$$S=\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}=1+\sum_{n=1}^{\infty}\frac{n\Gamma(n)\Gamma(n+1)}{\Gamma(2n+1)}=1+\sum_{n=1}^{\infty}nB(n,n+1)$$

$$S-1=\sum_{n=1}^{\infty}n\int\limits_0^1 t^n(1-t)^{n-1}\,dt=\int\limits_0^1 t\sum_{n=1}^{\infty}n(t(1-t))^{n-1}\,dt=\left[\begin{array}{c} z=t(1-t)\\ |z|<1\end{array}\right]=$$

$$=\int\limits_0^1 t\frac{d}{dz}\sum_{n=1}^{\infty}z^n\,dt=\int\limits_0^1 t\frac{d}{dz}\sum_{n=0}^{\infty}z^n\,dt=\int\limits_0^1 t\frac{d}{dz}\left(\frac{1}{1-z}\right)\,dt=$$

$$=\int\limits_0^1\frac{t}{(1-z)^2}\,dt=\int\limits_0^1\frac{t}{(1-t(1-t))^2}\,dt=\int\limits_0^1\frac{t}{(t^2-t+1)^2}\,dt=$$

$$=\int\limits_0^1\frac{t-\frac{1}{2}+\frac{1}{2}}{((t-\frac{1}{2})^2+\frac{3}{4})^2}\,dt=\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\frac{u+\frac{1}{2}}{(u^2+\frac{3}{4})^2}\,du=\frac{1}{2}\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\frac{du}{(u^2+\frac{3}{4})^2}=$$

$$=\left[\begin{array}{c} u=\frac{\sqrt{3}}{2}\tan\theta\\ du=\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta\end{array}\right]=\frac{1}{2}\int\limits_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\frac{\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta}{\frac{9}{16}\sec^4\theta}=\frac{4\sqrt{3}}{9}\int\limits_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\cos^2\theta\,d\theta=$$

$$=\frac{4\sqrt{3}}{9}\int\limits_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\frac{1+\cos 2\theta}{2}\,d\theta=\frac{2\sqrt{3}}{9}\cdot\frac{\pi}{3}+\frac{\sqrt{3}}{9}\cdot 2\cdot\frac{\sqrt{3}}{2}=\frac{2\pi\sqrt{3}}{27}+\frac{1}{3}$$

$$S=\frac{2\pi\sqrt{3}}{27}+\frac{4}{3}$$