \frac{n}{\lfloor 1}x^{n-1} + \&c$

apply IV 9.

Cor. If $x=0$, the value of the generating function of the Series $x^n\phi(0) + \frac{n}{\lfloor 1}x^{n-1}\phi(1) + \frac{n(n-1)}{\lfloor 2}x^{n-2}\phi(2) + \&c = \phi(n)$.

Ex. 1. When $x=0$

$$\frac{\phi(1)}{x} - \frac{\phi(2)}{x^2} + \frac{\phi(3)}{x^3} - \&c = \phi(0)$$

e.g. Let $\phi(n) = \frac{1}{n}\sin\frac{\pi n}{2}$, then $\phi(0) = \frac{\pi}{2}$.

$\therefore$ When $x=0$ $\quad \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \&c = \frac{\pi}{2}$ which is same as saying $\tan^{-1}\infty = \frac{\pi}{2}$.

2. If $x=0$, then $\frac{1}{x} - \frac{\lfloor\underline{1}}{x^2} + \frac{\lfloor\underline{2}}{x^3} - \&c = \infty$

Here $L.S = \cfrac{1}{x+1-\cfrac{1^2}{x+3-\cfrac{2^2}{x+5-\&c}}} = \cfrac{1}{1-\cfrac{1^2}{3-\cfrac{2^2}{5-\&c}}}$ when $x=0$

$= 1 + \frac{1}{2} + \frac{1}{3} + \&c = \infty$.

or $L.S = \frac{x}{\lfloor\underline{1}} + \frac{x^2}{\lfloor\underline{2}}\left(1+\frac{1}{2}\right) + \frac{x^3}{\lfloor\underline{3}}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \&c - e^x(C+\log_e x)$

$= \infty$ when $x=0$.

3. If $x=0$, then $x^n + \frac{n}{\lfloor\underline{1}}x^{n-1} + n(n-1)x^{n-2} + n(n-1)(n-2)x^{n-3} + \&c = \lfloor\underline{n}$ for all values of n.

4. If $x=0$, shew that

$\frac{1}{x} - \frac{\lfloor\underline{2}}{x^3} + \frac{\lfloor\underline{4}}{x^5} - \frac{\lfloor\underline{6}}{x^7} + \&c = \frac{\pi}{2}$.

Sol. Write $\lfloor\underline{n-1}\,\sin\frac{\pi n}{2}$ in ex. 1. Then $\phi(0) =$

$\lfloor\underline{n}\,.\,\frac{\sin\frac{\pi n}{2}}{n}$, when $n=0$, $= \frac{\pi}{2}$.

N.B. Thus we are able to find exact values when $x=0$, though the generating functions may be too difficult to find.

The generating function in ex. 4.

$= \frac{\pi}{2}\cos x + (C+\log_e x)\sin x -$

$\left\{\frac{x}{\lfloor\underline{1}} - \frac{x^3}{\lfloor\underline{3}}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \frac{x^5}{\lfloor\underline{5}}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right) - \&c\right\}$

$= \frac{\pi}{2}$ when $x=0$.

11. $\int_0^\infty e^{-x} x^n \, dx = \lfloor\underline{n}$ and hence

$$\int_0^\infty x^{n-1}\left\{\phi(0) - \frac{x}{\lfloor\underline{1}}\phi(1) + \frac{x^2}{\lfloor\underline{2}}\phi(2) - \&c\right\} dx = \lfloor\underline{n-1}\,\phi(-n).$$

Sol. $\int_0^\infty e^{-x} x^n \, dx = e^{-x}\{x^n + nx^{n-1} + n(n-1)x^{n-2} + \&c\}$

when $x = 0$ $= \lfloor\underline{n}$ by IV 10 Cor.

$$f(0)\int_0^\infty e^{-x} x^{n-1} \, dx = \lfloor\underline{n-1}\, f(0)$$

$$\frac{f'(0)}{\lfloor\underline{1}}\int_0^\infty e^{-rx} x^{n-1} \, dx = \frac{\lfloor\underline{n-1}}{r^n}\cdot\frac{f'(0)}{\lfloor\underline{1}}$$

$$\frac{f''(0)}{\lfloor\underline{2}}\int_0^\infty e^{-r^2x} x^{n-1} \, dx = \frac{\lfloor\underline{n-1}}{r^{2n}}\cdot\frac{f''(0)}{\lfloor\underline{2}}$$

and so on.

Adding up all the results we have

$$\int_0^\infty x^{n-1}\left\{f(1) - \frac{x}{\lfloor\underline{1}}f(r) + \frac{x^2}{\lfloor\underline{2}}f(r^2) - \&c\right\} dx = \lfloor\underline{n-1}\, f\left(\frac{1}{r^n}\right).$$

Let $f(r^n) = \phi(n)$ then $f\left(\frac{1}{r^n}\right) = \phi(-n)$.

Cor 1. $$\int_0^\infty x^{n-1}\{\phi(0) - x\phi(1) + x^2\phi(2) - \&c\} dx = \frac{\pi\,\phi(-n)}{\sin \pi n}$$

Cor 2. $$\int_0^\infty x^{n-1}\left\{\phi(0) - \frac{x^2}{\lfloor\underline{2}}\phi(2) + \frac{x^4}{\lfloor\underline{4}}\phi(4) - \&c\right\} dx = \lfloor\underline{n-1}\,\phi(-n) \times \cos\frac{\pi n}{2}.$$

Cor 3. $$\int_0^\infty \left\{\phi(0) - \frac{x}{\lfloor\underline{1}}\phi(1) + \frac{x^2}{\lfloor\underline{2}}\phi(2) - \&c\right\} \cos nx \, dx$$

$$= \phi(-1) - n^2\phi(-3) + n^4\phi(-5) - \&c.$$

Cor 4. $$\int_0^\infty \{\phi(0) - x^2\phi(2) + x^4\phi(4) - \&c\} \cos nx \, dx$$

$$= \frac{\pi}{2}\left\{\phi(-1) - \frac{n}{\lfloor\underline{1}}\phi(-2) + \frac{n^2}{\lfloor\underline{2}}\phi(-3) - \frac{n^3}{\lfloor\underline{3}}\phi(-5) + \&c\right\}$$

42.

Cor 5. $\int_0^1 x^m(1-x)^n\,dx = \dfrac{\underline{|m}\;\underline{|n}}{\underline{|m+n+1}}$

Sol. change x to $\dfrac{y}{1+y}$ and apply IV 11.

12. From IV 11 Cor 3 & 4 we see that

if $\int_0^\infty \phi(x)\cos nx\,dx = \psi(n)$, then

(i) $\int_0^\infty \psi(x)\cos nx\,dx = \dfrac{\pi}{2}\phi(n)$.

(ii) $\int_0^\infty \psi^2(x)\,dx = \dfrac{\pi}{2}\int_0^\infty \phi^2(x)\,dx$.

13. (i) $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx = \dfrac{\underline{|\frac{m-1}{2}}\;\underline{|\frac{n-1}{2}}}{2\,\underline{|\frac{m+n}{2}}}$.

(ii) $\int_0^{\frac{\pi}{2}} \cos^m x \cos nx\,dx = \dfrac{\pi\,\underline{|m}}{2^{m+1}\,\underline{|\frac{m+n}{2}}\;\underline{|\frac{m-n}{2}}}$

14. $\left(1+\dfrac{x^6}{1^6}\right)\left(1+\dfrac{x^6}{2^6}\right)\left(1+\dfrac{x^6}{3^6}\right)\left(1+\dfrac{x^6}{4^6}\right)$ &c

$$= \frac{\sinh 2\pi x - 2\sinh \pi x \cos \pi x\sqrt{3}}{4\pi^3 x^3}.$$

Sol. L.S $= \left(1+\dfrac{x^2}{1^2}\right)\left(1+\dfrac{x^2}{2^2}\right)\left(1+\dfrac{x^2}{3^2}\right)\left(1+\dfrac{x^2}{4^2}\right)$ &c

$\times \left(1+\dfrac{x^2\omega^2}{1^2}\right)\left(1+\dfrac{x^2\omega^2}{2^2}\right)\left(1+\dfrac{x^2\omega^2}{3^2}\right)$ &c

$\times \left(1+\dfrac{x^2\omega^4}{1^2}\right)\left(1+\dfrac{x^2\omega^4}{2^2}\right)\left(1+\dfrac{x^2\omega^4}{3^2}\right)$ &c

Apply II 9 Cor 1.

15. $e^x\left(\dfrac{x}{1\underline{|1}} - \dfrac{2x^2}{2\underline{|2}} + \dfrac{2^2x^3}{3\underline{|3}} - \dfrac{2^3x^4}{4\underline{|4}} + \&c\right)$

$= \dfrac{x}{\underline{|1}} + \dfrac{x^2}{\underline{|2}} + \dfrac{x^3}{\underline{|3}}\left(1+\frac{1}{3}\right) + \dfrac{x^4}{\underline{|4}}\left(1+\frac{1}{3}\right) + \dfrac{x^5}{\underline{|5}}\left(1+\frac{1}{3}+\frac{1}{5}\right) + \&c$

Sol. L.S $= e^x\int_0^1 \dfrac{1-e^{-2xz}}{2z}\,dz = \int_0^1 \dfrac{e^x - e^{x(1-2z)}}{2z}\,dz$ = R.S

CHAPTER V

If $f(x+h) - f(x) = h\,\phi'(x)$, then

$$f(x) = \phi(x) - \frac{h}{2}\phi'(x) + \frac{B_2}{\underline{\lvert 2}} h^2 \phi''(x) - \frac{B_4}{\underline{\lvert 4}} h^4 \phi^{IV}(x) + \&c$$

If $f(x+h) + f(x) = h\,\phi'(x)$, then

$$f(x) = \frac{h}{2}\phi'(x) - (2^2-1) B_2 \frac{h^2}{\underline{\lvert 2}} \phi''(x) + (2^4-1) B_4 \frac{h^4}{\underline{\lvert 4}} \phi^{IV}(x) - \&c$$

Sol. If we write e^x for $\phi(x)$, we see that the coeff^{ts} in R.S. are the same as those in the expansion of $\frac{h}{e^h-1}$ and $\frac{h}{e^h+1}$ respectively.

If

$$F_n(x) = \phi(x) - \frac{n-1}{n+1}\{\phi(x+h)+\phi(x-h)\} + \frac{(n-1)(n-3)}{(n+1)(n+3)} \times \{\phi(x+2h)+\phi(x-2h)\} - \frac{(n-1)(n-3)(n-5)}{(n+1)(n+3)(n+5)}\{\phi(x+3h)+\phi(x-3h)\} + \&c.$$

then,

i. If $f(x+h) - f(x-h) = 2h\,\phi'(x)$, then

$$f(x) = F_1(x) + \frac{1}{3}F_3(x) + \frac{1}{5}F_5(x) + \frac{1}{7}F_7(x) + \&c$$

ii. If $f(x+h) + f(x-h) = 2\phi(x)$, then

$$f(x) = F_1(x) + \frac{1}{\underline{\lvert 1}}F_3(x) + \frac{1.3}{\underline{\lvert 2}}F_5(x) + \frac{1.3.5}{\underline{\lvert 3}}F_7(x) + \&c.$$

3. If $f(x+h) + p f(x) = \phi(x)$, then

$$f(x) = \frac{\phi(x)\,\psi_0(p)}{p+1} - \frac{h}{\underline{\lvert 1}}\cdot\frac{\phi'(x)\,\psi_1(p)}{(p+1)^2} + \frac{h^2}{\underline{\lvert 2}}\cdot\frac{\phi''(x)\,\psi_2(p)}{(p+1)^3}$$

$- \&c.$ where $\psi(p)$ can be found from the expansion

$$\frac{1}{e^x+p} = \frac{\psi_0(p)}{p+1} - \frac{x}{\underline{\lvert 1}}\cdot\frac{\psi_1(p)}{(p+1)^2} + \frac{x^2}{\underline{\lvert 2}}\cdot\frac{\psi_2(p)}{(p+1)^3} - \&c$$

Sol. let $\phi(x) = e^x$, then $\frac{e^x}{e^h+p} = f(x)$.

4. $1^n - 2^n p + 3^n p^2 - 4^n p^3 + 5^n p^4 - \&c = \dfrac{\psi_n(p)}{(p+1)^{n+1}}$

Sol. $\dfrac{1}{e^x+p} = e^{-x} - pe^{-2x} + p^2 e^{-3x} - p^3 e^{-4x} + \&c$; equate the coeff^{ts} of x^n.

5. $$\psi_0(p) - \frac{n}{\lfloor\underline{1}}\cdot\frac{\psi_1(p)}{p+1} + \frac{n(n-1)}{\lfloor\underline{2}}\cdot\frac{\psi_2(p)}{(p+1)^2} - \cdots + (-1)^n\frac{\psi_n(p)}{(p+1)^n}$$
$$= (-1)^{n+1}\frac{p\,\psi_n(p)}{(p+1)^n}.$$

Sol. Multiply both sides in $\underline{V}$ 3. by e^x+p; then the coeff^t of $x^n = 0$.

6. If $\psi_n(p) = F_1(n) - pF_2(n) + p^2F_3(n) - p^3F_4(n) + \&c$
$+ (-1)^{n+1}F_n(n)\,p^{n-1}$, then i. $F_{n-r}(n) = F_{r+1}(n)$,

ii. $F_r(n-1) + nF_{r-1}(n-1) + \dfrac{n(n+1)}{\lfloor\underline{2}}F_{r-2}(n-1) + \cdots\cdots$
$+ \dfrac{\lfloor\underline{n+r-1}}{\lfloor\underline{r}\,\lfloor\underline{n-1}}F_0(n-1) = r^n.$

Sol. equate the coeff^{ts} of p^{r-1} in $\underline{V}$ 4.

iii. $F_r(n-1) = r^{n-1} - \dfrac{n}{\lfloor\underline{1}}(r-1)^{n-1} + \dfrac{n(n-1)}{\lfloor\underline{2}}(r-2)^{n-1} - \&c$
to $r+1$ terms.

Sol. multiply both sides in $\underline{V}$ 4 by $(p+1)^{n+1}$ and equate the coeff^{ts} of p^{r-1}.

7. $\psi_n(x-1)$ is the integral part of
$$\frac{x^{n+1}}{1-x}\left\{\frac{\lfloor\underline{n}}{\left(\log\frac{1}{1-x}\right)^{n+1}} - \frac{B_{n+1}}{n+1}\sin\frac{\pi n}{2}\right\}$$

Sol. $e^{-x}+e^{-2x}+e^{-3x}+\&c = \frac{1}{e^x-1} = \frac{1}{x} - \&c$

Differentiating n times we have

$1^n e^{-x} + 2^n e^{-2x} + 3^n e^{-3x} + \&c = \frac{\lfloor n}{x^{n+1}} \pm \&c$

writing $\log\frac{1}{1-x}$ for x we have

$$1^n(1-x) + 2^n(1-x)^2 + 3^n(1-x)^3 + \&c = \frac{\lfloor n}{\left(\log\frac{1}{1-x}\right)^{n+1}} \pm \&c$$

Apply V 4.

8. $\psi_n(-1) = \lfloor n$; $\psi_n(1) = 2^{n+1}(2^{n+1}-1)\frac{B_{n+1}}{n+1}\sin\frac{\pi n}{2}$. $\psi_0(p)=1$.

$\psi_1(p) = 1$

$\psi_2(p) = 1-p$

$\psi_3(p) = 1-4p+p^2$

$\psi_4(p) = 1-11p+11p^2-p^3$

$\psi_5(p) = 1-26p+66p^2-26p^3+p^4$

$\psi_6(p) = 1-57p+302p^2-302p^3+57p^4-p^5$

$\psi_7(p) = 1-120p+1191p^2-2416p^3+1191p^4-120p^5+p^6$.

Write under each term the sum of the product of its coefft. and the no. of terms from the left & the product of the coefft. of the preceding term and its no. of terms from above.

Cor. 1. $f(x)$ is the term independent of n in

$$\frac{\phi(x)+\frac{1}{n}\phi'(x)+\frac{1}{n^2}\phi''(x)+\frac{1}{n^3}\phi'''(x)+\&c}{e^{nh}+p}.$$

2. If $n \not> r$, then $F_r(x-1)$ is the coefft. of $\frac{x^{n-1}}{\lfloor n-1}$ in $e^{x(r-n)}(e^x-1)^{\alpha}$.

3. $\psi_n(p)$ is divisible by $1-p$ if n is even

4. $$\frac{p+\cos x}{1+2p\cos x+p^2} = \frac{\psi_0(p)}{p+1} - \frac{x^2}{|\underline{2}}\cdot\frac{\psi_2(p)}{(p+1)^3} + \frac{x^4}{|\underline{4}}\cdot\frac{\psi_4(p)}{(p+1)^5} - \&c$$

5. $$\frac{\sin x}{1+2p\cos x+p^2} = \frac{x}{|\underline{1}}\cdot\frac{\psi_1(p)}{(p+1)^2} - \frac{x^3}{|\underline{3}}\cdot\frac{\psi_3(p)}{(p+1)^4} + \frac{x^5}{|\underline{5}}\cdot\frac{\psi_5(p)}{(p+1)^6}$$

6. If $1^n(S_2-1) - 2^n(S_3-1) + 3^n(S_4-1) - \&c = \cos\pi n$

$- \frac{B_{n+1}}{n+1}(2^{n+1}-1)\sin\frac{\pi n}{2} + A_1 S_2 - A_2 S_3 + A_3 S_4 - \&c$

where $S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c$, then

i. $A_n + nA_{n-1} + \frac{n(n-1)}{|\underline{2}}A_{n-2} + \dots + A_0 = n^n$.

ii. $A_n = n^n - n(n-1)^n + \frac{n(n-1)}{|\underline{2}}(n-2)^n - \&c$

iii. $\frac{A_n}{|\underline{n}}$ is the coeff^t. of x^n in $(e^x-1)^n$.

iv. $\psi_n(p-1) = A_n - pA_{n-1} + p^2A_{n-2} - \&c$ to n terms

Ex. 1. $\frac{1^5}{2} + \frac{2^5}{2^2} + \frac{3^5}{2^3} + \&c = 1082$.

2. $\frac{1^5}{3} + \frac{2^5}{3^2} + \frac{3^5}{3^3} + \&c = 68\frac{1}{4}$.

9. $$\frac{x}{e^x-1} = 1 - \frac{x}{2} + B_2\frac{x^2}{|\underline{2}} - B_4\frac{x^4}{|\underline{4}} + B_6\frac{x^6}{|\underline{6}} - \&c$$

where B_n can be found from

10. $$\frac{x}{e^x+1} = \frac{x}{2} - B_2\frac{x^2}{|\underline{2}}(2^2-1) + B_4\frac{x^4}{|\underline{4}}(2^4-1) - \&c$$

Sol. $\frac{x}{e^x+1} = \frac{x}{e^x-1} - \frac{2x}{e^{2x}-1}$.

11. $$\log_e\frac{x}{e^x-1} = -\frac{x}{2} - B_2\frac{x^2}{2|\underline{2}} + B_4\frac{x^4}{4|\underline{4}} - \&c$$

Sol. $\log(e^x-1) = \int\frac{e^x}{e^x-1}dx$.

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12. $\log_e \frac{2}{e^x+1} = -\frac{x}{2} - B_2 \frac{x^2}{2\lfloor 2}(2^2-1) + B_4 \frac{x^4}{4\lfloor 4}(2^4-1) - \&c$

Sol. $\log(e^x+1) = \log(e^{2x}-1) - \log(e^x-1)$.

Ex. If P, Q, R, S &c be so small that $\frac{1}{120}$ of the sum of their cubes may be neglected, then

1. If $e^P + e^Q + e^R = 2 + e^{P+Q+R}$, then

$$\left(\frac{1}{P} + \frac{1}{Q} + \frac{1}{R}\right) + \frac{1}{12}(P+Q+R) = -\frac{1}{2}.$$

2. If $e^{P+Q+R+S} = \frac{e^P+e^Q+e^R+e^S-2}{e^{-P}+e^{-Q}+e^{-R}+e^{-S}-2}$, then

$$\left(\frac{1}{P} + \frac{1}{Q} + \frac{1}{R} + \frac{1}{S}\right) + \frac{1}{12}(P+Q+R+S) = 0.$$

3. If $2e^{P+Q+R+S+T} =$

$$\frac{(e^P+e^Q+e^R+e^S+e^T-2)^2 - (e^{2P}+e^{2Q}+e^{2R}+e^{2S}+e^{2T}-2)}{e^{-P}+e^{-Q}+e^{-R}+e^{-S}+e^{-T}-2}$$

then $\left(\frac{1}{P} + \frac{1}{Q} + \frac{1}{R} + \frac{1}{S} + \frac{1}{T}\right) + \frac{1}{12}(P+Q+R+S+T)$

$= \frac{1}{2}$

13. $x \cot x = 1 - B_2 \frac{(2x)^2}{\lfloor 2} - B_4 \frac{(2x)^4}{\lfloor 4} - B_6 \frac{(2x)^6}{\lfloor 6} - \&c$

Sol. Change x to xi in V 9.

14. $x \operatorname{Cosec} x = 1 + B_2 \frac{x^2(2^2-2)}{\lfloor 2} + B_4 \frac{x^4(2^4-2)}{\lfloor 4} + \&c$

Sol. $\operatorname{Cosec} x = \cot \frac{x}{2} - \cot x$

15. $x \tan x = B_2 \frac{2^2-1}{\lfloor 2}(2x)^2 + B_4 \frac{2^4-1}{\lfloor 4}(2x)^4 + \&c$

Sol. $\tan x = \cot x - 2\cot 2x$.

16. $\log_e \frac{x}{\sin x} = B_2 \frac{(2x)^2}{2\lfloor 2} + B_4 \frac{(2x)^4}{4\lfloor 4} + B_6 \frac{(2x)^6}{6\lfloor 6} + \&c$

Sol. $\log \sin x = \int \cot x \, dx$.

17. $\log_e \sec x = B_2 \frac{2^2-1}{2\lfloor 2}(2x)^2 + B_4 \frac{2^4-1}{4\lfloor 4}(2x)^4 + \&c$

Sol. $\log \sec x = \int \tan x \, dx$.

N. B. 1. From the nature of the coeff.ts, we see that $B_0 = -1$.

2. $\frac{B_n}{B_{n-2}} = \frac{n(n-1)}{4\pi^2}$ nearly if n is great.

Sol. Since $\cot \pi$ is $-\infty$, $B_{n-2} \frac{(2\pi)^{n-2}}{\lfloor n-2} \div B_n \frac{(2\pi)^n}{\lfloor n} = 1$ nearly if n is great.

Similarly we can prove that

3. $\frac{B_n}{B_{n-h}} = \frac{\lfloor n}{\lfloor n-h} \cdot \frac{1}{(2\pi)^h}$ nearly if n is great.

18. $(2n+1) B_{2n} = 2 B_2 B_{2n-2} \frac{2n(2n-1)}{\lfloor 2} + 2 B_4 B_{2n-4} \times \frac{2n(2n-1)(2n-2)(2n-3)}{\lfloor 4} + \&c$ the last term being $2 B_{n-1} B_{n+1} \frac{\lfloor 2n}{\lfloor n-1 \lfloor n+1}$ or $(B_n)^2 \frac{\lfloor 2n}{(\lfloor n)^2}$ according as n is odd or even.

Sol. $\cot^2 x = -\left(1 + \frac{d \cot x}{dx}\right)$; equate the coeff.ts of x^{2n-2}.

$B_0 = -1$; $B_2 = \frac{1}{6}$; $B_4 = \frac{1}{30}$; $B_6 = \frac{1}{42}$; $B_8 = \frac{1}{30}$; $B_{10} = \frac{5}{66}$.
$B_{12} = \frac{691}{2730}$; $B_{14} = \frac{7}{6}$; $B_{16} = \frac{3617}{510}$; $B_{18} = \frac{43867}{798}$; $B_{20} = \frac{174611}{330}$; $B_{22} = \frac{854513}{138}$; $B_{24} = \frac{236364091}{2730}$;
$B_{26} = \frac{8553103}{6}$; $B_{28} = \frac{23749461029}{870}$; $B_{30} = \frac{8615841276005}{14322}$; $B_{32} = \frac{7709321041217}{510}$;

$$B_{34} = \frac{2577687858367}{6};$$

$$B_{36} = \frac{26315271553053477373}{1919190}$$

$$B_{38} = \frac{2929993913841559}{6}; \text{ \&c } B_\infty = \infty$$

19. If n be an even integer,

i. B_n is a fraction & $2(2^n-1)B_n$ is an integer.

ii. The numerator of B_n in its lowest terms is divisible by the greatest odd measure of n prime to ~~(2^n-1)~~ the denominator and the quotient is a prime number.

iii. The denominator of B_n is the continued product of prime numbers next to the factors of n including unity and the number itself.

20. $B_n + (-1)^{\frac{n}{2}}(1-F_n)$ is an integer where F_n is the sum of the reciprocals of prime numbers next to the factors of n including unity and the no. itself. Let this integer be represented by I_n; then

$I_0 = I_2 = I_4 = I_6 = I_8 = I_{10} = I_{12} = 0$; $I_{14} = 1$; $I_{16} = 7$; $I_{18} = 55$

$I_{20} = 529$; $I_{22} = 6192$; $I_{24} = 86580$; $I_{26} = 1425517$.

Eg 1. Given that B_{22} lies between 6160 & 6200; find the true value of B_{22}.

Sol. The fractional part of $B_{22} = \left(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{23}\right) = \frac{17}{138}$

Since B_{22} is divisible by 11 it must be one of the numbers $6170\frac{17}{138}$, $6181\frac{17}{138}$, $6192\frac{17}{138}$; but the first two of these are composite even after divided by 11. $\therefore B_{22} = 6192\frac{17}{138} = \frac{854513}{138}$.

2. Find the fractional part of B_{200}.

Sol. The even factors of 200 are 2, 4, 8, 10, 20, 40, 50, 100, 200

$\therefore B_{200} + \left(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} - \frac{1}{11} - \frac{1}{41} - \frac{1}{101}\right)$ is an integer.

$\therefore$ The fractional part $= \frac{216641}{1366530}$.

21. To form prime numbers:—

2	3	5 7	11 13 17 19 23 29 31	7
	5	11 13	41 43 47 ~~49~~ 53 59 61	37
	7	17 19	71 73 ~~77~~ 79 83 89 ~~91~~	67
	~~9~~	~~23~~ ~~25~~	101 103 107 109 113 ~~119~~ ~~121~~	97
		29 31	131 ~~133~~ 137 139 ~~143~~ 149 151	127
		~~35~~	~~161~~ 163 167 ~~169~~ 173 179 181	157
			191 193 197 199 ~~203~~ ~~209~~ 211	~~187~~
				~~217~~

0	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67
0	71	73	79	83	89	97	1	3	7	9	13	27	31	37	39	49	51	57	63
1	67	73	79	81	91	93	97	99	11	23	27	29	33	39	41	51	57	63	69
2	71	77	81	83	93	7	11	13	17	31	37	47	49	53	59	67	73	79	83
3	89	97	1	9	19	21	31	33	39	43	49	57	61	63	67	79	87	91	99
5	3	9	21	23	41	47	57	63	69	71	77	87	93	99	1	7	13	17	19
6	31	41	43	47	53	59	61	73	77	83	91	1	9	19	27	33	39	43	51
7	57	61	69	73	87	97	9	11	21	23	27	29	39	53	57	59	63	77	81
8	83	87	7	11	19	29	37	41	47	53	67	71	77	83	91	97	9	13	19
10	21	31	33	39	49	51	61	63	69	87	91	93	97	3	9	17	23	29	51
11	53	63	71	81	87	93	1	13	17	23	29	31	37	49	59	77	79	83	89
12	91	97	1	3	7	19	21	27	61	67	73	81	99	9	23	27	29	33	39
14	47	51	53	59	71	81	83	87	89	93	99	11	23	31	43	49	53	59	67
15	71	79	83	97	1	7	9	13	19	21	27	37	57	63	67	69	93	97	99
17	9	21	23	33	41	47	53	59	77	83	87	89	1	11	23	31	47	61	67
18	71	73	77	79	89	1	7	13	31	[illegible]	49	51	73	79	87	93	97	99	3
20	11	17	27	29	39	53	63	69	81	83	87	89	99	11	13	29	31	37	41
21	43	53	61	79	3	7	13	21	37	39	43	51	67	69	73	81	87	93	97
23	9	11	33	39	41	47	51	57	71	77	81	83	89	93	99	11	17	23	37
24	41	47	59	67	73	77	3	21	31	39	43	49	51	57	79	91	93	9	17
26	21	33	47	57	59	63	71	77	83	87	89	93	99	7	11	13	19	29	31
27	41	49	53	67	77	89	91	97	1	3	19	33	37	43	51	57	61	79	87
28	97	3	9	17	27	39	53	57	63	69	71	99	1	7	19	23	37	41	49
30	61	67	79	83	89	9	19	21	37	53	67	69	81	87	91	3	9	17	21
32	29	51	53	57	59	71	99	1	7	13	19	23	29	31	43	47	59	61	71
33	73	89	91	7	13	33	49	57	61	63	67	69	91	99	11	17	27	29	33
35	39	41	47	57	59	71	81	83	93	7	13	17	23	31	37	43	59	71	73
36	77	91	97	1	9	19	27	33	39	61	67	69	79	93	97	3	21	23	33
38	47	51	53	63	77	81	89	7	11	17	19	23	29	31	43	47	67	89	1
40	3	7	13	19	21	27	49	51	57	73	79	91	93	99	11	27	29	33	39
41	53	57	59	77	1	11	17	19	29	31	41	43	53	59	61	71	73	83	89
42	97	27	37	39	49	57	63	73	91	97	9	21	23	41	47	51	57	63	81
44	83	93	7	13	17	19	23	47	49	61	67	83	91	97	3	21	37	39	43
46	49	51	57	63	73	79	91	3	21	23	29	33	51	59	83	87	89	93	99
48	1	13	17	31	61	71	77	89	3	9	19	31	33	37	43	51	57	67	69

22. If $\sec x = E_1 + \frac{x^2}{\underline{|2}}E_3 + \frac{x^4}{\underline{|4}}E_5 + \frac{x^6}{\underline{|6}}E_7 + \&c$ and consequently $\frac{2}{e^x + e^{-x}} = E_1 - \frac{x^2}{\underline{|2}}E_3 + \frac{x^4}{\underline{|4}}E_5 - \&c$, then

$$\frac{B_{2n}}{2n}2^{2n}(2^{2n}-1) = 2E_1E_{2n-1} + 2E_3E_{2n-3}\frac{(2n-2)(2n-3)}{\underline{|2}}$$

$+ \&c$ the last term being $2E_{n-1}E_{n+1}\frac{\underline{|2n-2}}{\underline{|n-2}\,\underline{|n}}$ or $(E_n)^2\frac{\underline{|2n-2}}{(\underline{|n-1})^2}$ according as n is even or odd.

$E_1 = 1$; $E_3 = 1$; $E_5 = 5$; $E_7 = 61$; $E_9 = 1385$; $E_{11} = 505$

$E_{13} = 2702765$; $E_{15} = 199360981$ &c & $E_\infty = \infty$

Sol. $\frac{d\tan x}{dx} = \sec^2 x$; equate the coeff^{ts} of x^{2n}

23. i. $\frac{1}{1^2-x^2} + \frac{1}{2^2-x^2} + \frac{1}{3^2-x^2} + \&c = \frac{1}{2x^2} - \frac{\pi}{2x}\cot \pi x$

ii. $\frac{1}{1^2-x^2} + \frac{1}{3^2-x^2} + \frac{1}{5^2-x^2} + \&c = \frac{\pi}{4x}\tan\frac{\pi x}{2}$

iii. $\frac{1}{1^2-x^2} - \frac{1}{2^2-x^2} + \frac{1}{3^2-x^2} - \&c = \frac{\pi}{2x}\operatorname{cosec}\pi x - \frac{1}{2x^2}$

iv. $\frac{1}{1^2-x^2} - \frac{3}{3^2-x^2} + \frac{5}{5^2-x^2} - \&c = \frac{\pi}{4}\sec\frac{\pi x}{2}$

Sol. change x to πx in II 60.

24. i. $\frac{1}{1^2+x^2} + \frac{1}{2^2+x^2} + \frac{1}{3^2+x^2} + \&c = \frac{\pi}{2x}\cdot\frac{e^{\pi x}+e^{-\pi x}}{e^{\pi x}-e^{-\pi x}} - \frac{1}{2x^2}$

ii. $\frac{1}{1^2+x^2} + \frac{1}{3^2+x^2} + \frac{1}{5^2+x^2} + \&c = \frac{\pi}{4x}\cdot\frac{e^{\pi x}-1}{e^{\pi x}+1}$

iii. $\frac{1}{1^2+x^2} - \frac{1}{2^2+x^2} + \frac{1}{3^2+x^2} - \&c = \frac{1}{2x^2} - \frac{\pi}{x(e^{\pi x}-e^{-\pi x})}$

iv. $\frac{1}{1^2+x^2} - \frac{3}{3^2+x^2} + \frac{5}{5^2+x^2} - \&c = \frac{\frac{\pi}{2}}{e^{\frac{\pi x}{2}} + e^{-\frac{\pi x}{2}}}$

Sol. change x to xi in V. 23

N. B. 1. If n be of the form $4m+1$, E_n ends in 5 and in 1 if of the form $4m-1$; $E_n - 1$ is always divisible by 4 if n be any positive integer.

$\frac{E_{n+2}}{E_n} = \frac{4n(n+1)}{\pi^2}$ nearly if n is great.

Sol. $\sec\frac{\pi}{2} = \infty$; $\therefore \frac{\left(\frac{\pi}{2}\right)^{n+1}}{\lfloor\underline{n+1}} E_{n+2} \div \frac{\left(\frac{\pi}{2}\right)^{n-1}}{\lfloor\underline{n-1}} E_n = 1$ nearly if n is great. Similarly we can prove that

$$\frac{E_{n+h+1}}{E_{n+1}} = \left(\frac{2}{\pi}\right)^h \cdot \frac{\lfloor\underline{n+h}}{\lfloor\underline{n}} \text{ (if } n \text{ is great) nearly}$$

25. i. $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c = \frac{(2\pi)^n}{2\lfloor\underline{n}} B_n = S_n$

ii. $\frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \&c = \frac{(2^n-1)\pi^n}{2\lfloor\underline{n}} B_n = s_n$

iii. $\frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \&c = \frac{(2^n-2)\pi^n}{2\lfloor\underline{n}} B_n = S'_n$

iv. $\frac{1}{1^n} - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \&c = \frac{\pi^n}{2^{n+1}\lfloor\underline{n-1}} E_n = s'_n$

N. B. From $\overline{V}$ 18 and 22 we know the values of B_n and E_n only for even and odd integers; but from 25 for all positive values of n. For the values of B_n & E_n if n be negative see chap. and to find $\lfloor\underline{n}$ for all values of n see chap.

Cor 1. If n be a positive quantity not less than unity the values of B_n are known from $\overline{V}$ 25. i & ii.

E.G. $B_1 = \infty$; $B_{1\frac{1}{2}} = \frac{3}{4\pi\sqrt{2}} S_{1\frac{1}{2}}$; $B_3 = \frac{3}{2\pi^3} S_3$ &c.

2. If n be not a negative quantity the values of B_n and E_n are known from 25. iii & iv.

E.G. $B_0 = -1$; $B_{\frac{1}{2}} = -\left(1+\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)$.

$E_0 = \infty$; $E_{\frac{1}{2}} = 2\sqrt{2}\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} - \&c\right)$.

$E_2 = \frac{8}{\pi^2}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c\right)$ &c &c.

21. $\frac{1}{(a+b)^n} + \frac{1}{(a+2b)^n} + \frac{1}{(a+3b)^n} + \&c = \frac{1}{b(n-1)a^{n-1}} - \frac{1}{2a^n}$

$+ B_2 \frac{n}{\underline{|2}} \cdot \frac{b}{a^{n+1}} - B_4 \frac{n(n+1)(n+2)}{\underline{|4}} \cdot \frac{b^3}{a^{n+3}} + \&c$

From this we can sum up the reciprocals of powers of all numbers in A.P. approximately.

Sol. Let $L.S = \phi(a)$, then $\phi(a-b) - \phi(a) = \frac{1}{a^n}$; apply V

N. B. $S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \cdots + \frac{1}{(r-1)^n} + \frac{1}{2r^n}$

$+ \frac{1}{(n-1)r^{n-1}} + B_2 \frac{n}{\underline{|2}} \cdot \frac{1}{r^{n+1}} - B_4 \frac{n(n+1)(n+2)}{\underline{|4}} \cdot \frac{1}{r^{n+3}} + \&c$

$S_2 = 1.6449340668$ $\qquad \frac{1}{B_1} = 0$; $\frac{1}{B_2} = 6$.

$S_3 = 1.2020569031$

$S_4 = 1.0823232337$ $\qquad \frac{1}{B_3} = 17.19624$.

$S_5 = 1.0369277551$ $\qquad \frac{1}{B_4} = 30$; $\frac{1}{B_5} = 39.34953$

$S_6 = 1.0173430620$

$S_7 = 1.0083492774$ $\qquad \frac{1}{B_6} = 42$; $\frac{1}{B_7} = 38.03538$

$S_8 = 1.0040773562$ $\qquad \frac{1}{B_8} = 30$; $\frac{1}{B_9} = 20.98719$

$S_9 = 1.0020083928$

$S_{10} = 1.0009945781$ $\qquad \frac{1}{B_{10}} = 13.2$

Cor. 1. $n S_{n+1} = 1$ if $n = 0$ and $S_{n+1} - \frac{1}{n} = .577$ nearly

Sol. Write $n+1$ for n and 1 for r in the above theorem, then we have $S_{n+1} - \frac{1}{n} = \frac{1}{2} + B_2 \frac{n+1}{\underline{|2}} - \&c$

$= \frac{1}{2} + \frac{1}{12} - \frac{1}{120} + \&c = .577$ nearly when n vanishes

2. $\pi n B_{n+1} = 1$ when $n = 0$

Sol. $n S_{n+1} = \frac{(2\pi)^n}{\underline{|n+1}} \pi n B_{n+1} = 1$ when $n = 0$

i.e. $\pi n B_{n+1} = 1$ when n approaches 0.

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$\cdot \frac{1}{1-a_2} \cdot \frac{1}{1-a_3} \cdot \frac{1}{1-a_5} \cdot \frac{1}{1-a_7} \cdot \frac{1}{1-a_{11}} \cdot \frac{1}{1-a_{13}} \cdot \frac{1}{1-a_{17}}$ &c

where 2, 3, 5, 7 &c are prime numbers.

$= 1 + a_2 + a_3 + a_2 a_2 + a_5 + a_2 a_3 + a_7 + a_2 a_2 a_2 +$ &c

where the suffixes are natural numbers resolved into prime numbers.

28. $(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})$ &c $= \frac{1}{S_n}$.

Sol. Write $\frac{1}{p^n}$ for a_p in 27. Similarly writing x^p for a_p we can get,

29. $$\cancel{\frac{1}{(1-x^2)(1-x^3)(1-x^5)(1-x^7)(1-x^{11})(1-x^{13})(1-x^{17})\ \&c}}$$

$$= \cancel{1+} \frac{x^2}{1-x} + \frac{x^{2+3}}{(1-x)(1-x^2)} + \frac{x^{2+3+5}}{(1-x)(1-x^2)(1-x^3)}$$

$$+ \frac{x^{2+3+5+7}}{\cancel{(1-x)(1-x^2)(1-x^3)(1-x^4)}} + \&c.$$

Cor. 1. $(1 + \frac{1}{2^n})(1 + \frac{1}{3^n})(1 + \frac{1}{5^n})$ &c $= \frac{S_n}{S_{2n}}$.

2. $\frac{2^n+1}{2^n-1} \cdot \frac{3^n+1}{3^n-1} \cdot \frac{5^n+1}{5^n-1}$ &c $= \frac{(S_n)^2}{S_{2n}}$.

3. $\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{8^n} + \frac{1}{11^n} + \frac{1}{12^n} +$ &c

where 2, 3, 5, 7 &c are natural numbers containing an odd number of prime factors

$= \frac{(S_n)^2 - S_{2n}}{2 S_n}$ where $S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} +$ &c

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Sol. Invert both sides in 28 and cor1. and find the difference after applying 27.

Ex. 1. i $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\&c = \frac{\pi^2}{6}$.

ii. $\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\&c = \frac{\pi^3}{32}$

iii. $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\&c = \frac{\pi^4}{96}$.

2. If 2, 3, 5, 7 &c be prime no.s.

i. $\frac{2^2+1}{2^2-1}\cdot\frac{3^2+1}{3^2-1}\cdot\frac{5^2+1}{5^2-1}\ \&c = \frac{5}{2}$.

ii. $(1+\frac{1}{2^4})(1+\frac{1}{3^4})(1+\frac{1}{5^4})\ \&c = \frac{105}{\pi^4}$.

3. If 2, 3, 5, 7, 8 &c be natural numbers containing an odd no. of prime factors.

i. $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}+\&c = \frac{\pi^2}{20}$.

ii. $\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{8^4}+\&c = \frac{\pi^4}{1260}$.

Cor 4. $\frac{3^n}{3^n+1}\cdot\frac{5^n}{5^n-1}\cdot\frac{7^n}{7^n+1}\cdot\frac{11^n}{11^n+1}\cdots\frac{p^n}{p^n-\sin\frac{\pi p}{2}}\cdots$ ad. inf.

where p is a prime number.

$= \frac{1}{1^n}-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\&c$

Cor 5. $$\frac{\frac{\log_e 1}{1^n}+\frac{\log_e 2}{2^n}+\frac{\log_e 3}{3^n}+\&c}{\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\&c} = \frac{\log 2}{2^n-1}+\frac{\log 3}{3^n-1}+\frac{\log 5}{5^n-1}+\&c$$ where 2, 3, 5, 7 are prime numbers.

~~Sol.~~ Differentiate both sides in 28.

Ex. $\frac{1}{2}\sin\frac{2\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \frac{1}{7}\sin\frac{7\pi}{2} + \&c$

is a Convergent Series, 2, 3, 5 being prime no. s.

30. $(1+a_2)(1+a_3)(1+a_5)(1+a_7)(1+a_{11})\ \&c$

$= 1 + a_2 + a_3 + a_5 + a_2 a_3 + a_7 + a_2 a_5 + a_{11} + a_{13} + \&c$

where the Suffixes are natural no. s resolved into prime factors no two of which are alike.

Cor 1. $\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \&c = \frac{S_n}{S_{2n}}$.

2. $\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \frac{1}{19^n} + \frac{1}{23^n}$
$+ \frac{1}{29^n} + \frac{1}{30^n} + \frac{1}{31^n} + \frac{1}{37^n} + \&c = \frac{(S_n)^2 - S_{2n}}{2 S_n S_{2n}}$.

where 2, 3, 5, 7 &c are natural no. s containing an odd no. of prime factors no two of which are alike.

3. $\frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{12^n} + \&c = \frac{S_n (S_{2n} - 1)}{S_{2n}}$,

where 4, 8, 9, 12 &c are Composite numbers containing at least two equal prime numbers.

Cor. 1. The Sum of the reciprocals of all prime numbers is infinite.

Sol. Putting $n = 1$ in V 28, we have,

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$$\frac{2}{2-1}\cdot\frac{3}{3-1}\cdot\frac{5}{5-1}\cdot\frac{7}{7-1}\ \&c = 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\&c$$

$$\therefore \log\frac{2}{2-1}+\log\frac{3}{3-1}+\log\frac{5}{5-1}+\&c = \log\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\&c\right).$$

i.e. $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\&c$ + a finite quantity $=\infty$

i.e The sum of the reciprocals of all prime nos $=\infty$

2. If 2, 3, 5, 7 &c be primes, then when n vanishes

$$\left(\log_e n+\frac{1}{2^{n+1}}+\frac{1}{3^{n+1}}+\frac{1}{5^{n+1}}+\&c\right) \text{ is finite.}$$

Sol. Changing n to $n+1$ in V 28 we have

$$\left(\frac{1}{1-\frac{1}{2^{n+1}}}\right)\left(\frac{1}{1-\frac{1}{3^{n+1}}}\right)\left(\frac{1}{1-\frac{1}{5^{n+1}}}\right)\&c = S_{n+1}$$

$$\therefore \log\left(1-\frac{1}{2^{n+1}}\right)+\log\left(1-\frac{1}{3^{n+1}}\right)+\log\left(1-\frac{1}{5^{n+1}}\right)+\&c = -\log S_{n+1}$$

$= -\log_e n$ when n is very small.

$$\therefore \log_e n+\frac{1}{2^{n+1}}+\frac{1}{3^{n+1}}+\frac{1}{5^{n+1}}+\&c = -.312 \text{ nearly when } n=0.$$

3. If P_n be the nth prime number, then

$\frac{P_n}{n}-\log_e n$ ~~is finite if n is infinite.~~

Sol. Let S_n be the sum of n prime numbers.

Then $S_2=5;\ S_4=17;\ S_6=41;\ S_8=77;\ S_{10}=129$ &c

$P_3=5;\ P_7=17;\ P_{13}=41;\ P_{21}=73;\ P_{31}=127$ &c

$\therefore \frac{P_{n^2+n+1}}{S_{2n}} = 1$ if n is very great.

$\therefore \frac{P_n}{n}-\log_e n$ ~~is finite if $n=\infty$.~~

CHAPTER VI

Let $f(1) + f(2) + f(3) + f(4) + \cdots\cdots + f(x) = \phi(x)$, then

$$\phi(x) = c + \int f(x)\,dx + \tfrac{1}{2}f(x) + \frac{B_2}{\lfloor 2}f'(x) - \frac{B_4}{\lfloor 4}f'''(x) + \frac{B_6}{\lfloor 6}f^{V}(x) - \frac{B_8}{\lfloor 8}f^{VII}(x) + \&c$$

Sol. $\phi(x) - \phi(x-1) = f(x)$; apply VI.

N.B. By giving any value to x, c can be found.

R.S. is not a terminating series except in some special cases. Consequently no constant can be found in $\frac{1}{2}f(x) + \frac{B_2}{\lfloor 2}f'(x) - \frac{B_4}{\lfloor 4}f'''(x) + \&c$ except in those special cases. If R.S be a terminating series, it must be some integral function of x. In this case there is no possibility of a constant (according to the ordinary sense) in $\phi(x)$; for $\phi(1) = f(1) + \phi(0)$: But $\phi(1) = f(1)$. $\therefore$ $\phi(0)$ is always 0 whether $\phi(x)$ is rational or irrational. $\therefore$ When $\phi(x)$ is a rational integral function of (x) it must be divisible and hence no constant but 0 can exist. The algebraic constant of a series is the constant obtained by completing the remaining part in the above theorem. We can substitute this constant which is like the centre of gravity of a body instead of its divergent infinite series.

E.G. The constant of the Series $1+1+1+$ &c $=-\frac{1}{2}$; for the Sum to x terms $= x = C + \int 1\,dx + \frac{1}{2}$. $\therefore C = -\frac{1}{2}$

We may also find the Constant thus:—

$$C = 1+2+3+4+ \text{ \&c}$$

$$\therefore 4C = \quad 4 \quad + 8 + \text{ \&c}$$

$$\therefore -3C = 1-2+3-4+ \text{ \&c} = \frac{1}{(1+1)^2} = \frac{1}{4}.$$

$$\therefore C = -\frac{1}{12}$$

2. $$\phi(x) + \sum_{n=0}^{n=\infty} \frac{B_n}{\lfloor n} f^{n-1}(x) \cos\frac{\pi n}{2} = 0$$

Sol. Let $\frac{B_n}{\lfloor n}\psi(n)$ be the coeff^{t}. of $f^{n-1}(x)$, then we see $\psi(0)=1$, $\psi(2)=-1$, $\psi(4)=1$, $\psi(6)=-1$ &c

$\psi(3)=0$, $\psi(5)=0$, $\psi(7)=0$; $\frac{B_1}{\lfloor 1}\psi(1)=\frac{1}{2}$; but $B_1=\infty$

$\therefore \psi(1)=0$. Again by V 26 cor 2, we have

$\pi(n-1)B_n = 1$ when $n=1$ $\therefore \frac{B_n\psi(n)}{\lfloor n} = \frac{\pi(n-1)B_n}{\lfloor n}\cdot\frac{\psi(n)}{\pi(n-1)}$

$=\frac{1}{2}$ when $n=1$, i.e $\frac{\psi(n)}{\pi(n-1)} = \frac{1}{2}$ when $n=1$.

$\therefore \psi(n) = -\cos\frac{\pi n}{2}$.

3. The sum to a negative number of terms is the sum with the sign changed, calculated backwards from the term previous to the first to the given number of terms with positive sign instead of negative.

Sol. $\phi(x) = f(1)+f(2)+\cdots\ +f(x+n)$

$\quad - f(1+n) - f(2+n) - \cdots\ - f(x+n)$.

change x to $-x$ and put $n = x$, then we have

$\phi(-x) = \phi(0) - \{f(0) + f(-1) + f(-2) + \cdots f(-x+1)\}$;

but $\phi(0) = 0$.

E.G. $1 + 2 + 3 +$ &c to -5 terms

$= -(0-1-2-3-4) = \underline{10}$

6. i. For finding the sum to a fractional number of terms assume the sum to be true always and if there is any difficulty in finding $\phi(x)$, take n any integer you choose, find $\phi(n+x)$ and then subtract $\{f(1+x) + f(2+x) + f(3+x) + \cdots + f(n+x)\}$ from the result

ii. $\phi(h) = \phi(n) - \{f(1+h) + f(2+h) + \cdots + f(n+h)\} + h\,f(n) + \frac{\Sigma h}{\lfloor 1}f'(n) + \frac{\Sigma h^2}{\lfloor 2}f''(n) +$ &c where n is any integer or infinity.

E.G. 1. $1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{h}$

$= \left(1 + \frac{1}{2} + \cdots \frac{1}{n}\right) - \left(\frac{1}{1+h} + \frac{1}{2+h} + \cdots \frac{1}{n+h}\right)$ when $n = \infty$

$= C_0 + \log_e n - \left(\frac{1}{1+h} + \frac{1}{2+h} + \cdots \frac{1}{n+h}\right)$ when $n = \infty$

where C_0 is the constant of $\Sigma \frac{1}{n}$

2. $\lfloor h = \dfrac{n^h}{(1+\frac{h}{1})(1+\frac{h}{2})\cdots(1+\frac{h}{n})}$ when $n = \infty$.

Sol. $\lfloor h = \dfrac{\lfloor n+h}{\lfloor n} \cdot \dfrac{\lfloor n \lfloor h}{\lfloor n+h} = \dfrac{n^h(1+\frac{1}{n})(1+\frac{2}{n})\cdots(1+\frac{h}{n})}{(1+\frac{h}{1})(1+\frac{h}{2})\cdots(1+\frac{h}{n})}$

$\therefore \lfloor h \div (1+\frac{1}{n})(1+\frac{2}{n})\cdots(1+\frac{h}{n}) = \dfrac{n^h}{(1+\frac{h}{1})(1+\frac{h}{2})\cdots(1+\frac{h}{n})}$

iii. $\phi(h) = x f(1) - x^{1+h} f(1+h) + x^2 f(2) - x^{2+h} f(2+h) + \&c$

5. Def. A series is said to be corrected when its constant is subtracted from it.

The differential Coeff^t. of a series is a corrected series

i.e $\dfrac{d\{\phi(1)+\phi(2)+\cdots+\phi(x)\}}{dx} = \phi'(1)+\phi'(2)+\cdots$
$+\phi'(x) - c'$ where c' is the constant of $\phi'(1)+\phi'(2)$
$+\phi'(3)+\cdots+\phi'(x)$.

Sol. In the diff^l. coeff^t. of $\phi(1)+\phi(2)+\cdots\phi(x)$ there can't be any constant. Therefore it should be corrected.

N.B. If $f(1)+f(2)+\cdots+f(x)$ be a convergent series then its constant is the sum of the series

E.G. 1. $\dfrac{d\left(1+\frac{1}{2}+\cdots+\frac{1}{x}\right)}{dx} = \dfrac{1}{(x+1)^2}+\dfrac{1}{(x+2)^2}+\dfrac{1}{(x+3)^2}+\&c$

Sol. $\dfrac{d\,\Sigma\frac{1}{x}}{dx} = -\frac{1}{1^2}-\frac{1}{2^2}-\cdots-\frac{1}{x^2}-c$

$= \dfrac{1}{(x+1)^2}+\dfrac{1}{(x+2)^2}+\&c$

2. If c_0 be the constant of $\Sigma\frac{1}{x}$, then

$\dfrac{d\,\lfloor x}{dx} = \lfloor x\left(\Sigma\frac{1}{x}-c_0\right)$

Sol. $\dfrac{d\,\lfloor x}{dx} = \lfloor x\,\dfrac{d\log_e\lfloor x}{dx} = \lfloor x\left(\Sigma\frac{1}{x}-c_0\right)$.

3. $\int_0^x \Sigma \frac{1}{x}\, dx = \log_e \lfloor x + x C_0$.

4. $\int_0^x (1^{13} + 2^{13} + \cdots + x^{13})\, dx = \frac{1}{14}(1^{14} + 2^{14} + \cdots x^{14}) - \frac{x}{12}$.

5. $$\frac{d(1^{10} + 2^{10} + \cdots x^{10})}{dx} = 10(1^9 + 2^9 + \cdots x^9) + \frac{10}{132}.$$

6. $$\int_0^x (\sqrt{1} + \sqrt{2} + \cdots \sqrt{x})\, dx = \frac{2}{3}(1\sqrt{1} + 2\sqrt{2} + \cdots x\sqrt{x})$$
$$- \frac{x}{4\pi}\left(\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \cdots \&c\right).$$

6. If $f^n(x)$ stands for the nth derivative of $f(x)$ and C_n be the constant of $\{f^n(1) + f^n(2) + \cdots f^n(x)\}$ then $\phi(x) = -C_1 x - C_2 \frac{x^2}{\lfloor 2} - C_3 \frac{x^3}{\lfloor 3} - C_4 \frac{x^4}{\lfloor 4} - \&c$

Sol. $\phi(x) = \phi(0) + \frac{x}{\lfloor 1}\phi'(0) + \frac{x^2}{\lfloor 2}\phi''(0) + \&c$

But from VI 5 we have $\phi(0) = 0$, $\phi'(0) = -C_1$, $\phi''(0) = -C_2$ &c

E.g. 1. $\log_e \lfloor x = -S_1 x + \frac{S_2}{2}x^2 - \frac{S_3}{3}x^3 + \&c$ where S_n is the constant of $\left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c\right)$.

2. $\Sigma \frac{1}{x} = S_2 x - S_3 x^2 + S_4 x^3 - \&c$ where $S_n = \frac{1}{1^n} + \frac{1}{2^n} + \&c$

N.B. This is very useful in finding $\phi(x)$ for fractional values of x.

7. If C'_n be the constant of
$$f\left(\tfrac{1}{n}\right) + f\left(\tfrac{2}{n}\right) + f\left(\tfrac{3}{n}\right) + \cdots + f\left(\tfrac{x}{n}\right), \text{ then}$$

$\phi\left(\frac{x}{n}\right) + \phi\left(\frac{x-1}{n}\right) + \phi\left(\frac{x-2}{n}\right) + \cdots + \phi\left(\frac{x-n+1}{n}\right) - xc$

$= f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots f\left(\frac{x}{n}\right) - c'_n$

Sol. Let $\psi(x) = \phi\left(\frac{x}{n}\right) + \phi\left(\frac{x-1}{n}\right) + \cdots + \phi\left(\frac{x-n+1}{n}\right)$, then

$\psi(x) - \psi(x-1) = \phi\left(\frac{x}{n}\right) - \phi\left(\frac{x-n}{n}\right) = f\left(\frac{x}{n}\right)$

$\therefore$ $\psi(x)$ & $f\left(\frac{x}{n}\right) + f\left(\frac{x-1}{n}\right) + \cdots + f\left(\frac{1}{n}\right)$ differ only by some constant; hence if these be corrected they must be equal. $\psi(x)$ contains n terms each each of which is of the form $\phi(y)$ whose constant is c. $\therefore$ The constant of $\psi(x)$ is nc & the constant of $f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots f\left(\frac{x}{n}\right)$ is c'_n by our supposition.

Cor. 1. $\phi\left(-\frac{1}{n}\right) + \phi\left(-\frac{2}{n}\right) + \cdots + \phi\left(-\frac{n-1}{n}\right) = nc - c'_n$

Sol. Put $x = 0$ in the above theorem.

2\. i. $\phi\left(-\frac{1}{2}\right) = 2c - c'_2$.

ii. $c = c_0 = c'_1$.

iii. $\phi\left(-\frac{1}{3}\right) + \phi\left(-\frac{2}{3}\right) = 3c - c'_3$

iv. $\phi\left(-\frac{1}{4}\right) + \phi\left(-\frac{3}{4}\right) = 2c + c'_2 - c'_4$.

v. $\phi\left(-\frac{1}{6}\right) + \phi\left(-\frac{5}{6}\right) = c + c'_2 + c'_3 - c'_4$.

8\. $\phi\left(x - \frac{1}{2}\right) = c + \int f(x)\,dx - \left(1 - \frac{1}{2}\right)\frac{B_2}{\underline{|2}} f'(x) + \left(1 - \frac{1}{2^3}\right)\frac{B_4}{\underline{|4}} f'''(x)$

$- \&c = \sum_{n=0}^{n=\infty} \left\{\left(1 - \frac{1}{2^{n-1}}\right)\frac{B_n}{\underline{|n}} f^{n-1}(x) \cos\frac{\pi n}{2}\right\}$

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Sol. Put $n=2$, change x to $2x$ and apply VI 1.

9. i. $S(a_1+a_2+a_3+$ &c) means that the series is a convergent series and its sum to infinity is required

ii. $C(a_1+a_2+a_3+$ &c) means that the series is a divergent series and its constant is reqd.

iii. $G(a_1+a_2+a_3+$ &c) means that the series is oscillating or divergent and the value of its generating function is required.

N.B. Hereafter the series will only be given omitting S, C or G and from the nature of the series we should infer whether C, S or G is reqd.; moreover if a series appear to be equal to a finite quantity we must select S, C, or G from the nature of the series.

10. i. The value of an oscillating series is only true when the series is deduced from a regular series. For example the series $1-1+1-1+$ &c $=\frac{1}{2}$ only when it is deduced from a regular series of the form $\phi(1)-\phi(2)+\phi(3)-$ &c. Again if we take an irregular series $a^x-b^x+c^x-d^x$ + &c we get the same series $1-1+1-1+$ &c when x becomes 0; yet its value is not $\frac{1}{2}$ in this case

ii. $a_1-a_2+a_3-a_4+$ &c is not equal to the series $(a_1-a_2)+(a_3-a_4)+(a_5-a_6)+$ &c or to the series

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$a_1 - (a_2 - a_3) - (a_4 - a_5) - (a_6 - a_7) - \&c$; but to the series $a_1 - (a_2 - a_3 + a_4 - \&c)$

e.g. $1 - 2 + 3 - 4 + \&c$ is not equal to $(1-2) + (3-4) + (5-6) + \&c$ or to $1 - (2-3) - (4-5) - \&c$

iii. $(a_1 - a_2 + a_3 - \&c) \pm (b_1 - b_2 + b_3 - \&c)$

$= (a_1 \pm b_1) - (a_2 \pm b_2) + (a_3 \pm b_3) - \&c$

Ex. 1. Shew that $(a_1 - a_2 + a_3 - \&c) + (b_1 - b_2 + \&c)$

$= a_1 + (b_1 - a_2) - (b_2 - a_3) + (b_3 - a_4) - \&c$

Sol. L. S $= a_1 + (b_1 - b_2 + b_3 - \&c) - (a_2 - a_3 + \&c)$

$= a_1 + (b_1 - a_2) - (b_2 - a_3) + \&c$

2. $a_1 - a_2 + a_3 - a_4 + \&c = \frac{a_1}{2} + \frac{1}{2}\{(a_1 - a_2) - (a_2 - a_3) + \&c\}$

3. $= \frac{3a_1 - a_2}{4} + \frac{1}{4}\{(a_1 - 2a_2 + a_3) - (a_2 - 2a_3 + a_4) + \&c\}$

4 $= \frac{7a_1 - 4a_2 + a_3}{8} + \frac{1}{8}\{(a_1 - 3a_2 + 3a_3 - a_4) - (a_2 - 3a_3 + 3a_4 - a_5) + (a_3 - 3a_4 + 3a_5 - a_6) - \&c\}$

11. $a_1 - a_2 + a_3 - a_4 + \&c$

$= \frac{a_1}{2} + \frac{a_1 - a_2}{4} + \frac{a_1 - 2a_2 + a_3}{8} + \&c$

$= xa_1 - x^2 a_2 + x^3 a_3 - x^4 a_4 + \&c$

$= x . \frac{a_1}{2} + x^2 . \frac{a_1 - a_2}{4} + x^3 \frac{a_1 - 2a_2 + a_3}{8} + \&c$

when x approaches unity.

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12. If $\frac{a_2}{a_3}$ lies between $\frac{a_1}{a_2}$ & $\frac{a_3}{a_4}$, then $a_1-a_2+a_3-a_4+$ &c lies between $\frac{a_1^2}{a_1+a_2}$ & $a_1-\frac{a_2^2}{a_2+a_3}$

e.g. $1-2+3-4+$ &c lies between $\frac{1}{3}$ & $\frac{1}{5}$ and its values is $\frac{1}{4}$. $\lfloor\underline{0}-\lfloor\underline{1}+\lfloor\underline{2}-\lfloor\underline{3}+$ &c lies between $\frac{1}{2}$ & $\frac{2}{3}$; its value is $\frac{3}{5}$ very nearly.

But $2-2\frac{1}{2}+3\frac{1}{3}-4\frac{1}{4}+5\frac{1}{5}-$ &c cannot lie between $\frac{2^2}{2+2\frac{1}{2}}$ & $2-\frac{(2\frac{1}{2})^2}{2\frac{1}{2}+3\frac{1}{3}}$ as $\frac{2\frac{1}{2}}{3\frac{1}{3}}$ is not lying between $\frac{2}{2\frac{1}{2}}$ & $\frac{3\frac{1}{3}}{4\frac{1}{4}}$. i.e it cannot lie between $.889$ & $.929$ as its value is 1.193

13. $\phi_1(x)+\phi_2(x)+\phi_3(x)+$ &c can be expanded in ascending powers of x, say $A_0+A_1x+A_2x^2+$ &c where each of $A_1, A_2,$ &c is a series.

Case I when A_n is a convergent series

(1) If $A_0+A_1x+A_2x^2+$ &c be a rapidly conver-gent series what is required is got.

(2) But if it is a slowly convergent or an oscillating series, convergent or divergent (at least for some values of x)

(a). Change x into a suitable function of y so that the new series in ascending powers of y may be a rapidly convergent series;

e.g. let $\frac{x}{1+\frac{x}{2}} = y$, then $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c$

$= y + \frac{y^3}{12} + \frac{y^5}{80} + \frac{y^7}{448} + \&c$

(b) or convert it into a continued fraction

e.g. $x - \frac{x^2}{3} + \frac{2}{15}x^3 - \frac{17}{315}x^4 + \&c = \frac{x}{1+}\frac{x}{3+}\frac{x}{5+}\&c$

$\frac{1}{x} - \frac{|1}{x^2} + \frac{|2}{x^3} - \frac{|3}{x^4} + \&c = \frac{1}{x+1-}\frac{1^2}{x+3-}\frac{2^2}{x+5-\&c}$

(c) or transform it into another series by applying III 8; e.g. $\frac{1}{x} - \frac{2}{x^2} + \frac{5}{x^3} - \frac{15}{x^4} + \&c$

$= \frac{1}{x+1} - \frac{1}{(x+1)(x+2)} + \frac{1}{(x+1)(x+2)(x+3)} - \&c$

(d) or take the reciprocal of the series and try to make it a rapidly convergent series in any way.

Case II When A_n is an oscillating (convergent or divergent) or a pure divergent series.

(1) Let C_n be the constant or the value of its generating function. Then the given series $= \psi(x) + C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \&c$ where $\psi(x)$ can be found in special cases.

(2) But if $C_0 + C_1 x + C_2 x^2 + \&c$ be a divergent series find some function of n (say P_n) such that the value of $P_0 + P_1 x + P_2 x^2 + \&c$ may be easily

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found and $c_n - P_n$ may rapidly diminish as n increases. Then the given series =

$$F(x) + (c_0 - P_0) + (c_1 - P_1)x + (c_2 - P_2)x^2 + \&c.$$

e.g. 1. $\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \&c = \frac{1}{x}(1-1+1-\&c)$

$- \frac{1}{x^2}(1-2+3-\&c) = \frac{1}{2x} - \frac{1}{4x^2} + \&c$

2. $\frac{1}{1^2-x^2} + \frac{1}{2^2-x^2} + \frac{1}{3^2-x^2} + \&c = -\frac{1}{x^2}(1+1+1+\&c)$

$- \frac{1}{x^4}(1^2+2^2+3^2+\&c) - \frac{1}{x^6}(1^4+2^4+3^4+\&c) = \psi(x)$

$+ \frac{1}{2x^2} = \frac{1}{2x^2} - \frac{\pi \cot \pi x}{2x}.$

3. $\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \&c = (1+1+1+\&c)$

$- x(\log 1 + \log 2 + \&c) + \&c = -\frac{1}{2} - x \log\sqrt{2\pi} - \&c$

$= \frac{1}{x-1} + 1 + x + x^2 + \&c - \frac{1}{2} - x\log\sqrt{2\pi} - \&c$

$= \frac{1}{x-1} + \frac{1}{2} + (1 - {\cdot}91894)x - \&c$

$= \frac{1}{x-1} + \frac{1}{2} + {\cdot}08106x - \&c$

4. i. $\frac{x}{e^x+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{3x}+1} + \&c$

$= \log 2 - \frac{x}{4} + (B_2)^2 \frac{x^2(2^2-1)}{2\,\underline{|2}} + (B_4)^2 \frac{x^4(2^4-1)}{4\,\underline{|4}} +$

$(B_6)^2 \frac{x^6(2^6-1)}{6\,\underline{|6}} + \&c$

Sol. $\frac{x}{e^x+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{3x}+1} + \frac{x}{e^{4x}+1} + \&c$

$= \frac{x}{2}(1+1+1+\&c) - B_2 \frac{x^2(2^2-1)}{\underline{|2}}(1+2+3+\&c)$

$+ B_4 \frac{x^4(2^4-1)}{\underline{|4}}(1^3+2^3+3^3+\&c) - \&c.$

$= \psi(x) - \frac{x}{4} + (B_2)^2 \frac{x^2(2^2-1)}{2\lfloor 2} + (B_4)^2 \frac{x^4(2^4-1)}{4\lfloor 4} + \&c$

Now it is reqd. to find $\psi(x)$.

The given series $= \frac{x}{e^x-1} - \frac{x}{e^{2x}-1} + \frac{x}{e^{3x}-1} - \&c$

$= \log_e 2$ + terms involving x & higher powers of x. $\therefore \psi(x) = \log_e 2$.

ii. $\frac{x}{e^x-1} + \frac{x}{e^{2x}-1} + \frac{x}{e^{3x}-1} + \frac{x}{e^{4x}-1} + \&c$

$= C - \log_e x + \frac{x}{4} - (B_2)^2 \frac{x^2}{2\lfloor 2} - B_4^2 \frac{x^4}{4\lfloor 4} - B_6^2 \frac{x^6}{6\lfloor 6} - \&c$

Sol. Proceeding as in the previous theorem we have the series $= \psi(x) + C + \frac{x}{4}$

$- B_2^2 \frac{x^2}{2\lfloor 2} - B_4^2 \frac{x^4}{4\lfloor 4} - \&c$.

But we know $\frac{x}{e^x+1} + \frac{x}{e^{2x}+1} + \frac{x}{e^{3x}+1} + \&c$

$= \left(\frac{x}{e^x-1} + \frac{x}{e^{2x}-1} + \&c\right) - \left(\frac{2x}{e^{2x}-1} + \frac{2x}{e^{4x}-1} + \&c\right)$

$\therefore \psi(x) - \psi(2x) = \log 2$; hence $\psi(x) = -\log_e x$.

Ex. 1: Shew that the constant in the series

$\sqrt[100]{1} + \sqrt[100]{2} + \sqrt[100]{3} + \sqrt[100]{4} + \cdots\cdots + \sqrt[100]{x}$

is $-.4909100$

2. $\frac{1}{2+1} + \frac{1}{2^2+1} + \frac{1}{2^3+1} + \&c \doteq \frac{3}{4} + \frac{\log_e 2}{48}$ nearly

3. $\frac{1}{1+\frac{10}{9}} + \frac{1}{1+(\frac{10}{9})^2} + \frac{1}{1+(\frac{10}{9})^3} + \&c = 6.331009$.

4. $\frac{1}{\frac{10}{9}-1} + \frac{1}{\left(\frac{10}{9}\right)^2-1} + \frac{1}{\left(\frac{10}{9}\right)^3-1} + \&c = 27$ nearly.

15. i. $\frac{1}{x-1} + \frac{1}{x^2-1} + \frac{1}{x^3-1} + \&c$

$= \frac{1}{x}\cdot\frac{x+1}{x-1} + \frac{1}{x^4}\cdot\frac{x^2+1}{x^2-1} + \frac{1}{x^9}\cdot\frac{x^3+1}{x^3-1} + \&c$

ii. $\frac{1}{x-1} - \frac{1}{x^2-1} + \frac{1}{x^3-1} - \frac{1}{x^4-1} + \&c$

$= \frac{1}{x}\cdot\frac{x^2+1}{x^2-1} - \frac{1}{x^4}\cdot\frac{x^4+1}{x^4-1} + \frac{1}{x^9}\cdot\frac{x^6+1}{x^6-1} - \&c$

Sol. $\frac{1}{x-1} = \frac{1}{x-1}$

$\pm\frac{1}{x^2-1} = \pm\left\{\frac{1}{x^2} + \frac{1}{x^2(x^2-1)}\right\}$

$\frac{1}{x^3-1} = \frac{1}{x^3} + \frac{1}{x^6} + \frac{1}{x^6(x^3-1)}$

$\pm\frac{1}{x^4-1} = \pm\left\{\frac{1}{x^4} + \frac{1}{x^8} + \frac{1}{x^{12}} + \frac{1}{x^{12}(x^4-1)}\right\}$

&c &c &c

Adding up all the terms we can get the results.

16. $\frac{r}{1-ax} + \frac{r^2}{1-ax^2} + \frac{r^3}{1-ax^3} + \&c$ to n terms

$= \frac{arx}{1-ax} + \frac{(arx^2)^2}{1-ax^2} + \frac{(arx^3)^3}{1-ax^3} + \&c$ to n terms

$+ \frac{r-r^{n+1}}{1-r} + a\,\frac{(rx)^2-(rx)^{n+1}}{1-rx} + a^2\,\frac{(rx^2)^3-(rx^2)^{n+1}}{1-rx^2} + \&c$ to n terms.

Sol. $\frac{r}{1-ax} = \frac{arx}{1-ax} + r.$

$\frac{r^2}{1-ax^2} = \frac{(arx^2)^2}{1-ax^2} + r^2 + ar^2x^2.$

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$$\frac{r^3}{1-ax^3} = \frac{(arx^3)^3}{1-ax^3} + r^3 + ar^3x^3 + a^2r^3x^6.$$

&c &c &c

Adding up all the terms in the n rows we can get the result.

Cor. $\frac{r}{1-ax} + \frac{r^2}{1-ax^2} + \frac{r^3}{1-ax^3} + \&c$

$$= \frac{arx}{1-ax} + \frac{(arx^2)^2}{1-ax^2} + \frac{(arx^3)^3}{1-ax^3} + \&c$$

$$+ \frac{r}{1-r} + \frac{a(rx)^2}{1-rx} + \frac{a^2(rx^2)^3}{1-rx^2} + \frac{a^3(rx^3)^4}{1-rx^3} + \&c$$

17. $\frac{a}{1-m} + \frac{(a+b)n}{1-mx} + \frac{(a+2b)n^2}{1-mx^2} + \frac{(a+3b)n^3}{1-mx^3} + \&c$

$$= a.\frac{1-mn}{(1-m)(1-n)} + (a+b)\frac{1-mnx^2}{(1-mx)(1-nx)}(mnx)$$

$$+ (a+2b)\frac{1-mnx^4}{(1-mx^2)(1-nx^2)}(mnx^2)^2 + (a+3b)\frac{1-mnx^6}{(1-mx^3)(1-nx^3)}$$

$$+ \&c + \frac{b}{m}\left\{\frac{mn}{(1-n)^2} + \frac{(mnx)^2}{(1-nx)^2} + \frac{(mnx^2)^3}{(1-nx^2)^2} + \&c\right.$$

Cor 1. $\frac{a}{1-n} + \frac{(a+b)n}{1-nx} + \frac{(a+2b)n^2}{1-nx^2} + \&c$

$$= a.\frac{1+n}{1-n} + (a+b)\frac{1+nx}{1-nx}.(n^2x) + (a+2b)\frac{1+nx^2}{1-nx^2}(n^2x^2)^2 + \&c$$

$$+ b\left\{\frac{n}{(1-n)^2} + \frac{n^3x^2}{(1-nx)^2} + \frac{n^5x^6}{(1-nx^2)^2} + \frac{n^7x^{12}}{(1-nx^3)^2} + \&c\right\}$$

2. If A_n denotes the no. of factors in n including 1 & n then $\frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \&c = \frac{1}{x-1} + \frac{1}{x^2-1} + \&c$ and hence deduce VI 15 i

CHAPTER VII

1. $1^n + 2^n + 3^n + 4^n + 5^n + \cdots + x^n = \phi_n(x)$

$$\phi_n(x) = \frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2} + \frac{x^{n+1}}{n+1} + \frac{x^n}{2} + B_2\frac{n}{\lfloor 2}x^{n-1} - B_4\frac{n(n-1)(n-2)}{\lfloor 4}$$

$$\times x^{n-3} + B_6\frac{n(n-1)(n-2)(n-3)}{\lfloor 6}x^{n-5} - \&c$$

Sol. The corrected series is found by applying $\overline{\text{VI}}$ 6.

The coeff.^{t} of $x^{n-n} = -\frac{\lfloor n}{\lfloor n-n \lfloor n+1}\cdot B_{n+1}\cos\frac{\pi(n+1)}{2}$.

$\therefore$ The values of the corrected series when $x=0$

$= -\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$ by $\overline{\text{IV}}$ 10 Cor. But $\phi_n(0) = 0$.

$\therefore$ The constant $= \frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$.

N.B. If C_n be the constant then $C_{-n} = S_n$ and consequently S_{-n} is invariably written for this constant.

2. $1^n - 2^n + 3^n - 4^n + \&c = (2^{n+1} - 1)\frac{B_{n+1}}{n+1}\sin\frac{\pi n}{2}$.

Sol. $(1 - 2^{n+1})C_n = (1^n + 2^n + \&c) - 2^{n+1}(1^n + 2^n + \&c)$

$= 1^n - 2^n + 3^n - \&c$

Cor. $\phi_{n-1}(-\tfrac{1}{2}) = 2\left(1 - \frac{1}{2^n}\right)\frac{B_n}{n}\cos\frac{\pi n}{2}$.

Sol. $\phi_n(-\tfrac{1}{2}) = 1^n - (\tfrac{1}{2})^n + 2^n - (1\tfrac{1}{2})^n + \&c$

$= -\frac{1}{2^n}(1^n - 2^n + 3^n - \&c) = \left(2 - \frac{1}{2^n}\right)\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2}$.

3. $(a+b)^n + (a+2b)^n + (a+3b)^n + \&c \ldots$ [illegible]

$= b^n\left\{\phi_n\left(x + \frac{a}{b}\right) - \phi_n\left(\frac{a}{b}\right)\right\}$

Sol. L. S $= b^n\left\{\left(1 + \frac{a}{b}\right)^n + \left(2 + \frac{a}{b}\right)^n + \cdots \left(x + \frac{a}{b}\right)^n\right\} =$ R. S.

4. $\frac{B_{1-n}}{1-n}\sin\frac{\pi n}{2} = S_n = \frac{(2\pi)^n}{2\lfloor n}B_n$

From this we can find B_n for negative values of n

Sol. $\frac{B_{1+n}}{1+n}\cos\frac{\pi(1+n)}{2}$ is the constant of $1^n+2^n+3^n+\&c$

$\therefore$ $\frac{B_{1-n}}{1-n}\cos\frac{\pi(1-n)}{2}$ is that of $\frac{1}{1^n}+\frac{1}{2^n}+\&c = S_n$.

$\therefore$ $\frac{B_{1-n}}{1-n}\sin\frac{\pi n}{2} = \frac{(2\pi)^n}{2\lfloor n}B_n$.

Cor. 1. $B_{-2}=2S_3$; $B_{-4}=-4S_5$; $B_{-6}=6S_7$; $B_{-8}=-8S_9$ &c

2. $\lfloor -\frac{1}{2} = \sqrt{\pi}$; Sol. $-\frac{B_{1\frac{1}{2}}}{1\frac{1}{2}}\sin\frac{\pi}{4} = \frac{(2\pi)^{-\frac{1}{2}}}{2\lfloor -\frac{1}{2}}B_{-\frac{1}{2}}$

Again $\frac{B_{-\frac{1}{2}}}{-\frac{1}{2}}\sin\frac{\pi}{4} = \frac{(2\pi)^{1\frac{1}{2}}}{2\lfloor 1\frac{1}{2}}B_{1\frac{1}{2}}$; multiplying the two results we have $\frac{2}{3} = \frac{2}{3}\cdot\frac{\pi}{(\lfloor -\frac{1}{2})^2}$ $\therefore$ $\lfloor -\frac{1}{2} = \sqrt{\pi}$.

3. In a similar manner we can prove that

$\lfloor n-1 \lfloor -n = \pi \operatorname{cosec} \pi n$.

4. $\pi\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{6}} - \frac{1}{\sqrt{6}+\sqrt{8}} + \&c\right)$

$= \frac{1}{1\sqrt{1}} + \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} + \frac{1}{7\sqrt{7}} + \&c$

Sol. L.S $= \frac{\pi}{\sqrt{2}}\left\{1-(\sqrt{2}-1)+(\sqrt{3}-\sqrt{2})-(\sqrt{4}-\sqrt{3})+\&c\right\}$

$= \pi\sqrt{2}\left(\sqrt{1}-\sqrt{2}+\sqrt{3}-\sqrt{4}+\&c\right) = 2(2\sqrt{2}-1)\frac{B_{1\frac{1}{2}}}{1\frac{1}{2}}\cdot\frac{\pi}{2}$

$= \left(1-\frac{1}{2\sqrt{2}}\right)\left(\frac{1}{1\sqrt{1}}+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\&c\right)$

$= \frac{1}{1\sqrt{1}} + \frac{1}{3\sqrt{3}} + \frac{1}{5\sqrt{5}} + \&c$

5. $$\frac{2\pi\left(\frac{1}{\sqrt[3]{1}}-\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}-\frac{1}{\sqrt[3]{4}}+\&c\right)}{\left(\sqrt[3]{4\pi}+\sqrt[3]{2\pi}\right)\left(\frac{1}{\sqrt[3]{1}}-\frac{1}{\sqrt[3]{4}}+\frac{1}{\sqrt[3]{9}}-\&c\right)}=\lfloor\!\underline{-\tfrac{1}{3}}$$

6. $$\sqrt{2+4x}-\left(\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{x}}\right)$$
$$=(\sqrt{2}+1)\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\&c\right)$$ when x is great

7. $$\frac{2}{3}\sqrt{(x+\tfrac{1}{4})(x+\tfrac{1}{2})(x+\tfrac{3}{4})}-(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{x})$$
$$=\frac{1}{4\pi}\left(\frac{1}{1\sqrt{1}}+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\&c\right)$$

8. $$\frac{2}{5}\sqrt{x(x+\tfrac{1}{4})(x+\tfrac{1}{2})(x+\tfrac{3}{4})(x+1)+\tfrac{5}{768}(x+\tfrac{1}{2})}$$
$$-(1\sqrt{1}+2\sqrt{2}+3\sqrt{3}+\cdots+x\sqrt{x})$$
$$=\frac{3}{16\pi^2}\left(\frac{1}{1\sqrt{1}}+\frac{1}{4\sqrt{2}}+\frac{1}{9\sqrt{3}}+\frac{1}{16\sqrt{4}}+\&c\right)$$

5. $$(a+b)^n-(a+2b)^n+(a+3b)^n-\&c=b^n\left\{\phi_n\left(\frac{a}{2b}\right)-\phi_n\left(\frac{a-b}{2b}\right)\right\}$$

6. i. $$\frac{(x^2+x)^n}{2}=\frac{n}{\lfloor\!\underline{1}}\phi_{2n-1}(x)+\frac{n(n-1)(n-2)}{\lfloor\!\underline{3}}\phi_{2n-3}(x)+\frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor\!\underline{5}}\phi_{2n-5}(x)+\&c$$

ii. $$\frac{(x+\frac{1}{2})(x^2+x)^n}{2}=\frac{(n+\frac{1}{2})}{\lfloor\!\underline{1}}\phi_{2n}(x)+\frac{n(n-1)(n-\frac{1}{2})}{\lfloor\!\underline{3}}\phi_{2n-2}(x)+\frac{n(n-1)(n-2)(n-3)(n-\frac{3}{2})}{\lfloor\!\underline{5}}\phi_{2n-4}(x)+\&c$$

Sol. $$\frac{(x^2+x)^n-(x^2-x)^n}{2}=\frac{n}{\lfloor\!\underline{1}}x^{2n-1}+\frac{n(n-1)(n-2)}{\lfloor\!\underline{3}}x^{2n-3}+\&c$$
change x to $x-1$, $x-2$ &c up to 1 & add up all the terms

$$\frac{(x+\frac{1}{2})(x^2+x)^n-(x-\frac{1}{2})(x^2-x)^n}{2}=\frac{x}{2}\left\{(x^2+x)^n-(x^2-x)^n\right\}+\frac{1}{4}\left\{(x^2+x)^n+(x^2-x)^n\right\}$$ & proceed as in i.

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Cor. If $x^2+x=y$ & $x+\frac{1}{2}=a$, then

1. $\phi_1(x)=\frac{y}{2}$; $\phi_2(x)=a\frac{y}{3}$; $\phi_3(x)=\frac{y^2}{4}$; $\phi_4(x)=\frac{a}{5}y(y-\frac{1}{3})$

$\phi_5(x)=\frac{y^2}{6}(y-\frac{1}{2})$; $\phi_6(x)=\frac{a}{7}y(y^2-y+\frac{1}{3})$; $\phi_7(x)=\frac{y^2}{8}(y^2-\frac{4}{3}y+\frac{2}{3})$

$\phi_8(x)=\frac{a}{9}y(y^3-2y^2+\frac{9}{5}y-\frac{3}{5})$; $\phi_9(x)=\frac{y^2}{10}(y-1)(y^2-\frac{3}{2}y+\frac{3}{2})$

$\phi_{10}(x)=\frac{a}{11}y(y-1)(y^3-\frac{7}{3}y^2+\frac{10}{3}y-\frac{5}{3})$

$\phi_{11}(x)=\frac{y^2}{12}(y^4-4y^3+8\frac{1}{2}y^2-10y+5)$

2\. i. $\left(\frac{1+\sqrt{5}}{2}\right)^9+\left(\frac{3+\sqrt{5}}{2}\right)^9+\cdots\cdots+\left(\frac{2x-1+\sqrt{5}}{2}\right)^9=\phi_9\left(\frac{2x-1+\sqrt{5}}{2}\right)$

ii. $\left(\frac{1+\sqrt{5}}{2}\right)^{10}+\left(\frac{3+\sqrt{5}}{2}\right)^{10}+\cdots\cdots+\left(\frac{2x-1+\sqrt{5}}{2}\right)^{10}=\phi_{10}\left(\frac{2x-1+\sqrt{5}}{2}\right)$

iii. If n be even then

$$1^n+3^n+5^n+7^n+\cdots\cdots+(2p-1)^n=2^n\phi_n(p-\tfrac{1}{2}).$$

7\. If n is a positive integer excluding zero

$$\phi_n(x-1)+(-1)^n\phi_n(-x)=0.$$

Sol. Let L. S $=\psi(x)$; then $\psi(x+1)-\psi(x)=0$

Cor. If $n>1$, then $\phi_n(x)$ is divisible by $\frac{x^2(x+1)^2}{4}$ or $\frac{x(x+\frac{1}{2})(x+1)}{3}$ according as n is odd or even

8\. $\phi_n(x)=-nxS_{1-n}-\frac{n(n-1)}{\lfloor 2}x^2S_{2-n}-\frac{n(n-1)(n-2)}{\lfloor 3}x^3S_{3-n}$

$-$ &c $=-B_n x\cos\frac{\pi n}{2}-\frac{n}{\lfloor 2}B_{n-1}x^2\sin\frac{\pi n}{2}+$

$\frac{n(n-1)}{\lfloor 3}B_{n-2}x^3\cos\frac{\pi n}{2}+\frac{n(n-1)(n-2)}{\lfloor 4}B_{n-3}x^4\sin\frac{\pi n}{2}$

$-$ &c; Sol, Apply VI 6.

9. $\phi_r(x) = 1^r - (1+x)^r + 2^r - (2+x)^r + \&c$

10. $\phi_r(x) - n^r \left\{ \phi_r\left(\frac{x}{n}\right) + \phi_r\left(\frac{x-1}{n}\right) + \phi_r\left(\frac{x-2}{n}\right) + \cdots + \phi_r\left(\frac{x-n+1}{n}\right) \right\}$

$= (n^{r+1} - 1)\frac{B_{r+1}}{r+1}\sin\frac{\pi r}{2}$ or $(1 - n^{r+1})S_{-r}$

Sol. Apply $\overline{\text{VI}}$ 7.

Cr. $\phi_r\left(-\frac{1}{n}\right) + \phi_r\left(-\frac{2}{n}\right) + \phi_r\left(-\frac{3}{n}\right) + \cdots + \phi_r\left(-\frac{n-1}{n}\right)$

$= (n - n^{-r})S_{-r}$

11. If r is a negative integer, then

$\phi_r(x-1) + (-1)^r\phi_r(-x) = \{1 + (-1)^r\}S_{-r} + \frac{(-1)^r}{\underline{|-r-1}}\, d^{\,\pi\cot\pi x}_{-(r+1)x}$

Sol. $\phi_1(x-1) - \phi_{-1}(-x) = -\pi\cot\pi x$ by $\overline{\text{II}}$ 10.

Differentiate both sides r times.

N.B. The above theorem is true even for positive integral values of r and hence $\overline{\text{VII}}$ 7 can be deduced from $\overline{\text{VII}}$ 11.

N.B. The following method is very useful in finding the derivatives of $\pi\cot\pi x$. Let $\pi\cot\pi x = y$; then the coeff.ts in the coeff.t of π^n are the same as those in the expansion of $(\tan^{-1}\frac{1}{y})^{-n}$.

Each derivative is divisible by y^2+1 so that the last term can be exactly found.

Write under each term the quotient obtained

18.

πy by dividing the sum of the products of the

$\pi^2(y^2+1)$ coefft. and the index of that term and of

$\pi^3(y^3+y)$ the preceding term by the index of π.

$\pi^4\left(y^4+\frac{4}{3}y^2+\frac{1}{3}\right)$

$\pi^5\left(y^5+\frac{5}{3}y^3+\frac{2}{3}y\right)$

$\pi^6\left(y^6+2y^4+\frac{17}{15}y^2+\frac{2}{15}\right)$

$\pi^7\left(y^7+\frac{7}{3}y^5+\frac{77}{45}y^3+\frac{17}{45}y\right)$

$\pi^8\left(y^8+\frac{8}{3}y^6+\frac{12}{5}y^4+\frac{248}{315}y^2+\frac{17}{315}\right)$

$\pi^9\left(y^9+3y^7+\frac{16}{5}y^5+\frac{88}{63}y^3+\frac{62}{315}y\right)$

$\pi^{10}\left(y^{10}+\frac{10}{3}y^8+\frac{37}{9}y^6+\frac{424}{189}y^4+\frac{1382}{2835}y^2+\frac{62}{2385}\right)$

Cr. For all values of n

i. $\phi_n(x) - 2^n\left\{\phi_n\left(\frac{x}{2}\right)+\phi_n\left(\frac{x-1}{2}\right)\right\} = (1-2^{n+1})\,S_{-n}$.

ii. $\phi_n\left(-\frac{1}{2}\right) = \left(2-\frac{1}{2^n}\right)S_{-n}$.

iii. $\phi_n\left(-\frac{1}{3}\right)+\phi_n\left(-\frac{2}{3}\right) = \left(3-\frac{1}{3^n}\right)S_{-n}$

iv. $\phi_n\left(-\frac{1}{4}\right)+\phi_n\left(-\frac{3}{4}\right) = \left(2+\frac{1}{2^n}-\frac{1}{4^n}\right)S_{-n}$

v. $\phi_n\left(-\frac{1}{6}\right)+\phi_n\left(-\frac{5}{6}\right) = \left(1+\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{6^n}\right)S_{-n}$

Ex. If n is a positive odd integer shew that

i. $\phi_n\left(-\frac{1}{3}\right) = \left(3-\frac{1}{3^n}\right)\frac{S_{-n}}{2}$.

ii. $\phi_n\left(-\frac{1}{4}\right) = \left(1+\frac{1}{2^{n+1}}-\frac{1}{2^{2n+1}}\right)S_{-n}$

iii. $\phi_n\left(-\frac{1}{6}\right) = \left(1+\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{6^n}\right)\frac{S_{-n}}{2}$

iv. $\phi_n\left(-\frac{1}{5}\right)+\phi_n\left(-\frac{2}{5}\right) = \left(5-\frac{1}{5^n}\right)\frac{S_{-n}}{2}$.

v. $\phi_n\left(-\frac{1}{8}\right)+\phi_n\left(-\frac{3}{8}\right) = \left(2+\frac{1}{2^{2n+1}}-\frac{1}{2^{3n+1}}\right)S_{-n}$.

vi. $\phi_n(-\frac{1}{10}) + \phi_n(-\frac{3}{10}) = (5 + \frac{1}{5^n} - \frac{1}{10^n})\frac{S_{-n}}{2}$.

vii. $\phi_n(-\frac{1}{12}) + \phi_n(-\frac{5}{12}) = (6 + \frac{1}{6^n} - \frac{1}{12^n})\frac{S_{-n}}{2}$.

12. $2^n\{\phi_n(-\frac{1}{6}) - \phi_n(-\frac{5}{6})\} = (2^n+1)\{\phi_n(-\frac{1}{3}) - \phi_n(-\frac{2}{3})\}$.

Sol. $\left.\begin{aligned} \phi_n(-\tfrac{1}{3}) - 2^n\{\phi_n(-\tfrac{1}{6}) + \phi_n(-\tfrac{2}{3})\} &= (2^{n+1}-1)S_{-n} \\ \phi_n(-\tfrac{2}{3}) - 2^n\{\phi_n(-\tfrac{1}{3}) + \phi_n(-\tfrac{5}{6})\} &= (2^{n+1}-1)S_{-n} \end{aligned}\right\}$ by $\overline{\text{vii}}$

$\therefore 2^n\{\phi_n(-\frac{1}{6}) - \phi_n(-\frac{5}{6})\} = (2^n+1)\{\phi_n(-\frac{1}{3}) - \phi_n(-\frac{2}{3})\}$

N.B. Since all these theorems and the following theorems are true for all values of n, the properties of $\Sigma\frac{1}{x}$, $\lfloor x$, $\frac{1}{1^n}+\frac{1}{2^n}+\cdots\frac{1}{x^n}$ &c &c are only their particular cases.

Ex. 1. $\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c = \frac{7}{8}S_3$

2. $\frac{1}{1^3} + \frac{1}{4^3} + \frac{1}{7^3} + \&c = \frac{2}{81\sqrt{3}}\pi^3 + \frac{13}{27}S_3$

3. $\frac{1}{1^3} + \frac{1}{5^3} + \frac{1}{9^3} + \&c = \frac{\pi^3}{64} + \frac{7}{16}S_3$

4. $\frac{1}{1^3} + \frac{1}{7^3} + \frac{1}{13^3} + \&c = \frac{\pi^3}{36\sqrt{3}} + \frac{91}{216}S_3$

13. If $C.$ be the constant of $\frac{(\log 1)^n}{1} + \frac{(\log 2)^n}{2} + \&c$ then $S_{n+1} = \frac{1}{n} + C_0 - \frac{n}{\lfloor 1}C_1 + \frac{n^2}{\lfloor 2}C_2 - \frac{n^3}{\lfloor 3}C_3 + \&c$

$= \frac{1}{n} + .5772156649 + .0728158455n - (.00485n^2 + .00034n^3) + E$,

where E, the error is less than $(\frac{n}{10})^4$.

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Sol. It is proved in V 26 Cor. 1. that $S_{n+1} - \frac{1}{n}$ is finite when $n = 0$; the remaining part is obtained from VI 13. N.B. The theorem is true for all values of n.

Ex. 1. $S_{1+n} + S_{1-n} = \dfrac{2C_0}{1 + .00839 n^2 + .0001 n^4 + \&c}$

2. $\dfrac{1}{1\sqrt[10]{1}} + \dfrac{1}{2\sqrt[10]{2}} + \dfrac{1}{3\sqrt[10]{3}} + \&c = 10.5844842$

3. $\dfrac{1}{1\sqrt{1}} + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + \&c = 2.6123752$ correct.

4. $\dfrac{1}{1^2\sqrt{1}} + \dfrac{1}{2^2\sqrt{2}} + \dfrac{1}{3^2\sqrt{3}} + \&c = 1.341490$

5. $B_{1\frac{1}{2}} = .4409932$; $B_{\frac{1}{2}} = -1.032627$

6. $B_{\frac{1}{3}} = -.9420745$; $B_{-\frac{1}{3}} = -1.3841347$

7. $B_{-\frac{1}{2}} = -1.847228$.

14. $\dfrac{1}{2(2^n - 1)} + \dfrac{1}{3(3^n - 1)} + \dfrac{1}{4(4^n - 1)} + \&c$

$= \dfrac{.7946786 - \log n}{n} + .2113922$

$- .0060680 n - .0000028 n^3 + \&c$

Sol. We can easily prove that $L.S = \dfrac{C - \log n}{n} + \&c$ where C is the constant in $\dfrac{1}{2\log 2} + \dfrac{1}{3\log 3} + \dfrac{1}{4\log 4} + \&c$. If $n = 1$ then $L.S = \dfrac{1}{1.2} + \dfrac{1}{2.3} + \&c = 1$; hence C is known

Cor. 1. $\dfrac{1}{2\log_e 2} + \dfrac{1}{3\log_e 3} + \cdots + \dfrac{1}{n\log_e n}$

$= .7946786 + \log\log(n + \tfrac{1}{2})$ nearly.

2. $\dfrac{1}{2^{n+1}\log_e 2} + \dfrac{1}{3^{n+1}\log_e 3} + \dfrac{1}{4^{n+1}\log_e 4} + \dfrac{1}{5^{n+1}\log 5} + \&c$

$= -\log_e x + \cdot 2174630 + \cdot 4227843x$

$- \cdot 0364079x^2 + \cdot 001617x^3 + \cdot 000085x^4$

$- \cdot 00002x^5 - \&c$

Sol. Integrate VII 13.

15. $$\frac{\phi_n(x-1) - \phi_n(-x)}{4\lfloor n} = -\cos\frac{\pi n}{2}\left\{\frac{\sin 2\pi x}{(2\pi)^{n+1}} + \frac{\sin 4\pi x}{(4\pi)^{n+1}} + \frac{\sin 6\pi x}{(6\pi)^{n+1}} + \&c\right\}$$

Sol. $\phi_n(x-1) - \phi_n(-x) = (1-x)^n - x^n + (2-x)^n - (1+x)^n + (3-x)^n - (2+x)^n + \&c$; then arrange the terms in ascending powers of x and substitute $\frac{B_n}{n}\cos\frac{\pi n}{2}$ for S_{1-n}. Similarly

16. $$\frac{\phi_n(x-1) + \phi_n(-x) - 2S_{-n}}{4\lfloor n} = \sin\frac{\pi n}{2}\left\{\frac{\cos 2\pi x}{(2\pi)^{n+1}} + \frac{\cos 4\pi x}{(4\pi)^{n+1}} + \frac{\cos 6\pi x}{(6\pi)^{n+1}} + \&c\right\}$$

N. B. The above two theorems are true for all values of x when n is an integer but when n is fractional they are true only when x lies between 0 and 1

Cor. If $\frac{p}{q}$ lies between 0 and 1 and p & q are integers

$$\therefore \frac{(2\pi q)^n}{4\lfloor n-1}\left\{\phi_{n-1}\left(\frac{p}{q}-1\right) - \phi_{n-1}\left(-\frac{p}{q}\right)\right\} = -\sin\frac{\pi n}{2}\left[\left\{S_n - \phi_{-n}\left(\frac{1}{q}-1\right)\right\} \times \sin\frac{2\pi p}{q} + \left\{S_n - \phi_{-n}\left(\frac{2}{q}-1\right)\right\}\sin\frac{4\pi p}{q} + \left\{S_n - \phi_{-n}\left(\frac{3}{q}-1\right)\right\}\right.$$

$$\times \sin\frac{6\pi p}{q} + \cdots + \left\{S_n - \phi_{-n}\left(\frac{q-1}{q}-1\right)\right\}\sin\frac{(2q-2)\pi p}{q}\Bigg]$$

ii. $$\frac{(2\pi q)^n}{4\underline{\lfloor n-1}}\left\{\phi_{n-1}\left(\frac{p}{q}-1\right)+\phi_{n-1}\left(-\frac{p}{q}\right)-2S_{1-n}\left(1-\frac{1}{q^n}\right)\right\}$$

$$= -\cos\frac{\pi n}{2}\Bigg[\left\{S_n-\phi_{-n}\left(\frac{1}{q}-1\right)\right\}\cos\frac{2\pi p}{q}+\left\{S_n-\phi_{-n}\left(\frac{2}{q}-1\right)\right\}\cos\frac{4\pi p}{q}$$

$$+\left\{S_n-\phi_{-n}\left(\frac{3}{q}-1\right)\right\}\cos\frac{6\pi p}{q}+\cdots+\left\{S_n-\phi_{-n}\left(\frac{q-1}{q}-1\right)\right\}\cos\frac{(2q-2)}{q}$$

17. $$\phi_n\left(-\tfrac{1}{4}\right)-\phi_n\left(-\tfrac{3}{4}\right) = 2\,\frac{E_{n+1}}{4^{n+1}}\cos\frac{\pi n}{2}$$

Sol. Put $x = \frac{1}{4}$ in $\overline{VII}$. 15.

Cor. $1^n - 3^n + 5^n - 7^n + \&c = \frac{1}{2}E_{n+1}\cos\frac{\pi n}{2}$.

18. $$E_{1-n}\cos\frac{\pi n}{2} = \left(\frac{\pi}{2}\right)^n\frac{E_n}{\underline{\lfloor n-1}}$$

Sol. change n to $-n$ in $\overline{VII}$ 17 Cor.

Cor. $$\pi\left\{\frac{1}{2}-\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}-\frac{1}{\sqrt{5}+\sqrt{7}}+\&c\right\}$$

$$= \frac{1}{1\sqrt{1}}-\frac{1}{3\sqrt{3}}+\frac{1}{5\sqrt{5}}-\frac{1}{7\sqrt{7}}+\&c.$$

19. If $\frac{p}{q}$ lies between 0 & 1, p being any integer & q an odd integer, then

i. $$\frac{(2\pi q)^n}{4\underline{\lfloor n-1}}\left\{\phi_{n-1}\left(\frac{p}{q}-1\right)-\phi_{n-1}\left(-\frac{p}{q}\right)\right\} = \sin\frac{\pi n}{2}\Bigg[\left\{\phi_{-n}\left(\frac{1}{q}-1\right)-\phi_{-n}\left(-\frac{1}{q}\right)\right\}$$

$$\times\sin\frac{2\pi p}{q}+\left\{\phi_{-n}\left(\frac{2}{q}-1\right)-\phi_{-n}\left(-\frac{2}{q}\right)\right\}\sin\frac{4\pi p}{q}+\&c \text{ to } \frac{q-1}{2} \text{ terms}$$

ii. $$\frac{(2\pi q)^n}{4\underline{\lfloor n-1}}\left\{\phi_{n-1}\left(\frac{p}{q}-1\right)+\phi_{n-1}\left(-\frac{p}{q}\right)-2S_{1-n}\left(1-\frac{1}{q^n}\right)\right\}$$

$$= \cos\frac{\pi n}{2}\Bigg[\left\{\phi_{-n}\left(\frac{1}{q}-1\right)-\phi_{-n}\left(-\frac{1}{q}\right)\right\}\cos\frac{2\pi p}{q}+\left\{\phi_{-n}\left(\frac{2}{q}-1\right)-\phi_{-n}\left(-\frac{2}{q}\right)\right\}$$

$\times \cos\frac{4\pi p}{q} + \text{\&c to } \frac{q-1}{2} \text{ terms}\Big]$

Cor. 1. $\dfrac{2^{n-1}}{\lfloor\underline{n}}\phi_n(-x) = \dfrac{\sin \pi x}{\pi^{n+1}}\cos\left(\pi x + \frac{\pi n}{2}\right) + \dfrac{\sin 2\pi x}{(2\pi)^{n+1}}\cos\left(2\pi x + \frac{\pi n}{2}\right)$

$+ \dfrac{\sin 3\pi x}{(3\pi)^{n+1}}\cos\left(3\pi x + \frac{\pi n}{2}\right) + \text{\&c}$

Sol. Combine the results of $\underline{\text{VII}}$ 15 & 16.

2. $\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{1-x}} + \dfrac{1}{\sqrt{1+x}} - \dfrac{1}{\sqrt{2-x}} + \dfrac{1}{\sqrt{2+x}} - \text{\&c}$

$= 2\left(\dfrac{\sin 2\pi x}{\sqrt{1}} + \dfrac{\sin 4\pi x}{\sqrt{2}} + \dfrac{\sin 6\pi x}{\sqrt{3}} + \text{\&c}\right).$

20. $\dfrac{(6\pi)^n}{2\lfloor\underline{n-1}\sqrt{3}}\left\{\phi_{n-1}\left(-\frac{1}{3}\right) - \phi_{n-1}\left(-\frac{2}{3}\right)\right\} = \left\{\phi_{-n}\left(-\frac{1}{3}\right) - \phi_{-n}\left(-\frac{2}{3}\right)\right\}\sin\frac{\pi n}{2}$

Sol. Put $p=1$ & $q=3$ in $\underline{\text{VII}}$ 19. i.

21. $\phi(0) + \dfrac{n}{\lfloor\underline{1}}\phi(1)x + \dfrac{n(n-1)}{\lfloor\underline{2}}\phi(2)x^2 + \dfrac{n(n-1)(n-2)}{\lfloor\underline{3}}\phi(3)x^3$

$+ \text{\&c} = (1+x)^n \phi_\infty\left(\dfrac{nx}{1+x}\right)$. where

$\phi_n(z) = \phi_{n-1}(z) + \dfrac{n P_{n-1}}{\lfloor\underline{1}\,\lfloor\underline{n}}\phi_{n-1}^{n}(z) + \dfrac{(n P_{n-1})^2}{\lfloor\underline{2}\,(\lfloor\underline{n})^2}\phi_{n-1}^{2n}(z)$

$+ \dfrac{(n P_{n-1})^3}{\lfloor\underline{3}\,(\lfloor\underline{n})^3}\phi_{n-1}^{3n}(z) + \text{\&c}$ and $\phi_1(z) = \phi(z)$.

and $P_n = 1^n x - 2^n x^2 + 3^n x^3 - 4^n x^4 + \text{\&c}$.

Sol. Prove the theorem by substituting e^{az} for $\phi(z)$ or proceed as in $\underline{\text{III}}$ 10.

Cor. $\left\{\phi(0) + \dfrac{n}{\lfloor\underline{1}}x\phi(1) + \dfrac{n(n-1)}{\lfloor\underline{2}}x^2\phi(2) + \text{\&c}\right\}(1+x)^{-n}$

$= \phi\left(\dfrac{nx}{1+x}\right) + \dfrac{nx}{(1+x)^2}\,\dfrac{\phi''\left(\frac{nx}{1+x}\right)}{\lfloor\underline{2}} + \text{\&c}.$

22. If $A_n = (1^n + 2^n + 3^n + \&c)(1 + \cos \pi n)$, then

$$2^n + 6^n + 12^n + 20^n + \&c = A_n + \frac{n}{\lfloor\underline{1}} A_{n+1} + \frac{n(n-1)}{\lfloor\underline{2}} A_{n+2}$$

Ex. $10 = \pi^2 + \frac{1}{2^3} + \frac{1}{6^3} + \frac{1}{12^3} + \frac{1}{20^3} + \&c$

23. $\log_e \lfloor\underline{x} = (x + \frac{1}{2}) \log x - x + \frac{1}{2} \log 2\pi + \frac{B_2}{1.2x} - \frac{B_4}{3.4x^3} + \frac{B_6}{5.6x^5} - \&c$

Sol. Equate the coeff^{ts} of r in $\overline{\text{VII}}$ 1; the coeff^t of r in S_r

$= \text{that in } - \frac{\lfloor\underline{r}}{\pi . (2\pi)^r} S_{r+1} \sin \frac{\pi r}{2} . = \text{that of } r \text{ in}$

$- \frac{r}{2}(1 - r \log_e 2\pi + \&c)(\frac{1}{r} + C_0 - \&c)(1 - r C_0 + \&c)$

$= \frac{1}{2} \log_e 2\pi$. or as follows

Let C be the constant in $\log_e \lfloor\underline{x}$. & $f(x) = \log \dfrac{\lfloor\underline{2x}}{\lfloor\underline{x} \lfloor\underline{\frac{x-1}{2}}}$

then we see that $f(x) - f(x-1) = \log 2$.

$\therefore \log \dfrac{\lfloor\underline{x}}{\lfloor\underline{\frac{x}{2}} \lfloor\underline{\frac{x-1}{2}}} - x \log_e 2 =$ some constant; by putting $x = 0$ we find this constant is $-\frac{1}{2} \log \pi$.

But the constant in $\log \dfrac{\lfloor\underline{x}}{\lfloor\underline{\frac{x}{2}} \lfloor\underline{\frac{x-1}{2}}} = \frac{1}{2} \log 2 - C$.

$\therefore C = \frac{1}{2} \log 2\pi = .9189385332046673$.

Cor. When x is great $\dfrac{e^x \lfloor\underline{x}}{x^x} = \sqrt{2\pi x + \frac{\pi}{3}}$ nearly.

24. $\lfloor\underline{x-1} \lfloor\underline{-x} = \pi \operatorname{cosec} \pi x$; Cor $\lfloor\underline{-\frac{1}{2}} = \sqrt{\pi}$.

25. $\lfloor\underline{\tfrac{x}{n}} \lfloor\underline{\tfrac{x-1}{n}} \lfloor\underline{\tfrac{x-2}{n}} \lfloor\underline{\tfrac{x-3}{n}} \ldots\ldots \lfloor\underline{\tfrac{x-n+1}{n}} = \dfrac{(2\pi)^{\frac{n-1}{2}}}{n^{x+\frac{1}{2}}} \lfloor\underline{x}$.

Cor. 1. $\lfloor\underline{-\tfrac{1}{n}} \lfloor\underline{-\tfrac{2}{n}} \lfloor\underline{-\tfrac{3}{n}} \ldots \lfloor\underline{-\tfrac{n-1}{n}} = \dfrac{(2\pi)^{\frac{n}{2}}}{\sqrt{2\pi n}}$.

2. $\lfloor\underline{-\tfrac{1}{3}} = \sqrt{\lfloor\underline{-\tfrac{1}{2}}} \sqrt[3]{2} \sqrt[4]{\frac{\pi}{3}}$

3. $\dfrac{\underline{|x}}{\underline{|\frac{x}{2}}\;\underline{|\frac{x-1}{2}}} = \dfrac{2^x}{\sqrt{\pi}}.$

4. $\log \underline{|x-\tfrac{1}{2}} = x\log x - x + \tfrac{1}{2}\log 2\pi + \left(1-\tfrac{1}{2}\right)\dfrac{B_2}{1.2x} - \left(1-\tfrac{1}{2^3}\right)\dfrac{B_4}{3.4x^3} + \left(1-\tfrac{1}{2^5}\right)\dfrac{B_6}{5.6x^5} - \&c$

26. $\log \underline{|x} = -C_0 x + \dfrac{S_2}{2}x^2 - \dfrac{S_3}{3}x^3 + \dfrac{S_4}{4}x^4 - \&c$

i.e $\log_e \dfrac{\underline{|x+2}}{2} = .9227843351x + .1974670334x^2 - .0256856344x^3 + .0049558084x^4 - .0011355510x^5 + .0002863437x^6 - .0000766825x^7 + .0000213883x^8 - .0000061409x^9 + .0000054047\dfrac{x^{10}}{3+x}$

Ex. 1 $\log_e \underline{|-\tfrac{1}{3}} = .5341990853.$

2. $\log_e \underline{|-\tfrac{1}{6}} = .1211436313$

3. $\log_e \underline{|-\tfrac{1}{10}} = .0663762397.$

27. $i2\pi x\left\{1+\left(\dfrac{x}{n+1}\right)^2\right\}\left\{1+\left(\dfrac{x}{n+2}\right)^2\right\}\left\{1+\left(\dfrac{x}{n+3}\right)^2\right\}$ &c ad inf.

$= \left(\dfrac{\underline{|n}}{x^n}\right)^2\left(e^{\pi x} - e^{\cancel{-\pi x}\,\theta}\right)e^{-\frac{S_2}{x^2}+\frac{S_4}{2x^4}-\frac{S_6}{3x^6}+\&c}$

where $S_p = 1^p + 2^p + 3^p + \dots + n^p$.

Sol. Let L.S $= f(n)$; then $\dfrac{f(n-1)}{f(n)} = 1+\left(\dfrac{x}{n}\right)^2$; find $f(n)$ by applying XI or in any way.

~~N.B. $\theta = \cos 2\pi n$ exactly or very nearly according as $2n$ is an integer or not.~~

~~Sol. For even values of $2n$, $e^{\pi x} - e^{-\pi x}$ appears i~~
~~R.S.; but for odd values $e^{\pi x} + e^{-\pi x}$~~

ii. $2\pi (x^2+n^2)^{n+\frac{1}{2}} \left\{1+\left(\frac{x}{n+1}\right)^2\right\}\left\{1+\left(\frac{x}{n+2}\right)^2\right\}\left\{1+\left(\frac{x}{n+3}\right)^2\right\}$ &c

$$= (\lfloor n)^2 \left(e^{\pi x} \cancel{- e^{-\pi x}\theta}\right) e^{2n - 2x\tan^{-1}\frac{n}{x} - \frac{B_2 S_2}{x} - \frac{B_4 S_4}{2x^3} - \frac{B_6 S_6}{3x^5} - \&c}$$

where $S_p = \frac{n}{x} - \frac{p(p+1)}{\lfloor 3}\left(\frac{n}{x}\right)^3 + \frac{p(p+1)(p+2)(p+3)}{\lfloor 5}\left(\frac{n}{x}\right)^5 - \&c.$

Sol. Find S_2, S_4, S_6 &c in the previous theorem by VII 1. and then simplify.

iii. $2\pi(n^2+x^2)^{n-\frac{1}{2}}\left\{1+\left(\frac{x}{n}\right)^2\right\}\left\{1+\left(\frac{x}{n+1}\right)^2\right\}\left\{1+\left(\frac{x}{n+2}\right)^2\right\}$ &c

$$= (\lfloor n-1)^2 e^{2n + 2x\beta - 2\frac{B_2\cos\beta}{1\cdot 2 r} + \frac{2B_4\cos 3\beta}{3\cdot 4 r^3} - \&c} \cancel{\times\left(1 - e^{-2\pi x}\theta\right)}$$

where $r^2 = n^2+x^2$ & $\tan\beta = \frac{x}{n}$.

Sol. $\lfloor n+xi \, \lfloor n-xi = \lfloor xi \, \lfloor -xi \, (1^2+x^2)(2^2+x^2)(3^2+x^2) \cdots (n^2+x^2) = \dfrac{(\lfloor n)^2}{\left\{1+\left(\frac{x}{n+1}\right)^2\right\}\left\{1+\left(\frac{x}{n+2}\right)^2\right\} \&c \text{ ad inf}}$

then find $\lfloor n+xi \, \lfloor n-xi$ by VII 23.

N.B. i. is useful only when x is great & n small
ii. when x is great when compared to n
iii. in all cases.

$\frac{B_r}{r}\cos\frac{\pi r}{2}+\frac{1}{r}$, when r vanishes, is a finite quantity which is invariably denoted by C_0; it is the constant of S, and its value is found from VIII 2 to be $\cdot 5772156649015\,33$ and ~~$\log C_0$~~ e^{-C_0} $= + \cdot 561459483 56$.

Sol. L.S in VII 1 is finite when $r=1$.

$\therefore \frac{B_r}{r}\cos\frac{\pi r}{2}+\frac{x^r}{r}$ is finite when $r=0$

i.e $\frac{B_r}{r}\cos\frac{\pi r}{2}+\frac{1}{r}+\frac{x^r-1}{r}$ is finite when $r=0$

But $\frac{x^r-1}{r}=\log_e x$ when $r=0$.

2. $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{x}=\Sigma\frac{1}{x}$ or $\phi(x)$. (Suppose).

$\Sigma\frac{1}{x}=C_0+\log_e x+\frac{1}{2x}-\frac{B_2}{2x^2}+\frac{B_4}{4x^4}-\frac{B_6}{6x^6}+\&c$

3. $\Sigma\frac{1}{x}=1-\frac{1}{x+1}+\frac{1}{2}-\frac{1}{x+2}+\frac{1}{3}-\frac{1}{x+3}+\&c$

$=\frac{x}{1(1+x)}+\frac{x}{2(2+x)}+\frac{x}{3(3+x)}+\&c$

4. $\Sigma\frac{1}{x}=xS_2-x^2S_3+x^3S_4-x^4S_5+\&c$

5. $\Sigma\frac{1}{x-1}-\Sigma\frac{1}{-x}=-\pi\cot\pi x$.

6. $n\Sigma\frac{1}{x}-\left\{\Sigma\frac{1}{x/n}+\Sigma\frac{1}{\frac{x-1}{n}}+\cdots\cdots+\Sigma\frac{1}{\frac{x-n+1}{n}}\right\}$
$=n\log n$.

Cor. 1. $\Sigma\frac{1}{x-\frac{1}{2}}=C_0+\log x+(1-\frac{1}{2})\frac{B_2}{2x^2}-(1-\frac{1}{2^3})\frac{B_4}{4x^4}+\&c$.

2. $\Sigma \frac{1}{-\frac{1}{n}} + \Sigma \frac{1}{-\frac{2}{n}} + \cdots\cdots + \Sigma \frac{1}{-\frac{n-1}{n}} = -n \log_e n.$

3. i. $\phi(-\frac{1}{2}) = -2\log 2$; ii $\phi(-\frac{2}{3}) = -\frac{3}{2}\log 3 - \frac{\pi}{2\sqrt{3}}$

iii. $\phi(-\frac{3}{4}) = -\frac{\pi}{2} - 3\log 2$; iv. $\phi(-\frac{5}{6}) = -\frac{\pi}{2}\sqrt{3} - 2\log 2 - \frac{3}{2}\log 3.$

v. $3\phi(-\frac{1}{2}) - 2\phi(-\frac{3}{4}) = \pi.$

4. $\phi(-\frac{1}{2n}) + \phi(-\frac{3}{2n}) + \cdots\cdots + \phi(-\frac{2n-1}{2n}) = -n\log_e 4n.$

7. $\frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \cdots\cdots + \frac{1}{a+xb} = \frac{1}{b}\left\{\phi(\frac{a}{b}+x) - \phi(\frac{a}{b}\right.$

8. $\frac{1}{a+b} - \frac{1}{a+2b} + \frac{1}{a+3b} - \&c = \frac{1}{2b}\left\{\phi(\frac{a}{2b}) - \phi(\frac{a-b}{2b})\right\}$

9. $\phi(\frac{1}{2x}) = \phi(\frac{1}{x}) - \log_e 2 + x\int_0^1 \frac{n^x}{1+n^x}\,dn.$

10. $\phi(-\frac{1}{x}) = -x\int_0^1 \frac{(1-n^x)^2}{n(n^x-1)}\,dn$

11. $\phi(\frac{1}{x}-1) + \phi(-\frac{1}{x}) = -x\left\{1 + \frac{2}{x^3-x} + \frac{2}{(2x)^3-2x} + \&c\right\}$

12. $\frac{2}{x^3-x} + \frac{2}{(2x)^3-2x} + \frac{2}{(3x)^3-3x} + \&c = \int_0^1 \frac{n^{x-2}(1-n)^2}{1-n^x}\,dn$

13. $1 + \frac{2}{(2x)^3-2x} + \frac{2}{(4x)^3-4x} + \frac{2}{(6x)^3-6x} + \&c$

$= \frac{1}{2}\left\{1 + \frac{2}{x^3-x} + \frac{2}{(2x)^3-2x} + \&c\right\} + \frac{\log_e 2}{x}$

$+$ Log^{mic} part of $\left(1 - \frac{1}{1+x} + \frac{1}{1+2x} - \frac{1}{1+3x} + \&c\right).$

N.B. i. $x - \frac{x^{1+n}}{1+n} + \frac{x^{1+2n}}{1+2n} - \&c = \int_0^x \frac{dx}{1+x^n}.$

ii. $x + \frac{x^{1+n}}{1+n} + \frac{x^{1+2n}}{1+2n} + \&c = \int_0^x \frac{dx}{1-x^n}.$

iii. If n is odd $\int_0^x \frac{dx}{1-x^n} = \int_0^x \frac{1}{1+(-x)^n}\,dx.$

iv. If n is even $\int_0^x \frac{dx}{1-x^n} = \frac{1}{2}\int_0^x \frac{dx}{1+x^{\frac{n}{2}}} + \frac{1}{2}\int_0^x \frac{dx}{1-x^{\frac{n}{2}}}$

V. If $l < n+1$

(a) If n is even $\int \frac{x^{l-1}}{x^n-1}\,dx = \frac{1}{n}\log(x-1) + \frac{(-1)^l}{n}\log(x+1) + \frac{1}{n}\,\Sigma \cos\frac{rl\pi}{n}\log\left(x^2 - 2x\cos\frac{\pi r}{n}+1\right)$

$- \frac{2}{n}\,\Sigma \sin\frac{rl\pi}{n}\tan^{-1}\frac{x-\cos\frac{r\pi}{n}}{\sin\frac{r\pi}{n}}.$

$r = 2, 4, 6,$ &c up to $n-2$.

(b) $\int_0 \frac{x^{l-1}}{x^n+1} = \frac{(-1)^{l-1}}{n}\log(x+1)$ n being odd.

$- \frac{1}{n}\Sigma\cos\frac{rl\pi}{n}\log\left(x^2 - 2x\cos\frac{r\pi}{n}+1\right)$

$+ \frac{2}{n}\Sigma\sin\frac{rl\pi}{n}\tan^{-1}\frac{x-\cos\frac{r\pi}{n}}{\sin\frac{\pi r}{n}}$

$r = 1, 3, 5.$ up to $n-2$.

VI. If $n+1$ be even

(a) $\int\frac{x^{l-1}}{x^n-1}\,dx = \frac{1}{n}\log(x-1) + \frac{1}{n}\Sigma\cos\frac{rl\pi}{n} \times$

$\log\left(x^2 - 2x\cos\frac{\pi r}{n}+1\right) - \frac{2}{n}\Sigma\sin\frac{rl\pi}{n} \times$

$\tan^{-1}\frac{x-\cos\frac{r\pi}{n}}{\sin\frac{\pi r}{n}}.$ $r = 2, 4, 6, \cdots (n-1).$

(b) $\int\frac{x^{l-1}}{x^n+1}\,dx = -\frac{1}{n}\Sigma\cos\frac{rl\pi}{n}\log\left(x^2-2x\cos\frac{\pi r}{n}+1\right)$

$+ \frac{2}{n}\sin\frac{\pi r l}{n}\tan^{-1}\frac{x-\cos\frac{\pi r}{n}}{\sin\frac{\pi r}{n}}$ n being even

$r = 1, 3, 5, \cdots (n-1).$

14. If $A_n = \int_0^x \frac{dx}{1+x^n}$, then

i. $A_1 = \log_e(1+x)$; ii $A_2 = \tan^{-1} x$.

iii $A_3 = \frac{1}{6}\log\frac{(1+x)^3}{1+x^3} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x\sqrt{3}}{2-x}$.

iv. $A_4 = \frac{1}{4\sqrt{2}}\log\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}$.

v. $A_5 = \frac{1}{20}\log\frac{(1+x)^5}{1+x^5} + \frac{1}{4\sqrt{5}}\log\frac{1+x.\frac{\sqrt{5}-1}{2}+x^2}{1-x.\frac{\sqrt{5}-1}{2}+x^2}$

$+ \frac{1}{10}\sqrt{10-2\sqrt{5}}\tan^{-1}\frac{x\sqrt{10-2\sqrt{5}}}{4-x(\sqrt{5}+1)} + \frac{\sqrt{10+2\sqrt{5}}}{10}\tan^{-1}\frac{x\sqrt{10+2\sqrt{5}}}{4+x(\sqrt{5}-1)}$

vi. $A_6 = \frac{1}{2}\tan^{-1}x + \frac{1}{6}\tan^{-1}x^3 + \frac{1}{4\sqrt{3}}\log\frac{1+x\sqrt{3}+x^2}{1-x\sqrt{3}+x^2}$.

vii. $A_8 = \frac{\sqrt{2+\sqrt{2}}}{16}\left\{\log_e\frac{1+x\sqrt{2+\sqrt{2}}+x^2}{1-x\sqrt{2+\sqrt{2}}+x^2} + 2\tan^{-1}\frac{x\sqrt{2+\sqrt{2}}}{1-x^2}\right\}$

$+ \frac{\sqrt{2-\sqrt{2}}}{16}\left\{\log_e\frac{1+x\sqrt{2-\sqrt{2}}+x^2}{1-x\sqrt{2-\sqrt{2}}+x^2} + 2\tan^{-1}\frac{x\sqrt{2-\sqrt{2}}}{1-x^2}\right\}$.

viii. $A_{10} = \frac{1}{4}\tan^{-1}x - \frac{1}{20}\tan^{-1}x^5 + \frac{1}{4\sqrt{5}}\tan^{-1}\frac{(x-x^3)\sqrt{5}}{1-3x^2+x^4}$

$+ \frac{1}{40}\sqrt{10-2\sqrt{5}}\log\frac{1+\frac{x}{2}\sqrt{10-2\sqrt{5}}+x^2}{1-\frac{x}{2}\sqrt{10-2\sqrt{5}}+x^2}$

$+ \frac{1}{40}\sqrt{10+2\sqrt{5}}\log\frac{1+\frac{x}{2}\sqrt{10+2\sqrt{5}}+x^2}{1-\frac{x}{2}\sqrt{10+2\sqrt{5}}+x^2}$

Ex. 1. i. $\frac{1}{1.2} - \frac{1}{4.2^4} + \frac{1}{7.2^7} - \&c = \frac{\pi}{6\sqrt{3}} + \frac{1}{6}\log 3$.

ii. $\frac{\sqrt{3}-1}{1} - \frac{(\sqrt{3}-1)^4}{4} + \frac{(\sqrt{3}-1)^7}{7} - \&c = \frac{\pi}{4\sqrt{3}} + \frac{1}{3}\log\frac{1+\sqrt{3}}{\sqrt{2}}$.

iii. $\frac{2-\sqrt{3}}{1} - \frac{(2-\sqrt{3})^3}{5} + \frac{(2-\sqrt{3})^5}{9} - \&c = \frac{\pi}{16}(\sqrt{3}-1)$

$- \frac{\sqrt{3}-1}{2}\log(\sqrt{3}-1)$.

2. If $A_n = 1 + \frac{2}{n^3 - n} + \frac{2}{(2n)^3 - 2n} + \frac{2}{(3n)^3 - 3n} + \&c$, then

$A_2 = 2\log_e 2$; $A_3 = \log_e 3$; $A_4 = \frac{3}{2}\log_e 2$; $A_6 = \frac{1}{2}\log 3 + \frac{1}{3}\log 4$.

$A_5 = \frac{1}{2}\log 5 + \frac{1}{\sqrt{5}}\log\frac{\sqrt{5}+1}{2}$; $A_8 = \log 2 + \frac{1}{2\sqrt{2}}\log(1+\sqrt{2})$.

$A_{10} = \frac{2}{5}\log 2 + \frac{1}{4}\log 5 + \frac{3}{2\sqrt{5}}\log\frac{1+\sqrt{5}}{2}$; $A_{12} = \frac{1}{2}\log 2 + \frac{1}{4}\log 3$

$- \frac{1}{\sqrt{8}}\log(\sqrt{3}-1)$; $A_{16} = \frac{5}{8}\log 2 + \frac{1}{4\sqrt{2}}\log(1+\sqrt{2})$

$$+ \frac{\sqrt{2+\sqrt{2}}}{16}\log\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}} + \frac{\sqrt{2-\sqrt{2}}}{16}\log\frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}.$$

$$A_{20} = \frac{1}{8}\log 5 + \frac{3}{10}\log 2 + \frac{3}{4\sqrt{5}}\log\frac{\sqrt{5}+1}{2}$$

$$+ \frac{\sqrt{10-2\sqrt{5}}}{40}\log\frac{4+\sqrt{10-2\sqrt{5}}}{4-\sqrt{10-2\sqrt{5}}} + \frac{\sqrt{10+2\sqrt{5}}}{40}\log\frac{4+\sqrt{10+2\sqrt{5}}}{4-\sqrt{10+2\sqrt{5}}}$$

15. If $\Sigma\frac{1}{x} = C_0 + \log_e a$, then

$$\left(\frac{x+\frac{1}{2}}{a}\right)^{4n} = 1 - \frac{n}{\lfloor\underline{1}}\cdot\frac{1}{6a^2} + \frac{n(n+1\frac{1}{10})}{\lfloor\underline{2}}\cdot\frac{1}{(6a^2)^2} - \frac{n(n^2 + 3\frac{3}{10}n + 12\frac{51}{70})}{\lfloor\underline{3}\,(6a^2)^3} + \&c$$

Cor. $\lfloor\underline{x}$ is minimum when $x = \frac{6}{13}$ very nearly.

Sol. $\lfloor\underline{x}$ is minimum when $\Sigma\frac{1}{x} = C_0$ i.e. $a = 1$

$\therefore x = \frac{1}{2} - \frac{1}{24} + \&c$ or $x = \frac{1}{2+\frac{1}{6}}$ very nearly.

16. $C_0 = \log_e 2 - 1\left(\frac{2}{3^3-3}\right) - 2\left(\frac{2}{6^3-6} + \frac{2}{9^3-9} + \frac{2}{12^3-12}\right) - \&c$

the last term in the nth group $= \dfrac{2}{\left(\frac{3^n+3}{2}\right)^3 - \frac{3^n+3}{2}}$.

17. i. $\frac{\log 1}{1} + \frac{\log 2}{2} + \frac{\log 3}{3} + \cdots + \frac{\log x}{x} = \phi(x)$

$\phi(x) = \left(\sum \frac{1}{x} - C_0\right)\log x - \frac{1}{2}(\log x)^2 + C_1 + \frac{B_2}{2x^2}\cdot 1$

$- \frac{B_4}{4x^4}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \frac{B_6}{6x^6}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right) - \&c$

where $C_1 = -\cdot 072815845483680$

Sol. Write $r-1$ for r in VII 1, then divide both sides by x^r and find the coeff^{ts} of r from both sides and equate them

Cor. When $x = \infty$, $\phi(x) - \frac{1}{2}\left(\sum \frac{1}{x} - C_0\right)^2 = C_1$

ii. $\phi(x) = \frac{\log 1}{1} - \frac{\log(1+x)}{1+x} + \frac{\log 2}{2} - \frac{\log(2+x)}{2+x} + \&c$

Cor. $\frac{\log 1}{1} - \frac{\log 3}{3} + \frac{\log 5}{5} - \&c = \frac{\pi}{2}\log 2 + \frac{1}{4}\left\{\phi\left(-\frac{1}{4}\right) - \phi\left(-\frac{3}{4}\right)\right\}$

iii. $n\phi(x) - \left\{\phi\left(\frac{x}{n}\right) + \phi\left(\frac{x-1}{n}\right) + \cdots + \phi\left(\frac{x-n+1}{n}\right)\right\}$

$= n\log n\left(\sum \frac{1}{x} - C_0\right) - \frac{n}{2}(\log n)^2$

Cor. $\phi\left(-\frac{1}{n}\right) + \phi\left(-\frac{2}{n}\right) + \cdots + \phi\left(-\frac{n-1}{n}\right)$

$= nC_0\log n + \frac{n}{2}(\log n)^2$

Ex. 1. $\frac{\sqrt[1]{1}}{\sqrt[2]{2}}\cdot\frac{\sqrt[3]{3}}{\sqrt[4]{4}}\cdot\frac{\sqrt[5]{5}}{\sqrt[6]{6}}\cdot\frac{\sqrt[7]{7}}{\sqrt[8]{8}}\cdot\frac{\sqrt[9]{9}}{\sqrt[10]{10}}\cdots$ ad inf $= 2^{\frac{1}{2}\log 2 - C_0}$

2. $\phi\left(-\frac{1}{2}\right) = (\log 2)^2 + 2C_0\log 2$

3. $\phi\left(-\frac{1}{3}\right) + \phi\left(-\frac{2}{3}\right) = \frac{3}{2}(\log 3)^2 + 3C_0\log 3$.

4. $\phi\left(-\frac{1}{4}\right) + \phi\left(-\frac{3}{4}\right) = 7(\log 2)^2 + 6C_0\log 2$

5. $\phi\left(-\frac{1}{6}\right) + \phi\left(-\frac{5}{6}\right) = C_0(3\log 3 + 4\log 2) + \frac{3}{2}(\log 12)^2 - (\log 4)^2$.

iv. When x lies between 0 & 1

$$\frac{\pi}{2}\left\{\log\frac{\lfloor x-1}{\lfloor -x} + (C_0+\log 2\pi)(2x-1)\right\}$$

$$= \frac{\log 1}{1}\sin 2\pi x + \frac{\log 2}{2}\sin 4\pi x + \frac{\log 3}{3}\sin 6\pi x + \&c$$

N.B. $\frac{\pi}{2} - \pi x = \sin 2\pi x + \frac{1}{2}\sin 4\pi x + \frac{1}{3}\sin 6\pi x + \&c$

v. $\phi(x-1) - \phi(-x) = (C_0+\log 2\pi)\,\pi\cot\pi x$ (for the same limits) $+ 2\pi\left\{\sin 2\pi x\log 1 + \sin 4\pi x\log 2 + \&c\right\}$

N.B. $\sin 2\pi x + \sin 4\pi x + \sin 6\pi x + \&c = \frac{1}{2}\cot\pi x.$

Ex. 1. Find $\phi(-\frac{1}{2})$, $\phi(-\frac{2}{3})$, $\phi(-\frac{3}{4})$ and $\phi(-\frac{5}{6})$

2. $\frac{\log 1}{1} - \frac{\log 3}{3} + \frac{\log 5}{5} - \&c = \frac{\pi}{4}\log\pi - \pi\log\lfloor -\frac{1}{4} - \frac{\pi}{4}C_0$

3. $$\frac{\left(\frac{\sqrt[1]{1}}{\sqrt{2}}\cdot\frac{\sqrt[3]{3}}{\sqrt[4]{4}}\cdot\frac{\sqrt[5]{5}}{\sqrt[6]{6}}\cdot\frac{\sqrt[7]{7}}{\sqrt[8]{8}}\,\&c\right)^{\frac{1}{\log 2}}}{\left(\frac{\sqrt[1]{1}}{\sqrt[3]{3}}\cdot\frac{\sqrt[5]{5}}{\sqrt[7]{7}}\cdot\frac{\sqrt[9]{9}}{\sqrt[11]{11}}\cdot\frac{\sqrt[13]{13}}{\sqrt[15]{15}}\,\&c\right)^{\frac{4}{\pi}}} = \frac{\sqrt{2}}{\pi}\left(\lfloor -\tfrac{1}{4}\right)^4.$$

18. $(\log 1)^2 + (\log 2)^2 + (\log 3)^2 + \cdots\cdots + (\log x)^2 = \phi(x).$

i. $$\phi(x) = 2\log x\log\frac{\lfloor x}{\sqrt{2\pi}} - (x+\tfrac{1}{2})(\log x)^2 + 2x + \tfrac{1}{2}C_1^2 + C_1 - \frac{\pi^2}{24} - \tfrac{1}{2}(\log 2\pi)^2 + 2\left\{\frac{B_4}{3\cdot 4}\cdot\frac{1+\frac{1}{2}}{x^3} - \frac{B_6}{5\cdot 6}\cdot\frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{x^5} + \&c\right\}$$

Sol. Equate the coeff^{ts}. of n^2 in VII 1.

ii. $\phi(x) - \left\{\phi\left(\frac{x}{n}\right) + \phi\left(\frac{x-1}{n}\right) + \cdots\cdots + \phi\left(\frac{x-n+1}{n}\right)\right\} =$

$$2\log n \log \frac{\lfloor x}{\sqrt{2\pi}} - x(\log n)^2 - (n-1)\left\{\tfrac12 C_0^2 + C_1 - \frac{\pi^2}{24} - \tfrac12(\log 2\pi)^2\right\}$$

$-\frac12(\log n)^2$. If C be the constant in this series then

Cor. $\phi(-\frac1n)+\phi(-\frac2n)+\phi(-\frac3n)+\cdots+\phi(-\frac{n-1}{n})$

$= \log n\log 2\pi + (n-1)C + \frac12(\log n)^2$.

Ex. 1. If x becomes infinite then

$$\frac{1^{\frac11}.2^{\frac12}.3^{\frac13}.4^{\frac14}\cdots x^{\frac1x}}{1^{\log 1}.2^{\log 2}.3^{\log 3}\cdots x^{\log x}} . x^{x\log x - 2x}$$

$$\times\, e^{2x+\frac12(\Sigma\frac1x - \log x)^2} = e^{\frac{\pi^2}{24}}(2\pi)^{\frac12\log 2\pi}.$$

2. Find $\phi(-\frac12)$, $\phi(-\frac13)+\phi(-\frac23)$, $\phi(-\frac14)+\phi(-\frac34)$ and $\phi(-\frac18)+\phi(-\frac58)$.

viii $\dfrac{\phi(x-1)+\phi(-x)}{2} = C_1 - \dfrac{\pi^2}{24} + \tfrac12(C_0+\log 2\pi)\left(C_0 - \log\dfrac{\pi}{2\sin\pi x}\right)$

$-\left\{\frac{\log 1}{1}\cos 2\pi x + \frac{\log 2}{2}\cos 4\pi x + \&c\right\}$

19. If C_n be the constant in $(\log 1)^n + (\log 2)^n + \cdots + (\log x)^n$ and if $\phi_n(x) = (\log 1)^n + (\log 2)^n + \cdots + (\log x)^n - C_n$, then

i The logarithmic part of $\phi_n(x) = n\log x\,\phi_{n-1}(x)$

$- \frac{n(n-1)}{\lfloor 2}(\log x)^2\phi_{n-2}(x) + \frac{n(n-1)(n-2)}{\lfloor 3}(\log x)^3\phi_{n-3}(x) - \&c$

and the non-logarithmic part can be found from VII 1.

ii $\phi_0(x)(\log x)^n - \frac{n}{\lfloor 1}\phi_1(x)(\log x)^{n-1} + \frac{n(n-1)}{\lfloor 2}\phi_2(x)(\log x)^{n-2} - \&c$

$$= x\,\lfloor n - \frac{1}{x^n}\cdot\frac{B_{n+1}}{n+1}\sin\frac{\pi n}{2} - \frac{n}{2}\cdot\frac{1}{x^{n+1}}\cdot\frac{B_{n+2}}{n+2}\cos\frac{\pi n}{2}$$

$$+ \frac{n(n+\frac{5}{3})}{2.4}\cdot\frac{1}{x^{n+2}}\,\frac{B_{n+3}}{n+3}\sin\frac{\pi n}{2} + \frac{n(n+2)(n+3)}{2.4.6}\cdot\frac{1}{x^{n+3}}$$

$$\times\ \frac{B_{n+4}}{n+4}\cos\frac{\pi n}{2} - \frac{n(n+2)(n+4)^2 + \frac{n(n+2)}{3} + \frac{4n}{5}}{2.4.6.8}\cdot\frac{1}{x^{n+4}}$$

$$\times\ \frac{B_{n+5}}{n+5}\sin\frac{\pi n}{2} - \frac{n(n+4)(n+5)\{(n+2)(n+4)+\frac{2}{3}(n+1)\}}{2.4.6.8.10}$$

$$\times\ \frac{1}{x^{n+5}}\ \frac{B_{n+6}}{n+6}\cos\frac{\pi n}{2} + \&c$$

iii. $\phi_n\left(\frac{x}{r}\right) + \phi_n\left(\frac{x-1}{r}\right) + \cdots + \phi_n\left(\frac{x-r+1}{r}\right)$

$$= \phi_n(x) - n\log r\,\phi_{n-1}(x) + \frac{n(n-1)}{\lfloor 2}(\log r)^2\phi_{n-2}(x) - \&c$$

Cr. 1. $\phi_n\left(-\frac{1}{r}\right) + \phi_n\left(-\frac{2}{r}\right) + \cdots + \phi_n\left(-\frac{r-1}{r}\right)$

$$= -\left\{C_n - n\log r\,C_{n-1} + \frac{n(n-1)}{\lfloor 2}(\log r)^2 C_{n-2} - \&c\right\}$$

Cor 2. There will be no logarithmic function in $\phi_n\left(\frac{1}{x}\right) + \phi_n\left(\frac{2}{x}\right) + \cdots + \phi_n\left(\frac{x}{x}\right)$.

20. Let $1^r + 2^r K + 3^r K^2 + \cdots + x^r K^{x-1} = K^x\phi(x) = F_K(x)$

i. $\phi(x) = C_r(K) + x^r\,\frac{\psi_0(-K)}{K-1} - \frac{r}{\lfloor 1}\cdot x^{r-1}\,\frac{\psi_1(-K)}{(K-1)^2} + \frac{r(r-1)}{\lfloor 2}\cdot\frac{\psi_2(-K)}{(K-1)^3} - \&c$ where ψ is the same ψ in

ii. $C_r(K) = \frac{\psi_r(-K)}{(1-K)^{r+1}}$ and $K\,\psi_r(-K) = K^r\psi\left(-\frac{1}{K}\right)$

iii. $F_K\left(\frac{x}{n}\right) + F_K\left(\frac{x-1}{n}\right) + F_K\left(\frac{x-2}{n}\right) + \cdots + F_K\left(\frac{x-n+1}{n}\right) - nC_r(K)$

$$= \frac{\sqrt[n]{K}}{K n^r}\left\{F_{\sqrt[n]{K}}(x) - C_r(\sqrt[n]{K})\right\}$$

Cor. $F_k(-\frac{1}{n}) + F_k(-\frac{2}{n}) + \cdots + F_k(-\frac{n-1}{n}) = n\,C_r(k) - \frac{n^{r/k}\,c_r(1/k)}{k\,n^r}$

21. Let $\frac{\log 1}{1^r} + \frac{\log 2}{2^r} + \frac{\log 3}{3^r} + \cdots + \frac{\log x}{x^r} = \phi_r(x)$ and let c'_r be the constant. Then,

i. $\phi_r(x) = c'_r - \left\{\frac{1}{(x+1)^r} + \frac{1}{(x+2)^r} + \frac{1}{(x+3)^r} + \&c\right\}\log x - \frac{1}{(r-1)^2 x^{r-1}}$

$+ B_2 \frac{r}{\underline{\lvert 2}} \cdot \frac{1}{r x^{r+1}} - B_4 \frac{r(r+1)(r+2)}{\underline{\lvert 4}}\left(\frac{1}{r} + \frac{1}{r+1} + \frac{1}{r+2}\right)\frac{1}{x^{r+3}}$

$+ B_6 \frac{r(r+1)(r+2)(r+3)(r+4)}{\underline{\lvert 6}}\left(\frac{1}{r} + \frac{1}{r+1} + \frac{1}{r+2} + \frac{1}{r+3} + \frac{1}{r+4}\right)$

$\times \frac{1}{x^{r+5}} - \&c$

ii. $\phi_r(x) = r x\, c'_{r+1} - \frac{r(r+1)}{\underline{\lvert 2}} x^2 c'_{r+2} + \frac{r(r+1)(r+2)}{\underline{\lvert 3}} x^3 c'_{r+3}$

$- \&c - r \cdot \frac{1}{r}\, x\, S_{r+1} + \frac{r(r+1)}{\underline{\lvert 2}}\left(\frac{1}{r} + \frac{1}{r+1}\right) x^2 S_{r+2} - \&c$

iii. $n^r \phi_r(x) - \left\{\phi_r\left(\frac{x}{n}\right) + \phi_r\left(\frac{x-1}{n}\right) + \cdots + \phi_r\left(\frac{x-n+1}{n}\right)\right\}$

$= c'_r(n^r - n) - n^r \log n \left\{\frac{1}{(x+1)^r} + \frac{1}{(x+2)^r} + \frac{1}{(x+3)^r} + \&c\right\}$

Cor. $\phi_r(-\frac{1}{n}) + \phi_r(-\frac{2}{n}) + \phi_r(-\frac{3}{n}) + \cdots + \phi_r(-\frac{n-1}{n})$

$= n^r \log n\, S_r - (n^r - n)\, c'_r$.

22. Let $(\log 1)^r + \frac{1}{2}(\log 2)^r + \frac{1}{3}(\log 3)^r + \&c$ to x terms $= \phi_r(x)$ and let C_r be its constant; then

i. $\phi_r(x) - \frac{1}{r+1}(\log x)^{r+1} = C_r$ when $x = \infty$

ii. $n\,\phi_r(x) - \left\{\phi_r\left(\frac{x}{n}\right) + \phi_r\left(\frac{x-1}{n}\right) + \phi_r\left(\frac{x-2}{n}\right) + \cdots + \phi_r\left(\frac{x-n+1}{n}\right)\right\}$

$= \frac{n}{r+1}(\log n)^{r+1}\cos \pi r + r\, n \log n\left\{\phi_{r-1}(x) - C_{r-1}\right\}$

$$-\frac{r(r-1)}{\lfloor\underline{2}}\, n(\log n)^2 \left\{ \phi_{r-2}(n) - C_{r-2} \right\} + \&c \text{ the last term being}$$

$$(-1)^{r-1}\, n(\log n)^r \left\{ \phi_0(n) - C_0 \right\}$$

23. $$\frac{(\log 1)^r}{1^{n+1}} + \frac{(\log 2)^r}{2^{n+1}} + \frac{(\log 3)^r}{3^{n+1}} + \&c$$

$$= \frac{\lfloor\underline{r}}{n^{r+1}} + C_r - \frac{n}{\lfloor\underline{1}} C_{r+1} + \frac{n^2}{\lfloor\underline{2}} C_{r+2} - \frac{n^3}{\lfloor\underline{3}} C_{r+3} + \&c.$$

Sol. Differentiate both sides r times in

Ex. 1. $$\frac{(\log 1)^3}{1\sqrt{1}} + \frac{(\log 2)^3}{2\sqrt{2}} + \frac{(\log 3)^3}{3\sqrt{3}} + \&c = 96.001 \text{ nearly}$$

2. $$\frac{\log 1}{1^2} + \frac{\log 2}{2^2} + \frac{\log 3}{3^2} + \&c = .9382 \text{ nearly}$$

3. $$\frac{(\log 1)^4}{1^2} + \frac{(\log 2)^4}{2^2} + \frac{(\log 3)^4}{3^2} + \&c = 24 \text{ nearly.}$$

4. $$\frac{(\log 1)^5}{1\sqrt{1}} + \frac{(\log 2)^5}{2\sqrt{2}} + \frac{(\log 3)^5}{3\sqrt{3}} + \&c = 7680 \text{ nearly.}$$

5. $$\frac{(\log 1)^5}{1^2}\sqrt{\log 1} + \frac{(\log 2)^5}{2^2}\sqrt{\log 2} + \&c = 288 \text{ nearly.}$$

24. $$\frac{\log 1}{\sqrt{1}} + \frac{\log 2}{\sqrt{2}} + \frac{\log 3}{\sqrt{3}} + \cdots + \frac{\log x}{\sqrt{x}} = \phi(x)$$

i. $$\phi(x) = \frac{\log 1}{\sqrt{1}} - \frac{\log(1+x)}{\sqrt{1+x}} + \frac{\log 2}{\sqrt{2}} - \frac{\log(2+x)}{\sqrt{2+x}} + \&c$$

ii. $$\phi(x) = \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{x}}\right)\log x$$

$$+ (\sqrt{2}+1)\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)\left(\log x + \tfrac{1}{2}C_0 + \tfrac{\pi}{4} + \tfrac{1}{2}\log 8\pi\right)$$

$$- 4\sqrt{x} + \frac{1}{2}\cdot\frac{B_2}{x\sqrt{x}} - \frac{1.3.5}{2.4.6}\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{B_4}{2x^3\sqrt{x}}$$

$$+ \frac{1.3.5.7.9}{2.4.6.8.10}\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}\right)\frac{B_6}{3x^5\sqrt{x}} - \&c.$$

iii. $\phi(x) - \frac{1}{\sqrt{n}}\left\{\phi\left(\frac{x}{n}\right) + \phi\left(\frac{x-1}{n}\right) + \cdots + \phi\left(\frac{x-n+1}{n}\right)\right\}$

$= \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{x}}\right)\log n$

$-(1+\sqrt{2})\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)\left\{(\sqrt{n}-1)\left(\frac{1}{2}C_0 + \frac{\pi}{4} + \log\sqrt{8\pi}\right) - \log n\right\}$

iv. If $\psi(x) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{x}}$, then

$\left\{\phi(x-1) + \phi(-x) - 2C\right\} + \left(C_0 + \frac{\pi}{2} + \log 8\pi\right)\left\{\psi(x-1) + \psi(-x) - 2C'\right\}$

$= 2\left\{\frac{\log 1}{\sqrt{1}}\cos 2\pi x + \frac{\log 2}{\sqrt{2}}\cos 4\pi x + \&c\right\}$

v. $\left\{\phi(x-1) - \phi(-x)\right\} + \left(C_0 - \frac{\pi}{2} + \log 8\pi\right)\left\{\psi(x-1) - \psi(-x)\right\}$

$= 2\left\{\frac{\log 1}{\sqrt{1}}\sin 2\pi x + \frac{\log 2}{\sqrt{2}}\sin 4\pi x + \&c\right\}$

In both cases C & C' are the constants of $\phi(x)$ and $\psi(x)$ respectively.

Ex. 1. Find the values of $\phi(-\frac{1}{2})$, $\phi(-\frac{2}{3})$, & $\phi(-\frac{3}{4})$.

2. Shew that the constant in $\phi(x)$

$= -\frac{1}{2}S_{\frac{1}{2}}\left(C_0 + \frac{\pi}{2} + \log 8\pi\right) = 3.92265$

$= 2\left\{2 - \frac{1}{2}\cdot\frac{B_2}{2} + \frac{1.3.5}{2.4.6}\left(1 + \frac{1}{3} + \frac{1}{5}\right)\frac{B_4}{4} - \&c\right\}$

Sol. Write $\frac{1+h}{2}$ for n in VII 4 and equate the coeff.^{ts} of h. Put $x = 1$ in VIII 24. ii; then the second result is at once obtained.

CHAPTER IX

1\. If $S_n = \frac{1}{(1-a)^n} - \frac{1}{(1+a)^n} + \frac{1}{(3-a)^n} - \frac{1}{(3+a)^n} + \&c$, then

i. If n is odd,

$$\frac{\cos(1-a)x}{(1-a)^n} - \frac{\cos(1+a)x}{(1+a)^n} + \frac{\cos(3-a)x}{(3-a)^n} - \frac{\cos(3+a)x}{(3+a)^n} + \&c$$

$= S_n - \frac{x^2}{\lfloor 2} S_{n-2} + \frac{x^4}{\lfloor 4} S_{n-4} - \&c$ as far as the term containing S_1

ii. If n is even

$$\frac{\sin(1-a)x}{(1-a)^n} - \frac{\sin(1+a)x}{(1+a)^n} + \frac{\sin(3-a)x}{(3-a)^n} - \frac{\sin(3+a)x}{(3+a)^n} + \&c$$

$= \frac{x}{\lfloor 1} S_{n-1} - \frac{x^3}{\lfloor 3} S_{n-3} + \frac{x^5}{\lfloor 5} S_{n-5} - \&c$ as far as the term containing S_1

2\. If $S_n = \frac{1}{(1-a)^n} + \frac{1}{(1+a)^n} + \frac{1}{(3-a)^n} + \frac{1}{(3+a)^n} + \&c$, then

i. If n is even

$$\frac{\cos(1-a)x}{(1-a)^n} + \frac{\cos(1+a)x}{(1+a)^n} + \frac{\cos(3-a)x}{(3-a)^n} + \frac{\cos(3+a)x}{(3+a)^n} + \&c$$

$= S_n - \frac{x^2}{\lfloor 2} S_{n-2} + \frac{x^4}{\lfloor 4} S_{n-4} - \&c$ as far as the term containing S_2

ii. If n is odd

$$\frac{\sin(1-a)x}{(1-a)^n} + \frac{\sin(1+a)x}{(1+a)^n} + \frac{\sin(3-a)x}{(3+a)^n} + \frac{\sin(3+a)x}{(3+a)^n} + \&c$$

$= \frac{x}{\lfloor 1} S_{n-1} - \frac{x^3}{\lfloor 3} S_{n-3} + \frac{x^5}{\lfloor 5} S_{n-5} - \&c$ as far as the term containing S_2

Sol. In both 1 & 2 expand the series in ascending powe[rs] of x and apply

3. If $\phi(n) = \dfrac{\cos x}{1^n} - \left(1+\tfrac{1}{2}\right)\dfrac{\cos 3x}{3^n} + \left(1+\tfrac{1}{2}+\tfrac{1}{3}\right)\dfrac{\cos 5x}{5^n} - \&c$

then if n is odd $\phi(n-2) - \phi(n) =$

$$x\left\{\left(\frac{\sin x}{1^{n-2}} - \frac{\sin 3x}{3^{n-2}} + \frac{\sin 5x}{5^{n-2}} - \&c\right) - \left(\frac{\sin x}{1^n} - \frac{\sin 3x}{3^n} + \frac{\sin 5x}{5^n} - \&c\right)\right\}$$

$$+ n\left\{\left(\frac{\cos x}{1^{n-1}} - \frac{\cos 3x}{3^{n-1}} + \frac{\cos 5x}{5^{n-1}} - \&c\right) - \left(\frac{\cos x}{1^{n+1}} - \frac{\cos 3x}{3^{n+1}} + \frac{\cos 5x}{5^{n+1}} - \&c\right)\right\}$$

4. Let $F(n) = \left\{\dfrac{\sin x}{1^n} - \dfrac{1}{2}\cdot\dfrac{\sin 3x}{3^n} + \dfrac{1.3}{2.4}\cdot\dfrac{\sin 5x}{5^n} - \&c\right\}$

$$-\cos \pi n\left\{\left(\frac{\sin 2x}{2^n} - \frac{1}{2}\cdot\frac{\sin 4x}{4^n} + \frac{1.3}{2.4}\cdot\frac{\sin 6x}{6^n} - \&c\right) - \left(\frac{\sin 2x}{2^{n+1}} - \frac{1}{2}\cdot\frac{\sin 4x}{4^{n+1}} + \frac{1.3}{2.4}\cdot\frac{\sin 6x}{6^{n+1}} - \&c\right)\right\}$$ and

$$\psi(n) = \left\{\frac{\cos x}{1^n} - \frac{1}{2}\cdot\frac{\cos 3x}{3^n} + \frac{1.3}{2.4}\,\frac{\cos 5x}{5^n} - \&c\right\}$$

$$+\cos \pi n\left\{\left(\frac{\cos 2x}{2^n} - \frac{1}{2}\cdot\frac{\cos 4x}{4^n} + \frac{1.3}{2.4}\cdot\frac{\cos 6x}{6^n} - \&c\right) - \left(\frac{\cos 2x}{2^{n+1}} - \frac{1}{2}\cdot\frac{\cos 4x}{4^{n+1}} + \frac{1.3}{2.4}\,\frac{\cos 6x}{6^{n+1}} - \&c\right)\right\}$$ then

If n is odd,

i. $$\frac{F(n)}{2}\sin\frac{\pi n}{2} = \frac{x^n}{\lfloor n}S_0\phi(0) - \frac{x^{n-2}}{\lfloor n-2}\left\{S_0\phi(2) + \frac{S_2}{2^2}\phi(0)\right\}$$
$$+ \frac{x^{n-4}}{\lfloor n-4}\left\{S_0\phi(4) + \frac{S_2}{2^2}\phi(2) + \frac{S_4}{2^4}\phi(0)\right\} - \&c$$

$$= \frac{A_{n-1}}{\lfloor n-1}\phi(0) - \frac{A_{n-3}}{\lfloor n-3}\phi(2) + \frac{A_{n-5}}{\lfloor n-5}\phi(4) - \&c \qquad \mathrm{I}.$$

ii. $$\frac{F_n(n+1)}{2}\sin\frac{\pi n}{2} = \frac{x^n}{\lfloor n}S_0\phi(1) - \frac{x^{n-2}}{\lfloor n-2}\left\{S_0\phi(3) + \frac{S_2}{2^2}\phi(1)\right\}$$
$$+ \frac{x^{n-4}}{\lfloor n-4}\left\{S_0\phi(5) + \frac{S_2}{2^2}\phi(3) + \frac{S_4}{2^4}\phi(1)\right\} - \&c.$$
$$= \frac{A_{n-1}}{\lfloor n-1}\phi(1) - \frac{A_{n-3}}{\lfloor n-3}\phi(3) + \frac{A_{n-5}}{\lfloor n-5}\phi(5) - \&c \qquad \mathrm{II}.$$

where $S_r = \frac{1}{1^r} - \frac{1}{2^r} + \frac{1}{3^r} - \frac{1}{4^r} + \&c,$

$$\frac{\pi}{2}\phi(r) = \frac{1}{1^{r+1}} + \frac{1}{2}\cdot\frac{1}{3^{r+1}} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5^{r+1}} + \&c$$

and $$\frac{2}{\pi}A_r = \left(\frac{\pi}{2}\right)^r + \left(\frac{3\pi}{2}\right)^r + \left(\frac{5\pi}{2}\right)^r + \cdots\cdots + \left(x - \frac{\pi}{2}\right)^r.$$

If n is even $\frac{\psi(n)}{2}\cos\frac{\pi n}{2} = \mathrm{I}$ & $\frac{\psi(n+1)}{2}\cos\frac{\pi n}{2} = \mathrm{II}$

S. l. From the following identities the I part of the theorem is obtained.

i. $$\sin x - \frac{1}{2}\cdot\sin 3x + \frac{1\cdot 3}{2\cdot 4}\sin 5x - \&c = \frac{1}{2}\sin 2x - \frac{1\cdot 3}{2\cdot 4}\sin 4x + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin 6x - \&c = \frac{\sin\frac{x}{2}}{\sqrt{2\cos x}}.$$

ii. $$\cos x - \frac{1}{2}\cos 3x + \frac{1\cdot 3}{2\cdot 4}\cos 5x - \&c = 1 - \frac{1}{2}\cos 2x + \frac{1\cdot 3}{2\cdot 4}\cos 4x - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos 6x + \&c = \frac{\cos\frac{x}{2}}{\sqrt{2\cos x}}.$$

iii. $$\sin 2x - \frac{1}{2}\sin 4x + \frac{1\cdot 3}{2\cdot 4}\sin 6x - \&c = \frac{\sin\frac{3x}{2}}{\sqrt{2\cos x}}.$$

iv. $$\cos 2x - \frac{1}{2}\cos 4x + \frac{1\cdot 3}{2\cdot 4}\cos 6x - \&c = \frac{\cos\frac{3x}{2}}{\sqrt{2\cos x}}.$$

v. $$\frac{\sin 2x}{2} - \frac{1}{2}\cdot\frac{\sin 4x}{4} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\sin 6x}{6} - \&c = \sin\frac{x}{2}\sqrt{2\cos x}.$$

vi. $$\frac{\cos 2x}{2} - \frac{1}{2}\cdot\frac{\cos 4x}{4} + \frac{1\cdot 3}{2\cdot 4}\frac{\cos 6x}{6} - \&c = \cos\frac{x}{2}\sqrt{2\cos x} - 1.$$

vii. $$\frac{\sin x}{1} - \frac{1}{2}\cdot\frac{\sin 3x}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\sin 5x}{5} - \&c = \sin^{-1}\left(\sqrt{2}\sin\frac{x}{2}\right)$$

viii. $$\frac{\cos x}{1} - \frac{1}{2}\cdot\frac{\cos 3x}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{\cos 5x}{5} - \&c = \log\left(\sqrt{\cos x} + \sqrt{2}\cos\frac{x}{2}\right)$$

ix $\dfrac{\sin 2x}{2^2} - \dfrac{1}{2}\cdot\dfrac{\sin 4x}{4^2} + \dfrac{1.3}{2.4}\cdot\dfrac{\sin 6x}{6^2} - \&c = \sin\dfrac{x}{2}\sqrt{2\cos x}$
$+ \sin^{-1}\left(\sqrt{2}\sin\dfrac{x}{2}\right) - x.$

x. $\dfrac{\cos 2x}{2^2} - \dfrac{1}{2}\cdot\dfrac{\cos 4x}{4^2} + \dfrac{1.3}{2.4}\cdot\dfrac{\cos 6x}{6^2} - \&c = \cos\dfrac{x}{2}\sqrt{2\cos x}$
$- \log\left(\sqrt{\cos x} + \sqrt{2}\cos\dfrac{x}{2}\right) - 1 + \log 2.$

5\. i. $\sin a\theta + \dfrac{n}{\underline{|1}}\sin(a+2)\theta + \dfrac{n(n-1)}{\underline{|2}}\sin(a+4)\theta + \&c$
$= 2^n\cos^n\theta\,\sin(a+n)\theta.$

ii. $\cos a\theta + \dfrac{n}{\underline{|1}}\cos(a+2)\theta + \dfrac{n(n-1)}{\underline{|2}}\cos(a+4)\theta + \&c$
$= 2^n\cos^n\theta\,\cos(a+n)\theta.$

6\. If $\phi(x) = \dfrac{x}{1^2} + \dfrac{x^2}{2^2} + \dfrac{x^3}{3^2} + \dfrac{x^4}{4^2} + \&c$, then

i. $\phi(1-x) + \phi\left(1-\dfrac{1}{x}\right) = -\dfrac{1}{2}(\log x)^2.$

ii. $\phi(-x) + \phi\left(-\dfrac{1}{x}\right) = -\dfrac{\pi^2}{6} - \dfrac{1}{2}(\log x)^2.$

iii. $\phi(x) + \phi(1-x) = \dfrac{\pi^2}{6} - \log x\log(1-x)$

iv. $\phi(x) + \phi(-x) = \dfrac{1}{2}\phi(x^2).$

v. If $\phi(x) - \phi(-x) = 2\psi(x) = 2\left(\dfrac{x}{1^2} + \dfrac{x^3}{3^2} + \dfrac{x^5}{5^2} + \&c\right)$, then
$\psi(x) + \psi\left(\dfrac{1-x}{1+x}\right) = \dfrac{\pi^2}{8} + \dfrac{1}{2}\log x\log\dfrac{1+x}{1-x}.$

vi. $\phi\left(\dfrac{x}{1-y}\right) + \phi\left(\dfrac{y}{1-x}\right) = \phi(x) + \phi(y) + \phi\dfrac{xy}{(1-x)(1-y)} + \log(1-x)\log(1-y)$

vii. $\phi(e^{-x}) = \dfrac{\pi^2}{6} + x\log x - x - \dfrac{x^2}{4} + \dfrac{B_2}{2\underline{|3}}x^3 - \dfrac{B_4}{4\underline{|5}}x^5 + \&c.$

viii. $\phi(1-e^{-x}) = x - \dfrac{x^2}{4} + B_2\dfrac{x^3}{\underline{|3}} - B_4\dfrac{x^5}{\underline{|5}} + B_6\dfrac{x^7}{\underline{|7}} - \&c.$

E.g. i $\phi\left(\dfrac{1}{2}\right) = \dfrac{\pi^2}{12} - \dfrac{1}{2}(\log 2)^2.$

ii. $\phi\left(\dfrac{\sqrt{5}-1}{2}\right) = \dfrac{\pi^2}{10} - \left(\log\dfrac{\sqrt{5}-1}{2}\right)^2$

iii. $\phi\left(\dfrac{3-\sqrt{5}}{2}\right) = \dfrac{\pi^2}{15} - \left(\log\dfrac{\sqrt{5}-1}{2}\right)^2$

iv. $\psi(\sqrt{2}-1) = \dfrac{\pi^2}{16} - \dfrac{1}{4}(\log\sqrt{2}-1)^2.$

v. $\psi\left(\dfrac{\sqrt{5}-1}{2}\right) = \dfrac{\pi^2}{12} - \dfrac{3}{4}\left(\log\dfrac{\sqrt{5}-1}{2}\right)^2$

vi. $\psi(\sqrt{5}-2) = \dfrac{\pi^2}{24} - \dfrac{3}{4}\left(\log\dfrac{\sqrt{5}-1}{2}\right)^2.$

7. If $\phi(x) = \frac{x}{1^3} + \frac{x^2}{2^3} + \frac{x^3}{3^3} + \&c$ then

i $\phi(1-x) + \phi\left(1-\frac{1}{x}\right) + \phi(x) = S_3 + \frac{\pi^2}{6}\log x - \frac{1}{2}(\log x)^2\log(1-x)$
$+ \frac{1}{6}(\log x)^3$

ii. $\phi(-x) - \phi\left(-\frac{1}{x}\right) = -\frac{1}{6}(\log x)^3 - \frac{\pi^2}{6}\log x.$

iii. $\phi(x) + \phi(-x) = \frac{1}{4}\phi(x^2).$

E.g. i $\phi\left(\frac{1}{2}\right) = \frac{1}{6}(\log 2)^3 - \frac{\pi^2}{12}\log 2 + \left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right).$

ii. $\phi\left(\frac{3-\sqrt{5}}{2}\right) = \frac{2}{3}\left(\log\frac{\sqrt{5}+1}{2}\right)^3 - \frac{2\pi^2}{15}\log\frac{\sqrt{5}+1}{2} + S_3.$

8. If $\phi(x) = x + \left(1+\frac{1}{3}\right)\frac{x^3}{3} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{x^5}{5} + \&c$, then

$\phi\left(\frac{x}{2-x}\right) = \frac{1}{8}\left(\log\overline{1-x}\right)^2 + \frac{1}{2}\left(\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \&c\right).$

E.g. i $\phi\left(\frac{1}{3}\right) = \frac{\pi^2}{24} - \frac{1}{8}(\log 2)^2.$

ii. $\phi\left(\frac{1}{\sqrt{5}}\right) = \frac{\pi^2}{20} - \frac{3}{8}\left(\log\frac{\sqrt{5}-1}{2}\right)^2.$

iii. $\phi(\sqrt{5}-2) = \frac{\pi^2}{30} - \frac{3}{4}\left(\log\frac{\sqrt{5}-1}{2}\right)^2.$

9. If $\phi(x) = \frac{x^2}{2^2} + \left(1+\frac{1}{2}\right)\frac{x^3}{3^2} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^4}{4^2} + \&c$ then

i. $\phi(1-x) = \frac{1}{2}\log(1-x)(\log x)^2 + \log x\left(\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \&c\right)$
$-\left(\frac{x}{1^3} + \frac{x^2}{2^3} + \frac{x^3}{3^3} + \&c\right) + S_3.$

ii. $\phi(1-x) - \phi\left(1-\frac{1}{x}\right) = \frac{1}{6}(\log x)^3.$

iii $\phi(1-x) = \frac{1}{2}\log(1-x)(\log x)^2 - \frac{1}{3}(\log x)^3 - \log x\left(\frac{1}{1^2 x} + \frac{1}{2^2 x^2} + \&c\right)$
$-\left(\frac{1}{1^3 x} + \frac{1}{2^3 x^2} + \frac{1}{3^3 x^3} + \&c\right) + S_3.$

iv. $\phi(-x) + \phi\left(-\frac{1}{x}\right) = -\frac{1}{6}(\log x)^3 + \log x\left(\frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c\right)$
$-\left(\frac{x}{1^3} - \frac{x^2}{2^3} + \frac{x^3}{3^3} - \&c\right) + S_3.$

10. If $\phi(x) = \frac{x^2}{2^3} + \left(1+\frac{1}{2}\right)\frac{x^3}{3^3} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^4}{4^3} + \&c$ then

i. $\phi(1-x) - \phi\left(1-\frac{1}{x}\right) = \frac{1}{24}(\log x)^4 - \frac{1}{6}(\log x)^3\log(1-x) - S_3\log x$

$$+ 2\left(\frac{x}{1^4} + \frac{x^2}{2^4} + \frac{x^3}{3^4} + \&c\right) - \log x\left(\frac{x}{1^3} + \frac{x^2}{2^3} + \frac{x^3}{3^3} + \&c\right) - \frac{\pi^4}{45}.$$

ii. $\phi(-x) - \phi\left(-\frac{1}{x}\right) = \frac{1}{24}(\log x)^4 - \log x\left(\frac{x}{1^3} - \frac{x^2}{2^3} + \frac{x^3}{3^3} - \&c\right)$

$$+ 2\left(\frac{x}{1^4} - \frac{x^2}{2^4} + \frac{x^3}{3^4} - \&c\right) - S_3 \log x - \frac{7\pi^4}{360}.$$

11. If $\phi(x) = \frac{x^2}{2^2} + \left(1+\frac{1}{3}\right)\frac{x^4}{4^2} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{x^6}{6^2} + \&c$ and

$\psi(x) = \frac{x^2}{2^3} + \left(1+\frac{1}{3}\right)\frac{x^4}{4^3} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{x^6}{6^3} + \&c$, then

i. $\phi\left(\frac{1-x}{1+x}\right) = \frac{1}{8}(\log x)^2 \log\frac{1-x}{1+x} + \frac{1}{2}\log x\left(\frac{x}{1^2} + \frac{x^3}{3^2} + \frac{x^5}{5^2} + \&c\right)$

$$+ \frac{1}{2}\left(\frac{1-x}{1^3} + \frac{1-x^3}{3^3} + \frac{1-x^5}{5^3} + \&c\right).$$

ii. $\psi(x) + \psi\left(\frac{1-x}{1+x}\right) = \phi(x)\log x + \phi\left(\frac{1-x}{1+x}\right)\log\frac{1-x}{1+x}$

$$- \frac{1}{16}(\log x)^2\left(\log\frac{1-x}{1+x}\right)^2 + \frac{\pi}{4}\left(\frac{1}{1^3} - \frac{1}{5^3} + \frac{1}{9^3} - \&c\right) - \frac{\pi}{3\sqrt{3}}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right).$$

12. If $\phi(x) = x + \left(1+\frac{1}{2}\right)\frac{x^3}{3} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^5}{5} + \&c$, then

$\phi\left(\frac{1-x}{1+x}\right) = -(1-\log 2)\log x + \frac{1+x}{1-x}\log\frac{4x}{(1+x)^2} + \frac{1}{4}(\log x)^2 + \frac{\pi^2}{12}$

$$- \left(\frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c\right).$$

E.g. i. $\frac{1}{2} + \frac{1+\frac{1}{2}}{2^2}\cdot\frac{1}{2^2} + \frac{1+\frac{1}{2}+\frac{1}{3}}{3^2}\cdot\frac{1}{2^3} + \&c = S_3 - \frac{\pi^2}{12}\log 2.$

ii $\frac{1}{1^2} + \frac{1+\frac{1}{2}+\frac{1}{3}}{3^2} + \frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}}{5^2} + \&c = \frac{3}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$

iii. $\frac{1}{1^2} + \frac{1+\frac{1}{3}}{2^2} + \frac{1+\frac{1}{3}+\frac{1}{5}}{3^2} + \&c = 2\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right).$

iv. $(\sqrt{5}-2) + \frac{1+\frac{1}{2}}{3}(\sqrt{5}-2)^3 + \frac{1+\frac{1}{2}+\frac{1}{3}}{5}(\sqrt{5}-2)^5 + \&c$

$$= \frac{\pi^2}{60} + \frac{3}{4}\left(\log\frac{\sqrt{5}-1}{2}\right)^2 + (\sqrt{5}+2)\log 4 + (3\sqrt{5}+5+\log 2)\log\frac{\sqrt{5}-1}{2}$$

13. $S_{n+1}\cos\frac{\pi n}{2}\,\underline{|n} = \int\frac{x^n}{2}\cot\frac{x}{2}\,dx + x^n\left(\frac{\cos x}{1} + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \&c\right)$

$$- n x^{n-1}\left(\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \&c\right)$$

$$- n(n-1)x^{n-2}\left(\frac{\cos x}{1^3} + \frac{\cos 2x}{2^3} + \frac{\cos 3x}{3^3} + \&c\right)$$

$$+ n(n-1)(n-2)x^{n-3}\left(\frac{\sin x}{1^4} + \frac{\sin 2x}{2^4} + \frac{\sin 3x}{3^4} + \&c\right) + \&c$$

where $S_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \&c$ and $S_1 = -\log 2$.

Sol. $\sin x + \sin 2x + \sin 3x + \&c = \frac{1}{2}\cot\frac{x}{2}$.

$\therefore \int x^n(\sin x + \sin 2x + \&c)\,dx = \int \frac{x^n}{2}\cot\frac{x}{2}\,dx$.

14. $$S_{n+1}\cos\frac{\pi n}{2}\,\lfloor n = \int \frac{x^n}{2\sin x}\,dx$$
$$+ x^n\left(\frac{\cos x}{1} + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \&c\right)$$
$$- nx^{n-1}\left(\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \&c\right)$$
$$- n(n-1)x^{n-2}\left(\frac{\cos x}{1^3} + \frac{\cos 3x}{3^3} + \frac{\cos 5x}{5^3} + \&c\right) + \&c.$$

Sol. $\sin x + \sin 3x + \sin 5x + \&c = \frac{1}{2}\operatorname{cosec} x$.

15. If $\int x^n \cot x\,dx = f_n(x)$ then
$$2^n f_n\left(\frac{\pi}{2} - x\right) = \pi^n\left\{f_0(2x) - f_0(x)\right\} - \frac{n}{\lfloor 1}\pi^{n-1}\left\{f_1(2x) - 2f_1(x)\right\}$$
$$+ \frac{n(n-1)}{\lfloor 2}\pi^{n-2}\left\{f_2(2x) - 2^2 f_2(x)\right\} - \frac{n(n-1)(n-2)}{\lfloor 3}\pi^{n-3} \times$$
$$\left\{f_3(2x) - 2^3 f_3(x)\right\} + \&c.$$

Sol. $\tan x = \cot x - 2\cot 2x$ and
$$f_n\left(\frac{\pi}{2} - x\right) = -\int\left(\frac{\pi}{2} - x\right)^n \cot\left(\frac{\pi}{2} - x\right)dx = -\int\left(\frac{\pi}{2} - x\right)^n \tan x\,dx.$$

N.B. Let $\sin x = y$ and $\tan x = z$ then
$$\int x^n \cot x\,dx = \int \frac{x^n}{\sin x}\cos x\,dx = \int \frac{(\sin^{-1} y)^n}{y}\,dy \text{ and}$$
$$\int \frac{2x^n}{\sin 2x}\,dx = \int \frac{x^n}{\cos x \sin x}\,dx = \int \frac{x^n}{\tan x}\sec^2 x\,dx$$
$$= \int \frac{(\tan^{-1} z)^n}{z}\,dz.$$

1. $$\frac{1}{2}(\tan^{-1} x)^2 = \frac{x^2}{2} - \left(1 + \frac{1}{3}\right)\frac{x^4}{4} + \left(1 + \frac{1}{3} + \frac{1}{5}\right)\frac{x^6}{6} - \&c.$$

ii. $\dfrac{1}{\underline{|2}}(\sin^{-1}x)^2 = \dfrac{x^2}{2} + \dfrac{2}{3}\cdot\dfrac{x^4}{4} + \dfrac{2\cdot 4}{3\cdot 5}\cdot\dfrac{x^6}{6} + \&c$

iii. $\dfrac{1}{\underline{|3}}(\sin^{-1}x)^3 = \dfrac{1}{2}\cdot\dfrac{x^3}{3} + \dfrac{1\cdot 3}{2\cdot 4}\cdot\dfrac{x^5}{5}\left(1+\dfrac{1}{3^2}\right) + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\dfrac{x^7}{7}\left(1+\dfrac{1}{3^2}+\dfrac{1}{5^2}\right) + \&c$

iv. $\dfrac{1}{\underline{|4}}(\sin^{-1}x)^4 = \dfrac{2}{3}\cdot\dfrac{x^4}{4}\cdot\dfrac{1}{2^2} + \dfrac{2\cdot 4}{3\cdot 5}\cdot\dfrac{x^6}{6}\left(\dfrac{1}{2^2}+\dfrac{1}{4^2}\right) + \dfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\dfrac{x^8}{8}\left(\dfrac{1}{2^2}+\dfrac{1}{4^2}+\dfrac{1}{6^2}\right) + \&c$

16. $\dfrac{\sin x}{1^2} + \dfrac{1}{2}\cdot\dfrac{\sin^3 x}{3^2} + \dfrac{1\cdot 3}{2\cdot 4}\cdot\dfrac{\sin^5 x}{5^2} + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\dfrac{\sin^7 x}{7^2} + \&c$

$= x\log 2\sin x + \dfrac{1}{2}\left(\dfrac{\sin 2x}{1^2} + \dfrac{\sin 4x}{2^2} + \dfrac{\sin 6x}{3^2} + \&c\right)$

e.g. i. $\dfrac{1}{1^2} + \dfrac{1}{2}\cdot\dfrac{1}{3^2} + \dfrac{1\cdot 3}{2\cdot 4}\cdot\dfrac{1}{5^2} + \&c = \dfrac{\pi}{2}\log 2.$

ii. $\dfrac{1}{1^2} + \dfrac{1}{2}\cdot\dfrac{1}{3^2}\cdot\dfrac{1}{2} + \dfrac{1\cdot 3}{2\cdot 4}\cdot\dfrac{1}{5^2}\cdot\dfrac{1}{2^2} + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\dfrac{1}{7^2}\cdot\dfrac{1}{2^3} + \&c$

$= \dfrac{\pi}{4\sqrt{2}}\log 2 + \dfrac{1}{\sqrt{2}}\left(\dfrac{1}{1^2} - \dfrac{1}{3^2} + \dfrac{1}{5^2} - \dfrac{1}{7^2} + \&c\right).$

iii. $\dfrac{1}{1^2}\cdot\dfrac{1}{2} + \dfrac{1}{2}\cdot\dfrac{1}{3^2}\cdot\dfrac{1}{2^3} + \dfrac{1\cdot 3}{2\cdot 4}\cdot\dfrac{1}{5^2}\cdot\dfrac{1}{2^5} + \&c$

$= \dfrac{3\sqrt{3}}{4}\left(\dfrac{1}{1^2} + \dfrac{1}{4^2} + \dfrac{1}{7^2} + \&c\right) - \dfrac{\pi^2}{6\sqrt{3}}.$

iv. $\dfrac{1}{1^2} + \dfrac{1}{2}\cdot\dfrac{1}{3^2}\cdot\dfrac{3}{4} + \dfrac{1\cdot 3}{2\cdot 4}\cdot\dfrac{1}{5^2}\cdot\left(\dfrac{3}{4}\right)^2 + \&c = \dfrac{\pi}{3\sqrt{3}}\log 3 - \dfrac{2\pi^2}{27}$

$+ \left(\dfrac{1}{1^2} + \dfrac{1}{4^2} + \dfrac{1}{7^2} + \&c\right).$

17. $\dfrac{\tan x}{1^2} - \dfrac{\tan^3 x}{3^2} + \dfrac{\tan^5 x}{5^2} - \&c$

$= x\log\tan x + \dfrac{\sin 2x}{1^2} + \dfrac{\sin 6x}{3^2} + \dfrac{\sin 10x}{5^2} + \&c$

e.g. i. $\displaystyle\int_0^{\frac{1}{\sqrt{3}}} \frac{\tan^{-1}x}{x}\,dx = -\frac{\pi}{12}\log 3 - \frac{5\pi^2}{18\sqrt{3}} + \frac{5\sqrt{3}}{4}\left(\frac{1}{1^2} + \frac{1}{4^2} + \frac{1}{7^2} + \&c\right)$

ii. $\displaystyle\int_0^{\sqrt{2}-1} \frac{\tan^{-1}x}{x}\,dx = \frac{\pi}{8}\log(\sqrt{2}-1) - \frac{\pi^2}{16} + \sqrt{2}\left(\frac{1}{1^2} - \frac{1}{5^2} + \frac{1}{9^2} - \&c\right)$

iii. $\displaystyle\int_0^{2-\sqrt{3}} \frac{\tan^{-1}x}{x}\,dx = \frac{\pi}{12}\log(2-\sqrt{3}) + \frac{2}{3}\int_0^1 \frac{\tan^{-1}x}{x}\,dx$

N. B. $\displaystyle\int_0^1 \frac{\tan^{-1}x}{x}\,dx = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c$

$= \cdot 915965594177$

18 When x lies between 0 and $\dfrac{\pi}{4}$,

$$\frac{\cos x-\sin x}{1^2}+\frac{1}{2}\cdot\frac{\cos^3 x-\sin^3 x}{3^2}+\frac{1.3}{2.4}\cdot\frac{\cos^5 x-\sin^5 x}{5^2}+\&c$$

$$=\frac{\pi}{2}\log 2\cos x-\frac{1}{2}\left\{\frac{\sin 2x}{1^2}+\frac{1}{2}\cdot\frac{\sin^3 2x}{3^2}+\frac{1.3}{2.4}\cdot\frac{\sin^5 2x}{5^2}+\&c\right\}$$

e.g. If $\psi(x)=\int_0 \frac{\sin^{-1}x}{x}\,dx$ then

$$\psi\left(\frac{3}{5}\right)-\frac{1}{2}\psi\left(\frac{2\sqrt{5}}{25}\right)=\frac{\pi}{2}\log 2+2\psi\left(\frac{1}{\sqrt{5}}\right)-2\psi\left(\frac{2}{\sqrt{5}}\right).$$

19. $$\frac{\cos x+\sin x}{1^2}+\frac{1}{2}\cdot\frac{\cos^3 x+\sin^3 x}{3^2}+\frac{1.3}{2.4}\cdot\frac{\cos^5 x+\sin^5 x}{5^2}+\&c$$

$$=\frac{\pi}{2}\log 2\cos x+\frac{\tan x}{1^2}-\frac{\tan^3 x}{3^2}+\frac{\tan^5 x}{5^2}-\&c.$$

e.g. $$\frac{1}{1^2}\cdot\frac{1+2}{5}+\frac{1}{2}\cdot\frac{1}{3^2}\cdot\frac{1+2^3}{5^2}+\frac{1.3}{2.4}\cdot\frac{1}{5^2}\cdot\frac{1+2^5}{5^3}+\&c.$$

$$=\frac{\pi}{2\sqrt{5}}\log\frac{4}{\sqrt{5}}+\frac{1}{\sqrt{5}}\left(\frac{1}{1^2}\cdot\frac{1}{2}-\frac{1}{3^2}\cdot\frac{1}{2^3}+\frac{1}{5^2}\cdot\frac{1}{2^5}-\&c\right)$$

20. $$\frac{\sin^2 x}{2^2}+\frac{2}{3}\cdot\frac{\sin^4 x}{4^2}+\frac{2.4}{3.5}\cdot\frac{\sin^6 x}{6^2}+\&c$$

$$=\frac{x^2}{2}\log 2\sin x+\frac{x}{2}\left(\frac{\sin 2x}{1^2}+\frac{\sin 4x}{2^2}+\frac{\sin 6x}{3^2}+\&c\right)$$

$$+\frac{1}{4}\left(\frac{\cos 2x}{1^3}+\frac{\cos 4x}{2^3}+\frac{\cos 6x}{3^3}+\&c\right)$$

e.g. i. $$\frac{1}{2^2}+\frac{2}{3}\cdot\frac{1}{4^2}+\frac{2.4}{3.5}\cdot\frac{1}{6^2}+\frac{2.4.6}{3.5.7}\cdot\frac{1}{8^2}+\&c$$

$$=\frac{\pi^2}{8}\log 2-\frac{1}{2}\left(\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\&c\right)$$

ii. $$\frac{1}{2^2}\cdot\frac{1}{2}+\frac{2}{3}\cdot\frac{1}{4^2}\cdot\frac{1}{2^2}+\frac{2.4}{3.5}\cdot\frac{1}{6^2}\cdot\frac{1}{2^3}+\&c$$

$$=\frac{\pi^2}{64}\log 2+\frac{\pi}{8}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\&c\right)-\frac{5}{16}\left(\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\&c\right)$$

21. $$\frac{\tan^2 x}{2^2}-\left(1+\frac{1}{3}\right)\frac{\tan^4 x}{4^2}+\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{\tan^6 x}{6^2}-\&c$$

$$=\frac{x^2}{2}\log\tan x+x\left(\frac{\sin 2x}{1^2}+\frac{\sin 6x}{3^2}+\frac{\sin 10x}{5^2}+\&c\right)$$

$$+\frac{1}{2}\left(\frac{\cos 2x}{1^3}+\frac{\cos 6x}{3^3}+\frac{\cos 10x}{5^3}+\&c\right)-\frac{1}{2}\left(\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\&c\right)$$

e.g. $\frac{1}{1^2} - \frac{1+\frac{1}{3}}{2^2} + \frac{1+\frac{1}{3}+\frac{1}{5}}{5^2} - \&c$

$= \frac{\pi}{4}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c\right) - \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$

22. $\frac{\cos^2 x + \sin^2 x}{2^2} + \frac{2}{3}\cdot\frac{\cos^4 x + \sin^4 x}{4^2} + \frac{2.4}{3.5}\cdot\frac{\cos^6 x + \sin^6 x}{6^2}$

$+ \frac{2.4.6}{3.5.7}\cdot\frac{\cos^8 x + \sin^8 x}{8^2} + \&c = -\frac{\pi^2}{8}\log 2\cos x$

$+ \frac{\pi}{2}\left\{\frac{\cos x}{1^2} + \frac{1}{2}\cdot\frac{\cos^3 x}{3^2} + \frac{1.3}{2.4}\cdot\frac{\cos^5 x}{5^2} + \&c\right\}$

$+ \frac{1}{4}\left\{\frac{\sin^2 2x}{2^2} + \frac{2}{3}\cdot\frac{\sin^4 2x}{4^2} + \frac{2.4}{3.5}\cdot\frac{\sin^6 2x}{6^2} + \&c\right\}$

$- \frac{1}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \&c\right)$.

23. $\frac{\tan^2 x}{2^2} - \left(1+\frac{1}{3}\right)\frac{\tan^4 x}{4^2} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{\tan^6 x}{6^2} - \&c$

$= 2\left\{\frac{\sin^2 x}{2^2} + \frac{2}{3}\cdot\frac{\sin^4 x}{4^2} + \frac{2.4}{3.5}\cdot\frac{\sin^6 x}{6^2} + \&c\right\}$

$- \frac{1}{4}\left\{\frac{\sin^2 2x}{2^2} + \frac{2}{3}\cdot\frac{\sin^4 2x}{4^2} + \frac{2.4}{3.5}\cdot\frac{\sin^6 2x}{6^2} + \&c\right\}$

24. If $x\cos\theta + y\cos\phi = 1$ and $x\sin\theta + y\sin\phi = 0$ then

i. $\frac{x}{1^2}\cos\theta + \frac{x^2}{2^2}\cos 2\theta + \frac{x^3}{3^2}\cos 3\theta + \&c$

$+ \frac{y}{1^2}\cos\phi + \frac{y^2}{2^2}\cos 2\phi + \frac{y^3}{3^2}\cos 3\phi + \&c$

$= \frac{\pi^2}{6} - \log x \log y + \theta\phi$.

ii. $\frac{x}{1^2}\sin\theta + \frac{x^2}{2^2}\sin 2\theta + \frac{x^3}{3^2}\sin 3\theta + \&c$

$+ \frac{y}{1^2}\sin\phi + \frac{y^2}{2^2}\sin 2\phi + \frac{y^3}{3^2}\sin 3\phi + \&c = -\phi\log x - \theta\log y$

25. If $x\cos\theta + y\cos\phi = xy\cos(\theta+\phi)$. & $x\sin\theta + y\sin\phi = xy\sin(\theta+\phi)$, then

i. $\frac{x}{1^2}\cos\theta + \frac{x^2}{2^2}\cos 2\theta + \frac{x^3}{3^2}\cos 3\theta + \&c$

$+ \frac{y}{1^2}\cos\phi + \frac{y^2}{2^2}\cos 2\phi + \frac{y^3}{3^2}\cos 3\phi + \&c$

$= \frac{1}{4}\log(1-2x\cos\theta+x^2)\log(1-2y\cos\phi+y^2)$

$- \frac{1}{2}\tan^{-1}\frac{x\sin\theta}{1-x\cos\phi}\tan^{-1}\frac{y\sin\phi}{1-y\cos\phi}$.

ii. $\frac{x}{1^2}\sin\theta + \frac{x^2}{2^2}\sin 2\theta + \frac{x^3}{3^2}\sin 3\theta + \&c$

$+ \frac{y}{1^2}\sin\phi + \frac{y^2}{2^2}\sin 2\phi + \frac{y^3}{3^2}\sin 3\phi + \&c$

$= -\frac{1}{2}\log(1 - 2\cos\theta . x + x^2)\tan^{-1}\frac{y\sin\phi}{1 - y\cos\phi}$

$- \frac{1}{2}\log(1 - 2y\cos\phi + y^2)\tan^{-1}\frac{x\sin\theta}{1 - x\cos\theta}$.

26. $x\cos\theta + y\cos\phi + xy\cos(\theta+\phi) = 1$ and

$x\sin\theta + y\sin\phi + xy\sin(\theta+\phi) = 0$, then

i. $\frac{x}{1^2}\cos\theta + \frac{x^3}{3^2}\cos 3\theta + \frac{x^5}{5^2}\cos 5\theta + \&c$

$+ \frac{y}{1^2}\cos\phi + \frac{y^3}{3^2}\cos 3\phi + \frac{y^5}{5^2}\cos 5\phi + \&c$

$= \frac{\pi^2}{8} - \frac{1}{2}\log x \log y + \frac{1}{2}\theta\phi$.

ii. $\frac{x}{1^2}\sin\theta + \frac{x^3}{3^2}\sin 3\theta + \frac{x^5}{5^2}\sin 5\theta + \&c$

$+ \frac{y}{1^2}\sin\phi + \frac{y^3}{3^2}\sin 3\phi + \frac{y^5}{5^2}\sin 5\phi + \&c$

$= -\frac{1}{2}\phi\log x - \frac{1}{2}\theta\log y$.

27. $1^r\log 1 + 2^r\log 2 + 3^r\log 3 + 4^r\log 4 + \cdots + x^r\log x = \phi_r(x)$

$\phi_r(x) = C_r + (1^r + 2^r + 3^r + \cdots + x^r - S_{-r})\log x - \frac{x^{r+1}}{(r+1)^2}$

$+ \frac{B_2}{\lfloor 2}.r.\frac{1}{r}.x^{r-1} - \frac{B_4}{\lfloor 4}r(r-1)(r-2)\left(\frac{1}{r} + \frac{1}{r-1} + \frac{1}{r-2}\right)x^{r-3} + \&c$.

and $C_{r-1} = \frac{B_r}{r}\left\{\cos\frac{\pi r}{2}\left(\sum\frac{1}{r-1} - C_0 - \log 2\pi\right) - \frac{\pi}{2}\sin\frac{\pi r}{2}\right\}$

$- 2\frac{\lfloor r-1}{(2\pi)^r}\cos\frac{\pi r}{2}\left\{\frac{\log 1}{1^r} + \frac{\log 2}{2^r} + \frac{\log 3}{3^r} + \&c\right\}$

Cor. If r is even $C_r = -\frac{\pi}{2}.\frac{B_{r+1}}{r+1}\cos\frac{\pi r}{2} = -\frac{\lfloor r}{2(2\pi)^r}S_{r+1}\cos\frac{\pi r}{2}$

$C_0 = \frac{1}{2}\log 2\pi$, $C_2 = \frac{S_3}{4\pi^2}$, $C_4 = -\frac{3S_5}{4\pi^4}$, $C_6 = \frac{45S_7}{8\pi^6}$ &c.

e.g. i. $\dfrac{(1^1.2^2.3^3.4^4\cdots x^x)^2 e^{\frac{1}{6}(x^2 - \sum\frac{1}{x})}}{\sqrt[3]{\dfrac{x^{(x+3)x}\lfloor x^2}{x+\frac{1}{4}}}}$ when $x = \infty$

$= 1^{\frac{1}{\pi^2}}.2^{\frac{1}{(2\pi)^2}}.3^{\frac{1}{(3\pi)^2}}.4^{\frac{1}{(4\pi)^2}}$ &c.

ii. $\left\{\left(\frac{1}{x}\right)^{1^2}\cdot\left(\frac{2}{x}\right)^{2^2}\left(\frac{3}{x}\right)^{3^2}\cdot\left(\frac{4}{x}\right)^{4^2}\cdots\left(\frac{x}{x}\right)^{x^2}\right\} e^{\frac{x^3}{9}-\frac{x}{12}}$ when $x=\infty$

$= e^{\frac{2}{\pi^2}\left(\frac{1}{2^2}+\frac{1}{4^3}+\frac{1}{6^3}+\frac{1}{8^2}+\&c\right)}$

28 $\phi_r(x) + r\,C_{r-1}x + \frac{r(r-1)}{\underline{|2}}C_{r-2}x^2 + \frac{r(r-1)(r-2)}{\underline{|3}}C_{r-3}x^3 + \cdots + C_0 x^r$

$+ S_1 \frac{x^{r+1}}{r+1}\underline{|0} - S_2\frac{x^{r+2}}{(r+1)(r+2)}\underline{|1} + S_3\frac{x^{r+3}}{(r+1)(r+2)(r+3)}\underline{|2} - \&c = f(x,r)$

where C_r is the constant of $1^r\log 1 + 2^r\log 2 + 3^r\log 3 + \&c$

& S_r is that of $\frac{1}{1^r}+\frac{1}{2^r}+\frac{1}{3^r}+\&c$. and

$f(x,r) = (1^r+2^r+3^r+\cdots+x^r - S_{-r})\,\Sigma\frac{1}{r} - \frac{r}{\underline{|2}}B_2x^{r-1}$

$+\frac{r(r-1)(r-2)}{\underline{|4}}B_4x^{r-3}\left(1+\frac{1}{2}+\frac{1}{3}\right) - \frac{r(r-1)(r-2)(r-3)(r-4)}{\underline{|6}}B_6 \times$

$x^{r-5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right) + \&c.$

$= \frac{1^r+2^r+3^r+\cdots+x^r}{r} + r.\int_0^x f(x,r-1)\,dx.$

29. $\phi_r(x) - n^r\left\{\phi_r\left(\frac{x}{n}\right) + \phi_r\left(\frac{x-1}{n}\right) + \phi_r\left(\frac{x-2}{n}\right) + \cdots + \phi_r\left(\frac{x-n+1}{n}\right)\right\}$

$= (1^r+2^r+3^r+\cdots+x^r - S_{-r})\log n - (n^{r+1}-1)C_r.$

Cor. 1. $\phi_r\left(-\frac{1}{n}\right) + \phi_r\left(-\frac{2}{n}\right) + \cdots + \phi_r\left(-\frac{n-1}{n}\right) = \frac{\log n}{n^r}S_{-r} + \left(n-\frac{1}{n^r}\right)C_r.$

Cor. 2. $\phi_r\left(-\frac{1}{2}\right) = \frac{\log 2}{2^r}S_{-r} + \left(2-\frac{1}{2^r}\right)C_r.$

30. i If r is even

$\phi_r(x-1) + \phi_r(-x) = 2C_r + \frac{\underline{|r}}{(2\pi)^r}\cos\frac{\pi r}{2}\left\{\frac{\cos 2\pi x}{1^{r+1}} + \frac{\cos 4\pi x}{2^{r+1}} + \&c\right\}$

ii. If r is odd

$\phi_r(x-1) - \phi_r(-x) = \frac{\underline{|r}}{(2\pi)^r}\sin\frac{\pi r}{2}\left\{\frac{\sin 2\pi x}{1^{r+1}} + \frac{\sin 4\pi x}{2^{r+1}} + \&c\right\}$

Sol. $\Sigma\frac{1}{x-1} - \Sigma\frac{1}{-x} = -\pi\cot\pi x = -\frac{\pi}{2}(\sin 2\pi x + \sin 4\pi x + \&c)$

Integrate both sides $r+1$ times.

N. B. More general theorems true for all values of r can be got by differentiating VII. 15. and 16 with respect to r.

31. If $1\log 1 + 2\log 2 + 3\log 3 + \&c \cdots + x\log x = \phi_1(x)$.

and $\pi\{\phi_1(x-1)-\phi_1(-x)\}+\pi x\log 2\sin\pi x=\psi(x)$ then

i. $\psi(x)=\sin\pi x+\frac{1}{2}\cdot\frac{\sin^3\pi x}{3^2}+\frac{1\cdot3}{2\cdot4}\frac{\sin^5\pi x}{5^2}+\&c$

$=\tan\pi x-(1+\frac{1}{3})\frac{\tan^3\pi x}{3}+(1+\frac{1}{3}+\frac{1}{5})\frac{\tan^5\pi x}{5}-\&c.$

ii. $\psi(x)+\psi(\frac{1}{2}-x)=\frac{\pi}{2}\log 2\cos\pi x$

$+\tan\pi x-\frac{\tan^3\pi x}{3^2}+\frac{\tan^5\pi x}{5^2}-\&c$

iii. $\psi(\frac{1}{2}-x)+\frac{1}{2}\psi(2x)-\psi(x)=\frac{\pi}{2}\log 2\cos\pi x.$

iv. $\psi(\frac{1}{2}-x)+\psi(\frac{1}{2}+x)=\pi\log 2\cos\pi x.$

e.g. i. $\psi(\frac{1}{2})=\frac{\pi}{2}\log 2$

ii. $\psi(\frac{1}{4})=(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\&c)+\frac{\pi}{4}\log 2$

iii. $\psi(\frac{1}{3})=\frac{\sqrt{3}}{2}(\frac{1}{1^2}+\frac{1}{4^2}+\frac{1}{7^2}+\&c)-\frac{\pi^2}{9\sqrt{3}}+\frac{\pi}{6}\log 3.$

iv. $\psi(\frac{1}{6})=\frac{3\sqrt{3}}{4}(\frac{1}{1^2}+\frac{1}{4^2}+\frac{1}{7^2}+\&c)-\frac{\pi^2}{6\sqrt{3}}$

v. $2\psi(x)-\frac{1}{2}\psi(2x)=\tan\pi x-\frac{\tan^3\pi x}{3^2}+\frac{\tan^5\pi x}{5^2}-\&c.$

Similarly we can find peculiarities for $\phi_2(x),\phi_3(x)$ &c.

32. $\sin 2x+\frac{2}{3^2}\sin^3 2x+\frac{2\cdot4}{3\cdot5^2}\sin^5 2x+\&c$

$=2(\tan x-\frac{\tan^3 x}{3^2}+\frac{\tan^5 x}{5^2}-\&c$

Cor. i. $1+\frac{2}{3^2}\cdot\frac{4x}{(1+x)^2}+\frac{2\cdot4}{3\cdot5^2}\left\{\frac{4x}{(1+x)^2}\right\}^2+\&c$

$=(1+x)(\frac{1}{1^2}-\frac{x}{3^2}+\frac{x^2}{5^2}-\frac{x^3}{7^2}+\&c).$

ii. $\tan 2x-\frac{2}{3^2}\tan^3 2x+\frac{2\cdot4}{3\cdot5^2}\tan^5 2x-\&c$

$=2(\tan x+\frac{\tan^3 x}{3^2}+\frac{\tan^5 x}{5^2}+\&c)$

e.g. i. $1+\frac{2}{3^2}+\frac{2\cdot4}{3\cdot5^2}+\&c=2(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\&c)$

ii. $1+\frac{2}{3^2}\cdot\frac{3}{4}+\frac{2\cdot4}{3\cdot5^2}\cdot(\frac{3}{4})^2+\&c=-\frac{\pi}{3\sqrt{3}}\log 3-\frac{10}{27}\pi^2+5(\frac{1}{1^2}+\frac{1}{4^2}+\frac{1}{7^2}+\&c$

iii. $\frac{1}{2}+\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2\cdot4}{3\cdot5^2}\cdot\frac{1}{2^5}+\&c=-\frac{\pi}{6}\log(2+\sqrt{3})$

$+\frac{2}{3}(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\&c)$

M 2.

iv. $1 + \frac{2}{3^2}\cdot\frac{1}{2} + \frac{2\cdot4}{3\cdot5^2}\cdot\frac{1}{2^2} + \frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\cdot\frac{1}{2^3} + \&c$

$= -\frac{\pi}{2\sqrt{2}}\log(1+\sqrt{2}) - \frac{\pi^2}{4\sqrt{2}} + 4\left(\frac{1}{1^2} - \frac{1}{5^2} + \frac{1}{9^2} - \&c\right).$

v. $\left(1-\frac{3}{4}\right) + \frac{2}{3^2}\left(1-\frac{3}{4^2}\right) + \frac{2\cdot4}{3\cdot5^2}\left(1-\frac{3}{4^3}\right) + \&c = \frac{\pi}{4}\log(2+\sqrt{3}).$

vi. $1 - \frac{2}{3^2} + \frac{2\cdot4}{3\cdot5^2} - \frac{2\cdot4\cdot6}{3\cdot5\cdot7^2} + \&c = \frac{\pi^2}{8} - \frac{1}{2}\left\{\log(1+\sqrt{2})\right\}^2.$

vii. $\frac{1}{2} - \frac{2}{3^2}\cdot\frac{1}{2^2} + \frac{2\cdot4}{3\cdot5^2}\cdot\frac{1}{2^3} - \&c = \frac{\pi^2}{12} - \frac{3}{4}\left(\log\frac{\sqrt{5}+1}{2}\right)^2.$

33. i. $\int_0^{\frac{\pi}{2}} x\cos^n x \sin nx\, dx = \frac{\pi}{2^{n+2}}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$

ii $\int_0^{\frac{\pi}{2}} \cos^n x \sin nx\, dx = \frac{1}{2^{n+1}}\left(\frac{2}{1}+\frac{2^2}{2}+\frac{2^3}{3}+\cdots+\frac{2^n}{n}\right)$

The above theorems are true for all values of n.

Cor. 1. $\frac{2-1}{1} + \frac{2^2-1}{2} + \frac{2^3-1}{3} + \cdots + \frac{2^n-1}{n}$ can be expanded in ascending powers of n in a convergent series the first two terms being $\frac{S_2}{2}n + \frac{S_3}{8}n^2 + \&c$

2. If $\phi(n) = \frac{2-1}{1} + \frac{2^2-1}{2} + \frac{2^3-1}{3} + \cdots + \frac{2^n-1}{n}$ then

$\phi(-n) + 2\frac{1}{n-1} + \frac{1}{n\cdot2^n} + \frac{1}{(n+1)2^{n+1}} + \frac{1}{(n+2)2^{n+2}} + \&c = 0$

and hence the values of the series $\frac{1}{1^n}\cdot\frac{1}{2} + \frac{1}{2^n}\cdot\frac{1}{2^2} + \&c.$

34. $\frac{x}{1+x} + \frac{1}{3^2}\cdot\left(\frac{x}{1+x}\right)^2 + \frac{1}{5^2}\cdot\left(\frac{x}{1+x}\right)^3 + \&c$

$= x - \frac{2}{3}\left(1+\frac{1}{3}\right)x^2 + \frac{2\cdot4}{3\cdot5}\cdot x^3\left(1+\frac{1}{3}+\frac{1}{5}\right) - \&c.$

35. If $A_n = (1^n+2^n+3^n+\&c)(1+\cos\pi n)$

$2^n + 6^n + 12^n + 20^n + 30^n + \&c$

$= A_n + \frac{n}{\lfloor 1}A_{n+1} + \frac{n(n-1)}{\lfloor 2}A_{n+2} + \frac{n(n-1)(n-2)}{\lfloor 3}A_{n+3} + \&c$

e.g. i. $\frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{12^2} + \&c = \frac{\pi^2-9}{3}.$

ii. $\frac{1}{2^3} + \frac{1}{6^3} + \frac{1}{12^3} + \&c = 10 - \pi^2.$

iii $\frac{1}{2^4} + \frac{1}{6^4} + \frac{1}{12^4} + \&c = \frac{\pi^4}{45} + \frac{10\pi^2}{3} - 35.$

iv. $\frac{1}{2^5} + \frac{1}{6^5} + \frac{1}{12^5} + \&c = 126 - \frac{35}{3}\pi^2 - \frac{\pi^4}{9}.$

CHAPTER X

1. If any one of x, y, z, u be a positive integer,

$$n \cdot \frac{\underline{|x+n}\ \underline{|y+n}\ \underline{|z+n}\ \underline{|u+n}\ \underline{|x+y+z+n}\ \underline{|y+z+u+n}\ \underline{|z+u+x+n}\ \underline{|x+y+u+n}}{\underline{|n}\ \underline{|x+y+n}\ \underline{|y+z+n}\ \underline{|z+u+n}\ \underline{|u+x+n}\ \underline{|x+z+n}\ \underline{|y+u+n}\ \underline{|x+y+z+u+n}}$$

$$= n - (n+2)\frac{n}{\underline{|1}} \cdot \frac{x}{x+n+1} \cdot \frac{y}{y+n+1} \cdot \frac{z}{z+n+1} \cdot \frac{u}{u+n+1} \cdot \frac{x+y+z+u+2n+1}{x+y+z+u+n}$$

$$+ (n+4)\frac{n(n+1)}{\underline{|2}}\ \frac{x(x-1)}{(x+n+1)(x+n+2)} \cdot \frac{y(y-1)}{(y+n+1)(y+n+2)} \cdot \frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$$\times \frac{u(u-1)}{(u+n+1)(u+n+2)} \cdot \frac{(x+y+z+u+2n+1)(x+y+z+u+2n+2)}{(x+y+z+u+n)(x+y+z+u+n-1)} - \&c$$

2. If any one of x, y, z be positive integers,

$$\frac{\underline{|n}\ \underline{|x+y+n}\ \underline{|y+z+n}\ \underline{|z+x+n}}{\underline{|x+n}\ \underline{|y+n}\ \underline{|z+n}\ \underline{|x+y+z+n}} = 1 + \frac{xyz}{\underline{|1}\,(n+1)(x+y+z+n)}$$

$$+ \frac{x(x-1)\,y(y-1)\,z(z-1)}{\underline{|2}\,(n+1)(n+2)(x+y+z+n)(x+y+z+n-1)} + \&c$$

3. If any one of x, y, z be positive integers,

$$\frac{(x+n)(y+n)(z+n)(x+y+z+n)}{(x+y+n)(y+z+n)(z+x+n)} = n + (n+2)\frac{x}{x+n+1} \cdot \frac{y}{y+n+1}$$

$$\times \frac{z}{z+n+1} \cdot \frac{x+y+z+2n}{x+y+z+n-1} + (n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)} \cdot \frac{y(y-1)}{(y+n+1)(y+n+2)}$$

$$\times \frac{z(z-1)}{(z+n+1)(z+n+2)}\ \frac{(x+y+z+2n)(x+y+z+2n+1)}{(x+y+z+n-1)(x+y+z+n-2)} + \&c.$$

4. If any one of x, y, z be a positive integer.

$$\Sigma\frac{1}{x+n} + \Sigma\frac{1}{y+n} + \Sigma\frac{1}{z+n} - \Sigma\frac{1}{x+y+n} - \Sigma\frac{1}{y+z+n}$$

$$- \Sigma\frac{1}{z+x+n} + \Sigma\frac{1}{x+y+z+n} - \Sigma\frac{1}{n}$$

$$= \left(1 + \frac{1}{n+1}\right)\frac{x}{x+n+1} \cdot \frac{y}{y+n+1} \cdot \frac{z}{z+n+1} \cdot \frac{x+y+z+2n+1}{x+y+z+n}$$

$$+ \left(\frac{1}{2} + \frac{1}{n+2}\right)\frac{x(x-1)}{(x+n+1)(x+n+2)}\ \frac{y(y-1)}{(y+n+1)(y+n+2)}\ \frac{z(z-1)}{(z+n+1)(z+n+2)}$$

$$\times \frac{(x+y+z+2n+1)(x+y+z+2n+2)}{(x+y+z+n)(x+y+z+n-1)} + \&c.$$

e.g. If x is a positive integer

i. $1 - 3\left(\frac{x-1}{x+1}\right)^4 \frac{4x-1}{4x-3} + 5\left(\frac{x-1}{x+1} \cdot \frac{x-2}{x+2}\right)^4 \frac{4x-1}{4x-3} \cdot \frac{2x}{4x-4} - \&c.$

$$= \frac{(\underline{|x}\,\underline{|3x-2})^4}{(\underline{|2x-1})^6\,\underline{|4x-3}}$$

ii. $$1\cdot\left(\frac{x-1}{x+1}\right)^3\frac{3x-1}{3x-3}+\frac{1}{2}\left\{\frac{x-1}{x+1}\cdot\frac{x-3}{x+3}\right\}^3\frac{(3x-1)(3x+1)}{(3x-3)(3x-5)}+\&c$$

$$=\frac{3}{2}\sum\frac{1}{\frac{x-1}{2}}-\frac{3}{2}\sum\frac{1}{x-1}+\frac{1}{2}\sum\frac{1}{\frac{3x-3}{2}}$$

iii. $$1+3\cdot\left(\frac{x-1}{x+1}\right)^3\frac{3x-1}{3x-3}+5\cdot\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^3\frac{(3x-1)3x}{(3x-3)(3x-4)}+\&c$$

$$=\left(\frac{x}{2x-1}\right)^3(3x-2)$$

iv. $$1+\left(\frac{x}{\underline{|1}}\right)^2\frac{x}{3x}+\left\{\frac{x(x-1)}{\underline{|2}}\right\}^2\frac{x(x-1)}{3x(3x-1)}+\&c=\left(\frac{\underline{|2x}}{\underline{|x}}\right)^3\underline{|3x}.$$

v. $$1+\frac{x}{\underline{|1}}\cdot\frac{x-1}{x+1}\cdot\frac{x}{4x-1}+\frac{x(x-1)}{\underline{|2}}\cdot\frac{(x-1)(x-2)}{(x+1)(x+2)}\cdot\frac{x(x-1)}{(4x-1)(4x-2)}+\&c$$

$$=\frac{8}{9}\left(\frac{\underline{|3x}}{\underline{|2x}}\right)^3\frac{\underline{|x}}{\underline{|4x}}.$$

5. $$n\frac{\underline{|x+n}\,\underline{|y+n}\,\underline{|z+n}\,\underline{|x+y+z+n}}{\underline{|n}\,\underline{|x+y+n}\,\underline{|y+z+n}\,\underline{|z+x+n}}=n-(n+2)\frac{n}{\underline{|1}}\frac{x}{(x+n+1)}\cdot\frac{y}{(y+n+1)}$$

$$\times\frac{z}{z+n+1}+(n+4)\frac{n(n+1)}{\underline{|2}}\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\times$$

$$\frac{z(z-1)}{(z+n+1)(z+n+2)}-\&c.$$

6. If $\alpha+\beta+\gamma+1=n$, then

$$(n+1)\frac{\underline{|n}}{\underline{|0}}\cdot\frac{\underline{|\alpha}\,\underline{|\beta}\,\underline{|\gamma}}{\underline{|n-\alpha}\,\underline{|n-\beta}\,\underline{|n-\gamma}}+(n+3)\frac{\underline{|n+1}}{\underline{|1}}\cdot\frac{\underline{|\alpha+1}\,\underline{|\beta+1}\,\underline{|\gamma+1}}{\underline{|n-\alpha+1}\,\underline{|n-\beta+1}\,\underline{|n-\gamma+1}}$$

$$+(n+5)\frac{\underline{|n+2}}{\underline{|2}}\cdot\frac{\underline{|\alpha+2}\,\underline{|\beta+2}\,\underline{|\gamma+2}}{\underline{|n-\alpha+2}\,\underline{|n-\beta+2}\,\underline{|n-\gamma+2}}+\&c \text{ to } K \text{ terms} - 2\log n$$

(when $K=\infty$) $$=-\sum\frac{1}{\alpha}-\sum\frac{1}{\beta}-\sum\frac{1}{\gamma}+2C_0.$$

Cor. $$\frac{\pi^2}{4}\left\{1+5\left(\frac{1}{2}\right)^4(1-x)+9\left(\frac{1\cdot3}{2\cdot4}\right)^4(1-x)^2+13\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^4(1-x)^3+\&c\right\}$$

$+\log x = 3\log 2$ when x vanishes.

7. $$1+\frac{n}{\underline{|1}}\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\cancel{\frac{1}{x+n}}+\frac{n(n+1)}{\underline{|2}}\frac{x(x-1)}{(x+n+1)(x+n+2)}\times$$

$$\frac{y(y-1)}{(y+n+1)(y+n+2)}+\&c=\frac{\underline{|x+n}\,\underline{|y+n}}{\underline{|n}\,\underline{|x+y+n}}\cdot\frac{\underline{|\frac{n}{2}}\,\underline{|x+y+\frac{n}{2}}}{\underline{|x+\frac{n}{2}}\,\underline{|y+\frac{n}{2}}},$$

Ex. 1. $\Sigma \frac{1}{x+n} + \Sigma \frac{1}{y+n} - \Sigma \frac{1}{x+y+n} - \Sigma \frac{1}{n}$

$$= \left(1 + \frac{1}{n+1}\right) \frac{x}{x+n+1} \cdot \frac{y}{y+n+1} + \left(\frac{1}{2} + \frac{1}{n+2}\right) \frac{x(x-1)}{(x+n+1)(x+n+2)} \times \frac{y(y-1)}{(y+n+1)(y+n+2)} + \&c$$

2. $$n \frac{\underline{\lvert x+y}}{\underline{\lvert x}\,\underline{\lvert y}} \cdot \frac{\underline{\lvert x+n}\,\underline{\lvert y+n}}{\underline{\lvert n}\,\underline{\lvert x+y+n}} = n + (n+2) \frac{n^2}{(\underline{\lvert 1})^2} \cdot \frac{xy}{(x+n+1)(y+n+1)} + (n+4) \frac{n^2(n+1)^2}{(\underline{\lvert 2})^2} \frac{x(x-1)}{(x+n+1)(x+n+2)} \cdot \frac{y(y-1)}{(y+n+1)(y+n+2)} + \&c$$

3. $$\frac{(x+n)(y+n)}{x+y+n} = n + (n+2) \frac{xy}{(x+n+1)(y+n+1)} + (n+4) \frac{x(x-1)}{(x+n+1)(x+n+2)} \cdot \frac{y(y-1)}{(y+n+1)(y+n+2)} + \&c$$

4. $$n \cdot \frac{\underline{\lvert x+n}\,\underline{\lvert y+n}}{\underline{\lvert n}\,\underline{\lvert x+y+n}} \cdot \frac{\underline{\lvert \tfrac{n-1}{2}}\,\underline{\lvert x+y+\tfrac{n-1}{2}}}{\underline{\lvert x+\tfrac{n-1}{2}}\,\underline{\lvert y+\tfrac{n-1}{2}}} = n + (n+2) \frac{n}{\underline{\lvert 1}} \frac{xy}{(x+n+1)(y+n+1)} + (n+4) \frac{n(n+1)}{\underline{\lvert 2}} \cdot \frac{x(x-1)}{(x+n+1)(x+n+2)} \times \frac{y(y-1)}{(y+n+1)(y+n+2)} + \&c$$

5. $$n \cdot \frac{\underline{\lvert x+n}\,\underline{\lvert y+n}}{\underline{\lvert n}\,\underline{\lvert x+y+n}} = n - (n+2) \frac{n}{\underline{\lvert 1}} \frac{xy}{(x+n+1)(y+n+1)} + (n+4) \frac{n(n+1)}{\underline{\lvert 2}} \frac{x(x-1)}{(x+n+1)(x+n+2)} \frac{y(y-1)}{(y+n+1)(y+n+2)} - \&c.$$

6. $$\left\{\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} + \&c\right\} - \left\{\frac{1}{(x+n+1)^2} + \frac{1}{(x+n+2)^2} + \&c\right\}$$
$$= \left(1 + \frac{1}{n+1}\right) \frac{x}{x+n+1} \cdot \frac{\underline{\lvert 1}}{n+1} - \left(\frac{1}{2} + \frac{1}{n+2}\right) \frac{\underline{\lvert 2}}{(n+1)(n+2)} \times \frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$$

7. $$\frac{\underline{\lvert x+n}\,\underline{\lvert x-n}}{(\underline{\lvert x})^2} \cdot \frac{\sin \pi n}{\pi} = n - (n+2) \frac{n^3}{(\underline{\lvert 1})^3} \frac{x}{x+n+1} + (n+4) \frac{n^3(n+1)^3}{(\underline{\lvert 2})^3} \cdot \frac{x(x-1)}{(x+n+1)(x+n+2)} - \&c.$$

8. $$\frac{x+n}{\underline{\lvert x}\,\underline{\lvert n}} \cdot \frac{\underline{\lvert \tfrac{n}{2}}\,\underline{\lvert x-\tfrac{n}{2}}}{\underline{\lvert -\tfrac{n}{2}}\,\underline{\lvert x+\tfrac{n}{2}}} = 1 - \frac{n^2}{(\underline{\lvert 1})^2} \cdot \frac{x}{x+n+1} +$$

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$$+\frac{n^2(n+1)^2}{(\lfloor\underline{2})^2}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} - \&c.$$

9. $$n\,\frac{\lfloor\underline{x-\tfrac{n+1}{2}}}{\lfloor\underline{x}\,\lfloor\underline{-\tfrac{n+1}{2}}}\cdot\frac{\lfloor\underline{x+n}\,\lfloor\underline{\tfrac{n-1}{2}}}{\lfloor\underline{x+\tfrac{n-1}{2}}\,\lfloor\underline{n}} = n-(n+2)\frac{n^2}{(\lfloor\underline{1})^2}\cdot\frac{x}{x+n+1} +$$

$$+(n+4)\cdot\frac{n^2(n+1)^2}{\lfloor\underline{2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$$

10. $$\frac{n\,\lfloor\underline{x+n}}{\lfloor\underline{x}\,\lfloor\underline{n}} = n+(n+2)\frac{n^2}{(\lfloor\underline{1})^2}\cdot\frac{x}{x+n+1} + \&c.$$

11. $$\frac{\lfloor\underline{x}\,\lfloor\underline{x+n}\left(\lfloor\underline{\tfrac{n}{2}}\right)^2}{n\,\lfloor\underline{n}\left(\lfloor\underline{x+\tfrac{n}{2}}\right)^2} = \frac{1}{n}-\frac{n}{\lfloor\underline{1}}\cdot\frac{x}{x+n+1}\cdot\frac{1}{n+2} +$$

$$\frac{n(n+1)}{\lfloor\underline{2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{1}{n+4} - \&c.$$

12. $$\frac{\lfloor\underline{2x}\,\lfloor\underline{x+n}}{\lfloor\underline{x}\,\lfloor\underline{2x+n}} = 1-\frac{n}{\lfloor\underline{1}}\cdot\frac{x}{x+n+1} + \frac{n(n+1)}{\lfloor\underline{2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} - \&c$$

13. $$\frac{\lfloor\underline{x+n}\,\lfloor\underline{\tfrac{n}{2}}}{\lfloor\underline{n}\,\lfloor\underline{x+\tfrac{n}{2}}} = 1+\frac{n}{\lfloor\underline{1}}\cdot\frac{x}{x+n+1} + \frac{n(n+1)}{\lfloor\underline{2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c$$

14. $$\frac{\lfloor\underline{x+n}\,\lfloor\underline{\tfrac{n-1}{2}}}{\lfloor\underline{n-1}\,\lfloor\underline{x+\tfrac{n-1}{2}}} = n+(n+2)\frac{n}{\lfloor\underline{1}}\cdot\frac{x}{x+n+1} +$$

$$(n+4)\frac{n(n+1)}{\lfloor\underline{2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)} + \&c.$$

15. $$\frac{(\lfloor\underline{n})^2}{\lfloor\underline{2n}}\cdot\frac{\sin\pi n\,\tan\pi n}{\pi^2 n} = n+(n+2)\frac{n^4}{(\lfloor\underline{1})^4}+(n+4)\frac{n^4(n+1)^4}{(\lfloor\underline{2})^4} + \&c$$

16. $$n+(n+2)\frac{n^3}{(\lfloor\underline{1})^3}+(n+4)\frac{n^3(n+1)^3}{(\lfloor\underline{2})^3} + \&c = \frac{\lfloor\underline{\tfrac{n-1}{2}}\,\lfloor\underline{-\tfrac{3n+1}{2}}}{\left(\lfloor\underline{-\tfrac{n+1}{2}}\right)^2}\,\frac{\sin\pi n}{\pi}$$

17. $$\frac{\sin\pi n}{\pi} = n-(n+2)\frac{n^3}{(\lfloor\underline{1})^3}+(n+4)\frac{n^3(n+1)^3}{(\lfloor\underline{2})^3} - \&c.$$

18. $$\frac{\left(\lfloor\underline{\tfrac{n}{2}}\right)^4}{(\lfloor\underline{n})^2}\cdot\frac{2\tan\frac{\pi n}{2}}{\pi n^2} = \frac{1}{n}+\frac{n^2}{(\lfloor\underline{1})^2}\cdot\frac{1}{n+2}+\frac{n^2(n+1)^2}{(\lfloor\underline{2})^2}\cdot\frac{1}{n+4} + \&c$$

19. $$\frac{\pi\left(\lfloor\underline{\tfrac{n}{2}}\right)^2}{2n\,\lfloor\underline{n}\,\sin\frac{\pi n}{2}} = \frac{1}{n^2}+\frac{n}{\lfloor\underline{1}}\cdot\frac{1}{(n+2)^2}+\frac{n(n+1)}{\lfloor\underline{2}}\cdot\frac{1}{(n+4)^2} + \&c$$

20. $$\Sigma\frac{1}{x+n}-\Sigma\frac{1}{n} = \left(1+\frac{1}{n+1}\right)\frac{n}{\lfloor\underline{1}}\cdot\frac{x}{x+n+1}+\left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{x(x-1)}{(x+n+1)(x+n+2)}$$

21. $$\Sigma\frac{1}{x}+\Sigma\frac{1}{n}-\Sigma\frac{1}{x+n} = \left(1+\frac{1}{n+1}\right)\frac{n}{\lfloor\underline{1}}\cdot\frac{x}{x+n+1} - \qquad + \&c$$

$$\left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{n(n+1)}{\underline{|2}}\cdot\frac{x(x-1)}{(x+n+1)(x+n+2)}+\&c.$$

$$22.\ 2n^2\left\{\frac{1}{n^3}+\frac{1}{(n+1)^3}+\frac{1}{(n+2)^3}+\frac{1}{(n+3)^3}+\&c\right\}$$

$$=\left(1+\frac{1}{n}\right)+\left(\frac{1}{2}+\frac{1}{n+1}\right)\left(\frac{1}{n+1}\right)^2+\left(\frac{1}{3}+\frac{1}{n+2}\right)\left(\frac{1}{n+1}\cdot\frac{2}{n+2}\right)^2+\&c$$

$$23.\ \left\{\frac{1}{\left(1+\frac{n}{2}\right)^2}+\frac{1}{\left(2+\frac{n}{2}\right)^2}+\frac{1}{\left(3+\frac{n}{2}\right)^2}+\&c\right\}-\left\{\frac{1}{(1+n)^2}+\frac{1}{(2+n)^2}+\&c\right\}$$

$$=\left(1-\frac{1}{n+1}\right)\frac{1}{n+1}+\left(\frac{1}{2}-\frac{1}{n+2}\right)\frac{\underline{|2}}{(n+1)(n+2)}+\&c$$

$$24.\ \Sigma\frac{1}{n}+\Sigma\frac{1}{-n}=\left(1+\frac{1}{n+1}\right)\frac{n^2}{(\underline{|1})^2}+\left(\frac{1}{2}+\frac{1}{n+2}\right)\frac{n^2(n+1)^2}{(\underline{|2})^2}+\&c$$

$$2\cdot y.\ 1.\ \frac{(\underline{|x})^3\,\underline{|3x-2}}{(\underline{|2x-1})^3}=1-3\cdot\left(\frac{x-1}{x+1}\right)^3+5\cdot\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^3-\&c.$$

$$2.\ \frac{x^2}{2x-1}=1+3\left(\frac{x-1}{x+1}\right)^2+5\cdot\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^2+\&c$$

$$3.\ \frac{(\underline{|x})^4\,\underline{|4x}}{(\underline{|2x})^4}\cdot\frac{2x}{4x-1}=1+\left(\frac{x-1}{x+1}\right)^2+\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^2+\&c$$

$$4.\ \frac{(\underline{|x})^2}{\underline{|2x-1}}=1-3\cdot\left(\frac{x-1}{x+1}\right)^2+5\cdot\left(\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}\right)^2-\&c$$

$$5.\ x=1+3\cdot\frac{x-1}{x+1}+5\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}+\&c.$$

$$6.\ \frac{\sqrt{\pi}}{2}\cdot\frac{\underline{|x}}{\underline{|x-\tfrac{1}{2}}}=1+\frac{x-1}{x+1}+\frac{(x-1)(x-2)}{(x+1)(x+2)}+\&c$$

$$7.\ \frac{x}{2x-1}=1-\frac{x-1}{x+1}+\frac{(x-1)(x-2)}{(x+1)(x+2)}+\&c$$

$$8.\ 1-3\cdot\frac{x-1}{x+1}+5\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}-\&c=0.$$

$$9.\ \frac{1}{2}\Sigma\frac{1}{x-1}+\frac{\left(2^{x-1}\,\underline{|x-1}\right)^2}{\underline{|2x-1}}=1+\frac{1}{2}\cdot\frac{x-1}{x+1}+\frac{1}{3}\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}+\&c$$

$$10.\ \Sigma\frac{1}{2x}-\Sigma\frac{1}{x}+\frac{1}{2x}=1-\frac{1}{2}\cdot\frac{x-1}{x+1}+\frac{1}{3}\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}-\&c.$$

$$11.\ \frac{2^{4x}(\underline{|x})^4}{4x(\underline{|2x})^2}=1-\frac{1}{3}\cdot\frac{x-1}{x+1}+\frac{1}{5}\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}-\&c.$$

12. $\frac{1}{2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{x^2}\right)+\frac{1}{x}\left(\frac{1}{x+1}+\frac{1}{x+2}+\cdots+\frac{1}{x+x}\right)$

$= 1-\frac{1}{2^2}\cdot\frac{x-1}{x+1}+\frac{1}{3^2}\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}-\&c.$

13. $x(4x-3)=1^3+3^3\cdot\frac{x-1}{x+1}+5^3\cdot\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}+\&c$

14. $\frac{2}{\pi}=1-5\cdot\left(\frac{1}{2}\right)^3+9\cdot\left(\frac{1\cdot3}{2\cdot4}\right)^3-13\cdot\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^3+\&c$

15. $1+9\cdot\left(\frac{1}{4}\right)^4+17\cdot\left(\frac{1\cdot5}{4\cdot8}\right)^4+\&c=\frac{2\sqrt{2}}{\sqrt{\pi}}\cdot\frac{1}{\left(\lfloor\underline{-\frac{3}{4}}\right)^2}$

16. $1+\left(\frac{1}{2}\right)^2\frac{1}{5}+\left(\frac{1\cdot3}{2\cdot4}\right)^2\frac{1}{9}+\&c=\frac{\pi^2}{4}\cdot\frac{1}{\left(\lfloor\underline{-\frac{3}{4}}\right)^4}$

17. $1+\frac{1}{2}\cdot\frac{1}{5^2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{9^2}+\&c=\frac{\pi^2}{8\sqrt{2}}+\frac{\sqrt{\pi}}{\left(\lfloor\underline{-\frac{3}{4}}\right)^2}$

18. $1+\left(\frac{1}{2}\right)^3+\left(\frac{1\cdot3}{2\cdot4}\right)^3+\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^3+\&c=\frac{\pi}{\left(\lfloor\underline{-\frac{3}{4}}\right)^4}$

19. $1-\left(\frac{1}{2}\right)^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2-\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2+\&c=\frac{\sqrt{\pi/2}}{\left(\lfloor\underline{-\frac{3}{4}}\right)^2}$

20. $1+\left(\frac{n}{\lfloor\underline{1}}\right)^3+\left\{\frac{n(n+1)}{\lfloor\underline{2}}\right\}^3+\left\{\frac{n(n+1)(n+2)}{\lfloor\underline{3}}\right\}^3+\&c=\frac{6\left(\lfloor\underline{\frac{n}{2}}\right)^3\sin\pi n\sin\frac{\pi n}{2}}{\pi^2 n^2(1+2\cos\pi n)\lfloor\underline{\frac{3n}{2}}}$

8. $\frac{\lfloor\underline{n}\,\lfloor\underline{x+y+n}}{\lfloor\underline{x+n}\,\lfloor\underline{y+n}}=1+\frac{x}{\lfloor\underline{1}}\cdot\frac{y}{n+1}+\frac{x(x-1)}{\lfloor\underline{2}}\cdot\frac{y(y-1)}{(n+1)(n+2)}+\&c$

Sol. Write $-n+m$ for z in $\overline{\text{X}}$ 5 and make n infinite or equate the coeff^{ts}. of u^n in $(1+u)^{y+n}\left(1+\frac{1}{u}\right)^x=\frac{(1+u)^{x+y+n}}{u^x}$.

9. $\Sigma\frac{1}{\alpha-1}-\Sigma\frac{1}{\alpha-\beta-1}=\frac{\beta}{\alpha}+\frac{\beta(\beta+1)}{\alpha(\alpha+1)}\cdot\frac{1}{2}+\frac{\beta(\beta+1)(\beta+2)}{\alpha(\alpha+1)(\alpha+2)}\cdot\frac{1}{3}+\&c$

10. $\frac{\lfloor\underline{n}\,\lfloor\underline{x}}{n\,\lfloor\underline{n+x}}=\frac{1}{n}-\frac{x}{\lfloor\underline{1}}\cdot\frac{1}{n+1}+\frac{x(x-1)}{\lfloor\underline{2}}\cdot\frac{1}{n+2}-\&c$

Ex. 1. $\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\frac{1}{(n+3)^2}+\frac{1}{(n+4)^2}+\&c$

$=\frac{1}{n+1}+\frac{\lfloor\underline{1}}{2(n+1)(n+2)}+\frac{\lfloor\underline{2}}{3(n+1)(n+2)(n+3)}+\&c$

2. $\frac{\pi}{\sin\pi n}=\frac{1}{n}+\frac{n}{\lfloor\underline{1}}\cdot\frac{1}{n+1}+\frac{n(n+1)}{\lfloor\underline{2}}\cdot\frac{1}{n+2}+\&c$

3. $\frac{\sqrt{\pi}\,\lfloor\underline{n}}{\lfloor\underline{n+\frac{1}{2}}}=\frac{1}{n+1}+\frac{1}{2}\cdot\frac{1}{n+2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{n+3}+\&c.$

4. $\frac{\sqrt{\pi}\,\lfloor\underline{n}}{2\lfloor\underline{n+\frac{1}{2}}} = 1 - \frac{n}{\lfloor\underline{1}}\cdot\frac{1}{3} + \frac{n(n-1)}{\lfloor\underline{2}}\cdot\frac{1}{5} - \frac{n(n-1)(n-2)}{\lfloor\underline{3}} + \&c.$

5. $\frac{\lfloor\underline{x}\,\lfloor\underline{n-1}}{\lfloor\underline{x+n}}\left(\Sigma\frac{1}{x+n} - \Sigma\frac{1}{n-1}\right) = \frac{1}{n^2} - \frac{x}{\lfloor\underline{1}}\cdot\frac{1}{(n+1)^2} + \frac{x(x-1)}{\lfloor\underline{2}}\cdot\frac{1}{(n+2)^2}$
$- \&c.$

6. $\frac{\sqrt{\pi}\,\lfloor\underline{n}}{\lfloor\underline{n+\frac{1}{2}}}\left(\Sigma\frac{1}{n+\frac{1}{2}} - \Sigma\frac{1}{n}\right) = \frac{1}{(n+1)^2} + \frac{1}{2}\cdot\frac{1}{(n+2)^2} + \frac{1.3}{2.4}\cdot\frac{1}{(n+3)^2}$
$+ \&c.$

7. $-\frac{\pi}{\sin\pi n}\,\Sigma\frac{1}{n-1} = \frac{1}{n^2} + \frac{n}{\lfloor\underline{1}}\cdot\frac{1}{(n+1)^2} + \frac{n(n+1)}{\lfloor\underline{2}}\cdot\frac{1}{(n+2)^2} + \&c.$

11. $\alpha^n = \left\{\alpha^n - (\beta+1)^n\right\} + \left\{(\alpha+1)^n - (\beta+2)^n\right\}\left(\frac{\beta+1}{\alpha+1}\right)^n +$
$\left\{(\alpha+2)^n - (\beta+3)^n\right\}\left(\frac{\beta+1}{\alpha+1}\cdot\frac{\beta+2}{\alpha+2}\right)^n + \&c.$

Cor. 1. $\frac{\beta}{\alpha-\beta-1} = \frac{\beta}{\alpha} + \frac{\beta(\beta+1)}{\alpha(\alpha+1)} + \frac{\beta(\beta+1)(\beta+2)}{\alpha(\alpha+1)(\alpha+2)} + \&c.$

2. $\frac{\beta^2}{\alpha-\beta-1} = (\alpha+\beta+1)\left(\frac{\beta}{\alpha}\right)^2 + (\alpha+\beta+3)\left(\frac{\beta}{\alpha}\cdot\frac{\beta+1}{\alpha+1}\right)^2 + \&c.$

12. If $e^{A_1x + A_2\frac{x^2}{2} + A_3\frac{x^3}{3} + \&c} = P_0 + P_1x + P_2x^2 + \&c$, then
$P_n = P_{n-1}A_1 + P_{n-2}A_2 + P_{n-3}A_3 + \&c$ to n terms and $P_0 = 1$
and consequently if $S_r = a_1^r + a_2^r + a_3^r + \cdots + a_n^r$ and
p_r denotes the sum of the products of $a_1, a_2, a_3, \ldots a_n$
taken r at a time then $rp_r = p_{r-1}S_1 - p_{r-2}S_2 + p_{r-3}S_3 -$
$p_{r-4}S_4 + \&c$ and $p_0 = 1$.

13. $\frac{1}{n^{r+1}} - \frac{x}{\lfloor\underline{1}}\cdot\frac{1}{(n+1)^{r+1}} + \frac{x(x-1)}{\lfloor\underline{2}}\cdot\frac{1}{(n+2)^{r+1}} - \&c = \frac{\lfloor\underline{n-1}\,\lfloor\underline{x}}{\lfloor\underline{n+x}}\phi(r)$
where $\phi(0) = 1$ and $r\phi(r) = S_1\phi(r-1) + S_2\phi(r-2) + S_3\phi(r-3) + \&c$
to r terms where $S_r = \frac{1}{n^r} - \frac{1}{(x+n+1)^r} + \frac{1}{(n+1)^r} - \frac{1}{(x+n+2)^r} + \&c$

Cor. 1. $1 + \frac{1}{2}\cdot\frac{1}{3^{r+1}} + \frac{1.3}{2.4}\cdot\frac{1}{5^{r+1}} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{7^{r+1}} + \&c = \frac{\pi}{2}\phi(r)$

120.

where $\phi(0) = 1$ and $n\,\phi(n) = S_1\,\phi(n-1) + S_2\,\phi(n-2) + S_3\,\phi(n-3) + \&c$ to n terms where $S_n = \frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \&c$

Cor. 2. $\frac{1}{2^{n+1}} + \frac{1}{2}\cdot\frac{1}{4^{n+1}} + \frac{1\cdot3}{2\cdot4}\cdot\frac{1}{6^{n+1}} + \&c = \phi(n)$ where $\phi(0)=1$ and $n\,\phi(n) = S_1\,\phi(n-1) + S_2\,\phi(n-2) + \&c$ where $S_n = \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \&c$. e.g. $1 + \frac{1}{2}\cdot\frac{1}{3^3} + \frac{1\cdot3}{2\cdot4}\cdot\frac{1}{5^3} + \&c = \frac{\pi^3}{48} + \frac{\pi}{4}(\log 2)^2$.

Ex. $\int_0^{\frac{\pi}{2}} \theta \cot\theta \log\sin\theta \, d\theta = -\frac{\pi^3}{48} - \frac{\pi}{4}(\log 2)^2$.

14. $\frac{1}{(x+1)^n} + \frac{1+\frac{1}{2}}{(x+2)^n} + \frac{1+\frac{1}{2}+\frac{1}{3}}{(x+3)^n} + \frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{(x+4)^n} + \&c$ $= \frac{n}{2} S_{n+1} - (S_1 S_n + S_2 S_{n-1} + S_3 S_{n-2} + \&c$ the last term being $S_{\frac{n}{2}} S_{\frac{n+2}{2}}$ or $\frac{1}{2} S_{\frac{n+1}{2}} S_{\frac{n+1}{2}}$ according as n is even or odd) where $S_n = \frac{1}{x^n} + \frac{1}{(x+1)^n} + \frac{1}{(x+2)^n} + \&c$ and $S_1 = -\Sigma\frac{1}{x-1}$.

Sol. $1\left(\frac{1}{2} - \frac{1}{n+2}\right) + \left(1+\frac{1}{2}\right)\left(\frac{1}{3} - \frac{1}{n+3}\right) + \left(1+\frac{1}{2}+\frac{1}{3}\right)\left(\frac{1}{4} - \frac{1}{n+4}\right) + \&c$ $= \frac{1}{2}\left\{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)^2 + \left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)\right\}$.

In the above identity write $n+x-1$ for n and equate the coeff^{ts} of n^n.

15. $\frac{\underline{|\alpha}\,\underline{|\beta}}{\underline{|\alpha+\beta+1}} + \frac{\underline{|\alpha+1}\,\underline{|\beta+1}}{\underline{|1}\,\underline{|\alpha+\beta+2}} + \frac{\underline{|\alpha+2}\,\underline{|\beta+2}}{\underline{|2}\,\underline{|\alpha+\beta+3}} + \&c$ to n terms $-\log n$ (when $n=\infty$) $= -\Sigma\frac{1}{\alpha} - \Sigma\frac{1}{\beta} + C_0$.

Cor. $\pi\left\{1 + \left(\frac{1}{2}\right)^2(1-x) + \left(\frac{1\cdot3}{2\cdot4}\right)^2(1-x)^2 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2(1-x)^3 + \&c\right\}$ $+ \log x = 4\log 2$ when $x=0$.

16. If $A_0 - n A_1 + \frac{n(n-1)}{\underline{|2}} A_2 - \frac{n(n-1)(n-2)}{\underline{|3}} A_3 + \&c = P_n$, then $P_0 - n P_1 + \frac{n(n-1)}{\underline{|2}} P_2 - \frac{n(n-1)(n-2)}{\underline{|3}} P_3 + \&c = A_n$.

17. $$\frac{A_0}{x^n} + \frac{n}{\lfloor 1} \cdot \frac{A_1}{x^{n+1}} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{A_2}{x^{n+2}} + \&c$$

$$= \frac{A_0}{(x+h)^n} + \frac{n}{\lfloor 1} \cdot \frac{A_1 + hA_0}{(x+h)^{n+1}} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{A_2 + 2hA_1 + h^2 A_0}{(x+h)^{n+2}} + \&c$$

18. If $\frac{A_0}{x^n} + \frac{n}{\lfloor 1} \cdot \frac{A_1}{x^{n+1}} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{A_2}{x^{n+2}} + \&c$

$= \frac{A_0}{(x-1)^n} - \frac{n}{\lfloor 1} \cdot \frac{A_1}{(x-1)^{n+1}} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{A_2}{(x-1)^{n+2}} - \&c$, then

i. $$e^x = \frac{A_0 + \frac{x}{\lfloor 1} A_1 + \frac{x^2}{\lfloor 2} A_2 + \frac{x^3}{\lfloor 3} A_3 + \&c}{A_0 - \frac{x}{\lfloor 1} A_1 + \frac{x^2}{\lfloor 2} A_2 - \frac{x^3}{\lfloor 3} A_3 + \&c}$$

ii. $$\frac{1}{\{\phi(x)\}^n}\left[A_0 + A_1 \frac{n}{\lfloor 1}\left\{\frac{\phi(x) - \phi(-x)}{\phi(x)}\right\} + A_2 \frac{n(n+1)}{\lfloor 2}\left\{\frac{\phi(x) - \phi(-x)}{\phi(x)}\right\}^2 + \&c\right]$$

is always an even function of x whatever be $\phi(x)$.

iii. If n is even, the value of A_{n+1} depends upon the value of A_n; but we may give for A_n any value we choose.

$$\frac{A_{n-1}}{2} = \frac{n-1}{\lfloor 2}(2^2-1) B_2 A_{n-2} - \frac{(n-1)(n-2)(n-3)}{\lfloor 4}(2^4-1) B_4 A_{n-4}$$

$$+ \frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{\lfloor 6}(2^6-1) B_6 A_{n-6} - \&c$$

19. $$\frac{1}{x^n} + \frac{n}{\lfloor 1} \cdot \frac{m}{n} \cdot \frac{1}{x^{n+1}} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{m(m+1)}{n(n+1)} \cdot \frac{1}{x^{n+2}} + \&c$$

$$= \frac{1}{(x-1)^n} + \frac{n}{\lfloor 1} \cdot \frac{m-n}{n} \cdot \frac{1}{(x-1)^{n+1}} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{(m-n)(m-n-1)}{n(n+1)} \cdot \frac{1}{(x-1)^{n+2}} + \&c$$

20. $$\phi(0) + \frac{m}{n} \cdot \frac{\phi'(0)}{\lfloor 1} + \frac{m(m+1)}{n(n+1)} \cdot \frac{\phi''(0)}{\lfloor 2} + \&c$$

$$= \phi(1) + \frac{m-n}{n} \cdot \frac{\phi'(1)}{\lfloor 1} + \frac{(m-n)(m-n-1)}{n(n+1)} \cdot \frac{\phi''(1)}{\lfloor 2} + \&c$$

21. $$e^x = \frac{1 + \frac{m}{n} \cdot \frac{x}{\lfloor 1} + \frac{m(m+1)}{n(n+1)} \cdot \frac{x^2}{\lfloor 2} + \&c}{1 + \frac{m-n}{n} \cdot \frac{x}{\lfloor 1} + \frac{(m-n)(m-n-1)}{n(n+1)} \cdot \frac{x^2}{\lfloor 2} + \&c}$$

22. $$\frac{1}{(x+1)^n} + \frac{n}{\lfloor 1} \cdot \frac{m}{2m} \cdot \frac{1}{(x+1)^{n+1}} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{m(m+1)}{2m(2m+1)} \cdot \frac{1}{(x+1)^{n+2}} + \&c$$

$$= \frac{1}{x^n} - \frac{n}{\lfloor 1} \cdot \frac{m}{2m} \cdot \frac{1}{x^{n+1}} + \frac{n(n+1)}{\lfloor 2} \cdot \frac{m(m+1)}{2m(2m+1)} \cdot \frac{1}{x^{n+2}} - \&c$$

23. $$e^{x} = \frac{1 + \frac{m}{2m}\cdot\frac{x}{\lfloor\!\underline{1}} + \frac{m(m+1)}{2m(2m+1)}\cdot\frac{x^2}{\lfloor\!\underline{2}} + \frac{m(m+1)(m+2)}{2m(2m+1)(2m+2)}\cdot\frac{x^3}{\lfloor\!\underline{3}} + \&c}{1 - \frac{m}{2m}\cdot\frac{x}{\lfloor\!\underline{1}} + \frac{m(m+1)}{2m(2m+1)}\cdot\frac{x^2}{\lfloor\!\underline{2}} - \frac{m(m+1)(m+2)}{2m(2m+1)(2m+2)}\cdot\frac{x^3}{\lfloor\!\underline{3}} + \&c}$$

Cor. 1. $$e^{x} = \frac{1 + \frac{1}{2}\cdot\frac{x}{\lfloor\!\underline{1}} + \frac{1.3}{2.4}\cdot\frac{x^2}{\lfloor\!\underline{2}} + \frac{1.3.5}{2.4.6}\cdot\frac{x^3}{\lfloor\!\underline{3}} + \&c}{1 - \frac{1}{2}\cdot\frac{x}{\lfloor\!\underline{1}} + \frac{1.3}{2.4}\cdot\frac{x^2}{\lfloor\!\underline{2}} - \frac{1.3.5}{2.4.6}\cdot\frac{x^3}{\lfloor\!\underline{3}} + \&c}$$

2. $$1 - \left(\tfrac{1}{2}\right)^2 \frac{x}{1-x} + \left(\tfrac{1.3}{2.4}\right)^2\left(\frac{x}{1-x}\right)^2 - \left(\tfrac{1.3.5}{2.4.6}\right)^2\left(\frac{x}{1-x}\right)^3 + \&c$$

$$= \sqrt{1-x}\left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1.3}{2.4}\right)^2 x^2 + \left(\tfrac{1.3.5}{2.4.6}\right)^2 x^3 + \&c\right\}$$

24. $$\frac{1}{nx^n} + \frac{m}{\lfloor\!\underline{1}}\cdot\frac{1}{(n+1)x^{n+1}} + \frac{m(m+1)}{\lfloor\!\underline{2}}\cdot\frac{1}{(n+2)x^{n+2}} + \&c$$

$$= \frac{1}{n(x-1)^n} + \frac{m-n-1}{\lfloor\!\underline{1}}\cdot\frac{1}{(n+1)(x-1)^{n+1}} + \frac{(m-n-1)(m-n-2)}{\lfloor\!\underline{2}}\,\frac{1}{(n+2)(x-1)^{n+2}}$$

25. $$\frac{1}{nx} + \frac{\lfloor\!\underline{1}}{n(n+1)x^2} + \frac{\lfloor\!\underline{2}}{n(n+1)(n+2)x^3} + \&c$$

$$= \frac{1}{n(x-1)} - \frac{1}{(n+1)(x-1)^2} + \frac{1}{(n+2)(x-1)^3} - \&c.$$

26. $$(1-x)^{\alpha+\beta}\left\{1 + \frac{\alpha}{\lfloor\!\underline{1}}\cdot\frac{\beta}{\gamma}x + \frac{\alpha(\alpha+1)}{\lfloor\!\underline{2}}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \&c\right\}$$

$$= (1-x)^{\gamma}\left\{1 + \frac{(\gamma-\alpha)(\gamma-\beta)}{\lfloor\!\underline{1}\,.\,\gamma}x + \frac{(\gamma-\alpha)(\gamma-\alpha+1)(\gamma-\beta)(\gamma-\beta+1)}{\lfloor\!\underline{2}\quad\gamma(\gamma+1)}x^2 + \&c\right\}$$

27. $$\frac{\lfloor\!\underline{x+y+n}}{\lfloor\!\underline{x+n}\,\lfloor\!\underline{y+n}} + \frac{PQ}{\lfloor\!\underline{1}}\,\frac{\lfloor\!\underline{x+y+n+1}}{\lfloor\!\underline{x+n+1}\,\lfloor\!\underline{y+n+1}} + \frac{P(P-1)\,Q(Q-1)}{\lfloor\!\underline{2}} \times$$

$$\frac{\lfloor\!\underline{x+y+n+2}}{\lfloor\!\underline{x+n+2}\,\lfloor\!\underline{y+n+2}} + \&c = \frac{\lfloor\!\underline{P+Q+n}}{\lfloor\!\underline{P+n}\,\lfloor\!\underline{Q+n}} + \frac{xy}{\lfloor\!\underline{1}}\,\frac{\lfloor\!\underline{P+Q+n+1}}{\lfloor\!\underline{P+n+1}\,\lfloor\!\underline{Q+n+1}}$$

$$+ \frac{x(x-1)\,y(y-1)}{\lfloor\!\underline{2}}\,\frac{\lfloor\!\underline{P+Q+n+2}}{\lfloor\!\underline{P+n+2}\,\lfloor\!\underline{Q+n+2}} + \&c.$$

28. $$\frac{1}{P+n} + \frac{x}{\lfloor\!\underline{1}}\cdot\frac{y}{n}\cdot\frac{1}{P+n+1} + \frac{x(x-1)}{\lfloor\!\underline{2}}\cdot\frac{y(y-1)}{n(n+1)}\cdot\frac{1}{P+n+2} + \&c$$

$$= \frac{\lfloor\!\underline{n-1}\,\lfloor\!\underline{x+y+n}}{\lfloor\!\underline{x+n}\,\lfloor\!\underline{y+n}} - P\,\frac{\lfloor\!\underline{n-1}\,\lfloor\!\underline{x+y+n+1}}{\lfloor\!\underline{x+n+1}\,\lfloor\!\underline{y+n+1}} + P(P-1)\times$$

$$\frac{\lfloor\!\underline{n-1}\,\lfloor\!\underline{x+y+n+2}}{\lfloor\!\underline{x+n+2}\,\lfloor\!\underline{y+n+2}} - \&c.$$

29. $\frac{\pi}{4}\left\{\frac{1}{n+1}+\left(\frac{1}{2}\right)^2\frac{1}{n+2}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{n+3}+\&c\right\}$

$= 1-\frac{n}{\lfloor 1}\cdot\left(\frac{2}{3}\right)^2+\frac{n(n-1)}{\lfloor 2}\cdot\left(\frac{2.4}{3.5}\right)^2-\frac{n(n-1)(n-2)}{\lfloor 3}\cdot\left(\frac{2.4.6}{3.5.7}\right)^2+\&c$

$= \frac{\pi}{4}\cdot\left(\frac{\lfloor n}{\lfloor n+\frac{1}{2}}\right)^2\left\{1+\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2+\&c \text{ to } n+1 \text{ terms}\right\}$

$= \frac{1}{2n+1}\left\{\frac{n+\frac{1}{2}}{n+1}\cdot 1+\frac{n+\frac{1}{2}}{n+1}\cdot\frac{n+1\frac{1}{2}}{n+2}\cdot\frac{1}{3}+\frac{n+\frac{1}{2}}{n+1}\cdot\frac{n+1\frac{1}{2}}{n+2}\cdot\frac{n+2\frac{1}{2}}{n+3}\cdot\frac{1}{5}\right.$
$\left.+\&c\right\}$

$= \frac{\sqrt{\pi}}{2}\cdot\frac{\lfloor n}{\lfloor n+\frac{1}{2}}\left\{1-\frac{n}{\lfloor 1}\cdot\frac{1}{2}\cdot\frac{1}{3}+\frac{n(n-1)}{\lfloor 2}\cdot\frac{1.3}{2.4}\cdot\frac{1}{5}-\&c\right\}.$

Cor. 1. $\frac{\pi}{4}\left\{1+\left(\frac{1}{2}\right)^2\frac{1}{3}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{5}+\left(\frac{1.3.5}{2.4.6}\right)^2\frac{1}{7}+\&c\right\}$

$= \frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\&c.$

2. $\pi n\left\{\frac{1}{n}+\left(\frac{1}{2}\right)^2\frac{1}{n+1}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{n+2}+\&c\right\}-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$

$= 4\log 2$ when n becomes infinite

30. $\frac{1}{y+n}-\frac{x}{n}\cdot\frac{1}{y+n+1}+\frac{x(x-1)}{n(n+1)}\cdot\frac{1}{y+n+2}-\&c$

$= \frac{1}{x+n}-\frac{y}{n}\cdot\frac{1}{x+n+1}+\frac{y(y-1)}{n(n+1)}\cdot\frac{1}{x+n+2}-\&c.$

31. $n-\frac{n}{\lfloor 1}\cdot(n+2)\frac{x}{x+n+1}\cdot\frac{y}{y+n+1}\cdot\frac{z}{z+n+1}\cdot\frac{u}{u+n+1}$

$+\frac{n(n+1)}{\lfloor 2}\cdot(n+4)\frac{x(x-1)}{(x+n+1)(x+n+2)}\cdot\frac{y(y-1)}{(y+n+1)(y+n+2)}\times$

$\frac{z(z-1)}{(z+n+1)(z+n+2)}\cdot\frac{u(u-1)}{(u+n+1)(u+n+2)}-\&c.$

$= n\cdot\frac{\lfloor x+n\,\lfloor y+n}{\lfloor n\,\lfloor x+y+n}\left\{1+\frac{xy}{\lfloor 1}\cdot\frac{z+u+n+1}{(z+n+1)(u+n+1)}+\frac{x(x-1)\,y(y-1)}{\lfloor 2}\times\right.$

$\left.\frac{(z+u+n+1)(z+u+n+2)}{(z+n+1)(z+n+2)(u+n+1)(u+n+2)}+\&c\right\}.$

32. $\frac{1}{n} + \frac{x}{\lfloor\underline{1}} \cdot \frac{y}{z} \cdot \frac{1}{n+1} + \frac{x(x-1)}{\lfloor\underline{2}} \cdot \frac{y(y-1)}{z(z+1)} \cdot \frac{1}{n+2} + \&c$

$= \frac{\lfloor\underline{x}\ \lfloor\underline{n-1}}{\lfloor\underline{x+n}} \left\{ 1 + \frac{n}{\lfloor\underline{1}} \cdot \frac{y+z}{z} + \frac{n(n+1)}{\lfloor\underline{2}} \cdot \frac{(y+z)(y+z+1)}{z(z+1)} + \&c \right.$

to $x+1$ terms $\Big\}$.

33. If $x+y+z=0$, then

$\frac{1}{n} + \frac{x}{\lfloor\underline{1}} \cdot \frac{y}{z} \cdot \frac{1}{n+1} + \frac{x(x-1)}{\lfloor\underline{2}} \cdot \frac{y(y-1)}{z(z+1)} \cdot \frac{1}{n+2} + \&c$

$= \frac{\lfloor\underline{n-1}\ \lfloor\underline{x+y+n}}{\lfloor\underline{x+n}\ \lfloor\underline{y+n}} \left\{ 1 + \frac{x}{\lfloor\underline{1}} \cdot \frac{y}{z} + \frac{x(x-1)}{\lfloor\underline{2}} \cdot \frac{y(y-1)}{z(z+1)} + \&c. \right.$

to $x+y+n+1$ terms $\Big\}$.

34. $\frac{\lfloor\underline{\frac{x+y-1}{2}}}{\lfloor\underline{\frac{x-1}{2}}\ \lfloor\underline{\frac{y-1}{2}}} \sqrt{\pi} = 1 + \frac{x}{\lfloor\underline{1}} \cdot \frac{y}{x+y+1} + \frac{x(x+1)}{\lfloor\underline{2}} \cdot \frac{y(y+1)}{(x+y+1)(x+y+3)}$

$+ \frac{x(x+1)(x+2)}{\lfloor\underline{3}} \cdot \frac{y(y+1)(y+2)}{(x+y+1)(x+y+3)(x+y+5)} + \&c$

Cor. $\frac{\lfloor\underline{\frac{n-1}{2}}}{\lfloor\underline{\frac{n+x-2}{4}}\ \lfloor\underline{\frac{n-x-2}{4}}} \sqrt{\frac{2\pi}{2^n}} = 1 + \frac{1^2-x^2}{4(n+1)} + \frac{(1^2-x^2)(3^2-x^2)}{4 \cdot 8(n+1)(n+3)} + \&c.$

e.g. 1. $\frac{\lfloor\underline{\frac{n-1}{2}}}{\left(\lfloor\underline{\frac{n-2}{4}}\right)^2} \sqrt{\frac{2\pi}{2^n}} = 1 + \frac{1^2}{4(n+1)} + \frac{1^2 \cdot 3^2}{4 \cdot 8 (n+1)(n+3)} + \&c.$

2. $\frac{\lfloor\underline{\frac{n-1}{2}}}{\lfloor\underline{\frac{2n-3}{8}}\ \lfloor\underline{\frac{2n-5}{8}}} \sqrt{\frac{2\pi}{2^n}} = 1 + \frac{1 \cdot 3}{16(n+1)} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{16 \cdot 32 (n+1)(n+3)} + \&c.$

35. If $\phi(n) = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 + \&c$ to n terms, then

i. $\pi \phi\left(\frac{n+1}{4}\right) = 3 \log 2 + \Sigma \frac{1}{\frac{n-1}{2}} + \frac{3}{4n^2} - \frac{99}{32n^4} + \frac{999}{32n^6} - \&c.$

ii. $1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2 \cdot 4}{3 \cdot 5}\right)^2 + \&c$ to n terms $= \frac{\pi^2}{4} \phi\left(n+\frac{1}{2}\right) - 2\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c\right)$

iii. $1 + \frac{16}{\pi^3}\left(\lfloor\underline{-\tfrac{1}{4}}\right)^4 \left\{ 1 + \left(\frac{3}{5}\right)^2 + \left(\frac{3 \cdot 7}{5 \cdot 9}\right)^2 + \&c \text{ to } n \text{ terms} \right\} = 2\phi\left(n+\frac{1}{4}\right).$

iv. $\phi\left(\frac{1}{4}\right) = \frac{1}{2}$ and $\frac{\pi^2}{8} \phi\left(\frac{1}{2}\right) = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c.$

CHAPTER XI

1. $\dfrac{1}{\{\phi(x)\}^n}\left[1+\dfrac{n}{\underline{|1}}\cdot\dfrac{m}{2m}\left\{1-\dfrac{\phi(-x)}{\phi(x)}\right\}+\dfrac{n(n+1)}{\underline{|2}}\cdot\dfrac{m(m+1)}{2m(2m+1)}\left\{1-\dfrac{\phi(-x)}{\phi(x)}\right\}\right.$
$\left.+\ \&c\right]$ is always an even function of x.

2. $1+\dfrac{n}{\underline{|1}}\cdot\dfrac{m}{2m}\cdot\dfrac{2x}{1+x}+\dfrac{n(n+1)}{\underline{|2}}\cdot\dfrac{m(m+1)}{2m(2m+1)}\left(\dfrac{2x}{1+x}\right)^2+\&c$

$=(1+x)^n\left\{1+\dfrac{n(n+1)}{2(2m+1)}x^2+\dfrac{n(n+1)(n+2)(n+3)}{2\cdot4\,(2m+1)(2m+3)}x^4+\&c\right\}$

3. $1+\dfrac{n}{\underline{|1}}\cdot\dfrac{m}{2m}\cdot\dfrac{4x}{(1+x)^2}+\dfrac{n(n+1)}{\underline{|2}}\cdot\dfrac{m(m+1)}{2m(2m+1)}\left\{\dfrac{4x}{(1+x)^2}\right\}^2+\&c.$

$=(1+x)^{2n}\left\{1+\dfrac{n}{\underline{|1}}\cdot\dfrac{n-m+\frac{1}{2}}{m+\frac{1}{2}}x^2+\dfrac{n(n+1)}{\underline{|2}}\cdot\dfrac{(n-m+\frac{1}{2})(n-m+1\frac{1}{2})}{(m+\frac{1}{2})(m+1\frac{1}{2})}x^4\right.$
$\left.+\ \&c\right\}$

4. $1+\dfrac{n(n+1)}{2(2m+1)}\cdot\dfrac{4x}{(1+x)^2}+\dfrac{n(n+1)(n+2)(n+3)}{2\cdot4\cdot(2m+1)(2m+3)}\cdot\left\{\dfrac{4x}{(1+x)^2}\right\}^2+\&c$

$=(1+x)^n\left\{1+\dfrac{n}{\underline{|1}}\cdot\dfrac{n-m+\frac{1}{2}}{m+\frac{1}{2}}x+\dfrac{n(n+1)}{\underline{|2}}\cdot\dfrac{(n-m+\frac{1}{2})(n-m+1\frac{1}{2})}{(m+\frac{1}{2})(m+1\frac{1}{2})}x^2\right.$
$\left.+\ \&c\right\}.$

5. $1+\dfrac{1}{2}\cdot\dfrac{n}{\underline{|1}}\cdot\dfrac{4x}{(1+x)^2}+\dfrac{1\cdot3}{2\cdot4}\cdot\dfrac{n(n+1)}{\underline{|2}}\cdot\left\{\dfrac{4x}{(1+x)^2}\right\}^2+\&c$

$=(1+x)^{2n}\left\{1+\left(\dfrac{n}{\underline{|1}}\right)^2x^2+\left[\dfrac{n(n+1)}{\underline{|2}}\right]^2x^4+\&c\right\}$

6. $1+\dfrac{n(n+1)}{2^2}\cdot\dfrac{4x}{(1+x)^2}+\dfrac{n(n+1)(n+2)(n+3)}{2^2\cdot4^2}\left\{\dfrac{4x}{(1+x)^2}\right\}^2+\&c$

$=(1+x)^n\left\{1+\left(\dfrac{n}{\underline{|1}}\right)^2x+\left[\dfrac{n(n+1)}{\underline{|2}}\right]^2x^2+\&c\right\}.$

7. $1+\dfrac{2x}{\underline{|1}}\cdot\dfrac{m}{2m}+\dfrac{(2x)^2}{\underline{|2}}\cdot\dfrac{m(m+1)}{2m(2m+1)}+\dfrac{(2x)^3}{\underline{|3}}\cdot\dfrac{m(m+1)(m+2)}{2m(2m+1)(2m+2)}+\&c$

$=e^x\left\{1+\dfrac{x^2}{2}\cdot\dfrac{1}{2m+1}+\dfrac{x^4}{2\cdot4}\cdot\dfrac{1}{(2m+1)(2m+3)}+\&c\right\}$

Cor. $1+\dfrac{1}{2}\cdot\dfrac{x}{\underline{|1}}+\dfrac{1\cdot3}{2\cdot4}\cdot\dfrac{x^2}{\underline{|2}}+\dfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\dfrac{x^3}{\underline{|3}}+\&c.\ =$

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$$e^{\frac{x}{2}}\left\{1+\frac{x^2}{4^2}+\frac{x^4}{4^2.8^2}+\frac{x^6}{4^2.8^2.12^2}+\&c\right\}.$$

8. $\phi(0)+\frac{2\phi'(0)}{\underline{|1}}\cdot\frac{m}{2m}+\frac{2^2\phi''(0)}{\underline{|2}}\cdot\frac{m(m+1)}{2m(2m+1)}+\&c.$

$$=\phi(1)+\frac{\phi''(1)}{2\underline{|1}}\cdot\frac{1}{2m+1}+\frac{\phi^{iv}(1)}{2^2\underline{|2}}\cdot\frac{1}{(2m+1)(2m+3)}+\&c.$$

9. $1+\frac{x^2}{2}\cdot\frac{1}{2n+1}+\frac{x^4}{2.4}\cdot\frac{1}{(2n+1)(2n+3)}+\&c.$

$$=\frac{2^{n-1}\underline{|n-\frac{1}{2}}}{x^n\sqrt{\pi}}\left[e^x\left\{1-\frac{n(n-1)}{2}\cdot\frac{1}{x}+\frac{(n+1)n(n-1)(n-2)}{2.4}\cdot\frac{1}{x^2}-\&c\right\}\right.$$

$$\left.+\,e^{-x}\cos\pi n\left\{1+\frac{n(n-1)}{2}\cdot\frac{1}{x}+\frac{(n+1)n(n-1)(n-2)}{2.4}\cdot\frac{1}{x^2}+\&c\right\}\right]$$

Cor. $1+\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}+\frac{x^6}{2^2.4^2.6^2}+\&c$

$$=\frac{e^x}{\sqrt{2\pi x}}\left(1+\frac{1^2}{8x}+\frac{1^2.3^2}{8.16x^2}+\frac{1^2.3^2.5^2}{8.16.24x^3}+\&c\right).$$

10. $1-\frac{x^2}{2}\cdot\frac{1}{2n+1}+\frac{x^4}{2.4}\cdot\frac{1}{(2n+1)(2n+3)}-\&c$

$$=\frac{2^{n}\underline{|n-\frac{1}{2}}}{x^n\sqrt{\pi}}\left[\cos\left(\frac{\pi n}{2}-x\right)\left\{1-\frac{(n+1)n(n-1)(n-2)}{2.4}\cdot\frac{1}{x^2}+\&c\right\}\right.$$

$$\left.+\sin\left(\frac{\pi n}{2}-x\right)\left\{\frac{n(n-1)}{2x}-\frac{(n+2)(n+1)n(n-1)(n-2)(n-3)}{2.4.6\ x^3}+\&c\right\}\right]$$

Cor. If $1-\frac{x^2}{2}\cdot\frac{1}{2n+1}+\frac{x^4}{2.4}\cdot\frac{1}{(2n+1)(2n+3)}-\&c=0$

then $x=\frac{\pi(\mu+n)}{2}-\frac{n(n-1)}{\pi(\mu+n)}-\frac{n(n-1)(7n.\overline{n-1}-6)}{3\pi^3(\mu+n)^3}-\&c$

where μ is any odd integer.

11. If $\int_0\frac{\sin x}{x}dx=\frac{\pi}{2}-r\cos(x-\theta)$, then

$$\int_0\frac{1-\cos x}{x}dx=c+\log x-r\sin(x-\theta)$$

where $r^2=\frac{\underline{|1}}{x^2}-\frac{\underline{|3}}{2x^4}+\frac{\underline{|5}}{3x^6}-\frac{\underline{|7}}{4x^8}+\&c,$

$r\cos\theta = \frac{1}{x} - \frac{\lfloor 2}{x^3} + \frac{\lfloor 4}{x^5} - \frac{\lfloor 6}{x^7} + \&c$ and

$r\sin\theta = \frac{\lfloor 1}{x^2} - \frac{\lfloor 3}{x^4} + \frac{\lfloor 5}{x^6} - \frac{\lfloor 7}{x^8} + \&c.$

Ex. 1. $\int_0^{\frac{\pi}{2}} \cos(\pi \sin^2\theta)\, d\theta = 0.$

2. $\int_0^{\frac{\pi}{2}} \cos(2\pi \sin^2\theta)\, d\theta = -\int_0^{\frac{\pi}{2}} \cos(\pi \sin\theta)\, d\theta.$

3. $\int_0^{\frac{\pi}{2}} \cos\left(\frac{2\pi}{3} \sin^2\theta\right) d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos\left(\frac{\pi}{\sqrt{3}} \sin\theta\right) d\theta.$

12. If $x + y + z = \frac{1}{2}$, then

$$1 + \frac{x}{\lfloor 1}\cdot\frac{y}{z}\,p + \frac{x(x-1)}{\lfloor 2}\cdot\frac{y(y-1)}{z(z+1)}\,p^2 + \frac{x(x-1)(x-2)}{\lfloor 3}\cdot\frac{y(y-1)(y-2)}{z(z+1)(z+2)}\,p^3 + \&c$$

$$= 1 + \frac{2x}{\lfloor 1}\cdot\frac{2y}{z}\cdot\frac{1-\sqrt{1-p}}{2} + \frac{2x(2x-1)}{\lfloor 2}\cdot\frac{2y(2y-1)}{z(z+1)}\cdot\left(\frac{1-\sqrt{1-p}}{2}\right)^2 + \&c.$$

Cor. $$1 + \frac{1^2+n}{4^2}\,x + \frac{(1^2+n)(5^2+n)}{4^2\cdot 8^2}\,x^2 + \frac{(1^2+n)(5^2+n)(9^2+n)}{4^2\cdot 8^2\cdot 12^2}\,x^3$$

$$+ \&c = 1 + \frac{1^2+n}{2^2}\cdot\frac{1-\sqrt{1-x}}{2} + \frac{(1^2+n)(3^2+n)}{2^2\cdot 4^2}\left(\frac{1-\sqrt{1-x}}{2}\right)^2 + \&c.$$

e.g. $$1 + \frac{1}{2}\left(1+\frac{1}{1^3}\right)\frac{1-\sqrt{1-x}}{2} + \frac{1}{3}\left(1+\frac{1}{1^3}\right)\left(1+\frac{1}{2^3}\right)\left(\frac{1-\sqrt{1-x}}{2}\right)^2 + \&c$$

$$= 1 + \frac{x}{2^2} + \frac{2}{1}\cdot\left(1-\frac{1}{2^3}\right)\frac{x^2}{4^2} + \frac{2\cdot 4}{1\cdot 3}\left(1-\frac{1}{2^3}\right)\left(1-\frac{1}{4^3}\right)\frac{x^3}{6^2} + \&c.$$

ex. $$\left(\frac{1+\sqrt{1-x}}{2}\right)^{\gamma}\left\{1 + \frac{(\alpha+\gamma)(\beta+\gamma)}{4\cdot(\gamma+1)}\,x + \frac{(\alpha+\gamma)(\alpha+\gamma+2)(\beta+\gamma)(\beta+\gamma+2)}{4\cdot 8\cdot(\gamma+1)(\gamma+2)}\,x^2 + \&c\right\}$$

$$= 1 + \frac{\alpha}{\lfloor 1}\cdot\frac{\beta}{\gamma+1}\cdot\frac{1-\sqrt{1-x}}{2} + \frac{\alpha(\alpha+1)}{\lfloor 2}\cdot\frac{\beta(\beta+1)}{(\gamma+1)(\gamma+2)}\left(\frac{1-\sqrt{1-x}}{2}\right)^2 + \&c.$$

13. If $\alpha + \beta + \gamma = 0$, then

$$\left\{1 + \frac{\alpha}{\lfloor 1}\cdot\frac{\beta}{\gamma+\frac{1}{2}}\,x + \frac{\alpha(\alpha-1)}{\lfloor 2}\cdot\frac{\beta(\beta-1)}{(\gamma+\frac{1}{2})(\gamma+1\frac{1}{2})}\,x^2 + \&c\right\}^2$$

$$= 1 + \frac{2\alpha}{\lfloor 1}\cdot\frac{2\beta}{\gamma+\frac{1}{2}}\cdot\frac{\gamma}{2\gamma}\,x + \frac{2\alpha(2\alpha-1)}{\lfloor 2}\cdot\frac{2\beta(2\beta-1)}{(\gamma+\frac{1}{2})(\gamma+1\frac{1}{2})}\cdot\frac{\gamma(\gamma+1)}{2\gamma(2\gamma+1)}\,x^2 + \&c$$

Cor. 1. $\left\{1+\frac{1^2+n}{4^2}x+\frac{(1^2+n)(5^2+n)}{4^2\cdot 8^2}x^2+\&c\right\}^2=$

$$1+\frac{1}{2}\cdot\frac{1^2+n}{2^2}x+\frac{1.3}{2.4}\cdot\frac{(1^2+n)(3^2+n)}{2^2\cdot 4^2}x^2+\&c.$$

Cor. 2. $\left\{1+\frac{x}{2^2}+\frac{x^2}{2^2\cdot 4^2}+\frac{x^3}{2^2\cdot 4^2\cdot 6^2}+\&c\right\}^2$

$$=1+\frac{1}{2}\cdot\frac{x}{(\underline{|1})^2}+\frac{1.3}{2.4}\cdot\frac{x^2}{(\underline{|2})^2}+\frac{1.3.5}{2.4.6}\cdot\frac{x^3}{(\underline{|3})^2}+\&c.$$

14. If $\alpha+\beta+1=\gamma+\delta$,

$$\left\{1+\frac{\alpha}{\underline{|1}}\cdot\frac{\beta}{\gamma}\cdot\frac{1-\sqrt{1-x}}{2}+\frac{\alpha(\alpha+1)}{\underline{|2}}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c\right\}$$

$$\times\left\{1+\frac{\alpha}{\underline{|1}}\cdot\frac{\beta}{\delta}\cdot\frac{1-\sqrt{1-x}}{2}+\frac{\alpha(\alpha+1)}{\underline{|2}}\cdot\frac{\beta(\beta+1)}{\delta(\delta+1)}\cdot\left(\frac{1-\sqrt{1-x}}{2}\right)^2+\&c\right\}$$

$$=1+\frac{\alpha}{\gamma}\cdot\frac{\beta}{\delta}\cdot\frac{(\alpha+\beta)(\gamma+\delta)}{2.\ (2\alpha+2\beta)}x+\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\cdot\frac{\beta(\beta+1)}{\delta(\delta+1)}\cdot\frac{(\alpha+\beta)(\alpha+\beta+2)}{2.4}\times$$

$$\frac{(\gamma+\delta)(\gamma+\delta+2)}{(2\alpha+2\beta)(2\alpha+2\beta+2)}x^2+\&c.$$

15. $$\left\{1+\frac{x}{\underline{|1}}\cdot\frac{1}{\gamma}+\frac{x^2}{\underline{|2}}\cdot\frac{1}{\gamma(\gamma+1)}+\frac{x^3}{\underline{|3}}\cdot\frac{1}{\gamma(\gamma+1)(\gamma+2)}+\&c\right\}$$

$$\times\left\{1+\frac{x}{\underline{|1}}\cdot\frac{1}{\delta}+\frac{x^2}{\underline{|2}}\cdot\frac{1}{\delta(\delta+1)}+\frac{x^3}{\underline{|3}}\cdot\frac{1}{\delta(\delta+1)(\delta+2)}+\&c\right\}$$

$$=1+\frac{x}{\underline{|1}}\cdot\frac{\gamma+\delta}{\gamma\delta}+\frac{x^2}{\underline{|2}}\cdot\frac{(\gamma+\delta+1)(\gamma+\delta+2)}{\gamma(\gamma+1)\delta(\delta+1)}+\frac{x^3}{\underline{|3}}\cdot\frac{(\gamma+\delta+2)(\gamma+\delta+3)}{\gamma(\gamma+1)(\gamma+2)\delta}\times$$

$$\frac{\gamma+\delta+3}{(\delta+1)(\delta+2)}+\frac{x^4}{\underline{|4}}\cdot\frac{(\gamma+\delta+3)(\gamma+\delta+4)(\gamma+\delta+5)(\gamma+\delta+6)}{\gamma(\gamma+1)(\gamma+2)(\gamma+3)\delta(\delta+1)(\delta+2)(\delta+3)}+\&c$$

16. $$\left\{1+\frac{x}{\underline{|1}}\cdot\frac{1}{m+1}\cdot\frac{1}{n+1}+\frac{x^2}{\underline{|2}}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n+1)(n+2)}+\&c\right\}\times$$

$$\left\{1-\frac{x}{\underline{|1}}\cdot\frac{1}{m+1}\cdot\frac{1}{n+1}+\frac{x^2}{\underline{|2}}\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n+1)(n+2)}-\&c\right\}$$

$$=1-\frac{x^2}{\underline{|1}}\cdot\frac{m+n+3}{(m+1)(n+1)}\cdot\frac{1}{(m+1)(m+2)}\cdot\frac{1}{(n+1)(n+2)}$$

$$+\frac{x^4}{\underline{|2}}\cdot\frac{(m+n+5)(m+n+6)}{(m+1)(m+2)(n+1)(n+2)}\cdot\frac{1}{(m+1)(m+2)(m+3)(m+4)(n+1)(n+2)}$$

$$\times \frac{1}{(n+3)(m+4)} - \frac{x^6}{\underline{|3}} \cdot \frac{(m+n+7)(m+n+8)(m+n+9)}{(m+1)(m+2)(m+3)(n+1)(n+2)(n+3)} \times$$

$$\frac{1}{(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)} \cdot \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} + \&c.$$

17. $$\left\{1 + \frac{x}{\underline{|1}} \cdot \frac{1}{m+n+1} \cdot \frac{1}{n+1} + \frac{x^2}{\underline{|2}} \cdot \frac{1}{(m+n+1)(m+n+2)} \cdot \frac{1}{(n+1)(n+2)} + \&c\right\}\left\{1 + \frac{x}{\underline{|1}} \cdot \frac{1}{m+1} \cdot \frac{1}{n-1} + \frac{x^2}{\underline{|2}} \cdot \frac{1}{(m+1)(m+2)} \cdot \frac{1}{(n-1)(n-2)} + \&c\right\}$$

$$= 1 + \frac{x}{\underline{|1}} \cdot \frac{2m+n+3}{m+n+1} \cdot \frac{1}{m+1} \cdot \frac{n}{n^2-1^2} + \frac{x^2}{\underline{|2}} \cdot \frac{(2m+n+4)(2m+n+6)}{(m+n+1)(m+n+2)} \times$$

$$\frac{1}{(m+1)(m+2)} \cdot \frac{1}{n^2-2^2} + \frac{x^3}{\underline{|3}} \cdot \frac{(2m+n+5)(2m+n+7)(2m+n+9)}{(m+n+1)(m+n+2)(m+n+3)} \times$$

$$\frac{1}{(m+1)(m+2)(m+3)} \cdot \frac{n}{(n^2-1^2)(n^2-3^2)} + \&c.$$

18. $$\left\{1 + \frac{\beta}{\gamma} \cdot \frac{x}{\underline{|1}} + \frac{\beta(\beta-1)}{\gamma(\gamma+1)} \cdot \frac{x^2}{\underline{|2}} + \frac{\beta(\beta-1)(\beta-2)}{\gamma(\gamma+1)(\gamma+2)} \cdot \frac{x^3}{\underline{|3}} + \&c\right\} \times$$

$$\left\{1 - \frac{\beta}{\gamma} \cdot \frac{x}{\underline{|1}} + \frac{\beta(\beta+1)}{\gamma(\gamma+1)} \cdot \frac{x^2}{\underline{|2}} - \frac{\beta(\beta-1)(\beta-2)}{\gamma(\gamma+1)(\gamma+2)} \cdot \frac{x^3}{\underline{|3}} + \&c\right\}$$

$$= 1 - \frac{\beta}{\gamma} \cdot \frac{\beta+\gamma}{\gamma(\gamma+1)} \cdot \frac{x^2}{\underline{|1}} + \frac{\beta(\beta-1)}{\gamma(\gamma+1)} \cdot \frac{(\beta+\gamma)(\beta+\gamma+1)}{\gamma(\gamma+1)(\gamma+2)(\gamma+3)(\gamma+4)} \cdot \frac{x^4}{\underline{|2}} - \&c$$

19. $$\left\{1 + \frac{x}{\underline{|1}} \alpha\beta + \frac{x^2}{\underline{|2}} \alpha(\alpha-1)\beta(\beta-1) + \&c\right\} \times$$

$$\left\{1 - \frac{x}{\underline{|1}} \alpha\beta + \frac{x^2}{\underline{|2}} \alpha(\alpha-1)\beta(\beta-1) - \&c\right\}$$

$$= 1 - \frac{x^2}{\underline{|1}} \alpha\beta(\alpha+\beta-1) + \frac{x^4}{\underline{|2}} \alpha(\alpha-1)\beta(\beta-1)(\alpha+\beta-2)(\alpha+\beta-3)$$

$$- \frac{x^6}{\underline{|3}} \alpha(\alpha-1)(\alpha-2)\beta(\beta-1)(\beta-2)(\alpha+\beta-3)(\alpha+\beta-4)(\alpha+\beta-5) + \&c$$

20. $$\left\{1 + \frac{x}{\underline{|1}} \cdot \frac{m}{n+1} + \frac{x^2}{\underline{|2}} \cdot \frac{m(m-1)}{(n+1)(n+2)} + \&c\right\} \times$$

$$\left\{1 + \frac{x}{\underline{|1}} \cdot \frac{m+n}{n-1} + \frac{x^2}{\underline{|2}} \cdot \frac{(m+n)(m+n-1)}{(n-1)(n-2)} + \&c\right\}$$

$$= 1 + \frac{x}{\underline{|1}} \cdot (2m+n+1) \frac{n}{n^2-1^2} + \frac{x^2}{\underline{|2}} \cdot (2m+n)(2m+n+2) \frac{1}{n^2-2^2} +$$

$$\frac{x^3}{\underline{|3}} \cdot (2m+n-1)(2m+n+1)(2m+n+3) \frac{n}{(n^2-1^2)(n^2-3^2)} + \&c.$$

e.g. 1. $$\left(1 + \frac{x^3}{\underline{|3}} + \frac{x^6}{\underline{|6}} + \frac{x^9}{\underline{|9}} + \&c\right)\left(1 - \frac{x^3}{\underline{|3}} + \frac{x^6}{\underline{|6}} - \frac{x^9}{\underline{|9}} + \&c\right) =$$

130. $\frac{1}{3} + \frac{2}{3}\left\{1 - \frac{(3x^2)^3}{\underline{|6}} + \frac{(3x^2)^6}{\underline{|12}} - \frac{(3x^2)^9}{\underline{|18}} + \&c\right\}.$

2. $\left\{1 + \frac{x}{(\underline{|1})^3} + \frac{x^2}{(\underline{|2})^3} + \frac{x^3}{(\underline{|3})^3} + \&c\right\}\left\{1 - \frac{x}{(\underline{|1})^3} + \frac{x^2}{(\underline{|2})^3} - \frac{x^3}{(\underline{|3})^3} + \&c\right\}$

$= 1 - \frac{x^2\,\underline{|3}}{(\underline{|1}\,\underline{|2})^3} + \frac{x^4\,\underline{|6}}{(\underline{|2}\,\underline{|4})^3} - \frac{x^6\,\underline{|9}}{(\underline{|3}\,\underline{|6})^3} + \&c.$

3. $\left(x + \frac{x^4}{\underline{|4}} + \frac{x^7}{\underline{|7}} + \frac{x^{10}}{\underline{|10}} + \&c\right)\left(x - \frac{x^4}{\underline{|4}} + \frac{x^7}{\underline{|7}} - \frac{x^{10}}{\underline{|10}} + \&c\right)$

$= \frac{2}{3}\left\{\frac{3x^2}{\underline{|2}} - \frac{(3x^2)^4}{\underline{|8}} + \frac{(3x^2)^7}{\underline{|14}} - \&c\right\}$

4. $\cos x \cosh x = 1 - \frac{(2x^2)^2}{\underline{|4}} + \frac{(2x^2)^4}{\underline{|8}} - \frac{(2x^2)^6}{\underline{|12}} + \&c$

5. $\sin x . \sinh x = \frac{2x^2}{\underline{|2}} - \frac{(2x^2)^3}{\underline{|6}} + \frac{(2x^2)^5}{\underline{|10}} - \&c.$

6. $\left\{1 + \frac{x}{(\underline{|1})^2} + \frac{x^2}{(\underline{|2})^2} + \frac{x^3}{(\underline{|3})^2} + \&c\right\}\left\{1 - \frac{x}{(\underline{|1})^2} + \frac{x^2}{(\underline{|2})^2} - \frac{x^3}{(\underline{|3})^2} + \&c\right\}$

$= 1 - \frac{x^2}{(\underline{|1})^2\,\underline{|2}} + \frac{x^4}{(\underline{|2})^2\,\underline{|4}} - \frac{x^6}{(\underline{|3})^2\,\underline{|6}} + \&c.$

7. $\left\{1 + \frac{1}{2}\cdot\frac{x}{\underline{|1}} + \frac{1.3}{2.4}\cdot\frac{x^2}{\underline{|2}} + \&c\right\}\left\{1 - \frac{1}{2}\cdot\frac{x}{\underline{|1}} + \frac{1.3}{2.4}\cdot\frac{x^2}{\underline{|2}} - \&c\right\}$

$= 1 + \frac{1}{2}\cdot\frac{x^2}{2^2} + \frac{1.3}{2.4}\cdot\frac{x^4}{2^2.4^2} + \frac{1.3.5}{2.4.6}\cdot\frac{x^6}{2^2.4^2.6^2} + \&c$

8. $\left(1 + \frac{x}{1.3} + \frac{x^2}{1.3.5} + \frac{x^3}{1.3.5.7} + \&c\right)\left(1 - \frac{x}{1.3} + \frac{x^2}{1.3.5} - \frac{x^3}{1.3.5.7} + \&c\right\}$

$= 1 + \frac{x^2}{1.3.5}\cdot\frac{1}{3} + \frac{x^4}{1.3.5.7.9}\cdot\frac{1}{5} + \&c.$

9. $\left\{\frac{1}{n} + \frac{x}{n(n+1)} + \frac{x^2}{n(n+1)(n+2)} + \&c\right\}\left\{\frac{1}{n} - \frac{x}{n(n+1)} + \&c\right\}$

$= \frac{1}{n}\cdot\frac{1}{n} + \frac{x^2}{n(n+1)(n+2)}\cdot\frac{1}{n+1} + \frac{x^4}{n(n+1)(n+2)(n+3)(n+4)}\cdot\frac{1}{n+2} + \&c.$

10. $\left\{1 + xn + x^2 n(n-1) + x^3 n(n-1)(n-2) + \&c\right\}\left\{1 - xn + x^2 n(n-1) - \&c\right\}$

$= \frac{1}{n}\cdot n + \frac{x^2}{n-1}\cdot n(n-1)(n-2) + \frac{x^4}{n-2}\, n(n-1)(n-2)(n-3)(n-4) + \&c$

21. $1 + \frac{1+x}{\underline{|1}}\cdot\frac{mn}{m+n+1} + \frac{(1+x)^2}{\underline{|2}}\cdot\frac{m(m+1)\,n(n+1)}{(m+n+1)(m+n+3)} + \&c$

$$= \sqrt{\pi}\,\frac{\underline{\left|\tfrac{m+n-1}{2}\right.}}{\underline{\left|\tfrac{m-1}{2}\right.}\;\underline{\left|\tfrac{n-1}{2}\right.}}\left\{1 + \frac{x^2}{\underline{|2}}\,mn + \frac{x^4}{\underline{|4}}\,m(m+2)\,n(n+2) + \&c\right\} +$$

$$2\sqrt{\pi}\,\frac{\underline{\left|\tfrac{m+n-1}{2}\right.}}{\underline{\left|\tfrac{m-2}{2}\right.}\;\underline{\left|\tfrac{n-2}{2}\right.}}\left\{\frac{x}{\underline{|1}} + \frac{x^3}{\underline{|3}}(m+1)(n+1) + \frac{x^5}{\underline{|5}}(m+1)(m+3)(n+1)(n+3) + \&c\right\}$$

22. $$e^{-mx}\left\{1 + \frac{1}{2}\cdot\frac{m}{\underline{|1}}\left(1-e^{-2x}\right) + \frac{1.3}{2.4}\cdot\frac{m(m+1)}{\underline{|2}}\left(1-e^{-2x}\right)^2 + \&c\right\}$$

$$= 1 + \frac{A_1}{2}\cdot\left(\frac{x}{\underline{|1}}\right)^2 + \frac{A_2}{2^2}\cdot\left(\frac{x^2}{\underline{|2}}\right)^2 + \frac{A_3}{2^3}\cdot\left(\frac{x^3}{\underline{|3}}\right)^2 + \&c$$

where $A_n = p^n - \frac{n(n-1)}{\underline{|3}}\,p^{n-1} + \frac{n(n-1)(n-2)(3n-1)}{\underline{|5}}\,p^{n-2}$

$- \cdots\cdots + (-1)^{n-1}\,2p\cdot\frac{\underline{|n-1}\,\left(2^{2n}-1\right)}{1.3.5.\cdots\cdots}\,\frac{B_{2n}}{(2n-1)}$; and $p = \frac{m(m-1)}{2}$.

Cor. If $A_n = K_p$, then $K_1 = \frac{\underline{|n}}{1.3.5.\cdots 2n-1}$; $K_3 = 3\,.\,2^{2(n-1)}\,K_1$; &c.

23. If $\phi(x)$ can be expressed in n different ways, the apparent value in the n th way being $c_n + v_n$ and if $c_1, c_2, c_3 \cdots c_n$ appear to be similar and $v_1, v_2, v_3, \cdots v_n$ are known to be dissimilar, then $c_1, c_2, c_3 \cdots c_n$ must be identically equal (say equal to c) and the real value of $\phi(x) = c + v_1 + v_2 + v_3 + \cdots + v_n$.

24. If $\phi(r) = \frac{1}{\underline{|n}}\left\{P_0 r^n + n P_1 r^{n-1} + \frac{n(n-1)}{\underline{|2}} P_2 r^{n-2} + \&c\right\}$

and $Q_r = \phi(r) + \frac{r+1}{\underline{|1}}\,\phi(r+1) + \frac{(r+1)(r+2)}{\underline{|2}}\,\phi(r+2) + \&c$, then

$$\phi(0) + (1-x)\,\phi(1) + (1-x)^2\,\phi(2) + (1-x)^3\,\phi(3) + \&c =$$

$$Q_0 - Q_1 x + Q_2 x^2 - Q_3 x^3 + \&c + \frac{1}{\left(\log\frac{1}{1-x}\right)^{n+1}}\left\{P_0 + P_1 \log\frac{1}{1-x}\right.$$

$$\left. + P_2\left(\log\frac{1}{1-x}\right)^2 + \&c\right\}.$$

Cor. 1. If $Q'_r = \frac{\underline{|m}}{\underline{|m-r}\;\underline{|r}}\,\phi(m) + \frac{\underline{|m+1}}{\underline{|m-r+1}\;\underline{|r}}\,\phi(m+1) + \frac{\underline{|m+2}}{\underline{|m-r+2}\;\underline{|r}}\,\phi(m+2)$

$+ \&c$, then $\phi(m)(1-x)^m + \phi(m+1)(1-x)^{m+1} + \phi(m+2)(1-x)^{m+2}$

$+ \&c = Q'_0 - Q'_1 x + Q'_2 x^2 - Q'_3 x^3 + Q'_4 x^4 - \&c +$

132. $\dfrac{1}{\left(\log\frac{1}{1-x}\right)^{n+1}}\left\{P_0 + P_1\log\frac{1}{1-x} + P_2\left(\log\frac{1}{1-x}\right)^2 + \&c\right\}$

Cor. 2. If $\alpha+\beta+\gamma+1=\delta+\epsilon$, then when x vanishes

$$\frac{\lfloor\underline{\alpha}\,\lfloor\underline{\beta}\,\lfloor\underline{\gamma}}{\lfloor\underline{0}\,\lfloor\underline{\delta}\,\lfloor\underline{\epsilon}} + (1-x)\frac{\lfloor\underline{\alpha+1}\,\lfloor\underline{\beta+1}\,\lfloor\underline{\gamma+1}}{\lfloor\underline{1}\,\lfloor\underline{\delta+1}\,\lfloor\underline{\epsilon+1}} + (1-x)^2\frac{\lfloor\underline{\alpha+2}\,\lfloor\underline{\beta+2}\,\lfloor\underline{\gamma+2}}{\lfloor\underline{2}\,\lfloor\underline{\delta+2}\,\lfloor\underline{\epsilon+2}} + \&c$$

$$+\log x + \Sigma\frac{1}{\alpha} + \Sigma\frac{1}{\beta} =$$

$$1.\ \frac{(\gamma-\delta)(\gamma-\epsilon)}{(\alpha+1)(\beta+1)} + \frac{1}{2}\cdot\frac{(\gamma-\delta)(\gamma-\delta-1)(\gamma-\epsilon)(\gamma-\epsilon-1)}{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)} + \&c.$$

25. $\lfloor\underline{a}\,\lfloor\underline{b}\left\{\dfrac{\lfloor\underline{a+n}\,\lfloor\underline{b+n}}{\lfloor\underline{a+b+n+1}} + \dfrac{1-x}{\lfloor\underline{1}}\cdot\dfrac{\lfloor\underline{a+n+1}\,\lfloor\underline{b+n+1}}{\lfloor\underline{a+b+n+2}} + \&c\right\}$

$$=\left\{\lfloor\underline{a+n}\,\lfloor\underline{b+n}\,\lfloor\underline{-n-1} - \frac{x}{\lfloor\underline{1}}\lfloor\underline{a+n+1}\,\lfloor\underline{b+n+1}\,\lfloor\underline{-n-2} + \&c\right\}$$

$$+\frac{1}{x^n}\left\{\lfloor\underline{a}\,\lfloor\underline{b}\,\lfloor\underline{n-1} - \frac{x}{\lfloor\underline{1}}\lfloor\underline{a+1}\,\lfloor\underline{b+1}\,\lfloor\underline{n-2} + \&c\right\}$$

N. B. Though the above theorem is true for all values of n yet if n is an Integer it assumes the form $\infty-\infty$; so we must write $n+h$ for n and then after simplification h should be made to vanish.

Cor. 1. If n is a positive integer,

$$\lfloor\underline{a}\,\lfloor\underline{b}\left\{\frac{\lfloor\underline{a+n}\,\lfloor\underline{b+n}}{\lfloor\underline{a+b+n+1}} + \frac{1-x}{\lfloor\underline{1}}\cdot\frac{\lfloor\underline{a+n+1}\,\lfloor\underline{b+n+1}}{\lfloor\underline{a+b+n+2}} + \&c\right\}$$

$$+(-1)^n\log x\left\{\frac{\lfloor\underline{a+n}\,\lfloor\underline{b+n}}{\lfloor\underline{n}} + \frac{x}{\lfloor\underline{1}}\cdot\frac{\lfloor\underline{a+n+1}\,\lfloor\underline{b+n+1}}{\lfloor\underline{n+1}} + \&c\right\}$$

$$+(-1)^n\left\{\frac{\lfloor\underline{a+n}\,\lfloor\underline{b+n}}{\lfloor\underline{n}}\left(\Sigma\frac{1}{a+n} + \Sigma\frac{1}{b+n} - \Sigma\frac{1}{n} - 0\right) + \right.$$

$$\frac{x}{\lfloor\underline{1}}\cdot\frac{\lfloor\underline{a+n+1}\,\lfloor\underline{b+n+1}}{\lfloor\underline{n+1}}\left(\Sigma\frac{1}{a+n+1} + \Sigma\frac{1}{b+n+1} - \Sigma\frac{1}{n+1} - \Sigma\frac{1}{1}\right) +$$

$$\left.\frac{x^2}{\lfloor\underline{2}}\cdot\frac{\lfloor\underline{a+n+2}\,\lfloor\underline{b+n+2}}{\lfloor\underline{n+2}}\left(\Sigma\frac{1}{a+n+2} + \Sigma\frac{1}{b+n+2} - \Sigma\frac{1}{n+2} - \Sigma\frac{1}{2}\right) + \&c\right\}$$

$$= \frac{1}{x^n}\left\{\lfloor\underline{a}\,\lfloor\underline{b}\,\lfloor\underline{n-1} - \frac{x}{\lfloor\underline{1}}\lfloor\underline{a+1}\,\lfloor\underline{b+1}\,\lfloor\underline{n-2} + \&c \text{ to } n \text{ terms}\right\}$$

Cor. 2. If n is a negative integer,

$$\lfloor\underline{a}\,\lfloor\underline{b}\left\{\frac{\lfloor\underline{a+n}\,\lfloor\underline{b+n}}{\lfloor\underline{a+b+n+1}} + \frac{1-x}{\lfloor\underline{1}}\cdot\frac{\lfloor\underline{a+n+1}\,\lfloor\underline{b+n+1}}{\lfloor\underline{a+b+n+2}} + \&c\right\}$$

$$+(-x)^{-n}\log x\left\{\frac{\lfloor\underline{a}\,\lfloor\underline{b}}{\lfloor\underline{-n}} + \frac{x}{\lfloor\underline{1}}\cdot\frac{\lfloor\underline{a+1}\,\lfloor\underline{b+1}}{\lfloor\underline{-n+1}} + \frac{x^2}{\lfloor\underline{2}}\cdot\frac{\lfloor\underline{a+2}\,\lfloor\underline{b+2}}{\lfloor\underline{-n+2}} + \&c\right\}$$

$$+(-x)^{-n}\left\{\frac{\lfloor\underline{a}\,\lfloor\underline{b}}{\lfloor\underline{-n}}\left(\Sigma\frac{1}{a} + \Sigma\frac{1}{b} - \Sigma\frac{1}{-n} - 0\right) + \right.$$

$$\frac{x}{\lfloor\underline{1}}\,\frac{\lfloor\underline{a+1}\,\lfloor\underline{b+1}}{\lfloor\underline{-n+1}}\left(\Sigma\frac{1}{a+1} + \Sigma\frac{1}{b+1} - \Sigma\frac{1}{-n+1} - \Sigma\frac{1}{1}\right) +$$

$$\left.\frac{x^2}{\lfloor\underline{2}}\cdot\frac{\lfloor\underline{a+2}\,\lfloor\underline{b+2}}{\lfloor\underline{-n+2}}\left(\Sigma\frac{1}{a+2} + \Sigma\frac{1}{b+2} - \Sigma\frac{1}{-n+2} - \Sigma\frac{1}{2}\right) + \&c\right\}$$

$$= \lfloor\underline{a+n}\,\lfloor\underline{b+n}\,\lfloor\underline{-n-1} - \frac{x}{\lfloor\underline{1}}\lfloor\underline{a+n+1}\,\lfloor\underline{b+n+1}\,\lfloor\underline{-n-2} + \&c \text{ to } -n$$

terms. N.B. We may put $n=0$ either in Cor. 1 or Cor. 2.

26. $$\lfloor\underline{a}\,\lfloor\underline{b}\left\{\frac{\lfloor\underline{a}\,\lfloor\underline{b}}{\lfloor\underline{a+b+1}} + \frac{(1-x)}{\lfloor\underline{1}}\,\frac{\lfloor\underline{a+1}\,\lfloor\underline{b+1}}{\lfloor\underline{a+b+2}} + \frac{(1-x)^2}{\lfloor\underline{2}}\cdot\frac{\lfloor\underline{a+2}\,\lfloor\underline{b+2}}{\lfloor\underline{a+b+3}} + \&c\right\}$$

$$+\log x\left\{\lfloor\underline{a}\,\lfloor\underline{b} + x\,\frac{\lfloor\underline{a+1}\,\lfloor\underline{b+1}}{\lfloor\underline{1}\,\lfloor\underline{1}} + x^2\,\frac{\lfloor\underline{a+2}\,\lfloor\underline{b+2}}{\lfloor\underline{2}\,\lfloor\underline{2}} + \&c\right\}$$

$$+\lfloor\underline{a}\,\lfloor\underline{b}\left(\Sigma\frac{1}{a} + \Sigma\frac{1}{b}\right) + x.\,\frac{\lfloor\underline{a+1}\,\lfloor\underline{b+1}}{\lfloor\underline{1}\,\lfloor\underline{1}}\left(\Sigma\frac{1}{a+1} + \Sigma\frac{1}{b+1} - 2\Sigma\frac{1}{1}\right)$$

$$+x^2\,\frac{\lfloor\underline{a+2}\,\lfloor\underline{b+2}}{\lfloor\underline{2}\,\lfloor\underline{2}}\left(\Sigma\frac{1}{a+2} + \Sigma\frac{1}{b+2} - 2\Sigma\frac{1}{2}\right) + \&c = 0.$$

Cor. $$\pi\left\{1 + \left(\tfrac{1}{2}\right)^2(1-x) + \left(\tfrac{1.3}{2.4}\right)^2(1-x)^2 + \left(\tfrac{1.3.5}{2.4.6}\right)^2(1-x)^3 + \&c\right\}$$

$$= \log\frac{16}{x}\left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1.3}{2.4}\right)^2 x^2 + \left(\tfrac{1.3.5}{2.4.6}\right)^2 x^3 + \&c\right\}.$$

$$-4\left\{\left(\tfrac{1}{2}\right)^2\frac{1}{1.2}x + \left(\tfrac{1.3}{2.4}\right)^2\left(\frac{1}{1.2} + \frac{1}{3.4}\right)x^2 + \&c\right\}.$$

ex. $$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\tan\frac{\phi}{2}}{\sqrt{1 - x\cos^2\theta\cos^2\phi}}\,d\theta\,d\phi = \frac{\pi}{4}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-(1-x)\sin^2\phi}}$$

$$+\frac{1}{4}\log x\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1 - x\sin^2\phi}}.$$

27. $$\left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1.3}{2.4}\right)^2\left(1 + \tfrac{1}{3}\right)x^2 + \left(\tfrac{1.3.5}{2.4.6}\right)^2\left(1 + \tfrac{1}{3} + \tfrac{1}{5}\right)x^3 + \&c =$$

$$-\frac{1}{4}\left\{1+\left(\tfrac{1}{2}\right)^2 x+\left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 x^2+\left(\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 x^3+\&c\right\}\log(1-x)$$

ex. 1. $$e^{-\pi\cdot\frac{1+\left(\frac{1}{2}\right)^2(1-x)+\&c}{1+\left(\frac{1}{2}\right)^2 x+\&c}}=\frac{1}{16}\left(x+\frac{x^2}{2}+\frac{21}{64}x^3+\&c\right)$$

2. $$e^{-\frac{2\pi}{\sqrt{3}}\cdot\frac{1+\frac{1\cdot 2}{3^2}(1-x)+\&c}{1+\frac{1\cdot 2}{3^2}x+\&c}}=\frac{1}{27}\left(x+\frac{5}{9}x^2+\&c\right)$$

3 $$e^{-\pi\sqrt{2}\,\frac{1+\frac{1\cdot 3}{4^2}(1-x)+\&c}{1+\frac{1\cdot 3}{4^2}x+\&c}}=\frac{1}{64}\left(x+\frac{5}{8}x^2+\&c\right)$$

4. $$e^{-2\pi\cdot\frac{1+\frac{1\cdot 5}{6^2}(1-x)+\&c}{1+\frac{1\cdot 5}{6^2}x+\&c}}=\frac{1}{432}\left(x+\frac{13}{18}x^2+\&c\right)$$

28. $$\phi(0)\lfloor\underline{a}\,\lfloor\underline{b}\,\lfloor\underline{n-1}-\frac{\phi(1)}{\lfloor\underline{1}}\lfloor\underline{a+1}\,\lfloor\underline{b+1}\,\lfloor\underline{n-2}+\&c$$

$$+\phi(n)\lfloor\underline{a+n}\,\lfloor\underline{b+n}\,\lfloor\underline{-n-1}-\frac{\phi(n+1)}{\lfloor\underline{1}}\lfloor\underline{a+n+1}\,\lfloor\underline{b+n+1}\,\lfloor\underline{-n-2}+\&c$$

$$=\lfloor\underline{a+n}\,\lfloor\underline{b+n}\left\{\phi(0)\frac{\lfloor\underline{a}\,\lfloor\underline{b}}{\lfloor\underline{a+b+n+1}}+\frac{\phi(0)-\phi(1)}{\lfloor\underline{1}}\,\frac{\lfloor\underline{a+1}\,\lfloor\underline{b+1}}{\lfloor\underline{a+b+n+2}}\right.$$

$$\left.+\frac{\phi(0)-2\phi(1)+\phi(2)}{\lfloor\underline{2}}\cdot\frac{\lfloor\underline{a+2}\,\lfloor\underline{b+2}}{\lfloor\underline{a+b+n+2}}+\&c\right\}$$

Cor. $$\lfloor\underline{a}\,\lfloor\underline{b}\left\{\phi(0)\frac{\lfloor\underline{a}\,\lfloor\underline{b}}{\lfloor\underline{a+b+1}}+\frac{\phi(0)-\phi(1)}{\lfloor\underline{1}}\cdot\frac{\lfloor\underline{a+1}\,\lfloor\underline{b+1}}{\lfloor\underline{a+b+2}}+\&c\right\}$$

$$+\phi'(0)\lfloor\underline{a}\,\lfloor\underline{b}+\phi'(1)\frac{\lfloor\underline{a+1}\,\lfloor\underline{b+1}}{\lfloor\underline{1}\,\lfloor\underline{1}}+\phi'(2)\frac{\lfloor\underline{a+2}\,\lfloor\underline{b+2}}{\lfloor\underline{2}\,\lfloor\underline{2}}+\&c$$

$$+\phi(0)\lfloor\underline{a}\,\lfloor\underline{b}\left(\Sigma\frac{1}{a}+\Sigma\frac{1}{b}\right)+\phi(1)\frac{\lfloor\underline{a+1}\,\lfloor\underline{b+1}}{\lfloor\underline{1}\,\lfloor\underline{1}}\left(\Sigma\frac{1}{a+1}+\Sigma\frac{1}{b+1}-2\Sigma 1\right).$$

$$+\phi(2)\frac{\lfloor\underline{a+2}\,\lfloor\underline{b+2}}{\lfloor\underline{2}\,\lfloor\underline{2}}\left(\Sigma\frac{1}{a+2}+\Sigma\frac{1}{b+2}-2\Sigma\frac{1}{2}\right)+\&c=0.$$

29. If $$F(\alpha,\beta,\gamma,\delta,\epsilon)=1+\frac{\alpha}{\lfloor\underline{1}}\cdot\frac{\beta}{\delta}\cdot\frac{\gamma}{\epsilon}+\frac{\alpha(\alpha+1)}{\lfloor\underline{2}}\cdot\frac{\beta(\beta+1)}{\delta(\delta+1)}\cdot\frac{\gamma(\gamma+1)}{\epsilon(\epsilon+1)}$$
$+\&c$, then

i. $$F(\alpha,\beta,\gamma,\delta,\epsilon)=\frac{\lfloor\underline{\delta-1}\,\lfloor\underline{\delta-\alpha-\beta-1}}{\lfloor\underline{\delta-\alpha-1}\,\lfloor\underline{\delta-\beta-1}}F(\alpha,\beta,\epsilon-\gamma,\alpha+\beta-\delta+1,\epsilon)$$

$$+\frac{\lfloor\underline{\delta-1}\,\lfloor\underline{\epsilon-1}\,\lfloor\underline{\alpha+\beta-\delta-1}\,\lfloor\underline{\delta+\epsilon-\alpha-\beta-\gamma-1}}{\lfloor\underline{\alpha-1}\,\lfloor\underline{\beta-1}\,\lfloor\underline{\epsilon-\gamma-1}\,\lfloor\underline{\delta+\epsilon-\alpha-\beta-1}}\cdot F(\delta-\alpha,\delta-\beta,\delta+\epsilon-\alpha-\beta-\gamma,$$
$$\delta-\alpha-\beta+1,\delta+\epsilon-\alpha-\beta.)$$

ii For integral values of α, β or γ,

$$F(-2\alpha, -2\beta, -\gamma, -\alpha-\beta+\tfrac{1}{2}, \delta)$$
$$= F(-\alpha, -\beta, -\gamma, \gamma+\delta, -\alpha-\beta+\tfrac{1}{2}, \tfrac{\delta}{2}, \tfrac{\delta+1}{2})$$

30. If $\alpha+\beta+1=\gamma+\delta$

and $y = \dfrac{\lfloor\underline{\alpha-1}\ \lfloor\underline{\beta-1}}{\lfloor\underline{\gamma-1}\ \lfloor\underline{\delta-1}} \cdot \dfrac{1+\frac{\alpha}{\lfloor\underline{1}}\cdot\frac{\beta}{\delta}(1-x)+\frac{\alpha(\alpha+1)}{\lfloor\underline{2}}\cdot\frac{\beta(\beta+1)}{\delta(\delta+1)}(1-x)^2+\&c}{1+\frac{\alpha}{\lfloor\underline{1}}\cdot\frac{\beta}{\gamma}x+\frac{\alpha(\alpha+1)}{\lfloor\underline{2}}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2+\&c}$

then,

$$\frac{dy}{dx} = -\frac{1}{\left\{1+\frac{\alpha}{\lfloor\underline{1}}\cdot\frac{\beta}{\gamma}x+\frac{\alpha(\alpha+1)}{\lfloor\underline{2}}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2+\&c\right\}^2}\cdot\frac{1}{x^{\gamma}(1-x)^{\delta}}.$$

Cor. If $y = \dfrac{\pi}{\sin \pi n} \cdot \dfrac{1+\frac{n}{\lfloor\underline{1}}\cdot\frac{1-n}{\lfloor\underline{1}}(1-x)+\frac{n(n+1)(1-n)(2-n)}{\lfloor\underline{2}\ \lfloor\underline{2}}(1-x)^2+\&c}{1+\frac{n}{\lfloor\underline{1}}\cdot\frac{1-n}{\lfloor\underline{1}}x+\frac{n(n+1)(1-n)(2-n)}{\lfloor\underline{2}\ \lfloor\underline{2}}x^2+\&c}$

then $\dfrac{dy}{dx} =$

$$-\frac{1}{x(1-x)\left\{1+\frac{n}{\lfloor\underline{1}}\cdot\frac{1-n}{\lfloor\underline{1}}x+\frac{n(n+1)(1-n)(2-n)}{\lfloor\underline{2}\ \lfloor\underline{2}}x^2+\&c\right\}^2}$$

31. If $y = 1+\dfrac{\alpha}{\lfloor\underline{1}}\cdot\dfrac{\beta}{\gamma}x+\dfrac{\alpha(\alpha+1)}{\lfloor\underline{2}}\cdot\dfrac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2+\&c$, then

i. $(\alpha-1)(\beta-1)\int y\,dx - x(1-x)\dfrac{dy}{dx} = (\gamma-1)(y-1)-(\alpha+\beta-1)xy.$

ii. $y\displaystyle\int\frac{\int x^{n-2}y\,dx}{x^{\gamma}(1-x)^{\delta}y^2}dx$ (where $\delta = \alpha+\beta+1-\gamma$)

$$= \frac{x^{n-\gamma}(1-x)^{1-\delta}}{(n-\gamma)(n-1)}\cdot 1+\frac{(n-\alpha)(n-\beta)}{n(n-\gamma+1)}x+\frac{(n-\alpha)(n-\alpha+1)(n-\beta)(n-\beta+1)}{n(n+1)(n-\gamma+1)(n-\gamma+2)}$$
$$\times x^2+\&c.$$

Cor. If $y = 1+\dfrac{n}{\lfloor\underline{1}}\cdot\dfrac{1-n}{\lfloor\underline{1}}x+\dfrac{n(n+1)}{\lfloor\underline{2}}\cdot\dfrac{(1-n)(2-n)}{\lfloor\underline{2}}x^2+\&c$, then

$$x(x-1)\frac{dy}{dx} = n(n-1)\int y\,dx.$$

32. i. $1+\left(\tfrac{1}{2}\right)^2\left\{1-\dfrac{\phi(-x)}{\phi(x)}\right\}+\left(\tfrac{1.3}{2.4}\right)^2\left\{1-\dfrac{\phi(-x)}{\phi(x)}\right\}^2+\&c$

$= \sqrt{\phi(x)}$ × an even function of x whatever be $\phi(x)$.

ii. $1 + \left(\frac{1}{2}\right)^2\left(1 - \frac{1}{x}\right) + \left(\frac{1.3}{2.4}\right)^2\left(1 - \frac{1}{x}\right)^2 + \&c$

$= \sqrt{x}\left\{1 + \left(\frac{1}{2}\right)^2(1-x) + \left(\frac{1.3}{2.4}\right)^2(1-x)^2 + \&c\right\}$

iii. $1 + \left(\frac{1}{2}\right)^2\left\{1 - \left(\frac{1-x}{1+x}\right)^2\right\} + \left(\frac{1.3}{2.4}\right)^2\left\{1 - \left(\frac{1-x}{1+x}\right)^2\right\}^2 + \&c$

$= (1+x)\left\{1 + \left(\frac{1}{2}\right)^2 x^2 + \left(\frac{1.3}{2.4}\right)^2 x^4 + \&c\right\}$

iv. $1 + \left(\frac{1}{2}\right)^2\left\{1 - \left(\frac{1-x}{1+x}\right)^4\right\} + \left(\frac{1.3}{2.4}\right)^2\left\{1 - \left(\frac{1-x}{1+x}\right)^4\right\}^2 + \&c$

$= (1+x)^2\left\{1 + \left(\frac{1}{2}\right)^2 x^4 + \left(\frac{1.3}{2.4}\right)^2 x^8 + \&c\right\}$

v. $\sqrt[4]{1+x^2}\left\{1 + \left(\frac{1}{2}\right)^2 \frac{1+ix}{2} + \left(\frac{1.3}{2.4}\right)^2\left(\frac{1+ix}{2}\right)^2 + \&c\right\}$

$= \frac{1+i}{2}\left\{1 + \left(\frac{1}{2}\right)^2 \frac{1 + \frac{x}{\sqrt{1+x^2}}}{2} + \left(\frac{1.3}{2.4}\right)^2\left(\frac{1 + \frac{x}{\sqrt{1+x^2}}}{2}\right)^2 + \&c\right\}$

$+ \frac{1-i}{2}\left\{1 + \left(\frac{1}{2}\right)^2 \frac{1 - \frac{x}{\sqrt{1+x^2}}}{2} + \left(\frac{1.3}{2.4}\right)^2\left(\frac{1 - \frac{x}{\sqrt{1+x^2}}}{2}\right)^2 + \&c\right\}$

33. i. $1 + \left(\frac{1}{2}\right)^2 \frac{2x}{1+x} + \left(\frac{1.3}{2.4}\right)^2\left(\frac{2x}{1+x}\right)^2 + \&c$

$= \sqrt{1+x}\left\{1 + \frac{1.3}{4^2}x^2 + \frac{1.3.5.7}{4^2.8^2}x^4 + \&c\right\}$

ii. $1 + \left(\frac{1}{2}\right)^2 \frac{1-\sqrt{1-x}}{2} + \left(\frac{1.3}{2.4}\right)^2\left(\frac{1-\sqrt{1-x}}{2}\right)^2 + \&c$

$= 1 + \left(\frac{1}{4}\right)^2 x + \left(\frac{1.5}{4.8}\right)^2 x^2 + \left(\frac{1.5.9}{4.8.12}\right)^2 x^3 + \&c$

iii. $1 + \left(\frac{1}{2}\right)^3 x + \left(\frac{1.3}{2.4}\right)^3 x^2 + \left(\frac{1.3.5}{2.4.6}\right)^3 x^3 + \&c$

$= \left\{1 + \left(\frac{1}{4}\right)^2 x + \left(\frac{1.5}{4.8}\right)^2 x^2 + \left(\frac{1.5.9}{4.8.12}\right)^2 x^3 + \&c\right\}^2.$

iv. $1 + \frac{1.3}{4^2}\frac{4x}{(1+x)^2} + \frac{1.3.5.7}{4^2.8^2}\left\{\frac{4x}{(1+x)^2}\right\}^2 + \&c$

$= \sqrt{1+x}\left\{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 x^3 + \&c\right\}$

v. $1 + \left(\frac{1}{4}\right)^2 x + \left(\frac{1.5}{4.8}\right)^2 x^2 + \left(\frac{1.5.9}{4.8.12}\right)^2 x^3 + \&c$

$= \sqrt{1-x}\left\{1 + \left(\frac{3}{4}\right)^2 x + \left(\frac{3.7}{4.8}\right)^2 x^2 + \left(\frac{3.7.11}{4.8.12}\right)^2 x^3 + \&c\right\}$

ex. i. $1 - \frac{1.3}{4^2}\cdot\frac{4x}{(1-x)^2} + \frac{1.3.5.7}{4^2.8^2}\left\{\frac{4x}{(1-x)^2}\right\}^2 - \&c =$

$$\sqrt{\frac{1-x}{1+x}}\left\{1+\left(\tfrac{1}{2}\right)^2\frac{x}{1+x}+\left(\tfrac{1.3}{2.4}\right)^2\left(\frac{x}{1+x}\right)^2+\&c\right\}$$

ii. $1-\left(\tfrac{1}{4}\right)^2\frac{4x}{(1-x)^2}+\left(\tfrac{1.5}{4.8}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}^2-\&c$

$$=\sqrt{1-x}\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1.3}{2.4}\right)^2x^2+\left(\tfrac{1.3.5}{2.4.6}\right)^2x^3+\&c\right\}$$

iii. $1-\left(\tfrac{3}{4}\right)^2\frac{4x}{(1-x)^2}+\left(\tfrac{3.7}{4.8}\right)^2\left\{\frac{4x}{(1-x)^2}\right\}^2-\&c$

$$=\frac{(1-x)\sqrt{1-x}}{1+x}\left\{1+\left(\tfrac{1}{2}\right)^2x+\left(\tfrac{1.3}{2.4}\right)^2x^2+\&c\right\}$$

34. If $\pi\mu\eta=1$ and $\mu=\dfrac{\sqrt{\pi}}{\left(-\frac{1}{4}\right)!^2}$ such that

$$\sqrt{\mu}=1.0864348112,1330801457,531612$$

$$\frac{1}{2\sqrt{2}\,\eta}=1.3110287771,46060$$

$$\mu=1.1803405990,16092$$

$$\eta=.2696763005,94191$$

$$\frac{1}{\eta}=3.7081493546,02731\text{, then}$$

i. $1+\left(\tfrac{1}{2}\right)^2\frac{1+x}{2}+\left(\tfrac{1.3}{2.4}\right)^2\left(\frac{1+x}{2}\right)^2+\left(\tfrac{1.3.5}{2.4.6}\right)^2\left(\frac{1+x}{2}\right)^3+\&c$

$$=\mu\left\{1+\frac{1^2}{2.4}x^2+\frac{1^2.5^2}{2.4.6.8}x^4+\frac{1^2.5^2.9^2}{2.4.6.8.10.12}x^6+\&c\right\}$$

$$+\eta\left\{x+\frac{3^2}{4.6}x^3+\frac{3^2.7^2}{4.6.8.10}x^5+\frac{3^2.7^2.11^2}{4.6.8.10.12.14}x^7+\&c\right\}$$

ii. $1+\left(\tfrac{1}{2}\right)^2\left(\frac{1}{2}+\frac{x}{1+x^2}\right)+\left(\tfrac{1.3}{2.4}\right)^2\left(\frac{1}{2}+\frac{x}{1+x^2}\right)^2+\&c$

$$=\mu\sqrt{1+x^2}\left\{1+\frac{1}{2}.\frac{1}{3}x^4+\frac{1.3}{2.4}.\frac{1.5}{3.7}x^8+\frac{1.3.5}{2.4.6}.\frac{1.5.9}{3.7.11}x^{12}+\&c\right\}$$

$$+\eta\sqrt{1+x^2}\left\{x+\frac{1}{2}.\frac{3}{5}x^5+\frac{1.3}{2.4}.\frac{3.7}{5.9}x^9+\frac{1.3.5}{2.4.6}.\frac{3.7.11}{5.9.13}x^{13}+\&c\right\}$$

iii. $\frac{\pi}{4}\left\{1+\left(\tfrac{1}{2}\right)^2\frac{1+x}{2}+\left(\tfrac{1.3}{2.4}\right)^2\left(\frac{1+x}{2}\right)^2+\&c\right\}^2$

$$-\frac{\pi}{4}\left\{1+\left(\tfrac{1}{2}\right)^2\frac{1-x}{2}+\left(\tfrac{1.3}{2.4}\right)^2\left(\frac{1-x}{2}\right)^2+\&c\right\}^2$$

$$=x+\frac{2}{3}x^3\left(1-\frac{1^2}{2^2}\right)+\frac{2.4}{3.5}x^5\left(1-2.\frac{1^2}{2^2}+\frac{1^2.3^2}{2^2.4^2}\right)+$$

$$\frac{2\cdot4\cdot6}{3\cdot5\cdot7}x^7\left(1-3\cdot\frac{1^2}{2^2}+3\cdot\frac{1^2\cdot3^2}{2^2\cdot4^2}-\frac{1^2\cdot3^2\cdot5^2}{2^2\cdot4^2\cdot6^2}\right)+\&c$$

$$=x+\frac{x^3}{2}+\frac{41x^5}{120}+\frac{21x^7}{80}+\&c=\frac{x}{1-x^2}-\frac{1}{2}\cdot\frac{x^3}{(1-x^2)^2}+\frac{41}{120}\cdot\frac{x^5}{(1-x^2)^3}-\&c.$$

ex. i. $1+\left(\frac12\right)^2\frac{1+x}{2}+\left(\frac{1\cdot3}{2\cdot4}\right)^2\left(\frac{1+x}{2}\right)^2+\&c$

$$=\frac{u}{(1-x^2)^{\frac14}}\left\{1-\frac{1^2}{2\cdot4}\cdot\frac{x^2}{1-x^2}+\frac{1^2\cdot5^2}{2\cdot4\cdot6\cdot8}\cdot\left(\frac{x^2}{1-x^2}\right)^2-\&c\right\}$$

$$+\frac{\eta x}{(1-x^2)^{\frac54}}\left\{1-\frac{3^2}{4\cdot6}\cdot\frac{x^2}{1-x^2}+\frac{3^2\cdot7^2}{4\cdot6\cdot8\cdot10}\left(\frac{x^2}{1-x^2}\right)^2-\&c\right\}$$

ii. $1+\left(\frac12\right)^2\left(\frac12+\frac{x}{1+x^2}\right)+\left(\frac{1\cdot3}{2\cdot4}\right)^2\left(\frac12+\frac{x}{1+x^2}\right)^2+\&c$

$$=\frac{u}{\sqrt{1-x^2}}\left\{1-\frac12\cdot\frac23\cdot\frac{x^4}{1-x^4}+\frac{1\cdot3}{2\cdot4}\cdot\frac{2\cdot6}{3\cdot7}\cdot\left(\frac{x^4}{1-x^4}\right)^2-\&c\right\}$$

$$+\frac{2\eta x}{\sqrt{1-x^2}}\left\{1-\frac12\cdot\frac25\cdot\frac{x^4}{1-x^4}+\frac{1\cdot3}{2\cdot4}\cdot\frac{2\cdot6}{5\cdot9}\cdot\left(\frac{x^4}{1-x^4}\right)^2-\&c\right\}.$$

35. i. $\cos(2n\sin^{-1}x)=1-\frac{n}{\underline{|1}}\cdot\frac{n}{\frac12}x^2+\frac{n(n-1)}{\underline{|2}}\cdot\frac{n(n+1)}{\frac12\cdot1\frac12}x^4-\&c$

ii. $\frac{\sin(2n\sin^{-1}x)}{2nx}=1+\frac{\frac12-n}{\underline{|1}}\cdot\frac{\frac12+n}{1\frac12}x^2+\frac{(\frac12-n)(1\frac12-n)}{\underline{|2}}\cdot\frac{(\frac12+n)(1\frac12+n)}{1\frac12\cdot2\frac12}x^4+\&c$

iii. $\frac{\cos(2n\sin^{-1}x)}{\sqrt{1-x^2}}=1+\frac{\frac12-n}{\underline{|1}}\cdot\frac{\frac12+n}{\frac12}x^2+\frac{(\frac12-n)(1\frac12-n)}{\underline{|2}}\cdot\frac{(\frac12+n)(1\frac12+n)}{\frac12\cdot1\frac12}x^4+\&c$

36. i. $(1+x)^n=1+nx(1+x)^{\frac{n-1}{2}}+\frac{n(n^2-1)}{4\underline{|3}}x^3(1+x)^{\frac{n-3}{2}}$

$$+\frac{n(n^2-1)(n^2-9)}{4^2\underline{|5}}x^5(1+x)^{\frac{n-5}{2}}+\&c.$$

ii. $\frac{1+(1+x)^n}{2}=(1+x)^{\frac{n}{2}}+\frac{n^2}{4\underline{|2}}x^2(1+x)^{\frac{n-2}{2}}+\frac{n^2(n^2-2^2)}{4^2\underline{|4}}x^4(1+x)^{\frac{n-4}{2}}+\&c.$

iii. $\left(\frac{1+\sqrt{1+4x}}{2}\right)^n=1+nx(1+x)^{\frac{n-3}{2}}+\frac{n(n-5)(n-7)}{4\underline{|3}}x^3(1+x)^{\frac{n-9}{2}}$

$$+\frac{n(n-7)(n-9)(n-11)(n-13)}{4^2\underline{|5}}x^5(1+x)^{\frac{n-15}{2}}+\&c$$

iv. $\frac12+\frac12\left(\frac{1+\sqrt{1+4x}}{2}\right)^n=(1+x)^{\frac{n}{2}}+\frac{n(n-4)}{4\underline{|2}}x^2(1+x)^{\frac{n-6}{2}}+\frac{n(n-6)(n-8)(n-10)}{4^2\underline{|4}}$

$$\times x^4(1+x)^{\frac{n-12}{2}}+\&c.$$

CHAPTER XII

1. $\dfrac{a_1}{b_1+}\dfrac{a_2}{b_2+}\cdots+\dfrac{a_n}{b_n} = a_1\dfrac{N_{n-1}}{D_n} = \dfrac{a_1}{D_0D_1} - \dfrac{a_1a_2}{D_1D_2} + \dfrac{a_1a_2a_3}{D_2D_3} - \&c$ to n terms, where

$N_{n-1} = b_nN_{n-2} + a_nN_{n-3}$ and $D_n = b_nD_{n-1} + a_nD_{n-2}$.

Cor. $a_1+a_2+a_3+\&c$ to n terms $= \dfrac{a_1}{1-}\dfrac{a_2}{a_1+a_2-}\dfrac{a_1a_3}{a_2+a_3-}\dfrac{a_2a_4}{a_3+a_4-}\dfrac{a_3a_5}{a_4+a_5-}\&c$ to n terms.

2. $x = (x-a_1) + \dfrac{xa_1}{x-a_2+}\dfrac{xa_2}{x-a_3+}\dfrac{xa_3}{x-a_4+}\&c.$

3. $x = a_1 + \sqrt{x^2 + a_1(a_1+2a_2) - 2a_1\sqrt{x^2 + a_2(a_2+2a_3) - 2a_3\sqrt{\&c}}}$

4. $x+n+a = \sqrt{ax + (n+a)^2 + x\sqrt{a(x+n) + (n+a)^2 + (x+n)\sqrt{\&c}}}$.

e.g. i. $3 = 1\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\&c}}}}$

ii. $4 = 1\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\&c}}}}$.

5. i. $2\cos\theta = \sqrt{2+2\cos2\theta} = \sqrt{2+\sqrt{2+2\cos4\theta}} = \sqrt{2+\sqrt{2+\sqrt{2+2\cos8\theta}}}$

$= \&c.$

ii $2\cos\theta = \sqrt[3]{2\cos3\theta + 3\sqrt[3]{2\cos3\theta + 3\sqrt[3]{2\cos3\theta + \&c}}}$.

$= \sqrt[3]{6\cos\theta + \sqrt[3]{6\cos3\theta + \sqrt[3]{6\cos9\theta + \sqrt[3]{6\cos27\theta + \&c}}}}$

6. $\sqrt{\dfrac{a(a-2)}{4} + \sqrt{\dfrac{a(a-2)}{4} + \sqrt{\dfrac{a(a-2)}{4} + \&c \text{ to } n \text{ terms} + h}}}$.

$= \dfrac{a}{2}\left\{1 - v/a^n + \dfrac{(v/a^n)^2}{2(a-1)} - \dfrac{(v/a^n)^3}{2(a-1)(a^2-1)} + \dfrac{(v/a^n)^4(a+5)}{8(a-1)(a^2-1)(a^3-1)} - \dfrac{(v/a^n)^5(2a^2+3a+7)}{8(a-1)(a^2-1)(a^3-1)(a^4-1)} + \&c\right\}$ where v is a function of a and h independent of n defined by the relation

$$\frac{2h}{a} = 1 - v + \frac{v^2}{2(a-1)} - \frac{v^3}{2(a-1)(a^2-1)} + \frac{v^4(a+5)}{8(a-1)(a^2-1)(a^3-1)} - \&c$$

the coeff^t. of $v^{n+1} = \dfrac{1}{2(a^n-1)} \times$ the coeff^t. of v^n in the Square of the Series.

7. $x = \dfrac{x+1}{x+}\dfrac{x+2}{x+1+}\dfrac{x+3}{x+2+}\&c.$ Cor. $1 = \dfrac{2}{1+}\dfrac{3}{2+}\dfrac{4}{3+}\dfrac{5}{4+}\&c.$

8. $\dfrac{1}{x+a} - \dfrac{1}{(x+a)(x+2a)} + \dfrac{1}{(x+a)(x+2a)(x+3a)} - \&c$ to n terms

$= \dfrac{1}{x+a+}\,\dfrac{x+a}{x+2a-1+}\,\dfrac{x+2a}{x+3a-1+}\,\dfrac{x+3a}{x+4a-1+}\,\&c$ to n terms.

Cor. $\dfrac{1}{e-1} = \dfrac{1}{1+}\,\dfrac{2}{2+}\,\dfrac{3}{3+}\,\dfrac{4}{4+}\,\&c$

9. $\dfrac{x+a+1}{x+1} = \dfrac{x+a}{x-1+}\,\dfrac{x+2a}{x+a-1+}\,\dfrac{x+3a}{x+2a-1+}\,\&c.$

e.g. 1. $\dfrac{4}{3} = \dfrac{3}{1+}\,\dfrac{4}{2+}\,\dfrac{5}{3+}\,\dfrac{6}{4+}\,\&c.$

2. $\dfrac{5}{3} = \dfrac{7}{1+}\,\dfrac{6}{3+}\,\dfrac{8}{5+}\,\dfrac{10}{7+}\,\&c.$

10. If n is a positive integer,

$n = \dfrac{1}{1-n+}\,\dfrac{2}{2-n+}\,\dfrac{3}{3-n+}\,\ldots\ldots + \dfrac{n}{0+}\,\dfrac{n+1}{1+}\,\dfrac{n+2}{2+}\,\dfrac{n+3}{3+}\,\&c.$

11. If a is a positive integer and $D = \phi(n-1)$ where $\phi(n) = N$ where N_{a+1} and N_a are the numerator and the denominator in the fraction

$n+2-a + \dfrac{a-1}{n+3-a+}\,\dfrac{a-2}{n+4-a+}\,\dfrac{a-3}{n+5-a+}\,\&c.$

Cor. 1. $\dfrac{n^2+n+1}{n^2-n+1} = \dfrac{n}{n-3+}\,\dfrac{n+1}{n-2+}\,\dfrac{n+2}{n-1+}\,\dfrac{n+3}{n+}\,\&c.$

2. $\dfrac{n^3+2n+1}{(n-1)^3+2(n-1)+1} = \dfrac{n}{n-4+}\,\dfrac{n+1}{n-3+}\,\dfrac{n+2}{n-2+}\,\dfrac{n+3}{n-1+}\,\&c.$

12. $1 = \dfrac{x+a}{a+}\,\dfrac{(x+a)^2-a^2}{a+}\,\dfrac{(x+2a)^2-a^2}{a+}\,\dfrac{(x+3a)^2-a^2}{a+\&c}$

13. If $a<b$, $a = \dfrac{ab}{a+b+d-}\,\dfrac{(a+d)(b+d)}{a+b+3d-}\,\dfrac{(a+2d)(b+2d)}{a+b+5d} - \&c$

14. $\dfrac{a_1}{x+}\,\dfrac{a_2}{1+}\,\dfrac{a_3}{x+}\,\dfrac{a_4}{1+}\,\&c$ to $2n$ terms

$= \dfrac{a_1}{x+a_2-}\,\dfrac{a_2a_3}{x+a_3+a_4}-\dfrac{a_4a_5}{x+a_5+a_6-}\,\&c$ to n terms.

15. $\dfrac{a_1+h}{1+}\,\dfrac{a_1}{x+}\,\dfrac{a_2+h}{1+}\,\dfrac{a_2}{x+}\,\dfrac{a_3+h}{1+}\,\&c$

$= h + \dfrac{a_1}{1+}\,\dfrac{a_1+h}{x+}\,\dfrac{a_2}{1+}\,\dfrac{a_2+h}{x+}\,\dfrac{a_3}{1+}\,\&c.$

16. $\dfrac{1}{(m+1)(n+1)} - \dfrac{1}{(m+2)(n+2)} + \dfrac{1}{(m+3)(n+3)} - \&c$

$$= \frac{1}{mn+m+n+1+}\;\frac{(m+1)^2(n+1)^2}{m+n+3+}\;\frac{(m+2)^2(n+2)^2}{m+n+5+}\;\&c.$$

17. $$\frac{1}{1+}\;\frac{a_1 x}{1+}\;\frac{a_2 x}{1+}\;\frac{a_3 x}{1+}\;\&c = 1 - A_1 x + A_2 x^2 - A_3 x^3 + \&c$$

let $P_n = a_1 a_2 a_3 \cdots a_{n-1} (a_1 + a_2 + \quad a_n)$, then

$P_1 = A_1$; $P_2 = A_2$; $P_3 = A_3 - a_1 A_2$; $P_4 = A_4 - (a_1 + a_2) A_3$

$P_5 = A_5 - (a_1 + a_2 + a_3) A_4 + a_1 a_3 A_3$

$P_6 = A_6 - (a_1 + a_2 + a_3 + a_4) A_5 + (a_1 a_3 + a_2 a_4 + a_1 a_4) A_4$

$P_n = \phi_0(n) A_n - \phi_1(n) A_{n-1} + \phi_2(n) A_{n-2} - \&c$

where $\phi_r(n+1) - \phi_r(n) = a_{n-1} \phi_{r-1}(n-1)$.

Cor. i. If $\dfrac{1}{1+b_1 x+}\;\dfrac{a_1 x}{1+b_2 x+}\;\dfrac{a_2 x}{1+b_3 x+}\;\&c = 1 - A_1 x + A_2 x^2 - \&c$

$P_n = a_1 a_2 a_3 \cdots a_{n-1} (\overline{a_1 + b_1} + \overline{a_2 + b_2} + \cdots + \overline{a_n + b_n})$

$= \phi_0(n) A_n - \phi_1(n) A_{n-1} + \phi_2(n) A_{n-2} - \&c$ where

$\phi_r(n+1) - \phi_r(n) = b_n \phi_{r-1}(n) + a_{n-1} \phi_{r-1}(n-1)$.

Cor. ii. In the above results $D_{n-1} = \phi_0(n) + x\,\phi_1(n) + x^2 \phi_2(n) + \&c$

ex. $\left\{1 + \left(\tfrac{1}{2}\right)^2 x + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^2 x^2 + \&c\right\}^2 = \dfrac{1}{1-}\;\dfrac{x}{2-}\;\dfrac{3x}{8-}\;\dfrac{5x}{2-}\;\dfrac{17x}{40}$

~~N. B. The peculiarity in this~~ $-\;\dfrac{23x}{2-}\;\dfrac{1895x}{3128-}\;\&c.$

~~continued fraction is if $x=1$~~

~~it assumes the form $1 + 1 + \frac{3}{5} + \frac{3}{5} + \&c$.~~

18. $$\frac{(x+1)^n - (x-1)^n}{(x+1)^n + (x-1)^n} = \frac{n}{x+}\;\frac{n^2-1^2}{3x+}\;\frac{n^2-2^2}{5x+}\;\frac{n^2-3^2}{7x+}\;\&c.$$

N. B. If V_n denotes the above fraction, then $V_n + \dfrac{1}{V_n} = \dfrac{2}{V_{2n}}$.

Cor. 1. $$\tan^{-1} x = \frac{x}{1+}\;\frac{(x)^2}{3+}\;\frac{(2x)^2}{5+}\;\frac{(3x)^2}{7+}\;\frac{(4x)^2}{9+}\;\&c$$

Cor. 2. $\log\frac{1+x}{1-x} = \frac{2x}{1-}\frac{(x)^2}{3-}\frac{(2x)^2}{5-}\frac{(3x)^2}{7-}$ &c.

3. $\tan x = \frac{x}{1-}\frac{x^2}{3-}\frac{x^2}{5-}\frac{x^2}{7-}$ &c.

4. $\frac{e^x-1}{e^x+1} = \frac{x}{2+}\frac{x^2}{6+}\frac{x^2}{10+}\frac{x^2}{14+}$ &c.

19. $$\frac{\frac{x}{n} + \frac{x^2}{\underline{|2}}\cdot\frac{1}{n(n+1)} + \frac{x^3}{\underline{|3}}\cdot\frac{1}{n(n+1)(n+2)} + \&c}{1 + \frac{x}{\underline{|1}}\cdot\frac{1}{n} + \frac{x^2}{\underline{|2}}\cdot\frac{1}{n(n+1)} + \&c}$$

$$= \frac{x}{n+}\frac{x}{n+1+}\frac{x}{n+2+}\frac{x}{n+3+}\ \&c$$

25. $$\frac{\frac{\beta}{\gamma}x + \frac{\alpha-\gamma}{\underline{|1}}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \frac{(\alpha-\gamma)(\alpha-\gamma-1)}{\underline{|2}}\cdot\frac{\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 + \&c}{1 + \frac{\alpha-\gamma}{\underline{|1}}\cdot\frac{\beta}{\gamma}x + \frac{(\alpha-\gamma)(\alpha-\gamma-1)}{\underline{|2}}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \&c}$$

$$= \frac{x\alpha\beta}{\gamma+}\frac{x(\alpha-\gamma)(\beta-\gamma)}{\gamma+1+}\frac{x(\alpha+1)(\beta+1)}{\gamma+2+}\frac{x(\alpha-\gamma-1)(\beta-\gamma-1)}{\gamma+3+}\ \&c.$$

21. $$\frac{\beta}{\gamma}x - \frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \frac{\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 - \&c$$

$$= \frac{\beta x}{\gamma+}\frac{\gamma(\beta+1)x}{\gamma+1+}\frac{1(\gamma-\beta)x}{\gamma+2+}\frac{(\gamma+1)(\beta+2)x}{\gamma+3+}\frac{2(\gamma-\beta+1)x}{\gamma+4+}\ \&c$$

$$= \frac{\beta x}{\gamma+}\frac{(\beta+1)x}{1+}\frac{1(1+x)}{\gamma+}\frac{(\beta+2)x}{1+}\frac{2(1+x)}{\gamma+}\ \&c.$$

$$= \frac{\beta x}{\gamma+x(\beta+1)-}\frac{1(\beta+1)x(1+x)}{\gamma+1+x(\beta+3)-}\frac{2(\beta+2)x(1+x)}{\gamma+2+x(\beta+5)-}\ \&c.$$

Cor. 1 $$\frac{x}{n} + \frac{x^2}{n(n+1)} + \frac{x^3}{n(n+1)(n+2)} + \&c$$

$$= \frac{x}{n-}\frac{nx}{n+1+}\frac{x}{n+2-}\frac{(n+1)x}{n+3+}\frac{2x}{n+4-}\ \&c.$$

$$= \frac{x}{n-x+}\frac{x}{n+1-x+}\frac{2x}{n+2-x+}\frac{3x}{n+3-x+}\ \&c$$

Cor. 2. $$1 + \frac{x}{x+1} + \frac{x^2}{(x+1)(x+2)} + \frac{x^3}{(x+1)(x+2)(x+3)} + \&c$$

$$= 1 + \frac{2x}{2+}\frac{3x}{3+}\frac{4x}{4+}\frac{5x}{5+}\frac{6x}{6+}\ \&c.$$

22. $$\frac{\frac{\beta}{\gamma}x + \frac{\alpha}{\underline{|1}}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha-1)}{\underline{|2}}\cdot\frac{\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 + \&c}{1 + \frac{\alpha}{\underline{|1}}\cdot\frac{\beta}{\gamma}x + \frac{\alpha(\alpha-1)}{\underline{|2}}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \&c}$$

$$= \frac{\beta x}{\gamma - (\alpha+\beta+1)x +}\;\frac{(\beta+1)(\alpha+\gamma+1)x}{\gamma+1-(\alpha+\beta+2)x +}\;\frac{(\beta+2)(\alpha+\gamma+2)x}{\gamma+2-(\alpha+\beta+3)x + \&c}$$

23. $$\frac{a_n}{b_n x +}\;\frac{a_{n+1}}{b_{n+1}x +}\;\frac{a_{n+2}}{b_{n+2}x + \&c} = C_n(1 - P_n x + Q_n x^2 - R_n x^3 + \&c)$$

where $C_n C_{n+1} = a_n$; $P_n + P_{n+1} = \frac{b_n}{C_{n+1}}$ or $\frac{b_n C_n}{a_n}$;

$Q_n + Q_{n+1} = (P_n)^2$; $R_n + R_{n+1} = P_n(Q_n - Q_{n+1})$;

$S_n + S_{n+1} = P_n(R_n - R_{n+1}) - Q_n Q_{n+1}$; generally

$Z_n + Z_{n+1} = P_n(Y_n - Y_{n+1}) - Q_n X_{n+1} - R_n W_{n+1}$

$- S_n V_{n+1} - \ldots\ldots - X_n Q_{n+1}$.

N.B. In some cases the above theorem is only approximately true.

ex. $$\sqrt{\frac{2x}{\pi}} - \frac{x}{1+}\;\frac{2x}{2+}\;\frac{3x}{3+}\;\frac{4x}{4+}\;\&c = \frac{2}{3\pi}$$ when $x = \infty$.

24. $$\frac{n}{n+}\;\frac{x}{n+1+}\;\frac{x}{n+2+}\;\frac{x}{n+3+}\;\cdots\;+\frac{x}{n+r}$$

$$= \Big\{1 + \frac{x}{\underline{|1}}\cdot\frac{r-1}{(n+1)(n+r)} + \frac{x^2}{\underline{|2}}\cdot\frac{(r-2)(r-3)}{(n+1)(n+2)(n+r)(n+r-1)}$$
$$+ \frac{x^3}{\underline{|3}}\cdot\frac{(r-3)(r-4)(r-5)}{(n+1)(n+2)(n+3)(n+r)(n+r-1)(n+r-2)} + \&c\Big\}$$

$$\div \Big\{1 + \frac{x}{\underline{|1}}\cdot\frac{r}{n(n+r)} + \frac{x^2}{\underline{|2}}\cdot\frac{(r-1)(r-2)}{n(n+1)(n+r)(n+r-1)} + \&c\Big\}$$

the no. of terms being limited.

25. $$\frac{\underline{\left|\frac{x+n-3}{4}\right.}\;\underline{\left|\frac{x-n-3}{4}\right.}}{\underline{\left|\frac{x+n-1}{4}\right.}\;\underline{\left|\frac{x-n-1}{4}\right.}} = \frac{4}{x-}\;\frac{n^2-1^2}{2x-}\;\frac{n^2-3^2}{2x-}\;\frac{n^2-5^2}{2x-}\;\&c.$$

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Cor. 1. $$\left(\frac{\underline{\left|\frac{x-3}{4}\right.}}{\underline{\left|\frac{x-1}{4}\right.}}\right)^2 = \frac{4}{x+}\,\frac{1^2}{2x+}\,\frac{3^2}{2x+}\,\frac{5^2}{2x+}\,\frac{7^2}{2x+}\,\&c$$

Cor. 2. $$\frac{\underline{\left|\frac{x-5}{8}\right.}\;\underline{\left|\frac{x-7}{8}\right.}}{\underline{\left|\frac{x-1}{8}\right.}\;\underline{\left|\frac{x-3}{8}\right.}} = \frac{8}{x+}\,\frac{1.3}{2x+}\,\frac{5.7}{2x+}\,\frac{9.11}{2x+}\,\&c.$$

26. $$\left\{\frac{\underline{\left|\frac{x+n-3}{4}\right.}\;\underline{\left|\frac{x-n-3}{4}\right.}}{\underline{\left|\frac{x+n-1}{4}\right.}\;\underline{\left|\frac{x-n-1}{4}\right.}}\right\}^2 = \frac{8}{\frac{x^2+n^2-1}{2}+}\,\frac{1^2-n^2}{1+}\,\frac{1^2}{x^2-1+}\,\frac{3^2-n^2}{1+}\,\frac{3^2}{x^2-1}+\&c$$

$$= \frac{8}{\frac{x^2-n^2-1}{2}+}\,\frac{1^2}{1+}\,\frac{1^2-n^2}{x^2-1+}\,\frac{3^2}{1+}\,\frac{3^2-n^2}{x^2-1}+\&c$$

Cor. $$\left\{\frac{\underline{\left|\frac{x-3}{4}\right.}}{\underline{\left|\frac{x-1}{4}\right.}}\right\}^4 = \frac{8}{\frac{x^2-1}{2}+}\,\frac{1^2}{1+}\,\frac{1^2}{x^2-1+}\,\frac{3^2}{1+}\,\frac{3^2}{x^2-1}+\&c$$

27. $$x+\frac{(1+y)^2+n}{2x+}\,\frac{(3+y)^2+n}{2x+}\,\frac{(5+y)^2+n}{2x+}\,\&c$$

$$= y+\frac{(1+x)^2+n}{2y+}\,\frac{(3+x)^2+n}{2y+}\,\frac{(5+x)^2+n}{2y+}\,\&c$$

28. $$x+\frac{n^2+1^2}{2x+}\,\frac{n^2+3^2}{2x+}\,\frac{n^2+5^2}{2x+}\,\&c$$

$$= n+\frac{x^2-1^2}{2n+}\,\frac{x^2-3^2}{2n+}\,\frac{x^2-5^2}{2n+}\,\&c$$ approximately if n is great.

29. $$\left(\frac{1}{x+n+1}-\frac{1}{x+n+3}+\frac{1}{x+n+5}-\&c\right)$$

$$+\left(\frac{1}{x-n+1}-\frac{1}{x-n+3}+\frac{1}{x-n+5}-\&c\right)$$

$$= \frac{1}{x+}\,\frac{1^2-n^2}{x+}\,\frac{2^2}{x+}\,\frac{3^2-n^2}{x+}\,\frac{4^2}{x+}\,\frac{5^2-n^2}{x+}\,\&c$$

Cor. $$2\left(\frac{1}{x+1}-\frac{1}{x+3}+\frac{1}{x+5}-\&c\right) = \frac{1}{x+}\,\frac{1^2}{x+}\,\frac{2^2}{x+}\,\frac{3^2}{x+}\,\&c$$

30. $$\left(\frac{1}{x-n+1}+\frac{1}{x-n+3}+\frac{1}{x-n+5}+\&c\right)$$

$$-\left(\frac{1}{x+n+1}+\frac{1}{x+n+3}+\frac{1}{x+n+5}+\&c\right)$$

$$= \frac{n}{x+}\,\frac{1^2(1^2-n^2)}{3x+}\,\frac{2^2(2^2-n^2)}{5x+}\,\frac{3^2(3^2-n^2)}{7x+}\,\&c.$$

Cor. $$2\left\{\frac{1}{(x+1)^2}+\frac{1}{(x+3)^2}+\frac{1}{(x+5)^2}+\&c\right\} = \frac{1}{x+}\,\frac{1^4}{3x+}\,\frac{2^4}{5x+}\,\frac{3^4}{7x}+\&c$$

31. $$\left(\frac{1}{x-n+1} - \frac{1}{x-n+3} + \frac{1}{x-n+5} - \&c\right)$$
$$-\left(\frac{1}{x+n+1} - \frac{1}{x+n+3} + \frac{1}{x+n+5} - \&c\right)$$
$$= \frac{n}{x^2-1+}\,\frac{2^2-n^2}{1+}\,\frac{2^2}{x^2-1+}\,\frac{4^2-n^2}{1+}\,\frac{4^2}{x^2-1}+\&c$$

Cor. $$2\left\{\frac{1}{(x+1)^2} - \frac{1}{(x+3)^2} + \frac{1}{(x+5)^2} - \&c\right\}$$
$$= \frac{1}{x^2-1+}\,\frac{2^2}{1+}\,\frac{2^2}{x^2-1+}\,\frac{4^2}{1+}\,\frac{4^2}{x^2-1}+\&c.$$

32. i. $$2x\left(\frac{1}{2x} - \frac{1}{x+2} + \frac{1}{x+4} - \frac{1}{x+6} + \&c\right)$$
$$= \frac{1}{x+}\,\frac{1.2}{x+}\,\frac{2.3}{x+}\,\frac{3.4}{x+}\,\frac{4.5}{x+}\,\&c$$

ii. $$2x^2\left\{\frac{1}{2x^2} - \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} - \frac{1}{(x+3)^2} + \&c\right\}$$
$$= \frac{1}{x+}\,\frac{1.2}{x+}\,\frac{1.2}{x+}\,\frac{2^2}{x+}\,\frac{2.3}{x+}\,\frac{3^2}{x+}\,\&c$$

iii. $$\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \&c$$
$$= \frac{1}{2x(x+1)+}\,\frac{1^3}{1+}\,\frac{1^3}{6x(x+1)+}\,\frac{2^3}{1+}\,\frac{2^3}{10x(x+1)}+\&c$$
$$= \frac{1}{2x^2+2x+1-}\,\frac{1^6}{3(2x^2+2x+3)-}\,\frac{2^6}{5(2x^2+2x+5)-}\,\frac{3^6}{7(2x^2+2x+7)}$$
$$-\&c.$$

33. $$\frac{\left(\frac{x+m+n-1}{2}\right)!\left(\frac{x-m-n-1}{2}\right)! - \left(\frac{x+m-n-1}{2}\right)!\left(\frac{x-m+n-1}{2}\right)!}{\left(\frac{x+m+n-1}{2}\right)!\left(\frac{x-m-n-1}{2}\right)! + \left(\frac{x+m-n-1}{2}\right)!\left(\frac{x-m+n-1}{2}\right)!}$$
$$= \frac{mn}{x+}\,\frac{(m^2-1^2)(n^2-1^2)}{3x+}\,\frac{(m^2-2^2)(n^2-2^2)}{5x+}\,\frac{(m^2-3^2)(n^2-3^2)}{7x}+\&c$$

34. If $$P = \frac{\left(\frac{x+l+n-3}{4}\right)!\left(\frac{x+l-n-3}{4}\right)!\left(\frac{x-l+n-1}{4}\right)!\left(\frac{x-l-n-1}{4}\right)!}{\left(\frac{x-l+n-3}{4}\right)!\left(\frac{x-l-n-3}{4}\right)!\left(\frac{x+l+n-1}{4}\right)!\left(\frac{x+l-n-1}{4}\right)!}$$

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then $\dfrac{1-P}{1+P} = \dfrac{l}{x+}\,\dfrac{1^2-n^2}{x+}\,\dfrac{2^2-l^2}{x+}\,\dfrac{3^2-n^2}{x+}\,\dfrac{4^2-l^2}{x+}$ &c

Cor. If $F(\alpha,\beta) = \tan^{-1}\dfrac{\alpha}{x+}\,\dfrac{\beta^2+\gamma^2}{x+}\,\dfrac{\alpha^2+(2\gamma)^2}{x+}\,\dfrac{\beta^2+(3\gamma)^2}{x+}$ &c and A be the average of α & β, then $F(A, A)$ is the average of $F(\alpha,\beta)$ and $F(\beta,\alpha)$

35. If $P = \dfrac{\left|\underline{\tfrac{x+l+m+n-1}{2}}\right.\left|\underline{\tfrac{x+l-m-n-1}{2}}\right.\left|\underline{\tfrac{x+m-n-l-1}{2}}\right.\left|\underline{\tfrac{x+n-l-m-1}{2}}\right.}{\left|\underline{\tfrac{x-l-m-n-1}{2}}\right.\left|\underline{\tfrac{x-l+m+n-1}{2}}\right.\left|\underline{\tfrac{x-m+n+l-1}{2}}\right.\left|\underline{\tfrac{x-n+l+m-1}{2}}\right.}$

then $\dfrac{1-P}{1+P} = \dfrac{2lmn}{x^2-l^2-m^2-n^2+1+}\,\dfrac{4(l^2-1^2)(m^2-1^2)(n^2-1^2)}{3(x^2-l^2-m^2-n^2+5)+}\,\dfrac{4(l^2-2^2)(m^2-2^2)(n^2-2^2)}{5(x^2-l^2-m^2-n^2+9)+}$ &c

$= \dfrac{2lmn}{y+t-2l^2m+}\,\dfrac{2(1-m)(1^2-n^2)}{1+}\,\dfrac{2(1+m)(1^2-l^2)}{3y+t+}\,\dfrac{2(2-m)(2^2-n^2)}{1+}\,\dfrac{2(2+m)(2^2-l^2)}{5y+t+}$ &c where $y = x^2-(1-m)^2$ & $t = (n^2-l^2)(1-2m)$

36. If $P = \dfrac{\left|\underline{\tfrac{x+l+n-1}{4}}\right.\left|\underline{\tfrac{x+l-n-3}{4}}\right.\left|\underline{\tfrac{x-l+n-3}{4}}\right.\left|\underline{\tfrac{x-l-n-1}{4}}\right.}{\left|\underline{\tfrac{x-l+n-1}{4}}\right.\left|\underline{\tfrac{x-l-n-3}{4}}\right.\left|\underline{\tfrac{x+l-n-1}{4}}\right.\left|\underline{\tfrac{x+l+n-3}{4}}\right.}$

then $\dfrac{1-P}{1+P} = \dfrac{ln}{x^2-1-l^2+}\,\dfrac{2^2-n^2}{1+}\,\dfrac{2^2-l^2}{x^2-1+}\,\dfrac{4^2-n^2}{1+}\,\dfrac{4^2-l^2}{x^2-1+}$ &c.

37. If $\phi(y) = \dfrac{1}{y+1}+\dfrac{1}{y+3}+\dfrac{1}{y+5}+$ &c, then

$\phi(x-l-n) - \phi(x+l-n) + \phi(x+l+n) - \phi(x-l+n)$

$= \dfrac{2ln}{x^2-1+n^2-l^2+}\,\dfrac{2(1^2-n^2)}{1+}\,\dfrac{2(1^2-l^2)}{3(x^2-1)+n^2-l^2+}\,\dfrac{4(2^2-n^2)}{1+}$

$$\frac{4(2^2-l^2)}{5(x^2-1)+n^2-l^2+\&c}$$

38. $\left\{\frac{1}{(x-n+1)^2}+\frac{1}{(x-n+3)^2}+\frac{1}{(x-n+5)^2}+\frac{1}{(x-n+7)^2}+\&c\right\}$

$$-\left\{\frac{1}{(x+n+1)^2}+\frac{1}{(x+n+3)^2}+\frac{1}{(x+n+5)^2}+\frac{1}{(x+n+7)^2}+\&c\right\}$$

$$=\frac{n}{x^2-1+n^2+}\;\frac{2(1^2-n^2)}{1+}\;\frac{2}{3(x^2-1)+n^2+}\;\frac{4(2^2-n^2)}{1+}\;\&c$$

$$=\frac{n}{x^2-n^2+1-}\;\frac{4(1^2-n^2)1^4}{3(x^2-n^2+5)-}\;\frac{4(2^2-n^2)2^4}{5(x^2-n^2+9)-}\;\&c$$

39. $$\frac{\left\lfloor\frac{x+l+n-3}{4}\right.\;\left\lfloor\frac{x-l+n-3}{4}\right.\;\left\lfloor\frac{x+l-n-3}{4}\right.\;\left\lfloor\frac{x-l-n-3}{4}\right.}{\left\lfloor\frac{x+l+n-1}{4}\right.\;\left\lfloor\frac{x-l+n-1}{4}\right.\;\left\lfloor\frac{x+l-n-1}{4}\right.\;\left\lfloor\frac{x-l-n-1}{4}\right.}$$

$$=\frac{8}{\frac{x^2-l^2+n^2-1}{2}+}\;\frac{1^2-n^2}{1+}\;\frac{1^2-l^2}{x^2-1+}\;\frac{3^2-n^2}{1+}\;\frac{3^2-l^2}{x^2-1}+\&c$$

40. If $P=\left\lfloor\frac{\alpha+\beta+\gamma+\delta+\epsilon-1}{2}\right.\;\left\lfloor\frac{\alpha+\beta+\gamma-\delta-\epsilon-1}{2}\right.\times$

$$\left\lfloor\frac{\alpha+\beta-\gamma-\delta+\epsilon-1}{2}\right.\;\left\lfloor\frac{\alpha-\beta-\gamma+\delta+\epsilon-1}{2}\right.\;\left\lfloor\frac{\alpha-\beta+\gamma+\delta-\epsilon-1}{2}\right.\times$$

$$\left\lfloor\frac{\alpha-\beta+\gamma-\delta+\epsilon-1}{2}\right.\;\left\lfloor\frac{\alpha+\beta-\gamma+\delta-\epsilon-1}{2}\right.\;\left\lfloor\frac{\alpha-\beta-\gamma-\delta-\epsilon-1}{2}\right..$$

and $Q=\left\lfloor\frac{\alpha+\beta+\gamma+\delta-\epsilon-1}{2}\right.\;\left\lfloor\frac{\alpha+\beta+\gamma-\delta+\epsilon-1}{2}\right.\times$

$$\left\lfloor\frac{\alpha+\beta-\gamma+\delta+\epsilon-1}{2}\right.\;\left\lfloor\frac{\alpha-\beta+\gamma+\delta+\epsilon-1}{2}\right.\;\left\lfloor\frac{\alpha+\beta-\gamma-\delta-\epsilon-1}{2}\right.\times$$

$$\left\lfloor\frac{\alpha-\beta+\gamma-\delta-\epsilon-1}{2}\right.\;\left\lfloor\frac{\alpha-\beta-\gamma+\delta-\epsilon-1}{2}\right.\;\left\lfloor\frac{\alpha-\beta-\gamma-\delta+\epsilon-1}{2}\right.,$$ then

$$\frac{P-Q}{P+Q}=\frac{8\alpha\beta\gamma\delta\epsilon}{\left\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\epsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2-1)^2-2^2\right\}\div}$$

$$\frac{64(\alpha^2-1)(\beta^2-1)(\gamma^2-1)(\delta^2-1)(\epsilon^2-1)}{3\left\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\epsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2-5)^2-6^2\right\}+}$$

$$\frac{64(\alpha^2-2^2)(\beta^2-2^2)(\gamma^2-2^2)(\delta^2-2^2)(\epsilon^2-2^2)}{5\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\epsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2-9)^2-10^2\}+\&c}$$

N. B. If any one of $\alpha, \beta, \gamma, \delta, \epsilon$ be an integer the theorem is true. The result will be permanently true if α is removed from the numerators or if it is expanded in powers of $\frac{1}{\alpha}$."

41. $$1+\frac{\beta}{\gamma+1}x+\frac{\beta(\beta-1)}{(\gamma+1)(\gamma+2)}x^2+\&c = \frac{\lfloor\underline{\beta}\,\lfloor\underline{\gamma}}{\lfloor\underline{\beta+\gamma}}\cdot\frac{(1+x)^{\beta+\gamma}}{x^\gamma} - \frac{\gamma}{(\beta+1)x+1-\gamma-}\ \frac{1(1-\gamma)(1+x)}{(\beta+2)x+3-\gamma-}\ \frac{2(2-\gamma)(1+x)}{(\beta+3)x+5-\gamma-}\ \&c.$$

42. $$1+\frac{x}{n+1}+\frac{x^2}{(n+1)(n+2)}+\frac{x^3}{(n+1)(n+2)(n+3)}+\&c$$

$$= \frac{e^x\lfloor\underline{n}}{x^n} - \frac{n}{x+}\ \frac{1-n}{1+}\ \frac{1}{x+}\ \frac{2-n}{1+}\ \frac{2}{x+}\ \frac{3-n}{1+}\ \&c.$$

$$= \frac{e^x\lfloor\underline{n}}{x^n} - \frac{n}{x+1-n-}\ \frac{1(1-n)}{x+3-n-}\ \frac{2(2-n)}{x+5-n-}\ \frac{3(3-n)}{x+7-n-}\ \&c.$$

Cor. $$\frac{1}{n}-\frac{x}{\lfloor\underline{1}}\cdot\frac{1}{n+1}+\frac{x^2}{\lfloor\underline{2}}\cdot\frac{1}{n+2}-\frac{x^3}{\lfloor\underline{3}}\cdot\frac{1}{n+3}+\&c$$

$$= \frac{\lfloor\underline{n-1}}{x^n} - \frac{e^{-x}}{x+}\ \frac{1-n}{1+}\ \frac{1}{x+}\ \frac{2-n}{1+}\ \frac{2}{x+}\ \&c.$$

43. $$1+\frac{x}{1.3}+\frac{x^2}{1.3.5}+\frac{x^3}{1.3.5.7}+\frac{x^4}{1.3.5.7.9}+\&c.$$

$$= \sqrt{\frac{\pi}{2x}}\,e^x - \frac{1}{x+}\ \frac{1}{1+}\ \frac{2}{x+}\ \frac{3}{1+}\ \frac{4}{x+}\ \frac{5}{1+}\ \&c$$

$$= \sqrt{\frac{\pi}{2x}}\,e^x - \frac{1}{x+1-}\ \frac{1.2}{x+5-}\ \frac{3.4}{x+9-}\ \frac{5.6}{x+13-}\ \&c$$

Cor. 1. $$\int_0^x e^{-x^2}dx = \frac{\sqrt{\pi}}{2} - \frac{e^{-x^2}}{2x+}\ \frac{1}{x+}\ \frac{2}{2x+}\ \frac{3}{x+}\ \frac{4}{2x+}\ \&c.$$

2. $$\int_0 \frac{\int_0 e^{-x^2}dx}{x}\,dx = \frac{\sqrt{\pi}}{2}\left(\frac{C}{2}+\log 2x\right)$$ when x is very great.

44. $$\int_0^x \frac{1-e^{-x}}{x}\,dx = \frac{x}{1\lfloor\underline{1}} - \frac{x^2}{2\lfloor\underline{2}} + \frac{x^3}{3\lfloor\underline{3}} - \&c = C+\log x+e^{-x}\phi(x).$$

i. $$\phi(x) = \frac{1}{x} - \frac{\lfloor\underline{1}}{x^2} + \frac{\lfloor\underline{2}}{x^3} - \frac{\lfloor\underline{3}}{x^4} + \&c$$

ii. $\phi(x)$ lies between $\frac{1}{x}$ & $\frac{1}{x+1}$ and very nearly equals $\sqrt{\frac{\phi(x+1)}{x}}$

iii. $\phi(x) = \frac{1}{x+}\,\frac{1}{1+}\,\frac{1}{x+}\,\frac{2}{1+}\,\frac{2}{x+}\,\frac{3}{1+}\,\frac{3}{x+}$ &c

$= \frac{1}{x+1-}\,\frac{1^2}{x+3-}\,\frac{2^2}{x+5-}\,\frac{3^2}{x+7-}$ &c

iv. $\phi(x) = \frac{1}{x} - \frac{\underline{|1}}{x^2} + \frac{\underline{|2}}{x^3} - \frac{\underline{|3}}{x^4} + \dots \pm \frac{\underline{|n}}{x^n}\cdot\frac{1}{x+n+1-}\,\frac{1(1+n)}{x+n+3-}\,\frac{2(2+n)}{x+n+5-}\,\frac{3(3+n)}{x+n+7-}$ &c

Cor. 1. $\frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}}(1+\tfrac{1}{2}) + \frac{x^3}{\underline{|3}}(1+\tfrac{1}{2}+\tfrac{1}{3}) +$ &c $= e^x(C_0 + \log x) + \phi(x)$.

Cor. 2. If $\int_0^{n(1-h)} \frac{1-e^{-x}}{x}\,dx = C + \log n$, then

$\phi(n) = h(e^n - 1) + \frac{h^2}{\underline{|2}}(e^n - 1 - \frac{n}{\underline{|1}}) + \frac{h^3}{\underline{|3}}(e^n - 1 - \frac{n}{\underline{|1}} - \frac{n^2}{\underline{|2}}) +$ &c

$\phi(1) = \cdot 5963474$; $\phi($[illegible]$) = \cdot 9229106$.

45. i. Den^r. in $\frac{1}{1+}\,\frac{x}{1+}\,\frac{x}{1+}\,\frac{2x}{1+}\,\frac{2x}{1+}\,\frac{3x}{1+}\dots + \frac{(n-1)x}{1+}\,\frac{nx}{1}$.

$= 1 + \frac{n^2}{\underline{|1}}x + \frac{n^2(n-1)^2}{\underline{|2}}x^2 + \frac{n^2(n-1)^2(n-2)^2}{\underline{|3}}x^3 +$ &c.

ii. Den^r. in $\frac{1}{1+}\,\frac{x}{1+}\,\frac{x}{1+}\,\frac{2x}{1+}\,\frac{2x}{1+}\dots + \frac{(n-1)x}{1+}\,\frac{(n-1)x}{1}$.

$= 1 + \frac{n^2}{\underline{|1}}(1-\frac{1}{n})x + \frac{n^2(n-1)^2}{\underline{|2}}(1-\frac{2}{n})x^2 + \frac{n^2(n-1)^2(n-2)^2}{\underline{|3}}(1-\frac{3}{n})x^3 +$ &c

46. i. $\frac{x}{1^n\underline{|1}} - \frac{x^2}{2^n\underline{|2}} + \frac{x^3}{3^n\underline{|3}} -$ &c $= \phi_n(x) + (-1)^{n-1}\psi_n(x)e^{-x}$,

where $\phi_n(x)$ is the term independent of p in $\frac{x^p\,\underline{|-p}}{p^n}$

and $\psi_n(x) - \psi'_n(x) = \frac{\psi_{n-1}(x)}{x}$.

ii. $\phi_n(x) = \frac{1}{\underline{|n}}\Big\{A_0(\log x)^n + \frac{n}{\underline{|1}}A_1(\log x)^{n-1} + \frac{n(n-1)}{\underline{|2}}A_2(\log x)^{n-2}$

$+ \dots + A_n\Big\}$ where $\underline{|x} = A_0 - A_1\frac{x}{\underline{|1}} + A_2\frac{x^2}{\underline{|2}} - A_3\frac{x^3}{\underline{|3}} +$ &c

$A_n = S_1 A_{n-1} + (n-1)S_2 A_{n-2} + (n-1)(n-2)S_3 A_{n-3} +$ &c

iii. $\underline{|x} = 1 - \cdot 5772156649\,x + \cdot 9890560173\,x^2 -$

$$\cdot 9074790803\, x^3 + \cdot 9817280965\, \frac{x^4}{1+\theta x}$$

$\theta_0 = 1{\cdot}00027$; $\theta_1 = \frac{51}{52}$; $\theta_2 = \frac{77}{82}$; $\theta_6 = \frac{5}{68}$; $\theta_7 = -\frac{1}{38}$ nearly.

iv. $$\psi_n(x) = \frac{x}{\left(x + \frac{n}{2+}\,\frac{5n+10}{6x+}\,\frac{41n+58}{10+}\,\&c\right)^{n+1}}$$

ex. $$\int \frac{\int \frac{1-e^{-x}}{x}\,dx}{x}\,dx - \frac{1}{2}\left\{\int \frac{1-e^{-x}}{x}\,dx\right\}^2 = \frac{\pi^2}{12} \text{ when } x \text{ is great.}$$

47. $$\int_0^\infty e^{-x}\left(1+\frac{x}{n}\right)^n dx = 1 + \frac{n}{1+}\,\frac{1(n-1)}{3+}\,\frac{2(n-2)}{5+}\,\frac{3(n-3)}{7+}\,\&c$$

$$= 2 + \frac{n-1}{2+}\,\frac{1(n-2)}{4+}\,\frac{2(n-3)}{6+}\,\frac{3(n-4)}{8+}\,\&c.$$

$$= \frac{e^n \underline{|n}}{n^n} - \frac{2n}{2+}\,\frac{3n}{3+}\,\frac{4n}{4+}\,\frac{5n}{5+}\,\&c$$

48. $$\int_0^\infty e^{-x}\left(1+\frac{x}{n}\right)^n dx = \frac{e^n \underline{|n}}{2n^n} + \frac{2}{3} - \frac{4}{135n} + \frac{8}{27.105n^2}$$

$$+ \frac{16}{105.81\, n^3} - \frac{32.281}{3^8.5^2.7.11\, n^4} - \&c.$$

Cor. $$1 + \frac{n}{\underline{|1}} + \frac{n^2}{\underline{|2}} + \cdots + \frac{n^n}{\underline{|n}}\theta = \frac{e^n}{2}.$$

where $\theta = \frac{4+15n}{8+45n}$ very nearly.

N. B.

	Real value of θ	App. value of θ.
$n=0$	$\cdot 50000$	$\cdot 50000$
$n=\frac{1}{2}$	$\cdot 37750$	$\cdot 37705$
$n=1$	$\cdot 35914$	$\cdot 35849$
$n=1\frac{1}{2}$	$\cdot 35146$	$\cdot 35099$
$n=2$	$\cdot 34726$	$\cdot 34694$
$n=\infty$	$\cdot 33333$	$\cdot 33333$.

49. $$C_0 + \log n + \frac{n}{1\underline{|1}} + \frac{n^2}{2\underline{|2}} + \frac{n^3}{3\underline{|3}} + \frac{n^4}{4\underline{|4}} + \&c$$

$$= e^n\left(\frac{1}{n} + \frac{\underline{|1}}{n^2} + \frac{\underline{|2}}{n^3} + \cdots + \frac{\underline{|n-1}}{n^n}\theta\right)$$

where $\theta = \frac{2}{3} + \frac{4}{135n} + \frac{8}{27.105n^2} - \&c.$

CHAPTER XIII

1. If N be the integer just greater than n or equal to n,

$$\int_0^\infty \frac{A_0 - A_1 x + A_2 x^2 - A_3 x^3 + \&c}{x^{n+1}}\, dx = \cos \pi N \int_0^\infty \frac{A_N x^N - A_{N+1} x^{N+1} + \&c}{x^{n+1}}\, dx$$

e.g. $\int_0^\infty \frac{e^{-x^2}}{x^4}\, dx = \frac{2}{3}\sqrt{\pi}$ really means that

$$\int_0^\infty \frac{e^{-x^2} - 1 + x^2}{x^4}\, dx = \frac{2}{3}\sqrt{\pi}.$$

Cor. Thus the meanings of the integrals $\int_0^\infty e^{-ax} x^{n-1} \begin{matrix}\cos bx \\ \sin bx\end{matrix}\, dx$

$$= \frac{\underline{|n-1}}{(a^2+b^2)^{\frac{n}{2}}} \begin{matrix}\cos(n \tan^{-1}\frac{b}{a}) \\ \sin(n \tan^{-1}\frac{b}{a})\end{matrix}$$ for negative values of n are known.

2. i. $$\int \phi(x) e^{-nx}\, dx = -e^{-nx}\left\{\frac{\phi(x)}{n} + \frac{\phi'(x)}{n^2} + \frac{\phi''(x)}{n^3} + \&c\right\}$$

ii. $$\int \phi(x) \cos nx\, dx = \sin nx \left\{\frac{\phi(x)}{n} - \frac{\phi''(x)}{n^3} + \&c\right\} + \cos nx \left\{\frac{\phi'(x)}{n^2} - \frac{\phi'''(x)}{n^4} + \&c\right\}$$

iii. $$\int \phi(x) \sin nx\, dx = \sin nx \left\{\frac{\phi'(x)}{n^2} - \frac{\phi'''(x)}{n^4} + \&c\right\} - \cos nx \left\{\frac{\phi(x)}{n} - \frac{\phi''(x)}{n^3} + \&c\right\}$$

3. $$\int_x^\infty e^{-x^2} \cos 2nx\, dx = e^{-x^2}\left\{\frac{\cos(2nx+\theta)}{2r} - \frac{1 \cos(2nx+3\theta)}{2^2 r^3} + \frac{1.3 \cos(2nx+5\theta)}{2^3 r^5} - \frac{1.3.5 \cos(2nx+7\theta)}{2^4 r^7} + \&c\right\}$$

where $\tan\theta = \frac{n}{x}$ and $r = \sqrt{n^2+x^2}$.

4. $$\int_0^\infty e^{-x^2}\left\{e^{2nx}\phi(x) + e^{-2nx}\phi(-x)\right\} dx$$

$$= \int_0^\infty e^{n^2-x^2}\left\{\phi(n+x) + \phi(n-x)\right\} dx =$$

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$$\sqrt{\pi}\, e^{n^2} \left\{ \phi(n) + \frac{\phi''(n)}{4} + \frac{\phi^{iv}(n)}{4.8} + \frac{\phi^{vi}(n)}{4.8.12} + \&c \right\}$$

5. $$\int_0^\infty e^{-\frac{1}{x^2}} \left\{ A_1 - \frac{x^2}{\lfloor 2} A_3 + \frac{x^4}{\lfloor 2} A_5 - \&c \right\} dx$$

$$= \frac{\sqrt{\pi}}{2} \left\{ A_0 - \frac{2}{\lfloor 1} A_1 + \frac{2^2}{\lfloor 2} A_2 - \frac{2^3}{\lfloor 3} A_3 + \&c \right\}$$

6. $$\int_0^\infty e^{-x} \left(1 + \frac{x}{n}\right)^{n-h} dx = 1 + \left(1 - \frac{h}{n}\right) + \left(1 - \frac{h}{n}\right)\left(1 - \frac{h+1}{n}\right)$$

$$+ \left(1 - \frac{h}{n}\right)\left(1 - \frac{h+1}{n}\right)\left(1 - \frac{h+2}{n}\right) + \&c$$

$$= \frac{e^n \lfloor n-h}{2 n^{n-h}} + A_0 - \frac{A_1}{n} + \frac{A_2}{n^2} - \&c \quad \text{where}$$

$$A_0 = \frac{2}{3} - h; \quad A_1 = \frac{4}{135} - \frac{h^2(1-h)}{3};$$

$$A_2 = \frac{8}{2835} + \frac{2h(1-h)}{135} - \frac{h(1-h^2)(2-3h^2)}{45} \quad \&c.$$

7. $$(m-n-1) \int_0^\infty \frac{\left(1 + \frac{x}{n}\right)^n}{\left(1 + \frac{x}{m}\right)^m} dx = \frac{m}{2} \cdot \frac{m^m \lfloor n}{n^n \lfloor m} \cdot \frac{\lfloor m-n}{(m-n)^{m-n}}$$

$$+ \frac{2}{3}(m+n) - \frac{4(m+n)(m-2n)(m-\frac{n}{2})}{135\, mn(m-n)}$$

$$+ \frac{8(m^3+n^3)(m-2n)(m-\frac{n}{2})}{2835\, m^2 n^2 (m-n)^2}$$

$$+ \frac{16(m^3+n^3)(m-2n)(m-\frac{n}{2})(m^2-mn+n^2)}{8505\, m^3 n^3 (m-n)^3} - \&c.$$

8. $$\int_0^\infty \left\{ \frac{n^x \lfloor n}{\lfloor n+x} + e^{-x}\left(1 + \frac{x}{n}\right)^n \right\} dx = \frac{e^n \lfloor n}{n^n} + \frac{6n}{12n+1}$$

very very nearly.

9. If $\int_0^\infty \frac{e^{-m^2 x^2}}{1+x^2} dx = \phi(m)$ and if $m \measuredangle n$, then

$$\int_0^\infty \frac{e^{-m^2x^2}}{1+x^2}\cos 2mnx\,dx = \frac{e^{-n^2}}{2}\left\{\phi(m+n)+\phi(m-n)\right\}.$$

10. $$1+\frac{\phi(h.\overline{\alpha+\delta})}{\phi(h.\overline{\beta+\gamma})}+\frac{\phi(h.\overline{\alpha+\delta})}{\phi(h.\overline{\beta+\gamma})}\cdot\frac{\phi(h.\overline{\alpha+2\delta})}{\phi(h.\overline{\beta+2\gamma})}+\&c$$

$$=\sqrt{\frac{\pi.\phi(0)}{2h(\gamma-\delta)\phi'(0)}}+\frac{1}{3}\cdot\frac{\gamma+\delta}{\gamma-\delta}\left\{1-\frac{\phi(0)}{\phi'(0)}\cdot\frac{\phi''(0)}{\phi'(0)}\right\}+\frac{\alpha-\beta}{\gamma-\delta}$$

if h is very small.

Cor. i. $$1+\left(\frac{x}{x+1}\right)^n+\left\{\frac{x^2}{(x+1)(x+2)}\right\}^n+\left\{\frac{x^3}{(x+1)(x+2)(x+3)}\right\}^n+\&c$$

$$=\sqrt{\frac{\pi x}{2n}}+\frac{1}{3n}$$ when x is very great

ii. $$1+\left(\frac{x}{\lfloor 1}\right)^n+\left(\frac{x^2}{\lfloor 2}\right)^n+\left(\frac{x^3}{\lfloor 3}\right)^n+\&c$$

$$=\frac{e^{nx+\frac{n^2-1}{24}\left(\frac{1}{nx}+\frac{1}{2n^2x^2}+\&c\right)}}{\sqrt{n}\,.\,(2\pi x)^{\frac{n-1}{2}}}$$

11. i $$1+\left(\frac{en}{1}\right)+\left(\frac{en}{2}\right)^2+\left(\frac{en}{3}\right)^3+\left(\frac{en}{4}\right)^4+\&c$$

$$=\sqrt{2\pi n}\;e^{n-\frac{1}{24n}-\frac{1}{48n^2}-\left(\frac{1}{36}+\frac{1}{5760}\right)\frac{1}{n^3}-\&c}$$ if n is great

ii. $$\int_0^\infty \frac{x^{n-1}\,dx}{1+\frac{x}{1}+\left(\frac{x}{2}\right)^2+\left(\frac{x}{3}\right)^3+\left(\frac{x}{4}\right)^4+\&c}=n^n\left(\frac{1}{n}+\frac{1}{2n^2}+\frac{1}{3n^3}+\frac{3}{8n^4}+\&c\right)$$ if n is great.

iii. $$\log 2\left(\frac{1}{2\log 2}-\frac{1}{3\log 3}+\frac{1}{4\log 4}-\frac{1}{5\log 5}+\&c\right)$$
$$+(\log 2)^2\left(\frac{1}{2\log 2\log 4}+\frac{1}{3\log 3\log 6}+\frac{1}{4\log 4\log 8}+\&c\right)=1.$$

12. The approximate value of $e^{-x}\left\{\phi(0)+\frac{x}{\lfloor 1}\phi(1)+\frac{x^2}{\lfloor 2}\phi(2)+\&c\right\}$ when x is great can be found by successive differentiation and by transforming the result applying III & ex. 1. if necessary

e.g. $\log 1 + \frac{x}{\underline{|1}}\log 2 + \frac{x^2}{\underline{|2}}\log 3 + \frac{x^3}{\underline{|3}}\log 4 + \&c$

$= e^x\left(\log x + \frac{1}{2x} + \frac{1}{12x^2} + \frac{1}{12x^3} + \frac{19}{120x^4} + \frac{9}{20}x^5 + \&c\right)$

13. $$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)(x^2+d^2)}$$

$$= \frac{\pi}{6}\,\frac{(a+b+c+d)^3 - (a^3+b^3+c^3+d^3)}{abcd(a+b)(b+c)(c+a)(a+d)(b+d)(c+d)}.$$

Cor. If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 - px^3 + qx^2 - rx + s = 0$

then $$\int_0^\infty \frac{dx}{(x^2+\alpha^2)(x^2+\beta^2)(x^2+\gamma^2)(x^2+\delta^2)} = \frac{\pi}{2s}\cdot\frac{1}{r - \frac{ps}{q - \frac{r}{p}}}$$

14. $$\frac{1}{a+\frac{x^2}{a}} - \frac{2a}{\underline{|1}}\cdot\frac{1}{a+1+\frac{x^2}{a+1}} + \frac{2a(2a+1)}{\underline{|2}}\cdot\frac{1}{a+2+\frac{x^2}{a+2}} - \&c$$

$$= \frac{\lambda\left(\underline{|a-1}\right)^2 / 2\underline{|2a-1}}{\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+2}\right)^2\right\}\left\{1+\left(\frac{x}{a+3}\right)^2\right\}\&c} \quad \text{and}$$

$\int_0^\infty \frac{\cos nx}{a+\frac{x^2}{a}}\,dx = \frac{\pi}{2}e^{-na}$; combining these results

15. $$\int_0^\infty \frac{\cos 2nx\,dx}{\left\{1+\left(\frac{x}{a}\right)^2\right\}\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+2}\right)^2\right\}\left\{1+\left(\frac{x}{a+3}\right)^2\right\}\&c} = \frac{\sqrt{\pi}}{2}\cdot\frac{\underline{|a-\frac{1}{2}}}{\underline{|a-1}}\,\mathrm{Sech}^{2a}$$

16. i $$\int_0^\infty \frac{\sinh ax}{\sinh \pi x}\cos nx\,dx = \frac{1}{2}\cdot\frac{\sin a}{\cosh n + \cos a}$$

ii. $$\int_0^\infty \frac{\cosh ax}{\sinh \pi x}\sin nx\,dx = \frac{1}{2}\cdot\frac{\sinh n}{\cosh n + \cos a}.$$

iii. $$\int_0^\infty \frac{\sin nx}{e^{2\pi x}-1}\,dx = \frac{1}{2}\left(\frac{1}{e^n-1} + \frac{1}{2} - \frac{1}{n}\right).$$

iv. $$\int_0^\infty \frac{x^{n-1}}{e^{2\pi x}-1}\,dx = \frac{B_n}{2n}. \qquad \int_0^\infty \frac{x^{n-1}}{\cosh\frac{\pi x}{2}}\,dx = E_n$$

17. $\phi(1) + \phi(2) + \phi(3) + \cdots + \phi(n)$

$$= \int^{n} \phi(x)\,dx + \tfrac{1}{2}\phi(n) \mp \int_0^\infty \frac{\phi(n+xi) - \phi(n-xi)}{i\,(e^{2\pi x}-1)}\,dx.$$

Cor. $\log \lfloor n = n\log n - n + \tfrac{1}{2}\log(2\pi n) + 2\int_0^\infty \frac{\tan^{-1}\frac{x}{n}}{e^{2\pi x}-1}\,dx$

18. i If $f(x) + \phi(x) = f(x+h)$, then

$$f(x) + \tfrac{1}{2}\phi(x) = \tfrac{1}{h}\int^{x}\phi(\xi)\,d\xi + 2\int_0^\infty \frac{\phi(x+ih\xi) - \phi(x-ih\xi)}{(e^{2\pi\xi}-1)\,i}\,d\xi$$

ii. If $f(x+h) + f(x-h) = \phi(x)$, then

$$2f(x) = \int_0^\infty \frac{\phi(x+ih\xi) + \phi(x-ih\xi)}{e^{\frac{\pi\xi}{2}} + e^{-\frac{\pi\xi}{2}}}\,d\xi$$

19. i. If $\int_0^{h} \phi(x)\cos nx\,dx = \psi(n)$ $\qquad m <=> h$

then $\int_0^\infty \psi(x)\cos mx\,dx = \frac{\pi}{2}\phi(m),\ \frac{\pi}{4}\phi(m),\ 0$

ii If $\int_0^{h} \phi(x)\sin nx\,dx = \psi(n)$

$m <=> h$

then $\int_0^\infty \psi(x)\sin mx\,dx = \frac{\pi}{2}\phi(m),\ \frac{\pi}{4}\phi(m),\ 0$

Cor. $\int_0^\infty \operatorname{Sech}^{2a} x \cos 2nx\,dx$

$$= \frac{\sqrt{\pi}\,\lfloor a-1 / 2\lfloor a-\frac{1}{2}}{\left\{1+\left(\frac{n}{a}\right)^2\right\}\left\{1+\left(\frac{n}{a+1}\right)^2\right\}\left\{1+\left(\frac{n}{a+2}\right)^2\right\}\ \&c}$$

20. $\int_0^\infty \frac{\sinh ax}{\sinh \pi x}\cdot\frac{dx}{1+n^2x^2}$ (a lying between 0 and π)

$$= \frac{\sin a}{1+n} - \frac{\sin 2a}{1+2n} + \frac{\sin 3a}{1+3n} - \frac{\sin 4a}{1+4n} + \&c$$

21. If $\int_{\alpha_1}^{\beta_1} \phi_1(p, x)\, F(nx)\, dx = \psi_1(p, n)$

& $\int_{\alpha_2}^{\beta_2} \phi_2(p, x)\, F(nx)\, dx = \psi_2(p, n)$, then

$$\int_{\alpha_1}^{\beta_1} \phi_1(p, x)\, \psi_2(q, nx)\, dx = \int_{\alpha_2}^{\beta_2} \phi_2(q, x)\, \psi_1(p, nx)\, dx.$$

Cor. If $\int_0^{\infty} \phi(p, x) \cos nx\, dx = \psi(p, n)$, then

$$\frac{\pi}{2} \int_0^{\infty} \phi(p, x)\, \phi(q, bx)\, dx = \int_0^{\infty} \psi(q, x)\, \psi(p, bx)\, dx$$

ex. If $\alpha\beta = \pi$, then $\sqrt{\alpha} \int_0^{\infty} \frac{e^{-x^2}}{e^{\alpha x} + e^{-\alpha x}}\, dx = \sqrt{\beta} \int_0^{\infty} \frac{e^{-x^2}}{e^{\beta x} + e^{-\beta x}}\, dx$

N.B. This can also be got from the theorem :- if $\alpha\beta = \frac{\pi}{2}$.

$$\sqrt{\alpha} \left\{ E_1 - E_3 \frac{\alpha^2}{\underline{|1}} + E_5 \frac{\alpha^4}{\underline{|2}} - \&c \right\} = \sqrt{\beta} \left\{ E_1 - E_3 \frac{\beta^2}{\underline{|1}} + E_5 \frac{\beta^4}{\underline{|2}} - \&c \right\}$$

which is obtained from the theorem :—

$$\phi(1) - \phi(2) + \phi(3) - \&c = \phi(0) - \phi(-1) + \phi(-2) - \&c.$$

22. i. $$\int_0^{\infty} \frac{1}{\left\{1 + \left(\frac{x}{a}\right)^2\right\}\left\{1 + \left(\frac{x}{a+1}\right)^2\right\}\left\{1 + \left(\frac{x}{a+2}\right)^2\right\} \&c} \cdot \frac{dx}{\left\{1 + \left(\frac{x}{b}\right)^2\right\}\left\{1 + \left(\frac{x}{b+1}\right)^2\right\}\left\{1 + \left(\frac{x}{b+2}\right)^2\right\} \&c}$$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{\underline{|a - \tfrac{1}{2}}\ \underline{|b - \tfrac{1}{2}}\ \underline{|a + b - 1}}{\underline{|a - 1}\ \underline{|b - 1}\ \underline{|a + b - \tfrac{1}{2}}}.$$

ii. $$\int_0^{\infty} \frac{1 + \left(\frac{x}{b+1}\right)^2}{1 + \left(\frac{x}{a}\right)^2} \cdot \frac{1 + \left(\frac{x}{b+2}\right)^2}{1 + \left(\frac{x}{a+1}\right)^2} \cdot \frac{1 + \left(\frac{x}{b+3}\right)^2}{1 + \left(\frac{x}{a+2}\right)^2} \&c\ dx$$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{\underline{|a - \tfrac{1}{2}}\ \underline{|b}\ \underline{|b - a - \tfrac{1}{2}}}{\underline{|a - 1}\ \underline{|b - \tfrac{1}{2}}\ \underline{|b - a}}$$

23. $$\int_0^\infty \frac{\lfloor x+a-1}{\lfloor x+a+n} \cdot \frac{dx}{x^m}$$
$$= \frac{\pi}{\lfloor n} \operatorname{cosec} \pi m \left\{ \frac{1}{a^m} - \frac{n}{\lfloor 1} \cdot \frac{1}{(a+1)^m} + \frac{n(n-1)}{\lfloor 2} \cdot \frac{1}{(a+2)^m} - \&c \right\}.$$

24. i. $A_0 + A_1 + A_2 + \cdots + A_n$
$= A_n + A_{n-1} + \&c.$ to infinity $- (A_{-1} + A_{-2} + A_{-3} + \&c)$.

Cor. $A_0 + \frac{A_1}{\lfloor 1} + \frac{A_2}{\lfloor 2} + \cdots + \frac{A_n}{\lfloor n}$
$= \frac{A_n}{\lfloor n} + \frac{A_{n-1}}{\lfloor n-1} + \&c$ ad. inf.

ii. $\phi(x) + \{\phi(x+1) + \phi(x-1)\} + \{\phi(x+2) + \phi(x-2)\} + \&c$
$= \phi(y) + \{\phi(y+1) + \phi(y-1)\} + \{\phi(y+2) + \phi(y-2)\} + \&c$

Cor. $\frac{x^h}{\lfloor h} + \left(\frac{x^{h+n}}{\lfloor h+n} + \frac{x^{h-n}}{\lfloor h-n}\right) + \left(\frac{x^{h+2n}}{\lfloor h+2n} + \frac{x^{h-2n}}{\lfloor h-2n}\right) + \&c$
$= 1 + \left(\frac{x^n}{\lfloor n} + \frac{x^{-n}}{\lfloor -n}\right) + \left(\frac{x^{2n}}{\lfloor 2n} + \frac{x^{-2n}}{\lfloor -2n}\right) + \&c = \frac{e^x}{n}.$
for all values of x, h and h. n being $\not> 1$.

iii. $$\int_{-\infty}^{\infty} \frac{\phi(x)}{\lfloor x}\, dx = \phi(0) + \frac{\phi(1)}{\lfloor 1} + \frac{\phi(2)}{\lfloor 2} + \frac{\phi(3)}{\lfloor 3} + \&c$$

Cor. 1. $\int_{-\infty}^{\infty} \frac{a^x}{\lfloor x}\, dx = e^a$. Cor. 2. $\int_{-\infty}^{\infty} \frac{a^x \lfloor n}{\lfloor x \lfloor n-x}\, dx = (1+a)^n$.

?? $n < 2\pi$ in the following examples ?

25. i $$\int_0^\infty \left(\frac{a^x}{\lfloor x} + \frac{a^{-x}}{\lfloor -x}\right) \cos nx\, dx = e^{a \cos n} \cos(a \sin n).$$

& $$\int_0^\infty \left(\frac{a^x}{\lfloor x} - \frac{a^{-x}}{\lfloor -x}\right) \sin nx\, dx = e^{a \cos n} \sin(a \sin n).$$

ii. $$\int_0^\infty \left(\frac{a^{b+x}}{\lfloor b+x} + \frac{a^{b-x}}{\lfloor b-x}\right) \cos nx\, dx = e^{a \cos n} \cos(a \sin n - b$$

158.

& $\int_0^\infty \left(\frac{a^{b+x}}{\underline{|b+x}} - \frac{a^{b-x}}{\underline{|b-x}}\right) \sin nx \, dx = e^{a\cos n} \sin(a \sin n - nb).$

N.B. i. The maximum value of $\frac{a^x}{\underline{|x}} = \frac{e^{\int \frac{x}{a} da}}{\sqrt{2\pi}}$

$$= \frac{a^{a-\frac{1}{2}}}{\underline{|a-\frac{1}{2}}} e^{\frac{1}{32a(36a^2+10\cdot 1)}} \text{ very nearly.}$$

ii. The following theorem is very useful in evaluating definite integrals:— $\int_a^b \phi(x)\,dx = h\left\{\frac{1}{2}\phi(a) + \phi(a+h) + \right.$

$\phi(a+2h) + \phi(a+3h) + \cdots + \phi(b-2h) + \phi(b-h) + \left.\frac{1}{2}\phi(b)\right\}$

$+ B_2\frac{h^2}{\underline{|2}}\left\{\phi'(a) - \phi'(b)\right\} - B_4\frac{h^4}{\underline{|4}}\left\{\phi'''(a) - \phi'''(b)\right\} + \&c.$

26. i. $\int_0^\infty \frac{\cos 2nx}{(1+x^2)^{m+1}} dx = \frac{\pi}{2}\cdot\frac{n^m}{\underline{|m}} e^{-2n}\left\{1 + \frac{m}{4}\cdot\frac{m+1}{n} + \right.$

$$+ \frac{m(m-1)}{4.8}\cdot\frac{(m+1)(m+2)}{n^2} + \frac{m(m-1)(m-2)}{4.8.12}\cdot\frac{(m+1)(m+2)(m+3)}{n^3} + \&c$$

ii $\int_0^\infty \frac{x^{2m}}{(1+x^2)^{n+1}} \cos px \, dx = \frac{\pi}{2}\cdot(-1)^m\cdot\frac{e^{-p}}{2^n\underline{|n}}\left\{p^n + A_1 p^{n-1} + A_2 p^{n-2} + \&c\right.$

where m is any positive integer and $A_r = \frac{\underline{|n+r}}{\underline{|n-r}}\cdot\frac{1}{2^r\underline{|r}} \times$

$$\left\{1 - \frac{4}{\underline{|1}}\cdot\frac{rmn}{(n+r)(n+r-1)} + \frac{4^2}{\underline{|2}}\cdot\frac{r(r-1)\,m(m-1)\,n(n-1)}{(n+r)(n+r-1)(n+r-2)(n+r-3)} - \&c\right\}$$

27. $\left\{1+\left(\frac{x}{1}\right)^n\right\}\left\{1+\left(\frac{x}{2}\right)^n\right\}\left\{1+\left(\frac{x}{3}\right)^n\right\}$ &c n being even

$$= \prod \sqrt{\frac{\cosh\left(2\pi x \sin\frac{\pi r}{n}\right) - \cos\left(2\pi x\cos\frac{\pi r}{n}\right)}{2\pi^2 x^2}} \text{ where}$$

$r = 1, 3, 5$ to $n-1$.

Cor. $\left\{1 + \left(\frac{2n}{n+1}\right)^3\right\}\left\{1 + \left(\frac{2n}{n+2}\right)^3\right\}\left\{1 + \left(\frac{2n}{n+3}\right)^3\right\}$ &c

$$= \frac{\left(\underline{|n}\right)^3}{\underline{|3n}}\cdot\frac{\sinh(\pi n\sqrt{3})}{\pi n\sqrt{3}}.$$

N.B. Thus it is possible to find the value of the product:—

$$\left\{1+\left(\frac{x}{a}\right)^3\right\}\left\{1+\left(\frac{x}{a+d}\right)^3\right\}\left\{1+\left(\frac{x}{a+2d}\right)^3\right\} \&c$$

Cor. 2. $\left\{1+\left(\frac{2n+1}{n+1}\right)^3\right\}\left\{1+\left(\frac{2n+1}{n+2}\right)^3\right\}\left\{1+\left(\frac{2n+1}{n+3}\right)^3\right\}$

$$= \cancel{\frac{(\lfloor 2n+1)^3 \; \lfloor 3n+1\tfrac{1}{2}}{\lfloor 6n+3 \; (\lfloor n+\tfrac{1}{2})^3}} \cosh\{\pi(n+\tfrac{1}{2})\sqrt{3}\} \times \frac{(\lfloor n)^3}{\pi \lfloor 3n+1}$$

28. $mn\left\{1+\left(\frac{x^n}{\lfloor n}+\frac{x^{-n}}{\lfloor -n}\right)+\left(\frac{x^{2n}}{\lfloor 2n}+\frac{x^{-2n}}{\lfloor -2n}\right)+\&c\right\}$

$$= e^x + e^{x\cos\frac{2\pi}{n}}\cos\left(x\sin\frac{2\pi}{n}\right) + e^{x\cos\frac{4\pi}{n}}\cos\left(x\sin\frac{4\pi}{n}\right)$$

$+ e^{x\cos\frac{6\pi}{n}}\cos\left(x\sin\frac{6\pi}{n}\right) + \&c.$ to mn terms where m is any arbitrary integer.

29. i $\displaystyle\int_0^\infty \frac{(-x^2)^{l}}{1+x^{2n}}\cos px\,dx = \frac{\pi}{2n}e^{-p} +$

$$\frac{\pi}{n}\sum e^{-p\cos\frac{\pi r}{n}}\cos\left\{(2l+1)\frac{\pi r}{n} - p\sin\frac{\pi r}{n}\right\}$$

where p is any quantity, l any integer, n any odd integer and $r = 1, 2, 3, 4$ up to $\frac{n-1}{2}$.

ii $\displaystyle\int_0^\infty \frac{(-x^2)^l}{1+x^{2n}}\cos px\,dx$ where n is even & $r = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ to $\frac{n-1}{2}$

$$= \frac{\pi}{n}\sum e^{-p\cos\frac{\pi r}{n}}\cos\left\{(2l+1)\frac{\pi r}{n} - p\sin\frac{\pi r}{n}\right\}.$$

30. i. $\displaystyle\int_0^\infty \frac{\sin^{2n+1}x}{x}\,dx = \int_0^\infty \frac{\sin^{2n+2}x}{x^2}\,dx = \frac{\sqrt{\pi}}{2}\cdot\frac{\lfloor n-\frac{1}{2}}{\lfloor n}.$

ii. If $\displaystyle\int_0^\infty \frac{\sin^n x}{x^p}\,dx = \phi(n,p)$, then $(p-1)(p-2)\,\phi(n,p) =$ $n(n-1)\,\phi(n-2,p-2) - n^2\phi(n,p-2)$. Thus it is possible

To find $\int_0^\infty \frac{\sin^{2n+b} x}{x^b}\,dx$ b being any integer.

Cor. 1. $\int_0^\infty \frac{\sin^{2n+3} x}{x^3}\,dx = \frac{\sqrt{\pi}}{4} \cdot \frac{|\underline{n-\frac{1}{2}}}{|\underline{n+1}} (n+1\tfrac{1}{2})$.

Cor 2. $\int_0^\infty \frac{\sin^{2n+4} x}{x^4}\,dx = \frac{\sqrt{\pi}}{6} \cdot \frac{|\underline{n-\frac{1}{2}}}{|\underline{n+1}} (n+2)$ &c &c &c.

N.B. The above theorems are obtained by combining XIII 22. i with the following theorems:—

i. $\int_0^\infty \frac{\sin^n x}{x^p}\,dx = \frac{1}{|\underline{p-1}} \int_0^\infty \int_0^\infty e^{-zx} z^{p-1} \sin^n x \, dz\, dx$.

ii. $\int_0^\infty e^{-ax} \sin^{2n+1} x\,dx = \frac{|\underline{2n+1}}{(a^2+1^2)(a^2+3^2)(a^2+5^2)\cdots\cdots(a^2+\overline{2n+1}^2)}$

iii. $\int_0^\infty e^{-ax} \sin^{2n} x\,dx = \frac{|\underline{2n}}{a(a^2+2^2)(a^2+4^2)(a^2+6^2)\cdots\cdots(a^2+\overline{2n}^2)}$

31. i. If $\int_0^h \phi(x) \cos nx\,dx = \psi(n)$ and $\alpha\beta = 2\pi$, then

$$\alpha\left\{\tfrac{1}{2}\phi(0) + \phi(\alpha)\cos n\alpha + \phi(2\alpha)\cos 2n\alpha + \cdots + \phi(m\alpha)\cos mn\alpha\right\}$$
$$= \psi(n) + \psi(\beta-n) + \psi(\beta+n) + \psi(2\beta-n) + \psi(2\beta+n) + \&c$$

where $m\alpha$ is the greatest multiple of α less than h and n lies between 0 & β. If h be a multiple of α the last term is $\frac{1}{2}\phi(h)\cos nh$. (Such conditions are required in similar theorems)

ii. $\int_0^h \frac{\sin nx}{\sin x}\phi(x)\,dx = \pi\left\{\tfrac{1}{2}\phi(0) - \phi(\pi)\cos n\pi + \phi(2\pi)\cos 2n\pi\right.$
$\left. - \phi(3\pi)\cos 3n\pi + \cdots\cdots \pm \phi(m\pi)\cos mn\pi\right\}$

$-2\psi(n+1) - 2\psi(n+3) - 2\psi(n+5) - \&c$ ad inf; the conditions being similar to that of i.

Cor. i. If $\int_0^\infty \phi(x)\cos nx\,dx = \psi(n)$ and $\alpha\beta = 2\pi$, then

$$\alpha\left\{\tfrac{1}{2}\phi(0) + \phi(\alpha) + \phi(2\alpha) + \phi(3\alpha) + \&c \text{ ad. inf.}\right\}$$
$$= \psi(0) + 2\psi(\beta) + 2\psi(2\beta) + 2\psi(3\beta) + \&c$$

Cor. ii. If n becomes infinitely great, $\int_0^h \frac{\sin nx}{\sin x}\phi(x)\,dx$

$$= \pi\left\{\tfrac{1}{2}\phi(0) - \phi(\pi)\cos n\pi + \phi(2\pi)\cos 2n\pi - \dots \pm \phi(m\pi)\cos mn\pi\right\}$$

where $m\pi$ is the greatest multiple of π less than h.

32. i. If $\int_0^h \phi(x)\sin nx\,dx = \psi(n)$ and $\alpha\beta = 2\pi$, then

$$\alpha\left\{\phi(\alpha)\sin n\alpha + \phi(2\alpha)\sin 2n\alpha + \phi(3\alpha)\sin 3n\alpha + \dots + \phi(m\alpha)\sin mn\alpha\right\}$$

$$= \psi(n) - \psi(\beta - n) + \psi(\beta + n) - \psi(2\beta - n) + \&c \text{ ad. inf.}$$

with the same condition as in 31.

ii. $\frac{1}{2}\phi(n) + \phi(n+x) + \phi(n+2x) + \&c$ ad inf.

$$= \frac{1}{x}\int_0^\infty \phi(n+z)\,dz - \frac{B_2}{\lfloor 2}x\phi'(n) + \frac{B_4}{\lfloor 4}x^3\phi'''(n) - \&c.$$

Cor. If $\int_0^\infty \phi(x)\sin nx\,dx = \psi(n)$ and $\alpha\beta = \frac{\pi}{2}$, then

$$\alpha\left\{\phi(\alpha) - \phi(3\alpha) + \phi(5\alpha) - \phi(7\alpha) + \&c \text{ ad. inf.}\right\}$$
$$= \psi(\beta) - \psi(3\beta) + \psi(5\beta) - \psi(7\beta) + \&c \text{ ad. inf.}$$

N. B. Just as in 31. ii. the following integrals can be found:

$$\int_0^h \frac{\cos nx}{\cos x}\phi(x)\,dx;\quad \int_0^h \frac{\sin nx}{\cos x}\phi(x)\,dx;\quad \int_0^h \frac{\cos nx}{\sin x}\phi(x)\,dx.$$

33. i. $\displaystyle\int_0^\infty \left\{\frac{(-x^2)^l}{1-x^{2n}} + \frac{(-1)^l}{n(x^2-1)}\right\} \cos px \, dx$

$\displaystyle = \frac{\pi}{2n} e^{-p} + \frac{\pi}{n} \sum e^{-p\cos\frac{\pi r}{n}} \cos\left\{(2l+1)\frac{\pi r}{n} - p\sin\frac{\pi r}{n}\right\}$

where n is even and $r = 1, 2, 3,$ up to $\frac{n-2}{2}$.

ii. $\displaystyle\int_0^\infty \left\{\frac{(-x^2)^l}{1-x^{2n}} + \frac{(-1)^l}{n(x^2-1)}\right\} \cos px \, dx$

$\displaystyle = \frac{\pi}{n} \sum e^{-p\cos\frac{\pi r}{n}} \cos\left\{(2l+1)\frac{\pi r}{n} - p\sin\frac{\pi r}{n}\right\}$

where n is odd and $r = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ up to $\frac{n-2}{2}$.

34. i. $\displaystyle\frac{\pi\cos\theta x}{x\sin\pi x} = \frac{1}{x^2} + \frac{2\cos\theta}{1^2-x^2} - \frac{2\cos 2\theta}{2^2-x^2} + \frac{2\cos 3\theta}{3^2-x^2} - \&c$

ii. $\displaystyle\frac{\pi\sin\theta x}{4x\cos\frac{\pi x}{2}} = \frac{\sin\theta}{1^2-x^2} - \frac{\sin 3\theta}{3^2-x^2} + \frac{\sin 5\theta}{5^2-x^2} - \&c$

Cor. i. $\displaystyle\frac{\pi\cosh\theta x}{x\sinh\pi x} = \frac{1}{x^2} - \frac{2\cos\theta}{1^2+x^2} + \frac{2\cos 2\theta}{2^2+x^2} - \&c$

ii. $\displaystyle\frac{\pi\sinh\theta x}{4x\cosh\frac{\pi x}{2}} = \frac{\sin\theta}{1^2+x^2} - \frac{\sin 3\theta}{3^2+x^2} + \frac{\sin 5\theta}{5^2+x^2} - \&c$

35. $\displaystyle\sqrt{\alpha}\left\{1 + \frac{2}{(1+\alpha^2)^{n+1}} + \frac{2}{(1+4\alpha^2)^{n+1}} + \frac{2}{(1+9\alpha^2)^{n+1}} + \&c\right\}$

$\displaystyle = \frac{\lfloor\underline{n-\tfrac{1}{2}}}{\lfloor\underline{n}}\sqrt{\beta}\left\{1 + 2e^{-2\beta}\phi(4\beta) + 2e^{-4\beta}\phi(8\beta) + \&c\right\}$ with

$\displaystyle\alpha\beta = \pi$ & $\displaystyle\phi(t)\lfloor\underline{2n} = t^n\lfloor\underline{n} + \frac{n}{\lfloor\underline{1}} t^{n-1}\lfloor\underline{n+1} + \frac{n(n-1)}{\lfloor\underline{2}} t^{n-2}\lfloor\underline{n+2} + \&c$

36. $\displaystyle m\left\{\frac{1}{2(m^2+n^2)} + \frac{1}{m^2+(n+1)^2} + \frac{1}{m^2+(n+2)^2} + \&c\right\}$

$\displaystyle = \tan^{-1}\frac{m}{n} + \frac{B_2}{2}\cdot\frac{\sin\left(2\tan^{-1}\frac{m}{n}\right)}{m^2+n^2} - \frac{B_4}{4}\cdot\frac{\sin\left(4\tan^{-1}\frac{m}{n}\right)}{(m^2+n^2)^2} + \&c.$

Cor. $\displaystyle n\left\{\frac{1}{4n^2} + \frac{1}{n^2+(n+1)^2} + \frac{1}{n^2+(n+2)^2} + \frac{1}{n^2+(n+3)^2} + \&c\right\}$

$\displaystyle = \frac{\pi}{4} + \frac{B_2}{2}\cdot\frac{1}{2n^2} - \frac{B_6}{6}\cdot\frac{1}{8n^6} + \frac{B_{10}}{10}\cdot\frac{1}{32n^{10}} - \&c$

CHAPTER XIV

1. $$\frac{1}{x^2\left(1+\frac{x^2}{1}\right)\left(1+\frac{x^2}{3}\right)\left(1+\frac{x^2}{6}\right)\left(1+\frac{x^2}{10}\right)\ \&c}$$
$$= \frac{1}{x^2} - \frac{3}{1+x^2} + \frac{5}{3+x^2} - \frac{7}{6+x^2} + \frac{9}{10+x^2} - \&c.$$

Cor. $$\frac{3}{\sqrt{2}\left(e^{2\pi x\sqrt{2}}-1\right)} - \frac{5}{\sqrt{6}\left(e^{2\pi x\sqrt{6}}-1\right)} + \frac{7}{\sqrt{12}\left(e^{2\pi x\sqrt{12}}-1\right)} - \&c$$
$$+ \frac{1}{x}\left\{\operatorname{Sech}\left(\frac{\pi}{x}\sqrt{1-\frac{x^2}{4}}\right) + \operatorname{Sech}\left(\frac{\pi}{x}\sqrt{4-\frac{x^2}{4}}\right) + \operatorname{Sech}\left(\frac{\pi}{x}\sqrt{9-\frac{x^2}{4}}\right) + \&c\right\}$$
$$= \frac{1}{2\pi x} + \frac{\pi x}{6} - C.$$ for all values of x

where $$C = \frac{1}{2} + \frac{1}{3+\sqrt{8}} - \frac{1}{5+\sqrt{24}} + \frac{1}{7+\sqrt{48}} - \&c$$
$$= 1 - \frac{\pi}{8} + \frac{1}{6(3+\sqrt{8})^2} - \frac{1}{10(5+\sqrt{24})^2} + \frac{1}{14(7+\sqrt{48})^2} - \&c$$

N.B. Similarly any function whose denominator is in the form of a product can be expressed as the sum of partial fractions and many other theorems may be deduced from the result.

2. $$\frac{\lfloor\!\underline{x}\ \lfloor\!\underline{y}}{\lfloor\!\underline{x+m}\ \lfloor\!\underline{y+n}} = \frac{1}{\lfloor\!\underline{m-1}}\left\{\frac{1}{x+1}\cdot\frac{\lfloor\!\underline{-\frac{y}{x}}}{\lfloor\!\underline{-\frac{y}{x}+n}} - \frac{m-1}{\lfloor\!\underline{1}}\cdot\frac{1}{x+2}\cdot\frac{\lfloor\!\underline{-\frac{2y}{x}}}{\lfloor\!\underline{-\frac{2y}{x}+n}}\right.$$
$$\left. + \frac{(m-1)(m-2)}{\lfloor\!\underline{2}}\cdot\frac{1}{x+3}\cdot\frac{\lfloor\!\underline{-\frac{3y}{x}}}{\lfloor\!\underline{-\frac{3y}{x}+n}} - \&c\right\} +$$
$$\frac{1}{\lfloor\!\underline{n-1}}\left\{\frac{1}{y+1}\cdot\frac{\lfloor\!\underline{-\frac{x}{y}}}{\lfloor\!\underline{-\frac{x}{y}+m}} - \frac{n-1}{\lfloor\!\underline{1}}\cdot\frac{1}{y+2}\,\frac{\lfloor\!\underline{-\frac{2x}{y}}}{\lfloor\!\underline{-\frac{2x}{y}+m}} + \&c\right\}.$$

Cor. 1. $$\frac{\pi}{\operatorname{Sin}\pi x}\cdot\frac{\lfloor\!\underline{m}\ \lfloor\!\underline{n}}{\lfloor\!\underline{m+x}\ \lfloor\!\underline{n-x}} = \frac{1}{x} + \frac{n}{m+1}\cdot\frac{1}{1-x} - \frac{n(n-1)}{(m+1)(m+2)}\cdot\frac{1}{2-x}$$
$$+ \frac{n(n-1)(n-2)}{(m+1)(m+2)(m+3)}\cdot\frac{1}{3-x} - \&c - \frac{m}{n+1}\cdot\frac{1}{1+x} + \frac{m(m-1)}{(n+1)(n+2)}\cdot\frac{1}{2+x}$$
$$- \frac{m(m-1)(m-2)}{(n+1)(n+2)(n+3)}\cdot\frac{1}{3+x} + \&c.$$

Cor. 2. $\dfrac{\frac{\pi}{2}\,\lfloor\!\underline{\alpha}\,\lfloor\!\underline{\beta}}{\lfloor\!\underline{\alpha-\frac12}\,\lfloor\!\underline{\beta-\frac12}} = \alpha\left\{1-\dfrac{\alpha-1}{\beta+1}\cdot\dfrac13+\dfrac{(\alpha-1)(\alpha-2)}{(\beta+1)(\beta+2)}\cdot\dfrac15-\&c\right\}$

$$+\beta\left\{1-\frac{\beta-1}{\alpha+1}\cdot\frac13+\frac{(\beta-1)(\beta-2)}{(\alpha+1)(\alpha+2)}\cdot\frac15-\&c\right\}$$

3. $$1+\frac{\alpha}{\gamma+1}\cdot\frac{\beta}{\delta+1}+\frac{\alpha(\alpha-1)}{(\gamma+1)(\gamma+2)}\cdot\frac{\beta(\beta-1)}{(\delta+1)(\delta+2)}+\&c$$

$$+\frac{\gamma}{\alpha+1}\cdot\frac{\delta}{\beta+1}+\frac{\gamma(\gamma-1)}{(\alpha+1)(\alpha+2)}\cdot\frac{\delta(\delta-1)}{(\beta+1)(\beta+2)}+\&c$$

$$=\frac{\lfloor\!\underline{\alpha}\,\lfloor\!\underline{\beta}\,\lfloor\!\underline{\gamma}\,\lfloor\!\underline{\delta}\,\lfloor\!\underline{\alpha+\beta+\gamma+\delta}}{\lfloor\!\underline{\alpha+\gamma}\,\lfloor\!\underline{\beta+\gamma}\,\lfloor\!\underline{\alpha+\delta}\,\lfloor\!\underline{\beta+\delta}}.$$

4. $$\frac{1}{1^2+x^2+\frac{x^4}{1^2}}+\frac{1}{2^2+x^2+\frac{x^4}{2^2}}+\frac{1}{3^2+x^2+\frac{x^4}{3^2}}+\&c$$

$$=\frac{\pi}{2x\sqrt3}\cdot\frac{\sinh\pi x\sqrt3-\sqrt3\sin\pi x}{\cosh\pi x\sqrt3-\cos\pi x}.$$

Cor. If n be any integer excluding 0,

$$\frac{1}{1^2+(2n)^2+\frac{(2n)^4}{1^2}}+\frac{1}{2^2+(2n)^2+\frac{(2n)^4}{2^2}}+\frac{1}{3^2+(2n)^2+\frac{(2n)^4}{3^2}}+\&c$$

$$=\frac{1}{12n^2}+\frac12\left(\frac{1}{1^2+3n^2}+\frac{1}{2^2+3n^2}+\frac{1}{3^2+3n^2}+\&c\right).$$

N.B. A great number of theorems like the above can be got from $\overline{\underline{\text{XIII}}}$ 29 & 33.

5. i If n is any integer greater than 0 and x lies between 0 and $\frac{\pi}{2n+1}$, then (both inclusive)

$$\frac{\sin^{2n+1}x}{1}+\frac{\sin^{2n+1}2x}{2}+\frac{\sin^{2n+1}3x}{3}+\&c=\frac{\sqrt\pi}{2}\cdot\frac{\lfloor\!\underline{n-\frac12}}{\lfloor\!\underline{n}}.$$

ii. ~~&~~ $$\frac{\sin^{2n+2}x}{x}+\frac{\sin^{2n+2}2x}{4x}+\frac{\sin^{2n+2}3x}{9x}+\&c=\frac{\sqrt\pi}{2}\cdot\frac{\lfloor\!\underline{n-\frac12}}{\lfloor\!\underline{n}}$$

if x lies between 0 and $\frac{\pi}{2n+2}$. (both inclusive).

N.B. Many series like the above can be got from $\overline{\underline{\text{XIII}}}$ 30.

6. $\sqrt{\alpha}\left\{\frac{1}{2} + \operatorname{Sech}^{2n}\alpha + \operatorname{Sech}^{2n}2\alpha + \operatorname{Sech}^{2n}3\alpha + \&c\right\}$

$= \frac{\lfloor n-1}{\lfloor n-\frac{1}{2}}\sqrt{\beta}\left\{\frac{1}{2} + \phi(\beta) + \phi(2\beta) + \phi(3\beta) + \&c\right\}$ with $\alpha\beta = \pi$

and $\phi(\beta) = \dfrac{1}{\left\{1 + \left(\frac{\beta}{n}\right)^2\right\}\left\{1 + \left(\frac{\beta}{n+1}\right)^2\right\}\left\{1 + \left(\frac{\beta}{n+2}\right)^2\right\}\&c}$

7. $e^{\frac{n^2}{4}}\sqrt{\alpha}\left\{\frac{1}{2} + e^{-\alpha^2}\cos n\alpha + e^{-4\alpha^2}\cos 2n\alpha + e^{-9\alpha^2}\cos 3n\alpha + \&c\right\}$

$= \sqrt{\beta}\left\{\frac{1}{2} + e^{-\beta^2}\cosh n\beta + e^{-4\beta^2}\cosh 2n\beta + e^{-9\beta^2}\cosh 3n\beta + \&c\right\}$

with $\alpha\beta = \pi$.

Cor. $\sqrt{\alpha}\left\{\frac{1}{2} + e^{-\alpha^2} + e^{-4\alpha^2} + e^{-9\alpha^2} + \&c\right\}$

$= \sqrt{\beta}\left\{\frac{1}{2} + e^{-\beta^2} + e^{-4\beta^2} + e^{-9\beta^2} + \&c\right\}$ with $\alpha\beta = \pi$.

8. i. If $\alpha\beta = \pi$, then $\frac{\alpha}{4}\coth n\alpha - \frac{\beta}{4}\cot n\beta$

$= \frac{n}{2} + \frac{\alpha\sinh 2n\alpha}{e^{2\alpha^2}-1} + \frac{\alpha\sinh 4n\alpha}{e^{4\alpha^2}-1} + \frac{\alpha\sinh 6n\alpha}{e^{6\alpha^2}-1} + \&c$

$+ \frac{\beta\sin 2n\beta}{e^{2\beta^2}-1} + \frac{\beta\sin 4n\beta}{e^{4\beta^2}-1} + \frac{\beta\sin 6n\beta}{e^{6\beta^2}-1} + \&c$

ii. If $\alpha\beta = \pi$, then $\frac{n^2}{2} + \frac{1}{2}\log\frac{\sin n\alpha}{\sinh n\beta}$

$= \left\{\frac{\alpha^2}{12} + \frac{\cos 2n\alpha}{e^{2\alpha^2}-1} + \frac{\cos 4n\alpha}{2(e^{4\alpha^2}-1)} + \frac{\cos 6n\alpha}{3(e^{6\alpha^2}-1)} + \&c\right\}$

$- \left\{\frac{\beta^2}{12} + \frac{\cosh 2n\beta}{e^{2\beta^2}-1} + \frac{\cosh 4n\beta}{2(e^{4\beta^2}-1)} + \frac{\cosh 6n\beta}{3(e^{6\beta^2}-1)} + \&c\right\}$

iii. If $\alpha\beta = \pi$, then $\frac{\alpha^2}{6}\phi(0) + \frac{\alpha n}{2}\phi'(0) + \frac{n^2}{4}\phi''(0) +$

$\frac{\phi(n\alpha)+\phi(-n\alpha)}{e^{2\alpha^2}-1} + \frac{\phi(2n\alpha)+\phi(-2n\alpha)}{2(e^{4\alpha^2}-1)} + \frac{\phi(3n\alpha)+\phi(-3n\alpha)}{3(e^{6\alpha^2}-1)} + \&c$

$+ \phi(n\alpha) + \frac{1}{2}\phi(2n\alpha) + \frac{1}{3}\phi(3n\alpha) + \&c$

$= \frac{\beta^2}{6}\phi(0) + \frac{\beta n i}{2}\phi'(0) + \frac{\pi i}{2}\phi(0) +$

$$\frac{\phi(n\beta i) + \phi(-n\beta i)}{e^{2\beta^2}-1} + \frac{\phi(2n\beta i) + \phi(-2n\beta i)}{2(e^{4\beta^2}-1)} + \&c$$

$$+ \phi(n\beta i) + \tfrac{1}{2}\phi(2n\beta i) + \tfrac{1}{3}\phi(3n\beta i) + \&c.$$

Cor. i. If $\alpha\beta = \pi^2$, then $\frac{\alpha+\beta}{12} = \frac{1}{2} + \frac{2\alpha}{e^{2\alpha}-1} + \frac{4\alpha}{e^{4\alpha}-1} + \frac{6\alpha}{e^{6\alpha}-1}$

$+ \frac{8\alpha}{e^{8\alpha}-1} + \&c + \frac{2\beta}{e^{2\beta}-1} + \frac{4\beta}{e^{4\beta}-1} + \frac{6\beta}{e^{6\beta}-1} + \frac{8\beta}{e^{8\beta}-1} + \&c$

ii. If $\alpha\beta = \pi^2$, then

$$e^{\frac{\alpha-\beta}{12}} = \frac{\sqrt[4]{\alpha}\,(1-e^{-2\alpha})(1-e^{-4\alpha})(1-e^{-6\alpha})\ \&c}{\sqrt[4]{\beta}\,(1-e^{-2\beta})(1-e^{-4\beta})(1-e^{-6\beta})\ \&c}$$

ex. $\frac{1}{24} - \frac{1}{8\pi} = \frac{1}{e^{2\pi}-1} + \frac{2}{e^{4\pi}-1} + \frac{3}{e^{6\pi}-1} + \frac{4}{e^{8\pi}-1} + \&c.$

9. i. If $\int_0^h \phi(x)\cos nx\,dx = \psi(n)$ and $\alpha\beta = \frac{\pi}{2}$, then

$$\alpha\{\phi(\alpha)\sin n\alpha - \phi(3\alpha)\sin 3n\alpha + \cdots \pm \phi(m\alpha)\sin mn\alpha\}$$

$$= \frac{\psi(\beta-n) - \psi(\beta+n)}{2} - \frac{\psi(3\beta-n) - \psi(3\beta+n)}{2} + \&c \text{ ad. inf.}$$

where $m\alpha$ is the greatest odd multiple of α less than h and n lies between $-\beta$ & β

ii If $\int_0^h \phi(x)\sin nx\,dx = \psi(n)$ and $\alpha\beta = \frac{\pi}{2}$, then

$$\alpha\{\phi(\alpha)\cos n\alpha - \phi(3\alpha)\cos 3n\alpha + \cdots \pm \phi(m\alpha)\cos mn\alpha\}$$

$$= \frac{\psi(\beta-n) + \psi(\beta+n)}{2} - \frac{\psi(3\beta-n) + \psi(3\beta+n)}{2} + \&c \text{ ad. inf.}$$

with the conditions in the first part.

10. $e^{\frac{n^2}{4}}\{e^{-\alpha^2}\sin n\alpha - e^{-9\alpha^2}\sin 3n\alpha + e^{-25\alpha^2}\sin 5n\alpha - \&c\}\sqrt{\alpha}$

$= \sqrt{\beta}\{e^{-\beta^2}\sinh n\beta - e^{-9\beta^2}\sinh 3n\beta + e^{-25\beta^2}\sinh 5n\beta - \&c\}$ with $\alpha\beta =$ [illegible]

11. If $\alpha\beta = \pi$,

$$\alpha\left\{\frac{1}{4}\sec n\alpha + \frac{\cos n\alpha}{e^{\alpha^2}-1} - \frac{\cos 3n\alpha}{e^{3\alpha^2}-1} + \frac{\cos 5n\alpha}{e^{5\alpha^2}-1} - \&c\right\}$$

$$= \beta\left\{\frac{1}{4} + \frac{\cosh 2n\beta}{e^{\beta^2}+e^{-\beta^2}} + \frac{\cosh 4n\beta}{e^{2\beta^2}+e^{-2\beta^2}} + \frac{\cosh 6n\beta}{e^{3\beta^2}+e^{-3\beta^2}} + \&c\right\}.$$

12. If $\alpha\beta = \frac{\pi}{2}$

$$\alpha\left\{\frac{\sin n\alpha}{e^{\alpha^2}+e^{-\alpha^2}} - \frac{\sin 3n\alpha}{e^{3\alpha^2}+e^{-3\alpha^2}} + \frac{\sin 5n\alpha}{e^{5\alpha^2}+e^{-5\alpha^2}} - \&c\right\}$$

$$= \beta\left\{\frac{\sinh n\beta}{e^{\beta^2}+e^{-\beta^2}} - \frac{\sinh 3n\beta}{e^{3\beta^2}+e^{-3\beta^2}} + \frac{\sinh 5n\beta}{e^{5\beta^2}+e^{-5\beta^2}} - \&c\right\}$$

Cor. If $\alpha\beta = \frac{\pi}{2}$

$$\alpha\left\{\frac{\phi(\alpha)-\phi(-\alpha)}{e^{\alpha^2}+e^{-\alpha^2}} - \frac{\phi(3\alpha)-\phi(-3\alpha)}{e^{3\alpha^2}+e^{-3\alpha^2}} + \&c\right\}$$

$$+ i\beta\left\{\frac{\phi(i\beta)-\phi(-i\beta)}{e^{\beta^2}+e^{-\beta^2}} - \frac{\phi(3\beta i)-\phi(-3\beta i)}{e^{3\beta^2}+e^{-3\beta^2}} + \&c\right\} = 0.$$

13. If $\alpha\beta = \pi^2$ and n is a positive integer greater than unity,

$$\alpha^n\left\{\frac{B_{2n}}{4n}\cos\pi n + \frac{1^{2n-1}}{e^{2\alpha}-1} + \frac{2^{2n-1}}{e^{4\alpha}-1} + \frac{3^{2n-1}}{e^{6\alpha}-1} + \&c\right\}$$

$$= (-\beta)^n\left\{\frac{B_{2n}}{4n}\cos\pi n + \frac{1^{2n-1}}{e^{2\beta}-1} + \frac{2^{2n-1}}{e^{4\beta}-1} + \frac{3^{2n-1}}{e^{6\beta}-1} + \&c\right\}$$

Cor. i. $\frac{1^5}{e^{2\pi}-1} + \frac{2^5}{e^{4\pi}-1} + \frac{3^5}{e^{6\pi}-1} + \frac{4^5}{e^{8\pi}-1} + \&c = \frac{1}{504}$.

ii. $\frac{1^9}{e^{2\pi}-1} + \frac{2^9}{e^{4\pi}-1} + \frac{3^9}{e^{6\pi}-1} + \frac{4^9}{e^{8\pi}-1} + \&c = \frac{1}{264}$

iii. $\frac{1^{13}}{e^{2\pi}-1} + \frac{2^{13}}{e^{4\pi}-1} + \frac{3^{13}}{e^{6\pi}-1} + \frac{4^{13}}{e^{8\pi}-1} + \&c = \frac{1}{24}$

iv. $\frac{1^{4n+1}}{e^{2\pi}-1} + \frac{2^{4n+1}}{e^{4\pi}-1} + \frac{3^{4n+1}}{e^{6\pi}-1} + \frac{4^{4n+1}}{e^{8\pi}-1} + \&c = \frac{B_{4n+2}}{8n+4}$.

14. If $\alpha\beta = \pi^2$ and n is a positive integer,

$$\alpha^{n+1}\left\{\frac{1^{2n+1}}{e^{\frac{\alpha}{2}}+e^{-\frac{\alpha}{2}}} - \frac{3^{2n+1}}{e^{\frac{3\alpha}{2}}+e^{-\frac{3\alpha}{2}}} + \frac{5^{2n+1}}{e^{\frac{5\alpha}{2}}+e^{-\frac{5\alpha}{2}}} - \&c\right\}$$

$$+(-\beta)^{n+1}\left\{\frac{1^{2n+1}}{e^{\frac{\beta}{2}}+e^{-\frac{\beta}{2}}} - \frac{3^{2n+1}}{e^{\frac{3\beta}{2}}+e^{-\frac{3\beta}{2}}} + \frac{5^{2n+1}}{e^{\frac{5\beta}{2}}+e^{-\frac{5\beta}{2}}} - \&c\right\} = 0.$$

Cor. If n is a positive integer excluding 0,

$$\frac{1^{4n-1}}{e^{\frac{\pi}{2}}+e^{-\frac{\pi}{2}}} - \frac{3^{4n-1}}{e^{\frac{3\pi}{2}}+e^{-\frac{3\pi}{2}}} + \frac{5^{4n-1}}{e^{\frac{5\pi}{2}}+e^{-\frac{5\pi}{2}}} - \&c = 0.$$

15. If $\alpha\beta = \frac{\pi^2}{4}$, then

$$\frac{\operatorname{sech}\alpha}{1} - \frac{\operatorname{sech}3\alpha}{3} + \frac{\operatorname{sech}5\alpha}{5} - \frac{\operatorname{sech}7\alpha}{7} + \&c$$

$$+\frac{\operatorname{sech}\beta}{1} - \frac{\operatorname{sech}3\beta}{3} + \frac{\operatorname{sech}5\beta}{5} - \frac{\operatorname{sech}7\beta}{7} + \&c$$

$$= 2\left\{\begin{matrix}\tan^{-1}e^{-\alpha} - \tan^{-1}e^{-3\alpha} + \tan^{-1}e^{-5\alpha} - \&c \\ +\tan^{-1}e^{-\beta} - \tan^{-1}e^{-3\beta} + \tan^{-1}e^{-5\beta} - \&c\end{matrix}\right\} = \frac{\pi}{4}.$$

Cor. $\tan^{-1}e^{-\frac{\pi}{2}} - \tan^{-1}e^{-\frac{3\pi}{2}} + \tan^{-1}e^{-\frac{5\pi}{2}} - \&c = \frac{\pi}{16}.$

16. If m and n are positive integers,

i. $$\int_0^\infty \frac{\sin^{2n+1}x}{x}\cos^{2m}x\,dx = \frac{\lfloor m-\frac{1}{2}\;\lfloor n-\frac{1}{2}}{2\lfloor m+n}$$

$$= \int_0^\infty \frac{\sin^{2n+2}x}{x^2}\cos^{2m}x\,dx.$$

ii. If ~~m~~, n and p are positive integers, $(-1)^p\frac{\sqrt{\pi}}{2}\cdot\frac{\lfloor n\;\lfloor n-\frac{1}{2}}{\lfloor n-p\;\lfloor n+p} =$

$$\int_0^\infty \frac{\sin^{2n+1}x}{x}\cos 2px\,dx = \int_0^\infty \frac{\sin^{2n+2}x}{x^2}\cos 2px\,dx \cancel{=}$$

71\. i. If $\alpha\beta = 2\pi$ and $m\alpha$ is the greatest multiple of α less than $\frac{\pi}{2}$, then for all values of n and p,

$$\alpha\left\{\frac{1}{2} + \cos^n\alpha\cos p\alpha + \cos^n 2\alpha\cos 2p\alpha + \dots + \cos^n m\alpha\cos pm\alpha\right\}$$

$$= \frac{\pi\left|\underline{n}\right.}{2^{n+1}}\left\{\frac{1}{\left|\underline{\tfrac{n+p}{2}}\right.\left|\underline{\tfrac{n-p}{2}}\right.} + \left(\frac{1}{\left|\underline{\tfrac{n+\beta-p}{2}}\right.\left|\underline{\tfrac{n-\beta+p}{2}}\right.} + \frac{1}{\left|\underline{\tfrac{n+\beta+p}{2}}\right.\left|\underline{\tfrac{n-\beta-p}{2}}\right.}\right)\right.$$

$$\left. + \left(\frac{1}{\left|\underline{\tfrac{n+2\beta-p}{2}}\right.\left|\underline{\tfrac{n-2\beta+p}{2}}\right.} + \frac{1}{\left|\underline{\tfrac{n+2\beta+p}{2}}\right.\left|\underline{\tfrac{n-2\beta-p}{2}}\right.}\right) + \text{\&c to } \infty\right\}.$$

ii. $\alpha\left\{\cos^n\alpha\sin p\alpha - \cos^n 3\alpha\sin 3p\alpha + \dots \pm \cos^n m\alpha\sin mp\alpha\right.$

$$= \frac{\pi\left|\underline{n}\right.}{2^{n+2}}\left\{\left(\frac{1}{\left|\underline{\tfrac{n+\beta-p}{2}}\right.\left|\underline{\tfrac{n-\beta+p}{2}}\right.} - \frac{1}{\left|\underline{\tfrac{n+\beta+p}{2}}\right.\left|\underline{\tfrac{n-\beta-p}{2}}\right.}\right)\right.$$

$$\left. - \left(\frac{1}{\left|\underline{\tfrac{n+3\beta-p}{2}}\right.\left|\underline{\tfrac{n-3\beta+p}{2}}\right.} - \frac{1}{\left|\underline{\tfrac{n+3\beta+p}{2}}\right.\left|\underline{\tfrac{n-3\beta-p}{2}}\right.}\right) + \text{\&c}\right\}$$

where $\alpha\beta = \frac{\pi}{2}$ and $m\alpha$ is the greatest odd multiple of α less than $\frac{\pi}{2}$. In both the cases if $m\alpha$ be an exact multiple $\frac{1}{2}$ the last term must be taken but there is no such necessity here.

Cor. 1. If α lies between 0 & $\frac{\pi}{n+1}$ (both inclusive)

$$\alpha\left\{\frac{1}{2} + \cos^{2n}\alpha + \cos^{2n}2\alpha + \cos^{2n}3\alpha + \dots + \cos^{2n}m\alpha\right\}$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\left|\underline{n-\tfrac{1}{2}}\right.}{\left|\underline{n}\right.} \text{ where } n \text{ is an integer and } m\alpha \not> \frac{\pi}{2}.$$

Cor. 2. But if it lies between $\frac{\pi}{n}$ & $\frac{2\pi}{n+1}$ the value is

$$\frac{\sqrt{\pi}}{2}\cdot\frac{\lfloor n-\frac{1}{2}}{\lfloor n}\left(1+\frac{2\lfloor n\,\lfloor n}{\lfloor n+\frac{\pi}{x}\;\lfloor n-\frac{\pi}{x}}\right).$$

18. If $\phi(x)=\sum\frac{P_n}{p_n-a_nx}$ and $\psi(x)=\sum\frac{Q_n}{q_n-b_nx}$, then

$$\phi(x)\,\psi(y)=\sum\frac{P_n}{p_n-a_nx}\,\psi\left(\frac{p_n}{a_n}\cdot\frac{y}{x}\right)+\sum\frac{Q_n}{q_n-b_ny}\,\phi\left(\frac{q_n}{b_n}\cdot\frac{x}{y}\right).$$

Cor. 1. $\pi^2xyn^2\,\dfrac{\cos\theta nx}{\sin\pi nx}\cdot\dfrac{\cosh\phi ny}{\sinh\pi ny}$

$$=1-2\pi xyx^2\left\{\frac{\cos\phi}{1^2+n^2y^2}\cdot\frac{\cosh\frac{\theta x}{y}}{\sinh\frac{\pi x}{y}}-\frac{2\cos2\phi}{2^2+n^2y^2}\cdot\frac{\cosh\frac{2\theta x}{y}}{\sinh\frac{2\pi x}{y}}+\frac{3\cos3\phi}{3^2+n^2y^2}\cdot\frac{\cosh\frac{3\theta x}{y}}{\sinh\frac{3\pi x}{y}}-\&c\right\}$$

$$+2\pi xyn^2\left\{\frac{\cos\theta}{1^2-n^2x^2}\cdot\frac{\cosh\frac{\phi y}{x}}{\sinh\frac{\pi y}{x}}-\frac{2\cos2\theta}{2^2-n^2x^2}\cdot\frac{\cosh\frac{2\phi y}{x}}{\sinh\frac{2\pi y}{x}}+\frac{3\cos3\theta}{3^2-n^2x^2}\cdot\frac{\cosh\frac{3\phi y}{x}}{\sinh\frac{3\pi y}{x}}-\&c\right\}.$$

Cor. 2. $\dfrac{\pi}{4n^2}\cdot\dfrac{\sin\theta nx}{\cos\frac{\pi nx}{2}}\cdot\dfrac{\sinh\phi ny}{\cosh\frac{\pi ny}{2}}$

$$=y^2\left\{\frac{\sin\phi}{1^2+n^2y^2}\cdot\frac{\sinh\frac{\theta x}{y}}{\cosh\frac{\pi x}{2y}}-\frac{\sin3\phi}{3^2+n^2y^2}\cdot\frac{\sinh\frac{3\theta x}{y}}{3\cosh\frac{3\pi x}{2y}}+\&c\right\}$$

$$+x^2\left\{\frac{\sin\theta}{1^2-n^2x^2}\cdot\frac{\sinh\frac{\phi y}{x}}{\cosh\frac{\pi y}{2x}}-\frac{\sin3\theta}{3^2-n^2x^2}\cdot\frac{\sinh\frac{3\phi y}{x}}{3\cosh\frac{3\pi y}{2x}}+\&c\right\}$$

Cor. 3. $\dfrac{\pi}{4}\cdot\dfrac{\cos\theta nx}{\sin\frac{\pi nx}{2}}\cdot\dfrac{\sinh\phi ny}{\cosh\frac{\pi ny}{2}}$

$$=\frac{\phi y}{2x}-\left\{\frac{\sin\phi}{1^2+y^2n^2}\cdot\frac{\cosh\theta\frac{x}{y}}{\sinh\frac{\pi x}{2y}}-\frac{3\sin3\phi}{3^2+y^2n^2}\cdot\frac{\cosh\frac{3\theta x}{y}}{3\sinh\frac{3\pi x}{2y}}+\&c\right\}n^2y^2$$

$$+n^2x^2\left\{\frac{\cos2\theta}{2^2-x^2n^2}\cdot\frac{\sinh\frac{2\phi y}{x}}{2\cosh\frac{\pi y}{x}}-\frac{\cos4\theta}{4^2-n^2x^2}\cdot\frac{\sinh\frac{4\phi y}{x}}{4\cosh\frac{2\pi y}{x}}+\&c\right\}$$

7. i $\pi^2 xy \cot \pi x \coth \pi y$

$$= 1 + 2\pi xy \left\{ \frac{\coth \frac{\pi x}{y}}{1^2+y^2} + \frac{2\coth\frac{2\pi x}{y}}{2^2+y^2} + \frac{3\coth\frac{3\pi x}{y}}{3^2+y^2} + \&c\right\}$$

$$- 2\pi xy \left\{ \frac{\coth\frac{\pi y}{x}}{1^2-x^2} + \frac{2\coth\frac{2\pi y}{x}}{2^2-x^2} + \frac{3\coth\frac{3\pi y}{x}}{3^2-x^2} + \&c\right\}$$

ii. $\dfrac{\pi^2 xy}{\sin\pi x \sinh \pi y}$

$$= 1 - 2\pi xy\left\{ \frac{1}{1^2+y^2}\cdot\frac{1}{\sinh\frac{\pi x}{y}} - \frac{2}{2^2+y^2}\cdot\frac{1}{\sinh\frac{2\pi x}{y}} + \&c\right\}$$

$$+ 2\pi xy\left\{\frac{1}{1^2-x^2}\cdot\frac{1}{\sinh\frac{\pi y}{x}} - \frac{2}{2^2-x^2}\cdot\frac{1}{\sinh\frac{2\pi y}{x}} + \&c\right\}$$

iii. $\dfrac{\pi}{4}\tan\dfrac{\pi x}{2}\tanh\dfrac{\pi y}{2}$

$$= y^2\left\{\frac{\tanh\frac{\pi x}{2y}}{(1^2+y^2)} + \frac{\tanh\frac{3\pi x}{2y}}{3(3^2+y^2)} + \frac{\tanh\frac{5\pi x}{2y}}{5(5^2+y^2)} + \&c\right\}$$

$$+ x^2\left\{\frac{\tanh\frac{\pi y}{2x}}{1(1^2-x^2)} + \frac{\tanh\frac{3\pi y}{2x}}{3(3^2-x^2)} + \frac{\tanh\frac{5\pi y}{2x}}{5(5^2-x^2)} + \&c\right\}$$

iv. $\dfrac{\pi}{4}\sec\dfrac{\pi x}{2}\operatorname{sech}\dfrac{\pi y}{2}$

$$= \frac{\operatorname{sech}\frac{\pi x}{2y}}{1^2+y^2} - \frac{3\operatorname{sech}\frac{3\pi x}{2y}}{3^2+y^2} + \frac{5\operatorname{sech}\frac{5\pi x}{2y}}{5^2+y^2} - \&c$$

$$+ \frac{\operatorname{sech}\frac{\pi y}{2x}}{1^2-x^2} - \frac{3\operatorname{sech}\frac{3\pi y}{2x}}{3^2-x^2} + \frac{5\operatorname{sech}\frac{5\pi y}{2x}}{5^2-x^2} - \&c.$$

v. $\dfrac{\pi}{4}\cot\dfrac{\pi x}{2}\operatorname{sech}\dfrac{\pi y}{2}$

$$= \frac{1}{2x} - y\left\{\frac{\coth\frac{\pi x}{2y}}{1^2+y^2} - \frac{\coth\frac{3\pi x}{2y}}{3^2+y^2} + \frac{\coth\frac{5\pi x}{2y}}{5^2+y^2} - \&c\right\}$$

$$- x\left\{\frac{\operatorname{sech}\frac{\pi y}{x}}{2^2-x^2} + \frac{\operatorname{sech}\frac{2\pi y}{x}}{4^2-x^2} + \frac{\operatorname{sech}\frac{3\pi y}{x}}{6^2-x^2} + \&c\right\}$$

N.B. Similarly for $\tan\frac{\pi x}{2}\coth\frac{\pi y}{2}$ and $\sec\frac{\pi x}{2}\coth\frac{\pi y}{2}$

20. i. $\pi^2 x^2 \cot \pi x \coth \pi x$

$$= 1 - 4\pi x^4 \left\{ \frac{\coth \pi}{1^4 - x^4} + \frac{2\coth 2\pi}{2^4 - x^4} + \frac{3\coth 3\pi}{3^4 - x^4} + \&c \right\}$$

Cor. $(\pi x)^2 \dfrac{\cosh \pi x\sqrt{2} + \cos \pi x\sqrt{2}}{\cosh \pi x\sqrt{2} - \cos \pi x\sqrt{2}}$

$$= 1 + 4\pi x^4 \left\{ \frac{\coth \pi}{1^4 + x^4} + \frac{2\coth 2\pi}{2^4 + x^4} + \frac{3\coth 3\pi}{3^4 + x^4} + \&c \right\}$$

ii. $\pi^2 x^2 \operatorname{cosec} \pi x \operatorname{cosech} \pi x$

$$= 1 + 4\pi x^4 \left\{ \frac{\operatorname{cosech} \pi}{1^4 - x^4} - \frac{2\operatorname{cosech} 2\pi}{2^4 - x^4} + \frac{3\operatorname{cosech} 3\pi}{3^4 - x^4} - \&c \right\}$$

Cor. $\dfrac{2\pi^2 x^2}{\cosh \pi x\sqrt{2} - \cos \pi x\sqrt{2}}$

$$= 1 - 4\pi x^4 \left\{ \frac{\operatorname{cosech} \pi}{1^4 + x^4} - \frac{2\operatorname{cosech} 2\pi}{2^4 + x^4} + \frac{3\operatorname{cosech} 3\pi}{3^4 + x^4} - \&c \right\}$$

iii. $\dfrac{\pi}{8x^2} \tan \dfrac{\pi x}{2} \tanh \dfrac{\pi x}{2}$

$$= \frac{\tanh \frac{\pi}{2}}{1^4 - x^4} + \frac{3\tanh \frac{3\pi}{2}}{3^4 - x^4} + \frac{5\tanh \frac{5\pi}{2}}{5^4 - x^4} + \&c$$

Cor. $\dfrac{\pi}{8x^2} \cdot \dfrac{\cosh \frac{\pi x}{\sqrt{2}} - \cos \frac{\pi x}{\sqrt{2}}}{\cosh \frac{\pi x}{\sqrt{2}} + \cos \frac{\pi x}{\sqrt{2}}}$

$$= \frac{\tanh \frac{\pi}{2}}{1^4 + x^4} + \frac{3\tanh \frac{3\pi}{2}}{3^4 + x^4} + \frac{5\tanh \frac{5\pi}{2}}{5^4 + x^4} + \&c$$

iv. $\dfrac{\pi}{8} \sec \dfrac{\pi x}{2} \operatorname{sech} \dfrac{\pi x}{2}$

$$= \frac{1^3 \operatorname{sech} \frac{\pi}{2}}{1^4 - x^4} - \frac{3^3 \operatorname{sech} \frac{3\pi}{2}}{3^4 - x^4} + \frac{5^3 \operatorname{sech} \frac{5\pi}{2}}{5^4 - x^4} - \&c$$

Cor. $\dfrac{\pi/4}{\cosh \frac{\pi x}{\sqrt{2}} + \cos \frac{\pi x}{\sqrt{2}}}$

$$= \frac{1^3 \operatorname{sech} \frac{\pi}{2}}{1^4 + x^4} - \frac{3^3 \operatorname{sech} \frac{3\pi}{2}}{3^4 + x^4} + \frac{5^3 \operatorname{sech} \frac{5\pi}{2}}{5^4 + x^4} - \&c.$$

21 i. If $\alpha\beta = \pi^2$ and n any integer,

$$(4\alpha)^{1-n}\left\{\tfrac{1}{2} S_{2n-1} + \frac{1}{1^{2n-1}(e^{2\alpha}-1)} + \frac{1}{2^{2n-1}(e^{4\alpha}-1)} + \&c\right\}$$

$$-(-4\beta)^{1-n}\left\{\tfrac{1}{2} S_{2n-1} + \frac{1}{1^{2n-1}(e^{2\beta}-1)} + \frac{1}{2^{2n-1}(e^{4\beta}-1)} + \&c\right\}$$

$$= \frac{B_{2n}}{\lfloor 2n}\left\{(-\alpha)^n + \beta^n\right\} + \pi^2 \frac{B_2}{\lfloor 2}\,\frac{B_{2n-2}}{\lfloor 2n-2}\left\{(-\alpha)^{n-2} + \beta^{n-2}\right\}$$

$$- \pi^4 \frac{B_4}{\lfloor 4}\cdot\frac{B_{2n-4}}{\lfloor 2n-4}\left\{(-\alpha)^{n-4} + \beta^{n-4}\right\} + \&c$$ the last term

being $-\pi^n \frac{B_n}{\lfloor n}\cdot\frac{B_n}{\lfloor n}(-1)^{\frac{n}{2}}$ or $\pi^{n-1}\frac{B_{n-1}}{\lfloor n-1}\,\frac{B_{n+1}}{\lfloor n+1}(-1)^{\frac{n+1}{2}}\left\{(-\alpha)+\beta\right\}$

according as n is even or odd.

ii. If $\alpha\beta = \pi^2$ and n any integer, then

$$\alpha^{1-n}\left\{\frac{1}{1^{2n-1}(e^{\frac{\alpha}{2}} + e^{-\frac{\alpha}{2}})} - \frac{1}{3^{2n-1}(e^{\frac{3\alpha}{2}} + e^{-\frac{3\alpha}{2}})} + \&c\right\}\cdot\frac{2^{2n+1}}{\pi}$$

$$+(-\beta)^{1-n}\left\{\frac{1}{1^{2n-1}(e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}})} - \frac{1}{3^{2n-1}(e^{\frac{3\beta}{2}} + e^{-\frac{3\beta}{2}})} + \&c\right\}\cdot\frac{2^{2n+1}}{\pi}$$

$$= \frac{E_1 E_{2n-1}}{\lfloor 2n-2}\left\{(-\alpha)^{n-1} + \beta^{n-1}\right\} - \frac{E_3 E_{2n-3}}{\lfloor 2\,\lfloor 2n-4}\left\{(-\alpha)^{n-3} + \beta^{n-3}\right\}$$

$$+ \frac{E_5 E_{2n-5}}{\lfloor 4\,\lfloor 2n-6}\left\{(-\alpha)^{n-5} + \beta^{n-5}\right\} - \&c$$ the last term being

$(-1)^{\frac{n-1}{2}}\left(\frac{E_n}{\lfloor n-1}\right)^2$ or $(-1)^{\frac{n}{2}}\frac{E_{n-1}}{\lfloor n-2}\cdot\frac{E_{n+1}}{\lfloor n}(\alpha-\beta)$ according as n is

odd or even.

iii. If $\alpha\beta = \pi^2$ and n any integer, $\frac{\sqrt{\alpha}}{\alpha^n}\left\{\frac{1}{2}\left(\frac{1}{1^{2n}} - \frac{1}{3^{2n}} + \frac{1}{5^{2n}} - \&c\right)\right.$

$$\left. + \frac{1}{1^{2n}(e^{\alpha}-1)} - \frac{1}{3^{2n}(e^{3\alpha}-1)} + \frac{1}{5^{2n}(e^{5\alpha}-1)} - \&c\right\} =$$

$$\frac{\sqrt{\beta}}{\beta^n}\Bigg[(-1)^n\left\{\frac{1}{2^{2n}(e^{\beta}+e^{-\beta})}+\frac{1}{4^{2n}(e^{2\beta}+e^{-2\beta})}+\frac{1}{6^{2n}(e^{3\beta}+e^{-3\beta})}+\&c\right.$$

$$+\frac{1}{4}\left\{\frac{\left(\frac{\beta}{2}\right)^{2n}}{\lfloor 2n}E_{2n+1}+\frac{B_2}{\lfloor 2}\cdot\frac{E_{2n-1}}{\lfloor 2n-2}\left(\frac{\beta}{2}\right)^{2n-1}(2\alpha)-\frac{B_4}{\lfloor 4}\cdot\frac{E_{2n-3}}{\lfloor 2n-4}\left(\frac{\beta}{2}\right)^{2n-2}(2\alpha)^2\right.$$

$$\left.+\frac{B_6}{\lfloor 6}\cdot\frac{E_{2n-5}}{\lfloor 2n-6}\left(\frac{\beta}{2}\right)^{2n-3}(2\alpha)^3-\cdots\cdots-\frac{(-\alpha\beta)^n}{\lfloor 2n}B_{2n}E_1\right\}\Bigg]$$

22\. i. $$\frac{\pi^2 xy}{2}\cdot\frac{\cosh\pi(x+y)\sqrt{2}\cos\pi(x-y)\sqrt{2}-\cosh\pi(x-y)\sqrt{2}\cos\pi(x+y)\sqrt{2}}{(\cosh\pi x\sqrt{2}-\cos\pi x\sqrt{2})(\cosh\pi y\sqrt{2}-\cos\pi y\sqrt{2})}$$

$$=1+2\pi x^3 y\left\{\frac{\coth\frac{\pi y}{x}}{1^4+x^4}+\frac{2\coth\frac{2\pi y}{x}}{2^4+x^4}+\frac{3\coth\frac{3\pi y}{x}}{3^4+x^4}+\&c\right\}$$

$$+2\pi x y^3\left\{\frac{\coth\frac{\pi x}{y}}{1^4+y^4}+\frac{2\coth\frac{2\pi x}{y}}{2^4+y^4}+\frac{3\coth\frac{3\pi x}{y}}{3^4+y^4}+\&c\right\}.$$

ii. $$\int_0^\infty\frac{\cos 2\pi x}{\cosh\pi\sqrt{x}+\cos\pi\sqrt{x}}\,dx=\frac{e^{-n}}{\cosh\frac{\pi}{2}}-\frac{3e^{-9n}}{\cosh\frac{3\pi}{2}}+\frac{5e^{-25n}}{\cosh\frac{5\pi}{2}}-\&c$$

Cor. If $\alpha\beta=\frac{\pi^3}{4}$, then

$$\frac{1}{\cosh\sqrt{\alpha}+\cos\sqrt{\alpha}}-\frac{1}{3}\frac{1}{\cosh\sqrt{3\alpha}+\cos\sqrt{3\alpha}}+\frac{1}{5}\frac{1}{\cosh\sqrt{5\alpha}+\cos\sqrt{5\alpha}}-\&c$$

$$+\frac{1}{\cosh\frac{\pi}{2}\cosh\beta}-\frac{1}{3}\frac{1}{\cosh\frac{3\pi}{2}\cosh 9\beta}+\frac{1}{5}\frac{1}{\cosh\frac{5\pi}{2}\cosh 25\beta}-\&c$$

$$=\frac{\pi}{8}.$$

iii. If $\alpha\beta=4\pi^3$ and $k=\frac{C_0+\log 2\pi}{4}+\frac{1}{e^{2\pi}-1}+\frac{1}{2(e^{4\pi}-1)}+\frac{1}{3(e^{6\pi}-1)}+\&c$

$$\frac{7\alpha}{720}+\frac{\cos\sqrt{\alpha}}{1(e^{\sqrt{\alpha}}-2\cos\sqrt{\alpha}+e^{-\sqrt{\alpha}})}+\frac{\cos\sqrt{2\alpha}}{2(e^{\sqrt{2\alpha}}-2\cos\sqrt{2\alpha}+e^{-\sqrt{2\alpha}})}+\&c$$

$$=k+\frac{\beta}{48\pi}-\frac{1}{4}\log\beta+\frac{\coth\pi}{1(e^{\beta}-1)}+\frac{\coth 2\pi}{2(e^{4\beta}-1)}+\frac{\coth 3\pi}{3(e^{9\beta}-1)}+\&c$$

N.B. $4k=C_0+3\log 2-\frac{\pi}{3}+\log\lfloor -\frac{1}{4}$, where C_0 is the constant of $\Sigma\frac{1}{x}$

23\. i. $$\frac{\phi(0)}{4\pi} + \coth\pi\left\{\phi(0) - x^4\phi(4) + x^8\phi(8) - \&c\right\}$$
$$+ 2\coth 2\pi\left\{\phi(0) - (2x)^4\phi(4) + (2x)^8\phi(8) - \&c\right\}$$
$$+ 3\coth 3\pi\left\{\phi(0) - (3x)^4\phi(4) + (3x)^8\phi(8) - \&c\right\}$$
$$+ \&c\ \&c\ \&c = \frac{\pi}{2x^2}\left\{\tfrac{1}{2}\phi(-2) + h\right\},$$

where h the error is very nearly equal to

$\phi(-2) - \frac{2\pi}{x\underline{|1}}\frac{\phi(-3)}{\sqrt{2}} + \frac{(2\pi)^3}{x^3\underline{|3}}\cdot\frac{\phi(-5)}{\sqrt{2}} + \&c$ the general term being $\frac{(2\pi)^n}{x^n\underline{|n}}\cos\frac{3n\pi}{4}$ if x is small. Similarly

ii. $$\operatorname{Sech}\frac{\pi}{2}\left\{\phi(0) - x^4\phi(4) + x^8\phi(8) - \&c\right\}$$
$$- \frac{\operatorname{Sech}\frac{3\pi}{2}}{3}\left\{\phi(0) - (3x)^4\phi(4) + (3x)^8\phi(8) - \&c\right\}$$
$$+ \frac{\operatorname{Sech}\frac{5\pi}{2}}{5}\left\{\phi(0) - (5x)^4\phi(4) + (5x)^8\phi(8) - \&c\right\}$$
$$- \&c\ \&c\ \&c = \frac{\pi}{8}\phi(0) - \frac{\pi}{2}h$$

where h is very nearly equal to $\phi(0) - \frac{\pi/\sqrt{2}}{x\underline{|1}}\phi(-1) + \frac{(\pi/\sqrt{2})^2}{x^2\underline{|2}}\phi(-2) - \&c$ if x is small.

24\. $$\frac{1}{4n^2} + \frac{1}{n^2+(n+1)^2} + \frac{1}{n^2+(n+2)^2} + \frac{1}{n^2+(n+3)^2} + \&c$$
$$= \frac{\pi}{4n} + \frac{1}{8\pi n^3} - \frac{\pi}{n}\cdot\frac{1}{e^{4\pi n} - 2e^{2\pi n}\cos 2\pi n + 1}$$
$$+ 4n\left\{\frac{1}{e^{2\pi}-1}\cdot\frac{1}{1^4+4n^4} + \frac{2}{e^{4\pi}-1}\cdot\frac{1}{2^4+4n^4} + \frac{3}{e^{6\pi}-1}\cdot\frac{1}{3^4+4n^4} + \&c\right\}$$

N.B. i. $$\frac{1}{2n^2} + \frac{1}{n^2+1^2} + \frac{1}{n^2+2^2} + \&c = \frac{\pi}{2n} + \frac{\pi}{n}\cdot\frac{1}{e^{2\pi n}-1}.$$

ii. $$\frac{1}{n^2+1^2} + \frac{1}{n^2+3^2} + \frac{1}{n^2+5^2} + \&c = \frac{\pi}{4n} - \frac{\pi}{2n}\cdot\frac{1}{e^{\pi n}+1}.$$

25. $$\frac{1}{n^2+(n+1)^2}+\frac{1}{n^2+(n+3)^2}+\frac{1}{n^2+(n+5)^2}+\&c$$

$$+4n\left\{\frac{1}{e^{\pi}+1}\cdot\frac{1}{1^4+4n^4}+\frac{3}{e^{3\pi}+1}\cdot\frac{1}{3^4+4n^4}+\frac{5}{e^{5\pi}+1}\cdot\frac{1}{5^4+4n^4}+\&c\right.$$

$$=\frac{\pi}{8n}-\frac{\pi}{2n}\cdot\frac{1}{e^{2\pi n}+2e^{\pi n}\cos\pi n+1}.$$

ex. i. $$\frac{\coth\pi}{1^3}+\frac{\coth 2\pi}{2^3}+\frac{\coth 3\pi}{3^3}+\&c=\frac{7\pi^3}{180}$$

ii. $$\frac{\coth\pi}{1^7}+\frac{\coth 2\pi}{2^7}+\frac{\coth 3\pi}{3^7}+\&c=\frac{19\pi^7}{56700}.$$

iii. $$\frac{\tanh\frac{\pi}{2}}{1^3}+\frac{\tanh\frac{3\pi}{2}}{3^3}-\frac{\tanh\frac{5\pi}{2}}{5^3}+\&c=\frac{\pi^3}{32}.$$

iv. $$\frac{\tanh\frac{\pi}{2}}{1^7}+\frac{\tanh\frac{3\pi}{2}}{3^7}+\frac{\tanh\frac{5\pi}{2}}{5^7}+\&c=\frac{7\pi^7}{23040}.$$

v. $$\frac{\operatorname{cosech}\pi}{1^3}-\frac{\operatorname{cosech}2\pi}{2^3}+\frac{\operatorname{cosech}3\pi}{3^3}-\&c=\frac{\pi^3}{360}.$$

vi. $$\frac{\operatorname{cosech}\pi}{1^7}-\frac{\operatorname{cosech}2\pi}{2^7}+\frac{\operatorname{cosech}3\pi}{3^7}-\&c=\frac{13\pi^7}{453600}.$$

vii. $$\frac{\operatorname{sech}\frac{\pi}{2}}{1}-\frac{\operatorname{sech}\frac{3\pi}{2}}{3}+\frac{\operatorname{sech}\frac{5\pi}{2}}{5}-\&c=\frac{\pi}{8}.$$

viii. $$\frac{\operatorname{sech}\frac{\pi}{2}}{1^5}-\frac{\operatorname{sech}\frac{3\pi}{2}}{3^5}+\frac{\operatorname{sech}\frac{5\pi}{2}}{5^5}-\&c=\frac{\pi^5}{768}.$$

ix. $$\frac{\operatorname{sech}\frac{\pi}{2}}{1^9}-\frac{\operatorname{sech}\frac{3\pi}{2}}{3^9}+\frac{\operatorname{sech}\frac{5\pi}{2}}{5^9}-\&c=\frac{23\pi^9}{1720320}.$$

x. $$\frac{1}{1^2(e^{\pi}-1)}-\frac{1}{3^2(e^{3\pi}-1)}+\frac{1}{5^2(e^{5\pi}-1)}-\&c$$

$$+\frac{1}{2^2(e^{\pi}+e^{-\pi})}+\frac{1}{4^2(e^{2\pi}+e^{-2\pi})}+\&c=\frac{5\pi^2}{96}-\frac{1}{2}\int_0^1\frac{\tan^{-1}x}{x}\,dx.$$

xi. $$\frac{1}{(1^2+2^2)(\sinh 3\pi-\sinh\pi)}+\frac{1}{(2^2+3^2)(\sinh 5\pi-\sinh\pi)}+\&c$$

$$=\left(\frac{1}{\pi}+\coth\pi-\frac{\pi}{2}\tanh^2\frac{\pi}{2}\right)\Big/(2\sinh\pi).$$

xii. $$\frac{1}{25\cdot01(e^{\pi}+1)}+\frac{3}{25\cdot81(e^{2\pi}+1)}+\frac{5}{31\cdot25(e^{5\pi}+1)}=\frac{\pi}{8}\coth^2\frac{5\pi}{2}-\frac{4689}{11890}.$$

CHAPTER XV

1. $h\,\phi(h) + h\,\phi(2h) + h\,\phi(3h) + h\,\phi(4h) + h\,\phi(5h) + \&c$

$= \int_0^\infty \phi(x)\,dx + F(h)$. where $F(h)$ can be found by expanding the left and writing the constant instead of a series and $F(0) = 0$.

Cor. If $h\,\phi(h) = ah^p + bh^q + ch^r + dh^s + \&c$, then

$h\,\phi(h) + h\,\phi(2h) + h\,\phi(3h) + h\,\phi(4h) + \&c$

$= \int_0^\infty \phi(x)\,dx + a\frac{B_p}{p}h^p \cos\frac{\pi p}{2} + b\frac{B_q}{q}h^q\cos\frac{\pi q}{2} + \&c.$

N.B. If the expansion of $\phi(h)$ be an infinite series, then that of $F(h)$ also will be an infinite series; but if most of the numbers p, q, r, s, t &c be odd integers $F(h)$ appears to terminate. In this case the hidden part of $F(h)$ can't be expanded in ascending powers of h and is very rapidly diminishing when h is slowly diminishing and consequently can be neglected for practical purposes when h is small. e.g. If $\phi(h) = \frac{1}{1+h^2}$ then $F(h)$ $= \frac{2\pi}{e^{\frac{2\pi}{h}}-1}$ and hence $F(\frac{1}{10}) = \frac{2\pi}{e^{20\pi}-1}$. If $\phi(h) = e^{-h^2}$ then $F(\frac{1}{10})$ is very nearly $10\sqrt{\pi}\,e^{-100\pi^2}$.

2. $\frac{1^{n-1}}{e^x} + \frac{2^{n-1}}{e^{2x}} + \frac{3^{n-1}}{e^{3x}} + \frac{4^{n-1}}{e^{4x}} + \frac{5^{n-1}}{e^{5x}} + \&c$

$= \frac{\lfloor\underline{n-1}}{x^n} + \frac{B_n}{n}\cos\frac{\pi n}{2} - \frac{x}{\lfloor\underline{4}}\cdot\frac{B_{n+1}}{n+1}\cos\frac{\pi(n+1)}{2} + \&c$

ex i. $\frac{C_0 + \log x}{x} + \frac{\log 1}{e^x} + \frac{\log 2}{e^{2x}} + \frac{\log 3}{e^{3x}} + \&c = \frac{1}{2}\log(2\pi)$ when x vanishes.

ii. The sum of the nos of factors (including unity and the number) of the first n natural nos divided by n when n is very great $= 2C_0 - 1 + \log n$.

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iii. $\log n + n^2\left(\frac{2}{e^n} + \frac{3}{e^{2n}} + \frac{5}{e^{3n}} + \frac{7}{e^{4n}} + \frac{11}{e^{5n}} + \&c\right)$ is finite when n vanishes, 2, 3, 5, 7 &c being prime numbers

iv If $I(n)$ ~~be the nearest integer to~~ is of the order $\frac{1}{4n}\left\{\cosh \pi\sqrt{n} - \frac{\sinh \pi\sqrt{n}}{\pi\sqrt{n}}\right\}$

when ~~then~~ $I(0) + xI(1) + x^2 I(2) + x^3 I(3) + \&c$

$$= 1/(1 - 2x + 2x^4 - 2x^9 + 2x^{16} - \&c).$$

3. $$\frac{1^{m-1}}{e^{1^n x}} + \frac{2^{m-1}}{e^{2^n x}} + \frac{3^{m-1}}{e^{3^n x}} + \frac{4^{m-1}}{e^{4^n x}} + \&c$$

$$= \frac{\left|\frac{m}{n}\right.}{m x^{\frac{m}{n}}} + \frac{B_m}{m}\cos\frac{\pi m}{2} - \frac{x}{\lfloor 1}\cdot\frac{B_{m+n}}{m+n}\cos\frac{\pi(m+n)}{2}$$
$$+ \frac{x^2}{\lfloor 2}\cdot\frac{B_{m+2n}}{m+2n}\cos\frac{\pi(m+2n)}{2} - \&c$$

Cor. $$\frac{e^{-1^n x}}{1} + \frac{e^{-2^n x}}{2} + \frac{e^{-3^n x}}{3} + \frac{e^{-4^n x}}{4} + \&c$$

$$= \frac{-C_0 - \log x}{n} + C_0 - \frac{x}{\lfloor 1}\cdot\frac{B_n}{n}\cos\frac{\pi n}{2} + \frac{x^2}{\lfloor 2}\cdot\frac{B_{2n}}{2n}\cos \pi n - \&c$$

ex. i $$\frac{e^{-x}}{1} + \frac{e^{-4x}}{2} + \frac{e^{-9x}}{3} + \frac{e^{-16x}}{4} + \&c$$

$$= \frac{C_0 - \log x}{2} + \frac{x}{12} + \frac{x^2}{240} + \frac{x^3}{1512} + \frac{x^4}{5760} + \frac{x^5}{15840} + \&c$$

ii. $$e^{-x} + 2e^{-16x} + 3e^{-81x} + 4e^{-256x} + 5e^{-625x} + \&c$$

$$= \frac{1}{4}\sqrt{\frac{\pi}{x}} - \frac{1}{12} + \frac{x}{252} - \frac{x^2}{264} + \frac{x^3}{72} - \&c$$

iii. $$e^{-x} + 2e^{-8x} + 3e^{-27x} + 4e^{-64x} + \&c$$

$$= \frac{\left|-\frac{2}{3}\right.}{3x^{\frac{2}{3}}} - \frac{1}{12} + \frac{x^2}{480} - \frac{x^4}{288} + \&c$$

iv. $$\frac{e^{-x}}{1} + \frac{e^{-4x}}{4} + \frac{e^{-9x}}{9} + \frac{e^{-16x}}{16} + \&c$$

$$= \frac{\pi^2}{6} - \sqrt{\pi x} + \frac{x}{2} \text{ very nearly.}$$

$$1^2 e^{-1^6 x} + 2^2 e^{-2^6 x} + 3^2 e^{-3^6 x} + \&c = \frac{1}{6}\sqrt{\frac{\pi}{x}} \text{ very nearly.}$$

4. $$\frac{1^{l-1}}{(1+x^n)^m} + \frac{2^{l-1}}{(1+2^n x^n)^m} + \frac{3^{l-1}}{(1+3^n x^n)^m} + \&c$$

$$= \frac{\lfloor \frac{l}{n} \; \lfloor m-\frac{l}{n}-1}{l x^l \; \lfloor m-1} + \frac{B_l}{l}\cos\frac{\pi l}{2} - \frac{m}{\lfloor 1}\cdot x^n \frac{B_{l+n}}{l+n}\cos\frac{\pi(l+n)}{2}$$

$$+ \frac{m(m+1)}{\lfloor 2} x^{2n} \frac{B_{l+2n}}{l+2n}\cos\frac{\pi(l+2n)}{2} - \&c.$$

ex. $$\frac{1}{\sqrt{1+x^8}} + \frac{2}{\sqrt{1+(2x)^8}} + \frac{3}{\sqrt{1+(3x)^8}} + \&c$$

$$= \frac{\pi}{4x^2}\cdot\frac{\sqrt{\pi}}{(\lfloor -\frac{3}{4})^2} - \frac{1}{12} + \frac{x^8}{264} - \&c.$$

5. $$\frac{1^{m-1}}{e^{1^n x}-1} + \frac{2^{m-1}}{e^{2^n x}-1} + \frac{3^{m-1}}{e^{3^n x}-1} + \&c$$

$$= \frac{1}{m}\cdot\frac{\lfloor \frac{m}{n}}{x^{\frac{m}{n}}} S_{\frac{m}{n}} + \frac{S_{1+n-m}}{x} - \frac{1}{2}\cdot\frac{B_m}{m}\cos\frac{\pi m}{2}$$

$$+ \frac{x}{\lfloor 1}\cdot\frac{B_2}{2}\cdot\frac{B_{m+n}}{m+n}\cos\frac{\pi(m+n)}{2} - \frac{x^3}{\lfloor 3}\frac{B_4}{4}\cdot\frac{B_{m+3n}}{m+3n}\cos\frac{\pi(m+3n)}{2}$$

$$+ \frac{x^5}{\lfloor 5}\cdot\frac{B_6}{6}\cdot\frac{B_{m+5n}}{m+5n}\cos\frac{\pi(m+5n)}{2} - \&c.$$

Cor. $$\frac{1^{n-1}}{e^{1^n x}-1} + \frac{2^{n-1}}{e^{2^n x}-1} + \frac{3^{n-1}}{e^{3^n x}-1} + \&c.$$

$$= \frac{C_0 - \frac{1}{n}\log x}{x} - \frac{1}{2}\cdot\frac{B_n}{n}\cos\frac{\pi n}{2} + \frac{x}{\lfloor 1}\cdot\frac{B_2}{2}\,\frac{B_{2n}}{2n}\cos\pi n$$

$$- \frac{x^3}{\lfloor 3}\cdot\frac{B_4}{4}\cdot\frac{B_{4n}}{4n}\cos 2\pi n + \frac{x^5}{\lfloor 5}\,\frac{B_6}{6}\cdot\frac{B_{6n}}{6n}\cos 3\pi n - \&c$$

6. If $$\phi(t) = \frac{1^{m-1}}{(e^{1^p x})^t - q} + \frac{2^{m-1}}{(e^{2^p x})^t - q} + \frac{3^{m-1}}{(e^{3^p x})^t - q} + \&c$$

then $$1^{n-1}\phi(1) + 2^{n-1}\phi(2) + 3^{n-1}\phi(3) + 4^{n-1}\phi(4) + \&c$$

180. $$= \left\{ \frac{\underline{\left|\frac{m}{p}\right.}}{m x^{\frac{m}{p}}} S_{1 + \frac{m}{p} \cdot q - n} \right\} + \left\{ \frac{\underline{\left|\frac{n}{q}\right.}}{n x^{\frac{n}{q}}} S_{1 + \frac{n}{q} p - m} \right\} +$$

$$\frac{B_m}{m} \cdot \frac{B_n}{n} \cos\frac{\pi m}{2} \cos\frac{\pi n}{2} - \frac{x}{\underline{|1}} \cdot \frac{B_{m+p}}{m+p} \cdot \frac{B_{n+q}}{n+q} \cos\frac{\pi(m+p)}{2} \cos\frac{\pi(n+q)}{2}$$

$$+ \frac{x^2}{\underline{|2}} \cdot \frac{B_{m+2p}}{m+2p} \cdot \frac{B_{n+2q}}{n+2q} \cos\frac{\pi(m+2p)}{2} \cos\frac{\pi(n+2q)}{2} - \&c.$$

N. B. If $\frac{m}{p} = \frac{n}{q} = k$ the right side becomes

$$\frac{\underline{|k}}{mn x^k} \left\{ k \left(\Sigma \frac{1}{k-1} - C_0 - \log x \right) + C_0 (m+n) \right\} + \frac{B_m}{m} \cdot \frac{B_n}{n} \cos\frac{\pi m}{2} \cos\frac{\pi n}{2} - \&c.$$

ex. i. $$\frac{1}{1(e^x - 1)} + \frac{1}{2(e^{4x} - 1)} + \frac{1}{3(e^{9x} - 1)} + \&c.$$

$$= \frac{S_3}{x} - \frac{C_0 + \log\frac{2\pi}{x}}{4} - \frac{x}{144} + \frac{x^3}{181440} - \frac{x^5}{3991680} + \frac{x^7}{145152000} - \&c.$$

ii. $$\frac{1^2}{e^x - 1} + \frac{2^2}{e^{2x} - 1} + \frac{3^2}{e^{3x} - 1} + \frac{4^2}{e^{4x} - 1} + \&c$$

$$= \frac{2 S_3}{x^3} - \frac{1}{12x} + \frac{x}{1440} + \frac{x^3}{181440} + \frac{x^5}{7257600} + \frac{x^7}{159667200} + \&c$$

7. $$\frac{1^m \cdot 1^n}{e^x - 1} + \frac{2^m (2^n + 1^n)}{e^{2x} - 1} + \frac{3^m (3^n + 1^n)}{e^{3x} - 1} + \frac{4^m (4^n + 2^n + 1^n)}{e^{4x} - 1} + \&c$$

(the numerator in the rth term being $r^m \times$ the sum of the nth powers of the factors of r.)

$$= \frac{\underline{|m}}{x^{m+1}} S_{m+1} S_{m-n+1} + \frac{\underline{|n}}{x^{n+1}} S_{n+1} S_{n-m+1} + \frac{1}{x} S_{1-m} S_{1-n}$$

$$- \frac{1}{2} S_{-m} S_{-n} + \frac{B_2}{\underline{|2}} x . S_{-1-m} S_{-1-n} - \frac{B_4}{\underline{|4}} x^3 S_{-3-m} S_{-3-n} + \&c.$$

ex. $$1^4 \left(\frac{1^2}{e^x - 1} + \frac{2^2}{e^{2x} - 1} + \frac{3^2}{e^{3x} - 1} + \&c \right)$$

$$+ 2^4 \left(\frac{1^2}{e^{2x} - 1} + \frac{2^2}{e^{4x} - 1} + \frac{3^2}{e^{6x} - 1} + \&c \right)$$

$$+ 3^4 \left(\frac{1^2}{e^{3x} - 1} + \frac{2^2}{e^{6x} - 1} + \frac{3^2}{e^{9x} - 1} + \&c \right) + \&c \; \&c =$$

$$\left(\frac{24 S_5}{x^5} - 6x^3\right) S_3 - \frac{x}{\lfloor 9} + \frac{x^3}{12\lfloor 10} - \frac{x^5}{24\lfloor 11} + \&c$$

8. If $F(h)$ in XI 1. terminates we do not know how far the result is true. But from the following and similar ways we can calculate the error in such cases; let us take

$$\frac{1}{e^n-1} + \frac{1}{e^{4n}-1} + \frac{1}{e^{9n}-1} + \frac{1}{e^{16n}-1} + \&c$$

$$= \frac{\pi^2}{6n} + \frac{1}{2}\sqrt{\frac{\pi}{n}}\, S_{\frac{1}{2}} + \frac{1}{4} \quad \text{very nearly}$$

But $\int_0^\infty (e^{-x} + e^{-4x} + e^{-9x} + \&c) \operatorname{Cos} ax\, dx$

$$= \frac{1^2}{1^4+a^2} + \frac{2^2}{2^4+a^2} + \frac{3^2}{3^4+a^2} + \frac{4^2}{4^4+a^2} + \&c.$$

$$= \frac{\pi}{2\sqrt{2a}} \cdot \frac{\sinh \pi\sqrt{2a} - \sin \pi\sqrt{2a}}{\cosh \pi\sqrt{2a} - \cos \pi\sqrt{2a}}.$$

Therefore $\dfrac{1}{e^n-1} + \dfrac{1}{e^{4n}-1} + \dfrac{1}{e^{9n}-1} + \dfrac{1}{e^{16n}-1} + \&c$

$$= \frac{\pi^2}{6n} + \frac{1}{2}\sqrt{\frac{\pi}{n}}\, S_{\frac{1}{2}} + \frac{1}{4} + \sqrt{\frac{\pi}{2n}}\left\{\frac{\cos(\frac{\pi}{4}+\sqrt{t}) - e^{-\sqrt{t}}\cos\frac{\pi}{4}}{\cosh\sqrt{t} - \cos\sqrt{t}}\right.$$

$$+ \frac{1}{\sqrt{2}} \cdot \frac{\cos(\frac{\pi}{4}+\sqrt{2t}) - e^{-\sqrt{2t}}\cos\frac{\pi}{4}}{\cosh\sqrt{2t} - \cos\sqrt{2t}} + \frac{1}{\sqrt{3}} \cdot \frac{\cos(\frac{\pi}{4}+\sqrt{3t}) - e^{-\sqrt{3t}}\cos\frac{\pi}{4}}{\cosh\sqrt{3t} - \cos\sqrt{3t}}$$

$$\left. + \frac{1}{\sqrt{4}} \frac{\cos(\frac{\pi}{4}+\sqrt{4t}) - e^{-\sqrt{4t}}\cos\frac{\pi}{4}}{\cosh\sqrt{4t} - \cos\sqrt{4t}} + \&c \text{ ad inf.}\right\}$$

where $t = \dfrac{4\pi^3}{n}$.

9. $$1^m\left\{1^n e^{-x} + 2^n e^{-2x} + 3^n e^{-3x} + 4^n e^{-4x} + \&c\right\}$$

$$+ 2^m\left\{1^n e^{-2x} + 2^n e^{-4x} + 3^n e^{-6x} + 4^n e^{-8x} + \&c\right\}$$

$$+ 3^m\left\{1^n e^{-3x} + 2^n e^{-6x} + 3^n e^{-9x} + 4^n e^{-12x} + \&c\right\}$$

$$+ 4^m\left\{1^n e^{-4x} + 2^n e^{-8x} + 3^n e^{-12x} + 4^n e^{-16x} + \&c\right\}$$

$$+ 5^m\left\{1^n e^{-5x} + 2^n e^{-10x} + 3^n e^{-15x} + 4^n e^{-20x} + \&c\right\}$$

$$+ \&c \quad \&c \quad \&c \quad \&c \quad \&c \quad =$$

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$$\frac{\underline{|m}}{x^{m+1}} S_{1+m-n} + \frac{\underline{|n}}{x^{n+1}} S_{1+n-m} + S_{-m} S_{-n}$$

$$- \frac{x}{\underline{|1}} S_{-m-1} S_{-n-1} + \frac{x^2}{\underline{|2}} S_{-m-2} S_{-n-2} - \&c.$$

N. B. The value of the above series can be exactly found if $m+n$ be a positive odd integer. For in that case it can always be expressed in terms of three primary series viz

i. $1 - 24\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \&c\right) = L.$

ii $1 + 240\left(\frac{1^3 x}{1-x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} + \frac{4^3 x^4}{1-x^4} + \&c\right) = M.$

iii. $1 - 504\left(\frac{1^5 x}{1-x} + \frac{2^5 x^2}{1-x^2} + \frac{3^5 x^3}{1-x^3} + \frac{4^5 x^4}{1-x^4} + \&c\right) = N.$

10\. i. $\frac{B_{2n}}{4n} \cos \pi n + \frac{1^{2n-1} x}{1-x} + \frac{2^{2n-1} x^2}{1-x^2} + \frac{3^{2n-1} x^3}{1-x^3} + \&c$ can be expressed in terms of M and N only and the series

$\frac{1^{2n} x}{(1-x)^2} + \frac{2^{2n} x^2}{(1-x^2)^2} + \frac{3^{2n} x^3}{(1-x^3)^2} + \&c$ (the diff^{l}. of the above series)

$$- \frac{nL}{6} \left\{ \frac{B_{2n}}{4n} \cos \pi n + \frac{1^{2n-1} x}{1-x} + \frac{2^{2n-1} x^2}{1-x^2} + \frac{3^{2n-1} x^3}{1-x^3} + \&c \right\}$$

can also be expressed in terms of M & N only by using indeterminate coeff^{ts}. paying attention to the degree. Thus by successive differentiations the double series in XV^{c} can be expressed in terms of L, M and N.

11\. The <u>degree</u> of a series is the sum of the highest powers of the n th terms together with unity if the series contains all the powers of x or if the powers of x be in A.P.

If the coeff^{t}. of each nth term is homogeneous the series is said to be <u>pure</u> and in other cases <u>mixed</u>.

The theory of indices holds good in terms of degrees of series.

If $F(h)$ in XV 1. terminates the series is said to be <u>perfect</u> if not it is said to be <u>imperfect</u>.

If $F(h) = 0$ the series is said to be complete in other cases incomplete.

A series is said to be absolutely complete when it remains complete when transformed or split up.

A linear se[illegible] can only be expressed by linear, double by double, treble by treble, pure by pure, perfect by perfect, imperfect by imperfect, and absolutely complete by absolutely complete adhering to the laws of indices in all cases. But a mixed series can be split up into a number of pure series of different degrees.

e.g. $1^n x + 2^n x^2 + 3^n x^3 + \&c$ is an imperfect, incomplete, pure, linear series of the $(n+1)$th degree.

$\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \&c$ is a perfect, incomplete, pure, linear series of 0 degree.

The series in XV 9 is a perfect, incomplete, pure, double series of $(m+n+1)$th degree if $m+n$ be odd and imperfect if $m+n$ be even.

The series in XVI 7 is a perfect, incomplete, pure, treble series of $(m+n+1)$th degree except when both m & n be even.

$\frac{1^m}{(e^x+e^{-x})^n} + \frac{2^m}{(e^{2x}+e^{-2x})^n} + \frac{3^m}{(e^{3x}+e^{-3x})^n} + \&c$ is always an mixed, incomplete double series of $(m+n)$th degree if $n > 2$.

$\frac{1}{2} + x + x^4 + x^9 + \&c$ is a perfect, complete, pure double series of $\frac{1}{2}$ a degree.

L, M and N are perfect, pure double series of 2nd, 4th and 6th degree respectively, M & N being complete and L incomplete.

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11. If $\alpha\beta = \pi^2$, then

$$\frac{1}{1^2(e^{\alpha}-e^{-\alpha})^2} + \frac{1}{2^2(e^{2\alpha}-e^{-2\alpha})^2} + \frac{1}{3^2(e^{3\alpha}-e^{-3\alpha})^2} + \&c$$

$$+ \frac{1}{1^2(e^{\beta}-e^{-\beta})^2} + \frac{1}{2^2(e^{2\beta}-e^{-2\beta})^2} + \frac{1}{3^2(e^{3\beta}-e^{-3\beta})^2} + \&c$$

$$-2\alpha\left\{1^2\log(1-e^{-2\alpha}) + 2^2\log(1-e^{-4\alpha}) + 3^2\log(1-e^{-6\alpha}) + \&c\right\}$$

$$-2\beta\left\{1^2\log(1-e^{-2\beta}) + 2^2\log(1-e^{-4\beta}) + 3^2\log(1-e^{-6\beta}) + \&c\right\}$$

$$= \frac{\alpha^2+\beta^2}{120} - \frac{\alpha\beta}{72}.$$

N.B. The theorem in $\overline{\text{XIII}}$ 24. ii and similar theorems are true only in case of a linear series but approximately in case of other series.

12. i. $M^3 - N^2 = 1728\, x(1-x)^{24}(1-x^2)^{24}(1-x^3)^{24}(1-x^4)^{24}\&c$

ii. $1 + 480\left(\frac{1^7x}{1-x} + \frac{2^7x^2}{1-x^2} + \frac{3^7x^3}{1-x^3} + \&c\right) = M^2.$

iii. $1 - 264\left(\frac{1^9x}{1-x} + \frac{2^9x^2}{1-x^2} + \frac{3^9x^3}{1-x^3} + \&c\right) = MN.$

iv. $1 - 24\left(\frac{1^{13}x}{1-x} + \frac{2^{13}x^2}{1-x^2} + \frac{3^{13}x^3}{1-x^3} + \&c\right) = M^2N.$

v. $\frac{1^2x}{(1-x)^2} + \frac{2^2x^2}{(1-x^2)^2} + \frac{3^2x^3}{(1-x^3)^2} + \&c = \frac{M-L^2}{288}.$

vi. $\frac{1^4x}{(1-x)^2} + \frac{2^4x^2}{(1-x^2)^2} + \frac{3^4x^3}{(1-x^3)^2} + \&c = \frac{LM-N}{720}.$

vii. $\frac{1^6x}{(1-x)^2} + \frac{2^6x^2}{(1-x^2)^2} + \frac{3^6x^3}{(1-x^3)^2} + \&c = \frac{M^2-LN}{1008}.$

viii. $\frac{1^8x}{(1-x)^2} + \frac{2^8x^2}{(1-x^2)^2} + \frac{3^8x^3}{(1-x^3)^2} + \&c = \frac{LM^2-MN}{720}$

ix. $L = \dfrac{1^3 - 3^3x + 5^3x^3 - 7^3x^6 + 9^3x^{10} - \&c}{1 - 3x + 5x^3 - 7x^6 + 9x^{10} - \&c}$

x. $M = \left\{\frac{1^5x}{1-x} + \frac{3^5x^2}{1-x^3} + \frac{5^5x^3}{1-x^5} + \frac{7^5x^4}{1-x^7} + \&c\right\}$

$\div\left\{\frac{x}{1-x} + \frac{3x^2}{1-x^3} + \frac{5x^3}{1-x^5} + \frac{7x^4}{1-x^7} + \&c\right\}$

13. i. $691 + 65520\left(\frac{1^{11}x}{1-x} + \frac{2^{11}x^2}{1-x^2} + \frac{3^{11}x^3}{1-x^3} + \&c\right)$

$= 441M^3 + 250N^2.$

ii. $3617 + 16320\left(\frac{1^{15}x}{1-x} + \frac{2^{15}x^2}{1-x^2} + \frac{3^{15}x^3}{1-x^3} + \&c\right)$

$= 1617M^4 + 2000MN^2.$

iii. $43867 - 28728\left(\frac{1^{17}x}{1-x} + \frac{2^{17}x^2}{1-x^2} + \frac{3^{17}x^3}{1-x^3} + \&c\right)$

$= 38367M^3N + 5500N^3.$

iv. $174611 + 13200\left(\frac{1^{19}x}{1-x} + \frac{2^{19}x^2}{1-x^2} + \frac{3^{19}x^3}{1-x^3} + \&c\right)$

$= 53361M^5 + 121250M^2N^2.$

v. $77683 - 552\left(\frac{1^{21}x}{1-x} + \frac{2^{21}x^2}{1-x^2} + \frac{3^{21}x^3}{1-x^3} + \&c\right)$

$= 57183M^4N + 20500MN^3.$

vi. $236364091 + 131040\left(\frac{1^{23}x}{1-x} + \frac{2^{23}x^2}{1-x^2} + \frac{3^{23}x^3}{1-x^3} + \&c\right)$

$= 49679091M^6 + 176400000M^3N^2 + 10285000N^4.$

vii. $657931 - 24\left(\frac{1^{25}x}{1-x} + \frac{2^{25}x^2}{1-x^2} + \frac{3^{25}x^3}{1-x^3} + \&c\right)$

$= 392931M^5N + 265000M^2N^3.$

viii. $3392780147 + 6960\left(\frac{1^{27}x}{1-x} + \frac{2^{27}x^2}{1-x^2} + \frac{3^{27}x^3}{1-x^3} + \&c\right)$

$= 489693897M^7 + 2507636250M^4N^2 + 395450000MN^4.$

ix. $1723168255201 - 171864\left(\frac{1^{29}x}{1-x} + \frac{2^{29}x^2}{1-x^2} + \frac{3^{29}x^3}{1-x^3} + \&c\right)$

$\cancel{= 6742202481M^6N + 1716211002720M^3N^3 + 215050000N^5}$

$= 815806500201M^6N + 881340705000M^3N^3$

$+ 26021050000N^5.$

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8. $7709321041217 + 32640\left(\frac{1^{31}x}{1-x} + \frac{2^{31}x^2}{1-x^2} + \frac{3^{31}x^3}{1-x^3} + \&c\right)$

$$= 764412173217 M^8 + 5323905468000 M^5N^2$$
$$+ 1621003400000 M^2N^4.$$

N.B. $x\frac{dL}{dx} = \frac{L^2-M}{12}$; $x\frac{dM}{dx} = \frac{LM-N}{3}$ and $x\frac{dN}{dx} = \frac{LN-M^2}{2}$.

ex. i. $1^5(1^4x + 2^4x^2 + 3^4x^3 + 4^4x^4 + \&c)$

$+ 2^5(1^4x^2 + 2^4x^4 + 3^4x^6 + 4^4x^8 + \&c)$

$+ 3^5(1^4x^3 + 2^4x^6 + 3^4x^9 + 4^4x^{12} + \&c)$

$+ 4^5(1^4x^4 + 2^4x^8 + 3^4x^{12} + 4^4x^{16} + \&c)$

$+ \&c \quad \&c \quad \&c \quad \&c$

$= (15LM^2 + 10L^3M - 20L^2N - 4MN - L^5)/12^4$

ii. $1^2(1^7x + 2^7x^2 + 3^7x^2 + 4^7x^4 + \&c)$

$+ 2^2(1^7x^2 + 2^7x^4 + 3^7x^6 + 4^7x^8 + \&c)$

$+ 3^2(1^7x^3 + 2^7x^6 + 3^7x^9 + 4^7x^{12} + \&c)$

$+ 4^2(1^7x^4 + 2^7x^8 + 3^7x^{12} + 4^7x^{16} + \&c)$

$+ \&c \quad \&c \quad \&c \quad \&c$

$$= \frac{2LM^2 - MN - L^2N}{12^3}.$$

iii. $1^3(1^6x + 2^6x^2 + 3^6x^3 + 4^6x^4 + \&c)$

$+ 2^3(1^6x^2 + 2^6x^4 + 3^6x^6 + 4^6x^8 + \&c)$

$+ 3^3(1^6x^3 + 2^6x^6 + 3^6x^9 + 4^6x^{12} + \&c)$

$+ 4^3(1^6x^4 + 2^6x^8 + 3^6x^{12} + 4^6x^{16} + \&c)$

$+ \&c \quad \&c \quad \&c \quad \&c$

$= (L^3M - 3L^2N + 3LM^2 - MN)/3456.$

14. If n is any even integer greater than 6 and $S_n = \frac{B_n}{2n} + (-1)^{\frac{n}{2}}\left\{\frac{1^{n-1}x}{1-x} + \frac{2^{n-1}x^2}{1-x^2} + \frac{3^{n-1}x^3}{1-x^3} + \&c\right\}$ such that $S_8 = 120\, S_4^2$

then

$$\frac{(n+2)(n+3)}{2} S_{n+2} + \frac{n(n-1)(n-2)(n-3)}{\underline{|2}}(n-2)(n-13)\, S_4 S_{n-2}$$

$$+ \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{\underline{|4}}(n-7)(n-18)\, S_6 S_{n-4}$$

$$+ \frac{n(n-1)\cdots\cdots(n-7)}{\underline{|6}}\left\{(n-12)(n-23) - 5.6\right\} S_8 S_{n-6}$$

$$+ \frac{n(n-1)\cdots\cdots(n-9)}{\underline{|8}}\left\{(n-17)(n-28) - 10.7\right\} S_{10} S_{n-8}$$

$$+ \frac{n(n-1)\cdots\cdots(n-11)}{\underline{|10}}\left\{(n-22)(n-33) - 15.8\right\} S_{12} S_{n-10} + \&c$$

N.B. If the last term be a perfect square then half the term must be taken.

15. i. $1 + \frac{1}{2}\cdot\frac{2t}{1+t} + \frac{1.3}{2.4}\cdot\left(\frac{2t}{1+t}\right)^2 + \frac{1.3.5}{2.4.6}\cdot\left(\frac{2t}{1+t}\right)^3 + \&c$

$= (1+t)\left\{1 + \frac{1}{2}\cdot t^2 + \frac{1.3}{2.4}t^4 + \frac{1.3.5}{2.4.6}t^6 + \&c\right\}$; thus we see if $\alpha = \frac{2t}{1+t}$ and $\beta = t^2$, β is in the 2nd degree of α.

By supposing $t^2 = \frac{2u}{1+u}$, $\frac{2t}{1+t} = \alpha$ and $u^2 = \beta$ we [illegible] that β is in the 4th degree of α and so on. The relation between α and β is the modular equation of the degree of β, and the ratio between the two series is denoted by M. Thus for the

2nd —— $M = 1 + \sqrt{\beta} = \sqrt{\frac{1-\beta}{1-\alpha}} = \sqrt{(1-\alpha)(1-\beta)} + 2\sqrt{\beta}$

3rd —— $M = 1 + 2\sqrt{\frac{\beta}{\alpha}} = \sqrt{\frac{1-\beta}{1-\alpha}}$.

nth —— $\beta = \frac{4\alpha^n}{\left\{(1+\sqrt{1-\alpha})^n + (1-\sqrt{1-\alpha})^n\right\}^2}$.

Cor. If 2nd be $\alpha^2 + 2\alpha = \beta$, then the nth is $\beta = (\alpha+1)^n - 1$.

ii. If pth and qth be $\phi(x)$ and $\psi(x)$ and rth be $f(x)$, then if pth and qth be $\phi F(x)$ and $\psi F(x)$ then rth is $f F(x)$. and also if pth and qth be $F\phi(x)$ and $F\psi(x)$ then rth is $F f(x)$.

Cor. Thus we may add or subtract any constant and multiply or divide by any constant to x in each function or to each function.

Cor. i. If 1st is x and 2nd x^2+4x then rth $=\left\{\left(\frac{\sqrt{x+4}+\sqrt{x}}{2}\right)^{r}-\left(\frac{\sqrt{x+4}-\sqrt{x}}{2}\right)^{r}\right\}^2$

ii ..... x ..... x^2-2 ..... $=\left(\frac{x+\sqrt{x^2-4}}{2}\right)^{r}+\left(\frac{x-\sqrt{x^2-4}}{2}\right)^{r}$.

iii. If $f(x)$ and $F(x)$ be of the pth and qth degree, find $\phi(x)$ such that $\sqrt[p]{\phi f(x)}=\sqrt[q]{\phi F(x)}=\chi(x)$ suppose, then the function for the rth degree $=\phi^{-1}\{\chi(x)\}^{r}$ and the self-repeating series is $\sqrt[n]{\frac{\phi(x)}{\psi(x)\,\phi'(x)}}$ where n is any quantity and $\psi(x)$ any suitable function. Supposing the series to be $S(x)$ we have $\frac{S\,F(x)}{S\,f(x)}=\sqrt[n]{\frac{p}{q}\cdot\frac{\psi f(x)}{\psi F(x)}\cdot\frac{F'(x)}{f'(x)}}$.

ex. If $\mathrm{I}=x$ and $\mathrm{II}=x^2+2nx$, then if x is great

$\mathrm{III}=x^3+3nx^2+\frac{3n(n+1)}{2}x-\frac{n(n-1)(n-2)x}{2x+\frac{3(n+1)}{2}}$ nearly.

16. If the modular equation for the $(n-1)$th degree be

$$\sqrt[n]{\alpha\beta}+\sqrt[n]{(1-\alpha)(1-\beta)}=1$$

then that of the $(n-1)^2$th is $\left\{\sqrt[n]{\alpha(1-\beta)}-\sqrt[n]{\beta(1-\alpha)}\right\}^{n}=$

$$\left(\sqrt[n]{\alpha}-\sqrt[n]{\beta}\right)^{n}+\left(\sqrt[n]{1-\beta}-\sqrt[n]{1-\alpha}\right)^{n}\cancel{=1}$$

N. B. The above result is got by eliminating γ from the equations $\sqrt[n]{\alpha\gamma}+\sqrt[n]{(1-\alpha)(1-\gamma)}=1$ & $\sqrt[n]{\beta\gamma}+\sqrt[n]{(1-\beta)(1-\gamma)}=1$

CHAPTER XVI

1. Let $\Pi(a,x) = (1+a)(1+ax)(1+ax^2)(1+ax^3)(1+ax^4)$ &c.

i. $\dfrac{\Pi(a,x)}{\Pi(ax^n,x)} = (1+a)^n$ when $x=1$.

ii. $\dfrac{\Pi(-x,x)}{(1-x)^n\,\Pi(-x^{n+1},x)} = \lfloor n$ when $x=1$.

iii. $\Pi(a,x) = \Pi(a,x^n)\,\Pi(ax,x^n)\,\Pi(ax^2,x^n)\cdots\cdots\Pi(ax^{n-1},x^n)$.

iv. $\Pi(a,x) = \dfrac{\Pi(a,\sqrt{x})}{\Pi(a\sqrt{x},x)}$.

2. $$\frac{\Pi(b,x)}{\Pi(-a,x)} = 1 + \frac{a+b}{1-x} + \frac{(a+b)(a+bx)}{(1-x)(1-x^2)} + \frac{(a+b)(a+bx)(a+bx^2)}{(1-x)(1-x^2)(1-x^3)}$$
$$+ \frac{(a+b)(a+bx)(a+bx^2)(a+bx^3)}{(1-x)(1-x^2)(1-x^3)(1-x^4)} + \text{\&c}.$$

3. $$\frac{1}{\Pi(-ax,x)} = 1 + \frac{ax}{(1-x)(1-ax)} + \frac{a^2x^4}{(1-x)(1-x^2)(1-ax)(1-ax^2)} +$$
$$\frac{a^3x^9}{(1-x)(1-x^2)(1-x^3)(1-ax)(1-ax^2)(1-ax^3)} + \text{\&c}$$

4. $$\frac{\Pi(-ab,x)\,\Pi(-ac,x)}{\Pi(-a,x)\,\Pi(-abc,x)} = 1 + a.\frac{(1-b)(1-c)}{(1-a)(1-x)} +$$
$$a^2\frac{(1-b)(x-b)(1-c)(x-c)}{(1-a)(1-ax)(1-x)(1-x^2)} + a^3\frac{(1-b)(x-b)(x^2-b)(1-c)(x-c)(x^2-c)}{(1-a)(1-ax)(1-ax^2)(1-x)(1-x^2)(1-x^3)}$$
$$+ \text{\&c}$$

5. $$\frac{\Pi(-a,x)\,\Pi(-abc,x)\,\Pi(-abd,x)\,\Pi(-acd,x)}{\Pi(-ab,x)\,\Pi(-ac,x)\,\Pi(-ad,x)\,\Pi(-abcd,x)}$$
$$= 1 - a\frac{(1-b)(1-c)(1-d)}{(1-ab)(1-ac)(1-ad)}\cdot\frac{1-ax}{1-x} + a^2\frac{(1-b)(x-b)(1-c)(x-c)}{(1-ab)(1-abx)(1-ac)(1-acx)}$$
$$\times\frac{(1-d)(x-d)}{(1-ad)(1-adx)}\cdot\frac{(1-ax^3)(1-a)}{(1-x)(1-x^2)} - a^3\frac{(1-b)(x-b)(x^2-b)}{(1-ab)(1-abx)(1-abx^2)}\cdot\frac{(1-c)(x-c)}{(1-ac)(1-acx)}$$
$$\times\frac{(x^2-c)\,(1-d)(x-d)(x^2-d)}{(1-acx^2)(1-ad)(1-adx)(1-adx^2)}\cdot\frac{(1-ax^5)(1-a)(1-ax)}{(1-x)(1-x^2)(1-x^3)} + \text{\&c}.$$

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6. $1 + \frac{a-b}{1-x} \cdot \frac{1-c}{1-d} + \frac{(a-b)(a-bx)}{(1-x)(1-x^2)} \cdot \frac{(1-c)(1-cx)}{(1-d)(1-dx)} + \&c$

$= \frac{\Pi(-b, x)\,\Pi(-c, x)}{\Pi(-a, x)\,\Pi(-d, x)} \cdot \left\{ 1 + \frac{c-d}{1-x} \cdot \frac{1-a}{1-b} + \frac{(c-d)(c-dx)}{(1-x)(1-x^2)} \cdot \frac{(1-a)(1-ax)}{(1-b)(1-bx)} + \&c \right\}$

7. $\frac{\Pi(-a, x)\,\Pi(-d; x)}{\Pi(-b, x)\,\Pi(-c, x)} \left\{ 1 + \frac{a-b}{1-x} \cdot \frac{a-c}{a-d} + \frac{(a-b)(a-bx)(a-c)(a-cx)}{(1-x)(1-x^2)(a-d)(a-dx)} + \&c \right\}$

$= 1 + \frac{1-dx}{1-x} \cdot \frac{1-a}{a-d} \cdot \frac{b-d}{1-b} \cdot \frac{c-d}{1-c} + x^2 \frac{(1-dx^3)(1-d)}{(1-x)(1-x^2)} \frac{(1-a)(1-ax)}{(a-d)(a-dx)} \times$

$\frac{(b-d)(b-dx)(c-d)(c-dx)}{(1-b)(1-bx)(1-c)(1-cx)} + x^6 \frac{(1-dx^5)(1-d)(1-dx)}{(1-x)(1-x^2)(1-x^3)} \frac{(1-a)(1-ax)(1-ax^2)}{(a-d)(a-dx)(a-dx^2)}$

$\times \frac{(b-d)(b-dx)(b-dx^2)}{(1-b)(1-bx)(1-bx^2)} \cdot \frac{(c-d)(c-dx)(c-dx^2)}{(1-c)(1-cx)(1-cx^2)} + \&c$

8. $\frac{\Pi(-a, x)}{\Pi(-b, x)} \left\{ 1 + \frac{a-b}{1-x} \cdot \frac{1-c}{1-d} + \frac{(a-b)(a-bx)}{(1-x)(1-x^2)} \cdot \frac{(1-c)(1-cx)}{(1-d)(1-dx)} + \&c \right\}$

$= 1 + \frac{1}{1-b} \cdot \frac{a-b}{1-x} \cdot \frac{d-c}{1-d} + \frac{x}{(1-b)(1-bx)} \cdot \frac{(a-b)(a-bx)}{(1-x)(1-x^2)} \cdot \frac{(d-c)(dx-c)}{(1-d)(1-dx)} +$

$\frac{x^3}{(1-b)(1-bx)(1-bx^2)} \cdot \frac{(a-b)(a-bx)(a-bx^2)}{(1-x)(1-x^2)(1-x^3)} \cdot \frac{(d-c)(dx-c)(dx^2-c)}{(1-d)(1-dx)(1-dx^2)} + \&c$

9. $\Pi(-ax, x) \left\{ 1 + \frac{bx}{(1-x)(1-ax)} + \frac{b^2x^4}{(1-x)(1-x^2)(1-ax)(1-ax^2)} + \&c \right\}$

$= 1 - x \cdot \frac{a-b}{1-x} + x^3 \cdot \frac{(a-b)(a-bx)}{(1-x)(1-x^2)} - x^6 \cdot \frac{(a-b)(a-bx)(a-bx^2)}{(1-x)(1-x^2)(1-x^3)} + \&c$

Cor. $1 + \frac{x^2}{(1-x)^2} + \frac{x^6}{(1-x)^2(1-x^2)^2} + \frac{x^{12}}{(1-x)^2(1-x^2)^2(1-x^3)^2} + \&c$

$= \frac{1 - x + x^3 - x^6 + x^{10} - x^{15} + x^{21} - \&c}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)\ \&c}$

11. $1 + \frac{x^3}{(1-x)(1-x^2)} + \frac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)} + \&c$

$= \frac{1 - x + x^4 - x^9 + x^{16} - \&c}{(1-x)(1-x^3)(1-x^5)(1-x^7)\ \&c}$

10. If
$$P = \frac{\left|\underline{\frac{x+l+n-m-1}{2}}\right. \left|\underline{\frac{x+l-n-m-1}{2}}\right. \left|\underline{\frac{x-l+n+m-1}{2}}\right. \left|\underline{\frac{x-l-n+m-1}{2}}\right.}{\left|\underline{\frac{x-l+n-m-1}{2}}\right. \left|\underline{\frac{x-l-n-m-1}{2}}\right. \left|\underline{\frac{x+l+n+m-1}{2}}\right. \left|\underline{\frac{x+l-n+m-1}{2}}\right.}$$

then
$$\frac{1-P}{1+P} = \frac{2lmx}{x^2+l^2+m^2-n^2-1+} \; \frac{4(x^2-1)(l^2-1)(m^2-1)}{3(x^2+l^2+m^2-n^2-5)+} \; \&c$$

N.B. Here the expansion in ascending powers of $\frac{1}{x}$ is true. But if x be removed from the numerators then the result will be true always.

11.
$$\frac{\Pi(a, x)\,\Pi(-b, x) - \Pi(-a, x)\,\Pi(b, x)}{\Pi(a, x)\,\Pi(-b, x) + \Pi(-a, x)\,\Pi(b, x)}$$
$$= \frac{a-b}{1-x+} \; \frac{(a-bx)(ax-b)}{1-x^3+} \; \frac{x(a-bx^2)(ax^2-b)}{1-x^5+} \; \frac{x^2(a-bx^3)(ax^3-b)}{1-x^7+} \; \&c$$

12.
$$\frac{\Pi(-a^2x^3, x^4)\,\Pi(-b^2x^3, x^4)}{\Pi(-a^2x, x^4)\,\Pi(-b^2x, x^4)}$$
$$= \frac{1}{1-ab+} \; \frac{(a-bx)(b-ax)}{(1+x^2)(1-ab)+} \; \frac{(a-bx^3)(b-ax^3)}{(1+x^4)(1-ab)+} \; \&c$$

13. $1 - ax + a^2x^3 - a^3x^6 + a^4x^{10} - \&c$
$$= \frac{1}{1+} \; \frac{ax}{1+} \; \frac{a(x^2-x)}{1+} \; \frac{ax^3}{1+} \; \frac{a(x^4-x^2)}{1+} \; \frac{ax^5}{1+} \; \&c$$

$$D_{2n} = 1 + ax^n \cdot \frac{1-x^n}{1-x} + a^2x^{2n} \cdot \frac{(1-x^n)(1-x^{n-1})}{(1-x)(1-x^2)} + a^3x^{3n} \cdot \frac{(1-x^n)(1-x^{n-1})(1-x^{n-2})}{(1-x)(1-x^2)(1-x^3)} + \&c$$

$$D_{2n+1} = 1 + (ax)\,x^n \cdot \frac{1-x^n}{1-x} + (ax)^2 x^{2n} \frac{(1-x^n)(1-x^{n-1})}{(1-x)(1-x^2)} + (ax)^3 x^{3n} \frac{(1-x^n)(1-x^{n-1})(1-x^{n-2})}{(1-x)(1-x^2)(1-x^3)} + \&c$$

14.
$$\int_0^\infty \frac{\Pi(ax, r)}{x^n\,\Pi(x, r)}\,dx = \frac{\pi}{\sin \pi n} \cdot \frac{\Pi(-a, r)\,\Pi(-r^n, r)}{\Pi(-r, r)\,\Pi(-ar^{n-1}, r)}.$$

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15. $$\frac{1+\frac{bx}{(1-x)(1-ax)}+\frac{b^2x^4}{(1-x)(1-x^2)(1-ax)(1-ax^2)}+\&c}{1+\frac{bx^2}{(1-x)(1-ax)}+\frac{b^2x^6}{(1-x)(1-x^2)(1-ax)(1-ax^2)}+\&c}$$

$$=1+\frac{bx}{1-ax+}\;\frac{bx^2}{1-ax^2+}\;\frac{bx^3}{1-ax^3+}\&c.$$

Cor. $$\frac{1+\frac{ax^2}{1-x}+\frac{a^2x^6}{(1-x)(1-x^2)}+\frac{a^3x^{12}}{(1-x)(1-x^2)(1-x^3)}+\&c}{1+\frac{ax}{1-x}+\frac{a^2x^4}{(1-x)(1-x^2)}+\frac{a^3x^9}{(1-x)(1-x^2)(1-x^3)}+\&c}$$

$$=\frac{1}{1+}\;\frac{ax}{1+}\;\frac{ax^2}{1+}\;\frac{ax^3}{1+}\;\frac{ax^4}{1+}\&c$$

16. If $$u=1+ax.\frac{1-x^n}{1-x}+a^2x^4.\frac{(1-x^{\overset{n-1}{n}})(1-x^{\overset{n-2}{n-1}})}{(1-x)(1-x^2)}+$$
$$a^3x^9.\frac{(1-x^{\overset{n-2}{n}})(1-x^{\overset{n-3}{n-1}})(1-x^{\overset{n-4}{n-2}})}{(1-x)(1-x^2)(1-x^3)}+\&c$$

& $$v=1+ax.\frac{x-x^n}{1-x}+a^2x^4.\frac{(x-x^{\overset{n-1}{n}})(x-x^{\overset{n-2}{n-1}})}{(1-x)(1-x^2)}+\&c,$$ then

$$\frac{u}{v}=1+\frac{ax}{1+}\;\frac{ax^2}{1+}\;\frac{ax^3}{1+}\cdots\cdots+\frac{ax^n}{1}.$$

17. $$\frac{\Pi(xy,x^2)\,\Pi(\frac{x}{y},x^2)\,\Pi(-x^2,x^2)\,\Pi(-\alpha\beta x^2,x^2)}{\Pi(\alpha xy,x^2)\,\Pi(\frac{\beta x}{y},x^2)\,\Pi(-\alpha x^2,x^2)\,\Pi(-\beta x^2,x^2)}$$

$$=1+\left\{xy\,\frac{1-\alpha}{1-\beta x^2}+\frac{x}{y}.\frac{1-\beta}{1-\alpha x^2}\right\}+$$
$$\left\{(xy)^2\frac{(1-\alpha)(x^2-\alpha)}{(1-\beta x^2)(1-\beta x^4)}+\left(\frac{x}{y}\right)^2\frac{(1-\beta)(x^2-\beta)}{(1-\alpha x^2)(1-\alpha x^4)}\right\}+$$
$$\left\{(xy)^3\frac{(1-\alpha)(x^2-\alpha)(x^4-\alpha)}{(1-\beta x^2)(1-\beta x^4)(1-\beta x^6)}+\left(\frac{x}{y}\right)^3\frac{(1-\beta)(x^2-\beta)(x^4-\beta)}{(1-\alpha x^2)(1-\alpha x^4)(1-\alpha x^6)}\right\}+\&c.$$

Cor. $$\frac{\Pi(xy,x^2)\,\Pi(\frac{x}{y},x^2)\,\Pi(-x^2,x^2)\,\Pi(-n^2x^2,x^2)}{\Pi(nxy,x^2)\,\Pi(\frac{nx}{y},x^2)\,\Pi(-nx^2,x^2)\,\Pi(-nx^2,x^2)}=$$

$$1+x(y+\tfrac{1}{y}).\frac{1-n}{1-nx^2}+x^2(y^2+\tfrac{1}{y^2}).\frac{(1-n)(x^2-n)}{(1-nx^2)(1-nx^4)}+$$

$$x^3(y^3+\tfrac{1}{y^3}).\frac{(1-n)(x^2-n)(x^4-n)}{(1-nx^2)(1-nx^4)(1-nx^6)}+\&c.$$

18. Let $f(a,b) = 1 + (a+b) + ab(a^2+b^2) + (ab)^3(a^3+b^3) + (ab)^6(a^4+b^4) + \&c$

then i. $f(a,b) = f(b,a)$; ii. $f(1,a) = 2f(a,a^3)$; iii. $f(-1,a) = 0$

iv. If n is any integer $f(a,b) = a^{\frac{n(n+1)}{2}}\, b^{\frac{n(n-1)}{2}}\, f\{a(ab)^n,\ b(ab)^{-n}\}$

N.B. If n is not an integer the result is approximately true.

19. $f(a,b) = \Pi(a,ab)\,\Pi(b,ab)\,\Pi(-ab,ab)$.

N.B. This result can be got like XVI 17 cor or as follows:—

We see from iv. that if $a(ab)^n$ or $b(ab)^n$ be equal to -1 then $f(a,b)=0$

and also if $(ab)^n = 1$, $f(a,b)\left\{1-\left(\frac{a}{b}\right)^{\frac{n}{2}}\right\} = 0$ & hence $f(a,b)=0$

Therefore $\Pi(a,ab)$, $\Pi(b,ab)$ & $\Pi(-ab,ab)$ are the factors of $f(a,b)$.

20. If $\alpha\beta = \pi$, then $\sqrt{\alpha}\, f\left(e^{-\alpha^2+n\alpha},\ e^{-\alpha^2-n\alpha}\right) =$

$$e^{\frac{n^2}{4}}\sqrt{\beta}\, f\left(e^{-\beta^2+in\beta},\ e^{-\beta^2-in\beta}\right).$$

21. $\log \Pi(a,x) = \frac{a}{1-x} - \frac{a^2}{2(1-x^2)} + \frac{a^3}{3(1-x^3)} - \frac{a^4}{4(1-x^4)} + \&c$

and consequently $\log f(a,b) = \log \Pi(-ab,ab) +$

$$\frac{a+b}{1-ab} - \frac{a^2+b^2}{2(1-a^2b^2)} + \frac{a^3+b^3}{3(1-a^3b^3)} - \frac{a^4+b^4}{4(1-a^4b^4)} + \&c$$

22. Let i. $\phi(x) = f(x,x) = 1 + 2x + 2x^4 + 2x^9 + 2x^{16} + \&c$

$$= \frac{1+x}{1-x}\cdot\frac{1-x^2}{1+x^2}\cdot\frac{1+x^3}{1-x^3}\cdot\frac{1-x^4}{1+x^4}\ \&c$$

ii. $\psi(x) = f(x,x^3) = 1 + x + x^3 + x^6 + x^{10} + x^{15} + \&c$

$$= \frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^3}\cdot\frac{1-x^6}{1-x^5}\cdot\frac{1-x^8}{1-x^7}\ \&c$$

iii. $f(-x) = f(-x,-x^2) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \&c$

$$= (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)\ \&c$$

iv. $\chi(x) = \Pi(x,x^2) = (1+x)(1+x^3)(1+x^5)(1+x^7)\ \&c.$

23\. i. $\log \phi(x) = 2\left\{ \frac{x}{1+x} + \frac{x^3}{3(1+x^3)} + \frac{x^5}{5(1+x^5)} + \&c \right\}$

ii. $\log \psi(x) = \frac{x}{1+x} + \frac{x^2}{2(1+x^2)} + \frac{x^3}{3(1+x^3)} + \&c$

iii. $\log f(-x) = -\left\{ \frac{x}{1-x} + \frac{x^2}{2(1-x^2)} + \frac{x^3}{3(1-x^3)} + \&c \right\}$

iv. $\log \chi(x) = \frac{x}{1-x^2} - \frac{x^2}{2(1-x^4)} + \frac{x^3}{3(1-x^6)} - \&c$

v. $\frac{\psi(x)}{\phi(x)} = \frac{1+x^2}{1+x} \cdot \frac{1+x^4}{1+x^3} \cdot \frac{1+x^6}{1+x^5} \cdot \&c$

ex. $\frac{11}{10} \cdot \frac{1111}{1110} \cdot \frac{111111}{111110} \&c = 1{\cdot}1010010001000010000010000001 + \ldots\ldots$

24\. i. $\frac{f(x)}{f(-x)} = \frac{\psi(x)}{\psi(-x)} = \frac{\chi(x)}{\chi(-x)} = \sqrt{\frac{\phi(x)}{\phi(-x)}}$.

ii. $f^3(-x) = \phi^2(-x)\,\psi(x) = 1 - 3x + 5x^3 - 7x^6 + 9x^{10} - \&c$

iii. $\chi(x) = \frac{f(x)}{f(-x^2)} = \sqrt[3]{\frac{\phi(x)}{\psi(-x)}} = \frac{\phi(x)}{f(x)} = \frac{f(-x^2)}{\psi(-x)}$.

iv. $f^3(-x^2) = \phi(-x)\,\psi^2(x)$ and $\chi(x)\chi(-x) = \chi(-x^2)$.

25\. i. $\phi(x) + \phi(-x) = 2\phi(x^4)$

ii. $\phi(x) - \phi(-x) = 4x\,\psi(x^8)$

iii. $\phi(x)\phi(-x) = \phi^2(-x^2)$ and $\psi(x)\psi(-x) = \psi(x^2)\,\phi(-x^2)$.

iv. $\phi(x)\,\psi(x^2) = \psi^2(x)$

v. $\phi^2(x) - \phi^2(-x) = 8x\,\psi^2(x^4)$

vi. $\phi^2(x) + \phi^2(-x) = 2\phi^2(x^2)$

vii. $\phi^4(x) - \phi^4(-x) = 16x\,\psi^4(x^2)$

Cor. If $\left(\frac{1-t}{1+t}\right)^2 = \left\{\frac{\phi(-x)}{\phi(x)}\right\}^4$, then $1 - t^2 = \left\{\frac{\phi(-x^2)}{\phi(x^2)}\right\}^4$.

26\. $x^{\frac{(m-n)^2}{8(m+n)}} \cdot f(x^m, x^n)$ is a perfect, complete, pure, double series of $\frac{1}{2}$ a degree.

Cor. i. $\phi(x)$, $\sqrt[8]{x}\,\psi(x)$ & $\sqrt[24]{x}\,f(x)$ are complete series of $\frac{1}{2}$ a degree

ii. $\frac{\chi(x)}{\sqrt[24]{x}}$ is a complete series of 0 degree.

27. i. $\sqrt{\alpha}\,\phi(e^{-\alpha^2}) = \sqrt{\beta}\,\phi(e^{-\beta^2})$ with $\alpha\beta = \pi$.

ii. $2\sqrt{\alpha}\,\psi(e^{-2\alpha^2}) = \sqrt{\beta}\,e^{\frac{\alpha^2}{4}}\,\phi(-e^{-\beta^2})$ with $\alpha\beta = \pi$.

iii. $e^{-\frac{\alpha}{12}}\sqrt[4]{\alpha}\,f(-e^{-2\alpha}) = e^{-\frac{\beta}{12}}\sqrt[4]{\beta}\,f(-e^{-2\beta})$ with $\alpha\beta = \pi^2$.

iv. $e^{-\frac{\alpha}{24}}\sqrt[4]{\alpha}\,f(e^{-\alpha}) = e^{-\frac{\beta}{24}}\sqrt[4]{\beta}\,f(e^{-\beta})$ with $\alpha\beta = \pi^2$.

v. $e^{\frac{\alpha}{24}}\chi(e^{-\alpha}) = e^{\frac{\beta}{24}}\chi(e^{-\beta})$ with $\alpha\beta = \pi^2$.

28. $f(a, p^{n-1}b)\,f(ap, p^{n-2}b)\,f(ap^2, p^{n-3}b) \cdots\cdots f(ap^{n-1}, b)$

$= f(a,b)\cdot\dfrac{\{f(-p^n)\}^n}{f(-p)}$. where $p = ab$.

Cor. $f(-x^2,-x^3)\,f(-x,-x^4) = f(-x)\,f(-x^5)$

$f(-x,-x^6)\,f(-x^2,-x^5)\,f(-x^3,-x^4) = f(-x)\,f^2(-x^7)$

and so on.

29. If $ab = cd$, then

i. $f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)\,f(ad,bc)$.

ii. $f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2a\,f\left(\frac{b}{c}, \frac{c}{b}\cdot abcd\right) f\left(\frac{b}{d}, \frac{d}{b}\cdot abcd\right)$

30. i. $f(a, ab^2)\,f(b, a^2b) = f(a,b)\,\psi(ab)$.

ii. $f(a,b) + f(-a,-b) = 2f(a^3b, ab^3)$.

iii. $f(a,b) - f(-a,-b) = 2a\,f\left(\frac{b}{a}, \frac{a}{b}\cdot a^4b^4\right)$

iv. $f(a,b)\,f(-a,-b) = f(-a^2,-b^2)\,\phi(-ab)$

v. $f^2(a,b) + f^2(-a,-b) = 2f^2(a^2,b^2)\,\phi(ab)$

vi. $f^2(a,b) - f^2(-a,-b) = 4a\,f\left(\frac{b}{a}, \frac{a}{b}\cdot a^2b^2\right)\psi(a^2b^2)$

Cor. If $ab = cd$, then

$f(a,b)f(c,d)f\left(an, \frac{b}{n}\right)f\left(cn, \frac{d}{n}\right) -$

$f(-a,-b)f(-c,-d)f\left(-an,-\frac{b}{n}\right)f\left(-cn,-\frac{d}{n}\right)$.

$= 2a\,f\left(\frac{c}{a}, ad\right) f\left(\frac{d}{an}, acn\right) f\left(n, \frac{ab}{n}\right)\psi(ab)$.

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31. If $u_n = a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}$ and $v_n = a^{\frac{n(n-1)}{2}} b^{\frac{n(n+1)}{2}}$, so that

$f(u_1, v_1) = 1 + (u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) + \&c$, then

$$f(u, v_1) = f(u_n, v_n) + u_1 f\left(\frac{v_{n-1}}{u_1}, \frac{u_{n+1}}{u_1}\right) + v_1 f\left(\frac{u_{n-1}}{v_1}, \frac{v_{n+1}}{v_1}\right)$$

$$+ u_2 f\left(\frac{v_{n-2}}{u_2}, \frac{u_{n+2}}{u_2}\right) + v_2 f\left(\frac{u_{n-2}}{v_2}, \frac{v_{n+2}}{v_2}\right)$$

$$+ u_3 f\left(\frac{v_{n-3}}{u_3}, \frac{u_{n+3}}{u_3}\right) + v_3 f\left(\frac{u_{n-3}}{v_3}, \frac{v_{n+3}}{v_3}\right)$$

$$+ \&c\ \&c \qquad + \&c\ \&c$$

e.g. i. $\phi(x) = \phi(x^9) + 2x f(x^3, x^{15}) = \phi(x^{25}) + 2x f(x^{15}, x^{35})$

$+ 2x^4 f(x^5, x^{45}) = \&c$

ii. $\psi(x) = f(x^3, x^6) + x\psi(x^9) = f(x^6, x^{10}) + x f(x^2, x^{14})$

$= f(x^{10}, x^{15}) + x f(x^5, x^{20}) + x^3 \psi(x^{25})$.

$= f(x^{15}, x^{21}) + x \cancel{\psi(x^9)} + x^3 f(x^3, x^{33}) + \&c\ \&c$

ex. i. $$\frac{\phi^2(x)}{\phi^2(-x)} + \frac{\phi^2(y)}{\phi^2(-y)} + \frac{\phi^2(z)}{\phi^2(-z)} + \frac{\phi^2(x)\,\phi^2(y)\,\phi^2(z)}{\phi^2(-x)\,\phi^2(-y)\,\phi^2(-z)}$$

$$= 4. \frac{\phi^2(x^2)\,\phi^2(y^2)\,\phi^2(z^2)}{\phi^2(-x^2)\,\phi^2(-y^2)\,\phi^2(-z^2)} + 256xyz.\frac{\psi^2(x^4)\,\psi^2(y^4)\,\psi^2(z^4)}{\phi^2(-x)\,\phi^2(-y)\,\phi^2(-z)}.$$

ii. $\dfrac{1}{\phi(x^4)} = \dfrac{1}{\phi(x) \pm \phi(x^2)} + \dfrac{1}{\phi(-x) \pm \phi(x^2)}$ and

$\dfrac{1}{\phi(-x^4)} = \dfrac{1}{\phi(-x^2) \pm \phi(x)} + \dfrac{1}{\phi(-x^2) \pm \phi(-x)}$.

iii. The coeff.t of x^n in the expansion of $\frac{x}{1-x}\psi(x^2)$ is the nearest integer to $\sqrt{n}$.

iv. $\phi(-x) + \phi(x^2) = 2.\dfrac{f^2(x^3, x^5)}{\psi(x)}$ and

$\phi(-x) - \phi(x^2) = -2x.\dfrac{f^2(x, x^7)}{\psi(x)}$.

v. $f(x, x^5) = \psi(-x^3)\chi(x)$.

32. i. $\dfrac{\phi'(x)}{\phi(x)} - \dfrac{\psi'(x)}{\psi(x)} = \dfrac{1-\phi^4(-x)}{8x}$ 197

ii. $\dfrac{\psi'(x)}{\psi(x)} - 2x\dfrac{\psi'(x^2)}{\psi(x^2)} = \dfrac{1-\phi^4(-x)}{8x}$

iii. $\dfrac{\phi'(x)}{\phi(x)} + \dfrac{\phi'(-x)}{\phi(x)} = \dfrac{\phi^4(x)-\phi^4(-x)}{4x}$

iv. $\dfrac{\phi'(x)}{\phi(x)} - \dfrac{\phi'(-x)}{\phi(-x)} = -4x\dfrac{\phi'(-x^2)}{\phi(-x^2)}$.

33. i. $\log(1+2x\cos\theta+2x^4\cos 2\theta+2x^9\cos 3\theta+\&c) - \log f(-x^2) = 2\left\{\dfrac{x}{1-x^2}\cos\theta - \dfrac{x^2}{2(1-x^2)}\cos 2\theta + \dfrac{x^3}{3(1-x^3)}\cos 3\theta - \&c\right\}$

ii. $\dfrac{1}{4}\log\dfrac{\sin n - x\sin 3n + x^3\sin 5n - x^6\sin 7n + \&c}{\sin n\,(1-3x+5x^3-7x^6+9x^{10}-\&c)}$

$= \dfrac{x\sin^2 n}{1(1-x)} + \dfrac{x^2\sin^2 2n}{2(1-x^2)} + \dfrac{x^3\sin^2 3n}{3(1-x^3)} + \&c$

iii. $1+\dfrac{4x\cos n}{1+x^2}+\dfrac{4x^2\cos 2n}{1+x^4}+\dfrac{4x^3\cos 3n}{1+x^6}+\&c$

$=\phi^2(-x^2)\,\dfrac{1+2x\cos n+2x^4\cos 2n+2x^9\cos 3n+\&c}{1-2x\cos n+2x^4\cos 2n-2x^9\cos 3n+\&c}$

Cor. $\dfrac{f(a,b)}{f(-a,-b)}\phi^2(-ab) = 1+2\left\{\dfrac{a+b}{1+ab}+\dfrac{a^2+b^2}{1+a^2b^2}+\dfrac{a^3+b^3}{1+a^3b^3}+\&c\right\}$

34. i. $\log\dfrac{\phi^2(x)}{1+4\cos n\left(\dfrac{x\cos n}{1-x}-\dfrac{x^3\cos 3n}{1-x^3}+\dfrac{x^5\cos 5n}{1-x^5}-\&c\right)}$

$=4\left\{\dfrac{x\sin^2 n}{1(1+x)}-\dfrac{x^2\sin^2 2n}{2(1+x^2)}+\dfrac{x^3\sin^2 3n}{3(1+x^3)}-\&c\right\}$

ii. $\dfrac{1}{8}\log\dfrac{\phi^2(x)}{1+\dfrac{4x\cos 2n}{1+x^2}+\dfrac{4x^2\cos 4n}{1+x^4}+\&c}$

$=\dfrac{x\sin^2 n}{1(1-x^2)}+\dfrac{x^3\sin^2 3n}{3(1-x^6)}+\dfrac{x^5\sin^2 5n}{5(1-x^{10})}+\&c$

Cor. i. $\frac{1}{4}\log\frac{\sin 2n - x\sin 4n + x^5\sin 8n - x^8\sin 10n + \&c}{\sin 2n\,(1 - 2x + 4x^5 - 5x^8 + 7x^{16} - \&c)}$

$$= \frac{x}{1+x}\sin^2 n + \frac{x^2}{2(1+x^2)}\sin^2 2n + \frac{x^3}{3(1+x^3)}\sin^2 3n + \&c$$

$$+ \frac{x^4}{1-x^4}\sin^2 2n + \frac{x^8}{2(1-x^8)}\sin^2 4n + \frac{x^{12}}{3(1-x^{12})}\sin^2 6n + \&c$$

ii. $\frac{1}{4}\log\frac{\sin n - x\sin 5n + x^2\sin 7n - x^5\sin 11n + \&c}{\sin n\,(1 - 5x + 7x^2 - 11x^5 + 13x^7 - \&c)}$

$$= \frac{x\sin^2 n}{1-x} + \frac{x^2\sin^2 2n}{2(1-x^2)} + \frac{x^3\sin^2 3n}{3(1-x^3)} + \&c$$

$$+ \frac{x\sin^2 2n}{1+x} + \frac{x^2\sin^2 4n}{2(1+x^2)} + \frac{x^3\sin^2 6n}{3(1+x^3)} + \&c.$$

35. i. If $P_n = \frac{B_n}{2n}\cos\frac{\pi n}{2} + \frac{1^{n-1}x}{1-x} + \frac{2^{n-1}x^2}{1-x^2} + \frac{3^{n-1}x^3}{1-x^3} + \&c$

and $Q_n = \frac{1}{n+1}\cdot\frac{1^{n+1} - 3^{n+1}x + 5^{n+1}x^3 - 7^{n+1}x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c}$

then $\frac{1}{2}Q_n = -2^n P_n - \frac{(n-1)(n-2)}{\lfloor 2}2^{n-2}P_{n-2}Q_2 -$

$$\frac{(n-1)(n-2)(n-3)(n-4)}{\lfloor 4}2^{n-4}P_{n-4}Q_4 - \&c$$

ii. If $P_n = \frac{B_n}{2n}(2^n - 1)\cos\frac{\pi n}{2} + \frac{1^{n-1}x}{1+x} - \frac{2^{n-1}x^2}{1+x^2} + \frac{3^{n-1}x^3}{1+x^3} - \&c$

and $Q_n = \frac{\frac{1}{4}E_{n+1}\cos\frac{\pi n}{2} + \frac{1^n x}{1-x} - \frac{3^n x^3}{1-x^3} + \frac{5^n x^5}{1-x^5} - \&c}{\frac{1}{4}E_1 + \frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \&c}$

then $\frac{1}{2}Q_n = 2^n P_n - \frac{(n-1)(n-2)}{\lfloor 2}2^{n-2}P_{n-2}Q_2 +$

$$\frac{(n-1)(n-2)(n-3)(n-4)}{\lfloor 4}2^{n-4}P_{n-4}Q_4 - \&c$$

N.B. Thus the series $1^{2n+1} - 3^{2n+1}x + 5^{2n+1}x^3 - 7^{2n+1}x^6 + \&c$ can be expressed in terms of L, M and N.

ex. i. $\dfrac{1^3 - 3^3 x + 5^3 x^3 - 7^3 x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c} = L.$

ii. $\dfrac{1^5 - 3^5 x + 5^5 x^3 - 7^5 x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c} = \dfrac{5L^2 - 2M}{3}.$

iii. $\dfrac{1^7 - 3^7 x + 5^7 x^3 - 7^7 x^6 + \&c}{1 - 3x + 5x^3 - 7x^6 + \&c} = \dfrac{35L^3 - 42LM + 16N}{9}.$

36. If $\frac{ab}{cd} = p$, then

i. $\frac{1}{2}\{f(a,b)f(c,d) + f(-a,-b)f(-c,-d)\}$

$$= f(ac,bd) + ad\, f\left(acp, \frac{bd}{p}\right) + bc\, f\left(bdp, \frac{ac}{p}\right)$$
$$+ (ad)^3 bc\, f\left(acp^2, \frac{bd}{p^2}\right) + (bc)^3 ad\, f\left(bdp^2, \frac{ac}{p^2}\right)$$
$$+ (ad)^6 (bc)^3 f\left(acp^3, \frac{bd}{p^3}\right) + (bc)^6 (ad)^3 f\left(bdp^3, \frac{ac}{p^3}\right)$$
$$+ \&c \quad \&c \qquad + \&c \quad \&c$$

ii. $\frac{1}{2}\{f(a,b)f(c,d) - f(-a,-b)f(-c,-d)\}$

$$= a f\left(\frac{c}{a}, \frac{a}{c}.abcd\right) + d f\left(\frac{b}{d}, \frac{d}{b}.abcd\right)$$
$$+ a^3 bc\, f\left(\frac{c}{ap}, \frac{ap}{c}.abcd\right) + d^3 bc\, f\left(\frac{bp}{d}, \frac{d}{bp}.abcd\right)$$
$$+ a^6 d (bc)^3 f\left(\frac{c}{ap^2}, \frac{ap^2}{c}.abcd\right) + ad^6 (bc)^3 f\left(\frac{bp^2}{d}, \frac{d}{bp^2}.abcd\right)$$
$$+ \&c \quad \&c \qquad + \&c \quad \&c$$

37. i. $\frac{1}{2}\{\phi(a)\phi(b) + \phi(-a)\phi(-b)\}$

$$= \phi(ab) + 2ab\, f\left(\frac{a^3}{b}, \frac{b^3}{a}\right) + 2(ab)^4 f\left(\frac{a^5}{b^3}, \frac{b^5}{a^3}\right) +$$
$$2(ab)^9 f\left(\frac{a^7}{b^5}, \frac{b^7}{a^5}\right) + \&c$$

ii. $\frac{1}{2}\{\phi(a)\phi(b) - \phi(-a)\phi(-b)\} = 2a\, f\left(\frac{b}{a}, a^3 b\right) +$

$$2a^4 b\, f\left(\frac{b^3}{a^3}, \frac{a^5}{b}\right) + 2a^9 b^4 f\left(\frac{b^5}{a^5}, \frac{a^7}{b^3}\right) + \&c$$

iii. $\psi(a)\psi(b) = \psi(ab) + a f\left(\frac{b}{a}, a^2\right) + a^3 b f\left(\frac{b^2}{a^2}, \frac{a^3}{b}\right) +$

$a^6 b^3 f\left(\frac{b^3}{a^3}, \frac{a^4}{b^2}\right) + \&c$

Cor. i. $\psi(x^3)\psi(x^{13}) - \psi(-x^3)\psi(-x^{13}) = x^3\{\psi(x)\psi(x^{39}) + \psi(-x)\psi(-x^{39})\}$

ii $\psi(x^5)\psi(x^{11}) - \psi(-x^5)\psi(-x^{11}) = x^5\{\psi(x)\psi(x^{55}) + \psi(-x)\psi(-x^{55})\}$

iii $\psi(x^7)\psi(x^9) - \psi(-x^7)\psi(-x^9) = x^6\{\psi(x)\psi(x^{63}) - \psi(-x)\psi(-x^{63})\}$

ex. $\psi(x)\psi(x^5) - \psi(-x)\psi(-x^5) = 2x \cdot \dfrac{\phi(-x^6)\phi(-x^{120})}{\chi(-x^2)\chi(-x^{40})} + 4x^{15}\psi(x^6)\psi(x^{120})$.

38. i. $\dfrac{f(-x^5)}{f(-x, -x^4)} = 1 + \dfrac{x}{1-x} + \dfrac{x^4}{(1-x)(1-x^2)} + \dfrac{x^9}{(1-x)(1-x^2)(1-x^3)} + \&c$

ii. $\dfrac{f(-x^5)}{f(-x^2, -x^3)} = 1 + \dfrac{x^2}{1-x} + \dfrac{x^6}{(1-x)(1-x^2)} + \dfrac{x^{12}}{(1-x)(1-x^2)(1-x^3)} + \&c$

iii. $\dfrac{f(-x, -x^4)}{f(-x^2, -x^3)} = \dfrac{1}{1+}\,\dfrac{x}{1+}\,\dfrac{x^2}{1+}\,\dfrac{x^3}{1+}\,\dfrac{x^4}{1+}\,\dfrac{x^5}{1+}\,\&c$

iv. $f^2(-x^2, -x^3) - \sqrt[5]{x^2}\, f^2(-x, -x^4) = f(-x)\{f(-\sqrt[5]{x}) + \sqrt[5]{x}\, f(-x^5)\}$

39. i. $\left\{\dfrac{\sqrt{5}+1}{2} + \dfrac{e^{-\frac{2\alpha}{5}}}{1+}\,\dfrac{e^{-2\alpha}}{1+}\,\dfrac{e^{-4\alpha}}{1+}\,\dfrac{e^{-6\alpha}}{1+}\,\dfrac{e^{-8\alpha}}{1+}\,\&c\right\} \times$

$\left\{\dfrac{\sqrt{5}+1}{2} + \dfrac{e^{-\frac{2\beta}{5}}}{1+}\,\dfrac{e^{-2\beta}}{1+}\,\dfrac{e^{-4\beta}}{1+}\,\dfrac{e^{-6\beta}}{1+}\,\dfrac{e^{-8\beta}}{1+}\,\&c\right\} = \dfrac{5+\sqrt{5}}{2}$.

ii $\left\{\dfrac{\sqrt{5}-1}{2} + \dfrac{e^{-\frac{\alpha}{5}}}{1-}\,\dfrac{e^{-\alpha}}{1+}\,\dfrac{e^{-2\alpha}}{1-}\,\dfrac{e^{-3\alpha}}{1+}\,\dfrac{e^{-4\alpha}}{1-}\,\&c\right\} \times$

$\left\{\dfrac{\sqrt{5}-1}{2} + \dfrac{e^{-\frac{\beta}{5}}}{1-}\,\dfrac{e^{-\beta}}{1+}\,\dfrac{e^{-2\beta}}{1-}\,\dfrac{e^{-3\beta}}{1+}\,\dfrac{e^{-4\beta}}{1-}\,\&c\right\} = \dfrac{5-\sqrt{5}}{2}$

with $\alpha\beta = \pi^2$ in both the cases.

Cor. i. $\dfrac{e^{-\frac{\pi}{5}}}{1-}\,\dfrac{e^{-\pi}}{1+}\,\dfrac{e^{-2\pi}}{1-}\,\dfrac{e^{-3\pi}}{1+}\,\dfrac{e^{-4\pi}}{1-}\,\&c = \sqrt{\dfrac{5-\sqrt{5}}{2}} - \dfrac{\sqrt{5}-1}{2}$

ii. $\dfrac{e^{-\frac{2\pi}{5}}}{1+}\,\dfrac{e^{-2\pi}}{1+}\,\dfrac{e^{-4\pi}}{1+}\,\dfrac{e^{-6\pi}}{1+}\,\dfrac{e^{-8\pi}}{1+}\,\&c = \sqrt{\dfrac{5+\sqrt{5}}{2}} - \dfrac{\sqrt{5}+1}{2}$

1. $$\int_0^{\frac{\pi}{2}} \frac{\cos\{(1-2n)\sin^{-1}(\sqrt{x}\sin\phi)\}}{\sqrt{1-x\sin^2\phi}}\,d\phi$$

$$= \frac{\pi}{2}\left\{1 + \frac{n(1-n)}{(\underline{|1})^2}x + \frac{n(n+1)(1-n)(2-n)}{(\underline{|2})^2}x^2 + \&c\right\}$$

Cor. i. $$\int_0^{\frac{\pi}{2}} \frac{\cos\{(1-2n)\sin^{-1}\left(\frac{\sin\phi}{\sqrt{2}}\right)\}}{\sqrt{1-\frac{1}{2}\sin^2\phi}}\,d\phi = \frac{\pi}{2}\cdot\frac{\sqrt{\pi}}{\underline{|-\frac{n}{2}}\;\underline{|\frac{n-1}{2}}}$$

ii. If $\int_0^{\frac{\pi}{2}} \frac{\cos(1-2n)\sin^{-1}(\sqrt{x}\sin\phi)}{\sqrt{1-x\sin^2\phi}}\,d\phi = u_x$, then

$$e^{-\pi\frac{u_{1-x}}{u_x}\operatorname{cosec}\pi n} = e^{\Sigma\frac{1}{-n}+\Sigma\frac{1}{n-1}} \times \{x + x^2(1 - 2.\overline{n-n^2})$$

$$+ x^3(1 - 7.\frac{n-n^2}{2} + 13.\overline{\frac{n-n^2}{2}}^2) + \&c\}$$

2. Let $F(x) = e^{-\pi\cdot\frac{1+(\frac{1}{2})^2(1-x)+(\frac{1.3}{2.4})^2(1-x)^2+\&c}{1+(\frac{1}{2})^2x+(\frac{1.3}{2.4})^2x^2+\&c}}$, then

i. $F(x) = \frac{x}{16}\cdot e^{4\frac{(\frac{1}{2})^2(\frac{1}{1.2})x + (\frac{1.3}{2.4})^2(\frac{1}{1.2}+\frac{1}{3.4})x^2 + \&c}{1+(\frac{1}{2})^2x+(\frac{1.3}{2.4})^2x^2+(\frac{1.3.5}{2.4.6})^2x^3+\&c}}$

ii. $F(1-e^{-x}) = \frac{x}{10+\sqrt{36+x^2}}$ very nearly.

iii. $\log F(x)\log F(1-x) = \pi^2$.

iv. $F(1-x) + F(1-\frac{1}{x}) = 0$

v. $F\left\{\frac{4x}{(1+x)^2}\right\} = \sqrt{F(x^2)}$

N.B. Suppose we know the expansion of $F(\frac{2x}{1+x})$ to n terms. Changing x to $\frac{x^2}{2-x^2}$ for x we have the expansion of $F(x^4)$ to $2n$ terms. i.e. that of $\left\{F\frac{4x}{(1+x)^2}\right\}^2$ to $2n$ terms. Extracting the square root and expanding the result in ascending powers of $\frac{2x}{1+x^2}$ we can find the expansions of $F\frac{2x}{1+x}$ to $2n$ terms.

Vi. $8F\left(\frac{2x}{1+x}\right) = x + \frac{5}{16}x^3 + \frac{369}{2048}x^5 + \frac{4097}{32768}x^7 + \frac{15948895}{16777216}x^9 + \&c$

Vii. $2F(1-e^{-8x}) = x - \frac{x^3}{3} + \frac{31}{120}x^5 - \frac{661}{2520}x^7 + \frac{219677}{725760}x^9 - \&c$

N.B. $F(0)=0$; $F(\tfrac{1}{2}) = e^{-\pi}$; $F(1)=1$; $F(\sqrt{2}-1)^2 = e^{-\pi\sqrt{2}}$; $F(\sqrt{2}-1)^4 = e^{-2\pi}$.

ex. $2F\left(1-e^{-\frac{8x}{1-x^2}}\right) = x + \frac{2}{3}x^3 + \frac{31}{120}x^5 + \frac{37}{1260}x^7 + \frac{5981}{725760}x^9 + \&c$

3. $\phi^2(x) = 1 + \left(\tfrac{1}{2}\right)^2\left\{1 - \frac{\phi^4(-x)}{\phi^4(x)}\right\} + \left(\tfrac{1.3}{2.4}\right)^2\left\{1 - \frac{\phi^4(-x)}{\phi^4(x)}\right\}^2 + \&c$

N.B. We know that $1 + \left(\tfrac{1}{2}\right)^2\left\{1 - \left(\frac{1-t}{1+t}\right)^2\right\} + \left(\tfrac{1.3}{2.4}\right)^2\left\{1 - \left(\frac{1-t}{1+t}\right)^4\right\} + \&c$

$= (1+t)\left\{1 + \left(\tfrac{1}{2}\right)^2 t^2 + \left(\tfrac{1.3}{2.4}\right)^2 t^4 + \&c\right\}$ and also that

$1 + \left(\tfrac{1}{2}\right)^2\left(\frac{1-t}{1+t}\right)^2 + \left(\tfrac{1.3}{2.4}\right)^2\left(\frac{1-t}{1+t}\right)^4 + \&c = \left(\frac{1+t}{2}\right)\left\{1 + \left(\tfrac{1}{2}\right)^2 t^2 + \&c\right\}$

Hence by $\overline{\underline{XVI}}$ 25 Cor. we have $1 + \left(\tfrac{1}{2}\right)^2\left\{1 - \frac{\phi^4(-x)}{\phi^4(x)}\right\} + \&c$

$$= \frac{\phi^2(x)}{\phi^2(x^2)}\left\{1 + \left(\tfrac{1}{2}\right)^2\left[1 - \frac{\phi^4(-x^2)}{\phi^4(x^2)}\right] + \&c\right\}$$

Consequently $1 + \left(\tfrac{1}{2}\right)^2\left\{1 - \frac{\phi^4(-x)}{\phi^4(x)}\right\} + \&c$

$$= \frac{\phi^2(x)}{\phi^2(x^n)}\left\{1 + \left(\tfrac{1}{2}\right)^2\left[1 - \frac{\phi^4(-x^n)}{\phi^4(x^n)}\right] + \&c\right\}$$

By making n infinite the above result is got.

In a similar manner we can show that

$$1 + \left(\tfrac{1}{2}\right)^2\left\{\frac{\phi^4(-x)}{\phi^4(x)}\right\} + \left(\tfrac{1.3}{2.4}\right)^2\left\{\frac{\phi^4(-x)}{\phi^4(x)}\right\}^2 + \&c = \frac{\phi^2(x)}{n\,\phi^2(x^n)}\left\{1 + \left(\tfrac{1}{2}\right)^2\frac{\phi^4(-x^n)}{\phi^4(x^n)} + \&c\right\}$$

from which we have

4. i. $F\left\{\frac{\phi^4(-x^n)}{\phi^4(x^n)}\right\} = \sqrt[n]{F\left\{\frac{\phi^4(-x)}{\phi^4(x)}\right\}}$ and similarly

ii. $F\left\{1 - \frac{\phi^4(-x)}{\phi^4(x)}\right\} = \sqrt[n]{F\left\{1 - \frac{\phi^4(-x^n)}{\phi^4(x^n)}\right\}}$ hence we have

5. $F\left\{1 - \frac{\phi^4(-x)}{\phi^4(x)}\right\} = x.$

6. $\phi^2\{F(x)\} = 1+\left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \left(\frac{1.3.5}{2.4.6}\right)^2 x^3 + \&c$ i.e.

If $Z = 1+\left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \&c$ and

$$y = \pi\,\frac{1+\left(\frac{1}{2}\right)^2(1-x)+\left(\frac{1.3}{2.4}\right)^2(1-x)^2+\&c}{1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1.3}{2.4}\right)^2 x^2+\&c}, \text{ then}$$

$$1+2e^{-y}+2e^{-4y}+2e^{-9y}+2e^{-16y}+\&c = \sqrt{Z}.$$

Cor. If $\alpha\beta=\pi$, then $\sqrt{\alpha}\left\{\frac{1}{2}+e^{-\alpha^2}+e^{-4\alpha^2}+\&c\right\} = \sqrt{\beta}\left\{\frac{1}{2}+e^{-\beta^2}+e^{-4\beta^2}+\&c\right\}$

ex. i. $1+2e^{-\pi}+2e^{-4\pi}+2e^{-9\pi}+\&c = \dfrac{\sqrt[4]{\pi}}{\underline{\lfloor -\tfrac{1}{4}}}$

ii. $1+2e^{-\pi\sqrt{2}}+2e^{-4\pi\sqrt{2}}+2e^{-9\pi\sqrt{2}}+\&c = \dfrac{\underline{\lfloor \tfrac{1}{8}}}{\sqrt{\pi}\,\underline{\lfloor \tfrac{1}{4}}}$.

iii. $1+2e^{-2\pi}+2e^{-8\pi}+2e^{-18\pi}+\&c = \dfrac{\sqrt[4]{\pi}}{2\,\underline{\lfloor -\tfrac{1}{4}}}\sqrt{2+\sqrt{2}}$.

iv. $\dfrac{\pi-\frac{1}{4}}{e^{\pi}}+\dfrac{4\pi-\frac{1}{4}}{e^{4\pi}}+\dfrac{9\pi-\frac{1}{4}}{e^{9\pi}}+\&c = \dfrac{1}{8}$.

7. i. If $\dfrac{\sin\alpha}{\sin\beta}=\sqrt{x}$, then $\displaystyle\int_0^\alpha \frac{d\phi}{\sqrt{x-\sin^2\phi}} = \int_0^\beta \frac{d\phi}{\sqrt{1-x\sin^2\phi}}$.

ii. If $\dfrac{\tan\alpha}{\tan\beta}=\sqrt{1-x}$, then $\displaystyle\int_0^\alpha \frac{d\phi}{\sqrt{1-x\cos^2\phi}} = \int_0^\beta \frac{d\phi}{\sqrt{1-x\sin^2\phi}}$.

iii. If $\dfrac{\tan\alpha}{\tan\beta}=\sqrt{1-b}$, then $\displaystyle\int_0^\beta \frac{d\phi}{\sqrt{(1-a\sin^2\phi)(1-b\sin^2\phi)}}$

$$= \frac{1}{\sqrt{1-b}}\int_0^\alpha \frac{d\phi}{\sqrt{1-\frac{a-b}{1-b}\sin^2\phi}}$$

iv. If $\dfrac{\tan\alpha}{\tan\beta}=\sqrt{1+x}$ then $\displaystyle\int_0^\alpha \frac{d\phi}{\sqrt{1+x\cos 2\phi}} = \int_0^\beta \frac{d\phi}{\sqrt{1-x^2\sin^2\phi}}$

v. If $\cot\alpha\cot\beta=\sqrt{1-x}$, then $\displaystyle\int_0^\alpha \frac{d\phi}{\sqrt{1-x\sin^2\phi}} + \int_0^\beta \frac{d\phi}{\sqrt{1-x\sin^2\phi}}$

$$= \frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1.3}{2.4}\right)^2 x^2+\left(\frac{1.3.5}{2.4.6}\right)^2 x^3+\&c\right\}$$

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vi. If $\cot\alpha \tan\frac{\beta}{2} = \sqrt{1-x\sin^2\alpha}$, then

$$2\int_0^{\alpha}\frac{d\phi}{\sqrt{1-x\sin^2\phi}} = \int_0^{\beta}\frac{d\phi}{\sqrt{1-x\sin^2\phi}}$$

vii If $\alpha = \log\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right)$, then

$$\int_0^{\alpha i}\frac{d\phi}{\sqrt{1-x\sin^2\phi}} = i\int_0^{\beta}\frac{d\phi}{\sqrt{1-(1-x)\sin^2\phi}}$$

viii If $\int_0^{\alpha}\frac{d\phi}{\sqrt{1-x\sin^2\phi}} + \int_0^{\beta}\frac{d\phi}{\sqrt{1-x\sin^2\phi}} = \int_0^{\gamma}\frac{d\phi}{\sqrt{1-x\sin^2\phi}}$, then

$$\tan\frac{\gamma}{2} = \frac{\sin\alpha\sqrt{1-x\sin^2\beta} + \sin\beta\sqrt{1-x\sin^2\alpha}}{\cos\alpha+\cos\beta}$$

$$\tan^{-1}\left(\tan\alpha\sqrt{1-x\sin^2\beta}\right) + \tan^{-1}\left(\tan\beta\sqrt{1-x\sin^2\alpha}\right) = \gamma$$

or $\cot\alpha\cot\beta = \frac{\cos\gamma}{\sin\alpha\sin\beta} + \sqrt{1-x\sin^2\gamma}$ or

$$\frac{\sqrt{x}}{2} = \frac{\sqrt{\sin s\sin(s-\alpha)\sin(s-\beta)\sin(s-\gamma)}}{\sin\alpha\sin\beta\sin\gamma},\text{ where } 2s = \alpha+\beta+\gamma$$

ix. $$\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1+x\sin\phi}} = \int_0^{\frac{\pi}{2}}\frac{\cos^{-1}(x\sin^2\phi)}{\sqrt{1-x^2\sin^4\phi}}\,d\phi$$

x. $$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{d\theta\,d\phi}{\sqrt{(1-x\sin^2\theta)(1-x\sin^2\theta\sin^2\phi)}} = \left\{\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-x\sin^2\phi}}\right\}^2.$$

xi. $$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{x\sin\phi\,d\theta\,d\phi}{\sqrt{1-x^2\sin^2\phi}\sqrt{1-x^2\sin^2\theta\sin^2\phi}}$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\sin^{-1}x}\frac{d\theta}{\sqrt{1-x^2\sin^2\phi-\sin^2\theta\cos^2\phi}}\,d\phi$$

$$= \frac{1}{2}\left\{\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\frac{1+x}{2}\sin^2\phi}}\right\}^2 - \frac{1}{2}\left\{\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\frac{1-x}{2}\sin^2\phi}}\right\}^2$$

xii. If $\dfrac{\sin\beta}{\sin\alpha} = \dfrac{1+x}{1+x\sin^2\alpha}$, then

$$(1+x)\int_0^\alpha \frac{d\phi}{\sqrt{1-x^2\sin^2\phi}} = \int_0^\beta \frac{d\phi}{\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\phi}}.$$

xiii. If $x\sin\alpha = \sin(2\beta-\alpha)$, then

$$(1+x)\int_0^\alpha \frac{d\phi}{\sqrt{1-x^2\sin^2\phi}} = 2\int_0^\beta \frac{d\phi}{\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\phi}}.$$

8. i. $\phi^2(x) = 1 + 4\left(\frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \&c\right)$

ii. $\phi^4(x) = 1 + 8\left(\frac{x}{1-x} + \frac{2x^2}{1+x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1+x^4} + \&c\right)$

iii. $\phi(x)\phi(x^2) = 1 + \frac{2x}{1-x} + \frac{2x^3}{1-x^3} - \frac{2x^5}{1-x^5} - \frac{2x^7}{1-x^7} + \&c$

iv. $\phi(x)\phi(x^3) = 1 + \frac{2x}{1-x} - \frac{2x^2}{1+x^2} + \frac{2x^4}{1+x^4} - \frac{2x^5}{1-x^5} + \frac{2x^7}{1-x^7} - \&c$

v. $\phi^2(-x) = 1 - \frac{4x}{1+x} + \frac{4x^3}{1+x^2} - \frac{4x^6}{1+x^3} + \frac{4x^{10}}{1+x^4} - \&c$

vi. $\psi(x)\phi(x^2) = \frac{1+x}{1-x} - x\cdot\frac{1+x^3}{1-x^3} + x^3\cdot\frac{1+x^5}{1-x^5} - x^6\cdot\frac{1+x^7}{1-x^7} + \&c$

vii. $\psi^2(x) = \frac{1+x}{1-x} - x^2\cdot\frac{1+x^3}{1-x^3} + x^6\cdot\frac{1+x^5}{1-x^5} - x^{12}\cdot\frac{1+x^7}{1-x^7} + \&c$

viii. $\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \&c$

$= x\cdot\frac{1+x}{(1-x)^2} - x^3\cdot\frac{1+x^2}{(1-x^2)^2} + x^6\cdot\frac{1+x^3}{(1-x^3)^2} - x^{10}\cdot\frac{1+x^4}{(1-x^4)^2} + \&c$

ix. $\phi^2(-x)f(-x) = 1 - 5x + 7x^2 - 11x^5 + 13x^7 - \&c$

x. $\psi(x^2)f^2(-x) = 1 - 2x + 4x^5 - 5x^8 + 7x^{16} - \&c$

xi. $f(-x)f(-x^2) = \phi(-x)\psi(x)$

xii. $\dfrac{f(-x)}{f(-x^4)} = \dfrac{\phi(-x^2)}{\psi(x)}$.

ex. $\psi(x^4)f^2(-x) + 2x\psi(x^8)f^2(-x^4) = \phi^2(-x^8)f(-x^8)$.

9. Let $y = \pi \cdot \dfrac{1 + \left(\frac{1}{2}\right)^2 (1-x) + \left(\frac{1.3}{2.4}\right)^2 (1-x)^2 + \&c}{1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \&c}$

and $z = 1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \&c$ such that $e^{-y} = F(x)$, then

i. $\dfrac{dy}{dx} = -\dfrac{1}{x(1-x)\,z^2}$. ii. $\dfrac{dz}{dx} = \dfrac{\int_0 z\,dx}{4x(1-x)}$.

iii. $z \int\int x^n (1-x)\, z^3 (dy)^2 = \dfrac{x^n}{n^2}\left\{1 + \left(\dfrac{n+\frac{1}{2}}{n+1}\right)^2 x + \left(\dfrac{n+\frac{1}{2}}{n+1} \cdot \dfrac{n+1\frac{1}{2}}{n+2}\right)^2 x^2 + \&c\right\}$

iv. $1 - 24\left(\dfrac{1}{e^{2y}-1} + \dfrac{2}{e^{4y}-1} + \dfrac{3}{e^{6y}-1} + \dfrac{4}{e^{8y}-1} + \&c\right)$

$= (1-2x)\,z^2 + 6x(1-x)\,z \cdot \dfrac{dz}{dx}$.

ex. $16^{11} e^{-11y} = x^{11} + \frac{11}{2} x^{12} + \frac{1111}{64} x^{13} + \frac{111111}{2688} x^{14} + \&c$

10. i. $\phi(e^{-y}) = \sqrt{z}$. ii $\phi(-e^{-y}) = \sqrt{z} \cdot \sqrt[4]{1-x}$.

iii. $\phi(-e^{-2y}) = \sqrt{z}\,\sqrt[8]{1-x}$. iv. $\phi(e^{-2y}) = \sqrt{z}\,\sqrt{\dfrac{1+\sqrt{1-x}}{2}}$.

v. $\phi(e^{-4y}) = \sqrt{z} \cdot \dfrac{1+\sqrt[4]{1-x}}{2}$.

vi $\phi(e^{-\frac{y}{2}}) = \sqrt{z}\,\sqrt{1+\sqrt{x}}$. vii $\phi(-e^{-\frac{y}{2}}) = \sqrt{z}\,\sqrt{1-\sqrt{x}}$

viii $\phi(e^{-\frac{y}{4}}) = \sqrt{z}\,(1+\sqrt[4]{x})$ ix. $\phi(-e^{-\frac{y}{4}}) = \sqrt{z}\,(1-\sqrt[4]{x})$.

11. i. $\psi(e^{-y}) = \sqrt{\dfrac{z}{2}} \cdot \sqrt[8]{x e^{y}}$. ii. $\psi(-e^{-y}) = \sqrt{\dfrac{z}{2}}\,\sqrt[8]{x(1-x)e^{y}}$.

iii. $\psi(e^{-2y}) = \frac{1}{2}\sqrt{z}\,\sqrt[4]{x e^{y}}$ iv. $\psi(e^{-4y}) = \frac{1}{2}\sqrt{\dfrac{z}{2}}\,\sqrt{(1-\sqrt{1-x})e^{y}}$.

v. $\psi(e^{-8y}) = \dfrac{\sqrt{z}}{4}(1-\sqrt[4]{1-x})e^{y}$.

vi. $\psi(e^{-\frac{y}{2}}) = \sqrt{z} \cdot \sqrt[4]{\dfrac{1+\sqrt{x}}{2}}\,\sqrt[16]{x e^{y}}$.

vii. $\psi(-e^{-\frac{y}{2}}) = \sqrt{z}\,\sqrt[4]{\dfrac{1-\sqrt{x}}{2}}\,\sqrt[16]{x e^{y}}$

viii. $\psi(e^{-\frac{y}{4}}) = \sqrt{z}\,\sqrt{1+\sqrt[4]{x}}\,\sqrt[8]{\dfrac{1+\sqrt{x}}{2}}\,\sqrt[32]{x e^{y}}$.

ix. $\psi(-e^{-\frac{y}{4}}) = \sqrt{z}\,\sqrt{1-\sqrt[4]{x}}\,\sqrt[8]{\dfrac{1+\sqrt{x}}{2}}\,\sqrt[32]{x e^{y}}$.

12. i. $f(e^{-y}) = \dfrac{\sqrt{z}}{\sqrt[6]{2}}\,\sqrt[24]{x(1-x)e^{y}}$. ii. $f(-e^{-y}) =$

$\dfrac{\sqrt{z}}{\sqrt[6]{2}}\,\sqrt[6]{1-x}\,\sqrt[24]{x e^{y}}$.

iii $f(-e^{-2y}) = \frac{\sqrt{2}}{\sqrt[3]{2}} \sqrt[12]{x(1-x)e^y}$. iv. $f(-e^{-4y}) = \frac{\sqrt{2}}{\sqrt[3]{4}} \sqrt[24]{1-x}\,\sqrt[6]{x e^y}$.

v. $\chi(e^{-y}) = \frac{\sqrt[6]{2}}{\sqrt[24]{x(1-x)e^y}}$. vi $\chi(-e^{-y}) = \frac{\sqrt[6]{2}\,\sqrt[12]{1-x}}{\sqrt[24]{x e^y}}$.

vii. $\chi(-e^{-2y}) = \frac{\sqrt[3]{2}\cdot\sqrt[24]{1-x}}{\sqrt[12]{x e^y}}$.

13. i. $1 + 240\left(\frac{1^3}{e^{2y}-1} + \frac{2^3}{e^{4y}-1} + \frac{3^3}{e^{6y}-1} + \frac{4^3}{e^{8y}-1} + \&c\right\}$

$= z^4(1 - x + x^2)$.

ii $1 - 504\left(\frac{1^5}{e^{2y}-1} + \frac{2^5}{e^{4y}-1} + \frac{3^5}{e^{6y}-1} + \frac{4^5}{e^{8y}-1} + \&c\right\}$

$= z^6(1+x)(1-\frac{x}{2})(1-2x)$.

iii $1 + 240\left(\frac{1^3}{e^y-1} + \frac{2^3}{e^{2y}-1} + \frac{3^3}{e^{3y}-1} + \&c\right)$

$= z^4(1 + 14x + x^2)$

iv. $1 - 504\left(\frac{1^5}{e^y-1} + \frac{2^5}{e^{2y}-1} + \frac{3^5}{e^{3y}-1} + \&c\right)$

$= z^6(1+x)(1 - 34x + x^2)$.

v. $1 + 240\left(\frac{1^3}{e^{4y}-1} + \frac{2^3}{e^{8y}-1} + \frac{3^3}{e^{12y}-1} + \&c\right)$

$= z^4(1 - x + \frac{x^2}{16})$

vi. $1 - 504\left(\frac{1^5}{e^{4y}-1} + \frac{2^5}{e^{8y}-1} + \frac{3^5}{e^{12y}-1} + \&c\right)$

$= z^6(1-\frac{x}{2})(1 - x - \frac{x^2}{32})$.

vii. If x is changed to $\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2$ then y is changed to $2y$

viii $1 + 24\left(\frac{1}{e^y+1} + \frac{2}{e^{2y}+1} + \frac{3}{e^{3y}+1} + \frac{4}{e^{4y}+1} + \&c\right)$

$= z^2(1+x)$.

ix. $1 + 24\left(\frac{1}{e^{2y}+1} + \frac{2}{e^{4y}+1} + \frac{3}{e^{6y}+1} + \&c\right)$

$= z^2(1-\frac{x}{2})$.

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x. $1-240\left(\frac{1^3}{e^y+1}+\frac{2^3}{e^{2y}+1}+\frac{3^3}{e^{3y}+1}+\&c\right)$

$=z^4(1-16x+x^2)$.

xi. $1+504\left(\frac{1^5}{e^y+1}+\frac{2^5}{e^{2y}+1}+\frac{3^5}{e^{3y}+1}+\&c\right)$

$=z^6(1+x)(1+29x+x^2)$.

xii. $1-240\left(\frac{1^3}{e^{2y}+1}+\frac{2^3}{e^{4y}+1}+\frac{3^3}{e^{6y}+1}+\&c\right)$

$=z^4\left(1-x-\frac{7}{8}x^2\right)$

xiii. $1+504\left(\frac{1^5}{e^{2y}+1}+\frac{2^5}{e^{4y}+1}+\frac{3^5}{e^{6y}+1}+\&c\right)$

$=z^6\left(1-\frac{x}{2}\right)\left(1-x+\frac{31}{16}x^2\right)$.

14. i. $1-8\left(\frac{1}{e^y+1}-\frac{2}{e^{2y}+1}+\frac{3}{e^{3y}+1}-\&c\right)=z^2(1-x)$.

ii. $1+16\left(\frac{1^3}{e^y+1}-\frac{2^3}{e^{2y}+1}+\frac{3^3}{e^{3y}+1}-\&c\right)=z^4(1-x^2)$.

iii. $1-8\left(\frac{1^5}{e^y+1}-\frac{2^5}{e^{2y}+1}+\frac{3^5}{e^{3y}+1}-\&c\right)=z^6(1-x)(1-x+x^2)$.

iv. $17+32\left(\frac{1^7}{e^y+1}-\frac{2^7}{e^{2y}+1}+\frac{3^7}{e^{3y}+1}-\&c\right)=z^8(1-x^2)(17-32x+x^2)$

v. $1-16\left(\frac{1^3}{e^y-1}-\frac{2^3}{e^{2y}-1}+\frac{3^3}{e^{3y}-1}-\&c\right)=z^4(1-x)^2$

vi. $1+8\left(\frac{1^5}{e^y-1}-\frac{2^5}{e^{2y}-1}+\frac{3^5}{e^{3y}-1}-\&c\right)=z^6(1+x)(1-x^2)$

vii $17-32\left(\frac{1^7}{e^y-1}-\frac{2^7}{e^{2y}-1}+\frac{3^7}{e^{3y}-1}-\&c\right)=$

$z^8(1-x)^2(17-2x+17x^2)$.

viii. $31+8\left(\frac{1^9}{e^y-1}-\frac{2^9}{e^{2y}-1}+\frac{3^9}{e^{3y}-1}-\&c\right)=$

$z^{10}(1-x)(1-x^2)(31-46x+31x^2)$

ix. $1-16\left(\frac{1^3}{e^{2y}-1}-\frac{2^3}{e^{4y}-1}+\&c\right)=z^4(1-x)$

IX. $1+8\left(\frac{1^5}{e^{2y}-1}-\frac{2^5}{e^{4y}-1}+\&c\right)=z^6(1-x)(1-\frac{x}{2})$.

xi. $17-32\left(\frac{1^7}{e^{2y}-1}-\frac{2^7}{e^{4y}-1}+\frac{3^7}{e^{6y}-1}-\&c\right)=$

$$z^8(1-x)(17-17x+2x^2)$$

xii. If x is changed to $-\frac{x}{1-x}$ then e^{-y} is changed to $-e^{-y}$.

15. i. $\frac{1^3}{e^y-e^{-y}}+\frac{2^3}{e^{2y}-e^{-2y}}+\frac{3^3}{e^{3y}-e^{-3y}}+\&c=z^4\,\frac{x}{16}$.

ii. $\frac{1^5}{e^y-e^{-y}}+\frac{2^5}{e^{2y}-e^{-2y}}+\frac{3^5}{e^{3y}-e^{-3y}}+\&c=z^6.\,\frac{x(1+x)}{16}$.

iii. $\frac{1^7}{e^y-e^{-y}}+\frac{2^7}{e^{2y}-e^{-2y}}+\frac{3^7}{e^{3y}-e^{-3y}}+\&c=$

$$z^8.\,\frac{x(1+6\tfrac{1}{2}x+x^2)}{16}.$$

iv. $\frac{1^9}{e^y-e^{-y}}+\frac{2^9}{e^{2y}-e^{-2y}}+\frac{3^9}{e^{3y}-e^{-3y}}+\&c$

$$=z^{10}.\,\frac{x(1+x)(1+29x+x^2)}{16}.$$

v. $\frac{1^3}{e^{2y}-e^{-2y}}+\frac{2^3}{e^{4y}-e^{-4y}}+\frac{3^3}{e^{6y}-e^{-6y}}+\&c=z^4\,\frac{x^2}{256}$.

vi. $\frac{1^5}{e^{2y}-e^{-2y}}+\frac{2^5}{e^{4y}-e^{-4y}}+\frac{3^5}{e^{6y}-e^{-6y}}+\&c=z^6\,\frac{x^2}{256}(1-\frac{x}{2})$.

vii. $\frac{1^7}{e^{2y}-e^{-2y}}+\frac{2^7}{e^{4y}-e^{-4y}}+\frac{3^7}{e^{6y}-e^{-6y}}+\&c=z^8\,\frac{x^2(1-x+\frac{17}{32}x^2)}{256}$

viii. $\frac{1^9}{e^{2y}-e^{-2y}}+\frac{2^9}{e^{4y}-e^{-4y}}+\frac{3^9}{e^{6y}-e^{-6y}}+\&c=$

$$z^{10}.\,\frac{x^2\cancel{(1-x+\frac{31}{16}x^2)}}{256}.\left(1-\frac{x}{2}\right)$$

ix. $\frac{1}{e^y-e^{-y}}+\frac{3}{e^{3y}-e^{-3y}}+\frac{5}{e^{5y}-e^{-5y}}+\&c=z^2.\,\frac{x}{16}$

x. $\frac{1^3}{e^y-e^{-y}}+\frac{3^3}{e^{3y}-e^{-3y}}+\frac{5^3}{e^{5y}-e^{-5y}}+\&c=z^4.\,\frac{x}{16}(1-\frac{x}{2})$.

xi. $$\frac{1^5}{e^{y}-e^{-y}}+\frac{3^5}{e^{3y}-e^{-3y}}+\frac{5^5}{e^{5y}-e^{-5y}}+\&c = z^6 \cdot \frac{x}{16}(1-x+x^2).$$

xii. $$\frac{1^7}{e^{y}-e^{-y}}+\frac{3^7}{e^{3y}-e^{-3y}}+\frac{5^7}{e^{5y}-e^{-5y}}+\&c = z^8\, \frac{x}{16}\left(1-\frac{x}{2}\right)\left(1-x+\frac{17}{2}x^2\right).$$

xiii. $$\frac{1}{e^{\frac{y}{2}}-e^{-\frac{y}{2}}}+\frac{3}{e^{\frac{3y}{2}}-e^{-\frac{3y}{2}}}+\frac{5}{e^{\frac{5y}{2}}-e^{-\frac{5y}{2}}}+\&c = z^2 \cdot \frac{\sqrt{x}}{4}$$

xiv. $$\frac{1^3}{e^{\frac{y}{2}}-e^{-\frac{y}{2}}}+\frac{3^3}{e^{\frac{3y}{2}}-e^{-\frac{3y}{2}}}+\frac{5^3}{e^{\frac{5y}{2}}-e^{-\frac{5y}{2}}}+\&c = z^4 \cdot \frac{\sqrt{x}}{4}(1+x).$$

xv. $$\frac{1^5}{e^{\frac{y}{2}}-e^{-y/2}}+\frac{3^5}{e^{3y/2}-e^{-3y/2}}+\frac{5^5}{e^{5y/2}-e^{-5y/2}}+\&c = z^6 \cdot \frac{\sqrt{x}}{4}(1+14x+x^2).$$

xvi. $$\frac{1^7}{e^{y/2}-e^{-y/2}}+\frac{3^7}{e^{3y/2}-e^{-3y/2}}+\frac{5^7}{e^{5y/2}-e^{-5y/2}}+\&c =$$
$$z^8\, \frac{\sqrt{x}}{4}(1+x)(1+134x+x^2)$$

16. i. $$\frac{1}{e^{y/2}+e^{-y/2}}-\frac{3}{e^{3y/2}+e^{-3y/2}}+\frac{5}{e^{5y/2}+e^{-5y/2}}-\&c = \frac{z^2}{4}\sqrt{x(1-x)}.$$

ii. $$\frac{1^3}{e^{y/2}+e^{-y/2}}-\frac{3^3}{e^{3y/2}+e^{-3y/2}}+\frac{5^3}{e^{5y/2}+e^{-5y/2}}-\&c = \frac{z^4}{4}\sqrt{x(1-x)}\,(1-2x)$$

iii. $$\frac{1^5}{e^{y/2}+e^{-y/2}}-\frac{3^5}{e^{3y/2}+e^{-3y/2}}+\frac{5^5}{e^{5y/2}+e^{-5y/2}}-\&c = \frac{z^6}{4}\sqrt{x(1-x)}\,\{1-16x(1-x)\}$$

iv. $$\frac{1^7}{e^{y/2}+e^{-y/2}}-\frac{3^7}{e^{3y/2}+e^{-3y/2}}+\frac{5^7}{e^{5y/2}+e^{-5y/2}}-\&c$$
$$= \frac{z^8}{4}\sqrt{x(1-x)}\,(1-2x)\{1-136x(1-x)\}$$

v. $$\frac{1^9}{e^{y/2}+e^{-y/2}}-\frac{3^9}{e^{3y/2}+e^{-3y/2}}+\frac{5^9}{e^{5y/2}+e^{-5y/2}}-\&c$$
$$= \frac{z^{10}}{4}\sqrt{x(1-x)}\,\{1-1232x(1-x)+7936x^2(1-x)^2\}$$

vi. $$\frac{1^{11}}{e^{y/2}+e^{-y/2}}-\frac{3^{11}}{e^{3y/2}+e^{-3y/2}}+\frac{5^{11}}{e^{5y/2}+e^{-5y/2}}-\&c$$
$$= \frac{z^{12}}{4}\sqrt{x(1-x)}\,(1-2x)\{1-11072x(1-x)+176896x^2(1-x)^2\}$$

vii. $\tan^{-1} e^{-y/2} - \tan^{-1} e^{-3y/2} + \tan^{-1} e^{-5y/2} - \&c = \frac{1}{2}\sin^{-1}\sqrt{x}$.

viii. $\tan^{-1} e^{-y/4} - \tan^{-1} e^{-3y/4} + \tan^{-1} e^{-5y/4} - \&c = \frac{1}{2}\tan^{-1}\sqrt[4]{x}$

ix $\frac{1}{e^{\frac{y}{2}} + e^{-\frac{y}{2}}} + \frac{1}{e^{3y/2} + e^{-3y/2}} + \frac{1}{e^{5y/2} + e^{-5y/2}} = Z.\frac{\sqrt{x}}{4}$.

x. $\frac{1^2}{e^{y/2} + e^{-y/2}} + \frac{3^2}{e^{3y/2} + e^{-3y/2}} + \frac{5^2}{e^{5y/2} + e^{-5y/2}} = Z^3\ \frac{\sqrt{x}}{4}$.

xi. $\frac{1^4}{e^{y/2} + e^{-y/2}} + \frac{3^4}{e^{3y/2} + e^{-3y/2}} + \frac{5^4}{e^{5y/2} + e^{-5y/2}} = Z^5\frac{\sqrt{x}}{4}(1+4x)$

xii $\frac{1^6}{e^{y/2} + e^{-y/2}} + \frac{3^6}{e^{3y/2} + e^{-3y/2}} + \frac{5^6}{e^{5y/2} + e^{-5y/2}} + \&c =$

$Z^7\frac{\sqrt{x}}{4}\left\{1 + 11(4x) + (4x)^2\right\}$

xiii. $\frac{1^8}{e^{y/2} + e^{-y/2}} + \frac{3^8}{e^{3y/2} + e^{-3y/2}} + \frac{5^8}{e^{5y/2} + e^{-5y/2}} + \&c$

$= Z^9\frac{\sqrt{x}}{4}\left\{1 + 57(4x) + 102(4x)^2 + (4x)^3\right\}$.

17. i. $1 + 4\left(\frac{1}{e^{y} + e^{-y}} + \frac{1}{e^{2y} + e^{-2y}} + \frac{1}{e^{3y} + e^{-3y}} + \&c\right) = Z$.

ii. $4\left(\frac{1^2}{e^{y} + e^{-y}} + \frac{2^2}{e^{2y} + e^{-2y}} + \frac{3^2}{e^{3y} + e^{-3y}} + \&c\right) = Z^3.\frac{x}{4}$.

iii. $4\left(\frac{1^4}{e^{y} + e^{-y}} + \frac{2^4}{e^{2y} + e^{-2y}} + \frac{3^4}{e^{3y} + e^{-3y}} + \&c\right) = Z^5\left\{\frac{x}{4} + \left(\frac{x}{4}\right)^2\right\}$

iv. $4\left(\frac{1^6}{e^{y} + e^{-y}} + \frac{2^6}{e^{2y} + e^{-2y}} + \frac{3^6}{e^{3y} + e^{-3y}} + \&c\right) = Z^7\left\{\frac{x}{4} + 11\left(\frac{x}{4}\right)^2 + \left(\frac{x}{4}\right)^3\right\}$

v. $4\left(\frac{1^8}{e^{y} + e^{-y}} + \frac{2^8}{e^{2y} + e^{-2y}} + \frac{3^8}{e^{3y} + e^{-3y}} + \&c\right) = Z^9\left\{\frac{x}{4} + 57\left(\frac{x}{4}\right)^2 + 102\left(\frac{x}{4}\right)^3 + \left(\right.\right.$

vi. $1 + 4\left(\frac{1}{e^{y} - 1} - \frac{1}{e^{3y} - 1} + \frac{1}{e^{5y} - 1} - \&c\right) = Z$.

vii $1 - 4\left(\frac{1^2}{e^{y} - 1} - \frac{3^2}{e^{3y} - 1} + \frac{5^2}{e^{5y} - 1} - \&c\right) = Z^3(1-x)$

viii $5 + 4\left(\frac{1^4}{e^{y} - 1} - \frac{3^4}{e^{3y} - 1} + \frac{5^4}{e^{5y} - 1} - \&c\right) = Z^5(5-x)(1-x)$.

ix. $61 - 4\left(\frac{1^6}{e^y - 1} - \frac{3^6}{e^{3y} - 1} + \frac{5^6}{e^{5y} - 1} - \&c\right) = z^7(1-x)(61 - 46x + x^2)$

ex. i. $\phi^8(x) = 1 + 16\left(\frac{1^3 x}{1+x} + \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1+x^3} + \frac{4^3 x^4}{1-x^4} + \&c\right)$

ii $x\psi^8(x) = \frac{1^3 x}{1-x^2} + \frac{2^3 x^2}{1-x^4} + \frac{3^3 x^3}{1-x^6} + \frac{4^3 x^4}{1-x^8} + \&c$

iii. $x\psi^4(x^2) = \frac{x}{1-x^2} + \frac{3x^3}{1-x^6} + \frac{5x^5}{1-x^{10}} + \frac{7x^7}{1-x^{14}} + \&c$

iv. $\psi^2(x^2) = \frac{1}{1+x} + \frac{x}{1+x^3} + \frac{x^2}{1+x^5} + \frac{x^3}{1+x^7} + \&c$

v. $\phi^2(x)\psi^4(x) = \frac{1^2}{1+x} + \frac{3^2 x}{1+x^3} + \frac{5^2 x^2}{1+x^5} + \frac{7^2 x^3}{1+x^7} + \&c$

vi. $\frac{1^9 x}{1-x^2} + \frac{2^9 x^2}{1-x^4} + \frac{3^9 x^3}{1-x^6} + \&c = x\psi^8(x)\left\{1 + 504\left(\frac{1^5 x}{1+x} + \frac{2^5 x^2}{1+x^2} + \&c\right)\right\}$

18. i. $\frac{1}{\cosh\frac{\pi\sqrt{3}}{2}} - \frac{1}{3\cosh\frac{3\pi\sqrt{3}}{2}} + \frac{1}{5\cosh\frac{5\pi\sqrt{3}}{2}} - \&c = \frac{\pi}{24}$

ii. $\frac{1}{\cosh\frac{\pi}{2\sqrt{3}}} - \frac{1}{3\cosh\frac{3\pi}{2\sqrt{3}}} + \frac{1}{5\cosh\frac{5\pi\sqrt{3}}{2}} - \&c = \frac{5\pi}{24}$.

iii. $\frac{1^{6n-1}}{\cosh\frac{\pi\sqrt{3}}{2}} - \frac{3^{6n-1}}{\cosh\frac{3\pi\sqrt{3}}{2}} + \frac{5^{6n-1}}{\cosh\frac{5\pi\sqrt{3}}{2}} - \&c =$

$\frac{1^{6n-1}}{\cosh\frac{\pi}{2\sqrt{3}}} - \frac{3^{6n-1}}{\cosh\frac{3\pi\sqrt{3}}{2}} + \frac{5^{6n-1}}{\cosh\frac{5\pi}{2\sqrt{3}}} - \&c = 0$

n being any positive integer excluding 0.

ex. i. If $\frac{1^7}{1+x} - \frac{3^7 x}{1+x^3} + \frac{5^7 x^2}{1+x^5} - \frac{7^7 x^3}{1+x^7} + \&c = 0$, then

$\chi(x) = \sqrt[4]{2}\,\sqrt[24]{x}$. or $\sqrt[4]{2}\cdot\sqrt[24]{34x}$.

ii If $\frac{1^9}{1+x} - \frac{3^9 x}{1+x^3} + \frac{5^9 x^2}{1+x^5} - \frac{7^9 x^3}{1+x^7} + \&c = 0$, then

$\chi(x) = \sqrt[4]{2}\cdot\sqrt[24]{(154 \pm 6\sqrt{645})x}$.

iii If $\frac{1^{11}}{1+x} - \frac{3^{11} x}{1+x^3} + \frac{5^{11} x^2}{1+x^5} - \frac{7^{11} x^3}{1+x^7} + \&c = 0$, then

$(1+x)(1+x^3)(1+x^5)(1+x^7)(1+x^9)\ \&c$ or $\chi(x) =$

$\sqrt[4]{2}\cdot\sqrt[24]{x}$ or $\sqrt[4]{2}\cdot\sqrt[24]{4x}$ or $\sqrt[4]{2}\,\sqrt[24]{2764x}$

CHAPTER XVIII

1. $1+\left(\frac{1}{2}\right)^2 x+\left(\frac{1.1}{2.4}\right)^2 x^2+\left(\frac{1.1.3}{2.4.6}\right)^2 x^3+\left(\frac{1.1.3.5}{2.4.6.8}\right)^2 x^4+ \&c$

$= z(1-x)+\int z\,dx = \frac{z}{3}(1+x)+\frac{2}{3z}\left\{1-24\left(\frac{1}{e^{2y}-1}+\frac{2}{e^{4y}-1}+\&c\right)\right\}$

2. $1-\frac{1}{2^2}x-\frac{1^2.3}{2^2.4^2}x^2-\frac{1^2.3^2.5}{2^2.4^2.6^2}x^3-\frac{1^2.3^2.5^2.7}{2^2.4^2.6^2.8^2}x^4- \&c$

$= z(1-x)+\frac{1}{2}\int z\,dx = \frac{z}{3}(2-x)+\frac{1}{3z}\left\{1-24\left(\frac{1}{e^{2y}-1}+\frac{2}{e^{4y}-1}+\&c\right)\right\}$

3. *The perimeter of an ellipse whose eccentricity is h, is*

$2a\pi\left\{1-\frac{1}{2^2}h^2-\frac{1^2.3}{2^2.4^2}h^4-\frac{1^2.3^2.5}{2^2.4^2.6^2}h^6- \&c\right\}$

$= \pi(a+b)\left\{1+\left(\frac{1}{2}\right)^2\left(\frac{a-b}{a+b}\right)^2+\left(\frac{1.1}{2.4}\right)^2\left(\frac{a-b}{a+b}\right)^4+\left(\frac{1.1.3}{2.4.6}\right)^2\left(\frac{a-b}{a+b}\right)^6+ \&c\right\}$

$= \pi\left\{3(a+b)-\sqrt{(a+3b)(3a+b)}\right\}$ *nearly*

$= \pi(a+b)\left\{1+\frac{3x}{10+\sqrt{4-3x}}\right\}$ *very nearly where* $x=\left(\frac{a-b}{a+b}\right)^2$.

N.B. i. $\pi = 3.14159265358,9793238462,6434$.

ii. $\log 10 = 2.302585092994,0456840179$.

iii. $e^{-\pi} = .0432139182,6377225$

iv. $e^{\frac{\pi}{4}} = 4.810477380965,3516555473$

Cr. $\pi = \frac{355}{113}\left(1-\frac{.0003}{3533}\right)$ *very nearly.*

$= \sqrt[4]{97\frac{1}{2}-\frac{1}{11}}$ *nearly.*

4. $\frac{\sqrt{x}}{2}\left\{1+\left(\frac{1}{2}\right)^2\frac{x}{3}+\left(\frac{1.3}{2.4}\right)^2\frac{x^2}{5}+\left(\frac{1.3.5}{2.4.6}\right)^2\frac{x^3}{7}+ \&c\right\}$

$= \log\frac{1+e^{-y/2}}{1-e^{-y/2}}-3\log\frac{1+e^{-3y/2}}{1-e^{-3y/2}}+5\log\frac{1+e^{-5y/2}}{1-e^{-5y/2}}- \&c$

5. $\log\frac{16}{x}-\left(\frac{1}{2}\right)^2\frac{x}{1}-\left(\frac{1.3}{2.4}\right)^2\frac{x^2}{2}-\left(\frac{1.3.5}{2.4.6}\right)^2\frac{x^3}{3}- \&c$

$= y-4\left\{\log(1-e^{-y})-3\log(1-e^{-3y})+5\log(1-e^{-5y})- \&c\right\}$

6. $\dfrac{1}{1^2(e^{y/2}+e^{-y/2})} + \dfrac{1}{3^2(e^{3y/2}+e^{-3y/2})} + \dfrac{1}{5^2(e^{5y/2}+e^{-5y/2})} + \&c$

$= \dfrac{\sqrt{x}}{4z}\left\{1 + \left(\frac{2}{3}\right)^2 x + \left(\frac{2\cdot 4}{3\cdot 5}\right)^2 x^2 + \left(\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\right)^3 x^3 + \&c\right\}$

7. $\dfrac{1}{1^2(e^{y}+1)} - \dfrac{1}{3^2(e^{3y}+1)} + \dfrac{1}{5^2(e^{5y}+1)} - \&c$

$= \frac{1}{2}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \&c\right) - \frac{\pi}{16} y$

$+ \dfrac{\sqrt{1-x}}{4z}\left\{1 + \left(\frac{2}{3}\right)^2 (1-x) + \left(\frac{2\cdot 4}{3\cdot 5}\right)^2 (1-x)^2 + \left(\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\right)^2 (1-x)^3 + \&c\right\}$

N.B. $\dfrac{1}{1(e^{y/2}-e^{-y/2})} + \dfrac{1}{3(e^{3y/2}-e^{-3y/2})} + \dfrac{1}{5(e^{5y/2}-e^{-5y/2})} + \&c = \frac{1}{8}\log\dfrac{1+\sqrt{x}}{1-\sqrt{x}}$

8. i. $\dfrac{\cos\theta + 2\cos\frac{\theta}{2}\cosh\frac{\theta\sqrt{3}}{2}}{\cosh\frac{\pi\sqrt{3}}{2}} - \dfrac{\cos 3\theta + 2\cos\frac{3\theta}{2}\cosh\frac{3\theta\sqrt{3}}{2}}{3\cosh\frac{3\pi\sqrt{3}}{2}}$

$+ \dfrac{\cos 5\theta + 2\cos\frac{5\theta}{2}\cosh\frac{5\theta\sqrt{3}}{2}}{5\cosh\frac{5\pi\sqrt{3}}{2}} - \&c = \dfrac{\pi}{8}.$

ii. $\dfrac{\cos\theta}{\cosh\frac{\pi\sqrt{3}}{2}}(\cos\theta + \cosh\theta\sqrt{3}) - \dfrac{\cos 3\theta}{3\cosh\frac{3\pi\sqrt{3}}{2}}(\cos 3\theta + \cosh 3\theta\sqrt{3})$

$+ \dfrac{\cos 5\theta}{5\cosh\frac{5\pi\sqrt{3}}{2}}(\cos 5\theta + \cosh 5\theta\sqrt{3}) - \&c = \dfrac{\pi}{12}.$

iii. $\dfrac{\sin\theta}{1^4\cosh\frac{\pi\sqrt{3}}{2}}(\cos\theta - \cosh\theta\sqrt{3}) - \dfrac{\sin 3\theta}{3^4\cosh\frac{3\pi\sqrt{3}}{2}}(\cos 3\theta - \cosh 3\theta\sqrt{3})$

$+ \dfrac{\sin 5\theta}{5^4\cosh\frac{5\pi\sqrt{3}}{2}}(\cos 5\theta - \cosh 5\theta\sqrt{3}) - \&c = \dfrac{\pi}{12}\theta^3.$

iv. $\dfrac{\cos\theta}{1^7\cosh\frac{\pi\sqrt{3}}{2}}(\cos\theta + \cosh\theta\sqrt{3}) - \dfrac{\cos 3\theta}{3^7\cosh\frac{3\pi\sqrt{3}}{2}}(\cos 3\theta + \cosh 3\theta\sqrt{3})$

$+ \dfrac{\cos 5\theta}{5^7\cosh\frac{5\pi\sqrt{3}}{2}}(\cos 5\theta + \cosh 5\theta\sqrt{3}) - \&c$

$= \dfrac{\pi^7}{11520} - \dfrac{\pi\theta^6}{180}.$

9. $\frac{1^5}{1^6-x^6}\cdot\frac{1}{\cosh\frac{\pi\sqrt{3}}{2}} - \frac{3^5}{3^6-x^6}\cdot\frac{1}{\cosh\frac{3\pi\sqrt{3}}{2}} + \&c$

$= \frac{\pi}{12}\cdot\frac{1}{\cos\frac{\pi x}{2}\left\{\cos\frac{\pi x}{2}+\cosh\frac{\pi x\sqrt{3}}{2}\right\}}$

N.B. i. $\frac{1}{2}\cos\frac{x}{2}\left\{\cos\frac{x}{2}+\cosh\frac{x\sqrt{3}}{2}\right\}$

$= 1 - \frac{3}{4}\left\{\frac{x^6}{6!} - \frac{x^{12}}{12!} + \frac{x^{18}}{18!} - \frac{x^{24}}{24!} + \&c\right\}$

$= \left(1-\frac{x^6}{\pi^6}\right)\left(1-\frac{x^6}{3^6\pi^6}\right)\left(1-\frac{x^6}{5^6\pi^6}\right)\left(1-\frac{x^6}{7^6\pi^6}\right)\&c$

ii. $\frac{1}{2}\sin\frac{x}{2}\left\{\cos\frac{x}{2}-\cosh\frac{x\sqrt{3}}{2}\right\}$

$= -\frac{3}{4}\left\{\frac{x^3}{3!} - \frac{x^9}{9!} + \frac{x^{15}}{15!} - \frac{x^{21}}{21!} + \&c\right\}$

$= -\frac{x^3}{8}\left(1-\frac{x^6}{2^6\pi^6}\right)\left(1-\frac{x^6}{4^6\pi^6}\right)\left(1-\frac{x^6}{6^6\pi^6}\right)\&c$

10. $\frac{1^5}{1^6-x^6}\cdot\frac{1}{\cosh\frac{\pi}{2\sqrt{3}}} - \frac{3^5}{3^6-x^6}\cdot\frac{1}{\cosh\frac{3\pi}{2\sqrt{3}}} + \&c$

$= \frac{\pi}{12}\cdot\frac{4\cosh\frac{\pi x}{2\sqrt{3}}\left(\cos\frac{\pi x}{2}+\cosh\frac{\pi x}{2\sqrt{3}}\right)-3}{\cos\frac{\pi x}{2}\left\{\cos\frac{\pi x}{2}+\cosh\frac{\pi x\sqrt{3}}{2}\right\}}$

N.B. $\frac{1^2x^3}{1^6-x^6} + \frac{3^2x^3}{3^6-x^6} + \frac{5^2x^3}{5^6-x^6} + \frac{7^2x^3}{7^6-x^6} + \&c$

$= \frac{\pi}{12}\cdot\frac{\cosh\frac{\pi x\sqrt{3}}{2} - \cos\frac{\pi x}{2}}{\cosh\frac{\pi x\sqrt{3}}{2} + \cos\frac{\pi x}{2}}\tan\frac{\pi x}{2}$

ex. $\frac{1}{1^7\cosh\frac{\pi\sqrt{3}}{2}} - \frac{1}{3^7\cosh\frac{3\pi\sqrt{3}}{2}} + \frac{1}{5^7\cosh\frac{5\pi\sqrt{3}}{2}} - \&c$

$= \frac{\pi^7}{23040}$

11. i. $\left\{1+2\left(\frac{\cos\theta}{\cosh\pi}+\frac{\cos 2\theta}{\cosh 2\pi}+\frac{\cos 3\theta}{\cosh 3\pi}+\&c\right)\right\}^{-2}$

$+\left\{1+2\left(\frac{\cosh\theta}{\cosh\pi}+\frac{\cosh 2\theta}{\cosh 2\pi}+\frac{\cosh 3\theta}{\cosh 3\pi}+\&c\right)\right\}^{-2} = \frac{2}{(\sqrt[4]{\pi}/\Gamma$

ii. $\left\{\frac{\cos\theta}{\cosh\frac{y}{2}}+\frac{\cos 3\theta}{\cosh\frac{3y}{2}}+\frac{\cos 5\theta}{\cosh\frac{5y}{2}}+\&c\right\}\times$

$\left\{\frac{\cosh\theta}{\cosh\frac{y'}{2}}+\frac{\cosh 3\theta}{\cosh\frac{3y'}{2}}+\frac{\cosh 5\theta}{\cosh\frac{5y'}{2}}+\&c\right\} = \frac{zz'}{4}\sqrt{x(1-x)}$

12. i. $\frac{1}{2} + \frac{\operatorname{sech} y}{1+n^2} + \frac{\operatorname{sech} 2y}{1+(2n)^2} + \frac{\operatorname{sech} 3y}{1+(3n)^2} + \&c$

$$= \cfrac{z}{2+\cfrac{(nz)^2x}{2+\cfrac{(2nz)^2}{2+\cfrac{(3nz)^2x}{2+\cfrac{(4nz)^2}{2+\&c}}}}}$$

ii. $\frac{\operatorname{sech} y/2}{1+n^2} + \frac{\operatorname{sech} 3y/2}{1+(3n)^2} + \frac{\operatorname{sech} 5y/2}{1+(5n)^2} + \&c$

$$= \frac{1}{2}\cdot\cfrac{z\sqrt{x}}{1+\cfrac{(nz)^2}{1+\cfrac{(2nz)^2x}{1+\cfrac{(3nz)^2}{1+\cfrac{(4nz)^2x}{1+\&c}}}}}$$

Cor. If A and G be the A.M and G.M between α and β and $F(\alpha,\beta) = \cfrac{\alpha}{n+\cfrac{(\beta)^2}{n+\cfrac{(2\alpha)^2}{n+\cfrac{(3\beta)^2}{n+\cfrac{(4\alpha)^2}{n+\&c}}}}}$, then

$F(A,G)$ is the A.M between $F(\alpha,\beta)$ and $F(\beta,\alpha)$.

13. i. $\frac{\operatorname{cosech} y/2}{1+n^2} - \frac{\operatorname{cosech} 3y/2}{1+(3n)^2} + \frac{\operatorname{cosech} 5y/2}{1+(5n)^2} - \&c$

$$= \frac{1}{2}\cdot\cfrac{z\sqrt{x}}{1+\cfrac{(1-x)(nz)^2}{1-\cfrac{x(2nz)^2}{1+\cfrac{(1-x)(3nz)^2}{1-\&c.}}}}$$

ii. $\frac{\operatorname{sech} y/2}{1+n^2} - \frac{3\operatorname{sech} \frac{3y}{2}}{1+(3n)^2} + \frac{5\operatorname{sech} \frac{5y}{2}}{1+(5n)^2} - \&c$

$$= \frac{1}{2}\cdot\cfrac{z^2\sqrt{x(1-x)}}{1+(nz)^2(1-2x)+\cfrac{2^2(2^2-1)x(1-x)(nz)^4}{1+(3nz)^2(1-2x)+\cfrac{4^2(4^2-1)x(1-x)(nz)^4}{1+(5nz)^2(1-2x)+\&c}}}$$

iii. $\frac{\operatorname{cosech} y/2}{1+n^2} + \frac{3\operatorname{cosech} 3y/2}{1+(3n)^2} + \frac{5\operatorname{cosech} 5y/2}{1+(5n)^2} + \&c$

$$= \frac{1}{2}\cdot\cfrac{z^2\sqrt{x}}{1+(nz)^2(1+x)-\cfrac{2^2(2^2-1)x(nz)^4}{1+(3nz)^2(1+x)-\cfrac{4^2(4^2-1)x(nz)^4}{1+(5nz)^2(1+x)-\&c}}}$$

Cor. $\frac{\operatorname{sech} \frac{\pi}{2}}{1+n^2} - \frac{3\operatorname{sech} \frac{3\pi}{2}}{1+(3n)^2} + \frac{5\operatorname{sech} \frac{5\pi}{2}}{1+(5n)^2} - \frac{7\operatorname{sech} \frac{7\pi}{2}}{1+(7n)^2} + \&c$

$$= \frac{1}{4} \cdot \cfrac{\mu^2}{1 + \cfrac{1.3.(n\mu)^4}{1 + \cfrac{6.10(n\mu)^4}{1 + \cfrac{15.21(n\mu)^4}{1 + \&c}}}} \quad \text{where } \mu = \frac{\sqrt{\pi}}{(1-x)^2}.$$

14. Let $S = \dfrac{\sin\frac{\theta}{2}}{\sinh\frac{y}{2}} + \dfrac{\sin\frac{3\theta}{2}}{\sinh\frac{3y}{2}} + \dfrac{\sin\frac{5\theta}{2}}{\sinh\frac{5y}{2}} + \&c$

$C = \dfrac{\cos\frac{\theta}{2}}{\cosh\frac{y}{2}} + \dfrac{\cos\frac{3\theta}{2}}{\cosh\frac{3y}{2}} + \dfrac{\cos\frac{5\theta}{2}}{\cosh\frac{5y}{2}} + \&c$

and $C_1 = \dfrac{1}{2} + \dfrac{\cos\theta}{\cosh y} + \dfrac{\cos 2\theta}{\cosh 2y} + \dfrac{\cos 3\theta}{\cosh 3y} + \&c$, then

we see that $C^2 + S^2 = \dfrac{x}{4}z^2$ and $C_1^2 + S_1^2 = \dfrac{z^2}{4}$.

and $CS = \dfrac{\sin\theta}{\cosh y} + \dfrac{2\sin 2\theta}{\cosh 2y} + \dfrac{3\sin 3\theta}{\cosh 3y} + \&c$

$\therefore\ Cs + \dfrac{dC_1}{d\theta} = 0$; $C_1 S + \dfrac{dC}{d\theta} = 0$ and $CC_1 = \dfrac{dS}{d\theta}$

Let $C = \dfrac{\sqrt{x}}{2} z\cos\phi$ and $S = \dfrac{\sqrt{x}}{2} z\sin\phi$.

$\therefore\ C_1 = \dfrac{z}{2}\sqrt{1 - x\sin^2\phi}$.

$\therefore\ \dfrac{z}{2}\cos\phi\sqrt{1 - x\sin^2\phi} = \dfrac{d\sin\phi}{d\theta} = \cos\phi\dfrac{d\phi}{d\theta}$

$\therefore\ \theta = \dfrac{2}{z}\displaystyle\int_0^\phi \frac{d\phi}{\sqrt{1 - x\sin^2\phi}}$

15. Let $z\theta = \displaystyle\int_0^\phi \frac{d\phi}{\sqrt{1 - x\sin^2\phi}}$; $y = \pi\cdot\dfrac{z'}{z}$; $y' = \pi\cdot\dfrac{z}{z'}$;

$z' = 1 + \left(\frac{1}{2}\right)^2(1-x) + \left(\frac{1.3}{2.4}\right)^2(1-x)^2 + \&c$ and $z = 1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \&c$

i. $1 + 2\left(\dfrac{\cos 2\theta}{\cosh y} + \dfrac{\cos 4\theta}{\cosh 2y} + \dfrac{\cos 6\theta}{\cosh 3y} + \&c\right) = z\sqrt{1 - x\sin^2\phi}$

ii. $\dfrac{\cos\theta}{\cosh\frac{y}{2}} + \dfrac{\cos 3\theta}{\cosh\frac{3y}{2}} + \dfrac{\cos 5\theta}{\cosh\frac{5y}{2}} + \&c = \dfrac{\sqrt{x}}{2} z\cos\phi$.

iii. $\dfrac{\sin\theta}{\sinh\frac{y}{2}} + \dfrac{\sin 3\theta}{\sinh\frac{3y}{2}} + \dfrac{\sin 5\theta}{\sinh\frac{5y}{2}} + \&c = \dfrac{\sqrt{x}}{2} z\sin\phi$.

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iv. $\theta + \frac{\sin 2\theta}{\cosh y} + \frac{\sin 4\theta}{2\cosh 2y} + \frac{\sin 6\theta}{3\cosh 3y} + \&c = \phi.$

v. $\frac{\sin\theta}{\cosh\frac{y}{2}} + \frac{\sin 3\theta}{3\sinh\frac{3y}{2}} + \frac{\sin 5\theta}{5\sinh\frac{5y}{2}} + \&c = \frac{1}{2}\sin^{-1}(\sqrt{x}\sin\phi)$

v. $\frac{\cos\theta}{\sinh\frac{y}{2}} + \frac{\cos 3\theta}{3\cosh\frac{3y}{2}} + \frac{\cos 5\theta}{5\cosh\frac{5y}{2}} + \&c = \frac{1}{2}\log\frac{\sqrt{1-x\sin^2\phi} - \sqrt{x}}{\sqrt{1-x}}$

16. If θ is changed to $\frac{\pi}{2}-\theta$, then $\cot\phi$ to $\sqrt{1-x}\tan\phi$;

$\sin\phi$ to $\frac{\cos\phi}{\sqrt{1-x\sin^2\phi}}$; $\cos\phi$ to $\frac{\sin\phi}{\sqrt{1-x\sin^2\phi}}\sqrt{1-x}$ and

$\sqrt{1-x\sin^2\phi}$ to $\frac{\sqrt{1-x}}{\sqrt{1-x\sin^2\phi}}$.

i. $\frac{\cos\theta}{\sinh\frac{y}{2}} - \frac{\cos 3\theta}{\sinh\frac{3y}{2}} + \frac{\cos 5\theta}{\sinh\frac{5y}{2}} - \&c = \frac{\sqrt{x}}{2}z.\frac{\cos\phi}{\sqrt{1-x\sin^2\phi}}$

ii. $\frac{\sin\theta}{\cosh\frac{y}{2}} - \frac{\sin 3\theta}{\cosh\frac{3y}{2}} + \frac{\sin 5\theta}{\cosh\frac{5y}{2}} - \&c = \frac{\sqrt{x(1-x)}}{2}z\frac{\sin\phi}{\sqrt{1-x\sin^2\phi}}$

iii. $\operatorname{cosec}\theta + 4\left(\frac{\sin\theta}{e^{y}-1} + \frac{\sin 3\theta}{e^{3y}-1} + \frac{\sin 5\theta}{e^{5y}-1} + \&c\right)$

$= z\operatorname{cosec}\phi.$

iv. $\sec\theta + 4\left(\frac{\cos\theta}{e^{y}-1} - \frac{\cos 3\theta}{e^{3y}-1} + \frac{\cos 5\theta}{e^{5y}-1} - \&c\right)$

$= z\sec\phi\sqrt{1-x\sin^2\phi}.$

v. $\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + 4\left\{\frac{\sin\theta}{e^{y}-1} - \frac{\sin 3\theta}{3(e^{3y}-1)} + \frac{\sin 5\theta}{5(e^{5y}-1)} - \&c\right\}$

$= \log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right).$

17. i. $\frac{\cos\theta}{\sin^3\theta} - 8\left(\frac{1^2\sin 2\theta}{e^{2y}-1} + \frac{2^2\sin 4\theta}{e^{4y}-1} + \frac{3^2\sin 6\theta}{e^{6y}-1} + \&c\right)\}$

$= z^3.\frac{\cos\phi}{\sin^3\phi}\sqrt{1-x\sin^2\phi}.$

ii. $\frac{1}{\sin^2\theta} - 8\left(\frac{\cos 2\theta}{e^{2y}-1} + \frac{2\cos 4\theta}{e^{4y}-1} + \frac{3\cos 6\theta}{e^{6y}-1} + \&c\right)$

$= \frac{z^2}{\sin^2\phi} - z^2 \cdot \frac{1+x}{3} + \frac{1}{3}\left\{1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \frac{3}{e^{6y}-1} + \&c\right)\right.$

iii. $\cot\theta + 4\left(\frac{\sin 2\theta}{e^{2y}-1} + \frac{\sin 4\theta}{e^{4y}-1} + \frac{\sin 6\theta}{e^{6y}-1} + \&c\right)$

$= z\cot\phi\sqrt{1-x\sin^2\phi} + z\int_0^\phi \sqrt{1-x\sin^2\phi}\,d\phi - \frac{2\theta z}{\pi}\int_0^{\frac{\pi}{2}}\sqrt{1-x\sin^2\phi}\,d\phi$

iv. $\frac{\sin 2\theta}{\sinh y} + \frac{\sin 4\theta}{\sinh 2y} + \frac{\sin 6\theta}{\sinh 3y} + \&c$

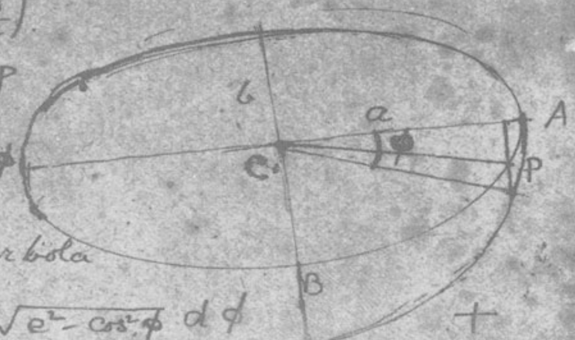
$= \frac{z}{2}\int_0^\phi \sqrt{1-x\sin^2\phi}\,d\phi - \frac{\theta z}{\pi}\int_0^{\frac{\pi}{2}}\sqrt{1-x\sin^2\phi}\,d\phi.$

18. i. If θ is changed to $\frac{\theta}{2}$ and y to $\frac{y}{2}$, then x must be changed to $\frac{4\sqrt{x}}{(1+\sqrt{x})^2}$ and 2ϕ to $\phi + \sin^{-1}(\sqrt{x}\sin\phi)$ and z to $(1+\sqrt{x})\,z$.

ii. If θ is changed to $\frac{\pi}{2} - \theta$ and e^{-y} to $-e^{-y}$, then x must be changed to $-\frac{x}{1-x}$; ϕ to $\frac{\pi}{2} - \phi$; & z to $z\sqrt{1-x}$

iii. If e^{-y} is changed to $-e^{-y}$, then change x to $-\frac{x}{1-x}$; z to $z\sqrt{1-x}$ and $\cot\phi$ to $\cot\phi\sqrt{1-x}$.

iv. If θ is changed to $i\theta\frac{z}{z'}$, and y to y', then change x to $1-x$; z to z'; $\sin\phi$ to $i\tan\phi$; $\cos\phi$ to $\sec\phi$; and ϕ to $i\log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$.



19. i. The length of the arc AP in an ellipse $= a\int_0^\phi \sqrt{1-e^2\cos^2\phi}\,d\phi$ where e is the eccentricity.

ii. The length of AP in a hyperbola

$= a\tan\phi\sqrt{e^2-\cos^2\phi} - a\int_0^\phi \sqrt{e^2-\cos^2\phi}\,d\phi$

$\frac{b^2}{a}\int_0^\phi \frac{d\phi}{\sqrt{e^2-\cos^2\phi}}$ where $x = a\sec\phi$ and $y = b\tan\phi$

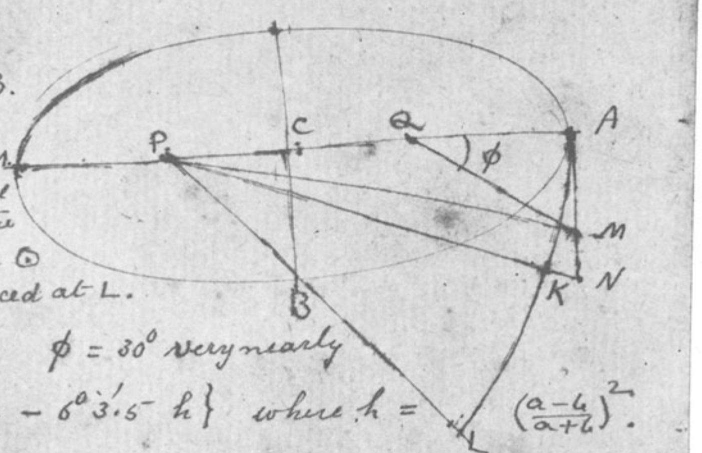
iii. If the perimeter of an ellipse $= \pi(a+b)\left(1+h\sin^2\frac{\theta}{2}\right)$ where $\sin\theta = \frac{a-b}{a+b}\sin\phi$. When $e=1$, $\phi = 30^\circ\ 18'\ 6''$ and very rapidly diminishes to 30° when e becomes 0.

iv. If the perimeter of an ellipse $= \pi(a+b)\left\{1+\frac{\sin^2\theta}{2+\cos^2\frac{\theta}{2}}\right\}$ where $\sin\theta = \frac{a-b}{a+b}\sin\phi$. When $e=1$, $\phi = 60^\circ\ 4'\ 55''$ and suddenly falls to 60° when e becomes 0.

Cor. 1. If $l = (a-b)\cos\phi = (a+b)\tan\theta$, then $\frac{\pi l}{\theta}$ will be the perimeter of the ellipse; where ϕ diminishes from 30° to 0° when e increases from 0 to 1.

$$\phi = \frac{2\sqrt{ab}}{a+b}\left\{30^\circ + 6^\circ\,18'{\cdot}8\,\frac{(\sqrt{a}-\sqrt{b})^2}{a+b} - 1^\circ\,10'{\cdot}9\left(\frac{a-b}{a+b}\right)^2\right\}$$

Cor. 2. Draw AN perp to AC.
Make CP & CQ equal to CB.
Draw QM making an $\angle\phi$ with AQ & meeting AN at M.
Join PM & make NPN equal to $\frac{1}{2}$ of APM. With P as centre and PA as radius desc. a ⊙ cutting PN at K & PB produced at L.



Then $\frac{\text{arc } AL}{\text{arc } AK} = \frac{\text{arc } AB}{AN}$. $\phi = 30^\circ$ very nearly

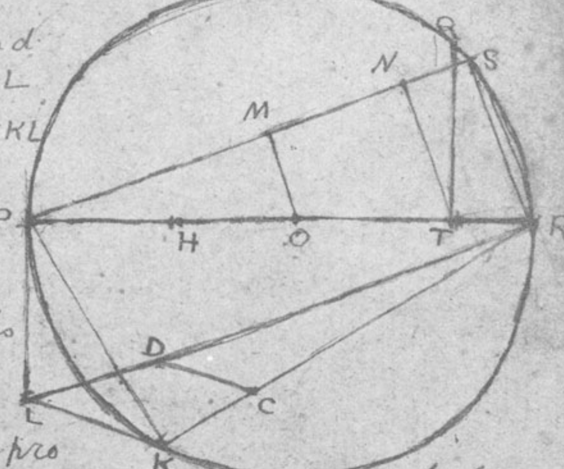
$$\phi = 30^\circ + h(1-h)\left\{5^\circ\,19'{\cdot}4 - 6^\circ\,3'{\cdot}5\,h\right\} \text{ where } h = \left(\frac{a-b}{a+b}\right)^2.$$

N.B. i. $\phi = 30^\circ$ when $e = 0, 1$ or $\cdot 99948$

ii. when $e = \cdot 999886$, ϕ assumes the minimum value of $29^\circ\ 58\tfrac{3}{4}'$ and when $e = \cdot 9589$, ϕ has the maximum value of $30^\circ\ 44\tfrac{1}{4}'$

20. i. To construct a square equal to a given circle.
Let O be the centre and PR any diameter.
Bisect OP at H and trisect OR at T. Draw TQ perp to OR

Draw $RS = TQ$. Join PS.
Draw OM & TN $\parallel^l$ to RS.
Draw $PK = PM$; & $PL = MN$ and perp to OP. Join RL, RK & KL
Cut off $RC = RH$. Draw CD $\parallel^l$ to KL
Then $RD^2 = \odot PQR$.



N.B. RD is $\frac{1}{100}$th of an inch greater than the true length if the given $\odot$ is 14 Sq. miles in area.

Cor. 1. One of the two mean pro-portionals between a side of an equilateral triangle inscribed in the $\odot$ and the length PS is only less than 30000 th part of it than the true length.

Cor. 2. The app. length got by assuming $\pi = \sqrt[4]{97\frac{1}{2} - \frac{1}{11}}$ is $\frac{1}{100}$th of an inch less than the true length if the $\odot$ is a million square miles in area.

ii. $\left\{6n^2 + (3n^3 - n)\right\}^3 + \left\{6n^2 - (3n^3 - n)\right\}^3 = \left\{6n^2(3n^2+1)\right\}^2$

iii. $\left\{m^7 - 3m^4(1+p) + m(3\cdot\overline{1+p}^2 - 1)\right\}^3$
$+ \left\{2m^6 - 3m^3(1+2p) + (1+3p+3p^2)\right\}^3$
$+ \left\{m^6 - (1+3p+3p^2)\right\}^3 = \left\{m^7 - 3m^4p + m(3p^2-1)\right\}^3$

ex. $(11\frac{1}{2})^3 + (\frac{1}{2})^3 = 39^2$; $(3 - \frac{1}{105})^3 + (\frac{1}{105})^3 = (5\frac{6}{35})^2$.
$(3\frac{1}{7})^3 - (\frac{1}{7})^3 = (5\frac{4}{7})^2$; $(3\frac{1}{104})^3 - (\frac{1}{104})^3 = (5\frac{23}{104})^2$.
$3^3 + 4^3 + 5^3 = 6^3$; $1^3 + 12^3 = 9^3 + 10^3$; $1^3 + 75^3 = (70\frac{1}{2})^3 + (41\frac{1}{2})$
$3^3 + 509^3 + 34^6 = 1188^3$; $18^3 + 19^3 + 21^3 = 28^3$.
$7^3 + 14^3 + 17^3 = 20^3$; $19^3 + 60^3 + 69^3 = 82^3$; $15^3 + 82^3 + 89^3$
$= 108^3$; $3^3 + 36^3 + 37^3 = 46^3$; $1^3 + 135^3 + 138^3 = 172^3$;

$23^3+134^3=95^3+116^3$; $133^3+174^3=45^3+196^3$;

$1^3+6^3+8^3=9^3$; $11^3+37^3=228^2$; $71^3-23^3=588^2$.

21\. i. $\frac{1}{2\pi\sqrt{3}\,x^4}+\frac{1}{1^4+1^2x^2+x^4}+\frac{2}{2^4+2^2x^2+x^4}+\frac{3}{3^4+3^2x^2+x^4}+\&c$

$=\frac{\pi}{3x^2\sqrt{3}}\cdot\frac{\cosh\pi x\sqrt{3}+2\cos\pi x}{\cosh\pi x\sqrt{3}-\cos\pi x}+2\left\{\frac{1}{e^{\pi\sqrt{3}}+1}\cdot\frac{1}{1^4+1^2x^2+x^4}\right.$

$\left.-\frac{2}{e^{2\pi\sqrt{3}}-1}\cdot\frac{1}{2^4+2^2x^2+x^4}+\frac{3}{e^{3\pi\sqrt{3}}+1}\cdot\frac{1}{3^4+3^2x^2+x^4}-\&c\right\}$

ii. $\frac{\sqrt{3}}{2\pi x^4}+\frac{1}{1^4+1^2x^2+x^4}+\frac{2}{2^4+2^2x^2+x^4}+\frac{3}{3^4+3^2x^2+x^4}+\&c$

$=\frac{\pi}{3x^2\sqrt{3}}\cdot\frac{\cosh\pi x\sqrt{3}+2\cos\pi x+6\cosh\frac{\pi x}{\sqrt{3}}}{\cosh\pi x\sqrt{3}-\cos\pi x}+2\left\{\frac{1}{e^{\pi/\sqrt{3}}+1}\cdot\frac{1}{1^4+1^2x^2+x^4}\right.$

$\left.-\frac{2}{e^{2\pi/\sqrt{3}}-1}\cdot\frac{1}{2^4+2^2x^2+x^4}+\frac{3}{e^{3\pi/\sqrt{3}}+1}\cdot\frac{1}{3^4+3^2x^2+x^4}-\&c\right\}$

iii. $\frac{1}{2n^2}+\frac{1}{1^2+n+n^2}+\frac{1}{2^2+2n+n^2}+\frac{1}{3^2+3n+n^2}+\&c$

$+2n\left\{\frac{1}{e^{\pi\sqrt{3}}+1}\cdot\frac{1}{1^4+1^2n^2+n^4}-\frac{2}{e^{2\pi\sqrt{3}}-1}\cdot\frac{1}{2^4+2^2n^2+n^4}+\&c\right\}$

$=\frac{1}{2\pi n^3\sqrt{3}}+\frac{2\pi}{3n\sqrt{3}}-\frac{2\pi}{n\sqrt{3}}\cdot\frac{1}{e^{2\pi n\sqrt{3}}-2e^{\pi n\sqrt{3}}\cos\pi n+1}$.

iv. $\frac{1}{6n^2}+\frac{1}{1^2+3n+3n^2}+\frac{1}{2^2+6n+3n^2}+\frac{1}{3^2+9n+3n^2}+\&c$

$+6n\left\{\frac{1}{e^{\pi\sqrt{3}}+1}\cdot\frac{1}{1^4-3n^2+9n^4}-\frac{2}{e^{2\pi\sqrt{3}}-1}\cdot\frac{1}{2^4-2^2.3n^2+9n^4}+\&c\right\}$

$=\frac{1}{6\pi n^3\sqrt{3}}+\frac{\pi}{3n\sqrt{3}}-\frac{2\pi}{n\sqrt{3}}\cdot\frac{1}{e^{2\pi n\sqrt{3}}-2e^{\pi n\sqrt{3}}\cos 3\pi n+1}$.

ex. $\frac{1}{7.13(e^{\pi\sqrt{3}}+1)}-\frac{2}{7.19(e^{2\pi\sqrt{3}}-1)}+\frac{3}{9.27(e^{3\pi\sqrt{3}}+1)}$

$-\frac{4}{13.37(e^{4\pi\sqrt{3}}-1)}+\&c=\frac{1}{324\pi\sqrt{3}}+\frac{25}{756}-\frac{\pi}{54\sqrt{3}}$

$+\frac{\pi}{18\sqrt{3}}\cdot\frac{1}{1+\cosh 3\pi\sqrt{3}}$.

N.B. The series $\frac{1}{1^2+n+y}+\frac{1}{2^2+2n+y}+\frac{1}{3^2+3n+y}$ + &c can be exactly found if n is any integer and y any quantity.

22. i. $\int_0^\infty e^{-n\int_0^\phi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}}\, d\phi = \frac{1}{n} + \frac{x}{n} + \frac{4}{n} + \frac{9x}{n} + \frac{16}{n} + \&c$

ii $\int_0^\infty e^{-n\int_0^\phi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}}\, \frac{\cos\phi}{\sqrt{1-x\sin^2\phi}}\, d\phi = \frac{1}{n} + \frac{1}{n} + \frac{4x}{n} + \frac{9}{n} + \frac{16x}{n} + \&c$

iii. $\int_0^\infty e^{-n\int_0^\phi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}}\, \frac{\cos\phi}{1-x\sin^2\phi}\, d\phi = \frac{1}{n} + \frac{1-x}{n} - \frac{4x}{n} + \frac{9(1-x)}{n} - \&c$

23. i. $\sqrt{2}\left\{\frac{1}{2} + e^{-\frac{\pi x}{x^2+y^2}}\cos\frac{\pi y}{x^2+y^2} + e^{-\frac{4\pi x}{x^2+y^2}}\cos\frac{4\pi y}{x^2+y^2} + e^{-\frac{9\pi x}{x^2+y^2}}\cos\frac{9\pi y}{x^2+y^2} + \&c\right\}$

$= \sqrt{\sqrt{x^2+y^2}+x}\left\{\frac{1}{2} + e^{-\pi x}\cos\pi y + e^{-4\pi x}\cos 4\pi y + e^{-9\pi x}\cos 9\pi y + \&c\right\}$

$+ \sqrt{\sqrt{x^2+y^2}-x}\left\{e^{-\pi x}\sin\pi y + e^{-4\pi x}\sin 4\pi y + e^{-9\pi x}\sin 9\pi y + \&c\right\}$

ii. $\sqrt{2}\left\{e^{-\frac{\pi x}{x^2+y^2}}\sin\frac{\pi y}{x^2+y^2} + e^{-\frac{4\pi x}{x^2+y^2}}\sin\frac{4\pi y}{x^2+y^2} + \&c\right\}$

$= \sqrt{\sqrt{x^2+y^2}+x}\left\{\frac{1}{2} + e^{-\pi x}\cos\pi y + e^{-4\pi x}\cos 4\pi y + e^{-9\pi x}\cos 9\pi y + \&c\right\}$

$- \sqrt{\sqrt{x^2+y^2}-x}\left\{e^{-\pi x}\sin\pi y + e^{-4\pi x}\sin 4\pi y + e^{-9\pi x}\sin 9\pi y + \&c\right\}$

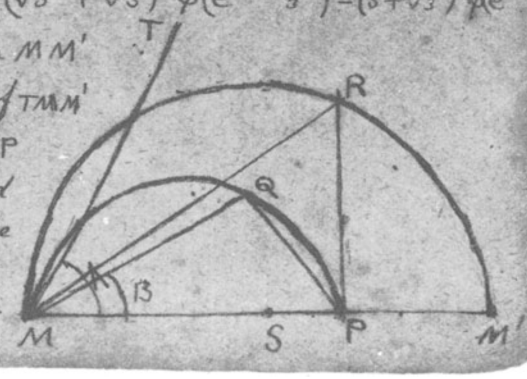
Cor. $\frac{1}{2} + e^{-\pi x}\cos\pi\sqrt{1-x^2} + e^{-4\pi x}\cos 4\pi\sqrt{1-x^2} + \&c$

$= \frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{1-x}}\left\{e^{-\pi x}\sin\pi\sqrt{1-x^2} + e^{-4\pi x}\sin 4\pi\sqrt{1-x^2} + \&c\right\}$

ex. $\phi(e^{-\pi}) = \phi(e^{-5\pi})\sqrt{5\sqrt{5}-10}$; $(\sqrt{5}+\sqrt{3})\,\phi(e^{-\frac{\pi\sqrt{5}}{3}}) = (3+\sqrt{3})\,\phi(e^{-3\pi\ldots}$

24. i. Let TMM' be any angle. On MM' desc. a semi ⊙. cutting the bisector of TMM' at R. Draw RP perp to MM'. On MP desc. a semi ⊙. In it place a chord PQ equal to PM'. Join MQ. Let S be the middle point of MM'.

V.13. If RP divides MM' in medial section then MQ coincides with MR



A pendulum oscillating through $\angle A$ takes $\frac{MM'}{MP}$ times the time required through $\angle B$. Let $\sin^2 A = \alpha$ & $\sin^2 B = \beta$. & let $\frac{MM'}{MP} = m$ then $2PS = m \cos A$ & $m = \dfrac{1 + \left(\frac{1}{2}\right)^2 \alpha + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \alpha^2 + \&c}{1 + \left(\frac{1}{2}\right)^2 \beta + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \beta^2 + \&c}$

N.B. Here β is in the second degree of α.

ii. 2nd degree :— $m\sqrt{1-\alpha} + \sqrt{\beta} = 1$ and $m^2\sqrt{1-\alpha} + \beta = 1$;

$$\frac{m^2}{2} = \frac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}} = \frac{1+\beta}{1+(1-\alpha)}$$

iii. 4th degree :— $\sqrt{m}\sqrt[4]{1-\alpha} + \sqrt[4]{\beta} = 1$ and $m\sqrt[4]{1-\alpha} + \sqrt{\beta} = 1$;

$$\frac{m}{2} = \frac{1+\sqrt[4]{\beta}}{1+\sqrt[4]{1-\alpha}} = \frac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}}.$$

iv. 8th degree :— $\sqrt{m}\sqrt[8]{1-\alpha} + \sqrt[4]{\beta} = 1$

16th degree :— $\dfrac{\sqrt{m}}{2} = \dfrac{1+\sqrt[4]{\beta}}{1+\sqrt[4]{1-\alpha}}$.

v. In any equation α may be changed to $1-\beta$, β to $1-\alpha$ and m to n/m where n is the degree of β; thus we see that

2nd degree :— $\frac{2}{m}\sqrt{\beta} + \sqrt{1-\alpha} = 1$ and $(1-\sqrt{1-\alpha})(1-\sqrt{\beta}) = 2\sqrt{\beta(1-\alpha)}$

4th degree :— $\frac{2}{\sqrt{m}}\sqrt[4]{\beta} + \sqrt[4]{1-\alpha} = 1$ and $(1-\sqrt[4]{1-\alpha})(1-\sqrt[4]{\beta}) = 2\sqrt[4]{\beta(1-\alpha)}$

8th degree :— $\frac{2\sqrt{2}}{\sqrt{m}}\sqrt[8]{\beta} + \sqrt[4]{1-\alpha} = 1$ and $(1-\sqrt[4]{1-\alpha})(1-\sqrt[4]{\beta}) = 2\sqrt{2}\sqrt[8]{\beta(1-\alpha)}$

vi. $$n\pi \cdot \frac{1 + \left(\frac{1}{2}\right)^2(1-\alpha) + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2(1-\alpha)^2 + \&c}{1 + \left(\frac{1}{2}\right)^2\alpha + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2\alpha^2 + \&c} = \pi \cdot \frac{1 + \left(\frac{1}{2}\right)^2(1-\beta) + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2(1-\beta)^2 + \&c}{1 + \left(\frac{1}{2}\right)^2\beta + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2\beta^2 + \&c}$$

Differentiating both sides we have,

$n \cdot \dfrac{d\alpha}{d\beta} = \dfrac{\alpha(1-\alpha)}{\beta(1-\beta)} m^2$. Again by differentiating any equation we know $\dfrac{d\alpha}{d\beta}$ and hence m is known.

vii. Equations in terms of ψ functions can be transformed to those of ϕ functions and vice versa while those of f and χ functions remains unchanged. e.g. the identity $\dfrac{\psi(x^{\frac{1}{3}})}{\sqrt[3]{x} \cdot \psi(x^3)} = 1 + \sqrt[3]{\dfrac{\psi^4(x)}{x\,\psi^4(x^3)} - 1}$ becomes $\dfrac{\phi(x^{\frac{1}{3}})}{\phi(x^3)} = 1 + \sqrt[3]{\dfrac{\phi^4(x)}{\phi^4(x^3)} - 1}$.

CHAPTER XIX

1. i. $\sqrt[8]{x}\,\dfrac{\psi(x)}{\phi(x)} = \dfrac{\sqrt[8]{x}}{1+}\,\dfrac{x}{1+x+}\,\dfrac{x^2}{1+x^2+}\,\dfrac{x^3}{1+x^3+}\,\dfrac{x^4}{1+x^4+}$ &c

ii Let $v = \sqrt[8]{x}\cdot\dfrac{f(-x,-x^7)}{f(-x^3,-x^5)}$, then

$$v = \frac{\sqrt{x}}{1+x+}\,\frac{x^2}{1+x^3+}\,\frac{x^4}{1+x^5+}\,\frac{x^6}{1+x^7+}\,\frac{x^8}{1+x^9+}\ \&c$$

$$\frac{1}{v} - v = \frac{\phi(x^2)}{\sqrt{x}\,\psi(x^4)} \text{ and } \frac{1}{v} + v = \frac{\phi(x)}{\sqrt{x}\,\psi(x^4)}.$$

2. i. $f(-x,-x^4)\,f^3(-x^{15}) = f(-x^5)\,f(-x^6,-x^9)\,f(-x,-x^{14})\,f(-x^4,-x^{11})$;

$f(-x^2,-x^3)\,f^3(-x^{15}) = f(-x^5)\,f(-x^3,-x^{12})\,f(-x^2,-x^{13})\,f(-x^7,-x^8)$.

ii $f(-x,-x^6)\,f^3(-x^{21}) = f(-x^7)\,f(-x^6,-x^{15})\,f(-x,-x^{20})\,f(-x^8,-x^{13})$;

$f(-x^2,-x^5)\,f^3(-x^{21}) = f(-x^7)\,f(-x^9,-x^{12})\,f(-x^2,-x^{19})\,f(-x^5,-x^{16})$;

$f(-x^3,-x^4)\,f^3(-x^{21}) = f(-x^7)\,f(-x^3,-x^{18})\,f(-x^4,-x^{17})\,f(-x^{10},-x^{11})$.

and so on.

3. i. $x\,\psi(x^2)\,\psi(x^6) = \dfrac{x}{1-x^2} - \dfrac{x^5}{1-x^{10}} + \dfrac{x^7}{1-x^{14}} - \dfrac{x^{11}}{1-x^{22}} +$ &c

ii. $\phi(x)\,\phi(x^3) = 1 + 2\left(\dfrac{x}{1-x} - \dfrac{x^2}{1+x^2} + \dfrac{x^4}{1+x^4} - \dfrac{x^5}{1-x^5} + \dfrac{x^7}{1-x^7} - \right.$ &c$\left.\right)$

iii. $x\,\psi^2(x)\,\psi^2(x^3) = \dfrac{x}{1-x^2} + \dfrac{2x^2}{1-x^4} + \dfrac{4x^4}{1-x^8} + \dfrac{5x^5}{1-x^{10}} +$ &c

iv. $\phi^2(x)\,\phi^2(x^3) = 1 + 4\left(\dfrac{x}{1-x} + \dfrac{4x^4}{1-x^4} + \dfrac{5x^5}{1-x^5} + \dfrac{7x^7}{1-x^7} + \dfrac{8x^8}{1-x^8} + \right.$ &c$\left.\right)$

4. i. $x\,\psi^5(x)\,\psi(x^3) - 9x^2\,\psi(x)\,\psi^5(x^3)$

$$= \frac{x}{1-x^2} - \frac{2^2x^2}{1-x^4} + \frac{4^2x^4}{1-x^8} - \frac{5^2x^5}{1-x^{10}} + \&c$$

ii. $9\,\phi(x)\,\phi^5(x^3) - \phi^5(x)\,\phi(x^3)$

$$= 8\left\{1 + \frac{x}{1+x} - \frac{2^2x^2}{1-x^2} + \frac{4^2x^4}{1-x^4} - \frac{5^2x^5}{1+x^5} + \frac{7^2x^7}{1+x^7} - \&c\right\}$$

iii. $\dfrac{\psi^3(x)}{\psi(x^3)} = 1 + 3\left(\dfrac{x}{1-x} - \dfrac{x^5}{1-x^5} + \dfrac{x^7}{1-x^7} - \dfrac{x^{11}}{1-x^{11}} + \text{\&c}\right)$

iv. $\dfrac{\phi^3(x)}{\phi(x^3)} = 1 + 6\left(\dfrac{x}{1-x} + \dfrac{x^2}{1+x^2} - \dfrac{x^4}{1+x^4} - \dfrac{x^5}{1-x^5} + \text{\&c}\right)$

5. From these we get the following results.

326. If β be of the 3rd degree,

i. $$\sqrt[8]{\frac{\alpha^3}{\beta}} - \sqrt[8]{\frac{(1-\alpha)^3}{1-\beta}} = \sqrt[8]{\frac{(1-\beta)^3}{1-\alpha}} - \sqrt[8]{\frac{\beta^3}{\alpha}} = 1.$$

ii $$\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} = 1.$$

iii. $$m = 1 + 2\sqrt[8]{\frac{\beta^3}{\alpha}} \text{ and } \frac{3}{m} = 1 + 2\sqrt[8]{\frac{(1-\alpha)^3}{1-\beta}}$$

iv. $$m^2\left(\sqrt[8]{\frac{\alpha^3}{\beta}} - \alpha\right) = \sqrt[8]{\frac{\alpha^3}{\beta}} - \ldots$$

v. $$m = \frac{1 - 2\sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}}}{1 - 2\sqrt[4]{\alpha\beta}} = \sqrt{1 + 4\sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}}} \text{ and}$$

$$3/m = \frac{2\sqrt[8]{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}} - 1}{1 - 2\sqrt[4]{\alpha\beta}} = \sqrt{1 + 4\sqrt[8]{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}}}.$$

vi. If $\alpha = p\left(\frac{2+p}{1+2p}\right)^3$ then $\beta = p^3 \cdot \frac{2+p}{1+2p}$. so that

$$1-\alpha = (1+p)\left(\frac{1-p}{1+2p}\right)^3 \;\&\; 1-\beta = (1+p)^3 \cdot \frac{1-p}{1+2p}$$

vii. $$m^2 = \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{1-\beta}{1-\alpha}} - \sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} \text{ and hence}$$

$$9/m^2 = \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{1-\alpha}{1-\beta}} - \sqrt{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}}.$$

viii. $$\sqrt[8]{\alpha\beta^5} + \sqrt[8]{(1-\alpha)(1-\beta)^5} = 1 - \sqrt[8]{\frac{\beta^3(1-\alpha)^3}{\alpha(1-\beta)}}$$

$$= \sqrt[8]{\alpha^5\beta} + \sqrt[8]{(1-\alpha)^5(1-\beta)} = \sqrt{\frac{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}}{2}}$$

ix. $$\sqrt{\alpha(1-\beta)} + \sqrt{\beta(1-\alpha)} = 2\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}.$$

~~x.~~ $$m^2\sqrt{\alpha(1-\alpha)} + \sqrt{\beta(1-\beta)} = 9/m^2 \cdot \sqrt{\beta(1-\beta)} + \sqrt{\alpha(1-\alpha)}.$$

x. $$m\sqrt{1-\alpha} + \sqrt{1-\beta} = \frac{3}{m}\sqrt{1-\beta} - \sqrt{1-\alpha} = 2\sqrt[8]{(1-\alpha)(1-\beta)} \text{ and}$$

$$m\sqrt{\alpha} - \sqrt{\beta} = \frac{3}{m}\sqrt{\beta} + \sqrt{\alpha} = 2\sqrt[8]{\alpha\beta}.$$

xi. $$m - \frac{3}{m} = 2\left\{\sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}\right\} \text{ and}$$

$$m + \frac{3}{m} = 4\sqrt{\frac{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}}{2}}.$$

xii If $P = \sqrt[8]{16\alpha\beta(1-\alpha)(1-\beta)}$ and $Q = \sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$, then

$$Q + \frac{1}{Q} + 2\sqrt{2}\left(P - \frac{1}{P}\right) = 0$$

xiii. If $P = \sqrt[8]{\alpha\beta}$ and $Q = \sqrt[4]{\frac{\beta}{\alpha}}$, then

$$Q - \frac{1}{Q} = 2\left(P - \frac{1}{P}\right).$$

xiv. If $\alpha = \sin^2(u+v)$ and $\beta = \sin^2(u-v)$, then $\sin 2u = 2\sin v$

xv. If $\alpha(1-\alpha) = p\cdot\left(\frac{2-p}{1+4p}\right)^3$ then $\beta(1-\beta) = p^3\cdot\frac{2-p}{1+4p}$.

6. i. $$1 + \left(\tfrac{1}{2}\right)^2 p\cdot\left(\frac{2+p}{1+2p}\right)^3 + \left(\frac{1.3}{2.4}\right)^2 p^2\cdot\left(\frac{2+p}{1+2p}\right)^6 + \&c$$

$$= (1+2p)\left\{1 + \left(\tfrac{1}{2}\right)^2 p^3\cdot\frac{2+p}{1+2p} + \left(\frac{1.3}{2.4}\right)^2 p^6\cdot\left(\frac{2+p}{1+2p}\right)^2 + \&c\right\}$$

ii. $$1 + \left(\tfrac{1}{4}\right)^2 4p\left(\frac{2-p}{1+4p}\right)^3 + \left(\frac{1.5}{4.8}\right)^2 16p^2\cdot\left(\frac{2-p}{1+4p}\right)^6 + \&c$$

$$= \sqrt{1+4p}\left\{1 + \left(\tfrac{1}{4}\right)^2 4p^3\cdot\frac{2-p}{1+4p} + \left(\frac{1.5}{4.8}\right)^2 16p^6\left(\frac{2-p}{1+4p}\right)^2 + \&c\right\}.$$

iii. If $\tan\frac{A+B}{2} = (1+p)\tan A$, then

$$(1+2p)\int_0^A \frac{d\phi}{\sqrt{1-p^3\cdot\frac{2+p}{1+2p}\sin^2\phi}} = \int_0^B \frac{d\phi}{\sqrt{1-p\cdot\left(\frac{2+p}{1+2p}\right)^3\sin^2\phi}}$$

iv. If $\tan\frac{A-B}{2} = \frac{1-p}{1+2p}\tan B$, then

$$(1+2p)\int_0^A \frac{d\phi}{\sqrt{1-p^3\cdot\frac{2+p}{1+2p}\sin^2\phi}} = 3\int_0^B \frac{d\phi}{\sqrt{1-p\cdot\left(\frac{2+p}{1+2p}\right)^3\sin^2\phi}}.$$

v. If $\tan\frac{A+B}{2} = \frac{2\tan B + 2\tan^3 B\,(1-x)}{1-\tan^4 B\,(1-x)}$, then

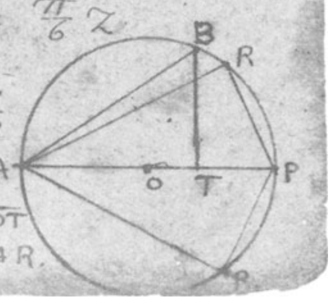
$$\int_0^A \frac{d\phi}{\sqrt{1-x\sin^2\phi}} = 3\int_0^B \frac{d\phi}{\sqrt{1-x\sin^2\phi}}.$$

7. If $x = p\cdot\left(\frac{2+p}{1+2p}\right)^3$ and $z = 1 + \left(\tfrac{1}{2}\right)^2 x + \left(\frac{1.3}{2.4}\right)^2 x^2 + \&c$

i If $\cos A = \frac{1-p}{2+p}$ then $\int_0^A \frac{d\phi}{\sqrt{1-x\sin^2\phi}} = \frac{\pi}{3}z$.

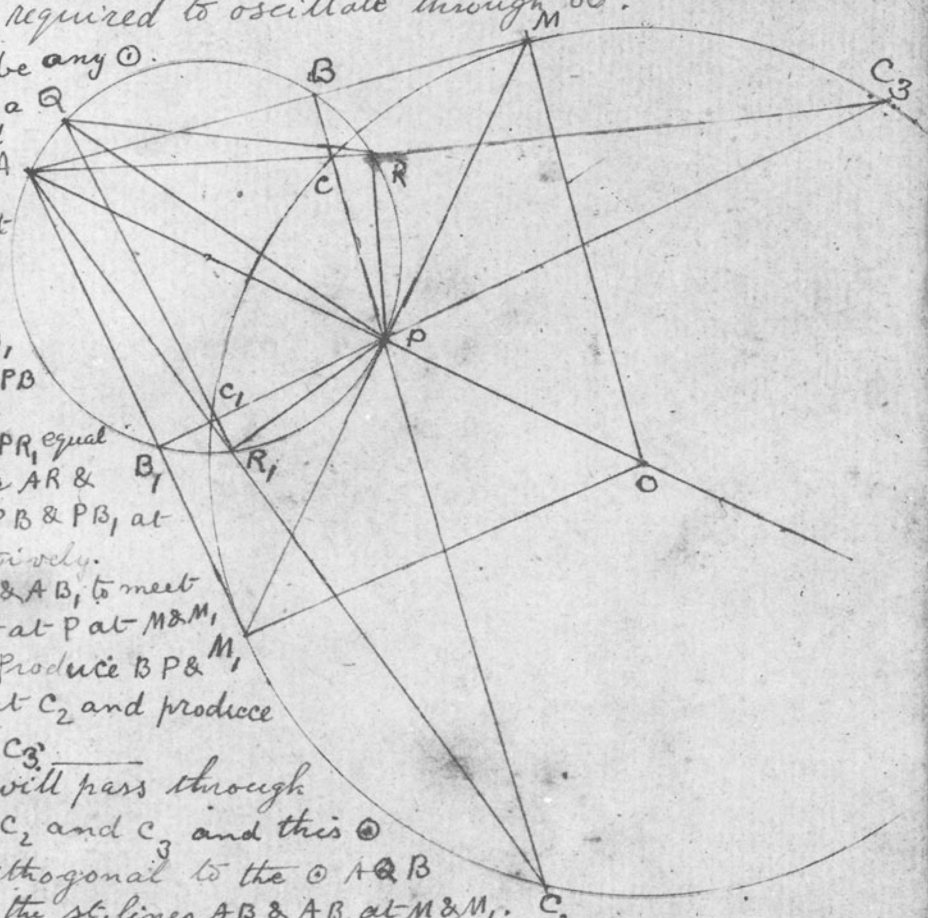
ii If $\sin A = \frac{1+2p}{2+p}$, then $\int_0^A \frac{d\phi}{\sqrt{1-x\sin^2\phi}} = \frac{\pi}{6}z$

iii. PA is any diameter of a ⊙ whose centre is O. Draw TB any perp to AP and PR & PR_1 equal to TB. Join AB, AR & AR_1. Then a pendulum oscillating through $\angle BAR_1$ takes $\frac{AR+OT}{AO}$ or $\frac{3AO}{AR+OT}$ times the time required to oscillate through $\angle BAR$.



Cor. If T coincides with O, $\angle BAR = 15^\circ$ & $\angle BAR_1 = 75^\circ$ and TO =
so that $\frac{AR-OT}{AO}$ or $\frac{3AO}{AR+OT} = \sqrt{3}$ that is
A pendulum oscillating through 300° takes $\sqrt{3}$ times the time required to oscillate through 60°.

IV. Let AQP be any ⊙.
Let AP & PQ be a diameter and a chord.
Let B be the middle point of the arc PQ.
Join AB & PB
Draw AB_1 & PB_1 equal to AB & PB respectively.
Draw PR & PR_1 equal to $\frac{1}{2}$PQ. Join AR & AR_1 cutting PB & PB_1 at C & C_1 respectively.
Produce AB & AB_1 to meet the tangent at P at M & M_1 respectively. Produce BP & AR_1 to meet at C_2 and produce B_1P & AR at C_3.



Then a ⊙ will pass through M, C, C_1, M_1, C_2 and C_3 and this ⊙ will be orthogonal to the ⊙ AQB and touch the st. lines AB & AB_1 at M & M_1.
Let O be the centre of the new ⊙. Join OM & OM_1 and QR & QR_1.

The ⊙ MCM_1 passes also through the intersections of the ⊙s whose centres are A & P and radii AB & PR respectively.
The distances of any pt. on the ⊙ce of MCM_1 from A & P bear a constant ratio. $QR.QR_1 = 3RP^2$.
A pendulum oscillating through 4 times BAR_1 takes $\frac{QR}{RP}$ or $\frac{3R_1P}{R_1Q}$ times the time required to oscillate through 4 times BAR.

7. The properties in page 226 can be proved geometrically as follows:— $\sqrt{\alpha} = \frac{BC_2}{AC_2}$; $\sqrt{\beta} = \frac{BC}{AC}$; $\sqrt{1-\alpha} = \frac{AB}{AC_2}$; $\sqrt{1-\beta} = \frac{AB}{AC}$.

$\sqrt[8]{\alpha\beta} = \sqrt[4]{\frac{BC \cdot BC_2}{AC_1 \cdot AC_2}} = \sqrt{\frac{BM}{AM}} = \frac{BP}{AP}$. similarly $\sqrt[8]{(1-\alpha)(1-\beta)} = \sqrt{\frac{AB}{AM}}$

$= \frac{AB}{AP}$. $m = \frac{QR}{RP}$ and $\frac{3}{m} = \frac{QR_1}{R_1P}$.

$$(1)\quad \sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} = \frac{BM}{AM} + \frac{AB}{AM} = 1.$$

$$(2)\quad \sqrt[8]{\frac{\alpha^3}{\beta}} - \sqrt[8]{\frac{(1-\alpha)^3}{1-\beta}} = \frac{\sqrt{\alpha}}{\sqrt[8]{\alpha\beta}} - \frac{\sqrt{1-\alpha}}{\sqrt[8]{(1-\alpha)(1-\beta)}} = \frac{BC_2}{BP} \cdot \frac{AP}{AC_2} - \frac{AP}{AC_2}$$

$$= \frac{PC_2}{BP} \cdot \frac{AP}{AC_2} = \frac{PC_2}{AC_2} \cdot \frac{AM}{PM} = 1.$$

$$(3)\quad \sqrt[8]{\frac{(1-\beta)^3}{1-\alpha}} - \sqrt[8]{\frac{\beta^3}{\alpha}} = \frac{\sqrt{1-\beta}}{\sqrt[8]{(1-\alpha)(1-\beta)}} - \frac{\sqrt{\beta}}{\sqrt[8]{\alpha\beta}} = \frac{AP}{AC} - \frac{BC}{AC} \cdot \frac{AP}{BP}$$

$$= \frac{AP}{AC} \cdot \frac{CP}{BP} = \frac{CP}{AC} \cdot \frac{AM}{PM} = 1.$$

and so on.

8\. i $x\psi^3(x)\psi(x^5) - 5x^2\psi(x)\psi^3(x^5)$

$$= \frac{x}{1-x^2} - \frac{2x^2}{1-x^4} - \frac{3x^3}{1-x^6} + \frac{4x^4}{1-x^8} + \frac{6x^6}{1-x^{12}} - \&c$$

ii. $5\phi(x)\phi^3(x^5) - \phi^3(x)\phi(x^5)$

$$= 4\left\{1 + \frac{x}{1+x} - \frac{2x^2}{1-x^2} - \frac{3x^3}{1+x^3} + \frac{4x^4}{1-x^4} + \frac{6x^6}{1-x^6} - \&c\right\}.$$

iii. $25\phi(x)\phi^3(x^5) - \frac{\phi^5(x)}{\phi(x^5)}$

$$= 24 + 40\left(\frac{x}{1+x} - \frac{3x^3}{1+x^3} - \frac{7x^7}{1+x^7} + \frac{9x^9}{1+x^9} + \&c\right)$$

iv. $\frac{\psi^5(x)}{\psi(x^5)} - 25x^2\psi(x)\psi^3(x^5)$

$$= 1 + 5\left(\frac{x}{1+x} - \frac{2x^2}{1+x^2} - \frac{3x^3}{1+x^3} + \frac{4x^4}{1+x^4} + \&c\right)$$

9\. i. $$\frac{f^5(-x)}{f(-x^5)} = 1 - 5\left(\frac{x}{1+x} - \frac{3x^3}{1+x^3} + \frac{4x^4}{1+x^4} - \frac{7x^7}{1+x^7} + \frac{9x^9}{1+x^9} + \frac{11x^{11}}{1+x^{11}} - \frac{12x^{12}}{1+x^{12}} - \&c\right)$$

ii. $4x\frac{f^5(-x^5)}{f(-x)} + \frac{\phi^5(x^5)}{\phi(x)} = \phi(x)\phi^3(x^5)$.

iii. $\phi^2(x) - \phi^2(x^5) = 4x\,\chi(x)\,f(-x^5)\,f(-x^{20})$.

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iv. $\left\{\phi(x^5) + 2x^{\frac{1}{5}} f(x^3, x^7)\right\}^2 + \left\{\phi(x^5) + 2x^{\frac{4}{5}} f(x, x^9)\right\}^2$

$= \phi^2(x^{\frac{1}{5}}) - 2\phi^2(x) + 3\phi^2(x^5).$

v. $1 - \dfrac{f^5(-x)}{f(-x^5)} = 5x\, d \log \dfrac{f(-x^2, -x^3)}{f(-x, -x^4)} \Big/ dx$

vi $\dfrac{\psi^5(x)}{\psi(x^5)} - 25x^2 \psi(x) \psi^3(x^5) = 1 - 5x\, d \log \dfrac{f(x^2, x^3)}{f(x, x^4)} \Big/ dx$

vii. $f(x, x^4) f(x^2, x^3) = \dfrac{\phi(-x^5) f(-x^5)}{\chi(-x)}$. & $f(-x, -x^4) f(-x^2, -x^3) =$ $f(-x) f(-x^5)$ and $f(x, x^9) f(x^3, x^7) = \chi(x) f(-x^5) f(-x^{20})$.

10. i. $\psi(x^{\frac{1}{5}}) - x^{\frac{3}{5}} \psi(x^5) = f(x^2, x^3) + x^{\frac{1}{5}} f(x, x^4)$

ii $\phi(x^{\frac{1}{5}}) - \phi(x^5) = 2x^{\frac{1}{5}} f(x^3, x^7) + 2x^{\frac{4}{5}} f(x, x^9)$

iii. $f(-x)\left\{f(-x^{\frac{1}{5}}) + x^{\frac{1}{5}} f(-x^5)\right\} = f^2(-x^2, -x^3) - \sqrt[5]{x^2}\, f^2(-x, -x^4)$

iv. $\phi^2(x) - \phi^2(x^5) = 4x f(x, x^9) f(x^3, x^7)$

v. $\psi^2(x) - x\psi^2(x^5) = f(x, x^4) f(x^2, x^3)$.

vi. $f^5(x^2, x^3) + x f^5(x, x^4) = \left\{\dfrac{\psi^2(x)}{\psi(x^5)} - x\psi(x^5)\right\} \times$

$\left\{\psi^4(x) - 4x\psi^2(x)\psi^2(x^5) + 11x^2\psi^4(x^5)\right\}$.

& hence

vii. $32x f^5(x^3, x^7) + 32x^4 f^5(x, x^9) = \left\{\dfrac{\phi^2(x)}{\phi(x^5)} - \phi(x^5)\right\} \times$

$\left\{\phi^4(x) - 4\phi^2(x)\phi^2(x^5) + 11\phi^4(x^5)\right\}$

viii. $f^{10}(-x^2, -x^3) - x^2 f^{10}(-x, -x^4) = \dfrac{f^{11}(-x)}{f(-x^5)} + 11x f^5(-x) f^5(-x^5)$.

11. i. $\phi(x^{\frac{1}{5}}) = \phi(x^5) + \sqrt[5]{u} + \sqrt[5]{v}$ where

$u + v = \dfrac{\phi^2(x) - \phi^2(x^5)}{\phi(x^5)} \cdot \left\{\phi^4(x) - 4\phi^2(x)\phi^2(x^5) + 11\phi^4(x^5)\right\}$.

$u - v = \dfrac{\phi^2(x) - \phi^2(x^5)}{\phi(x^5)} \left\{5\phi^2(x^5) - \phi^2(x)\right\} \sqrt{\phi^4(x) - 2\phi^2(x)\phi^2(x^5) + 5\phi^4(x}$

$\sqrt[5]{uv} = \phi^2(x) - \phi^2(x^5)$.

ii. $x^{\frac{1}{40}} \psi(x^{\frac{1}{5}}) = x^{\frac{5}{8}} \psi(x^5) + \sqrt[5]{u} + \sqrt[5]{v}$ where

$$u+v = x^{\frac{1}{8}}\,\frac{\psi^2(x)-x\psi^2(x^5)}{\psi(x^5)}\left\{\psi^4(x)-4x\,\psi^2(x)\psi^2(x^5)+11x^2\,\psi^4(x^5)\right\}$$

$$u-v = x^{\frac{1}{8}}.\,\frac{\psi^2(x)-x\,\psi^2(x^5)}{\psi(x^5)}\left\{\psi^2(x)-5x\,\psi^2(x^5)\right\}\times\sqrt{\psi^4(x)-2x\,\psi^2(x)\psi^2(x^5)+5x^2\psi^4(x^5)}$$

$$\sqrt[5]{uv} = x^{\frac{1}{4}}\left\{\psi^2(x)-x^2\psi^2(x^5)\right\}$$

iii. ~~If~~ $2u = 11 + \dfrac{f^6(-x)}{xf^6(-x^5)}$ and $2v = 1+\dfrac{f(-x^{\frac{1}{5}})}{x^{\frac{1}{5}}f(-x^5)}$, then

$$\sqrt[5]{\sqrt{u^2+1}-u} = \sqrt{v^2+1}-v = \frac{\sqrt[5]{x}}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+}\,\frac{x^4}{1+}\,\&c$$

$$= x^{\frac{1}{5}}\,\frac{f(-x,-x^4)}{f(-x^2,-x^3)}$$

iv. $\dfrac{f(-x^{\frac{1}{5}})}{x^{\frac{1}{5}}f(-x^5)} = \sqrt[3]{5+\sqrt[5]{u}-\sqrt[5]{v}}$ when $\sqrt[5]{uv} = 25 + 3\cdot\dfrac{f^6(-x)}{xf^6(-x^5)}$

and $\cancel{v} - u = 5^5\cdot 11 + 75^2\cdot\dfrac{f^6(-x)}{xf^6(-x^5)} + 15^2\dfrac{f^{12}(-x)}{x^2f^{12}(-x^5)} - \dfrac{f^{18}(-x)}{x^3f^{18}(-x^5)}$

12\. i. $1+5x\,\dfrac{f(-x^{25})}{f(-x)} = \sqrt[5]{u}-\sqrt[5]{v}$ where $uv = 1$ and

$$u - v = 11 + 125x\,\frac{f^6(-x^5)}{f^6(-x)}.$$

ii $x\,\dfrac{f(-x^{25})}{f(-x)} = \sqrt[3]{\dfrac{1+\sqrt[5]{u}-\sqrt[5]{v}}{25}}$ where $\sqrt[5]{uv} = 1 + 15x\,\dfrac{f^6(-x^5)}{f^6(-x)}$

and $v - u = 11 + 15^2x\,\dfrac{f^6(-x^5)}{f^6(-x)} + 5\cdot 15^2x^2\dfrac{f^{12}(-x^5)}{f^{12}(-x)} - 25^2x^3\dfrac{f^{18}(-x^5)}{f^{18}(-x)}$

iii. $5\,\dfrac{\phi(x^{25})}{\phi(x)} = 1+\sqrt[5]{u}+\sqrt[5]{v}$ where $\sqrt[5]{uv} = 5\,\dfrac{\phi^2(x^5)}{\phi^2(x)} - 1$

$\&\ u+v = \left\{5\,\dfrac{\phi^2(x^5)}{\phi^2(x)}-1\right\}\left\{11-20\,\dfrac{\phi^2(x^5)}{\phi^2(x)}+25\,\dfrac{\phi^4(x^5)}{\phi^4(x)}\right\}$

iv. $5x^3\,\dfrac{\psi(x^{25})}{\psi(x)} = 1-\sqrt[5]{u}+\sqrt[5]{v}$ when $\sqrt[5]{uv} = 1-5x\,\dfrac{\psi^2(x^5)}{\psi^2(x)}$

$\&\ u-v = \left\{1-5x\,\dfrac{\psi^2(x^5)}{\psi^2(x)}\right\}\left\{11-20x\,\dfrac{\psi^2(x^5)}{\psi^2(x)}+25x^2\,\dfrac{\psi^4(x^5)}{\psi^4(x)}\right\}$

v. $\dfrac{f(-x^{\frac{1}{5}})}{f(-x^5)} = \dfrac{f(-x^2,-x^3)}{f(-x,-x^4)} - x^{\frac{1}{5}} - x^{\frac{2}{5}}\,\dfrac{f(-x,-x^4)}{f(-x^2,-x^3)}$

vi. $\dfrac{\phi(-x^{\frac{2}{5}})\,\phi(-x^{10})}{\phi^2(-x^2)} + x^{\frac{2}{5}}\left\{\dfrac{\psi(x^{\frac{1}{5}})\,\psi(x^5)}{\psi^2(x)} + \dfrac{\psi(-x^{\frac{1}{5}})\,\psi(-x^5)}{\psi^2(-x)}\right\} = 1$

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13. *If β be of the fifth degree,*

i. $\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}+2\sqrt[6]{16\alpha\beta(1-\alpha)(1-\beta)}=1.$

ii. $\sqrt[8]{\frac{\alpha^5}{\beta}}-\sqrt[8]{\frac{(1-\alpha)^5}{1-\beta}}=1+\sqrt[3]{2}\sqrt[24]{\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}}$

iii. $\sqrt[8]{\frac{(1-\beta)^5}{1-\alpha}}-\sqrt[8]{\frac{\beta^5}{\alpha}}=1+\sqrt[3]{2}\sqrt[24]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}}$

iv. $m=1+2\sqrt[3]{2}\sqrt[24]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}}$ & $\frac{5}{m}=1+2\sqrt[3]{2}\sqrt[24]{\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}}$

v. $$m=\frac{1+\sqrt[8]{\frac{(1-\beta)^5}{1-\alpha}}}{1+\sqrt[8]{(1-\alpha)^3(1-\beta)}}=\frac{1-\sqrt[8]{\frac{\beta^5}{\alpha}}}{1-\sqrt[8]{\alpha^3\beta}}.$$

vi. $$\frac{5}{m}=\frac{1+\sqrt[8]{\frac{\alpha^5}{\beta}}}{1+\sqrt[8]{\alpha\beta^3}}=\frac{1-\sqrt[8]{\frac{(1-\alpha)^5}{1-\beta}}}{1-\sqrt[8]{(1-\alpha)(1-\beta)^3}}$$

vii. $$\sqrt[8]{\alpha\beta^3}+\sqrt[8]{(1-\alpha)(1-\beta)^3}=1-\sqrt[3]{2}\sqrt[24]{\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}}=$$

$$\sqrt[8]{\alpha^3\beta}+\sqrt[8]{(1-\alpha)^3(1-\beta)}=\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$$

viii. *For all values of a and b*

$$m=\frac{a+2(a-b)\sqrt[3]{2}\sqrt[24]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}}+b\sqrt[3]{4}\sqrt[12]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}}}{a-b\sqrt[6]{16\alpha\beta(1-\alpha)(1-\beta)}}$$

$$=\frac{1-\sqrt[3]{2}\sqrt[24]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}}-\sqrt[3]{4}\sqrt[12]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}}}{\sqrt{1-3\sqrt[6]{16\alpha\beta(1-\alpha)(1-\beta)}+\sqrt[3]{16\alpha\beta(1-\alpha)(1-\beta)}}}$$

ix. $$1+\sqrt[3]{4}\sqrt[12]{\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}}=m\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}\text{ \& }$$

$$1+\sqrt[3]{4}\sqrt[12]{\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}}=\frac{5}{m}\cdot\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}.$$

x. $$\sqrt[4]{\alpha(1-\beta)}+\sqrt[4]{\beta(1-\alpha)}=\sqrt[3]{4}\sqrt[24]{\alpha\beta(1-\alpha)(1-\beta)}$$

$$=m\sqrt[4]{\alpha(1-\alpha)}+\sqrt[4]{\beta(1-\beta)}=\sqrt[4]{\alpha(1-\alpha)}+\frac{5}{m}\sqrt[4]{\beta(1-\beta)}.$$

xi. $$\sqrt[8]{\frac{(1-\beta)^5}{1-\alpha}}+\sqrt[8]{\frac{\beta^5}{\alpha}}=m\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}\text{ and }$$

$$\sqrt[8]{\frac{(1-\alpha)^5}{1-\beta}}+\sqrt[8]{\frac{\alpha^5}{\beta}}=\frac{5}{m}\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}.$$

ii. $m = \sqrt[4]{\frac{\beta}{\alpha}} + \sqrt[4]{\frac{1-\beta}{1-\alpha}} - \sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$ and hence

$$\frac{5}{m} = \sqrt[4]{\frac{\alpha}{\beta}} + \sqrt[4]{\frac{1-\alpha}{1-\beta}} - \sqrt[4]{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}}$$

xiii. $m - \frac{5}{m} = 4\left\{\sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}\right\} \Big/ \sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$

and

$$m + \frac{5}{m} = 2\left\{2 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}\right\}$$

xiv. If $P = \sqrt[12]{16\alpha\beta(1-\alpha)(1-\beta)}$ and $Q = \sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$, then

$$Q + \frac{1}{Q} + 2\left(P - \frac{1}{P}\right) = 0.$$

xv. If $P = \sqrt[4]{\alpha\beta}$ and $Q = \sqrt[8]{\frac{\beta}{\alpha}}$, then

$$\left(Q - \frac{1}{Q}\right)^3 + 8\left(Q - \frac{1}{Q}\right) = 4\left(P - \frac{1}{P}\right).$$

14. i. If $\alpha = \sin^2(u+v)$ and $\beta = \sin^2(u-v)$, then

$$\sin 2u = \sin v\,(1 + \cos^2 v).$$

ii. If $4\alpha(1-\alpha) = p\left(\frac{2-p}{1+2p}\right)^5$, then $4\beta(1-\beta) = p^5\,\frac{2-p}{1+2p}$.

iii. If $1 - 2\alpha = \frac{1-11p-p^2}{(1+2p)^2}\sqrt{\frac{1+p^2}{1+2p}}$, then $1 - 2\beta = (1+p-p^2)\sqrt{\frac{1+p^2}{1+2p}}$

iv. $1 + \left(\frac{1}{2}\right)^2 \dfrac{1 - \frac{1-11p-p^2}{(1+2p)^2}\sqrt{\frac{1+p^2}{1+2p}}}{2} + \&c.$

$$= (1+2p)\left\{1 + \left(\tfrac{1}{2}\right)^2 \frac{1 - (1+p-p^2)\sqrt{\frac{1+p^2}{1+2p}}}{2} + \&c.\right.$$

v. $1 + \left(\frac{1}{4}\right)^2 p\cdot\left(\frac{2-p}{1+2p}\right)^5 + \left(\frac{1\cdot 5}{4\cdot 8}\right)^2 p^2\cdot\left(\frac{2-p}{1+2p}\right)^{10} + \&c$

$$= (1+2p)\left\{1 + \left(\tfrac{1}{4}\right)^2 p^5\cdot\frac{2-p}{1+2p} + \left(\tfrac{1\cdot 5}{4\cdot 8}\right)^2 p^{10}\cdot\left(\frac{2-p}{1+2p}\right)^2 + \&c\right\}$$

15. If γ be of the 25th degree,

i. $\sqrt[8]{\frac{\gamma}{\alpha}} + \sqrt[8]{\frac{1-\gamma}{1-\alpha}} - \sqrt[8]{\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}} - 2\sqrt[12]{\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}} = \sqrt{\frac{1+\left(\frac{1}{2}\right)^2\alpha + \&c}{1+\left(\frac{1}{2}\right)^2\gamma + \&c}}$

ii. $\sqrt[8]{\frac{\alpha}{\gamma}} + \sqrt[8]{\frac{1-\alpha}{1-\gamma}} - \sqrt[8]{\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}} - 2\sqrt[12]{\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}} = 5\sqrt{\frac{1+\left(\frac{1}{2}\right)^2\gamma + \&c}{1+\left(\frac{1}{2}\right)^2\alpha + \&c}}$

iii. $\sqrt[8]{\frac{\alpha\gamma}{\beta^2}} + \sqrt[8]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}} + \sqrt[8]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}} = \dfrac{1+\left(\frac{1}{2}\right)^2\beta + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\beta^2 + \&c}{\sqrt{1+\left(\frac{1}{2}\right)^2\alpha + \&c}\ \sqrt{1+\left(\frac{1}{2}\right)^2\gamma + \&c}}$

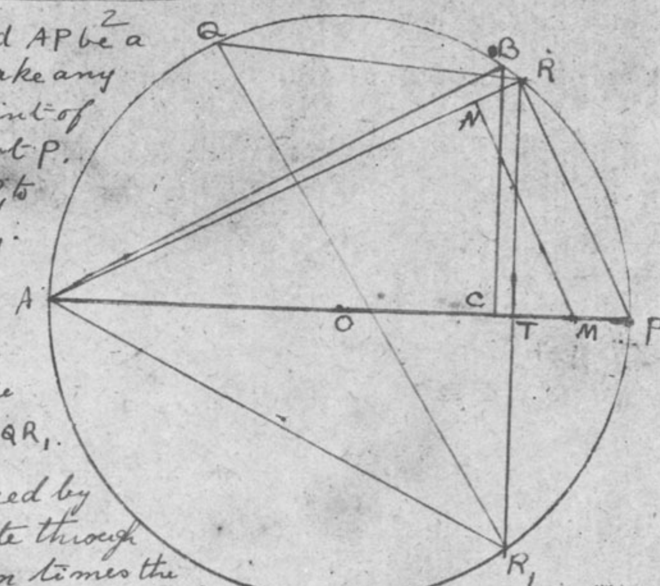
iv. $\sqrt[4]{\frac{\beta^2}{\alpha\gamma}} + \sqrt[4]{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}} + \sqrt[4]{\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}} - 2\sqrt[8]{\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}}\left\{1+\sqrt[8]{\frac{\beta^2}{\alpha\gamma}}+\sqrt[8]{\frac{(1-}{(1-}}\right.$?

$$= 5 \cdot \frac{1+\left(\frac{1}{2}\right)^2\alpha + \&c}{1+\left(\frac{1}{2}\right)^2\beta + \&c} \cdot \frac{1+\left(\frac{1}{2}\right)^2\gamma + \&c}{1+\left(\frac{1}{2}\right)^2\beta + \&c}.$$

v. $$\frac{1+\sqrt[3]{4}\sqrt[24]{\frac{\beta^{10}(1-\beta)^{10}}{\alpha\gamma(1-\alpha)(1-\gamma)}}}{1+\sqrt[3]{4}\sqrt[24]{\frac{\alpha^5\gamma^5(1-\alpha)^5(1-\gamma)^5}{\beta^2(1-\beta)^2}}} = \frac{1+\left(\frac{1}{2}\right)^2\alpha + \&c}{1+\left(\frac{1}{2}\right)^2\beta + \&c} \cdot \frac{1+\left(\frac{1}{2}\right)^2\gamma + \&c}{1+\left(\frac{1}{2}\right)^2\beta + \&c}.$$

16. i. If $\int_0^A \frac{d\phi}{\sqrt{1-\alpha\sin^2\phi}} = m\int_0^B \frac{d\phi}{\sqrt{1-\beta\sin^2\phi}}$, then

$$\tan\frac{A-B}{2} = \frac{p\tan B}{1+\frac{1+p+\sqrt{(1+2p)(1+p^2)}}{2}\tan^2 B}.$$



ii. Let O be the centre and AP be a diameter of the ⊙ PAQ. Take any point T between the point of medial section & the point P. Through T draw a perp. RR_1 to AP and join PR, RA & AR_1. Through M draw MN ∥^{l} to PR, M being the middle pt. of TP. Draw BC perp. to AP and equal to MN. Cut off the arc BQ = BP. Join AB, QR & QR_1.

Then if the time required by a pendulum to oscillate through 4 times the ∠ BAR, be m times the time required to oscillate through 4 times the ∠ BAR_1, then

$$1+m = 2\,\frac{QR}{RT} \text{ and } 1+\frac{5}{m} = \frac{2\,QR_1}{R_1T} \cdot \text{ and } \frac{5}{m} - m = 8\cdot\frac{OC}{AR}.$$

N.B. i. Taking AP = 1, we see that $TP = \sqrt[4]{16\alpha\beta(1-\alpha)(1-\beta)}$ & $CT = \sqrt{\alpha\beta}$ and $OC + OT = \sqrt{(1-\alpha)(1-\beta)}$ so that $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 2\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)} = 1$

N.B. ii. If T be the point of medial section of AP, then C will coincide with the centre O and the ratio between the times to oscillate through ∠ BAR & ∠ BAR_1 is $1 : \sqrt{5}$.

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7. i. $x\psi(x)\psi(x^7) = \frac{x}{1-x} - \frac{x^3}{1-x^3} - \frac{x^5}{1-x^5} + \frac{x^9}{1-x^9} + \frac{x^{11}}{1-x^{11}} - \frac{x^{13}}{1-x^{13}}$

$+ \frac{x^{15}}{1-x^{15}} - \frac{x^{17}}{1-x^{17}} - \frac{x^{19}}{1-x^{19}} + \frac{x^{23}}{1-x^{23}} + \&c$

ii. $\phi(x)\phi(x^7) = 1 + 2\Big(\frac{x}{1-x} - \frac{x^2}{1-x^2} - \frac{x^3}{1-x^3} + \frac{x^4}{1-x^4} - \frac{x^5}{1-x^5} +$

$\frac{x^6}{1-x^6} + \frac{x^8}{1-x^8} + \frac{x^9}{1-x^9} + \frac{x^{10}}{1-x^{10}} + \&c\Big)$

iii. $\phi(x^{\frac{1}{7}}) - \phi(x^7) = 2x^{\frac{1}{7}}f(x^5, x^9) + 2x^{\frac{4}{7}}f(x^3, x^{11}) + 2x^{\frac{9}{7}}f(x, x^{13})$.

iv. $\psi(x^{\frac{1}{7}}) - x^{\frac{6}{7}}\psi(x^7) = f(x^3, x^4) + x^{\frac{1}{7}}f(x^2, x^5) + x^{\frac{3}{7}}f(x, x^6)$.

v. $\frac{f(-x^{\frac{1}{7}})}{f(-x^7)} = \frac{f(-x^2, -x^5)}{f(-x, -x^6)} - x^{\frac{1}{7}}\frac{f(-x^3, -x^4)}{f(-x^2, -x^5)} - x^{\frac{2}{7}} + x^{\frac{5}{7}}\frac{f(-x, -x^6)}{f(-x^3, -x^4)}$.

18. i. $1 + \frac{f(-x^{\frac{1}{7}})}{x^{\frac{2}{7}}f(-x^7)} = \sqrt[7]{u} - \sqrt[7]{v} + \sqrt[7]{w}$ where

$u - v + w = 57 + 14\frac{f^4(-x)}{xf^4(-x^7)} + \frac{f^8(-x)}{x^2f^8(-x^7)}$.

$uv - uw + vw = 289 + 126\frac{f^4(-x)}{xf^4(-x^7)} + 19\frac{f^8(-x)}{x^2f^8(-x^7)}$

$uvw = 1.$ $\qquad + \frac{f^{12}(-x)}{x^3f^{12}(-x^7)}$.

ii. $1 + 7x^2\frac{f(-x^{49})}{f(-x)} = \sqrt[7]{u} - \sqrt[7]{v} + \sqrt[7]{w}$ where

$u - v + w = 57 + 2.7^3x\frac{f^4(-x^7)}{f^4(-x)} + 7^4x^2\frac{f^8(-x^7)}{f^8(-x)}$

$uv - uw + vw = 289 + 18.7^3x\frac{f^4(-x^7)}{f^4(-x)} + 19.7^4x^2\frac{f^8(-x^7)}{f^8(-x)}$

$+ 7^6x^3\frac{f^{12}(-x^7)}{f^{12}(-x)}$.

iii. $f(x, x^6)f(x^2, x^5)f(x^3, x^4) = \frac{f^2(-x^7)}{\chi(-x)}\phi(-x^7)$.

iv. $f(-x, -x^6)f(-x^2, -x^5)f(-x^3, -x^4) = f(-x)f^2(-x^7)$.

v. $f(x, x^{13})f(x^3, x^{11})f(x^5, x^9) = \chi(x)\psi(-x^7)f^2(-x^{14})$.

vi. If $u = \frac{f^4(-x)}{xf^4(-x^7)}$ and $v = \frac{f(-x^{\frac{1}{7}})}{x^{\frac{2}{7}}f(-x^7)}$, then

$2u = 7(v^3 + 5v^2 + 7v) + (v^2 + 7v + 7)\sqrt{4v^3 + 21v^2 + 28v}$

19. If β be of the 7th degree,

i. $\sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)} = 1$ so that $\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$

$$= 1 - \sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}.$$

ii. $m = \dfrac{1-4\sqrt[24]{\dfrac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}}}{\sqrt[8]{(1-\alpha)(1-\beta)} - \sqrt[8]{\alpha\beta}}$ and $\dfrac{7}{m} = \dfrac{1-4\sqrt[24]{\dfrac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}}}{\sqrt[8]{\alpha\beta} - \sqrt[8]{(1-\alpha)(1-\beta)}}.$

iii. $\sqrt[8]{\dfrac{(1-\beta)^7}{1-\alpha}} - \sqrt[8]{\dfrac{\beta^7}{\alpha}} = m\sqrt{\dfrac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$ and

$$\sqrt[8]{\frac{\alpha^7}{\beta}} - \sqrt[8]{\frac{(1-\alpha)^7}{1-\beta}} = \frac{7}{m}\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$$

iv. $\sqrt[8]{\dfrac{(1-\beta)^7}{1-\alpha}} - 1 = \sqrt[8]{\alpha\beta}\left\{\sqrt[8]{\dfrac{(1-\beta)^7}{1-\alpha}} - \sqrt[8]{\dfrac{\beta^7}{\alpha}}\right\}$ and

$$\sqrt[8]{\frac{\alpha^7}{\beta}} - 1 = \sqrt[8]{(1-\alpha)(1-\beta)}\left\{\sqrt[8]{\frac{\alpha^7}{\beta}} - \sqrt[8]{\frac{(1-\alpha)^7}{1-\beta}}\right\}$$

v. $m^2 = \sqrt{\dfrac{\beta}{\alpha}} + \sqrt{\dfrac{1-\beta}{1-\alpha}} - \sqrt{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 8\sqrt[3]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}$ and

$$\frac{49}{m^2} = \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{1-\alpha}{1-\beta}} - \sqrt{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}} - 8\sqrt[3]{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}}.$$

vi. $\sqrt[4]{\dfrac{(1-\beta)^3}{1-\alpha}} + \sqrt[4]{\dfrac{\beta^3}{\alpha}} - \sqrt[4]{\dfrac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}} = m^2 \cdot \dfrac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}$

& $\sqrt[4]{\dfrac{(1-\alpha)^3}{1-\beta}} + \sqrt[4]{\dfrac{\alpha^3}{\beta}} - \sqrt[4]{\dfrac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}} = \dfrac{49}{m^2} \cdot \dfrac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}$

vii. $\sqrt[8]{\dfrac{(1-\beta)^7}{1-\alpha}} + \sqrt[8]{\dfrac{\beta^7}{\alpha}} + 2\sqrt[24]{\dfrac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}} = \dfrac{3}{4} + \dfrac{m^2}{4}$ and

$$\sqrt[8]{\frac{(1-\alpha)^7}{1-\beta}} + \sqrt[8]{\frac{\alpha^7}{\beta}} + 2\sqrt[24]{\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}} = \frac{3}{4} + \frac{49}{4m^2}$$

viii. $m - \dfrac{7}{m} = 2\left(\sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}\right)\left(2+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}\right).$

ix. If $P = \sqrt[8]{16\alpha\beta(1-\alpha)(1-\beta)}$ and $Q = \sqrt[6]{\dfrac{\beta(1-\beta)}{\alpha(1-\alpha)}}$, then

$$Q + \frac{1}{Q} + 7 = 2\sqrt{2}\left(P+\frac{1}{P}\right).$$

x. If $P = \sqrt{\alpha\beta}$ and $Q = \sqrt{\dfrac{\beta}{\alpha}}$, then

$$P + \frac{1}{P} = Q + \frac{1}{Q} + \left(\sqrt[8]{P} - \frac{1}{\sqrt[8]{P}}\right)^8.$$

xi. If $\alpha = \sin^2(u+v)$ & $\beta = \sin^2(u-v)$, then $\cos 2u = (2\cos v - 1)\sqrt{4\cos v - 3}$

CHAPTER XX

i. Let $v = \sqrt[3]{x}\,\dfrac{\chi(-x)}{\chi^3(-x^3)} = \dfrac{\sqrt[3]{x}}{1+}\;\dfrac{x+x^2}{1+}\;\dfrac{x^2+x^4}{1+}\;\dfrac{x^3+x^6}{1+}\;\dfrac{x^4+x^8}{1+}\;\&c$ then

$1+\dfrac{1}{v} = \dfrac{\psi(x^{\frac{1}{3}})}{x^{\frac{1}{3}}\psi(x^3)}$ & $1+\dfrac{1}{v^3} = \dfrac{\psi^4(x)}{x\,\psi^4(x^3)}$.

ii. $1+\dfrac{\psi(-x^{\frac{1}{3}})}{x^{\frac{1}{3}}\psi(-x^3)} = \sqrt[3]{1+\dfrac{\psi^4(-x)}{x\,\psi^4(-x^3)}}$ and $2v = 1 - \dfrac{\phi(-x^{\frac{1}{3}})}{\phi(-x^3)}$

$1+3x\,\dfrac{\psi(-x^9)}{\psi(-x)} = \sqrt[3]{1+9x\,\dfrac{\psi^4(-x^3)}{\psi^4(-x)}}$

$\cos 40 + \cos 80 = \cos 20$

iii. $\dfrac{\phi(x^{\frac{1}{3}})}{\phi(x^3)} = 1+\sqrt[3]{\dfrac{\phi^4(x)}{\phi^4(x^3)}-1}$ and $\dfrac{1}{\cos 40}+\dfrac{1}{\cos 80} = \dfrac{1}{\cos 20}+6$

$\dfrac{\phi(x^9)}{\phi(x)} = \dfrac{1+\sqrt[3]{9\dfrac{\phi^4(x^3)}{\phi^4(x)}-1}}{3}$.

iv. $3+\dfrac{f^3(-x^{\frac{1}{3}})}{x^{\frac{1}{3}}f^3(-x^3)} = \sqrt[3]{27+\dfrac{f^{12}(-x)}{x\,f^{12}(-x^3)}}$ and $= \dfrac{1}{v}+4v^2$.

$1+9x\,\dfrac{f^3(-x^9)}{f^3(-x)} = \sqrt[3]{1+27x\,\dfrac{f^{12}(-x^3)}{f^{12}(-x)}}$

v. $f^3(-x^{\frac{1}{3}}) + 3x^{\frac{1}{3}}.f^3(-x^3) = f(-x)\left\{1+6\left(\dfrac{x}{1-x}-\dfrac{x^2}{1-x^2}+\dfrac{x^4}{1-x^4}-\dfrac{x^5}{1-x^5}+\&c\right)\right\}$

2\. i. $\phi(x)\phi(x^9) - \phi^2(x^3) = 2x\,\phi(-x^2)\,\psi(x^9)\,\chi(x^3)$.

ii. $\psi(x) - 3x\,\psi(x^9) = \dfrac{\phi(-x)}{\chi(-x^3)}$.

iii. $\phi(x)\phi(x^9) + \phi^2(x^3) = 2\psi(x)\,\phi(-x^{18})\,\chi(x^3)$.

iv. $\psi(x^{\frac{1}{9}}) - x^{\frac{1}{9}}\psi(x) = f(x^4, x^5) + x^{\frac{1}{3}}f(x^2, x^7) + x^{\frac{2}{3}}f(x, x^8)$.

v. $f(-x^{\frac{1}{3}}) = f(-x^4, -x^5) - x^{\frac{1}{3}}f(-x^2, -x^7) - x^{\frac{2}{3}}f(-x, -x^8)$.

vi. $f(-x, -x^8)\,f(-x^2, -x^7)\,f(-x^4, -x^5) = \dfrac{f(-x)\,f^3(-x^9)}{f(-x^3)}$.

vii. $\dfrac{f(-x^4, -x^5)}{f(-x^2, -x^7)} + x\,\dfrac{f(-x, -x^8)}{f(-x^4, -x^5)} = \dfrac{f(-x^2, -x^7)}{f(-x, -x^8)}$.

viii. $\dfrac{f(-x^4, -x^5)}{f(-x, -x^8)} + x\,\dfrac{f(-x^2, -x^7)}{f(-x^4, -x^5)} = x\,\dfrac{f(-x, -x^8)}{f(-x^2, -x^7)} + \dfrac{f^4(-x^3)}{f(-x)\,f^3(-x^9)}$

ix. $\phi(x^{\frac{1}{9}}) - x^{\frac{1}{9}}\phi(x) = 2x^{\frac{1}{9}}f(x^7, x^{11}) + 2x^{\frac{4}{9}}f(x^5, x^{13}) + 2x^{\frac{16}{9}}f(x, x^{17})$

3. If β be of the 3rd degree and γ of the 9th degree then.

i. $$1+\sqrt[3]{4}\,\sqrt[24]{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}} = 3\sqrt{\frac{1+\left(\frac{1}{2}\right)^2\gamma+\left(\frac{1.3}{2.4}\right)^2\gamma^2+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\left(\frac{1.3}{2.4}\right)^2\alpha^2+\&c}}$$

ii. $$1+\sqrt[3]{4}\,\sqrt[24]{\frac{\gamma^3(1-\gamma)^3}{\beta(1-\beta)}} = 3\sqrt{\frac{1+\left(\frac{1}{2}\right)^2\alpha+\&c}{1+\left(\frac{1}{2}\right)^2\gamma+\&c}}.$$

iii. $$1-2\sqrt[3]{2}\,\sqrt[24]{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}}\,\sqrt[24]{\frac{\gamma^3(1-\gamma)^3}{\beta(1-\beta)}} = \frac{1+\left(\frac{1}{2}\right)^2\beta+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\&c}\cdot\frac{1+\left(\frac{1}{2}\right)^2\beta+\&c}{1+\left(\frac{1}{2}\right)^2\gamma+\&c}$$

iv. $$1-\sqrt[3]{4}\,\sqrt[24]{\frac{\gamma^3(1-\alpha)^3}{\beta(1-\beta)}} = \sqrt[3]{4}\,\sqrt[24]{\frac{\alpha^3(1-\gamma)^3}{\beta(1-\beta)}}-1$$
$$= \sqrt{\frac{1+\left(\frac{1}{2}\right)^2\beta+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\&c}}\,\sqrt{\frac{1+\left(\frac{1}{2}\right)^2\beta+\&c}{1+\left(\frac{1}{2}\right)^2\gamma+\&c}}.$$

v. $$\sqrt{\alpha\gamma}+\sqrt{(1-\alpha)(1-\gamma)}+2\sqrt[3]{4\beta(1-\beta)} = 1+8\sqrt[4]{\beta(1-\beta)}\,\sqrt[8]{\alpha\gamma(1-\alpha)(1-\gamma)}$$

vi. $$\sqrt[8]{\alpha(1-\gamma)}+\sqrt[8]{\gamma(1-\alpha)} = \sqrt[3]{2}\,\sqrt[24]{\beta(1-\beta)}.$$

vii. $$\frac{-1+\sqrt[4]{\frac{(1-\beta)^3}{1-\alpha}}}{1-\sqrt[4]{(1-\alpha)(1-\beta)}} = \frac{1-\sqrt[4]{\frac{\beta^3}{\alpha}}}{1-\sqrt[4]{\alpha\beta}} = \frac{1+\left(\frac{1}{2}\right)^2\alpha+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\&c}.$$

viii. $$1+\sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}} = \frac{1+\left(\frac{1}{2}\right)^2\alpha+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\&c}\cdot\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$$

ix. $$1+\sqrt[8]{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}} = 3\cdot\frac{1+\left(\frac{1}{2}\right)^2\beta+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\&c}\cdot\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$$

x. $$\sqrt[8]{\frac{\gamma}{\alpha}}+\sqrt[8]{\frac{1-\gamma}{1-\alpha}}-\sqrt[8]{\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}} = \sqrt{\frac{1+\left(\frac{1}{2}\right)^2\alpha+\&c}{1+\left(\frac{1}{2}\right)^2\gamma+\&c}}.$$

xi. $$\sqrt[8]{\frac{\alpha}{\gamma}}+\sqrt[8]{\frac{1-\alpha}{1-\gamma}}-\sqrt[8]{\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}} = 3\sqrt{\frac{1+\left(\frac{1}{2}\right)^2\gamma+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\&c}}$$

xii. $$\sqrt[4]{\frac{\beta^2}{\alpha\gamma}}+\sqrt[4]{\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}}-\sqrt[4]{\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}}$$
$$= -3\cdot\frac{1+\left(\frac{1}{2}\right)^2\alpha+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\&c}\cdot\frac{1+\left(\frac{1}{2}\right)^2\gamma+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\&c}$$

xiii. $$\sqrt[4]{\frac{\alpha\gamma}{\beta^2}}+\sqrt[4]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}}-\sqrt[4]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}}$$
$$= \frac{1+\left(\frac{1}{2}\right)^2\beta+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\&c}\cdot\frac{1+\left(\frac{1}{2}\right)^2\beta+\&c}{1+\left(\frac{1}{2}\right)^2\gamma+\&c}.$$

xiv. $$\frac{\sqrt[3]{2}\,\sqrt[24]{\beta(1-\beta)}}{\sqrt[8]{\alpha(1-\gamma)}-\sqrt[8]{\gamma(1-\alpha)}} = \sqrt{\frac{1+\left(\frac{1}{2}\right)^2\alpha+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\&c}}\,\sqrt{\frac{1+\left(\frac{1}{2}\right)^2\gamma+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\&c}}$$

xv. $$\left(\sqrt[4]{\alpha}-\sqrt[4]{\gamma}\right)^4+\left(\sqrt[4]{1-\gamma}-\sqrt[4]{1-\alpha}\right)^4 = \left\{\sqrt[4]{\alpha(1-\gamma)}-\sqrt[4]{\gamma(1-\alpha)}\right\}^4.$$

xvi. $$1 = \sqrt{\alpha\gamma}+\sqrt{(1-\alpha)(1-\gamma)}+2\sqrt[3]{4\beta(1-\beta)}\cdot\frac{1+\left(\frac{1}{2}\right)^2\beta+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\&c}\cdot\frac{1+\left(\frac{1}{2}\right)^2\beta+\&c}{1+\left(\frac{1}{2}\right)^2\gamma+\&c}$$

i. $\dfrac{\phi(-x^{18})}{\phi(-x^{2})} + x\left\{\dfrac{\psi(x^{9})}{\psi(x)} - \dfrac{\psi(-x^{9})}{\psi(-x)}\right\} = 1.$

ii. $\dfrac{\phi(-x^{2})}{\phi(-x^{18})} + \dfrac{1}{x}\left\{\dfrac{\psi(x)}{\psi(x^{9})} - \dfrac{\psi(-x)}{\psi(-x^{9})}\right\} = 3.$

iii. $\dfrac{\phi(-x^{2})\,\phi(-x^{54})}{\phi(-x^{6})\,\phi(-x^{18})} + \dfrac{1}{x^{2}}\left\{\dfrac{\psi(x)\,\psi(x^{27})}{\psi(x^{3})\,\psi(x^{9})} + \dfrac{\psi(-x)\,\psi(-x^{27})}{\psi(-x^{3})\,\psi(-x^{9})}\right\} = 1$

iv. $\phi(x)\phi(x^{27}) - \phi(-x)\phi(-x^{27}) = 4x f(-x^{6})f(-x^{18}) + 4x^{7}\psi(x^{4})\psi(x^{54})$

5\. i. If $\alpha, \beta, \gamma, \delta$ be of the 1st, 3rd, 9th and 27th degree respectively,

$$\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} + \sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} = \sqrt{\frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}}\sqrt{\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}}$$

ii. $$\sqrt[4]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[4]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} + \sqrt[4]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} -$$
$$2\sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}\left\{1 + \sqrt[8]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}}\right\}$$
$$= -3.\ \frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}\cdot\frac{1+(\frac{1}{2})^2\delta+\&c}{1+(\frac{1}{2})^2\gamma+\&c}$$

iii. $$\frac{1-\sqrt[4]{\alpha\delta}-\sqrt[4]{(1-\alpha)(1-\delta)}}{2\sqrt[12]{16\beta\gamma(1-\beta)(1-\gamma)}} = \sqrt{\frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}}\sqrt{\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}}$$

iv. $$= \frac{\sqrt[24]{16\beta\gamma(1-\beta)(1-\gamma)} + \sqrt[8]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[24]{16\beta\gamma(1-\beta)(1-\gamma)} + \sqrt[8]{16\alpha\delta(1-\alpha)(1-\delta)}}$$

6\. i. $\psi(x^{\frac{1}{11}}) - x^{\frac{15}{11}}\psi(x^{11}) = f(x^{5}, x^{6}) + x^{\frac{1}{11}}f(x^{4}, x^{7}) + x^{\frac{3}{11}}f(x^{3}, x^{8})$
$\qquad + x^{\frac{6}{11}}f(x^{2}, x^{9}) + x^{\frac{10}{11}}f(x, x^{10}).$

ii. $\phi(x^{\frac{1}{11}}) - \phi(x^{11}) = 2x^{\frac{1}{11}}f(x^{9}, x^{13}) + 2x^{\frac{4}{11}}f(x^{7}, x^{15}) +$
$\qquad 2x^{\frac{9}{11}}f(x^{5}, x^{17}) + 2x^{\frac{16}{11}}f(x^{3}, x^{19}) + 2x^{\frac{25}{11}}f(x, x^{21})$

iii. $$\frac{f(-x^{\frac{1}{11}})}{f(-x^{11})} = \frac{f(-x^{4}, -x^{7})}{f(-x^{2}, -x^{9})} - x^{\frac{1}{11}}\frac{f(-x^{2}, -x^{9})}{f(-x, -x^{10})} - x^{\frac{2}{11}}\frac{f(-x^{5}, -x^{6})}{f(-x^{3}, -x^{8})}$$
$$+ x^{\frac{5}{11}} + x^{\frac{7}{11}}\frac{f(-x^{3}, -x^{8})}{f(-x^{4}, -x^{7})} - x^{\frac{15}{11}}\frac{f(-x, -x^{10})}{f(-x^{5}, -x^{6})}$$

7\. If β be of the 11th degree,

i. $\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)} + 2\sqrt[12]{16\alpha\beta(1-\alpha)(1-\beta)} = 1.$

$$\text{ii. } m - \frac{11}{m} = 2\left(\sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}\right)\left(4 + \sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right)$$

$$\text{iii. } m + \frac{11}{m} = 4\left(2 + \sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right)\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$$

$$\text{iv. } \sqrt[8]{\frac{(1-\beta)^3}{1-\alpha}} - \sqrt[8]{\frac{\beta^3}{\alpha}} - \sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}} = m\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$$

$$\text{v. } \sqrt[8]{\frac{\alpha^3}{\beta}} - \sqrt[8]{\frac{(1-\alpha)^3}{1-\beta}} - \sqrt[8]{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}} = \frac{11}{m}\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$$

$$\text{vi. } \frac{1}{m}\left\{1 + 8\sqrt[3]{2}\sqrt[24]{\frac{\beta^{11}(1-\beta)^{11}}{\alpha(1-\alpha)}}\right\} - \frac{m}{11}\left\{1 + 8\sqrt[3]{2}\sqrt[24]{\frac{\alpha^{11}(1-\alpha)^{11}}{\beta(1-\beta)}}\right\}$$

$$= 2\left(\sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}\right).$$

$$\text{vii. } \frac{1}{m}\left\{1 + 8\sqrt[3]{2}\sqrt[24]{\frac{\beta^{11}(1-\beta)^{11}}{\alpha(1-\alpha)}}\right\} + \frac{m}{11}\left\{1 + 8\sqrt[3]{2}\sqrt[24]{\frac{\alpha^{11}(1-\alpha)^{11}}{\beta(1-\beta)}}\right\}$$

$$= 4\left(\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right)\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$$

$$\text{viii. } \sqrt[3]{2}\sqrt[24]{\frac{\alpha^{11}(1-\beta)^{11}}{\beta(1-\alpha)}} - \sqrt[3]{2}\sqrt[24]{\frac{\beta^{11}(1-\alpha)^{11}}{\alpha(1-\beta)}}$$

$$= \left(3 + \sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right)\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$$

$$\text{ix. } 4 - \sqrt[3]{2}\cdot\sqrt[24]{\frac{\alpha^{11}(1-\beta)^{11}}{\beta(1-\alpha)}} - \sqrt[3]{2}\sqrt[24]{\frac{\beta^{11}(1-\alpha)^{11}}{\alpha(1-\beta)}}$$

$$= 2\sqrt[12]{16\alpha\beta(1-\alpha)(1-\beta)}\left\{2 + \sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right\}.$$

$$\text{8. i } \frac{f(-x^{\frac{1}{13}})}{x^{\frac{7}{13}}f(-x^{13})} = \frac{f(-x^4,-x^9)}{x^{\frac{7}{13}}f(-x^2,-x^{11})} - \frac{f(-x^6,-x^7)}{x^{\frac{6}{13}}f(-x^3,-x^{10})} - \frac{f(-x^2,-x^{11})}{x^{\frac{5}{13}}f(-x,-x^{12})}$$

$$+ \frac{f(-x^5,-x^8)}{x^{\frac{2}{13}}f(-x^4,-x^9)} + 1 - x^{\frac{5}{13}}\frac{f(-x^3,-x^{10})}{f(-x^5,-x^8)} + x^{\frac{15}{13}}\frac{f(-x,-x^{12})}{f(-x^6,-x^7)}$$

$= u_1 - u_2 - u_3 + u_4 + 1 - u_5 + u_6$, where

$$u_1u_2 - u_3u_5 - u_4u_6 = 1 + \frac{f^2(-x)}{xf^2(-x^{13})}$$

$$\frac{1}{u_1u_2} - \frac{1}{u_3u_5} - \frac{1}{u_4u_6} = -4 - \frac{f^2(-x)}{xf^2(-x^{13})}.$$

$u_2u_3u_4 - u_1u_5u_6 = 3 + \dfrac{f^2(-x)}{xf^2(-x^{13})}$ & $u_1u_2u_3u_4u_5u_6 = 1.$

ii. $f(-x,-x^{12})f(-x^2,-x^{11})f(-x^3,-x^{10})f(-x^4,-x^9)f(-x^5,-x^8)f(-x^6,-x^7)$

$$= f(-x)f^5(-x^{13}).$$

If β be of the 13th degree,

iii. $m = \sqrt[4]{\frac{\beta}{\alpha}} + \sqrt[4]{\frac{1-\beta}{1-\alpha}} - \sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 4\sqrt[6]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$ and

iv. $\frac{13}{m} = \sqrt[4]{\frac{\alpha}{\beta}} + \sqrt[4]{\frac{1-\alpha}{1-\beta}} - \sqrt[4]{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}} - 4\sqrt[6]{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}}$

9\. i. $\psi(x^3)\psi(x^5) - \psi(-x^3)\psi(-x^5) = 2x^3\psi(x^2)\psi(x^{30}).$

ii. $\phi(-x^6)\phi(-x^{10}) + 2x\psi(x^3)\psi(x^5) = \phi(x)\phi(x^{15}).$

iii. $\phi(-x^2)\phi(-x^{30}) + 2x^2\psi(x)\psi(x^{15}) = \phi(x^3)\phi(x^5).$

iv. $\psi(x)\psi(x^{15}) + \psi(-x)\psi(-x^{15}) = 2\psi(x^6)\psi(x^{10}).$

v. $\phi(x)\phi(x^{15}) - \phi(x^3)\phi(x^5) = 2xf(-x^2)f(-x^{30})\chi(x^3)\chi(x^5)$

vi. $\phi(x)\phi(x^{15}) + \phi(x^3)\phi(x^5) = 2f(-x^6)f(-x^{10})\chi(x)\chi(x^{15})$

vii. $\{\psi(x^3)\psi(x^5) - x\psi(x)\psi(x^{15})\}\phi(-x^3)\phi(-x^5)$

$$= \{\psi(x^3)\psi(x^5) + x\psi(x)\psi(x^{15})\}\phi(-x)\phi(-x^{15})$$

$$= f(-x)f(-x^3)f(-x^5)f(-x^{15}).$$

10\. i. $f(-x^7,-x^8) + xf(-x^2,-x^{13}) = \frac{f(-x^2,-x^3)}{f(-x,-x^4)}f(-x^5)$

ii. $f(-x^4,-x^{11}) - xf(-x,-x^{14}) = \frac{f(-x,-x^4)}{f(-x^2,-x^3)}f(-x^5)$

iii $f(-x^7,-x^8) - xf(-x^2,-x^{13})$

$$= f(-x^{\frac{2}{3}},-x) + x^{\frac{2}{3}}f(-x^3,-x^{12})$$

iv. $\{f(-x^4,-x^{11}) + xf(-x,-x^{14})\}\frac{1}{3}x^{\frac{1}{3}}$

$$= f(-x^6,-x^9) - f(-x^{\frac{1}{3}},-x^{\frac{4}{3}})$$

v. $x\psi(x^3)\psi(x^5) + x^2\psi(x)\psi(x^{15}) = \frac{x}{1-x} - \frac{x^7}{1-x^7} - \frac{x^{11}}{1-x^{11}} - \frac{x^{13}}{1-x^{13}}$

$$+ \frac{x^{17}}{1-x^{17}} + \frac{x^{19}}{1-x^{19}} + \&c$$

vi. $\phi(x^3)\phi(x^5) + \phi(x)\phi(x^{15}) = 2\left(1 + \frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} - \frac{x^7}{1-x^7} + \&c\right)$

11. If $\alpha, \beta, \gamma, \delta$ be of the 1st, 3rd, 5th & 15th degree

i. $$\sqrt[8]{\alpha\delta} + \sqrt[8]{(1-\alpha)(1-\delta)} = \sqrt{\frac{1+\left(\frac{1}{2}\right)^2\beta + \&c}{1+\left(\frac{1}{2}\right)^2\alpha + \&c}}\sqrt{\frac{1+\left(\frac{1}{2}\right)^2\gamma + \&c}{1+\left(\frac{1}{2}\right)^2\delta + \&c}}$$

ii. $$\sqrt[8]{\beta\gamma} + \sqrt[8]{(1-\beta)(1-\gamma)} = \sqrt{\frac{1+\left(\frac{1}{2}\right)^2\alpha + \&c}{1+\left(\frac{1}{2}\right)^2\beta + \&c}}\sqrt{\frac{1+\left(\frac{1}{2}\right)^2\delta + \&c}{1+\left(\frac{1}{2}\right)^2\gamma + \&c}}$$

$$= \frac{\sqrt[8]{\beta\gamma} - \sqrt[8]{\beta\gamma(1-\beta)(1-\gamma)}}{\sqrt[4]{\alpha\delta}} = \frac{\sqrt[8]{(1-\beta)(1-\gamma)} - \sqrt[8]{\beta\gamma(1-\beta)(1-\gamma)}}{\sqrt[4]{(1-\alpha)(1-\delta)}}.$$

iii. $$\sqrt[8]{\alpha\delta} - \sqrt[8]{(1-\alpha)(1-\delta)} = \sqrt[8]{\beta\gamma} - \sqrt[8]{(1-\beta)(1-\gamma)}.$$

iv. $$1 + \sqrt[8]{\beta\gamma} + \sqrt[8]{(1-\beta)(1-\gamma)} = \sqrt[3]{4}\sqrt[24]{\frac{\beta^2\gamma^2(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)(1-\delta)}},$$

v. $$1 - \sqrt[8]{\alpha\delta} - \sqrt[8]{(1-\alpha)(1-\delta)} = \sqrt[3]{4}\sqrt[24]{\frac{\alpha^2\delta^2(1-\alpha)^2(1-\delta)^2}{\beta\gamma(1-\beta)(1-\gamma)}},$$

vi. $$\sqrt[16]{\alpha\delta}\left(\sqrt[4]{1+\sqrt{\alpha}}\sqrt[4]{1+\sqrt{\delta}} + \sqrt[4]{1-\sqrt{\alpha}}\sqrt[4]{1-\sqrt{\delta}}\right)$$

$$+ \sqrt[16]{(1-\alpha)(1-\delta)}\left(\sqrt[4]{1+\sqrt{1-\alpha}}\sqrt[4]{1+\sqrt{1-\delta}} + \sqrt[4]{1-\sqrt{1-\alpha}}\sqrt[4]{1-\sqrt{1-\delta}}\right) = \sqrt{2}.$$

vii. $$\sqrt[16]{\beta\gamma}\left(\sqrt[4]{1+\sqrt{\beta}}\sqrt[4]{1-\sqrt{\gamma}} - \sqrt[4]{1-\sqrt{\beta}}\sqrt[4]{1-\sqrt{\gamma}}\right)$$

$$+ \sqrt[16]{(1-\beta)(1-\gamma)}\left(\sqrt[4]{1+\sqrt{1-\beta}}\sqrt[4]{1+\sqrt{1-\gamma}} + \sqrt[4]{1-\sqrt{1-\beta}}\sqrt[4]{1-\sqrt{1-\gamma}}\right) = \sqrt{2}.$$

viii. $$\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} - \sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} = \sqrt{\frac{1+\left(\frac{1}{2}\right)^2\beta + \&c}{1+\left(\frac{1}{2}\right)^2\alpha + \&c}}\sqrt{\frac{1+\left(\frac{1}{2}\right)^2\gamma + \&c}{1+\left(\frac{1}{2}\right)^2\delta + \&c}}$$

ix. $$\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} = -\sqrt{\frac{1+\left(\frac{1}{2}\right)^2\alpha + \&c}{1+\left(\frac{1}{2}\right)^2\beta + \&c}}\sqrt{\frac{1+\left(\frac{1}{2}\right)^2\delta + \&c}{1+\left(\frac{1}{2}\right)^2\gamma + \&c}}$$

x. $$\sqrt[4]{\frac{\beta\delta}{\alpha\gamma}} + \sqrt[4]{\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}} - \sqrt[4]{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}} - 4\sqrt[6]{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}}$$

$$= \frac{1+\left(\frac{1}{2}\right)^2\alpha + \&c}{1+\left(\frac{1}{2}\right)^2\beta + \&c} \cdot \frac{1+\left(\frac{1}{2}\right)^2\gamma + \&c}{1+\left(\frac{1}{2}\right)^2\delta + \&c}$$

xi. $$\sqrt[4]{\frac{\alpha\gamma}{\beta\delta}} + \sqrt[4]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}} - \sqrt[4]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}} - 4\sqrt[6]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}}$$

$$= 9 \cdot \frac{1+\left(\frac{1}{2}\right)^2\beta + \&c}{1+\left(\frac{1}{2}\right)^2\alpha + \&c} \cdot \frac{1+\left(\frac{1}{2}\right)^2\delta + \&c}{1+\left(\frac{1}{2}\right)^2\gamma + \&c}.$$

xii. $$-\sqrt[4]{\frac{\gamma\delta}{\alpha\beta}} + \sqrt[4]{\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}} + \sqrt[4]{\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}}$$

$$- 2\sqrt[8]{\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}}\left\{1 + \sqrt[8]{\frac{\gamma\delta}{\alpha\beta}} + \sqrt[8]{\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}}\right\} = \frac{1+\left(\frac{1}{2}\right)^2\alpha + \&c}{1+\left(\frac{1}{2}\right)^2\gamma + \&c} \cdot \frac{1+\left(\frac{1}{2}\right)^2\beta + \&c}{1+\left(\frac{1}{2}\right)^2\delta + \&}$$

xiii. $\sqrt[4]{\frac{\alpha\beta}{\gamma\delta}} + \sqrt[4]{\frac{(1-\alpha)(1-\beta)}{(1-\gamma)(1-\delta)}} + \sqrt[4]{\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}} - 2\sqrt[8]{\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}} \times$

$$\left\{1 + \sqrt[8]{\frac{\alpha\beta}{\gamma\delta}} + \sqrt[8]{\frac{(1-\alpha)(1-\beta)}{(1-\gamma)(1-\delta)}}\right\} = 25 \cdot \frac{1 + (\frac{1}{2})^2\gamma + \&c}{1 + (\frac{1}{2})^2\alpha + \&c} \cdot \frac{1 + (\frac{1}{2})^2\delta + \&c}{1 + (\frac{1}{2})^2\beta + \&c}$$

xiv. $\sqrt[8]{\alpha\beta\gamma\delta} + \sqrt[8]{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)} + \sqrt[3]{2}\sqrt[24]{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}$

$$= 1.$$

xv. If $P = \sqrt[48]{256\,\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}$ and

$Q = \sqrt[16]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}$, then $Q + \frac{1}{Q} = \sqrt{2}\left(P + \frac{1}{P}\right)$.

12. i. $$\frac{f(-x^{\frac{1}{17}})}{x^{\frac{12}{17}} f(-x^{17})} = \frac{f(-x^6, -x^{11})}{x^{\frac{12}{17}} f(-x^3, -x^{14})} - \frac{1}{x^{\frac{11}{17}}} \frac{f(-x^4, -x^{13})}{f(-x^2, -x^{15})}$$

$$- \frac{1}{x^{\frac{10}{17}}} \frac{f(-x^8, -x^9)}{f(-x^4, -x^{13})} + \frac{1}{x^{\frac{7}{17}}} \frac{f(-x^2, -x^{15})}{f(-x, -x^{16})} + \frac{1}{x^{\frac{5}{17}}} \frac{f(-x^7, -x^{10})}{f(-x^5, -x^{12})}$$

$$-1 - x^{\frac{3}{17}} \frac{f(-x^5, -x^{12})}{f(-x^6, -x^{11})} + x^{\frac{14}{17}} \frac{f(-x^3, -x^{14})}{f(-x^7, -x^{10})} - x^{\frac{28}{17}} \frac{f(-x, -x^{16})}{f(-x^8, -x^9)}$$

$= u_1 - u_2 - u_3 + u_4 + u_5 - 1 - u_6 + u_7 - u_8$ where

$u_1 u_5 = u_6 u_7 = u_2 u_8 = u_3 u_4 = 1$. and

$u_1 u_5 + u_2 u_8 - u_3 u_4 - u_6 u_7 = -1$.

ii. $f(-x, -x^{16}) f(-x^2, -x^{15}) f(-x^3, -x^{14}) f(-x^4, -x^{13}) f(-x^5, -x^{12}) \times$

$$f(-x^6, -x^{11}) f(-x^7, -x^{10}) f(-x^8, -x^9) = f(-x) f^7(-x^{17}).$$

iii If β be of the 17th degree,

$$m = \sqrt[4]{\frac{\beta}{\alpha}} + \sqrt[4]{\frac{1-\beta}{1-\alpha}} + \sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 2\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left\{1 + \sqrt[8]{\frac{\beta}{\alpha}} + \sqrt[8]{\frac{1-\beta}{1-\alpha}}\right\}.$$

iv. $$\frac{17}{m} = \sqrt[4]{\frac{\alpha}{\beta}} + \sqrt[4]{\frac{1-\alpha}{1-\beta}} + \sqrt[4]{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}} - 2\sqrt[8]{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}}\left\{1 + \sqrt[8]{\frac{\alpha}{\beta}} + \sqrt[8]{\frac{1-\alpha}{1-\beta}}\right\}.$$

N.B. Thus we see that $\phi(x^{\frac{1}{n}})$, $\psi(x^{\frac{1}{n}})$ or $f(-x^{\frac{1}{n}})$ n being any prime number can be expressed as the sum of $\frac{n-1}{2}$, nth roots of several functions and $\phi(x^n)$, $\psi(x^n)$ or $f(-x^n)$. In finding the values of these functions, quadratics only apear in case of the 5th, 17th, 257th &c degrees, and cubics in case of the 7th 13th, 19th, 37th, 73rd, 97th, 109th, 163rd, 193rd &c degrees not as cube roots but as $\sin(\frac{1}{3}\sin^{-1}\theta)$ and quintics in case of the 11th, 41st, 101st &c degrees. $f^3(-x^{\frac{1}{n}})$ can also be similarly expressed.

13. If α, β, γ & δ be of the 1st 3rd 7th and 21st degree,

i. $$\sqrt[4]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[4]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[4]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} + 4\sqrt[6]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$
$$= \frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}\cdot\frac{1+(\frac{1}{2})^2\delta+\&c}{1+(\frac{1}{2})^2\gamma+\&c}.$$

ii. $$\sqrt[4]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[4]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} - \sqrt[4]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} + 4\sqrt[6]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}$$
$$= \frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}\cdot\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}.$$

iii. $$\sqrt[8]{\frac{\gamma\delta}{\alpha\beta}} + \sqrt[8]{\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}} - \sqrt[8]{\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}} - 2\sqrt[12]{\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}}$$
$$= \sqrt{\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\gamma+\&c}}\sqrt{\frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\delta+\&c}}$$

iv. $$\sqrt[8]{\frac{\alpha\beta}{\gamma\delta}} + \sqrt[8]{\frac{(1-\alpha)(1-\beta)}{(1-\gamma)(1-\delta)}} - \sqrt[8]{\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}} - 2\sqrt[12]{\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}}$$
$$= 7\sqrt{\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\alpha+\&c}}\sqrt{\frac{1+(\frac{1}{2})^2\delta+\&c}{1+(\frac{1}{2})^2\beta+\&c}}$$

v. $$\sqrt[4]{\frac{\beta\delta}{\alpha\gamma}} + \sqrt[4]{\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}} + \sqrt[4]{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}} - 2\sqrt[8]{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}}\left\{1+\sqrt[8]{\frac{\beta\delta}{\alpha\gamma}} + \sqrt[8]{\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}}\right\}$$
$$= \frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}\cdot\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}.$$

vi. $$\sqrt[4]{\frac{\alpha\gamma}{\beta\delta}} + \sqrt[4]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}} + \sqrt[4]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}} - 2\sqrt[8]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}}\left\{1+\sqrt[8]{\frac{\alpha\gamma}{\beta\delta}} + \sqrt[8]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}}\right\}$$
$$= 9\cdot\frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}\cdot\frac{1+(\frac{1}{2})^2\delta+\&c}{1+(\frac{1}{2})^2\gamma+\&c}$$

14. If $\alpha, \beta, \gamma, \delta$ be of the 1st, 3rd, 11th and 33rd degree.

i. $$\sqrt[8]{\frac{\beta\delta}{\alpha\gamma}} + \sqrt[8]{\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}} - \sqrt[8]{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}} - 2\sqrt[12]{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}}$$
$$= \sqrt{\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}}\sqrt{\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}}$$

ii. $$\sqrt[8]{\frac{\alpha\gamma}{\beta\delta}} + \sqrt[8]{\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}} - \sqrt[8]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}} - 2\sqrt[12]{\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}}$$
$$= 3\sqrt{\frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}}\sqrt{\frac{1+(\frac{1}{2})^2\delta+\&c}{1+(\frac{1}{2})^2\gamma+\&c}}.$$

15. If β be of the 23rd degree,

i. $$\sqrt[8]{\alpha\beta} + \sqrt[8]{(1-\alpha)(1-\beta)} + \sqrt[3]{4}\,\sqrt[24]{\alpha\beta(1-\alpha)(1-\beta)} = 1.$$

ii. $$\frac{1+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}+2\sqrt[3]{2}\,\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}}{2} =$$

$$\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}.$$

$$\cancel{\text{ii. } m - \frac{23}{m} = 2\left(\sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}\right)\left\{11 - 2\sqrt[3]{4}\sqrt[24]{\alpha\beta(1-\alpha)(1-\beta)}\right.}$$

$$\cancel{\left. + 14\sqrt[3]{2}\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}\right\}}.$$

16. If β be of the 19th degree,

$$\text{i. } \sqrt[8]{\frac{(1-\beta)^3}{1-\alpha}} - \sqrt[8]{\frac{\beta^3}{\alpha}} + \sqrt[8]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}} - 2\sqrt[16]{\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}} \times$$

$$\sqrt{\sqrt[8]{\frac{(1-\beta)^3}{1-\alpha}} - 1 - \sqrt[8]{\frac{\beta^3}{\alpha}}} = m\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$$

$$\text{ii. } \sqrt[8]{\frac{\alpha^3}{\beta}} - \sqrt[8]{\frac{(1-\alpha)^3}{1-\beta}} + \sqrt[8]{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}} - 2\sqrt[16]{\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}} \times$$

$$\sqrt{\sqrt[8]{\frac{\alpha^3}{\beta}} - 1 - \sqrt[8]{\frac{(1-\alpha)^3}{1-\beta}}} = \frac{19}{m}\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}.$$

17. i. $\phi(x)\phi(x^{35}) = \phi(-x)\phi(-x^{35}) + 4x f(-x^{10}) f(-x^{14})$

$$+ 4x^9\psi(x^2)\psi(x^{70}).$$

ii. $\phi(x^5)\phi(x^7) = \phi(-x^5)\phi(-x^7) + 4x^3\psi(x^{10})\psi(x^{14})$

$$- 4x^3 f(-x^2) f(-x^{70}).$$

iii. $\phi(-x^{10})\phi(-x^{54}) + 2x f(x^9) f(x^{15}) + 2x^4\psi(x^3)\psi(x^{27})$

$$= \phi(x)\phi(x^{135}).$$

iv. $\phi(-x^2)\phi(-x^{270}) + 2x^{17}\psi(x)\psi(x^{135}) + 2x^2 f(x^3) f(x^{45})$

$$= \phi(x^5)\phi(x^{27}).$$

18. If α, β, γ & δ be of the 1st, 5th, 7th & 35th degree,

$$\text{i. } \sqrt[4]{\alpha\delta} + \sqrt[4]{(1-\alpha)(1-\delta)} + 2\sqrt[3]{2}\sqrt[12]{\alpha\delta(1-\alpha)(1-\delta)} +$$

$$\sqrt[4]{\beta\gamma} + \sqrt[4]{(1-\beta)(1-\gamma)} + 2\sqrt[3]{2}\sqrt[12]{\beta\gamma(1-\beta)(1-\gamma)} =$$

$$1 + \left\{1 + 2\sqrt[3]{2}\sqrt[24]{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}\right\}^2.$$

$$\text{ii. } \left\{\sqrt[4]{\alpha\delta} + \sqrt[4]{(1-\alpha)(1-\delta)} + 2\sqrt[3]{2}\sqrt[12]{\alpha\delta(1-\alpha)(1-\delta)}\right\} \times$$

$$\left\{\sqrt[4]{\beta\gamma} + \sqrt[4]{(1-\beta)(1-\gamma)} + 2\sqrt[3]{2}\sqrt[12]{\beta\gamma(1-\beta)(1-\gamma)}\right\} =$$

$$1 - 4\sqrt[3]{2}\sqrt[24]{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}\left\{\sqrt[8]{\alpha\beta\gamma\delta} + \sqrt[8]{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)}\right.$$

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iii. $\sqrt[4]{\alpha\delta} + \sqrt[4]{(1-\alpha)(1-\delta)} + 2\sqrt[3]{2}\,\sqrt[12]{\beta\gamma(1-\beta)(1-\gamma)}\cdot\sqrt{\frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}\cdot\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}}$
$= 1.$

iv. $\sqrt[4]{\beta\gamma} + \sqrt[4]{(1-\beta)(1-\gamma)} - 2\sqrt[3]{2}\,\sqrt[12]{\alpha\delta(1-\alpha)(1-\delta)}\,\sqrt{\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}\cdot\frac{1+(\frac{1}{2})^2\delta+\&c}{1+(\frac{1}{2})^2\gamma+\&c}}$
$= 1.$

v. $$\sqrt{\frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}\cdot\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}} = \frac{\sqrt[24]{16\beta\gamma(1-\beta)(1-\gamma)} - \sqrt[8]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[24]{16\beta\gamma(1-\beta)(1-\gamma)} + \sqrt[8]{16\beta\gamma(1-\beta)(1-\gamma)}}$$

$$= \frac{\sqrt[8]{16\alpha\delta(1-\alpha)(1-\delta)} + \sqrt[24]{16\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[8]{16\beta\gamma(1-\beta)(1-\gamma)} - \sqrt[24]{16\alpha\delta(1-\alpha)(1-\delta)}}$$

vi. $$\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} - \sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} + 2\sqrt[12]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}$$
$$= \sqrt{\frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}\cdot\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}}$$

vii. $$\sqrt[8]{\frac{\beta\gamma}{\alpha\delta}} + \sqrt[8]{\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}} - \sqrt[8]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}} + 2\sqrt[12]{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}}$$
$$= \sqrt{\frac{1+(\frac{1}{2})^2\alpha+\&c}{1+(\frac{1}{2})^2\beta+\&c}\cdot\frac{1+(\frac{1}{2})^2\delta+\&c}{1+(\frac{1}{2})^2\gamma+\&c}}.$$

19. i. $\phi(x)\phi(x^{63}) - \phi(x^7)\phi(x^9) = 2x f(x^3) f(x^{21})$

ii. $\psi(x^7)\psi(x^9) - x^6\psi(x)\psi(x^{63}) = f(-x^8) f(-x^{42})$.

iii. If $\alpha, \beta, \gamma, \delta$ be of the 1st 3rd 13th and 39th degree or 1st 5th 11th and 55th degree or 1st 7th 9th and 63rd degree.

$$\frac{1+\sqrt[4]{(1-\alpha)(1-\delta)}+\sqrt[4]{\alpha\delta}}{1+\sqrt[4]{(1-\beta)(1-\gamma)}+\sqrt[4]{\beta\gamma}} = \frac{\sqrt[8]{(1-\alpha)(1-\delta)} - \sqrt[8]{\alpha\delta}}{\sqrt[8]{(1-\beta)(1-\gamma)} - \sqrt[8]{\beta\gamma}} =$$

$$\sqrt{\frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}\cdot\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}} = \frac{\sqrt[8]{\alpha\delta} \pm \sqrt[8]{\alpha\delta(1-\alpha)(1-\delta)}}{\sqrt[8]{\beta\gamma} - \sqrt[8]{\beta\gamma(1-\beta)(1-\gamma)}}$$

(+ in the first two cases and − in the last case).

iv. If $\alpha, \beta, \gamma, \delta$ be of the 1st 3rd 13th and 39th degree or 1st 5th 7th and 35th degree

$$\sqrt[8]{\frac{\alpha\delta}{\beta\gamma}} + \sqrt[8]{\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}} - \sqrt[8]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}} + 2\sqrt[12]{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}}$$
$$= \sqrt{\frac{1+(\frac{1}{2})^2\beta+\&c}{1+(\frac{1}{2})^2\alpha+\&c}}\,\sqrt{\frac{1+(\frac{1}{2})^2\gamma+\&c}{1+(\frac{1}{2})^2\delta+\&c}} \text{ and}$$

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i. $\sqrt[32]{\alpha\beta}\left\{\sqrt[8]{(1+\sqrt{\alpha})(1+\sqrt{\beta})}\sqrt{1+\sqrt[4]{\alpha\beta}+\sqrt{(1-\sqrt{\alpha})(1-\sqrt{\beta})}}+\right.$

$$\left.\sqrt[8]{(1-\sqrt{\alpha})(1-\sqrt{\beta})}\sqrt{1+\sqrt[4]{\alpha\beta}+\sqrt{(1+\sqrt{\alpha})(1+\sqrt{\beta})}}\right\}$$

$$+\sqrt[32]{(1-\alpha)(1-\beta)}\left\{\cdots\cdots\cdots\cdots\right\}=\sqrt[4]{8}.$$

ii. $1+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}-2\left(\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}+\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\right)$

$$=2\sqrt[16]{\alpha\beta(1-\alpha)(1-\beta)}\sqrt{1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}}.$$

iii. $1+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}-\sqrt{\dfrac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$

$$=\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}+\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}.$$

23\. i. If β be of the 47th degree,

$$2\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}=\left(1+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}\right)$$

$$+\sqrt[3]{4}\,\sqrt[24]{\alpha\beta(1-\alpha)(1-\beta)}\left\{1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\right\}$$

ii. If β be of the 71st degree

$$1+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}-\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}$$

$$=\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\mp\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}$$

$$+\sqrt[3]{4}\,\sqrt[24]{\alpha\beta(1-\alpha)(1-\beta)}\left\{1\mp\sqrt[8]{\alpha\beta}\mp\sqrt[8]{(1-\alpha)(1-\beta)}\right\}$$

24\. If $\alpha, \beta, \gamma, \delta$ be of the 1st 3rd 29th 87th or 1st 5th 27th 135th or 1st 11th 21st 231st or 1st 13th 19th 247th or 1st 7th 25th 175th or 1st 9th 23rd 207th or 1st 15th 17th 255th degree, then

i. $\sqrt{\dfrac{1+\sqrt{\beta\gamma}+\sqrt{(1-\beta)(1-\gamma)}}{2}}+\sqrt[8]{\beta\gamma}+\sqrt[8]{(1-\beta)(1-\gamma)}+\sqrt[8]{\beta\gamma(1-\beta)(1-\gamma)}$

$$=\left(1+\sqrt[4]{\alpha\delta}+\sqrt[4]{(1-\alpha)(1-\delta)}\right)\sqrt{\frac{1+\left(\frac{1}{2}\right)^2\alpha+\&c}{1+\left(\frac{1}{2}\right)^2\beta+\&c}\cdot\frac{1+\left(\frac{1}{2}\right)^2\delta+\&c}{1+\left(\frac{1}{2}\right)^2\gamma+\&c}}.$$

ii. $\sqrt{\dfrac{1+\sqrt{\alpha\delta}+\sqrt{(1-\alpha)(1-\delta)}}{2}}+\sqrt[8]{\alpha\delta}+\sqrt[8]{(1-\alpha)(1-\delta)}\pm\sqrt[8]{\alpha\delta(1-\alpha)(1-\delta)}.$

$$=\left(1+\sqrt[4]{\beta\gamma}+\sqrt[4]{(1-\beta)(1-\gamma)}\right)\sqrt{\frac{1+\left(\frac{1}{2}\right)^2\beta+\&c}{1+\left(\frac{1}{2}\right)^2\alpha+\&c}\cdot\frac{1+\left(\frac{1}{2}\right)^2\gamma+\&c}{1+\left(\frac{1}{2}\right)^2\delta+\&c}}.$$

(– in the 1st 4 cases and + in the last 3).

i. $1-\frac{3}{y}-24\left(\frac{1}{e^{2y}-1}+\frac{2}{e^{4y}-1}+\frac{3}{e^{6y}-1}+\frac{4}{e^{8y}-1}+\&c\right)$ is a complete series which when divided by z^2 can be expressed as radicals precisely in the same manner as the series $1+240\left(\frac{1^3}{e^{2y}-1}+\frac{2^3}{e^{4y}-1}+\frac{3^3}{e^{6y}-1}+\frac{4^3}{e^{8y}-1}+\&c\right)$ and the series $1-504\left(\frac{1^5}{e^{2y}-1}+\frac{2^5}{e^{4y}-1}+\frac{3^5}{e^{6y}-1}+\frac{4^5}{e^{8y}-1}+\&c\right)$ when divided by z^4 and z^6 respectively.

ii. $1-24\left(\frac{1}{e^{y}+1}+\frac{3}{e^{3y}+1}+\frac{5}{e^{5y}+1}+\&c\right)=z^2(1-2x)$.

iii. $1-240\left(\frac{1^3}{e^{y}+1}-\frac{2^3}{e^{2y}-1}+\frac{3^3}{e^{3y}+1}-\&c\right)=z^4(1-16x.\overline{1-x})$

iv. $1+504\left(\frac{1^5}{e^{y}+1}-\frac{2^5}{e^{2y}-1}+\frac{3^5}{e^{3y}+1}-\&c\right)=z^6(1-2x)(1+32x.\overline{1-x})$.

i. $12x\frac{d\,\phi(x)}{dx}\Big/\phi(x)=\left\{1-24\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)\right\}$
$-\left\{1-24\left(\frac{x}{1+x}+\frac{3x^3}{1+x^3}+\&c\right)\right\}$

ii. $24x\frac{d\,x^{\frac{1}{8}}\psi(x)}{dx}\Big/x^{\frac{1}{8}}\psi(x)=4\left\{1-24\left(\frac{x^4}{1-x^4}+\frac{2x^8}{1-x^8}+\&c\right)\right\}$
$-\left\{1-24\left(\frac{x}{1+x}+\frac{3x^3}{1+x^3}+\&c\right)\right\}$

iii. $24x\frac{d\,x^{\frac{1}{24}}f(-x)}{dx}\Big/x^{\frac{1}{24}}f(-x)=1-24\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\&c\right)$

iv. $24x\frac{d\,x^{\frac{1}{24}}/\chi(x)}{dx}\Big/\frac{x^{\frac{1}{24}}}{\chi(x)}=1-24\left(\frac{x}{1+x}+\frac{3x^3}{1+x^3}+\frac{5x^5}{1+x^5}+\&c\right)$.

v. By differentiating the equation for m once or the equation for α, β twice we can calculate the value of the first series.

3. i. $1+12\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\&c\right)-12\left(\frac{3x^3}{1-x^3}+\frac{6x^6}{1-x^6}+\&c\right)$

$=\left\{1+6\left(\frac{x}{1-x}-\frac{x^2}{1-x^2}+\frac{x^4}{1-x^4}-\&c\right)\right\}^2$

$=\left\{1+24\left(\frac{1^3x}{1-x}+\frac{2^3x^2}{1-x^2}+\frac{3^3x^3}{1-x^3}+\&c\right)+8\left(\frac{3^3x^3}{1-x^3}+\frac{6^3x^6}{1-x^6}+\&c\right)\right\}^{\frac{1}{2}}$

$=\left\{\frac{\psi^4(x)+3x\psi^4(x^3)}{\psi(x)\psi(x^3)}\right\}^2=\left\{\frac{f^{12}(-x)+27xf^{12}(-x^3)}{f^3(-x)f^3(-x^3)}\right\}^{\frac{2}{3}}$

ii. $1+12\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-12\left(\frac{3x^6}{1-x^6}+\frac{6x^{12}}{1-x^{12}}+\&c\right)$

$=\left\{\frac{\phi^4(x)+3\phi^4(x^3)}{4\phi(x)\phi(x^3)}\right\}^2=\phi^2(x)\phi^2(x^3)-4x.\psi^2(-x)\psi^2(-x^3)$.

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iii. $1 + 12\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right) - 12\left(\frac{3}{e^{6y}-1} + \frac{6}{e^{12y}-1} + \&c\right)$

$= \phi^2(e^{-y})\,\phi^2(e^{-3y}) \cdot \frac{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}}{2}$.

4. i. $1 + 6\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right) - 6\left(\frac{5x^5}{1-x^5} + \frac{10x^{10}}{1-x^{10}} + \&c\right)$

$= \sqrt{f^{12}(-x) + 22x f^6(-x) f^6(-x^5) + 125x^2 f^{12}(-x^5)} \,/\, f(-x) f(-x^5)$

$= \frac{\psi^4(x) + 2x\psi^2(x)\psi^2(x^5) + 5x^2\psi^4(x^5)}{\psi(x)\psi(x^5)} \sqrt{\psi^4(x) - 2x\psi^2(x)\psi^2(x^5) + 5x^2\psi^4(x^5)}$

ii. $1 + 6\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \frac{3x^6}{1-x^6} + \&c\right) - 6\left(\frac{5x^{10}}{1-x^{10}} + \frac{10x^{20}}{1-x^{20}} + \&c\right)$

$= \left\{\phi^2(x)\phi^2(x^5) - 2x f^2(-x^2) f^2(-x^{10})\right\} \sqrt{1 - 4x / \chi^4(x)\chi^4(x^5)}$.

iii. $1 + 6\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right) - 6\left(\frac{5}{e^{10y}-1} + \frac{10}{e^{20y}-1} + \&c\right)$

$= \phi^2(e^{-y})\,\phi^2(e^{-5y}) \cdot \frac{3 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}}{4} \sqrt{\frac{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}}{2}}$

$= \phi^2(e^{-y})\,\phi^2(e^{-5y}) \sqrt{\frac{1 + \alpha\beta + (1-\alpha)(1-\beta)}{2} - \frac{3}{4}\sqrt[3]{16\alpha\beta(1-\alpha)(1-\beta)}}$

5. i. $1 + 4\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \&c\right) - 4\left(\frac{7x^7}{1-x^7} + \frac{14x^{14}}{1-x^{14}} + \&c\right)$

$= \left\{1 + 2\left(\frac{x}{1-x} + \frac{x^2}{1-x^2} - \frac{x^3}{1-x^3} + \frac{x^4}{1-x^4} - \frac{x^5}{1-x^5} - \frac{x^6}{1-x^6} + \frac{x^8}{1-x^8} + \&c\right)\right\}^2$.

$= \left\{\frac{f^8(-x) + 13x f^4(-x) f^4(-x^7) + 49x^2 f^8(-x^7)}{f(-x) f(-x^7)}\right\}^{\frac{2}{3}}$

ii. $1 + 4\left(\frac{x^2}{1-x^2} + \frac{2x^4}{1-x^4} + \frac{3x^6}{1-x^6} + \&c\right) - 4\left(\frac{7x^{14}}{1-x^{14}} + \frac{14x^{28}}{1-x^{28}} + \&c\right)$

$= \left\{\phi(x)\phi(x^7) - 2x\psi(-x)\psi(-x^7)\right\}^2$.

iii. $1 + 4\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right) - 4\left(\frac{7}{e^{14y}-1} + \frac{14}{e^{28y}-1} + \&c\right)$

$= \phi^2(e^{-y})\,\phi^2(e^{-7y}) \cdot \frac{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}}{2}$.

6 i. $1 - 12\left(\frac{1}{e^y+1} - \frac{2}{e^{2y}-1} + \frac{3}{e^{3y}+1} - \&c\right) + 12\left(\frac{3}{e^{3y}+1} - \frac{6}{e^{6y}-1} + \&c\right)$

$= \phi^2(e^{-y})\,\phi^2(e^{-3y}) \left\{\sqrt[4]{\alpha\beta} - \sqrt[4]{(1-\alpha)(1-\beta)}\right\}^2$.

ii. $1 - 6\left(\frac{1}{e^y+1} - \frac{2}{e^{2y}-1} + \frac{3}{e^{3y}+1} - \&c\right) + 6\left(\frac{5}{e^{5y}+1} - \frac{10}{e^{10y}-1} + \&c\right)$

$$= \phi^2(e^{-y})\,\phi^2(e^{-5y})\left(\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\right)\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}}.$$

ii. $$1-4\left(\frac{1}{e^{y}+1}-\frac{2}{e^{2y}-1}+\frac{3}{e^{3y}+1}-\&c\right)+4\left(\frac{7}{e^{7y}+1}-\frac{14}{e^{14y}-1}+\&c\right)$$

$$= \phi^2(e^{-y})\,\phi^2(e^{-7y})\left\{\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}\right\}^2.$$

7. i. $$1+3\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\&c\right)-3\left(\frac{9x^9}{1-x^9}+\frac{18x^{18}}{1-x^{18}}+\&c\right)$$

$$= \frac{f^6(-x^3)}{f^2(-x)f^2(-x^9)}\left\{f^6(-x)+9xf^3(-x)f^3(-x^9)+27x^2f^6(-x^9)\right\}^{\frac{1}{3}}$$

$$= \left\{\frac{\psi^4(x^3)+3x\psi^2(x)\psi^2(x^9)}{\psi(x)\psi(x^9)}\right\}^2\cdot\frac{\psi^2(x^3)}{\psi(x)\psi(x^9)}.$$

ii. $$1+3\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\&c\right)-3\left(\frac{9x^{18}}{1-x^{18}}+\frac{18x^{36}}{1-x^{36}}+\&c\right)$$

$$= \left\{\frac{\phi^4(x^3)+3\phi^2(x)\phi^2(x^9)}{4}\right\}^2\cdot\frac{\phi^2(x^3)}{\phi^3(x)\phi^3(x^9)}.$$

iii. $$1+\left(\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\&c\right)-\left(\frac{25x^{25}}{1-x^{25}}+\frac{50x^{50}}{1-x^{50}}+\&c\right)$$

$$= \frac{f^5(-x^5)}{f(-x)f(-x^{25})}\sqrt{f^2(-x)+2xf(-x)f(-x^{25})+5x^2f^2(-x^{25})}.$$

8. i. $$5+12\left(\frac{x^2}{1-x^2}+\frac{2x^4}{1-x^4}+\frac{3x^6}{1-x^6}+\&c\right)-12\left(\frac{11x^{22}}{1-x^{22}}+\frac{22x^{44}}{1-x^{44}}+\&c\right)$$

$$= 5\phi^2(x)\phi^2(x^{11})-20xf^2(x)f^2(x^{11})+32x^2f^2(-x^2)f^2(-x^{22})$$
$$-\,20x^3\psi^2(-x)\psi^2(-x^{11}).$$

ii. $$5+12\left(\frac{1}{e^{2y}-1}+\frac{2}{e^{4y}-1}+\frac{3}{e^{6y}-1}+\&c\right)-12\left(\frac{11}{e^{22y}-1}+\frac{22}{e^{44y}-1}+\&c\right)$$

$$= \phi^2(e^{-y})\,\phi^2(e^{-11y})\left\{2\left(1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\right)\right.$$
$$\left.+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}-\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}\right\}$$

iii. $$3+4\left(\frac{1}{e^{2y}-1}+\frac{2}{e^{4y}-1}+\frac{3}{e^{6y}-1}+\&c\right)-4\left(\frac{19}{e^{38y}-1}+\frac{38}{e^{76y}-1}+\&c\right)$$

$$= \phi^2(e^{-y})\,\phi^2(e^{-19y})\left\{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\right.$$
$$\left.+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}-\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}\right\}$$

9. i. $$11+12\left(\frac{1}{e^{2y}-1}+\frac{2}{e^{4y}-1}+\&c\right)-12\left(\frac{23}{e^{46y}-1}+\frac{46}{e^{92y}-1}+\&c\right)$$

$$= \phi^2(e^{-y})\,\phi^2(e^{-23y})\left\{\frac{11}{2}\left(1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\right)\right.$$
$$\left.-\,8\sqrt[12]{16\alpha\beta(1-\alpha)(1-\beta)}\left(1+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}\right)-10\sqrt[6]{16\alpha\beta(1-\alpha)(1-\beta)}\right\}$$

ii. $$7+12\left(\frac{1}{e^{2y}-1}+\frac{2}{e^{4y}-1}+\&c\right)-12\left(\frac{15}{e^{30y}-1}+\frac{30}{e^{60y}-1}+\&c\right)$$

$$= \phi^2(e^{-y})\,\phi^2(e^{-15y})\left\{\tfrac{1}{2}\left(1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\right)^4 - \frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-}}{2}\right.$$

iii. $$5+4\left(\frac{1}{e^{2y}-1}+\frac{2}{e^{4y}-1}+\&c\right)-4\left(\frac{31}{e^{62y}-1}+\frac{62}{e^{124y}-1}+\&c\right)$$

$$= \phi^2(e^{-y})\,\phi^2(e^{-31y})\left\{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}+\left(1+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}\right)^2\right.$$

$$\left. - 2\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\left(1+\sqrt[8]{\alpha\beta}+\sqrt[8]{(1-\alpha)(1-\beta)}\right)\right\}$$

10. i. $$1+6\left(\frac{1}{e^{2y}-1}+\frac{2}{e^{4y}-1}+\&c\right)-6\left(\frac{5}{e^{10y}-1}+\frac{10}{e^{20y}-1}+\&c\right)$$

$$= \phi^2(e^{-y})\,\phi^2(e^{-5y})\sqrt{\left\{\frac{1+\alpha\beta+(1-\alpha)(1-\beta)}{2}-\frac{3}{16}\left(1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\right)^2\right\}}$$

ii. $$1+3\left(\frac{1}{e^{2y}-1}+\frac{2}{e^{4y}-1}+\&c\right)-3\left(\frac{9}{e^{18y}-1}+\frac{18}{e^{36y}-1}+\&c\right)$$

$$= \phi^2(e^{-y})\,\phi^2(e^{-9y})\sqrt{\left\{\frac{1+\alpha\beta+(1-\alpha)(1-\beta)}{2}-\frac{9}{32}\left(1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\right)^2\right.}$$

$$\left. + \frac{\frac{3}{2}\sqrt{\alpha\beta(1-\alpha)(1-\beta)}}{1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}}\right\}$$

iii. $$2+3\left(\frac{1}{e^{2y}-1}+\frac{2}{e^{4y}-1}+\&c\right)-3\left(\frac{17}{e^{34y}-1}+\frac{34}{e^{68y}-1}+\&c\right)$$

$$= \phi^2(e^{-y})\,\phi^2(e^{-17y})\sqrt{\left\{2\left(1+\alpha\beta+(1-\alpha)(1-\beta)\right)-\frac{21}{16}\left(1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\right)^2\right.}$$

$$\left. - \frac{51}{32}\left(1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\right)\sqrt[4]{16\alpha\beta(1-\alpha)(1-\beta)} - 3\sqrt[3]{16\alpha\beta(1-\alpha)(1-\beta)}\right\}$$

11. i. $$17+12\left(\frac{1}{e^{2y}-1}+\frac{2}{e^{4y}-1}+\&c\right)-12\left(\frac{35}{e^{70y}-1}+\frac{70}{e^{140y}-1}+\&c\right)$$

$$= \phi^2(e^{-y})\,\phi^2(e^{-35y})\left\{\frac{\left(1-\sqrt[4]{\alpha\beta}-\sqrt[4]{(1-\alpha)(1-\beta)}\right)^3}{2\sqrt[12]{16\alpha\beta(1-\alpha)(1-\beta)}}+\sqrt[4]{\alpha\beta}+\sqrt[4]{(1-\alpha)(1-\beta)}\right.$$

$$\left. - \sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}\right\}$$

$$\frac{\phi(x)-\phi(-x)}{\phi(x)+\phi(-x)} = \sqrt{\frac{\phi^2(x^2)-\phi^2(-x^2)}{\phi^2(x^2)+\phi^2(-x^2)}} = \sqrt[4]{\frac{\phi^4(x^4)-\phi^4(-x^4)}{\phi^4(x^4)}}$$

$$\sqrt{\phi(x)+i\,\phi(-x)} = \sqrt{\frac{\phi(x)+\phi(x^2)\sqrt{2}}{2}} + i\sqrt{\frac{\phi(x)-\phi(x^2)\sqrt{2}}{2}}$$

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$$y = e^{-\frac{2\pi}{\sqrt{3}} \cdot \frac{1 + \frac{1.2}{3^2}(1-x) + \frac{1.2.4.5}{3^2.6^2}(1-x)^2 + \&c}{1 + \frac{1.2}{3^2} x + \frac{1.2.4.5}{3^2.6^2} x^2 + \&c}}$$

$$1 + 240\left(\frac{1^3 y}{1-y} + \frac{2^3 y^2}{1-y^2} + \frac{3^3 y^3}{1-y^3} + \&c\right) = z^4(1+8x)$$

where $z = 1 + \frac{1.2}{3^2} x + \frac{1.2.4.5}{3^2.6^2} x^2 + \&c$

$$1 - 504\left(\frac{1^5 y}{1-y} + \frac{2^5 y^2}{1-y^2} + \frac{3^5 y^3}{1-y^3} + \&c\right)$$

$$= z^6(1 - 20x - 8x^2).$$

$$\sqrt[24]{y}\,(1-y)(1-y^2)(1-y^3)(1-y^4)\,\&c = \frac{\sqrt[24]{x}\,\sqrt[8]{1-x}}{\sqrt[8]{3}}\,.\sqrt{z}.$$

$$\sqrt[8]{y}\,(1-y^3)(1-y^6)(1-y^9)(1-y^{12})\,\&c = \sqrt[8]{\frac{x}{27}}\,\sqrt[24]{1-x}\,\sqrt{z}.$$

$$1 + 240\left(\frac{1^3 y^3}{1-y^3} + \frac{2^3 y^6}{1-y^6} + \frac{3^3 y^9}{1-y^9} + \&c\right) = z^4\left(1 - \frac{8}{9}x\right).$$

$$1 - 504\left(\frac{1^5 y^3}{1-y^3} + \frac{2^5 y^6}{1-y^6} + \&c\right) = z^6\left(1 - \frac{4}{3}x + \frac{8}{27}x^2\right)$$

$$1 + 6\left(\frac{1}{e^y + e^{-y} + 1} + \frac{1}{e^{2y} + e^{-2y} + 1} + \&c\right) = z.$$

$$\frac{1^2}{e^y + e^{-y} + 1} + \frac{2^2}{e^{2y} + e^{-2y} + 1} + \frac{3^2}{e^{3y} + e^{-3y} + 1} + \&c = \frac{x}{27} z^3.$$

$$\frac{1^4}{e^y + e^{-y} + 1} + \frac{2^4}{e^{2y} + e^{-2y} + 1} + \&c = \frac{x}{27} z^5.$$

$$\frac{1^6}{e^y + e^{-y} + 1} + \frac{2^6}{e^{2y} + e^{-2y} + 1} + \&c = \frac{x}{27}\left(1 + \frac{4x}{3}\right) z^7.$$

$$\frac{1^8}{e^y + e^{-y} + 1} + \frac{2^8}{e^{2y} + e^{-2y} + 1} + \&c = \frac{x}{27}(1 + 8x) z^9.$$

If $\theta z = \int_0^\phi \left\{1 + \frac{1.2}{3^2} \cdot \frac{2}{1} \cdot x \sin^2\phi + \frac{1.2.4.5}{3^2.6^2} \cdot \frac{2.4}{1.3} x^2 \sin^4\phi + \&c\right\} d\phi$

then $\phi = \theta + 3\left\{\frac{\sin 2\theta}{1 + 2\cosh y} + \frac{\sin 4\theta}{2(1 + 2\cosh 2y)} + \&c\right\}$

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$$y = e^{-\frac{2\pi}{\sqrt{3}} \cdot \frac{1 + \frac{1.2}{3^2}(1-x) + \frac{1.2.4.5}{3^2.6^2}(1-x)^2 + \&c}{1 + \frac{1.2}{3^2}x + \frac{1.2.4.5}{3^2.6^2}x^2 + \&c}}$$

$$1 + 240\left(\frac{1^3 y}{1-y} + \frac{2^3 y^2}{1-y^2} + \frac{3^3 y^3}{1-y^3} + \&c\right) = z^4(1+8x)$$

where $z = 1 + \frac{1.2}{3^2}x + \frac{1.2.4.5}{3^2.6^2}x^2 + \&c$

$$1 - 504\left(\frac{1^5 y}{1-y} + \frac{2^5 y^2}{1-y^2} + \frac{3^5 y^3}{1-y^3} + \&c\right)$$

$$= z^6(1 - 20x - 8x^2).$$

$$\sqrt[24]{y}\,(1-y)(1-y^2)(1-y^3)(1-y^4)\,\&c = \frac{\sqrt[24]{x}\,\sqrt[8]{1-x}}{\sqrt[8]{3}} . \sqrt{z}.$$

$$\sqrt[8]{y}\,(1-y^3)(1-y^6)(1-y^9)(1-y^{12})\,\&c = \sqrt[8]{\frac{x}{27}}\,\sqrt[24]{1-x}\,\sqrt{z}.$$

$$1 + 240\left(\frac{1^3 y^3}{1-y^3} + \frac{2^3 y^6}{1-y^6} + \frac{3^3 y^9}{1-y^9} + \&c\right) = z^4\left(1 - \frac{8}{9}x\right).$$

$$1 - 504\left(\frac{1^5 y^3}{1-y^3} + \frac{2^5 y^6}{1-y^6} + \&c\right) = z^6\left(1 - \frac{4}{3}x + \frac{8}{27}x^2\right)$$

$$1 + 6\left(\frac{1}{e^y + e^{-y} + 1} + \frac{1}{e^{2y} + e^{-2y} + 1} + \&c\right) = z.$$

$$\frac{1^2}{e^y + e^{-y} + 1} + \frac{2^2}{e^{2y} + e^{-2y} + 1} + \frac{3^2}{e^{3y} + e^{-3y} + 1} + \&c = \frac{x}{27} z^3.$$

$$\frac{1^4}{e^y + e^{-y} + 1} + \frac{2^4}{e^{2y} + e^{-2y} + 1} + \&c = \frac{x}{27} z^5.$$

$$\frac{1^6}{e^y + e^{-y} + 1} + \frac{2^6}{e^{2y} + e^{-2y} + 1} + \&c = \frac{x}{27}\left(1 + \frac{4x}{3}\right) z^7.$$

$$\frac{1^8}{e^y + e^{-y} + 1} + \frac{2^8}{e^{2y} + e^{-2y} + 1} + \&c = \frac{x}{27}(1 + 8x) z^9.$$

If $\theta z = \int_0^{\phi} \left\{1 + \frac{1.2}{3^2} \cdot \frac{2}{1} x \sin^2\phi + \frac{1.2.4.5}{3^2.6^2} \cdot \frac{2.4}{1.3} x^2 \sin^4\phi + \&c\right\} d\phi$

then $\phi = \theta + 3\left\{\frac{\sin 2\theta}{1 + 2\cosh y} + \frac{\sin 4\theta}{2(1 + 2\cosh 2y)} + \&c\right\}$

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then $1 + \frac{1\cdot2}{3^2}\beta + \frac{1\cdot2\cdot4\cdot5}{3^2\cdot6^2}\beta^2 + \&c$

$$= \left\{1 + \left(\tfrac{1}{2}\right)^2\alpha + \left(\tfrac{1\cdot3}{2\cdot4}\right)^2\alpha^2 + \&c\right\}\cdot\frac{1+p+p^2}{\sqrt{1+2p}}.$$

$$1 + 6\left(\frac{y}{1-y} - \frac{y^2}{1-y^2} + \frac{y^4}{1-y^4} - \frac{y^5}{1-y^5} + \&c\right) = z.$$

$$1 + 12\left(\frac{y}{1-y} + \frac{2y^2}{1-y^2} + \frac{4y^4}{1-y^4} + \frac{5y^5}{1-y^5} + \&c\right) = z^2$$

$$z = \frac{\phi^3(y^3)}{\phi(y)}(1+4p+p^2). = 4\,\frac{\psi^3(y^2)}{\psi(y^6)} - 3\cdot\frac{\phi^3(y^3)}{\phi(y)}.$$

$$\cancel{1 + 4\left(\frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{2x^3}{}\right.}$$

If $\alpha = \frac{p\,(2+p)^2}{2(1+p)^3}$ and $\beta = \frac{p^2(3+p)}{4}$.

or $1-\alpha = \frac{(1-p)^2(2+p)}{2(1+p)^3}$ and $1-\beta = \frac{(1-p)(2+p)^2}{4}$.

then $1 + \frac{1\cdot2}{3^2}\alpha + \frac{1\cdot2\cdot4\cdot5}{3^2\cdot6^2}\alpha^2 + \&c$

$$= (1+p)\left\{1 + \frac{1\cdot2}{3^2}\beta + \frac{1\cdot2\cdot4\cdot5}{3^2\cdot6^2}\beta^2 + \&c\right\}$$

$$1 + \frac{1\cdot2}{3^2}\left\{1 - \left(\frac{1-t}{1+2t}\right)^3\right\} + \frac{1\cdot2\cdot4\cdot5}{3^2\cdot6^2}\left\{1 - \left(\frac{1-t}{1+2t}\right)^3\right\}^2 + \&c$$

$$= (1+2t)\left\{1 + \frac{1\cdot2}{3^2}t^3 + \frac{1\cdot2\cdot4\cdot5}{3^2\cdot6^2}t^6 + \&c\right\}$$

If $\alpha = \frac{27p(1+p)^4}{2(1+4p+p^2)^3}$ and $\beta = \frac{27p^4(1+p)}{2(2+2p-p^2)^3}$

then $\left(1+p-\frac{p^2}{2}\right)\left\{1 + \frac{1\cdot2}{3^2}\alpha + \frac{1\cdot2\cdot4\cdot5}{3^2\cdot6^2}\alpha^2 + \&c\right\}$

$$= (1+4p+p^2)\left\{1 + \frac{1\cdot2}{3^2}\beta + \frac{1\cdot2\cdot4\cdot5}{3^2\cdot6^2}\beta^2 + \&c\right\}$$

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II degree $\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} = 1$

$$\sqrt[3]{\frac{\alpha^2}{\beta}} - \sqrt[3]{\frac{(1-\alpha)^2}{1-\beta}} = \frac{2}{1+p} = \frac{2}{m}.$$

$$\sqrt[3]{\frac{(1-\beta)^2}{1-\alpha}} - \sqrt[3]{\frac{\beta^2}{\alpha}} = m.$$

$$\sqrt[3]{\frac{\alpha^2}{\beta}} + \sqrt[3]{\frac{(1-\alpha)^2}{1-\beta}} = \frac{4}{m^2}.$$

$$\sqrt[3]{\frac{(1-\beta)^2}{1-\alpha}} + \sqrt[3]{\frac{\beta^2}{\alpha}} = m^2.$$

V degree $\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} + 3\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}$
$= 1.$

XI degree $\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)} + 6\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}$
$+ 3\sqrt{3}\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}\left\{\sqrt[6]{\alpha\beta} + \sqrt[6]{(1-\alpha)(1-\beta)}\right\} = 1.$

~~VII degree.~~ $\sqrt[4]{\frac{(1-\beta)^3}{1-\alpha}} + \sqrt[4]{\frac{\beta^3}{\alpha}} - .$

IX degree $m = 3 \cdot \dfrac{1+2\sqrt[3]{\beta}}{1+2\sqrt[3]{1-\alpha}}.$

IV degree $m = \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{1-\beta}{1-\alpha}} - \frac{4}{m}\sqrt[3]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}.$

VII degree $m = \sqrt[3]{\frac{\beta}{\alpha}} + \sqrt[3]{\frac{1-\beta}{1-\alpha}} - \frac{7}{m}\sqrt[3]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}} - 3\sqrt[6]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$

I, II, IV and VIII.

$$\frac{1 - \sqrt[3]{\alpha\delta} - \sqrt[3]{(1-\alpha)(1-\delta)}}{3\sqrt[6]{\beta\gamma(1-\beta)(1-\gamma)}} = \frac{1+\frac{1.2}{3^2}\beta + \&c}{1+\frac{1.2}{3^2}\alpha + \&c} \cdot \frac{1+\frac{1.2}{3^2}\gamma + \&c}{1+\frac{1.2}{3^2}\delta + \&c}$$

I, II, VII, XIV or I, IV, V, XX.

$$\frac{1+2\left(\sqrt[3]{\alpha\delta} + \sqrt[3]{(1-\alpha)(1-\delta)}\right)}{1+2\left(\sqrt[3]{\beta\gamma} + \sqrt[3]{(1-\beta)(1-\gamma)}\right)} = \frac{1+\frac{1.2}{3^2}\beta + \&c}{1+\frac{1.2}{3^2}\alpha + \&c} \cdot \frac{1+\frac{1.2}{3^2}\gamma + \&c}{1+\frac{1.2}{3^2}\delta + \&c}$$

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$$y = e^{-\frac{2\pi}{\sqrt{2}} \cdot \frac{1+\frac{1\cdot 3}{4^2}(1-x)+\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}(1-x)^2+\&c}{1+\frac{1\cdot 3}{4^2}x+\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}x^2+\&c}} = F\left(\frac{2\sqrt{x}}{1+\sqrt{x}}\right)$$

$$1+240\left(\frac{1^3 y}{1-y}+\frac{2^3 y^2}{1-y^2}+\frac{3^3 y^3}{1-y^3}+\&c\right) = z^4(1+3x)$$

$$1-504\left(\frac{1^5 y}{1-y}+\frac{2^5 y^2}{1-y^2}+\frac{3^5 y^3}{1-y^3}+\&c\right) = z^6(1-9x).$$

$$\sqrt[24]{y}\,(1-y)(1-y^2)(1-y^3)\,\&c = \frac{\sqrt[24]{x}\,\sqrt[12]{1-x}}{\sqrt[4]{2}}\sqrt{z}.$$

$$\sqrt[12]{y}\,(1-y^2)(1-y^4)(1-y^6)\,\&c = \frac{\sqrt[12]{x}\,\sqrt[24]{1-x}}{\sqrt{2}}\sqrt{z}.$$

$$1+240\left(\frac{1^3 y^2}{1-y^2}+\frac{2^3 y^4}{1-y^4}+\frac{3^3 y^6}{1-y^6}+\&c\right) = z^4\left(1-\frac{3}{4}x\right)$$

$$1-504\left(\frac{1^5 y^2}{1-y^2}+\frac{2^5 y^4}{1-y^4}+\frac{3^5 y^6}{1-y^6}+\&c\right) = z^6\left(1-\frac{9}{8}x\right).$$

$$1+\frac{1\cdot 3}{4^2}\left\{1-\left(\frac{1-t}{1+3t}\right)^2\right\}+\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}\left\{1-\left(\frac{1-t}{1+3t}\right)^2\right\}^2+\&c$$

$$=\sqrt{1+3t}\left\{1+\frac{1\cdot 3}{4^2}t^2+\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}t^4+\&c\right\}$$

If $\alpha = \dfrac{64t}{(3+6t-t^2)^2}$ and $\beta = \dfrac{64t^3}{(27-18t-t^2)^2}$

then $\sqrt{1+2t-\dfrac{t^2}{3}}\left(1+\dfrac{1\cdot 3}{4^2}\beta+\&c\right)$

$$=\sqrt{1-\frac{2}{3}t-\frac{t^2}{27}}\left(1+\frac{1\cdot 3}{4^2}\alpha+\&c\right).$$

$$1+\left(\frac{1}{2}\right)^2\frac{2x}{1+x}+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{2x}{1+x}\right)^2+\&c$$

$$=\sqrt{1+x}\left\{1+\frac{1\cdot 3}{4^2}x^2+\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}x^4+\&c\right\}.$$

$$\left(z = 1+\frac{1\cdot 3}{4^2}x+\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}x^2+\&c\right).$$

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III degree $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 4\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)} = 1$

VII degree $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 20\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}$

$$+ 8\sqrt{2}\sqrt[8]{\alpha\beta(1-\alpha)(1-\beta)}\left(\sqrt[4]{\alpha\beta} + \sqrt[4]{(1-\alpha)(1-\beta)}\right) = 1.$$

V degree $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} +$

$$8\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}\left(\sqrt[6]{\alpha\beta} + \sqrt[6]{(1-\alpha)(1-\beta)}\right) = 1.$$

XI degree. $\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 68\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)}$

$$+ 16\sqrt[12]{\alpha\beta(1-\alpha)(1-\beta)}\left(\sqrt[3]{\alpha\beta} + \sqrt[3]{(1-\alpha)(1-\beta)}\right)$$

$$+ 48\sqrt[6]{\alpha\beta(1-\alpha)(1-\beta)}\left(\sqrt[6]{\alpha\beta} + \sqrt[6]{(1-\alpha)(1-\beta)}\right) = 1.$$

III degree $m^2 = \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{1-\beta}{1-\alpha}} - \frac{9}{m^2}\sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}.$

V degree $m = \sqrt[4]{\frac{\beta}{\alpha}} + \sqrt[4]{\frac{1-\beta}{1-\alpha}} - \frac{5}{m}\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}.$

IX degree $\sqrt{m} = \sqrt[8]{\frac{\beta}{\alpha}} + \sqrt[8]{\frac{1-\beta}{1-\alpha}} - \frac{3}{\sqrt{m}}\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}.$

VII degree $m^2 = \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{1-\beta}{1-\alpha}} - \frac{49}{m^2}\sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$

$$- 8\sqrt[6]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left(\sqrt[6]{\frac{\beta}{\alpha}} + \sqrt[6]{\frac{1-\beta}{1-\alpha}}\right).$$

XIII degree $m = \sqrt[4]{\frac{\beta}{\alpha}} + \sqrt[4]{\frac{1-\beta}{1-\alpha}} - \frac{13}{m}\sqrt[4]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$

$$- 4\sqrt[12]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left(\sqrt[12]{\frac{\beta}{\alpha}} + \sqrt[12]{\frac{1-\beta}{1-\alpha}}\right).$$

XXV degree $\sqrt{m} = \sqrt[8]{\frac{\beta}{\alpha}} + \sqrt[8]{\frac{1-\beta}{1-\alpha}} - \frac{5}{\sqrt{m}}\sqrt[8]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}$

$$- 2\sqrt[24]{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}}\left(\sqrt[24]{\frac{\beta}{\alpha}} + \sqrt[24]{\frac{1-\beta}{1-\alpha}}\right).$$

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$$\text{If } y = e^{-2\pi \cdot \frac{1+\frac{1\cdot5}{6^2}(1-x)+\frac{1\cdot5\cdot7\cdot11}{6^2\cdot12^2}(1-x)^2+\&c}{1+\frac{1\cdot5}{6^2}x+\frac{1\cdot5\cdot7\cdot11}{6^2\cdot12^2}x^2+\&c}}$$

and $z = 1+\frac{1\cdot5}{6^2}x+\frac{1\cdot5\cdot7\cdot11}{6^2\cdot12^2}x^2+\&c$, then

$$1+240\left(\frac{1^3y}{1-y}+\frac{2^3y^2}{1-y^2}+\frac{3^3y^3}{1-y^3}+\&c\right)=z^4.$$

$$1-504\left(\frac{1^5y}{1-y}+\frac{2^5y^2}{1-y^2}+\frac{3^5y^3}{1-y^3}+\&c\right)=z^6(1-2x)$$

$$\sqrt[24]{y}\,(1-y)(1-y^2)(1-y^3)(1-y^4)\,\&c=\sqrt[24]{\frac{x(1-x)}{432}}\,\sqrt{z}.$$

If $u = x(1-x)$ and $v = y(1-y)$

and $u = \frac{27v^2}{16(1-v)^3}$, then

$$1+\frac{1\cdot5}{6^2}x+\frac{1\cdot5\cdot7\cdot11}{6^2\cdot12^2}x^2+\&c$$

$$=\left\{1+\left(\tfrac{1}{2}\right)^2y+\left(\tfrac{1\cdot3}{2\cdot4}\right)^2y^2+\&c\right\}\sqrt[4]{1-y+y^2}$$

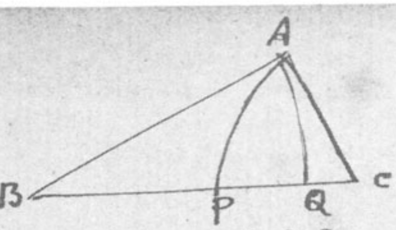
If $y = \frac{p(2+p)}{1+2p}$, then $x = \frac{27}{4}\cdot\frac{(p+p^2)^2}{(1+p+p^2)^3}$

$$\sqrt{21}.\ \tfrac{1}{2}\cdot\left(\frac{3-\sqrt7}{\sqrt2}\right)^2\left(\sqrt{\frac{5+\sqrt7}{4}}-\sqrt{\frac{1+\sqrt7}{4}}\right)^4\left(\sqrt{\frac{3+\sqrt7}{4}}-\sqrt{\frac{\sqrt7-1}{4}}\right)^4\left(\frac{\sqrt7-\sqrt3}{2}\right)^3$$

$$\sqrt{33}.\ \tfrac{1}{2}(2-\sqrt3)^3\left(\sqrt{\frac{7+3\sqrt3}{4}}-\sqrt{\frac{3+3\sqrt3}{4}}\right)^4\left(\sqrt{\frac{5+\sqrt3}{4}}-\sqrt{\frac{1+\sqrt3}{4}}\right)^4\left(\frac{\sqrt{11}-3}{\sqrt2}\right)^2$$

$$\sqrt{45}.\ \tfrac{1}{2}(\sqrt5-2)^3\left(\sqrt{\frac{7+3\sqrt5}{4}}-\sqrt{\frac{3+3\sqrt5}{4}}\right)^4\left(\sqrt{\frac{3+\sqrt5}{2}}-\sqrt{\frac{1+\sqrt5}{2}}\right)^4\left(\frac{\sqrt5-\sqrt3}{\sqrt2}\right)^4$$

$$\sqrt{15}.\ =\ \frac{1}{16}\cdot\left(\frac{\sqrt5-1}{2}\right)^4\cdot(2-\sqrt3)^2(4-\sqrt{15}).$$



$$PQ^2 = 2\, BP\cdot QC.$$

$$\left(a+b-\sqrt{a^2+b^2}\right)^2 = 2\left(\sqrt{a^2+b^2}-a\right)\left(\sqrt{a^2+b^2}-b\right)$$

$$\left\{\sqrt[3]{(a+b)^2}-\sqrt[3]{a^2-ab+b^2}\right\}^3 = 3\left(\sqrt[3]{a^3+b^3}-a\right)\left(\sqrt[3]{a^3+b^3}-b\right)$$

$$\sqrt{A+B\sqrt[3]{p}} = \sqrt{\frac{B}{p+k^3}}\left(\frac{k^2}{2}+k\sqrt[3]{p}-\sqrt[3]{p^2}\right)$$

where $Bk^4-4Ak^3-8Bkp-4Ap=0$.

$F\dfrac{1-\sqrt{1-t^{24}}}{2} = e^{-\pi\sqrt{29}}$, then

$$t^{24}+9t^{20}+5t^{16}-2t^{12}-5t^8+9t^4-1=0$$

$$\frac{t^6+t^2}{1-t^4} = \sqrt{\frac{\sqrt{29}-5}{2}}$$

$$\frac{t^3+t\sqrt{\sqrt{29}-2}}{1+t^2\sqrt{\sqrt{29}+2}} = \sqrt[4]{\frac{\sqrt{29}-5}{2}}.$$

if $\sqrt[4]{1-t^8} = t(1+u^2)$, then $u^3+u=\sqrt{2}$.

$F\dfrac{1-\sqrt{1-\frac{1}{64}t^{24}}}{2} = e^{-\pi\sqrt{79}}$. then

$$t^5-t^4+t^3-2t^2+3t-1=0.$$

$F\dfrac{1-\sqrt{1-\frac{4}{64}t^{24}}}{2} = e^{-\pi\sqrt{47}}$, then

$$t^5+2t^4+2t^3+t^2-1=0$$

$$(1)\quad \phi^2(-x) = 1 - \frac{4x}{1+x} + \frac{4x^3}{1+x^2} - \frac{4x^6}{1+x^3} + \frac{4x^{10}}{1+x^4} - \&c$$

$$(2)\quad \psi(x)\phi(x^2) = \frac{1+x}{1-x} - x\cdot\frac{1+x^3}{1-x^3} + x^3\cdot\frac{1+x^5}{1-x^5} - x^6\cdot\frac{1+x^7}{1-x^7} + \&c$$

$$(3)\quad \psi^2(x) = \frac{1+x}{1-x} - x^2\cdot\frac{1+x^3}{1-x^3} + x^6\cdot\frac{1+x^5}{1-x^5} - x^{12}\cdot\frac{1+x^7}{1-x^7} + \&c$$

$$(4)\quad \frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \&c$$

$$= x\cdot\frac{1+x}{(1-x)^2} - x^3\cdot\frac{1+x^2}{(1-x^2)^2} + x^6\cdot\frac{1+x^3}{(1-x^3)^2} - x^{10}\cdot\frac{1+x^4}{(1-x^4)^2} + \&c$$

$$(5).\quad x\psi(x)\psi(x^4) = \frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^6}{1-x^5} - \frac{x^{10}}{1-x^7} + \&c$$

$$(6).\quad \frac{1^2x}{1+x} - \frac{2^2x^3}{1+x^2} + \frac{3^2x^6}{1+x^3} - \frac{4^2x^{10}}{1+x^4} + \&c$$

$$= \phi^2(-x)\left\{x\cdot\frac{1+x}{(1-x)^2} + x^6\cdot\frac{1+x^3}{(1-x^3)^2} + x^{10}\cdot\frac{1+x^5}{(1-x^5)^2} + \&c\right\}$$

$$(7)\quad \frac{1+x}{1-x} - 3^2x^2\cdot\frac{1+x^3}{1-x^3} + 5^2x^6\cdot\frac{1+x^5}{1-x^5} - \&c$$

$$= \psi^2(x)\left\{1 - \frac{8x^2}{(1+x)^2} + \frac{8x^6}{(1+x^2)^2} - \frac{8x^{12}}{(1+x^3)^2} + \&c\right\}.$$

$$(8).\quad \frac{x^2}{(1-x)^2} - \frac{3x^6}{(1-x^3)^2} + \frac{5x^{12}}{(1-x^5)^2} - \frac{7x^{20}}{(1-x^7)^2} + \&c$$

$$= x\psi^2(x)\left\{x\cdot\frac{1+x^2}{1-x^2} - 2x^4\cdot\frac{1+x^4}{1-x^4} + 3x^9\cdot\frac{1+x^6}{1-x^6} + \&c\right\}$$

$$(9)\quad x\cdot\frac{1-x}{(1+x)^2} - 2x^3\cdot\frac{1-x^2}{(1+x^2)^2} + 3x^6\cdot\frac{1-x^3}{(1+x^3)^2} - \&c$$

$$= \phi^2(-x)\left(\frac{x}{1-x} + \frac{2x^3}{1-x^2} + \frac{3x^6}{1-x^3} + \frac{4x^{10}}{1-x^4} + \&c\right).$$

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$$\frac{1}{2\log 2}+\frac{1}{3\log 3}+\cdots+\frac{1}{x\log x}$$
$$=.1015314+\log\log(x^2+x+\theta)$$

$x=\infty \quad \theta=\frac{4}{3}$

$x=1 \quad \theta=.46811$

$$\int_0^\infty e^{-nx}\sin nx\cot x\,dx$$
$$=\frac{n^2}{2}\left\{\frac{1}{1^4+\frac{n^4}{4}}+\frac{2}{2^4+\frac{n^4}{4}}+\frac{3}{3^4+\frac{n^4}{4}}+\cdots\right\}$$

$$\cancel{\int_0^\infty e^{-x}\frac{\sin x}{x}\left\{A_0-\frac{2B_2}{\lfloor 2}A_2x^2-\frac{2^2B_4}{\lfloor 4}A_4x^4\cdots\right.}$$
$$\cancel{=(A_2-A_6+A_{10}-\&c)}$$
$$\cancel{+(2A_2-2^5A_6+2^9A_{10}-\&c)}$$
$$\cancel{+(3A_2-3^5A_6+3^9A_{10}-\&c)}$$

130489 quadrillion terms to get the value of 5

$$\int_0^\infty e^{-ax}\sin ax\coth x\,dx$$
$$=\frac{1}{2a}+2a\left\{\frac{1}{a^2+(a+4)^2}+\frac{1}{a^2+(a+2)^2}+\frac{1}{a^2+(a+6)^2}+\&c\right\}$$

$$\int_0^\infty e^{-ax}\sin ax(\cot x+\coth x)\,dx$$
$$=\frac{\pi}{2}\cdot\frac{\sinh\pi a}{\cosh\pi a-\cos\pi a}$$

If $x + na^2 = y + nab = z + nb^2 = (a+b)^2$

then $x^2 + (n-2)xz + z^2 = ny^2$

If p, q, r are quantities so taken that

$$p + 3a^2 = q + 3ab = r + 3b^2 = (a+b)^2.$$

and m & n are any two quantities, then

$$n(mp+nq)^3 + m(mq+nr)^3$$
$$= m(np+mq)^3 + n(nq+mr)^3.$$

A particular case of the above theorem is

$$\cancel{(3a^2+5ab-5b^2)^3 + (4a^2-4ab+6b^2)^3 + (3a^2+5ab-7b^2)}$$

$$\cancel{= 6a^2 + 16ab + 1}$$

$$(3a^2+5ab-5b^2)^3 + (4a^2-4ab+6b^2)^3 + (5a^2-5ab-3b^2)^3$$
$$= (6a^2-4ab+4b^2)^3$$

$$(2x^2+3xy+5y^2)(2p^2+3pq+5q^2)$$
$$= 2u^2+3uv+5v^2 \quad \text{where}$$
$$u = \tfrac{5}{2}(x+y)(p+q) - 2xp \;\&\; v = 2qy - (x+y)(p+q).$$

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Let $I(P)$ be the integer equal to or just less than P and $G(P)$, equal to or just greater than P, and let $N(P)$ be the nearest integer to P. Then,

(1) $N(P) = I(P+\frac{1}{2})$.

(2) $I\left(\frac{n}{p}\right)$ is the coeff.^t of x^n in $\dfrac{x^p}{(1-x)(1-x^p)}$.

(3) $I\,\phi(n)$ is the coeff.^t of x^n in $\displaystyle\sum_{n=1}^{n=\infty} \frac{x^{G\phi^{-1}(n)}}{1-x}$

(4) The sum of the nos. of divisors of P natural nos.

$= I\left(\frac{P}{1}\right) + I\left(\frac{P}{2}\right) + I\left(\frac{P}{3}\right) + I\left(\frac{P}{4}\right) + \cdots\cdots + I\left(\frac{P}{P}\right)$.

$= 2\left\{I\left(\frac{P}{1}\right) + I\left(\frac{P}{2}\right) + \text{\&c to } I(\sqrt{P}) \text{ terms}\right\} - \left(I\sqrt{P}\right)^2$.

(5) The above sum is odd or even according as $I(\sqrt{P})$ is odd or even and is approximately equal to $P(2C-1+\log P)$ $+ \frac{1}{2}$ the no of factors of $P + \frac{1}{4}$.

(6) If $n > \sqrt{P}$ and $m = I\left(\frac{P}{n}\right)$, then

$I\left(\frac{P}{1}\right) + I\left(\frac{P}{2}\right) + I\left(\frac{P}{3}\right) + \cdots\cdots + I\left(\frac{P}{m}\right)$

$= n\,I\left(\frac{P}{n}\right) + I\left(\frac{P}{1+m}\right) + I\left(\frac{P}{2+m}\right) + I\left(\frac{P}{3+m}\right) + \cdots\cdots + I\left(\frac{P}{P}\right)$.

(7) If P be the nth Prime no. then $\dfrac{dP}{dn} = \log P$ nearly and hence $n = \dfrac{P}{\log P - 1}$ nearly.

(8). $\phi(2) + \phi(3) + \phi(5) + \phi(7) + \phi(11) + \text{\&c}$ and $\dfrac{\phi(2)}{\log 2} + \dfrac{\phi(3)}{\log 3} + \dfrac{\phi(4)}{\log 4} + \text{\&c}$ are both convergent or both divergent.

(1) If $\alpha\beta = \pi^2$, then $\frac{1}{\sqrt[4]{\alpha}}\left\{1 + 4\alpha\int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\,dx\right\}$

$= \frac{1}{\sqrt[4]{\beta}}\left\{1 + 4\beta\int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x}-1}\,dx\right\} = \sqrt[4]{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{2}{3}}$ near

(2) If $\alpha\beta = \pi^2$, then

$\frac{1}{\sqrt[4]{\alpha}}\left\{\phi(0) + \frac{\alpha}{\underline{|1}}\phi(2)B_2 - \frac{\alpha^2}{\underline{|2}}\phi(4)B_4 + \frac{\alpha^3}{\underline{|3}}\phi(6)B_6 - \&c\right\}$

$= \frac{1}{\sqrt[4]{\beta}}\left\{\phi(1) + \frac{\beta}{\underline{|1}}\phi(-1)B_2 - \frac{\beta^2}{\underline{|2}}\phi(-3)B_4 + \frac{\beta^3}{\underline{|3}}\phi(-5)B_6 - \&c\right\}$

(3) If $\alpha\beta = 4\pi^2$, then $2\alpha^{\frac{m+1}{2}}\int_0^\infty \frac{x^m}{e^{2\pi x}-1}\cdot\frac{dx}{e^{\alpha x}-1} =$

$\alpha^{\frac{m-1}{2}}\left\{\frac{B_m}{m} - \frac{\alpha}{2}\cdot\frac{B_{m+1}}{m+1} + \alpha^2\cdot\frac{B_2}{\underline{|2}}\cdot\frac{B_{m+2}}{m+2} - \alpha^4\frac{B_4}{\underline{|4}}\cdot\frac{B_{m+4}}{m+4}\right.$

$= \beta^{\frac{m-1}{2}}\left\{\frac{B_m}{m} - \frac{\beta}{2}\cdot\frac{B_{m+1}}{m+1} + \beta^2\cdot\frac{B_2}{\underline{|2}}\cdot\frac{B_{m+2}}{m+2} - \beta^4\cdot\frac{B_4}{\underline{|4}}\cdot\frac{B_{m+4}}{m+4}\right.$

(4) If $\alpha\beta = \pi^2$, then $-\frac{\pi}{2}\cdot\frac{\alpha^{\frac{x}{2}}}{\sin\frac{\pi x}{2}}\cdot\frac{B_x}{\underline{|x}}\phi(x) =$

$\frac{\phi(0)}{x} - \frac{\alpha}{\underline{|1}}\cdot\frac{\phi(2)}{2-x}B_2 + \frac{\alpha^2}{\underline{|2}}\cdot\frac{\phi(4)}{4-x}B_4 - \frac{\alpha^3}{\underline{|3}}\cdot\frac{\phi(6)}{6-x}B_6 + \&c$

$+\sqrt{\frac{\alpha}{\pi}}\left\{\frac{\phi(1)}{1-x} - \frac{\beta}{\underline{|1}}\cdot\frac{\phi(-1)}{1+x}B_2 + \frac{\beta^2}{\underline{|2}}\cdot\frac{\phi(-3)}{3+x}B_4 - \&c\right\}$

(5) $\frac{\pi}{2}\cdot\frac{a^x B_x\phi(x)}{\sin\frac{\pi x}{2}} + \frac{\phi(0)}{x} + \frac{a\,\phi(1)}{2(1-x)} =$

$\frac{a^2\phi(2)B_2}{2-x} - \frac{a^4\phi(4)B_4}{4-x} + \frac{a^6B_6\phi(6)}{6-x} - \&c$

$+ \frac{S_2\phi(-1)}{a(1+x)} - \frac{2S_3\phi(-2)}{a^2(2+x)} + \frac{3S_4\phi(-3)}{a^3(3+x)} - \&c.$

(1) If $\alpha\beta = 4\pi^2$, then

$$\sqrt{\alpha^n}\left\{\frac{B_n}{2n} + \cos\frac{\pi n}{2}\left(\frac{1^{n-1}}{e^{\alpha}-1} + \frac{2^{n-1}}{e^{2\alpha}-1} + \frac{3^{n-1}}{e^{3\alpha}-1} + \&c\right)\right\}$$

$$= \sqrt{\beta^n}\left\{\frac{B_n}{2n}\cos\frac{\pi n}{2} - \sin\frac{\pi n}{2}\int_0^\infty \frac{x^{n-1}\cot\frac{\beta x}{2}}{e^{2\pi x}-1}\,dx + \right.$$

$$\left.\frac{1^{n-1}}{e^{\beta}-1} + \frac{2^{n-1}}{e^{2\beta}-1} + \frac{3^{n-1}}{e^{3\beta}-1} + \&c\right\}$$

(2). $$\frac{1^{n+1}}{1^4+4x^4} + \frac{2^{n+1}}{2^4+4x^4} + \frac{3^{n+1}}{3^4+4x^4} + \frac{4^{n+1}}{4^4+4x^4} + \&c$$

$$= \frac{\pi}{4}(x\sqrt{2})^{n-2}\sec\frac{\pi n}{4} - 2\cos\frac{\pi n}{2}\int_0^\infty \frac{z^{n+1}}{e^{2\pi z}-1}\cdot\frac{dz}{z^4+4x^4}$$

$$+ \frac{\pi}{2}(x\sqrt{2})^{n-2}\frac{\cos\left(\frac{\pi n}{4}+2\pi x\right) - e^{-2\pi x}\cos\frac{\pi n}{4}}{\cosh 2\pi x - \cos 2\pi x}$$

(3). $$\int_0^\infty \frac{x\sin 2nx}{e^{x^2}-1}\,dx = \frac{n\sqrt{\pi}}{2}\left(\frac{e^{-n^2}}{1\sqrt{1}} + \frac{e^{-\frac{n^2}{2}}}{2\sqrt{2}} + \&c\right)$$

$$= \frac{\pi}{2}\left(1 + 2e^{-2n\sqrt{\pi}}\cos 2n\sqrt{\pi} + 2e^{-2n\sqrt{2\pi}}\cos 2n\sqrt{2\pi} + \&c\right)$$

(4) $$\int_0^\infty \frac{x\sin 2nx}{e^{x^2}+e^{-x^2}}\,dx = \frac{n\sqrt{\pi}}{2}\left(e^{-n^2} - \frac{e^{-\frac{n^2}{3}}}{3\sqrt{3}} + \frac{e^{-\frac{n^2}{5}}}{5\sqrt{5}} - \&c\right)$$

$$= \frac{\pi}{2}\left(e^{-n\sqrt{\pi}}\sin n\sqrt{\pi} - e^{-n\sqrt{3\pi}}\sin n\sqrt{3\pi} + \&c\right).$$

(5) If n is a positive integer, then

$$\frac{1^{4n}}{(e^{\pi}-e^{-\pi})^2} + \frac{2^{4n}}{(e^{2\pi}-e^{-2\pi})^2} + \frac{3^{4n}}{(e^{3\pi}-e^{-3\pi})^2} + \&c =$$

$$\frac{n}{\pi}\left(\frac{B_{4n}}{8n} + \frac{1^{4n-1}}{e^{2\pi}-1} + \frac{2^{4n-1}}{e^{4\pi}-1} + \frac{3^{4n-1}}{e^{6\pi}-1} + \&c\right)$$

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(1) $\dfrac{1}{p+1} + \dfrac{1}{(p+2)^2} + \dfrac{3}{(p+3)^3} + \dfrac{4^2}{(p+4)^4} + \dfrac{5^3}{(p+5)^5} + \text{\&c}$

$$= \frac{1-e^{-p}}{p} + e^{-(p+1)}\left\{\frac{1}{p+1}\right\} + \frac{e^{-(p+2)}}{\underline{|1}}\left\{\frac{1}{p+2} + \frac{1}{(p+2)^2}\right\}$$

$$+ \frac{3e^{-(p+3)}}{\underline{|2}}\left\{\frac{1}{p+3} + \frac{2}{(p+3)^2} + \frac{2}{(p+3)^3}\right\} + \text{\&c};$$ the nth term within the brackets being: $\dfrac{1}{p+n} + \dfrac{n-1}{(p+n)^2}$

$$+ \frac{(n-1)(n-2)}{(p+n)^3} + \frac{(n-1)(n-2)(n-3)}{(p+n)^4} + \frac{(n-1)(n-2)(n-3)(n-4)}{(p+n)^5} + \text{\&c}$$

$$= \frac{1}{p} - \frac{1}{p^2} + \frac{2}{p^3} - \frac{6}{p^4} + \frac{24}{p^5} - \frac{120}{p^6} + \text{\&c}$$

$$- n\left(\frac{1}{p^3} - \frac{5}{p^4} + \frac{26}{p^5} - \frac{154}{p^6} + \text{\&c}\right)$$

$$+ n^2\left(\frac{3}{p^5} - \frac{35}{p^6} + \frac{340}{p^7} - \frac{3304}{p^8} + \text{\&c}\right)$$

$$- n^3\left(\frac{15}{p^7} - \frac{315}{p^8} + \text{\&c}\right) + \text{\&c \&c}.$$

$154 = 4.6 + 5.26$; $340 = 5.26 + 6.35$; $3304 = 6.154 + 7.34$

&c &c &c

1, 1, 1, 1, 1, &c &c

$\frac{1}{2}$, $\frac{1}{2}+\frac{1}{3}$, $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$, &c &c

$\frac{1}{2}\cdot\frac{1}{4}$, $\frac{1}{2}\cdot\frac{1}{4} + \left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{5}$, &c &c

$\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{6}$, $\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{6} + \left\{\frac{1}{2}\cdot\frac{1}{4} + \left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{5}\right\}\frac{1}{7}$, &c

&c &c &c &c &c

(2). $\dfrac{1}{p+1}+\dfrac{1}{(p+2)^2}+\dfrac{3}{(p+3)^3}+\dfrac{4^2}{(p+4)^4}+\dfrac{5^3}{(p+5)^5}+\&c$ 269 267

$$=\frac{1-e^{-p}}{p}+e^{-p}\left\{\frac{1}{p+1}-\frac{1}{(p+1)(p+2)}+\frac{4}{3(p+1)(p+2)(p+4)}\right.$$

$$\left.-\frac{4}{(p+1)(p+2)(p+4)(3p+23+\theta)}\right\}$$

where $\theta_{-1}=-2\cdot 5856$; $\theta_0=\cdot 0069$; $\theta_1=\cdot 4137$

and $\theta_\infty=\frac{3}{5}$.

(3). $\dfrac{1}{a(p+a)}+\dfrac{1}{(p+a+1)^2}+\dfrac{a+2}{(p+a+2)^3}+\dfrac{(a+3)^2}{(p+a+3)^4}+\&c$

$$=u_0(a)-\frac{p}{\lfloor\underline{1}}u_1(a)+\frac{p^2}{\lfloor\underline{2}}u_2(a)-\frac{p^3}{\lfloor\underline{3}}u_3(a)+\&c$$

where $u_n(a)=\dfrac{\lfloor\underline{n}}{a^{n+2}}+\dfrac{\lfloor\underline{n+1}}{(a+1)^{n+2}\lfloor\underline{1}}+\dfrac{\lfloor\underline{n+2}}{(a+2)^{n+2}\lfloor\underline{2}}+\&c$

and $\dfrac{u_{n-1}(a)-u_n(a+1)}{u_n(a)-u_n(a+1)}=\dfrac{a}{n}$.

(4) $u_n(a)=\dfrac{1}{a(n+1)}+\dfrac{1}{2a^2}+\dfrac{1}{a^3}\left(\dfrac{1}{6}+\dfrac{n}{4}\right)+\dfrac{1}{a^4}\left\{\dfrac{n}{4}+\dfrac{n(n-1)}{8}\right\}$

$$+\frac{1}{a^5}\left\{-\frac{1}{30}+\frac{n}{12}+\frac{n(n-1)}{4}+\frac{n(n-1)(n-2)}{16}\right\}$$

$$+\frac{1}{a^6}\left\{-\frac{n}{12}+\frac{5n(n-1)}{24}+\frac{5n(n-1)(n-2)}{24}+\frac{n(n-1)(n-2)(n-3)}{32}\right\}$$

$$+\frac{1}{a^7}\left\{\frac{1}{42}-\frac{n}{12}-\frac{n(n-1)}{18}+\frac{5n(n-1)(n-2)}{16}+\right.$$

$$\left.\frac{5n(n-1)(n-2)(n-3)}{32}+\frac{n(n-1)(n-2)(n-3)(n-4)}{64}\right\}$$

$$+\frac{1}{a^8}\left\{\frac{n}{12}-\frac{7n(n-1)}{24}+\frac{7n(n-1)(n-2)}{72}+\frac{35n(n-1)(n-2)(n-3)}{96}\right.$$

$$\left.+\frac{7n(n-1)(n-2)(n-3)(n-4)}{64}+\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{128}\right\}$$

$+\ \&c \quad \&c \quad \&c \quad \&c$

(5).[268] $$\frac{1}{a(2p+a)} + \frac{1}{(2p+a+1)^2} + \frac{a+2}{(2p+a+2)^3} + \frac{(a+3)^2}{(2p+a+3)^4} + \&c$$

$$= \frac{1}{2ap} - e^{-2p}\left\{\frac{1}{2p(a+p)} - \frac{P_2}{(a+p)^3} + \frac{P_4}{(a+p)^5} - \&c\right\}$$

$$P_2 = \frac{1}{6}$$

$$P_4 = \frac{1}{30} + \frac{p}{6}$$

$$P_6 = \frac{1}{42} + \frac{p}{6} + \frac{5p^2}{18}$$

$$P_8 = \frac{1}{30} + \frac{3p}{10} + \frac{7p^2}{9} + \frac{35p^3}{54}$$

$$P_{10} = \frac{5}{66} + \frac{5p}{6} + \frac{17p^2}{6} + \frac{35p^3}{9} + \frac{35p^4}{18}$$

$$P_{12} = \frac{691}{2730} + \frac{691p}{210} + \frac{616p^2}{45} + \frac{451p^3}{18} + \frac{385p^4}{18} + \frac{385p^5}{54}.$$

$$P_{14} = \frac{7}{6} + \frac{35p}{2} + \frac{7709p^2}{90} + \frac{26026p^3}{185} + \frac{9002p^4}{9} + \frac{7007p^5}{54} + \frac{5005p^6}{162}$$

&c &c &c.

$$P_{2n} = \frac{1.3.5.\cdots(2n-1)}{2.3^n}\left\{p^{n-1} + \frac{n(n-1)}{10}p^{n-2} + \right.$$

$$\frac{n(n-1)(n-2)}{200}\left[(n-3) + \frac{20}{7}\right]p^{n-3} +$$

$$\frac{n(n-1)(n-2)(n-3)}{6000}\left[(n-4)(n-5) + \frac{60}{7}(n-4) + \frac{90}{7}\right]p^{n-4} +$$

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{240000}\left[\begin{matrix}(n-5)(n-6)(n-7) + \frac{120}{7}(n-5)(n- \\ + \frac{3720}{49}(n-5) + \frac{6000}{77}\end{matrix}\right]p^{n-}$$

$$\left. + \&c \quad \&c \quad \&c \right\}.$$

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$$P_n = B_n + (n+1)B_n p + \left\{\frac{(n+1)(n+2)}{3}B_n - \frac{n(n-1)}{6}B_{n-2}\right\}p^2$$

$$+ \left\{\frac{(n+1)(n+2)(n+3)}{18}B_n - \frac{n^2(n-1)}{9}B_{n-2}\right\}p^3 +$$

$$\left\{\frac{(n+1)(n+2)(n+3)(n+4)}{180}B_n - \frac{n^2(n^2-1)}{36}B_{n-2} + \frac{n(n-1)(n-2)(n-3)}{120}B_{n-4}\right\}p^4$$

$$+ \left\{\frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{2700}B_n - \frac{n^2(n^2-1)(n+2)}{270}B_{n-2}\right.$$

$$\left. + \frac{n(n-1)(n-2)(n-3)(23n-25)}{5400}B_{n-4}\right\}p^5 + \&c$$

which is got from $\dfrac{(a-p+n)^{n-1}}{(a+p+n)^{n+1}} = \dfrac{1}{(a+n)^2}\exp\left\{\dfrac{2ap}{a+n}\right.$

$$\left. + \frac{p^2}{(a+n)^2} - \frac{2np^3}{3(a+n)^3} + \frac{p^4}{2(a+n)^4} - \frac{2np^5}{5(a+n)^5} + \frac{p^6}{3(a+n)^6} - \&c\right\}$$

$$= 1 + 2p\cdot\frac{a}{a+n} + 2p^2\cdot\frac{a^2+\frac{1}{2}}{(a+n)^2} + \frac{4p^3}{3}\left\{\frac{a^3+2a}{(a+n)^3} - \frac{1}{2(a+n)^2}\right\}$$

$$+ \frac{2p^4}{3}\left\{\frac{a^4+5a^2+\frac{3}{2}}{(a+n)^4} - \frac{2a}{(a+n)^3}\right\} +$$

$$+ \frac{4p^5}{15}\left\{\frac{a^5+10a^3+\frac{23}{2}}{(a+n)^5} - \frac{5a^2+4}{(a+n)^4}\right\} + \&c.$$

). $\dfrac{\coth\pi}{1^n} + \dfrac{\coth 2\pi}{2^n} + \dfrac{\coth 3\pi}{3^n} + \dfrac{\coth 4\pi}{4^n} + \&c$

$= \frac{1}{2}\left(\frac{3}{\pi}S_{n+1} + \frac{\pi}{3}S_{n-1}\right) + \dfrac{2^{n-3}}{|\underline{n-3}}\pi^n\,\dfrac{v_{n+1}}{270}$ where

$v_4 = -\frac{3}{2}$, $v_8 = 0$, $v_{12} = \frac{1}{2730}$, $v_{16} = \frac{1}{340}$,

$v_{20} = \frac{191}{2310}$, $v_{24} = \dfrac{907}{\cancel{294}\ 147}$, &c &c.

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(1) $\frac{\theta}{\lfloor 1} B_2 - \frac{\theta^3}{\lfloor 3} B_6 + \frac{\theta^5}{\lfloor 5} B_{10} - \&c$

$$= \sqrt{\frac{\theta}{2\pi}} \left\{ 1 + \frac{\pi^4}{\theta^2 \lfloor 2} B_4 - \frac{\pi^8}{\theta^4 \lfloor 4} B_8 + \frac{\pi^{12}}{\theta^6 \lfloor 6} B_{12} - \&c \right\}$$

$$- \sqrt{\frac{\theta}{2\pi}} \left\{ \frac{\pi^2}{\theta \lfloor 1} B_2 - \frac{\pi^6}{\theta^3 \lfloor 3} B_6 + \frac{\pi^{10}}{\theta^5 \lfloor 5} B_{10} - \&c \right\}$$

(2) If $\int_0^\infty \frac{\cos nx}{e^{2\pi\sqrt{x}}-1}\, dx = \phi(n)$, then

$$\int_0^\infty \frac{\sin nx}{e^{2\pi\sqrt{x}}-1}\, dx = \phi(n) - \frac{1}{2n} + \phi\left(\frac{\pi^2}{n}\right)\sqrt{\frac{2\pi^3}{n^3}}$$

(3) $\frac{1}{4\pi} + \frac{2\cos n}{e^{2\pi}-1} + \frac{4\cos 4n}{e^{4\pi}-1} + \frac{6\cos 9n}{e^{6\pi}-1} + \&c$

$= \phi(n) + \psi(n)$, where,

$$\int_0^\infty e^{-2a^2 n}\, \psi(n)\, dn = \frac{\pi}{e^{4\pi a} - 2e^{2\pi a}\cos 2\pi a + 1}.$$

(4) The part without the transcendental part of $\phi(\pi n)$ can be found from the series

$$\frac{1}{n\sqrt{2n}} \left\{ \sin\left(\frac{\pi}{4} + \frac{\pi}{n}\right) + 2\sin\left(\frac{\pi}{4} + \frac{4\pi}{n}\right) + 3\sin\left(\frac{\pi}{4} + \frac{9\pi}{n}\right) + \&c \right.$$

$$- \left(\cos \pi n + 2\cos 4\pi n + 3\cos 9\pi n + \&c\right).$$

$\phi(0) = \frac{1}{12}$; $\phi\left(\frac{\pi}{2}\right) = \frac{1}{4\pi}$; $\phi(\pi) = \frac{2-\sqrt{3}}{8}$; $\phi(2\pi) = \frac{1}{16}$

$\phi\left(\frac{2\pi}{5}\right) = \frac{8-3\sqrt{5}}{16}$; $\phi\left(\frac{\pi}{5}\right) = \frac{6+\sqrt{5}}{4} - \frac{5\sqrt{10}}{8}$; $\phi(\infty) = 0$.

$$\phi\left(\frac{2\pi}{3}\right) = \frac{1}{3} - \sqrt{3}\left(\frac{3}{16} - \frac{1}{8\pi}\right).$$

5). If $\int_0^\infty e^{-2a^2x} f(x)\,dx = \pi e^{-4ap}$

then $f(x) = \frac{p\sqrt{2\pi}}{x\sqrt{x}}\, e^{-\frac{2p^2}{x}}$. $n\sqrt{\frac{n}{2}}\,\psi(n\pi) =$

6) $\cancel{\frac{n\sqrt{2n}}{\pi\sqrt{\pi}}\phi(n\pi) =} \Big\{ e^{-\frac{2\pi}{n}} + e^{-\frac{4\pi}{n}}\left(3\cos\frac{3\pi}{n} + \sin\frac{3\pi}{n}\right)$

$+ e^{-\frac{6\pi}{n}}\left(4\cos\frac{8\pi}{n} + 2\sin\frac{8\pi}{n}\right) + e^{-\frac{8\pi}{n}}\left(5\cos\frac{15\pi}{n} + 3\sin\frac{15\pi}{n}\right)$

$+ e^{-\frac{10\pi}{n}}\left(6\cos\frac{24\pi}{n} + 4\sin\frac{24\pi}{n}\right) + \&c \Big\} + \Big\{ 2e^{-\frac{8\pi}{n}} +$

$e^{-\frac{12\pi}{n}}\left(5\cos\frac{5\pi}{n} + \sin\frac{5\pi}{n}\right) + e^{-\frac{16\pi}{n}}\left(6\cos\frac{12\pi}{n} + 2\sin\frac{12\pi}{n}\right)$

$+ e^{-\frac{20\pi}{n}}\left(7\cos\frac{21\pi}{n} + 3\sin\frac{21\pi}{n}\right) + \&c \Big\} + \Big\{ 3e^{-\frac{18\pi}{n}}$

$+ e^{-\frac{24\pi}{n}}\left(7\cos\frac{7\pi}{n} + \sin\frac{7\pi}{n}\right) + \&c \Big\} + \Big\{ 4e^{-\frac{32\pi}{n}} + \&c \Big\} + \&c$

The pth term being,

$p e^{-\frac{2\pi p^2}{n}} + e^{-\frac{2\pi p(p+1)}{n}}\left\{(2p+1)\cos\frac{\pi(2p+1)}{n} + \sin\frac{\pi(2p+1)}{n}\right\}$

$+ e^{-\frac{2\pi p(p+2)}{n}}\left\{(2p+2)\cos\frac{2\pi(2p+2)}{n} + 2\sin\frac{2\pi(2p+2)}{n}\right\}$

$+ e^{-\frac{2\pi p(p+3)}{n}}\left\{(2p+3)\cos\frac{3\pi(2p+3)}{n} + 3\sin\frac{3\pi(2p+3)}{n}\right\}$

$+ e^{-\frac{2\pi p(p+4)}{n}}\left\{(2p+4)\cos\frac{4\pi(2p+4)}{n} + 4\sin\frac{4\pi(2p+4)}{n}\right\}$

$+$ &c &c &c &c &c

(1). $\frac{\pi}{2}\cdot\frac{a^x S_x}{\sin\frac{\pi x}{2}} + \frac{1}{2x} + \frac{\pi a}{2(1-x)} = \frac{a^2 S_2}{2-x} - \frac{a^4 S_4}{4-x} + \&c$

$+ \frac{e^{-2\pi a}}{2}\phi(2\pi a) + \frac{e^{-4\pi a}}{4}\phi(4\pi a) + \frac{e^{-6\pi a}}{6}\phi(6\pi a) + \&c$

where $\phi(z) = 1 - \frac{x}{z} + \frac{x(x+1)}{z^2} - \frac{x(x+1)(x+2)}{z^3} + \&c$

(2). $x\left\{\frac{1}{2} + e^{-ax-bx^2} + e^{-2ax-4bx^2} + e^{-3ax-9bx^2} + \&c\right\}$

$= \frac{1}{a+}\,\frac{2b}{a+}\,\frac{4b}{a+}\,\frac{6b}{a+\&c} + \frac{B_2}{\lfloor 2}x^2 A_1 - \frac{B_4}{\lfloor 4}x^4 A_3 + \&c$

where $A_n = a^n - \frac{n(n-1)}{\lfloor 1}a^{n-2}b + \frac{n(n-1)(n-2)(n-3)}{\lfloor 2}a^{n-4}b^2 -$

(3) when x is small, $\frac{1}{1+}\,\frac{1}{1+}\,\frac{2}{1+}\,\frac{3}{1+}\,\frac{4}{1+\&c} =$

$x e^{\frac{1}{2}}\left\{e^{-\frac{(1+x)^2}{2}} + e^{-\frac{(1+3x)^2}{2}} + e^{-\frac{(1+3x)^2}{2}} + \&c\right\} +$

$\frac{x}{2} - \frac{x^2}{12} - \frac{x^4}{360} - \frac{x^6}{5040} - \frac{x^8}{60480} - \frac{x^{10}}{1710720}$, near

(4) $2(2^2-1)\frac{B_2}{x} - 2(2^6-1)\frac{B_6}{3x^3} + 2(2^{10}-1)\cdot\frac{B_{10}}{5x^5} - \&c$

$= \frac{1}{x+}\,\frac{1}{x+}\,\frac{30}{x+}\,\frac{150}{x+}\,\frac{493}{x+\&c}$

(5). If $m = \frac{n(n+1)}{2}$, then $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} =$

$C + \frac{1}{2}\log 2m + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4}$

$+ \frac{1}{2310m^5} - \frac{191}{360360m^6} + \frac{29}{30030m^7} - \frac{2833}{1166880}$

$+ \frac{140051}{17459442m^9} - \&c.$

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(6) $x \coth x = 1 + \frac{x^2}{3} - \frac{x^2}{9} \cdot \frac{x^2}{5+} \; \frac{\frac{4.5}{2.3}x^2}{7+} \; \frac{\frac{2.3}{4.5}x^2}{9+} \; \frac{\frac{6.7}{4.5}x^2}{11+} \; \frac{\frac{4.5}{6.7}x^2}{13+}$ &c

(7) $\frac{x}{n+} \; \frac{x}{n+1+} \; \frac{x}{n+2+} \; \frac{x}{n+3+}$ &c

$= \frac{x}{n} - \frac{x}{n^2} \cdot \frac{x}{n+1+} \; \frac{\frac{2(n+1)}{1(n)}x}{n+2+} \; \frac{\frac{1.n}{2(n+1)}x}{n+3+} \; \frac{\frac{3(n+2)}{2(n+1)}x}{n+4+} \; \frac{\frac{2(n+1)}{3(n+2)}x}{n+5+}$ &c

(8) $\frac{1}{a(p+a)} + \frac{n}{(p+a+1)^2} + \frac{n^2(a+2)}{(p+a+2)^3} + \frac{n^3(a+3)^2}{(p+a+3)^4} + \frac{n^4(a+4)^3}{(p+a+4)^5} +$ &c

$$= \int_0^1 \frac{x^{a-1}\left(1 - x^{\frac{p}{1-nx}}\right)}{p}\,dx$$

(9) $\frac{1^{n-1}}{e^{2\pi}-1} + \frac{2^{n-1}}{e^{4\pi}-1} + \frac{3^{n-1}}{e^{6\pi}-1} +$ &c

$= \frac{B_n}{2n} + \frac{B_n}{n}\cos\frac{\pi n}{4}\left\{\frac{1}{2^{\frac{n}{2}}} + \frac{2\cos(n\tan^{-1}\frac{4}{3})}{5^{\frac{n}{2}}} + \frac{2\cos(n\tan^{-1}\frac{4}{2})}{10^{\frac{n}{2}}} + \frac{2\cos(n\tan^{-1}\frac{12}{5})}{13^{\frac{n}{2}}} + \text{\&c}\right\}$ where:

2, 5, 10, 13 &c are sum of sqrs. of numbers that are prime to each other

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(10). $$\frac{1^{n-1}}{\cosh\frac{\pi}{2}} - \frac{3^{n-1}}{\cosh\frac{3\pi}{2}} + \frac{5^{n-1}}{\cosh\frac{5\pi}{2}} - \&c$$

$$= 2^n(2^n-1)\frac{B_n}{n}\sin\frac{\pi n}{4}\left\{\frac{1}{2^{\frac{n}{2}}} - \frac{2\cos(n\tan^{-1}\frac{1}{2})}{10^{\frac{n}{2}}} + \frac{2\cos(n\tan^{-1}\frac{2}{3})}{26^{\frac{n}{2}}} - \&c\right\}$$

(11) $$\frac{1^{n-1}}{e^{\pi}-e^{-\pi}} - \frac{2^{n-1}}{e^{2\pi}-e^{-2\pi}} + \frac{3^{n-1}}{e^{3\pi}-e^{-3\pi}} - \&c$$

$$= -(2^n-1)\frac{B_n}{n}\cos\frac{\pi n}{4}\left\{\frac{1}{2^{\frac{n}{2}}} + \frac{2\cos(n\tan^{-1}\frac{1}{3})}{10^{\frac{n}{2}}} + \frac{2\cos(n\tan^{-1}\frac{2}{3})}{26^{\frac{n}{2}}} + \&c\right\}$$

(12) $$\frac{1^{n-1}}{\cosh\frac{\pi\sqrt{3}}{2}} - \frac{3^{n-1}}{\cosh\frac{3\pi\sqrt{3}}{2}} + \frac{5^{n-1}}{\cosh\frac{5\pi\sqrt{3}}{2}} - \&c$$

$$= (2^n-1)\frac{B_n}{n}\sin\frac{\pi n}{6}\left\{1 - \frac{2\cos\frac{\pi n}{6}}{3^{\frac{n}{2}}} + \frac{2\cos(n\tan^{-1}\frac{\sqrt{3}}{2})}{7^{\frac{n}{2}}}\right.$$

(14) $$\frac{1^{n-1}}{e^{\pi\sqrt{3}}+1} - \frac{2^{n-1}}{e^{2\pi\sqrt{3}}-1} + \frac{3^{n-1}}{e^{3\pi\sqrt{3}}+1} - \frac{4^{n-1}}{e^{4\pi\sqrt{3}}-1} + \&c$$

$$= -\frac{B_n}{n}\cos\frac{\pi n}{6} - \frac{B_n}{n}\left(\frac{1}{2}+\cos\frac{\pi n}{3}\right)\left\{\frac{1}{3^{\frac{n}{2}}} + \frac{2\cos(n\tan^{-1}}{7^{\frac{n}{2}}} + \&\right.$$

(15). $$\frac{1^{6n}}{\cosh\pi\sqrt{3}+1} - \frac{2^{6n}}{\cosh 2\pi\sqrt{3}-1} + \frac{3^{6n}}{\cosh 3\pi\sqrt{3}-1} - \&c$$

$$+ \frac{2n\sqrt{3}}{\pi}\left\{\frac{B_{6n}}{12n}\cos 3\pi n - \left(\frac{1^{6n-1}}{e^{\pi\sqrt{3}}+1} - \frac{2^{6n-1}}{e^{2\pi\sqrt{3}}-1} + \&c\right)\right.$$

n being a positive integer.

(1). $\int_0^\infty e^{-x} \frac{\cos x}{\sin x} x^{n-1}\, dx = \frac{\lfloor n-1}{2^{\frac{n}{2}}} \cdot \frac{\cos \frac{\pi n}{4}}{\sin \frac{\pi n}{4}}$.

(2). $\int_0^\infty \frac{\sinh ax}{\sinh \pi x} \cdot \frac{dx}{n^2+x^2} = \int_0^1 \frac{x^n}{n} \cdot \frac{\sin a}{1+2x\cos a+x^2}\, dx$.

(3) $\frac{1}{2} \log\left[2\pi(n^2+x^2)^{\frac{1}{2}} \left\{1+\left(\frac{x}{n+1}\right)^2\right\} \left\{1+\left(\frac{x}{n+2}\right)^2\right\} \left\{1+\left(\frac{x}{n+3}\right)^2\right\} \&c\right]$

$= \log \lfloor n + n + \tan^{-1}\frac{x}{n} - \frac{n}{2}\log(n^2+x^2) - \int_0^\infty \frac{\tan^{-1}\frac{2nz}{n^2+x^2-z^2}}{e^{2\pi z}-1}\, dz$

(4) If $\alpha\beta = 2\pi$, then $\alpha\left\{e^{-n} + e^{-ne^{\alpha}} + e^{-ne^{2\alpha}} + \&c\right\}$

$= \alpha\left\{\frac{1}{2} + \frac{n}{\lfloor 1} \cdot \frac{1}{e^{\alpha}-1} - \frac{n^2}{\lfloor 2} \cdot \frac{1}{e^{2\alpha}-1} + \frac{n^3}{\lfloor 3} \cdot \frac{1}{e^{3\alpha}-1} - \&c\right\}$

$+ C - \log n + 2\phi(\beta) + 2\phi(2\beta) + 2\phi(3\beta) + \&c$

where $\phi(\beta) = \sqrt{\frac{\pi}{\beta \sinh \pi\beta}} \cos\left(\beta \log \frac{\beta}{n} - \beta - \frac{\pi}{4} - \frac{B_2}{1.2\beta} - \&c\right)$

(5) If $\alpha\beta = \frac{\pi}{2}$, then $\alpha\left\{e^{-ne^{\alpha}} - e^{-ne^{3\alpha}} + e^{-ne^{5\alpha}} - \&c\right\}$

$= \alpha\left\{\frac{1}{2} - \frac{n}{\lfloor 1} \cdot \frac{1}{e^{\alpha}+e^{-\alpha}} + \frac{n^2}{\lfloor 2} \cdot \frac{1}{e^{2\alpha}+e^{-2\alpha}} - \&c\right\}$

$+ \phi(\beta) - \phi(3\beta) + \phi(5\beta) - \phi(7\beta) + \&c$, where

$\phi(\beta) = \sqrt{\frac{\pi}{\beta \sinh \pi\beta}} \sin\left(\beta \log \frac{\beta}{n} - \beta - \frac{\pi}{4} - \frac{B_2}{1.2\beta} - \frac{B_4}{3.4\beta^3} - \&c\right)$

(6). $\frac{(\lfloor n)^2}{\lfloor n-1+xi \; \lfloor n-xi} = \left\{1+\frac{x^2}{(n+1)^2}\right\} \left\{1+\frac{x^2}{(n+2)^2}\right\} \left\{1+\frac{x^2}{(n+3)^2}\right\}$ &c

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(1) $$\frac{1}{2x^2}+\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\&c$$

$$=\frac{1}{2\pi x^3}+\frac{\pi}{3x}-\frac{\pi^2}{\sin^2\pi x\,(e^{2\pi x}-1)}+$$

$$4x\left\{\frac{1}{e^{2\pi}-1}\cdot\frac{1}{(1^2-x^2)^2}+\frac{2}{e^{4\pi}-1}\cdot\frac{1}{(2^2-x^2)^2}+\&c\right\}$$

$$+8\pi x^3\left\{\frac{1}{(e^{\pi}-e^{-\pi})^2}\cdot\frac{1}{1^4-x^4}+\frac{1}{(e^{2\pi}-e^{-2\pi})^2}\cdot\frac{1}{2^4-x^4}+\&c\right\}$$

(2) $$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{x}=C+\frac{\pi}{3}\log x+\frac{1}{2x}-\frac{1}{4\pi x}$$

$$+\frac{\pi\cot\pi x}{e^{2\pi x}-1}+\frac{2\pi\log(2\sin\pi x)}{(e^{\pi x}-e^{-\pi x})^2}+$$

$$2\left(\frac{1}{e^{2\pi}-1}\cdot\frac{1}{1^2-x^2}+\frac{2}{e^{4\pi}-1}\cdot\frac{1}{2^2-x^2}+\frac{3}{e^{6\pi}-1}\cdot\frac{1}{3^2-x^2}+\&c\right)$$

$$-2\pi\left\{\frac{\log(1^4-x^4)}{(e^{\pi}-e^{-\pi})^2}+\frac{\log(2^4-x^4)}{(e^{2\pi}-e^{-2\pi})^2}+\frac{\log(3^4-x^4)}{(e^{3\pi}-e^{-3\pi})^2}+\&c\right\}$$

$$-2\pi\sum_{n=1}^{m=\infty}e^{-2\pi nx}\left\{n^2\left(\frac{\sin 2\pi x}{1^2+n^2}+\frac{\sin 4\pi x}{2^2+n^2}+\frac{\sin 6\pi x}{3^2+n^2}+\&c\right)\right.$$

$$\left.-n^3\left(\frac{\cos 2\pi x}{1^2+n^2}+\frac{1}{2}\cdot\frac{\cos 4\pi x}{2^2+n^2}+\frac{1}{3}\cdot\frac{\cos 6\pi x}{3^2+n^2}+\&c\right)\right\}$$

(3) $$\frac{\pi}{x^2\sqrt{3}}\cdot\frac{\sinh\pi x\sqrt{3}\,\sinh\pi x+\sin\pi x\sqrt{3}\,\sin\pi x}{(\cosh\pi x\sqrt{3}-\cos\pi x)(\cosh\pi x-\cos\pi x\sqrt{3})}=$$

$$\frac{1}{2\pi x^4}+\coth\pi\left(\frac{1}{1+x^2+x^4}+\frac{1}{1-x^2+x^4}\right)+2\coth 2\pi$$

$$\times\left(\frac{1}{2^4+2^2x^2+x^4}+\frac{1}{2^4-2^2x^2+x^4}\right)+3\coth 3\pi\left(\frac{1}{3^4+3^2x^2+x^4}+\frac{1}{3^4-3^2x^2+x^4}\right)$$

$$+\&c$$

(4). If $S_n = \frac{B_n}{2n} + \frac{1^{n-1}}{e^{2\pi}-1} + \frac{2^{n-1}}{e^{4\pi}-1} + \frac{3^{n-1}}{e^{6\pi}-1} + \&c$,

then if $n-2$ be a multiple of 4,

$$\frac{(n+3)(n-4)}{24} S_{n+2} = \frac{n(n-1)(n-2)(n-3)}{\lfloor\underline{2}} S_4 S_{n-2} +$$

$$\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)}{\lfloor\underline{6}} S_8 S_{n-6} + \&c$$

(5). $\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \cdots\cdots + \sqrt{x} = C + \frac{2}{3}x\sqrt{x} + \frac{1}{2}\sqrt{x}$

$$+ \frac{1}{6}\left\{\frac{1}{(\sqrt{x}+\sqrt{x+1})^3} + \frac{1}{(\sqrt{x+1}+\sqrt{x+2})^3} + \&c\right\}.$$

(6) $1\sqrt{1} + 2\sqrt{2} + 3\sqrt{3} + \cdots\cdots + x\sqrt{x} = C + \frac{2}{5}x^2\sqrt{x} + \frac{x}{2}\sqrt{x} + \frac{1}{8}\sqrt{x}$

$$+ \frac{1}{40}\left\{\frac{1}{(\sqrt{x}+\sqrt{x+1})^5} + \frac{1}{(\sqrt{x+1}+\sqrt{x+2})^5} + \&c\right\}.$$

(7) $(1^2\sqrt{1} + 2^2\sqrt{2} + 3^2\sqrt{3} + \cdots + x^2\sqrt{x}) + \frac{1}{16}(\sqrt{1} + \sqrt{2} + \cdots + \sqrt{x})$

$$= C + \frac{2}{7}x^3\sqrt{x} + \frac{x^2}{2}\sqrt{x} + \frac{x}{4}\sqrt{x} + \frac{1}{32}\sqrt{x} +$$

$$\frac{1}{224}\left\{\frac{1}{(\sqrt{x}+\sqrt{x+1})^7} + \frac{1}{(\sqrt{x+1}+\sqrt{x+2})^7} + \&c\right\}.$$

(8). $\Sigma\frac{1}{x} - \Sigma\frac{1}{x/3} + \frac{1}{x} - \log 3 =$

$$\frac{2}{3}\cdot\frac{1}{x^2+}\,\frac{2^3-2}{6+}\,\frac{4^3-4}{3x^2+}\,\frac{5^3-5}{6+}\,\frac{7^3-7}{5x^2+}\,\&c$$

(9). $\int_0^\infty \cos nx \log(1+x^2)\,dx = -\frac{\pi}{n}e^{-n}.$

(1). $\frac{x^n}{\underline{|n}}\left\{1+\frac{x^2}{\underline{|1}}\cdot\frac{1}{(1+n)}+\frac{x^4}{\underline{|2}}\cdot\frac{1}{(1+n)(2+n)}+\&c\right\}$

$-\frac{x^{-n}}{\underline{|-n}}\left\{1+\frac{x^2}{\underline{|1}}\cdot\frac{1}{(1-n)}+\frac{x^4}{\underline{|2}}\cdot\frac{1}{(1-n)(2-n)}+\&c\right\}$

$=-\frac{e^{-2x}}{\sqrt{\pi x}}\sin\pi n\left\{1+\frac{n^2-\frac{1^2}{2^2}}{4x}+\frac{(n^2-\frac{1^2}{2^2})(n^2-\frac{3^2}{2^2})}{4.8x^2}+\&c\right.$

$=-\frac{\sin\pi n}{\pi}\int_0^\infty z^{n-1}e^{-x(z+1/z)}\,dz.$

(2). $\int_0^\infty x^{2n}e^{-x^2-\frac{a^2}{x^2}}dx=\frac{\sqrt{\pi}}{2}e^{-2a}a^n\left\{1+\right.$

$\frac{n(n+1)}{4a}+\frac{(n-1)n(n+1)(n+2)}{4.8.a^2}+$

$\left.\frac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{4.8.12\,a^3}+\&c\right\}.$

N.B. Integrate partially and add.

(3) $\log(1+\frac{x^2}{1^2})-3\log(1+\frac{x^2}{3^2})+5\log(1+\frac{x^2}{5^2})-\&c$

$+2x\tan^{-1}e^{-\frac{\pi x}{2}}.=$

$\frac{4}{\pi}\left(\frac{1-e^{-\frac{\pi x}{2}}}{1^2}-\frac{1-e^{-\frac{3\pi x}{2}}}{3^2}+\frac{1-e^{-\frac{5\pi x}{2}}}{5^2}-\&c\right)$

(4). $\log\left\{1+\left(\frac{2}{\pi}\log\overline{2+\sqrt{3}}\right)^2\right\}-3\log\left\{1+\left(\frac{2}{3\pi}\log\overline{2+\sqrt{3}}\right)^2\right\}+$

$\log\left\{1+\left(\frac{2}{5\pi}\log\overline{2+\sqrt{3}}\right)^2\right\}-\&c=\frac{4}{3\pi}\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\&c\right).$

5). $\frac{1^n}{1^2-x^2}+\frac{2^n}{2^2-x^2}+\frac{3^n}{3^2-x^2}+\&c$

$= \frac{\pi}{2}x^{n-1}\left(\tan\frac{\pi n}{2}-\cot\pi x\right)+2\sin\frac{\pi n}{2}\int_0^\infty\frac{z^n}{e^{2\pi z}-1}\cdot\frac{dz}{z^2+x^2}$

6) $\left(\frac{1^n}{1^2+x^2}+\frac{2^n}{2^2+x^2}+\frac{3^n}{3^2+x^2}+\&c\right)-\frac{\pi}{2}x^{n-1}\sec\frac{\pi n}{2}.$

$= \frac{\pi x^{n-1}\cos\frac{\pi n}{2}}{e^{2\pi x}-1}+2\sin\frac{\pi n}{2}\int_0^\infty\frac{z^n}{e^{2\pi z}-1}\cdot\frac{dz}{z^2-x^2}.$

(7) If $\int_0^\infty e^{-p^2x}\phi(x)\,dx=\frac{e^{-2ap}}{p^{n+1}}$, then

$\phi(x)=\frac{x^n}{\sqrt{\pi x}}e^{-\frac{a^2}{x}}\int_0^\infty e^{-az-\frac{xz^2}{4}}\cdot\frac{z^{n-1}}{\underline{\lfloor n-1}}\,dz.$

$=\frac{x^n}{a^n\sqrt{\pi x}}e^{-\frac{a^2}{x}}\left\{1-\frac{n(n+1)}{4a^2}x+\frac{n(n+1)(n+2)(n+3)}{4.8.a^4}x^2\right.$

$\left.-\frac{n(n+1)(n+2)(n+3)(n+4)(n+5)}{4.8.12a^6}x^3+\&c\right\}$

(8) If $\frac{2}{3}\theta u=v+\frac{1}{2}\cdot\frac{v^7}{7}+\frac{1.3}{2.4}\cdot\frac{v^{13}}{13}+\&c$

then $\frac{4}{9}\cdot\frac{u^2}{v^2}=\frac{1}{\sin^2\theta}-\frac{2}{\pi\sqrt3}+8\left(\frac{\cos2\theta}{e^{\pi\sqrt3}+1}-\frac{2\cos4\theta}{e^{2\pi\sqrt3}-1}\right.$

$\left.+\frac{3\cos6\theta}{e^{3\pi\sqrt3}+1}-\&c\right)$ where $u=\frac{\sqrt{\pi}}{\underline{\lfloor -\frac{4}{3}}\,\underline{\lfloor -\frac{4}{6}}}.$

(9) $\frac{B_4}{4}-\frac{B_8}{8}+\&c$, $\frac{B_2}{2}\cos+\frac{B_4}{4}\cos$ &c

(10) $\int_0^n\left(\frac{n}{x}\right)^x dx=\frac{n}{1^1}+\frac{n^2}{2^2}+\frac{n^3}{3^3}+\frac{n^4}{4^4}+\&c$ —

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(11). The difference between $\dfrac{\lfloor\underline{\beta-n}}{\lfloor\underline{\alpha+\beta-n}}$ and

$$\frac{\lfloor\underline{\beta}}{\lfloor\underline{\alpha+\beta}} + \frac{\alpha}{\lfloor\underline{1}}\cdot m\cdot \frac{\lfloor\underline{\beta+n}}{\lfloor\underline{\alpha+\beta+n+1}} + \frac{\alpha(\alpha+1)}{\lfloor\underline{2}}\cdot m(m+2n+1)$$

$$\times \frac{\lfloor\underline{\beta+2n}}{\lfloor\underline{\alpha+\beta+2n+2}} + \frac{\alpha(\alpha+1)(\alpha+2)}{\lfloor\underline{3}}\, m(m+3n+1)(m+3n+2) \quad (1)$$

$$\times \frac{\lfloor\underline{\beta+3n}}{\lfloor\underline{\alpha+\beta+3n+3}} + \&c.$$

(12). $e^{-\frac{x}{24}}(1-e^{-a})^{\frac{1}{2}}(1-e^{-a-x})(1-e^{-a-2x})(1-e^{-a-3x})\,\&c$

$$= \frac{\left(\frac{a}{x}\right)^{\frac{a}{x}}}{e^{\frac{a}{x}}\lfloor\underline{\tfrac{a}{x}}}\sqrt{\frac{2\pi a}{x}}\; e^{-\frac{1}{x}\left(\frac{e^{-a}}{1^2}+\frac{e^{-2a}}{2^2}+\&c\right)-\theta}$$

where $\theta = \displaystyle\sum_{n=1}^{n=\infty} \frac{B_{2n}}{2n}\cdot\frac{B_{2n}}{2n}\cdot\frac{a}{\lfloor\underline{1}}\cdot\frac{x^{2n-1}}{\lfloor\underline{2n-1}}$

$$\frac{B_{2n}}{\lfloor\underline{2n}}\,x^{2n-1}\left\{\frac{B_{2n}}{2n}\cdot\frac{a}{\lfloor\underline{1}} - \frac{B_{2n+2}}{2n+2}\cdot\frac{a^3}{\lfloor\underline{3}} + \&c\right\}$$

(13). The property of the function

$$\frac{\log 1}{1^2+x^2} + \frac{\log 2}{2^2+x^2} + \frac{\log 3}{3^2+x^2} + \&c \text{ and}$$

the integral $\displaystyle\int_0^\infty \frac{z}{e^{2\pi z}-1}\cdot\frac{dz}{z+x}$.

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(1) If $\frac{\theta u}{\sqrt{2}} = v + \frac{1}{2}\cdot\frac{v^5}{5} + \frac{1.3}{2.4}\cdot\frac{v^9}{9} + \&c$ where u is the constant obtained by putting $v=1$ and $\theta = \frac{\pi}{2}$, then

(1) $\frac{u^2}{2v^2} = \frac{1}{\sin^2\theta} - \frac{1}{\pi} - 8\left(\frac{\cos 2\theta}{e^{2\pi}-1} + \frac{2\cos 4\theta}{e^{4\pi}-1} + \&c\right)$

(2) $\frac{u}{\sqrt{2}}\left(\frac{1}{v} - \frac{1}{2}\cdot\frac{v^3}{3} - \frac{1.3}{2.4}\cdot\frac{v^7}{7} - \frac{1.3.5}{2.4.6}\cdot\frac{v^{11}}{11} - \&c\right)$

$= \cot\theta + \frac{\theta}{\pi} + 4\left(\frac{\sin 2\theta}{e^{2\pi}-1} + \frac{\sin 4\theta}{e^{4\pi}-1} + \frac{\sin 6\theta}{e^{6\pi}-1} + \&c\right)$

(3) $\log\frac{v\sqrt{2}}{u} + \frac{1}{3}\cdot\frac{v^4}{4} + \frac{1.5}{3.7}\cdot\frac{v^8}{8} + \frac{1.5.9}{3.7.11}\cdot\frac{v^{12}}{12} + \&c$

$= \log\sin\theta + \frac{\theta^2}{2\pi} - 2\left\{\frac{\cos 2\theta}{1(e^{2\pi}-1)} + \frac{\cos 4\theta}{2(e^{4\pi}-1)} + \&c\right\}$

(4) $\frac{1}{2}\tan^{-1} v = \frac{\sin\theta}{\cosh\frac{\pi}{2}} + \frac{\sin 3\theta}{3\cosh\frac{3\pi}{2}} + \frac{\sin 5\theta}{5\cosh\frac{5\pi}{2}} + \&c$

(5) $\frac{1}{2}\cos^{-1} v^2 = \frac{\cos\theta}{\cosh\frac{\pi}{2}} - \frac{\cos 3\theta}{3\cosh\frac{3\pi}{2}} + \frac{\cos 5\theta}{5\cosh\frac{5\pi}{2}} - \&c$

(6) $\frac{\sqrt{2}}{4u}\left\{\frac{v^3}{3} + \frac{2}{3}\cdot\frac{v^7}{7} + \frac{2.4}{3.5}\cdot\frac{v^{11}}{11} + \&c\right\}$

$= \frac{\pi\theta}{8} - \frac{\sin\theta}{1^2\cosh\frac{\pi}{2}} + \frac{\sin 3\theta}{3^2\cosh\frac{3\pi}{2}} - \frac{\sin 5\theta}{5^2\cosh\frac{5\pi}{2}} + \&c$

If $\frac{\theta u}{2} = v - \frac{1}{2}\cdot\frac{v^5}{5} + \frac{1.3}{2.4}\cdot\frac{v^9}{9} - \&c$, then

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(7) $2\tan^{-1} v = \theta + \frac{\sin 2\theta}{\cosh \pi} + \frac{\sin 4\theta}{2\cosh 2\pi} + \&c$ (1)

(8) $\frac{\pi}{8} - \frac{1}{2}\tan^{-1} v^2 = \frac{\cos\theta}{\cosh\frac{\pi}{2}} - \frac{\cos 3\theta}{3\cosh\frac{3\pi}{2}} + \&c$

(9) $\frac{1}{2}\log\frac{1+v}{1-v} = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + 4\left\{\frac{\sin\theta}{e^{\pi}-1} - \frac{\sin 3\theta}{3(e^{3\pi}-1)} + \&c\right.$

(10) $\log\left(1-\frac{x^2}{1^2}\right) - 3\log\left(1-\frac{x^2}{3^2}\right) + 5\log\left(1-\frac{x^2}{5^2}\right) - \&c$

$= \frac{4}{\pi}\left\{\frac{1-\cos\frac{\pi x}{2}}{1^2} - \frac{1-\cos\frac{3\pi x}{2}}{3^2} + \&c\right\} +$

$x\log\tan\frac{\pi-\pi x}{4}.$

$= \frac{4}{\pi}\left\{\frac{1-\tan\left(\frac{\pi-\pi x}{4}\right)}{1^2} - \frac{1-\tan^3\left(\frac{\pi-\pi x}{4}\right)}{3^2} + \&c\right\}$

$+ \log\tan\frac{\pi-\pi x}{4}.$

(11) If $\frac{\pi\alpha}{2} = \log\tan\left(\frac{\pi}{4}+\frac{\pi\beta}{4}\right)$, then

$\log\left(1+\frac{\alpha^2}{1^2}\right) - 3\log\left(1+\frac{\alpha^2}{3^2}\right) + 5\log\left(1+\frac{\alpha^2}{5^2}\right) - \&c$

$= \frac{\pi\alpha\beta}{2} + \log\left(1-\frac{\beta^2}{1^2}\right) - 3\log\left(1-\frac{\beta^2}{3^2}\right) + 5\log\left(1-\frac{\beta^2}{5^2}\right)$

$- \&c$

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(1) If $\phi(m, n) = \left\{1+\left(\frac{m+n}{1+m}\right)^3\right\}\left\{1+\left(\frac{m+n}{2+m}\right)^3\right\}$ &c

then $\phi(m, n)\,\phi(n, m) =$

$$\frac{(\lfloor m)^3(\lfloor n)^3}{\lfloor 2m+n\,\lfloor 2n+m} \cdot \frac{\cosh \pi(m+n)\sqrt{3} - \cos \pi(m-n)}{2\pi^2(m^2+mn+n^2)}.$$

(2) $\left\{1+\left(\frac{n}{1}\right)^3\right\}\left\{1+\left(\frac{n}{2}\right)^3\right\}\left\{1+\left(\frac{n}{3}\right)^3\right\}$ &c

$\times\left\{1+3\cdot\left(\frac{n}{n+2}\right)^4\right\}\left\{1+3\left(\frac{n}{n+4}\right)^4\right\}\left\{1+3\left(\frac{n}{n+6}\right)^4\right\}$ &c

$$= \frac{\lfloor \frac{n}{2}-1 \ [illegible]}{[illegible]} \cdot \frac{\cosh \pi n\sqrt{3} - \cos \pi n}{2^{n+2}\,\pi n\sqrt{\pi}}.$$

(3) $\frac{3}{2}\log 2\pi n + \log\left(1+\frac{n^3}{1^3}\right)\left(1+\frac{n^3}{2^3}\right)\left(1+\frac{n^3}{3^3}\right)$ &c

$- \log\left(e^{\pi n\sqrt{3}} + e^{-\pi n\sqrt{3}} - 2\cos \pi n\right)$

$$= -\frac{\pi n}{\sqrt{3}} + \frac{B_4}{4}\cdot\frac{1}{n^3} - \frac{B_{10}}{10}\cdot\frac{1}{3n^9} + \frac{B_{16}}{16}\cdot\frac{4}{5n^{15}} - \text{\&c}$$

(4) $\frac{B_2}{1.2.2n} + \frac{B_4}{3.4.2^2n^3} - \frac{B_6}{5.6.2^2n^5} - \frac{B_8}{7.8.2^4n^7} + \text{\&c}$

$$= \log\frac{e^n\lfloor n}{n^n\sqrt{2\pi n}} + \frac{n}{2}\left(\frac{\pi}{2}-\log 2\right) - \frac{1}{4}\log 2$$

$- \frac{1}{2}\log\left\{1+\left(\frac{n}{n+1}\right)^4\right\}\left\{1+\left(\frac{n}{n+2}\right)^4\right\}\left\{1+\left(\frac{n}{n+3}\right)^4\right\}$ &c

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(1) The diff^ce between the two series ($\alpha\beta=\frac{\pi^2}{16}$)

$$\alpha^2\left\{\frac{\operatorname{sech}\frac{\pi}{2}}{\cosh\alpha+\cos\alpha}-\frac{3^3\operatorname{sech}\frac{3\pi}{2}}{\cosh 3\alpha+\cos 3\alpha}+\&c\right\} \text{ and}$$

$$\beta^2\left\{\frac{\operatorname{sech}\frac{\pi}{2}}{\cosh\beta+\cos\beta}-\frac{3^3\operatorname{sech}\frac{3\pi}{2}}{\cosh 3\beta+\cos 3\beta}\right\} \text{ is } 0?$$

(2) $$\int_0^\infty \frac{\sin 2nx}{x(\cosh\pi x+\cos\pi x)}\,dx=\frac{\pi}{4}-2\left(\frac{e^{-n}\cos n}{\cosh\frac{\pi}{2}}\right.$$

$$\left.-\frac{e^{-3n}\cos 3n}{3\cosh\frac{3\pi}{2}}+\frac{e^{-5n}\cos 5n}{5\cos\frac{5\pi}{2}}-\&c\right)$$

(3) If $\alpha\beta=\frac{\pi^2}{4}$, then, $$\frac{1}{\cosh\alpha+\cos\alpha}$$

$$-\frac{1}{3(\cosh 3\alpha+\cos 3\alpha)}+\frac{1}{5(\cosh 5\alpha+\cos 5\alpha)}-\&c$$

$$=\frac{\pi}{8}-\frac{2\cos\beta\cosh\beta}{\cosh\frac{\pi}{2}(\cosh 2\beta+\cos 2\beta)}+$$

$$\frac{2\cos 3\beta\cosh 3\beta}{3\cosh\frac{3\pi}{2}(\cosh 6\beta+\cos 6\beta)}+\&c$$

(4) If $\alpha\beta=\frac{\pi^2}{2}$, then $$\frac{\pi}{8}-\frac{\pi^3}{32\alpha^2}+$$

$$\frac{\cos\alpha}{\cosh\alpha-\cos\alpha}-\frac{\cos 3\alpha}{3(\cosh 3\alpha-\cos 3\alpha)}+\&c=$$

$$\frac{\sin\beta\sinh\beta}{\cosh 2\beta+\cos 2\beta}\cdot\frac{\coth\pi}{1}+\frac{\sin 2\beta\sinh 2\beta}{\cosh 4\beta+\cos 4\beta}\cdot\frac{\coth 2\pi}{2}$$

(1) The difference between the series

$$\frac{\theta}{8\pi}+\frac{\sin\theta}{1(e^{2\pi}-1)}+\frac{\sin 4\theta}{2(e^{4\pi}-1)}+\frac{\sin 9\theta}{3(e^{6\pi}-1)}+\&c$$

and $\frac{1}{4}\left\{\frac{B_2}{1\lfloor 1}\cdot\theta-\frac{B_6}{3\lfloor 3}\theta^3+\frac{B_{10}}{5\lfloor 5}\theta^5-\&c\right\}$

(2) $$\frac{\pi}{2n}\cdot\frac{\sec\frac{\pi m}{2n}}{e^{\frac{\pi}{n}}-1}+\frac{1}{m+n}-\frac{1}{m+3n}+\frac{1}{m+5n}-\&c$$

$$=\frac{1}{2}+\frac{\operatorname{sech}\pi m}{1+(2n)^2}+\frac{\operatorname{sech}2\pi m}{1+(4n)^2}+\&c$$

$$-2m\left\{\frac{1}{m^2-n^2}\cdot\frac{1}{e^{\frac{\pi}{m}}-1}-\frac{1}{m^2-(3n)^2}\cdot\frac{1}{e^{\frac{3\pi}{m}}-1}+\&c\right\}$$

(3) If $\phi=\frac{x}{1+}\frac{x^5}{1+}\frac{x^{10}}{1+}\frac{x^{15}}{1+}\&c$ and

$f=\frac{\sqrt[5]{x}}{1+}\frac{x}{1+}\frac{x^2}{1+}\frac{x^3}{1+}\&c$, then

$$f^5=\phi\cdot\frac{1-2\phi+4\phi^2-3\phi^3+\phi^4}{1+3\phi+4\phi^2+2\phi^3+\phi^4}.$$

(4) $$\left.\begin{aligned}&1-\frac{ax}{1+}\frac{a^2}{1-}\frac{a^3x}{1+}\frac{a^4}{1-}\frac{a^5x}{1+\&c}\\&=\frac{a}{x+}\frac{a^4}{x+}\frac{a^8}{x+}\frac{a^{12}}{x+\&c}\ \text{nearly}\end{aligned}\right\}\ \text{Conventional only.}$$

(5) $$\frac{\pi}{2}\int_0^\infty\frac{dx}{e^{x^n}+e^{-x^n}}=$$

$$\sqrt[n]{\frac{\pi}{2}}\,\lfloor\tfrac{1}{n}-1\,\cos\frac{\pi}{2n}\int_0^\infty\frac{x^{n-2}}{e^{x^n}+e^{-x^n}}\,dx$$

(1) $\frac{x}{4n+2+}\frac{x^2}{4n+6+}\frac{x^2}{4n+10+}\&c$

$+\frac{2n}{x+}\frac{n-1}{1-}\frac{n+1}{x+}\frac{n-2}{1-}\frac{n+2}{x+}\&c$

$=1$ nearly.

(2). $1-\frac{ax}{1+a+}\frac{a^3x}{1+a^2-}\frac{a^2x}{1+a^3+}\frac{a^6x}{1+a^4-}\frac{a^3x}{1+a^5+}$

$\frac{a^9x}{1+a^6-\&c}=\frac{1}{x+}\frac{a}{x+}\frac{a^3}{x+}\frac{a^5}{x+\&c}$ nearly.

(3) $\frac{1-a^2}{1-a}\cdot\frac{1-a^5}{1-a^4}\cdot\frac{1-a^8}{1-a^7}\cdot\frac{1-a^{11}}{1-a^{10}}\&c$

$=\frac{1}{1-}\frac{a}{1+a-}\frac{a^3}{1+a^2-}\frac{a^5}{1+a^3-}\frac{a^7}{1+a^4-}\&c$

(4) $\frac{1-a^3}{1-a}\cdot\frac{1-a^7}{1-a^5}\cdot\frac{1-a^{11}}{1-a^9}\&c=$

$\frac{1}{1-}\frac{a}{1+a^2-}\frac{a^3}{1+a^4-}\frac{a^5}{1+a^6-}\&c$

(5) $\frac{1+a^2}{1+a}\cdot\frac{1+a^4}{1+a^3}\cdot\frac{1+a^6}{1+a^5}\&c=$

$\frac{1}{1+}\frac{a}{1+}\frac{a^2+a}{1+}\frac{a^3}{1+}\frac{a^4+a^2}{1+}\frac{a^5}{1+}\&c$

(6) $\frac{(1-a)(1-a^7)(1-a^9)(1-a^{15})\&c}{(1-a^3)(1-a^5)(1-a^{11})(1-a^{13})\&c}=$

$\frac{1}{1+}\frac{a+a^2}{1+}\frac{a^4}{1+}\frac{a^3+a^6}{1+}\frac{a^8}{1+}\&c$

(1) If $\phi(\alpha,\beta) = \dfrac{\pi}{e^{4\pi\alpha} - 2e^{2\pi\alpha}\cos 2\pi\beta + 1} +$ 2087

$\alpha\left\{\dfrac{1}{2(\alpha^2+\beta^2)} + \dfrac{1}{\alpha^2+(1+\beta)^2} + \dfrac{1}{\alpha^2+(2+\beta)^2} + \&c\right\}$

$- 4\alpha\beta\left\{\dfrac{1}{e^{4\pi}-1}\cdot\dfrac{1}{\alpha^2+(1+\beta)^2}\cdot\dfrac{1}{\alpha^2+(1-\beta)^2} + \right.$

$\left.\dfrac{2}{e^{4\pi}-1}\cdot\dfrac{1}{\alpha^2+(2+\beta)^2}\cdot\dfrac{1}{\alpha^2+(2-\beta)^2} + \&c\right\}$

then $\phi(\alpha,\beta) + \phi(\beta,\alpha) = \dfrac{\pi}{2} + \dfrac{\alpha\beta}{\pi(\alpha^2+\beta^2)^2} +$

$\dfrac{\pi}{2}\cdot\dfrac{\cosh 2\pi(\alpha-\beta) - \cos 2\pi(\alpha-\beta)}{(\cosh 2\pi\alpha - \cos 2\pi\beta)(\cosh 2\pi\beta - \cos 2\pi\alpha)}$

(3) If $\phi(\alpha,\beta) = \dfrac{\pi/2}{e^{2\pi\alpha} + 2e^{\pi\alpha}\cos\pi\beta + 1} +$

$\alpha\left\{\dfrac{1}{\alpha^2+(1+\beta)^2} + \dfrac{1}{\alpha^2+(3+\beta)^2} + \dfrac{1}{\alpha^2+(5+\beta)^2} + \&c\right\}$

$+ 4\alpha\beta\left\{\dfrac{1}{e^{\pi}+1}\cdot\dfrac{1}{\alpha^2+(1+\beta)^2}\cdot\dfrac{1}{\alpha^2+(1-\beta)^2} + \&c\right\}$

then $\phi(\alpha,\beta) + \phi(\beta,\alpha) = \dfrac{\pi}{4} +$

$\dfrac{\pi}{4}\cdot\dfrac{\cosh\pi(\alpha-\beta) - \cos\pi(\alpha-\beta)}{(\cosh\pi\alpha + \cos\pi\beta)(\cosh\pi\beta + \cos\pi\alpha)}$

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If $y = \frac{\sqrt{1+x^2}-1}{x}$ and $m = \frac{n}{\sqrt{1+x^2}}$, then

(1) $\frac{x}{1+n+} \frac{(x)^2}{3+n+} \frac{(2x)^2}{5+n+} \frac{(3x)^2}{7+n} + \&c$

$= 2\left(\frac{y}{m+1} - \frac{y^3}{m+3} + \frac{y^5}{m+5} - \&c\right)$

(2) $\frac{x}{2+n+} \frac{1.2x^2}{4+n+} \frac{2.3x^2}{6+n+} \frac{3.4x^2}{8+n} + \&c$

$= y - m(y+1/y)\left(\frac{y^2}{m+2} - \frac{y^4}{m+4} + \frac{y^6}{m+8} - \&c\right)$

(3) $\frac{1}{n+} \frac{1.p}{n+} \frac{2(p+1)}{n+} \frac{3(p+2)}{n+} \frac{4(p+3)}{n} + \&c$

$= 2^p\left\{\frac{1}{n+p} - \frac{p}{\lfloor 1}\cdot\frac{1}{n+p+2} + \frac{p(p+1)}{\lfloor 2}\cdot\frac{1}{n+p+4} - \&c\right.$

(4). $\frac{x}{p+n+} \frac{1.px^2}{p+2+n+} \frac{2(p+1)x^2}{p+4+n+} \frac{3(p+2)x^2}{p+6+n} + \&c$

$= (1+1/x^2)^{\frac{p-1}{2}}(2y)^p\left\{\frac{1}{m+p} - \frac{p}{\lfloor 1}\cdot\frac{y^2}{m+p+2} + \frac{p(p+1)}{\lfloor 2}\cdot\frac{y^4}{m+p+4} - \&c\right\}.$

(5) $\frac{1}{2x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \&c$

$= 1/x + 1/2x^2 \cdot \frac{1}{3x+} \frac{3}{5x+} \frac{18}{7x+} \frac{60}{9x} + \&c$

$3 = 2^2(2^2-1)/4$; $18 = 3^2(3^2-1)/4$; $60 = 4^2(4^2-1)/4$ &c

(6). $\frac{1}{2x^3}+\frac{1}{(x+1)^3}+\frac{1}{(x+2)^3}+\frac{1}{(x+3)^3}+\&c$

$$=\frac{1}{2x^2}+\frac{1}{4x^3}\cdot\frac{1}{x+}\,\frac{1}{3x+}\,\frac{2}{x+}\,\frac{6}{5x+}\,\frac{9}{x+}\,\frac{18}{7x+}\&c$$

(7) $1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n}=\frac{1}{2n}-\frac{1}{2\pi n^2}+$

$$\frac{\pi\cot\pi n}{e^{2\pi n}-1}+\frac{n^2}{1(1^2+n^2)}+\frac{n^2}{2(2^2+n^2)}+\frac{n^2}{3(3^2+n^2)}+\&c$$

$$+\frac{4n^2}{1^4-n^4}\cdot\frac{1}{e^{2\pi}-1}+\frac{8n^2}{2^4-n^4}\cdot\frac{1}{e^{4\pi}-1}+\frac{12n^2}{3^4-n^4}\cdot\frac{1}{e^{6\pi}-1}+\&c$$

(8) $1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{n-1}=\frac{\pi}{2}\cdot\frac{\tan\frac{\pi n}{2}}{e^{\pi n}+1}$

$$+\frac{n^2}{1(1^2+n^2)}+\frac{n^2}{3(3^2+n^2)}+\frac{n^2}{5(5^2+n^2)}+\&c$$

$$-\left(\frac{4n^2}{1^4-n^4}\cdot\frac{1}{e^{\pi}+1}+\frac{12n^2}{3^4-n^4}\cdot\frac{1}{e^{3\pi}+1}+\&c\right).$$

(9). $\frac{1}{n+1}-\frac{1}{n+3}+\frac{1}{n+5}-\frac{1}{n+7}+\&c$

$$=\frac{1}{2n}-\frac{\pi}{2}\cdot\frac{\sec\frac{\pi n}{2}}{e^{\pi n}-1}+$$

$$2n\left\{\frac{1}{1^2-n^2}\cdot\frac{1}{e^{\pi}-1}-\frac{1}{3^2-n^2}\cdot\frac{1}{e^{3\pi}-1}+\&c\right\}$$

$$+2n\left\{\frac{1}{2^2+n^2}\cdot\frac{1}{e^{\pi}+e^{-\pi}}+\frac{1}{4^2+n^2}\cdot\frac{1}{e^{2\pi}+e^{-2\pi}}+\&c\right\}$$

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$$(1+e^{-\pi n})(1+e^{-3\pi n})(1+e^{-5\pi n}) \text{ \&c}$$

$$= \frac{\sqrt[4]{2}}{\sqrt[24]{G_n e^{\pi n}}}.$$

$$(1-e^{-\pi n})(1-e^{-3\pi n})(1-e^{-5\pi n}) \text{ \&c}$$

$$= \frac{\sqrt[4]{2}}{\sqrt[24]{g_n e^{\pi n}}}. \quad \text{then}$$

$$g_n G_n = 64 g_{2n} \text{ and } 4 = 4\sqrt[3]{\frac{G}{g}} + \sqrt[3]{gG^2}$$

✓1. $G = 1.$

✓3. $G = \frac{1}{4}$

✓5 $G = (\sqrt{5}-2)^2$

✓7 $G = \frac{1}{64}$

✓9. $G = (2-\sqrt{3})^4$

$\{$✓11 $G^3 - G^2 + G = \frac{1}{2}\}$

✓13 $G = \left(\frac{\sqrt{13}-3}{2}\right)^6.$

✓15. $G = \frac{1}{64}\cdot\left(\frac{\sqrt{5}\pm 1}{2}\right)^8$

✓17. $G = \left(\sqrt{\frac{5+\sqrt{17}}{8}} - \sqrt{\frac{\sqrt{17}-3}{8}}\right)^{2}$

$\{$✓19 $G^3 + G^2 = \frac{1}{2}.\}$

✓21. $G = (8-3\sqrt{7})^2\left(\frac{5\pm\sqrt{21}}{2}\right)$

$\{$✓23. $G^3 + G^2 = 1\}$

✓25 $G = (\sqrt{5}-2)^8$

✓27 $G = \frac{1}{4}(\sqrt[3]{2}-1)^8.$

or $\{G^3 + G^2\sqrt[3]{3} = \frac{1}{2}\}$

$\sqrt{31}.$ $\left\{ G^3 + G = 1 \right\}$

$\sqrt{33}$ $G = (2-\sqrt{3})^6 (10 \pm 3\sqrt{11})^2.$

$\sqrt{37}.$ $G = (\sqrt{37}-6)^6.$

$\sqrt{39}$ $G = \frac{1}{64} \cdot \left(\frac{\sqrt{13}-3}{2}\right)^4 \left(\sqrt{\frac{5+\sqrt{13}}{8}} \pm \sqrt{\frac{\sqrt{13}-3}{8}}\right)^{24}$

$\sqrt{43}$ $\left\{ G^3 + G = \frac{4}{2} \right\}$

$\sqrt{45}$ $G = (\sqrt{5}-2)^6 (4 \pm \sqrt{15})^4.$

$\sqrt{49}$ $G = \left(\frac{\sqrt{4+\sqrt{7}} - \sqrt[4]{7}}{2}\right)^{24}$

$\sqrt{55}$ $G = \frac{1}{64} (\sqrt{5}-2)^4 \left(\sqrt{\frac{7+\sqrt{5}}{8}} \pm \sqrt{\frac{\sqrt{5}-1}{8}}\right)^{24}$

$\sqrt{57}$ $G = \left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^4 (2 \pm \sqrt{3})^6.$

$\sqrt{63}$ $G = \frac{1}{64} \cdot \left(\frac{5-\sqrt{21}}{2}\right)^4 \left(\sqrt{\frac{5+\sqrt{21}}{8}} - \sqrt{\frac{\sqrt{21}-3}{8}}\right)^{24}$

$\sqrt{65}$ $G = \left(\frac{\sqrt{13} \pm 3}{2}\right)^6 (\sqrt{5} \pm 2)^2 \left(\sqrt{\frac{9+\sqrt{65}}{8}} - \sqrt{\frac{1+\sqrt{65}}{8}}\right)^{12}$

$\sqrt{67}$ $\left\{ G^3 + G^2 + G = \frac{4}{2} \right\}$

$\sqrt{69}.$ $G = \left(\frac{5 \pm \sqrt{23}}{\sqrt{2}}\right)^2 \left(\frac{3\sqrt{3} \pm \sqrt{23}}{2}\right)^3 \left(\sqrt{\frac{6+3\sqrt{3}}{4}} - \sqrt{\frac{2+3\sqrt{3}}{4}}\right)^{12}$

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$\sqrt{73}.\quad G = \left(\sqrt{\frac{9+\sqrt{73}}{8}} - \sqrt{\frac{1+\sqrt{73}}{8}}\right)^{24}$

$\sqrt{77}.\quad G = (8 \pm 3\sqrt{7})^{3}\left(\frac{\sqrt{11} \pm \sqrt{7}}{2}\right)^{3}\left(\sqrt{\frac{6+\sqrt{11}}{4}} - \sqrt{\frac{2+\sqrt{11}}{4}}\right)^{12}$

$\sqrt{81}\quad G = \left(\frac{\sqrt[3]{2(\sqrt{3}-1)} - 1}{\sqrt[3]{2(\sqrt{3}+1)} + 1}\right)^{8}.$

$\sqrt{85}\quad G = (\sqrt{5} \pm 2)^{8}\left(\frac{\sqrt{85}-9}{2}\right)^{6}$

$\sqrt{93}\quad G = \left(\frac{39-7\sqrt{31}}{\sqrt{2}}\right)^{4}\left(\frac{\sqrt{31} \pm 3\sqrt{3}}{2}\right)^{6}$

$\sqrt{97}\quad G = \left(\sqrt{\frac{13+\sqrt{97}}{8}} - \sqrt{\frac{5+\sqrt{97}}{8}}\right)^{24}$

$\sqrt{105}.\quad \left(\frac{5-\sqrt{21}}{2}\right)^{6}(2 \pm \sqrt{3})^{6}(\sqrt{5} \pm 2)^{4}(6 \pm \sqrt{35})^{2}$

$\sqrt{165}.\quad (4-\sqrt{15})^{6}(3\sqrt{5} \pm 2\sqrt{11})^{4}\left(\frac{\sqrt{15} \pm \sqrt{11}}{2}\right)^{6}(\sqrt{5} \pm 2)^{4}.$

$\sqrt{273}.\quad \left(\frac{15\sqrt{7}-11\sqrt{13}}{\sqrt{2}}\right)^{4}\left(\frac{\sqrt{13} \pm 3}{2}\right)^{12}\left(\frac{\sqrt{7} \pm \sqrt{3}}{2}\right)^{12}(2 \pm \sqrt{3})^{6}.$

$\sqrt{301}.\quad (8 \pm 3\sqrt{7})^{3}\left(\frac{23\sqrt{43} \pm 57\sqrt{7}}{2}\right)^{3} \times$

$$\left(\sqrt{\frac{46+7\sqrt{43}}{4}} - \sqrt{\frac{42+7\sqrt{43}}{4}}\right)^{12}$$

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$$\sqrt{141}.\ (4\sqrt{3}\pm\sqrt{47})^3\left(\frac{7\pm\sqrt{47}}{\sqrt{2}}\right)^2 \times \left(\sqrt{\frac{18+9\sqrt{3}}{4}}-\sqrt{\frac{14+9\sqrt{3}}{4}}\right)^{12}.$$

$$\sqrt{345}.\ \left(\frac{3\sqrt{3}-\sqrt{23}}{2}\right)^{12}\left(\frac{7\sqrt{23}\pm15\sqrt{5}}{\sqrt{2}}\right)^4(\sqrt{5}\pm2)^8(2\pm\sqrt{3})^6$$

$$\sqrt{289}.\ \left\{\sqrt{\frac{17+\sqrt{17}+(5+\sqrt{17})\sqrt[4]{17}}{16}}-\sqrt{\frac{1+\sqrt{17}+(5+\sqrt{17})\sqrt[4]{17}}{16}}\right\}^{48}$$

$$\sqrt{357}.\ \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{24}(8\pm3\sqrt{7})^6\left(\frac{11\pm\sqrt{119}}{\sqrt{2}}\right)^4\left(\frac{\sqrt{21}\pm\sqrt{17}}{2}\right)^6$$

$$\sqrt{385}.\ (10-3\sqrt{11})^6(6\pm\sqrt{35})^6\left(\frac{\sqrt{11}\pm\sqrt{7}}{2}\right)^{12}(\sqrt{5}\pm2)^8.$$

$$\sqrt{445}.\ (\sqrt{5}-2)^{12}\left(\frac{\sqrt{445}-21}{2}\right)^6\left(\sqrt{\frac{13+\sqrt{89}}{8}}\pm\sqrt{\frac{5+\sqrt{89}}{8}}\right)^{24}$$

$$\sqrt{505}\ (\sqrt{5}-2)^{14}(\sqrt{101}-10)^6\left(\frac{5\sqrt{5}+\sqrt{101}}{4}-\sqrt{\frac{105+5\sqrt{105}}{8}}\right)^{12}$$

$$\sqrt{441}.\ \left(\frac{\sqrt{4+\sqrt{7}}-\sqrt[4]{7}}{2}\right)^{24}\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^{12}(2-\sqrt{3})^4 \times \left(\frac{\sqrt{3+\sqrt{7}}-\sqrt[4]{6\sqrt{7}}}{\sqrt{3+\sqrt{7}}+\sqrt[4]{6\sqrt{7}}}\right)^{12}$$

$$\sqrt{553}.$$

$$\left(\sqrt{\frac{143+16\sqrt{79}}{2}}-\sqrt{\frac{141+16\sqrt{79}}{2}}\right)^{12}\left(\sqrt{\frac{100+11\sqrt{79}}{4}}\pm\sqrt{\frac{96+11\sqrt{79}}{4}}\right)^{12}$$

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$$\sqrt{117}.\quad \left(\frac{\sqrt{13}-3}{2}\right)^6 (\sqrt{13}-2\sqrt{3})^4 \left(\frac{\sqrt{4+\sqrt{3}}\pm\sqrt[4]{3}}{2}\right)^{24}$$

$$\sqrt{133}.\quad (8-3\sqrt{7})^6\left(\frac{5\sqrt{7}\pm3\sqrt{19}}{2}\right)^6$$

$$\sqrt{153}\quad \left(\sqrt{\frac{5+\sqrt{17}}{8}}-\sqrt{\frac{\sqrt{17}-3}{8}}\right)^{48}\left(\sqrt{\frac{37+9\sqrt{17}}{4}}\pm\sqrt{\frac{33+9\sqrt{17}}{4}}\right)$$

$$\sqrt{145}\quad (\sqrt{5}-2)^6\left(\frac{\sqrt{29}-5}{2}\right)^6\left(\sqrt{\frac{17+\sqrt{145}}{8}}\pm\sqrt{\frac{9+\sqrt{145}}{8}}\right)^{12}$$

$$\sqrt{177}\quad \left(\frac{3\sqrt{59}\pm23}{\sqrt{2}}\right)^4(2-\sqrt{3})^{18}.$$

$$\sqrt{213}\quad \left(\frac{59\pm7\sqrt{71}}{\sqrt{2}}\right)^2\left(\frac{5\sqrt{3}\pm\sqrt{71}}{2}\right)^3\left(\sqrt{\frac{21+12\sqrt{3}}{2}}-\sqrt{\frac{19+12\sqrt{3}}{2}}\right)$$

$$\sqrt{217}\quad \left(\sqrt{\frac{11+4\sqrt{7}}{2}}-\sqrt{\frac{9+4\sqrt{7}}{2}}\right)^{12}\left(\sqrt{\frac{16+5\sqrt{7}}{4}}\pm\sqrt{\frac{12+5\sqrt{7}}{4}}\right)$$

$$\sqrt{205}\quad (\sqrt{5}-2)^8\left(\frac{3\sqrt{5}-\sqrt{41}}{2}\right)^6\left(\sqrt{\frac{7+\sqrt{41}}{8}}\pm\sqrt{\frac{\sqrt{41}-1}{8}}\right)$$

$$\sqrt{253}\quad (24-5\sqrt{23})^6\left(\frac{9\sqrt{23}\pm13\sqrt{11}}{2}\right)^6.$$

$$\sqrt{265}.\quad \left(\frac{\sqrt{53}\pm7}{2}\right)^6(\sqrt{5}\pm2)^6\left(\sqrt{\frac{89+5\sqrt{265}}{8}}-\sqrt{\frac{81+5\sqrt{265}}{8}}\right)$$

$$\sqrt{147}.\quad \frac{1}{4}\left\{\frac{1\pm\left(2\sqrt[6]{\frac{28}{27}}-\sqrt{\frac{7}{3}}\right)}{2}\right\}^{24}.$$

$g_2 = 1$; $g_6 = (\sqrt{2}-1)^4$; $g_{10} = (\sqrt{5}-2)^4$;

$\sqrt{14}$. $\left(\sqrt{\frac{3+\sqrt{2}}{4}} - \sqrt{\frac{\sqrt{2}-1}{4}}\right)^{24}$.

$\sqrt{18}$ $(5-2\sqrt{6})^4$. $\sqrt{22}$. $(\sqrt{2}-1)^{12}$.

$\sqrt{30}$. $(\sqrt{5}-2)^4(\sqrt{10}-3)^4$. $\sqrt{58}$. $\left(\frac{\sqrt{29}-5}{2}\right)^{12}$.

$\sqrt{70}$. $(\sqrt{5}-2)^8(\sqrt{2}-1)^{12}$. $\sqrt{46}$. $\left(\sqrt{\frac{5+\sqrt{2}}{4}} - \sqrt{\frac{1+\sqrt{2}}{4}}\right)^{24}$

$\sqrt{42}$. $\left(\frac{5-\sqrt{21}}{2}\right)^6(2\sqrt{2}-\sqrt{7})^4$. $\sqrt{82}$. $\left(\sqrt{\frac{13+\sqrt{41}}{8}} - \sqrt{\frac{5+\sqrt{41}}{8}}\right)^{24}$

$\sqrt{78}$. $\left(\frac{\sqrt{13}-3}{2}\right)^{12}(\sqrt{26}-5)^4$.

$\sqrt{102}$. $(\sqrt{2}-1)^{12}(3\sqrt{2}-\sqrt{17})^8$

$\sqrt{34}$. $\left(\sqrt{\frac{7+\sqrt{17}}{8}} - \sqrt{\frac{\sqrt{17}-1}{8}}\right)^{24}$

$\sqrt{130}$. $\left(\frac{\sqrt{13}-3}{2}\right)^{12}(\sqrt{5}-2)^{12}$

$\sqrt{190}$. $(\sqrt{5}-2)^{12}(\sqrt{10}-3)^{12}$.

$\sqrt{142}$. $\left(\sqrt{\frac{11+5\sqrt{2}}{4}} - \sqrt{\frac{7+5\sqrt{2}}{4}}\right)^{24}$.

$\sqrt{90}$. $(\sqrt{5}-2)^4 \cdot (\sqrt{6}-\sqrt{5})^4\left(\sqrt{\frac{3+\sqrt{6}}{4}} - \sqrt{\frac{\sqrt{6}-1}{4}}\right)^{24}$

$\sqrt{198}$. $(\sqrt{2}-1)^{12}(4\sqrt{2}-\sqrt{33})^4\left(\sqrt{\frac{9+\sqrt{33}}{8}} - \sqrt{\frac{1+\sqrt{33}}{8}}\right)^{24}$.

79 th $\sqrt{\frac{1}{t} - t} = u$

then $u^5 - 2u^4 + u^3 + 2u - 3 = 0$

163. $t^3 - 2t^2 + 3t = \frac{1}{2}$.

$$\sqrt{2}\cdot\sqrt{\begin{array}{c}\phi(x)\,\phi(x^7)\,\phi(x^9)\,\phi(x^{63}) + \phi(-x)\,\phi(-x^7)\,\phi(-x^9)\,\phi(-x^{63}) \\ + 4x^4 f^2(-x^6) f^2(-x^{42})\end{array}}$$

$$= \phi(x)\phi(x^{63}) + \phi(-x)\phi(-x^{63}) + 4x^{16}\psi(x^2)\psi(x^{126}).$$

If $\phi(x) = 1 + 6\left(\frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} - \frac{x^5}{1-x^5} + \cdots\right)$

then $\phi(x) + \phi(-x) = 2\phi(x^4)$

$\phi^2(x) + \phi(x)\phi(-x) + \phi^2(-x) = 3\phi^2(x^{2})$.

$$\frac{1}{x^{\frac{3}{7}}}\frac{f(-x^3,-x^4)}{f(-x,-x^6)} - 1 - x^{\frac{1}{7}}\frac{f(-x^2,-x^5)}{f(-x^3,-x^4)} + x^{\frac{2}{7}}\frac{f(-x,-x^6)}{f(-x^2,-x^5)}$$

$$= \frac{1}{2}\left\{\frac{3f(-x^{\frac{1}{7}})}{x^{\frac{2}{7}}f(-x^7)} + \sqrt{\frac{4f^3(-x^{\frac{1}{7}})}{x^{\frac{6}{7}}f^3(-x^7)} + \frac{21f^2(-x^{\frac{1}{7}})}{x^{\frac{4}{7}}f^2(-x^7)} + \frac{28f(-x^{\frac{1}{7}})}{x^{\frac{2}{7}}f(-x^7)}}\right.$$

VII :—

$$\frac{u^2}{w} + \frac{v^2}{u} - \frac{w^2}{v} = 8 + \frac{f^4(-x)}{xf^4(-x^7)}$$

$$\frac{v}{w^2} - \frac{u}{v^2} - \frac{w}{u^2} = 5 + \frac{f^4(-x)}{xf^4(-x^7)}.$$

$u = x^{\frac{1}{56}}f(-x^3,-x^4)$ $\quad v = x^{\frac{9}{56}}f(-x^2,-x^5)$, $\quad w = x^{\frac{25}{56}}f(-x,-x^6)$

then $\frac{u^2}{v} - \frac{v^2}{w} + \frac{w^2}{u} = 0$

$$uvw = x^{\frac{5}{8}}f(-x)f^2(-x^7)$$

$$\frac{v}{u^2} - \frac{w}{v^2} + \frac{u}{w^2} = \frac{f(-x)}{x^{\frac{3}{8}}f^2(-x^7)}\cdot\sqrt[3]{\frac{f^4(-x)}{f^4(-x^7)} + 13x + 49x^2\frac{f^4(-x^7)}{f^4(-x)}}$$

1

$$P = \frac{f(-x)}{x^{\frac{1}{4}} f(-x^7)}; \quad Q = \frac{f(-x^5)}{x^{\frac{5}{4}} f(-x^{35})}.$$

$$(PQ)^2 - 5 + \frac{49}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - 5\left(\frac{Q}{P}\right)^2 - 5\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^3$$

(2)

$$\sqrt{a-\sqrt{a+\sqrt{a+\sqrt{a-\sqrt{a+\sqrt{a+\sqrt{a-\sqrt{a+\&c}}}}}}}}$$

$$=\frac{\sqrt{4a-7}-1}{6}+\frac{2}{3}\sqrt{4a+\sqrt{4a-7}}\;\sin\left(\frac{1}{3}\tan^{-1}\frac{1+2\sqrt{4a-7}}{3\sqrt{3}}\right)$$

$$\sqrt{a+\sqrt{a-\sqrt{a+\sqrt{a+\sqrt{a-\&c}}}}}$$

$$=\frac{\sqrt{4a-7}-1}{6}+\frac{2}{3}\sqrt{4a+\sqrt{4a-7}}\;\sin\left(\frac{\pi}{3}-\frac{1}{3}\tan^{-1}\frac{1+2\sqrt{4a-7}}{3\sqrt{3}}\right)$$

$$\sqrt{a+\sqrt{a+\sqrt{a-\sqrt{a+\sqrt{a+\sqrt{a-\&c}}}}}}$$

$$=\frac{1-\sqrt{4a-7}}{6}+\frac{2}{3}\sqrt{4a+\sqrt{4a-7}}\;\sin\left(\frac{\pi}{3}+\frac{1}{3}\tan^{-1}\frac{1+2\sqrt{4a-7}}{3\sqrt{3}}\right)$$

$$\sqrt{a\,\cancel{+}\sqrt{a+\sqrt{a-\sqrt{a\mp\sqrt{a+\sqrt{a-\&c}}}}}}$$

$$=\frac{1+\sqrt{4a-7}}{6}+\frac{2}{3}\sqrt{4a-\sqrt{4a-7}}\;\sin\left(\frac{1}{3}\tan^{-1}\frac{2\sqrt{4a-7}-1}{3\sqrt{3}}\right)$$

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(3)

$$\sqrt{a+\sqrt{a\mp\sqrt{a-\sqrt{a+\&c}}}}$$

$$= \frac{1+\sqrt{4a-7}}{6} + \frac{2}{3}\sqrt{4a-\sqrt{4a-7}}\,\sin\left(\frac{\pi}{3}-\frac{1}{3}\tan^{-1}\frac{2\sqrt{4a-7}-1}{3\sqrt{3}}\right)$$

$$\sqrt{a-\sqrt{a-\sqrt{a+\sqrt{a-\&c}}}}$$

$$= -\frac{1+\sqrt{4a-7}}{6} + \frac{2}{3}\sqrt{4a-\sqrt{4a-7}}\,\sin\left(\frac{\pi}{3}+\frac{1}{3}\tan^{-1}\frac{2\sqrt{4a-7}-1}{3\sqrt{3}}\right)$$

$$\frac{1^2}{1^3+x^3} - \frac{2^2}{2^3+x^3} + \frac{3^2}{3^3+x^3} - \frac{4^2}{4^3+x^3} + \&c$$

$$= \frac{1}{3}\left(\frac{1}{1+x} - \frac{1}{2+x} + \frac{1}{3+x} - \&c\right)$$

$$+\frac{4}{3}\left\{\frac{2-x}{(2-x)^2+3x^2} - \frac{4-x}{(4-x)^2+3x^2} + \frac{6-x}{(6-x)^2+3x^2} - \&c\right\}$$

$$e^{-x} + e^{-ax} + e^{-a^2x} + \&c = -\frac{C + \log\frac{x}{\sqrt{a}}}{\log a} \text{ nearly}$$

$$\left.\begin{matrix} e^{-x} + e^{-ax} + \cdots \\ + e^{-bx} + e^{-abx} \\ + e^{-b^2x} + \cdots \end{matrix}\right\} = \frac{\left(C + \log\frac{x}{\sqrt{ab}}\right)^2}{2\log a\log b} - \frac{(\log a)^2 + (\log b)^2}{24\log a\log b}$$

$$\left.\begin{matrix} e^{-x} + e^{-ax} \\ e^{-bx} \\ e^{-cx} \end{matrix}\right\} = -\frac{\left(C + \log\frac{x}{\sqrt{abc}}\right)^3}{6\log a\log b\log c} + \frac{(\log a)^2 + (\log b)^2 + (\log c)^2}{24\log a\log b\log c}\left(C + \log\frac{x}{\sqrt{abc}}\right)$$

$x^2 = a + y$, $y^2 = a + z$ and $z^2 = a + x$ [illegible]

$$x^3 + x^2\,\frac{1+\sqrt{4a-7}}{2} - x\,\frac{2a+1-\sqrt{4a-7}}{2} - \text{[illegible]}$$

$$x^3 + x^2\,\frac{1-\sqrt{4a-7}}{2} - x\,\frac{2a+1+\sqrt{4a-7}}{2} - \text{[illegible]}$$

The no. of prime nos. between A and B

$$= \int_A^B \frac{dx}{\log x} \text{ nearly}$$

The no. of numbers of two squares between A and B

$$= C\int_A^B \frac{dx}{\sqrt{\log x}} \text{ nearly} \quad \text{where } C = .764\ldots$$

[illegible]

~~If a prime~~

Primes of the form. &c.

$4n-1$ —— $4n+1$
$4n+1$ —— $4n+1$ } Hence the no of prime nos of the form $4n+1$ is less.

$6n-1$ —— $6n+1$
$6n+1$ —— $6n+1$ } Similarly for $6n+1$.

$8n+1$ —— $8n+1$
$8n+3$ —— $8n+1$
$8n+5$ —— $8n+1$
$8n+7$ —— $8n+1$ } $8n+1$ —— the least
The rest are equal.

$10n+1$ —— $10n+1$
$10n+3$ —— $10n+9$
$10n+7$ —— $10n+9$
$10n+9$ —— $10n+1$ } $\left.\begin{matrix}10n+1\\10n+9\end{matrix}\right\}$ less than $\left\{\begin{matrix}10n+3\\10n+7\end{matrix}\right.$

$12n+1$ —— $12n+1$
$12n+5$ —— $12n+1$
$12n+7$ —— $12n+1$
$12n+11$ —— $12n+1$ } $12n+1$ —— the least
The rest are equal

$24n+1$
$24n+5$
$24n+7$
$24n+11$
$24n+13$
$24n+17$
$24n+19$
$24n+23$ } —— $24n+1$ —— the least
and the rest are equal

when x becomes unity

$$\frac{f(-x^m, -x^n)}{\phi\left(-x^{\frac{m+n}{2}}\right)} = \sin\frac{\pi m}{m+n}$$

6

$$\sqrt{2\left(1-\tfrac{1}{3^2}\right)\left(1-\tfrac{1}{7^2}\right)\left(1-\tfrac{1}{11^2}\right)\left(1-\tfrac{1}{19^2}\right)}$$

$$= \left(1+\tfrac{1}{7}\right)\left(1+\tfrac{1}{11}\right)\left(1+\tfrac{1}{19}\right).$$

All nos of the form $a^p b^q$ within n

$$= \frac{1}{2}\ \frac{\log an \ \log bn}{\log a \ \log b} \ \left(+\tfrac{1}{2} \text{ if } n \text{ of the reqd f}\right)$$

Sol.

$$1 + \frac{1}{a^p} + \frac{1}{a^{2p}} + \frac{1}{a^{3p}} +$$

$$= \frac{1}{p\log a} + \frac{1}{2} + \&c$$

$$1 + \frac{1}{b^p} + \frac{1}{b^{2p}} + \&c$$

$$= \frac{1}{p\log b} + \frac{1}{2} + \&c$$

If the reqd no of such nos $= x$

then $\displaystyle\int^\infty \frac{dx}{n^p} = \int^\infty \frac{dn}{n^{p+1}}\left(\frac{\log n}{\log a \log b} + \frac{1}{2}\log a + \frac{1}{2}\log b\right)$

when $p = 0$

$$\therefore \frac{dx}{dn} = \frac{\log n}{n\log a \log b} + \frac{1}{2n}\log a + \frac{1}{2n}\log b.$$

$$\therefore x = \frac{1}{2}\ \frac{\log an \ \log bn}{\log a \ \log b}$$

If $m+n = p+q = k$

$$\frac{f(-x^m, -x^n)}{f(x^p, -x^q)} = \frac{\sin \frac{\pi m}{k}}{\sin \frac{\pi p}{k}} \text{ when } x = 1.$$

Correct

I Of n continuous numbers at least $5/6$ of them can be expressed as the sum of 3 sqrs.

II Of n even numbers at least $\frac{11}{12}$ of them can be expressed as the sum of 3 sqrs.

III Of n odd numbers at least $\frac{3}{4}$ of them can be expressed as the sum of 3 sqrs.

If $a_1 e^{-x} + a_2 e^{-2x} + a_3 e^{-3x} + \&c$

$$= \int_0^\infty e^{-nx} u_n \, dx + (a_1 + a_2 + a_3 + \&c) + \&c$$

then the average value of $a_n = u_n$ exactly

The average value of the no of divisors of n

$$= 2C + \log n \text{ exactly}$$

and that of the sum of the divisors of n

$$= \frac{\pi^2}{6} n - \frac{1}{2} \text{ exactly}$$

The no of prime nos less than N is less than $\sqrt{\frac{eN}{\log N}}$. The no of primes less than $\frac{N}{e}$.

$$\frac{\phi^3(x^{\frac{1}{3}})}{\phi(x)} = \frac{\phi^3(x)}{\phi(x^3)} + 6x^{\frac{1}{3}} \frac{f^3(x^3)}{f(x)} + 12x^{\frac{2}{3}} \frac{f^3(-x^6)}{f(-x^2)}$$

$$\frac{\psi^3(x^{\frac{1}{3}})}{\psi(x)} = \frac{\psi^3(x)}{\psi(x^3)} + 3x^{\frac{1}{3}} \frac{f^3(-x^3)}{f(-x)} + 3x^{\frac{2}{3}} \frac{f^3(-x^6)}{f(-x^2)}$$

If a prime no of the form $An+B$ can be expressed as ax^2-by^2, then a prime no of the form $An-B$ can be expressed as bx^2-ay^2. 8

All nos can be expressed as the sum of 4 p. squares

All nos except of the form $2^{\nu}(8n-1)$ can be expressed as the sum of 3 perfect sqs.

All nos of the form $2^p \cdot 3^{2q} \cdot 5^r \cdot 7^{2s} \cdot 11^{2t} \cdot 13^u$ &c can be expressed as the sum of 2 sqs.

p, q, r, s, t may have all integral values including 0

A prime no of the form	can be expressed as.
$4n+1$	x^2+y^2
$8n+1,\ 8n+3$	x^2+2y^2
$8n+1,\ 8n-1$	x^2-2y^2
$6n+1$	x^2+3y^2
$12n+1$	x^2-3y^2
$20n+1,\ 20n+9$	x^2+5y^2
$10n+1,\ 10n+9$	x^2-5y^2
$14n+1,\ 14n+9,\ 14n+25$	x^2+7y^2
$28n+1,\ 28n+9,\ 28n+25$	x^2-7y^2

$$\frac{1}{p^2+1} - \frac{2}{p^2+4} - \frac{3}{p^2+9} - \frac{5}{p^2+25} + \frac{6}{p^2+36} - \&c$$

$$= \frac{\pi}{p}\left(e^{-\frac{2\pi}{p}} - \frac{1}{2}e^{-\frac{2\pi}{2p}} - \frac{1}{3}e^{-\frac{2\pi}{3p}} \&c\right)$$

$$e^{-\frac{p}{1}} - \frac{1}{2}e^{-\frac{p}{4}} - \frac{1}{3}e^{-\frac{p}{9}} - \frac{1}{5}e^{-\frac{p}{25}} + \&c$$

$$= \sqrt{\frac{\pi}{p}}\left(e^{-\frac{\pi^2}{p}} - \frac{1}{2}e^{-\frac{\pi^2}{4p}} - \frac{1}{3}e^{-\frac{\pi^2}{9p}} - \&c\right)$$

If $\int_0^\infty \phi(x)\cos nx\,dx = \psi(n)$

then $\frac{\alpha}{2}\left\{\phi(\alpha) - \frac{1}{2}\phi\left(\frac{\alpha}{2}\right) - \frac{1}{3}\phi\left(\frac{\alpha}{3}\right) - \frac{1}{5}\phi\left(\frac{\alpha}{5}\right) + \&c\right\}$

$$= \psi(\beta) - \frac{1}{2}\psi\left(\frac{\beta}{2}\right) - \frac{1}{3}\psi\left(\frac{\beta}{3}\right) - \frac{1}{5}\psi\left(\frac{\beta}{5}\right) + \&c$$

with the condition $\alpha\beta =$

$$\frac{p}{S_1}\phi(1) - \frac{p^3}{S_3}\phi(3) + \frac{p^5}{S_5}\phi(5) - \frac{p^7}{S_7}\phi(7) + \&c$$

$$= \pi\left\{\frac{\phi(0)}{S_1} - \frac{2\pi}{p}\cdot\frac{\phi(-1)}{S_2\,\underline{|1}} + \left(\frac{2\pi}{p}\right)^2\frac{\phi(-2)}{S_3\,\underline{|2}} - \&c\right\}$$

$$\frac{p}{S_1}\cdot\frac{\phi(1)}{\underline{|0}} - \frac{p^3}{S_3}\frac{\phi(3)}{\underline{|1}} + \frac{p^5}{S_5}\frac{\phi(5)}{\underline{|2}} - \&c$$

$$= \sqrt{\pi}\left\{\frac{\phi(0)}{S_1} - \left(\frac{\pi}{p}\right)^2\frac{\phi(-2)}{S_3\,\underline{|1}} + \left(\frac{\pi}{p}\right)^4\frac{\phi(-4)}{S_5\,\underline{|2}} - \&c\right\}$$

$30n+1,\ 30n+49$ ——— x^2+15y^2

$60n+1,\ 60n+49$ ——— x^2-15y^2

$30n-7,\ 30n+17$ ——— $5x^2+3y^2$

$60n-7,\ 60n+17$ ——— $5x^2-3y^2$

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$24n+1,\ 24n+7$ ——— x^2+6y^2

$24n+1,\ 24n+19$ ——— x^2-6y^2

$24n+5,\ 24n+11$ ——— $2x^2+3y^2$

$24n+5,\ 24n-1$ ——— $2x^2-3y^2$

If G be the G.C.M of any one of $2^p-1,\ 2^p,\ 2^p+1$ and any one of $3^p-1,\ 3^p,\ 3^p+1$

Then the pth power of any integer ± 1 is divisible by G.

If $u_1+u_2+u_3+$ &c and $v_1+v_2+v_3+$ &c are both divergent but the diff.ce between the 2 series finite then u_n and v_n are nearly equal and also $\frac{1}{u_n}$ and $\frac{1}{v_n}$ are nearly equal for when n is great.

If $P=\dfrac{f(-x)f(-x^2)}{x^{\frac{1}{2}}f(-x^5)f(-x^{10})}$ and $Q=\dfrac{f(-x^3)f(-x^6)}{x^{\frac{3}{2}}f(-x^{15})f(-x^{30})}$

then $PQ+\dfrac{25}{PQ}=\left(\dfrac{Q}{P}\right)^2+\left(\dfrac{P}{Q}\right)^2-3\left(\dfrac{Q}{P}+\dfrac{P}{Q}+2\right)$

$$\frac{\pi}{2}\cdot\frac{p^x\,\phi(x)}{S_x\cos\frac{\pi x}{2}} = \frac{p}{S_1}\cdot\frac{\phi(1)}{1-x} - \frac{p^3}{S_3}\cdot\frac{\phi(3)}{3-x} + \cdots$$

$$+\pi\left\{\frac{\phi(0)}{S_1 x} - \left(\frac{2\pi}{p}\right)\cdot\frac{\phi(-1)}{S_2\lfloor 1\,(1+x)} + \left(\frac{2\pi}{p}\right)^2\frac{\phi(-2)}{S_3\lfloor 2\,(2+x)}\right.$$

+ terms involving roots of S_x

$$\frac{\pi}{2}\cdot\frac{p^x\,\phi(x)}{S_x\,\lfloor\frac{x-1}{2}\,\cos\frac{\pi x}{2}} = \text{Terms involving roots of } S_x$$

$$+\frac{p}{S_1}\,\frac{\phi(1)}{1-x} - \frac{p^3}{S_3\lfloor 1}\,\frac{\phi(3)}{3-x} + \frac{p^5}{S_5\lfloor 2}\,\frac{\phi(5)}{5-x}$$

$$+\sqrt{\pi}\left\{\frac{\phi(0)}{S_1 x} - \left(\frac{\pi}{p}\right)^2\frac{\phi(-2)}{S_3\lfloor 1\,(2+x)} + \left(\frac{\pi}{p}\right)^4\frac{\phi(-4)}{S_5\lfloor 2\,(4+x)} - \cdots\right.$$

If the perimeter of an ellipse $= \pi(a+b)(1+h)$, then

$$\left(\frac{a-b}{a+b}\right)^2 = 4h - \frac{3h^2}{2+\sqrt{1-3h}}$$ very nearly.

according to the above app^n the perimeter of a parabola $= 3.99944(a+b)$ for $4(a+b)$.

If $u = \dfrac{f(-x)f(-x^2)}{x^{\frac{1}{4}}f(-x^3)f(-x^6)}$ and $v = \dfrac{f(-x^{\frac{1}{3}})f(-x^{\frac{2}{3}})}{x^{\frac{1}{3}}f(-x^3)f(-x^6)}$, then

$$u^4 = v^3 + 3v^2 + 9v$$

$$x\log 1 + x^2\log 2 + x^3\log 3 + x^4\log 4 + \&c$$

$$= \left(\frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} + \frac{x^8}{1-x^8} + \&c\right)\log 2$$

$$+ \left(\frac{x^3}{1-x^3} + \frac{x^9}{1-x^9} + \frac{x^{27}}{1-x^{27}} + \&c\right)\log 3$$

$$+ \left(\frac{x^5}{1-x^5} + \frac{x^{25}}{1-x^{25}} + \frac{x^{125}}{1-x^{125}} + \&c\right)\log 5$$

$$+ \quad \&c \qquad \&c \qquad \&c$$

$$\log 3\left(\frac{1}{e^{3x}+1} + \frac{1}{e^{9x}+1} + \frac{1}{e^{27x}+1} + \&c\right)$$

$$+ \log 5\left(\frac{1}{e^{5x}+1} + \frac{1}{e^{25x}+1} + \frac{1}{e^{125x}+1} + \&c\right)$$

$$+ \log 7\left(\frac{1}{e^{7x}+1} + \frac{1}{e^{49x}+1} + \frac{1}{e^{343x}+1} + \&c\right)$$

$$+ \log 11\left(\frac{1}{e^{11x}+1} + \&c \qquad \&c\right)$$

$$\&c \qquad \&c$$

$$= \log 2\left(\frac{1}{e^{2x}-1} + \frac{1}{e^{4x}-1} + \frac{1}{e^{8x}-1} + \&c\right)$$

$$+ e^{-x}\log 1 - e^{-2x}\log 2 + e^{-3x}\log 3 \ldots$$

$$\log 2\,(e^{-2x}+e^{-4x}+e^{-8x}+\&c)$$
$$+\log 3\,(e^{-3x}+e^{-9x}+e^{-27x}+\&c)$$
$$+\log 5\,(e^{-5x}+e^{-25x}+e^{-125x}+\&c)$$
$$+\quad \&c \quad \&c \quad \&c$$

$\left\{ 2, 3, 5, 7 \ \&c \right.$ being prime nos

$$= \log 2\,(2e^{-2x}+4e^{-4x}+8e^{-8x}+16e^{-16x}+\&c) + \phi(x).$$

where $\phi(x) - \phi(2x) + \phi(3x) - \phi(4x) + \&c$

$$= e^{-x}\log 1 - e^{-2x}\log 2 + e^{-3x}\log 3 - e^{-4x}\log 4 + \&c$$

$$\frac{\log 2}{2^n-1} + \frac{\log 3}{3^n-1} + \frac{\log 5}{5^n-1} + \frac{\log 7}{7^n-1} + \frac{\log 11}{11^n-1} + \&c$$

$= \frac{1}{n-1}$, nearly.

Hence $\frac{\log 2}{2^n} + \frac{\log 3}{3^n} + \frac{\log 5}{5^n} + \frac{\log 7}{7^n} + \&c =$

$$\frac{1}{n-1} - \frac{1}{2n-1} - \frac{1}{3n-1} - \frac{1}{5n-1} + \frac{1}{6n-1} - \frac{1}{7n-1} + \frac{1}{10n-1}$$

From which we infer

$$\int \frac{\log p}{p^k}\,dn = \int \frac{dp}{p^k} - \frac{1}{2}\int \frac{dp}{p^{k+\frac{1}{2}}} - \frac{1}{3}\int \frac{dp}{p^{k+\frac{2}{3}}} - \&c$$

Hence $\frac{dn}{dp} = \frac{1}{p\log p}\left(p - \frac{\sqrt{p}}{2} - \frac{\sqrt[3]{p}}{3} - \frac{\sqrt[5]{p}}{5} + \frac{\sqrt[6]{p}}{6} - \&c\right.$

If $f(x)=0$ and $\phi(z,x)=0$, then $\phi(\alpha,x)\,\phi(\beta,x)\,\&c$ –

where α, β, &c are the roots of $f(z)=0$

If P be a function of x such that

$$\int_0^\infty \left(e^{-aP}+e^{-aP^2}+e^{-aP^3}+\&c\right)\log P\,dx = \frac{1}{a}$$

then there will be x prime numbers within 1 and P.

$$x = \int^P \frac{dx}{\log x} - \frac{1}{2}\int^{\sqrt{P}} \frac{dx}{\log x} - \frac{1}{3}\int^{\sqrt[3]{P}} \frac{dx}{\log x} - \frac{1}{5}\int^{\sqrt[5]{P}} \frac{dx}{\log x}$$

$$+\frac{1}{6}\int^{\sqrt[6]{P}} \frac{dx}{\log x} - \frac{1}{7}\int^{\sqrt[7]{P}} \frac{dx}{\log x} + \frac{1}{10}\int^{\sqrt[10]{P}} \frac{dx}{\log x} - \&c$$

$$= \frac{2}{\pi}\left\{ \frac{2}{1 . B_2}\cdot\left(\frac{\log P}{2\pi}\right) + \frac{6}{5B_6}\cdot\left(\frac{\log P}{2\pi}\right)^5 + \frac{10}{9B_{10}}\cdot\left(\frac{\log P}{2\pi}\right)^9 + \&c\right\}$$

$$+\frac{4}{3B_4}\left(\frac{\log P}{2\pi}\right)^3 + \frac{8}{7B_8}\left(\frac{\log P}{2\pi}\right)^7$$

If $\phi(p)+\phi(2p)+\phi(3p)+\phi(4p)+\&c = \psi(p)$

then $\phi(p) = \psi(p) - \psi(2p) - \psi(3p) - \psi(5p) + \psi(6p) - \&c$

If $\phi(p) - \phi(2p) + \phi(3p) - \&c = \psi(p) + \psi(2p) + \psi(3p) + \&c$

then $\phi(p) = \psi(p) + 2\psi(2p) + 4\psi(4p) + 8\psi(8p) + \&c$

and $\psi(p) = \phi(p) - 2\phi(2p)$

$$\int_{\mu}^{x} \frac{dt}{\log t} = x\left\{ \frac{1}{\log x} + \frac{\lfloor 1}{(\log x)^2} + \frac{\lfloor 2}{(\log x)^3} + \cdots + \frac{\lfloor n-1}{(\log x)^n}\theta \right\}$$

where $\mu = 1.45136380$.

and $\theta = \left(\frac{2}{3} - \delta\right) + \frac{1}{\log x}\left\{ \frac{4}{135} - \frac{\delta^2(1-\delta)}{3} \right\} +$

$$\frac{1}{(\log x)^2}\left\{ \frac{8}{2835} + \frac{2\delta(1-\delta)}{135} - \frac{\delta(1-\delta^2)(2-3\delta^2)}{45} \right\} - \&c$$

where $n - \log x = \delta$.

$$\int_{\mu}^{x} \frac{dt}{\log t} = c + \log\log x + \frac{\log x}{1\lfloor 1} + \frac{(\log x)^2}{2\lfloor 2} + \frac{(\log x)^3}{3\lfloor 3} + \&c$$

where $c = .5772\cdots$

The no of prime nos less than $e^a = \int_0^\infty \frac{a^x}{x\lfloor x\, S_{x+1}}\,dx$

——— do ——— $e^{2\pi a} = \int_0^\infty \frac{a^x(1+x)}{2\pi x\, B_{1+x}}\,dx$

If

$$\left(1+\tfrac{1}{p}\right) + \tfrac{1}{2}\left(1+\tfrac{1}{p}\right)^2 + \tfrac{1}{3}\left(1+\tfrac{1}{p}\right)^3 + \cdots\cdots + \tfrac{1}{n}\left(1+\tfrac{1}{p}\right)^n = \log p$$

then $n = \left(p+\tfrac{1}{2}\right)\log \mu - \tfrac{1}{2}$ very nearly.

$$\int_0^\infty \frac{\phi(x)}{x^x}\,dx = \phi(0) + \frac{\phi(1)}{1^1} + \frac{\phi(2)}{2^2} + \frac{\phi(3)}{3^3} + \frac{\phi(4)}{4^4} + \&c$$

$$- \phi(-1) + 2^2\phi(-2) - 3^3\phi(-3) + \&c$$

$$\int_a^\infty \left(\frac{a}{x}\right)^x dx = 1 - \frac{1}{a} + \left(\frac{2}{a}\right)^2 - \left(\frac{3}{a}\right)^3 + \left(\frac{4}{a}\right)^4 - \&c$$

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$1 - 2 - 3 - 5 \mid + 6 - 7 \mid + 10 - 11 \mid - 13 + 14 \mid + 15 - 17 \mid$
$- 19 + 21 \mid + 22 - 23 \mid + 26 - 29 \mid - 30 - 31 + 33 + 34 \mid + 35 - 37 \mid$
$+ 38 + 39 - 41 - 42 \mid - 43 + 46 \mid - 47 + 51 \mid - 53 + 55 \mid + 57 + 58$
$- 59 - 61 \mid + 62 + 65 - 66 - 67 \mid + 69 - 70 \mid - 71 - 73 + 74 + 77 \mid$
$- 78 - 79 + 82 - 83 + 85 + 86 \mid + 87 - 89 \mid + 91 + 93 + 94 + 95 - 97$
$- 101 - 102 - 103 \mid - 105 + 106 \mid - 107 - 109 - 110 + 111 - 113 - 114 + 115$
$+ 118 + 119 + 122 \mid + 123 - 127 \mid + 129 - 130 \mid - 131 + 133 \mid + 134 - 137 \mid - 138$
$- 139 + 141 + 142 \mid - 143 + 145 + 146 - 149 - 151 - 154 \mid + 155 - 157 \mid + 158$
$+ 159 + 161 - 163 - 165 + 166 - 167 - 170 \mid - 173 - 174 + 177 + 178 \mid - 179$
$- 181 - 182 + 183 + 185 - 186 + 187 - 190 - 191 - 193 + 194 - 195 - 197 - 199$
$+ 201 + 202 + 203 + 205 + 206 + 209 \mid + 210 - 211 \mid + 213 + 214 + 215 + 217$
$+ 218 + 219 + 221 - 222 - 223 + 226 - 227 - 229 - 230 - 231 - 233 + 235$
$+ 237 - 238 - 239 - 241 \mid - 246 + 247 \mid + 249 - 251 \mid + 253 + 254 - 255 - 257 \mid$
$- 258 + 259 \mid + 262 - 263 \mid + 265 - 266 \mid + 267 - 269 \mid - 271 - 273 + 274 - 277$
$+ 278 - 281 - 282 - 283 - 285 - 286 + 287 - 290 + 291 - 293 + 295 + 298 + 299$
$+ 301 + 302 + 303 \mid + 305 - 307 \mid + 309 - 310 \mid - 311 - 313 + 314 - 317 - 318$
$- 319 + 321 - 322 + 323 + 326 + 327 \mid + 329 + 330 - 331 + 334 + 335 - 337$
$+ 339 + 341 + 345 + 346 - 347 - 349 - 353 - 354 \mid + 355 - 357 \mid + 358 - 359 \mid$
$+ 362 + 365 - 366 - 367 \mid - 370 + 371 \mid - 373 - 374 + 377 - 379 + 381 \mid + 382 \mid$
$- 383 - 385 + 386 - 387 - 389 + 390 + 391 + 393 \mid + 394 + 395 - 397 + 398 - 399$

The no of prime nos between 4 and 1000 $= 166$.

of which those

of the form $4n+1 = 80$	of the form $6n+1 = 80$
——— $4n-1 = 86$	——— $6n-1 = 86$
——— $8n+1 = 37$	——— $12n+1 = 36$
——— $8n+3 = 43$	——— $12n+5 = 44$
——— $8n+5 = 43$	——— $12n+7 = 44$
——— $8n+7 = 43$	——— $12n+11 = 42$

If $x = \alpha + \beta + \gamma(n-1)$
$y = \beta + \gamma + \alpha(n-1)$
and $z = \gamma + \alpha + \beta(n-1)$

$$\left.\begin{aligned} x &= \alpha+\beta+\gamma(n-1) \\ y &= \beta+\gamma+\alpha(n-1) \\ z &= \gamma+\alpha+\beta(n-1) \end{aligned}\right\}$$

then, $x^2 + nyz = (n^3 - 3n^2 + 4)\alpha\beta +$

$(n^2 - n + 1)(\alpha^2 + \beta^2 + \gamma^2) + (n^2 + 2n - 2)(\alpha\beta + \beta\gamma + \gamma\alpha)$

Similarly for $y^2 + nzx$ and $z^2 + nxy$ also

$$\left.\begin{aligned} x^2 + 2yz &= a \\ y^2 + 2zx &= b \\ z^2 + 2xy &= c \end{aligned}\right\}$$

Let $\theta = xy + yz + zx$ and $t = (x-y)(y-z)(z-x)$

then,
$$\left.\begin{aligned} x - y &= \frac{t}{\theta - c} \\ y - z &= \frac{t}{\theta - a} \\ z - x &= \frac{t}{\theta - b} \end{aligned}\right\} \text{ and } t^2 = (\theta - a)(\theta - b)(\theta - c).$$

$$\frac{1}{\theta - a} + \frac{1}{\theta - b} + \frac{1}{\theta - c} = 0$$

$$\left\{\begin{aligned} a + b + c &= (x + y + z)^2 \\ a^3 + b^3 + c^3 - 3abc &= (x^3 + y^3 + z^3 - 3xyz)^2. \end{aligned}\right.$$

If $x = Ap + Bq + Cr$, $y = Bp + Cq + Ar$, & $z = Cp + Aq + Br$

then $x^2 + 2yz = (A^2 + 2BC)(p^2 + 2qr) + (B^2 + 2CA)(q^2 + 2rp)$
$+ (C^2 + 2AB)(r^2 + 2pq)$.

Hence if
$$\left.\begin{aligned} x &= \frac{p - 2q - 2r}{3} \\ y &= \frac{q - 2r - 2p}{3} \\ z &= \frac{r - 2p - 2q}{3} \end{aligned}\right\} \text{ then, } \left.\begin{aligned} x^2 + 2yz &= p^2 + 2qr \\ y^2 + 2zx &= q^2 + 2rp \\ z^2 + 2xy &= r^2 + 2pq \end{aligned}\right\}$$

If $u = \cfrac{\sqrt[5]{x}}{1+\cfrac{x}{1+\cfrac{x^2}{1+\cfrac{x^3}{1+\&c}}}}$

and $v = \cfrac{\sqrt[5]{x^3}}{1+\cfrac{x^3}{1+\cfrac{x^6}{1+\cfrac{x^9}{1+\&c}}}}$

then $(v-u^3)(1+uv^3) = 3u^2v^2$.

$$\frac{f(-x^{\frac{3}{5}})f^3(-x^5)}{f(-x^{15})} = f^3(-x^7,-x^8) - x^{\frac{3}{5}}f^3(-x^5) + x^3f^3(-x^2,-x^{13}) + x^{\frac{24}{5}}f^3(-x,-x^{14}) - x^{\frac{6}{5}}f^3(-x^4,-x^{11})$$

$$f^3(-x^7,-x^8) + x^3f^3(-x^2,-x^{13}) = f^3(-x^5)\frac{f(-x^6,-x^9)}{f(-x^3,-x^{12})}$$

$$f^3(-x^4,-x^{11}) - x^3f^3(-x,-x^{14}) = f^3(-x^5)\frac{f(-x^3,-x^{12})}{f(-x^6,-x^9)}$$

$$f^3(-x^6,-x^9) - xf^3(-x^4,-x^{11}) - x^4f^3(-x,-x^{14})$$
$$= f(-x,-x^4)\left\{1+6\left(\frac{x^5}{1+x^5+x^{10}}+\&c\right)\right\}$$

$$f^3(-x^7,-x^8) - x^2f^3(-x^3,-x^{12}) - x^8f^3(-x^2,-x^{13})$$
$$= f(-x^2,-x^3)\left\{1+6\left(\frac{x^5}{1+x^5+x^{10}}+\&c\right)\right\}.$$

$$f^3(-x^4,-x^5) - xf^3(-x^2,-x^7) - x^2f^3(-x,-x^8)$$
$$= f(-x)\left\{1+6\left(\frac{x^3}{1+x^3+x^6}+\&c\right)\right\}$$

$$f(a,b)\left\{1+6\left(\frac{ab}{1-ab}-\frac{a^2b^2}{1-a^2b^2}+\frac{a^4b^4}{1-a^4b^4}-\&c\right)\right\}$$
$$= f^3(a^2b,ab^2) + af^3(b,a^3b^2) + bf^3(a,a^2b^3).$$

If $u = \cfrac{\sqrt[5]{x}}{1-\cfrac{x}{1+\cfrac{x^2}{1-\&c}}}$ and $v = \cfrac{\sqrt[5]{x}}{1+\cfrac{x}{1+\cfrac{x^2}{1+\&c}}}$

then $uv(u-v)^4 - u^2v^2(u-v)^2 + 2u^3v^3 = (u-v)(1+u^5v^5)$

If $\int_0^\infty F(ax)f(bx)\,dx = \frac{1}{a+b}$ for all values of a an[d b]

or $\int_0^\infty x^{p-1}F(x)\,dx \times \int_0^\infty x^{-p}f(x)\,dx = \frac{\pi}{\sin \pi p}$ for [all]

values of p, then

If $\int_0^\infty \phi(x) . \frac{F(nxi) + F(-nxi)}{2}\,dx = \psi(n)$

then $\int_0^\infty \psi(x) . \frac{f(nxi) + f(-nxi)}{2}\,dx = \frac{\pi}{2}\phi(n)$.

If $\frac{a}{m} = \frac{n-b}{n} = p$.

and $\int_0^\infty F(\alpha x) f(\beta x)\,dx = \frac{1}{\alpha+\beta}$

then $\int_0^\infty x^{a-1}F(x^m)\,dx \int_0^\infty x^{b-1}f(x^n)\,dx$

$$= \frac{\pi}{mn \sin \pi p}.$$

If $P = \frac{f(-x)}{x^{\frac{1}{2}}f(-x^{13})}$ and $Q = \frac{f(-x^3)}{x^{\frac{3}{2}}f(-x^{39})}$, then

$$PQ + \frac{13}{PQ} = \left(\frac{Q}{P}\right)^2 - 3\frac{Q}{P} - 3 - 3\frac{P}{Q} + \left(\frac{P}{Q}\right)^2.$$

e no of primes   less than   according to formula

15 ———— 50 ———— 14.9

300 ———— 62 ———— 61.9

1000 ———— 168 ———— 168.2

$$\int^{a} \frac{dx}{\log x} = C + \log\log a + \sqrt{a}\left\{ \frac{\log a}{\underline{|1}} - \frac{(\log a)^2}{\underline{|2}}\cdot\frac{1}{2} + \frac{(\log a)^3}{\underline{|3}}\cdot\frac{1+\frac{1}{3}}{2^2} - \frac{(\log a)^4}{\underline{|4}}\cdot\frac{1+\frac{1}{3}}{2^3} + \frac{(\log a)^5}{\underline{|5}}\cdot\frac{1+\frac{1}{3}+\frac{1}{5}}{2^4} - \&c \right\}$$

If $P = \dfrac{f(-x^3)}{x^{1/12} f(-x^5)}$ and $Q = \dfrac{f(-x)}{x^{7/12} f(-x^{15})}$

then $(PQ)^3 - \dfrac{125}{(PQ)^3} = \left(\dfrac{Q}{P}\right)^4 + \left(\dfrac{Q}{P}\right)^2 - 9\cdot\left(\dfrac{P}{Q}\right)^2 - 81\cdot\left(\dfrac{P}{Q}\right)^4$

If $P = \dfrac{f(-x)}{x^{1/4} f(-x^7)}$ and $Q = \dfrac{f(-x^3)}{x^{3/4} f(-x^{21})}$, then

$$PQ + \frac{7}{PQ} = \left(\frac{Q}{P}\right)^2 - 3 + \left(\frac{P}{Q}\right)^2.$$

If $P = \dfrac{f(-x)}{x^{1/12} f(-x^3)}$ and $Q = \dfrac{f(-x^7)}{x^{7/12} f(-x^{21})}$, then

$$(PQ)^3 + \frac{27}{(PQ)^3} = \left(\frac{Q}{P}\right)^4 - 7\cdot\left(\frac{Q}{P}\right)^2 + 7\cdot\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^4.$$

If $P = \dfrac{f(-x^3)}{x^{1/6} f(-x^7)}$ and $Q = \dfrac{f(-x)}{x^{5/6} f(-x^{21})}$, then

$$\left(\frac{Q}{P}\right)^3 - 27\cdot\left(\frac{P}{Q}\right)^3 = (PQ)^2 - PQ + \frac{7}{PQ} - \frac{49}{(PQ)^2}$$

No. of the form $p^2 q^3$

$$= 2.1732542\sqrt{Z} - 1.458455\sqrt[3]{Z}$$

$$= \sqrt{4.723034\,Z} - \sqrt[3]{3.10227\,Z}$$

$$\int_0^\infty e^{-ax} \cot x \sinh bx \, dx$$

$$= \int_0^\infty \frac{2 \sinh bx \sin ax}{e^{2x}-1}\, dx$$

$$\frac{1}{2^3-2} + \frac{1}{6^3-6} + \frac{1}{10^3-10} + \dots \text{ to } n \text{ terms.}$$

$$= \frac{1}{4n+2} + \frac{1}{4n+6} + \frac{1}{4n+10} + \dots \text{ to } n \text{ terms.}$$

$$\tan^{-1}\frac{1}{2n+1} + \tan^{-1}\frac{1}{2n+3} + \text{ \&c to } n \text{ terms}$$

$$\cancel{= \tan^{-1}\frac{1}{2\cdot 1^3+1} + \tan^{-1}\frac{1}{2\cdot 2^3+1} + \text{ \&c to } n \text{ terms}}$$

$$\tan^{-1}\frac{1}{(2n+1)\sqrt{3}} + \tan^{-1}\frac{1}{(2n+3)\sqrt{3}} + \text{ \&c to } n \text{ terms}$$

$$= \tan^{-1}\frac{1}{(\sqrt{3})^3} + \tan^{-1}\frac{1}{(3\sqrt{3})^3} + \tan^{-1}\frac{1}{(5\sqrt{3})^3} + \text{ \&c to } n \text{ terms}$$

$$2\left\{1 - \left(\frac{1-t}{1+t}\right)^2 + \left(\frac{1-t}{1+t}\right)^6 - \left(\frac{1-t}{1+t}\right)^{12} + \left(\frac{1-t}{1+t}\right)^{20} - \text{\&c}\right\}$$

$$= 1 + t + t^2 + 2t^3 + 5t^4 + 17t^5 + \text{ \&c asymp.}$$

If $P = \dfrac{f(-x)}{x^{1/12} f(-x^3)}$ and $Q = \dfrac{f(-x^5)}{x^{5/12} f(-x^{15})}$, then

$$(PQ)^2 + 5 + \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3.$$

$$\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\frac{1}{(n+3)^2}+\frac{1}{(n+4)^2}+\&c$$

$$=\left(n+\tfrac{1}{2}\right)\left\{\frac{1}{n^2+n+}\;\frac{1^2}{3+}\;\frac{1^2}{5(n^2+n)+}\;\frac{6^2}{7+}\;\frac{6^2}{9(n^2+n)+\&c}\right\}$$

$$\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\frac{1}{n+4}+\&c$$

$$=\left(n+\tfrac{1}{2}\right)\left\{\frac{1}{2n^2+2n+}\;\frac{1}{1+}\;\frac{1}{2n^2+2n+}\;\frac{6}{1+}\;\frac{6}{2n^2+2n+}\right\}$$

1.1, 2.3, 3.5 &c is the order.

If α, β, γ be the roots of the equation

$$x^3-ax^2+bx-1=0$$

then
$$\left.\begin{aligned}\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}&=\sqrt[3]{a+6+3t}\\ \text{and }\sqrt[3]{\alpha\beta}+\sqrt[3]{\beta\gamma}+\sqrt[3]{\gamma\alpha}&=\sqrt[3]{b+6+3t}\end{aligned}\right\}$$

where $t^3-3t(a+b+3)-(ab+6\overline{a+b}+9)=0$

If $P=\dfrac{f^3(-x)}{x^{\frac{1}{2}}f^3(-x^5)}$ and $Q=\dfrac{f^3(-x^3)}{x^{1\frac{1}{2}}f^3(-x^{15})}$, then

$$PQ+\frac{125}{PQ}=\left(\frac{Q}{P}\right)^2-9\cdot\left(\frac{Q}{P}\right)-9\left(\frac{P}{Q}\right)-\left(\frac{P}{Q}\right)^2.$$

* If $P=\dfrac{f(-x^{\frac{1}{5}})}{x^{\frac{1}{5}}f(-x^5)}$ and $Q=\dfrac{f(-x^{\frac{2}{5}})}{x^{\frac{2}{5}}f(-x^{10})}$, then

$$PQ+\frac{25}{PQ}=\left(\frac{Q}{P}\right)^3-4\cdot\left(\frac{Q}{P}\right)^2-4\cdot\left(\frac{P}{Q}\right)^2+\left(\frac{P}{Q}\right)^3.$$

If $P=\dfrac{f(-x)}{x^{\frac{1}{6}}f(-x^5)}$ and $Q=\dfrac{f(-x^2)}{\sqrt[3]{x}\,f(-x^{10})}$, then

$$PQ+\frac{5}{PQ}=\left(\frac{Q}{P}\right)^3+\left(\frac{P}{Q}\right)^3.$$

* If $P=\dfrac{f(-x^{\frac{1}{5}})}{x^{\frac{1}{5}}f(-x^5)}$ and $Q=\dfrac{f(-x^{\frac{2}{5}})}{x^{\frac{2}{5}}f(-x^{10})}$, then

$$P^2Q^2+5PQ=P^3-2P^2Q-2PQ^2+Q^3.$$

If $u = \frac{\sqrt[5]{x}}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+\&c} = \sqrt[5]{U}$

and $v = \frac{\sqrt[5]{x^2}}{1+}\,\frac{x^2}{1+}\,\frac{x^4}{1+}\,\frac{x^6}{1+\&c} = \sqrt[5]{V}$

then

(i) $\frac{v-u^2}{v+u^2} = uv^2$.

(ii) $UV^2(U^2+V)+U^2-V$
$+10UV(UV-U+V+1)=0$

(iii) If $U = n\left(\frac{1-n}{1+n}\right)^2$ then $V = n^2\cdot\frac{1+n}{1-n}$.

If $u = \frac{\sqrt[5]{x}}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+\&c}$ and

$v = \frac{\sqrt[5]{x^4}}{1+}\,\frac{x^4}{1+}\,\frac{x^8}{1+}$, then

$(u^5+v^5)(uv-1)+u^5v^5+uv$
$= 5u^2v^2(uv-1)^2$.

If $u = \frac{\sqrt[5]{x}}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+\&c} = \sqrt[5]{U}$

and $v = \frac{\sqrt[5]{x^3}}{1+}\,\frac{x^3}{1+}\,\frac{x^6}{1+\&c} = \sqrt[5]{V}$ and also

$m = x^{\frac{2}{5}}\cdot\frac{f(-x^4,-x^{11})}{f(-x^7,-x^8)}\,v$ and $n = x^{\frac{2}{5}}\,\frac{f(-x,-x^{14})}{f(-x^4,-x^{11})}\,v$.

then $m-n = mn = \frac{m^2}{1+m} = \frac{n^2}{1-n} = uv^3$.

If $P = \dfrac{f^2(-x)}{x^{\frac{1}{6}} f^2(-x^3)}$ and $Q = \dfrac{f^2(-x^2)}{x^{\frac{1}{3}} f^2(-x^6)}$, then

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3.$$

If $P = \dfrac{f(-x^2)}{x^{\frac{1}{24}} f(-x^3)}$ and $Q = \dfrac{f(-x)}{x^{\frac{5}{24}} f(-x^6)}$, then

$$(PQ)^2 - \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - 8\cdot\left(\frac{P}{Q}\right)^3.$$

If $P = \dfrac{f(-x^2)}{x^{\frac{1}{8}} f(-x^5)}$ and $Q = \dfrac{f(-x)}{x^{\frac{3}{8}} f(-x^{10})}$, then

$$PQ - \frac{5}{PQ} = \left(\frac{Q}{P}\right)^2 - 4\cdot\left(\frac{P}{Q}\right)^2.$$

If $P = \dfrac{f^2(-x)}{x^{\frac{1}{2}} f^2(-x^7)}$ and $Q = \dfrac{f^2(-x^2)}{x^{\frac{3}{2}} f^2(-x^{14})}$, then

$$PQ + \frac{49}{PQ} = \left(\frac{Q}{P}\right)^3 - 8\cdot\frac{Q}{P} - 8\cdot\frac{P}{Q} + \left(\frac{P}{Q}\right)^3.$$

If $P = \dfrac{f(-x)}{x^{\frac{1}{2}} f(-x^{13})}$ and $Q = \dfrac{f(-x^2)}{x f(-x^{26})}$, then

$$PQ + \frac{13}{PQ} = \left(\frac{Q}{P}\right)^2 - 4\cdot\frac{Q}{P} - 4\cdot\frac{P}{Q} + \left(\frac{P}{Q}\right)^2.$$

If $P = \dfrac{f(-x)}{x^{\frac{1}{3}} f(-x^9)}$ and $Q = \dfrac{f(-x^2)}{x^{\frac{2}{3}} f(-x^{18})}$, then

$$P^3 + Q^3 = P^2Q^2 + 3PQ.$$

If $P = \dfrac{\psi(x)}{x^{\frac{1}{2}} \psi(x^5)}$ and $Q = \dfrac{\psi(x^3)}{x^{\frac{3}{2}} \psi(x^{15})}$, then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^2 + 3\cdot\frac{Q}{P} + 3\cdot\frac{P}{Q} - \left(\frac{P}{Q}\right)^2.$$

If $P = \dfrac{\phi(x)}{\phi(x^5)}$ and $Q = \dfrac{\phi(x^3)}{\phi(x^{15})}$, then

$$5PQ + \frac{1}{PQ} = \left(\frac{P}{Q}\right)^2 + 3\cdot\frac{P}{Q} + 3\cdot\frac{Q}{P} - \left(\frac{Q}{P}\right)^2.$$

If $x = y + \sqrt[3]{\frac{z}{y} - y^3}$

then $3y = x + \sqrt[3]{\frac{9z}{x} - x^3}$

$$f(a,b) - f(a^6b^3, a^3b^6) = \sqrt[3]{\frac{f(a^3,b^3)}{f(a^6b^3, a^3b^6)} \cdot f^3(a^2b, ab^2) - f^3(a^6b^3, a^3b^6)}$$

If $1 + \frac{x}{y} = \sqrt[5]{\alpha} - \sqrt[5]{\beta}$ where $\alpha\beta = 1$ and $\alpha - \beta = 11 + \frac{z}{y^6}$.

then $1 + 5\frac{y}{x} = \sqrt[5]{\gamma} - \sqrt[5]{\delta}$ where $\gamma\delta = 1$ and $\gamma - \delta = 11 + 125 \cdot \frac{z}{x^6}$

If $x = y + \sqrt{z - y^2}$, then $2y = x + \sqrt{2z - x^2}$.

$$f(a,b) - f(a^2b, ab^3) = \sqrt{f(a^2,b^2)\,\phi(ab) - f^2(a^2b, ab^3)}$$

$$f(ai, bi) = \frac{1+i}{2} f(a,b) + \frac{1-i}{2} f(-a,-b).$$

$$f(ai, bi)\, f(ci, di) - f(-ai, -bi)\, f(-ci, -di) = i\{f(a,b)\, f(c,d) - f(-a,-b)\, f(-c,-d)\}$$

$$f(ai, bi)\, f(ci, di) + f(-ai, -bi)\, f(-ci, -di) = f(a,b)\, f(-c,-d) + f(-a,-b)\, f(c,d).$$

$$\phi(x)\,\phi(x^3) + 4x\,\psi(x^2)\,\psi(x^6) = 1 + 6\left(\frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} - \frac{x^5}{1-x^5} + \&c\right)$$

$$f(a\omega, b\omega) = \omega f(a,b) + (1-\omega)\, f(a^6b^3, a^3b^6)$$

$f(a, b) = f(a^{15}b^{10}, a^{10}b^{15}) + \sqrt[5]{u} + \sqrt[5]{v}$ where

$\sqrt[5]{uv} = f(a^4b, ab^4)\, f(a^3b^2, a^2b^3) - f^2(a^{15}b^{10}, a^{10}b^{15})$.

$$u + v = \frac{f(a^5, b^5)}{f(a^{15}b^{10}, a^{10}b^{15})} f^5(a^3b^2, a^2b^3) - 5 f^2(a^4b, ab^4)\, f^2(a^3b^2, a^2b^3)\, f(a^{15}b^{10}, a^{10}b^{15})$$
$$+ 15 f(a^4b, ab^4)\, f(a^3b^2, a^2b^3)\, f^3(a^{15}b^{10}, a^{10}b^{15}) - 11 f^5(a^{15}b^{10}, a^{10}b^{15})$$

$$\psi(a)\,\psi(b) = f(a, b) + ab\, f\left(\frac{a^2}{b}, \frac{b^2}{a}\right) + (ab)^3 f\left(\frac{a^3}{b^2}, \frac{b^3}{a^2}\right) + (ab)^6 f\left(\frac{a^4}{b^3}, \frac{b^4}{a^3}\right) + \&c$$

$$\left.\begin{array}{l} f(a, b)\, f(a^3, b^3) - f(-a, -b)\, f(-a^3, -b^3) = 2a\, f\left(\frac{b}{a}, a^5\right) \psi(a^3b^3) \\ f(a, b)\, f(a^2b, ab^2) - f(-a, -b)\, f(-a^2b, -ab^2) = 2a\, f\left(\frac{b}{a}, a^4b^2\right) \psi(ab) \end{array}\right\}$$

$$(a - b) + 2ab(a^2 - b^2) + 3a^3b^3(a^3 - b^3) + 4a^6b^6(a^4 - b^4) + \&c$$
$$= f(a, b)\left\{\frac{a - b}{1 - ab} - \frac{a^2 - b^2}{1 - a^2b^2} + \frac{a^3 - b^3}{1 - a^3b^3} - \frac{a^4 - b^4}{1 - a^4b^4} + \&c\right\}$$

$$f(a, b) + c\, f(ap, bq) + d\, f(aq, bp) + c^3 d\, f(ap^3q, bpq^3) + cd^3 f(apq^3, bp^3q)$$
$$+ c^6d^3 f(ap^6q^3, bp^3q^6) + c^3d^6 f(ap^3q^6, bp^6q^3) + \&c \quad \&c$$

is unaltered by interchanging a and c and at the same time b and d. It is better to take p and q to be of the form x^m and x^n where m and n are of opposite signs.

If $P = \dfrac{f(-x^3)f(-x^5)}{x^{\frac{1}{3}} f(-x) f(-x^{15})}$ and $Q = \dfrac{f(-x^6)f(-x^{10})}{x^{\frac{2}{3}} f(-x^2) f(-x^{30})}$

then $PQ + \dfrac{1}{PQ} = \left(\dfrac{Q}{P}\right)^3 + \left(\dfrac{P}{Q}\right)^3 + 4.$

If $P = \dfrac{f(-x)f(-x^5)}{x^{\frac{1}{2}} f(-x^3) f(-x^{15})}$ and $Q = \dfrac{f(-x^2)f(-x^{10})}{x f(-x^6) f(-x^{30})}$

then $PQ + \dfrac{9}{PQ} = \left(\dfrac{Q}{P}\right)^3 - 4 \cdot \dfrac{Q}{P} - 4 \cdot \dfrac{P}{Q} + \left(\dfrac{P}{Q}\right)^3.$

If $u = \dfrac{f(-x)f(-x^5)}{x^{\frac{1}{2}} f(-x^3) f(-x^{15})}$ & $v = \dfrac{f(-x^{\frac{1}{3}})f(-x^{\frac{5}{3}})}{x^{\frac{2}{3}} f(-x^3) f(-x^{15})}$

then $u^4 - 3u^2 v = v^3 + 3v^2 + 9v$

If $P = \dfrac{f(-x^6)f(-x^{5})}{x^{\frac{1}{4}} f(-x^2) f(-x^{15})}$ and $Q = \dfrac{f(-x^3)f(-x^{10})}{x^{\frac{3}{4}} f(-x) f(-x^{30})}$

then $PQ + 1 + \dfrac{1}{PQ} = \left(\dfrac{Q}{P}\right)^2 + \left(\dfrac{P}{Q}\right)^2.$

If $\alpha = \dfrac{p(2+p)^3}{(1+p)^3}$ and $\beta = p^2(2+p)$

then $\sqrt[3]{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} = 1.$

$$1 + \left(\tfrac{1}{4}\right)^2 \cdot 64t \cdot \left(\frac{1-t}{1+2t}\right)^8 \cdot \frac{1-t^3}{1+8t^3} + \&c$$

$$= \frac{(1+2t)^2}{\sqrt{1+8t^3}} \left\{1 + \left(\tfrac{1}{4}\right)^2 \cdot 64t^3 \cdot \left(\frac{1-t^3}{1+8t^3}\right)^3 + \&c\right\}$$

$$= (1+2t)^2 \left\{1 + \left(\tfrac{1}{4}\right)^2 64t^9 \cdot \frac{1-t^3}{1+8t^3} + \&c\right\}$$

If α and β be of the I and IX degree

then $4\alpha(1-\alpha) = 64t \cdot \left(\dfrac{1-t}{1+2t}\right)^8 \cdot \dfrac{1-t^3}{1+8t^3}$

$4\beta(1-\beta) = 64t^9 \cdot \dfrac{1-t^3}{1+8t^3}$ and

$m = (1+2t)^2.$

Let ϕ be a function defined by the relation $\phi(x) = f'(x)\,\phi\{f(x)\}$ where f is a known function.

Then $\displaystyle\int_a^{f(a)} \phi(x)\,dx$ is always constant whatever be the value of a. Call this C.

Denote $ff(x)$ by $f^2(x)$, $fff(x)$ by $f^3(x)$ etc.

then (1) $f^n f^m(x) = f^{m+n}(x)$.

(2) $\displaystyle\int_{f^n(x)}^{f^m(x)} \phi(x)\,dx = (m-n)C$.

(3). $\int \phi(x)\,dx$ is of an order lower than $f^{\pm n}(x)$.

If $f^n(x) = \psi_0(x) + \frac{n}{\underline{1}}\cdot\psi_1(x) + \frac{n^2}{\underline{2}}\psi_2(x) + \&c.$

then (1) $\psi_0(x) = f^0(x) = x$.

(2). $\dfrac{d f^n(x)}{dn} = \psi_1(x)\cdot\dfrac{d f^n(x)}{dx}$.

(3). $\psi_n(x) = \psi_1(x)\cdot\dfrac{d\,\psi_{n-1}(x)}{dx}$.

(4). $\psi_1(x) = \dfrac{C}{\phi(x)}$.

If ψ be a function defined by the relations $\psi_0(x) = x$ and $\psi_n(x)\,\phi(x) = \dfrac{d\,\psi_{n-1}(x)}{dx}$ where ϕ is a known function, then if $v = \displaystyle\int_\alpha^\beta \phi(x)\,dx$

then $\beta = \psi_0(\alpha) + \frac{v}{\underline{1}}\psi_1(\alpha) + \frac{v^2}{\underline{2}}\psi_2(\alpha) + \frac{v^3}{\underline{3}}\psi_3(\alpha) + \&c.$

$$\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} + 2a\sqrt[4]{\alpha\beta(1-\alpha)(1-\beta)} = 1$$

$$\sqrt{\alpha(1-\beta)} = \frac{4ap}{(1-p^2)^2 + (2ap)^2}$$

$$\sqrt{\beta(1-\alpha)} = \frac{4ap^3}{(1-p^2)^2 + (2ap)^2}$$

$\alpha = \sin^2(u+v)$ and $\beta = \sin^2(u-v)$

$\sin v = \dfrac{\sin\phi}{\sqrt{1-\frac{1}{a}}}$ and $\sin 2u = \dfrac{\sin 2\phi}{\sqrt{1-\frac{b}{a}}}$

If $\displaystyle\int_0^\infty \frac{\phi(bx) - \psi(ax)}{x}\,dx$ be finite, then its value

$$= \phi(0)\left[\frac{d\log\dfrac{\text{coeff}^{t} \text{ of } x^n \text{ in } \psi(x)}{\text{coeff}^{t} \text{ of } x^n \text{ in } \phi(x)}}{dn}\right]_{n=0} + \phi(0)\log\frac{b}{a}.$$

with the condition $\phi(0) = \psi(0)$.

If $\displaystyle\int_0^\infty e^{-a^2x^2} F(x)\,dx = \frac{\pi}{2a}\phi(a)$; then

$$F(x) = \int_0^\infty e^{-u^2}\left\{\phi\left(\frac{ui}{x}\right) + \phi\left(-\frac{ui}{x}\right)\right\}du.$$

$$\frac{1}{n^2+1} - \frac{a-1}{a+1}\cdot\frac{3}{n^2+9} + \frac{(a-1)(a-2)}{(a+1)(a+2)}\cdot\frac{5}{n^2+25} - \&c$$

$$= \frac{\pi \lfloor a \lfloor a-1 / 4 \lfloor a-\tfrac{1}{2} \lfloor a-\tfrac{1}{2}}{\left(1+\frac{n^2}{1^2}\right)\left(1+\frac{n^2}{3^2}\right)\left(1+\frac{n^2}{5^2}\right)\cdots\left(1+\frac{n^2}{(2a-1)^2}\right)}$$

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$$\frac{1}{8n^4}+\frac{1}{1^4+4n^4}+\frac{1}{2^4+4n^4}+\frac{1}{3^4+4n^4}+\cdots$$

$$=\frac{\pi}{8n^3}\cdot\frac{\sinh 2\pi n+\sin 2\pi n}{\cosh 2\pi n-\cos 2\pi n}.$$

$$\frac{1^2}{1^4+4n^4}+\frac{2^2}{2^4+4n^4}+\frac{3^2}{3^4+4n^4}+\cdots$$

$$=\frac{\pi}{4n}\cdot\frac{\sinh 2\pi n-\sin 2\pi n}{\cosh 2\pi n-\cos 2\pi n}.$$

$$\frac{1}{1^4+4n^4}+\frac{2}{2^4+4n^4}+\frac{3}{3^4+4n^4}+\cdots$$

$$=\frac{\pi}{4n^2}\frac{\sinh 2\pi n}{\cosh 2\pi n-\cos 2\pi n}-\frac{1}{2n}\left\{\frac{1}{4n^2}+\frac{1}{n^2+(n+1)^2}+\frac{1}{n^2+(n+2)^2}+\&c\right\}$$

If $u_n=\underline{|n}\left\{e^{-\frac{n^2x}{4}}-\frac{n+2}{\underline{|1}}e^{-\frac{(n+2)^2}{4}x}+(n+4)\cdot\frac{n+1}{\underline{|2}}e^{-\frac{(n+4)^2}{4}x}-(n+6)\cdot\frac{(n+1)(n+2)}{\underline{|3}}e^{-\frac{(n+6)^2}{4}x}+\&c\right\}$

then $u_{n+2}=\frac{n^2}{4}u_n+\frac{du_n}{dx}$.

$$u_n=\frac{\left(\frac{\pi}{x}\right)^{n+\frac{1}{2}}}{2^{n-1}}e^{-\frac{\pi^2}{4x}}\left\{1-\frac{n(n-1)x}{\pi^2}+\&c\right\}$$ nearly

$$\int_0^\infty\frac{\cos nx\,dx}{\left(1+\frac{x^2}{1^2}\right)\left(1+\frac{x^2}{3^2}\right)\left(1+\frac{x^2}{5^2}\right)\cdots\cdots\left(1+\frac{x^2}{(2a-1)^2}\right)}$$

$$=\frac{2\left(\underline{|a-\tfrac{1}{2}}\right)^2}{\underline{|a}\;\underline{|a-1}}\left\{e^{-n}-\frac{a-1}{a+1}\cdot e^{-3n}+\frac{(a-1)(a-2)}{(a+1)(a+2)}e^{-5n}-\cdots\right\}$$

$$\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2} \cdot \&c \cdot \cos nx\, dx$$

$$= \pi \cdot \frac{\lfloor 2a-1}{(\lfloor a)^2} \frac{(\lfloor b)^2}{\lfloor b+a \lfloor b-a} \left\{ e^{-an} - \frac{2a}{\lfloor 1} \cdot \frac{b-a}{b+a+1} e^{-(a+1)n} \right.$$

$$+ \frac{2a(2a+1)}{\lfloor 2} \cdot \frac{(b-a)(b-a-1)}{(b+a+1)(b+a+2)} e^{-(a+2)n} +$$

$$\phi(1-a) - \phi(1+a) + \phi(3-a) - \phi(3+a) + \phi(5-a) - \&c$$

$$= \frac{1}{2} \int_0^\infty \frac{\phi(xi) + \phi(-xi)}{\cosh \pi x + \cos \pi a} \sin \pi a\, dx$$

$$\int_0^\infty n x^{n-1} \phi(x)\, dx = \phi(0) - \phi(\infty) \text{ when } n = 0$$

$$\int_0^\infty x^{n-1} \cdot \frac{\phi(ix) + \phi(-ix)}{2}\, dx = \cos \frac{\pi n}{2} \int_0^\infty x^{n-1} \phi(x)\, dx$$

$$\int_0^\infty x^{n-1} \cdot \frac{\phi(ix) - \phi(-ix)}{2i}\, dx = \sin \frac{\pi n}{2} \int_0^\infty x^{n-1} \phi(x)\, dx$$

The product of the two series $(a_1 - a_2 + a_3 - \&c)(b_1 - b_2 + b_3 - b_4 + \&c)$ is convergent, divergent or oscillating according as $\underset{n=\infty}{Lt}\, n a_n b_n$ is zero, infinite or finite. when a_n and b_n do not contain any log. function.

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$\frac{1}{2}\phi(0) + \phi(1) + \phi(2) + \phi(3) + \&c$

$$= \int_0^\infty \phi(x)\,dx + i\int_0^\infty \frac{\phi(xi) - \phi(-xi)}{e^{2\pi x} - 1}\,dx.$$

Cor. $1^{n-1}e^{-x} + 2^{n-1}e^{-2x} + 3^{n-1}e^{-3x} + \&c$

$$= \int_0^\infty e^{-zx} z^{n-1}\,dz + 2\int_0^\infty \frac{z^{n-1}\cos\left(\frac{\pi n}{2} - xz\right)}{e^{2\pi z} - 1}\,dz.$$

$\frac{1}{2}\phi(0) - \phi(1) + \phi(2) - \phi(3) + \&c$

$$= i\int_0^\infty \frac{\phi(xi) - \phi(-xi)}{e^{\pi x} - e^{-\pi x}}\,dx$$

$\phi(1) + \phi(3) + \phi(5) + \phi(7) + \&c$

$$= \frac{1}{2}\int_0^\infty \phi(x)\,dx - \frac{i}{2}\int_0^\infty \frac{\phi(xi) - \phi(-xi)}{e^{\pi x} + 1}\,dx.$$

$\phi(1) - \phi(3) + \phi(5) - \phi(7) + \&c$.

$$= \frac{1}{2}\int_0^\infty \frac{\phi(xi) + \phi(-xi)}{e^{\frac{\pi x}{2}} + e^{-\frac{\pi x}{2}}}\,dx.$$

$a_1^2 - 2a_1a_2 + (2a_1a_3 + a_2^2) - (2a_1a_4 + 2a_2a_3) + \&c$ oscillates between $(a_1 - a_2 + a_3 - \&c)^2 \pm \frac{\pi}{2}\{\mathrm{Lt}_{n=\infty} n(a_n)^2\}$

e.g. $1 - \frac{2}{\sqrt{2}} + \left(\frac{2}{\sqrt{3}} + \frac{1}{2}\right) - \left(\frac{2}{\sqrt{4}} + \frac{2}{\sqrt{6}}\right) + \left(\frac{2}{\sqrt{5}} + \frac{2}{\sqrt{8}} + \frac{1}{3}\right) - \&c$ oscillates between $\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)^2 + \frac{\pi}{2}$ and $\left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \&c\right)^2 - \frac{\pi}{2}$.

If $\phi(x)$ vanishes for a, b, c, d &c of x, then

(1) the coeff^{t} of x^{n-1} in the expansion of $\frac{1}{\phi(x)}$

$$= -\frac{1}{a^n \phi'(a)} - \frac{1}{b^n \phi'(b)} - \frac{1}{c^n \phi'(c)} - \&c - \theta(n).$$

where $\underset{n=\infty}{Lt}\, K^n \theta(n) = 0$ for any value of K. and

$\theta(n)$ is 0 in many cases.

(2). The expansion of the function

$$\frac{1}{\phi(x)} + \frac{1}{(a-x)\phi'(a)} + \frac{1}{(b-x)\phi'(b)} + \frac{1}{(c-x)\phi'(c)} + \&c$$

is convergent for all values of x.

If $\phi(x) = \infty$ for the values of a, b, c, d &c of x,

and if $|a|$ be the nearest to 0, then

(1) The expansion of $\phi(x)$ is convergent if $x < |a|$

and divergent if $x > |a|$.

(2) $\underset{n=\infty}{Lt}\, a^n$. coeff^{t} of x^{n-1} in the expansion $= \frac{\phi'(a)}{\{\phi(a)\}^2}$

$$= -\left[\frac{d\,\frac{1}{\phi(x)}}{dx}\right]_{x=a}.$$

If $\phi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 +$ &c where the coeff^{ts} are positive at least after some finite no of them, then $\underset{n=\infty}{Lt}\, \phi\left(\frac{a_n}{a_{n+1}}\right) = \infty$.

I. $aF(x+p)+bF(x+q)+cF(x+r)+\&c = \phi(x)$. 24

Write $\phi(x)$ as $\int_{\alpha}^{\beta} u^x v\,dz$ where u and v are functions of z, then

$$F(x) = \int_{\alpha}^{\beta} \frac{u^x v\,dz}{au^p+bu^q+cu^r+\&c}.$$

II. $\phi(x)F(x+p)+\psi(x)F(x+q) = f(x)$.

Find a function $\chi(x)$ so that $\dfrac{\chi(x+p)}{\chi(x+q)} = \dfrac{\psi(x)}{\phi(x)}$.

Now let $F(x) = F_1(x)\chi(x)$; then we have

$$F_1(x+p)+F_1(x+q) = \frac{f(x)}{\phi(x)\chi(x+p)}.$$

III. If a, b, c are constants in A.P and u, v, w are functions of x in G.P

Solve $uF(x+a)+vF(x+b)+wF(x+c) = \phi(x)$.

Find $\chi(x)$ so that $\dfrac{\chi(x+\frac{3a}{2})}{\chi(x+\frac{3c}{2})} = \sqrt{\dfrac{w^3}{u^3}}$ or $\dfrac{v^3}{u^3}$

and substitute $F(x) =$

$$\chi(x)\left\{\sqrt{u}\,F_1\left(x+\frac{a}{2}\right)-\sqrt{w}\,F_1\left(x+\frac{c}{2}\right)\right\}.$$

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IV. $F(x+p)\left\{\phi(x) + \psi(x)F(x+q)\right\} = f(x)$

Substitute $F(x) = \dfrac{F_1(x+q)}{F_1(x+p)}$, then

$$\phi(x)F_1(x+p+q) + \psi(x)F_1(x+2q) = f(x)F_1(x+2p).$$

$$\left.\begin{array}{ll}\text{I.} & \dfrac{x^5-a}{x^2-y} = \dfrac{y^5-b}{y^2-x} = 5(xy-1) \\ \text{II.} & \dfrac{x^7-a}{(x^2-y)^2+x} = \dfrac{y^7-b}{(y^2-x)^2+y} = 7(xy-1)\end{array}\right\} \begin{array}{l}\text{Suppose} \\ x = \alpha+\beta+\gamma \\ y = \alpha\beta+\beta\gamma+\gamma\alpha \\ 1 = \alpha\beta\gamma.\end{array}$$

III.
$$\begin{aligned} x+y+z+\&c &= a \\ px+qy+rz+\&c &= b \\ p^2x+q^2y+r^2z+\&c &= c \\ p^3x+q^3y+r^3z+\&c &= d \\ \&c \quad \&c \quad \&c & \end{aligned}$$

$$\frac{x}{1-tp} + \frac{y}{1-tq} + \frac{z}{1-tr} + \&c = a+bt+ct^2+dt^3+\&c$$

Find the sum of the right hand side by converting it into a continued fraction or by using indeterminate coeff'ts and then split up the result into partial fractions.

$$\text{IV.}\quad x^2+ay=b\ ;\ y^2+cx=d \left\}\begin{array}{l} x = \alpha+\beta+\gamma \\ y = -\frac{2}{a}(\alpha\beta+\beta\gamma+\gamma\alpha) \\ \alpha\beta\gamma = \text{suitable value}\end{array}\right.$$

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$$(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \text{\&c}$$

$$= a_1 - \underset{n=\infty}{Lt}\, a_n$$

$$\phi(\infty) = \phi(0) - \{\phi(0) - \phi(1)\} - \{\phi(1) - \phi(2)\} - \text{\&c}$$

$$= \phi(a) - \{\phi(a) - \phi(b)\} - \{\phi(b) - \phi(c)\} - \text{\&c}$$

where a, b, c are increasing quantities

If $S = a_1 - a_2 + a_3 - a_4 + \ldots$

then $\underset{n=\infty}{Lt}\{S_{2n+1} - S_n\} = \underset{n=\infty}{Lt}\, a_n$

$$= a_1 - (a_1 - a_2) + (a_2 - a_3) - (a_3 - a_4) - \text{\&c}$$

$$\begin{cases} (1+x^2)^{\frac{n}{2}} \operatorname{Sin}(n \tan^{-1} x) = \\ n x \left\{1 - \dfrac{x^2}{\tan^2 \frac{\pi}{n}}\right\}\left\{1 - \dfrac{x^2}{\tan^2 \frac{3\pi}{n}}\right\}\left\{1 - \dfrac{x^2}{\tan^2 \frac{3\pi}{n}}\right\} \text{\&c} \\ \operatorname{Sin}(n \sin^{-1} x) = n x \left\{1 - \dfrac{x^2}{\sin^2 \frac{\pi}{n}}\right\}\left\{1 - \dfrac{x^2}{\sin^2 \frac{2\pi}{n}}\right\} \end{cases}$$

to limited factors only.

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \text{\&c}$$

$$= \cfrac{1}{a_1 - \cfrac{a_1^2}{a_1 + a_2 - \cfrac{a_2^2}{a_2 + a_3 - \cfrac{a_3^2}{a_3 + a_4 - \cfrac{a_4^2}{a_4 + a_5 - \text{\&c}}}}}}$$

$$\cfrac{1}{1 - \cfrac{a_1}{1 - \cfrac{a_2}{1 - \cfrac{a_3}{1 - \cfrac{a_4}{1 - \text{\&c}}}}}}$$ is intelligible or not according as $\underset{n=\infty}{Lt}\, a_n < \text{or} > \frac{1}{4}$.

(3) $\dfrac{1}{p+}\,\dfrac{a_1}{p+}\,\dfrac{a_2}{p+}\,\dfrac{a_3}{p+\&c}$ tends to two limits

or one limit according as $\Sigma \dfrac{1}{\sqrt{a_n}}$ is conver-gent or divergent

$$(a_1+b_0-b_1)+(a_2+b_1-b_2)+(a_3+b_2-b_3)+\&c$$

$$= \underset{n=\infty}{Lt}\,(a_1+a_2+\cdots a_n+b_0-b_n).$$

$\dfrac{a_1}{1-}\,\dfrac{a_2}{1-}\,\dfrac{a_3}{1-}\,\dfrac{a_4}{1-\&c}$ is intelligible when

(1). $1-4a$ is positive (if $\underset{n=\infty}{Lt}\,a_n$ tends to the Lt a)

(2). $\{1-(a+b)\}^2-4ab$ is positive (if a_n tends to two limits a & b)

(3). $\{1-(a+b+c)\}^2-4abc$ is positive (a, b & c).

(4.) $\{1-(a+b+c+d)+(ac+bd)\}^2-4abcd$

(5) $\{1-(a+b+c+d+e)+a(c+d)+b(d+e)+ce\}^2$ $-4abcde$.

If $V=(a_1-a_2+a_3-\&c)^p$ then the expansion of V oscillates between $V \pm \frac{1}{2}\underset{n=\infty}{Lt}\, n^{p-1}(a_n)^{\frac{1}{p}-1})^p$.

If $V = (a_1 - a_2 + \&c)(b_1 - b_2 + \&c)(c_1 - c_2 + \&c) \times$
$(d_1 - d_2 + \&c)$ &c &c to p factors.

Then the expansion of V oscillates between

$$V \pm \frac{1}{2} \operatorname*{Lt}_{n=\infty} n^{p-1} a_n b_n c_n \&c \left\lfloor \frac{\log a_n}{\log n} \right. \left\lfloor \frac{\log b_n}{\log n} \right. \left\lfloor \frac{\log c_n}{\log n} \right. \&c$$

The series $a_1 - a_2 + a_3 - a_4 + \&c$ is deranged into

The first $\phi(1)$ positive terms $-a_2$

+ the next $\phi(2)$ — do — $-a_4$

+ — $\phi(3)$ — do — $-a_6$

+ &c &c &c &c

and consequently the sum is made greater by

$$\frac{1}{2}\int_n^{\int \phi\left(\frac{n}{2}\right)dn} a_x \, dx \quad \text{when } n \text{ becomes } \infty$$

$\sqrt[3]{4} - \sqrt[3]{5}$

$$\sqrt{m\sqrt[3]{4m-8n} + n\sqrt[3]{4m+n}} = \frac{\sqrt[3]{(4m+n)^2} + \sqrt[3]{4(m-2n)(4m+n)} - \sqrt[3]{2(m-2n)^2}}{3}$$

$\sqrt[3]{4}-1$, $5-\sqrt[3]{4}$, $\sqrt[3]{5}-\sqrt[3]{4}$, $\sqrt[3]{2^7}-\sqrt[3]{7}$, $2^5-\sqrt[3]{7}$, $2^3-\sqrt[3]{5}$, $8+\sqrt[2]{17}$

$\sqrt[3]{2^7}-5$, $2^4-5\sqrt[3]{3}$, $\sqrt[3]{28}-\sqrt[3]{27}$, $3\sqrt[2]{7}-\sqrt[3]{20}$. &c are $5+\sqrt[3]{44}$, $11+\sqrt[3]{2^8}$. perfect squares

$$\sqrt[3]{(m^2+mn+n^2)\sqrt[3]{(m-n)(m+2n)(2m+n)}+3mn^2+n^3-m^3}$$

$$= \sqrt[3]{\frac{(m-n)(m+2n)^2}{9}} - \sqrt[3]{\frac{(2m+n)(m-n)^2}{9}} + \sqrt[3]{\frac{(m+2n)(2m+n)^2}{9}}$$

$$x + \cfrac{(1+y)^2+n}{2x + \cfrac{(3+y)^2+n}{2x + \cfrac{(5+y)^2+n}{2x + \&c}}}$$

$$= y + \cfrac{(1+x)^2+n}{2y + \cfrac{(3+x)^2+n}{2y + \cfrac{(5+x)^2+n}{2y + \&c}}}$$

$$= x + \cfrac{(y+1)^2+n}{x+y+2 + \cfrac{(x+1)^2+n}{x+y+4 + \cfrac{(y+3)^2+n}{x+y+6+\&c}}}$$

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \&c}}} \quad \text{to } 2n \text{ terms}$$

$$= \cfrac{a_1 b_2}{b_1 b_2 + a_2 - \cfrac{a_2 a_3 b_4}{b_2 b_3 b_4 + a_3 b_4 + a_4 b_2 - \cfrac{a_4 a_5 b_2 b_6}{b_4 b_5 b_6 + a_5 b_6 + a_6 b_4 - \cfrac{a_6 a_7 b_4 b_8}{\&c}}}}$$

$$\frac{1}{n} - \frac{x}{n(n+1)} + \frac{x^2}{n(n+1)(n+2)} - \&c$$

$$= \cfrac{1}{n + \cfrac{x}{1 + \cfrac{1}{n + \cfrac{x}{1 + \cfrac{2}{n + \cfrac{x}{1 + \cfrac{3}{n + \&c}}}}}}}$$

$$= \cfrac{1}{n+x - \cfrac{x}{n+x+1 - \cfrac{2x}{n+x+2 - \cfrac{3x}{n+x+3 - \&c}}}}$$

$$\frac{e^{2x}-1}{2} = \cfrac{x}{1 - \cfrac{x}{1 + \cfrac{x}{3 - \cfrac{x}{1 + \cfrac{x}{5 - \cfrac{x}{1 + \&c}}}}}}$$

$$\frac{x}{1-e^{-x}} = 1 + \frac{x}{1+}\,\frac{1}{1+}\,\frac{x}{1+}\,\frac{2}{1+}\,\frac{x}{1+}\,\frac{3}{1+}\,\frac{x}{1+}\,\&c$$

$$x - n + \frac{n}{1 + \frac{x}{n+1} + \frac{x^2}{(n+1)(n+2)} + \&c}$$

$$= \frac{x}{1+}\,\frac{n}{1+}\,\frac{x}{1+}\,\frac{n+1}{1+}\,\frac{x}{1+}\,\frac{n+2}{1+}\,\&c$$

$$\frac{\sinh^{-1}x}{\sqrt{1+x^2}} = \frac{x}{1+}\,\frac{2x^2}{1+}\,\frac{2(1+x^2)}{1+}\,\frac{4x^2}{1+}\,\frac{4(1+x^2)}{1-}$$

$$\tan^{-1}x = \frac{x}{1+}\,\frac{x^2}{1+}\,\frac{2(1+x^2)}{1+}\,\frac{3x^2}{1+}\,\frac{4(1+x^2)}{1+}\,\&c$$

$$\left(\frac{\lfloor\frac{x-3}{4}}{\lfloor\frac{x-1}{4}}\right)^2 \frac{\left\{1+\frac{n^2}{(x+3)^2}\right\}\left\{1+\frac{n^2}{(x+7)^2}\right\}\left\{1+\frac{n^2}{(x+11)^2}\right\}\&c}{\left\{1+\frac{n^2}{(x+1)^2}\right\}\left\{1+\frac{n^2}{(x+5)^2}\right\}\left\{1+\frac{n^2}{(x+9)^2}\right\}\&c}$$

$$= \frac{4}{x+}\,\frac{n^2+1^2}{2x+}\,\frac{n^2+3^2}{2x+}\,\frac{n^2+5^2}{2x+}\,\&c$$

$$\frac{\tanh\frac{\pi n}{4}}{n} = \frac{1}{1+}\,\frac{1^2+n^2}{2+}\,\frac{3^2+n^2}{2+}\,\frac{5^2+n^2}{2+}\,\&c$$

$$2\left\{\frac{x+1}{(x+1)^2+n^2} - \frac{x+3}{(x+3)^2+n^2} + \frac{x+5}{(x+5)^2+n^2} - \&c\right\}$$

$$= \frac{1}{x+}\,\frac{1^2+n^2}{x+}\,\frac{2^2}{x+}\,\frac{3^2+n^2}{x+}\,\frac{4^2}{x+}\,\frac{5^2+n^2}{x+}\,\&c$$

$$\frac{1}{1^2+n^2} - \frac{2}{2^2+n^2} + \frac{3}{3^2+n^2} - \&c$$

$$= \frac{1}{1+}\,\frac{1^2+n^2}{1+}\,\frac{2^2}{1+}\,\frac{3^2+n^2}{1+}\,\frac{4^2}{1+\&c}$$

$$2\left\{\frac{1}{(x+1)^2+n^2} + \frac{1}{(x+3)^2+n^2} + \frac{1}{(x+5)^2+n^2} + \&c\right\}$$

$$= \frac{1}{x+}\,\frac{1^2(1^2+n^2)}{3x+}\,\frac{2^2(2^2+n^2)}{5x+\&c}$$

$$\frac{\pi n}{2}\cdot\frac{e^{\pi n}+1}{e^{\pi n}-1} = 1 + \frac{n^2}{1+}\,\frac{1^2(1^2+n^2)}{3+}\,\frac{2^2(2^2+n^2)}{5+\&c}$$

$$2\left\{\frac{1}{(x+1)^2+n^2} - \frac{1}{(x+2)^2+n^2} + \frac{1}{(x+3)^2+n^2} - \&c\right\}$$

$$= \frac{1}{x^2+x+}\,\frac{1^2+n^2}{1+}\,\frac{1^2}{x^2+x+}\,\frac{2^2+n^2}{1+}\,\frac{2^2}{x^2+x+\&c}$$

If $u = \left\{1+\left(\frac{m+n}{x+1}\right)^2\right\}\left\{1+\left(\frac{m+n}{x+3}\right)^2\right\}\ \&c$ and

$v = \left\{1+\left(\frac{m-n}{x+1}\right)^2\right\}\left\{1+\left(\frac{m-n}{x+3}\right)^2\right\}\ \&c$

then $$\frac{u-v}{u+v} = \frac{mn}{x+}\,\frac{(m^2+1^2)(n^2+1^2)}{3x+}\,\frac{(m^2+2^2)(n^2+2^2)}{5x+\&c}$$

$$\frac{m\tanh\frac{\pi n}{2} - n\tanh\frac{\pi m}{2}}{m\tanh\frac{\pi m}{2} - n\tanh\frac{\pi n}{2}} = \frac{mn}{1+}\,\frac{(m^2+1^2)(n^2+1^2)}{3+\&c}$$

If $u = \left\{1+\left(\frac{2m}{x+1}\right)^2\right\}\left\{1+\left(\frac{2m}{x+3}\right)^2\right\}$ &c and

$$v = \frac{\left(\lfloor\frac{x-1}{2}\right)^2}{\lfloor\frac{x-1}{2}+m\ \lfloor\frac{x-1}{2}-m}, \text{ then}$$

$$\frac{u-v}{u+v} = \frac{2m^2}{x+}\;\frac{1^4+4m^4}{3x+}\;\frac{2^4+4m^4}{5x+}\;\frac{3^4+4m^4}{7x+}\;\&c$$

$$\frac{\sinh \pi n - \sin \pi n}{\sinh \pi n + \sin \pi n} = \frac{2n^2}{1+}\;\frac{1^4+4n^4}{3+}\;\frac{2^4+4n^4}{5+}\;\&c$$

If $u = \left\{1+\left(\frac{n}{x+1}\right)^3\right\}\left\{1+\left(\frac{n}{x+2}\right)^3\right\}$ &c

& $v = \left\{1-\left(\frac{n}{x+1}\right)^3\right\}\left\{1-\left(\frac{n}{x+2}\right)^3\right\}$ &c

then $$\frac{u-v}{u+v} = \frac{n^3}{2x^2+2x+1+}\;\frac{n^6-1^6}{3(2x^2+2x+3)+}\;\frac{n^6-2^6}{5(2x^2+2x+5)+}\;\&c$$

~~If $a+b+c+$ &c~~

If the sum of n quantities a, b, c &c $=$ the sum of any other n quantities p, q, r &c

then $$\frac{\lfloor x+a\ \lfloor x+b\ \lfloor x+c\ \&c}{\lfloor x+p\ \lfloor x+q\ \lfloor x+r\ \&c} = 1 \text{ when } x=\infty$$

and approximately equal to

$$1+\frac{\Sigma a^2-\Sigma p^2+\frac{1}{6}}{2x} \text{ when } x \text{ is great}$$

The ratio between

$$\lfloor Ax+a \;\lfloor Bx+b \;\lfloor Cx+c \text{ \&c to } m \text{ factors}$$

and $\lfloor Px+p \;\lfloor Qx+q \;\lfloor Rx+r$ &c to n factors

will tend to a finite limit

$$(2\pi)^{\frac{m-n}{2}} \cdot \frac{A^{a+\frac{1}{2}} \cdot B^{b+\frac{1}{2}} \cdot C^{c+\frac{1}{2}} \text{ \&c}}{P^{p+\frac{1}{2}} \cdot Q^{q+\frac{1}{2}} \cdot R^{r+\frac{1}{2}} \text{ \&c}}$$ only when

If the following conditions are satisfied.

$$A+B+C+\text{\&c} = P+Q+R+\text{\&c} \text{ ——— (1)}$$

$$A^A \cdot B^B \cdot C^C \text{ \&c} = P^P \cdot Q^Q \cdot R^R \text{ \&c} \text{ ——— (2)}$$

$$\frac{m}{2}+a+b+c+\text{\&c} = \frac{n}{2}+p+q+r+\text{\&c} \text{ ——— (3)}$$

A, B, C &c as well as P, Q, R &c should all be positive but a, b, c &c and p, q, r &c may be any quantities whatever.

N.B. a, b, c &c and p, q, r &c are easily determined from the condition (3) and A, B, C &c and P, Q, R &c can be found thus:

From (2) alone find the quantities first

e.g. $2^2 \cdot 6^6 = 3^3 \cdot 3^3 \cdot 4^4$

and multiply the result by as many 1^1 as to satisfy (1). e.g. $1^1 \cdot 1^1 \cdot 2^2 \cdot 6^6 = 3^3 \cdot 3^3 \cdot 4^4$.

$1^1 \cdot 1^1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \cdot 5^5 \cdot 6^6 = 3^3 \cdot 3^3 \cdot 3^3 \cdot 4^4 \cdot 4^4 \cdot 5^5$.

$1^1 . 8^8 . 9^9 = 3^3 . 3^3 . 12^{12}$; $1^1 . 3^3 . 12^{12} . 20^{20} = 5^5 . 15^{15} . 16^{16}$

$1^1 . 4^4 . 20^{20} . 30^{30} = 6^6 . 24^{24} . 25^{25}$.

If a, b, c be any three quantities, then

$$\frac{\lfloor\underline{3x+a}\ \lfloor\underline{3x+b}\ \lfloor\underline{12x+3c}}{\lfloor\underline{x+\frac{a+b-c}{2}}\ \lfloor\underline{8x+2c}\ \lfloor\underline{9x+\frac{a+b+3c}{2}}} = \sqrt{\frac{3}{2}} \text{ when } x=\infty$$

$$\frac{\lfloor\underline{x+a-b}\ \lfloor\underline{8x+2b}\ \lfloor\underline{9x+a+b}}{\lfloor\underline{3x+a-c}\ \lfloor\underline{3x+a-b+c}\ \lfloor\underline{12x+3b}} = \sqrt{\frac{2}{3}} \text{ when } x$$

$$\lfloor\underline{x+iy} = \frac{\lfloor\underline{x}\ e^{i \lim_{n=\infty} \left\{ y\log n - \left(\tan^{-1}\frac{y}{x+1} + \tan^{-1}\frac{y}{x+2} + \dots \tan^{-1}\frac{y}{x+n}\right)\right\}}}{\sqrt{\left\{1+\left(\frac{y}{x+1}\right)^2\right\}\left\{1+\left(\frac{y}{x+2}\right)^2\right\}\left\{1+\left(\frac{y}{x+3}\right)^2\right\} \&c}}$$

$$\left\{\tan^{-1}\frac{m}{x-n+1} + \tan^{-1}\frac{m}{x-n+3} + \&c\right\}$$

$$-\left\{\tan^{-1}\frac{m}{x+n+1} + \tan^{-1}\frac{m}{x+n+3} + \&c\right\}$$

$$= \tan^{-1}\cfrac{mn}{x + \cfrac{(1^2+m^2)(1^2-n^2)}{3x + \cfrac{(2^2+m^2)(2^2-n^2)}{5x + \&c}}}$$

$$\tan^{-1}\frac{m+n}{x+1} - \tan^{-1}\frac{m+n}{x+3} + \tan^{-1}\frac{m+n}{x+5} - \&c$$

$$+\tan^{-1}\frac{m-n}{x+1} - \tan^{-1}\frac{m-n}{x+3} + \tan^{-1}\frac{m-n}{x+5} - \&c$$

$$= \tan^{-1}\frac{m}{x+}\ \frac{1^2+n^2}{x+}\ \frac{2^2+m^2}{x+}\ \frac{3^2+n^2}{x+} \&c$$

$$\tan^{-1}\frac{2n}{x+1} - \tan^{-1}\frac{2n}{x+3} + \tan^{-1}\frac{2n}{x+5} - \&c$$

$$= \tan^{-1}\frac{n}{x+}\ \frac{1^2+n^2}{x+}\ \frac{2^2+n^2}{x+}\ \frac{3^2+n^2}{x+} \&c$$

45 e^{ax} can be expanded in ascending powers of $e^{bx}-e^{cx}$ and consequently e^{ax} can be expanded in ascending powers of $e^{bx}\sin x$ and hence many transcendental equations can be solved.

$$\frac{1}{x+a_1}+\frac{a_1}{(x+a_1)(x+a_2)}+\frac{a_1a_2}{(x+a_1)(x+a_2)(x+a_3)}+\&c \text{ to } n \text{ terms}$$

$$=\frac{1}{x}-\frac{1}{x\left(1+\frac{x}{a_1}\right)\left(1+\frac{x}{a_2}\right)\cdots\left(1+\frac{x}{a_n}\right)}.$$

$$(a_0^n-b_1^n)+\left(\frac{b_1}{a_1}\right)^n(a_1^n-b_2^n)+\left(\frac{b_1}{a_1}\cdot\frac{b_2}{a_2}\right)^n(a_2^n-b_3^n)$$

$$+\left(\frac{b_1b_2b_3}{a_1a_2a_3}\right)^n(a_3^n-b_4^n)+\&c \text{ to } r \text{ terms}.$$

$$=a_0^n-\left(\frac{b_1b_2b_3\cdots b_n}{a_1a_2a_3\cdots a_{n-1}}\right)^n.$$

$\underset{n=\infty}{Lt}\, u_n$ is said to be c when u_n-c cannot be made greater than any arbitrary small quantity h by making n sufficiently great.

The expansion of $\phi(x)$ is said to be a legitimate convergent series if $\phi(x)=\underset{n=\infty}{Lt}$ Sum of the first ~~n~~ terms of the expansion of $\phi(x)$. The ~~remaining~~ cases are, illegitimate convergent, legitimate divergent and illegitimate divergent series.

$\frac{1}{n}\left\{\phi(x-n+1)+\phi(x-n+3)+\cdots\qquad\phi(x+n-1)\right\}$ 46

$= \phi(x)$, as the first approximation.

$= \dfrac{\phi\left(x+\sqrt{\frac{n^2-1}{3}}\right)+\phi\left(x-\sqrt{\frac{n^2-1}{3}}\right)}{2}$ as the 2nd

$\doteq \dfrac{5(n^2-1)\left\{\phi\left(x+\sqrt{\frac{3n^2-7}{5}}\right)+\phi\left(x-\sqrt{\frac{3n^2-7}{5}}\right)\right\}+8(n^2-4)\phi(x)}{6(3n^2-7)}$.

$= \left(\frac{1}{4}-\frac{n^2-16}{6\beta}\right)\left\{\phi\left(x+\sqrt{\frac{\alpha+\beta}{7}}\right)+\phi\left(x-\sqrt{\frac{\alpha+\beta}{7}}\right)\right\}$

$+\left(\frac{1}{4}+\frac{n^2-16}{6\beta}\right)\left\{\phi\left(x+\sqrt{\frac{\alpha-\beta}{7}}\right)+\phi\left(x-\sqrt{\frac{\alpha-\beta}{7}}\right)\right\}$

where $\alpha = 3n^2-13$ and $\beta = \sqrt{\frac{4}{5}(6n^4-45n^2+164)}$.

Examples...

$u_1+u_2+\cdots\cdots+u_{13}$

$= \frac{13}{25}(7u_2+11u_7+7u_{12})$

$u_1+u_2+\cdots\cdots+u_{22}$

$= \frac{11}{289}(161u_3+256u_{11\frac{1}{2}}+161u_{20})$.

$\phi(1)+\phi(2)+\cdots\cdots+\phi(21)$

$= \frac{7}{958}\left[506\{\phi(2)+\phi(20)\}+931\left\{\phi\left(11+2\sqrt{\frac{22}{7}}\right)+\phi\left(11-2\sqrt{\frac{22}{7}}\right)\right\}\right]$

$u_1+u_2+\cdots+u_7 = \frac{7}{2}(u_2+u_6)$

$u_1+u_2+\cdots+u_{26} = 13(u_6+u_{21})$

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$$e^{nx} = 1 + \frac{n}{\underline{|1}} e^{-bx}\frac{\sin cx}{c} + \frac{n(n+2b)}{\underline{|2}} e^{-2bx}\left(\frac{\sin cx}{c}\right)^2$$

$$+ \frac{n\{(n+3b)^2+c^2\}}{\underline{|3}} e^{-3bx}\left(\frac{\sin cx}{c}\right)^3 \quad *$$

$$+ \frac{n(n+4b)\{(n+4b)^2+2^2c^2\}}{\underline{|4}} e^{-4bx}\left(\frac{\sin cx}{c}\right)^4$$

$$+ \frac{n\{(n+5b)^2+c^2\}\{(n+5b)^2+3^2c^2\}}{\underline{|5}} e^{-5bx}\left(\frac{\sin cx}{c}\right)^5$$

$$+ \frac{n(n+6b)\{(n+6b)^2+2^2c^2\}\{(n+6b)^2+4^2c^2\}}{\underline{|6}} e^{-6bx}\left(\frac{\sin cx}{c}\right)^6$$

$$+ \cdots\cdots\cdots$$

The series is convergent or divergent according as $\left|e^{-bx}\frac{\sin cx}{c}\right|$ is $<$ or $>$ $\frac{e^{-\frac{b}{c}\tan^{-1}\frac{c}{b}}}{\sqrt{b^2+c^2}}$

The series is legitimate when x is less than $\frac{1}{c}\tan^{-1}\frac{c}{b}$ (inclusive). and convergent when b is not −ve.

$$a\left\{1+\frac{a^2}{\phi(1)^2}\right\}\left\{1+\frac{a^2}{\phi(2)^2}\right\}\left\{1+\frac{a^2}{\phi(3)^2}\right\} \&c$$

$$= C e^{2\int_{\frac{\phi(0)}{a}}^{\infty}\frac{\phi^{-1}(ax)}{x(1+x^2)}dx} \quad \text{when } a \text{ is very great.}$$

$$a^{\frac{n}{2}}\left\{1+\left(\frac{a}{\phi(1)}\right)^n\right\}\left\{1+\left(\frac{a}{\phi(2)}\right)^n\right\}\left\{1+\left(\frac{a}{\phi(3)}\right)^n\right\} \&c$$

$$= C e^{n\int_{\frac{\phi(0)}{a}}^{\infty}\frac{\phi^{-1}(ax)}{x(1+x^n)}dx} \quad \text{when } a \text{ is very great}$$

The above theorem is very useful to know

$$1-\frac{x^3}{\lfloor 3}+\frac{x^6}{\lfloor 6}-\frac{x^9}{\lfloor 9}+\&c=0$$ 48

If n is any odd positive integer and $h=e^{-\frac{\pi n\sqrt{3}}{2}}$.

Then all the real roots of x are included in the following formula and all the imaginary roots can be found by multiplying all the real roots by ω and ω^2 $(=\sqrt[3]{1})$.

$$x=\frac{\pi n}{\sqrt{3}}-\frac{1}{2}\left\{\frac{h^2}{\lfloor 1}+\frac{13}{\lfloor 3}h^4+\frac{28.31}{\lfloor 5}h^6\right.$$

$$\left.+\frac{49.52.57}{\lfloor 7}h^8+\frac{76.79.84.93}{\lfloor 9}h^{10}+\&c\right\}$$

$$+\frac{(-1)^{\frac{n-1}{2}}}{\sqrt{3}}\left\{\frac{h}{\lfloor 1}+\frac{7}{\lfloor 3}h^3+\frac{19.21}{\lfloor 5}h^5+\frac{37.39.43}{\lfloor 7}h^7\right.$$

$$\left.+\frac{61.63.67.73}{\lfloor 9}h^9+\frac{91.93.97.103.111}{\lfloor 11}h^{11}+\&c\right\}$$

This series is convergent if $h<e^{\frac{\pi\sqrt{3}}{6}}$ and the greatest value of $h=e^{-\frac{\pi\sqrt{3}}{2}}$ which is $<\frac{1}{15}$

If $\alpha,\beta,\gamma,$ &c are real roots of $1-\frac{x^3}{\lfloor 3}+\frac{x^6}{\lfloor 6}-$&c

then $1+\frac{x^3}{\lfloor 3}+\frac{x^6}{\lfloor 6}+\frac{x^9}{\lfloor 9}+\frac{x^{12}}{\lfloor 12}+$ &c

$$=\left(1+\frac{x^3}{\alpha^3}\right)\left(1+\frac{x^3}{\beta^3}\right)\left(1+\frac{x^3}{\gamma^3}\right)\left(1+\frac{x^3}{\delta^3}\right)\&c.$$

the nature of roots and to check the product.

If an nth degree series can be expressed in terms of M and N only, then,

$$x\frac{du}{dx} - \frac{nLu}{12}$$ can be expressed in terms of M and N only.

<u>Dem</u>. Let $u = M^{\frac{n}{4}} f\left(\frac{M^3}{N^2}\right)$. Find $x \cdot \frac{du}{dx}$ assuming that $x \cdot \frac{d\frac{M^3}{N^2}}{dx} = \frac{M^2}{N^3}(M^3 - N^2)$.

Cor. $\frac{d\,L^4/M}{dN} = \frac{2L^3}{3M}$ and $\frac{d\,L^6/N}{dM} = \frac{3L^5 M}{2N^2}$.

The set of simultaneous equations are useful to find the conditions for as well as the method for expressing a function as the sum of a given no. of squares. (Areas and approxns also).

$$\phi(0) + \frac{x}{\lfloor 1}\phi(1) + \frac{x^2}{\lfloor 2}\phi(2) + \frac{x^3}{\lfloor 3}\phi(3) + \cdots$$
$$= e^x \phi(x) \,.\, e^{\frac{D^2}{\lfloor 2}x + \frac{D^3}{\lfloor 3}x^2 + \&c} = e^x\phi(x) \text{ as the first approximation.}$$
$$= e^x\left\{\frac{\sqrt{1+4x}-1}{2\sqrt{1+4x}}\,\phi\left(x + \frac{1+\sqrt{1+4x}}{2}\right) + \frac{\sqrt{1+4x}+1}{2\sqrt{1+4x}}\,\phi\left(x+\frac{1-\sqrt{1+4x}}{2}\right)\right\}$$
$$= e^x\left\{\frac{2}{3}\phi(x) + \frac{\sqrt{1+12x}-1}{6\sqrt{1+12x}}\,\phi\left(x + \frac{1+\sqrt{1+12x}}{2}\right) + \frac{\sqrt{1+12x}+1}{6\sqrt{1+12x}}\,\phi\left(x + \frac{1-\sqrt{1+12x}}{2}\right)\right\}.$$

6	30	42	66						
B_2	B_4	B_6	B_{10}	1	7	13	17	19	31
B_{14}	B_8	B_{114}	B_{50}	37	43	47	49	59	61
B_{26}	B_{68}	B_{186}	B_{170}	67	71	73	79	91	97
B_{34}	B_{76}	B_{258}	B_{370}	101	103	107	109	127	
B_{38}	B_{124}	B_{294}	B_{470}	133	137	139	149	151	
B_{62}	B_{152}	B_{354}	B_{590}	157	167	169	179	181	
B_{14}	B_{188}	B_{402}	B_{610}	193	197	199	211	217	
B_{76}	B_{236}	B_{426}	B_{670}						
$B_{9[illegible]}$	B_{244}	B_{474}	B_{710}						
B_{98}	B_{248}	B_{582}	B_{730}						

B_2 —— log 2

B_4 —— log 4 & 8

B_6 —— log 6

B_{10} —— log 10 & 50

$$2\left\{\frac{1}{x^2+a^2}-\frac{1}{x^2+(a+1)^2}+\frac{1}{x^2+(a+2)^2}-\right\}$$

$$=\frac{1}{x^2+a^2}+\frac{\lfloor\underline{2}\,a}{(x^2+a^2)(x^2+\overline{a+1}^2)}$$

$$+\frac{\lfloor\underline{2}\,a(a+1)}{(x^2+a^2)(x^2+\overline{a+1}^2)(x^2+\overline{a+2}^2)}$$

$$\frac{(\lfloor\underline{x})^2(\lfloor\underline{y})^2\lfloor\underline{2x+2y}}{(\lfloor\underline{x+y})^4}\cdot\frac{\left\{1+\left(\frac{m+n}{x+y+1}\right)^2\right\}\left\{1+\left(\frac{m+n}{x+y+2}\right)^2\right\}\&c}{\left\{1+\left(\frac{m}{x+1}\right)^2\right\}\left\{1+\left(\frac{m}{x+2}\right)^2\right\}\&c}$$

$$\times\frac{\left\{1+\left(\frac{m-n}{x+y+1}\right)^2\right\}\left\{1+\left(\frac{m-n}{x+y+2}\right)^2\right\}\&c}{\left\{1+\left(\frac{n}{y+1}\right)^2\right\}\left\{1+\left(\frac{n}{y+2}\right)^2\right\}\&c}$$

$$1+\frac{x^2+m^2}{(y+1)^2+n^2}+\frac{x^2+m^2}{(y+1)^2+n^2}\cdot\frac{(x-1)^2+m^2}{(y+2)^2+n^2}+\&c$$

$$+\frac{y^2+n^2}{(x+1)^2+m^2}+\frac{y^2+n^2}{(x+1)^2+m^2}\cdot\frac{(y-1)^2+n^2}{(x+2)^2+m^2}+\&c$$

$$\frac{f\left(\frac{a}{n},bn\right)f^3(-ab)}{nf(-a,-b)f\left(nab,\frac{1}{n}\right)}=\frac{1}{1+n}+\left(\frac{a}{n+ab}+\frac{b}{1+nab}\right)$$

$$+\left(\frac{a^2}{n+a^2b^2}+\frac{b^2}{1+na^2b^2}\right)+\&c.$$

$$\frac{\Pi(ax,x)}{\Pi(-bx,x)}=1+\frac{(a+b)x}{(1-x)(1-bx)}+\frac{(a+b)(a+bx)x^3}{(1-x)(1-x^2)(1-bx)(1-bx^2)}$$

$$+\frac{(a+b)(a+bx)(a+bx^2)x^6}{(1-x)(1-x^2)(1-x^3)(1-bx)(1-bx^2)(1-bx^3)}+\&c$$

$$\frac{ax}{1-x}+\frac{a^2x^2}{1-x^2}+\frac{a^3x^3}{1-x^3}+\frac{a^4x^4}{1-x^4}+\&c$$

$$=\frac{ax}{1-ax}\cdot\frac{1}{1-x}-\frac{a^2x^3}{(1-ax)(1-ax^2)}\cdot\frac{1}{1-x^2}+\frac{a^3x^6}{\cdots}$$

$$\Pi(ax,x)\left\{\frac{ax}{(1-x)(1-ax)}+\frac{2a^2x^4}{(1-x)(1-x^2)(1-ax)(1-ax^2)}+\&\right]$$

$$=\frac{ax}{1-x}-\frac{a^2x^3}{1-x^2}+\frac{a^3x^6}{1-x^3}-\frac{a^4x^{10}}{1-x^4}+\&c$$

$$\frac{a-b}{1-x}+\frac{a^2-b^2}{(1-x^2)}+\frac{a^3-b^3}{1-x^3}+\frac{a^4-b^4}{1-x^4}+\&c$$

$$=\frac{1}{1-x}\cdot\frac{a-b}{1-b}+\frac{1}{1-x^2}\cdot\frac{(a-b)(a-bx)}{(1-b)(1-bx)}+\frac{1}{1-x^3}\times$$

$$\frac{(a-b)(a-bx)(a-bx^2)}{(1-b)(1-bx)(1-bx^2)}+\&c$$

$$\frac{a}{1-x} + \frac{2a^2}{1-x^2} + \frac{3a^3}{1-x^3} + \frac{4a^4}{1-x^4} + \&c$$

$$= \frac{a}{1-x}\cdot\frac{1}{1-a} + \frac{a^2}{1-x^2}\cdot\frac{1-x}{(1-a)(1-ax)} +$$

$$\frac{a^3}{1-x^3}\cdot\frac{(1-x)(1-x^2)}{(1-a)(1-ax)(1-ax^2)} + \&c$$

$$2\cdot\frac{\psi(x)\psi(x^2)\psi(x^3)}{\psi(x^6)} - \phi^2(x^3)$$

$$= 1 + 2\left(\frac{x}{1-x} + \frac{x^5}{1-x^5} - \frac{x^7}{1-x^7} - \frac{x^{11}}{1-x^{11}} + \&c\right)$$

$$\psi(x)\psi(x^3) + \psi(-x)\psi(-x^3) = 2\psi(x^4)\phi(x^6)$$

$$\phi(x)\phi(x^3) + \phi(-x)\phi(-x^3) =$$

$$2\left\{1 + 6\left(\frac{x^4}{1-x^4} - \frac{x^8}{1-x^8} + \frac{x^{16}}{1-x^{16}} - \&c\right)\right\}.$$

$$1 + \left(\tfrac{1}{2}\right)^3 4x(1-x) + \left(\tfrac{1\cdot 3}{2\cdot 4}\right)^3\{4x(1-x)\}^2 + \&c = z^2.$$

$$1 + 4\cdot\left(\tfrac{1}{2}\right)^3 4x(1-x) + 7\cdot\left(\tfrac{1\cdot 3}{2\cdot 4}\right)^3\{4x(1-x)\}^2 + \&c$$

$$= \frac{1}{1-2x}\left\{1 - 24\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \&c\right)\right\}.$$

$$\frac{4}{\pi} = 1 + \frac{7}{4}\cdot\left(\tfrac{1}{2}\right)^3 + \frac{13}{4^2}\cdot\left(\tfrac{1\cdot 3}{2\cdot 4}\right)^3 + \frac{19}{4^3}\cdot\left(\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + \&c$$

$$\frac{16}{\pi} = 5 + \frac{47}{64}\left(\tfrac{1}{2}\right)^3 + \frac{89}{64^2}\left(\tfrac{1\cdot 3}{2\cdot 4}\right)^3 + \frac{131}{64^3}\left(\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + \&c$$

$$\frac{8(1+\sqrt{5})}{\pi} = (6+\sqrt{5}) + (66+19\sqrt{5})\cdot\left(\tfrac{1}{2}\right)^3\cdot\frac{\left(\frac{\sqrt{5}-1}{2}\right)^8}{64} + \&c$$

$$\frac{x}{1-x} + \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} + \frac{x^7}{1-x^7} + \cdots$$

$$= \frac{x}{1-x} + \frac{x^3}{1-x^2} + \frac{x^6}{1-x^3} + \frac{x^{10}}{1-x^4} + \frac{x^{15}}{1-x^5} + \cdots$$

If $\left.\begin{array}{l} x^3+ax+b=y \\ y^3+ay+b=x \end{array}\right\}$

then $(x^3+\overline{a-1}\,x+b)(x^2+\alpha x+\overline{\alpha^2+1+a})\times$
$(x^2+\beta x+\beta^2+1+a)(x^2+\gamma x+\gamma^2+1+a)=0$

where α, β, and γ are the roots of the equation

$$z^3+z(a+2)+b=0$$

If $p+q+r+s=x$

$qr+ps=a$

$pq^2+qs^2+rp^2+sr^2=b$

$pr^3+qp^3+rs^3+sq^3=a^2+c-3pqrs$

and $p^5+q^5+r^5+s^5=d+5(qr-ps)(pq^2-qs^2-rp^2+sr^2)$

then $x^5=5ax^3+5bx^2+5cx+d.$

$$\sqrt[3]{\cos 40}+\sqrt[3]{\cos 80}=\sqrt[3]{\cos 20}+\sqrt[3]{\tfrac{3}{2}(\sqrt[3]{9}-2)}$$

$$\sqrt[3]{\sec 40}+\sqrt[3]{\sec 80}=\sqrt[3]{\sec 20}+\sqrt[3]{6(\sqrt[3]{9}-1)}$$

If α, β, γ be the roots of $x^3-ax^2+bx-1=0$

then $\sqrt[3]{\frac{\alpha}{\beta}}+\sqrt[3]{\frac{\beta}{\gamma}}+\sqrt[3]{\frac{\gamma}{\alpha}}$ and
$\sqrt[3]{\frac{\beta}{\alpha}}+\sqrt[3]{\frac{\gamma}{\beta}}+\sqrt[3]{\frac{\alpha}{\gamma}}$ are the roots of

(1) $\begin{cases} z^2-tz+a+b+3 \text{ where} \\ t^3-3t(a+b+3)-(ab+6\overline{a+b}+9)=0. \end{cases}$

(2) $y^6-y^3(ab+6\overline{a+b}+9)+(a+b+3)^3=0$

$\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}=\sqrt[3]{6+10a+9b+\frac{ab+9}{2}}$

$=\sqrt[3]{a+6+3\sqrt[3]{\frac{ab+9}{2}+3(a+b)+\sqrt{\left(\frac{ab+9}{2}\right)^2-(a^2+\ldots+27}}}$
$+3\sqrt[3]{\ldots do-\sqrt{do}}$

NOTEBOOK 3

1

1. $a_1 e^{-b_1 x} + a_2 e^{-b_2 x} + a_3 e^{-b_3 x} + \cdots\cdots$
$- \int^{\infty} a_z e^{-b_z x}\, dz$ is finite when x is 0?

2. $\frac{1}{a_1^{\,n+1}} + \frac{1}{a_2^{\,n+1}} + \frac{1}{a_3^{\,n+1}} + \cdots\cdots - \int \frac{dz}{a_z^{\,n+1}}$
is finite when n is 0?

3. If $a_1 e^{-x} + a_2 e^{-2x} + a_3 e^{-3x} + \cdots\cdots - \frac{c}{x^r}$
is finite when x is 0,
then the average value of a_n is $\frac{c\, n^{r-1}}{\lfloor r-1}$.

4. If $\int_0^{\infty} \phi(z)\, e^{-az}\, dz = \frac{1}{a^{r+1}}$ then $\phi(z) = \frac{z^r}{\lfloor r}$.

5. i. The coeff.t of x^{100} in $\frac{x^7}{(1-x^2)(1-x^3)}$ = coeff.t of x^{95}
in $\frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)} = I\left(\frac{95}{2}\right) - I\left(\frac{95}{3}\right) = 16$.

ii. $I\left(\frac{n+4}{6}\right) - I\left(\frac{n+3}{6}\right) + I\left(\frac{n+2}{6}\right) = I\left(\frac{n}{2}\right) - I\left(\frac{n}{3}\right)$

proof. $\frac{x^2}{(1-x^2)(1-x^3)} = \frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)}$

or $\frac{x^2 - x^3 + x^4}{(1-x)(1-x^6)}$.

iii. $I\left(\sqrt{n+1} + \sqrt{n}\right) = I\left(\sqrt{4n+2}\right)$.

iv. The coeff.t of x^n in $\frac{\psi(x^2)}{1-x} = I\left(\frac{1}{2} + \sqrt{n+\frac{1}{2}}\right)$
$= I\left(\frac{1}{2} + \sqrt{n+\frac{1}{4}}\right)$.

6. If $N = a^p . b^q . c^r \ldots$ where a, b, c are primes, the no. of divisors of n is
$(p+1)(q+1)(r+1)(s+1)\ldots\ldots$

2

7. If N is formed of the prime no a and its powers alone then

i) The no of divisors will never exceed $\frac{\log aN}{\log a}$

ii) if formed of primes a, b then it will not exceed $\left(\frac{\log abN}{2}\right)^2 / \log a \log b$

iii) if formed of a, b, c ..... n primes the no. of divisors will never exceed

$$\frac{\left\{\frac{\log(Nabc\text{-----})}{n}\right\}^n}{\log a \log b \log c \text{ ------}}$$

8. If A, B, C are quantities so taken that

$$\frac{1}{A^k} + \frac{1}{B^k} + \frac{1}{C^k} + \text{-----} = \frac{a}{(k-\alpha)^r}$$

when $k = \alpha$ (the only pole) then the no. of such quantities less than z is

$$\int \frac{a\,(\log z)^{r-1}}{z^{1-\alpha}\,\lfloor r-1}\,dz$$

For $\int^\infty \frac{dn}{z^k} = \frac{a}{(k-\alpha)^r}$ and $\int \frac{\infty\, dz}{z^{k-\alpha+1}} = \frac{1}{k-\alpha}$

Differentiating r times with respect to k. we get the above result.

9. No. of the form p^2+q^2

$$\frac{1}{1^k}+\frac{1}{2^k}+\frac{1}{4^k}+\frac{1}{5^k}+\frac{1}{8^k}+\cdots$$

$$=\frac{1}{1-2^{-k}}\cdot\frac{1}{1-5^{-k}}\cdot\frac{1}{1-13^{-k}}\cdot\frac{1}{1-17^{-k}}\cdots$$

$$\times\frac{1}{1-3^{-2k}}\cdot\frac{1}{1-7^{-2k}}\cdot\frac{1}{1-11^{-2k}}\cdots$$

$$=\frac{\sqrt{s_k\cdot s'_k}}{1-2^{-k}}\sqrt{\frac{1}{1-3^{-2k}}\cdot\frac{1}{1-7^{-2k}}\cdot\frac{1}{1-11^{-2k}}\cdots}$$

where $s_k=\frac{1}{1^k}+\frac{1}{3^k}+\frac{1}{5^k}+\cdots$

and $s'_k=\frac{1}{1^k}-\frac{1}{3^k}+\frac{1}{5^k}-\cdots$

$$\frac{s_k}{s'_k}=\frac{1+3^{-k}}{1-3^{-k}}\cdot\frac{1+7^{-k}}{1-7^{-k}}\cdot\frac{1+11^{-k}}{1-11^{-k}}\cdots$$

Hence the series =

$$\frac{s'_k}{1-2^{-k}}\sqrt{\frac{s_k}{s'_k}}\cdot\sqrt[4]{\frac{s_{2k}}{s'_{2k}}}\sqrt[8]{\frac{s_{4k}}{s'_{4k}}}\sqrt[16]{\frac{s_{8k}}{s'_{8k}}}\cdots$$

$$=\frac{A}{\sqrt{k-1}}+\frac{B}{\sqrt[4]{2k-1}}+\frac{C}{\sqrt[8]{4k-1}}+\frac{D}{\sqrt[16]{8k-1}}+\cdots$$

where $A=\sqrt{\frac{\pi}{2\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\cdots}}$ and

B, C, D are depending upon A.

Hence the reqd nos between m and n

is $C\int_n^m\frac{dx}{\sqrt{\log x}}+\theta(x)$ where $C=\frac{1}{\sqrt{2\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{7^2}\right)\cdots}}$

and $\theta(x)$ is of the order $\frac{\sqrt{x}}{(\log x)^{\frac{3}{4}}}$.

obs. $\sqrt{2\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\left(1-\frac{1}{19^2}\right)}=\left(1+\frac{1}{7}\right)\left(1+\frac{1}{11}\right)\left(1+\frac{1}{19}\right)$

$$(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \cdots\cdots = a_1 - \underset{n=\infty}{Lt}\, a_n$$

$$\frac{a_1}{a_2} \cdot \frac{a_2}{a_3} \cdot \frac{a_3}{a_4} \cdots\cdots = \underset{n=\infty}{Lt}\left(\frac{a_1}{a_n}\right)$$

$$\frac{1}{x} - \cot x = \frac{1}{2}\tan\frac{x}{2} + \frac{1}{4}\tan\frac{x}{4} + \frac{1}{8}\tan\frac{x}{8} +$$

$$\frac{1}{2x} - \frac{1}{4}\cot\frac{x}{2} = \frac{\sin\frac{x}{3}}{3\left(1 + 2\cos\frac{x}{3}\right)} + \frac{\sin\frac{x}{9}}{9\left(1 + 2\cos\frac{x}{9}\right)} + \cdots$$

$$\frac{3}{4}\sin x = \sin^3 x + \frac{\sin^3 3x}{3} + \frac{\sin^3 9x}{9} + \cdots$$

$$\frac{1}{x} - \cot x = \left(\frac{1}{\sin x} - \frac{1}{x}\right) + \left(\frac{1}{\sin\frac{x}{2}} - \frac{2}{x}\right) +$$

$$\left(\frac{1}{\sin\frac{x}{4}} - \frac{4}{x}\right) + \cdots\cdots$$

$$\frac{2x}{\sin 2x} = \frac{\tan x}{x} \cdot \left(\frac{2}{x}\tan\frac{x}{2}\right)^2 \left(\frac{4}{x}\tan\frac{x}{4}\right)^4 \cdots$$

$$\frac{1}{\log x} + \frac{1}{1-x} = \frac{1}{2(1+\sqrt{x})} + \frac{1}{4(1+\sqrt[4]{x})} + \frac{1}{8(1+\sqrt[8]{x})} + \cdots$$

$$\frac{1}{\log x} + \frac{1}{1-x} = \frac{2 + \sqrt[3]{x}}{3\left(1 + \sqrt[3]{x} + \sqrt[3]{x^2}\right)} + \frac{2 + \sqrt[9]{x}}{9\left(1 + \sqrt[9]{x} + \sqrt[9]{x^2}\right)} + \cdots$$

$$\left\{1^2 \log\left(1 + \frac{x^2}{1^2}\right) - x^2\right\} + \left\{2^2 \log\left(1 + \frac{x^2}{2^2}\right) - x^2\right\} +$$

$$\left\{3^2 \log\left(1 + \frac{x^2}{3^2}\right) - x^2\right\} + \cdots\cdots = \frac{x^2}{2} - \frac{\pi x^3}{3}$$

$$+ \frac{x}{\pi}\left(\frac{e^{-2\pi x}}{1^2} + \frac{e^{-4\pi x}}{2^2} + \frac{e^{-6\pi x}}{3^2} + \cdots\cdots\right)$$

$$- x^2 \log\left(1 - e^{-2\pi x}\right) - \frac{1}{2\pi^2}\left\{\frac{1 - e^{-2\pi x}}{1^3} + \frac{1 - e^{-4\pi x}}{2^3}\right.$$

$$\left. + \frac{1 - e^{-6\pi x}}{3^3} + \cdots\cdots\cdots\right\}$$

$$\frac{1}{2}\left\{\log\left(1+\frac{1}{n}\right)\right\}^2 = \frac{\frac{1}{n}-\frac{1}{n+1}}{1^2} - \frac{\frac{1}{n^2}+\frac{1}{(n+1)^2}}{2^2}$$

$$+\frac{\frac{1}{n^3}-\frac{1}{(n+1)^3}}{3^2} - \frac{\frac{1}{n^4}+\frac{1}{(n+1)^4}}{4^2} + \cdots$$

If $x = \left(\dfrac{\log\frac{1+\sqrt{5}}{2}}{\pi}\right)^2$ then $e^{\frac{x}{2}} = \dfrac{1+x}{e^{x}} \cdot \dfrac{\left(1+\frac{x}{4}\right)^4}{e^x} \cdot \dfrac{\left(1+\frac{x}{9}\right)^9}{e^x}$

$$\lfloor x\, e^{Cx} = \frac{e^x}{1+x} \cdot \frac{e^{\frac{x}{2}}}{1+\frac{x}{2}} \cdot \frac{e^{\frac{x}{3}}}{1+\frac{x}{3}} \cdot \frac{e^{\frac{x}{4}}}{1+\frac{x}{4}} \cdots$$

If $\phi(x) = \dfrac{x}{1^2} + \dfrac{x^2}{2^2} + \dfrac{x^3}{3^2} + \dfrac{x^4}{4^2} + \cdots$

then $\displaystyle\int \frac{\log(p+qx)}{r+sx}\,dx$, $\displaystyle\int \log(p+qx)\log(r+sx)\,dx$ and similar integrals, as well as the values of $\phi\left(\frac{1}{3}\right) - \frac{1}{6}\phi\left(\frac{1}{9}\right)$, $\phi\left(-\frac{1}{2}\right) + \frac{1}{6}\phi\left(\frac{1}{9}\right)$, $\phi\left(\frac{1}{4}\right) + \frac{1}{3}\phi\left(\frac{1}{9}\right)$, $\phi\left(-\frac{1}{3}\right) - \frac{1}{3}\phi\left(\frac{1}{9}\right)$, $\phi\left(\frac{1}{8}\right) + \phi\left(\frac{1}{9}\right)$, &c can be found. $\displaystyle\int_0^1 \frac{\log\frac{1+\sqrt{1+4a}}{2}}{a}\,da = \frac{\pi^2}{15}$ &c

$$\sqrt{1+a}\left\{1 + \frac{a e^{-x}}{1-e^{-x}} + \frac{a^2 e^{-3x}}{(1-e^{-x})(1-e^{-2x})} + \cdots\right\}$$

$$= e^{\frac{1}{x}\left(\frac{a}{1^2} - \frac{a^2}{2^2} + \frac{a^3}{3^2} - \right) + \frac{B_2}{\lfloor 2}\cdot\frac{x}{1+a}\cdot a - \frac{B_4}{\lfloor 4}\cdot\left(\frac{x}{1+a}\right)^3(a-a^2) + \frac{B_6}{\lfloor 6}\left(\frac{x}{1+a}\right)^5(a-11a^2+11a^3-a^4) - \&c}\ ?$$

$$1 + \frac{a e^{-x}}{1-e^{-x}} + \frac{a^2 e^{-4x}}{(1-e^{-x})(1-e^{-2x})} + \frac{a^3 e^{-9x}}{(1-e^{-x})(1-e^{-2x})(1-e^{-3x})} \cdots$$

$$= \sqrt{\frac{1+h}{1+2h}}\; e^{\frac{1}{x}\left\{\frac{1}{2}(\log 1+h)^2 + \frac{h}{1^2} - \frac{h^2}{2^2} + \frac{h^3}{3^2} - \&c\right\}}$$

where $h + h^2 = a$

6

If $az^{2n}+z=1$, then

when x is very small the value of the series

$$1+\frac{ae^{-nx-mx}}{1-e^{-x}}+\frac{a^2e^{-4nx-2mx}}{(1-e^{-x})(1-e^{-2x})}+\frac{a^3e^{-9nx-3mx}}{(1-e^{-x})(1-e^{-2x})(1-e^{-3x})}+\cdots$$

$$=\frac{z^m e^{\frac{1}{x}\int_0 \frac{\log\frac{1}{z}}{a}da}}{\sqrt{z+2n(1-z)}}+(Ax+Bx^2+\cdot)$$

$$\frac{2\pi}{\log 2}=9.0647203;\quad \left\{\begin{array}{l}\log\left(\frac{2\pi}{\log 2}\right)=2.20437894\\ \frac{2\pi^2}{\log 2}=28.4776587\end{array}\right.$$

$$\log 2\left\{e^{-x}+2e^{-2x}+4e^{-4x}+8e^{-8x}+\cdots+1-\frac{x}{3\lfloor 1}+\frac{x^2}{7\lfloor 2}-\frac{x^3}{15\lfloor 3}+\frac{x^4}{31\lfloor 4}-\&c\right\}$$

$$=\frac{1+.0000098844\cos\left(\frac{2\pi\log x}{\log 2}+.872811\right)}{x}$$

If $(a_1-a_2+a_3-\&c)(b_1-b_2+b_3-\&c)=u_1-u_2+\cdots$

then u_n is known when n is very great

from (1) $\int_c^\infty u_n e^{-nh}dn=\int_c^\infty a_n e^{-nh}dn\times$

$\int_0^\infty b_n e^{-nh}dn$ where h is a very small quantity

(2) $\int_0^{n-1} a_{1+z}b_{n-z}\,dz = u_n$ $\Big|$ $\frac{a_n\int b_n dn}{u_n}$ or

$\frac{\int a_n dn\int b_n dn}{n u_n}$ is finite when $n=\infty$??

The equation $\left.\begin{aligned} x^2 &= a+y \\ y^2 &= a+z \\ z^2 &= a+u \\ u^2 &= a+x \end{aligned}\right\}$

is a 16th degree equation which can be reduced to four quartics of which one quartic is known by inspection and each of the three remaining quartics is of the form $\left(x^2+px+\frac{p^2-2a-\frac{4}{p}}{2}\right)\left(x^2+qx+\frac{q^2-2a-\frac{4}{q}}{2}\right)$

where $pq=-1$ and $p+q$ is a root of the

equation $z^3+3z=4(1+az)$.

or p is a root of $z^6-4az^4-4z^3+4az^2-1=0$

$$\sqrt{5+\sqrt{5+\sqrt{5-\sqrt{5+\sqrt{5+\sqrt{5+\sqrt{5-\&c}}}}}}}$$
$$=\frac{2+\sqrt{5}+\sqrt{15-6\sqrt{5}}}{2}$$

$$\sqrt{5+\sqrt{5-\sqrt{5-\sqrt{5+\sqrt{5+\sqrt{5-\sqrt{5-\sqrt{5+\&c}}}}}}}}$$
$$=\frac{\sqrt{5}-2+\sqrt{13-4\sqrt{5}}+\sqrt{50+12\sqrt{5}-2\sqrt{65-20\sqrt{5}}}}{4}$$

Similarly for the product of three or more series; the value of u_n is found from the value of the series $u_1e^{-x}+u_2e^{-2x}+\cdots$ when $x=0$; e.g. $\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\&c\right)\left(1-\frac{1}{2}+\frac{1}{3}-\&c\right)$ is found from $u_1e^{-x}+u_2e^{-2x}+u_3e^{-3x}+\cdots$
$=\left(\frac{e^{-x}}{\sqrt{1}}+\frac{e^{-2x}}{\sqrt{2}}+\frac{e^{-3x}}{\sqrt{3}}-\cdots\right)\left(\frac{e^{-x}}{1}+\frac{e^{-2x}}{2}+\frac{e^{-3x}}{3}+\cdots\right)$.

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1	2	3	4	5	6	7	8	9	10
12	14	15	16	18	20	21	24	25	27
28	30	32	35	36	40	42	45	48	49
50	54	56	60	63	64	70	72	75	80
81	84	90	96	98	100	105	108	112	120
125	126	128	135	140	144	147	150	160	162
168	175	180	189	192	196	200	210	216	224
225	240	243	245	250	252	256	270	280	288
294	300	315	320	324	336	343	350	360	375
378	384	392	400	405	420	432	441	448	450
480	486	490	500	504	512	525	540	560	567
576	588	600	625	630	640	648	672	675	686
700	720	729	735	750	756	768	784	800	810
840	864	875	882	896	900	945	960	972	980
1000	1008	1024	1029	1050	1080	1120	1125	1134	1152
1176	1200	1215	1225	1250	1260	1280	1296	1323	1344
1350	1372	1400	1440	1458	1470	1500	1512	1536	1568
1575	1600	1620	1680	1701	1715	1728	1750	1764	1792
1800	1875	1890	1920	1944	1960	2000	2016	2025	2048
2058	2100	2160	2187	2205	2240	2250	2268	2304	2352
2400	2401	2430	2450	2500	2520	2560	2592	2625	2646
2688	2700	2744	2800	2835	2880	2916	2940	3000	3024
3072	3087	3125	3136	3150	3200	3240	3360	3375	3402
3430	3456	3500	3528	3584	3600	3645	3675	3750	3780
3840	3888	3920	3969	4000	4032	4050	4096	4116	4200
4320	4374	4375	4410	4480	4500	4536	4608	4704	4725
4800	4802	4860	4900	5000	5040	5103	5120	5145	5184
5250	5292	5376	5400	5488	5600	5625	5670	5760	5832

5880	6000	6048	6075	6125	6144	6174	6250	6272	6300
6400	6480	6561	6615	6720	6750	6804	6860	6912	7000
7056	7168	7200	7203	7290	7350	7500	7560	7680	7776
7840	7875	7938	8000	8064	8100	8192	8232	8400	8505
8575	8640	8748	8750	8820	8960	9000	9072	9216	9261
9375	9408	9450	9600	9604	9720	9800	10000	10080	10125
10206	10240	10290	10368	10500	10584	10752	10800	10935	10976
11025	11200	11250	11340	11520	11664	11760	11907	12000	12005

Let $\sqrt[7]{1} = 1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$ such that $\alpha+\alpha^2+\alpha^4$ & $\alpha^3+\alpha^6+\alpha^5$ are $\dfrac{-1\pm i\sqrt{7}}{2}$

$$\phi(x) = \sin x + \sin x\alpha + \sin x\alpha^2 + \&c$$
$$= -7\left(\frac{x^7}{\lfloor 7} - \frac{x^{21}}{\lfloor 21} + \frac{x^{35}}{\lfloor 35} - \&c\right)$$

$$64 \sin x \sin x\alpha \sin x\alpha^2 \&c$$
$$= -\phi(2x) - \phi(2x.\overline{\alpha+\alpha^2+\alpha^4}) - \phi(2x.\overline{\alpha^3+\alpha^6+\alpha^5})$$
$$+\phi(2x.\overline{\alpha+\alpha^6}) + \phi(2x.\overline{\alpha^2+\alpha^5}) + \phi(2x.\overline{\alpha^3+\alpha^4})$$
$$+\phi\left(\frac{2x}{\alpha+\alpha^6}\right) + \phi\left(\frac{2x}{\alpha^2+\alpha^5}\right) + \phi\left(\frac{2x}{\alpha^3+\alpha^4}\right)$$

Similarly $\psi(x) = \cos x + \ldots$

$\psi(2x) + \psi(2x.\overline{\alpha+\alpha^2+\alpha^4}) + \&c$ all plus.

$= 64 \cos x . \cos x\alpha \cos x\alpha^2$ & ...

from which 7 interval formula can be found

6 interval formula

If $a_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$

so that $a_m a_n = a_{m+n} + a_{m-n}$, then

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$$6\left(B_0 + B_{12}\frac{x^{12}}{\underline{|12}} + B_{24}\frac{x^{24}}{\underline{|24}} + \&c.\right)$$

$$= -x.\ \frac{\frac{x^5}{\underline{|5}}(a_3 + 2^3) - \frac{x^{17}}{\underline{|17}}(a_9 + 2^9) + \&c}{\frac{x^6}{\underline{|6}}(a_3 + 2^3) - \frac{x^{18}}{\underline{|18}}(a_9 + 2^9) + \&c}$$

$$6.\left(B_2\frac{x^2}{\underline{|2}} + B_{14}\frac{x^{14}}{\underline{|14}} + B_{26}\frac{x^{26}}{\underline{|26}} + \&c.\right)$$

$$= x.\ \frac{\frac{x^7}{\underline{|7}}.(a_4 + 2^4) - \frac{x^{19}}{\underline{|19}}(a_{10} + 2^{10}) + \cdots}{\frac{x^6}{\underline{|6}}(a_3 + 2^3) - \frac{x^{18}}{\underline{|18}}(a_9 + 2^9) + \&c}$$

$$6\left(B_6\frac{x^6}{\underline{|6}} + B_{18}\frac{x^{18}}{\underline{|18}} + \&c\right)$$

$$= x.\ \frac{\frac{x^{11}}{\underline{|11}}(a_5 - 2^6) - \frac{x^{23}}{\underline{|23}}(a_{11} - 2^{12}) + \cdots}{\frac{x^6}{\underline{|6}}(a_3 + 2^3) - \frac{x^{18}}{\underline{|18}}(a_9 + 2^9) + \cdots}$$

$$6\left(B_8\frac{x^8}{\underline{|8}} + B_{20}\frac{x^{20}}{\underline{|20}} + \&c\right)$$

$$= x.\ \frac{\frac{x^{13}}{\underline{|13}}(a_6 - 2^7) - \frac{x^{25}}{\underline{|25}}(a_{12} - 2^{13}) + \&c}{\frac{x^6}{\underline{|6}}(a_3 + 2^3) - \frac{x^{18}}{\underline{|18}}(a_9 + 2^9) + \&c}$$

$$\frac{x}{1+x} - \frac{x^2}{1+x^3} + \frac{x^4}{1+x^5} - \frac{x^5}{1+x^7} + \&c$$

$$= \frac{1}{4} + \frac{1}{4\sqrt{2}}\log(1+\sqrt{2}) - \frac{\pi}{8\sqrt{2}} \quad \text{when } x=1$$

$$\frac{1^3 x}{1-x} - \frac{2^3 x^2}{1-x^2} + \frac{3^3 x^3}{1-x^3} - \frac{4^3 x^4}{1-x^4} + \cdots = \frac{1}{16}$$

(Examples): when $x = 1$.

Numbers of the form $x^3 \pm y^3$. 11

1 2 7 8 9 16 19 26 27 28 35 37 54 56 61
63 64 65 72 91 98 117 124 125 126 127 128 133 152 169
189 208 215 216 217 218 224 243 250 271 279 280 296 316 331
335 341 342 343 344 351 370 386 387 397 407 432 448 469 485
488 504 511 512 513 520 539 547 559 576 602 604 631 637 657
665 686 702 721 728 729 730 737 756 784 793 817 819 854 855
866 875 919 936 945 973 989 992 999 1000.

Long intervals of Composite nos

23 to 29; 89 to 97; 113 to 127; 523 to 541; 887 to 907;
1129 to 1151; 1327 to 1361; 9551 to 9587;
15683 to 15727; 19609 to 19661.
31397 to 31469; 265621 to 265703
360653 to 360749; 492113 to 492227
1561919 to 1562051. (below 2000000).
(370261 to 370373; 1357201 to 1357333; 2010733 to 2010881)

2 to 3; 3 to 5; and 7 to 11

No of primes

10^5	9592	6.10^5	49098	2.10^6	148931
2.10^5	17984	7.10^5	56543	3.10^6	216816
3.10^5	25997	8.10^5	63951	10^7	664579
4.10^5	33860	9.10^5	71274	10^8	5761460
5.10^5	41538	10^6	78498	2.10^4	2262

If P be any prime no. and there are k primes between P and $P + \phi(P, k)$, to find the max., min, & average value of ϕ.

1√

The highest Composite no. can be found from

$$\mp\, 2^{\left[\frac{\log n}{\log 2}\right]-1}\; 3^{\left[\frac{\log n}{\log 3}\right]-1}\; 5^{\left[\frac{\log n}{\log 5}\right]-1}\; 7^{\left[\frac{\log n}{\log 7}\right]-1}$$

where n is any positive quantity and [ ] mean-ing that the ~~integral~~ nearest integer is taken.

The highest Composite no. near the region N is

$$2^{\frac{\log K}{\log 2}-1}\cdot 3^{\frac{\log K}{\log 3}-1}\cdot 5^{\frac{\log K}{\log 5}-1}\cdot 7^{\frac{\log K}{\log 7}-1}\cdot$$

where $K = \dfrac{\log(N.\,2.\,3.\,5.\ \ldots\ \theta K)}{n}$ where ?

θK is a prime no. between K and $\frac{K}{e}$?

and n the no. of primes from 2 to θK ?

~~$K = \dfrac{\log(N.\,2.\,3.\,5.\ \ldots\ Z)}{n}$ where~~ { ~~Z is a prime just greater than $\log N$? and n the no. of primes from 2 to Z~~

The order of the no. of divisors of a highly Composite no. $N = e^{\frac{\log N}{\log\log N}}$?

If N be of the form $\lfloor x$ then the order is $e^{\frac{\log N}{(\log\log N)^2}}$.

$$\frac{\chi(-x^2)\, f(-x^5)}{f(-x, -x^4)} =$$

$$x.\ \frac{\chi(-x^4)\, f(-x^5)}{f(-x^2, -x^3)}$$

$$\frac{f(-x,-x^5)}{f(-x^3,-x^3)} = \frac{1}{1+}\,\frac{x+x^2}{1+}\,\frac{x^2+x^4}{1+}$$

$$\begin{cases} \text{Num}^r = \dfrac{\phi(-x^3)}{f(-x)} \\ \text{Den}^r = \dfrac{\psi(x^3)}{f(-x^4)} \end{cases}$$

$$\frac{f(-x,-x^7)}{f(-x^3,-x^5)} = \frac{1}{1+}\,\frac{x+x^2}{1+}\,\frac{x^4}{1+}\,\frac{x^3+x^6}{1+}\cdots$$

If $\phi(a) = 1 + \dfrac{ax}{(1-x)(1-bx)} + \dfrac{a^2x^4}{(1-x)(1-x^2)(1-bx)(1-bx^2)}$

then $\dfrac{\phi(a)}{\phi(ax)} = 1 + \dfrac{ax}{1+}\,\dfrac{ax^2-bx}{1+}\,\dfrac{ax^3}{1+}\,\dfrac{ax^4-bx^2}{1+}$

If $u^n - u^{n-1} = v$

and $\displaystyle\int_0^1 \frac{\log u}{v}\,dv = \phi(n)$

then $\phi(0) = \dfrac{\pi^2}{6}$; $\phi(1) = \dfrac{\pi^2}{12}$; $\phi(2) = \dfrac{\pi^2}{15}$

$$\phi(n) + \phi\left(\tfrac{1}{n}\right) = \frac{\pi^2}{6}.$$

$$\frac{f(x,x^9)}{f(-x^4,-x^{16})} = 1 + \frac{x}{1-x^4} + \frac{x^4}{(1-x^4)(1-x^8)} + \frac{x^9}{(1-x^4)\cdots}$$

$$x\frac{f(x^3,x^7)}{f(-x^8,-x^{12})} = x + \frac{x^4}{1-x^4} + \frac{x^9}{(1-x^4)(1-x^8)} + \frac{x^{16}}{(1-x^4)\cdots}$$

and the ratio

$$= \frac{x}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+}\,\frac{x^4}{1+}\cdots$$

$$\text{If } \phi(a) = 1 + \frac{ax}{(1-x)(1+bx)} + \frac{a^2x^3}{(1-x)(1-x^2)(1+bx)(1+bx^2)}$$

$$\text{then } \frac{\phi(a)}{\phi(ax)} = 1 + \frac{ax}{1+}\,\frac{bx}{1+}\,\frac{ax^2}{1+}\,\frac{bx^2}{1+}\,\frac{ax^3}{1+bx^3}\,\frac{}{1+}$$

If $x > 1$ $\quad \dfrac{1}{1+}\,\dfrac{x}{1+}\,\dfrac{x^2}{1+}\,\dfrac{x^3}{1+}$

oscillates between

$$1 - \frac{x^{-1}}{1+}\,\frac{x^{-2}}{1-}\,\frac{x^{-3}}{1+} - - -$$

and $\quad \dfrac{x^{-1}}{1+}\,\dfrac{x^{-4}}{1+}\,\dfrac{x^{-8}}{1+}\,\dfrac{x^{-12}}{1+}$

$$e^{\frac{\pi}{4}\sqrt{30}} = 4\sqrt{3}\,(5 + 4\sqrt{2})$$

$$e^{\frac{\pi}{4}\sqrt{34}} = 12\,(4 + \sqrt{17})$$

$$e^{\frac{\pi}{2}\sqrt{46}} = 12^2\,(147 + 104\sqrt{2})$$

$$e^{\frac{\pi}{4}\sqrt{42}} = 4\,(21 + 8\sqrt{6})$$

$$e^{\frac{\pi}{4}\sqrt{70}} = 12\sqrt{7}\,(5\sqrt{5} + 8\sqrt{2})$$

$$e^{\frac{\pi}{4}\sqrt{78}} = 4\sqrt{3}\,(75 + 52\sqrt{2})$$

$$e^{\frac{\pi}{4}\sqrt{102}} = 4\sqrt{3}\,(200 + 49\sqrt{17})$$

$$e^{\frac{\pi}{4}\sqrt{130}} = 12\,(323 + 40\sqrt{65})$$

$$\pi = \frac{12}{\sqrt{130}} \log \frac{(3+\sqrt{13})(\sqrt{8}+\sqrt{10})}{2} \quad \text{to 15 dec.}$$

$$= \frac{24}{\sqrt{142}} \log\left(\frac{\sqrt{10+11\sqrt{2}}+\sqrt{10+7\sqrt{2}}}{2}\right) \quad \text{to 16 dec.}$$

$$= \frac{12}{\sqrt{190}} \log (3+\sqrt{10})(\sqrt{8}+\sqrt{10}) \quad \text{to 18 dec.}$$

$$\sqrt[4]{3^4+2^4+\frac{1}{2+\left(\frac{2}{3}\right)^2}} = 3.14159265262\ldots$$

$$\pi = 3.14159265358\ldots$$

$$+\frac{12}{\sqrt{n}}\left\{\frac{1}{\sinh \pi\sqrt{n}}+\frac{1}{2\sinh 2\pi\sqrt{n}}+\frac{1}{3\sinh 3\pi\sqrt{n}}+\right\}$$

$$\left.\begin{array}{l} \pi = 3.141592\ldots \\ \frac{9}{5}+\sqrt{\frac{9}{5}} = 3.14164\ldots \end{array}\right\} \text{error } .00005.$$

$$e^{\pi\sqrt{22}} = 2508951.9982$$

$$e^{\pi\sqrt{37}} = 199148647.999978$$

$$e^{\pi\sqrt{58}} = 24591257751.9999982\ldots$$

16

$$\frac{63}{25} \cdot \frac{17+15\sqrt{5}}{7+15\sqrt{5}} = 3.14159265380.$$

$$\frac{7}{3}\left(1+\frac{\sqrt{3}}{5}\right) = 3.14162$$

$$\frac{19}{16}\sqrt{7} = 3.14180.$$

17

3 — 5 — 2
7 — 11 — 4
[illegible]
[illegible] — 97 — 8
113 — 127 — 14

$$\frac{27}{4\pi} = 2 + 17 \cdot \frac{1}{2} \cdot \frac{1.2}{3^2} \cdot \left(\frac{2}{27}\right) + \cdots$$

$$\frac{15\sqrt{3}}{2\pi} = 4 + 37 \cdot \frac{1}{2} \cdot \frac{1.2}{3^2} \cdot \left(\frac{4}{125}\right) + \cdots$$

$$\frac{5\sqrt{5}}{2\pi\sqrt{3}} = 1 + 12 \cdot \frac{1}{2} \cdot \frac{1.5}{6^2} \cdot \left(\frac{4}{125}\right) + \cdots$$

$$\frac{85\sqrt{85}}{18\pi\sqrt{3}} = 8 + 141 \cdot \frac{1}{2} \cdot \frac{1.5}{6^2} \left(\frac{4}{85}\right)^3 + \cdots$$

$$\frac{4}{\pi} = \frac{3}{2} - \frac{23}{2^3} \cdot \frac{1}{2} \cdot \frac{1.3}{4} + \cdots$$

$$\frac{4}{\pi\sqrt{3}} = \frac{3}{4} - \frac{31}{3 \cdot 4^3} \cdot \frac{1}{2} \cdot \frac{1.3}{4^2} + \cdots$$

$$\frac{4}{\pi} = \frac{23}{18} - \frac{283}{18^3} \cdot \frac{1}{2} \cdot \frac{1.3}{4^2} + \cdots$$

$$\frac{4}{\pi\sqrt{5}} = \frac{41}{72} - \frac{685}{5 \cdot 72^3} \cdot \frac{1}{2} \cdot \frac{1.3}{4^2} + \cdots$$

$$\frac{4}{\pi} = \frac{1123}{882} - \frac{22583}{882^3} \cdot \frac{1}{2} \cdot \frac{1.3}{4^2} + \cdots$$

$$\frac{2}{\pi\sqrt{3}} = \frac{1}{3} + \frac{9}{3^3} \cdot \frac{1}{2} \cdot \frac{1.3}{4^2} + \cdots$$

$$\frac{1}{2\pi\sqrt{2}} = \frac{1}{9} + \frac{11}{9^3} \cdot \frac{1}{2} \cdot \frac{1.3}{4^2} + \cdots$$

$$\frac{1}{3\pi\sqrt{3}} = \frac{3}{49} + \frac{43}{49^3} \cdot \frac{1}{2} \cdot \frac{1.3}{4^2} + \cdots$$

$$\frac{2}{\pi\sqrt{11}} = \frac{19}{99} + \frac{299}{99^3} \cdot \frac{1}{2} \cdot \frac{1.3}{4^2} + \cdots$$

$$\frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2} + \frac{27493}{99^6} \cdot \frac{1}{2} \cdot \frac{1.3}{4^2} + \cdots$$

$I\left(\frac{n}{a}\right) + I\left(\frac{n}{a^2}\right) + \cdots$ lies between

$\frac{n-1}{a-1}$ and $\frac{n}{a-1} - \frac{\log(n+1)}{\log a}$

a and n being integers.

$\underline{n = 2}$

$$2\left\{1 - 24\left(\frac{q^4}{1-q^4} + \frac{2q^8}{1-q^8} \cdots\right)\right\}$$
$$- \left\{1 - 24\left(\frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} \cdots\right)\right\}$$
$$= \frac{4KL}{\pi^2}(k' + l)$$

$n = 4$

$$4\left\{1 - 24\left(\frac{q^8}{1-q^8} + \frac{2q^{16}}{1-q^{16}} + \cdots\right)\right\}$$
$$- \left\{1 - 24\left(\frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} + \cdots\right)\right\}$$
$$= \frac{12KL}{\pi^2}\left(\sqrt{k'} + \sqrt{l}\right)^2$$

If $x > 1$, then $\frac{1}{1+}\ \frac{x}{1+}\ \frac{x^2}{1+} \cdots$ oscillates between

$$1 - \frac{x-1}{1+}\ \frac{x-2}{1-}\ \frac{x-3}{1+} \cdots$$

and $\frac{x-1}{1+}\ \frac{x-4}{1+}\ \frac{x-8}{1+}\ \frac{x-12}{1+} \cdots$

41. $z = x + 4_x \quad 4_x = 6/2kk'$

$$z^2 - z \cdot \frac{5 \pm \sqrt{41}}{2} + \frac{7 \pm \sqrt{41}}{2}$$

$$I\left(\frac{x}{1^n}\right) + I\left(\frac{x}{2^n}\right) + I\left(\frac{x}{3^n}\right) + \cdots$$

$$= x S_n + x^{\frac{1}{n}} S_{\frac{1}{n}} + O(x^{\varepsilon}).$$

$$= I\left(\sqrt[n]{\tfrac{x}{1}}\right) + I\left(\sqrt[n]{\tfrac{x}{2}}\right) + I\left(\sqrt[n]{\tfrac{x}{3}}\right) + \cdots$$

If ~~a_1, a_2, a_3 &c are increasing + ve numbers, then~~ and $a_n - a_{n-1}$ is not always finite when $n \to 0$, then $\frac{a_n}{n} \to \infty$

If the no. of such no.s as $a_1\ a_2\ a_3\ a_4 \ldots$ within $n = O\,\phi(n)$.

then $\cos a_1 x + \cos a_2 x + \ldots \cos a_n x$

$$= O\left\{\frac{n}{\phi(n)}\right\}.$$

The sum of the ~~no of~~ divisors of ~~N~~

$$= e^{\gamma} N (\log\log N)\ ?$$

$$\frac{2^{p+1}-1}{2-1} \cdot \frac{3^{q+1}-1}{3-1} \cdot \frac{5^{r+1}-1}{5-1} \ldots$$

$$\frac{355}{113}\left(1 - \frac{\cdot 0003}{3533}\right) = 3\cdot 14159265358979432.$$

If $u = \frac{\sqrt[5]{x}}{1+}\ \frac{x}{1+}\ \frac{x^2}{1+}\ \frac{x^3}{1+} \ldots$

then $u^5 + u - 1 = 0$ when $x^n = 1$, where n is any positive integer except multiples of 5 in which case u is not definite

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$4^4+6^4+8^4+9^4+14^4=15^4$

$1^4+2^4+12^4+24^4+44^4=45^4$

$4^4+21^4+22^4+26^4+28^4=35^4$

$4^4+8^4+13^4+28^4+54^4=55^4$

$1^4+8^4+12^4+32^4+64^4=65^4$

$22^4+28^4+63^4+72^4+94^4=105^4$

$4^5+5^5+6^5+7^5+9^5+11^5=12^5$

$5^5+10^5+11^5+16^5+19^5+29^5=30^5$

$2^4+39^4+44^4+46^4+52^4=65^4$

$22^4+52^4+57^4+74^4+76^4=95^4$

$$(8s^2+40st-24t^2)^4+(6s^2-44st-18t^2)^4$$
$$+(14s^2-4st-42t^2)^4+(9s^2+27t^2)^4+(4s^2+12t^2)^4$$
$$=(15s^2+45t^2)^4.$$

$$(4m^2-12n^2)^4+(3m^2+9n^2)^4+(2m^2-12mn-6n^2)^4$$
$$+(4m^2+12n^2)^4+(2m^2+12mn-6n^2)^4$$
$$=(5m^2+15n^2)^4.$$

$$\sqrt[3]{2}=1.259921,049894,873164,767208$$
$$=\frac{5}{4}\left(1+\frac{24}{1000}\right)^{\frac{1}{3}}=\frac{63}{50}\left(1+\frac{188}{1000000}\right)^{-\frac{1}{3}}$$

If $a+b+c=0$, then

$2(ab+ac+bc)^2 = a^4+b^4+c^4,$

(i) $2(ab+ac+bc)^4 = a^4(b-c)^4 + b^4(c-a)^4 + c^4(a-b)^4,$

(ii) $2(ab+ac+bc)^6 = (a^2b+b^2c+c^2a)^4 + (ab^2+bc^2+ca^2)^4$

(iii) $\qquad + (3abc)^4.$

(iv) $2(ab+ac+bc)^8 = (a^3+2abc)^4(b-c)^4 +$
$(b^3+2abc)^4(c-a)^4 + (c^3+2abc)^4(a-b)^4.$

and so on.

If $\frac{a}{b} = \frac{c}{d}$, then

$$\left.\begin{aligned}&(a+b+c)^4 + (b+c+d)^4 + (a-d)^4\\ &= (c+d+a)^4 + (d+a+b)^4 + (b-c)^4.\end{aligned}\right\}$$ 4 may be replaced by 2 also.

$2^4+4^4+7^4 = 3^4+6^4+6^4.$

$3^4+7^4+8^4 = 1^4+2^4+9^4.$

$6^4+9^4+12^4 = 2^4+2^4+13^4.$

$3^4+9^4 = 5^4+5^4+6^4+8^4.$

$2^4+2^4+7^4 = 4^4+4^4+5^4+6^4$

$3^4+9^4+14^4 = 7^4+8^4+10^4+13^4$

$7^4+10^4+13^4 = 5^4+5^4+6^4+14^4$

$1^4+2^4+4.2^4 = 3^4$

$3^4+6^4+4.4^4 = 7^4$

$7^4+8^4+4.2^4 = 9^4.$

$3^4+14^4+4.2^4 = 10^4+13^4$

$3^4+7^4+4.2^4 = 5^4+5^4+6^4.$

$$64\left\{(a+b+c)^6+(b+c+d)^6-(c+d+a)^6-(d+a+b)^6+(a-d)^6-(b-c)^6\right\}$$

$$\times\left\{(a+b+c)^{10}+(b+c+d)^{10}-(c+d+a)^{10}-(d+a+b)^{10}+(a-d)^{10}-(b-c)^{10}\right\}$$

$$=45\left\{(a+b+c)^8+(b+c+d)^8-(c+d+a)^8-(d+a+b)^8+(a-d)^8-(b-c)^8\right\}^2$$

If $3k=a^2+ab+b^2$, then

$$(ax^3+k)^3-(bx^3+k)^3=k\left\{(x^4+ax)^3-(x^4+bx)^3\right\}$$

$$(x^4+1)^4+4\cdot\left(\frac{x^5-5x}{2}\right)^4+5(x^4-2)^4$$

$$=3^4+4\left(\frac{x^5+x}{2}\right)^4.$$

$$(4x^5-5x)^4+(4x^4+1)^4+5(4x^4-2)^4$$

$$=3^4+(4x^5+x)^4.$$

$$3^4+(2x^4-1)^4+(4x^5+x)^4$$

$$=(4x^4+1)^4+(6x^4-3)^4+(4x^5-5x)^4.$$

$$\cancel{\frac{(x+a)^4-(x+b)^4}{2(a-b)}} = (x+p)^3 + (x+q)^3,$$

$$\cancel{\text{where } \quad p = \frac{a}{3-\sqrt{3}} + \frac{b}{3+\sqrt{3}}}$$

$$\cancel{q = \frac{a}{3+\sqrt{3}} + \frac{b}{3-\sqrt{3}}}.$$

$$\left(x+\frac{1}{4}\right)^4 - \left(x-\frac{1}{4}\right)^4 = \left(x+\frac{1}{\sqrt{3}}\right)^3 + \left(x-\frac{1}{\sqrt{3}}\right)^3$$

If $\alpha^2+\alpha\beta+\beta^2 = 3\lambda\gamma^2$, then

$$(\alpha+\lambda^2\gamma)^3 + (\lambda\beta+\gamma)^3 = (\lambda\alpha+\gamma)^3 + (\beta+\lambda^2\gamma)^3.$$

$$\sum p_n x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}$$

$$n p_n = \sigma(1)\, p(n-1) + \sigma(2)\, p(n-2) + \sigma(3)\, p(n-3) + \cdots$$

$$\frac{1}{(1-a_1)(1-a_2)(1-a_3)\cdots}$$

$$= 1 + \frac{a_1 + a_2 + \cdots}{1-a_1} + \frac{a_2^2 + a_2 a_3 + a_3^2 + \cdots}{(1-a_1)(1-a_2)}$$

$$+ \frac{a_3^3 + a_3^2 a_4 \cdots}{(1-a_1)(1-a_2)(1-a_3)} + \frac{a_4^4 \cdots}{(1-a_1)(1-a_2)(1-a_3)}$$

$$\frac{1}{4} + 2 = \left(1\tfrac{1}{2}\right)^2$$

$$\frac{1}{4} + 2 \cdot 3 = \left(2\tfrac{1}{2}\right)^2$$

$$\frac{1}{4} + 2 \cdot 3 \cdot 5 = \left(5\tfrac{1}{2}\right)^2$$

$$\frac{1}{4} + 2 \cdot 3 \cdot 5 \cdot 7 = \left(14\tfrac{1}{2}\right)^2$$

$$\frac{1}{4} + 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = \left(714\tfrac{1}{2}\right)^2.$$

$1 + 2 = 3$.

$2 + 3 = 5, \quad 1 + 5 = 2 \cdot 3.$

$3 + 7 = 2 \cdot 5, \quad 1 + 2 \cdot 7 = 3 \cdot 5.$

$2 \cdot 5 + 11 = 3 \cdot 7, \quad 3 \cdot 5 + 7 = 2 \cdot 11, \quad 2 + 3 \cdot 11 = 5 \cdot 7$

$2 \cdot 3 \cdot 7 + 13 = 5 \cdot 11$

$3 \cdot 5 \cdot 11 + 17 = 2 \cdot 7 \cdot 13, \quad 1 + 2 \cdot 3 \cdot 7 \cdot 17 = 5 \cdot 11 \cdot 13.$

$a_1^{a_2^{a_3^{\cdot^{\cdot^{\cdot}}}}}$ is convergent when

$$1 + \log\log a_n \leq \frac{1}{2}\left\{\frac{1}{n^2} + \frac{1}{(n\log n)^2} + \frac{1}{(n\log n\log\log n)^2} + \frac{1}{(n\log n\log\log n\log\log\log n)^2} + \cdots\right\}$$

divergent when it is greater than the right-hand side when any 1 is replaced by $1+\varepsilon$.

Contour integration $\quad \int_0^\infty \frac{\cos nx}{1+x^2} \log x \, dx + \frac{\pi}{2} \int_0^\infty \frac{\sin nx}{1+x^2} dx = 0.$

$(\log x)^2 \qquad \log x$

Elliptic function formulae.

$$\int_0^\infty e^{-x} x^{n-1} dx = \Gamma(n), \quad \int_{-\infty}^\infty \frac{n^{x-1}}{\Gamma(x)} dx = e^n.$$

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(x)}{n^x} dx = e^{-n}, \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^x}{x^n} dx = \frac{1}{\Gamma(n)}.$$

$$\int_0^\infty \cancel{x}^{n-1} \phi(x) dx = \psi(n), \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} n^{-x} \psi(x) dx = \phi(n)$$

$$\int_{-\infty}^\infty n^{x-1} \phi(x) dx = \psi(n), \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-n} \psi(x) dx = \phi(n)$$

$$\int_0^\infty \phi(x) e^{-nx} dx = \psi(n), \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \psi(x) e^{nx} dx = \phi(n)$$

$$\int_{-\infty}^\infty \phi(x) e^{nx} dx = \psi(n), \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \psi(x) e^{-nx} dx = \phi(n)$$

$$\int_0^\infty x^{n-1} \{ \qquad \} dx = \Gamma(n) \phi(-n)$$

$$\int_{-\infty}^\infty \frac{\phi(x)}{\Gamma(x+1)} dx = \phi(0) + \frac{\phi(1)}{1!} + \frac{\phi(2)}{2!} + $$

$$J = \frac{1-16\alpha(1-\alpha)}{8\sqrt[3]{4\alpha(1-\alpha)}} \qquad J = \frac{\sqrt[3]{4t}}{2^3}$$

$J_3 = 0$, $J_{11} = 1$, $J_{19} = 3$, $J_{27} = 5\sqrt[3]{3}$, $J_{35} = \sqrt{5}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^4$

$J_{43} = 30$, $J_{51} = 3.\left(\frac{5+\sqrt{17}}{2}\right)\sqrt[3]{(4+\sqrt{17})^2}$

$$t = \frac{(1-p^8)^3}{p^8}$$

$$J_{59} = . \; (p^9 - 7p^8 + 22p^7 - 34p^6 + 40p^5 - 28p^4 + 22p^3 - 10p^2 + 11p - 1) = 0$$

cubic in t.

$J_{67} = 165$, $J_{75} = 3\cdot\left(\frac{69+31\sqrt{5}}{2}\right)\sqrt[6]{5}$

J_{83}. $\beta = \frac{1-2s-2s^2-2s^3}{2s^2}$.

$$\beta^3 + 4\beta^2 + 2\beta - 5 = 0$$

$$t = \frac{(1-256s^{24})^3}{256s^{24}}$$

$J_{91} = 3\left(\frac{227+63\sqrt{13}}{2}\right)$ $J_{99} = \ldots$

$J_{115} = 3\left(\frac{785+351\sqrt{5}}{2}\right)$, $J_{163} = 20010$

$$1 + 5\cdot\frac{1}{2}\cdot\frac{1\cdot 3}{4^2}4x(1-x) + \cdots,$$

$$= \frac{1}{1-2x}\left\{1 - \left[\frac{4}{y}\right] - 8\left(\frac{1}{e^{y}-1} + \frac{2}{e^{2y}-1} + \cdots\right) - 16\left(\frac{1}{e^{2y}-1} + \frac{2}{e^{4y}-1} + \cdots\right)\right\}$$

$$1 + 4\cdot\frac{1}{2}\cdot\frac{1\cdot 2}{3^2}4x(1-x) + \cdots$$

$$= \frac{1}{1-2x}\left\{1 - \left[\frac{3}{y}\right] - 6\left(\frac{1}{e^{y}-1} + \frac{2}{e^{2y}-1} + \cdots\right) - 18\left(\frac{1}{e^{3y}-1} + \frac{2}{e^{6y}-1} + \cdots\right)\right\}$$

$$1 + \frac{1\cdot 2}{3^2}t + \cdots$$

$$\sqrt[3]{t(1-t)} = \frac{2}{3}\cdot\frac{2-\sqrt[6]{GG'}}{\sqrt[6]{GG'}} = \frac{2}{3}\cdot\frac{2+\sqrt[6]{gg'}}{\sqrt[6]{gg'}}.$$

$8J+3$		$64J^2-24J+9$
11 —	11	7^2
19 —	27	$27\cdot 19$
43 —	$27\cdot 3^2$	$27\cdot 7^2\cdot 43$
67 —	$27\cdot 7^2$	$27\cdot 31^2\cdot 67$
163 —	$27\cdot 77^2$	$27\cdot 2413^2\cdot 163.$

$$\sqrt[6]{2\sqrt{64J^2-24J+9}-(16J-3)} = t = \sqrt{3}\,x^{\frac{1}{18}}\frac{f(x^{\frac{1}{3}})f(x^3)}{f^2(x)}$$

11 — $t-1=0$

35 — $t^2+t-1=0$

59 — $t^3+2t-1=0$

83 — $t^3+2t^2+2t-1=0$

107 — $t^3-2t^2+4t-1=0.$

$$\frac{1}{3}\sqrt{1+\tfrac{8}{3}J} = t \qquad \frac{e^{\frac{\pi\sqrt{n}}{6}}+6e^{-\frac{\pi\sqrt{n}}{6}}}{6\sqrt{3}} \text{ appr}$$

19 — $t-1=0$

43 — $t-3=0$

67 — $t-7=0$

91 — $t=7+2\sqrt{13}.\quad t^2-14t-3=0$

115 — $t=13+6\sqrt{5}\quad t^2-26t-11=0$

163 — $t-77=0$

$$R = x^{\frac{1}{36}}\left(\frac{f(x)}{f(x^{\frac{1}{3}})}\right)^{1/3}$$

$$\frac{3\sqrt{3}}{R^6} = \sqrt{8J+3}+\sqrt{2\sqrt{64J^2-24J+9}-8J+6}$$