1 Special functions

1.1 Definition and basic properties of Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt \quad \mathrm{Re}(s) > 0$$

$$\Gamma(s+1) = \int_0^\infty t^{(s+1)-1} e^{-t} dt = \int_0^\infty t^s e^{-t} dt$$

$$= -\left. e^{-t} t^s \right|_0^\infty - \int_0^\infty \frac{dt^s}{dt} \left(-e^{-t}\right) dt$$

$$= \int_0^\infty s t^{s-1} e^{-t} dt = s \int_0^\infty t^{s-1} e^{-t} dt$$

$$\Gamma(s+1) = s \cdot \int_0^\infty t^{s-1} e^{-t} dt$$

$$\Gamma(s+1) = s \cdot \Gamma(s)$$

$$\Gamma(n+1) = n \cdot \Gamma(n) = n \cdot (n-1) \cdot \Gamma(n-1)$$

$$\Gamma(n+1) = n!$$

1.2 Gauss Representation

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt \quad \mathrm{Re}(s) > 0$$

$$e^{-t} = \lim_{n \to \infty} \left(1 + \frac{-t}{n}\right)^n$$

$$\Gamma(s) = \int_0^\infty t^{s-1} \lim_{n \to \infty} \left(1 - \frac{t}{n}\right)^n dt$$

$$\Gamma(s) = \lim_{n \to \infty} \int_0^n t^{s-1} \left(1 - \frac{t}{n}\right)^n dt$$

$$\int_0^n t^{s-1} \left(1 - \frac{t}{n}\right)^n dt = \left. \frac{1}{s} t^s \left(1 - \frac{t}{n}\right)^n \right|_0^n$$

$$- \int_0^n \frac{1}{s} t^s n \left(1 - \frac{t}{n}\right)^{n-1} \left(-\frac{1}{n}\right) dt$$

$$= \frac{n}{s \cdot n} \int_0^n t^s \left(1 - \frac{t}{n}\right)^{n-1} dt$$

$$= \frac{n}{s \cdot n} \frac{n-1}{(s+1)n} \int_0^n t^{s+1} \left(1 - \frac{t}{n}\right)^{n-2} dt$$

$$= \frac{n}{s \cdot n} \frac{n-1}{(s+1)n} \int_0^n t^{s+1} \left(1 - \frac{t}{n}\right)^{n-2} dt$$

$$= \frac{n}{s \cdot n} \frac{n-1}{(s+1)n} \frac{n-2}{(s+2)n} \int_0^n t^{s+2} \left(1 - \frac{t}{n}\right)^{n-3} dt$$

$$= \frac{n}{s \cdot n} \frac{n-1}{(s+1)n} \frac{n-2}{(s+2)n} \cdots \frac{1}{(s+n-1)n} \int_0^n t^{s+n-1} dt$$

$$= \frac{n}{s \cdot n} \frac{n-1}{(s+1)n} \frac{n-2}{(s+2)n} \cdots \frac{1}{(s+n-1)n} \frac{t^{s+n}}{(s+n)} \bigg|_0^n$$

$$\int_0^n t^{s-1} \left(1 - \frac{t}{n}\right)^n dt = \frac{n!}{n^n} n^{s+n} \prod_{i=0}^{n} (s+i)^{-1}$$

$$\int_0^n t^{s-1} \left(1 - \frac{t}{n}\right)^n dt = \frac{n! n^s}{\prod_{i=0}^{n} (s+i)}$$

$$\Gamma(s) = \lim_{n\to\infty} \left(\frac{n^s}{s} \prod_{i=1}^{n} \frac{i}{(s+i)} \right)$$

1.3 Weierstrass Representation

$$\Gamma(s) = \lim_{n\to\infty} \left(\frac{n^s}{s} \prod_{i=1}^{n} \frac{i}{(s+i)} \right)$$

$$\Gamma(s) = \lim_{n\to\infty} \left(\frac{n^s}{s} \prod_{i=1}^{n} \left(1 + \frac{s}{i}\right)^{-1} \right)$$

$$\Gamma(s) = \lim_{n\to\infty} \left(\frac{1}{s} e^{s \log n} \prod_{i=1}^{n} \left(1 + \frac{s}{i}\right)^{-1} \right)$$

$$\Gamma(s) = \lim_{n\to\infty} \left(\frac{1}{s} e^0 e^{s \log n} \prod_{i=1}^{n} \left(1 + \frac{s}{i}\right)^{-1} \right)$$

$$\Gamma(s) = \lim_{n\to\infty} \left(\frac{1}{s} \exp\left(\sum_{i=1}^{n} \frac{s}{i} - \sum_{i=1}^{n} \frac{s}{i} + s \log n \right) \prod_{i=1}^{n} \left(1 + \frac{s}{i}\right)^{-1} \right)$$

$$\gamma = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \log n \right)$$

$$\Gamma(s) = \lim_{n\to\infty} \left(\frac{1}{s} \exp\left(\sum_{i=1}^{n} \frac{s}{i} - \gamma s \right) \prod_{i=1}^{n} \left(1 + \frac{s}{i}\right)^{-1} \right)$$

$$\exp\left(\sum_{i=1}^{n}\frac{s}{i}\right)=\prod_{i=1}^{n}e^{\frac{s}{i}}$$

$$\Gamma(s)=\lim_{n\to\infty}\left(\frac{1}{s}e^{-\gamma}\prod_{i=1}^{n}e^{\frac{s}{i}}\left(1+\frac{s}{i}\right)^{-1}\right)$$

$$\Gamma(s)=\frac{1}{s}e^{-\gamma s}\prod_{i=1}^{\infty}e^{\frac{s}{i}}\left(1+\frac{s}{i}\right)^{-1}$$

1.4 Relationship to Sine

$$\Gamma(s)=\lim_{n\to\infty}\left(\frac{n^s}{s}\prod_{i=1}^{n}\frac{i}{i\left(1+\frac{s}{i}\right)}\right)$$

$$\Gamma(-s)=\lim_{n\to\infty}\left(\frac{n^{-s}}{-s}\prod_{i=1}^{n}\frac{i}{i\left(1-\frac{s}{i}\right)}\right)$$

$$\Gamma(s)\Gamma(-s)=\lim_{n\to\infty}\left(\frac{n^s}{s}\frac{n^{-s}}{-s}\prod_{i=1}^{n}\frac{1}{\left(1+\frac{s}{i}\right)\left(1-\frac{s}{i}\right)}\right)$$

$$\Gamma(s)\Gamma(-s)\cdot(-s)=\lim_{n\to\infty}\left(\frac{1}{s}\prod_{i=1}^{n}\frac{1}{\left(1-\frac{s^2}{i^2}\right)}\right)$$

$$\Gamma(s)\Gamma(1-s)=\frac{1}{s}\prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{s^2}{i^2}\right)}$$

$$\frac{\pi}{\sin\pi s}=\frac{1}{s}\prod_{i=1}^{\infty}\frac{1}{\left(1-\frac{s^2}{i^2}\right)}$$

$$\Gamma\left(s\right)\Gamma(1-s)=\frac{\pi}{\sin\pi s}$$

1.5 Example: Gamma of $\frac{1}{2}$

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s}$$

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right)=\frac{\pi}{\sin\frac{\pi}{2}}$$

$$\Gamma^2\left(\frac{1}{2}\right)=\pi$$

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{\frac{1}{2}-1} e^{-t} dt = \int_0^\infty \frac{1}{\sqrt{t}} e^{-t} dt$$
$$\sqrt{\pi} = \int_0^\infty \frac{1}{\sqrt{t}} e^{-t} dt$$
$$t = pu^2 \Rightarrow dt = 2pudu \quad \text{p} > 0$$
$$\int_0^\infty \frac{1}{\sqrt{pu^2}} e^{-pu^2} 2pudu = 2\sqrt{p} \int_0^\infty e^{-pu^2} du = \sqrt{\pi}$$
$$\sqrt{\frac{\pi}{p}} = \int_{-\infty}^\infty e^{-pu^2} du$$

1.6 Stirling's approximation

$$\Gamma(n+1) = n! = \int_0^\infty t^n e^{-t} dt$$
$$\log\left(t^n e^{-t}\right) = n \log t - t$$
$$n \log t - t = n \log(n+\varepsilon) - (n+\varepsilon)$$
$$\log(n+\varepsilon) = \log n \left(1 + \frac{\varepsilon}{n}\right) = \log n + \log\left(1 + \frac{\varepsilon}{n}\right)$$
$$\text{For very large}\, n, \text{ we can guarantee that } \frac{\varepsilon}{n} < 1$$
$$\log\left(1 + \frac{\varepsilon}{n}\right) = \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k} \frac{\varepsilon^k}{n^k}$$
$$\log(n+\varepsilon) - (n+\varepsilon) = n\left(\log n + \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k} \frac{\varepsilon^k}{n^k}\right) - n - \varepsilon$$
$$\log n - n + \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k} \frac{\varepsilon^k}{n^{k-1}} - \varepsilon$$
$$\log n - n + \sum_{k=2}^\infty \frac{(-1)^{k+1}}{k} \frac{\varepsilon^k}{n^{k-1}}$$
$$\log n - n - \frac{\varepsilon^2}{2n} + \frac{\varepsilon^3}{3n^2} - \frac{\varepsilon^4}{4n^3} \pm \dots$$
$$\log\left(t^n e^{-t}\right) \approx n \log n - n - \frac{\varepsilon^2}{2n} \quad t^n e^{-t} \approx \frac{n^n}{e^n} e^{-\frac{\varepsilon^2}{2n}}$$

$$t^n e^{-t} \approx \frac{n^n}{e^n} e^{-\frac{\varepsilon^2}{2n}}$$
$$n! = \int_0^\infty t^n e^{-t} dt \approx \int_{-n}^\infty \frac{n^n}{e^n} e^{-\frac{\varepsilon^2}{2n}} d\varepsilon$$
$$\sqrt{\frac{\pi}{p}} = \int_{-\infty}^\infty e^{-px^2} dx$$
$$n! \approx \frac{n^n}{e^n}\sqrt{2n\pi}$$

1.7 Euler Integral I

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$
$$t = pu^n \to dt = pnu^{n-1} du$$
$$\Gamma(s) = \int_0^\infty p^{s-1} u^{n(s-1)} e^{-pu^n} pnu^{n-1} du$$
$$\Gamma(s) = \int_0^\infty np^s u^{ns-1} e^{-pu^n} du$$
$$\frac{\Gamma(s)}{np^s} = \int_0^\infty u^{ns-1} e^{-pu^n} du \quad \frac{\Gamma(s)}{n\bar{p}^s} = \int_0^\infty u^{ns-1} e^{-\bar{p}u^n} du$$
$$\frac{\Gamma(s)}{n\bar{p}^s} \pm \frac{\Gamma(s)}{np^s} = \int_0^\infty \left\{u^{ns-1} e^{-\bar{p}u^n} \pm u^{ns-1} e^{-pu^n}\right\} du$$
$$\frac{\Gamma(s)}{n}\left(\frac{1}{\bar{p}^s} \pm \frac{1}{p^s}\right) = \int_0^\infty u^{ns-1}\left(e^{-\bar{p}u^n} \pm e^{-pu^n}\right) du$$
$$p = a + ib \quad p = |p|e^{i\alpha} \quad p^s = |p|^s e^{i\alpha s}$$
$$\bar{p} = a - ib \quad \bar{p} = |p|e^{-i\alpha} \quad \bar{p}^s = |p|^s e^{-i\alpha s}$$
$$\frac{\Gamma(s)}{n|p|^s}\left(\frac{1}{e^{-i\alpha s}} \pm \frac{1}{e^{i\alpha s}}\right) = \int_0^\infty u^{ns-1}\left(e^{-(a-ib)u^n} \pm e^{-(a+ib)u^n}\right) du$$
$$\frac{\Gamma(s)}{n|p|^s}\left(e^{i\alpha s} \pm e^{-i\alpha s}\right) = \int_0^\infty u^{ns-1}\left(e^{-au^n+ibu^n} \pm e^{-auu^n-ibu^n}\right) du$$
$$\frac{\Gamma(s)}{n|p|^s}\left(e^{i\alpha s} \pm e^{-i\alpha s}\right) = \int_0^\infty u^{ns-1}\left(e^{-au^n+ibu^n} \pm e^{-au^n-ibu^n}\right) du$$
$$\frac{\Gamma(s)}{n|p|^s}\left(e^{i\alpha s} \pm e^{-i\alpha s}\right) = \int_0^\infty u^{ns-1} e^{-au^n}\left(e^{+ibu^n} \pm e^{-ibu^n}\right) du$$
$$\frac{\Gamma(s)}{n|p|^s} 2\cos(\alpha s) = \int_0^\infty u^{ns-1} e^{-au^n} 2\cos\left(bu^n\right) du$$
$$\frac{\Gamma(s)}{n|p|^s} 2i\sin(\alpha s) = \int_0^\infty u^{ns-1} e^{-au^n} 2i\sin\left(bu^n\right) du$$

$$\frac{\Gamma(s)}{n|p|^s}\cos(\alpha s) = \int_0^\infty u^{ns-1}e^{-au^n}\cos\left(bu^n\right)du$$
$$\frac{\Gamma(s)}{n|p|^s}\sin(\alpha s) = \int_0^\infty u^{ns-1}e^{-au^n}\sin\left(bu^n\right)du$$
$$p = a + ib \quad |p| = \sqrt{a^2+b^2}\tan\alpha = \frac{b}{a}$$
$$\tan\alpha = \frac{b}{0} \Rightarrow \alpha = \frac{\pi}{2}$$

1.8 Euler Integral II. The Sinc-Function

$$\int_0^\infty \frac{\sin x}{x}dx$$
$$\frac{\Gamma(s)}{n|p|^s}\sin(\alpha s) = \int_0^\infty u^{ns-1}e^{-au^n}\sin\left(bu^n\right)du$$
$$s = 0 \quad a = 0 \quad b = 1 \quad n = 1$$
$$|p| = \sqrt{a^2+b^2} = 1$$
$$\tan\alpha = \frac{1}{0} = \infty \Rightarrow \alpha = \frac{\pi}{2}$$
$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}$$
$$\sin\frac{\pi s}{2}\cdot\frac{\pi}{\sin\pi s\cdot\Gamma(1-s)} = \int_0^\infty \left\{u^{s-1}\sin u\right\}du$$
$$\frac{\pi}{2}\frac{\sin\frac{\pi s}{2}}{\frac{\pi s}{2}}\cdot\frac{\pi s}{\sin\pi s\cdot\Gamma(1-s)} = \int_0^\infty \left\{u^{s-1}\sin u\right\}du$$
$$\int_0^\infty \left\{u^{-1}\sin u\right\}du = \frac{\pi}{2}\lim_{s\to 0}\left\{\frac{\pi}{\frac{\pi s}{2}}\cdot\frac{\pi s}{\sin\pi s\cdot\Gamma(1-s)}\right\}$$
$$\int_0^\infty \left\{u^{-1}\sin u\right\}du = \frac{\pi}{2}\frac{1}{\Gamma(1)}\lim_{s\to 0}\frac{\sin\frac{\pi s}{2}}{\frac{\pi s}{2}}\cdot\lim_{s\to 0}\frac{\pi s}{\sin\pi s}$$
$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$
$$\int_0^\infty \left\{u^{-1}\sin u\right\}du = \frac{\pi}{2}\frac{1}{0!}$$
$$\int_0^\infty \frac{\sin u}{u}du = \frac{\pi}{2}$$

1.9 Euler Integral III. Fresnel Integral

$$\int_0^\infty \sin u^2 du$$

$$\frac{\Gamma(s)}{n|p|^s}\sin(\alpha s) = \int_0^\infty u^{ns-1}e^{-au^n}\sin\left(bu^n\right)du$$

$$s = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad a = 0 \quad b = 1 \quad n = 2$$

$$|p| = \sqrt{a^2+b^2} = 1 \quad \tan\alpha = \frac{1}{0} = \infty \Rightarrow \alpha = \frac{\pi}{2}$$

$$\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\sin\left(\frac{\pi}{4}\right) = \int_0^\infty \sin u^2 du$$

$$\int_0^\infty \sin u^2 du = \frac{\sqrt{2\pi}}{4}$$

$$\int_0^\infty \cos u^2 du$$

$$\frac{\Gamma(s)}{n|p|^s}\cos(\alpha s) = \int_0^\infty u^{ns-1}e^{-au^n}\cos\left(bu^n\right)du$$

$$s = \frac{1}{2} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad a = 0 \quad b = 1 \quad n = 2$$

$$|p| = \sqrt{a^2+b^2} = 1 \quad \tan\alpha = \frac{1}{0} = \infty \Rightarrow \alpha = \frac{\pi}{2}$$

$$\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cos\left(\frac{\pi}{4}\right) = \int_0^\infty \cos u^2 du$$

$$\int_0^\infty \cos u^2 du = \frac{\sqrt{2\pi}}{4}$$

1.10 Beta Function

$$\Gamma\left(s_1\right)\cdot\Gamma\left(s_2\right) = \int_0^\infty x^{s_1-1}e^{-x}dx \cdot \int_0^\infty y^{s_2-1}e^{-y}dy$$

$$x = u^2 dx = 2udu$$

$$y = v^2 dy = 2vdv$$

$$\int_0^\infty u^{2s_1-2}e^{-u^2}2udu \cdot \int_0^\infty v^{2s_2-2}e^{-v^2}2vdv$$

$$4\int_0^\infty u^{2s_1-1}e^{-u^2}du \cdot \int_0^\infty v^{2s_2-1}e^{-v^2}dv$$

$$4\int_0^\infty \left(\int_0^\infty v^{2s_2-1}e^{-v^2}dv\right)u^{2s_1-1}e^{-u^2}du$$
$$4\int_0^\infty \int_0^\infty v^{2s_2-1}e^{-v^2}u^{2s_1-1}e^{-u^2}dvdu$$
$$4\int_0^\infty \int_0^\infty v^{2s_2-1}u^{2s_1-1}e^{-\left(v^2+u^2\right)}dvdu$$
$$v = r\cos\alpha \quad u = r\sin\alpha$$
$$dvdu = rd\alpha dr \quad r : |_0^\infty \quad \alpha : |_0^{\frac{\pi}{2}}$$
$$4\int_0^\infty \int_0^{\frac{\pi}{2}} r^{2s_2-1}(\cos\alpha)^{2s_2-1}r^{2s_1-1}(\sin\alpha)^{2s_1-1}e^{-r^2}rd\alpha dr$$
$$4\int_0^\infty \int_0^{\frac{\pi}{2}} r^{2s_2-1}r^{2s_1-1}e^{-r^2}r(\cos\alpha)^{2s_2-1}(\sin\alpha)^{2s_1-1}d\alpha dr$$
$$4\int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^2 (\cos\alpha)^{2s_2-1}(\sin\alpha)^{2s_1-1}d\alpha\right) r^{2s_1+2s_2-2}e^{-r^2}rdr$$
$$4\int_0^\infty r^{2s_1+2s_2-2}e^{-r^2}rdr\cdot\int_0^{\frac{\pi}{2}}(\cos\alpha)^{2s_2-1}(\sin\alpha)^{2s_1-1}d\alpha$$
$$\int_0^\infty r^{2s_1+2s_2-2}e^{-r^2}2rdr\cdot 2\int_0^{\frac{\pi}{2}}(\cos\alpha)^{2s_2-1}(\sin\alpha)^{2s_1-1}d\alpha$$
$$\int_0^\infty \left(r^2\right)^{s_1+s_2-1}e^{-r^2}2rdr\cdot 2\int_0^{\frac{\pi}{2}}(\cos\alpha)^{2s_2-1}(\sin\alpha)^{2s_1-1}d\alpha$$
$$r^2 = u \quad 2rdr = du$$
$$\int_0^\infty u^{s_1+s_2-1}e^{-u}du\cdot 2\int_0^{\frac{\pi}{2}}(\cos\alpha)^{2s_2-1}(\sin\alpha)^{2s_1-1}d\alpha$$
$$\Gamma\left(s_1+s_2\right)\cdot 2\int_0^{\frac{\pi}{2}}(\cos\alpha)^{2s_2-1}(\sin\alpha)^{2s_1-1}d\alpha$$
$$\Gamma\left(s_1\right)\cdot\Gamma\left(s_2\right) = \Gamma\left(s_1+s_2\right)\cdot 2\int_0^{\frac{\pi}{2}}(\sin\alpha)^{2s_1-1}(\cos\alpha)^{2s_2-1}d\alpha$$
$$\frac{\Gamma\left(s_1\right)\cdot\Gamma\left(s_2\right)}{\Gamma\left(s_1+s_2\right)} = 2\int_0^{\frac{\pi}{2}}(\sin\alpha)^{2s_1-1}(\cos\alpha)^{2s_2-1}d\alpha$$
$$\frac{\Gamma\left(s_1\right)\cdot\Gamma\left(s_2\right)}{\Gamma\left(s_1+s_2\right)} = \int_0^{\frac{\pi}{2}}(\sin\alpha)^{2s_1-2}(\cos\alpha)^{2s_2-2}2\sin\alpha\cos\alpha d\alpha$$
$$u = (\sin\alpha)^2 du = 2\sin\alpha\cos\alpha d\alpha$$
$$\frac{\Gamma\left(s_1\right)\cdot\Gamma\left(s_2\right)}{\Gamma\left(s_1+s_2\right)} = \int_0^1 u^{s_1-1}(1-u)^{s_2-1}du$$

$$\mathrm{B}\left(s_1, s_2\right)=\frac{\Gamma\left(s_1\right) \cdot \Gamma\left(s_2\right)}{\Gamma\left(s_1+s_2\right)}=2 \int_0^{\frac{\pi}{2}}(\sin \alpha)^{2 s_1-1}(\cos \alpha)^{2 s_2-1} d \alpha$$
$$\mathrm{B}\left(s_1, s_2\right)=\frac{\Gamma\left(s_1\right) \cdot \Gamma\left(s_2\right)}{\Gamma\left(s_1+s_2\right)}=\int_0^1 u^{s_1-1}(1-u)^{s_2-1} d u$$

1.11 Legendre Duplication Formula

$$\mathrm{B}\left(s_1, s_2\right)=\frac{\Gamma\left(s_1\right) \Gamma\left(s_2\right)}{\Gamma\left(s_1+s_2\right)}=\int_0^1 u^{s_1-1}(1-u)^{s_2-1} d u$$
$$\frac{\Gamma(s) \Gamma(s)}{\Gamma(2 s)}=\int_0^1 u^{s-1}(1-u)^{s-1} d u$$
$$u=\frac{1+x}{2} \quad \mathrm{du}=\left.\frac{1}{2} d x\right|_0^1 \rightarrow \Big|_{-1}^1$$
$$\int_{-1}^1\left(\frac{1+x}{2}\right)^{s-1}\left(\frac{1-x}{2}\right)^{s-1} \frac{d x}{2}$$
$$\frac{1}{2^{2 s-1}} \int_{-1}^1\left(1-x^2\right)^{s-1} d x$$
$$\frac{\Gamma(s) \Gamma(s)}{\Gamma(2 s)}=\frac{1}{2^{2 s-1}} \int_{-1}^1\left(1-x^2\right)^{s-1} d x$$
$$2^{2 s-1} \Gamma(s) \Gamma(s)=2 \Gamma(2 s) \int_0^1\left(1-x^2\right)^{s-1} d x$$
$$\mathrm{B}\left(s_1, s_2\right)=\int_0^1 u^{s_1-1}(1-u)^{s_2-1} d u$$
$$u=x^2 d u=2 x d x$$
$$\mathrm{B}\left(s_1, s_2\right)=\int_0^1 x^{2 s_1-2}\left(1-x^2\right)^{s_2-1} 2 x d x$$
$$\mathrm{B}\left(\frac{1}{2}, s\right)=2 \int_0^1\left(1-x^2\right)^{s-1} d x$$
$$2^{2 s-1} \Gamma(s) \Gamma(s)=2 \Gamma(2 s) \int_0^1\left(1-x^2\right)^{s-1} d t$$
$$\mathrm{B}\left(\frac{1}{2}, s\right)=2 \int_0^1\left(1-x^2\right)^{s-1} d x$$
$$2^{2 s-1} \Gamma(s) \Gamma(s)=\Gamma(2 s) \mathrm{B}\left(\frac{1}{2}, s\right)$$
$$2^{2 s-1} \Gamma(s) \Gamma(s)=\Gamma(2 s) \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(s)}{\Gamma\left(\frac{1}{2}+s\right)}$$

$$2^{2s-1}\Gamma(s)\Gamma(s) = \Gamma(2s)\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(s)}{\Gamma\left(\frac{1}{2}+s\right)}$$

$$2^{2s-1}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2s)$$

$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)$$

1.12 Zeta Function

1.12.1 Definition and convergence

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

$$|\zeta(s)| = \left|\sum_{n=1}^{\infty}\frac{1}{n^s}\right| \le \sum_{n=1}^{\infty} | \frac{1}{n^s} = \sum_{n=1}^{\infty}\frac{1}{|n^s|}$$

$$|\zeta(s)| \le \sum_{n=1}^{\infty}\frac{1}{|n^s|} = \sum_{n=1}^{\infty}\frac{1}{|n^{x+iy}|} = \sum_{n=1}^{\infty}\frac{1}{|n^x|\,|n^{iy}|}$$

$$|\zeta(s)| \le \sum_{n=1}^{\infty}\frac{1}{|n^x|\,n^{iy}\,|} = \sum_{n=1}^{\infty}\frac{1}{n^x\left|e^{iy\log(n)}\right|} = \sum_{n=1}^{\infty}\frac{1}{n^x}$$

$$|\zeta(s)| \le \sum_{n=1}^{\infty}\frac{1}{n^x}$$

Cauchy-Integral Theorem

Sum is converging/diverging, if the corresponding Integral is converging / diverging.

$$\sum_{n=1}^{\infty}\frac{1}{n^x} \leftrightarrow \int_1^{\infty}\frac{1}{t^x}dt = \frac{1}{1-x}t^{1-x}\Big|_1^{\infty} \quad x = \mathrm{Re}(s) > 1$$

1.12.2 Euler Product Representation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} \frac{1}{1 - p^{-s}}$$

$$\begin{aligned}
\prod_{p \in P} \frac{1}{1 - p^{-s}} &= \prod_{i=1}^{\infty} \sum_{k_i=0}^{\infty} \left(\frac{1}{p_i^s}\right)^{k_i} = \prod_{i=1}^{\infty} \sum_{k_i=0}^{\infty} \frac{1}{p_i^{sk_i}} \\
&= \prod_{i=1} \sum_{k_i=0}^{\infty} \frac{1}{p_i^{sk_i}} = \sum_{k_1=0}^{\infty} \frac{1}{p_1^{sk_1}} \cdot \sum_{k_2=0}^{\infty} \frac{1}{p_2^{sk_2}} \cdot \ldots \cdot \sum_{k_N=0}^{\infty} \frac{1}{p_N^{sk_N}} \cdot \ldots \\
&= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \left(\frac{1}{p_1^{k_1}} \frac{1}{p_2^{k_2}} \cdots \frac{1}{p_N^{k_N}} \cdot \ldots \right)^s
\end{aligned}$$

$$\prod_{p \in P} \frac{1}{1 - p^{-s}} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \left(\frac{1}{p_1^{k_1}} \frac{1}{p_2^{k_2}} \cdots \frac{1}{p_N^{k_N}} \cdot \ldots \right)^s$$

Fundamental Theorem of Arithmetic Let n be a natural number, then it is possible to write n as a product of primes with corresponding powers. This representation is unique!

$$n = \prod_{1} p_1^{k_1} p_2^{k_2} \cdot \ldots \cdot p_{N(n)}^{k_{N(n)}}$$

$$\prod_{p \in P} \frac{1}{1 - p^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

There is also easier proof. This sketch of a proof makes use of simple algebra only. This was the method by which Euler originally discovered the formula. There is a certain sieving property that we can use to our advantage:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \ldots$$
$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \ldots$$

Subtracting the second equation from the first we remove all elements that have a factor of 2 :

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \frac{1}{13^s} + \ldots$$

Repeating for the next term:

$$\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} + \frac{1}{33^s} + \ldots$$

Subtracting again we get:

$$\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} + \ldots$$

where all elements having a factor of 3 or 2 (or both) are removed. It can be seen that the right side is being sieved. Repeating infinitely for $\frac{1}{p^s}$ where p is prime, we get:

$$\ldots\left(1 - \frac{1}{11^s}\right)\left(1 - \frac{1}{7^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1$$

Dividing both sides by everything but the $\zeta(s)$ we obtain:

$$\zeta(s) = \frac{1}{\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{7^s}\right)\left(1 - \frac{1}{11^s}\right)\ldots}$$

This can be written more concisely as an infinite product over all primes p :

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

To make this proof rigorous, we need only to observe that when $\Re(s) > 1$, the sieved right-hand side approaches 1 , which follows immediately from the convergence of the Dirichlet series for $\zeta(s)$.

1.12.3 Infinitude of Prime Numbers

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} \frac{1}{1-p^{-s}}$$

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \ldots$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \ldots = \infty$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} \frac{1}{1-p^{-s}}$$

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

A finite Set of Primes would make the product converge, but it has to diverge! That concludes that there must be infinite many primes. Q.e.d.

1.12.4 Prime Zeta Function

$$\zeta(s) = \prod_{p \in P} \frac{1}{1-p^{-s}}$$

$$\log \zeta(s) = \log \prod_{p \in P} \frac{1}{1-p^{-s}}$$

$$\log \zeta(s) = \sum_{p \in P} \log \frac{1}{1-p^{-s}}$$

$$\log \frac{1}{1-x} = -\log(1-x) = -\left(-\sum_{n=1}^{\infty} \frac{x^n}{n}\right)$$

$$\log \frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$\log \zeta(s) = \sum_{p \in P} \log \frac{1}{1-p^{-s}}$$

$$\log \zeta(s) = \sum_{p \in P} \sum_{n=1}^{\infty} \frac{(p^{-s})^n}{n}$$

$$\log \zeta(s) = \sum_{p \in P} \sum_{n=1}^{\infty} \frac{p^{-sn}}{n}$$

$$\log \zeta(s) = \sum_{p \in P} \sum_{n=1}^{\infty} \frac{p^{-sn}}{n}$$

$$\log \zeta(s) = \sum_{p \in P} p^{-s} + \sum_{p \in P} \sum_{n=2}^{\infty} \frac{p^{-sn}}{n}$$

$$\log \zeta(s) = \sum_{p \in P} \frac{1}{p^s} + \sum_{n=2}^{\infty} \frac{1}{n} \sum_{p \in P} \frac{1}{p^{sn}}$$

$$\left| \sum_{p \in P} \frac{1}{p^{sn}} \right| \leq \sum_{p \in P} \frac{1}{|p^{sn}|} = \sum_{p \in P} \frac{1}{\left|p^{(x+iy)n}\right|} = \sum_{p \in P} \frac{1}{|p^{xn+iyn}|}$$

$$\left| \sum_{p \in P} \frac{1}{p^{sn}} \right| \leq \sum_{p \in P} \frac{1}{|p^{xn}||p^{iyn}|} = \sum_{p \in P} \frac{1}{p^{xn} \left|e^{iyn \log p}\right|} = \sum_{p \in P} \frac{1}{p^{xn}}$$

$$\left| \sum_{p \in P} \frac{1}{p^{sn}} \right| \leq \sum_{p \in P} \frac{1}{p^{xn}} \leq \sum_{p \in P} \frac{1}{p^n} \leq \sum_{k=2}^{\infty} \frac{1}{k^n}$$

$$\left| \sum_{p \in P} \frac{1}{p^{sn}} \right| \leq \sum_{k=2}^{\infty} \frac{1}{k^n} < \sum_{k=2}^{\infty} \int_{k-1}^{k} \frac{1}{t^n} dt = \int_{1}^{\infty} \frac{1}{t^n} dt$$

$$\left| \sum_{p \in P} \frac{1}{p^{sn}} \right| \leq \left. \frac{1}{1-n} t^{1-n} \right|_1^{\infty} = \frac{1}{n-1}$$

$$\left| \sum_{n=2}^{\infty} \frac{1}{n} \sum_{p \in P} \frac{1}{p^{sn}} \right| \leq \sum_{n=2}^{\infty} \frac{1}{n} \left| \sum_{p \in P} \frac{1}{p^{sn}} \right| \leq \sum_{n=2}^{\infty} \frac{1}{n} \frac{1}{n-1} = 1$$

$$\log\zeta(s) = \sum_{p\in P}\frac{1}{p^s} + \sum_{n=2}^{\infty}\frac{1}{n}\sum_{p\in P}\frac{1}{p^{sn}}$$

$$s\to 1 \quad |\sum_{n=2}^{\infty}\frac{1}{n}\sum_{p\in P}\frac{1}{p^{sn}}| \le 1$$

$$|\log\zeta(s)| \to \infty$$

$$\sum_{p\in P}\frac{1}{p} \to \infty$$

1.12.5 The Prime Counting Function

$$\zeta(s) = \prod_{p\in P}\frac{1}{1-p^{-s}}$$

$$\log\zeta(s) = \log\prod_{p\in P}\frac{1}{1-p^{-s}}$$

$$\log\zeta(s) = \sum_{p\in P}\log\frac{1}{1-p^{-s}}$$

$$\log\zeta(s) = \sum_{n=2}^{\infty}\{\pi(n)-\pi(n-1)\}\log\frac{1}{1-n^{-s}}$$

$$\begin{aligned}
&\log\zeta(s) = \sum_{n=2}^{\infty}\{\pi(n)-\pi(n-1)\}\log\frac{1}{1-n^{-s}}\\
&= \sum_{n=2}^{\infty}\pi(n)\log\frac{1}{1-n^{-s}} - \sum_{n=2}^{\infty}\pi(n-1)\log\frac{1}{1-n^{-s}}\\
&= \sum_{n=2}^{\infty}\pi(n)\log\frac{1}{1-n^{-s}} - \sum_{n=2}^{\infty}\pi(n)\log\frac{1}{1-(n+1)^{-s}}\\
&= \sum_{n=2}^{\infty}\pi(n)\left(\log\left(1-(n+1)^{-s}\right) - \log\left(1-n^{-s}\right)\right)\\
&\frac{d}{dx}\log\left(1-x^{-s}\right) = \frac{1}{1-x^{-s}}\left(sx^{-s-1}\right) = \frac{s}{x\left(x^s-1\right)}
\end{aligned}$$

$$\log\zeta(s)=\sum_{n=2}^{\infty}\pi(n)\left(\log\left(1-(n+1)^{-s}\right)-\log\left(1-n^{-s}\right)\right)$$

$$\log\left(1-x^{-s}\right)=\int\frac{s}{x\left(x^{s}-1\right)}dx+c$$

$$\log\zeta(s)=\sum_{n=2}^{\infty}\pi(n)\int_{n}^{n+1}\frac{s}{x\left(x^{s}-1\right)}dx$$

$$\log\zeta(s)=\sum_{n=2}^{\infty}\pi(n)\int_{n}^{n+1}\frac{s}{x\left(x^{s}-1\right)}dx$$

$$\log\zeta(s)=\sum_{n=2}^{\infty}\int_{n}^{n+1}\frac{s\pi(x)}{x\left(x^{s}-1\right)}dx$$

$$\log\zeta(s)=\int_{2}^{\infty}\frac{s\pi(x)}{x\left(x^{s}-1\right)}dx$$

1.12.6 Zeta of 2 aka The Basel Problem

$$\sin s=s-\frac{s^3}{3!}+\frac{s^5}{5!}-\frac{s^7}{7!}\pm\dots$$

$$\frac{\sin\pi s}{\pi s}=1-\frac{(\pi s)^2}{3!}+\frac{(\pi s)^4}{5!}-\frac{(\pi s)^6}{7!}\pm\dots$$

$$\frac{\sin\pi s}{\pi s}=1-\frac{\pi^2}{3!}s^2+\frac{\pi^4}{5!}s^4-\frac{\pi^6}{7!}s^6\pm\dots$$

$$\frac{\sin\pi s}{\pi s}=\left(1-\frac{s^2}{1^2}\right)\left(1-\frac{s^2}{2^2}\right)\left(1-\frac{s^2}{3^2}\right)\dots$$

$$\frac{\sin\pi s}{\pi s}=1-\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right)s^2+$$

$$+\left(\frac{1}{1^22^2}+\frac{1}{1^23^2}+\frac{1}{2^23^2}+\dots\right)s^4-\left(\frac{1}{1^22^23^2}+\dots\right)s^6+\dots$$

"De Summis Serierum Reciprocarum"(1735)

$$\zeta(2)=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots=\frac{\pi^2}{6}$$

1.12.7 Zeta of $2n$. Part 1

$$\tfrac{\sin \pi s}{\pi s} = \left(1 - \tfrac{s^2}{1^2}\right)\left(1 - \tfrac{s^2}{2^2}\right)\left(1 - \tfrac{s^2}{3^2}\right)\dots$$

$$\tfrac{\sin \pi s}{\pi s} = \textstyle\prod_{k=1}^{\infty}\left(1 - \tfrac{s^2}{k^2}\right)$$

$$\log \tfrac{\sin \pi s}{\pi s} = \log \textstyle\prod_{k=1}^{\infty}\left(1 - \tfrac{s^2}{k^2}\right)$$

$$\log \sin \pi s = \log \pi s + \textstyle\sum_{k=1}^{\infty} \log\left(1 - \tfrac{s^2}{k^2}\right)$$

$$\tfrac{d}{ds} \log \sin \pi s = \tfrac{d}{ds}\left\{\log \pi s + \textstyle\sum_{k=1}^{\infty} \log\left(1 - \tfrac{s^2}{k^2}\right)\right\}$$

$$\frac{d}{ds} \log \sin \pi s = \frac{d}{ds}\left\{\log \pi s + \sum_{k=1}^{\infty} \log\left(1 - \frac{s^2}{k^2}\right)\right\}$$

$$\frac{\cos \pi s}{\sin \pi s}\pi = \frac{1}{s} + \sum_{k=1}^{\infty} \frac{1}{(1 - \frac{s^2}{k^2})}\left(-\frac{2s}{k^2}\right)$$

$$\pi s \cot \pi s = 1 + \sum_{k=1}^{\infty} \frac{1}{(1 - \frac{s^2}{k^2})}\left(-\frac{2s^2}{k^2}\right)$$

$$\pi s \cot \pi s = 1 + 2s^2 \sum_{k=1}^{\infty} \frac{1}{(s^2 - k^2)}$$

$$\pi \cot \pi s = \frac{1}{s} + \sum_{k=1}^{\infty}\left(\frac{1}{s-k} + \frac{1}{s+k}\right) = \sum_{k=-\infty}^{\infty} \frac{1}{s+k}$$

$$\pi s \cot \pi s = 1 + \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\left(\frac{s^2}{k^2}\right)^n\left(-\frac{2s^2}{k^2}\right)$$

$$\pi s \cot \pi s = 1 - 2\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\left(\frac{s^2}{k^2}\right)^{n+1}$$

$$\pi s \cot \pi s = 1 - 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^{2n}}s^{2n}$$

$$\pi s \cot \pi s = 1 - 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k^{2n}}\right)s^{2n}$$

$$\pi s \cot \pi s = 1 - 2\sum_{n=1}^{\infty}\zeta(2n)s^{2n}$$

1.12.8 Zeta of $2n$. Part 2

$$\pi s\cot\pi s=\pi s\frac{\cos\pi s}{\sin\pi s}=\pi s\frac{e^{i\pi s}+e^{-i\pi s}}{2}\frac{2i}{e^{i\pi s}-e^{-i\pi s}}$$

$$\pi s\cot\pi s=\pi si\frac{e^{i\pi s}+e^{-i\pi s}}{e^{i\pi s}-e^{-i\pi s}}$$

$$\pi s\cot\pi s=\pi si\frac{e^{2i\pi s}+1}{e^{2i\pi s}-1}$$

$$\pi s\cot\pi s=\pi si\frac{e^{2i\pi s}-1+2}{e^{2i\pi s}-1}=i\pi s\left(1+\frac{2}{e^{2i\pi s}-1}\right)$$

$$\pi s\cot\pi s=i\pi s+\frac{2i\pi s}{e^{2i\pi s}-1}$$

$$\frac{s}{e^s-1}=\sum_{n=0}^{\infty}\frac{\beta_n}{n!}s^n \qquad \frac{s}{\sum_{n=1}^{\infty}\frac{1}{n!}s^n}=\sum_{n=0}^{\infty}\frac{\beta_n}{n!}s^n$$

$$s=\sum_{n=0}^{\infty}\frac{\beta_n}{n!}s^n\sum_{n=1}^{\infty}\frac{1}{n!}s^n \qquad 1=\sum_{n=0}^{\infty}\frac{\beta_n}{n!}s^n\sum_{n=1}^{\infty}\frac{1}{n!}s^{n-1}$$

$$1=\sum_{n=0}^{\infty}\frac{\beta_n}{n!}s^n\sum_{n=0}^{\infty}\frac{1}{(n+1)!}s^n \qquad 1=\sum_{n=0}^{\infty}\sum_{\mu=0}^{n}\frac{\beta_\mu}{\mu!}\frac{1}{(n-\mu+1)!}s^n$$

$$1=\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\sum_{\mu=0}^{n}\left(\begin{array}{c}n+1\\ \mu\end{array}\right)\beta_\mu s^n$$

$$1=\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\sum_{\mu=0}^{n}\left(\begin{array}{c}n+1\\ \mu\end{array}\right)\beta_\mu s^n$$

$$\sum_{\mu=0}^{n}\left(\begin{array}{c}n+1\\ \mu\end{array}\right)\beta_\mu=0 \quad \beta_0=1 \quad \beta_1=-\frac{1}{2}$$

$$\pi s\cot\pi s=i\pi s+\frac{2i\pi s}{e^{2i\pi s}-1} \quad \frac{s}{e^s-1}=\sum_{n=0}^{\infty}\frac{\beta_n}{n!}s^n$$

$$\pi s\cot\pi s=i\pi s+\sum_{n=0}^{\infty}\frac{\beta_n}{n!}(2i\pi s)^n$$

$$\pi s\cot\pi s=1-2\sum_{n=1}^{\infty}\zeta(2n)s^{2n}$$

1.12.9 Zeta of $2n$. Part 3

$$\sum_{\mu=0}^{n} \left(\begin{array}{c} n+1 \\ \mu \end{array} \right) \beta_\mu = 0 \quad \beta_0 = 1, \beta_1 = -\frac{1}{2}$$

$$\pi s \cot \pi s = i\pi s + \frac{\beta_o}{0!} + \frac{\beta_1}{1!}(2i\pi s) - 2\sum_{n=2}^{\infty} \frac{\beta_n}{n!} \left(-\frac{1}{2}\right) (2i\pi s)^n$$

$$\pi s \cot \pi s = 1 - 2\sum_{n=2}^{\infty} \frac{\beta_n}{n!} \left(-\frac{1}{2}\right) (2i\pi s)^n$$

$$\pi s \cot \pi s = 1 - 2\sum_{n=1}^{\infty} \zeta(2n) s^{2n}$$

$$\pi s \cot \pi s = 1 - 2\sum_{n=1}^{\infty} \frac{\beta_{2n}}{2n!} \left(-\frac{1}{2}\right) (2i\pi s)^{2n}$$

$$\pi s \cot \pi s = 1 - 2\sum_{n=1}^{\infty} \zeta(2n) s^{2n}$$

$$\pi s \cot \pi s = 1 - 2\sum_{n=1}^{\infty} \frac{\beta_{2n}}{2n!} \left(-\frac{1}{2}\right) (2i\pi s)^{2n}$$

$$\pi s \cot \pi s = 1 - 2\sum_{n=1}^{\infty} \frac{\beta_{2n}}{2n!}(-1)\frac{(2\pi)^{2n} i^{2n}}{2} s^{2n}$$

$$\pi s \cot \pi s = 1 - 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2\pi)^{2n}\beta_{2n}}{2 \cdot 2n!} s^{2n}$$

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}\beta_{2n}}{2 \cdot 2n!}$$

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}\beta_{2n}}{2 \cdot 2n!}$$

$$\sum_{\mu=0}^{n} \left(\begin{array}{c} n+1 \\ \mu \end{array} \right) \beta_\mu = 0 \quad \beta_0 = 1, \beta_1 = -\frac{1}{2}$$

$$\zeta(2) = \frac{4\pi^2}{2 \cdot 2!}\frac{1}{6} = \frac{\pi^2}{6}$$

$$\zeta(4) = -\frac{16\pi^4}{2 \cdot 24}\left(-\frac{1}{30}\right) = \frac{\pi^4}{90}$$

$$\zeta(6) = \frac{32\pi^6}{2 \cdot 720}\frac{1}{42} = \frac{\pi^6}{945}$$

1.12.10 Relation to Gamma Function. Bose Integral

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

$$t = nu \Rightarrow dt = ndu$$

$$\Gamma(s) = \int_0^\infty (nu)^{s-1} e^{-nu} ndu$$

$$\Gamma(s) = \int_0^\infty n^s u^{s-1} e^{-nu} du$$

$$\Gamma(s)\frac{1}{n^s} = \int_0^\infty u^{s-1} e^{-nu} du$$

$$\Gamma(s)\sum_{n=1}^{\infty}\frac{1}{n^s} = \sum_{n=1}^{\infty}\int_0^\infty u^{s-1} e^{-nu} du$$

$$\Gamma(s)\zeta(s) = \int_0^\infty u^{s-1} \sum_{n=1}^{\infty} e^{-nu} du$$

$$\Gamma(s)\zeta(s) = \int_0^\infty u^{s-1} \left(\frac{1}{1-e^{-u}} - 1\right) du$$

$$\Gamma(s)\zeta(s) = \int_0^\infty u^{s-1} \left(\frac{1}{1-e^{-u}} - \frac{1-e^{-u}}{1-e^{-u}}\right) du$$

$$\Gamma(s)\zeta(s) = \int_0^\infty u^{s-1} \frac{e^{-u}}{1-e^{-u}} du$$

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{u^{s-1}}{e^u - 1} du$$

1.12.11 Jacobi Theta Function

$$\vartheta(x) = \sum_{n\in Z} e^{-mn^2 x}$$

$$\sum_{n\in Z} f(n) = \sum_{k\in Z}\int_{-\infty}^{+\infty} f(y) e^{-2\pi k y} dy$$

$$\sum_{n\in Z} e^{-\pi\eta^2 x} = \sum_{k\in Z}^{+\infty}\int_{-\infty}^{+\infty} e^{-\pi y^2 x} e^{-2\pi k y} dy = \sum_{k\in Z}\int_{-\infty}^{+\infty} e^{-\pi y^2 x - 2\pi i k y} dy$$

$$= \sum_{k\in Z}\int_{-\infty}^{+\infty} e^{-\pi x\left(y^2 + 2i\frac{k}{x}y + i^2\frac{k^2}{x^2} - i^{-\frac{k^2}{x^2}}\right)} dy$$

$$= \sum_{k\in Z}^{+\infty}\int_{-\infty}^{+\infty} e^{-\pi x\left(\left(y + i\frac{k}{x}\right)^2 - i^2\frac{k^2}{x^2}\right)} dy$$

$$\sum_{n\in Z} e^{-\pi n^2 x} = \sum_{k\in Z}\int_{-\infty}^{+\infty} e^{-\pi x\left(\left(y+i\frac{k}{x}\right)^2 - i^2\frac{k^2}{x^2}\right)}dy$$
$$\sum_{n\in Z} e^{-\pi n^2 x} = \sum_{k\in Z}\int_{-\infty}^{+\infty} e^{-\pi k^2\frac{1}{x}}e^{-\pi x\left(y+i\frac{k}{x}\right)^2}dy$$
$$\sum_{n\in Z} e^{-\pi n^2 x} = \sum_{k\in Z} e^{-\pi k^2\frac{1}{x}}\int_{-\infty}^{+\infty} e^{-\pi x\left(y+i\frac{k}{x}\right)^2}dy$$
$$y + i\frac{k}{x} = z \Rightarrow \mathrm{dy} = \mathrm{dz}|_{-\infty}^{+\infty} \rightarrow|_{-\infty+i\frac{k}{x}}^{+\infty+i\frac{k}{x}}$$
$$\sum_{n\in Z} e^{-\pi n^2 x} = \sum_{k\in Z} e^{-\pi k^2\frac{1}{x}}\int_{-\infty+i\frac{k}{x}}^{+\infty+i\frac{k}{x}} e^{-\pi x z^2}dz$$

$$\sum_{n\in Z} e^{-\pi n^2 x} = \sum_{k\in Z} e^{-\pi k^2\frac{1}{x}}\int_{-\infty+i\frac{k}{x}}^{+\infty+i\frac{k}{x}} e^{-\pi x z^2}dz$$
$$\int_{-R}^{R} e^{-\pi z z^2}dz = \int_{-R}^{-R+i\frac{k}{x}} e^{-\pi z z^2}dz + \int_{-R+i\frac{k}{x}}^{R+i\frac{k}{x}} e^{-\pi z z^2}dz + \int_{R+i\frac{k}{x}}^{R} e^{-\pi z z^2}dz$$
$$\sum_{n\in Z} e^{-\pi n^2 x} = \sum_{k\in Z} e^{-\pi k^2\frac{1}{x}}\int_{-\infty}^{+\infty} e^{-\pi z^2}dz = \sum_{k\in Z} e^{-\pi k^2\frac{1}{x}}\sqrt{\frac{\pi}{\pi x}}$$
$$\sum_{n\in Z} e^{-\pi n^2 x} = \sum_{k\in Z} e^{-\pi k^2\frac{1}{x}}\sqrt{\frac{1}{x}}$$

$$\vartheta(x) = \sum_{n\in Z} e^{-\pi n^2 x}$$
$$\vartheta(x) = \frac{1}{\sqrt{x}}\vartheta\left(\frac{1}{x}\right)$$

1.12.12 Riemann Functional Equation *I*

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$$
$$s \rightarrow \frac{s}{2} : \Gamma\left(\frac{s}{2}\right) = \int_0^\infty t^{\frac{s}{2}-1}e^{-t}dt$$
$$Sub : t = \pi n^2 x \rightarrow dt = \pi n^2 dx$$
$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty \left(\pi n^2 x\right)^{\frac{s}{2}-1} e^{-\pi n^2 x}\pi n^2 dx$$
$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty \pi^{\frac{s}{2}} n^s x^{\frac{s}{2}-1} e^{-\pi n^2 x}dx$$

$$\Gamma\left(\frac{s}{2}\right)=\int_0^\infty \pi^{\frac{s}{2}}n^s x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\frac{1}{n^s}=\int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx$$

$$\sum_{n=1}^{\infty}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\frac{1}{n^s}=\sum_{n=1}^{\infty}\int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^s}=\int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^{\infty}e^{-\pi n^2 x}dx$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^s}=\int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^{\infty}e^{-\pi n^2 x}dx$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^{\infty}e^{-\pi n^2 x}dx$$

$$\vartheta(x)=\sum_{n\in Z}e^{-\pi n^2 x}=1+2\sum_{n=1}^{\infty}e^{-\pi n^2 x}=1+2\psi(x)$$

$$\int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^{\infty}e^{-\pi n^2 x}dx=\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx$$

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx=\int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx+\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx$$

$$\vartheta(x)=\frac{1}{\sqrt{x}}\vartheta\left(\frac{1}{x}\right)\text{ or }2\psi(x)+1=\frac{1}{\sqrt{x}}\left(2\psi\left(\frac{1}{x}\right)+1\right)$$

$$\psi(x)=\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right)+\frac{1}{2\sqrt{x}}-\frac{1}{2}$$

$$\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx=\int_0^1 x^{\frac{s}{2}-1}\left(\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right)+\frac{1}{2\sqrt{x}}-\frac{1}{2}\right)dx$$

$$\int_0^1\left[x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right)+\frac{1}{2}\left(x^{\frac{s}{2}-\frac{3}{2}}-x^{\frac{s}{2}-1}\right)\right]$$

$$\int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right)dx+\frac{1}{2}\int_0^1\left(x^{\frac{s}{2}-\frac{3}{2}}-x^{\frac{s}{2}-1}\right)dx$$

$$\int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right)dx+\frac{1}{2}\left(\frac{1}{\frac{s}{2}-\frac{1}{2}}x^{\frac{s}{2}-\frac{1}{2}}-\frac{1}{\frac{s}{2}}x^{\frac{s}{2}}\right)_0^1 dx$$

$$\int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right)dx+\frac{1}{s(s-1)}$$

$$\int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right)dx+\frac{1}{s(s-1)}$$

$$x=\frac{1}{u}\rightarrow dx=-\frac{1}{u^2}du \quad \Big|_0^1\rightarrow\Big|_\infty^1$$

$$\int_\infty^1\left(\frac{1}{u}\right)^{\frac{s}{2}-\frac{3}{2}}\psi(u)\left(-\frac{du}{u^2}\right)+\frac{1}{s(s-1)}$$

$$\int_1^\infty\left(\frac{1}{x}\right)^{\frac{s}{2}-\frac{3}{2}}\psi(x)\left(\frac{dx}{x^2}\right)+\frac{1}{s(s-1)}$$

$$\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx=\int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\psi(x)dx+\frac{1}{s(s-1)}$$

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx=\int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx+\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx$$

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx=\int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx+\int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\psi(x)dx+\frac{1}{s(s-1)}$$

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx=\int_1^s\left(x^{\frac{s}{2}-1}+x^{-\frac{s}{2}-\frac{1}{2}}\right)\psi(x)dx+\frac{1}{s(s-1)}$$

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx=\int_1^\infty\left(x^{\frac{s}{2}-1}+x^{-\frac{s}{2}-\frac{1}{2}}\right)\psi(x)dx+\frac{1}{s(s-1)}$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_1^\infty\left(x^{\frac{s}{2}}+x^{\frac{1-s}{2}}\right)\psi\frac{\psi(x)}{x}dx-\frac{1}{s(1-s)}$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

1.12.13 Riemann Functional Equation II

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

$$\frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s)=\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$$

$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right)=\frac{\pi}{\cos\frac{\pi s}{2}}$$

$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)$$
$$s \to \frac{s}{2}$$
$$\Gamma\left(2\frac{s}{2}\right) = \frac{2^{2\frac{s}{2}-1}}{\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)$$
$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$$
$$\frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s) = \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$
$$s \to \frac{s+1}{2} = \frac{s}{2}+\frac{1}{2}$$
$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi s}{2}+\frac{\pi}{2}\right)}$$
$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right) = \frac{\pi}{\cos\frac{\pi s}{2}}$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$
$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$
$$\pi^{-\frac{s}{2}}\frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s)\zeta(s) = \pi^{-\frac{1-s}{2}}\frac{\pi}{\cos\frac{\pi s}{2}}\zeta(1-s)$$
$$\zeta(1-s) = \frac{2}{(2\pi)^s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s)$$

$$\zeta(1-s) = \frac{2}{(2\pi)^s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s)$$
$$1-s \to s$$
$$\zeta(s) = \frac{2}{(2\pi)^{1-s}}\cos\frac{\pi(1-s)}{2}\Gamma(1-s)\zeta(1-s)$$
$$\zeta(s) = 2^s\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s)$$

$$\zeta(1-s) = \frac{2}{(2\pi)^s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s)$$
$$\zeta(s) = 2^s\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s)$$

1.12.14 Trivial Zeros of the Zeta Function

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s)$$
$$\zeta(-2k) = -2^{-2k} 2k! \pi^{-2k-1} \sin \pi k \zeta(1+2k)$$
$$\zeta(-2k) = 0$$
$$\zeta(-2) = 0 \neq 1 + 2^2 + 3^2 + 4^2 + \ldots$$
$$\zeta(-4) = 0 \neq 1 + 2^4 + 3^4 + 4^4 + \ldots$$

1.12.15 Riemann Xi Function

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_1^\infty \left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right) \frac{\psi(x)}{x} dx - \frac{1}{s(1-s)}$$
$$\frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{2} s(s-1) \int_1^\infty \left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right) \frac{\psi(x)}{x} dx + \frac{1}{2}$$
$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$
$$\xi(s) = \xi(1-s)$$

Symmetry of the ξ-Function

$$s = \frac{1}{2} + it \quad t \in C$$
$$\xi\left(\frac{1}{2} + it\right) = \xi\left(1 - \left(\frac{1}{2} + it\right)\right)$$
$$\xi\left(\frac{1}{2} + it\right) = \xi\left(\frac{1}{2} - it\right)$$

1.12.16 Why $1+2+3+4+5+ ...$ not equals $-1/12 = \zeta(-1)$

NOT TRUE!!!

$$1 + 2 + 3 + 4 + \cdots \neq -\tfrac{1}{12}$$
$$1 + 2 + 3 + 4 + \cdots + n = \tfrac{n(n+1)}{2}$$

TRUE!!!

$$\zeta(-1) = -\frac{1}{12}$$

Representation of Sine function:

$$
\begin{aligned}
f(z) &= \sin(z) \\
f(z) &= \frac{e^{iz} - e^{-iz}}{2i} \\
f(z) &= \cdots \\
f(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \pm \cdots \\
f(z) &= z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right)
\end{aligned}
$$

Representation of Zeta function:

$$
\begin{aligned}
\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \mathrm{Re}(s) > 1 \\
\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots \\
\zeta(-1) &= 1 + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \frac{1}{4^{-1}} + \cdots \\
\zeta(-1) &= 1 + 2 + 3 + 4 + \cdots
\end{aligned}
$$

But $\mathrm{Re}(s) = -1 < 1$! Not allowed!!!

$$
\begin{aligned}
\zeta(s) &= 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \quad s \in C \backslash \{0{,}1\} \\
\zeta(-1) &= 2^{-1}\pi^{-1-1} \sin\left(-\frac{\pi}{2}\right) \Gamma(1-(-1))\zeta(1-(-1)) \\
\zeta(-1) &= 2^{-1}\pi^{-2}(-1)\Gamma(2)\zeta(2)
\end{aligned}
$$

$$
\zeta(-1) \neq 1 + 2 + 3 + 4 + \cdots
$$

But $\mathrm{Re}(s) = -1 < 1$! Not allowed!!!

$$
\zeta(-1) = 2^{-1}\pi^{-2}(-1)\Gamma(2)\zeta(2)
$$

$$
\Gamma(2) = (2-1)! = 1 \quad \zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}
$$

$$
\zeta(-1) = 2^{-1}\pi^{-2}(-1) \cdot 1 \cdot \frac{\pi^2}{6} = -\frac{1}{12}
$$