1. $\Pi(t_1, x_1^i; t_2, x_2^i)$ $\quad x_1^i = x^i(t_1) \quad x_2^i = x^i(t_2)$

$$S[x(t)] = \int_{t_1}^{t_2} dt\, L(x(t), \dot{x}(t), \ddot{x}(t))$$

$$\delta S = \frac{1}{\varepsilon}\int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial x}\varepsilon\delta x + \frac{\partial L}{\partial \dot{x}}\varepsilon\delta\dot{x} + \frac{\partial L}{\partial \ddot{x}}\varepsilon\delta\ddot{x} + O(\varepsilon^2)\right\}$$

$$= \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L}{\partial \ddot{x}}\delta\ddot{x} \right\}$$

$$\frac{\partial L}{\partial \dot{x}}\delta\dot{x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right) - \delta x \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$$

$$\frac{\partial L}{\partial \ddot{x}}\delta\ddot{x} = \left(\frac{d}{dt}\right)^2\left[\frac{\partial L}{\partial \ddot{x}}\delta x\right] - \delta x\left(\frac{d}{dt}\right)^2\frac{\partial L}{\partial \ddot{x}} - 2\delta\dot{x}\frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}$$

$$= \cancel{\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\delta x\right)}\ \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\delta\dot{x}\right) - \delta\dot{x}\frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}$$

$$\delta\dot{x}\frac{d}{dt}\frac{\partial L}{\partial \ddot{x}} = \frac{d}{dt}\left(\delta x\frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}\right) - \delta x\left(\frac{d}{dt}\right)^2\frac{\partial L}{\partial \ddot{x}}$$

$$\delta S = \left(\frac{\partial L}{\partial \dot{x}}\delta x + \frac{\partial L}{\partial \ddot{x}}\dot{x} - \delta x\frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}\right)\Bigg|_{t_1}^{t_2} +$$

$$+ \int_{t_1}^{t_2} dt\,\delta x\left\{\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \left(\frac{d}{dt}\right)^2\frac{\partial L}{\partial \ddot{x}}\right\}$$

$$\delta x(t_1) = \delta x(t_2) = 0$$

$$\delta\dot{x}(t_1) = \delta\dot{x}(t_2) = 0$$

$$\delta S = \int_{t_1}^{t_2} dt\, \delta x \left\{ \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \left(\frac{d}{dt}\right)^2 \frac{\partial L}{\partial \ddot{x}} \right\}$$

$\delta S = 0$ (экстремаль)

$$\boxed{\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \left(\frac{d}{dt}\right)^2 \frac{\partial L}{\partial \ddot{x}} = 0}$$

2. $L(x, \dot{x}, v, \dot{v}) = m\dot{x}v - \frac{mv^2}{2} - U(x)$

$$\cancel{\delta S = \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial x}\delta x - \delta x \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial v}\delta v - \delta v \frac{d}{dt}\frac{\partial L}{\partial \dot{v}} \right\}}$$

$$\cancel{= \int_{t_1}^{t_2} dt \left\{ \delta x\left(-U'(x) - m\dot{v}\right) + \delta v\left(m\dot{x} - mv\right) \right\}}$$

по x: $\quad -U'(x) - m\dot{v} = 0$

по v: $\quad m\dot{x} - mv = 0 \Rightarrow v = \dot{x}$

Тогда $L = \frac{m\dot{x}^2}{2} - U(x)$

$$-U'(x) = m\ddot{x}$$

3. $S[x(t), \varepsilon(t)] = \frac{1}{2}\int_{t_1}^{t_2} dt \sqrt{\varepsilon}\left[\frac{\dot{x}^2}{\varepsilon(t)} - m^2\right]$

$$L = \frac{\sqrt{\varepsilon}}{2}\left[\frac{\dot{x}^2}{\varepsilon(t)} - m^2\right]$$

$$\frac{\partial L}{\partial \dot{\varepsilon}} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\varepsilon}} = 0 = \frac{\partial L}{\partial \varepsilon} = \frac{1}{2}\left[-\frac{\dot{x}^2}{2\varepsilon^{3/2}} - \frac{m^2}{2\sqrt{\varepsilon}}\right]$$

$$= -\frac{1}{4\varepsilon^{3/2}}\left[\dot{x}^2 + m^2\varepsilon\right] = 0$$

$$\varepsilon = -\frac{\dot{x}^2}{m^2}$$

$$L = \frac{1}{2m}\sqrt{-\dot{x}^2}\left[-m^2 - m^2\right] = -m\sqrt{-\dot{x}^2}$$

$$S_{eff}[x(t)] = -m\int_{t_1}^{t_2} dt\,\sqrt{-\dot{x}^2}$$

$$L = -mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}} \underset{\hbar=c=k=1}{=\!=\!=} -m\sqrt{1-\dot{x}^2}$$

$$\dot{x}^2 = \dot{x}^2 - 1$$

данное эф-д. соотв. мех. сист. частицы дв. с околосветовой скоростью (рел. сист.)

4. $$\frac{\partial L}{\partial x^\mu} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\mu}$$

$$L = (\bar{r}, \dot{\bar{r}}, t) = \frac{m\dot{\bar{r}}^2}{2} + \frac{e}{c}\left(\dot{\bar{r}} \cdot \bar{A}(\bar{r}, t)\right) - e A_0(\bar{r}, t)$$

$$\frac{\partial L}{\partial x^\mu} = \frac{e}{c}\left[\dot{x}^k \frac{\partial A^k}{\partial x^\mu}\right] - e\frac{\partial A_0}{\partial x^\mu}$$

$$\frac{\partial L}{\partial \dot{x}^\mu} = m\dot{x}^\mu + \frac{e}{c}\left(A^\mu(\bar{r}, t)\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\mu} = m\ddot{x}^\mu + \frac{e}{c}\left(\dot{A}^\mu + \frac{\partial A^\mu}{\partial x^k}\dot{x}^k\right)$$

$$\frac{e}{c}\left[\dot{x}^k \frac{\partial A^k}{\partial x^\mu}\right] - e\frac{\partial A_0}{\partial x^\mu} = m\ddot{x}^\mu + \frac{e}{c}\left(\dot{A}^\mu + \frac{\partial A^\mu}{\partial x^k}\dot{x}^k\right)$$

$$\frac{e}{c}\frac{\partial A^k}{\partial x^\mu} - \frac{e}{c}\frac{\partial A^\mu}{\partial x^k}\dot{x}^k - e\frac{\partial A_0}{\partial x^\mu} - \frac{e}{c}\dot{A}^\mu = m\ddot{x}^\mu =$$

$$= e\vec{E} + \frac{e}{c}\left(\vec{v} \times \vec{H}\right)$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \frac{\partial A_0}{\partial \vec{r}}$$

$$\vec{H} = \operatorname{rot}\vec{A}$$

$$m\ddot{x}^\mu = -e\left(\frac{\partial A_0}{\partial x^\mu} + \frac{\dot{A}^\mu}{c}\right)$$

5. $$\delta^{(2)} S = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_{t_1}^{t_2} dt\, \delta x \left[\frac{\partial L(x+\varepsilon\delta x, \dot{x}+\varepsilon\delta\dot{x}, t)}{\partial x} - \frac{d}{dt}\frac{\partial L(\dots)}{\partial \dot{x}} \right]$$

$$= \int_{t_1}^{t_2} \delta x \left\{ \frac{\partial^2 L}{\partial x^2}\delta x + \frac{\partial^2 L}{\partial \dot{x}\partial x}\delta\dot{x} - \right.$$

$$\left. - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial x \partial \dot{x}}\delta x + \frac{\partial^2 L}{\partial \dot{x}^2}\delta\dot{x} \right) \right\} =$$

$$= \int_{t_1}^{t_2} \delta x \left\{ \frac{\partial^2 L}{\partial x^2}\delta x + \cancel{\frac{\partial^2 L}{\partial \dot{x}\partial x}\frac{d}{dt}\delta x} - \right.$$

$$- \delta x \frac{d}{dt}\frac{\partial^2 L}{\partial x \partial \dot{x}} - \cancel{\frac{\partial^2 L}{\partial x \partial \dot{x}}\frac{d}{dt}\delta x} -$$

$$\left. - \frac{d}{dt}\frac{\partial^2 L}{\partial \dot{x}^2}\frac{d}{dt}\delta x - \frac{\partial^2 L}{\partial \dot{x}^2}\left(\frac{d}{dt}\right)^2 \delta x \right\} =$$

$$= \int_{t_1}^{t_2} \delta x \left\{ -\frac{\partial^2 L}{\partial \dot{x}^2}\left(\frac{d}{dt}\right)^2 - \frac{d}{dt}\frac{\partial^2 L}{\partial \dot{x}^2}\frac{d}{dt} + \frac{\partial^2 L}{\partial x^2} - \frac{d}{dt}\frac{\partial^2 L}{\partial x \partial \dot{x}} \right\} \delta x$$

$$= \int_{t_1}^{t_2} \delta x \, \Gamma \, \delta x$$

$$L = \frac{m\dot{x}^2}{2} - U(x) \quad (\text{cf. 2.})$$

$$\frac{\partial^2 L}{\partial \dot{x}^2} = m \qquad \frac{\partial^2 L}{\partial x^2} = -U''(x)$$

$$\frac{\partial^2 L}{\partial x \partial \dot{x}} = 0$$

$$\delta^{(2)} S = \int_{t_1}^{t_2} \delta x \left\{ -m\left(\frac{d}{dt}\right)^2 - U''(x) \right\} \delta x$$

$$L = \frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2}$$ (гарм. осц.)

$$\delta^{(2)} S = \int_{t_1}^{t_2} \delta x \left\{ -m\left(\frac{d}{dt}\right)^2 - m\omega^2 \right\} \delta x$$

II. 6. $x_a = x + a$, $t_a = t$ (простр. трансляции)

$$J = \frac{\partial L}{\partial \dot{x}} = p_x$$

$$\bar{p} = \frac{\partial L}{\partial \dot{\bar{r}}}$$

$x_a = x$, $t_a = t - a$ (вр. трансл.)

$$J = \frac{\partial L}{\partial \dot{x}} \dot{x} - L = E$$

$$E = \frac{\partial L}{\partial \dot{\bar{r}}} \dot{\bar{r}} - L = \bar{p}\dot{\bar{r}} - L$$

$$\bar{r} = (x, y, z) \xrightarrow[\text{Ox}]{\substack{\text{поворот}\\ \text{вокруг}}} \bar{r}_\varphi = (\cancel{x} x,\ y\cos\varphi - z\sin\varphi,\ y\sin\varphi + z\cos\varphi)$$

$$J = \frac{\partial L}{\partial \dot{x}^i} \frac{\partial x^i_\varphi}{\partial \varphi}\Big|_{\varphi=0} = \frac{\partial L}{\partial y}(-y\cancel{\sin\varphi}^{\,0} - z\cos\varphi) + \frac{\partial L}{\partial z}(y\cos\varphi - z\cancel{\sin\varphi}^{\,0})$$

$$= p_z y - p_y z = (\bar{r} \times \bar{p})_x = \text{const}$$

$$\overline{r \times p} = \overline{\text{const}}$$

7. $S[x(t)] = \int_{t_1}^{t_2} dt\, L(x(t), \dot{x}(t))$

$$x(t) = y(s(t)) \quad , \quad S[y(s(t))] = \int_{t_1}^{t_2} dt \frac{ds}{dt} L(y(s(t)), \dot{y}(s(t)))$$

$$s(t) = t + \tau$$

$$\cancel{x(t) = y(t) + \tau \dot{y}(t)} \quad \cancel{y(s(t)) = y}$$

$$ds = dt(1 + \dot{\tau})$$

$$\frac{d}{ds} = (1 - \dot{\tau}) \frac{d}{dt}$$

$$\int L\, dt = S[y(s(t))] = \int_{t_1}^{t_2} dt\, (1 + \dot{\tau}) L(x(t), \dot{x} - \dot{\tau}\dot{x}) =$$

$$= \int_{t_1}^{t_2} dt\, (1 + \dot{\tau}) \left(L - \frac{\partial L}{\partial \dot{x}} \dot{\tau} \dot{x} \right) =$$

$$= \int L\, dt + \int_{t_1}^{t_2} dt\, \dot{\tau} \left(L - \frac{\partial L}{\partial \dot{x}} \dot{x} \right)$$

$$\int_{t_1}^{t_2} dt\, \dot{\tau} \underbrace{\left(L - \frac{\partial L}{\partial \dot{x}} \dot{x} \right)}_{= E} = 0$$

$$\Downarrow$$

$$E = 0$$

8. $x \to x_u = \dfrac{x - ut}{\sqrt{1-u^2}}$ $\quad t \to t_u = \dfrac{t - ux}{\sqrt{1-u^2}}$

$$X = \frac{\partial x_u}{\partial u}\Big|_{u=0} = \frac{x-ut}{\sqrt{1-u^2}}\left(\frac{-t}{x-ut} + \frac{u}{1-u^2}\right)\Big|_{u=0} = -t$$

$$T = \frac{\partial t_u}{\partial u}\Big|_{u=0} = \frac{t-ux}{\sqrt{1-u^2}}\left(\frac{-x}{t-ux} + \frac{u}{1-u^2}\right)\Big|_{u=0} = -x$$

$$\mathcal{J} = \left(L - \frac{\partial L}{\partial \dot{x}}\dot{x}\right)T + \frac{\partial L}{\partial \dot{x}}X = \left(\frac{\partial L}{\partial \dot{x}}\dot{x} - L\right)x - t\frac{\partial L}{\partial \dot{x}}$$

инв. отн ~~пр~~. пр. $\Rightarrow$ $\dfrac{\partial L}{\partial \dot{x}} = p = const$

инв. отн. врем. пр. $\Rightarrow$ $-L + \dfrac{\partial L}{\partial \dot{x}}\dot{x} = E = const$

значит $\quad Ex - pt = const$

$$E\,dx = p\,dt$$

$$\dot{x}E = p$$

9. $$\frac{dS}{du} = \int_{t_1}^{t_2} dt\,\frac{d}{dt}(-Ex + pt) = \int_{t_1}^{t_2} dt\,(-E\dot{x} + p)$$

$$-E\dot{x} + p = \frac{df(x,t)}{dt} = \underbrace{\frac{\partial f}{\partial t}}_{0} + \frac{\partial f}{\partial x}\dot{x} = m\dot{x}$$

$$L = L(\dot{x}^2)$$

$$-(L'\dot{x} - L)\dot{x} + L' = m\dot{x}$$

$$L'(1-\dot{x}^2) = \dot{x}(m - L)$$

$$\int \frac{dL}{m-L} = \int \frac{\dot{x}\,d\dot{x}}{1-\dot{x}^2}$$

$$-\ln(m-L) = -\frac{1}{2}\ln(1-\dot{x}^2)+C$$

$$m-L = C\sqrt{1-\dot{x}^2}$$

~~$L \to L + \frac{df(x,t)}{dt} = L - m$ (ур-е дв. не изменяются)~~

$$m-L = c\sqrt{1-\dot{x}^2}$$

$$\dot{x}\to 0 \text{ (нерел.)} \Rightarrow m-\frac{m\dot{x}^2}{2} \approx c - \frac{c}{2}\dot{x}^2$$

(до I порядка)

$$c = m$$

$$L = m - m\sqrt{1-\dot{x}^2}$$

$$L \to L + \frac{df(x,t)}{dt} = L - m \text{ (ур-е дв. не изменяются)}$$

$$L = -m\sqrt{1-\dot{x}^2}$$

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{m\dot{x}}{\sqrt{1-\dot{x}^2}}$$

$$E = \frac{\partial L}{\partial \dot{x}}\dot{x} - L = \frac{m\dot{x}^2}{\sqrt{1-\dot{x}^2}} + m\sqrt{1-\dot{x}^2} = \frac{m}{\sqrt{1-\dot{x}^2}}$$

$$E^2 - p^2 = \frac{m^2(1-\dot{x}^2)}{1-\dot{x}^2} = m^2$$

$$E^2 = p^2 + m^2$$

III. 10. $\mathcal{L} = \frac{m\dot{\vec{r}}^2}{2} - U(r)$

$(\vec{\ell})_z = \text{const}$

$$\begin{cases} x = r\cos\varphi & \dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \\ y = r\sin\varphi & \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \\ z = 0 & \dot{z} = 0 \end{cases}$$

$$\dot{\vec{r}}^2 = \dot{x}^2 + \dot{y}^2 + \cancel{\dot{z}^2} = \dot{r}^2 + r^2\dot{\varphi}^2$$

$$\mathcal{L} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r) = \frac{m\dot{r}^2}{2} + \underbrace{\frac{m}{2}r^2\dot{\varphi}^2 - U(r)}_{-V_{eff}}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \underbrace{m r^2 \dot{\varphi}}_{= \ell} = \text{const}$$

$$\cancel{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r}}$$

$$\cancel{\frac{d}{dt} m\dot{r} = m\ddot{r} = m r\dot{\varphi}^2}$$

$$E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + U(r)$$

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\frac{\ell^2}{m^2 r^4}\right) + U(r) = \frac{m\dot{r}^2}{2} + \underbrace{\frac{\ell^2}{2mr^2} + U(r)}_{V_{eff}}$$

$$\dot{r} = \sqrt{\frac{2}{m}(E - U) - \frac{\ell^2}{m^2 r^2}}$$

$$\cancel{\int dt = \int \frac{dr}{\sqrt{\frac{2}{m}(E-U) - \frac{\ell^2}{m^2r^2}}}}$$

$$\frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = \frac{dr}{d\varphi}\cdot\frac{\ell}{mr^2}$$

$$\frac{\frac{l}{mr^2}\,dr}{\sqrt{\frac{2}{m}(E-U)-\frac{l^2}{m^2r^2}}} = d\varphi, \quad U=-\frac{\alpha}{r}$$

$$\cancel{\xi =}\quad d\varphi = \frac{\frac{l}{mr^2}\,dr}{\sqrt{\frac{2}{m}E+\frac{2\alpha}{mr}-\frac{l^2}{m^2r^2}}}$$

$$\xi = -\frac{\alpha}{l}+\frac{l}{mr}$$

$$\int d\varphi = \int\frac{-d\xi}{\sqrt{\left(\frac{2E}{m}+\frac{\alpha^2}{l^2}\right)-\xi^2}}$$

$$\varphi + \underbrace{\cancel{const}}_{\varphi_0} = \cos^{-1}\left(\frac{-\frac{\alpha}{l}+\frac{l}{mr}}{\sqrt{\frac{2E}{m}+\frac{\alpha^2}{l^2}}}\right)$$

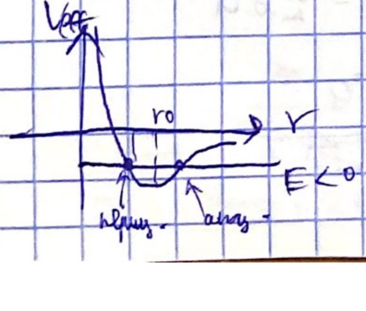
$$\sqrt{\frac{2E}{m}+\frac{\alpha^2}{l^2}}\cos\varphi + \frac{\alpha}{l} = \frac{l}{mr}$$

$$r = \frac{l}{m}\,\frac{1}{\frac{\alpha}{l}\left(\sqrt{1+\frac{2El^2}{m\alpha^2}}\cos\varphi+1\right)} = \frac{\overbrace{l^2/m\alpha}^{p_{orb}}}{\underbrace{\sqrt{1+\frac{2El^2}{m\alpha^2}}}_{e}\cos\varphi+1}$$

$$r = \frac{p_{orb}}{1+e\cos\varphi}$$

$$V_{eff} = +\frac{m}{2}r^2\dot{\varphi}^2 + U(r) = +\frac{m}{2}r^2\frac{l^2}{m^2r^4}+\left(-\frac{\alpha}{r}\right) = \frac{l^2}{2mr^2}-\frac{\alpha}{r}$$

$$V'_{eff} = -\frac{l^2}{mr^3}+\frac{\alpha}{r^2}=0 \Rightarrow r_0 = \frac{l^2}{\alpha m}$$



$E = V_{eff} \Rightarrow \dot{r}=0$ (т. поворота)

$E<0$ — финит. дв. (эллипс)

$E=0$ — инф. (пар.) $\quad E<0$ — инф. (гип)

$$a = \frac{p_{orb}}{1-e^2} = \frac{\alpha}{2|E|}$$

$$b = \frac{p_{orb}}{\sqrt{1-e^2}} = \frac{\ell}{\sqrt{2m|E|}} = \sqrt{a} \cdot \ell \cdot \sqrt{\frac{1}{\alpha m}}$$

$$r_p = \frac{p_{orb}}{1+e} \qquad r_a = \frac{p_{orb}}{1-e}$$

$$dA = \frac{1}{2} r^2 d\varphi$$

$$\dot{A} = \frac{1}{2} r^2 \dot{\varphi} = \frac{1}{2m} \ell = const$$

$$\frac{A}{T} = \frac{\ell}{2m}$$

$$T = \frac{2m}{\ell} A = \frac{2m}{\ell} \pi a b = 2m\pi a^{3/2} \sqrt{\frac{1}{\alpha m}} = 2\pi a^{3/2} \sqrt{\frac{m}{\alpha}}$$

$$\frac{T^2}{a^3} = const$$

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91. $$\delta U(r) = -\frac{\gamma}{r^2} = -\frac{\beta \ell^2}{r^3}$$

$$\int d\varphi = \int \frac{\ell / mr^2 \, dr}{\sqrt{\frac{2}{m}(E-U) - \frac{\ell^2}{m^2 r^2} - \frac{2}{m}\delta U}} =$$

$$= \cancel{\int \frac{\ell / mr^2 \, dr}{\sqrt{\dots}}}_{2\pi} + \int \frac{1}{2} \frac{\ell \, dr}{mr^2} \cdot \frac{\frac{2}{m}\delta U}{\sqrt{\dots}^{\,3}}$$

$$\int d\varphi = -\frac{\partial}{\partial \ell}\int \sqrt{\frac{2}{m}(E-U) - \frac{\ell^2}{m^2 r^2} - \frac{2}{m}\delta U}\; dr$$

$$= -\frac{\partial}{\partial\ell}\left[\underbrace{\int\sqrt{\frac{2}{m}(E-U)-\frac{\ell^2}{m^2r^2}}\,dr}_{2\pi} - \int \frac{\frac{\delta U}{m}}{\sqrt{\dots}}\,dr\right]$$

$$= \frac{\partial}{\partial \ell}\int \frac{r^2}{\ell}\,\delta U\, d\varphi = \frac{\partial}{\partial\ell}\int \frac{-\gamma}{\ell r}\, d\varphi =$$

$$= -\frac{\gamma}{p_{orb}}\frac{\partial}{\partial\ell}\int_0^{2\pi}\frac{1+\cos\varphi}{\ell}\,d\varphi = \frac{2\pi\gamma}{\ell^2 p_{orb}} = \frac{2\pi\beta}{p_{orb}}$$

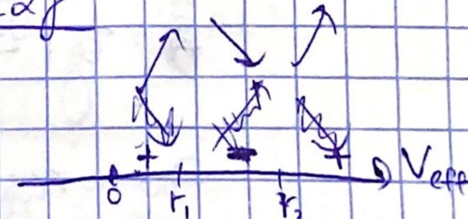
12. $$V_{eff} = \frac{m}{2}r^2\dot\varphi^2 + U(r) + \delta U(r) =$$

$$= \frac{m}{2}r^2\frac{\ell^2}{m^2r^4} - \frac{\alpha}{r} - \frac{\gamma}{r^3} = \frac{\ell^2}{2mr^2} - \frac{\alpha}{r} - \frac{\gamma}{r^3}$$

$$V'_{eff} = \frac{-\ell^2}{mr^3} + \frac{\alpha}{r^2} + \frac{3\gamma}{r^4} = 0$$

$$m\alpha r_0^2 - \ell^2 r_0 + 3\gamma m = 0$$

$$r_{1,2} = \frac{\ell^2 \pm\sqrt{\ell^4 - 12\alpha\gamma}}{2\alpha}$$



$$V_{eff} = -\frac{2m\alpha r^2 - r\ell^2 + 2m\gamma}{2mr^3}$$

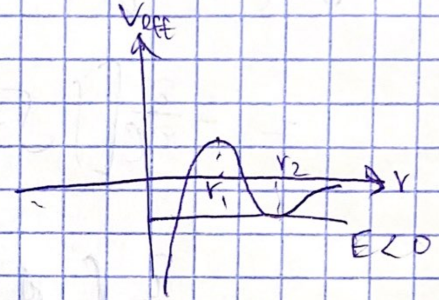
$$= -\frac{1}{2mr^3}\left(2m\alpha r^2 - 2\ell^2 r + 6\gamma m + r\ell^2 - 4\gamma m\right)$$

$$V_{eff}(r_1) = -\frac{1}{2mr_1^3}\left(\ell^2 r_1 - 4\gamma m\right) =$$

$$= -\frac{1}{2mr_1^3}\left(\ell^2 \frac{\ell^2 - \sqrt{\ell^4 - 12\alpha\gamma}}{2\alpha} - 4\gamma m\right) =$$

$$= -\frac{1}{2mr_1^3}\left(\ldots \right)$$

$$r_{isco} = r_2 = \frac{\ell^2 + \sqrt{\ell^4 - 12\alpha\gamma}}{2\alpha}$$



$E = V_{eff}$

13. $$d\varphi = \frac{\ell/mr^2\, dr}{\sqrt{\frac{2}{m}(E-U) - \frac{\ell^2}{m^2 r^2}}}$$

$$d\varphi = \frac{\ell/r^2\, dr}{\sqrt{2m\left(E + \frac{\alpha}{r^2}\right) - \frac{\ell^2}{r^2}}} = \frac{-\ell\, d\frac{1}{r}}{\sqrt{2mE + \frac{2m\alpha - \ell^2}{r^2}}} =$$

$$= \frac{-\ell\, d\frac{1}{r}}{\sqrt{2m\alpha - \ell^2}\sqrt{\left(\sqrt{\frac{2mE}{2m\alpha - \ell^2}}\right)^2 + \left(\frac{1}{r}\right)^2}}$$

$$-\varphi + \varphi_0 = \frac{\ell}{\sqrt{2m\alpha - \ell^2}}\left(\sqrt{\frac{2mE}{2m\alpha - \ell^2}}\right)^{-1} \tan^{-1}\left(\frac{1/r}{\sqrt{\frac{2mE}{2m\alpha - \ell^2}}}\right)$$

$$\mathrm{tg}\left(\frac{\sqrt{2mE}}{\ell}(\varphi_0 - \varphi)\right)\sqrt{\frac{2mE}{2m\alpha - \ell^2}} = \frac{1}{r}$$

Это если $2m\alpha - \ell^2 > 0$

Можно подобрать φ: $\frac{\sqrt{2mE}}{\ell}(\varphi_0 - \varphi) = \frac{\pi}{2}$. Тогда $r \to 0$

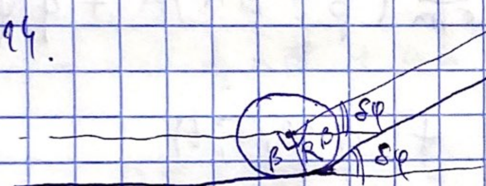
Итого $\varphi = \frac{l}{\sqrt{l^2-2m\alpha}} \cos^{-1} \frac{1/r}{\sqrt{\frac{2mE}{l^2-2m\alpha}}}$

$$\sqrt{\frac{2mE}{l^2-2m\alpha}} \cos\left(\sqrt{1-\frac{2m\alpha}{l^2}}\,\varphi\right) = \frac{1}{r}$$

$$r = \sqrt{\frac{l^2-2m\alpha}{2mE}} \frac{1}{\cos\sqrt{1-\frac{2m\alpha}{l^2}}\,\varphi}$$

Ответ: частица падает на центр, если $2m\alpha - l^2 > 0$

14.



$$\beta = \frac{\pi}{2} + \frac{\delta\varphi}{2}$$

$$U = \frac{-GMm}{r} \Rightarrow \alpha = GMm$$

$$E = V_{eff} = \frac{m}{2}R^2\dot{\varphi}^2 - \frac{GMm}{R} =$$

(вт. разв.)

$$= \frac{m}{2}R^2\frac{l^2}{m^2R^4} - \frac{GMm}{R} =$$

$$= \frac{l^2}{2mR^2} - \frac{GMm}{R} \overset{l=mcR}{=\!=} \frac{mc^2}{2} - \frac{GMm}{R} \sim \frac{mc^2}{2}$$

$$r = \frac{p_{orb}}{1+e\cos\varphi} \quad R = r_p = r(\varphi=0) = \frac{p_{orb}}{1+e} = \frac{l^2/m\alpha}{1+\sqrt{\frac{2El^2}{m\alpha^2}}}$$

$$r_a \Rightarrow \infty \Rightarrow \cos\varphi = \cos\beta = -\frac{1}{e} = -\frac{1}{\sqrt{\frac{2El^2}{m\alpha^2}}}$$

$$-\sin\frac{\delta\varphi}{2} \approx -\frac{\delta\varphi}{2}$$

$$\sqrt{\frac{2El^2}{m\alpha^2}} \to \infty$$

$$\delta\varphi = 2\sqrt{\frac{m\alpha^2}{2El^2}} = 2\frac{GmM}{mcR}\sqrt{\frac{m}{2\cdot\frac{mc^2}{2}}} = \boxed{\frac{2GM}{Rc^2}}$$

по ОТО: $\delta\varphi = 2\delta\varphi = \frac{4GM}{Rc^2}$

IV. 15. $\mathcal{L} = \frac{m\dot{\vec{r}}^2}{2} + \frac{q}{c}(\dot{\vec{r}}\cdot\vec{A}(\vec{r},t)) - q\varphi(\vec{r},t)$

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} + \frac{q}{c}\vec{A}(\vec{r},t)$$

$$\dot{\vec{r}} = \frac{1}{m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)$$

$$\mathcal{H}(\vec{r},\vec{p},t) = \vec{p}\cdot\dot{\vec{r}} - \mathcal{L}\Big|_{\dot{\vec{r}}\to\vec{p}} = \frac{\vec{p}}{m}\left(\vec{p} - \frac{q}{c}\vec{A}\right) -$$

$$- \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 - \frac{q}{cm}\left(\vec{p} - \frac{q}{c}\vec{A}\right)\vec{A} + q\varphi =$$

$$= \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}(\vec{r},t)\right)^2 + q\varphi(\vec{r},t)$$

$$\frac{\partial \mathcal{H}}{\partial \vec{p}} = +\dot{\vec{r}}$$

$$\frac{\vec{p}}{m} - \frac{q}{mc}\vec{A}(\vec{r},t) = +\dot{\vec{r}}$$

$$\frac{\partial \mathcal{H}}{\partial \vec{r}} = -\dot{\vec{p}}$$

$$-\frac{q\vec{p}}{mc}\frac{\partial \vec{A}}{\partial \vec{r}} + \frac{q^2}{mc^2}\vec{A}\frac{\partial \vec{A}}{\partial \vec{r}} + q\frac{\partial \varphi}{\partial \vec{r}} = -\dot{\vec{p}}$$

$$-\dot{p}_i = q\frac{\partial \varphi}{\partial x^i} + \frac{1}{m}\left(p_k - \frac{q}{c}A^k\right)\left(-\frac{q}{c}\frac{\partial A^k}{\partial x^i}\right)$$

$$\dot{p}_i = -q\frac{\partial \varphi}{\partial x^i} + \frac{q}{mc}\left(p_k - \frac{q}{c}A^k\right)\frac{\partial A^k}{\partial x^i}$$

16. $$S[\bar{r}(t)] = \int_{t_1}^{t_2} dt \left\{ \frac{1}{2} m (\bar{r} \cdot \ddot{\bar{r}}) - a\cos(\bar{k}\cdot\bar{r}) \right\}$$

$$\delta S = \int_{t_1}^{t_2} dt \left\{ \frac{1}{2} m (\delta \bar{r} \cdot \ddot{\bar{r}}) + a\sin(\bar{k}\cdot\bar{r})(\bar{k}\cdot\delta\bar{r}) \right\} =$$

$$L(\bar{r}, t)$$

$$= \int_{t_1}^{t_2} dt\, \delta \bar{r} \left\{ \frac{1}{2} m \ddot{\bar{r}} + a\bar{k}\sin(\bar{k}\cdot\bar{r}) \right\}$$

$$\frac{\partial L}{\partial \vec{r}} = \frac{d}{dt} \cancelto{0}{\frac{\partial L}{\partial \dot{\vec{r}}}}$$

$$\frac{1}{2} m \ddot{\vec{r}} + a\vec{k}\sin(\vec{k}\cdot\vec{r}) = 0$$

$$p = \frac{\partial L}{\partial \dot{\vec{r}}} = 0$$

$$\mathcal{H} = \vec{p}\cdot\dot{\vec{r}} - L \Big|_{\dot{r} \to p} = a\cos(\bar{k}\cdot\bar{r}) - \frac{1}{2} m (\bar{r}\cdot\ddot{\bar{r}})$$

Если сказать, что $L = L(\vec{r}, \ddot{\vec{r}}, t)$ и потребовать $\delta\vec{r} = \delta\dot{\vec{r}} = 0$ на концах, то

$$\frac{\partial L}{\partial \vec{r}} - \frac{d}{dt} \underbrace{\frac{\partial L}{\partial \dot{\vec{r}}}}_{0} + \left(\frac{d}{dt}\right)^2 \frac{\partial L}{\partial \ddot{\vec{r}}} = 0$$

$$\frac{1}{2} m \ddot{\vec{r}} + a\vec{k}\sin(\bar{k}\cdot\bar{r}) + \left(\frac{d}{dt}\right)^2 \left(\frac{m\bar{r}}{2}\right) = 0$$

$$m\ddot{\vec{r}} + a\vec{k}\sin(\bar{k}\cdot\bar{r}) = 0$$

17. $q = x - \frac{p}{m} t \qquad q(t=0) = q_0 = x_0$

$\mathcal{H} = \frac{p^2}{2m}$

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + \{q, H\} = -\frac{p}{m} + \{q, H\}$$

$$\{q, H\} = \frac{\partial q}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial q}{\partial p}\cancel{\frac{\partial H}{\partial x}}^{0} = \frac{p}{m}$$

$$\frac{dq}{dt} = 0$$

q сохраняется и имеет физ. смысл нач. координаты

$\downarrow$

физ. инв.

18. $m, \quad \vec{L} = (\vec{r} \times \vec{p})$

$$\{x^i, p_j\} = \frac{\partial x^i}{\partial x^k}\frac{\partial p_j}{\partial p_k} - \cancel{\frac{\partial x^i}{\partial p_k}}^{0}\frac{\partial p_j}{\partial x^k} = \delta_{ik}\delta_{jk} = \boxed{\delta_{ij}}$$

$\{p_i, l^j\}$

$\epsilon_{123} = 1$ fix

$$l^j = \cancel{\epsilon_{ijk} x^i p_k} \quad \epsilon_{jkl} x_k p_l$$

$$\{p_i, l^j\} = \cancel{\frac{\partial p_i}{\partial x^n}}^{0}\frac{\partial l^j}{\partial p^n} - \frac{\partial p_i}{\partial p^n}\frac{\partial l^j}{\partial x^n} =$$

$$= -\epsilon_{jkl}\delta_{in}\frac{\partial}{\partial x^m} x_k p_l = -\epsilon_{jkl}\delta_{in}\delta_{km} p_l =$$

$$= -\epsilon_{jml}\delta_{in} p_l = -\cancel{\delta_{in}\frac{l^j}{x^m}} - \epsilon_{jil} p_l = \boxed{\epsilon_{ijl} p_l}$$

$$\{l^i, l^j\} = \frac{\partial l^i}{\partial x^n}\frac{\partial l^j}{\partial p^n} - \frac{\partial l^j}{\partial x^n}\frac{\partial l^i}{\partial p^n} =$$

$$= \varepsilon_{ikl}\,\varepsilon_{jab}\left[\cancel{\delta_{in}\delta_{jn}}\ \frac{\partial x_k}{\partial x^n} p_l \frac{\partial p_b}{\partial p^n} x_a - \frac{\partial x_a}{\partial x_n} p_b \frac{\partial p_l}{\partial p_n} x_k\right]$$

$$= \varepsilon_{ikl}\,\varepsilon_{jab}\left[\delta_{kn}\delta_{bn} x_a p_l - \delta_{an}\delta_{ln} x_k p_b\right] =$$

$$= \varepsilon_{inl}\,\varepsilon_{jan}\, x_a p_l - \varepsilon_{ikn}\,\varepsilon_{jnb}\, x_k p_b \ \ominus$$

$$\varepsilon_{ijk}\,\varepsilon_{abk} = \begin{vmatrix} \delta_{ia} & \delta_{ib} \\ \delta_{ja} & \delta_{jb} \end{vmatrix}$$

$$\ominus\ \varepsilon_{ikn}\,\varepsilon_{jbn}\, x_k p_b - \varepsilon_{iln}\,\varepsilon_{jan}\, x_a p_l =$$

$$= \begin{vmatrix} \delta_{ij} & \delta_{ib} \\ \delta_{kj} & \delta_{kb} \end{vmatrix} x_k p_b - \begin{vmatrix} \delta_{ij} & \delta_{ia} \\ \delta_{lj} & \delta_{la} \end{vmatrix} x_a p_l =$$

$$= \cancel{\delta_{ij} x_k p_k} - x_j p_i - \cancel{\delta_{ij} x_a p_a} + x_i p_j =$$

$$= -x_j p_i + x_i p_j = \boxed{\varepsilon_{ijk}\, l_k}$$

$$\{l^i, \bar{l}^2\} = \frac{\partial l^i}{\partial x^n}\frac{\partial \bar{l}^2}{\partial p^n} - \frac{\partial l^i}{\partial p^n}\frac{\partial l^2}{\partial x^n} =$$

$$= \varepsilon_{ikl}\, p_l \frac{\partial x^k}{\partial x^n}(x^n)^2(2p^n) - \varepsilon_{ikl}\, x_k \frac{\partial p_l}{\partial p_n} p_n^2\, 2x_n =$$

$$= 2\,\varepsilon_{ikl}\,\delta_{kn}\, p_l\, p_n\, x_n^2 - 2\,\varepsilon_{ikl}\,\delta_{ln}\, x_n x_k\, p_n^2 =$$

$$= 2\varepsilon_{ikl} x_k p_l (x_k p_k - x_l p_l) \overset{?}{=} 0$$

19. $\quad$ gue m, $U(r) = -\frac{\alpha}{r}$

$$A_i = \frac{x_i}{r} - \frac{1}{m\alpha}\varepsilon_{iab} p_a l_b$$

$$\bar{A} = \frac{\bar{r}}{r} - \frac{\bar{p}\times\bar{L}}{m\alpha}$$

$$l_j = \varepsilon_{jkl} x_k p_l$$

$$A_i = \frac{x_i}{r} - \frac{1}{m\alpha}\varepsilon_{iab}\varepsilon_{bcd} p_a x_c p_d$$

$$\{A_i, l_j\} = \frac{\partial A_i}{\partial x^n}\cdot\frac{\partial l_j}{\partial p^n} - \frac{\partial A_i}{\partial p^n}\cdot\frac{\partial l_j}{\partial x^n} =$$

$$= \varepsilon_{jkl} x_k \delta_{ln}\left(\frac{x_i}{r}\left(\frac{\delta_{ni}}{x_i} - \frac{x_n}{r^2}\right) - \frac{\varepsilon_{iab}\varepsilon_{bcd}}{m\alpha} p_a p_d \delta_{cn}\right) +$$

$$+ \varepsilon_{jkl} p_l \delta_{kn}\left(\frac{\varepsilon_{iab}\varepsilon_{bcd}\, x_c}{m\alpha}\left(\delta_{an} p_d + \delta_{dn} p_a\right)\right) =$$

$$= \varepsilon_{jkl}\delta_{ln} x_k \frac{\delta_{ni}}{r} - \varepsilon_{jkl} x_k \delta_{ln}\frac{x_i x_n}{r^3} -$$

$$- \frac{\varepsilon_{jkl}\delta_{ln} x_k \varepsilon_{iab}\varepsilon_{bcd} p_a p_d \delta_{cn}}{m\alpha} +$$

$$+ \frac{1}{m\alpha}\Big(\varepsilon_{jkl} p_l \delta_{kn}\varepsilon_{iab}\varepsilon_{bcd} x_c \delta_{an} p_d +$$

$$+ \varepsilon_{jkl} p_l \delta_{kn}\varepsilon_{iab}\varepsilon_{bcd} x_c \delta_{dn} p_a\Big) =$$

$$= \varepsilon_{jkl}\delta_{li}\frac{x_k}{r} - \varepsilon_{jkl}\frac{x_k x_i x_l}{r^3} +$$

$$+ \frac{1}{m\alpha}\Big(-\varepsilon_{jkl}\varepsilon_{iab}\varepsilon_{bcd}\delta_{cl} x_k p_a p_d + \varepsilon_{jkl}\varepsilon_{iab}\varepsilon_{bcd}\delta_{ak} x_c p_l p_d +$$

$$+ \varepsilon_{jkl}\varepsilon_{iab}\varepsilon_{bcd}\,\delta_{kd}\,x_c\,p_a\,p_l\Big) =$$

$$= \varepsilon_{jki}\frac{x_k}{r} - \cancel{\varepsilon_{jkl}\frac{x_k x_i x_l}{r^3}}^{\,0} +$$

$$+ \frac{1}{m\alpha}\Big(-\varepsilon_{jkl}\varepsilon_{iab}\varepsilon_{bld}\,x_k\,p_a p_d + \varepsilon_{jkl}\varepsilon_{ikb}\varepsilon_{bcd}\,x_c\,p_l p_d +$$

$$+ \varepsilon_{jkl}\varepsilon_{iab}\varepsilon_{bck}\,x_c\,p_a p_l\Big) =$$

$$= \varepsilon_{ijk}\frac{x_k}{r} - \frac{1}{m\alpha}\Big(\;\dots\;\Big)$$

$$\varepsilon_{iab}\varepsilon_{bld} = \varepsilon_{iab}\varepsilon_{ldb} = \begin{vmatrix} \delta_{il} & \delta_{id} \\ \delta_{al} & \delta_{ad} \end{vmatrix} = \delta_{il}\delta_{ad} - \delta_{id}\delta_{al}$$

$$\varepsilon_{ikb}\varepsilon_{bcd} = \varepsilon_{ikb}\varepsilon_{cdb} = \begin{vmatrix} \delta_{ic} & \delta_{id} \\ \delta_{kc} & \delta_{kd} \end{vmatrix} = \delta_{ic}\delta_{kd} - \delta_{id}\delta_{kc}$$

$$\varepsilon_{iab}\varepsilon_{bck} = \varepsilon_{iab}\varepsilon_{ckb} = \begin{vmatrix} \delta_{ic} & \delta_{ik} \\ \delta_{ac} & \delta_{ak} \end{vmatrix} = \delta_{ic}\delta_{ak} - \delta_{ik}\delta_{ac}$$

$$(\dots) = \varepsilon_{jkl}\Big[-\delta_{il}\delta_{ad}\,x_k\,p_a p_d + \delta_{id}\delta_{al}\,x_k p_a p_d +$$

$$+ \delta_{ic}\delta_{kd}\,x_c\,p_l p_d - \delta_{id}\delta_{kc}\,x_c\,p_l p_d + \delta_{ic}\delta_{ak}\,x_c\,p_a p_l -$$

$$- \delta_{ik}\delta_{ac}\,x_c\,p_a p_l\Big] = \varepsilon_{jkl}\Big[-\delta_{il}\,x_k\,p_a^2 + \cancel{x_k p_l p_i} +$$

$$+ x_i p_l p_k - \cancel{x_k p_l p_i} + x_i p_k p_l - \delta_{ik} x_a p_a p_l\Big]$$

$$= -\varepsilon_{jki}\,x_k\,p_a^2 + \cancel{2x_i\varepsilon_{jkl}p_k p_l}^{\,0} - \varepsilon_{jil}\,x_a p_a p_l =$$

$$\varepsilon_{ijl} \, x_a p_a p_l - \varepsilon_{ijk} \, x_k p_a^2 =$$

$$\varepsilon_{ijk} (x_a p_a p_k - x_k p_a^2) \xleftarrow{\varepsilon} p_a \quad (1)$$

$$(p \times l)_k = \varepsilon_{kij} \, p_i l_j = \varepsilon_{kij} \varepsilon_{jnm} \, p_i x_n p_m =$$

$$= \delta_{kn} \delta_{im} \, p_i x_n p_m - \delta_{km} \delta_{in} \, p_i x_n p_m =$$

$$= p_m x_k p_m - p_n x_n p_k \quad (2)$$

$$(1) = (2)$$

Тогда $\{A_i, l_j\} = \varepsilon_{ijk} \left(\frac{x_k}{r} - \frac{(\bar{p} \times \bar{l})_k}{m\alpha} \right) = \varepsilon_{ijk} A^k$

$A \neq A(t)$

$\{A, H\} = 0 \Rightarrow A$ – инт. дв.

$$\frac{\partial A^k}{\partial x^n} \frac{\partial H}{\partial p^n} - \frac{\partial A^k}{\partial p^n} \frac{\partial H}{\partial x^n} = \cancel{\left(\frac{1}{r} - \frac{\varepsilon_{kij} \, p_i \varepsilon_{jab} \, p_b \, \delta_{an}}{m\alpha} \right) \frac{\partial H}{\partial p^n}}$$

f . . .

$$\mathcal{H} = p \dot{x} - L \big|_{\dot{x} \to p}$$

$$L = \frac{m \dot{x}^2}{2} - U(r)$$

$$\frac{\partial L}{\partial \dot{x}} = p = m \dot{x} \Rightarrow \dot{x} = \frac{p}{m}$$

$$\mathcal{H} = \frac{p^2}{m} - \frac{\cancel{m} p^2}{2m} + U(r) = \frac{p^2}{2m} + U(r)$$

$$\frac{\partial H}{\partial p^n} = \frac{p^n}{m} \quad , \quad \frac{\partial H}{\partial x_n} = U'(r)\,\frac{x_n}{r}$$

$$\frac{\partial A^k}{\partial x^n} = \frac{x^k}{r}\left(\frac{\delta_{kn}}{x_k} - \frac{x_n}{r^2}\right) - \frac{1}{m\alpha}\,\varepsilon_{kab}\varepsilon_{bcd}\,\delta_{cn}\,p_a\,p_d$$

$$\frac{\partial A^k}{\partial p_n} = -\frac{1}{m\alpha}\,\varepsilon_{kab}\varepsilon_{bcd}\,x_c\left(\delta_{an}\,p_d + \delta_{dn}\,p_a\right)$$

$$\{A, H\} = \cancel{\frac{p_n x_k}{x_k r}}\ \frac{\delta_{kn}\,p_n}{m r} - \frac{p_n\,x_k\,x_n}{m r^3} - \frac{p_n}{\alpha m^2}\,\varepsilon_{kab}\varepsilon_{bcd}\,\delta_{cn}\,p_a\,p_d$$

$$+ \ U'(r)\,\frac{x_n}{m\alpha r}\,\varepsilon_{kab}\varepsilon_{bcd}\,x_c\left(\delta_{an}\,p_d + \delta_{dn}\,p_a\right) =$$

$$= \frac{p_k}{m r} - \frac{p_n x_k x_n}{m r^3} - \frac{p_n}{\alpha m^2}\,\varepsilon_{kab}\varepsilon_{bnd}\,p_a p_d \ +$$

$$+ \ U'(r)\,\frac{x_n}{m\alpha r}\left[\varepsilon_{knb}\,\varepsilon_{bcd}\,x_c\,p_d + \varepsilon_{kab}\,\varepsilon_{bcn}\,x_c\,p_a\right]$$

$$\varepsilon_{kab}\,\varepsilon_{bnd} = \varepsilon_{kab}\,\varepsilon_{ndb} = \begin{vmatrix} \delta_{kn} & \delta_{kd} \\ \delta_{an} & \delta_{ad} \end{vmatrix} = \delta_{kn}\delta_{ad} - \delta_{kd}\delta_{an}$$

$$\varepsilon_{knb}\,\varepsilon_{bcd} = \varepsilon_{knb}\,\varepsilon_{cdb} = \delta_{kc}\,\delta_{nd} - \delta_{kd}\,\delta_{nc}$$

$$\varepsilon_{kab}\,\varepsilon_{bcn} = \varepsilon_{kab}\,\varepsilon_{cnb} = \delta_{kc}\,\delta_{an} - \delta_{kn}\,\delta_{ac}$$

$$\{A, H\} = \frac{p_k}{m r} - \frac{p_n x_k x_n}{m r^3} - \frac{p_n}{\alpha m^2}\left[\delta_{kn}\,\delta_{ad}\,p_a\,p_d - \delta_{kd}\,\delta_{an}\,p_a\,p_d\right]$$

$$+ \ U'(r)\,\frac{x_n}{m\alpha r}\Big[\delta_{kc}\,\delta_{an}\,x_c\,p_d - \delta_{kd}\,\delta_{nc}\,x_c\,p_d + \delta_{kc}\,\delta_{an}\,x_c\,p_a$$

$$- \ \delta_{kn}\,\delta_{ac}\,x_c\,p_a\Big] =$$

$$= \frac{p_k}{mr} - \frac{p_n x_n x_k}{mr^3} - \frac{1}{\alpha m^2}\left(p_k p_a^2 - p_k p_n^2\right) +$$

$$+ U'(r)\,\frac{x_n}{m\alpha r}\left(x_k p_d \delta_{an} - x_n p_k + x_k p_n - x_a p_a \delta_{kn}\right) =$$

$$\overset{?}{=} \frac{p_k}{mr} - \frac{p_n x_n x_k}{mr^3} + \frac{U'(r)}{m\alpha r}\left(x_a x_k p_d - x_n^2 p_k + \right.$$

$$\left. + x_k x_n p_n - x_k x_a p_a\right) =$$

$$\overset{?}{=} \frac{1}{mr^3}\left(p_k r^2 - p_n x_n x_k\right) = \frac{1}{mr^3}\left(p_k x_n^2 - p_n x_n x_k\right) \overset{?}{=} 0$$

20. $\alpha > 0$

$$I_{2n}(\alpha) = \int_{\mathbb{R}} dx\, x^{2n} e^{-\alpha x^2}, \quad n \in \mathbb{N} \cup \{0\}$$

$$+\alpha x^2 = t$$

$$2\alpha x\, dx = dt$$

$$\int_0^\infty \frac{dt}{2\alpha\sqrt{\frac{t}{\alpha}}} \left(\frac{t}{\alpha}\right)^n e^{-t} = \frac{1}{2\alpha^{n+\frac{1}{2}}} \int_0^\infty t^{n-\frac{1}{2}} e^{-t}\, dt =$$

$$= \frac{\Gamma\left(n+\frac{1}{2}\right)}{2\alpha^{n+1/2}} = \frac{\left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)}{2\alpha^{n+1/2}} = \dots =$$

$$= \frac{\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdot\ldots\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}{2\alpha^{n+1/2}} = \frac{\left(\frac{1}{2}\right)_n \sqrt{\pi}}{2\alpha^{n+1/2}} =$$

$$= \frac{(2n-1)!!\sqrt{\pi}}{2^{n+1}\alpha^{n+\frac{1}{2}}}$$

21. $$I(\omega) \overset{f(t)\equiv 1}{=} \int_{\mathbb{R}} dt\, e^{i\omega t} = \lim_{\alpha\to+0} \int_{\mathbb{R}} dt\, e^{i\omega t - \alpha t^2} =$$

$$= \lim_{\alpha\to+0} \int_{\mathbb{R}} dt\, e^{-\alpha\left(t^2 - \frac{i\omega}{\alpha}t + \left(\frac{i\omega}{2\alpha}\right)^2\right)} e^{-\frac{\omega^2}{4\alpha}} =$$

$$= \lim_{\alpha\to+0} \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\omega^2}{4\alpha}} = \begin{cases} 0, & \omega \neq 0 \\ +\infty, & \omega = 0 \end{cases} = 2\pi\delta(\omega)$$

$$A = \int_{\mathbb{R}} d\omega\, J(\omega) f(\omega) = \lim_{\alpha \to +0} \int_{\mathbb{R}} d\omega \cdot \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\omega^2}{4\alpha}} f(\omega) =$$

$$= \lim_{\varepsilon \to +0} \int_{-\varepsilon}^{+\varepsilon} d\omega \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\omega^2}{4\alpha}} f(\omega)$$

$\exists$ f непр. на ~~$U_\varepsilon(0)$~~ $[-\varepsilon, \varepsilon]$. Тогда $m \leq f(\omega) \leq M$ $\forall\, \omega \in [-\varepsilon, \varepsilon]$

$$2\pi m = \lim_{\varepsilon \to +0} m \int_{-\varepsilon}^{\varepsilon} d\omega \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\omega^2}{4\alpha}} \leq A \leq \lim_{\varepsilon \to +0} M \int_{-\varepsilon}^{\varepsilon} d\omega \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\omega^2}{4\alpha}} = 2\pi M$$

в силу непр. $m = M = f(0)$

Тогда $A = 2\pi f(0)$

22. $\theta(x) = \begin{cases} 1, & x>0 \\ 0, & x<0 \end{cases}$

$$\theta(x) = \frac{1}{2\pi} \lim_{\alpha \to +0} \int_{-\infty}^{x} dy \int_{\mathbb{R}} dz\, e^{iyz - \alpha z^2} =$$

$$= \int_{-\infty}^{x} dy\, \delta(y) = \begin{cases} 1, & x>0 \\ 0, & x<0 \end{cases}$$

✓.

$$\frac{d}{dx}\theta(x) = \frac{d}{dx}\int_{-\infty}^{x} \delta(y)\, dy = \delta(x)$$

23*. $f(x) = x^2$

$$\theta(x) = \lim_{n\to\infty} \frac{1}{1+e^{-2nx}}$$

$$\mathcal{J}(w) = \int_{\mathbb{R}} x^2 e^{iwx}\, dx = -\frac{\partial^2}{\partial w^2} \int_{\mathbb{R}} e^{iwx}\, dx =$$

$$= \boxed{-2\pi\, \delta''(w)}$$

$$\delta'(w) = \lim_{h\to 0} \frac{\delta(w+h) - \delta(w)}{h} = \begin{cases} 0, & w \neq 0 \\ \lim_{h\to 0} \frac{-\delta(w)}{h}, & w = 0 \end{cases}$$

$$\delta'(w) = \frac{-\delta(w)}{w}$$

$$\delta''(w) = -\frac{-\frac{\delta(w)}{w} w - \delta(w)}{w^2} = \frac{2\,\delta(w)}{w^2}$$

$$\mathcal{J}(w) = -\frac{4\pi\,\delta(w)}{w^2}$$

$$\mathcal{J}_1(w) = \int_{\mathbb{R}} \frac{e^{iwx}}{1+e^{-2nx}}\, dx = \int_{\mathbb{R}_-} e^{iwx} \frac{e^{2nx}}{1+e^{2nx}}\, dx +$$

$$+ \int_{\mathbb{R}_+} \frac{e^{iwx}}{1+e^{-2nx}}\, dx = \cancel{\int_{\mathbb{R}} e^{iwx}\, dx \sum_{k=0}^{\infty} (-1)^k \left(e^{2nx(k+1)} + e^{-2nxk} \right)}$$

$$= \sum_{k=0}^{\infty} (-1)^k \int_{\mathbb{R}_+} e^{ix(w+2nk)}\, dx =$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{1}{2} \cdot 2\pi\, \delta(w + 2ink) = \boxed{\pi\,\delta(w)}$$

24. $\int_{\mathbb{R}} \frac{d\omega}{2\pi} \tilde{f}(\omega)\tilde{g}(\omega) e^{-i\omega x} = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{d\omega\, dz\, dy}{2\pi} f(z) g(y) e^{i\omega(z+y-x)} =$

$= \iint_{\mathbb{R}^2} \frac{dy\, dz}{2\pi} f(z) g(y) \cdot 2\pi\, \delta(z+y-x) =$

$= \iint_{\mathbb{R}^2} dy\, dz\, f(z) g(y)\, \delta(z+y-x) = \left[\begin{array}{l} x-y=t \\ -dy = dt \end{array} \right] =$

$= \iint_{\mathbb{R}^2} (-dt)\, dz\, f(z)\, g(x-t)\, \delta(z-t) =$

$\cancel{\int_{\mathbb{R}} dt\, f(z) \int_{\mathbb{R}}} \int_{\mathbb{R}} f(z)\,\delta(z-t)\, dz \int_{\mathbb{R}} g(x-t)\, dt =$

$= \int_{\mathbb{R}} f(t)\, g(x-t)\, dt = \left[\begin{array}{l} x-t = z \\ -dt = dz \end{array} \right]$

$= \int_{-\mathbb{R}} f(x-z)\, g(z)\, (-dz) = [\, z = t \,]$

$= \int_{\mathbb{R}} g(t)\, f(x-t)\, dt$

$z+g$!

25. $\delta[f(x)] = \lim\limits_{\alpha\to+0} \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{f^2(x)}{4\alpha}}$

$\exists$ x_0 - корень $f(x)=0$

$$f(x) = f'(x_0)(x-x_0) + o(x-x_0)$$

$$\cancel{\delta[f(x)]} = \lim_{\alpha\to+0} \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{f^2(x)}{4\alpha}} =$$

$$= \lim_{\alpha\to+0} \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{(f'(x_0))^2 (x-x_0)^2}{4\alpha}} \qquad \underline{\alpha/(f'(x_0))^2 = \beta}$$

$$= \lim_{\beta\to+0} \sqrt{\frac{\pi}{\beta}} \cdot \frac{1}{|f'(x_0)|}\, e^{-\frac{(x-x_0)^2}{4\beta}} = \frac{\delta(x-x_0)}{|f'(x_0)|}$$

$$\delta[f(x)] = \sum_{x_0} \frac{\delta(x-x_0)}{|f'(x_0)|}$$

26. $\int\limits_{\mathbb{R}} dx\, \delta^{[n]}(x-x_0) f(x) =$

$$= \sum_{k=0}^{n-1} (-1)^k f^{(k)}(x)\, \delta^{[n-k-1]}(x-x_0) \Big|_{-\infty}^{\infty}$$

$$+ (-1)^n \int\limits_{\mathbb{R}} \frac{\partial^n f}{\partial x^n} \delta(x-x_0)\, dx =$$

$$= (-1)^n \frac{\partial^n f}{\partial x^n}(x_0)$$

$$\left[\begin{array}{lll} & \mathcal{D} & \mathrm{I} \\ + & f(x) & \delta^{[n]}(x-x_0) \\ - & f^{(1)}(x) & \delta^{[n-1]}(x-x_0) \\ + & f^{(2)}(x) & \delta^{[n-2]}(x-x_0) \\ & \vdots & \vdots \\ (-1)^n & f^{(n)}(x) & \delta(x-x_0) \end{array}\right]$$

27. $$\int_{\mathbb{R}} dx\ |f(x)|^2 \overset{?}{=} \int_{\mathbb{R}} \frac{dw}{2\pi} |\tilde{f}(w)|^2$$

$$\int_{\mathbb{R}} \frac{dw}{2\pi} |\tilde{f}(w)|^2 = \int_{\mathbb{R}} \frac{dw}{2\pi} \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)\, f(y)\, e^{iw(x+y)}\, dx\, dy =$$

$$= \iint_{\mathbb{R}^2} f(x)\, f(y)\, \delta(-x+y)\, dx\, dy = \int_{\mathbb{R}} f(x) f(+x)\, dx$$

$$= \int_{\mathbb{R}} |f(x)|^2\, dx$$

270g!