$$I=\int\limits_0^\infty \frac{(\ln x)^2 \operatorname{arcctg} x}{x^2+1}\,dx = \left[\begin{array}{c} x=1/t \\ dx=-dt/t^2 \end{array}\right] =$$

$$=\int\limits_\infty^0 \frac{(\ln t)^2(\frac{\pi}{2}-\operatorname{arcctg} t)}{\frac{1}{t^2}+1}\cdot\Big(-\frac{1}{t^2}\Big)\,dt = \frac{\pi}{2}\int\limits_0^\infty \frac{(\ln t)^2}{t^2+1}\,dt - I$$

$$I=\frac{\pi}{4}\int\limits_0^\infty \frac{(\ln t)^2}{t^2+1}\,dt$$

$$J(a)=\int\limits_0^\infty \frac{t^a}{t^2+1}\,dt$$

$$I=\frac{\pi}{4}J''(0)$$

$$J(a)=\int\limits_0^\infty \frac{t^a}{t^2+1}\,dt = \left[\begin{array}{c} t=\tan\theta \\ dt=\sec^2\theta d\theta \end{array}\right] = \int\limits_0^{\frac{\pi}{2}} \tan^a\theta\,d\theta =$$

$$\frac{1}{2}\cdot 2\int\limits_0^{\frac{\pi}{2}} \sin^{2\frac{a+1}{2}-1}\theta\cos^{2\frac{1-a}{2}-1}\theta\,d\theta = \frac{1}{2}B\Big(\frac{a+1}{2};\frac{1-a}{2}\Big) =$$

$$=\frac{1}{2}\frac{\Gamma(\frac{a+1}{2})\Gamma(\frac{a+1}{2})}{\Gamma(1)} = \frac{1}{2}\Gamma\Big(\frac{a+1}{2}\Big)\Gamma\Big(\frac{a+1}{2}\Big) =$$

$$=\frac{\pi}{2\sin(\pi\cdot\frac{a+1}{2})} = \frac{\pi}{2}\cdot\frac{1}{\sin(\frac{\pi}{2}(a+1))}$$

$$J'=\frac{\pi}{2}\cdot\frac{-\frac{\pi}{2}\cdot\cos(\frac{\pi}{2}(a+1))}{\sin^2(\frac{\pi}{2}(a+1))} = -\frac{\pi^2}{4}\cdot\frac{\cos(\frac{\pi}{2}(a+1))}{\sin^2(\frac{\pi}{2}(a+1))}$$

$$J''=-\frac{\pi^2}{4}\cdot\frac{-\frac{\pi}{2}\cdot\sin^3(\frac{\pi}{2}(a+1))-\cos(\frac{\pi}{2}(a+1))\cdot\partial_a\sin^2(\frac{\pi}{2}(a+1))}{\sin^4(\frac{\pi}{2}(a+1))}$$

$$J''(0)=\frac{\pi^3}{8}$$

$$I=\frac{\pi}{4}\cdot\frac{\pi^3}{8}=\frac{\pi^4}{32}$$