№1.

$$\partial_t \begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = r$$

$$r = u e^{\lambda t}$$

$$\lambda u = J u$$

$$J = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda +$$

$$+ a_{11} a_{22} - a_{12} a_{21} = 0$$

$$\lambda^2 - \lambda \operatorname{tr} J + \det J = 0$$

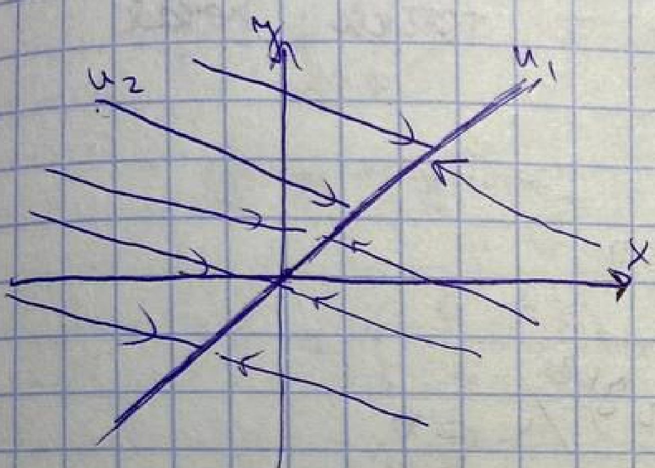
1) $|J| = 0 \quad \operatorname{tr} J < 0$

$$\lambda(\lambda - \operatorname{tr} J) = 0$$

$$\left[\begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = \operatorname{tr} J < 0 \end{array}\right.$$

$$r = c_1 u_1 + c_2 u_2 e^{\lambda_2 t}$$

$$\frac{\dot y}{\dot x} = \frac{+ c_2 u_{22} \lambda_2 e^{\lambda_2 t}}{c_2 u_{21} \lambda_2 e^{\lambda_2 t}} = \frac{u_{22}}{u_{21}}$$



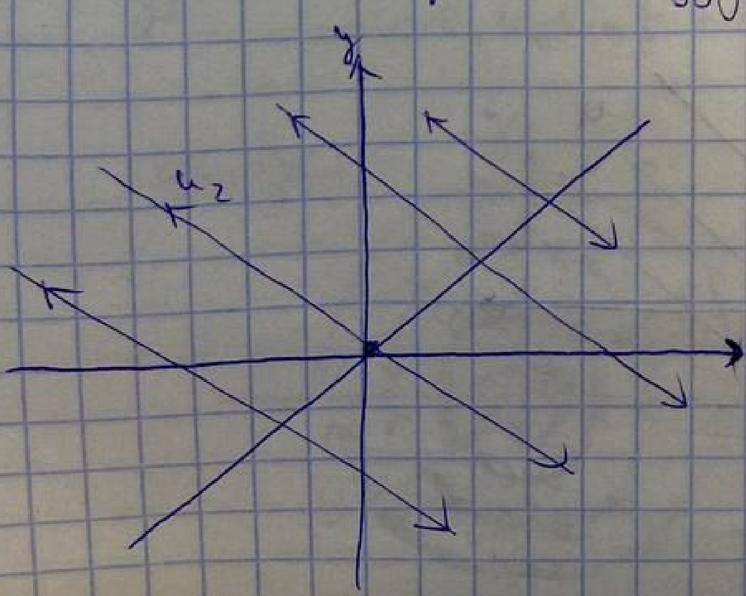
$$\begin{pmatrix} a_{11}-\lambda_2 & a_{12} \\ a_{21} & a_{22}-\lambda_2 \end{pmatrix} \begin{pmatrix} u_{21} \\ u_{22} \end{pmatrix} = 0$$

$$\lambda_2 = \mathrm{tr}\, J = a_{11} + a_{22}$$

$$\frac{u_{22}}{u_{21}} = \frac{a_{11}-\lambda_2}{a_{12}} = -\frac{a_{22}}{a_{12}}$$

2) $|J| = 0 \quad \mathrm{tr}\, J > 0$

Аналогично (1), но по Ляпунову положения равн. будут неуст.



3) $|J| = 0 \quad tr\, J = 0$

$\lambda^2 = 0 \quad , \lambda = 0 ,$

$\Rightarrow$ Все точки пл-ти – точки покоя

4) $|J| = (tr\, J)^2/4 \neq 0$

$$\lambda^2 - \lambda\, tr\, J + (tr\, J)^2/4 = 0$$

$$\lambda = \frac{tr\, J}{2}$$

$$r = c_1 u e^{\lambda t} + c_2 u t e^{\lambda t}$$

$$\frac{\dot{y}}{\dot{x}} = \frac{u_{12} e^{\lambda t}}{u_{11} e^{\lambda t}} \cdot \frac{c_1 \lambda + c_2 \lambda t + c_2 t}{c_1 \lambda + c_2 \lambda t + c_2 t} = \frac{u_{12}}{u_{11}}$$

№2.

1) $\begin{cases} \dot{x} = x^2 - y \\ \dot{y} = x - 1 \end{cases}$ $\qquad \begin{pmatrix} x \\ y \end{pmatrix} = r$

$\cancel{\dot{r} = \begin{pmatrix} 0 & -1 \\ 1 & \end{pmatrix}}$ $\qquad \cancel{x - 1 = z}$

$\cancel{\dot{z} = 2z - y}$

$\dot{r} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} r + o(r)$

$\mathcal{Y}$

$\operatorname{tr} \mathcal{Y} = 0 \quad , \quad |\mathcal{Y}| = 1$

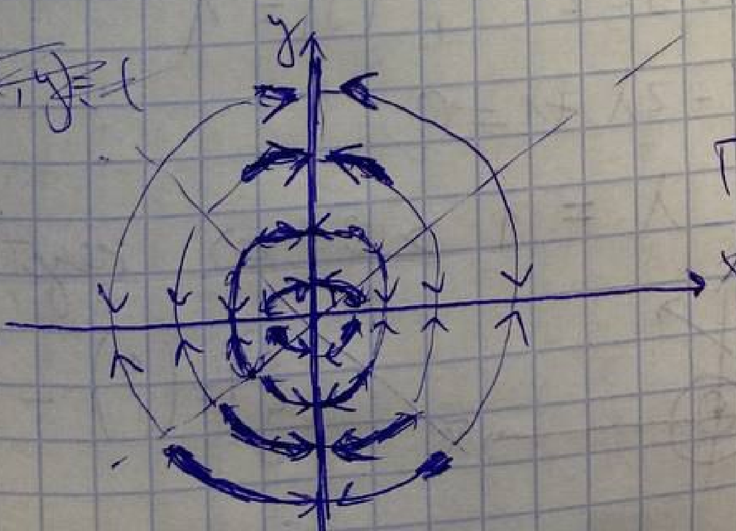
$\lambda^2 + 1 = 0 \quad , \quad \lambda = \pm i$

$x = c_1 u_{11} \cos t + c_2 u_{21} \sin t =$

$y = c_1 u_{12} \cos t + c_2 u_{22} \sin t$

$\frac{\dot{y}}{\dot{x}} = \frac{-c_1 u_{12} \sin t + c_2 u_{22} \cos t}{-c_1 u_{11} \sin t + c_2 u_{21} \cos t} = \frac{+x}{-y}$

~~(x, y) = (~~



Тип: центр.

ПР: $\left[\begin{array}{l} x = 0 \\ y = 0 \end{array}\right.$

№2.

1) $\begin{cases} \dot{x} = x^2 - y \\ \dot{y} = x - 1 \end{cases}$

$\cancel{y = -y + z}$ $\begin{cases} x - 1 = \tilde{x} \\ y = \tilde{y} + 1 \end{cases}$

$\begin{cases} \dot{\tilde{x}} = \tilde{x}^2 + 2\tilde{x} + \cancel{1} - \tilde{y} - \cancel{1} \\ \dot{\tilde{y}} = \tilde{x} \end{cases}$

$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = r$

$\dot{r} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} r + o(r)$

J

$\operatorname{tr} J = 2$, $|J| = 1$

$\lambda^2 - 2\lambda + 1 = 0$

$\lambda = 1 > 0 \Rightarrow$ неуст. фокус

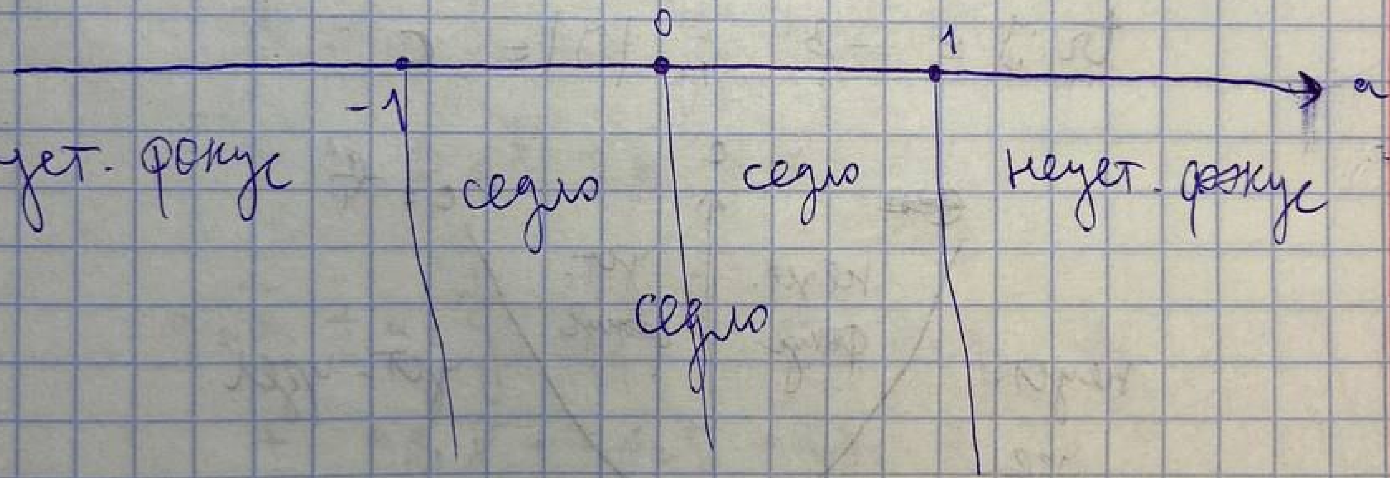
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ – неуст. П.Р.

2) $\begin{cases} \dot{x} = ax + y \\ \dot{y} = x + ay \end{cases}$

$$\dot{r} = \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix} r$$

$$\operatorname{tr} J = 2a, \quad |J| = a^2 - 1$$

$$r^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



$a = 1$ (см. задачу 1.2)

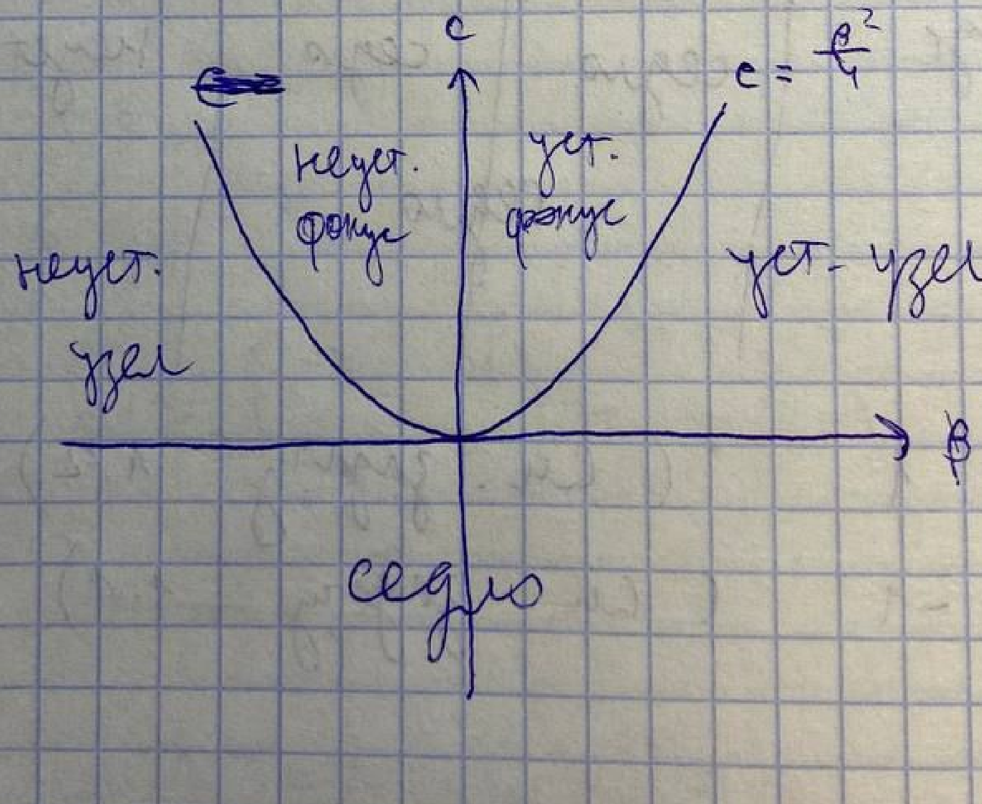
$a = -1$ (см. задачу 1.1)

3. $\ddot{x} + \beta\dot{x} + cx = 0$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\beta y - cx \end{cases}$$

$$\dot{r} = \begin{pmatrix} 0 & 1 \\ -c & -\beta \end{pmatrix} r$$

$\operatorname{tr} J = -\beta \quad , \quad |J| = c$



4. $\begin{cases} \dot{x} = -x + ay + x^2 y \\ \dot{y} = b - ay - x^2 y \end{cases}$, $a, b > 0$

$$\cancel{b - ay = \tilde{y}}$$

$$\cancel{x = \tilde{x} + b}$$

$$\dot{\tilde{x}} = \cancel{-\tilde{x} - b + b - \tilde{y} + o(\tilde{x}^2)}$$

$$x = \alpha_1 \tilde{x} + \alpha_2 \tilde{y} + \alpha_3$$

$$y = \beta_1 \tilde{x} + \beta_2 \tilde{y} + \beta_3$$

$$\alpha_1 \dot{\tilde{x}} + \alpha_2 \dot{\tilde{y}} = -\alpha_1 \tilde{x} - \alpha_2 \tilde{y} - \underline{\alpha_3}$$

$$+ a\beta_1 \tilde{x} + a\beta_2 \tilde{y} + \underline{a\beta_3} +$$

$$+ (\alpha_1 \tilde{x} + \alpha_2 \tilde{y} + \underline{\alpha_3})^2 (\beta_1 \tilde{x} + \beta_2 \tilde{y} + \underline{\beta_3})$$

Хочу: $-\alpha_3 + a\beta_3 + \alpha_3^2 \beta_3 = 0$

А ещё: $b - a\beta_3 - \alpha_3^2 \beta_3 = 0$

$$\alpha_3 = b$$

$$\beta_3 (a + b^2) = b, \quad \beta_3 = \frac{b}{a + b^2}$$

$$\cancel{\tilde{x} = x + b}$$

$$x = \bar{x} + b$$

$$y = \bar{y} + \frac{b}{a+b^2}$$

$$\dot{\bar{x}} = -(\bar{x}+b) + a\left(\bar{y} + \frac{b}{a+b^2}\right) + (\bar{x}+b)^2\left(\bar{y} + \frac{b}{a+b^2}\right)$$

$$\dot{\bar{y}} = b - (a)\left(\bar{y} + \frac{b}{a+b^2}\right) - (\bar{x}+b)^2\left(\bar{y} + \frac{b}{a+b^2}\right)$$

$$\dot{r} = J r + o(r)$$

$$J: \quad a_{11} = -1 + \frac{2b^2}{a+b^2} = \frac{b^2-a}{a+b^2}$$

$$a_{12} = a + b^2$$

$$a_{21} = -\frac{2b^2}{a+b^2}$$

$$a_{22} = -\frac{b^2-a}{a+b^2}$$

$$\operatorname{tr} J = a_{11} + a_{22} = 0$$

$$|J| = -\frac{(b^2-a)^2}{(a+b^2)^2} + 2b^2$$

$$|J| = 0 \Rightarrow \sqrt{2}\, b = \pm \frac{b^2-a}{a+b^2}$$

$$\pm\sqrt{2}\, b a \pm \sqrt{2}\, b^3 = b^2 - a$$

$$a(1 \pm b\sqrt{2}) = b^2(1 \mp b\sqrt{2})$$

1) $\cancel{a(1 - b\sqrt{2}}$

$$a = \frac{b^2(1 \mp b\sqrt{2})}{1 \pm b\sqrt{2}} > 0$$

$$1 - 2b^2 > 0$$

$$|b| < \frac{1}{\sqrt{2}}$$

$$\cancel{a_b =} \qquad \partial_b a = \frac{b^2(1 \mp b\sqrt{2})}{1 \pm b\sqrt{2}}\left(\frac{\mp\sqrt{2}}{1 \mp b\sqrt{2}} + \frac{2}{b} + \frac{\mp\sqrt{2}}{1 \pm b\sqrt{2}}\right)$$

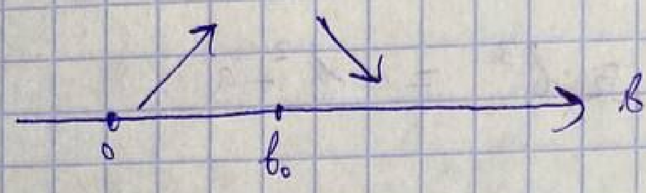
$$\partial_b a = a\left(\frac{2}{b} \mp \sqrt{2}\cdot\frac{2}{1-2b^2}\right) =$$

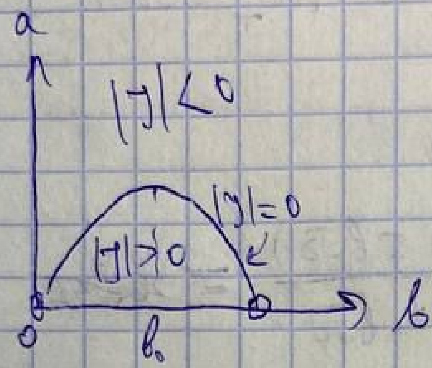
$$= \frac{a}{b(1-2b^2)}\left(2 - 4b^2 \mp 2\sqrt{2}\, b\right) =$$

$$= \frac{2a(2b^2 \pm \sqrt{2}\, b - 1)}{b(2b^2-1)} = \frac{-4b\left(b - \frac{1}{2}(\pm\sqrt{2} + \sqrt{10})\right)\left(b - \frac{1}{2}(\pm\sqrt{2} - \sqrt{10})\right)}{(1 \pm b\sqrt{2})^2}$$

$$b > 0 \Rightarrow \frac{-4b\left(b - \frac{1}{2}(\pm\sqrt{2} - \sqrt{10})\right)}{(1 \pm b\sqrt{2})^2} < 0$$

$$b_0 = \frac{1}{2}(\pm\sqrt{2} + \sqrt{10}) \; ; \quad \operatorname{sgn} \partial_b a = \operatorname{sgn}(b - b_0)$$





$$r^* = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ \dfrac{b}{a+b^2} \end{pmatrix} \ - \ \text{ПР}$$