Задание II.

I. Основы тензорной алгебры.

1. $l_c = \{x : \mathbb{N} \to \mathbb{R} \mid \exists N \in \mathbb{N} : \forall n \geq N : x(n) = 0\}$

Пусть $a, b \in l_c$

Тогда $a + b \in l_c$ (если взять $N = \max\{N_a, N_b\}$)

$\lambda \in \mathbb{R}$ $\lambda a \in l_c$

Единственность очевидна.

Так как мы имеем дело с числами, то аксиомы вект. пр-ва тоже очевидно выполнены.

Значит l_c – вещ. вект. пр-во.

Предположим, что $\dim l_c = m < \infty$

Тогда $\exists$ базис финит. посл-тей в l_c $\{e_1, \dots, e_m\}$

Положим $N = \max\{N_1, \dots, N_m\} + 1$

и возьмем такую посл-ть a, что $\forall n \leq N$ $a_n \neq 0$,

$\forall n > N$ $a_n = 0$.

Заметим, что a нельзя разложить по базису,

т.к. $\forall k \leq m$ $e_{k_N} = 0 \Rightarrow \lambda_i e_{i_N} = 0$,

при этом $a_N \neq 0$. Противоречие. Значит $\dim l_c = \infty$

Д-ть, что $e_k(n) = \delta_{kn}$ – базис Гамеля в l_c

Выберем $\forall a \in l_c$.

$$a_n = a_k \delta_{kn} = a_k e_k(n)$$

Единственность очевидна в силу невырожденности матрицы δ_{kn}.

2. Δ $Mat_n(\mathbb{R})$

$f: Mat_n(\mathbb{R}) \to \mathbb{R}$

$$f(A) = trA = A_{\alpha\alpha}$$

f – лин. ф-л!

$$f(A+B) = (A+B)_{\alpha\alpha} = A_{\alpha\alpha} + B_{\alpha\alpha} = f(A) + f(B)$$

$\lambda \in \mathbb{R}$

$$f(\lambda A) = (\lambda A)_{\alpha\alpha} = \lambda A_{\alpha\alpha} = \lambda f(A)$$

$\Rightarrow f$ – лин. ф-л на $Mat_n(\mathbb{R})$

3. Д-ть: $\forall f \in V^*$ на $Mat_n(\mathbb{R})$:

$$f(A) = tr(\sigma A) \qquad \sigma \in Mat_n(\mathbb{R}),\ \sigma = \sigma_f$$

$$f(A) = f(A_{ij} E_{ij}) = A_{ij} f(E_{ij})$$

$$\sigma_{ij} = f(E_{ij})$$

$$(\sigma A)_{\alpha\beta} = \sigma_{\alpha\gamma} A_{\gamma\beta}$$

$$(\sigma A)_{\alpha\alpha} = \sigma_{\alpha\gamma} A_{\gamma\alpha} = A_{ij} f(E_{ij}) = f(A)$$

$$f(A) = tr(\sigma A)$$

чтд!

4. $V = \langle v_1, v_2, v_3 \rangle = \mathbb{R}[x]_2 \qquad \langle h^1, h^2, h^3 \rangle = V^*$

$$h^\alpha(v_\beta) = \delta^\alpha_\beta$$

$$h^1(f) = f(0)$$

$$h^2(f) = f'(0)$$

$$h^3(f) = f(1)$$

$$h^1(v_1) = h^1(a_{1\alpha} x^\alpha) = a_{1\alpha} h^1(x^\alpha), \quad \alpha = \overline{0,2}$$

$$a_{1\alpha} h^1(x^\alpha) = a_{10} = 1$$

$$h^2(v_1) = a_{1\alpha} h^2(x^\alpha) = a_{11} = 0$$

$$h^3(v_1) = a_{1\alpha} h^3(x^\alpha) = a_{10} + a_{11} + a_{12} = 0$$

$$\Rightarrow a_{12} = -1$$

$$v_1 = 1 - x^2$$

Аналогично $v_2 = x - x^2$

$$v_3 = x^2$$

Нетрудно убедиться, что это действительно базис!

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

5. $f_1, f_2 \in V^*$ $\dim V = n < \infty$

$\operatorname{Ker} f_1 = \operatorname{Ker} f_2$

fix $u \in V \setminus \operatorname{Ker} f_1$

$\forall v \in V \hookrightarrow$ $v = w_1 + \frac{f_1(v)}{f_1(u)} u =$

$= w_2 + \frac{f_2(v)}{f_2(u)} u$, $w_1, w_2 \in \operatorname{Ker} f_1$

$w_1 - w_2 \in \operatorname{Ker} f_1$

$\overset{\shortparallel}{u \left[\frac{f_2(v)}{f_2(u)} - \frac{f_1(v)}{f_1(u)} \right]}$ $\Rightarrow \frac{f_1(v)}{f_1(u)} = \frac{f_2(v)}{f_2(u)}$

$f_2(v) = \frac{f_2(u)}{f_1(u)} f_1(v) = \lambda f_1(v)$

$\Rightarrow f_2 = \lambda f_1$

Дифгем

6. Д-ть! $\beta(A, B) = tr(AB)$ – невырожд. билин. форма на $Mat_n(\mathbb{R})$

$$\beta(A + \tilde{A}, B) = tr(AB + \tilde{A}B) = tr(AB) + tr(\tilde{A}B) =$$
$$= \beta(A, B) + \beta(\tilde{A}, B)$$

$$\beta(\lambda A, B) = tr(\lambda AB) = \lambda\, tr(AB) =$$
$$= \lambda \beta(A, B)$$

По второму аргументу аналогично.

$$\beta(A, B) = tr(A_{\alpha_1\alpha_2} E_{\alpha_1\alpha_2} B_{\beta_1\beta_2} E_{\beta_1\beta_2}) =$$
$$= A_{\alpha_1\alpha_2} B_{\beta_1\beta_2} tr(E_{\alpha_1\alpha_2} E_{\beta_1\beta_2}) =$$
$$= A_{\alpha_1\alpha_2} B_{\beta_1\beta_2} tr(E_{\alpha_1\beta_2} \delta_{\alpha_1\beta_1}) =$$
$$= A_{\alpha_1\alpha_2} B_{\alpha_1\beta_2} tr(E_{\alpha_1\beta_2}) = A_{\alpha_1\alpha_2} B_{\alpha_1\beta_2} \delta_{\alpha_1\beta_1} =$$
$$= A_{\beta_1\alpha_2} B_{\beta_1\beta_2}.$$

$$\beta(A,B) = A_{\alpha_1\alpha_2} B_{\beta_1\beta_2} \operatorname{tr}(E_{\beta_1\beta_2}) =$$

$$= A_{\alpha_1\alpha_2} B_{\beta_1\beta_2} \underbrace{\delta_{\beta_1\beta_2}}_{\text{базис формы } \beta}$$

$$\delta_{\beta_1\beta_2} = \left(\underline{11}\right)^{\beta_1}_{\beta_2} \to \text{невырожден.}$$

Значит $\beta(A,B) = \operatorname{tr}(AB)$ является невырожд. билин. формой на $Mat_n(\mathbb{R})$

ч.т.д.!

8. $\{e_1, e_2, e_3\}$ – базис в $V = \mathbb{R}^3$

$\{h^1, h^2, h^3\}$ – дв. базис в V^*

$$\tau = (3h^1 + h^2) \otimes (h^2 - 2h^3) \otimes h^2$$

$$\tau_{122} = (3h^1(e_1) + h^2(e_1)) \cdot (h^2(e_2) - 2h^3(e_2)) \cdot h^2(e_2)$$

$$= 3 \cdot 1 \cdot 1 = 3$$

$e_1' = e_1 - e_2$, $e_2' = e_2 + e_3$, $e_3' = e_3 - e_2$

$$t'_{122} = (3 - 1) \cdot (1 - 2) \cdot 1 = -2$$

11. $T^{\mu\nu}_{\alpha\beta} = \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}$ $\quad \dim V = n$

1) $\operatorname{tr} T = \cancel{\sum_{\alpha\leq n} \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta} =}$

$$= T^{\alpha\nu}_{\alpha\beta} = \delta^{\alpha}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\alpha}_{\beta} =$$

$$= n\,\delta^{\nu}_{\beta} - n\delta^{\nu}_{\beta} = 0$$

Аналогично по β и ν.

$\Rightarrow$ Только 1 полная свертка

12. $\tau_{\mu\nu} = \tau_{\nu\mu}$, $s^{\alpha\beta} = -s^{\beta\alpha}$

$$\tau_{\alpha\beta}\, s^{\alpha\beta} = A$$

$$A = -\tau_{\beta\alpha}\, s^{\beta\alpha} = -A \Rightarrow A = 0$$

13. $f \otimes v$, $v \in V$, $f \in V^*$

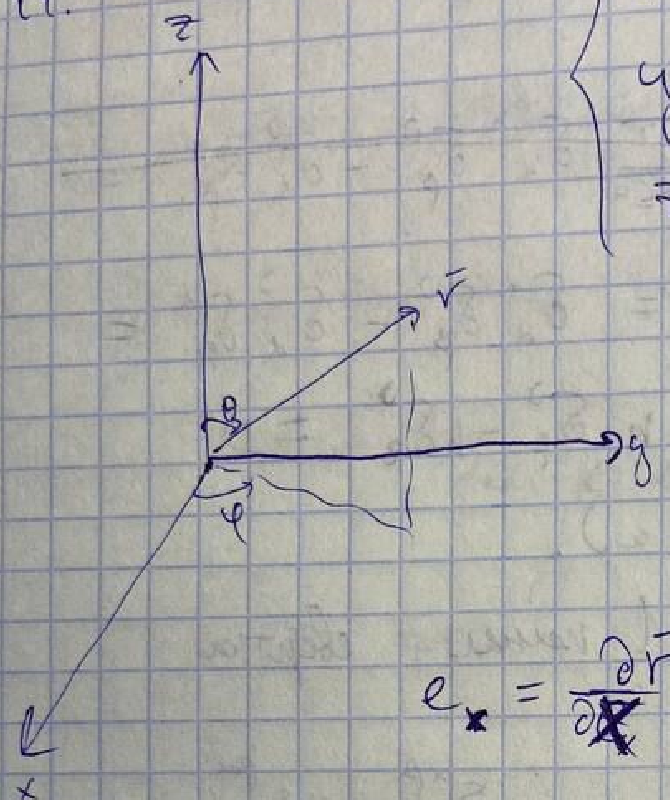
$$v = v^{\alpha} e_{\alpha}$$

$$f = f_{\beta} h^{\beta}$$

$$f \otimes v = f_{\beta} v^{\alpha}\, h^{\beta} \otimes e_{\alpha} = f_{\beta} v^{\alpha}\, \delta^{\beta}_{\alpha} =$$

$$= f_{\alpha} v^{\alpha} = f(v)$$

14.



$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$e_x = \frac{\partial \bar{r}}{\partial \cancel{x}} \left|\frac{\partial \bar{r}}{\partial \cancel{x}}\right|^{-1}$$

$$\cancel{\frac{\partial x}{\partial x}}$$

$$e_\varphi = \frac{\partial \bar{r}}{\partial \varphi} \left|\frac{\partial \bar{r}}{\partial \varphi}\right|^{-1}$$

$$\frac{\partial x}{\partial \varphi} = -r\sin\theta\sin\varphi$$

$$\frac{\partial y}{\partial \varphi} = r\sin\theta\cos\varphi$$

$$\frac{\partial z}{\partial \varphi} = 0$$

$$\left|\frac{\partial \bar{r}}{\partial \varphi}\right| = \left|\frac{\partial \bar{r}}{\partial \varphi}\right| = r\sin\theta$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\varphi$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta\sin\varphi$$

$$\frac{\partial z}{\partial \theta} = -r\sin\theta$$

$$\left|\frac{\partial \bar{r}}{\partial \theta}\right| = r$$

$$\left|\frac{\partial \bar{r}}{\partial r}\right| = 1$$

$$e_\varphi = \frac{\partial \bar{r}}{\partial \varphi} \left| \frac{\partial \bar{r}}{\partial \varphi} \right|^{-1}$$

$$e_\varphi = \left(\frac{\partial x}{\partial \varphi} e_x + \frac{\partial y}{\partial \varphi} e_y + \frac{\partial z}{\partial \varphi} e_z \right) \cdot \frac{1}{r \sin\theta} =$$

$$= -\sin\varphi \, e_x + \cos\varphi \, e_y .$$

$$e_\theta = \frac{1}{r} \left(r\cos\theta \cos\varphi \, e_x + r\cos\theta \sin\varphi \, e_y - r\sin\theta \, e_z \right)$$

$$\begin{cases} e_\theta = e_x \cos\theta \cos\varphi + e_y \cos\theta \sin\varphi - e_z \sin\theta \\ e_r = e_x \sin\theta \cos\varphi + e_y \sin\theta \sin\varphi + e_z \cos\theta \\ e_\varphi = -e_x \sin\varphi + e_y \cos\varphi \end{cases}$$

$$(e_\theta, e_r) = \sin\theta \cos\theta \cos^2\varphi + \sin\theta \cos\theta \sin^2\varphi -$$

$$- \sin\theta \cos\theta = 0$$

$$(e_\theta, e_\varphi) = -\cos\theta \sin\varphi \cos\varphi + \cos\theta \sin\varphi \cos\varphi = 0$$

$$(e_r, e_\varphi) = -\sin\theta \sin\varphi \cos\varphi + \sin\theta \sin\varphi \cos\varphi = 0$$

15. $X = \underbrace{X^a}_{\text{гл. ф-ции}}(x) \underbrace{\frac{\partial}{\partial x^a}}_{\text{базис}}$

$x^a \to x'^a(x)$

$X = X'^a(x') \frac{\partial}{\partial x'^a}$

$\frac{\partial}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a} \cdot \frac{\partial}{\partial x^b}$

$X^b(x) \frac{\partial}{\partial x^b} = X'^a(x') \frac{\partial x^b}{\partial x'^a} \frac{\partial}{\partial x^b}$

$\underbrace{\left[X'^a(x') \frac{\partial x^b}{\partial x'^a} - X^b(x)\right]}_{0} \underbrace{\frac{\partial}{\partial x^b}}_{\text{ЛНЗ}} = 0$

$X'^a(x') \underbrace{\frac{\partial x^b}{\partial x'^a}}_{(\Lambda^{-1})^b_a} = X^b(x) \quad \Big| \cdot \Lambda^c_b$

$X'^a(x) \cdot \Lambda^c_b (\Lambda^{-1})^b_a = X^b(x) \Lambda^c_b$

$$X'^{a}(x)\,\delta^{c}_{a} = X^{b}(x)\,\Lambda^{c}_{b}$$

$$X'^{c}(x') = \Lambda^{c}_{b}\,X^{b}(x)$$

$$\Lambda^{a}_{b} = \frac{\partial x'^{a}}{\partial x^{b}}$$

$\omega \in \Omega^1(M)$

$$\omega = \omega_a(x)\, dx^a = \omega'_b(x')\, dx'^b =$$

$$= \omega'_b(x') \frac{\partial x'^b}{\partial x^a} dx^a$$

$$\parallel$$

$$\Lambda^b_{\ a}$$

$$\omega_a(x) = \omega'_b(x')\, \Lambda^b_{\ a}$$

17. $f(x,y,z) = xyz + x^3y^2$

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$

$df(X) = X(f)$

$X = 3y^2\frac{\partial}{\partial y} - 2xy\frac{\partial}{\partial x}$

$df = \frac{\partial f}{\partial x^a} dx^a = (yz + 3x^2y^2)\,dx + (xz + 2yx^3)\,dy$

$+ xy\,dz$

$df\left(3y^2\frac{\partial}{\partial y} - 2xy\frac{\partial}{\partial x}\right) = -2xy(yz + 3x^2y^2) +$

$+ (xz + 2yx^3)\,3y^2$

$dx^a\left(\frac{\partial}{\partial x^b}\right) = \delta^a_b$

$df\left(\frac{\partial}{\partial z}\right) = xy$

19. $dx^a \wedge dx^b =$

$$\underset{\Omega^k(M)}{\omega} \wedge \underset{\Omega^m(M)}{\lambda} (X_1, \ldots, X_{k+m}) = \frac{1}{k!\,m!} \sum_{\sigma \in S_{k+m}} (\operatorname{sgn} \sigma)\, \omega(X_{\sigma(1)} \ldots X_{\sigma(k)}) \cdot \lambda(X_{\sigma(k+1)} \ldots X_{\sigma(k+m)})$$

$$\underbrace{dx^a}_{\Omega^1(M)} \wedge \underbrace{dx^b}_{\Omega^1(M)} = dx^a \otimes dx^b - dx^b \otimes dx^a$$

21. $$\omega = \sum_{a_1 < \dots < a_k} \omega_{a_1 \dots a_k}(x)\, dx^{a_1} \wedge \dots \wedge dx^{a_k}$$

$$d\omega = \sum_{a,\, a_1 < \dots < a_k} \frac{\partial \omega_{a_1 \dots a_k}}{\partial x^a}\, dx^a \wedge dx^{a_1} \wedge \dots dx^{a_k}$$

$$\omega = f(x^2+y^2)\,(x\,dx + y\,dy)$$

$$f : \mathbb{R}^2 \setminus \{0,0\} \to \mathbb{R}$$

$$d\omega = \frac{\partial f(x^2+y^2)x)}{\partial y}\, dy \wedge dx + \frac{\partial}{\partial x}(f(x^2+y^2)y)\, dx \wedge dy$$

23. $d\omega = 0$ — замк. форма

$\omega = d\lambda$ — точная форма

$\omega \wedge \sigma$ = точная ?

$\overset{\uparrow}{\text{замк.}}$ $\overset{\uparrow}{\text{точная}}$

$$\sigma = d\lambda$$

$$\omega \wedge \sigma = \omega \wedge d\lambda$$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$$

$$(-1)^{\deg \alpha}\left[d(\alpha \wedge \beta) - d\alpha \wedge \beta \right] = \alpha \wedge d\beta$$

$$\omega \wedge d\lambda = (-1)^{\deg \omega}\left[d(\omega \wedge \lambda) - \underbrace{d\omega}_{0} \wedge \lambda \right]$$

$$\omega \wedge d\lambda = d\left((-1)^{\deg \omega}\, \omega \wedge \lambda \right)$$

$$\Downarrow$$

$\omega \wedge d\lambda$ – точ. форма.

25. $\mathbb{R}^4(t, x, y, z)$

$$F = E_x\, dx \wedge dt + E_y\, dy \wedge dt + E_z\, dz \wedge dt + H_x\, dy \wedge dz + H_y\, dz \wedge dx + H_z\, dx \wedge dy$$

$F \in \Omega^2(\mathbb{R}^4)$

E – эл. поле. H – маг. поле.

$dF = 0$. На $\mathbb{R}^4$ нет дыр

$$\Downarrow$$

$F = dA$

$A \in \Omega^1(\mathbb{R}^4)$

$$A = A_t\, dt - A_x\, dx - A_y\, dy - A_z\, dz$$

$$dF = 0 = \frac{\partial E_x}{\partial y} dy \wedge dx \wedge dt + \frac{\partial E_x}{\partial z} dz \wedge dx \wedge dt +$$

$$+ \frac{\partial E_y}{\partial x} dx \wedge dy \wedge dt + \frac{\partial E_y}{\partial z} dz \wedge dy \wedge dt +$$

$$+ \frac{\partial E_z}{\partial x} dx \wedge dz \wedge dt + \frac{\partial E_z}{\partial y} dy \wedge dz \wedge dt +$$

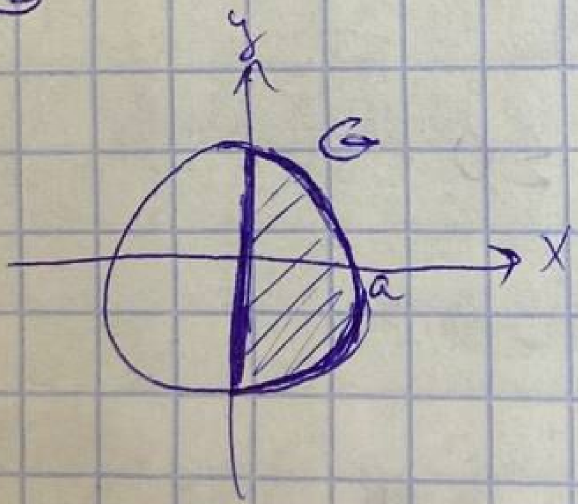
$$+ \frac{\partial H_x}{\partial x} dx \wedge dy \wedge dz + \frac{\partial H_x}{\partial t} dt \wedge dy \wedge dz$$

$$+ \frac{\partial H_y}{\partial y} dy \wedge dz \wedge dx + \frac{\partial H_y}{\partial t} dt \wedge dz \wedge dx +$$

$$+ \frac{\partial H_z}{\partial z} dz \wedge dx \wedge dy + \frac{\partial H_z}{\partial t} dt \wedge dx \wedge dy$$

$$\begin{array}{l} \mathrm{rot}\,\vec{E} = \frac{\partial \vec{H}}{\partial t}(-) \\ \mathrm{div}\,\vec{H} = 0 \end{array} \begin{cases} \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \\ \frac{\partial H_z}{\partial t} + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \\ \frac{\partial H_y}{\partial t} + \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0 \\ \frac{\partial H_x}{\partial t} + \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0 \end{cases}$$

27. $\int\limits_{G} xy^2 \, dx \wedge dy$, $G = \{x^2 + y^2 \leq a^2, \quad x \geq 0\}$



$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

$$\begin{cases} dx = dr\cos\varphi - r\sin\varphi\, d\varphi \\ dy = dr\sin\varphi + r\cos\varphi\, d\varphi \end{cases}$$

$$dx \wedge dy = dr \wedge d\varphi\,(r\cos^2\varphi + r\sin^2\varphi) =$$
$$= r\, dr \wedge d\varphi$$

$$\int_G r^4 \cos\varphi \sin^2\varphi \; dr \wedge d\varphi =$$

$$= \int_0^a r^4\, dr \int_{-\frac{\pi}{2}}^{\pi/2} \cos\varphi \sin^2\varphi\, d\varphi = \frac{a^5}{5} \cdot 2 \cdot \int_0^{\frac{\pi}{2}} \cos\varphi \sin^2\varphi\, d\varphi =$$
$$= \frac{2}{15} a^5$$

28. $\cancel{\alpha_E = E_x\,dx + E_y\,dy + E_z\,dz}$

$$\alpha_E = E_x\,dy\wedge dz + E_y\,dz\wedge dx + E_z\,dx\wedge dy$$

$$\int_{\partial M} \alpha_E = \int_M d\alpha_E = \int \Big(\underbrace{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}}_{\operatorname{div}\vec{E}} \Big)\, dx\wedge dy\wedge dz$$

2-форма потока

$$\beta_H = H_x\,dx + H_y\,dy + H_z\,dz$$

$$\int_M \beta_H = \int_{\partial M} d\beta_H = \int \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) dx \wedge dy$$

$$+ \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) dy \wedge dz \; +$$

$$+ \left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right) dx \wedge dz$$

32. $\omega = x\,dy + y\,dx$

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

$$L_X \omega = L_X(x\,dy + y\,dx) =$$

$$= (L_X\, dy)\,x + \underbrace{X(x)\cdot dy}\ \oplus$$

$$// \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(x)\,dy = x\,dy \ //$$

$$\oplus\ y\,L_X\, dx + y\,dx =$$

$$= \omega + x\,L_X\, dy + y\,L_X\, dx \ \ominus$$

$$L_{fX}\,\omega = f\,L_X\,\omega + df \wedge i_X\,\omega$$

$$\ominus\ x\left(L_{x\frac{\partial}{\partial x}} dy + L_{y\frac{\partial}{\partial y}} dy\right) + y\left(L_{x\frac{\partial}{\partial x}} dx + L_{y\frac{\partial}{\partial y}} dx\right) + \omega =$$

$$= x\left(+ dx \wedge \cancelto{0}{i_{\frac{\partial}{\partial x}} dy} + dy \wedge \cancelto{1}{i_{\frac{\partial}{\partial y}} dy}\right) +$$

$$+ y\left(dx \wedge \cancelto{1}{i_{\frac{\partial}{\partial x}} dx} + dy \wedge \cancelto{0}{i_{\frac{\partial}{\partial y}} dx}\right) + \omega = 2\omega$$

16. $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

$$\frac{\partial}{\partial y} = \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi} + \frac{\partial r}{\partial y}\frac{\partial}{\partial r} =$$

$$= \frac{1}{r\cos\varphi}\frac{\partial}{\partial \varphi} + \frac{1}{\sin\varphi}\frac{\partial}{\partial r}$$

$$\frac{\partial}{\partial x} = \frac{-1}{r\sin\varphi}\frac{\partial}{\partial \varphi} + \frac{1}{\cos\varphi}\frac{\partial}{\partial r}$$

$$X = r\cos\varphi\left(\frac{1}{r\cos\varphi}\frac{\partial}{\partial\varphi} + \frac{1}{\sin\varphi}\frac{\partial}{\partial r}\right) -$$

$$- r\sin\varphi\left(\frac{-1}{r\sin\varphi}\frac{\partial}{\partial\varphi} + \frac{1}{\cos\varphi}\frac{\partial}{\partial r}\right) =$$

$$= 2\frac{\partial}{\partial\varphi} + r(\cot\varphi - \mathrm{tg}\varphi)\frac{\partial}{\partial r}$$

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = r\cos\varphi\left(-\frac{1}{r\sin\varphi}\frac{\partial}{\partial\varphi} + \frac{1}{\cos\varphi}\frac{\partial}{\partial r}\right)$$

$$+ r\sin\varphi\left(\frac{1}{r\cos\varphi}\frac{\partial}{\partial\varphi} + \frac{1}{\sin\varphi}\frac{\partial}{\partial r}\right) =$$

$$= (\mathrm{tg}\varphi - \cot\varphi)\frac{\partial}{\partial\varphi} + r\frac{\partial}{\partial r}$$

18. $$df(X) = df\left(X^a \frac{\partial}{\partial x^a}\right) =$$

$$= X^a \, df\left(\frac{\partial}{\partial x^a}\right) =$$

$$= X^a f_b \, \underbrace{dx^b \left(\frac{\partial}{\partial x^a}\right)}_{\delta_{ab}} = X^a f_a =$$

$$= X^a f_b \left(\frac{\partial}{\partial x^a}\right) dx^b = X^a \left(\frac{\partial}{\partial x^a}\right) f_b \, dx^b =$$

$$= X(f)$$