Bruce C. Berndt

Ramanujan's Notebooks

Part V

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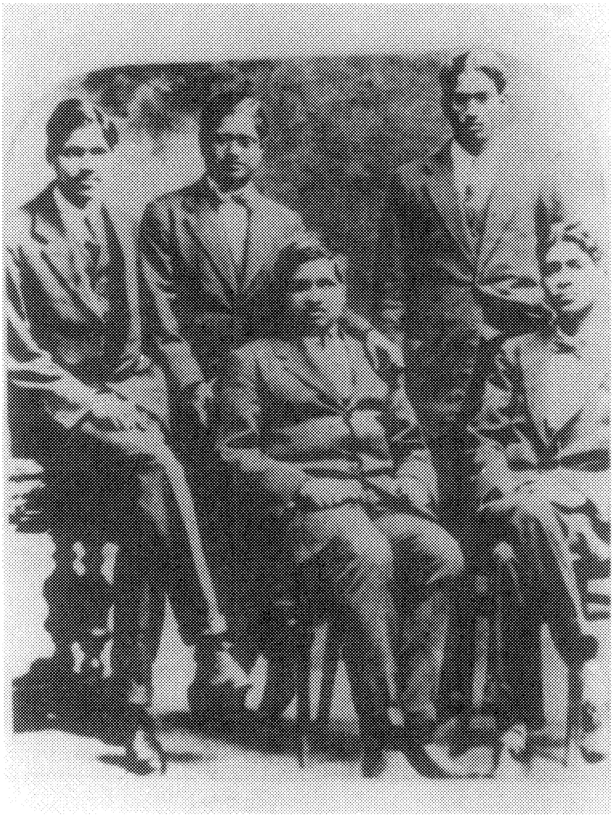
Dedicated to the mathematicians who
generously assisted by their many
contributions to these five volumes



To the author's knowledge, only three photographs of Ramanujan are extant. Variations of his passport photo appear in our books, Parts I and IV. A group photo with Ramanujan appearing in cap and gown can be found as the frontispiece of the publication of Ramanujan's lost notebook [11], and has been excised in several cropped versions, often with Ramanujan standing alone. The photograph above is also one of several renditions, the most frequent being one with Ramanujan sitting alone.

Preface

During the years 1903–1914, Ramanujan recorded most of his mathematical discoveries without proofs in notebooks. Although many of his results had already been published by others, most had not. Almost a decade after Ramanujan's death in 1920, G. N. Watson and B. M. Wilson began to edit Ramanujan's notebooks, but, despite devoting over ten years to this project, they never completed their task. An unedited photostat edition of the notebooks was published by the Tata Institute of Fundamental Research in Bombay in 1957.

This book is the fifth and final volume devoted to the editing of Ramanujan's notebooks. Parts I–III, published, respectively, in 1985, 1989, and 1991, contain accounts of Chapters 1–21 in the second notebook, a revised enlarged edition of the first. Part IV, published in 1994, contains results from the 100 unorganized pages in the second notebook and the 33 unorganized pages comprising the third notebook. Also examined in Part IV are the 16 organized chapters in the first notebook, which contain very little that is not found in the second notebook. In this fifth volume, we examine the remaining contents from the 133 unorganized pages in the second and third notebooks, and the claims in the 198 unorganized pages of the first notebook that cannot be found in the succeeding notebooks. In contrast to the organized portion of the first notebook, the unorganized material in the first notebook contains several results, particularly about class invariants, singular moduli, and values of theta-functions, which are not recorded in the second and third notebooks.

As in the first four volumes, either proofs are provided for claims not previously established in the literature, or citations are given for results already proved in the literature.

Urbana, Illinois
September, 1997

Bruce C. Berndt

Contents

Introduction

Knowledge comes, but wisdom lingers.

Alfred, Lord Tennyson, Locksley Hall

This book constitutes the fifth and final volume of our attempts to establish all the results claimed by the great Indian mathematician Srinivasa Ramanujan in his *Notebooks*, first published in a photostat edition by the Tata Institute of Fundamental Research in 1957 [9]. Although each of the five volumes contains many deep results, perhaps the average depth in this volume is greater than in the first four. As will be seen in the following paragraphs, several mathematicians made important contributions to the completion of this volume. However, I particularly extend my deepest gratitude to Heng Huat Chan and Liang–Cheng Zhang without whose contributions this volume would have been woefully deficient. This volume, however, should not be regarded as the closing chapter on Ramanujan's notebooks. Instead, it is just the first milestone on our journey to understanding Ramanujan's ideas. Many of the proofs given here and in other volumes certainly do not reflect Ramanujan's motivation, insights, proofs, and wisdom. It is our fervent wish that these volumes will serve as springboards for further investigations by mathematicians intrigued by Ramanujan's remarkable ideas. As in the other four volumes, for each correct claim, we either provide a proof or cite references in the literature where proofs can be found. We emphasize that Ramanujan made extremely few errors, and that most "mistakes" are either minor misprints, or, in fact, they are errors made by the author arising from misinterpretations of Ramanujan's claims, which are occasionally fuzzy.

The second notebook is a revised, enlarged edition of the first, and, as with G. N. Watson and B. M. Wilson, who made the first attempts at editing Ramanujan's notebooks, the second was our initial focus. It was therefore quite surprising for us to discover that the unorganized pages of the first notebook contain many beautiful results, especially in the areas of class invariants, singular moduli, and explicit values of theta–functions, that Ramanujan failed to record in his second notebook. The material examined in this volume arises from the unorganized pages in all three notebooks, and we provide now brief descriptions of the contents of each of the eight chapters.

Ramanujan loved continued fractions, and many of his most beautiful results involve continued fractions. Chapter 32 contains about 70 results on continued fractions scattered among the unorganized pages in his second and third notebooks, and four evaluations of the Rogers–Ramanujan continued fraction from his first notebook. Several modular equations for the Rogers–Ramanujan continued fraction $R(q)$ can be found; in less technical language these are functional equations relating $R(q)$ at two different arguments. Other q–continued fractions were also examined by Ramanujan in these unorganized pages. Several results arise from Ramanujan's beautiful continued fractions for quotients of gamma functions found in Chapter 12 of his second notebook. The present chapter primarily constitutes a reorganized and partially rewritten version of the memoir published by G. E. Andrews, the author, L. Jacobsen, and R. L. Lamphere [2]; a preview and discussion of some of the results was published by the four of us in [1]. The four evaluations of the Rogers–Ramanujan continued fraction from the first notebook first appeared in a paper with H. H. Chan [1], and in Chan's doctoral thesis [2].

In the classical theory of elliptic functions, the ordinary hypergeometric function ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$ plays a very important role. In his famous paper [3], in the course of stating without proofs some remarkable series representations for $1/\pi$, Ramanujan remarked that several of his series arose from alternative theories of elliptic functions wherein the aforementioned hypergeometric function is replaced by either ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; x)$, ${}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; x)$, or ${}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; x)$. These theories were never developed, but the first six pages in the unorganized section at the end of his second notebook are devoted to these theories. This is the content of Chapter 33, most of which was first published in a paper with S. Bhargava and F. G. Garvan [1]. A few results from the first notebook have been added to this presentation. The first of the three alternative theories is the most interesting and the most important, and we feel that a large body of work remains to be discovered here.

Like Chapter 33, it took several years for us to satisfactorily examine all of the material in Chapter 34, which is devoted to class invariants and singular moduli. Most of this work has appeared in papers with Chan and L.–C. Zhang [1]–[3], [5] and with Chan [3], [4]. A summary, written with Chan and Zhang appears in [6], and several results were established in Chan's Ph.D. thesis [2]. Ramanujan did a prodigious amount of work in calculating over 100 class invariants. For reasons that are unclear to us, he failed to record many of these values in his second notebook. To establish most of Ramanujan's hitherto unproved class invariants, we had to develop methods that were completely unknown to Ramanujan. Thus, Ramanujan's methods and insights into class invariants remain largely a mystery to us. It is also puzzling to us that, except for four values, Ramanujan did not record in his second notebook the more than 30 representations for singular moduli found in his first notebook. For even n, Ramanujan left us a beautiful formula to aid in the calculation of singular moduli, although we are uncertain how he found it, but for odd n, Ramanujan's methods are unknown to us.

The values of the classical theta-function $\sum_{k=-\infty}^{\infty} e^{-2\pi k^2\sqrt{n}}$ are beautiful algebraic numbers when n is a positive rational number. Chapter 35 is devoted to Ramanujan's explicit values of theta–functions found in his first notebook. Most of this chapter previously appeared in papers with Chan [2] and Chan and Zhang [4].

Chapters 19–21 in Ramanujan's second notebook contain several hundred modular equations. Surprisingly, some of his deepest results on modular equations in the forms of Russell, Schläfli, or Weber appear only in the first notebook. In Chapter 36 we establish all the results on modular equations found in the first notebook but not in the second. Some are easy variations of results in the second notebook that we proved in our third volume [3].

We return in Chapter 37 to the second notebook. All the results in this chapter pertain to infinite series. Many were very difficult for us to prove, and we owe our thanks to R. A. Askey, G. Bachman, P. Bialek, D. Bradley, R. J. Evans, J. L. Hafner, and A. Hildebrand for important contributions. Although it is impossible to summarize in one brief paragraph the contents of this long and varied chapter, we mention just a few highlights. The first several sections of the chapter are devoted to interesting variants of the Abel–Plana summation formula and examples thereof. In Section 21, we examine Ramanujan's surprising transformation formulas for two certain doubly exponential cousins of two classical theta–functions. In Section 22, an intriguing formula for the logarithmic derivative of the gamma function is derived. In Section 42, Ramanujan offers some very remarkable theorems on the explicit behavior of partial sums of certain divergent alternating series.

In previous volumes we marveled about Ramanujan's insights into asymptotic analysis; see, in particular, Chapter 13 in our second book [2]. In Chapter 38, we gather all of Ramanujan's approximations and asymptotic expansions found in the unorganized pages of his second and third notebooks. We are very grateful to Askey, R. P. Brent, Evans, and M. L. Glasser for their valuable contributions to this chapter. Again it is difficult to succinctly summarize this work. The first several sections are devoted to the asymptotic analysis of series which are hybrids of the Riemann zeta–function and hypergeometric series.

Last, in Chapter 39 we collect together results from the unorganized pages of the first notebook which do not fall under the purviews of Chapters 32–36. Most are from analysis, but some are elementary.

In Part IV we provided a chapter documenting each entry in the 16 organized chapters of the first notebook. The vast majority of these results can be found in the second and third notebooks, but for those that are not we gave proofs. At the end of this volume, we provide a similar account for all the claims made in the 198 unorganized pages of the first notebook. Thus, for each entry we indicate where a proof can be found in Parts I–V.

Except for the massive amount of material in Chapters 33 and 34 related to Ramanujan's paper on modular equations and approximations to π [3], [10, pp. 23–39], in contrast to the first four volumes, very few claims in this volume pertain to Ramanujan's published papers and problems. The following table summarizes these connections:

Paper	Related Material in this Volume
[1]	See final chapter on location of entries, in particular, pp. 347–349
[2]	Entry 18 in Chapter 37
[3]	Most of Chapters 33 and 34
[4]	Entry 30 in Chapter 37
[5]	Entries 27–30 in Chapter 37
[6]	Entries 24–26 in Chapter 37
[7]	Entry 23 in Chapter 37
[8]	First section in Chapter 32

The following table gives the number of results in each of the eight chapters in this volume:

Chapter	Number of Results
32	73
33	62
34	196
35	24
36	87
37	53
38	46
39	24
Total	565

The next table summarizes our reckonings of the results examined in each of the five volumes.

Volume	Number of Results
I	759
II	605
III	834
IV	491
V	565
Total	3254

This total is in agreement with Hardy's original estimate that Ramanujan's notebooks contain the statements of approximately 3000–4000 theorems.

In the sequel, equation numbers always refer to equalities in the same chapter, unless otherwise indicated. Several references will be made to our first four volumes [1]–[4], and we almost always use the abbreviations Part I, Part II, Part III, and Part IV, respectively. In chapters where claims from both the first and second notebooks are discussed, we append the abbreviations NB 1 or NB 2 to a page number to indicate that the result is in the first or second notebook, respectively. Throughout the text, the residue of a meromorphic function $F(z)$ at a pole z_0 is denoted by R_{z_0}.

To prove Ramanujan's claims in Chapters 33 and 34, it was necessary to derive many ancillary results. Thus, in contrast to the remaining chapters, the formats in these two chapters are different.

The road through Ramanujan's notebooks has been a long one, and many people on this journey deserve my gratitude.

After taking a course in modular forms from my subsequent thesis advisor, J. R. Smart, my first introduction to the work of Ramanujan came in a course on modular forms with applications to number theory taught by Marvin Knopp at the University of Wisconsin in the Fall of 1964. Here I learned about the Hardy–Ramanujan asymptotic formula for the partition function $p(n)$ and Ramanujan's congruences for $p(n)$. Knopp's book [1] arose from this course.

I learned of the existence of Ramanujan's notebooks one day in early 1967 in Robert Rankin's office at the University of Glasgow. However, when Rankin asked me if I would be interested in examining his copy of the Tata Institute's publication of the notebooks [9], I declined.

I owe a huge debt to the late Emil Grosswald for my initial interest in the notebooks. It was on a cold winter day in early February, 1974, while I was on leave at the Institute for Advanced Study, that I was reading two papers by Grosswald [1], [2] in which he proved some formulas from Ramanujan's notebooks. I suddenly realized that I could also prove these formulas by using some transformation formulas for Eisenstein series that I had proved about two years earlier. I was naturally curious to determine if there were other formulas in the notebooks that I could prove with my methods. Fortunately, the library at Princeton University possesses a copy of the Tata Institute's publication [9]. I found a few more formulas which I could prove, but I also found a few thousand others which I could not prove. My papers [6] and [7] contain proofs of my initial findings and several other formulas in the same genre.

At the close of the spring semester in May, 1977, at the University of Illinois, I decided to attempt to find proofs for all of the formulas (a total of 87) in Chapter 14 of Ramanujan's second notebook, where the formulas which Grosswald proved can be found. After working on this project for nearly a year, George Andrews visited the University of Illinois and informed me that the attempts of G. N. Watson and B. M. Wilson to edit Ramanujan's notebooks in the late 1920s and 1930s had been preserved in the library at Trinity College, Cambridge. Thinking that with a copy of their notes, I could edit further chapters, I wrote Trinity College. Indeed, Watson and Wilson's notes were very useful, and, in particular, Watson's work on modular equations in Chapters 19–21 of the second notebook was invaluable. We are pleased to record here our thanks to the Master and Fellows of Trinity College, Cambridge, for providing us with a copy of these notes. Thus, to bring us to the end of the story, since May, 1977, I have devoted all of my research time to proving the claims made by Ramanujan in his notebooks.

This work could not have been completed without the help of several people. First, C. Adiga, G. E. Andrews, S. Bhargava, A. J. Biagioli, P. Bialek, H. H. Chan, R. J. Evans, F. G. Garvan, J. L. Hafner, P. T. Joshi, R. L. Lamphere, L. Lorentzen, J. M. Purtilo, and L.–C. Zhang collaborated with me in writing papers on chapters or sets of formulas from the notebooks, and I extend to them my sincere gratitude for their collaborations.

Others have made contributions in papers that they have individually written, or in work that appeared only in our accounts. Thus, I wish to thank the following mathematicians without whose proofs these volumes could not have been completed: G. E. Andrews, R. A. Askey, G. Bachman, J. M. Borwein, P. B. Borwein, D. Bradley, H. Cohen, M. L. Glasser, A. Hildebrand, L. Lorentzen, R. McIntosh, K. S. Williams, and D. Zagier. Some names appear in each of the past two paragraphs, because these mathematicians also made contributions independent of any collaboration with me.

Finally, I emphasize the enormous help given by three people. For several years, when I became stymied for months or perhaps years over one of Ramanujan's enigmatic formulas, I turned to Ron Evans. On each occasion, he was able to supply a proof, and some of the most difficult proofs in these five volumes are due to Evans. Preliminary versions of many of the chapters were read by Dick Askey, who found mistakes, supplied references, gave insights, and sometimes provided a proof. Lastly, in recent years, Heng Huat Chan not only served as a valuable collaborator, but he offered many additional comments and insights, supplied further proofs, and critically read preliminary versions of several chapters.

Jaebum Sohn also read in detail several chapters in this volume, and I thank him for the several errors he uncovered.

I have given hundreds of lectures on Ramanujan's work to graduate students and colleagues at the University of Illinois during the past two decades, and I am grateful for the many meaningful comments they have provided.

Others who offered important comments and insights are cited in the Introductions to the first four volumes [1]–[4].

It is important that mathematicians are cited for the relevant contributions they have made to a subject. At times, it is difficult to unearth their work, and I thank Nancy Anderson, Mathematics Librarian at the University of Illinois, for helping me locate many obscure papers.

One day in the early 1980s Heini Halberstam called me to his office here at Illinois to meet Springer–Verlag's Mathematics Editor, Walter Kaufmann–Bühler, who suggested that I compile my work on Ramanujan's notebooks into volumes for Springer–Verlag. Thus, I express my sincere thanks to the late Kaufmann–Bühler and the current Mathematics Editor, Ina Lindemann, for their support of my work.

In the early years devoted to Ramanujan's notebooks, I received support from the Vaughn Foundation, and I express my deep gratitude to James Vaughn for his financial support. In more recent years, the National Science Foundation, the Sloan Foundation, and the National Security Agency have provided grants, and I thank these agencies for their support. I also am pleased to thank The Center for Advanced Study at The University of Illinois for three appointments which provided me with time that I could exclusively devote to Ramanujan's notebooks.

The author bears the responsibility for all errors and would appreciate being notified of such, whether they be minor or serious.

32

Continued Fractions

Chapter 12 in Ramanujan's second notebook is devoted almost entirely to continued fractions. Further continued fractions can be found in other chapters, especially in Chapter 16. See Parts II [2] and III [3] for accounts of Chapters 12 and 16, respectively. The 100 pages of unorganized material at the end of the second notebook and the 33 unorganized pages in the third notebook contain about 60 further results on continued fractions. These and four evaluations of the Rogers–Ramanujan continued fraction from the first notebook will be examined in this chapter.

We have divided the entries into five categories. Section 1 is devoted to the famous Rogers–Ramanujan continued fraction, the only continued fraction appearing in Ramanujan's published papers [8], [10, pp. 214–215], and certain generalizations. Other q–continued fractions are examined in Section 2. In Section 3 we establish several continued fractions arising from Ramanujan's beautiful continued fractions for quotients of Γ–functions in Chapter 12 of his second notebook. Most of the continued fractions in Section 4 arise from special functions, in particular, hypergeometric functions. General theorems are the focus of Section 5.

We next describe a few highlights in this chapter.

Let, for $|q| < 1$,

$$R(q) := \frac{q^{1/5}}{1}\,+\,\frac{q}{1}\,+\,\frac{q^2}{1}\,+\,\frac{q^3}{1}\,+\cdots$$

and

$$S(q) := -R(-q)$$

denote the famous Rogers–Ramanujan continued fractions. Entries 1–6 provide beautiful equations relating $R(q)$ with each of $R(-q)$, $R(q^2)$, $R(q^3)$, $R(q^4)$, and $R(q^5)$. In both his first and second letters to Hardy, Ramanujan [10, pp. xxvii, xxviii] communicated theorems about $R(q)$ and $S(q)$. In particular, in his first letter, he asserted that

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2} \tag{0.1}$$

and

$$S(e^{-\pi}) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}. \tag{0.2}$$

The evaluation (0.1) follows easily from a reciprocity theorem for $R(q)$, which Ramanujan gave in his second letter, and which was first proved by G. N. Watson [2]. The evaluation (0.2) follows from a similar reciprocity theorem for $S(q)$, which apparently Ramanujan did not communicate to Hardy, but which is found in his notebooks [9, p. 204]; see also Part III [3, p. 83]. The latter theorem was first proved by K. G. Ramanathan [2], but (0.2) was first established by Watson [1] in a different manner. In his second letter, Ramanujan also claimed that

$$R(e^{-2\pi\sqrt{5}}) = \frac{\sqrt{5}}{1+\left(5^{3/4}\left(\frac{\sqrt{5}-1}{2}\right)^{5/2}-1\right)^{1/5}} - \frac{\sqrt{5}+1}{2}, \tag{0.3}$$

which was also first proved in print by Watson [2]; Ramanathan [2] also established (0.3). Entries 7–10 offer four particular evaluations of $R(q)$ from page 311 of Ramanujan's first notebook. Several further evaluations of $R(q)$ and $S(q)$ were recorded by Ramanujan in his "lost notebook" [11], and these have been proved by the author, H. H. Chan, and L.–C. Zhang [3]. Entry 11 is a fascinating theorem concerning the oscillating behavior of the divergent Rogers–Ramanujan continued fraction for $q > 1$. Ramanujan offers some "approximating" continued fractions, with the most interesting results being Entries 13 and 14 involving modest generalizations of the Rogers–Ramanujan continued fraction. In each case, one continued fraction is approximated by another continued fraction. D. Zagier [1] has discovered the proper interpretations for these fascinating results, and we briefly describe his work. Some elegant continued fractions, for example, Entry 15, are instances of more general continued fractions found later by Ramanujan and recorded in his "lost notebook" [11].

Some q–continued fractions give representations for certain q–products. A few of these results were established by A. Selberg [1], [2, pp. 1–23] in 1936. We particularly call attention to Entry 19. To prove this, we employ a continued fraction found by G. E. Andrews [1] which may be the unidentified continued fraction to which Ramanujan alludes in his first letter to Hardy [9, p. xxviii].

Ramanujan had a tremendous facility for extracting interesting, important, and elegant special cases from his theorems. The unorganized material contains numerous corollaries arising from his amazing work on continued fractions for products of gamma functions in Chapter 12. Two of the most curious results in this vein are Entries 41 and 42.

Before commencing our examination of Ramanujan's continued fractions, we must offer several remarks about notation. All "chapter" references refer to chapters in Ramanujan's second notebook [9].

We employ the notation

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots \tag{0.4}$$

for the continued fraction

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}.$$

Occasionally, for brevity, we shall use the notation $\mathbf{K}(a_n/b_n)$ instead of (0.4). We let A_n and B_n denote the nth numerator and denominator, respectively, for (0.4). Thus, for $n \geq 1$,

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n} = \frac{A_n}{B_n},$$

where

$$A_n = b_n A_{n-1} + a_n A_{n-2} \tag{0.5}$$

and

$$B_n = b_n B_{n-1} + a_n B_{n-2}, \tag{0.6}$$

where $A_{-1} = 1 = B_0$ and $A_0 = 0 = B_{-1}$ (H. S. Wall [1, p. 15]).

The set of natural numbers is denoted by $\mathbb{N}$, while the set of complex numbers is designated by $\mathbb{C}$. Furthermore, set $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The set of real numbers is denoted by $\mathbb{R}$, and we set $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. The set of integers is denoted by $\mathbb{Z}$.

If $a_N = 0$ for some $N \in \mathbb{N}$, we say that the continued fraction (0.4) terminates, and we assign to it the value

$$f := \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_{N-1}}{b_{N-1}} = \frac{A_{N-1}}{B_{N-1}},$$

if $a_n \neq 0$ for $n < N$. If $a_n \neq 0$, $1 \leq n < \infty$, then the continued fraction (0.4) converges if $\lim_{n\to\infty} A_n/B_n$ exists in $\widehat{\mathbb{C}}$. Its value is then given by $f = \lim_{n\to\infty} A_n/B_n$, and we write

$$f = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots. \tag{0.7}$$

(Note that (0.7) includes the case "$\infty = \infty$.") If $\lim_{n\to\infty} A_n/B_n$ does not exist in $\widehat{\mathbb{C}}$ (and $a_n \neq 0$, $1 \leq n < \infty$), we say that (0.4) diverges.

Many of Ramanujan's continued fractions arise from equivalence transformations or from the "even parts" of continued fractions. In contrast to many contemporary authors, we employ equality signs when invoking these ideas.

Several results depend upon continued fractions from Chapter 12, and so we frequently make reference to our account [2] of this chapter. Our convergence statements for many of the continued fractions of this chapter are based upon a theorem of L. Jacobsen [4, Theorem 2.3], which is a consequence of the parabola theorems (W. B. Jones and W. J. Thron [1]).

Several entries below concern theta–functions, and we employ the notation Ramanujan introduced in Chapter 16 [9] (Part III [3]). For $|ab| < 1$, put

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{0.8}$$

Also set

$$\varphi(q) := f(q,q), \qquad \psi(q) := f(q,q^3), \quad \text{and} \quad f(-q) := f(-q,-q^2). \tag{0.9}$$

For each nonnegative integer n, let

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \tag{0.10}$$

and, if $|q| < 1$, set

$$(a;q)_\infty := \lim_{n\to\infty} (a;q)_n.$$

We employ the contemporary convention for Bernoulli numbers and not that used by Ramanujan in his notebooks. Thus, the Bernoulli numbers B_n, $n \geq 0$, are here defined by

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n, \qquad |x| < 2\pi.$$

We always employ the principal branch of each multivalued relation such as $z^{-1/2}$, $\log z$, and $\tan^{-1} z$.

Lastly, as customary, set

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

1. The Rogers–Ramanujan Continued Fraction

Entry 1 (p. 326). *Let* $|q| < 1$,

$$U := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+\cdots} =: U^{1/5}$$

and

$$v := \frac{q^{2/5}}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^4}{1} \, \frac{}{+} \, \frac{q^6}{1} \, \frac{}{+\cdots} =: V^{1/5}.$$

Then

(i) $\dfrac{v-u^2}{v+u^2} = uv^2$

and

(ii) $UV^2(U^2+V) + U^2 - V + 10UV(UV - U + V + 1) = 0.$

(iii) *If* $U = n\left(\frac{1-n}{1+n}\right)^2$, *then* $V = n^2\frac{1+n}{1-n}$.

Proof of (i). In Part III [3, p. 80, eq. (39.1)], we showed that

$$\frac{1}{u} - u - 1 = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \quad \text{and} \quad \frac{1}{v} - v - 1 = \frac{f(-q^{2/5})}{q^{2/5}f(-q^{10})}. \tag{1.1}$$

In order to derive a relation involving u and v, we need to rely on Ramanujan's work on modular equations.

Let $\sqrt{\alpha}$ and $\sqrt{\beta}$ be the moduli associated with the variables $q^{1/5}$ and q^5, respectively. (Ordinarily, of course, these variables would be designated by q and q^{25}, respectively.) Let m denote the multiplier associated with α and β. Furthermore, set

$$P = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)}, \quad Q = \frac{f(-q^{2/5})}{q^{2/5}f(-q^{10})}, \quad \text{and} \quad R = \frac{f(q^{1/5})}{q^{1/5}f(q^5)}.$$

It will be easier to first derive a relation involving Q and R. It will then be an easy matter to deduce a formula connecting P and Q.

From Entries 12(i), (iii) of Chapter 17 (Part III [3, p. 124]), we easily deduce that

$$R = m^{1/2}\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/24} \quad \text{and} \quad Q = m^{1/2}\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12}.$$

It follows that

$$\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/24} = \frac{Q}{R} \quad \text{and} \quad m^{1/2} = \frac{R^2}{Q}. \tag{1.2}$$

Now rewrite Entries 15(i), (ii) of Chapter 19 (Part III [3, p. 291]) in the respective forms

$$\left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} - 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12} = m^{1/2}$$

and

$$\left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} - 2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12} = \frac{5}{m^{1/2}}.$$

These two equalities are, in fact, modular equations of degree 25. Multiplying the former equality by $(\alpha(1-\alpha))^{1/8}$ and the latter by $(\beta(1-\beta))^{1/8}$, we see that we can combine the resulting two modular equations to obtain the single equation

$$\begin{aligned}(\alpha(1-\alpha))^{1/8}&\left\{\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} + 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12} + m^{1/2}\right\}\\ &= (\beta(1-\beta))^{1/8}\left\{\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} + 2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12} + \frac{5}{m^{1/2}}\right\}.\end{aligned}$$

Using (1.2), we write this equality in the form

$$\frac{R^3}{Q^3}+2\frac{R^2}{Q^2}+\frac{R^2}{Q}=\frac{R^3}{Q^3}\left\{\frac{Q^3}{R^3}+2\frac{Q^2}{R^2}+5\frac{Q}{R^2}\right\}.$$

Clearing denominators and rearranging, we find that

$$R^3+2R^2Q-2RQ^2-Q^3=5QR-Q^2R^2.$$

We now replace $q^{1/5}$ by $-q^{1/5}$. Then R is replaced by $-P$, and Q remains unaffected. So,

$$Q^3-2PQ^2-2P^2Q+P^3=P^2Q^2+5PQ. \tag{1.3}$$

Next, set $P_1=1+P$ and $Q_1=1+Q$. After some elementary tedious algebra, we deduce, from (1.3), that

$$(P_1^2+2)(Q_1^2+2)-P_1^3-Q_1^3+P_1Q_1-4P_1-4Q_1=0.$$

By (1.1), this last equality may be rewritten in the form

$$\left(\frac{1}{u^2}+u^2\right)\left(\frac{1}{v^2}+v^2\right)-\left(\frac{1}{u}-u\right)^3-\left(\frac{1}{v}-v\right)^3+\left(\frac{1}{u}-u\right)\left(\frac{1}{v}-v\right)$$
$$-4\left(\frac{1}{u}-u\right)-4\left(\frac{1}{v}-v\right)=0,$$

which can be written in the shape

$$u^2v^3(u^2+v)(v^2+u)+(u^4v^3+u^3v^4)+(u^5v+uv^5-u^4v^2-u^2v^4)$$
$$-(u^3v^2+u^2v^3)+(u^2-v)(v^2-u)=0.$$

Upon factorization, we arrive at

$$(uv^2(u^2+v)+u^2-v)(u^2v(v^2+u)+v^2-u)=0.$$

From the definitions of u and v, it is clear that $u=O(q^{1/5})$ and $v=O(q^{2/5})$ as q tends to 0. Hence, the first factor (and not the second) vanishes for q sufficiently small. By the identity theorem, the first factor vanishes for $|q|<1$. This proves (i).

A proof of (i) was also given by L. J. Rogers [4, eq. (5.4)].

It will be convenient to prove (iii) before (ii). We shall alter Ramanujan's formulation of (iii) by *defining* n by uv^2. We shall then establish the two proffered formulas for U and V.

Proof of (iii). From part (i),

$$n=\frac{v-u^2}{v+u^2}.$$

Solving for u^2, we find that

$$u^2=\frac{v(1-n)}{1+n}.$$

But since $u^2v^4 = n^2$, we deduce that

$$v^5 = \frac{n^2(1+n)}{1-n},$$

which proves the formula for V. Now,

$$U = \frac{n^5}{V^2} = n\left(\frac{1-n}{1+n}\right)^2,$$

which completes the proof.

Proof of (ii). From part (iii), we obtain the two cubic equations in n,

$$n^3 + n^2 + Vn - V = 0 \tag{1.4}$$

and

$$n^3 - (U+2)n^2 - (2U-1)n - U = 0.$$

By subtraction,

$$(U+3)n^2 + (2U+V-1)n + (U-V) = 0. \tag{1.5}$$

We now obtain three equations, in addition to (1.4) and (1.5), by multiplying (1.4) by n and (1.5) by n and n^2. Thus, we obtain five homogeneous equations in the "unknowns," 1, n, n^2, n^3, and n^4. Since there exists a solution ($n = uv^2$), the determinant of the coefficients is equal to 0, i.e.,

$$0 = \begin{vmatrix} 1 & 1 & V & -V & 0 \\ 0 & 1 & 1 & V & -V \\ U+3 & 2U+V-1 & U-V & 0 & 0 \\ 0 & U+3 & 2U+V-1 & U-V & 0 \\ 0 & 0 & U+3 & 2U+V-1 & U-V \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & V & -V & 0 \\ 0 & 1 & 1 & V & -V \\ 0 & 0 & 4-5V-UV & 7V-V^2 & UV+V^2-4V \\ 0 & 0 & U+V-4 & U-UV-4V & UV+3V \\ 0 & 0 & U+3 & 2U+V-1 & U-V \end{vmatrix}$$

$$= 4(10U^2V^2 - 10U^2V + 10UV^2 + UV + U^2 + U^3V^2 + UV^3 - V).$$

This completes the proof of (ii).

Entry 2 (p. 321). *Let* $|q| < 1$. *If*

$$u := \frac{q^{1/5}}{1} \; - \; \frac{q}{1} \; + \; \frac{q^2}{1} \; - \; \frac{q^3}{1} \; + \cdots$$

and

$$v := \frac{q^{1/5}}{1} \; + \; \frac{q}{1} \; + \; \frac{q^2}{1} \; + \; \frac{q^3}{1} \; + \cdots,$$

then

$$uv(u-v)^4 - u^2v^2(u-v)^2 + 2u^3v^3 = (u-v)(1+u^5v^5). \tag{2.1}$$

Proof. Let

$$w := \frac{q^{2/5}}{1} \; + \; \frac{q^2}{1} \; + \; \frac{q^4}{1} \; + \; \frac{q^6}{1} \; + \cdots.$$

By a direct application of Entry 1(i), we find that

$$\frac{w - v^2}{w + v^2} = vw^2. \tag{2.2}$$

Replacing q by $-q$ in Entry 1(i), we also find that

$$\frac{w - u^2}{w + u^2} = -uw^2. \tag{2.3}$$

These two equalities yield, respectively,

$$vw^3 + v^3w^2 - w + v^2 = 0$$

and

$$uw^3 + u^3w^2 + w - u^2 = 0.$$

Multiply the first of the last two equalities by u^2 and the second by v^2 and then add the resulting two equalities. Second, multiply the first equality by u and the second by v and then subtract them. Upon cancelling the nonvanishing factor $u+v$ in each case, we find that, respectively,

$$uvw^2 + u^2v^2w - (u - v) = 0 \tag{2.4}$$

and

$$uv(u-v)w^2 + w - uv = 0. \tag{2.5}$$

Next, divide (2.2) by (2.3) and cancel the nonvanishing factor $u + v$ to obtain the quadratic equation

$$w^2 + (u-v)^2w - u^2v^2 = 0. \tag{2.6}$$

The three quadratic equations (2.4)–(2.6) may be written in the more succinct form

$$\begin{pmatrix} uv & u^2v^2 & v-u \\ uv(u-v) & 1 & -uv \\ 1 & (u-v)^2 & -u^2v^2 \end{pmatrix} \begin{pmatrix} w^2 \\ w \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The determinant of the 3×3 matrix on the left side must therefore equal 0, and this is equivalent to (2.1). This completes the proof.

Entry 3 (p. 321). *Let* $|q| < 1$. *If*

$$u := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots$$

and

$$v := \frac{q^{3/5}}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \, \frac{q^6}{1} \, \frac{}{+} \, \frac{q^9}{1} \, \frac{}{+} \cdots,$$

then

$$(v - u^3)(1 + uv^3) = 3u^2v^2.$$

Entry 3 was established by Rogers [4, p. 392, eq. (6.2)]. (In fact, (6.2) should be designated by (6.3), as apparently there is a misprint.)

Entry 4 (p. 326). *Let, for* $|q| < 1$,

$$u := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots = U^{1/5},$$

$$v := \frac{q^{3/5}}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \, \frac{q^6}{1} \, \frac{}{+} \, \frac{q^9}{1} \, \frac{}{+} \cdots = V^{1/5},$$

$$m := q^{2/5} \frac{f(-q^4, -q^{11})}{f(-q^7, -q^8)} v,$$

and

$$n := q^{2/5} \frac{f(-q, -q^{14})}{f(-q^2, -q^{13})} v.$$

Then

$$m - n = mn = \frac{m^2}{1+m} = \frac{n^2}{1-n} = uv^3.$$

Proof. Simple algebra shows that $mn = m^2/(1+m)$ if and only if $m - n = mn$ and that $mn = n^2/(1-n)$ if and only if $m - n = mn$.

By Entry 38(iii) of Chapter 16 (Part III [3, p. 79]), $mn = uv^3$ if and only if

$$\frac{f(-q^4, -q^{11}) f(-q, -q^{14}) f^2(-q^3, -q^{12})}{f(-q^7, -q^8) f(-q^2, -q^{13}) f^2(-q^6, -q^9)} = \frac{f(-q, -q^4) f^3(-q^3, -q^{12})}{f(-q^2, -q^3) f^3(-q^6, -q^9)},$$

or

$$\begin{aligned} &f(-q^4, -q^{11}) f(-q, -q^{14}) f(-q^2, -q^3) f(-q^6, -q^9) \\ &\quad = f(-q, -q^4) f(-q^3, -q^{12}) f(-q^7, -q^8) f(-q^2, -q^{13}). \end{aligned} \tag{4.1}$$

By applying the Jacobi triple product identity, Entry 19 of Chapter 16 (Part III [3, p. 35]), to each of the eight theta-functions in (4.1), we, indeed, verify that (4.1) is valid. Thus, we have shown that $mn = uv^3$.

It remains to prove that $m - n = mn$. Using Entry 38(iii) of Chapter 16 as before, we find that the proposed identity is equivalent to the identity

$$\frac{f(-q^4,-q^{11})}{f(-q^7,-q^8)}-\frac{f(-q,-q^{14})}{f(-q^2,-q^{13})}=q\frac{f(-q^4,-q^{11})f(-q,-q^{14})f(-q^3,-q^{12})}{f(-q^7,-q^8)f(-q^2,-q^{13})f(-q^6,-q^9)},$$

which is equivalent to

$$\frac{f(-q^2,-q^{13})}{f(-q,-q^{14})}-\frac{f(-q^7,-q^8)}{f(-q^4,-q^{11})}=\frac{f(-q^3,-q^{12})}{f(-q^6,-q^9)}. \tag{4.2}$$

To prove (4.2), we shall apply the quintuple product identity three times. Referring to Part III [3, p. 80, eq. (38.2)], we replace q by $q^{15/2}$ and set, in turn, $B = -q^{13/2}$, $-q^{7/2}$, and $-q^{3/2}$. Accordingly, we find that

$$f(-q^{27},-q^{18})-q^{13}f(-q^{-12},-q^{57})=f(-q^{15})\frac{f(-q^2,-q^{13})}{f(-q,-q^{14})},$$

$$f(-q^{27},-q^{18})-q^{7}f(-q^{-3},-q^{48})=f(-q^{15})\frac{f(-q^7,-q^{8})}{f(-q^4,-q^{11})},$$

and

$$f(-q^{12},-q^{33})-q^{3}f(-q^{3},-q^{42})=f(-q^{15})\frac{f(-q^3,-q^{12})}{f(-q^6,-q^{9})}.$$

Thus, (4.2) is equivalent to the identity

$$\begin{aligned} &-q^{13}f(-q^{-12},-q^{57})+q^7f(-q^{-3},-q^{48}) \\ &\quad = qf(-q^{12},-q^{33})-q^4f(-q^3,-q^{42}). \end{aligned} \tag{4.3}$$

However, from a basic property of theta-functions (Part III [3, p. 34, Entry 18(iv)]),

$$-q^{13}f(-q^{-12},-q^{57})=qf(-q^{33},-q^{12})$$

and

$$q^7f(-q^{-3},-q^{48})=-q^4f(-q^{42},-q^3).$$

Substituting these facts into (4.3), we see that (4.3) reduces to a tautology. This then completes the proof of the equality $m - n = mn$.

Entry 5 (p. 326). *If* $|q| < 1$,

$$u := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots$$

and

$$v := \frac{q^{4/5}}{1} \, \frac{}{+} \, \frac{q^4}{1} \, \frac{}{+} \, \frac{q^8}{1} \, \frac{}{+} \, \frac{q^{12}}{1} \, \frac{}{+} \cdots,$$

then

$$(u^5+v^5)(uv-1)+u^5v^5+uv=5u^2v^2(uv-1)^2.$$

Proof. Let

$$w:=\frac{q^{2/5}}{1}\,\frac{}{+}\,\frac{q^2}{1}\,\frac{}{+}\,\frac{q^4}{1}\,\frac{}{+}\,\frac{q^6}{1}\,\frac{}{+}\cdots.$$

From Entry 1(i), it follows that

$$uw^3+u^3w^2-w+u^2=0$$

and

$$v^2w^3+w^2+v^3w-v=0.$$

Eliminating w^3 from this pair of equations and then cancelling the nonvanishing factor $uv+1$, we find that

$$u(uv-1)w^2-v^2w+uv=0. \tag{5.1}$$

We now take this pair of cubic equations in w and eliminate the "constant" terms from the pair. After dividing out the nonvanishing factor $w(uv+1)$, we deduce that

$$uvw^2+u^2w+v(uv-1)=0. \tag{5.2}$$

Next, we take (5.1) and (5.2) and eliminate the "constant" terms. Second, we eliminate the terms quadratic in w from (5.1) and (5.2). We then obtain the pair of equations

$$\frac{w^2}{u^3v+v^3(uv-1)}=\frac{w}{uv(uv-1)^2-u^2v^2}=\frac{1}{-uv^3-u^3(uv-1)}.$$

We thus can derive two formulas for w^2, namely,

$$\frac{u^3v+v^3(uv-1)}{-uv^3-u^3(uv-1)}=w^2=\left(\frac{uv(uv-1)^2-u^2v^2}{uv^3+u^3(uv-1)}\right)^2.$$

Therefore,

$$(u^3+v^2(uv-1))(v^3+u^2(uv-1))+uv(u^2v^2-3uv+1)^2=0.$$

The desired result now follows after some elementary algebra.

Entry 6 (Formula (3), p. 289). *For* $|q|<1$, *let*

$$\varphi:=\frac{q}{1}\,\frac{}{+}\,\frac{q^5}{1}\,\frac{}{+}\,\frac{q^{10}}{1}\,\frac{}{+}\,\frac{q^{15}}{1}\,\frac{}{+}\cdots$$

and

$$f:=\frac{q^{1/5}}{1}\,\frac{}{+}\,\frac{q}{1}\,\frac{}{+}\,\frac{q^2}{1}\,\frac{}{+}\,\frac{q^3}{1}\,\frac{}{+}\cdots.$$

Then

$$f^5 = \varphi \frac{1 - 2\varphi + 4\varphi^2 - 3\varphi^3 + \varphi^4}{1 + 3\varphi + 4\varphi^2 + 2\varphi^3 + \varphi^4}.$$

This result was communicated by Ramanujan [10, p. xxvii] in his first letter to Hardy. The first proof of Entry 6 is due to Rogers [4, p. 392, eq. (7.1)]. A second proof has been given by Watson [1], and another proof has been found by Ramanathan [2]. Entry 6 is connected with modular equations of degree 5.

Rogers [4] also derived a modular equation relating $R(q)$ and $R(q^{11})$.

We now establish the four values for $R(q)$ stated on page 311 in Ramanujan's first notebook. Each of our proofs employs an eta-function identity from the unorganized portions of the second notebook [9] (Part IV [4, Chap. 25]). Ramanathan [3] gave a different proof of Entry 7, but proofs of Theorems 8, 9, and 10 were first given by the author and Chan [1]. In order to state the first four theorems, we set

$$2c := 1 + \frac{a+b}{a-b}\sqrt{5}, \tag{7.1}$$

where a and b are certain real numbers to be specified below.

Entry 7. *Let $a = 5^{1/4}$, $b = 1$, and c be given by (7.1). Then*

$$R(e^{-4\pi}) = \sqrt{c^2+1} - c. \tag{7.2}$$

Entry 8. *Let $a = 3 + \sqrt{2} - \sqrt{5}$, $b = (20)^{1/4}$, and c be given by (7.1). Then*

$$R(e^{-8\pi}) = \sqrt{c^2+1} - c. \tag{8.1}$$

Entry 9. *Let $a = 5^{1/4}(4 - \sqrt{2})$, $b = 1 + \sqrt{2} + \sqrt{5} - 2^{1/4}(3 - \sqrt{2} + \sqrt{5} - \sqrt{10})$, and c be given by (7.1). Then*

$$R(e^{-16\pi}) = \sqrt{c^2+1} - c. \tag{9.1}$$

Entry 10. *Let $a = (60)^{1/4}$, $b = 2 - \sqrt{3} + \sqrt{5}$, and c be given by (7.1). Then*

$$R(e^{-6\pi}) = \sqrt{c^2+1} - c. \tag{10.1}$$

Before proving Entries 7–10, we offer some needed notation and preliminary results. Recall that $f(-q)$ is defined in (0.9). We shall need two related transformation formulas for f (Part III [3, p. 43, Entry 27(iii), (iv)]). If $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$, then

$$e^{-\alpha/12}\alpha^{1/4} f(-e^{-2\alpha}) = e^{-\beta/12}\beta^{1/4} f(-e^{-2\beta}) \tag{7.3}$$

and

$$e^{-\alpha/24}\alpha^{1/4} f(e^{-\alpha}) = e^{-\beta/24}\beta^{1/4} f(e^{-\beta}). \tag{7.4}$$

Following Ramanujan [3], [10, p. 23], we define the two class invariants

$$G_n = 2^{-1/4}q^{-1/24}(-q;q^2)_\infty \quad \text{and} \quad g_n = 2^{-1/4}q^{-1/24}(q;q^2)_\infty, \tag{7.5}$$

where $n > 0$ and $q = e^{-\pi\sqrt{n}}$. At the beginning of Section 2 of Chapter 34, we establish two simple relations for these invariants,

$$g_{4n} = 2^{1/4}g_n G_n \tag{7.6}$$

and

$$(g_n G_n)^8(G_n^8 - g_n^8) = \tfrac{1}{4}. \tag{7.7}$$

First Proof of Entry 7. Recall that (Part III, [3, p. 84, eq. (39.1)])

$$\frac{1}{R(e^{-\alpha})} - R(e^{-\alpha}) - 1 = e^{-\alpha/5}\frac{f(-e^{-\alpha/5})}{f(-e^{-5\alpha})}, \tag{7.8}$$

where $\alpha > 0$ and f is defined by (0.9). After some elementary algebraic manipulation, we find that (7.2) is equivalent to the identity

$$\frac{1}{R(e^{-\alpha})} - R(e^{-\alpha}) = 2c, \tag{7.9}$$

with $\alpha = 4\pi$. Thus, from (7.8), (7.9), and (7.1), we must prove that

$$e^{4\pi/5}\frac{f(-e^{-4\pi/5})}{f(-e^{-20\pi})} = \frac{a+b}{a-b}\sqrt{5}, \tag{7.10}$$

where a and b are as stated in Entry 7.

We shall employ Entry 58 of Chapter 25 of Part IV [4, pp. 212–213]. Let

$$P = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \quad \text{and} \quad Q = \frac{f(-q^{2/5})}{q^{2/5}f(-q^{10})}.$$

Then

$$(PQ)^2 + 5PQ = P^3 - 2P^2Q - 2PQ^2 + Q^3. \tag{7.11}$$

Let $q = e^{-2\pi}$. Then, by (7.3),

$$P = \sqrt{5}. \tag{7.12}$$

Using (7.12) in (7.11), we find that

$$5Q^2 + 5\sqrt{5}Q = 5\sqrt{5} - 10Q - 2\sqrt{5}Q^2 + Q^3. \tag{7.13}$$

It will be convenient to set $Q = \sqrt{5}T$, so that (7.13) takes the form

$$\sqrt{5}T^2 + \sqrt{5}T = 1 - 2T - 2T^2 + T^3 = (T^2 - 3T + 1)(T + 1).$$

Since clearly $T \neq -1$,

$$\sqrt{5}T = T^2 - 3T + 1.$$

Solving this quadratic equation, we find that

$$T=\frac{3+\sqrt{5}\pm\sqrt{10+6\sqrt{5}}}{2}.$$

If we took the minus sign above, we would find that $Q=\sqrt{5}T<1$. But clearly $Q>1$, and so we deduce that

$$Q=\sqrt{5}\left(\frac{3+\sqrt{5}+\sqrt{10+6\sqrt{5}}}{2}\right). \tag{7.14}$$

By (7.10) and (7.14), it remains to show that

$$\frac{3+\sqrt{5}+\sqrt{10+6\sqrt{5}}}{2}=\frac{5^{1/4}+1}{5^{1/4}-1}. \tag{7.15}$$

However,

$$\begin{aligned}3+\sqrt{5}+\sqrt{10+6\sqrt{5}}&=3+\sqrt{5}+5^{1/4}\sqrt{2\sqrt{5}+6}\\&=3+\sqrt{5}+5^{1/4}(\sqrt{5}+1)\\&=\frac{(3+5^{1/4}+\sqrt{5}+5^{3/4})(5^{1/4}-1)}{5^{1/4}-1}\\&=\frac{2(5^{1/4}+1)}{5^{1/4}-1},\end{aligned}$$

and thus (7.15) has been shown to complete the proof.

Ramanathan [3] gave a more difficult proof of Entry 7 in which class invariants were employed. We have also discovered a proof of Entry 7 that utilizes class invariants. Since our proof is simpler than that of Ramanathan and much different from our proof above, we give it below. Like Ramanathan's proof, our proof requires the value of G_{25}, and so we give a simple derivation of this evaluation next.

Lemma 7.1.

$$G_{25}=\frac{1+\sqrt{5}}{2}.$$

Proof. We employ a modular equation of degree 5 found in Entry 13(xiv) of Chapter 19 of Ramanujan's second notebook (Part III [3, p. 282]). Let

$$P=2^{1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}$$

and

$$Q=\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8},$$

where β has degree 5 over α. Then

$$Q+\frac{1}{Q}+2\left(P-\frac{1}{P}\right)=0. \tag{7.16}$$

Recalling the definition of G_n in (7.5), recalling that (Part III [3, p. 37]) $\chi(q) := (-q;q^2)_\infty$, and using Entry 12(v) of Chapter 17 of Ramanujan's second notebook (Part III [3, p. 124]), we find that

$$G_{25}=2^{-1/4}e^{\pi/24}2^{1/6}e^{-\pi/24}\{\beta(1-\beta)\}^{-1/24}=2^{-1/12}\{\beta(1-\beta)\}^{-1/24}.$$

Since it is well known and easy to prove that $G_1 = 1$, we have, by the same reasoning as above,

$$G_1=2^{-1/12}\{\alpha(1-\alpha)\}^{-1/24}=1.$$

Hence, it follows that

$$P=\frac{1}{G_{25}^2} \quad \text{and} \quad Q=\frac{1}{G_{25}^3}.$$

Therefore, by (7.16), if $x = G_{25}$,

$$\begin{aligned}
\frac{1}{x^3}+x^3+2\left(\frac{1}{x^2}-x^2\right) &= \left(\frac{1}{x}+x\right)^3-3\left(\frac{1}{x}+x\right)+2\left(\frac{1}{x}+x\right)\left(\frac{1}{x}-x\right)\\
&= \left(\frac{1}{x}+x\right)\left\{\left(\frac{1}{x}-x\right)^2+2\left(\frac{1}{x}-x\right)+1\right\}\\
&= \left(\frac{1}{x}+x\right)\left\{\left(\frac{1}{x}-x\right)+1\right\}^2\\
&= 0.
\end{aligned}$$

Since $x + 1/x \neq 0$ and $x > 0$, we conclude that

$$G_{25}=x=\frac{1+\sqrt{5}}{2},$$

and so the proof is complete.

We remark that the value of G_{25} is given without proof in Ramanujan's paper [3], [10, p. 26].

Second Proof of Entry 7. Recall from our first proof that it suffices to prove (7.10).

Set $\alpha = 2\pi/5$ in (7.3), so that $\beta = 5\pi/2$. After some simplification, we find that

$$f(-e^{-4\pi/5})=\sqrt{\tfrac{5}{2}}\,e^{-7\pi/40}f(-e^{-5\pi}).$$

Thus,

$$Q = e^{4\pi/5} \frac{f(-e^{-4\pi/5})}{f(-e^{-20\pi})} = \sqrt{\tfrac{5}{2}} e^{5\pi/8} \frac{f(-e^{-5\pi})}{f(-e^{-20\pi})}. \tag{7.17}$$

Since

$$\frac{(q;q)_\infty}{(q^4;q^4)_\infty} = (q;q^2)_\infty (q^2;q^4)_\infty,$$

we deduce from (7.17) that

$$Q = \sqrt{5} g_{25} g_{100}, \tag{7.18}$$

where g_n is defined in (7.5). We thus must determine g_{25} and g_{100}. Since G_{25} was computed in Lemma 7.1, we see from (7.6) and (7.7) that Q can be calculated.

For brevity, set $x = g_{25}^8$ and $a = G_{25}^8$. Thus, from (7.7),

$$ax^2 - a^2 x + \tfrac{1}{4} = 0, \tag{7.19}$$

Since $x > 0$, from (7.19), we deduce that

$$\begin{aligned} x &= \frac{a^2 + \sqrt{a^4 - a}}{2a} \\ &= \frac{1}{2}\left(G_{25}^8 + G_{25}^2 \sqrt{G_{25}^{12} - G_{25}^{-12}} \right) \\ &= \frac{1}{4} G_{25}^2 \left(\sqrt{G_{25}^6 + G_{25}^{-6}} + \sqrt{G_{25}^6 - G_{25}^{-6}} \right)^2 . \end{aligned} \tag{7.20}$$

By Lemma 7.1, $G_{25}^6 = (2+\sqrt{5})^2 = 9 + 4\sqrt{5}$, and so $G_{25}^{-6} = 9 - 4\sqrt{5}$. Hence, by (7.20) and Lemma 7.1,

$$\begin{aligned} x = g_{25}^8 &= \frac{1}{4}\left(\frac{1+\sqrt{5}}{2}\right)^2 \left(\sqrt{18} + \sqrt{8\sqrt{5}}\right)^2 \\ &= \frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^2 (3 + 2\cdot 5^{1/4})^2. \end{aligned} \tag{7.21}$$

Thus, from (7.18), (7.21), and Lemma 7.1,

$$\begin{aligned} Q &= \sqrt{5}\, 2^{1/4} g_{25}^2 G_{25} \\ &= \sqrt{5}\left(\frac{1+\sqrt{5}}{2}\right)^{3/2} (3 + 2\cdot 5^{1/4})^{1/2} \\ &= \sqrt{5}\frac{\sqrt{2}}{\sqrt{5}-1}(1+\sqrt{5})^{1/2}(3+2\cdot 5^{1/4})^{1/2} \\ &= \sqrt{5}\frac{\{(1+5^{1/4})^4\}^{1/2}}{\sqrt{5}-1} \\ &= \sqrt{5}\frac{5^{1/4}+1}{5^{1/4}-1}. \end{aligned}$$

Thus, (7.10) is established, and the proof is complete.

Proof of Entry 8. We will again employ Entry 58 of Chapter 25 in Part IV [4, p. 212–213], but now with

$$P = e^{4\pi/5}\frac{f(-e^{-4\pi/5})}{f(-e^{-20\pi})} \quad \text{and} \quad Q = e^{8\pi/5}\frac{f(-e^{-8\pi/5})}{f(-e^{-40\pi})}.$$

By the same reasoning as that used in the first proof of (7.2), in order to prove (8.1), it suffices to prove that

$$Q = \frac{a+b}{a-b}\sqrt{5}, \tag{8.2}$$

where a and b are given in the statement of Entry 8.

Write (7.11) in the form

$$PQ + 5 = \frac{P^2}{Q} - 2P - 2Q + \frac{Q^2}{P}. \tag{8.3}$$

From Entry 7, $P = (5^{1/4}+1)\sqrt{5}/(5^{1/4}-1)$. Putting this in (8.3) and setting $Q = \sqrt{5}T$, we find that

$$\frac{5^{1/4}+1}{5^{1/4}-1}\sqrt{5}T + \sqrt{5} = \left(\frac{5^{1/4}+1}{5^{1/4}-1}\right)^2 \frac{1}{T} - 2\frac{5^{1/4}+1}{5^{1/4}-1} - 2T + \frac{5^{1/4}-1}{5^{1/4}+1}T^2. \tag{8.4}$$

By an elementary verification, it is easily checked that $T = 1$ is a root of (8.4). Since clearly $Q > \sqrt{5}$, this root is not the one that we seek. Writing (8.4) in the form $a_3T^3 + a_2T^2 + a_1T + a_0 = 0$, and dividing by $T-1$, we find that

$$(5^{1/4}-1)^2T^2 - 2(1+5^{1/4}+\sqrt{5}+5^{3/4})T - (9+6\cdot5^{1/4}+3\sqrt{5}+2\cdot5^{3/4}) = 0. \tag{8.5}$$

Now set

$$T = \frac{a+b}{a-b}$$

in (8.5) to deduce, with the help of *Mathematica,* that

$$-(5+3\sqrt{5})a^2 + (6\cdot5^{1/4} + 2\cdot5^{3/4})ab + (3-3\sqrt{5})b^2 = 0.$$

Solving for a, we find that

$$a = \frac{5^{1/4}b \pm \sqrt{(7\sqrt{5}-15)b^2}}{\sqrt{5}}.$$

We now set $b = 5^{1/4}\sqrt{2}$ and choose the plus sign above, because if we had chosen the minus sign, we would find that $T < 0$, which is impossible. Hence,

$$\begin{aligned} a &= \sqrt{2} + \frac{\sqrt{70-30\sqrt{5}}}{\sqrt{5}} \\ &= \sqrt{2} + \sqrt{14-6\sqrt{5}} = \sqrt{2} + \sqrt{(3-\sqrt{5})^2} = \sqrt{2} + 3 - \sqrt{5}. \end{aligned}$$

Hence,

$$Q = T\sqrt{5} = \frac{a+b}{a-b}\sqrt{5} = \frac{\sqrt{2}+3-\sqrt{5}+5^{1/4}\sqrt{2}}{\sqrt{2}+3-\sqrt{5}-5^{1/4}\sqrt{2}}\sqrt{5},$$

and so (8.2) has been shown to complete the proof.

Proof of Entry 9. We again employ Entry 58 of Chapter 25 in Part IV, but now we set

$$P = e^{8\pi/5}\frac{f(-e^{-8\pi/5})}{f(-e^{-40\pi})} \quad \text{and} \quad Q = e^{16\pi/5}\frac{f(-e^{-16\pi/5})}{f(-e^{-80\pi})}.$$

By the same argument that we used in the proofs of Entries 7 and 8, to prove (9.1), it suffices to prove that

$$Q = \frac{a+b}{a-b}\sqrt{5}, \tag{9.2}$$

where a and b are prescribed in the statement of Entry 9.

Set $A = 3+\sqrt{2}-\sqrt{5}$ and $B = (20)^{1/4}$. As in the last proof, let $Q = \sqrt{5}T$. Thus, by Entry 8 and (8.3), we know that

$$\frac{A+B}{A-B}\sqrt{5}T + \sqrt{5} = \left(\frac{A+B}{A-B}\right)^2\frac{1}{T} - 2\frac{A+B}{A-B} - 2T + \frac{A-B}{A+B}T^2. \tag{9.3}$$

Let

$$T = \frac{a+b}{a-b}$$

in (9.3). Clearing fractions and simplifying with the help of *Mathematica*, we find that

$$\begin{aligned}
&(-10-7\sqrt{2}+4\sqrt{5}+2\sqrt{10})a^3 + 5^{1/4}(8+9\sqrt{2}-2\sqrt{5}-2\sqrt{10})a^2b \\
&\quad + (-20-15\sqrt{2}+6\sqrt{5}+4\sqrt{10})ab^2 \\
&\quad + 5^{1/4}(10+15\sqrt{2}-4\sqrt{5}-6\sqrt{10})b^3 = 0.
\end{aligned}$$

Let $a = 5^{1/4}d$, cancel $5^{3/4}\sqrt{2}$, and simplify to deduce that

$$\begin{aligned}
&(-7-5\sqrt{2}+2\sqrt{5}+2\sqrt{10})d^3 + (9+4\sqrt{2}-2\sqrt{5}-\sqrt{10})bd^2 \\
&\quad + (4+3\sqrt{2}-3\sqrt{5}-2\sqrt{10})b^2d \\
&\quad + (-6-2\sqrt{2}+3\sqrt{5}+\sqrt{10})b^3 = 0.
\end{aligned} \tag{9.4}$$

Observe that $d = b$ is a root of (9.4). If this were the root that we are seeking, then Q would equal $(5^{1/4}+1)\sqrt{5}/(5^{1/4}-1)$. Thus, with P and Q interchanged, we have the same solutions to (8.3) that we had in the proof of Entry 8. Clearly, this is not the solution that we want. Hence, dividing (9.4) by $(d-b)$, we find that

$$\begin{aligned}
&(4+6\sqrt{2}-2\sqrt{5}-3\sqrt{10})b^2 + 2(-1+\sqrt{2}+\sqrt{5})bd \\
&\quad + (-10-7\sqrt{2}+4\sqrt{5}+2\sqrt{10})d^2 = 0.
\end{aligned}$$

Solving for b, we find that

$$b = \frac{2(1-\sqrt{2}-\sqrt{5})d \pm 2\sqrt{2}\sqrt{(116+83\sqrt{2}-52\sqrt{5}-37\sqrt{10})d^2}}{2(4+6\sqrt{2}-2\sqrt{5}-3\sqrt{10})}. \tag{9.5}$$

Since

$$\begin{aligned}\frac{2(1-\sqrt{2}-\sqrt{5})}{2(4+6\sqrt{2}-2\sqrt{5}-3\sqrt{10})} &= \frac{(1-\sqrt{2}-\sqrt{5})}{(2+3\sqrt{2})(2-\sqrt{5})}\\ &= \frac{(6+5\sqrt{2}+4\sqrt{5}+\sqrt{10})}{14}\\ &= \frac{(1+\sqrt{2}+\sqrt{5})(4+\sqrt{2})}{14},\end{aligned}$$

we are motivated to set $d = 4-\sqrt{2}$. Therefore, $a = 5^{1/4}(4-\sqrt{2})$ in agreement with what Ramanujan claimed. Thus, by (9.5),

$$b = \frac{6-5\sqrt{2}-4\sqrt{5}+\sqrt{10} \pm 2^{5/4}\sqrt{283+190\sqrt{2}-125\sqrt{5}-86\sqrt{10}}}{4+6\sqrt{2}-2\sqrt{5}-3\sqrt{10}}. \tag{9.6}$$

Observe that

$$\frac{6-5\sqrt{2}-4\sqrt{5}+\sqrt{10}}{4+6\sqrt{2}-2\sqrt{5}-3\sqrt{10}} = 1+\sqrt{2}+\sqrt{5}. \tag{9.7}$$

We next wish to write

$$4(283+190\sqrt{2}-125\sqrt{5}-86\sqrt{10}) = (w+x\sqrt{2}+y\sqrt{5}+z\sqrt{10})^2,$$

for certain rational integers w, x, y, and z. Thus,

$$\begin{aligned} w^2+2x^2+5y^2+10z^2 &= 1132, \\ wx+5yz &= 380, \\ wy+2xz &= -250, \\ wz+xy &= -172. \end{aligned} \tag{9.8}$$

David Bradley kindly wrote a program to determine the 24 positive solutions to (9.8). We then found the unique solution of the system of four diophantine equations to be

$$w=20, \quad x=9, \quad y=-8, \quad \text{and} \quad z=-5.$$

Thus,

$$\begin{aligned}\frac{2\sqrt{283+190\sqrt{2}-125\sqrt{5}-86\sqrt{10}}}{4+6\sqrt{2}-2\sqrt{5}-3\sqrt{10}} &= \frac{20+9\sqrt{2}-8\sqrt{5}-5\sqrt{10}}{4+6\sqrt{2}-2\sqrt{5}-3\sqrt{10}}\\ &= 3-\sqrt{2}+\sqrt{5}-\sqrt{10}.\end{aligned} \tag{9.9}$$

Putting (9.7) and (9.9) in (9.6), we find that

$$b = 1 + \sqrt{2} + \sqrt{5} \pm 2^{1/4}(3 - \sqrt{2} + \sqrt{5} - \sqrt{10}).$$

If we choose the plus sign above, we would find that $a - b < 0$ and $T < 0$, which is impossible. Thus, we conclude that

$$b = 1 + \sqrt{2} + \sqrt{5} - 2^{1/4}(3 - \sqrt{2} + \sqrt{5} - \sqrt{10}),$$

which is what Ramanujan asserted. Thus, (9.2) is proved, and the proof of Entry 9 is complete.

Our proof of Entry 9 heavily relies on computation in the later stages. Although Ramanujan possibly used Entry 58 of Chapter 25, he undoubtedly found a less computational proof.

Proof of Entry 10. By the same reasoning in the proofs of Entries 7–9, to prove (10.1), it suffices to prove that

$$e^{6\pi/5}\frac{f(-e^{-6\pi/5})}{f(-e^{-30\pi})} = \frac{a+b}{a-b}\sqrt{5}, \tag{10.2}$$

where a and b are specified in the statement of Entry 10.

Apply the transformation formula (7.3) with $\alpha = 3\pi/5$ and $\beta = 5\pi/3$. After some simplification,

$$f(-e^{-6\pi/5}) = \sqrt{\tfrac{5}{3}}e^{-4\pi/45}f(-e^{-10\pi/3}).$$

Thus,

$$e^{6\pi/5}\frac{f(-e^{-6\pi/5})}{f(-e^{-30\pi})} = \sqrt{\tfrac{5}{3}}e^{10\pi/9}\frac{f(-e^{-10\pi/3})}{f(-e^{-30\pi})} =: \sqrt{\tfrac{5}{3}}A. \tag{10.3}$$

Because $30 = 9 \cdot \frac{10}{3}$, we are led to Ramanujan's cubic continued fraction

$$G(q) := \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \cdots, \qquad |q| < 1.$$

From Part III [3, p. 345, Entry 1(i)],

$$G(-q) = -q^{1/3}\frac{\chi(q)}{\chi^3(q^3)},$$

where

$$\chi(q) = (-q;q^2)_\infty.$$

In particular,

$$G(-e^{-5\pi}) = -e^{-5\pi/3}\frac{\chi(e^{-5\pi})}{\chi^3(e^{-15\pi})} = -\frac{G_{25}}{\sqrt{2}G_{225}^3}, \tag{10.4}$$

by (7.5). Recalling the value $G_{25} = (1+\sqrt{5})/2$ from Lemma 7.1 and the value

$$G_{225} = \tfrac{1}{4}(1+\sqrt{5})(2+\sqrt{3})^{1/3}\left\{\sqrt{4+\sqrt{15}} + (15)^{1/4}\right\}$$

from Ramanujan's paper [3], [10, p. 28], or from the table of Section 2 of Chapter 34, but apparently first proved in print by Watson [7], we find that, from (10.4),

$$\begin{aligned} G(-e^{-5\pi}) &= -\frac{16\sqrt{2}}{(1+\sqrt{5})^2(2+\sqrt{3})\left\{\sqrt{4+\sqrt{15}}+(15)^{1/4}\right\}} \\ &= -\frac{2\sqrt{2}(2-\sqrt{3})(3-\sqrt{5})}{(4+\sqrt{15})^{3/2}+3(4+\sqrt{15})(15)^{1/4}+3\sqrt{4+\sqrt{15}}\sqrt{15}+(15)^{3/4}} \\ &= -\frac{(2-\sqrt{3})(3-\sqrt{5})}{\sqrt{2}\left(\sqrt{4+\sqrt{15}}(1+\sqrt{15})+(15)^{1/4}(3+\sqrt{15})\right)}. \end{aligned}$$

Now

$$\begin{aligned} \sqrt{4+\sqrt{15}}(1+\sqrt{15}) &= \sqrt{4+\sqrt{15}(1+\sqrt{15})^2} \\ &= \sqrt{94+24\sqrt{15}} = \sqrt{2}\sqrt{\tfrac{1}{4}(6\sqrt{3}+4\sqrt{5})^2} \\ &= \frac{1}{\sqrt{2}}(6\sqrt{3}+4\sqrt{5}). \end{aligned}$$

Thus,

$$\begin{aligned} G(-e^{-5\pi}) &= -\frac{(2-\sqrt{3})(3-\sqrt{5})}{6\sqrt{3}+4\sqrt{5}+(60)^{1/4}(3+\sqrt{5})} \\ &= -\frac{(2-\sqrt{3})(3-\sqrt{5})(6\sqrt{3}+4\sqrt{5}-3(60)^{1/4}-\sqrt{15}(60)^{1/4})}{8}. \end{aligned} \tag{10.5}$$

Now Chan [1, Theorem 1] has shown that $G(q)$ satisfies the modular equation

$$G^2(q) + 2G^2(q^2)G(q) - G(q^2) = 0.$$

Replacing q by $-q$ and solving for $G(q^2)$, we find that

$$G(q^2) = \frac{1-\sqrt{1-8G^3(-q)}}{4G(-q)}. \tag{10.6}$$

Set $q = e^{-5\pi}$, as above, and $v = G(e^{-10\pi})$. Recall the definition of A from (10.3). Then Entry 1(iv) of Chapter 20 in Ramanujan's second notebook (Part III [3, p. 345]) can be written in the form

$$3 + A^3 = \frac{1}{v} + 4v^2. \tag{10.7}$$

Thus, by (10.7) and (10.6), with $w := G(-e^{-5\pi})$ given by (10.5),

$$A^3 = 4\left(\frac{1-\sqrt{1-8w^3}}{4w}\right)^2 + \frac{4w}{1-\sqrt{1-8w^3}} - 3 = -\frac{(w+1)^2(2w-1)}{w^2}, \tag{10.8}$$

by a somewhat lengthy, but straightforward, calculation.

Hence, by (10.2), (10.3), and (10.8), it remains to show that

$$\left(\frac{(w+1)^2(1-2w)}{w^2}\right)^{1/3} = \frac{a+b}{a-b}\sqrt{3}, \tag{10.9}$$

where a and b are specified in the statement of Entry 10. We used *Mathematica* to verify (10.9) and complete the proof.

Another proof of Entry 10 has been given by the author, Chan, and Zhang in [3].

The next entry is actually recorded twice by Ramanujan in his notebooks. We quote Ramanujan.

Entry 11 (pp. 374, 382). *If* $q > 1$,

$$\frac{1}{1}+\frac{q}{1}+\frac{q^2}{1}+\frac{q^3}{1}+\cdots \tag{11.1}$$

oscillates between

$$1-\frac{q^{-1}}{1}+\frac{q^{-2}}{1}-\frac{q^{-3}}{1}+\cdots \tag{11.2}$$

and

$$\frac{q^{-1}}{1}+\frac{q^{-4}}{1}+\frac{q^{-8}}{1}+\frac{q^{-12}}{1}+\cdots. \tag{11.3}$$

From the general theory of continued fractions, if all the elements of a divergent continued fraction are positive, then the even and odd approximants approach distinct limits. Thus, since (11.1) diverges for $q > 1$, Ramanujan is indicating that its odd approximants tend to (11.2) while its even approximants approach (11.3).

In fact, we shall prove Entry 11 for $|q| > 1$.

First Proof. Recall the definition of the Gaussian binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}},$$

where $|q| < 1$ and n and k are integers with $0 \le k \le n$. Define

$$c_n(a;q) = \frac{1}{1}+\frac{aq}{1}+\frac{aq^2}{1}+\cdots+\frac{aq^n}{1}.$$

Then by a result of Ramanujan (Part III [3, p. 31, Entry 16]),

$$c_n(a;q) = \frac{D_n(aq;q)}{D_{n+1}(a;q)}, \tag{11.4}$$

where

$$D_n(a;q) = \sum_{0\le 2j\le n} \begin{bmatrix} n-j \\ j \end{bmatrix} a^j q^{j^2}.$$

We will need two further results of Ramanujan (Part III [3, p. 77, Entries 38(i), (ii)]), viz.,

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \tag{11.5}$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{11.6}$$

These are the famous Rogers–Ramanujan identities first established by Rogers [1]. Lastly, we require the following identities due to Rogers [1] and Watson [11], [14] independently:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}} = \frac{1}{(q;q^2)_\infty} G(q^4), \tag{11.7}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n+1}} = \frac{1}{(q;q^2)_\infty} H(-q), \tag{11.8}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n}} = \frac{1}{(q;q^2)_\infty} G(-q), \tag{11.9}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q;q)_{2n+1}} = \frac{1}{(q;q^2)_\infty} H(q^4). \tag{11.10}$$

It will be convenient to replace q by $1/q$ in Entry 11. Letting $c(q)$ denote the infinite continued fraction $\lim_{n\to\infty} c_n(1;q)$, we may rephrase Entry 11 in the following way. For $0 < q < 1$,

$$\text{the odd approximants of } c(q^{-1}) \text{ tend to } 1/c(-q), \tag{11.11}$$

while

$$\text{the even approximants of } c(q^{-1}) \text{ tend to } qc(q^4). \tag{11.12}$$

We first examine the odd approximants. Using (11.4), replacing j by $n-j$, and utilizing the fact

$$\begin{bmatrix} A \\ B \end{bmatrix}_{q^{-1}} = q^{-B(A-B)} \begin{bmatrix} A \\ B \end{bmatrix}_q, \tag{11.13}$$

we find that

$$c_{2n}(1;q^{-1}) = \frac{D_{2n}(q^{-1};q^{-1})}{D_{2n+1}(1;q^{-1})} = \frac{\sum_{j=0}^{n} \begin{bmatrix} 2n-j \\ j \end{bmatrix}_{q^{-1}} q^{-j^2-j}}{\sum_{j=0}^{n} \begin{bmatrix} 2n+1-j \\ j \end{bmatrix}_{q^{-1}} q^{-j^2}}$$

$$= \frac{\sum_{j=0}^{n} \begin{bmatrix} n+j \\ 2j \end{bmatrix}_{q^{-1}} q^{-(n-j)^2-(n-j)}}{\sum_{j=0}^{n} \begin{bmatrix} n+j+1 \\ 2j+1 \end{bmatrix}_{q^{-1}} q^{-(n-j)^2}} = \frac{\sum_{j=0}^{n} \begin{bmatrix} n+j \\ 2j \end{bmatrix} q^{j^2+j}}{\sum_{j=0}^{n} \begin{bmatrix} n+j+1 \\ 2j+1 \end{bmatrix} q^{j^2+j}}.$$

Hence, by (11.8) and (11.9),

$$\lim_{n\to\infty} c_{2n}(1;q^{-1}) = \frac{\sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(q;q)_{2j}}}{\sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(q;q)_{2j+1}}} = \frac{G(-q)}{H(-q)} = \frac{1}{c(-q)},$$

where in the last step we employed (11.5), (11.6), and the Rogers–Ramanujan continued fraction (Part III [3, p. 79, Entry 38(iii)]). This establishes (11.11).

We next examine the even approximants. Employing (11.4), reversing the order of summation in both the numerator and denominator, and applying (11.13), we find that

$$c_{2n-1}(1;q^{-1}) = \frac{D_{2n-1}(q^{-1};q^{-1})}{D_{2n}(1;q^{-1})} = \frac{\sum_{j=0}^{n-1} \begin{bmatrix} 2n-1-j \\ j \end{bmatrix}_{q^{-1}} q^{-j^2-j}}{\sum_{j=0}^{n} \begin{bmatrix} 2n-j \\ j \end{bmatrix}_{q^{-1}} q^{-j^2}}$$

$$= \frac{\sum_{j=0}^{n-1} \begin{bmatrix} n+j \\ 2j+1 \end{bmatrix}_{q^{-1}} q^{-(n-1-j)^2-(n-1-j)}}{\sum_{j=0}^{n} \begin{bmatrix} n+j \\ 2j \end{bmatrix}_{q^{-1}} q^{-(n-j)^2}} = q\frac{\sum_{j=0}^{n-1} \begin{bmatrix} n+j \\ 2j+1 \end{bmatrix} q^{j^2+2j}}{\sum_{j=0}^{n} \begin{bmatrix} n+j \\ 2j \end{bmatrix} q^{j^2}}.$$

Thus, by (11.7) and (11.10),

$$\lim_{n\to\infty} c_{2n-1}(q^{-1}) = q\frac{\sum_{j=0}^{\infty} \frac{q^{j^2+2j}}{(q;q)_{2j+1}}}{\sum_{j=0}^{\infty} \frac{q^{j^2}}{(q;q)_{2j}}} = q\frac{H(q^4)}{G(q^4)} = qc(q^4),$$

where we used (11.5), (11.6), and the Rogers–Ramanujan continued fraction. Thus, (11.12) is proved.

Entry 11 is a truly amazing result. It is very remarkable that the Rogers–Ramanujan continued fraction reappears in determining the limits of both the even and odd approximants of the divergent Rogers–Ramanujan continued fraction. It also should be noted that the continued fractions (11.2) and (11.3) are "near" each other in the sense of Entry 13 below. More precisely, if we set $x = 1$ in (13.1) and (13.2), we obtain (11.2) and (11.3), respectively, but with q replaced by $1/q$.

We will now give a second proof of Entry 11. This proof shows that Entry 11 arises from infinitely many Bauer–Muir transformations.

Second Proof. The even part of (11.1) is given by (see (64.1))

$$\begin{aligned}&\frac{1}{1+q}-\frac{q^3}{1+q^2+q^3}-\frac{q^7}{1+q^4+q^5}-\frac{q^{11}}{1+q^6+q^7}-\cdots\\&=\frac{q^{-1}}{1+q^{-1}}-\frac{q^{-1}}{1+q^{-1}+q^{-3}}-\frac{q^{-1}}{1+q^{-1}+q^{-5}}-\frac{q^{-1}}{1+q^{-1}+q^{-7}}-\cdots.\end{aligned} \tag{11.14}$$

The latter continued fraction is a limit 1–periodic continued fraction for $|q| > 1$. Since the linear fractional transformation

$$t(w) = -\frac{q^{-1}}{1+q^{-1}+w}$$

has the two fixed points $-q^{-1}$ and -1, it follows that (11.14) converges for $|q| > 1$. Moreover, the modified approximants

$$S_n(w_n) = \frac{q^{-1}}{1+q^{-1}}-\frac{q^{-1}}{1+q^{-1}+q^{-3}}-\cdots-\frac{q^{-1}}{1+q^{-1}q^{-2n+1}+w_n}$$

also converge to the same value if $\{w_n\}$ does not have a limit point at -1. (For instance, see Jacobsen's paper [1].) Applying the Bauer–Muir transformation (13.7) to the second continued fraction in (11.14), with $w_0 = 0$ and $w_n = -1/q$, $n \geq 1$, we obtain the continued fraction

$$\frac{q^{-1}}{1}+\frac{q^{-4}}{1+q^{-3}}-\frac{q^{-3}}{1+q^{-3}+q^{-5}}-\frac{q^{-3}}{1+q^{-3}+q^{-7}}-\cdots, \tag{11.15}$$

which converges to the same value for $|q| > 1$. Again, this is a limit periodic continued fraction. The attractive fixed point of

$$t_1(w) = -\frac{q^{-3}}{1+q^{-3}+w}$$

is $-q^{-3}$. Hence, with $w_0 = w_1 = 0$ and $w_n = -q^{-3}$ for $n \geq 2$, the Bauer–Muir transformation of (11.15) is given by

$$\frac{q^{-1}}{1} + \frac{q^{-4}}{1} + \frac{q^{-8}}{q+q^{-5}} - \frac{q^{-5}}{1+q^{-5}+q^{-7}} - \frac{q^{-5}}{1+q^{-5}+q^{-9}} - \cdots,$$

which converges to the same value. Repeating this process k times with

$$w_0 = w_1 = \cdots = w_{k-1} = 0 \quad \text{and} \quad w_n = -q^{-2k+1}, \quad n \geq k, \tag{11.16}$$

we obtain the limit periodic continued fraction

$$\begin{aligned} &\frac{q^{-1}}{1} + \frac{q^{-4}}{1} + \frac{q^{-8}}{1} + \cdots + \frac{q^{-4k}}{1+q^{-2k-1}} - \frac{q^{-2k-1}}{1+q^{-2k-1}+q^{-2k-3}} \\ &\quad - \frac{q^{-2k-1}}{1+q^{-2k-1}+q^{-2k-5}} - \cdots, \end{aligned} \tag{11.17}$$

which converges to the same value as (11.14). Repeating this process infinitely many times, we obtain (11.3). Since (11.17) converges uniformly with respect to k by the uniform parabola theorem (Jones and Thron [1, p. 99]), and since (11.3) converges, we conclude that (11.3) converges to the same value as (11.14). We have thus proved that the even part of (11.1) converges to the value of (11.3) for $|q| > 1$.

To prove that the odd part of (11.1) converges to the value of (11.2) for $|q| > 1$, we can use the same idea, and even the same choices (11.16) for w_n. We then find that the odd part (Jones and Thron [1, p. 43, eq. (2.4.29)], where the first minus sign is misplaced) of (11.1) equals

$$\begin{aligned} &1 - \frac{q}{1+q+q^2} - \frac{q^5}{1+q^3+q^4} - \frac{q^9}{1+q^5+q^6} - \cdots \\ &= 1 - \frac{q^{-1}}{1+q^{-1}+q^{-2}} - \frac{q^{-1}}{1+q^{-1}+q^{-4}} - \frac{q^{-1}}{1+q^{-1}+q^{-6}} - \cdots \\ &= 1 - \frac{q^{-1}}{1+q^{-2}} + \frac{q^{-5}}{1+q^{-4}} - \frac{q^{-3}}{1+q^{-3}+q^{-6}} - \frac{q^{-3}}{1+q^{-3}+q^{-8}} - \cdots \\ &= 1 - \frac{q^{-1}}{1+q^{-2}} + \frac{q^{-5}}{1-q^{-3}+q^{-4}} + \frac{q^{-9}}{1+q^{-6}} - \frac{q^{-5}}{1+q^{-5}+q^{-8}} \\ &\quad - \frac{q^{-5}}{1+q^{-5}+q^{-10}} - \cdots \\ &= \cdots \\ &= 1 - \frac{q^{-1}}{1+q^{-2}} + \frac{q^{-5}}{1-q^{-3}+q^{-4}} + \frac{q^{-9}}{1-q^{-5}+q^{-6}} + \frac{q^{-13}}{1-q^{-7}+q^{-8}} + \cdots. \end{aligned}$$

The last continued fraction is the even part of (11.2). Since (11.2) converges for $|q| > 1$, the second proof of Entry 11 is complete by the same argument as above.

K. Alladi [1] has given another proof of Entry 11 that is similar to our first proof. However, he related his proof to the continued fraction

$$r(q) := q + \frac{1}{q^3} + \frac{1}{q^5} + \frac{1}{q^7} + \cdots.$$

To be more precise, let $P_n(q)$ and $Q_n(q)$ denote the nth numerator and denominator, respectively, of the nth convergent of $r(q)$. Let $T(q) := q^{-1/5}R(q)$. Then Alladi [1, pp. 225–229] proved that, for $|q| < 1$,

$$\lim_{n\to\infty} \frac{P_{2n-1}(q)}{Q_{2n-1}(q)} = \frac{G(q^{16})}{q^3 H(q^{16})} = \frac{T(q^{16})}{q^3}$$

and

$$\lim_{n\to\infty} \frac{P_{2n}(q)}{Q_{2n}(q)} = \frac{qH(-q^4)}{G(-q^4)} = \frac{q}{T(-q^4)}.$$

Moreover, in the sense of modified convergence, Alladi proved that $r(q)$ tends to $T(q)$, i.e.,

$$\lim_{n\to\infty} 1 + \frac{1}{q} + \frac{1}{q^3} + \frac{1}{q^5} + \cdots + \frac{1}{q^{2n-1}} + \frac{1}{q^{2n+1}+1} = T(q).$$

For the definition, importance, and historical background of modified convergence, see Jacobsen's paper [2].

For the last entry on continued fractions found in the third notebook, we quote Ramanujan.

Entry 12 (p. 383). *If*

$$u := \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots,$$

then $u^2 + u - 1 = 0$ when $q^n = 1$, where n is any positive integer except multiples of 5 in which case u is not definite.

This statement is not quite correct. However, I. Schur [1], [2, pp. 117–136] has established the following theorem.

Theorem 12.1. *Let*

$$K(q) = 1 + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots,$$

where q is a primitive nth root of unity. If n is a multiple of 5, $K(q)$ diverges. When n is not a multiple of 5, let $\lambda = \left(\frac{n}{5}\right)$, the Legendre symbol. Furthermore, let ρ denote the least positive residue of n modulo 5. Then, for $n \not\equiv 0 \pmod 5$,

$$K(q) = \lambda q^{(1-\lambda\rho n)/5} K(\lambda).$$

Note that it is elementary that $K(1) = (\sqrt{5}+1)/2$ and $K(-1) = (\sqrt{5}-1)/2$. We provide a short table of further values of $K(q)$.

n	$K(q)$
3	$-q^{2}\frac{\sqrt{5}-1}{2}$
4	$q\frac{\sqrt{5}+1}{2}$
6	$q^{5}\frac{\sqrt{5}+1}{2}$
7	$-q^{3}\frac{\sqrt{5}-1}{2}$

Ramanujan's lost notebook [11, p. 57] contains some claims on finite, generalized Rogers–Ramanujan continued fractions, and these results have recently been proved by S.–S. Huang [1]. Perhaps the main result gives a formula for evaluating certain finite generalized Rogers–Ramanujan continued fractions at primitive roots of unity x. At the bottom of page 57 is a table of general formulas arranged according to residue classes of n modulo 5, when x is a primitive nth root of unity. However, the table contains some errors. When this table is used in Ramanujan's primary formula, specialized to the ordinary Rogers–Ramanujan continued fraction, we obtain Entry 12 as Ramanujan recorded it. This is evidence that these results in the lost notebook were derived before Entry 12. For more details, see Huang's paper [1].

The next result is stated exactly as Ramanujan recorded it.

Entry 13 (Formula (4), p. 289).

$$\left.\begin{aligned} &1-\frac{qx}{1}+\frac{q^2}{1}-\frac{q^3x}{1}+\frac{q^4}{1}-\frac{q^5x}{1}+\cdots \\ &=\frac{q}{x}+\frac{q^4}{x}+\frac{q^8}{x}+\frac{q^{12}}{x}+\cdots \qquad \textit{nearly} \end{aligned}\right\} \quad \textit{conventional only}$$

Both continued fractions converge to meromorphic functions of x in $\mathbb{C}-\{0\}$ for $|q|<1$. We shall indeed prove that

$$1-\frac{qx}{1}+\frac{q^2}{1}-\frac{q^3x}{1}+\frac{q^4}{1}-\cdots = \frac{\sum\limits_{n=0}^{\infty}\dfrac{(-x)^n q^{n^2}}{(q^4;q^4)_n}}{\sum\limits_{n=0}^{\infty}\dfrac{(-x)^n q^{n^2+2n}}{(q^4;q^4)_n}}, \qquad |q|<1, \tag{13.1}$$

and

$$\frac{q}{x}+\frac{q^4}{x}+\frac{q^8}{x}+\frac{q^{12}}{x}+\cdots = \frac{q}{x}\,\frac{\sum\limits_{n=0}^{\infty}\dfrac{x^{-2n}q^{4n(n+1)}}{(q^4;q^4)_n}}{\sum\limits_{n=0}^{\infty}\dfrac{x^{-2n}q^{4n^2}}{(q^4;q^4)_n}}, \qquad |q|<1. \tag{13.2}$$

Let $f(x;q)$ and $g(x;q)$, respectively, denote the values of the continued fractions in (13.1) and (13.2). We shall then prove that both $f(x;q)$ and $g(x;q)$ satisfy the same functional equation

$$F(xq^2;q) = \frac{1}{xq + F(x;q)}, \tag{13.3}$$

for $0 < |q| < 1$. Furthermore, we shall prove that $f(q;q) = g(q;q)$. It then follows from (13.3) that

$$f(q^{2n+1};q) = g(q^{2n+1};q) \tag{13.4}$$

for every nonnegative integer n. Lastly, we shall describe some work of Zagier [1] which indicates in what sense $f(x;q)$ and $g(x;q)$ are "nearly" equal.

The continued fraction on the left side of (13.1) may be identified by employing a continued fraction found in Ramanujan's "lost notebook" [11] and proved by Andrews [5], M. D. Hirschhorn [2], S. Bhargava and C. Adiga [1], [2], Bhargava, Adiga, and D. D. Somashekara [1], and others. Let

$$F(a,b,\lambda,q) = 1 + \frac{aq+\lambda q}{1} + \frac{bq+\lambda q^2}{1} + \frac{aq^2+\lambda q^3}{1} + \frac{bq^2+\lambda q^4}{1} + \cdots$$

and

$$G(a,b,\lambda,q) = \sum_{n=0}^{\infty} \frac{(-\lambda/a)_n q^{(n^2+n)/2} a^n}{(q;q)_n(-bq;q)_n}, \qquad |q| < 1.$$

Then

$$F(a,b,\lambda,q) = \frac{G(a,b,\lambda,q)}{G(aq,b,\lambda q,q)}. \tag{13.5}$$

Setting $b = 1$ and $\lambda = 0$, replacing q by q^2, and setting $a = -x/q$, we establish (13.1).

The continued fraction on the right side of Entry 13 is equivalent to

$$\frac{q/x}{1} + \frac{q^4/x^2}{1} + \frac{q^8/x^2}{1} + \frac{q^{12}/x^2}{1} + \cdots.$$

Thus, (13.2) follows from the corollary to Entry 15 of Chapter 16 (Part III [3, p. 30]). This corollary also appears in Ramanujan's [10, p. xxviii] second letter to Hardy.

To prove (13.3) for $F(x;q) = f(x;q)$, we shall use the Bauer–Muir transformation found in Perron's book [1, p. 47]. Briefly, a Bauer–Muir transformation of a continued fraction $b_0 + \mathbf{K}(a_n/b_n)$ is a (new) continued fraction whose approximants have the values

$$S_k(w_k) := b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_k}{b_k + w_k}, \qquad k = 0, 1, 2, \ldots .$$

Such a transformation exists if

$$\lambda_k := a_k - w_{k-1}(b_k + w_k) \neq 0, \quad k \geq 1, \tag{13.6}$$

and it is given by

$$b_0 + w_0 + \frac{\lambda_1}{b_1 + w_1} + \frac{a_1\lambda_2/\lambda_1}{b_2 + w_2 - w_0\lambda_2/\lambda_1} + \frac{a_2\lambda_3/\lambda_2}{b_3 + w_3 - w_1\lambda_3/\lambda_2} + \cdots. \tag{13.7}$$

Suppose that $b_0 + \mathbf{K}(a_n/b_n)$ converges to a value $f \in \widehat{\mathbb{C}}$. Let $t_0 \neq f$ be arbitrarily chosen from $\widehat{\mathbb{C}}$, and define $t_k = S_k^{-1}(t_0)$, $k \geq 1$. Then $S_k(w_k)$ tends to f as well, unless the chordal distance $d(w_k, t_k)$ has a limit point at 0. Hence, the Bauer–Muir transformation (13.7) converges to f as well, if $\lim_{k\to\infty} \inf d(w_k, t_k) > 0$ (Jacobsen [5]).

We shall apply this to the odd part

$$1 - qx + \frac{q^3x}{1 + q^2(1 - qx)} + \frac{q^7x}{1 + q^4(1 - qx)} + \frac{q^{11}x}{1 + q^6(1 - qx)} + \cdots$$
$$= f(x; q) \tag{13.8}$$

of (13.1). Let

$$a_n = q^{4n-1}x \quad (n \geq 1) \quad \text{and} \quad b_n = 1 + q^{2n}(1 - qx) \quad (n \geq 0),$$

so that (13.8) can be written as $-1 + b_0 + \mathbf{K}(a_n/b_n) = f(x; q)$. Let $w_k = -q^{2k}$ for each $k \in \mathbb{N}$. Then the modified approximants $-1 + S_k(w_k)$ of (13.8) also converge to $f(x; q)$ by the following argument. The continued fraction $-1 + b_0 + \mathbf{K}(a_n/b_n)$ is limit periodic since $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 1$. It is then well known (see, for instance, Jacobsen's paper [3]) that $-1 + b_0 + \mathbf{K}(a_n/b_n)$ converges to a value φ and that t_k tends to -1 for every tail sequence of $-1 + b_0 + \mathbf{K}(a_n/b_n)$ with $t_0 \neq \varphi$. In particular, this implies that $\lim_{k\to\infty} \inf d(w_k, t_k) = d(0, -1) \neq 0$, so that $\lim_{k\to\infty} S_k(w_k) = f(x; q)$. Since (13.1) obviously holds for $q = 0$, we may assume that $q \neq 0$.

A simple calculation shows that, for $k \geq 0$,

$$\lambda_k = a_k - w_{k-1}(b_k + w_k) = q^{2k-2} \neq 0,$$

so that the Bauer–Muir transformation exists, converges to $f(x; q)$, and is given by

$$f(x; q) = -qx + \frac{1}{1 - q^3x} + \frac{q^5x}{1 + q^2(1 - q^3x)} + \frac{q^9x}{1 + q^4(1 - q^3x)} + \cdots. \tag{13.9}$$

Comparing (13.8) and (13.9), we find that

$$f(x; q) = -qx + \frac{1}{f(q^2x; q)},$$

which proves (13.3) for $F(x; q) = f(x; q)$.

To prove (13.3) for $F(x;q)=g(x;q)$, we just observe that replacing x by q^2x in (13.2) yields

$$g(q^2x;q)=\frac{q}{q^2x}\,{}_{+}\,\frac{q^4}{q^2x}\,{}_{+}\,\frac{q^8}{q^2x}\,{}_{+}\cdots=\frac{1/q}{x}\,{}_{+}\,\frac{1}{x}\,{}_{+}\,\frac{q^4}{x}\,{}_{+}\,\frac{q^8}{x}\,{}_{+}\cdots$$
$$=\frac{1/q}{x+\frac{1}{q}g(x;q)}=\frac{1}{xq+g(x;q)},$$

which is what we wanted to show.

For $x=q$, the continued fractions in Entry 13 reduce to, respectively,

$$f(q;q)=1-\frac{q^2}{1}\,{}_{+}\,\frac{q^2}{1}\,{}_{-}\,\frac{q^4}{1}\,{}_{+}\,\frac{q^4}{1}\,{}_{-}\cdots \tag{13.10}$$

and

$$g(q;q)=\frac{1}{1}\,{}_{+}\,\frac{q^2}{1}\,{}_{+}\,\frac{q^6}{1}\,{}_{+}\,\frac{q^{10}}{1}\,{}_{+}\cdots. \tag{13.11}$$

Thus, (13.11) is the even part of (13.10), and so $f(q;q)=g(q;q)$.

In order to examine how "close" the continued fractions $f(x;q)$ and $g(x;q)$ are to each other, by (13.1) and (13.2), we are led to examine

$$F(x;q):=\sum_{n=0}^{\infty}\frac{x^{-2n}q^{4n^2}}{(q^4;q^4)_n}\sum_{n=0}^{\infty}\frac{(-x)^nq^{n^2}}{(q^4;q^4)_n}-\sum_{n=0}^{\infty}\frac{x^{-2n-1}q^{4n(n+1)}}{(q^4;q^4)_n}\sum_{n=0}^{\infty}\frac{(-x)^nq^{(n+1)^2}}{(q^4;q^4)_n}.$$

Quite remarkably, Ramanujan stated an identity for $F(x;q)$ in his lost notebook [11], namely

$$F(x;q)=\frac{\sum_{n=-\infty}^{\infty}(-x)^nq^{n^2}}{(q^4;q^4)_\infty}=\frac{(q^2;q^2)_\infty(qx;q^2)_\infty(q/x;q^2)_\infty}{(q^4;q^4)_\infty}, \tag{13.12}$$

where the last equality is obtained from an application of the Jacobi triple product identity. The identity (13.12) was proved by Andrews in [6, pp. 25–32] and is mentioned by him in his Introduction to the lost notebook [11, p. xxi, eq. (10.6)]. Zagier [1] independently also established (13.12). From (13.12), it is obvious that $F(q^{2n+1};q)=0$ for each nonnegative integer n. We thus have obtained a second proof of (13.4). Of course, the second proof is shorter than the first, but the first proof is more elementary than the second, because (13.12) is somewhat difficult to prove.

If x is not an odd power of q, in what sense are $f(x;q)$ and $g(x;q)$ near each other? H. Cohen performed extensive calculations to answer this question, and Zagier [1] established Cohen's conjectures as well as much more. We give a brief summary of some of Zagier's results. We always assume here that $0<q<1$ and $x>0$.

Let $Q = \exp\left(\dfrac{\pi^2/5}{\log q}\right)$. Then, for $x = 1$,

$$f(1;q) = q^{1/5}\frac{\sqrt{5}-1}{2}\left(1 + \sqrt{5}Q + \frac{5-\sqrt{5}}{2}Q^2 - \frac{5-3\sqrt{5}}{2}Q^3 - \cdots\right)$$

and

$$g(1;q) = q^{1/5}\frac{\sqrt{5}-1}{2}\left(1 - \sqrt{5}Q + \frac{5-\sqrt{5}}{2}Q^2 + \frac{5-3\sqrt{5}}{2}Q^3 - \cdots\right).$$

In particular, as q tends to $1-$,

$$f(1;q) - g(1;q) = (5-\sqrt{5})q^{1/5}Q + O(Q^2). \tag{13.13}$$

However, as x tends to 0,

$$f(x;q) - g(x;q) = O\left(\exp\left(\frac{\pi^2/4}{\log q}\right)\right). \tag{13.14}$$

Note that the asymptotic behaviors for $x = 1$ and x near 0 are different.

In general, Zagier [1] has shown that

$$f(x;q) - g(x;q) = \exp\left(\frac{c(x) + o(1)}{\log q}\right)\cos\theta$$

as q tends to $1-$, where $\theta = (\pi \log x)/(2\log q)$ and

$$\begin{aligned} c(x) = \frac{\pi^2}{6} + \frac{1}{2}\operatorname{Li}_2\left(\left(\sqrt{1+x^2/4} - x/2\right)^2\right) \\ + \frac{1}{2}\log^2\left(\sqrt{1+x^2/4}+x/2\right) - \log x \log\left(\sqrt{1+x^2/4}+x/2\right), \end{aligned}$$

where $\operatorname{Li}_2(t)$ denotes the dilogarithm

$$\operatorname{Li}_2(t) = \sum_{n=1}^{\infty} t^n/n^2, \quad |t| \le 1.$$

Since $\operatorname{Li}_2(1) = \pi^2/6$ and $\operatorname{Li}_2\left((3-\sqrt{5})/2\right) = \pi^2/15 - \log^2\left((1+\sqrt{5})/2\right)$ (Lewin [1, p. 7]), we find that $c(0) = \pi^2/4$ and $c(1) = \pi^2/5$, in agreement with (13.13) and (13.14).

Zagier's analysis shows that, for instance, $f(x;0.99)$ and $g(x;0.99)$ agree to about 85 decimal places if x is near 1, about 96 places if x is near $\frac{1}{2}$, and about 107 places if x is close to 0. The function $c(x)$ becomes negative for x larger than about 6.177. Thus, for x larger than this, $f(x;q) - g(x;q)$ becomes exponentially large as q tends to $1-$.

We do not know the meaning of the words "conventional only" in Entry 13.

For Entry 14, we again quote Ramanujan.

Entry 14 (Formula (2), p. 290).

$$1-\frac{qx}{1+q}+\frac{q^3x}{1+q^2}-\frac{q^2x}{1+q^3}+\frac{q^6x}{1+q^4}-\frac{q^3x}{1+q^5}+\frac{q^9x}{1+q^6}-\cdots$$
$$=\frac{1}{x}+\frac{q}{x}+\frac{q^3}{x}+\frac{q^5}{x}+\cdots \quad \textit{nearly}.$$

The analysis for Entry 14 is very similar to that for Entry 13. Both continued fractions converge to meromorphic functions of x in $\mathbb{C}-\{0\}$ for $|q|<1$. We shall prove that

$$1-\frac{qx}{1+q}+\frac{q^3x}{1+q^2}-\frac{q^2x}{1+q^3}+\frac{q^6x}{1+q^4}-\frac{q^3x}{1+q^5}+\frac{q^9x}{1+q^6}-\cdots$$
$$=\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-x)^n q^{(n^2+n)/2}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-x)^n q^{(n^2+3n)/2}}{(q^2;q^2)_n}}, \quad |q|<1, \tag{14.1}$$

and

$$\frac{1}{x}+\frac{q}{x}+\frac{q^3}{x}+\frac{q^5}{x}+\cdots=\frac{1}{x}\frac{\displaystyle\sum_{n=0}^{\infty}\frac{x^{-2n}q^{2n^2+n}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{x^{-2n}q^{2n^2-n}}{(q^2;q^2)_n}}, \qquad |q|<1. \tag{14.2}$$

Let $f(x;q)$ and $g(x;q)$, respectively, denote the continued fractions on the left sides of (14.1) and (14.2). We shall prove that $f(q^n;q)=g(q^n;q)$, for every nonnegative integer n, by invoking the same theorem from Ramanujan's lost notebook that we used in Section 13. Thus, we shall easily see that the "closeness" of $f(x;q)$ and $g(x;q)$ can be determined merely by changing the variables in the analysis of Section 13.

Proof. For $|q|<1$ and each nonnegative integer m define

$$P_{2m}(x)=\sum_{n=0}^{\infty}\frac{(-q^m x)^n q^{(n^2+n)/2}}{(-q^{2m};q)_n(q;q)_n} \text{ and } P_{2m+1}(x)=\sum_{n=0}^{\infty}\frac{(-q^{m+1}x)^n q^{(n^2+n)/2}}{(-q^{2m+1};q)_n(q;q)_n}.$$

By straightforward calculations, we then find that

$$P_{2m}(x)-P_{2m+1}(x)=-\frac{q^{m+1}x}{(1+q^{2m})(1+q^{2m+1})}P_{2m+2}(x)$$

and

$$P_{2m-1}(x)-P_{2m}(x)=\frac{q^{3m}x}{(1+q^{2m-1})(1+q^{2m})}P_{2m+1}(x).$$

Since $P_m(0) = 1$ for each $m \in \mathbb{N}$, we thus deduce that

$$\frac{P_0(x)}{P_1(x)} = 1 - \frac{\frac{qx}{(1+1)(1+q)}}{1} + \frac{\frac{q^3x}{(1+q)(1+q^2)}}{1} - \frac{\frac{q^2x}{(1+q^2)(1+q^3)}}{1} + \frac{\frac{q^6x}{(1+q^3)(1+q^4)}}{1} + \cdots$$

$$= 1 - \frac{1}{2}\left\{\frac{qx}{1+q} + \frac{q^3x}{1+q^2} - \frac{q^2x}{1+q^3} + \frac{q^6x}{1+q^4} - \cdots\right\}. \tag{14.3}$$

Hence, the first continued fraction in Entry 14 converges for $|q| < 1$ to the value

$$f(x;q) = -1 + 2\frac{P_0(x)}{P_1(x)} = -1 + 2\frac{\sum_{n=0}^{\infty} \frac{(-x)^n q^{(n^2+n)/2}}{(q;q)_n(-1;q)_n}}{\sum_{n=0}^{\infty} \frac{(-qx)^n q^{(n^2+n)/2}}{(q;q)_n(-q;q)_n}}$$

$$= -1 + \frac{\sum_{n=0}^{\infty} \frac{(-x)^n q^{(n^2+n)/2}(1+q^n)}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty} \frac{(-x)^n q^{(n^2+3n)/2}}{(q^2;q^2)_n}},$$

which immediately proves (14.1).

By the corollary of Entry 15 in Chapter 16 (Part III [3, p. 30]),

$$\frac{1}{x} + \frac{q}{x} + \frac{q^3}{x} + \frac{q^5}{x} + \cdots = \frac{1}{x}\left(\frac{1}{1} + \frac{q^2\frac{1}{qx^2}}{1} + \frac{q^4\frac{1}{qx^2}}{1} + \frac{q^6\frac{1}{qx^2}}{1} + \cdots\right)$$

$$= \frac{1}{x}\frac{\sum_{n=0}^{\infty} \frac{x^{-2n}q^{2n^2+n}}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty} \frac{x^{-2n}q^{2n^2-n}}{(q^2;q^2)_n}}, \qquad |q| < 1,$$

and so the proof of (14.2) is completed.

From (14.1) and (14.2),

$$f(x;q) - g(x;q)$$

$$= \frac{\sum_{n=0}^{\infty} \frac{x^{-2n}q^{2n^2-n}}{(q^2;q^2)_n} \sum_{n=0}^{\infty} \frac{(-x)^n q^{(n^2+n)/2}}{(q^2;q^2)_n} - \sum_{n=0}^{\infty} \frac{x^{-2n-1}q^{2n^2+n}}{(q^2;q^2)_n} \sum_{n=0}^{\infty} \frac{(-x)^n q^{(n^2+3n)/2}}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty} \frac{(-x)^n q^{(n^2+3n)/2}}{(q^2;q^2)_n} \sum_{n=0}^{\infty} \frac{x^{-2n}q^{2n^2-n}}{(q^2;q^2)_n}}$$

$$= \frac{(q;q)_\infty (xq;q)_\infty (1/x;q)_\infty}{(q^2;q^2)_\infty \sum_{n=0}^\infty \frac{(-x)^n q^{(n^2+3n)/2}}{(q^2;q^2)_n} \sum_{n=0}^\infty \frac{x^{-2n} q^{2n^2-n}}{(q^2;q^2)_n}},$$

where we have employed (13.12) with q^2 replaced by q and x replaced by $x\sqrt{q}$. It follows that $f(q^n;q) = g(q^n;q)$, for each nonnegative integer n. We also see that Zagier's analysis can be applied *mutatis mutandi*, after the aforementioned changes of variables are made.

It is interesting to note that the continued fractions in (13.1) and (14.1) are connected to the basic hypergeometric functions

$$_2\varphi_1(a,b;c;q;z) = \sum_{n=0}^\infty \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} z^n, \qquad |z| < 1.$$

For example, we shall show that the continued fraction in (14.1) can be derived from E. Heine's [1] continued fraction expansion

$$\frac{_2\varphi_1(a,b;c;q;z)}{_2\varphi_1(a,bq;cq;q;z)} = 1 - \frac{a_1 z}{1} - \frac{a_2 z}{1} - \frac{a_3 z}{1} - \cdots, \tag{14.4}$$

where

$$a_{2k+1} = \frac{q^k(1-aq^k)(b-cq^k)}{(1-cq^{2k})(1-cq^{2k+1})} \quad \text{and} \quad a_{2k} = \frac{q^{k-1}(1-bq^k)(a-cq^k)}{(1-cq^{2k-1})(1-cq^{2k})}.$$

Let $a = 0$, $c = -1$, and $z = xq/b$, and let b approach ∞. We then find that

$$-a_{2k+1}z = -\frac{q^{k+1}x}{(1+q^{2k})(1+q^{2k+1})} \quad \text{and} \quad -a_{2k}z = \frac{q^{3k}x}{(1+q^{2k-1})(1+q^{2k})}.$$

Thus, the continued fraction in (14.4) reduces to the one in (14.3). Likewise, since

$$\lim_{b\to\infty} (b;q)_n b^{-n} = (-1)^n q^{(n^2-n)/2} \quad \text{and} \quad \lim_{b\to\infty} (bq;q)_n b^{-n} = (-1)^n q^{(n^2+n)/2},$$

the left side of (14.4) reduces to the left side of (14.3). The identity (14.3) follows then, since the continued fraction (14.4) converges uniformly with respect to b in a neighborhood of $b = \infty$. A rigorous proof of this statement can be given along the same lines as that given for (24.5) below.

Entry 15 (p. 373). *Let a, b, and q be complex numbers with $|q| < 1$. Define*

$$\varphi(a) = \sum_{n=0}^\infty \frac{q^{n^2} a^n}{(q;q)_n (bq;q)_n}.$$

Then

$$\frac{\varphi(a)}{\varphi(aq)} = 1 + \frac{aq}{1} + \frac{aq^2 - bq}{1} + \frac{aq^3}{1} + \frac{aq^4 - bq^2}{1} + \cdots.$$

Proof. In (13.5), let $a = 0$, replace b by $-b$, and set $\lambda = a$. Upon observing that

$$\lim_{a\to 0} (-\lambda/a;q)_n a^n = \lambda^n q^{n(n-1)/2},$$

we see immediately that Entry 15 follows, since the continued fraction expansion of $F(a, b, \lambda, q)$ in the proof of Entry 13 converges locally uniformly in our domain.

Observe that another continued fraction for $\varphi(a)/\varphi(aq)$ is given in Entry 15 of Chapter 16 (Part III [3, p. 30]). Furthermore, another representation for $\varphi(a)$ can be found in Entry 9 of Chapter 16 (Part III [3, p. 18]). With the help of these two observations, Ramanathan [4] has found another proof of Entry 15.

Entry 16 (p. 373). *For* $|q| < 1$,

$$\frac{\chi(-q^2)f(-q^5)}{f(-q,-q^4)} = \frac{f(q,q^9)}{f(-q^4,-q^{16})} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} \tag{16.1}$$

and

$$\frac{q\chi(-q^2)f(-q^5)}{f(-q^2,-q^3)} = \frac{qf(q^3,q^7)}{f(-q^8,-q^{12})} = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^4;q^4)_n}, \tag{16.2}$$

where, as before, $\chi(q) = (-q;q^2)_\infty$. *Moreover,*

$$\frac{qf(q^3,q^7)}{f(-q^8,-q^{12})} \Big/ \frac{f(q,q^9)}{f(-q^4,-q^{16})} = \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots. \tag{16.3}$$

Proof. First, (16.3) follows immediately from (16.1), (16.2), and a standard representation for the Rogers–Ramanujan continued fraction given in Entry 38(iii) of Chapter 16 (Part III [3, p. 79]).

We next demonstrate the first equalities of (16.1) and (16.2). By Entry 19 (the Jacobi triple product identity), (22.2) (Euler's identity), and Entry 22, all in Chapter 16 (Part III [3, pp. 35, 37, 36]), we find that

$$\begin{aligned}
\frac{f(q,q^9)}{f(-q^4,-q^{16})} &= \frac{(-q;q^{10})_\infty(-q^9;q^{10})_\infty(q^{10};q^{10})_\infty}{(q^4;q^{20})_\infty(q^{16};q^{20})_\infty(q^{20};q^{20})_\infty} \\
&= \frac{(-q;q^{10})_\infty(-q^9;q^{10})_\infty}{(-q^2;q^{10})_\infty(q^2;q^{10})_\infty(-q^8;q^{10})_\infty(q^8;q^{10})_\infty(-q^{10};q^{10})_\infty} \\
&= \frac{(-q^4;q^{10})_\infty(-q^6;q^{10})_\infty(-q;q^{10})_\infty(-q^9;q^{10})_\infty}{(-q^2;q^2)_\infty(q;q^5)_\infty(-q;q^5)_\infty(q^4;q^5)_\infty(-q^4;q^5)_\infty} \\
&= \frac{1}{(-q^2;q^2)_\infty(q;q^5)_\infty(q^4;q^5)_\infty} \qquad (16.4) \\
&= \frac{(q^2;q^4)_\infty(q^5;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty} \\
&= \frac{\chi(-q^2)f(-q^5)}{f(-q,-q^4)},
\end{aligned}$$

which establishes the first equality of (16.1). The proof of the first equality in (16.2) is completely analogous.

By (16.4) and its analogue, in order to prove the second equalities of (16.1) and (16.2), it suffices to show that, respectively,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} = \frac{1}{(-q^2;q^2)_\infty (q;q^5)_\infty (q^4;q^5)_\infty}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^4;q^4)_n} = \frac{q}{(-q^2;q^2)_\infty (q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

These last two identities have been proved by L. J. Rogers [1, pp. 330, 331]. Hence, the proof of Entry 16 is complete.

Entry 17 (p. 374). *Let a, b, and q be complex numbers with $|q| < 1$. Define*

$$\varphi(a) = \sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2} a^n}{(q;q)_n(-bq;q)_n}.$$

Then

$$\frac{\varphi(a)}{\varphi(aq)} = 1 + \frac{aq}{1}\, + \frac{bq}{1}\, + \frac{aq^2}{1}\, + \frac{bq^2}{1}\, + \frac{aq^3}{1}\, + \frac{bq^3}{1}\, + \cdots.$$

Proof. The result follows immediately from setting $\lambda = 0$ in (13.5).

2. Other q–Continued Fractions

Entry 18 (p. 373). *For $|q| < 1$,*

$$\frac{f(-q,-q^5)}{f(-q^3,-q^3)} = \frac{1}{1}\, + \frac{q+q^2}{1}\, + \frac{q^2+q^4}{1}\, + \cdots.$$

Beneath this continued fraction, Ramanujan writes

$$\text{Num?} = \frac{\varphi(-q^3)}{f(-q)} \quad \text{and Den?} = \frac{\psi(q^3)}{f(-q^2)}.$$

In fact, he has incorrectly inverted the identifications of the "numerator" and "denominator" on the left side of Entry 18.

By Entry 22 and Example (v), Section 31 of Chapter 16 (Part III [3, pp. 36, 51]),

$$\frac{f(-q,-q^5)}{f(-q^3,-q^3)} = \frac{\chi(-q)\psi(q^3)}{\varphi(-q^3)} = \frac{(q;q^2)_\infty (q^6;q^6)_\infty}{(q^3;q^6)^2_\infty (q^3;q^3)_\infty} = \frac{(q;q^2)_\infty}{(q^3;q^6)^3_\infty},$$

where $\chi(q) = (-q; q^2)_\infty$. On the other hand, by Entry 22 of Chapter 16 (Part III [3, p. 36]),

$$\frac{\psi(q^3)/f(-q^2)}{\varphi(-q^3)/f(-q)} = \frac{(q^6; q^6)_\infty (q; q)_\infty}{(q^3; q^6)_\infty^2 (q^2; q^2)_\infty (q^3; q^3)_\infty} = \frac{(q; q^2)_\infty}{(q^3; q^6)_\infty^3}.$$

Hence, we have shown that Ramanujan has mistakenly confused the roles of the "numerator" and "denominator." Moreover, we now see that Entry 18 can be written in the more transparent form

$$G(q) := \frac{(q; q^2)_\infty}{(q^3; q^6)_\infty^3} = \frac{1}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \cdots. \tag{18.1}$$

The first proofs of (18.1) in print are due to Watson [2] in 1929 and Selberg [1, p. 19] in 1936. B. Gordon [1] and Andrews [1] found proofs in 1965 and 1968, respectively, while Hirschhorn [3] has shown that (18.1) can be deduced from Ramanujan's continued fraction (13.5). Ramanathan [2], [3] has briefly discussed (18.1). L.–C. Zhang [1] has examined (18.1) when q is a root of unity.

A detailed study of $G(q)$ has been made by H. H. Chan [1]. He has derived modular equations relating $G(q)$ with each of $G(-q)$, $G(q^2)$, and $G(q^3)$. Using these and other modular equations, he has determined values for $G(\pm e^{-\pi\sqrt{n}})$ for several positive rational numbers n. The author, Chan, and L.–C. Zhang [1] have found general formulas that enable one to evaluate $G(\pm e^{-\pi\sqrt{n}})$ in terms of class invariants.

Entry 19 (Formula (3), p. 290). *For* $|q| < 1$,

$$\frac{(q^2; q^3)_\infty}{(q; q^3)_\infty} = \frac{1}{1} - \frac{q}{1+q} - \frac{q^3}{1+q^2} - \frac{q^5}{1+q^3} - \frac{q^7}{1+q^4} - \cdots.$$

Proof. We apply Theorem 6 in Andrews' paper [1] with $a_1 = \omega x q$, $a_2 = \omega^{-1} x q$, $a = -1/(x^2q^2)$, and $b = 1/(xq)$, where $\omega = \exp(2\pi i/3)$. Thus, $1/(a_1a_2) = -a$ and $1/a_1 + 1/a_2 = -b$, as required in Theorem 6. Accordingly, by the same argument as in the justification of the limiting procedures in Entry 24, using the uniform parabola theorem (Jones and Thron [1, p. 99]), we find that

$$\lim_{x\to 0} \frac{H_{2,1}(\omega x q, \omega^{-1} x q; x; q)}{H_{2,1}(\omega x q, \omega^{-1} x q; xq; q)} = 2 - \frac{q}{1+q} - \frac{q^3}{1+q^2} - \frac{q^5}{1+q^3} - \cdots, \tag{19.1}$$

where the identification of $H_{2,1}$ will be made shortly. Comparing Entry 19 and (19.1), we see that it remains to show that

$$\lim_{x\to 0} \frac{1}{\dfrac{H_{2,1}(\omega x q, \omega^{-1} x q; x; q)}{H_{2,1}(\omega x q, \omega^{-1} x q; xq; q)} - 1} = \frac{(q^2; q^3)_\infty}{(q; q^3)_\infty}. \tag{19.2}$$

By Andrews' paper [1, eq. (1.1)],

$$H_{2,1}(a_1, a_2; x; q) = \frac{(xq/a_1; q)_\infty (xq/a_2; q)_\infty}{(xq; q)_\infty} C_{2,1}(a_1, a_2; x; q). \tag{19.3}$$

Furthermore, by [1, eq. (1.1)],

$$\begin{aligned}
&\lim_{x\to 0} C_{2,1}(\omega xq, \omega^{-1}xq; x; q)\\
&= \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n-1)/2}}{(q;q)_n(\omega^{-1};q)_n(\omega;q)_n}\\
&= \frac{1}{(1-\omega^{-1})(1-\omega)} \sum_{n=0}^{\infty} \frac{(1-\omega^{-1}q^n)(1-\omega q^n)(-1)^n q^{3n(n-1)/2}}{(q^3;q^3)_n}\\
&= \frac{1}{3}\left(\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n-1)/2}}{(q^3;q^3)_n} - (\omega^{-1}+\omega)\sum_{n=0}^{\infty} \frac{(-q)^n q^{3n(n-1)/2}}{(q^3;q^3)_n}\right.\\
&\quad \left. + \sum_{n=0}^{\infty} \frac{(-q^2)^n q^{3n(n-1)/2}}{(q^3;q^3)_n}\right)\\
&= \frac{1}{3}(0 + (q;q^3)_\infty + (q^2;q^3)_\infty),
\end{aligned} \tag{19.4}$$

where we have employed an identity of Euler (Andrews [4, p. 19]). By a similar argument,

$$\begin{aligned}
\lim_{x\to 0} C_{2,1}(\omega xq, \omega^{-1}xq; xq; q) &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n(n-1)/2+2n}}{(q;q)_n(\omega^{-1}q;q)_n(\omega q;q)_n}\\
&= \sum_{n=0}^{\infty} \frac{(-q^2)^n q^{3n(n-1)/2}}{(q^3;q^3)_n} = (q^2;q^3)_\infty,
\end{aligned} \tag{19.5}$$

by another application of Euler's identity.

Putting each of (19.4) and (19.5) in (19.3) and using the results on the left side of (19.2), we find that

$$\begin{aligned}
&\lim_{x\to 0} \frac{1}{\dfrac{H_{2,1}(\omega xq, \omega^{-1}xq; x; q)}{H_{2,1}(\omega xq, \omega^{-1}xq; xq; q)} - 1}\\
&= \frac{1}{\dfrac{(1-\omega^{-1})(1-\omega)\frac{1}{3}((q;q^3)_\infty + (q^2;q^3)_\infty)}{(q^2;q^3)_\infty} - 1} = \frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}.
\end{aligned}$$

This completes the proof of (19.2) and hence also of Entry 19.

The functions $C_{k,i}$ studied by Andrews in [1] also appear prominently in his paper [3, Sect. 2]. These functions play a crucial role in the full three parameter general Rogers–Ramanujan theorem (Andrews [2]).

In Ramanujan's [10, p. xxviii] first letter to Hardy, he states the Rogers–Ramanujan continued fraction and some identities involving it. Ramanujan continues by claiming "The above theorem is a particular case of a theorem on the continued fraction

$$\frac{1}{1}+\frac{ax}{1}+\frac{ax^2}{1}+\frac{ax^3}{1}+\frac{ax^4}{1}+\frac{ax^5}{1}+\cdots,$$

which is a particular case of the continued fraction

$$\frac{1}{1}+\frac{ax}{1+bx}+\frac{ax^2}{1+bx^2}+\frac{ax^3}{1+bx^3}+\cdots,$$

which is a particular case of a general theorem on continued fractions." It seems possible that Andrews' Theorem 6 of [1] giving an evaluation of

$$1+bxq+\frac{xq(1+axq^2)}{1+bxq^2}+\frac{xq^2(1+axq^3)}{1+bxq^3}+\cdots$$

is the "general theorem" about which Ramanujan writes. However, Hirschhorn [1], [3] has also put forth a very good candidate for this "general theorem." The most general continued fraction containing the Rogers–Ramanujan continued fraction as a special case is undoubtedly that of Andrews and D. Bowman [1].

Entry 20 (Formula (4), p. 290). *For* $|q|<1$,

$$\frac{(q^3;q^4)_\infty}{(q;q^4)_\infty}=\frac{1}{1}-\frac{q}{1+q^2}-\frac{q^3}{1+q^4}-\frac{q^5}{1+q^6}-\cdots.$$

This result is simply the case $a=1, b=0$ of Entry 12 of Chapter 16 (Part III [3, p. 24]). Entry 20 can also be found in the "lost notebook" [11]. Ramanathan [4] has also given a proof of Entry 20. Another continued fraction for the left side of Entry 20 is found in the "lost notebook" and has been proved by Andrews [7] as well as by Ramanathan [4].

It is interesting to note that the continued fraction in Entry 20 also converges for $|q|>1$. In fact, set $q=1/a$, so that $|a|<1$. Then

$$\begin{aligned}
&\frac{1}{1}-\frac{q}{1+q^2}-\frac{q^3}{1+q^4}-\frac{q^5}{1+q^6}-\cdots\\
&=\frac{1}{1}-\frac{1/a}{1+1/a^2}-\frac{1/a^3}{1+1/a^4}-\frac{1/a^5}{1+1/a^6}-\cdots\\
&=\frac{1}{1}-\frac{a}{a^2+1}-\frac{a^3}{a^4+1}-\frac{a^5}{a^6+1}-\cdots\\
&=\frac{(a^3;a^4)_\infty}{(a;a^4)_\infty}=\frac{(1/q^3;1/q^4)_\infty}{(1/q;1/q^4)_\infty}.
\end{aligned}\tag{20.1}$$

This is, indeed, a beautiful example of symmetry.

It follows more generally from Entry 12 of Chapter 16 that

$$\frac{(bq^3;q^4)_\infty}{(bq;q^4)_\infty} = \frac{1}{1} - \frac{bq}{1+q^2} - \frac{bq^3}{1+q^4} - \frac{bq^5}{1+q^6} - \cdots, \qquad |q| < 1.$$

Now let $|q| > 1$ and set $q = 1/a$, so that $|a| < 1$. Then, as in (20.1),

$$\frac{1}{1} - \frac{bq}{1+q^2} - \frac{bq^3}{1+q^4} - \frac{bq^5}{1+q^6} - \cdots = \frac{(ba^3;a^4)_\infty}{(ba;a^4)_\infty} = \frac{(b/q^3;1/q^4)_\infty}{(b/q;1/q^4)_\infty}.$$

Although the continued fraction above is symmetric in q and $1/q$, the product $(bq^3;q^4)_\infty/(bq;q^4)_\infty$ does not share this invariance. However, if $b = -1$, then

$$\frac{(-q^3;q^4)_\infty}{(-q;q^4)_\infty} = \frac{(q;q^8)_\infty(q^5;q^8)_\infty(q^6;q^8)_\infty}{(q^2;q^8)_\infty(q^3;q^8)_\infty(q^7;q^8)_\infty},$$

and the latter quotient is invariant when q is replaced by $1/q$. These observations are due to K. Alladi and B. Gordon [1, p. 298].

The convergence of (20.1) when q is a primitive root of unity has been examined by Zhang [1].

Entry 21 (Formula (5), p. 290). *For* $|q| < 1$,

$$\frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty} = \frac{1}{1} + \frac{q}{1} + \frac{q^2+q}{1} + \frac{q^3}{1} + \frac{q^4+q^2}{1} + \frac{q^5}{1} + \cdots. \tag{21.1}$$

Entry 21 was first proved in print by Selberg [1, eq. (54)]. Another proof has been given by Ramanathan [4]. We provide yet another proof based on Entry 15.

Proof. Applying Entry 15 with $a = 1$ and $b = -1$, we find that

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n}} = 1 + \frac{q}{1} + \frac{q^2+q}{1} + \frac{q^3}{1} + \frac{q^4+q^2}{1} + \cdots.$$

(Alternatively, this can also be proved by using (13.5) in Chapter 16 of Part III [3, p. 28] with $a = -1$ and $b = 1$.) Using Euler's identity (Andrews [4, p. 19]) once again, we find that the numerator and denominator on the left side above are, respectively, $(-q;q^2)_\infty$ and $(-q^2;q^2)_\infty$. The desired result now follows.

If $Q(q)$ denotes the left side of (21.1), then, by using Entries 22(i), (ii) and 25(vii) of Chapter 16 (Part III [3, pp. 36, 40]), we can easily show that

$$Q^8(q) = \frac{\psi^4(q^2)}{\varphi^4(q)} = \frac{x}{16q},$$

in the notation of Entries 5 and 6 of Chapter 17 (Part III [3, pp. 100–102]). Thus, modular equations for $Q(q)$ can be trivially derived from any of Ramanujan's modular equations.

Entry 22 (Formula (6), p. 290). *For* $|q| < 1$,

$$\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty} = \frac{1}{1+}\ \frac{q+q^2}{1}\ \frac{}{+}\ \frac{q^4}{1}\ \frac{}{+}\ \frac{q^3+q^6}{1}\ \frac{}{+}\ \frac{q^8}{1}\ \frac{}{+\cdots}.$$

The first published proof known to us is by Selberg [1, eq. (53)]. Other proofs have been given by Andrews [5] and Ramanathan [4]. Entry 22 also appears in Ramanujan's "lost notebook" [11]. Another continued fraction for the left side of Entry 22 has been found by Andrews [1] and Gordon [1]. Chan and Huang [1] have developed an extensive elegant theory for these continued fractions, including modular equations and explicit evaluations.

Entry 23 (p. 373). *For* $|q| < 1$,

$$\frac{f(-q,-q^7)}{f(-q^3,-q^5)} = \frac{1}{1+}\ \frac{q+q^2}{1}\ \frac{}{+}\ \frac{q^4}{1}\ \frac{}{+}\ \frac{q^3+q^6}{1}\ \frac{}{+}\ \frac{q^8}{1}\ \frac{}{+\cdots}.$$

By the Jacobi triple product identity, we may rewrite Entry 23 in the form

$$\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty} = \frac{1}{1+}\ \frac{q+q^2}{1}\ \frac{}{+}\ \frac{q^4}{1}\ \frac{}{+}\ \frac{q^3+q^6}{1}\ \frac{}{+}\ \frac{q^8}{1}\ \frac{}{+\cdots}.$$

Hence, Entry 23 is equivalent to Entry 22.

3. Continued Fractions Arising from Products of Gamma Functions

For the first result, we quote Ramanujan [9, vol. 2, p. 281]. We have

$$\begin{aligned}&\sum\frac{1}{x}-\sum\frac{1}{x/3}+\frac{1}{x}-\log 3\\&=\frac{2/3}{x^2}\ \frac{}{+}\ \frac{2^3-2}{6}\ \frac{}{+}\ \frac{4^3-4}{3x^2}\ \frac{}{+}\ \frac{5^3-5}{6}\ \frac{}{+}\ \frac{7^3-7}{5x^2}\ \frac{}{+\cdots}.\end{aligned}\tag{24.1}$$

The symbol $\sum 1/x$ denotes $\sum_{k\le x} 1/k$. However, (24.1) is clearly incorrect with this interpretation, because the left side is discontinuous for positive, integral x, while the right side is continuous for such x. Now if x is a nonnegative integer, then (M. Abramowitz and I. A. Stegun [1, p. 259])

$$\psi(x+1)+\gamma=\sum_{k\le x}1/k,\tag{24.2}$$

where γ denotes Euler's constant and $\psi(x)=\Gamma'(x)/\Gamma(x)$. As in Chapter 6 (Part I [1, p. 138]), we shall then take the left side of (24.2) as our interpretation of $\sum 1/x$ for all positive numbers x.

We now reinterpret (24.1).

Entry 24 (p. 281). *If* $\operatorname{Re} x > 0$,

$$\begin{aligned}
&\psi(x+1) - \psi\left(\tfrac{1}{3}x + 1\right) + \frac{1}{x} - \log 3 \\
&= \frac{2/3}{x^2} + \frac{2^3-2}{6} + \frac{4^3-4}{3x^2} + \frac{5^3-5}{6} + \frac{7^3-7}{5x^2} + \cdots \\
&\quad + \frac{(3k-1)^3-(3k-1)}{6} + \frac{(3k+1)^3-(3k+1)}{(2k+1)x^2} + \cdots.
\end{aligned} \tag{24.3}$$

Proof. If ℓ, m, or n is an integer or if $\operatorname{Re} x > 0$, then by Entry 35 of Chapter 12 (Part II [2, pp. 156, 157], Jacobsen [4]),

$$\begin{aligned}
\frac{1-P}{1+P} &= \frac{2\ell mn}{x^2-\ell^2-m^2-n^2+1} \\
&\quad + \mathop{\mathrm{K}}_{j=1}^{\infty} \frac{4(\ell^2-j^2)(m^2-j^2)(n^2-j^2)}{(2j+1)\{x^2-\ell^2-m^2-n^2+2j^2+2j+1\}},
\end{aligned} \tag{24.4}$$

where

$$P = \frac{\Gamma(\frac{1}{2}\{x+\ell+m+n+1\})\Gamma(\frac{1}{2}\{x+\ell-m-n+1\})\Gamma(\frac{1}{2}\{x-\ell+m-n+1\})\Gamma(\frac{1}{2}\{x-\ell-m+n+1\})}{\Gamma(\frac{1}{2}\{x-\ell-m-n+1\})\Gamma(\frac{1}{2}\{x-\ell+m+n+1\})\Gamma(\frac{1}{2}\{x+\ell-m+n+1\})\Gamma(\frac{1}{2}\{x+\ell+m-n+1\})}.$$

Dividing both sides of (24.4) by m and letting m tend to 0, we find, upon an application of L'Hospital's rule, that

$$\begin{aligned}
&\tfrac{1}{2}\left\{\psi\left(\tfrac{1}{2}\{x+\ell-n+1\}\right) + \psi\left(\tfrac{1}{2}\{x-\ell+n+1\}\right)\right. \\
&\quad \left.-\psi\left(\tfrac{1}{2}\{x+\ell+n+1\}\right) - \psi\left(\tfrac{1}{2}\{x-\ell-n+1\}\right)\right\} \\
&= \frac{2\ell n}{x^2-\ell^2-n^2+1} + \mathop{\mathrm{K}}_{j=1}^{\infty} \frac{-4j^2(\ell^2-j^2)(n^2-j^2)}{(2j+1)\{x^2-\ell^2-n^2+2j^2+2j+1\}}.
\end{aligned} \tag{24.5}$$

We must justify the limiting procedure. Let ℓ, n, and x be fixed with $\operatorname{Re} x > 0$. Of course, if the continued fraction terminates, no justification is necessary. So assume that ℓ and n are not integral and that $|m| < 1$. Let $\mathbf{K}(a_k(m)/b_k(m))$ denote the continued fraction in (24.4), but with the first partial numerator divided by m, and let $f_k(m)$ and $f(m)$ denote the continued fraction's kth approximant and value, respectively. Without loss of generality, we assume that $f(m) \neq \infty$ in a neighborhood of $m = 0$. (Otherwise, consider $1/(1 + \mathbf{K}(a_k(m)/b_k(m)))$.) Furthermore, let g_k denote the kth approximant of the continued fraction in (24.5). We want to prove that

$$\lim_{k\to\infty} g_k = \lim_{m\to 0} f(m) =: f(0). \tag{24.6}$$

Suppose that the convergence of $f_k(m)$ to $f(m)$ is uniform with respect to m in a neighborhood of $m = 0$, i.e., for some $\epsilon > 0$ and $|m| < \epsilon$,

$$|f(m) - f_k(m)| < \lambda_k, \tag{24.7}$$

where λ_k tends to 0 as k tends to ∞. Also suppose that $f_k(m)$ tends to g_k as m tends to 0, uniformly with respect to k in a neighborhood of $k = \infty$, i.e., there exists an integer k_0 such that for $k \geq k_0$,

$$|g_k - f_k(m)| < r(m), \tag{24.8}$$

where $r(m)$ approaches 0 as m tends to 0. Now,

$$|f(0) - g_k| \leq |f(0) - f(m)| + |f(m) - f_k(m)| + |f_k(m) - g_k|.$$

Thus, (24.6) follows provided that (24.7) and (24.8) hold. Indeed, these two statements of uniform convergence follow from the uniform parabola theorem (W. J. Thron [2]).

Next, the even part (see (64.1) below) of the continued fraction in (24.3) is given by

$$\begin{aligned}
\mathrm{CFE}(x) :=& \frac{4}{2^3 - 2 + 6x^2} - \frac{6(2^3-2)(4^3-4)}{6\{4^3 - 4 + 5^3 - 5 + 6\cdot 3x^2\}} \\
&- \frac{6^2(5^3-5)(7^3-7)}{6\{7^3 - 7 + 8^3 - 8 + 6\cdot 5x^2\}} - \cdots \\
&- \frac{6^2\{(3k-1)^3 - (3k-1)\}\{(3k+1)^3 - (3k+1)\}}{6\{(3k+1)^3 - (3k+1) + (3k+2)^3 - (3k+2) + 6(2k+1)x^2\}} - \cdots \\
=& \frac{4/6}{x^2+1} - \frac{4^3-4}{4^3 - 4 + 5^3 - 5 + 18x^2} - \frac{5\cdot 7(5^2-1)(7^2-1)}{7^3 - 7 + 8^3 - 8 + 30x^2} - \cdots \\
&- \frac{9k^2(9k^2-1)(9k^2-4)}{3(2k+1)\{9k^2 + 9k + 2 + 2x^2\}} - \cdots \\
=& \frac{8/27}{4(x^2+1)/9} - \frac{4\cdot 1^2(1^2 - \frac{1}{9})(1^2 - \frac{4}{9})}{3\{2\cdot 1^2 + 2\cdot 1 + 4(x^2+1)/9\}} \\
&- \frac{4\cdot 2^2(2^2 - \frac{1}{9})(2^2 - \frac{4}{9})}{5\{2\cdot 2^2 + 2\cdot 2 + 4(x^2+1)/9\}} - \cdots \\
&- \frac{4k^2\left(k^2 - \left(\frac{1}{3}\right)^2\right)\left(k^2 - \left(\frac{2}{3}\right)^2\right)}{(2k+1)\{2k^2 + 2k + 4(x^2+1)/9\}} - \cdots.
\end{aligned}$$

Hence, by (24.5), with $\ell = \frac{1}{3}$, $n = \frac{2}{3}$, and x replaced by $2x/3$,

$$\mathrm{CFE}(x) = \frac{2}{3}\frac{1}{2}\left\{\psi\left(\tfrac{1}{3}(x+1)\right) + \psi\left(\tfrac{1}{3}(x+2)\right) - \psi\left(\tfrac{1}{3}x + 1\right) - \psi\left(\tfrac{1}{3}x\right)\right\}. \tag{24.9}$$

Since (Abramowitz and Stegun [1, p. 259])

$$\psi(z) = -\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k-1+z}\right),$$

where γ denotes Euler's constant, we find that

$$\begin{aligned}
&\psi\left(\tfrac{1}{3}(x+1)\right)+\psi\left(\tfrac{1}{3}(x+2)\right)-\psi\left(\tfrac{1}{3}x+1\right)-\psi\left(\tfrac{1}{3}x\right)\\
&=3\sum_{k=1}^{\infty}\left(-\frac{1}{3k+x-2}-\frac{1}{3k+x-1}+\frac{1}{3k+x}+\frac{1}{3k+x-3}\right)\\
&=3\sum_{k=1}^{\infty}\left(-\frac{1}{3k+x-2}-\frac{1}{3k+x-1}-\frac{1}{3k+x}+\frac{3}{3k+x}\right)+\frac{3}{x}\\
&=3\sum_{k=1}^{\infty}\left(-\frac{1}{3k+x-2}-\frac{1}{3k+x-1}-\frac{1}{3k+x}+\frac{1}{3k-2}+\frac{1}{3k-1}+\frac{1}{3k}\right)\\
&\quad+3\sum_{k=1}^{\infty}\left(\frac{1}{3k+x}-\frac{1}{3k-2}\right)+3\sum_{k=1}^{\infty}\left(\frac{1}{3k+x}-\frac{1}{3k-1}\right)\\
&\quad+3\sum_{k=1}^{\infty}\left(\frac{1}{3k+x}-\frac{1}{3k}\right)+\frac{3}{x}\\
&=3\sum_{k=1}^{\infty}\left(-\frac{1}{k+x}+\frac{1}{k}\right)+3\sum_{k=1}^{\infty}\left(\frac{1}{k+x/3}-\frac{1}{k}\right)+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k-2/3}\right)\\
&\quad+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k-1/3}\right)+\frac{3}{x}\\
&=3\psi(x+1)-3\psi\left(\frac{1}{3}x+1\right)+\frac{3}{x}\\
&\quad+3\lim_{N\to\infty}\sum_{k=1}^{N}\left(\frac{1}{k}-\frac{1}{3k-2}-\frac{1}{3k-1}-\frac{1}{3k}\right)\\
&=3\psi(x+1)-3\psi\left(\frac{1}{3}x+1\right)+\frac{3}{x}+3\lim_{N\to\infty}(\log N-\log 3N)\\
&=3\psi(x+1)-3\psi\left(\frac{1}{3}x+1\right)+\frac{3}{x}-3\log 3.
\end{aligned}$$

Using the foregoing calculation in (24.9), we complete the proof.

Entry 25 (Formula (5), p. 292). *If* $\operatorname{Re} x > 0$, *then*

$$\begin{aligned}
&\frac{1}{2x^2}+\sum_{k=1}^{\infty}\frac{1}{(x+k)^2}\\
&=\frac{1}{x}+\frac{1}{2x^2}\left\{\frac{1}{3x}+\frac{3}{5x}+\frac{18}{7x}+\cdots+\frac{k^2(k^2-1)/4}{(2k+1)x}+\cdots\right\}.
\end{aligned}$$

Proof. Replacing x by $2x$ in Entry 30 of Chapter 12 (Part II [2, p. 149]) we find that

$$T(n,x) := \sum_{k=0}^{\infty} \left\{ \frac{1}{2x-n+2k+1} - \frac{1}{2x+n+2k+1} \right\}$$
$$= \frac{(n/2)}{x} + \frac{1^2(1^2-n^2)/4}{3x} + \frac{2^2(2^2-n^2)/4}{5x} + \cdots$$
$$= \frac{(n/2)}{x+(1-n)F(n,x)},$$

where

$$F(n,x) = \frac{1^2(1+n)/4}{3x} + \frac{2^2(2^2-n^2)/4}{5x} + \cdots.$$

After some elementary algebra, we find that

$$\frac{n-2xT(n,x)}{2T(n,x)(1-n)} = F(n,x).$$

Letting n tend to 1, applying L'Hospital's rule, and using the facts $T(1,x) = 1/(2x)$ and $\frac{\partial T}{\partial n}(1,x) \neq 0$, we find that

$$\lim_{n\to 1} F(n,x) = \lim_{n\to 1} \frac{1-2x\frac{\partial T}{\partial n}(n,x)}{-2T(n,x)+2(1-n)\frac{\partial T}{\partial n}(n,x)}$$
$$= \frac{1-2x\frac{\partial T}{\partial n}(1,x)}{-1/x}$$
$$= -x + 2x^2 \sum_{k=0}^{\infty} \left\{ \frac{1}{(2x+2k)^2} + \frac{1}{(2x+2k+2)^2} \right\}$$
$$= -x + \frac{1}{2} + x^2 \sum_{k=1}^{\infty} \frac{1}{(x+k)^2}.$$

Since

$$F(1,x) = \frac{1}{2}\frac{1}{3x} + \frac{2^2(2^2-1)/4}{5x} + \frac{3^2(3^2-1)/4}{7x} + \cdots,$$

we see that the proof is completed after a little algebraic manipulation and an appeal to uniform convergence as in the proof of Entry 24.

Entry 26 (Formula (6), p. 293). *If* $\operatorname{Re} x > 0$, *then*

$$\frac{1}{2x^3} + \sum_{k=1}^{\infty} \frac{1}{(x+k)^3} = \frac{1}{2x^2} + \frac{1}{4x^3} \left\{ \frac{1}{x} + \frac{p_1}{x} + \frac{q_1}{x} + \frac{p_2}{x} + \frac{q_2}{x} + \cdots \right\},$$

where, for $k \geq 1$, $p_k = k^2(k+1)/(4k+2)$ *and* $q_k = k(k+1)^2/(4k+2)$.

Entry 26 is, in fact, due to T. J. Stieltjes [1], [3, pp. 378–391]. It is also given in Wall's book [1, p. 37], where $4z^2$ should be replaced by $4z^3$.

Entry 27 (p. 325). *Let n be a complex number such that* $\operatorname{Re} n > -\frac{1}{2}$. *Then*

$$\sum_{k=1}^{\infty} \frac{1}{(n+k)^2} = \left(n+\frac{1}{2}\right)\left\{\frac{1}{n^2+n}+\frac{(1\cdot 1)^2}{3}+\frac{(1\cdot 1)^2}{5(n^2+n)}+\frac{(2\cdot 3)^2}{7}\right.$$
$$\left.+\frac{(2\cdot 3)^2}{9(n^2+n)}+\frac{(3\cdot 5)^2}{11}+\frac{(3\cdot 5)^2}{13(n^2+n)}+\cdots\right\}. \tag{27.1}$$

Proof. From the corollary to Entry 30 of Chapter 12 of the second notebook (Part II [2, p. 150]), we find that, for $x = 2n+1$ and $\operatorname{Re} x = \operatorname{Re}(2n+1) > 0$,

$$\sum_{k=1}^{\infty} \frac{1}{(n+k)^2} = 4\sum_{k=1}^{\infty} \frac{1}{(x+2k-1)^2} = 2\left\{\frac{1}{x}+\frac{1^4}{3x}+\frac{2^4}{5x}+\frac{3^4}{7x}+\cdots\right\}$$
$$= 4\left(n+\frac{1}{2}\right)\left\{\frac{1}{1(4n^2+4n+1)}+\frac{1^4}{3}+\frac{2^4}{5(4n^2+4n+1)}+\frac{3^4}{7}+\cdots\right\}. \tag{27.2}$$

We shall use the Bauer–Muir transformation (13.7) to prove that this continued fraction converges to the same value as the one presented in this entry.

Choose $\omega_{2k} = 0$ and $\omega_{2k+1} = -(2k+1)^2$ for each nonnegative integer k. From (13.6),

$$\lambda_1 = 1, \quad \lambda_{2k} = 4k^2(2k-1)^2, \quad \text{and} \quad \lambda_{2k+1} = 16k^4,$$

for each positive integer k. Moreover,

$$a_{2k}\lambda_{2k+1}/\lambda_{2k} = 4k^2(2k-1)^2,$$
$$a_{2k+1}\lambda_{2k+2}/\lambda_{2k+1} = 4(k+1)^2(2k+1)^2,$$
$$b_{2k}+\omega_{2k}-\omega_{2k-2}\lambda_{2k}/\lambda_{2k-1} = 4k-1,$$

and

$$b_{2k+1}+\omega_{2k+1}-\omega_{2k-1}\lambda_{2k+1}/\lambda_{2k} = (4k+1)(4n^2+4n).$$

Hence, the continued fraction in (27.2) is transformed by (13.7) into the continued fraction

$$\frac{1}{4n^2+4n}+\frac{4(1\cdot 1)^2}{3}+\frac{4(1\cdot 1)^2}{5(4n^2+4n)}+\frac{4(2\cdot 3)^2}{7}+\frac{4(2\cdot 3)^2}{9(4n^2+4n)}+\cdots$$
$$= \frac{1}{n^2+n}+\frac{(1\cdot 1)^2}{3}+\frac{(1\cdot 1)^2}{5(n^2+n)}+\frac{(2\cdot 3)^2}{7}+\frac{(2\cdot 3)^2}{9(n^2+n)}+\cdots.$$

Since the even approximants of the two continued fractions coincide and since both continued fractions converge for $\operatorname{Re} n > -\frac{1}{2}$, the proof of Entry 27 is complete.

In the foregoing proof we have seen that the even approximants of the continued fractions in (27.1) and (27.2) are identical. Thus, an alternative proof can be derived by showing that the even parts of (27.1) and (27.2) agree. Such a proof would be shorter and simpler but not as instructive as the constructive approach via the Bauer–Muir transformation.

If we let $n = 1$ in Entry 27, we find that

$$\frac{\pi^2}{6} = \zeta(2) = 1 + \frac{3}{2}\left\{\frac{1}{2+}\,\frac{1^2\cdot 1^2}{3}\,\frac{}{+}\,\frac{1^2\cdot 1^2}{10}\,\frac{}{+}\,\frac{2^2\cdot 3^2}{7}\,\frac{}{+}\,\frac{2^2\cdot 3^2}{18}\,\frac{}{+\cdots}\right\},$$

where ζ denotes the Riemann zeta-function.

Letting $n = 1$ in Entry 28 below, we deduce that

$$\log 2 = 1 - \frac{3}{2}\left\{\frac{1}{4+}\,\frac{1\cdot 1}{1}\,\frac{}{+}\,\frac{1\cdot 1}{4}\,\frac{}{+}\,\frac{2\cdot 3}{1}\,\frac{}{+}\,\frac{2\cdot 3}{4}\,\frac{}{+\cdots}\right\}.$$

Entry 28 (p. 325). *Let n be a complex number such that* $\operatorname{Re} n > -\frac{1}{2}$. *Then*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{n+k} = \left(n+\frac{1}{2}\right)\left\{\frac{1}{2n^2+2n}\,\frac{}{+}\,\frac{1\cdot 1}{1}\,\frac{}{+}\,\frac{1\cdot 1}{2n^2+2n}\,\frac{}{+}\,\frac{2\cdot 3}{1}\right.$$
$$\left.\frac{}{+}\,\frac{2\cdot 3}{2n^2+2n}\,\frac{}{+}\,\frac{3\cdot 5}{1}\,\frac{}{+}\,\frac{3\cdot 5}{2n^2+2n}\,\frac{}{+\cdots}\right\}.$$

Proof. From the corollary to Entry 29 of Chapter 12 of Ramanujan's second notebook (Part II [2, p. 149]), it follows that

$$2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{x+2k-1} = \frac{1}{x+}\,\frac{1^2}{x+}\,\frac{2^2}{x+}\,\frac{3^2}{x+}\cdots,$$

for $\operatorname{Re} x > 0$. Setting $x = 2n+1$, we find, via equivalence transformations, that

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{n+k} &= \frac{1}{2n+1+}\,\frac{1^2}{2n+1+}\,\frac{2^2}{2n+1+}\,\frac{3^2}{2n+1+}\cdots\\ &= (2n+1)\left\{\frac{1}{(2n+1)^2+}\,\frac{1^2}{1+}\,\frac{2^2}{(2n+1)^2+}\,\frac{3^2}{1+}\cdots\right\}\\ &= \left(n+\frac{1}{2}\right)\left\{\frac{1}{2n^2+2n+\frac{1}{2}+}\,\frac{\frac{1}{2}\cdot 1^2}{1+}\,\frac{\frac{1}{2}\cdot 2^2}{2n^2+2n+\frac{1}{2}+}\,\frac{\frac{1}{2}\cdot 3^2}{1+}\cdots\right\},\end{aligned} \tag{28.1}$$

for $\operatorname{Re} n > -\frac{1}{2}$.

We apply the Bauer–Muir transformation (13.7). Let $\omega_{2k} = 0$ and $\omega_{2k+1} = -(2k+1)/2$ for each integer $k \ge 0$. Then from (13.6),

$$\lambda_1 = 1, \quad \lambda_{2k} = k(2k-1), \quad \text{and} \quad \lambda_{2k+1} = 2k^2,$$

for each positive integer k. Furthermore,

$$a_{2k}\lambda_{2k+1}/\lambda_{2k} = k(2k-1),$$
$$a_{2k+1}\lambda_{2k+2}/\lambda_{2k+1} = (k+1)(2k+1),$$
$$b_{2k} + \omega_{2k} - \omega_{2k-2}\lambda_{2k}/\lambda_{2k-1} = 1,$$

and

$$b_{2k+1} + \omega_{2k+1} - \omega_{2k-1}\lambda_{2k+1}/\lambda_{2k} = 2n^2 + 2n.$$

Thus, the Bauer–Muir transformation transforms the continued fraction in (28.1) into the continued fraction

$$\frac{1}{2n^2+2n}\,+\,\frac{1\cdot 1}{1}\,+\,\frac{1\cdot 1}{2n^2+2n}\,+\,\frac{2\cdot 3}{1}\,+\,\frac{2\cdot 3}{2n^2+2n}\,+\cdots.$$

Since the even approximants of the two continued fractions coincide and since both continued fractions converge for $\operatorname{Re} n > -\frac{1}{2}$, the proof is complete.

Alternatively, the even part of the last continued fraction in (28.1) is precisely the continued fraction in Entry 28. This gives an even shorter proof.

Entry 29 (p. 343). *Let x and n be complex numbers such that either* $\operatorname{Re} x > 0$ *or* $n = (2k+1)i$ *for some integer k. Then*

$$\frac{\Gamma^2\left(\frac{1}{4}(x+1)\right)}{\Gamma^2\left(\frac{1}{4}(x+3)\right)}\prod_{k=0}^{\infty}\frac{1+\dfrac{n^2}{(x+3+4k)^2}}{1+\dfrac{n^2}{(x+1+4k)^2}}$$
$$=\frac{4}{x}\,+\,\frac{n^2+1^2}{2x}\,+\,\frac{n^2+3^2}{2x}\,+\,\frac{n^2+5^2}{2x}\,+\cdots.$$

Proof. Replacing n by in in Entry 25 of Chapter 12 (Part II [2, p. 140]), we find that, under the conditions specified above,

$$\begin{aligned}&\frac{\Gamma\left(\frac{1}{4}(x+in+1)\right)\Gamma\left(\frac{1}{4}(x-in+1)\right)}{\Gamma\left(\frac{1}{4}(x+in+3)\right)\Gamma\left(\frac{1}{4}(x-in+3)\right)}\\ &=\frac{4}{x}\,+\,\frac{n^2+1^2}{2x}\,+\,\frac{n^2+3^2}{2x}\,+\,\frac{n^2+5^2}{2x}\,+\cdots.\end{aligned}\tag{29.1}$$

However, from Euler's product formula for the gamma function,

$$\Gamma\left(\tfrac{1}{4}(x+in+1)\right)\Gamma\left(\tfrac{1}{4}(x-in+1)\right) = \frac{\Gamma^2\left(\frac{1}{4}(x+1)\right)}{\prod_{k=0}^{\infty}\left\{1+\dfrac{n^2}{(x+1+4k)^2}\right\}}.$$

Using this formula and an analogous formula in (29.1), we complete the proof. (A product representation for $|\Gamma(x+iy)|^2$ is given in Gradshteyn and Ryzhik's tables [1, p. 945, formula 8.326, no. 1].)

Entry 30 (p. 343). *For all complex* n,

$$\frac{\tanh(\pi n/4)}{n}=\frac{1}{1+}\,\frac{n^2+1^2}{2}\,\frac{}{+}\,\frac{n^2+3^2}{2}\,\frac{}{+}\,\frac{n^2+5^2}{2}\,\frac{}{+\cdots}.$$

Proof. Setting $x=1$ in Entry 29, we deduce that

$$\frac{\pi}{4}\prod_{k=0}^{\infty}\frac{1+\left(\dfrac{n}{4(k+1)}\right)^2}{1+\left(\dfrac{n}{2(2k+1)}\right)^2}=\frac{1}{1+}\,\frac{n^2+1^2}{2}\,\frac{}{+}\,\frac{n^2+3^2}{2}\,\frac{}{+}\,\frac{n^2+5^2}{2}\,\frac{}{+\cdots}.$$

However, by a familiar product representation for $\tanh z$, the left side above equals $(1/n)\tanh(\pi n/4)$.

Entry 30 may also be found in O. Perron's book [1, p. 36, eq. (23)]. Entry 31 below is also in Perron's text [1, p. 33].

Entry 31 (p. 343). *Let* x *and* n *be complex numbers such that either* $\operatorname{Re} x>0$ *or* $n=(2k+1)i$ *for some integer* k. *Then*

$$\begin{aligned}&2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x+2k-1}{(x+2k-1)^2+n^2}\\&=\frac{1}{x+}\,\frac{n^2+1^2}{x}\,\frac{}{+}\,\frac{2^2}{x+}\,\frac{n^2+3^2}{x}\,\frac{}{+}\,\frac{4^2}{x+}\,\frac{n^2+5^2}{x}\,\frac{}{+\cdots}.\end{aligned}$$

Proof. Replacing n by in in Entry 29 of Chapter 12 (Part II [2, pp. 147, 148]), we find that, under the conditions given above,

$$\begin{aligned}&2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x+2k-1}{(x+2k-1)^2+n^2}\\&=\sum_{k=1}^{\infty}(-1)^{k+1}\left\{\frac{1}{x+ni+2k-1}+\frac{1}{x-ni+2k-1}\right\}\\&=\frac{1}{x+}\,\frac{n^2+1^2}{x}\,\frac{}{+}\,\frac{2^2}{x+}\,\frac{n^2+3^2}{x}\,\frac{}{+}\,\frac{4^2}{x+\cdots}.\end{aligned}$$

Entry 32 (p. 344). *For all complex numbers* n,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{k^2+n^2}=\frac{1}{1+}\,\frac{4n^2+1^2}{1}\,\frac{}{+}\,\frac{2^2}{1+}\,\frac{4n^2+3^2}{1}\,\frac{}{+}\,\frac{4^2}{1+\cdots}.$$

Proof. Set $x=1$ and replace n by $2n$ in Entry 31.

In the continued fraction of Entry 32, Ramanujan inadvertently wrote n^2 for $4n^2$.

Entry 33 (p. 344). *Let x and n be complex numbers such that either* $\operatorname{Re} x > 0$ *or* $n = (2k+1)i$ *for some integer k. Then*

$$2\sum_{k=0}^{\infty} \frac{1}{(x+2k+1)^2+n^2} = \frac{1}{x+}\,\frac{1^2(n^2+1^2)}{3x+}\,\frac{2^2(n^2+2^2)}{5x+}\,\frac{3^2(n^2+3^2)}{7x+}\cdots.$$

Proof. In Entry 30 of Chapter 12 (Part II [2, p. 149]) merely replace n by in, and the desired result immediately follows.

Entry 34 (p. 343). *For every complex number n,*

$$\frac{\pi n}{2}\coth\left(\frac{\pi n}{2}\right) = 1 + \frac{n^2}{1+}\,\frac{1^2(n^2+1^2)}{3+}\,\frac{2^2(n^2+2^2)}{5+}\,\frac{3^2(n^2+3^2)}{7+}\cdots.$$

Proof. It is clear that the identity holds for $n = 0$. Thus, assume that $n \neq 0$. Setting $x = 1$ in Entry 33, we find that

$$\frac{\pi}{2n}\left(\coth\left(\frac{\pi n}{2}\right) - \frac{2}{\pi n}\right) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(k+1)^2+n^2/4} = \frac{1}{1+}\,\frac{1^2(n^2+1^2)}{3+}\,\frac{2^2(n^2+2^2)}{5+}\cdots.$$

Upon multiplying both sides by n^2 and rearranging, we complete the proof.

Entry 35 (p. 344). *Let x and n be complex numbers such that either* $\operatorname{Re} x > -\frac{1}{2}$ *with $x \notin (-\frac{1}{2}, 0]$, or $n = ki$ for some integer k. Then*

$$2\sum_{k=0}^{\infty}\frac{(-1)^k}{(x+1+k)^2+n^2} = \frac{1}{x^2+x+}\,\frac{n^2+1^2}{1+}\,\frac{1^2}{x^2+x+}\,\frac{n^2+2^2}{1+}\,\frac{2^2}{x^2+x+}\cdots.$$

Proof. In Entry 31 of Chapter 12 (Part II [2, p. 150]), replace n by $2in$ and x by $2x+1$. Then, under the proposed hypotheses we find that

$$\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(x+1+k)^2+n^2} = \frac{1}{4x^2+4x+}\,\frac{4+4n^2}{1+}\,\frac{2^2}{4x^2+4x+}\,\frac{4^2+4n^2}{1+}\,\frac{4^2}{4x^2+4x+}\cdots,$$

which is equivalent to the continued fraction displayed in Entry 35.

Entry 36 (p. 344). *Let m, n, and x denote complex numbers such that either* $\operatorname{Re} x > 0$, *or* $m = ij$ *for an integer* j *and* $x \neq -(2k+1)$ *for any nonnegative integer* k, *or* $n = ij$ *for an integer* j *and* $x \neq -(2k+1)$ *for any nonnegative integer* k. *Furthermore, let*

$$u = \prod_{k=0}^{\infty}\left\{1 + \left(\frac{m+n}{x+2k+1}\right)^2\right\} \quad \textit{and} \quad v = \prod_{k=0}^{\infty}\left\{1 + \left(\frac{m-n}{x+2k+1}\right)^2\right\}.$$

Then

$$\frac{u-v}{u+v} = \frac{mn}{x\,+}\;\frac{(m^2+1^2)(n^2+1^2)}{3x\,+}\;\frac{(m^2+2^2)(n^2+2^2)}{5x\,+\cdots}.$$

Proof. Assume that x, m, and n are positive. In Entry 33 of Chapter 12 (Part II [2, p. 155]), replace m and n by im and in, respectively. Employing a product formula for $|\Gamma(x+iy)|^2$ (Gradshteyn and Ryzhik [1, p. 945]), we find that, under the given hypotheses,

$$\begin{aligned}
&\frac{-mn}{x\,+}\;\frac{(m^2+1^2)(n^2+1^2)}{3x\,+}\;\frac{(m^2+2^2)(n^2+2^2)}{5x\,+\cdots} \\
&= \frac{\left|\Gamma\left(\frac{1}{2}(x+1)+\frac{1}{2}(m+n)i\right)\right|^2 - \left|\Gamma\left(\frac{1}{2}(x+1)+\frac{1}{2}(m-n)i\right)\right|^2}{\left|\Gamma\left(\frac{1}{2}(x+1)+\frac{1}{2}(m+n)i\right)\right|^2 + \left|\Gamma\left(\frac{1}{2}(x+1)+\frac{1}{2}(m-n)i\right)\right|^2} \\
&= \frac{\prod_{k=0}^{\infty}\left\{1+\frac{(m+n)^2}{(x+2k+1)^2}\right\}^{-1} - \prod_{k=0}^{\infty}\left\{1+\frac{(m-n)^2}{(x+2k+1)^2}\right\}^{-1}}{\prod_{k=0}^{\infty}\left\{1+\frac{(m+n)^2}{(x+2k+1)^2}\right\}^{-1} + \prod_{k=0}^{\infty}\left\{1+\frac{(m-n)^2}{(x+2k+1)^2}\right\}^{-1}} \\
&= \frac{1/u - 1/v}{1/u + 1/v} = \frac{v-u}{v+u}.
\end{aligned}$$

This completes the proof for $x > 0$, $m > 0$, and $n > 0$. Since the continued fraction converges to a meromorphic function of x for $\operatorname{Re} x > 0$, the entry holds in this half plane by analytic continuation. Furthermore, since the continued fraction converges to a meromorphic function of m and of n, it follows that Entry 36 is true for all complex m and n. That the equality holds if the continued fraction terminates follows by straightforward computation.

Entry 37 (p. 344). *If m and n are complex numbers with $m \neq n$, then*

$$\begin{aligned}
&\frac{m\tanh\left(\frac{1}{2}\pi n\right) - n\tanh\left(\frac{1}{2}\pi m\right)}{m\tanh\left(\frac{1}{2}\pi m\right) - n\tanh\left(\frac{1}{2}\pi n\right)} \\
&= \frac{mn}{1\,+}\;\frac{(m^2+1^2)(n^2+1^2)}{3\,+}\;\frac{(m^2+2^2)(n^2+2^2)}{5\,+\cdots}.
\end{aligned}$$

Proof. Putting $x = 1$ in Entry 36, we see that it only remains to show that $(u - v)/(u + v)$ reduces to the left side of Entry 37.

Using a familiar product representation for sinh z, we see that, when $x = 1$,

$$\begin{aligned}\frac{u-v}{u+v} &= \frac{(m+n)^{-1}\sinh\left\{\frac{1}{2}\pi(m+n)\right\} - (m-n)^{-1}\sinh\left\{\frac{1}{2}\pi(m-n)\right\}}{(m+n)^{-1}\sinh\left\{\frac{1}{2}\pi(m+n)\right\} + (m-n)^{-1}\sinh\left\{\frac{1}{2}\pi(m-n)\right\}} \\ &= \frac{-2n\sinh\left(\frac{1}{2}\pi m\right)\cosh\left(\frac{1}{2}\pi n\right) + 2m\sinh\left(\frac{1}{2}\pi n\right)\cosh\left(\frac{1}{2}\pi m\right)}{2m\sinh\left(\frac{1}{2}\pi m\right)\cosh\left(\frac{1}{2}\pi n\right) - 2n\sinh\left(\frac{1}{2}\pi n\right)\cosh\left(\frac{1}{2}\pi m\right)} \\ &= \frac{m\tanh\left(\frac{1}{2}\pi n\right) - n\tanh\left(\frac{1}{2}\pi m\right)}{m\tanh\left(\frac{1}{2}\pi m\right) - n\tanh\left(\frac{1}{2}\pi n\right)},\end{aligned}$$

and so the proof is complete.

Entry 38 (p. 345). *Let m and x be complex numbers such that either* $\operatorname{Re} x > 0$, *or $m = k(1+i)/2$, or $m = k(1-i)/2$, for some integer k. Furthermore, set*

$$u = \prod_{k=0}^{\infty}\left\{1 + \left(\frac{2m}{x+2k+1}\right)^2\right\}$$

and

$$v = \frac{\Gamma^2\left(\frac{1}{2}(x+1)\right)}{\Gamma\left(\frac{1}{2}(x+2m+1)\right)\Gamma\left(\frac{1}{2}(x-2m+1)\right)}.$$

Then

$$\frac{u-v}{u+v} = \frac{2m^2}{x} \, + \, \frac{4m^4+1^4}{3x} \, + \, \frac{4m^4+2^4}{5x} \, + \, \frac{4m^4+3^4}{7x} \, + \cdots.$$

Note that if $m = k(1 \pm i)/2$ for some integer k, the continued fraction terminates.

Proof. We apply Entry 33 of Chapter 12 (Part II [2, p. 155]) with m and n replaced by $(1+i)m$ and $(1-i)m$, respectively. We then find that, for $m > 0$ and $x > 0$, or for $(1 \pm i)m = k$ for some integer k,

$$\begin{aligned}&\frac{2m^2}{x} \, + \, \frac{4m^4+1^4}{3x} \, + \, \frac{4m^4+2^4}{5x} \, + \, \frac{4m^4+3^4}{7x} \, + \cdots \\ &= \frac{\Gamma\left(\frac{1}{2}(x+2m+1)\right)\Gamma\left(\frac{1}{2}(x-2m+1)\right) - \Gamma^2\left(\frac{1}{2}(x+1)\right)\prod_{k=0}^{\infty}\left\{1+\left(\frac{2m}{x+2k+1}\right)^2\right\}^{-1}}{\Gamma\left(\frac{1}{2}(x+2m+1)\right)\Gamma\left(\frac{1}{2}(x-2m+1)\right) + \Gamma^2\left(\frac{1}{2}(x+1)\right)\prod_{k=0}^{\infty}\left\{1+\left(\frac{2m}{x+2k+1}\right)^2\right\}^{-1}} \\ &= \frac{u-v}{u+v},\end{aligned}$$

where in the penultimate step, we used the same formula for $|\Gamma(x+iy)|^2$ that we used in the proofs of Entries 29 and 36. The result is then valid for all complex m and all complex x with $\operatorname{Re} x > 0$ by analytic continuation.

Entry 39 (p. 345). *For arbitrary complex n,*

$$\frac{\sinh(\pi n)-\sin(\pi n)}{\sinh(\pi n)+\sin(\pi n)}=\frac{2n^2}{1}\,+\,\frac{4n^4+1^4}{3}\,+\,\frac{4n^4+2^4}{5}\,+\,\frac{4n^4+3^4}{7}\,+\cdots.$$

Proof. Setting $x=1$ and replacing m by n in Entry 38, we deduce that

$$\begin{aligned}&\frac{2n^2}{1}\,+\,\frac{4n^4+1^4}{3}\,+\,\frac{4n^4+2^4}{5}\,+\cdots\\&=\frac{\prod_{k=0}^{\infty}\left\{1+\left(\frac{n}{k+1}\right)^2\right\}-\frac{1}{\Gamma(1+n)\Gamma(1-n)}}{\prod_{k=0}^{\infty}\left\{1+\left(\frac{n}{k+1}\right)^2\right\}+\frac{1}{\Gamma(1+n)\Gamma(1-n)}}\\&=\frac{(\pi n)^{-1}\sinh(\pi n)-(\pi n)^{-1}\sin(\pi n)}{(\pi n)^{-1}\sinh(\pi n)+(\pi n)^{-1}\sin(\pi n)},\end{aligned}$$

where we have employed a familiar product representation for $\sinh z$ and the reflection formula for the gamma function. This completes the proof.

Entry 40 (p. 345). *Let x and n be complex numbers such that either* $\operatorname{Re} x>-\frac{1}{2}$*; or ρn is an integer, where ρ is a sixth root of unity, and x is not a negative integer. Furthermore, let*

$$u=\prod_{k=1}^{\infty}\left\{1+\left(\frac{n}{x+k}\right)^3\right\}\quad\textit{and}\quad v=\prod_{k=1}^{\infty}\left\{1-\left(\frac{n}{x+k}\right)^3\right\}.$$

Then

$$\frac{u-v}{u+v}=\frac{n^3}{2x^2+2x+1}\,+\,\frac{n^6-1^6}{3(2x^2+2x+3)}\,+\,\frac{n^6-2^6}{5(2x^2+2x+7)}\,+\cdots.$$

The constant term within parentheses in the kth denominator is given by k^2-k+1. In the notebooks, Ramanujan mistakenly indicated that this constant is equal to $2k-1$. Thus, Ramanujan wrote 5 instead of 7 in the third denominator displayed above. Ramanujan's error can be traced back to a scribal error in recording Entry 40 of Chapter 12. For a discussion of this error, see Part II [2, pp. 163, 164]. Note that if ρn is an integer, where ρ is a sixth root of unity, the continued fraction terminates.

Proof. We shall apply Entry 35 of Chapter 12 (Part II [2, pp. 156, 157], Jacobsen [4]) with $\ell=e^{2\pi i/3}n$, $m=e^{4\pi i/3}n$, and x replaced by $2x+1$. Observe that

$$(\ell^2-k^2)(m^2-k^2)(n^2-k^2)=n^6-k^6$$

and that $x^2-\ell^2-m^2-n^2+2k^2-2k+1$ is transformed into $4x^2+4x+2k^2-2k+2$. Thus, the continued fraction that arises from Entry 35 of Chapter 12 equals

$$\begin{aligned}&\frac{2n^3}{4x^2+4x+2}+\frac{4(n^6-1^6)}{3(4x^2+4x+6)}+\frac{4(n^6-2^6)}{5(4x^2+4x+14)}+\cdots\\&=\frac{n^3}{2x^2+2x+1}+\frac{n^6-1^6}{3(2x^2+2x+3)}+\frac{n^6-2^6}{5(2x^2+2x+7)}+\cdots.\end{aligned}\tag{40.1}$$

We now examine the gamma functions appearing in Entry 35 with the parametric designations given above. Letting $\omega = e^{2\pi i/3}$ and using Euler's product representation of the gamma function, we find that, in the notation of Entry 35 of Chapter 12,

$$\begin{aligned}P&=\prod_{j=0}^{2}\frac{\Gamma(x+1+\omega^j n)}{\Gamma(x+1-\omega^j n)}\\&=\lim_{m\to\infty}\prod_{k=0}^{m}\frac{(x+1+k-n)(x+1+k-\omega n)(x+1+k-\omega^2 n)}{(x+1+k+n)(x+1+k+\omega n)(x+1+k+\omega^2 n)}\\&=\prod_{k=0}^{\infty}\frac{(x+1+k)^3-n^3}{(x+1+k)^3+n^3}\\&=\prod_{k=0}^{\infty}\left\{\frac{1-\left(\dfrac{n}{x+1+k}\right)^3}{1+\left(\dfrac{n}{x+1+k}\right)^3}\right\}=\frac{v}{u}.\end{aligned}$$

Hence, by Entry 35 of Chapter 12, the continued fraction (40.1) equals

$$\frac{1-P}{1+P}=\frac{1-v/u}{1+v/u}=\frac{u-v}{u+v},$$

and the proof is complete.

Entry 41 (p. 347). *Suppose that m, n, and x are complex numbers such that* $\operatorname{Re} x > 0$, *or assume that n is an integer or that im is an integer. Then*

$$\begin{aligned}&\sum_{k=0}^{\infty}\left\{\tan^{-1}\left(\frac{m}{x-n+2k+1}\right)-\tan^{-1}\left(\frac{m}{x+n+2k+1}\right)\right\}\\&=\tan^{-1}\left\{\frac{mn}{x}+\frac{(1^2+m^2)(1^2-n^2)}{3x}+\frac{(2^2+m^2)(2^2-n^2)}{5x}+\cdots\right\}.\end{aligned}$$

Proof. In our proof below, we will temporarily ignore the fact that $\tan^{-1} z$ is multivalued. At the end of the proof, we shall show that the correct branches have been chosen.

Replacing m by im in Entry 33 of Chapter 12 (Part II [2, p. 155]), we find that, for $\operatorname{Re} x > 0$, or $n \in \mathbb{Z}$, or $im \in \mathbb{Z}$,

$$\frac{\Gamma(\frac{1}{2}(x+n+1)+\frac{1}{2}im)\Gamma(\frac{1}{2}(x-n+1)-\frac{1}{2}im)-\Gamma(\frac{1}{2}(x-n+1)+\frac{1}{2}im)\Gamma(\frac{1}{2}(x+n+1)-\frac{1}{2}im)}{\Gamma(\frac{1}{2}(x+n+1)+\frac{1}{2}im)\Gamma(\frac{1}{2}(x-n+1)-\frac{1}{2}im)+\Gamma(\frac{1}{2}(x-n+1)+\frac{1}{2}im)\Gamma(\frac{1}{2}(x+n+1)-\frac{1}{2}im)}$$
$$= \frac{imn}{x+}\;\frac{(1^2+m^2)(1^2-n^2)}{3x+}\;\frac{(2^2+m^2)(2^2-n^2)}{5x+\cdots}. \tag{41.1}$$

Suppose first that x, m, and n are positive, and let

$$\alpha = \Gamma\left(\tfrac{1}{2}(x+n+1)+\tfrac{1}{2}im\right)\Gamma\left(\tfrac{1}{2}(x-n+1)-\tfrac{1}{2}im\right) =: a+ib,$$

where a and b are real. Multiplying both sides of (41.1) by $-i$ and then taking the inverse tangent of each side, we see that

$$\begin{aligned}
&\tan^{-1}\left\{\frac{mn}{x+}\;\frac{(1^2+m^2)(1^2-n^2)}{3x+}\;\frac{(2^2+m^2)(2^2-n^2)}{5x+\cdots}\right\}\\
&= \tan^{-1}\left\{i\left(\frac{\bar{\alpha}-\alpha}{\bar{\alpha}+\alpha}\right)\right\} = \tan^{-1}(b/a) = \operatorname{Im}(\log\alpha) \qquad (41.2)\\
&= \operatorname{Im}\left(\log\left\{\Gamma\left(\tfrac{1}{2}(x+n+1)+\tfrac{1}{2}im\right)\Gamma\left(\tfrac{1}{2}(x-n+1)-\tfrac{1}{2}im\right)\right\}\right) =: T.
\end{aligned}$$

In order to calculate T, we employ Euler's product representation for the gamma function. Hence,

$$\begin{aligned}
T &= -\sum_{k=0}^{\infty}\left\{\operatorname{Im}\log\left(\tfrac{1}{2}(x+n+1)+k+\tfrac{1}{2}im\right)\right.\\
&\qquad \left.+\operatorname{Im}\log\left(\tfrac{1}{2}(x-n+1)+k-\tfrac{1}{2}im\right)\right\}\\
&= -\sum_{k=0}^{\infty}\left\{\tan^{-1}\left(\frac{m/2}{\frac{1}{2}(x+n+1)+k}\right)+\tan^{-1}\left(\frac{-m/2}{\frac{1}{2}(x-n+1)+k}\right)\right\}\\
&= \sum_{k=0}^{\infty}\left\{\tan^{-1}\left(\frac{m}{x-n+2k+1}\right)-\tan^{-1}\left(\frac{m}{x+n+2k+1}\right)\right\}.
\end{aligned} \tag{41.3}$$

Thus, formally, the proof has been completed.

To complete the proof, we first apply Stirling's formula to show easily that, for $x > 0$,

$$\lim_{x\to\infty}\frac{\Gamma\left(\frac{1}{2}(x+n+1)+\frac{1}{2}im\right)\Gamma\left(\frac{1}{2}(x-n+1)-\frac{1}{2}im\right)}{\Gamma\left(\frac{1}{2}(x-n+1)+\frac{1}{2}im\right)\Gamma\left(\frac{1}{2}(x+n+1)-\frac{1}{2}im\right)} = 1.$$

Thus, for the principal branch of $\tan^{-1} z$,

$$\lim_{x\to\infty}\tan^{-1}\left\{i\left(\frac{\bar{\alpha}-\alpha}{\bar{\alpha}+\alpha}\right)\right\} = 0.$$

On the other hand,

$$\lim_{x\to\infty}\sum_{k=0}^{\infty}\left\{\tan^{-1}\left(\frac{m}{x-n+2k+1}\right)-\tan^{-1}\left(\frac{m}{x+n+2k+1}\right)\right\}$$

$$=\lim_{x\to\infty}\sum_{k=0}^{\infty}\left\{\frac{m}{x-n+2k+1}-\frac{m}{x+n+2k+1}+O\left(\frac{1}{(x-|n|+2k+1)^3}\right)\right\}$$

$$=\lim_{x\to\infty}\sum_{k=0}^{\infty}\left\{\frac{2mn}{(x+2k+1)^2-n^2}+O\left(\frac{1}{(x-|n|+2k+1)^3}\right)\right\}=0,$$

for the limit as $x\to\infty$ can be taken under the summation sign, since the series converges uniformly for $|n|\le x<\infty$. Thus, our calculations in (41.2) and (41.3) demonstrate that Entry 41 is correct for x sufficiently large and positive. However, since both sides of Entry 41 are meromorphic for $\operatorname{Re} x>0$, the equality of Entry 41 must then be valid for all x with $\operatorname{Re} x>0$.

If n or im is an integer, the continued fraction terminates. A straightforward computation shows that the identity still holds for $x>0$. Hence, the proposed result follows by analytic continuation. Furthermore, since both sides are meromorphic functions of m and n, the entry holds for all $m, n\in\mathbb{C}$ by analytic continuation.

Entry 42 (p. 347). *Let m, n, and x denote complex numbers. Suppose that either* $\operatorname{Re} x>0$, *or* $m=2ir$ *for some integer* r, *or* $n=(2s+1)i$ *for some integer* s. *Then*

$$\sum_{k=0}^{\infty}(-1)^k\left\{\tan^{-1}\left(\frac{m+n}{x+2k+1}\right)+\tan^{-1}\left(\frac{m-n}{x+2k+1}\right)\right\}$$

$$=\tan^{-1}\left\{\frac{m}{x}\,{+}\,\frac{n^2+1^2}{x}\,{+}\,\frac{m^2+2^2}{x}\,{+}\,\frac{n^2+3^2}{x}\,{+}\,\frac{m^2+4^2}{x}\,{+}\cdots\right\}.$$

Proof. As in the previous proof, we temporarily ignore the fact that $\tan^{-1} z$ is multivalued, and assume that x, m, and n are positive.

In Entry 34 of Chapter 12 (Part II [2, p. 156], Jacobsen [4]) replace ℓ by im and n by in. Thus, under the given assumptions on x, m, and n,

$$\frac{im}{x}\,{+}\,\frac{n^2+1^2}{x}\,{+}\,\frac{m^2+2^2}{x}\,{+}\,\frac{n^2+3^2}{x}\,{+}\,\frac{m^2+4^2}{x}\,{+}\cdots=\frac{1-P}{1+P},\tag{42.1}$$

where

$$P=\frac{\Gamma(\frac14(x+1+i(m+n)))\Gamma(\frac14(x+1+i(m-n)))\Gamma(\frac14(x+3-i(m-n)))\Gamma(\frac14(x+3-i(m+n)))}{\Gamma(\frac14(x+1-i(m-n)))\Gamma(\frac14(x+1-i(m+n)))\Gamma(\frac14(x+3+i(m+n)))\Gamma(\frac14(x+3+i(m-n)))}$$

$$=\frac{\alpha}{\bar\alpha}=\frac{a+ib}{a-ib},$$

where $\alpha = a + ib$ is the numerator of P. Multiplying both sides of (42.1) by $-i$ and then applying the operator $\tan^{-1}$, we deduce that

$$\begin{aligned}
&\tan^{-1}\left\{\frac{m}{x+}\ \frac{n^2+1^2}{x}\ \frac{m^2+2^2}{+\ \ x}\ \frac{n^2+3^2}{+\ \ x}\ \frac{m^2+4^2}{+\ \ x}\ +\cdots\right\} \\
&= \tan^{-1} i\left(\frac{\alpha/\bar{\alpha}-1}{\alpha/\bar{\alpha}+1}\right) = -\tan^{-1}(b/a) = -\operatorname{Im}(\log\alpha) =: -T.
\end{aligned} \tag{42.2}$$

Employing Euler's product formula for the gamma function, we find that

$$\begin{aligned}
T &= \operatorname{Im}\log\Gamma\left(\tfrac{1}{4}(x+1+i(m+n))\right) + \operatorname{Im}\log\Gamma\left(\tfrac{1}{4}(x+1+i(m-n))\right) \\
&\quad + \operatorname{Im}\log\Gamma\left(\tfrac{1}{4}(x+3-i(m-n))\right) + \operatorname{Im}\log\Gamma\left(\tfrac{1}{4}(x+3-i(m+n))\right) \\
&= -\sum_{k=0}^{\infty}\Big\{\operatorname{Im}\log\left(\tfrac{1}{4}(x+1+i(m+n))+k\right) \\
&\quad + \operatorname{Im}\log\left(\tfrac{1}{4}(x+1+i(m-n))+k\right) \\
&\quad + \operatorname{Im}\log\left(\tfrac{1}{4}(x+3-i(m-n))+k\right) + \operatorname{Im}\log\left(\tfrac{1}{4}(x+3-i(m+n))+k\right)\Big\} \\
&= -\sum_{k=0}^{\infty}\left\{\tan^{-1}\left(\frac{m+n}{x+1+4k}\right) + \tan^{-1}\left(\frac{m-n}{x+1+4k}\right)\right. \\
&\quad \left. -\tan^{-1}\left(\frac{m-n}{x+3+4k}\right) - \tan^{-1}\left(\frac{m+n}{x+1+4k}\right)\right\} \\
&= -\sum_{k=0}^{\infty}(-1)^k\left\{\tan^{-1}\left(\frac{m+n}{x+1+2k}\right) + \tan^{-1}\left(\frac{m-n}{x+1+2k}\right)\right\}.
\end{aligned} \tag{42.3}$$

Using (42.3) in (42.2), we formally complete the proof.

To show that we have, indeed, chosen the correct branches in all our calculations, we use the same type of argument as in the proof of Entry 41 for the case $\operatorname{Re} x > 0$. Since the details are very similar, we omit them.

If $mi/2$ or $(ni - 1)/2$ is an integer, the continued fraction terminates. The proposed result again follows, as in the proof of Entry 41.

Entry 43 (p. 347). *Let x and n denote complex numbers. Assume either that $\operatorname{Re} x > 0$ or $n = ji$ for some integer j. Then*

$$\sum_{k=0}^{\infty}(-1)^k\tan^{-1}\left(\frac{2n}{x+2k+1}\right) = \tan^{-1}\left\{\frac{n}{x+}\ \frac{n^2+1^2}{x}\ \frac{n^2+2^2}{+\ \ x}\ +\cdots\right\}.$$

Proof. Set $m = n$ in Entry 42.

4. Other Continued Fractions

Entry 44 (Formula (2), p. 276). *Let a and b be complex numbers such that $a \neq 0$ and $|\arg(b/a^2)| < \pi$. Let B_n denote the nth Bernoulli number. For each* *nonnegative integer n, define*

$$A_n = \sum_{k=0}^{[n/2]} \frac{(-n)_{2k}}{k!} a^{n-2k} b^k.$$

Then, for each positive integer N, as x tends to 0 through values such that $\operatorname{Re}(bx^2) > 0$,

$$x\left\{\frac{1}{2} + \sum_{n=1}^{\infty} e^{-nax-n^2bx^2}\right\}$$
$$= \frac{1}{a+}\,\frac{2b}{a+}\,\frac{4b}{a+}\,\frac{6b}{a+}\cdots + \sum_{n=1}^{N-1} \frac{B_{2n}x^{2n}}{(2n)!} A_{2n-1} + O(x^{2N}).$$

Our version of Entry 44 is slightly more precise than that of Ramanujan. A proof of Entry 44 has been given by Watson [3] for the case when $a > 0, b > 0$, and $x > 0$. The extension to $b/a^2 \in \mathbb{C} - (-\infty, 0]$ and $\operatorname{Re}(bx^2) > 0$ follows by analytic continuation, since the continued fraction converges to a holomorphic function of (a, b) for $b/a^2 \in \mathbb{C} - (-\infty, 0]$, and the series on the left side converges to a holomorphic function of (a, b) for $\operatorname{Re}(bx^2) > 0$. The reader should note that the notations of Ramanujan, Watson, and the authors for Bernoulli numbers are different.

The next result was communicated by Ramanujan [10, p. 352] in his second letter to Hardy and is simply the case $a = 1, b = \frac{1}{2}$ of Entry 44. We precisely quote Ramanujan below, but, of course, a more accurate version can be formulated as above. Ramanujan tacitly assumed that $x > 0$. However, the result holds for all $x \in \mathbb{C}$ with $\operatorname{Re}(x^2) > 0$, i.e., for $|\arg x| < \pi/4$.

Corollary (Formula (3), p. 276). *When x is small,*

$$\frac{1}{1+}\,\frac{1}{1+}\,\frac{2}{1+}\,\frac{3}{1+}\,\frac{4}{1+}\cdots = x\sqrt{e}\sum_{n=1}^{\infty} e^{-(1+nx)^2/2}$$
$$+\frac{x}{2} - \frac{x^2}{12} - \frac{x^4}{360} - \frac{x^6}{5040} - \frac{x^8}{60480} - \frac{x^{10}}{1710720} \quad \textit{nearly}.$$

Entry 45 (Formula (4), p. 276). *The formal power series*

$$L(x) := \sum_{k=0}^{\infty} \frac{(-1)^k 2(2^{4k+2}-1)B_{4k+2}}{(2k+1)x^{2k+1}}$$

has the corresponding continued fraction

$$CF(x) := \frac{a_1}{x+}\,\frac{a_2}{x+}\,\frac{a_3}{x+}\cdots,$$

where $a_k > 0$ *for* $1 \le k < \infty$. *In particular,* $a_1 = a_2 = 1$, $a_3 = 30$, $a_4 = 150$, *and* $a_5 = 493$. *As before,* B_j, $0 \le j < \infty$, *denotes the jth Bernoulli number.*

For the definition of correspondence, we refer to the text of Jones and Thron [1, p. 148]. Continued fractions of the form above are called S–fractions or Stieltjes fractions. They have the property that their even and odd parts converge to analytic functions both of which have the asymptotic expansion $L(x)$ (Jones and Thron [1, pp. 136, 342]). Moreover, they converge for all x in the cut plane $|\arg x| < \pi$ if and only if they converge for one x in this region. It seems to be very difficult to determine if $CF(x)$ converges.

Proof. To see that $CF(x)$ is an S–fraction, we observe that $L(x)$ can be written in the form

$$L(x) = \sum_{k=0}^{\infty}(-1)^k c_k/x^{2k+1},$$

where

$$\begin{aligned} c_k &= 2(2^{4k+2}-1)\frac{B_{4k+2}}{2k+1} \\ &= \frac{4}{\pi^{4k+2}}\int_0^\infty \frac{u^{4k+1}}{\sinh u}du = 4\int_0^\infty \frac{u^{4k+1}du}{\sinh(\pi u)}, \end{aligned} \tag{45.1}$$

where we have used a familiar integral evaluation (E. T. Whittaker and G. N. Watson [1, p. 126]). Since $\sinh(\pi u) > 0$ for $u > 0$, the primary assertion of Entry 45 follows from Stieltjes' theory. (See, for instance, Wall's book [1, p. 363].)

To calculate the numerators a_k, we may use Entry 17 of Chapter 12 in Ramanujan's second notebook (Part II [2, pp. 124, 125]) Alternatively, Viskovatoff's algorithm (A. N. Khovanskii [1, pp. 27, 28]) can be employed. The calculations in the second method are somewhat easier, and, in either case, they are routine. Hence, we omit them.

It is tempting to conjecture that the numerators of $CF(x)$ are integers. However, $a_6 = 588456/493$ and $a_7 = 10101660478/4029289$.

Lastly, we find two functions that have the asymptotic expansion $L(x)$ as x tends to ∞.

From (45.1), for each positive integer n,

$$\begin{aligned} F_1(x) :&= 4\int_0^\infty \frac{t\,dt}{x(1+t^4/x^2)\sinh(\pi t)} \\ &= 4\sum_{k=0}^{n}\frac{(-1)^k}{x^{2k+1}}\int_0^\infty \frac{t^{4k+1}dt}{\sinh(\pi t)} - \frac{4(-1)^n}{x^{2n+3}}\int_0^\infty \frac{t^{4n+5}dt}{(1+t^4/x^2)\sinh(\pi t)} \\ &= 4\sum_{k=0}^{n}\frac{(-1)^k c_k}{x^{2k+1}} + O\left(\frac{1}{x^{2n+3}}\right), \end{aligned}$$

as x tends to ∞. Thus, $F_1(x)$ has the asymptotic expansion $L(x)$.

From the Laurent expansions of $\cot t$ and $\coth t$ about the origin (Gradshteyn and Ryzhik [1, p. 42]), we easily find that

$$\coth t - \cot t = \sum_{k=0}^{\infty} \frac{2^{4k+3} B_{4k+2}}{(4k+2)!} t^{4k+1}, \qquad |t| < \pi.$$

Applying Watson's Lemma (E. T. Copson [1, p. 49]), we deduce that

$$\int_0^{\infty} e^{-xt}(\coth t - \cot t)\, dt \sim \sum_{k=0}^{\infty} \frac{2^{4k+3} B_{4k+2}}{(4k+2)x^{4k+2}},$$

as x tends to ∞ with $\operatorname{Re} x > 0$. Replacing x by $2\sqrt{ix}$ and by $\sqrt{ix}$, where the principal branch of the square root is chosen, we find that, for $-3\pi/2 < \arg x < \pi/2$,

$$F_2(x) := 2i \int_0^{\infty} (e^{-\sqrt{ix}t} - e^{-2\sqrt{ix}t})(\coth t - \cot t)\, dt \sim L(x),$$

as x tends to ∞.

Unfortunately, neither F_1 nor F_2 has been of any use to us in determining the convergence or divergence of $CF(x)$.

Entry 46 (Formula (6), p. 277). *For each complex number x,*

$$x \coth x = 1 + \frac{x^2}{3} - \frac{x^2}{9}\left\{\frac{x^2}{5\,+}\;\frac{\frac{4\cdot 5}{2\cdot 3}x^2}{7\,+}\;\frac{\frac{2\cdot 3}{4\cdot 5}x^2}{9\,+}\;\frac{\frac{6\cdot 7}{4\cdot 5}x^2}{11\,+}\;\frac{\frac{4\cdot 5}{6\cdot 7}x^2}{13\,+\cdots}\right\}.$$

Entry 47 (Formula (7), p. 277). *For each complex number x and each complex number $n \neq 0, -1, -2, -3, \ldots,$*

$$\frac{x}{n\,+}\;\frac{x}{n+1\,+}\;\frac{x}{n+2\,+}\;\frac{x}{n+3\,+\cdots}$$
$$= \frac{x}{n} - \frac{x}{n^2}\left\{\frac{x}{n+1\,+}\;\frac{a_2 x}{n+2\,+}\;\frac{a_3 x}{n+3\,+\cdots+}\;\frac{a_{2k} x}{n+2k\,+}\;\frac{a_{2k+1} x}{n+2k+1\,+\cdots}\right\},$$

where, for $k \geq 1$,

$$a_{2k} = \frac{(k+1)(n+k)}{k(n+k-1)} \quad \textit{and} \quad a_{2k+1} = \frac{k(n+k-1)}{(k+1)(n+k)}.$$

We first remark that the continued fraction on the left side of Entry 47 is equal to

$$\frac{\sqrt{x}}{i} \frac{J_n(2i\sqrt{x})}{J_{n-1}(2i\sqrt{x})}$$

for all complex x, where J_ν denotes the ordinary Bessel function of order ν. See Wall's book [1, p. 349] or Part II [2, p. 133, Entry 19].

Next, we show that Entry 46 readily follows from Entry 47.

Proof of Entry 46. Recall that (Wall [1, p. 349]), for each complex number x,

$$x \coth x = 1 + \frac{x^2}{3} + \frac{x^2}{5} + \frac{x^2}{7} + \cdots, \tag{46.1}$$

which is due to J. H. Lambert [1]. This suggests that we let $n = \frac{3}{2}$ and replace x by $x^2/4$ in Entry 47. Accordingly, we find that

$$\begin{aligned}
&\frac{x^2/4}{3/2} + \frac{x^2/4}{5/2} + \frac{x^2/4}{7/2} + \frac{x^2/4}{9/2} + \cdots \\
&= \frac{x^2/4}{3/2} - \frac{x^2/4}{9/4} \\
&\quad \times \left\{ \frac{x^2/4}{5/2} + \frac{\frac{2\cdot 5/2}{1\cdot 3/2}\frac{x^2}{4}}{7/2} + \frac{\frac{1\cdot 3/2}{2\cdot 5/2}\frac{x^2}{4}}{9/2} + \frac{\frac{3\cdot 7/2}{2\cdot 5/2}\frac{x^2}{4}}{11/2} + \frac{\frac{2\cdot 5/2}{3\cdot 7/2}\frac{x^2}{4}}{13/2} + \cdots \right\},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{1}{2}\left\{ \frac{x^2}{3} + \frac{x^2}{5} + \frac{x^2}{7} + \cdots \right\} \\
&= \frac{1}{2}\frac{x^2}{3} - \frac{1}{2}\frac{x^2}{9} \left\{ \frac{x^2}{5} + \frac{\frac{4\cdot 5}{2\cdot 3}x^2}{7} + \frac{\frac{2\cdot 3}{4\cdot 5}x^2}{9} + \frac{\frac{6\cdot 7}{4\cdot 5}x^2}{11} + \frac{\frac{4\cdot 5}{6\cdot 7}x^2}{13} + \cdots \right\}.
\end{aligned}$$

Multiplying both sides by 2, adding 1 to each side, and employing (46.1), we complete the proof.

Proof of Entry 47. We shall employ a lemma of Rogers [2, p. 74]. If $f_1 = e_0 e_1$, $f_2 = e_1 + e_2$, $f_2 f_3 = e_2 e_3$, $f_3 + f_4 = e_3 + e_4$, $f_4 f_5 = e_4 e_5$, $f_5 + f_6 = e_5 + e_6, \ldots,$ then

$$\frac{e_0}{1} - \frac{e_1 x}{1} - \frac{e_2 x}{1} - \cdots = e_0 + \frac{f_1 x}{1} - \frac{f_2 x}{1} - \frac{f_3 x}{1} - \cdots, \tag{47.1}$$

in the sense that both continued fractions correspond to the same (formal) power series. This means that if both continued fractions converge in a neighborhood of $x = 0$, then they converge to the same value (Jones and Thron [1, p. 181]); that is, (47.1) expresses an identity between their values. Writing the left side of Entry 47 as the equivalent continued fraction

$$\frac{\frac{x}{n}}{1} - \frac{\frac{x}{n(n+1)}}{1} - \frac{\frac{x}{(n+1)(n+2)}}{1} - \cdots,$$

we see that, in the notation (47.1),

$$e_0 = \frac{x}{n}, \quad e_k = -\frac{1}{(n+k-1)(n+k)}, \quad k \geq 1.$$

We now calculate f_k, $k \geq 1$. Straightforward calculations show that

$$f_1 = -\frac{x}{n^2(n+1)}, \quad f_2 = -\frac{2}{n(n+2)}, \quad \text{and } f_3 = -\frac{n}{2(n+1)(n+2)(n+3)}.$$

By induction, we shall show that

$$f_{2k} = -\frac{(k+1)(n+k)}{k(n+k-1)(n+2k-1)(n+2k)}, \qquad k \geq 1, \tag{47.2}$$

and

$$f_{2k+1} = -\frac{k(n+k-1)}{(k+1)(n+k)(n+2k)(n+2k+1)}, \qquad k \geq 1. \tag{47.3}$$

We assume that (47.2) and (47.3) hold for $k = 1, 2, ..., m$. Simple algebraic calculations show that

$$f_{2m+2} = e_{2m+1} + e_{2m+2} - f_{2m+1} = -\frac{(m+2)(n+m+1)}{(m+1)(n+m)(n+2m+1)(n+2m+2)}$$

and

$$f_{2m+3} = \frac{e_{2m+2}e_{2m+3}}{f_{2m+2}} = -\frac{(m+1)(n+m)}{(m+2)(n+m+1)(n+2m+2)(n+2m+3)}.$$

This completes the proof.

For the next result, we again quote Ramanujan.

Entry 48 (Formula (1), p. 290).

$$\begin{aligned} &\frac{x}{4n+2} + \frac{x^2}{4n+6} + \frac{x^2}{4n+10} + \cdots \\ &+ \frac{2n}{x} + \frac{n-1}{1} - \frac{n+1}{x} + \frac{n-2}{1} - \frac{n+2}{x} + \cdots = 1 \quad \textit{nearly}. \end{aligned}$$

Proof. From our remark after Entry 47, it is not difficult to show that

$$\frac{x}{4n+2} + \frac{x^2}{4n+6} + \frac{x^2}{4n+10} + \cdots = i\frac{J_{n-3/2}(ix/2)}{J_{n-1/2}(ix/2)} - \frac{4n-2}{x}, \tag{48.1}$$

in the sense that the continued fraction converges to the function on the right side for all $(n, x) \in \mathbb{C}^2$, $x \neq 0$. It will be convenient to write the right side in terms of Bessel functions of imaginary argument (Watson [15, p. 77]). Thus, comparing Entry 48 with (48.1), we must show that

$$\begin{aligned} &\frac{I_{n-3/2}(x/2)}{I_{n-1/2}(x/2)} - \frac{4n-2}{x} - 1 \\ &= -\frac{2n}{x} + \frac{n-1}{1} - \frac{n+1}{x} + \frac{n-2}{1} - \frac{n+2}{x} + \cdots \qquad \text{nearly}, \end{aligned} \tag{48.2}$$

where I_ν denotes the Bessel function of imaginary argument of order ν.

As x tends to ∞ (Watson [15, p. 203]),

$$I_\nu(x/2) \sim \frac{e^{x/2}}{(\pi x)^{1/2}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\nu + k + \frac{1}{2})}{k!\, \Gamma(\nu - k + \frac{1}{2}) x^k}, \qquad \operatorname{Re} x > 0,$$

where we have ignored the exponentially decreasing terms in the complete asymptotic expansion. Using this expansion in (48.2), we find that the left side of (48.2) is asymptotically equal to the quotient

$$-\frac{2}{x} \frac{\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(n+k+1)}{k!\, \Gamma(n-k) x^k}}{\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(n+k)}{k!\, \Gamma(n-k) x^k}} = -\frac{2n}{x} \frac{{}_2F_0(1-n, n+1; 1/x)}{{}_2F_0(1-n, n; 1/x)},$$

where ${}_2F_0(a, b; z)$ denotes the (divergent) hypergeometric series

$${}_2F_0(a, b; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!} z^k.$$

From the book of Jones and Thron [1, p. 212], it follows that

$$\begin{aligned} &\frac{2n}{x} \, {}_{+} \, \frac{n-1}{1} \, {}_{-} \, \frac{n+1}{x} \, {}_{+} \, \frac{n-2}{1} \, {}_{-} \, \frac{n+2}{x} \, {}_{+} \cdots \\ &= \frac{2n/x}{1} \, {}_{-} \, \frac{(1-n)/x}{1} \, {}_{-} \, \frac{(n+1)/x}{1} \, {}_{-} \, \frac{(2-n)/x}{1} \, {}_{-} \, \frac{(n+2)/x}{1} \, {}_{+} \cdots \\ &= \frac{2n}{x} \frac{{}_2F_0(1-n, n+1; 1/x)}{{}_2F_0(1-n, n; 1/x)}, \end{aligned} \tag{48.3}$$

in the sense of correspondence. From Jacobsen's paper [4, Theorem 2.3(iii)], it follows that the continued fraction converges for $x \in \mathbb{C} - [0, \infty)$. For positive x, it is likely to diverge.

We are now able to properly interpret the word "nearly" in Entry 48, or, equivalently, (48.2). Replacing the left side of (48.2) by a quotient of asymptotic series as x tends to ∞, with $\operatorname{Re} x > 0$, we see, from (48.3), that the continued fraction on the right side of (48.2) equals this quotient of asymptotic series in the sense of correspondence of C-fractions.

Observe that, if n is an integer, the power series and continued fraction in (48.3) each terminate. Thus, in such an instance, we have equality in (48.3) in the usual sense.

Entry 49 (Formula (2), p. 292). *Let x and n denote complex numbers such that* $\operatorname{Re} x > 0$, *or such that* $\operatorname{Re} x = 0$ *and* $0 < |\operatorname{Im} x| < 1$. *Furthermore, let* $y = \{(1+x^2)^{1/2} - 1\}/x$ *and* $m = n(1+x^2)^{-1/2}$. *Then*

$$\frac{x}{2+n} \, {}_{+} \, \frac{1 \cdot 2x^2}{4+n} \, {}_{+} \, \frac{2 \cdot 3x^2}{6+n} \, {}_{+} \, \frac{3 \cdot 4x^2}{8+n} \, {}_{+} \cdots = y + m \left(y + \frac{1}{y} \right) \sum_{k=1}^{\infty} \frac{(-1)^k y^{2k}}{m+2k}.$$

Proof. Let $x \neq 0$. In Entry 22 of Chapter 12 (Part II [2, p. 136]), Ramanujan offers a continued fraction for a certain quotient of ordinary hypergeometric series. Setting $\alpha = v - 1$, $\beta = 0$, and $\gamma = u$, and replacing x by α/β with $|\alpha/\beta| < 1$, in that theorem, we readily deduce that

$$\begin{aligned}&\frac{1}{u}{}_2F_1(1-v, 1; 1+u; -\alpha/\beta)\\&= \frac{\beta}{u\beta - \alpha v} + \frac{1(u+v)\alpha\beta}{u\beta - \alpha v + 1(\beta - \alpha)} + \frac{2(u+v+1)\alpha\beta}{u\beta - \alpha v + 2(\beta - \alpha)} + \cdots.\end{aligned} \tag{49.1}$$

(This last result was also established by Preece [3].) We now set $r = (1+x^2)^{1/2}$, $\alpha = (r-1)/x^2$, $\beta = (r+1)/x^2$, $u = (pr+n)/(2r)$, and $v = (pr-n)/(2r)$, where p will be specified shortly. Then $|\alpha/\beta| = |(r-1)/(r+1)| < 1$, since $\operatorname{Re} r = \operatorname{Re}(1+x^2)^{1/2} > 0$, and (49.1) takes the simplified form

$$\begin{aligned}&\frac{1}{u}{}_2F_1(1-v, 1; 1+u; -\alpha/\beta)\\&= \frac{r+1}{p+n} + \frac{px^2}{p+n+2} + \frac{2(p+1)x^2}{p+n+4} + \frac{3(p+2)x^2}{p+n+6} + \cdots.\end{aligned} \tag{49.2}$$

By a fundamental result on hypergeometric series (Bailey [1, p. 2, eq. (2)]),

$$\frac{1}{u}{}_2F_1(1-v, 1; 1+u; -y^2) = \frac{1}{u}(1+y^2)^{v+u-1}{}_2F_1(u+v, u; 1+u; -y^2). \tag{49.3}$$

Thus, (49.2) may be recast in the form

$$\begin{aligned}&\frac{1}{u}(1+y^2)^{v+u-1}{}_2F_1(u+v, u; 1+u; -y^2)\\&= \frac{r+1}{p+n} + \frac{px^2}{p+n+2} + \frac{2(p+1)x^2}{p+n+4} + \frac{3(p+2)x^2}{p+n+6} + \cdots.\end{aligned} \tag{49.4}$$

We now put $p = 2$, and so $u + v = 2$. Since $y(r+1) = x$, we find from (49.4) that

$$\frac{y(1+y^2)}{u}\,{}_2F_1(2, u; 1+u; -y^2) = \frac{x}{2+n} + \frac{1\cdot 2x^2}{4+n} + \frac{2\cdot 3x^2}{6+n} + \cdots. \tag{49.5}$$

An elementary calculation shows that

$$(1-u)(1+y^2)\,{}_2F_1(1, u; 1+u; -y^2) = (1+y^2)\,{}_2F_1(2, u; 1+u; -y^2) - u.$$

It follows that

$$\begin{aligned}&\frac{y(1+y^2)}{u}\,{}_2F_1(2, u; 1+u; -y^2)\\&= y + \frac{1-u}{u}\left(y + \frac{1}{y}\right) y^2{}_2F_1(1, u; 1+u; -y^2)\\&= y - m\left(y + \frac{1}{y}\right)\frac{y^2}{m+2}\,{}_2F_1\left(1, \frac{m+2}{2}; \frac{m+4}{2}; -y^2\right).\end{aligned} \tag{49.6}$$

Combining (49.5) and (49.6) and then simplifying somewhat, we complete the proof.

Entry 50 (Formula (4), p. 292). *Let x, p, and n be complex numbers such that either* $\operatorname{Re} x > 0$, *or* $\operatorname{Re} x = 0$ *and* $0 < |\operatorname{Im} x| < 1$, *or p is a nonpositive integer. Furthermore, let $y = \{(1+x^2)^{1/2} - 1\}/x$ and let $m = n(1+x^2)^{-1/2}$. Then*

$$\frac{x}{p+n}+\frac{1\cdot px^2}{p+n+2}+\frac{2(p+1)x^2}{p+n+4}+\frac{3(p+2)x^2}{p+n+6}+\cdots$$
$$=(1+1/x^2)^{(p-1)/2}(2y)^p\sum_{k=0}^{\infty}\frac{(-1)^k(p)_k y^{2k}}{k!(m+p+2k)}.$$

Proof. Let $x^2 \in \mathbb{C} - (-\infty, 0]$. Multiply both sides of (49.4) by y. Note that $y(r+1) = x$, $u = (m+p)/2$, and $1 + y^2 = 2ry/x$. After some elementary algebraic simplification, we deduce Entry 50 in this first case. If, in addition, p is a nonpositive integer, then both the continued fraction and series terminate. We therefore have an identity between two rational functions of x for $\operatorname{Re} x > 0$. Hence, the identity holds for all complex x by analytic continuation, when p is a nonpositive integer.

In his second letter to Hardy, Ramanujan [10, pp. xxix, 353] asserted that

$$\text{“}\frac{a}{1+n}+\frac{a^2}{3+n}+\frac{(2a)^2}{5+n}+\frac{(3a)^2}{7+n}+\cdots$$
$$=2a\int_0^1 z^{n(1+a^2)^{-1/2}}\frac{dz}{\{(1+a^2)^{1/2}+1\}+z^2\{(1+a^2)^{1/2}-1\}},$$

which is a particular case of the continued fraction

$$\frac{a}{p+n}+\frac{pa^2}{p+n+2}+\frac{2(p+1)a^2}{p+n+4}+\cdots,$$

which is a particular case of a corollary to a theorem on transformation of integrals and continued fractions.” In Ramanujan's *Collected Papers* [10, p. 353], the third denominator above appears incorrectly as $p+n+3$.

Now C. T. Preece [3, p. 99] showed that, for $n, p > 0$,

$$\frac{a}{p+n}+\frac{pa^2}{p+n+2}+\frac{2(p+1)a^2}{p+n+4}+\cdots$$
$$=2^p a(1+a^2)^{(p-1)/2}\int_0^1\frac{t^{p-1+n(1+a^2)^{-1/2}}dt}{(\{(1+a^2)^{1/2}+1\}+t^2\{(1+a^2)^{1/2}-1\})^p}.$$

To see that this result is equivalent to Entry 50, replace a by x and write the right side above as

$$\begin{aligned}
&\frac{2^p x(1+x^2)^{(p-1)/2}}{\{(1+x^2)^{1/2}+1\}^p}\int_0^1 \frac{t^{p-1+m}\,dt}{\{1+t^2y^2\}^p}\\
&=\frac{2^p x(1+x^2)^{(p-1)/2}y^p}{x^p}\sum_{k=0}^{\infty}\frac{(-1)^k(p)_k y^{2k}}{k!}\int_0^1 t^{p+m-1+2k}\,dt\\
&=(1+1/x^2)^{(p-1)/2}(2y)^p\sum_{k=0}^{\infty}\frac{(-1)^k(p)_k y^{2k}}{k!\,(p+m+2k)}.
\end{aligned}$$

Entry 51 (Formula (1), p. 292). *Let x and n be complex numbers such that* $\operatorname{Re} x \neq 0$, *or such that* $\operatorname{Re} x = 0$ *and* $0 < |\operatorname{Im} x| < 1$. *Furthermore, let* $y = \{(1+x^2)^{1/2}-1\}/x$ *and let* $m = n(1+x^2)^{-1/2}$, *where the principal branch of* $(1+x^2)^{1/2}$ *is chosen. Then*

$$\frac{x}{1+n}\,+\,\frac{x^2}{3+n}\,+\,\frac{(2x)^2}{5+n}\,+\,\frac{(3x)^2}{7+n}\,+\cdots=2\sum_{k=0}^{\infty}\frac{(-1)^k y^{2k+1}}{m+2k+1}.$$

An elementary calculation shows that $|y| = 1$ if and only if $\operatorname{Re} x = 0$ and $|\operatorname{Im} x| \geq 1$. The choice of the principal branch of $(1+x^2)^{1/2}$ ensures that $|y| < 1$, and so the series on the right side above converges.

Entry 51 is simply the case $p = 1$ of Entry 50.

Entry 52 (Formula (3), p. 292). *Let n and p denote complex numbers such that either* $\operatorname{Re} n > 0$ *or p is a nonpositive integer. Then*

$$\begin{aligned}
&\frac{1}{n}\,+\,\frac{1\cdot p}{n}\,+\,\frac{2(p+1)}{n}\,+\,\frac{3(p+2)}{n}\,+\,\frac{4(p+3)}{n}\,+\cdots\\
&=2^p\sum_{k=0}^{\infty}\frac{(-1)^k(p)_k}{k!\,(n+p+2k)}.
\end{aligned}$$

Proof. Replace n by nx in Entry 50; thus, now $m = nx(1+x^2)^{-1/2}$. We then find that

$$\begin{aligned}
&\frac{x}{p+nx}\,+\,\frac{1\cdot px^2}{p+2+nx}\,+\,\frac{2(p+1)x^2}{p+4+nx}\,+\,\frac{3(p+2)x^2}{p+6+nx}\,+\cdots\\
&=\frac{1}{n+p/x}\,+\,\frac{1\cdot p}{n+(p+2)/x}\,+\,\frac{2(p+1)}{n+(p+4)/x}\,+\,\frac{3(p+2)}{n+(p+6)/x}\,+\cdots\\
&=\left(1+\frac{1}{x^2}\right)^{(p-1)/2}(2y)^p\sum_{k=0}^{\infty}\frac{(-1)^k(p)_k y^{2k}}{k!(m+p+2k)}.
\end{aligned}\tag{52.1}$$

Now let x tend to ∞. Then y tends to 1 and m approaches n. Thus, we see that the left and right sides of (52.1) approach, respectively, the left and right sides of

Entry 52. To see that equality still holds, we apply the uniform parabola theorem, just as we did in the proofs of Entries 24, 14, and 19.

Entry 53 (p. 342). *Let x and y be complex numbers with* $\operatorname{Re} x > 0$ *and* $\operatorname{Re} y > 0$. *Then*

$$x+\frac{(y+1)^2+n}{2x}\;+\;\frac{(y+3)^2+n}{2x}\;+\;\frac{(y+5)^2+n}{2x}\;+\cdots$$

$$=y+\frac{(x+1)^2+n}{2y}\;+\;\frac{(x+3)^2+n}{2y}\;+\;\frac{(x+5)^2+n}{2y}\;+\cdots$$

$$=x+\frac{(y+1)^2+n}{x+y+2}\;+\;\frac{(x+1)^2+n}{x+y+4}\;+\;\frac{(y+3)^2+n}{x+y+6}\;+\;\frac{(x+3)^2+n}{x+y+8}\;+\cdots.$$

We remark that, by symmetry, each of the continued fractions above is also equal to

$$y+\frac{(x+1)^2+n}{x+y+2}\;+\;\frac{(y+1)^2+n}{x+y+4}\;+\;\frac{(x+3)^2+n}{x+y+6}\;+\;\frac{(y+3)^2+n}{x+y+8}\;+\cdots.$$

The first equality in Entry 53 is actually the same as Entry 27 of Chapter 12 (Part II [2, p. 146]), for $x > 0$ and $y > 0$. As we remarked there, this elegant identity is found in Ramanujan's [10, p. xxix] second letter to Hardy. The first proof in print is by Preece [2], and the result can also be found in Perron's book [1, p. 37, eq. (31)]. This result has also been proved by Ramanathan [6]. Since both continued fractions converge locally uniformly for $\operatorname{Re} x > 0$ and $\operatorname{Re} y > 0$, the identity follows by analytic continuation for $\operatorname{Re} x > 0$ and $\operatorname{Re} y > 0$.

The first continued fraction diverges for $\operatorname{Re} x = 0$, while the second diverges for $\operatorname{Re} y = 0$. The identity does not hold if $\operatorname{Re} x < 0$ and/or $\operatorname{Re} y < 0$, since the first continued fraction is an odd function of x but not of y, whereas the second is an odd function of y but not of x.

The third (and fourth) continued fraction converges to a meromorphic function of x and y, since it is equivalent to a continued fraction $\mathbf{K}(c_k/1)$, where

$$c_k \sim \frac{k^2}{(2k)^2} = \frac{1}{4} > 0,$$

as k tends to ∞.

Proof. As just indicated, it suffices to establish the second equality.

In Gauss's continued fraction, Entry 20 of Chapter 12 (Part II [2, p. 134]), we set $x = 1$ and then replace α, β, and γ by $(x-n-1)/2$, $(x+n-1)/2$, and

$(x+y)/2$, respectively. After some simplification, we find that

$$(x+y)\frac{{}_2F_1\left(\frac{1}{2}(y+1+n), \frac{1}{2}(x-1+n); \frac{1}{2}(x+y); -1\right)}{{}_2F_1\left(\frac{1}{2}(y+1+n), \frac{1}{2}(x+1+n); \frac{1}{2}(x+y+2); -1\right)}$$
$$= x+y+\frac{(y+1)^2-n^2}{x+y+2}+\frac{(x+1)^2-n^2}{x+y+4}+\frac{(y+3)^2-n^2}{x+y+6}$$
$$+\frac{(x+3)^2-n^2}{x+y+8}+\cdots. \tag{53.1}$$

Second, we use Euler's continued fraction, Entry 22 of Chapter 12 (Part II [2, p. 136]), when $x=1$ and α, β, and γ are replaced by $-(y+1+n)/2$, $(x-1+n)/2$, and $(x+y)/2$, respectively. Upon simplification, we deduce that

$$(x+y)\frac{{}_2F_1\left(\frac{1}{2}(y+1+n), \frac{1}{2}(x-1+n); \frac{1}{2}(x+y); -1\right)}{{}_2F_1\left(\frac{1}{2}(y+1+n), \frac{1}{2}(x+1+n); \frac{1}{2}(x+y+2); -1\right)}$$
$$= 2y+\frac{(x+1)^2-n^2}{2y}+\frac{(x+3)^2-n^2}{2y}+\frac{(x+5)^2-n^2}{2y}+\cdots. \tag{53.2}$$

Comparing (53.1) and (53.2) and replacing n by $i\sqrt{n}$, we deduce the second equality of Entry 53.

It is interesting to note that the third continued fraction in Entry 53 can be obtained from the second one by repeated applications of the Bauer–Muir transformation with modifying factors $w_k = x-y+2k$. Also, the first continued fraction can be obtained from the second one by repeated use of equally simple Bauer–Muir transformations.

Entry 54 (p. 342). *For all complex numbers x and n,*

$$\frac{1}{n}\sum_{k=0}^{\infty}\frac{(-x)^k}{(n+1)_k} = \frac{1}{n}+\frac{x}{1}+\frac{1}{n}+\frac{x}{1}+\frac{2}{n}+\frac{x}{1}+\frac{3}{n}+\cdots$$
$$= \frac{1}{n+x}-\frac{x}{n+x+1}-\frac{2x}{n+x+2}-\frac{3x}{n+x+3}-\cdots.$$

Proof. The latter continued fraction in Entry 54 is merely the even part of the former continued fraction, a fact immediately seen from (64.1).

To establish the first part of Entry 54, replace x by x/β and set $\gamma = n$ in Part II [2, p. 134, Entry 21, eq. (21.2)]. Since the continued fraction converges uniformly with respect to β in a neighborhood of $\beta = \infty$, we may let β tend to ∞ to complete the proof. Alternatively, we can replace x by $-x$ in the second continued fraction of Corollary 1 of Entry 21 of Chapter 12 (Part II [2, p. 136]) to immediately achieve the desired result, since both continued fractions converge for all x and n.

Entry 55 (p. 343). *For every complex number x,*

$$\frac{x}{1-e^{-x}} = 1 + \frac{x}{1+}\,\frac{1}{1+}\,\frac{x}{1+}\,\frac{2}{1+}\,\frac{x}{1+}\,\frac{3}{1+}\,\frac{x}{1+\cdots}.$$

Proof. Setting $n = 1$ in Entry 54, we deduce that

$$\frac{1-e^{-x}}{x} = -\frac{1}{x}\sum_{k=1}^{\infty}\frac{(-x)^k}{k!} = \sum_{k=0}^{\infty}\frac{(-x)^k}{(k+1)!}$$
$$= \frac{1}{1+}\,\frac{x}{1+}\,\frac{1}{1+}\,\frac{x}{1+}\,\frac{2}{1+}\,\frac{x}{1+}\,\frac{3}{1+\cdots}.$$

Taking the reciprocal of both sides, we complete the proof.

Entry 56 (p. 342). *For all complex x,*

$$\tfrac{1}{2}(e^{2x}-1) = \frac{x}{1-}\,\frac{x}{1+}\,\frac{x}{3-}\,\frac{x}{1+}\,\frac{x}{5-}\,\frac{x}{1+\cdots}.$$

Proof. If we set $n = 1$ and replace x by $2x$ in the first continued fraction of Corollary 1 of Entry 21 of Chapter 12 (Part II [2, p. 136]), we find that, for all complex x,

$$\begin{aligned}\tfrac{1}{2}(e^{2x}-1) &= x\,{}_1F_1(1;2;2x)\\ &= \frac{x}{1-}\,\frac{2x}{2+}\,\frac{2x}{3-}\,\frac{4x}{4+}\,\frac{4x}{5-}\,\frac{6x}{6+}\,\frac{6x}{7-\cdots}\\ &= \frac{x}{1-}\,\frac{x}{1+}\,\frac{x}{3-}\,\frac{x}{1+}\,\frac{x}{5-}\,\frac{x}{1+}\,\frac{x}{7-\cdots}.\end{aligned}$$

Entry 56 also readily follows from a continued fraction for e^z found in Wall's book [1, p. 348].

Entry 57 (p. 343). *For all complex numbers x and n,*

$$x - n + \cfrac{n}{\displaystyle\sum_{k=0}^{\infty}\frac{x^k}{(n+1)_k}} = \frac{x}{1+}\,\frac{n}{1+}\,\frac{x}{1+}\,\frac{n+1}{1}\,+\,\frac{x}{1+}\,\frac{n+2}{1}\,+\cdots.$$

Proof. By a straightforward calculation, it is easily shown that the left side of Entry 57 is equal to

$$F(x) := \frac{x}{n+1}\,\frac{{}_1F_1(2;n+2;x)}{{}_1F_1(1;n+1;x)}.$$

To show that $F(x)$ has the given continued fraction, we require the continued fraction

$$\frac{{}_2F_1(\alpha,\beta;\gamma;x)}{{}_2F_1(\alpha+1,\beta;\gamma+1;x)} = 1 - \frac{1}{\gamma}\left\{\frac{\beta(\gamma-\alpha)x}{(\beta-\alpha)x+\gamma+1} - \frac{(\beta+1)(\gamma-\alpha+1)x}{(\beta-\alpha+1)x+\gamma+2} - \frac{(\beta+2)(\gamma-\alpha+2)x}{(\beta-\alpha+2)x+\gamma+3} - \cdots\right\}, \tag{57.1}$$

due to E. Frank [1] and valid for $|x| < 1$. Replacing x by x/β, letting $\beta \to \infty$, and using the fact that the resulting continued fraction in (57.1) converges uniformly with respect to β in a neighborhood of $\beta = \infty$, we find that, for all x,

$$\frac{{}_1F_1(\alpha+1;\gamma+1;x)}{{}_1F_1(\alpha;\gamma;x)} = \frac{\gamma}{\gamma} - \frac{(\gamma-\alpha)x}{x+\gamma+1} - \frac{(\gamma-\alpha+1)x}{x+\gamma+2} - \frac{(\gamma-\alpha+2)x}{x+\gamma+3} - \cdots.$$

Putting $\alpha = 1$ and $\gamma = n+1$, we deduce that, for all x,

$$\frac{1}{x}F(x) = \frac{1}{n+1} - \frac{nx}{x+n+2} - \frac{(n+1)x}{x+n+3} - \frac{(n+2)x}{x+n+4} - \cdots.$$

By (64.1), this last continued fraction is the even part of

$$CF(x) := \frac{1}{1} + \frac{n}{1} + \frac{x}{1} + \frac{n+1}{1} + \frac{x}{1} + \frac{n+2}{1} + \cdots.$$

It remains to show that $CF(x)$ converges to $F(x)/x$.

The odd part of $CF(x)$ is

$$\frac{1}{1} - \frac{n}{x+n+1} - \frac{(n+1)x}{x+n+2} - \frac{(n+2)x}{x+n+3} - \frac{(n+3)x}{x+n+4} - \cdots,$$

which converges for all x and n. Thus, the even and odd parts of $CF(x)$ both converge to meromorphic functions of x and n. The even part converges to $F(x)/x$, and so we want to show that the odd part also converges to $F(x)/x$. Now $CF(x)$ and thus the even and odd parts converge to the same values for $x > 0$ and $n > 0$. Therefore, by analytic continuation, they are equal for all x and n. This completes the proof.

Frank's continued fraction (57.1) can, in fact, be derived from Euler's continued fraction, Entry 22 of Chapter 12 (Part II [2, p. 136]).

Entry 58 (p. 343). *Let x be a complex number such that* $\operatorname{Re} x^2 > -\frac{1}{2}$. *Then*

$$\frac{\sinh^{-1} x}{(1+x^2)^{1/2}} = \frac{x}{1} + \frac{2x^2}{1} + \frac{2(1+x^2)}{1} + \frac{4x^2}{1} + \frac{4(1+x^2)}{1} + \cdots.$$

Proof. Using a familiar transformation for ${}_2F_1$ (Bailey [1, p. 2, eq. 2]), we find that (Gradshteyn and Ryzhik [1, p. 60])

$$\sinh^{-1} x = x\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; -x^2) = x(1+x^2)^{1/2}{}_2F_1(1, 1; \tfrac{3}{2}; -x^2).$$

We now apply Entry 21, eq. (21.2), of Chapter 12 (Part II [2, p. 134]) with $\beta = 0$, $\gamma = 1/2$, and x replaced by x^2. Hence, for $\operatorname{Re} x^2 > -\frac{1}{2}$,

$$\begin{aligned}\frac{\sinh^{-1} x}{(1+x^2)^{1/2}} &= x\,{}_2F_1(1, 1; \tfrac{3}{2}; -x^2)\\ &= \frac{x/2}{1/2}\,\frac{}{+}\,\frac{x^2}{1}\,\frac{}{+}\,\frac{1+x^2}{1/2}\,\frac{}{+}\,\frac{2x^2}{1}\,\frac{}{+}\,\frac{2(1+x^2)}{1/2}\,\frac{}{+}\,\frac{3x^2}{1}\,\frac{}{+}\,\frac{3(1+x^2)}{1/2}\,\frac{}{+}\cdots,\end{aligned}$$

which is easily seen to be equivalent to the proposed continued fraction.

Entry 59 (p. 343). *Let x be any complex number such that* $\operatorname{Re} x^2 > -\frac{1}{2}$. *Then*

$$\tan^{-1} x = \frac{x}{1}\,\frac{}{+}\,\frac{x^2}{1}\,\frac{}{+}\,\frac{2(1+x^2)}{1}\,\frac{}{+}\,\frac{3x^2}{1}\,\frac{}{+}\,\frac{4(1+x^2)}{1}\,\frac{}{+}\cdots.$$

Proof. We know that (A. Erdélyi [1, p. 102])

$$\tan^{-1} x = x\,{}_2F_1(\tfrac{1}{2}, 1; \tfrac{3}{2}; -x^2).$$

We again apply Entry 21, eq. (21.2), of Chapter 12 (Part II [2, p. 134]) but now with $\beta = -\frac{1}{2}$, $\gamma = \frac{1}{2}$, and x replaced by x^2. Thus, for $\operatorname{Re} x^2 > -\frac{1}{2}$,

$$\tan^{-1} x = \frac{x/2}{1/2}\,\frac{}{+}\,\frac{x^2/2}{1}\,\frac{}{+}\,\frac{1+x^2}{1/2}\,\frac{}{+}\,\frac{3x^2/2}{1}\,\frac{}{+}\,\frac{2(1+x^2)}{1/2}\,\frac{}{+}\cdots,$$

which is equivalent to the proposed continued fraction.

5. General Theorems

Entry 60 (p. 339). *For $1 \le k \le n$, assume that $a_k \neq 0$. Then*

$$\sum_{k=1}^{n} \frac{1}{a_k} = \frac{1}{a_1}\,\frac{}{-}\,\frac{a_1^2}{a_1+a_2}\,\frac{}{-}\,\frac{a_2^2}{a_2+a_3}\,\frac{}{-}\cdots\frac{}{-}\,\frac{a_{n-1}^2}{a_{n-1}+a_n}. \tag{60.1}$$

In fact, we have stated a finite version of Ramanujan's claim, i.e., Ramanujan's statement is for "$n = \infty$."

Proof. Entry 60 is easily established by induction on n. In fact, Entry 60 is a version of an identity

$$\sum_{k=1}^{n} b_1 b_2 \cdots b_k = \frac{b_1}{1}\,\frac{}{-}\,\frac{b_2}{1+b_2}\,\frac{}{-}\,\frac{b_3}{1+b_3}\,\frac{}{-}\cdots\frac{}{-}\,\frac{b_n}{1+b_n}, \tag{60.2}$$

due to Euler (Jones and Thron [1, p. 37]), where $b_k \neq 0$, $1 \leq k \leq n$. To derive (60.1) from (60.2), set $b_1 = 1/a_1$ and $b_k = a_{k-1}/a_k$, $2 \leq k \leq n$. After a simple equivalence transformation, we deduce (60.1).

In the following three entries, Ramanujan examines the convergence and divergence of limit k–periodic continued fractions of the form

$$\frac{a_1}{p}+\frac{a_2}{p}+\frac{a_3}{p}+\cdots, \tag{61.1}$$

where $\lim_{n\to\infty} a_{kn+j} = a_j^* \in \widehat{\mathbb{R}}$, for $1 \leq j \leq k$. The convergence behavior for the special periodic case, $a_{kn+j} = a_j^*$, $0 \leq n < \infty$, has been known since the 1880s (O. Stolz [1], Jones and Thron [1, p. 46]). If we think of the continued fraction (61.1) as being generated by the linear fractional transformations

$$s_n(w) = \frac{a_n}{p+w}, \qquad n = 1, 2, 3, \ldots,$$

such that

$$\begin{aligned} S_n(w) &:= s_1 \circ s_2 \circ \cdots \circ s_n(w) \\ &= \frac{a_1}{p}+\frac{a_2}{p}+\cdots+\frac{a_n}{p+w} = \frac{A_n + A_{n-1}w}{B_n + B_{n-1}w}, \end{aligned} \tag{61.2}$$

we may deduce the following information:

(1) For $k = 1$, the approximants $S_n(0)$ of the continued fraction

$$\frac{a}{p}+\frac{a}{p}+\frac{a}{p}+\cdots \tag{61.3}$$

are just iterations of the linear fractional transformation $s_1(w)$ evaluated at $w = 0$. Hence, (61.3) converges if $s_1(w)$ has one attractive fixed point and it has no repulsive fixed point at 0. The fixed points of $s_1(w)$ are $p(\pm\sqrt{1 + 4a/p^2} - 1)/2$. Hence, (61.3) converges if and only if

$$4a/p^2 \in \mathbb{C} - (-\infty, -1). \tag{61.4}$$

(2) For $k > 1$, we regard the periodic continued fraction as iterations of $S_k(w)$, given by (61.2), evaluated at the points

$$0, a_1 = S_1(0), a_1/(1 + a_2) = S_2(0), \ldots, S_{k-1}(0). \tag{61.5}$$

Hence, the k–periodic continued fraction converges if and only if $S_k(w)$ has an attractive fixed point and it has no repulsive fixed points at any of the points (61.5). It was first pointed out by T. N. Thiele [1] in 1879 that if $S_k(w)$ has a repulsive fixed point at one of the points (61.5), then the periodic continued fraction diverges. This phenomenon is therefore called Thiele oscillation (Perron [1, p. 87]).

For more details, we refer to Jones and Thron's book [1, p. 47]. The results quoted above were probably known to Ramanujan who most likely derived them himself, because they are not found in the books of G. Chrystal [1] or G. S. Carr [1], the two primary sources of information about continued fractions for

Ramanujan. He then must have realized that he could generalize these results to limit k–periodic continued fractions, and Entries 61 and 63 below are the results of his investigations. His first result is on limit 1–periodic continued fractions.

Entry 61 (p. 339).

$$\frac{1}{1}\,-\,\frac{a_1}{1}\,-\,\frac{a_2}{1}\,-\,\frac{a_3}{1}\,-\,\frac{a_4}{1}\,-\,\cdots \tag{61.6}$$

is intelligible or not according as $\lim_{n\to\infty} a_n$ $<$ *or* $>$ $1/4$.

Here we have precisely quoted Ramanujan. By "intelligible," Ramanujan evidently means "convergent." In the periodic case $a_n = a$, Entry 61 is true, since the condition (61.4) then reduces to $a \in \mathbb{C} - (\frac{1}{4}, \infty)$. It is also true that the limit periodic continued fraction converges if $a \in \mathbb{C} - [\frac{1}{4}, \infty)$. This was first proved by E. B. Van Vleck [1] and is beautifully presented in Perron's book [1, p. 93]. In the case $\lim_{n\to\infty} a_n = \frac{1}{4}$, the point wisely omitted by Ramanujan, the continued fraction may converge or diverge, according to how a_n tends to $\frac{1}{4}$. But what happens if $\lim_{n\to\infty} a_n = a > \frac{1}{4}$? It is easy to prove that if a_n tends to a "fast enough," then the continued fraction (61.6) diverges (J. Gill [1]). That it may converge otherwise was also shown by Gill [1], if one allows complex elements a_n.

In our *Memoir* [2] with Andrews, Jacobsen, and Lamphere, we asked if there exist convergent continued fractions (61.6) with $a_n > 0$ and $\lim_{n\to\infty} a_n = a > \frac{1}{4}$. In a lecture at a conference on continued fractions in Trondheim, Norway, on May 31, 1997, L. J. Lange answered this question. In particular, in 1985, he and N. J. Kalton [1, Theorem 8.1] had proved the following theorem.

Theorem 61.1. *If* a_n *is real,* $\lim_{n\to\infty} a_n = a > \frac{1}{4}$, *and*

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty,$$

then (61.6) diverges.

Moreover, if the last hypothesis above is dropped, Kalton and Lange [1, Theorems 6.1, 6.2] found specific classes of sequences $\{a_n\}$ for which (61.6) converges. For further results relevant to our question, see Theorems 2.1 and 3.2 of their paper [1]. Lange also raised a more difficult question. If a is *any* real number such that $a > \frac{1}{4}$, does there exist a real sequence $\{a_n\}$ such that $a_n \to a$ and (61.6) converges?

D. Masson [1] has communicated to us another theorem relevant to our original question in the *Memoir* [2]. Using Pincherle's theorem, he has shown that (61.6) diverges provided that a_n is real and

$$a_n = a\,(1 + g(n) + o(g(n))),$$

as n tends to ∞, where $a > \frac{1}{4}$, and where $g(n)$ is any positive function monotonically decreasing to 0 as $n \to \infty$, for example, $g(n) = cn^{-\alpha}$, for some constant c and positive number α.

For the next entry, we again quote Ramanujan.

Entry 62 (p. 340). *The continued fraction*

$$\frac{1}{p} + \frac{a_1}{p} + \frac{a_2}{p} + \frac{a_3}{p} + \cdots \tag{62.1}$$

tends to two limits or one limit according as $\sum 1/\sqrt{a_n}$ *is convergent or divergent.*

Ramanujan evidently considered a_n, $1 \le n < \infty$, to be positive and p to be real. From Stieltjes' classical work [2], [3, pp. 402–566], it follows that (62.1) converges if and only if

$$\sum_{n=1}^{\infty} \frac{a_1 a_3 \cdots a_{2n-1}}{a_2 a_4 \cdots a_{2n}} + \sum_{n=1}^{\infty} \frac{a_2 a_4 \cdots a_{2n}}{a_1 a_3 \cdots a_{2n+1}} = \infty; \tag{62.2}$$

otherwise, its even and odd parts converge to two distinct values. This coincides with the natural interpretation of Entry 62, except for one matter; the condition (62.2) is not equivalent to Ramanujan's condition

$$\sum_{n=1}^{\infty} 1/\sqrt{a_n} = \infty, \tag{62.3}$$

unless one makes further restrictions. Indeed, (62.3) is a sufficient condition for the convergence of (62.1) (Perron [1, p. 47]), but there exist convergent continued fractions (62.1) with $a_n > 0$, $p > 0$, and $\sum 1/\sqrt{a_n} < \infty$. For instance, the continued fraction

$$\frac{1^4}{1} + \frac{1^4}{1} + \frac{3^4}{1} + \frac{3^4}{1} + \frac{5^4}{1} + \frac{5^4}{1} + \cdots$$

converges since

$$\sum_{n=1}^{\infty} \frac{a_1 a_3 \cdots a_{2n-1}}{a_2 a_4 \cdots a_{2n}} = \sum_{n=1}^{\infty} 1 = \infty.$$

Ramanujan was not the only person to have made this mistake; for example, see Khovanskii's book [1, p. 45].

Entry 63 considers limit k–periodic continued fractions for $1 \le k \le 5$. We state a rather general version, although Ramanujan probably examined only real continued fractions.

Entry 63 (p. 340). *Consider*

$$CF := \frac{a_1}{1} - \frac{a_2}{1} - \frac{a_3}{1} - \frac{a_4}{1} - \cdots.$$

(1) *If CF is limit 1–periodic, then CF converges if* $\lim_{n\to\infty} a_n = a \in \mathbb{C} - [\frac{1}{4}, \infty)$.

(2) *If CF is limit 2–periodic with limits a and b, then CF converges if*

$$\frac{ab}{\{1-(a+b)\}^2} \in \mathbb{C} - [\tfrac{1}{4}, \infty),$$

where $a+b \neq 1$.

(3) *If CF is limit 3–periodic with $\lim_{n\to\infty} a_{3n+1} = a$, $\lim_{n\to\infty} a_{3n+2} = b$, and $\lim_{n\to\infty} a_{3n} = c$, then CF converges if*

$$\frac{abc}{\{1-(a+b+c)\}^2} \in \mathbb{C} - [\tfrac{1}{4}, \infty),$$

where $a+b+c \neq 0$, and if $|a| > |c|$ when $b = 1$, $|b| > |a|$ when $c = 1$, and $|c| > |b|$ when $a = 1$.

(4) *Suppose that CF is limit 4–periodic with a_{4n+1}, a_{4n+2}, a_{4n+3}, and a_{4n} tending to a, b, c, and d, respectively, as n tends to ∞. Then CF converges if*

$$\frac{abcd}{\{1-(a+b+c+d)+(ac+bd)\}^2} \in \mathbb{C} - [\tfrac{1}{4}, \infty),$$

where $a+b+c+d-ac-bd \neq 1$, and if $|ab| > |cd|$ when $b+c = 1$, $|bc| > |ad|$ when $c+d = 1$, $|cd| > |ab|$ when $a+d = 1$, and $|ad| > |bc|$ when $a+b = 1$.

(5) *Suppose that a_{5n+1}, a_{5n+2}, a_{5n+3}, a_{5n+4}, and a_{5n} approach a, b, c, d, and e, respectively, as n tends to ∞. Then CF converges if*

$$\frac{abcde}{\{1-(a+b+c+d+e)+a(c+d)+b(d+e)+ce\}^2} \in \mathbb{C} - [\tfrac{1}{4}, \infty),$$

where the denominator above is not equal to 0, and if

$$\begin{aligned}
|ab(1-d)| &> |de(1-b)|, \text{ when } & b+c+d-bd &= 1,\\
|bc(1-e)| &> |ea(1-c)|, \text{ when } & c+d+e-ce &= 1,\\
|cd(1-a)| &> |ab(1-d)|, \text{ when } & d+e+a-da &= 1,\\
|de(1-b)| &> |bc(1-e)|, \text{ when } & e+a+b-eb &= 1,
\end{aligned}$$

and

$$|ea(1-c)| > |cd(1-a)|, \text{ when } \quad a+b+c-ac = 1.$$

The "extra" conditions on the parameters $a, b, \ldots$ are not given by Ramanujan. Thus, for example, in (3), Ramanujan says that CF "is intelligible when $\{1-(a+b+c)\}^2 - 4abc$ is positive." The primary conditions insure that $S_k(w)$ has an attractive fixed point and a repulsive fixed point. The "extra" conditions eliminate the cases where the corresponding k–periodic continued fraction diverges by Thiele oscillation.

Such limit k–periodic continued fractions were first studied by M. von Pidoll [1] in 1912 and by O. Szász [1] in 1917. It is interesting to note that Ramanujan probably made his discoveries in the period 1912–1914.

Note that part (1) follows from Entry 61 and the remarks we made following it.

Proof. The corresponding periodic continued fraction is given by

$$\frac{-a}{1}-\frac{b}{1}-\frac{c}{1}-\cdots.$$

The corresponding linear fractional transformation

$$S_k(w)=\frac{-a}{1}-\frac{b}{1}-\cdots-\frac{*}{1+w}=\frac{A_k+A_{k-1}w}{B_k+B_{k-1}w}$$

has an attractive fixed point and a repulsive fixed point if either
(a)

$$B_{k-1}\neq 0,\ A_{k-1}+B_k\neq 0,\quad \text{and}\quad \left|\arg\left(1+\frac{4(A_kB_{k-1}-B_kA_{k-1})}{(A_{k-1}+B_k)^2}\right)\right|<\pi,$$

or

(b) $$B_{k-1}=0\quad\text{and}\quad |A_{k-1}|\neq|B_k|.$$

(See, for instance, the book by Jones and Thron [1, pp. 51–52].) (If S_k is singular, i.e., one or more of the elements $a, b, \ldots$ are equal to 0, then S_k is a constant function whose value we regard as the attractive fixed point of S_k. Its "repulsive fixed point" is then the point w for which S_k is not well defined. For more details, we refer to Jacobsen's paper [3].)

From the work of von Pidoll [1], Szász [1], and, in more generality, Jacobsen [3], it further follows that if condition (a) or (b) holds and none of the points (61.5) is the repulsive fixed point of S_k (Thiele oscillation), then the limit k–periodic continued fraction converges. The results in Entry 63 arise from the application of these criteria when $k=1,2,3,4,5$.

As noted earlier, the case $k=1$ was examined in Entry 61.

Let $k=2$. Then

$$S_2(w)=\frac{-a(1+w)}{1-b+w}.$$

For case (a) we require that $B_1=1\neq 0$, $A_1+B_2=1-a-b\neq 0$, and

$$\left|\arg\left(1+\frac{4(A_2B_1-B_2A_1)}{(A_1+B_2)^2}\right)\right|=\left|\arg\left(1-\frac{4ab}{(1-a-b)^2}\right)\right|<\pi,$$

i.e., $ab/\{1-(a+b)\}^2\in\mathbb{C}-[\frac{1}{4},\infty)$. Note that case (b) is impossible.

If one of the fixed points in (61.5) is the repulsive fixed point of $S_2(w)$, then this fixed point is either 0 or $-a$. Now $w_1=0$ is a fixed point of $S_2(w)$ if $A_2=-a=0$. Then

$$S_2(w)=\frac{0}{1-b+w}$$

is singular, and $w_1=0$ is the repulsive fixed point if and only if $1-b=0$. But this contradicts the requirement $1-a-b\neq 0$. Thus, 0 is not a repulsive fixed point. If $w_1=-a$ is a fixed point of $S_2(w)$, then $b=0$. It is easily seen that the other fixed point of $S_2(w)$ is $w_2 = -1$. Now $w_1 = -a$ is the attractive fixed point of $S_2(w)$ (and not the repulsive one) if and only if

$$|B_2 + B_1 w_1| > |B_2 + B_1 w_2| \tag{63.1}$$

(Jones and Thron [1, p. 52]), i.e., if and only if $|1 - a| > 0$. This is true since $1 - a \neq 0$. Thus, $w_1 = -a$ also is not a repulsive fixed point, and there is no Thiele oscillation. This completes the proof of (2).

Next, let $k = 3$. Then

$$S_3(w) = -\frac{a}{1} - \frac{b}{1} - \frac{c}{1+w} = \frac{A_3 + A_2 w}{B_3 + B_2 w},$$

where $A_3 = -a(1-c)$, $A_2 = -a$, $B_3 = 1 - b - c$, and $B_2 = 1 - b$, by the recursion formulas (0.5) and (0.6), or by direct calculation. For condition (a), we require $B_2 = 1 - b \neq 0$, i.e., $b \neq 1$; $A_2 + B_3 = 1 - (a + b + c) \neq 0$, i.e., $a + b + c \neq 1$; and $\left|\arg\left(1 + \dfrac{4(A_3B_2 - B_3A_2)}{(A_2 + B_3)^2}\right)\right| < \pi$, i.e.,

$$\frac{abc}{\{1 - (a + b + c)\}^2} \in \mathbb{C} - [\tfrac{1}{4}, \infty). \tag{63.2}$$

For case (b), we require $B_2 = 1 - b = 0$, i.e., $b = 1$, and $|A_2| \neq |B_3|$, i.e., $|a| \neq |1 - b - c| = |c|$. Now with $b = 1$

$$\frac{abc}{\{1 - (a + b + c)\}^2} = \frac{ac}{(a + c)^2} \in \mathbb{C} - [\tfrac{1}{4}, \infty)$$

if and only if $|a| \neq |c|$. Hence, S_3 has an attractive and a repulsive fixed point (in the extended sense if S_3 is singular, i.e., if $abc = 0$) if (63.2) holds.

We next need to determine the conditions that yield repulsive fixed points. If one of the fixed points in (61.5) is the repulsive fixed point of $S_3(w)$, then this fixed point is either 0, $-a$, or $-a/(1 - b)$.

First, $w_1 = 0$ is a fixed point of $S_3(w)$ if $S_3(0) = A_3 = 0$, i.e., if $-a(1-c) = 0$.

Case 1. $a = 0$. Then

$$S_3(w) = \frac{0}{1 - b - c + (1 - b)w}$$

is singular, and $w_1 = 0$ is "the repulsive fixed point" if and only if $1 - b - c = 0$. But this is impossible by (63.2).

Case 2. $c = 1, a \neq 0$. Then

$$S_3(w) = \frac{-aw}{-b + (1 - b)w},$$

and the other fixed point of $S_3(w)$ is given by $w_2 = (b - a)/(1 - b)$. Hence, in analogy with (63.1), $w_1 = 0$ is the attractive fixed point of $S_3(w)$ (and not the repulsive one) if and only if

$$|B_3 + B_2 w_1| > |B_3 + B_2 w_2|$$

(Jones and Thron [1, p. 52]), i.e., $|b| > |a|$. (The case $|B_3 + B_2 w_1| = |B_3 + B_2 w_2|$ is excluded by (63.2).) In conclusion, under the condition (63.2), if $c = 1$, we need to require that $|b| > |a|$ for the convergence of CF.

Next, $w_3 = -a$ is a repulsive fixed point of $S_3(w)$ if and only if $w_1 = 0$ is a repulsive fixed point of

$$S_3^{(1)}(w) := \frac{-b}{1} - \frac{c}{1} - \frac{a}{1+w}.$$

So, by symmetry, we need to require that $|c| > |b|$ if $a = 1$ in order for CF to converge. Likewise, $w_3 = -a/(1-b)$ is a repulsive fixed point of $S_3(w)$ if and only if $w_1 = 0$ is a repulsive fixed point of

$$S_3^{(2)}(w) := \frac{-c}{1} - \frac{a}{1} - \frac{b}{1+w},$$

which yields the requirement $|a| > |c|$ if $b = 1$. This concludes the proof of (3).

Cases $k = 4$ and $k = 5$ are proved in exactly the same manner as case $k = 3$. However, with increasing k, the details become more laborious. For these reasons, we shall provide only brief sketches of the proofs when $k = 4$ and $k = 5$.

Let $k = 4$. Then

$$S_4(w) = \frac{A_4 + A_3 w}{B_4 + B_3 w},$$

where $A_3 = a(c-1)$, $A_4 = a(c+d-1)$, $B_3 = 1-b-c$, and $B_4 = 1-b-c-d+bd$. For condition (a), we require that $b + c \neq 1$, $1 - (a+b+c+d) + ac + bd \neq 0$, and

$$\frac{abcd}{\{1-(a+b+c+d)+(ac+bd)\}^2} \in \mathbb{C} - [\tfrac{1}{4}, \infty). \tag{63.3}$$

In case (b), we need $b + c = 1$ and $|ab| \neq |cd|$. Now when $b + c = 1$,

$$\frac{abcd}{\{1-(a+b+c+d)+(ac+bd)\}^2} = \frac{abcd}{\{ab+cd\}^2} \in \mathbb{C} - [\tfrac{1}{4}, \infty)$$

if and only if $|ab| \neq |cd|$. Hence, S_4 has an attractive fixed point and a repulsive fixed point if (63.3) is valid.

We next need to determine when the points (61.5) are repulsive fixed points. Now $w_1 = 0$ is a fixed point if $A_4 = a(c+d-1) = 0$. It is easy to see that the case $a = 0$ is impossible. If $c + d = 1$, then

$$S_4(w) = \frac{A_3 w}{B_4 + B_3 w}$$

has the fixed point $w_2 = (A_3 - B_4)/B_3$. Thus, in analogy with (63.1), w_1 is not a repulsive fixed point if and only if

$$|B_4| > |B_4 + B_3 w_2| = |A_3|,$$

i.e., if and only if $|bc| > |ad|$.

The remaining three conditions listed in Entry 63 for the case $k = 4$ arise from the remaining three possible repulsive fixed points in (61.5) and considerations of symmetry.

Let $k = 5$. Then

$$S_5(w) = \frac{A_5 + A_4 w}{B_5 + B_4 w},$$

where $A_4 = a(c+d-1)$, $A_5 = a(c+d+e-ce-1)$, $B_4 = 1-b-c-d+bd$, and $B_5 = 1-b-c-d-e+ce+be+bd$. For condition (a), we require that $1-(b+c+d)+bd \neq 0$, $1-(a+b+c+d+e)+ad+ac+ce+be+bd \neq 0$, and

$$\frac{abcde}{\{1-(a+b+c+d+e)+a(c+d)+b(d+e)+ce\}^2} \in \mathbb{C}-[\tfrac{1}{4}, \infty). \quad (63.4)$$

Forgoing the calculations for condition (b), we conclude that S_5 has an attractive fixed point and a repulsive fixed point if (63.4) holds.

We now examine the five possible repulsive fixed points given by (61.5). Now, $w_1 = 0$ is a fixed point of $S_5(w)$ if $S_5(0) = A_5 = a(c+d+e+ce-1) = 0$. The case $a = 0$ is impossible, and so we assume that $c+d+e+ce-1 = 0$. The remaining fixed point of $S_5(w)$ is $(A_4 - B_5)/B_4$. It follows that w_1 is not a repulsive fixed point if and only if $|B_5| > |A_4|$, i.e., if and only if $|bc(1-e)| > |ae(1-c)|$. The remaining four possible repulsive fixed points yield the additional restrictions listed for the case $k = 5$ of Entry 63.

It may be remarked that the "extra" conditions in Entry 63 can be eliminated if we use the notion of general convergence (Jacobsen [2]).

Entry 64 (p. 342). *If n is even, then*

$$\begin{aligned}&\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}\\ &= \frac{a_1 b_2}{a_2 + b_1 b_2} - \frac{a_2 a_3 b_4}{a_3 b_4 + b_2(a_4 + b_3 b_4)} - \frac{a_4 a_5 b_2 b_6}{a_5 b_6 + b_4(a_6 + b_5 b_6)}\\ &\quad - \cdots - \frac{a_{n-2} a_{n-1} b_{n-4} b_n}{a_{n-1} b_n + b_{n-2}(a_n + b_{n-1} b_n)}.\end{aligned}$$

This is just the finite form of the even part of an infinite continued fraction, namely (Jones and Thron [1, p. 42]), the even part of

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots$$

is

$$b_0 + \frac{a_1 b_2}{a_2 + b_1 b_2} - \frac{a_2 a_3 b_4}{a_3 b_4 + b_2(a_4 + b_3 b_4)} - \frac{a_4 a_5 b_2 b_6}{a_5 b_6 + b_4(a_6 + b_5 b_6)} - \cdots. \quad (64.1)$$

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Ramanujan's Theories of Elliptic Functions to Alternative Bases

1. Introduction

In his famous paper [3], [10, pp. 23–39], Ramanujan offers several beautiful series representations for $1/\pi$. He first states three formulas, one of which is

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(6n+1)(\frac{1}{2})_n^3}{(n!)^3 4^n},$$

where $(a)_0 = 1$ and, for each positive integer n,

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1).$$

He then remarks that "There are corresponding theories in which q is replaced by one or other of the functions

$$q_1 = \exp\left(-\pi\sqrt{2}K_1'/K_1\right), \quad q_2 = \exp\left(-2\pi K_2'/(K_2\sqrt{3})\right),$$
$$q_3 = \exp\left(-2\pi K_3'/K_3\right),$$

where

$$K_1 = {}_2F_1\left(\tfrac{1}{4}, \tfrac{3}{4}; 1; k^2\right), \quad K_2 = {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; k^2\right), \quad K_3 = {}_2F_1\left(\tfrac{1}{6}, \tfrac{5}{6}; 1; k^2\right)."$$

Here $K_j' = K_j(k')$, where $1 \le j \le 3$, $k' = \sqrt{1-k^2}$, and $0 < k < 1$; k is called the modulus. In the classical theory, the hypergeometric functions above are replaced by ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; k^2)$. Ramanujan then offers 16 further formulas for $1/\pi$ that arise from these alternative theories, but he provides no details for his proofs. In an appendix at the end of Ramanujan's *Collected Papers* [10, p. 336], the editors, quoting L. J. Mordell, lament "It is unfortunate that Ramanujan has not developed in detail the corresponding theories referred to in ¶14."

Ramanujan's formulas for $1/\pi$ were not established until 1987, when they were first proved by J. M. and P. B. Borwein [1, pp. 177–188], [2], [3]. To prove these formulas, they needed to develop only a very small portion of the "corresponding theories" to which Ramanujan alluded. In particular, the main ingredients in their work are Clausen's formula and identities relating ${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)$, to each of the functions ${}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; x\right)$, ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right)$, and ${}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; x\right)$. The Borweins [4], [6] further developed their ideas by deriving several additional formulas for $1/\pi$. Ramanujan's ideas were also greatly extended by D. V. and G. V. Chudnovsky [1], [2] who showed that other transcendental constants could be represented by similar series and that an infinite class of such formulas existed.

Ramanujan's "corresponding theories" have not been heretofore developed. Initial steps were taken by K. Venkatachaliengar [1, pp. 89–95] who examined some of the entries in Ramanujan's notebooks [9] devoted to his alternative theories.

The greatest advances toward establishing Ramanujan's theories have been made by J. M. and P. B. Borwein [5]. In searching for analogues of the classical arithmetic–geometric mean of Gauss, they discovered an elegant cubic analogue. Playing a central role in their work is a cubic transformation formula for ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right)$, which, in fact, is found on page 258 of Ramanujan's second notebook [9], and which was rediscovered by the Borweins. A third major discovery by the Borweins is a beautiful and surprising cubic analogue of a famous theta-function identity of Jacobi for fourth powers. We shall describe these findings in more detail in the sequel.

As alluded in the foregoing paragraphs, Ramanujan had recorded some results in his three alternative theories in his second notebook [9]. In fact, six pages, pp. 257–262, are devoted to these theories. These are the first six pages in the 100 unorganized pages of material that immediately follow the 21 organized chapters in the second notebook. Our objective in this chapter is to establish all of these claims. In proving these results, it is very clear to us that Ramanujan had established further results that he unfortunately did not record either in his notebooks, unpublished papers, or published papers. Moreover, Ramanujan's work points the way to many additional theorems in these theories, and we hope that others will continue to develop Ramanujan's beautiful ideas.

The most important of the three alternative theories is the one arising from the hypergeometric function ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right)$. The theories in the remaining two cases are more easily extracted from the classical theory and so are of less interest.

We first review the classical terminology and theory, which can be found in Part III [3]. In particular, see Chapter 16, pages 34–37, Chapter 17, pages 101–102, and Chapter 18, pages 213–214.

The complete elliptic integral of the first kind $K = K(k)$ associated with the modulus k, $0 < k < 1$, is defined by

$$K := \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \tfrac{1}{2}\pi\ {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2\right), \tag{1.1}$$

where the latter representation is achieved by expanding the integrand in a binomial series and integrating termwise. For brevity, Ramanujan sets

$$z := \frac{2}{\pi}K = {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2\right). \tag{1.2}$$

The base (or nome) q is defined by

$$q := e^{-\pi K'/K}, \tag{1.3}$$

where $K' = K(k')$. Ramanujan sets x (or α) $= k^2$.

Let n denote a fixed positive integer, and suppose that

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-k^2\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\ell^2\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\ell^2\right)}, \tag{1.4}$$

where $0 < k, \ell < 1$. Then a modular equation of degree n is a relation between the moduli k and ℓ which is implied by (1.4). Following Ramanujan, we put $\alpha = k^2$ and $\beta = \ell^2$. We often say that β has degree n, or degree n over α. The multiplier m is defined by

$$m = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)}. \tag{1.5}$$

We employ analogous notation for the three alternative systems. The classical terminology described above is represented by the case $r = 2$ below. For $r = 2, 3, 4, 6$ and $0 < x < 1$, set

$$z(r) := z(r;x) := {}_2F_1\left(\frac{1}{r},\frac{r-1}{r};1;x\right) \tag{1.6}$$

and

$$q_r := q_r(x) := \exp\left(-\pi\csc(\pi/r)\frac{{}_2F_1\left(\frac{1}{r},\frac{r-1}{r};1;1-x\right)}{{}_2F_1\left(\frac{1}{r},\frac{r-1}{r};1;x\right)}\right).$$

In particular,

$$q_3 = \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-x\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x\right)}\right), \tag{1.7}$$

$$q_4 = \exp\left(-\pi\sqrt{2}\frac{{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;1-x\right)}{{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;x\right)}\right), \tag{1.8}$$

and

$$q_6 = \exp\left(-2\pi\frac{{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;1-x\right)}{{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;x\right)}\right). \tag{1.9}$$

(We consider the notation (1.7)–(1.9) to be more natural than that of Ramanujan quoted at the beginning of this chapter.)

Let n denote a fixed natural number, and assume that

$$n\frac{{}_2F_1\left(\frac{1}{r},\frac{r-1}{r};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{r},\frac{r-1}{r};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{r},\frac{r-1}{r};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{r},\frac{r-1}{r};1;\beta\right)}, \tag{1.10}$$

where $r = 2, 3, 4$, or 6. Then a modular equation of degree n is a relation between α and β induced by (1.10). The multiplier $m(r)$ is defined by

$$m(r) = \frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \alpha\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \beta\right)}, \tag{1.11}$$

for $r = 2, 3, 4$, or 6. When the context is clear, we omit the argument r in q_r, $z(r)$, and $m(r)$.

In the sequel, we say that these theories are of *signature* 2, 3, 4, and 6, respectively.

Theta–functions are at the focal point in Ramanujan's theories. His general theta–function $f(a, b)$ is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1.$$

If we set $a = qe^{2iz}$, $b = qe^{-2iz}$, and $q = e^{\pi i \tau}$, where z is an arbitrary complex number and $\operatorname{Im}(\tau) > 0$, then $f(a, b) = \vartheta_3(z, \tau)$, in the classical notation of Whittaker and Watson [1, p. 464]. In particular, we utilize three special cases of $f(a, b)$, namely,

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{1.12}$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \tag{1.13}$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = \prod_{n=1}^{\infty} (1 - q^n), \tag{1.14}$$

where $|q| < 1$. The last equality above is Euler's pentagonal number theorem, which is most easily derived from Jacobi's triple product identity (Part III [3, p. 35, Entry 19]).

One of the fundamental results in the theory of elliptic functions is the inversion formula (Whittaker and Watson [1, p. 500]; Part III [3, p. 101, eq. (6.4)])

$$z = {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; x\right) = \varphi^2(q). \tag{1.15}$$

We set

$$z_n = \varphi^2(q^n), \tag{1.16}$$

for each positive integer n, so that $z_1 = z$. Thus, by (1.5), (1.15), and (1.16),

$$m = \frac{z_1}{z_n} = \frac{\varphi^2(q)}{\varphi^2(q^n)}. \tag{1.17}$$

In the sequel, unattended page numbers, particularly after the statements of theorems, refer to the pagination of the Tata Institute's publication of Ramanujan's second notebook [9]. We employ many results from Ramanujan's second notebook in our proofs, in particular, from Chapters 17, 19, 20, and 21.

2. Ramanujan's Cubic Transformation, the Borweins' Cubic Theta–Function Identity, and the Inversion Formula

In classical notation, the identity

$$\vartheta_3^4(q) = \vartheta_4^4(q) + \vartheta_2^4(q)$$

is Jacobi's famous identity for fourth powers of theta–functions. In Ramanujan's notation (1.12) and (1.13), this identity has the form (Part III [3, p. 40, Entry 25(vii)])

$$\varphi^4(q) = \varphi^4(-q) + 16q\psi^4(q^2). \tag{2.1}$$

The Borweins [5] discovered an elegant cubic analogue which we now relate. For $\omega = \exp(2\pi i/3)$, let

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \tag{2.2}$$

$$b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}, \tag{2.3}$$

and

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}. \tag{2.4}$$

Then the Borweins [5] proved that

$$a^3(q) = b^3(q) + c^3(q). \tag{2.5}$$

They also established the alternative representations

$$a(q) = 1 + 6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right) \tag{2.6}$$

and

$$a(q) = \varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6). \tag{2.7}$$

Formula (2.6) can also be found in one of Ramanujan's letters to Hardy, written from the nursing home, Fitzroy House [11, p. 93], and is proved by us in [9]. The identity (2.7) is found on page 328 in the unorganized portions of Ramanujan's second notebook and was proved in Part III [3, p. 462, eq. (3.6)] in the course of proving some related identities in Section 3 of Chapter 21 in Ramanujan's second notebook. Furthermore, the Borweins [5] proved that

$$b(q) = \tfrac{1}{2}\left\{3a(q^3) - a(q)\right\} \tag{2.8}$$

and

$$c(q) = \tfrac{1}{2}\left\{a(q^{1/3}) - a(q)\right\}. \tag{2.9}$$

The Borweins' proof of (2.5) employs the theory of modular forms on the group generated by the transformations $t \to 1/t$ and $t \to t + i\sqrt{3}$. Shortly thereafter, they and F. G. Garvan [1] gave a simpler, more elementary proof that does not depend upon the theory of modular forms. Although Ramanujan does not state (2.5) in his notebooks, we shall show that (2.5) may be simply derived from results given by him in his notebooks. Our proof also does not utilize the theory of modular forms.

We first establish parametric representations for $a(q)$, $b(q)$, and $c(q)$.

Lemma 2.1. *Let $m = z_1/z_3$, as in (1.17). Then*

$$a(q) = \sqrt{z_1 z_3}\,\frac{m^2 + 6m - 3}{4m}, \tag{2.10}$$

$$b(q) = \sqrt{z_1 z_3}\,\frac{(3 - m)(9 - m^2)^{1/3}}{4m^{2/3}}, \tag{2.11}$$

and

$$c(q) = \sqrt{z_1 z_3}\,\frac{3(m + 1)(m^2 - 1)^{1/3}}{4m}. \tag{2.12}$$

Proof. From Entry 11(iii) of Chapter 17 in Ramanujan's second notebook (Part III [3, p. 123]),

$$\psi(q^2) = \tfrac{1}{2}\sqrt{z_1}(\alpha/q)^{1/4} \qquad \text{and} \qquad \psi(q^6) = \tfrac{1}{2}\sqrt{z_3}(\beta/q^3)^{1/4}, \tag{2.13}$$

where β has degree 3 over α. In proving Ramanujan's modular equations of degree 3 in Section 5 of Chapter 19 of Ramanujan's second notebook, we [3, p. 233, eq. (5.2)] derived the parametric representations

$$\alpha = \frac{(m - 1)(3 + m)^3}{16m^3} \tag{2.14}$$

and

$$\beta = \frac{(m - 1)^3(3 + m)}{16m}. \tag{2.15}$$

Thus, by (1.16), (2.7), (2.13), (2.14), and (2.15),

$$\begin{aligned} a(q) &= \sqrt{z_1 z_3}\left\{1 + (\alpha\beta)^{1/4}\right\} \\ &= \sqrt{z_1 z_3}\left\{1 + \frac{(m - 1)(m + 3)}{4m}\right\} = \sqrt{z_1 z_3}\,\frac{m^2 + 6m - 3}{4m}, \end{aligned}$$

and so (2.10) is established. (In fact, (2.10) is proved in Part III [3, p. 462, eq. (3.5)].)

Next, from (2.7) and (2.8),

$$2b(q) = \varphi(q)\varphi(q^3)\left(3\frac{\varphi(q^9)}{\varphi(q)} - 1\right) - 4q\psi(q^2)\psi(q^6)\left(1 - 3q^2\frac{\psi(q^{18})}{\psi(q^2)}\right) \tag{2.16}$$

and, from (2.7) and (2.9),

$$2c(q) = \varphi(q)\varphi(q^3)\left(\frac{\varphi(q^{1/3})}{\varphi(q^3)} - 1\right) - 4q\psi(q^2)\psi(q^6)\left(1 - \frac{\psi(q^{2/3})}{q^{2/3}\psi(q^6)}\right). \tag{2.17}$$

By Entry 1(iii) of Chapter 20 (Part III [3, p. 345]), (1.16), and (1.17),

$$3\frac{\varphi(q^9)}{\varphi(q)} - 1 = \left(9\frac{\varphi^4(q^3)}{\varphi^4(q)} - 1\right)^{1/3} = \left(\frac{9}{m^2} - 1\right)^{1/3} \tag{2.18}$$

and

$$\frac{\varphi(q^{1/3})}{\varphi(q^3)} - 1 = \left(\frac{\varphi^4(q)}{\varphi^4(q^3)} - 1\right)^{1/3} = \left(m^2 - 1\right)^{1/3}. \tag{2.19}$$

By Entry 1(ii) of Chapter 20 (Part III [3, p. 345]) and (2.13)–(2.15),

$$\begin{aligned}1 - 3q^2\frac{\psi(q^{18})}{\psi(q^2)} &= \left(1 - 9q^2\frac{\psi^4(q^6)}{\psi^4(q^2)}\right)^{1/3}\\ &= \left(1 - \frac{9}{m^2}\frac{\beta}{\alpha}\right)^{1/3} = 2\frac{m^{1/3}(3-m)^{1/3}}{(m+3)^{1/3}}\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}1 - \frac{\psi(q^{2/3})}{q^{2/3}\psi(q^6)} &= \left(1 - \frac{\psi^4(q^2)}{q^2\psi^4(q^6)}\right)^{1/3}\\ &= \left(1 - m^2\frac{\alpha}{\beta}\right)^{1/3} = -2\frac{(m+1)^{1/3}}{(m-1)^{2/3}}.\end{aligned} \tag{2.21}$$

Using (2.13) and (2.18)–(2.21) and then (2.14) and (2.15) in (2.16) and (2.17), we deduce that, respectively,

$$\begin{aligned}2b(q) &= \sqrt{z_1 z_3}\left\{\left(\frac{9}{m^2} - 1\right)^{1/3} - (\alpha\beta)^{1/4}\frac{2m^{1/3}(3-m)^{1/3}}{(m+3)^{2/3}}\right\}\\ &= \sqrt{z_1 z_3}\left\{\frac{(9-m^2)^{1/3}}{m^{2/3}} - \frac{(m-1)(m+3)}{4m}\frac{2m^{1/3}(3-m)^{1/3}}{(m+3)^{2/3}}\right\}\\ &= \sqrt{z_1 z_3}\frac{(3-m)(9-m^2)^{1/3}}{2m^{2/3}}\end{aligned}$$

and

$$\begin{aligned}
2c(q) &= \sqrt{z_1 z_3}\left\{(m^2-1)^{1/3} + (\alpha\beta)^{1/4}\frac{2(m+1)^{1/3}}{(m-1)^{2/3}}\right\} \\
&= \sqrt{z_1 z_3}\left\{(m^2-1)^{1/3} + \frac{(m-1)(m+3)(m+1)^{1/3}}{2m(m-1)^{2/3}}\right\} \\
&= \sqrt{z_1 z_3}\frac{3(m^2-1)^{1/3}(m+1)}{2m}.
\end{aligned}$$

Hence, (2.11) and (2.12) have been established.

Theorem 2.2. *The cubic theta-function identity (2.5) holds.*

Proof. From (2.11) and (2.12),

$$\begin{aligned}
b^3(q) + c^3(q) &= \frac{(z_1 z_3)^{3/2}}{64m^3}\left\{m(3-m)^3(9-m^2) + 27(m+1)^3(m^2-1)\right\} \\
&= \frac{(z_1 z_3)^{3/2}}{64m^3}(m^2+6m-3)^3 = a^3(q),
\end{aligned}$$

by (2.10). This completes the proof.

Our next task is to state a generalization of Ramanujan's beautiful cubic transformation for ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; x)$.

Theorem 2.3. *For $|x|$ sufficiently small,*

$$\begin{aligned}
&{}_2F_1\left(c, c+\frac{1}{3}; \frac{3c+1}{2}; 1-\left(\frac{1-x}{1+2x}\right)^3\right) \\
&\qquad = (1+2x)^{3c}\,{}_2F_1\left(c, c+\frac{1}{3}; \frac{3c+5}{6}; x^3\right).
\end{aligned} \tag{2.22}$$

Proof. Using MAPLE, we have shown that both sides of (2.22) are solutions of the differential equation

$$\begin{aligned}
&2x(1-x)(1+x+x^2)(1+2x)^2y'' \\
&\quad - (1+2x)[(4x^3-1)(3c+2x+1) + 18cx]y' - 6c(3c+1)(1-x)^2y = 0.
\end{aligned}$$

This equation has a regular singular point at $x = 0$, and the roots of the associated indicial equation are 0 and $(3c-1)/2$. Thus, in general, to verify that (2.22) holds, we must show that the values at $x = 0$ of the functions and their first derivatives on each side are equal. These values are easily seen to be equal, and so the proof is complete.

Corollary 2.4 (p. 258). *For $|x|$ sufficiently small,*

$$ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \left(\frac{1-x}{1+2x}\right)^3\right) = (1+2x)\,{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x^3\right). \tag{2.23} $$

Proof. Set $c = \frac{1}{3}$ in Theorem 2.3.

The Borweins [5] deduced Corollary 2.4 in connection with their cubic analogue of the arithmetic–geometric mean. Neither their proof nor our proof is completely satisfactory, because they depend upon prior knowledge of the identity and differential equations. Recently, H. H. Chan [4] has given a considerably more natural proof that depends upon rederiving some of the results in Sections 4–6 of this chapter without appealing to the theorems here in Section 2. The Eisenstein series $M(q)$ and $N(q)$, defined at the beginning of Section 4, play key roles. Chan [4] has also shown that Ramanujan's cubic transformation can be derived from two cubic transformations due to E. Goursat [1].

Our next goal is to prove a cubic analogue of (1.15). We accomplish this through a series of lemmas.

Lemma 2.5. *If $n = 3^m$, where m is a positive integer, then*

$$ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \frac{b^3(q)}{a^3(q)}\right) = \frac{a(q)}{a(q^n)}\,{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \frac{b^3(q^n)}{a^3(q^n)}\right). \tag{2.24} $$

Proof. Replacing x by $(1-x)/(1+2x)$ in (2.23), we find that

$$ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - x^3\right) = \frac{3}{1+2x}\,{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \left(\frac{1-x}{1+2x}\right)^3\right). \tag{2.25} $$

Setting $x = b(q)/a(q)$ and employing (2.8) and (2.9), we deduce that

$$ \begin{aligned} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \frac{b^3(q)}{a^3(q)}\right) &= \frac{3a(q)}{a(q)+2b(q)}\,{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \left(\frac{a(q)-b(q)}{a(q)+2b(q)}\right)^3\right) \\ &= \frac{a(q)}{a(q^3)}\,{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{c^3(q^3)}{a^3(q^3)}\right) \\ &= \frac{a(q)}{a(q^3)}\,{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \frac{b^3(q^3)}{a^3(q^3)}\right), \end{aligned} $$

by Theorem 2.2. Iterating this identity m times, we complete the proof of (2.24).

The next result is the Borweins' [5] form of the cubic inversion formula.

Lemma 2.6. *We have*

$$ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{c^3(q)}{a^3(q)}\right) = a(q). \tag{2.26} $$

Proof. Letting m tend to ∞ in (2.24), noting that, by (2.2) (or (2.6)) and (2.3) (or (2.8)), respectively,

$$\lim_{n\to\infty} a(q^n) = 1 = \lim_{n\to\infty} b(q^n),$$

and invoking Theorem 2.2, we deduce (2.26) at once.

Lemma 2.7. *If $n = 3^m$, where m is a positive integer, then*

$$ {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{b^3(q)}{a^3(q)}\right) = \frac{a(q)}{na(q^n)}\, {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{b^3(q^n)}{a^3(q^n)}\right). \tag{2.27}$$

Proof. By Theorem 2.2, (2.25) with $x = c(q)/a(q)$, (2.8), and (2.9),

$$\begin{aligned} {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{b^3(q)}{a^3(q)}\right) &= {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\frac{c^3(q)}{a^3(q)}\right) \\ &= \frac{3a(q)}{a(q)+2c(q)}\, {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\left(\frac{a(q)-c(q)}{a(q)+2c(q)}\right)^3\right) \\ &= \frac{3a(q)}{a(q^{1/3})}\, {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{b^3(q^{1/3})}{a^3(q^{1/3})}\right). \end{aligned} \tag{2.28}$$

Replacing q by q^3 in (2.28), and then iterating the resulting equality a total of m times, we deduce (2.27) to complete the proof.

Lemma 2.8. *Let q_3 be defined by (1.7), and put $F(x) = q_3$. Let $n = 3^m$, where m is a positive integer. Then*

$$F\left(\frac{b^3(q)}{a^3(q)}\right) = F^n\left(\frac{b^3(q^n)}{a^3(q^n)}\right) \tag{2.29}$$

and

$$F^n\left(\frac{c^3(q)}{a^3(q)}\right) = F\left(\frac{c^3(q^n)}{a^3(q^n)}\right). \tag{2.30}$$

Proof. Dividing (2.24) by (2.27), we deduce that

$$\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\frac{b^3(q)}{a^3(q)}\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{b^3(q)}{a^3(q)}\right)} = n\,\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\frac{b^3(q^n)}{a^3(q^n)}\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{b^3(q^n)}{a^3(q^n)}\right)}. \tag{2.31}$$

Multiplying both sides of (2.31) by $-2\pi/\sqrt{3}$ and then taking the exponential of each side, we obtain (2.29).

Multiply both sides of (2.31) by $-\sqrt{3}/(2\pi n)$, take the reciprocal of each side, use Theorem 2.2, and then take the exponential of each side. We then arrive at (2.30).

We now establish another fundamental inversion formula.

Lemma 2.9. *Let F be defined as in Lemma 2.8. Then*

$$F\left(\frac{c^3(q)}{a^3(q)}\right) = q.$$

Proof. Letting n tend to ∞ in (2.30) and employing Example 2 in Section 27 of Chapter 11 in Ramanujan's second notebook (Part II [2, p. 81]), we find that

$$\begin{aligned} F\left(\frac{c^3(q)}{a^3(q)}\right) &= \lim_{n\to\infty} F^{1/n}\left(\frac{c^3(q^n)}{a^3(q^n)}\right) \\ &= \lim_{n\to\infty}\left(\frac{c^3(q^n)}{27a^3(q^n)}\left(1+\frac{5}{9}\frac{c^3(q^n)}{a^3(q^n)}+\cdots\right)\right)^{1/n} \\ &= \lim_{n\to\infty}\left((q^n+\cdots)\left(1+\frac{5}{9}(q^n+\cdots)+\cdots\right)\right)^{1/n} \\ &= q, \end{aligned}$$

where in the penultimate line we used (2.6) and (2.9).

Theorem 2.10 (p. 258). *Let F be defined as in Lemma 2.8. Then*

$$z := {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; x\right) = a(F(x)) = a(q_3). \tag{2.32}$$

Proof. Let $u = u(x) = b^3(F(x))/a^3(F(x))$. Then by Lemma 2.6 and Theorem 2.2,

$$a(F(x)) = {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; 1-u\right). \tag{2.33}$$

On the other hand, by Lemma 2.9,

$$F(1-u) = F(x),$$

or

$$\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; u\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-u\right)} = \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right)}. \tag{2.34}$$

By the monotonicity of ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right)$ on (0,1), it follows that, for $0 < x < 1$,

$${}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; 1-u\right) = {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; x\right). \tag{2.35}$$

(The argument is given in more complete detail in Part III [3, p. 101] with ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right)$ replaced by ${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)$.) In conclusion, (2.32) now follows from (2.33) and (2.35).

Theorem 2.10 is an analogue of the classical theorem (1.15). Our proof followed along lines similar to those in Ramanujan's development of the classical theory, which is presented in Part III [3, Chap. 17, pp. 98–102].

Corollary 2.11 (p. 258). *If z is defined by (2.32) and q_3 is defined by (1.7), then*

$$z^2 = 1 + 12 \sum_{n=1}^{\infty} \frac{\chi_3(n) n q_3^n}{1 - q_3^n}, \tag{2.36}$$

where χ_3 denotes the principal character modulo 3.

Proof. In Part III [3, p. 460, Entry 3(i)], it is shown that

$$1 + 12 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} - 36 \sum_{n=1}^{\infty} \frac{nq^{3n}}{1 - q^{3n}} = a^2(q). \tag{2.37}$$

Since here q is arbitrary, we may replace q by q_3 in (2.37). Thus, (2.36) can be deduced from Theorem 2.10 and (2.37).

We conclude this section by offering three additional formulas for z.

Theorem 2.12 (p. 257). *Let $q = q_3$ and $z = z(3)$. Then*

$$z = 1 + 6 \sum_{n=1}^{\infty} \frac{q^n}{1 + q^n + q^{2n}}.$$

Proof. By Theorem 2.10 and (2.6),

$$\begin{aligned} z &= 1 + 6 \sum_{n=0}^{\infty} \left(\frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right) \\ &= 1 + 6 \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (q^{(3n+1)m} - q^{(3n+2)m}) \\ &= 1 + 6 \sum_{m=1}^{\infty} (q^m - q^{2m}) \sum_{n=0}^{\infty} q^{3mn} \\ &= 1 + 6 \sum_{m=1}^{\infty} \frac{q^m - q^{2m}}{1 - q^{3m}} \\ &= 1 + 6 \sum_{m=1}^{\infty} \frac{q^m}{1 + q^m + q^{2m}}, \end{aligned}$$

and the proof is complete.

In the middle of page 258, Ramanujan offers two representations for z, but one of them involves an unidentified parameter p. If q is replaced by $-q$ below, then the parameter p becomes identical to the parameter p in Lemma 5.5 below, as can be seen from (5.11).

Theorem 2.13 (p. 258). *Let z and q be as given above. Put $p = (m-1)/2$, where m is the multiplier of degree 3 in the classical sense. Then*

$$z = \frac{\varphi^3(q^3)}{\varphi(q)}(1+4p+p^2) = 4\frac{\psi^3(q^2)}{\psi(q^6)} - 3\frac{\varphi^3(q^3)}{\varphi(q)}. \tag{2.38}$$

Proof. Our proofs will be effected in the classical base q. We first assume that the second equality holds and then solve it for p. Let β have degree 3. By (2.13),

$$\begin{aligned} 1+4p+p^2 &= 4\frac{\psi^3(q^2)}{\psi(q^6)}\frac{\varphi(q)}{\varphi^3(q^3)} - 3 \\ &= m^2\frac{(\alpha/q)^{3/4}}{(\beta/q^3)^{1/4}} - 3 = m^2\left(\frac{3+m}{2m}\right)^2 - 3, \end{aligned} \tag{2.39}$$

by (2.14) and (2.15). Solving (2.39) for p, we easily find that $p = (m-1)/2$, as claimed.

Second, we prove the first equality in (2.38). By the same reasoning as used in (2.39),

$$\begin{aligned} 4\frac{\psi^3(q^2)}{\psi(q^6)} - 3\frac{\varphi^3(q^3)}{\varphi(q)} &= \frac{z_1^{3/2}}{z_3^{1/2}}\frac{\alpha^{3/4}}{\beta^{1/4}} - 3\frac{z_3^{3/2}}{z_1^{1/2}} \\ &= \sqrt{z_1 z_3}\left(m\left(\frac{3+m}{2m}\right)^2 - \frac{3}{m}\right) \\ &= \sqrt{z_1 z_3}\left(\frac{m^2+6m-3}{4m}\right) = a(q), \end{aligned}$$

by (2.10). Appealing to Theorem 2.10, we complete the proof.

3. The Principles of Triplication and Trimidiation

In Sections 3 and 4, for brevity, we set $q = q_3$, and $z = z(3; x)$ (unless otherwise stated).

In the classical theory of elliptic functions, the processes of *duplication* and *dimidiation*, which rest upon Landen's transformation, are very useful in obtaining formulas from previously derived formulas when q is replaced by q^2 or $\sqrt{q}$, respectively. These procedures are described in detail in Part III [3, Chap. 17, Section 13], where many applications are given. We now show that Ramanujan's cubic transformation, Corollary 2.4, can be employed to devise the new processes of *triplication* and *trimidiation.*

Theorem 3.1. *Let x, $q_3 = q = q(x)$, and $z(3; x) = z$ be as given in (1.7) and (1.6), respectively. Set $x = t^3$. Suppose that a relation of the form*

$$\Omega(t^3, q, z) = 0 \tag{3.1}$$

holds. Then we have the triplication formula

$$\Omega\left(\left\{\frac{1-(1-t^3)^{1/3}}{1+2(1-t^3)^{1/3}}\right\}^3, q^3, \frac{1}{3}\left\{1+2(1-t^3)^{1/3}\right\}z\right)=0 \tag{3.2}$$

and the trimidiation formula

$$\Omega\left(1-\left(\frac{1-t}{1+2t}\right)^3, q^{1/3}, (1+2t)z\right)=0. \tag{3.3}$$

Proof. Set

$$t'^3 = 1-\left(\frac{1-t}{1+2t}\right)^3. \tag{3.4}$$

Therefore,

$$t = \frac{1-(1-t'^3)^{1/3}}{1+2(1-t'^3)^{1/3}}. \tag{3.5}$$

By Corollary 2.4,

$$\begin{aligned} z' := z(t'^3) &= {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; t'^3\right) \\ &= {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\left(\frac{1-t}{1+2t}\right)^3\right) \\ &= (1+2t)\, {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; t^3\right) = (1+2t)z(t^3). \end{aligned} \tag{3.6}$$

Also, by (1.7), (3.4), (2.25), and (2.23),

$$\begin{aligned} q' := q(t'^3) &= \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-t'^3\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; t'^3\right)}\right) \\ &= \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \left(\frac{1-t}{1+2t}\right)^3\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\left(\frac{1-t}{1+2t}\right)^3\right)}\right) \\ &= \exp\left(-\frac{2\pi}{3\sqrt{3}}\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-t^3\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; t^3\right)}\right) = q^{1/3}(t^3). \end{aligned} \tag{3.7}$$

Thus, suppose that (3.1) holds. Then by (3.5)–(3.7), we obtain (3.2), but with t, q, and z replaced by t', q', and z', respectively.

On the other hand, suppose that (3.1) holds with t, q, and z replaced by t', q', and z', respectively. Then by (3.4), (3.6), and (3.7), it follows that (3.3) holds.

Corollary 3.2. *With q and z as above,*

$$b(q) = (1-x)^{1/3}z$$

and

$$c(q) = x^{1/3}z.$$

Proof. By (2.8), Theorem 2.10, and the process of triplication,

$$b(q) = \tfrac{1}{2}\left(3 \cdot \tfrac{1}{3}\left\{1 + 2(1-x)^{1/3}\right\}z - z\right) = (1-x)^{1/3}z,$$

while by (2.9), Theorem 2.10, and the process of trimidiation,

$$c(q) = \tfrac{1}{2}\left((1 + 2x^{1/3})z - z\right) = x^{1/3}z.$$

Theorem 3.3. *Recall that $f(-q)$ is defined by (1.14). Then for any base q,*

$$qf^{24}(-q) = \tfrac{1}{27}b^9(q)c^3(q). \tag{3.8}$$

Proof. All calculations below pertain to the classical base q.

By Entry 12(ii) of Chapter 17 of Ramanujan's second notebook (Part III [3, p. 124]),

$$qf^{24}(-q) = \tfrac{1}{16}z_1^{12}\alpha(1-\alpha)^4. \tag{3.9}$$

It thus suffices to show that the right side of (3.8) is equal to the right side of (3.9). To do this, we use the parametrizations for $b(q)$ and $c(q)$ given by (2.11) and (2.12), respectively. It will then be necessary to express the functions of m arising in (2.11) and (2.12) in terms of α and β. In addition to (2.14) and (2.15), we need the parametrizations

$$1 - \alpha = \frac{(m+1)(3-m)^3}{16m^3} \tag{3.10}$$

and

$$1 - \beta = \frac{(m+1)^3(3-m)}{16m}, \tag{3.11}$$

given in (5.5) of Chapter 19 of (Part III [3, p. 233]). Direct calculations yield

$$9 - m^2 = 4m^2\frac{\alpha^{3/8}(1-\alpha)^{3/8}}{\beta^{1/8}(1-\beta)^{1/8}}, \tag{3.12}$$

$$3 - m = 2m\frac{(1-\alpha)^{3/8}}{(1-\beta)^{1/8}}, \tag{3.13}$$

$$m + 1 = 2\frac{(1-\beta)^{3/8}}{(1-\alpha)^{1/8}}, \tag{3.14}$$

and

$$m^2 - 1 = 4\frac{\beta^{3/8}(1-\beta)^{3/8}}{\alpha^{1/8}(1-\alpha)^{1/8}}. \tag{3.15}$$

Hence, by (2.11), (3.12), and (3.13),

$$b(q) = \frac{z_1}{4m^{7/6}} 2m \frac{(1-\alpha)^{3/8}}{(1-\beta)^{1/8}} (4m^2)^{1/3} \frac{\alpha^{1/8}(1-\alpha)^{1/8}}{\beta^{1/24}(1-\beta)^{1/24}}$$
$$= \frac{z_1 m^{1/2} \alpha^{1/8}(1-\alpha)^{1/2}}{2^{1/3}\beta^{1/24}(1-\beta)^{1/6}}. \tag{3.16}$$

By (2.12), (3.14), and (3.15),

$$c(q) = \frac{3z_1}{4m^{3/2}} \frac{2(1-\beta)^{3/8}}{(1-\alpha)^{1/8}} \frac{4^{1/3}\beta^{1/8}(1-\beta)^{1/8}}{\alpha^{1/24}(1-\alpha)^{1/24}}$$
$$= \frac{3z_1\beta^{1/8}(1-\beta)^{1/2}}{2^{1/3}m^{3/2}\alpha^{1/24}(1-\alpha)^{1/6}}. \tag{3.17}$$

It now follows easily from (3.16) and (3.17) that

$$\tfrac{1}{27}b^9(q)c^3(q) = \tfrac{1}{16}z_1^{12}\alpha(1-\alpha)^4,$$

which, by (3.9), completes the proof.

Corollary 3.4 (p. 257). *Let $q = q_3$, and let z be as in Theorem 2.10. Then*

$$q^{1/24}f(-q) = \sqrt{z}3^{-1/8}x^{1/24}(1-x)^{1/8}. \tag{3.18}$$

Proof. By Theorem 3.3 and Corollary 3.2,

$$qf^{24}(-q) = \tfrac{1}{27}(1-x)^3xz^{12},$$

from which (3.18) follows.

Corollary 3.5 (p. 257). *With the same notation as in Corollary 3.4,*

$$q^{1/8}f(-q^3) = \sqrt{z}3^{-3/8}x^{1/8}(1-x)^{1/24}.$$

Proof. Applying to (3.18) the process of triplication enunciated in Theorem 3.1, we deduce that

$$q^{1/8}f(-q^3) = \sqrt{z}\frac{\{1+2(1-x)^{1/3}\}^{1/2}}{3^{5/8}} \left\{\frac{1-(1-x)^{1/3}}{1+2(1-x)^{1/3}}\right\}^{1/8}$$
$$\times \left\{\frac{\left(1+2(1-x)^{1/3}\right)^3 - \left(1-(1-x)^{1/3}\right)^3}{\left(1+2(1-x)^{1/3}\right)^3}\right\}^{1/8}$$
$$= \sqrt{z}3^{-3/8}\left\{1-(1-x)^{1/3}\right\}^{1/8}(1-x)^{1/24}$$
$$\times \left\{1+(1-x)^{1/3}+(1-x)^{2/3}\right\}^{1/8}$$
$$= \sqrt{z}3^{-3/8}\{1-(1-x)\}^{1/8}(1-x)^{1/24}$$
$$= \sqrt{z}3^{-3/8}(1-x)^{1/24}x^{1/8},$$

as desired.

4. The Eisenstein Series L, M, and N

We now recall Ramanujan's definitions of L, M, and N, first defined in Chapter 15 of his second notebook (Part II [2, p. 318]) and thoroughly studied by him, especially in his paper [7], [10, pp. 136–162], where the notations P, Q, and R are used instead of L, M, and N, respectively. Thus, for $|q| < 1$,

$$L(q) := 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n},$$

$$M(q) := 1 + 240\sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n},$$

and

$$N(q) := 1 - 504\sum_{n=1}^{\infty} \frac{n^5q^n}{1-q^n}.$$

We first derive an analogue of Entry 9(iv) of Chapter 17 in Ramanujan's second notebook (Part III [3, p. 120]).

Lemma 4.1. *Let $q = q_3$ be defined by (1.7), and let z be as in Theorem 2.10. Then*

$$L(q) = (1-4x)z^2 + 12x(1-x)z\frac{dz}{dx}. \tag{4.1}$$

Proof. By logarithmic differentiation,

$$\begin{aligned} q\frac{d}{dq}\log\left(qf^{24}(-q)\right) &= q\frac{d}{dq}\log\left(q\prod_{n=1}^{\infty}(1-q^n)^{24}\right) \\ &= 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} = L(q). \end{aligned} \tag{4.2}$$

On the other hand, by Corollary 3.4,

$$\begin{aligned} q\frac{d}{dq}\log\left(qf^{24}(-q)\right) &= q\frac{d}{dq}\log\left(\tfrac{1}{27}z^{12}x(1-x)^3\right) \\ &= q\frac{d}{dx}\log\left(\tfrac{1}{27}z^{12}x(1-x)^3\right)\frac{dx}{dq}. \end{aligned} \tag{4.3}$$

Now by Entry 30 of Chapter 11 of Ramanujan's second notebook (Part II [2, p. 87]),

$$\frac{d}{dx}\left(\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-x)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;x)}\right) = -\frac{1}{x(1-x)z^2}.$$

Thus,

$$\frac{dq}{dx}=\frac{q}{x(1-x)z^2}. \tag{4.4}$$

Using (4.4) in (4.3), we deduce that

$$\begin{aligned} q\frac{d}{dq}\log\left(qf^{24}(-q)\right) &= x(1-x)z^2\left(\frac{12}{z}\frac{dz}{dx}+\frac{1}{x}-\frac{3}{1-x}\right) \\ &= 12x(1-x)z\frac{dz}{dx}+(1-4x)z^2. \end{aligned} \tag{4.5}$$

Combining (4.2) and (4.5), we arrive at (4.1) to complete the proof.

Theorem 4.2 (p. 257). *We have*

$$M(q)=z^4(1+8x). \tag{4.6}$$

Proof. From Ramanujan's paper [7], [10, p. 330] or from Part II [2, p. 330, Entry 13],

$$q\frac{dL}{dq}=\tfrac{1}{12}\left\{L^2(q)-M(q)\right\}.$$

Thus, by (4.4) and Lemma 4.1,

$$\begin{aligned} M(q) &= L^2(q)-12x(1-x)z^2\frac{dL}{dx} \\ &= (1-4x)^2z^4+24x(1-x)(1-4x)z^3\frac{dz}{dx}+144x^2(1-x)^2z^2\left(\frac{dz}{dx}\right)^2 \\ &\quad -12x(1-x)z^2\left(-4z^2+2(1-4x)z\frac{dz}{dx}+12x(1-x)\left(\frac{dz}{dx}\right)^2+12z\frac{2}{9}z\right), \end{aligned}$$

where we have employed the differential equation for z (Bailey [1, p. 1])

$$\frac{d}{dx}\left\{x(1-x)\frac{dz}{dx}\right\}=\frac{2}{9}z. \tag{4.7}$$

Upon simplifying the equality above, we deduce (4.6).

Theorem 4.3 (p. 257). *We have*

$$N(q)=z^6(1-20x-8x^2). \tag{4.8}$$

Proof. From Ramanujan's paper [7], [10, p. 142] or from Part II [2, p. 330, Entry 13],

$$q\frac{dM}{dq}=\tfrac{1}{3}\{L(q)M(q)-N(q)\}.$$

Thus, by (4.4), Lemma 4.1, and Theorem 4.2,

$$\begin{aligned}
N(q) &= L(q)M(q) - 3x(1-x)z^2\frac{dM}{dx} \\
&= \left\{(1-4x)z^2 + 12x(1-x)z\frac{dz}{dx}\right\} z^4(1+8x) \\
&\quad - 3x(1-x)z^2\left\{4z^3(1+8x)\frac{dz}{dx} + 8z^4\right\} \\
&= z^6\{(1-4x)(1+8x) - 24x(1-x)\} \\
&= z^6(1-20x-8x^2),
\end{aligned}$$

and so (4.8) has been proved.

Theorem 4.4 (p. 257). *We have*

$$M(q^3) = z^4\left(1-\tfrac{8}{9}x\right).$$

Proof. Apply the process of triplication to (4.6). Thus, by Theorem 3.1,

$$\begin{aligned}
M(q^3) &= \tfrac{1}{81}z^4\left(1+2(1-x)^{1/3}\right)^4\left(1+8\left\{\frac{1-(1-x)^{1/3}}{1+2(1-x)^{1/3}}\right\}^3\right) \\
&= \tfrac{1}{9}z^4\left(1+2(1-x)^{1/3}\right)\left(1-2(1-x)^{1/3}+4(1-x)^{2/3}\right) \\
&= \tfrac{1}{9}z^4(1+8(1-x)) \\
&= \tfrac{1}{9}z^4(9-8x),
\end{aligned}$$

and the proof is complete.

Theorem 4.5 (p. 257). *We have*

$$N(q^3) = z^6\left(1-\tfrac{4}{3}x+\tfrac{8}{27}x^2\right).$$

Proof. Applying the process of triplication to (4.8), we find that

$$\begin{aligned}
N(q^3) &= \frac{z^6}{3^6}\left(1+2(1-x)^{1/3}\right)^6\left(1-20\left\{\frac{1-(1-x)^{1/3}}{1+2(1-x)^{1/3}}\right\}^3\right. \\
&\quad \left. -8\left\{\frac{1-(1-x)^{1/3}}{1+2(1-x)^{1/3}}\right\}^6\right) \\
&= \frac{z^6}{3^6}\left\{\left(1+2(1-x)^{1/3}\right)^6 - 20\left(1+2(1-x)^{1/3}\right)^3\left(1-(1-x)^{1/3}\right)^3\right. \\
&\quad \left. -8\left(1-(1-x)^{1/3}\right)^6\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{z^6}{3^6}\left\{-27 + 540(1-x) + 216(1-x)^2\right\} \\
&= \frac{z^6}{3^6}(729 - 972x + 216x^2) \\
&= z^6\left(1 - \tfrac{4}{3}x + \tfrac{8}{27}x^2\right).
\end{aligned}
$$

We complete this section by offering a remarkable formula for z^4 and an identity involving the sixth powers of the Borweins' cubic theta–functions $b(q)$ and $c(q)$.

Corollary 4.6. *We have*

$$10z^4 = 9M(q^3) + M(q). \tag{4.9}$$

Proof. Using Theorems 4.2 and 4.4 on the right side of (4.9), we easily establish the desired result.

Corollary 4.7. *We have*

$$28\left\{b^6(q) - c^6(q)\right\} = 27N(q^3) + N(q). \tag{4.10}$$

Proof. Using Theorems 4.3 and 4.5 on the right side of (4.10), and also employing Corollary 3.2, we readily deduce (4.10).

5. A Hypergeometric Transformation and Associated Transfer Principle

We shall prove a new transformation formula relating the hypergeometric functions $z(2)$ and $z(3)$ and employ it to establish a means for transforming formulas in the theory of signature 2 to that in signature 3, and conversely. We first need to establish several formulas relating the functions $\varphi(q)$, $\psi(q)$, and $f(-q)$ with $a(q)$, $b(q)$, and $c(q)$.

Let, as customary,

$$(a;q)_\infty := (1-a)(1-aq)(1-aq^2)\cdots, \qquad |q| < 1.$$

For any integer n, also set

$$(a;q)_n = \frac{(a;q)_\infty}{(a;q^n)_\infty}.$$

From the Jacobi triple product identity (Part III [3, pp. 36, 37]),

$$\varphi(-q) = (q;q)_\infty(q;q^2)_\infty, \tag{5.1}$$

$$\psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{5.2}$$

and

$$f(-q) = (q;q)_\infty. \tag{5.3}$$

Lemma 5.1. *We have*

$$b(q) = \frac{f^3(-q)}{f(-q^3)}, \tag{5.4}$$

$$c(q) = 3q^{1/3}\frac{f^3(-q^3)}{f(-q)}, \tag{5.5}$$

$$3\frac{c(q^2)}{c^2(q)} = \frac{\varphi(-q)}{\varphi^3(-q^3)}, \tag{5.6}$$

$$\frac{c^2(q^4)}{3c(q^2)} = q^2\frac{\psi^3(q^6)}{\psi(q^2)}, \tag{5.7}$$

and

$$\frac{c^2(q^4)}{c^2(q)} = q^2\frac{\psi^3(q^6)\varphi(-q)}{\psi(q^2)\varphi^3(-q^3)}. \tag{5.8}$$

Proof. First, (5.4) and (5.5) follow directly from Corollaries 3.2, 3.4, and 3.5.
Next, from (5.5) and (5.3),

$$3\frac{c(q^2)}{c^2(q)} = \frac{(q^6;q^6)^3_\infty(q;q)^2_\infty}{(q^2;q^2)_\infty(q^3;q^3)^6_\infty}.$$

Using (5.1), we readily find that the right side above equals $\varphi(-q)/\varphi^3(-q^3)$, and the proof of (5.6) is complete.

Again, from (5.5) and (5.3),

$$\frac{c^2(q^4)}{3c(q^2)} = q^2\frac{(q^{12};q^{12})^6_\infty(q^2;q^2)_\infty}{(q^4;q^4)^2_\infty(q^6;q^6)^3_\infty},$$

which, by (5.2), is seen to equal $q^2\psi^3(q^6)/\psi(q^2)$. Thus, (5.7) is proved.

Lastly, (5.8) follows from combining (5.6) and (5.7).

Lemma 5.2. *We have*

$$1 - \frac{\varphi^2(-q)}{\varphi^2(-q^3)} = 4\frac{c(q^4)}{c(q)}.$$

Proof. By (5.8), we want to prove that

$$\varphi^2(-q^3) - \varphi^2(-q) = 4\varphi^2(-q^3)\left(q^2\frac{\psi^3(q^6)\varphi(-q)}{\psi(q^2)\varphi^3(-q^3)}\right)^{1/2}.$$

By Entry 10(ii) of Chapter 17 of the second notebook (Part III [3, p. 122]),

$$\varphi(-q) = \sqrt{z_1}(1-\alpha)^{1/4} \qquad \text{and} \qquad \varphi(-q^3) = \sqrt{z_3}(1-\beta)^{1/4}, \tag{5.9}$$

where β has degree 3 over α. Thus, by (2.13) and (5.9), it suffices to prove that

$$z_3(1-\beta)^{1/2} - z_1(1-\alpha)^{1/2} = 2z_3(1-\beta)^{1/2}\left(\frac{\beta^3}{\alpha}\right)^{1/8}\left(\frac{1-\alpha}{(1-\beta)^3}\right)^{1/8}.$$

Since $m = z_1/z_3$, the last equality is equivalent to the equality,

$$1 - m\left(\frac{1-\alpha}{1-\beta}\right)^{1/2} = 2\left(\frac{\beta^3}{\alpha}\right)^{1/8}\left(\frac{1-\alpha}{(1-\beta)^3}\right)^{1/8}.$$

By (2.14), (2.15), (3.10), and (3.11), the last equality can be written entirely in terms of m, namely,

$$1 - m\frac{3-m}{m(m+1)} = 2\frac{m-1}{2}\frac{2}{m+1}.$$

This equality is trivially verified, and so the proof is complete.

Lemma 5.3. *We have*

$$1 + \frac{\psi^2(q^2)}{q\psi^2(q^6)} = \frac{c(q)}{c(q^4)}.$$

Proof. By (5.8), the proposed identity is equivalent to the identity

$$q\psi^2(q^6) + \psi^2(q^2) = \left(\frac{\psi(q^2)\psi(q^6)\varphi^3(-q^3)}{\varphi(-q)}\right)^{1/2}.$$

By (2.13) and (5.9), the previous identity is equivalent to the identity,

$$1 + m\left(\frac{\alpha}{\beta}\right)^{1/2} = 2\left(\frac{\alpha}{\beta^3}\right)^{1/8}\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8}.$$

By (2.14), (2.15), (3.10), and (3.11), the last equality can be expressed completely in terms of m as

$$1 + m\frac{3+m}{m(m-1)} = 2\frac{2}{m-1}\frac{m+1}{2}.$$

Since this last equality is trivially verified, the proof is complete.

Lemma 5.4. *We have*

$$a(q) = \frac{\varphi^3(-q^3)}{\varphi(-q)} + 4q\frac{\psi^3(q^3)}{\psi(q)}.$$

Proof. By Entry 11(i) in Chapter 17 (Part III [3, p. 123]),

$$\psi(q) = \sqrt{\tfrac{1}{2}z_1}(\alpha/q)^{1/8} \qquad \text{and} \qquad \psi(q^3) = \sqrt{\tfrac{1}{2}z_3}(\beta/q^3)^{1/8}. \tag{5.10}$$

Thus, by Entry 3(i) of Chapter 21 of Ramanujan's second notebook (Part III [3, p. 460]), (5.10), (2.14), (2.15), (3.10), and (3.11),

$$\begin{aligned}
a(q) &= \frac{\psi^3(q)}{\psi(q^3)} + 3q\frac{\psi^3(q^3)}{\psi(q)} \\
&= \frac{z_1^{3/2}}{2z_3^{1/2}}\left(\frac{\alpha^3}{\beta}\right)^{1/8} + 3q\frac{\psi^3(q^3)}{\psi(q)} \\
&= \frac{z_3^{3/2}}{z_1^{1/2}}\frac{m^2+3m}{4} + 3q\frac{\psi^3(q^3)}{\psi(q)} \\
&= \frac{z_3^{3/2}}{z_1^{1/2}}\left(\frac{(m+1)^2}{4} + \frac{m-1}{4}\right) + 3q\frac{\psi^3(q^3)}{\psi(q)} \\
&= \frac{z_3^{3/2}}{z_1^{1/2}}\left(\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/4} + \frac{1}{2}\left(\frac{\beta^3}{\alpha}\right)^{1/8}\right) + 3q\frac{\psi^3(q^3)}{\psi(q)}.
\end{aligned}$$

By (5.9) and (5.10), the right-hand side above equals

$$\frac{\varphi^3(-q^3)}{\varphi(-q)} + q\frac{\psi^3(q^3)}{\psi(q)} + 3q\frac{\psi^3(q^3)}{\psi(q)}.$$

The desired result now follows.

Lemma 5.5. *If* $p := p(q) := -2c(q^4)/c(q)$, *then*

$$1+2p = \frac{\varphi^2(-q)}{\varphi^2(-q^3)}, \tag{5.11}$$

$$2+p = 2\frac{c(q^4)}{c(q)}\frac{\psi^2(q^2)}{q\psi^2(q^6)}, \tag{5.12}$$

$$1+p = -\frac{c(-q)}{c(q)} = \frac{c^3(q^2)}{c^2(q)c(q^4)}, \tag{5.13}$$

and

$$1+p+p^2 = 3\frac{a(q^2)c(q^2)}{c^2(q)}. \tag{5.14}$$

Proof. Equations (5.11) and (5.12) follow from Lemmas 5.2 and 5.3, respectively.

As observed by the Borweins and F. G. Garvan [1], it follows from the definition (2.4) of $c(q)$ that

$$c(q) + c(-q) = 2c(q^4). \tag{5.15}$$

Thus, by (5.15), (5.5), and (5.3),

$$\begin{aligned}1+p=-\frac{c(-q)}{c(q)}&=\frac{(-q^3;-q^3)_\infty^3(q;q)_\infty}{(-q;-q)_\infty(q^3;q^3)_\infty^3}\\&=\frac{(-q^3;q^6)_\infty^3(q^6;q^6)_\infty^3(q;q)_\infty}{(-q;q^2)_\infty(q^2;q^2)_\infty(q^3;q^3)_\infty^3}\\&=\frac{(q^6;q^{12})_\infty^3(q^6;q^6)_\infty^3(q;q)_\infty(q;q^2)_\infty}{(q^3;q^6)_\infty^3(q^2;q^4)_\infty(q^2;q^2)_\infty(q^3;q^3)_\infty^3}\\&=\frac{(q^6;q^6)_\infty^9(q;q)_\infty^2(q^4;q^4)_\infty}{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^6(q^{12};q^{12})_\infty^3}\\&=\frac{c^3(q^2)}{c^2(q)c(q^4)}.\end{aligned}$$

Thus, (5.13) has been verified.

Lastly, by Lemma 5.4, (5.6), and (5.7),

$$a(q^2)=\frac{1}{3}\left(\frac{c^2(q^2)}{c(q^4)}+4\frac{c^2(q^4)}{c(q^2)}\right). \tag{5.16}$$

Hence, by (5.13) and (5.16),

$$\begin{aligned}1+p+p^2&=\frac{c^3(q^2)}{c^2(q)c(q^4)}+4\frac{c^2(q^4)}{c^2(q)}\\&=\frac{c(q^2)}{c^2(q)}\left(\frac{c^2(q^2)}{c(q^4)}+4\frac{c^2(q^4)}{c(q^2)}\right)=\frac{3c(q^2)a(q^2)}{c^2(q)},\end{aligned}$$

which proves (5.14).

Theorem 5.6 (p. 258). *If*

$$\alpha:=\frac{p^3(2+p)}{1+2p}\qquad \textit{and}\qquad \beta:=\frac{27p^2(1+p)^2}{4(1+p+p^2)^3},$$

then, for $0\le p<1$,

$$(1+p+p^2)\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;\alpha\right)=\sqrt{1+2p}\,{}_2F_1\left(\tfrac{1}{3},\tfrac{2}{3};1;\beta\right). \tag{5.17}$$

Proof. By Lemma 5.5 and (5.8),

$$\begin{aligned}\alpha&=-16\frac{c^4(q^4)}{c^4(q)}\frac{\varphi^2(-q^3)}{\varphi^2(-q)}\frac{\psi^2(q^2)}{q\psi^2(q^6)}\\&=-16q^3\frac{\psi^4(q^6)}{\varphi^4(-q^3)}=1-\frac{\varphi^4(q^3)}{\varphi^4(-q^3)},\end{aligned} \tag{5.18}$$

by Jacobi's identity for fourth powers, (2.1), with q replaced by q^3. Thus (Part III [3, p. 98, Entry 3]), with q replaced by $-q^3$,

$${}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;\alpha\right)=\varphi^2(-q^3). \tag{5.19}$$

Also, by Lemma 5.5,

$$\beta = \frac{c^3(q^2)}{a^3(q^2)}.$$

Thus, by Lemma 2.6,

$$ {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; \beta\right) = a(q^2). \tag{5.20}$$

By Lemma 5.5, (5.19), and (5.6),

$$\begin{aligned}(1+p+p^2)\,{}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \alpha\right) &= 3\frac{a(q^2)c(q^2)\varphi^2(-q^3)}{c^2(q)} \\ &= \frac{\varphi(-q)}{\varphi(-q^3)}a(q^2) \\ &= \sqrt{1+2p}\,a(q^2) \\ &= \sqrt{1+2p}\,{}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; \beta\right),\end{aligned}$$

by (5.20).

Lastly, we show that our proof of (5.17) above is valid for $0 \le p < 1$. Observe that

$$\frac{d\alpha}{dp} = \frac{6p^2(1+p)^2}{(1+2p)^2} \quad \text{and} \quad \frac{d\beta}{dp} = \frac{27p(1-p^2)(1+2p)(2+p)}{4(1+p+p^2)^4}.$$

Hence, $\alpha(p)$ and $\beta(p)$ are monotonically increasing on (0,1). Since $\alpha(0) = 0 = \beta(0)$ and $\alpha(1) = 1 = \beta(1)$, it follows that Theorem 5.6 is valid for $0 \le p < 1$.

We now prove a corresponding theorem with α and β replaced by $1-\alpha$ and $1-\beta$, respectively.

Corollary 5.7. *Let α and β be as defined in Theorem 5.6. Then, for $0 < p \le 1$,*

$$(1+p+p^2)\,{}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-\alpha\right) = \sqrt{3+6p}\,{}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; 1-\beta\right). \tag{5.21}$$

Proof. By (5.19) and (5.20), with q replaced by $-q$,

$$\frac{a(q^2)}{\varphi^2(q^3)} = \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{c^3(q^2)}{a^3(q^2)}\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\frac{\varphi^4(-q^3)}{\varphi^4(q^3)}\right)}. \tag{5.22}$$

Thus, by (5.21) and (5.22), it suffices to prove that

$$\frac{a(q^2)}{\varphi^2(q^3)} = \sqrt{3}\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\frac{c^3(q^2)}{a^3(q^2)}\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\varphi^4(-q^3)}{\varphi^4(q^3)}\right)}. \tag{5.23}$$

By Lemma 2.9,

$$q^2 = \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\frac{c^3(q^2)}{a^3(q^2)}\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{c^3(q^2)}{a^3(q^2)}\right)}\right) \tag{5.24}$$

and by Entry 5 of Chapter 17 in Ramanujan's second notebook (Part III [3, p. 100]),

$$q^3 = \exp\left(-\pi\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\varphi^4(-q^3)}{\varphi^4(q^3)}\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\frac{\varphi^4(-q^3)}{\varphi^4(q^3)}\right)}\right). \tag{5.25}$$

Combining (5.24) and (5.25), we find that

$$\sqrt{3}\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\frac{c^3(q^2)}{a^3(q^2)}\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{c^3(q^2)}{a^3(q^2)}\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\varphi^4(-q^3)}{\varphi^4(q^3)}\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\frac{\varphi^4(-q^3)}{\varphi^4(q^3)}\right)}. \tag{5.26}$$

We now see that (5.23) follows from combining (5.22) and (5.26).

Corollary 5.8. *With α and β as above, for $0 < p < 1$,*

$$\frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha)}{\sqrt{3}\,{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \alpha)} = \frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-\beta)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; \beta)}. \tag{5.27}$$

Proof. Divide (5.21) by (5.17).

The authors' first proof of Corollary 5.7 employed Theorem 5.6 and a lemma arising from the hypergeometric differential equation satisfied by ${}_2F_1(a, b; c; x)$ and ${}_2F_1(a, b; c; 1-x)$. We are grateful to Heng Huat Chan for providing the proof that is given above. He has also shown that still another proof of Corollary 5.7 can be effected by combining Theorem 5.6 with Entry 6(i) in Chapter 19 of Ramanujan's second notebook (Part III [3, p. 238]). We leave this proof as an exercise for readers.

Corollary 5.8 is important, for from (5.27) and (1.7),

$$q_3^3 =: q_3^3(\beta) = q^2(\alpha) := q^2, \tag{5.28}$$

where $q = q(\alpha)$ denotes the classical base. Thus, from Theorem 5.6 and (5.28), we can deduce the following theorem.

Theorem 5.9 (Transfer Principle). *Suppose that we have a formula*

$$\Omega\left(q^2(\alpha), z(2; \alpha(p))\right) = 0 \tag{5.29}$$

in the classical situation. Then

$$\Omega\left(q_3^3(\beta), \frac{\sqrt{1+2p}}{1+p+p^2}z(3; \beta(p))\right) = 0, \tag{5.30}$$

in the theory of signature 3.

If the formula (5.29) involves α, in addition to its appearance through q and z, it may be possible to convert (5.29) into a formula (5.30) involving only β, q_3, and $z(3)$. This good fortune is manifest in the next three instances, as we offer alternative proofs of Corollary 3.5, Theorem 4.4, and Theorem 4.5.

Second Proof of Corollary 3.5. By elementary calculations,

$$1-\alpha = \frac{(1-p)(1+p)^3}{1+2p} \qquad \text{and} \qquad 1-\beta = \frac{(2+p)^2(1+2p)^2(1-p)^2}{4(1+p+p^2)^3}. \tag{5.31}$$

From Entry 12(iii) of Chapter 17 (Part III [3, p. 124]), Theorem 5.6, (5.28), and (5.30),

$$\begin{aligned}
f(-q_3^3) = f(-q^2) &= \sqrt{z}2^{-1/3}\{\alpha(1-\alpha)/q\}^{1/12} \\
&= \frac{(1+2p)^{1/4}}{(1+p+p^2)^{1/2}}\sqrt{z(3)}2^{-1/3}\left(\frac{p^3(2+p)(1-p)(1+p)^3}{(1+2p)^2}\right)^{1/12}\frac{1}{q_3^{1/8}} \\
&= \frac{\sqrt{z(3)}}{q_3^{1/8}3^{3/8}}\left(\frac{27p^2(1+p)^2}{4(1+p+p^2)^3}\right)^{1/8}\left(\frac{(2+p)^2(1+2p)^2(1-p)^2}{4(1+p+p^2)^3}\right)^{1/24} \\
&= \frac{\sqrt{z(3)}}{q_3^{1/8}3^{3/8}}\beta^{1/8}(1-\beta)^{1/24},
\end{aligned}$$

by Theorem 5.6 and (5.31). This completes our second proof of Corollary 3.5.

Second Proof of Theorem 4.4. By Entry 13(i) of Chapter 17 (Part III [3, p. 126]), Theorem 5.6, (5.28), and (5.30),

$$\begin{aligned}
M(q_3^3) = M(q^2) &= z^4(1-\alpha+\alpha^2) \\
&= \frac{(1+2p)^2}{(1+p+p^2)^4}z^4(3)\left(1-\frac{p^3(2+p)}{1+2p}+\frac{p^6(2+p)^2}{(1+2p)^2}\right) \\
&= z^4(3)\frac{1+3p-5p^3+3p^5+p^6}{(1+p+p^2)^3} \\
&= z^4(3)\frac{\{(1+p+p^2)^3-6p^2(1+p)^2\}}{(1+p+p^2)^3} \\
&= z^4(3)\left(1-\tfrac{8}{9}\beta\right).
\end{aligned}$$

Second Proof of Theorem 4.5. By Entry 13(ii) of Chapter 17 (Part III [3, p. 126]), Theorem 5.6, (5.28), and (5.30),

$$\begin{aligned}
N(q_3^3) = N(q^2) &= z^6(1+\alpha)(1-\tfrac{1}{2}\alpha)(1-2\alpha) \\
&= \frac{z^6(3)}{2(1+p+p^2)^6}\left(1+2p+p^3(2+p)\right)\left(2(1+2p)-p^3(2+p)\right)
\end{aligned}$$

$$\times\left(1+2p-2p^3(2+p)\right)$$
$$=\frac{z^6(3)}{2(1+p+p^2)^6}\left\{2(1+p+p^2)^6-18p^2(1+p)^2(1+p+p^2)^3\right.$$
$$\left.+27p^4(1+p)^4\right\}$$
$$=z^6(3)\left(1-\tfrac{4}{3}\beta+\tfrac{8}{27}\beta^2\right).$$

Having thus proved Corollary 3.5, Theorem 4.4, and Theorem 4.5, we may use the process of trimidiation to reprove Corollary 3.4, Theorem 4.2, and Theorem 4.3.

6. More Higher Order Transformations for Hypergeometric Series

The first theorem will be used to prove Ramanujan's modular equations of degree 2 in the theory of signature 3.

Theorem 6.1 (p. 258). *If*

$$\alpha:=\alpha(p):=\frac{p(3+p)^2}{2(1+p)^3}\qquad\textit{and}\qquad\beta:=\beta(p):=\frac{p^2(3+p)}{4},\tag{6.1}$$

then, for $0\le p<1$,

$${}_2F_1\left(\tfrac13,\tfrac23;1;\alpha\right)=(1+p)\,{}_2F_1\left(\tfrac13,\tfrac23;1;\beta\right).\tag{6.2}$$

Proof. We first prove that

$$a(q)-a(q^2)=2\frac{c^2(q^2)}{c(q)}.\tag{6.3}$$

From Entry 3(i), (ii) of Chapter 21 of Ramanujan's second notebook (Part III [3, p. 460]),

$$a(q^2)=\frac{\varphi^3(q)}{4\varphi(q^3)}+\frac{3\varphi^3(q^3)}{4\varphi(q)}.\tag{6.4}$$

Thus, by (6.4), Theorem 2.13, (1.17), (2.13), (2.14), (2.15), (5.10), and (5.7),

$$a(q)-a(q^2)=4\frac{\psi^3(q^2)}{\psi(q^6)}-\frac{15\varphi^3(q^3)}{4\varphi(q)}-\frac{\varphi^3(q)}{4\varphi(q^3)}$$
$$=\frac{z_1^{3/2}}{z_3^{1/2}}\left(\frac{\alpha^3}{\beta}\right)^{1/4}-\frac{15}{4}\frac{z_3^{3/2}}{z_1^{1/2}}-\frac{z_1^{3/2}}{4z_3^{1/2}}$$
$$=\frac{z_3^{3/2}}{z_1^{1/2}}\left(m^2\frac{(3+m)^2}{4m^2}-\frac{15}{4}-\frac{1}{4}m^2\right)$$

$$= \frac{3z_3^{3/2}}{2z_1^{1/2}}(m-1) = 3\frac{z_3^{3/2}}{z_1^{1/2}}\left(\frac{\beta^3}{\alpha}\right)^{1/8}$$
$$= 6q\frac{\psi^3(q^3)}{\psi(q)} = 2\frac{c^2(q^2)}{c(q)},$$

which completes the proof of (6.3).

Second, we prove that

$$a(q) + 2a(q^2) = \frac{c^2(q)}{c(q^2)}. \tag{6.5}$$

By (6.4), Theorem 2.13, (1.17), (2.13), (2.14), (2.15), (3.10), (3.11), (5.9), and (5.6),

$$a(q) + 2a(q^2) = 4\frac{\psi^3(q^2)}{\psi(q^6)} - \frac{3\varphi^3(q^3)}{2\varphi(q)} + \frac{\varphi^3(q)}{2\varphi(q^3)}$$
$$= \frac{z_1^{3/2}}{z_3^{1/2}}\left(\frac{\alpha^3}{\beta}\right)^{1/4} - \frac{3}{2}\frac{z_3^{3/2}}{z_1^{1/2}} + \frac{z_1^{3/2}}{2z_3^{1/2}}$$
$$= \frac{z_3^{3/2}}{z_1^{1/2}}\left(\frac{(3+m)^2}{4} - \frac{3}{2} + \frac{1}{2}m^2\right)$$
$$= \frac{3z_3^{3/2}}{4z_1^{1/2}}(m+1)^2 = 3\frac{z_3^{3/2}}{z_1^{1/2}}\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/4}$$
$$= 3\frac{\varphi^3(-q^3)}{\varphi(-q)} = \frac{c^2(q)}{c(q^2)},$$

which proves (6.5).

Now let

$$p := p(q) := \frac{a(q)}{a(q^2)} - 1. \tag{6.6}$$

(Note that p tends to 0 as q tends to 0, by (2.2).) Then by (6.6), (6.3), and (6.5),

$$\alpha = \frac{p(3+p)^2}{2(1+p)^3} = \frac{1}{2}\left(\frac{a(q)}{a(q^2)} - 1\right)\left(\frac{a(q)}{a(q^2)} + 2\right)^2 \frac{a^3(q^2)}{a^3(q)}$$
$$= \frac{\left(a(q) - a(q^2)\right)\left(a(q) + 2a(q^2)\right)^2}{2a^3(q)}$$
$$= 2\frac{c^2(q^2)}{c(q)}\frac{c^4(q)}{c^2(q^2)}\frac{1}{2a^3(q)} = \frac{c^3(q)}{a^3(q)}. \tag{6.7}$$

Also, by (6.6), (6.3), and (6.5),

$$\beta = \frac{p^2(3+p)}{4} = \frac{1}{4}\left(\frac{a(q)}{a(q^2)} - 1\right)^2\left(\frac{a(q)}{a(q^2)} + 2\right)$$
$$= \frac{1}{4a^3(q^2)}\frac{4c^4(q^2)}{c^2(q)}\frac{c^2(q)}{c(q^2)} = \frac{c^3(q^2)}{a^3(q^2)}. \tag{6.8}$$

The desired result now follows immediately from Lemma 2.6, (6.6), (6.7), and (6.8).

We now determine those values of p for which our proof of (6.2) above holds. By (6.1),

$$\frac{d\alpha}{dp} = \frac{3(3+p)(1-p)}{2(1+p)^4} \quad \text{and} \quad \frac{d\beta}{dp} = \frac{3p(2+p)}{4}.$$

Thus, $\alpha(p)$ and $\beta(p)$ are monotonically increasing on (0, 1). Since $\alpha(0) = 0 = \beta(0)$ and $\alpha(1) = 1 = \beta(1)$, it follows that (6.2) holds for $0 \le p < 1$.

As functions of p, the left and right sides of (6.2) are solutions of the differential equation,

$$p(1-p)(1+p)^2(2+p)(3+p)u'' + 2(1+p)(3-4p-6p^2-4p^3-p^4)u' \\ = 2(1-p)(3+p)u.$$

Corollary 6.2. *Let α and β be defined by (6.1). Then, for $0 < p \le 1$,*

$${}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; 1-\alpha\right) = \tfrac{1}{2}(1+p)\, {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; 1-\beta\right). \tag{6.9}$$

Proof. By Lemma 2.9 and (6.7) and (6.8), respectively,

$$q = \exp\left(-\frac{2\pi}{\sqrt{3}} \frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-\alpha)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; \alpha)}\right) \tag{6.10}$$

and

$$q^2 = \exp\left(-\frac{2\pi}{\sqrt{3}} \frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-\beta)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; \beta)}\right). \tag{6.11}$$

The desired result now follows easily from (6.2), (6.10), and (6.11).

Corollary 6.3. *Let $\alpha(p)$ and $\beta(p)$ be given by (6.1). Then, for $0 < p < 1$,*

$$2\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \beta\right)} \tag{6.12}$$

and

$$m(3) = 1 + p, \tag{6.13}$$

where $m(3)$ is the multiplier of degree 2 for the theory of signature 3.

Proof. Divide (6.9) by (6.2). Since (6.12) is the defining relation for a modular equation of degree 2 in the theory of signature 3, (6.13) follows from (1.10) and (6.2).

The next transformation is useful in establishing Ramanujan's modular equation of degree 4.

Theorem 6.4 (p. 258). *Let*

$$\alpha := \alpha(p) := \frac{27p(1+p)^4}{2(1+4p+p^2)^3} \quad \textit{and} \quad \beta := \beta(p) := \frac{27p^4(1+p)}{2(2+2p-p^2)^3}. \tag{6.14}$$

Then, for $0 \le p < 1$,

$$(2+2p-p^2)\,{}_2F_1\left(\tfrac{1}{3},\tfrac{2}{3};1;\alpha\right) = 2(1+4p+p^2)\,{}_2F_1\left(\tfrac{1}{3},\tfrac{2}{3};1;\beta\right). \tag{6.15}$$

Proof. For brevity, set $z(x) = {}_2F_1\left(\tfrac{1}{3},\tfrac{2}{3};1;x\right)$. In view of Theorem 6.1, we want to find x and y so that

$$\begin{aligned} z\left(\frac{y(3+y)^2}{2(1+y)^3}\right) &= (1+y)z\left(\frac{y^2(3+y)}{4}\right) \\ &= (1+y)z\left(\frac{x(3+x)^2}{2(1+x)^3}\right) = (1+y)(1+x)z\left(\frac{x^2(3+x)}{4}\right), \end{aligned} \tag{6.16}$$

$$\frac{y(3+y)^2}{2(1+y)^3} = \frac{27p(1+p)^4}{2(1+4p+p^2)^3}, \tag{6.17}$$

$$\frac{x^2(3+x)}{4} = \frac{27p^4(1+p)}{2(2+2p-p^2)^3}, \tag{6.18}$$

and

$$(1+y)(1+x) = \frac{2(1+4p+p^2)}{2+2p-p^2}. \tag{6.19}$$

Solving (6.18) for x, or judiciously guessing the solution with the help of (6.19), we find that

$$x = \frac{3p^2}{2+2p-p^2}. \tag{6.20}$$

Substituting (6.20) into (6.19) and solving for y, we find that

$$y = \frac{3p}{1+p+p^2}. \tag{6.21}$$

Substituting (6.21) into the left side of (6.17), we see that, indeed, (6.17) holds. Lastly, it is easily checked that, with x and y as chosen above,

$$\frac{x(3+x)^2}{2(1+x)^3} = \frac{y^2(3+y)}{4},$$

i.e., the middle equality of (6.16) holds. Hence, (6.15) is valid, and the interval of validity, $0 \le p < 1$, follows by an elementary argument like that in the proof of Theorem 6.1.

Corollary 6.5. *Let* α *and* β *be defined by (6.14). Then, for* $0 < p \le 1$,

$$2(2+2p-p^2)\,{}_2F_1\left(\tfrac{1}{3},\tfrac{2}{3};1;1-\alpha\right) = (1+4p+p^2)\,{}_2F_1\left(\tfrac{1}{3},\tfrac{2}{3};1;1-\beta\right). \tag{6.22}$$

Proof. The proof is analogous to that for Corollary 6.2.

Corollary 6.6. *Let α and β be defined by (6.14). Then, for* $0 < p < 1$,

$$4\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right)}. \tag{6.23}$$

Proof. Divide (6.22) by (6.15).

7. Modular Equations in the Theory of Signature 3

We first show that Corollary 6.3 can be utilized to prove five modular equations of degree 2 offered by Ramanujan.

Theorem 7.1 (p. 259). *If β has degree 2 in the theory of signature 3, then*

(i)
$$(\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3} = 1,$$

(ii)
$$\left(\frac{\alpha^2}{\beta}\right)^{1/3} - \left\{\frac{(1-\alpha)^2}{1-\beta}\right\}^{1/3} = \frac{2}{1+p} = \frac{2}{m},$$

(iii)
$$\left\{\frac{(1-\beta)^2}{1-\alpha}\right\}^{1/3} - \left(\frac{\beta^2}{\alpha}\right)^{1/3} = m,$$

(iv)
$$\left(\frac{\alpha^2}{\beta}\right)^{1/3} + \left\{\frac{(1-\alpha)^2}{1-\beta}\right\}^{1/3} = \frac{4}{m^2},$$

and

(v)
$$\left\{\frac{(1-\beta)^2}{1-\alpha}\right\}^{1/3} + \left(\frac{\beta^2}{\alpha}\right)^{1/3} = m^2.$$

Proof. From (6.1), by elementary calculations,

$$1-\alpha = \frac{(1-p)^2(2+p)}{2(1+p)^3} \qquad \text{and} \qquad 1-\beta = \frac{(1-p)(2+p)^2}{4}. \tag{7.1}$$

Thus, from (6.1) and (7.1), respectively,

$$(\alpha\beta)^{1/3} = \frac{p(3+p)}{2(1+p)} \qquad \text{and} \qquad \{(1-\alpha)(1-\beta)\}^{1/3} = \frac{(1-p)(2+p)}{2(1+p)}.$$

Hence,

$$(\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3} = \frac{3p+p^2+2-p-p^2}{2(1+p)} = \frac{2p+2}{2(1+p)} = 1.$$

Similarly, by (6.1), (7.1), and (6.13),

$$\left(\frac{\alpha^2}{\beta}\right)^{1/3} - \left\{\frac{(1-\alpha)^2}{1-\beta}\right\}^{1/3} = \frac{3+p}{(1+p)^2} - \frac{1-p}{(1+p)^2} = \frac{2}{1+p} = \frac{2}{m},$$

$$\left\{\frac{(1-\beta)^2}{1-\alpha}\right\}^{1/3} - \left(\frac{\beta^2}{\alpha}\right)^{1/3} = \frac{(2+p)(1+p)}{2} - \frac{p(1+p)}{2} = 1+p = m,$$

$$\left(\frac{\alpha^2}{\beta}\right)^{1/3} + \left\{\frac{(1-\alpha)^2}{1-\beta}\right\}^{1/3} = \frac{3+p}{(1+p)^2} + \frac{1-p}{(1+p)^2} = \frac{4}{(1+p)^2} = \frac{4}{m^2},$$

and

$$\left\{\frac{(1-\beta)^2}{1-\alpha}\right\}^{1/3} + \left(\frac{\beta^2}{\alpha}\right)^{1/3} = \frac{(2+p)(1+p)}{2} + \frac{p(1+p)}{2} = (1+p)^2 = m^2.$$

Thus, the proofs of (i)–(v) have been completed.

Theorem 7.2 (p. 259). *Let β be of degree 4, and let m be the associated multiplier in the theory of signature 3. Then*

$$m = \left(\frac{\beta}{\alpha}\right)^{1/3} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/3} - \frac{4}{m}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3}.$$

Proof. From (6.14), we easily find that

$$1-\alpha = \frac{(1-p)^4(2+p)(1+2p)}{2(1+4p+p^2)^3} \quad \text{and} \quad 1-\beta = \frac{(1-p)(2+p)^4(1+2p)}{2(2+2p-p^2)^3}. \tag{7.2}$$

Thus, from (6.14) and (7.2),

$$\begin{aligned}\left(\frac{\beta}{\alpha}\right)^{1/3} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/3} &= \frac{p(1+4p+p^2)}{(1+p)(2+2p-p^2)} + \frac{(2+p)(1+4p+p^2)}{(1-p)(2+2p-p^2)} \\ &= \frac{2(1+4p+p^2)(1+2p)}{(2+2p-p^2)(1+p)(1-p)}. \end{aligned}\tag{7.3}$$

From Theorem 6.4,

$$m = \frac{2(1+4p+p^2)}{2+2p-p^2}. \tag{7.4}$$

Hence, by (7.2) and (7.4),

$$\begin{aligned}\frac{4}{m}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3} &= \frac{2(2+2p-p^2)}{1+4p+p^2}\,\frac{p(2+p)(1+4p+p^2)^2}{(1+p)(1-p)(2+2p-p^2)^2} \\ &= \frac{2p(2+p)(1+4p+p^2)}{(1+p)(1-p)(2+2p-p^2)}. \end{aligned}\tag{7.5}$$

Therefore, combining (7.3) and (7.5), we deduce that

$$\left(\frac{\beta}{\alpha}\right)^{1/3}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/3}-\frac{4}{m}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3}$$
$$=\frac{2(1+4p+p^2)(1+2p)}{(2+2p-p^2)(1+p)(1-p)}-\frac{2p(2+p)(1+4p+p^2)}{(1+p)(1-p)(2+2p-p^2)}$$
$$=\frac{2(1+4p+p^2)}{2+2p-p^2}=m,$$

by (7.4), and the proof is complete.

Theorem 7.3 (p. 204, NB 1). *Let α, β, and γ have degrees 1, 2, and 4, respectively. Let m_1 and m_2 denote the multipliers associated with the pairs α, β and β, γ, respectively. Then*

$$\frac{\sqrt{3}\{\beta(1-\beta)\}^{1/6}}{\{\alpha(1-\gamma)\}^{1/3}-\{\gamma(1-\alpha)\}^{1/3}}=\frac{m_1}{m_2}.$$

Proof. In (6.14), replace β by γ, so that, for $0 \le p < 1$,

$$\alpha=\frac{27p(1+p)^4}{2(1+4p+p^2)^3} \qquad \text{and} \qquad \gamma=\frac{27p^4(1+p)}{2(2+2p-p^2)^3}. \tag{7.6}$$

From the proof of Theorem 6.4, β has the representations

$$\frac{y^2(3+y)}{4} \qquad \text{and} \qquad \frac{x(3+x)^2}{2(1+x)^3},$$

where x and y are given by (6.20) and (6.21), respectively. In either case, a short calculation shows that

$$\beta=\frac{27p^2(1+p)^2}{4(1+p+p^2)^3}. \tag{7.7}$$

Using (7.6) and (7.7), we find that

$$1-\alpha=\frac{2p^6-3p^5-6p^4+14p^3-6p^2-3p+2}{2(1+4p+p^2)^3}$$
$$=\frac{(p-1)^4(2p^2+5p+2)}{2(1+4p+p^2)^3}, \tag{7.8}$$
$$1-\beta=\frac{4p^6+12p^5-3p^4-26p^3-3p^2+12p+4}{4(1+p+p^2)^3}$$
$$=\frac{(p-1)^2(2p^2+5p+2)^2}{4(1+p+p^2)^3}, \tag{7.9}$$

and

$$1-\gamma = \frac{-2p^6-15p^5-39p^4-32p^3+24p^2+48p+16}{2(2+2p-p^2)^3}$$
$$= -\frac{(p-1)(p+2)^3(2p^2+5p+2)}{2(2+2p-p^2)^3}. \tag{7.10}$$

Hence, from (7.6)–(7.10),

$$\frac{\sqrt{3}\{\beta(1-\beta)\}^{1/6}}{\{\alpha(1-\gamma)\}^{1/3}-\{\gamma(1-\alpha)\}^{1/3}}$$
$$= \frac{\sqrt{3}\left(\dfrac{27p^2(1+p)^2}{4(1+p+p^2)^3}\dfrac{(p-1)^2(2p^2+5p+2)^2}{4(1+p+p^2)^3}\right)^{1/6}}{\left\{\left(\dfrac{27p(1+p)^4}{2(1+4p+p^2)^3}\dfrac{(1-p)(p+2)^3(2p^2+5p+2)}{2(2+2p-p^2)^3}\right)^{1/3} - \left(\dfrac{27p^4(1+p)}{2(2+2p-p^2)^3}\dfrac{(p-1)^4(2p^2+5p+2)}{2(1+4p+p^2)^3}\right)^{1/3}\right\}}$$
$$= \frac{(1+4p+p^2)(2+2p-p^2)}{(1+p+p^2)\{(1+p)(2+p)-p(1-p)\}}$$
$$= \frac{(1+4p+p^2)(2+2p-p^2)}{2(1+p+p^2)^2}. \tag{7.11}$$

On the other hand, from the proof of Theorem 6.4, and from (6.20) and (6.21),

$$\frac{m_1}{m_2} = \frac{1+y}{1+x} = \frac{(1+4p+p^2)(2+2p-p^2)}{2(1+p+p^2)^2}. \tag{7.12}$$

Combining (7.11) and (7.12), we complete the proof.

Next, we show that Ramanujan's beautiful cubic transformation in Corollary 2.4 yields the defining relation for modular equations of degree 3. We then iterate the transformation in order to derive Ramanujan's modular equation of degree 9.

Lemma 7.4. *If*

$$\alpha := 1-\left(\frac{1-\beta^{1/3}}{1+2\beta^{1/3}}\right)^3, \tag{7.13}$$

then

$$\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right)} = 3\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha\right)}. \tag{7.14}$$

Furthermore, the multiplier m is equal to $1+2\beta^{1/3}$.

Proof. In (2.23) and (2.25), set $x=\beta^{1/3}$. Dividing (2.25) by (2.23), we deduce (7.14). The formula $m = 1+2\beta^{1/3}$ is an immediate consequence of (2.23).

Theorem 7.5 (p. 259). *If m is the multiplier for modular equations of degree 9, then*

$$m = 3\frac{1+2\beta^{1/3}}{1+2(1-\alpha)^{1/3}}, \tag{7.15}$$

where β has degree 9.

Proof. Let α be given by (7.13), but with β replaced by

$$t := 1 - \left(\frac{1-\beta^{1/3}}{1+2\beta^{1/3}}\right)^3.$$

Applying (2.23) twice, we find that

$$\begin{aligned} {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; \alpha\right) &= (1+2t^{1/3})\, {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; t\right) \\ &= (1+2t^{1/3})(1+2\beta^{1/3})\, {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; \beta\right). \end{aligned}$$

We want to express the multiplier

$$m = (1+2t^{1/3})(1+2\beta^{1/3}) \tag{7.16}$$

entirely in terms of α and β. Solving for $\beta^{1/3}$ in (7.13) and then replacing β by t, we find that

$$t^{1/3} = \frac{1-(1-\alpha)^{1/3}}{2(1-\alpha)^{1/3}+1}.$$

Thus,

$$1+2t^{1/3} = \frac{3}{2(1-\alpha)^{1/3}+1}.$$

Using this in (7.16), we deduce (7.15) to complete the proof.

Theorem 7.6 (p. 259). *If β has degree 5, then*

$$(\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3} + 3\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1. \tag{7.17}$$

Proof. By Corollary 3.2, we may rewrite (7.17) in the form

$$b(q)b(q^5) + c(q)c(q^5) + 3\sqrt{b(q)c(q)b(q^5)c(q^5)} = a(q)a(q^5). \tag{7.18}$$

From Theorem 2.2 and (5.4) and (5.5) of Lemma 5.1, we find that

$$a(q) = \left\{\frac{f^{12}(-q)+27qf^{12}(-q^3)}{f^3(-q)f^3(-q^3)}\right\}^{1/3}. \tag{7.19}$$

(In fact, (7.19) is given by Ramanujan in his second notebook (Part III [3, p. 460, Entry 3(i)].) Thus, by (5.4), (5.5), and (7.19), (7.18) is equivalent to the identity

$$\frac{f^3(-q)f^3(-q^5)}{f(-q^3)f(-q^{15})}+9q^2\frac{f^3(-q^3)f^3(-q^{15})}{f(-q)f(-q^5)}+9qf(-q)f(-q^3)f(-q^5)f(-q^{15})$$
$$=\frac{\left\{f^{12}(-q)+27qf^{12}(-q^3)\right\}^{1/3}\left\{f^{12}(-q^5)+27q^5f^{12}(-q^{15})\right\}^{1/3}}{f(-q)f(-q^3)f(-q^5)f(-q^{15})}. \tag{7.20}$$

Cubing both sides of (7.20), simplifying, and setting

$$A=f(-q),\qquad B=f(-q^3),\qquad C=f(-q^5),\qquad\text{and}\qquad D=f(-q^{15}),$$

we deduce that (7.13) is equivalent to the proposed identity

$$45q^2A^6B^6C^6D^6+10qA^8C^8B^4D^4+90q^3A^4C^4B^8D^8+A^{10}C^{10}B^2D^2$$
$$+81q^4A^2C^2B^{10}D^{10}=q^4A^{12}D^{12}+B^{12}C^{12}. \tag{7.21}$$

Setting

$$P=\frac{f(-q)}{q^{1/12}f(-q^3)}\qquad\text{and}\qquad Q=\frac{f(-q^5)}{q^{5/12}f(-q^{15})}$$

and dividing both sides of (7.21) by $q^2(ABCD)^6$, we find that (7.21) can be written in the equivalent form

$$45+10P^2Q^2+\frac{90}{P^2Q^2}+P^4Q^4+\frac{81}{P^4Q^4}=\frac{P^6}{Q^6}+\frac{Q^6}{P^6},$$

or

$$\left\{(PQ)^2+5+\frac{9}{(PQ)^2}\right\}^2=\left\{\left(\frac{Q}{P}\right)^3-\left(\frac{P}{Q}\right)^3\right\}^2. \tag{7.22}$$

By examining P and Q in a neighborhood of $q=0$, so that the proper square root can be taken on the right side of (7.22), we find that (7.22) is equivalent to the identity

$$(PQ)^2+5+\frac{9}{(PQ)^2}=\left(\frac{Q}{P}\right)^3-\left(\frac{P}{Q}\right)^3. \tag{7.23}$$

Now (7.23) is stated by Ramanujan on page 324 of his second notebook and has been proved in Part IV [4, p. 221, Entry 62]. (See also a paper by the author and L.–C. Zhang [1].) This therefore completes the proof of (7.17).

Theorem 7.7 (p. 259). *If β has degree 7, then*

$$m=\left(\frac{\beta}{\alpha}\right)^{1/3}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/3}-\frac{7}{m}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3}-3\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}. \tag{7.24}$$

Proof. Using Corollary 3.2 and recalling that $m = z_1/z_7$, we find that (7.24) is equivalent to the equality

$$1 = \frac{b(q^7)}{b(q)} + \frac{c(q^7)}{c(q)} - 7\frac{b(q^7)c(q^7)}{b(q)c(q)} - 3\sqrt{\frac{b(q^7)c(q^7)}{b(q)c(q)}}. \tag{7.25}$$

Employing (5.4) and (5.5) in (7.25) and then multiplying both sides of the resulting equality by $f(-q)f(-q^3)/\left(qf(-q^7)f(-q^{21})\right)$, we find that (7.25) may be written in the equivalent form

$$\begin{aligned}\frac{f(-q)f(-q^3)}{qf(-q^7)f(-q^{21})} &= \frac{qf^2(-q)f^2(-q^{21})}{f^2(-q^3)f^2(-q^7)} + \frac{f^2(-q^3)f^2(-q^7)}{qf^2(-q)f^2(-q^{21})}\\ &\quad - 7\frac{qf(-q^7)f(-q^{21})}{f(-q)f(-q^3)} - 3.\end{aligned} \tag{7.26}$$

If we set

$$P = \frac{f(-q)}{q^{1/4}f(-q^7)} \qquad \text{and} \qquad Q = \frac{f(-q^3)}{q^{3/4}f(-q^{21})},$$

we deduce that (7.26) is equivalent to the identity

$$PQ + \frac{7}{PQ} = \frac{P^2}{Q^2} + \frac{Q^2}{P^2} - 3. \tag{7.27}$$

However, (7.27) can be found on page 323 of Ramanujan's second notebook and has been proved by the author and Zhang [1, Theorem 4]. See also Part IV [4, p. 236, Entry 68]. This completes the proof of (7.24).

Theorem 7.8 (p. 259). *If β has degree 11, then*

$$\begin{aligned}&(\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3} + 6\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}\\ &+ 3\sqrt{3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}\left\{(\alpha\beta)^{1/6} + \{(1-\alpha)(1-\beta)\}^{1/6}\right\} = 1.\end{aligned} \tag{7.28}$$

Proof. Employing Corollary 3.2, we find that (7.28) is equivalent to the identity

$$\begin{aligned}&c(q)c(q^{11}) + b(q)b(q^{11}) + 6\sqrt{b(q)b(q^{11})c(q)c(q^{11})}\\ &\quad + 3\sqrt{3}\left\{b(q)b(q^{11})c(q)c(q^{11})\right\}^{1/4}\left\{\sqrt{c(q)c(q^{11})} + \sqrt{b(q)b(q^{11})}\right\}\\ &= a(q)a(q^{11}).\end{aligned} \tag{7.29}$$

By (5.4), (5.5), and (7.19), (7.29) can be transformed into the equivalent identity

$$\begin{aligned}&9\frac{q^4f^3(-q^3)f^3(-q^{33})}{f(-q)f(-q^{11})} + \frac{f^3(-q)f^3(-q^{11})}{f(-q^3)f(-q^{33})}\\ &\quad + 18q^2f(-q)f(-q^3)f(-q^{11})f(-q^{33})\\ &\quad + 9q\sqrt{f(-q)f(-q^3)f(-q^{11})f(-q^{33})}\end{aligned}$$

$$\times\left\{3q^2\sqrt{\frac{f^3(-q^3)f^3(-q^{33})}{f(-q)f(-q^{11})}}+\sqrt{\frac{f^3(-q)f^3(-q^{11})}{f(-q^3)f(-q^{33})}}\right\}$$
$$=\frac{\{f^{12}(-q)+27qf^{12}(-q^3)\}^{1/3}\{f^{12}(-q^{11})+27q^{11}f^{12}(-q^{33})\}^{1/3}}{f(-q)f(-q^3)f(-q^{11})f(-q^{33})}. \tag{7.30}$$

Setting

$$A=f(-q),\quad B=f(-q^3),\quad C=f(-q^{11}),\quad \text{and}\quad D=f(-q^{33}),$$

and multiplying both sides of (7.30) by $ABCD$, we find that it suffices to prove that

$$9q^4B^4D^4+A^4C^4+18q^2A^2B^2C^2D^2+27q^3ACB^3D^3+9qA^3C^3BD$$
$$=\{A^{12}+27qB^{12}\}^{1/3}\{C^{12}+27q^{11}D^{12}\}^{1/3}. \tag{7.31}$$

We next cube both sides of (7.31), simplify, and divide both sides of the resulting equality by $27(qABCD)^6$. After considerable algebra, we deduce that

$$\left(\frac{AC}{qBD}\right)^5+\left(\frac{3qBD}{AC}\right)^5+(3^2+2)\left\{\left(\frac{AC}{qBD}\right)^4+\left(\frac{3qBD}{AC}\right)^4\right\}$$
$$+(3^3+4\cdot3^2+3)\left\{\left(\frac{AC}{qBD}\right)^3+\left(\frac{3qBD}{AC}\right)^3\right\}$$
$$+(2\cdot3^4+2\cdot3^3+4\cdot3^2+1)\left\{\left(\frac{AC}{qBD}\right)^2+\left(\frac{3qBD}{AC}\right)^2\right\}$$
$$+(3^5+4\cdot3^4+4\cdot3^3+2\cdot3^2)\left\{\frac{AC}{qBD}+\frac{3qBD}{AC}\right\}$$
$$+(4\cdot3^5+2\cdot3^4+8\cdot3^3+4\cdot3^2)$$
$$=q^5\left(\frac{AD}{BC}\right)^6+\frac{1}{q^5}\left(\frac{BC}{AD}\right)^6. \tag{7.32}$$

Now set

$$P=\frac{A}{q^{1/12}B}=\frac{\eta(z)}{\eta(3z)}\quad\text{and}\quad Q=\frac{C}{q^{11/12}D}=\frac{\eta(11z)}{\eta(33z)},$$

where $\eta(z)=q^{1/24}f(-q)$ denotes the Dedekind eta–function, $q=\exp(2\pi iz)$, and $\operatorname{Im}(z)>0$. Then (7.32) is equivalent to the identity

$$(PQ)^5+\left(\frac{3}{PQ}\right)^5+11\left\{(PQ)^4+\left(\frac{3}{PQ}\right)^4\right\}+66\left\{(PQ)^3+\left(\frac{3}{PQ}\right)^3\right\}$$
$$+253\left\{(PQ)^2+\left(\frac{3}{PQ}\right)^2\right\}+693\left\{PQ+\frac{3}{PQ}\right\}+1386$$

$$= \left(\frac{P}{Q}\right)^6 + \left(\frac{Q}{P}\right)^6. \tag{7.33}$$

The beautiful eta–function identity (7.33) has the same shape as many eta–function identities found in Ramanujan's notebooks, but is apparently not in Ramanujan's work. In contrast to our proofs of most of these identities, we shall invoke the theory of modular forms to prove (7.33). All of the necessary theory is found in Part IV [4, pp. 237–239].

Let $\Gamma(1)$ denote the full modular group, and let $M(\Gamma, r, v)$ denote the space of modular forms of weight r and multiplier system v on Γ, where Γ is a subgroup of finite index in $\Gamma(1)$. As usual, let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : c \equiv 0 (\mathrm{mod} N) \right\}.$$

Then from Part IV [4, p. 237, Lemma 68.1], $PQ \in M(\Gamma_0(33), 0, 1)$, and from Part IV [4, p. 238, Lemma 68.2], $(P/Q)^6 \in M(\Gamma_0(33), 0, 1)$. Let $\mathrm{ord}(g; z)$ denote the invariant order of a modular form g at z. Let r/s, $(r, s) = 1$, denote a cusp for the group Γ. Then, for any pair of positive integers m, n,

$$\mathrm{ord}\left(\eta(mnz); \frac{r}{s}\right) = \frac{(mn, s)^2}{24mn}, \tag{7.34}$$

where η denotes the Dedekind eta–function.

A complete set of cusps for $\Gamma_0(33)$ is $\{0, \frac{1}{3}, \frac{1}{11}, \infty\}$. Using (7.34), we calculate the orders of PQ and P/Q at each finite cusp. The following table summarizes these calculations:

Function g	Cusp ζ	$\mathrm{ord}(g; \zeta)$
PQ	0	$\frac{1}{33}$
	$\frac{1}{3}$	$-\frac{1}{11}$
	$\frac{1}{11}$	$\frac{1}{3}$
P/Q	0	$\frac{5}{198}$
	$\frac{1}{3}$	$-\frac{5}{66}$
	$\frac{1}{11}$	$-\frac{5}{18}$

Let L and R denote the left and right sides, respectively, of (7.33). Using the valence formula (R. A. Rankin [1, p. 98, Theorem 4.1.4]) and the table above, we find that

$$\begin{aligned} 0 &\geq \mathrm{ord}(L; \infty) + \mathrm{ord}(L; 0) + \mathrm{ord}(L; \tfrac{1}{3}) + \mathrm{ord}(L; \tfrac{1}{11}) \\ &= \mathrm{ord}(L; \infty) - \tfrac{5}{33} - \tfrac{5}{11} - \tfrac{5}{3} = \mathrm{ord}(L; \infty) - \tfrac{25}{11} \end{aligned} \tag{7.35}$$

and

$$0 \geq \operatorname{ord}(R;\infty) - \tfrac{5}{33} - \tfrac{5}{11} - \tfrac{5}{3} = \operatorname{ord}(R;\infty) - \tfrac{25}{11}. \tag{7.36}$$

By (7.35) and (7.36), if we can show that

$$L - R = O(q^3), \tag{7.37}$$

as q tends to 0, then we shall have completed the proof of (7.33). Using *Mathematica*, we find that

$$L = q^{-5}+6q^{-4}+27q^{-3}+92q^{-2}+279q^{-1}+756+1913q+4536q^2+O(q^3) = R.$$

Thus, the proof of (7.37), and hence also of (7.28), is complete.

At the bottom of page 259 in his second notebook, Ramanujan records three modular equations of composite degrees. Unfortunately, we have been unable to prove them by methods familiar to Ramanujan and so have resorted to the theory of modular forms for our proofs. It would be of considerable interest if more instructive proofs could be found. Because the proofs are similar, we give the three together.

Theorem 7.9 (p. 259). *If α, β, γ, and δ are of degrees 1, 2, 4, and 8, respectively, and if m_1 and m_2 are the multipliers associated with the pairs α, β and γ, δ, respectively, then*

$$\frac{1-(\alpha\delta)^{1/3}-\{(1-\alpha)(1-\delta)\}^{1/3}}{3\{\beta\gamma(1-\beta)(1-\gamma)\}^{1/6}} = \frac{m_2}{m_1}. \tag{7.38}$$

Theorem 7.10 (p. 259). *If α, β, γ, and δ have degrees 1, 2, 7, and 14 or 1, 4, 5, and 20, respectively, and if m_1 and m_2 are as in the previous theorem, then*

$$\frac{1+2\left((\alpha\delta)^{1/3}+\{(1-\alpha)(1-\delta)\}^{1/3}\right)}{1+2\left((\beta\gamma)^{1/3}+\{(1-\beta)(1-\gamma)\}^{1/3}\right)} = \frac{m_2}{m_1}. \tag{7.39}$$

Proofs of Theorems 7.9 and 7.10. Transcribing (7.38) and (7.39) via Corollary 3.2, we see that it suffices to prove that

$$a(q)a(q^8) - c(q)c(q^8) - b(q)b(q^8) = 3\sqrt{c(q^2)c(q^4)b(q^2)b(q^4)}, \tag{7.40}$$

$$\begin{aligned} &a(q)a(q^{14}) + 2c(q)c(q^{14}) + 2b(q)b(q^{14}) \\ &\quad = a(q^2)a(q^7) + 2c(q^2)c(q^7) + 2b(q^2)b(q^7), \end{aligned} \tag{7.41}$$

and

$$\begin{aligned} &a(q)a(q^{20}) + 2c(q)c(q^{20}) + 2b(q)b(q^{20}) \\ &\quad = a(q^4)a(q^5) + 2c(q^4)c(q^5) + 2b(q^4)b(q^5). \end{aligned} \tag{7.42}$$

Next, employing (5.4), (5.5), and (7.12), we translate (7.40)–(7.42) into the equivalent eta–function identities,

$$\begin{aligned}
&\frac{\{f^{12}(-q)+27qf^{12}(-q^3)\}^{1/3}\{f^{12}(-q^8)+27q^8f^{12}(-q^{24})\}^{1/3}}{f(-q)f(-q^3)f(-q^8)f(-q^{24})}\\
&\qquad -9q^3\frac{f^3(-q^3)f^3(-q^{24})}{f(-q)f(-q^8)}-\frac{f^3(-q)f^3(-q^8)}{f(-q^3)f(-q^{24})}\\
&\quad =9qf(-q^2)f(-q^4)f(-q^6)f(-q^{12}),
\end{aligned}\tag{7.43}$$

$$\begin{aligned}
&\frac{\{f^{12}(-q)+27qf^{12}(-q^3)\}^{1/3}\{f^{12}(-q^{14})+27q^{14}f^{12}(-q^{42})\}^{1/3}}{f(-q)f(-q^3)f(-q^{14})f(-q^{42})}\\
&\qquad +18q^5\frac{f^3(-q^3)f^3(-q^{42})}{f(-q)f(-q^{14})}+2\frac{f^3(-q)f^3(-q^{14})}{f(-q^3)f(-q^{42})}\\
&\quad =\frac{\{f^{12}(-q^2)+27q^2f^{12}(-q^6)\}^{1/3}\{f^{12}(-q^7)+27q^7f^{12}(-q^{21})\}^{1/3}}{f(-q^2)f(-q^6)f(-q^7)f(-q^{21})}\\
&\qquad +18q^3\frac{f^3(-q^6)f^3(-q^{21})}{f(-q^2)f(-q^7)}+2\frac{f^3(-q^2)f^3(-q^7)}{f(-q^6)f(-q^{21})},
\end{aligned}\tag{7.44}$$

and

$$\begin{aligned}
&\frac{\{f^{12}(-q)+27qf^{12}(-q^3)\}^{1/3}\{f^{12}(-q^{20})+27q^{20}f^{12}(-q^{60})\}^{1/3}}{f(-q)f(-q^3)f(-q^{20})f(-q^{60})}\\
&\qquad +18q^7\frac{f^3(-q^3)f^3(-q^{60})}{f(-q)f(-q^{20})}+2\frac{f^3(-q)f^3(-q^{20})}{f(-q^3)f(-q^{60})}\\
&\quad =\frac{\{f^{12}(-q^4)+27q^4f^{12}(-q^{12})\}^{1/3}\{f^{12}(-q^5)+27q^5f^{12}(-q^{15})\}^{1/3}}{f(-q^4)f(-q^{12})f(-q^5)f(-q^{15})}\\
&\qquad +18q^3\frac{f^3(-q^{12})f^3(-q^{15})}{f(-q^4)f(-q^5)}+2\frac{f^3(-q^4)f^3(-q^5)}{f(-q^{12})f(-q^{15})}.
\end{aligned}\tag{7.45}$$

Recall that if $q=\exp(2\pi iz)$, where $\operatorname{Im}(z)>0$, then $\eta(z)=q^{1/24}f(-q)$ is a modular form on $\Gamma(1)$ of weight $\frac{1}{2}$. If $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma(1)$ and d is odd, the multiplier system $v_\eta\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ is given by (Knopp [1, p. 51])

$$v_\eta\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\pm\left(\frac{c}{|d|}\right)e^{2\pi i\{ac(1-d^2)+d(b-c)+3(d-1)\}/24},\tag{7.46}$$

where $\left(\frac{c}{|d|}\right)$ denotes the Legendre symbol, the plus sign is taken if $c\geq 0$ or $d\geq 0$, and the minus sign is chosen if $c<0$ and $d<0$. Using (7.46), we find that each of the four expressions in (7.43) and each of the six expressions in both (7.44) and (7.45) has a multiplier system identically equal to 1, provided, of course, that

c is divisible by 24, 42, and 60, respectively. Hence, both sides of (7.43)–(7.45) belong to $M(\Gamma_0(n), 2, 1)$, where $n = 24$, 42, and 60, respectively.

If σ_∞ denotes the number of inequivalent cusps of a fundamental region for $\Gamma_0(n)$, then (B. Schoeneberg [1, p. 102])

$$\sigma_\infty = \sum_{d|n} \varphi((d, n/d)),$$

where φ denotes Euler's φ–function and (a, b) denotes the greatest common divisor of a and b. Thus, for $n = 24$, 42, and 60, there are 8, 8, and 12 cusps, respectively. Using a procedure found in Schoeneberg's book [1, pp. 86–87], we find that $\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{12}, \infty\}$; $\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \frac{1}{7}, \frac{1}{14}, \frac{1}{21}, \infty\}$; and $\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{10}, \frac{1}{12}, \frac{1}{15}, \frac{1}{20}, \frac{1}{30}, \infty\}$ constitute complete sets of inequivalent cusps for $\Gamma_0(24)$, $\Gamma_0(42)$, and $\Gamma_0(60)$, respectively. Employing (7.34), we calculate the order of each expression in (7.43)–(7.45) at each finite cusp. In each instance, we find that each order is nonnegative.

Let F_{24}, F_{42}, and F_{60} denote the differences of the left and right sides of (7.43)–(7.45), respectively. Since the order of F_{24}, F_{42}, and F_{60} at each point of a fundamental set is nonnegative, we deduce from the valence formula that

$$r\rho_{\Gamma_0(n)} \geq \operatorname{ord}(F_n; \infty), \qquad n = 24, 42, 60, \tag{7.47}$$

provided that F_n is not constant, where r is the weight of F_n and

$$\rho_{\Gamma_0(n)} := \tfrac{1}{12}[\Gamma(1) : \Gamma_0(n)]. \tag{7.48}$$

Now (Schoeneberg [1, p. 79]),

$$[\Gamma(1) : \Gamma_0(n)] = n \prod_{p|n} \left(1 + \frac{1}{p}\right), \tag{7.49}$$

where the product is over all primes p dividing n. Thus, by (7.48) and (7.49), $\rho_{\Gamma_0(24)} = 4$, $\rho_{\Gamma_0(42)} = 8$, and $\rho_{\Gamma_0(60)} = 12$. Since $r = 2$ in each case, by (7.47),

$$\operatorname{ord}(F_{24}; \infty) \leq 8, \qquad \operatorname{ord}(F_{42}; \infty) \leq 16, \qquad \operatorname{ord}(F_{60}; \infty) \leq 24, \tag{7.50}$$

unless F_{24}, F_{42}, or F_{60}, respectively, is constant.

Using the pentagonal number theorem, (1.14), i.e.,

$$f(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + \cdots,$$

and *Mathematica*, we expanded the left and right sides of (7.43)–(7.45) about the cusp ∞ ($q = 0$). We found that the left and right sides of (7.43)–(7.45) are equal to, respectively,

$$\begin{aligned} &9q - 9q^3 - 18q^5 + O(q^9), \\ &3 + 18q^3 + 18q^5 + 18q^6 + 36q^7 + 54q^9 + 18q^{10} + 36q^{11} \\ &\qquad + 18q^{12} + 18q^{13} + 36q^{14} + 90q^{15} + O(q^{17}), \end{aligned} \tag{7.51}$$

and

$$3 + 18q^3 + 18q^7 + 18q^8 + 54q^9 + 36q^{11} + 36q^{12} + 36q^{13} + 18q^{15}$$
$$+ 36q^{16} + 36q^{17} + 36q^{19} + 126q^{21} + 54q^{23} + 90q^{24} + O(q^{25}). \tag{7.52}$$

Thus, $F_{24} = O(q^9)$, $F_{42} = O(q^{17})$, and $F_{60} = O(q^{25})$, which contradicts (7.50) unless F_{24}, F_{42}, and F_{60} are each constant. These constants are obviously equal to 0, and hence (7.43)–(7.45) are established. This completes the proofs of Theorems 7.9 and 7.10.

On page 328 in his first notebook, Ramanujan gives another modular equation of degree 8 in the theory of signature 3. This equation is quite interesting, because it is the only known modular equation of Weber type (H. Weber [1]) in the alternative theories.

Theorem 7.11 (p. 328, NB 1). *Let*

$$P := 1 - (\alpha\beta)^{1/3} - \{(1-\alpha)(1-\beta)\}^{1/3},$$
$$T := (\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3},$$

and

$$R := 9\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/3}.$$

Then

$$P^4 - RP(5P + 9T) - 2R^2 = 0. \tag{7.53}$$

Proof. As with the proof of Theorem 7.9, we utilize the theory of modular forms. Transcribing (7.53) via Corollary 3.2, we determine that it suffices to prove that

$$p^4 - rp(5p + 9t) - 2r^2 = 0, \tag{7.54}$$

where $p := z_1 z_8 P$, $t := z_1 z_8 T$, and $r = z_1^2 z_8^2 R$. Employing (5.4), (5.5), and (7.19), we find that (7.54) is equivalent to an eta-function identity in the spirit of (7.43)–(7.45). Because the identity contains the same eta-function products and quotients as (7.43), it follows from our previous work that each term has multiplier system identically equal to 1. Furthermore, it is easy to see that $r = 8$; recall also from the proof of Theorem 7.9 that $\rho_{\Gamma_0(24)} = 4$. Thus, from the valence formula, in order to prove (7.54), it suffices to show that

$$p^4 - rp(5p + 9t) - 2r^2 = O(q^{33}) \tag{7.55}$$

as q tends to 0. Indeed, we have used the pentagonal number theorem (1.14) in connection with *Mathematica* to prove (7.55). This completes the proof.

8. The Inversion of an Analogue of $K(k)$ in Signature 3

Theorem 8.1 (p. 257). *Let $q = q_3$ be defined by (1.7), and let z be defined by (1.6) with $r = 3$. For $0 \le \varphi \le \pi/2$, define $\theta = \theta(\varphi)$ by*

$$\theta z = \int_0^{\varphi} {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; x\sin^2 t\right) dt. \tag{8.1}$$

Then, for $0 \le \theta \le \pi/2$,

$$\varphi = \theta + 3\sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n(1+2\cosh(ny))} = \theta + 3\sum_{n=1}^{\infty} \frac{\sin(2n\theta)q^n}{n(1+q^n+q^{2n})} =: \Phi(\theta), \tag{8.2}$$

where $q =: e^{-y}$.

Recall from Entry 35(iii) of Chapter 11 (Part II [2, p. 99]) that

$${}_2F_1\left(\tfrac{1}{2}+n, \tfrac{1}{2}-n; \tfrac{1}{2}; x^2\right) = (1-x^2)^{-1/2}\cos\left(2n\sin^{-1}x\right), \tag{8.3}$$

where n is arbitrary. With $n = \frac{1}{6}$ in (8.3), we see that the integral in (8.1) is an analogue of the incomplete integral of the first kind, which arises from the case $n = 0$ in (8.3). Since ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; x\sin^2 t\right)$ is a nonnegative, monotonically increasing function on $[0, \pi/2]$, there exists a unique inverse function $\varphi = \varphi(\theta)$. Thus, (8.2) gives the "Fourier series" of this inverse function and is analogous to familiar Fourier series for the Jacobian elliptic functions (Whittaker and Watson [1, pp. 511–512]). The function φ may therefore be considered a cubic analogue of the Jacobian functions. Theorem 8.1 is also remindful of some new inversion formulas in the classical setting which are found on pages 283, 285, and 286 in Ramanujan's second notebook and which have been proved by the author and S. Bhargava [1]. (See also Part IV [4, Chap. 26].)

When $\varphi = 0 = \theta$, (8.2) is trivial. When $\varphi = \pi/2$,

$$\begin{aligned}
\int_0^{\varphi} {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; x\sin^2 t\right) dt &= \sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{(\frac{1}{2})_n n!} x^n \int_0^{\pi/2} \sin^{2n} t\, dt \\
&= \sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{(\frac{1}{2})_n n!} x^n \frac{(\frac{1}{2})_n}{n!}\frac{\pi}{2} \\
&= \tfrac{1}{2}\pi\, {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; x\right).
\end{aligned}$$

Thus, $\theta = \pi/2$, which is implicit in our statement of Theorem 8.1.

We now give an outline of the proof of Theorem 8.1. Returning to (8.3), we observe that

$$S(x) := {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; x^2\right) = (1-x^2)^{-1/2}\cos\left(\tfrac{1}{3}\sin^{-1}x\right), \qquad |x| < 1, \tag{8.4}$$

is that unique, real-valued function on $(-1, 1)$ satisfying the properties

$$S(x) \text{ is continuous on } (-1,1), \tag{8.5}$$

$$S(0) = 1, \tag{8.6}$$

and

$$4(1-x^2)S^3(x) - 3S(x) - 1 = 0. \tag{8.7}$$

Properties (8.5) and (8.6) are obvious, and (8.7) follows from the elementary identity $4\cos^3\theta = 3\cos\theta + \cos(3\theta)$. To see that $S(x)$ is unique, set $y = S(x)$ in (8.7) and solve for x^2. Thus,

$$x^2 = \frac{(y-1)(1+2y)^2}{4y^3}.$$

Since $S(x)$ is real valued, either $y < 0$ or $y \geq 1$. Since $S(0) = 1$ and S is continuous, we conclude that $y \geq 1$. Hence

$$x = \pm g(y),$$

where

$$g(y) := \frac{\sqrt{1-1/y}\,(2y+1)}{2y}.$$

Now, $g(y)$ is monotonically increasing on $[1,\infty)$. Thus, $g^{-1}(x)$ exists, and if $0 \leq x < 1, y = g^{-1}(x)$, while if $-1 < x < 0$, we have $y = g^{-1}(-x)$. Thus, $S(x)$ is uniquely determined.

We fix q, $0 < q < 1$. Set $x = c^3(q)/a^3(q)$, so by (2.5), $0 < x < 1$. Then, by Lemma 2.6, $Z := {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right) = a(q)$. (We use the notation Z instead of $z = z(3)$, because in this section z will denote a complex variable.) With $\Phi(\theta)$ defined in (8.2), we shall prove that

$$\frac{d\Phi(\theta)}{d\theta} > 0, \qquad 0 < \theta < \pi/2, \tag{8.8}$$

and

$$4x\sin^2(\Phi(\theta)) = 4 - \frac{1}{Z^3}\left(\frac{d\Phi}{d\theta}\right)^3 - \frac{3}{Z^2}\left(\frac{d\Phi}{d\theta}\right)^2. \tag{8.9}$$

By (8.8), we may define $\Theta := \Phi^{-1} : [0,\pi/2] \to [0,\pi/2]$. Setting $S := Z\,d\Theta/d\varphi$, we see from (8.9) that

$$4S^3(1 - x\sin^2\varphi) - 3S - 1 = 0.$$

Hence, (8.7) holds with x^2 replaced by $x\sin^2\varphi$. Now, from (8.2),

$$\Phi'(0) = 1 + 6\sum_{n=1}^{\infty}\frac{q^n}{1+q^n+q^{2n}} = Z,$$

by Theorem 2.12. Thus,

$$S(0) = Z/\Phi'(0) = 1.$$

Hence, by (8.4)–(8.7), we conclude that

$$Z\frac{d\Theta}{d\varphi} = {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; x\sin^2\varphi\right), \qquad 0 < \varphi < \pi/2,$$

and so

$$Z\Theta(\varphi) = \int_0^{\varphi} {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; x\sin^2 t\right) dt, \qquad 0 \le \varphi \le \pi/2,$$

since $\Theta(0) = 0$. It follows that $\Theta(\varphi) = \theta(\varphi)$ and that $\varphi = \Phi(\theta)$. Hence, the desired result (8.2) holds.

Proof of Theorem 8.1. For $|q| < |z| < 1/|q|$, define

$$v(z,q) := 1 + 3\sum_{n=1}^{\infty} \frac{(z^n + z^{-n})q^n}{1 + q^n + q^{2n}}. \tag{8.10}$$

We note, by Theorem 2.12, that $v(1,q) = a(q)$, and, by (8.2), that

$$V(\theta) := v(e^{2i\theta}, q) = \frac{d\Phi}{d\theta}. \tag{8.11}$$

By expanding $1/(1-q^{3n})$ in a geometric series and inverting the order of summation, we find that

$$\begin{aligned} v(z,q) = 1 + 3\sum_{n=0}^{\infty} \Bigg\{ & \frac{zq^{3n+1}}{1 - zq^{3n+1}} - \frac{zq^{3n+2}}{1 - zq^{3n+2}} \\ & + \frac{z^{-1}q^{3n+1}}{1 - z^{-1}q^{3n+1}} - \frac{z^{-1}q^{3n+2}}{1 - z^{-1}q^{3n+2}} \Bigg\}. \end{aligned} \tag{8.12}$$

As a function of z, $v(z,q)$ can be analytically continued to $\mathbb{C} \setminus \{0\}$, where the analytic continuation $v(z,q)$ is analytic except for simple poles at $z = q^m$, where m is an integer such that $m \not\equiv 0 \pmod 3$. Using (8.12), we find, by a straightforward calculation, that

$$v(zq^3, q) = v(z,q). \tag{8.13}$$

M. Hirschhorn, F. Garvan, and J. M. Borwein [1] have studied generalizations of $a(q)$, $b(q)$, and $c(q)$ in two variables. In particular, they defined

$$b(z,q) := \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2} z^n$$

and showed that [1, eq. (1.22)]

$$\begin{aligned} b(z,q) &= (q;q)_\infty (q^3;q^3)_\infty \frac{(zq;q)_\infty (z^{-1}q;q)_\infty}{(zq^3;q^3)_\infty (z^{-1}q^3;q^3)_\infty} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{1-q^{3n}} \prod_{3\nmid n}^{\infty} \frac{(1-zq^n)(1-z^{-1}q^n)}{(1-q^n)^2} \end{aligned} \tag{8.14}$$

and [1, eq. (1.17)]

$$b(zq^3, q) = z^{-2}q^{-3} b(z,q). \tag{8.15}$$

We next show that $v(z,q)$ can be written in terms of $b(z,q)$ and $b(-z,q)$.

Lemma 8.2. *If*

$$\alpha(q) := \frac{(q;q)_\infty^2 (q^3;q^3)_\infty^2}{(q^2;q^2)_\infty (q^6;q^6)_\infty}$$

and

$$\beta(q) := \frac{(q;q)_\infty^6 (q^6;q^6)_\infty}{(q^2;q^2)_\infty^3 (q^3;q^3)_\infty^2},$$

then

$$v(z,q) = \frac{3}{2}\alpha(q)\frac{b(-z,q)}{b(z,q)} - \frac{1}{2}\beta(q). \tag{8.16}$$

The case $z = 1$ of (8.16) follows from the paper by Hirschhorn, Garvan, and Borwein [1, eq. (1.29)]. To prove Lemma 8.2, we employ the following lemma due to A. O. L. Atkin and P. Swinnerton–Dyer [1].

Lemma 8.3. *Let q, $0 < q < 1$, be fixed. Suppose that $f(z)$ is an analytic function of z, except for possibly a finite number of poles, in every region, $0 < z_1 \le |z| \le z_2$. If*

$$f(zq) = Az^k f(z)$$

for some integer k (positive, zero, or negative) and some constant A, then either $f(z)$ has k more poles than zeros in the region $|q| < |z| \le 1$, or $f(z)$ vanishes identically.

Proof of Lemma 8.2. Define

$$F(z) := b(z,q)v(z,q) - \tfrac{3}{2}\alpha(q)b(-z,q) + \tfrac{1}{2}\beta(q)b(z,q). \tag{8.17}$$

Examining (8.16), we see that our goal is to prove that $F(z) \equiv 0$. From (8.13) and (8.15),

$$F(zq^3) = z^{-2}q^{-3}F(z).$$

From our previous identification of the poles of $v(z,q)$ and from the definition (8.14) of $b(z,q)$, we see that the singularities of $F(z)$ are removable. Thus, by Lemma 8.3, to show that $F(z) \equiv 0$, we need only show that $F(z) = 0$ for three distinct values of z in the region $|q|^3 < |z| \le 1$. We choose the values $z = -1, \omega, \omega^2$, where $\omega = \exp(2\pi i/3)$.

For $z = -1$, by (8.17) and (8.14), we want to prove that

$$v(-1,q) = \frac{3}{2}\alpha(q)\,\frac{(q;q)_\infty^4 (q^6;q^6)_\infty^2}{(q^3;q^3)_\infty^4 (q^2;q^2)_\infty^2} - \frac{1}{2}\beta(q) = \beta(q). \tag{8.18}$$

But this has been proved by N. J. Fine [1, p. 84, eq. (32.64)].

For the values $z = \omega, \omega^2$, we need the evaluations

$$v(\omega,q) = v(\omega^2,q) = b(q). \tag{8.19}$$

The first equality follows from the representation of $v(z, q)$ in (8.12). To prove the second, we first find from (8.10) that

$$\frac{1}{2}v(1, q) + v(\omega, q) = \frac{3}{2} + 9\sum_{n=1}^{\infty} \frac{q^{3n}}{1 + q^{3n} + q^{6n}} = \frac{3}{2}a(q^3),$$

by Theorem 2.12. Since $v(1, q) = a(q)$, by Theorem 2.12, we deduce that

$$v(\omega, q) = \tfrac{3}{2}a(q^3) - \tfrac{1}{2}a(q).$$

By (2.8), we conclude that $b(q) = v(\omega, q)$ to complete the proof of (8.19).

Now setting $z = \omega$, we see from (8.17) that we are required to prove that

$$\begin{aligned} b(q) &= \frac{3}{2}\alpha(q)\frac{b(-\omega, q)}{b(\omega, q)} - \frac{1}{2}\beta(q) \\ &= \frac{3}{2}\frac{(q;q)_\infty^4 (q^6;q^6)_\infty (q^9;q^9)_\infty^2}{(q^2;q^2)_\infty^2 (q^3;q^3)_\infty^2 (q^{18};q^{18})_\infty} - \frac{1}{2}\beta(q) \\ &= \frac{\varphi^2(-q)}{2\varphi(-q^3)}\left(3\varphi(-q^9) - \varphi(-q)\right), \end{aligned} \tag{8.20}$$

where we employed (8.14), much simplification, and (5.1). From Entry 1(iii) of Chapter 20 of Ramanujan's second notebook (Part III [3, p. 345]),

$$3\frac{\varphi(-q^9)}{\varphi(-q)} - 1 = \left(9\frac{\varphi^4(-q^3)}{\varphi^4(-q)} - 1\right)^{1/3}.$$

Thus, by (5.9), (1.17), (3.10), and (3.11),

$$\begin{aligned} \frac{\varphi^3(-q)}{2\varphi(-q^3)}\left(3\frac{\varphi(-q^9)}{\varphi(-q)} - 1\right) &= \frac{\varphi^3(-q)}{2\varphi(-q^3)}\left(9\frac{\varphi^4(-q^3)}{\varphi^4(-q)} - 1\right)^{1/3} \\ &= \frac{z_1^{3/2}(1-\alpha)^{3/4}}{2z_3^{1/2}(1-\beta)^{1/4}}\left(\frac{9}{m^2}\frac{1-\beta}{1-\alpha} - 1\right)^{1/3} \\ &= \frac{z_1^{3/2}(3-m)^2}{8z_3^{1/2}m^2}\left(\frac{9}{m^2}\frac{m^2(m+1)^2}{(3-m)^2} - 1\right)^{1/3} \\ &= \frac{(3-m)^{4/3}\sqrt{z_1z_3}}{4m}(m^2+3m)^{1/3} \\ &= b(q), \end{aligned} \tag{8.21}$$

by (2.11). (See also the Borweins' book [1, p. 143, Theorem 4.11(b)].) Thus, (8.20) has been proved. (Note that in (8.21) α and β are squares of moduli and are not to be confused with the definitions of $\alpha(q)$ and $\beta(q)$ in Lemma 8.2.)

In conclusion, we have shown that $F(z) = 0$ for $z = -1, \omega, \omega^2$, and so the proof of Lemma 8.2 is complete.

We shall need some further relations among $a(q)$, $c(q)$, $\alpha(q)$, and $\beta(q)$. First, from (8.21),

$$b^3(q) = \frac{\varphi^9(-q)}{8\varphi^3(-q^3)}\left(9\frac{\varphi^4(-q^3)}{\varphi^4(-q)} - 1\right) = \frac{1}{8}\beta\left(9\alpha^2 - \beta^2\right).$$

Since, from (8.14) and (8.16), with $z = 1$,

$$a(q) = \frac{3\alpha^2}{2\beta} - \frac{1}{2}\beta, \tag{8.22}$$

it follows from (2.5) that

$$c^3(q) = \frac{27\alpha^4}{8\beta^3}\left(\alpha^2 - \beta^2\right). \tag{8.23}$$

Recall from (8.11) that $V(\theta) = d\Phi/d\theta$, where $\Phi(\theta)$ is defined by (8.2). Our next task is to derive an infinite product representation for $dV/d\theta$. To do this, we employ Bailey's ${}_6\psi_6$ summation (G. Gasper and M. Rahman [1, p. 239]).

Lemma 8.4. *Let*

$$\prod\begin{bmatrix} a_1, \dots, a_m \\ b_1, \dots, b_n \end{bmatrix} := \frac{(a_1; q)_\infty \cdots (a_m; q)_\infty}{(b_1; q)_\infty \cdots (b_n; q)_\infty}$$

and, for $|z| < 1$,

$${}_6\psi_6\begin{bmatrix} a_1, \dots, a_6 \\ b_1, \dots, b_6 \end{bmatrix}; q; z\Bigg] := \sum_{n=-\infty}^{\infty} \frac{(a_1; q)_n \cdots (a_6; q)_n}{(b_1; q)_n \cdots (b_6; q)_n} z^n.$$

Then, for $|a^2q/(bcde)| < 1$,

$$\begin{aligned}
&{}_6\psi_6\left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, b, c, d, e \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e \end{matrix}; q; \frac{a^2q}{bcde}\right] \\
&= \prod\begin{bmatrix} aq, aq/(bc), aq/(bd), aq/(be), aq/(cd), aq/(ce), aq/(de), q, q/a \\ q/b, q/c, q/d, q/e, aq/b, aq/c, aq/d, aq/e, a^2q/(bcde) \end{bmatrix}.
\end{aligned} \tag{8.24}$$

Lemma 8.5. *With* $z = e^{2i\theta}$, *we have*

$$\begin{aligned}
\frac{dV}{d\theta} = \frac{d^2\Phi}{d\theta^2} &= -12\sum_{n=1}^{\infty} \frac{n\sin(2n\theta)q^n}{1 + q^n + q^{2n}} \\
&= q(z - 1/z)\frac{(z^2q^3; q^3)_\infty(z^{-2}q^3; q^3)_\infty(q; q^3)_\infty(q^2; q^3)_\infty(q^3; q^3)_\infty^4}{(zq; q^3)_\infty^2(z^{-1}q; q^3)_\infty^2(zq^2; q^3)_\infty^2(z^{-1}q^2; q^3)_\infty^2} \\
&= -12q\sin(2\theta) \\
&\quad\times\prod_{n=1}^{\infty}\frac{(1 - 2\cos(4\theta)q^{3n} + q^{6n})(1 - q^n)(1 - q^{3n})^3}{(1 - 2\cos(2\theta)q^{3n-1} + q^{6n-2})^2(1 - 2\cos(2\theta)q^{3n-2} + q^{6n-4})^2}.
\end{aligned} \tag{8.25}$$

Proof. From (8.10),

$$\frac{z}{3}\frac{dv}{dz}=\sum_{n=1}^{\infty}\frac{n(z^n-z^{-n})q^n}{1+q^n+q^{2n}}. \tag{8.26}$$

By expanding $1/(1-q^{3n})$ in a geometric series and inverting the order of summation, we find that

$$\begin{aligned}\frac{z}{3}\frac{dv}{dz}=\sum_{n=0}^{\infty}\Bigg(&\frac{zq^{3n+1}}{(1-zq^{3n+1})^2}-\frac{zq^{3n+2}}{(1-zq^{3n+2})^2}\\&-\frac{z^{-1}q^{3n+1}}{(1-z^{-1}q^{3n+1})^2}+\frac{z^{-1}q^{3n+2}}{(1-z^{-1}q^{3n+2})^2}\Bigg).\end{aligned}$$

Replacing n by $-n-1$ in the second and fourth sums and then combining the first and fourth sums and the second and third sums, we find that

$$\begin{aligned}\frac{z}{3}\frac{dv}{dz}&=\sum_{n=-\infty}^{\infty}\left(\frac{zq^{3n+1}}{(1-zq^{3n+1})^2}-\frac{z^{-1}q^{3n+1}}{(1-z^{-1}q^{3n+1})^2}\right)\\&=q(z-1/z)\sum_{n=-\infty}^{\infty}\frac{(1-q^{3n+1})(1+q^{3n+1})}{(1-zq^{3n+1})^2(1-z^{-1}q^{3n+1})^2}q^{3n}\\&=\frac{q(z-1/z)(1-q^2)}{(1-zq)^2(1-z^{-1}q)^2}\,{}_6\psi_6\left[\begin{matrix}q^4,-q^4,zq,zq,z^{-1}q,z^{-1}q\\q,-q,zq^4,zq^4,z^{-1}q^4,z^{-1}q^4\end{matrix};q^3;q^3\right]\\&=\frac{q(z-1/z)(1-q^2)}{(1-zq)^2(1-z^{-1}q)^2}\\&\quad\times\prod\left[\begin{matrix}q^5,q^3/z^2,q^3,q^3,q^3,q^3,z^2q^3,q^3,q\\q^2/z,q^2/z,q^2z,q^2z,q^4/z,q^4/z,q^4z,q^4z,q^3\end{matrix}\right]\\&=q(z-1/z)\frac{(z^2q^3;q^3)_\infty(z^{-2}q^3;q^3)_\infty(q;q^3)_\infty(q^2;q^3)_\infty(q^3;q^3)_\infty^4}{(zq;q^3)_\infty^2(z^{-1}q;q^3)_\infty^2(zq^2;q^3)_\infty^2(z^{-1}q^2;q^3)_\infty^2},\end{aligned} \tag{8.27}$$

by (8.24). With $z=e^{2i\theta}$, we thus have shown, by (8.26) and (8.27), that

$$\begin{aligned}\frac{dV}{d\theta}=2iz\frac{dv}{dz}&=-12\sum_{n=1}^{\infty}\frac{n\sin(2n\theta)q^n}{1+q^n+q^{2n}}=-12q\sin(2\theta)\\&\times\prod_{n=1}^{\infty}\frac{(1-2\cos(4\theta)q^{3n}+q^{6n})(1-q^n)(1-q^{3n})^3}{(1-2\cos(2\theta)q^{3n-1}+q^{6n-2})^2(1-2\cos(2\theta)q^{3n-2}+q^{6n-4})^2},\end{aligned}$$

which is (8.25).

Next, define

$$\Psi(\theta):=\frac{1}{4x}\left(4-\left(\frac{V(\theta)}{Z}\right)^3-3\left(\frac{V(\theta)}{Z}\right)^2\right)=\frac{1}{4xZ^3}(Z-V)(2Z+V)^2. \tag{8.28}$$

Thus, (8.9) is equivalent to the identity

$$\Psi(\theta) = \sin^2(\Phi(\theta)). \tag{8.29}$$

As a first step in proving (8.29), we establish the following lemma.

Lemma 8.6. *With Ψ defined by (8.28),*

$$\left(\frac{d\Psi}{d\theta}\right)^2 = 4\Psi(1-\Psi)\left(\frac{d\Phi}{d\theta}\right)^2. \tag{8.30}$$

Proof. Differentiating (8.28), we find that

$$\frac{d\Psi}{d\theta} = -\frac{3V}{4xZ^3}(V+2Z)\frac{dV}{d\theta}. \tag{8.31}$$

Putting (8.31) in (8.30), recalling that $V(\theta) = d\Phi/d\theta$, and simplifying, we see that it suffices to prove that

$$9\left(\frac{dV}{d\theta}\right)^2 = 4(Z-V)\left(4xZ^3-(Z-V)(V+2Z)^2\right). \tag{8.32}$$

Employing (8.25) above and using (8.14), we now find that it suffices to prove that

$$\begin{aligned} &81q^2(z-1/z)^2(z^2q^3;q^3)_\infty^2(z^{-2}q^3;q^3)_\infty^2(q;q)_\infty^6(q^3;q^3)_\infty^{10} \\ &\quad = b^4(z,q)(v-Z)\left(4xZ^3-(Z-v)(v+2Z)^2\right), \end{aligned} \tag{8.33}$$

where $v = v(z,q)$. By a direct calculation, it can be shown that the left side of (8.33) satisfies the functional equation

$$F(zq^3,q) = z^{-8}q^{-12}F(z,q). \tag{8.34}$$

By (8.13) and (8.15), it is obvious that the right side of (8.33) is also a solution of (8.34). As observed in (8.12), $v(z,q)$ has simple poles at $z = q^m$, where m is an integer such that $m \not\equiv 0 \pmod 3$. From (8.14), we see that $b(z,q)$ has simple zeros at these same points. Thus, the singularities on the right side of (8.33) are removable. In view of Lemma 8.3, it suffices to show that (8.33) is valid for at least nine values of z in the region $|q^3| < |z| \le 1$.

It is clear that (8.33) holds for $z = 1$, since $v(1,q) = a(q) = Z$. In fact, this zero is of order at least 2 on each side, since, by (8.10), $\partial v(z,q)/\partial z$ vanishes at $z = 1$.

Next, we show that (8.33) holds for $z = q$. Since $b(z,q)$ has a simple zero at $z = q$, and $v(z,q)$ has a simple pole at $z = q$, we see that we must show that

$$\begin{aligned} \lim_{z\to q}(b(z,q)v(z,q))^4 &= 81q^2(q-1/q)^2(q^5;q^3)_\infty^2(q;q^3)_\infty^2(q;q)_\infty^6(q^3;q^3)_\infty^{10} \\ &= \left(3(q;q)_\infty^2(q^3;q^3)_\infty^2\right)^4. \end{aligned} \tag{8.35}$$

But, by (8.12) and (8.14),

$$\begin{aligned}\lim_{z\to q} b(z,q)v(z,q) &= 3\frac{z^{-1}q(q;q)_\infty(q^3;q^3)_\infty(zq;q)_\infty(z^{-1}q;q)_\infty}{(1-z^{-1}q)(zq^3;q^3)_\infty(z^{-1}q^3;q^3)_\infty}\\ &= 3(q;q)_\infty^2(q^3;q^3)_\infty^2,\end{aligned}$$

which establishes (8.35). A similar argument shows that (8.33) holds for $z=q^2$.

Next, we examine the case $z=\omega$. We shall need to use the equality

$$a(q)-b(q)=3c(q^3), \tag{8.36}$$

which is a consequence of (2.8) and (2.9). We shall also need the equalities

$$b(\omega,q)=b(q^3)=b(\omega^2,q), \tag{8.37}$$

which are readily verified by means of (8.14).

Recall that $x=c^3(q)/a^3(q)$ (see the beginning of the paragraph immediately prior to the proof of Theorem 8.1) and that $Z=a(q)$. Also, by (8.19), $v(\omega,q)=b(q)$. Hence, by (2.5),

$$\begin{aligned}4xZ^3-(Z-v(\omega,q))(2Z+v(\omega,q))^2 &= 4c^3-(a-b)(2a+b)^2\\ &= 4(a^3-b^3)-(a-b)(2a+b)^2\\ &= 3b^2(a-b).\end{aligned}$$

Thus, by (8.36) and (8.37), the proposed equality (8.33) for $z=\omega$ reduces to the equality

$$-243q^2(q^9;q^9)_\infty^2(q;q)_\infty^6(q^3;q^3)_\infty^8=-27b^4(q^3)c^2(q^3)b^2(q).$$

But this equality is readily verified by using (5.4) and (5.5).

An almost identical argument shows that (8.33) also holds for $z=\omega^2$.

From (8.16), (8.22), (8.23), and considerable algebra, we find that

$$\begin{aligned}&b^4(z,q)(v(z,q)-Z)\left(4c^3(q)-(Z-v(z,q))(2Z+v(z,q))^2\right)\\ &=-\frac{81\alpha^2}{16\beta^2}(\alpha b(z,q)-\beta b(-z,q))(\alpha b(-z,q)-\beta b(z,q))\\ &\quad\times\left(b(z,q)b(-z,q)\left\{3\alpha^2-\beta^2\right\}+\alpha\beta\left\{b^2(z,q)+b^2(-z,q)\right\}\right).\end{aligned}$$

Hence, both sides of (8.33) are even functions of z. Therefore, we have shown that (8.33) holds for 12 values of z, namely, ± 1 (with multiplicities 2), $\pm q$, $\pm q^2$, $\pm\omega$, and $\pm\omega^2$. Thus, (8.33) holds for all values of z, and the proof of Lemma 8.6 is complete.

We are now ready to complete the proof of Theorem 8.1.

Proof of Theorem 8.1 (continued). From (8.25),

$$\frac{dV}{d\theta}<0, \qquad 0<\theta<\pi/2. \tag{8.38}$$

From (8.18),

$$V(\pi/2) = v(-1, q) = \beta(q). \tag{8.39}$$

It follows from (8.38) and (8.39) that

$$0 < \beta(q) = V(\pi/2) < V(\theta) < V(0) = v(1, q) = Z, \qquad 0 < \theta < \pi/2. \tag{8.40}$$

Observe that (8.8) follows from (8.11) and (8.40).

Combining (8.31) and (8.38), we find that

$$0 < \Psi(\theta) < \Psi(\pi/2). \tag{8.41}$$

We now calculate $\Psi(\frac{1}{2}\pi)$. By (8.39) and (8.22),

$$Z - V(\pi/2) = a(q) - \beta(q) = \frac{3\alpha^2}{2\beta} - \frac{3}{2}\beta \tag{8.42}$$

and

$$V(\pi/2) + 2Z = \frac{3\alpha^2}{\beta}. \tag{8.43}$$

Thus, by (8.28), (8.42), and (8.43),

$$\Psi(\pi/2) = \frac{1}{4c^3(q)}\left(\frac{3\alpha^2}{2\beta} - \frac{3\beta}{2}\right)\frac{9\alpha^4}{\beta^2} = \frac{27\alpha^4}{8c^3(q)\beta^3}\left(\alpha^2 - \beta^2\right) = 1,$$

by (8.23). So, from (8.41),

$$0 < \Psi(\theta) < 1, \qquad 0 < \theta < \pi/2. \tag{8.44}$$

Now, by (8.38) and (8.31), $d\Psi/d\theta > 0$, and again by (8.11) and (8.40), $d\Phi/d\theta > 0$. Also noting (8.44), we conclude from (8.30) that

$$\frac{1}{2\sqrt{\Psi(\theta)}\sqrt{1-\Psi(\theta)}}\frac{d\Psi}{d\theta} = \frac{d\Phi}{d\theta}, \qquad 0 < \theta < \pi/2. \tag{8.45}$$

Since $V(0) = Z$, from (8.28), we see that $\Psi(0) = 0$. By definition, $\Phi(0) = 0$. Hence, by (8.45),

$$\begin{aligned}\Phi(\theta) = \int_0^\theta \frac{d\Phi}{dt}dt &= \frac{1}{2}\int_0^\theta \frac{1}{\sqrt{\Psi(t)}\sqrt{1-\Psi(t)}}\frac{d\Psi}{dt}\,dt \\ &= \frac{1}{2}\int_0^{\Psi(\theta)} \frac{du}{\sqrt{u}\sqrt{1-u}} \\ &= \int_0^{\sqrt{\Psi(\theta)}} \frac{dx}{\sqrt{1-x^2}} = \arcsin\left(\sqrt{\Psi(\theta)}\right),\end{aligned}$$

i.e.,

$$\Psi(\theta) = \sin^2(\Phi(\theta)), \qquad 0 \le \theta \le \pi/2.$$

This proves (8.29). Thus, (8.8) and (8.9) have been proved, and this finally completes the proof of Theorem 8.1.

Theorem 8.1 will be used to prove Ramanujan's next result, Theorem 8.7 below. For each positive integer r, define

$$S_{2r} := \sum_{n=1}^{\infty} \frac{n^{2r} q^n}{1 + q^n + q^{2n}}.$$

Ramanujan evaluated S_{2r} in closed form for $r = 1, 2, 3, 4$. Ramanujan's claimed value for S_8, namely,

$$S_8 = \tfrac{1}{27} x(1 + 8x) z^9,$$

is actually incorrect. We shall prove a general formula for S_{2r} from which the values for S_2, S_4, S_6, and S_8 follow. J. M. and P. B. Borwein [5] have evaluated S_2 but give no details.

Theorem 8.7 (p. 257). *For $r \geq 1$,*

$$S_{2r} = \frac{(-1)^r}{6 \cdot 2^{2r}} s_{2r}(Z, x, Z),$$

where the polynomials $s_{2r}(V, x, Z)$ are defined by (8.49)–(8.52) below. In particular,

$$S_2 = \frac{x}{27} Z^3,$$
$$S_4 = \frac{x}{27} Z^5,$$
$$S_6 = \frac{x(3 + 4x)}{81} Z^7,$$

and

$$S_8 = \frac{x(81 + 648x + 80x^2)}{3^7} Z^9.$$

Proof. Define

$$q(V, x, Z) := -\tfrac{4}{9}\left(4xZ^3 + (V - Z)(V + 2Z)^2\right)(V - Z).$$

Then, by (8.32),

$$q(V, x, Z) = \left(\frac{dV}{d\theta}\right)^2. \tag{8.46}$$

By a straightforward calculation,

$$\begin{aligned} p(V, x, Z) &:= \frac{1}{2}\frac{\partial}{\partial V} q(V, x, Z) \\ &= -\tfrac{8}{9}V^3 - \tfrac{4}{3}V^2 Z + \tfrac{4}{3} V Z^2 - \tfrac{8}{9} x Z^3 + \tfrac{8}{9} Z^3 \\ &= -\tfrac{4}{9}\left(2xZ^3 + (V - Z)(V + 2Z)(2V + Z)\right). \end{aligned} \tag{8.47}$$

From (8.46),

$$2\frac{dV}{d\theta}\frac{d^2V}{d\theta^2} = \frac{\partial q(V,x,Z)}{\partial V}\frac{dV}{d\theta},$$

and so, from (8.47),

$$\frac{d^2V}{d\theta^2} = p(V,x,Z). \tag{8.48}$$

For $n \geq 2$, define two sequences of polynomials $s_n(V,x,Z)$ and $t_n(V,x,Z)$ by

$$s_2(V,x,Z) := p(V,x,Z), \tag{8.49}$$

$$t_2(V,x,Z) := 0, \tag{8.50}$$

and, for $n \geq 3$,

$$s_n(V,x,Z) := t_{n-1}(V,x,Z)p(V,x,Z) + \frac{\partial}{\partial V}t_{n-1}(V,x,Z)q(V,x,Z) \tag{8.51}$$

and

$$t_n(V,x,Z) := \frac{\partial}{\partial V}s_{n-1}(V,x,Z). \tag{8.52}$$

By using (8.48)–(8.52), we may prove by induction that

$$\frac{d^nV}{d\theta^n} = s_n(V,x,Z) + t_n(V,x,Z)\frac{dV}{d\theta}, \qquad n \geq 2. \tag{8.53}$$

Now, from (8.2), (8.11), and the definition of S_{2r}, we readily see by induction on r that

$$\left.\frac{d^{2r}V(\theta)}{d\theta^{2r}}\right|_{\theta=0} = 6(-1)^r 2^{2r} S_{2r}. \tag{8.54}$$

Since $V'(0) = 0$ and $V(0) = Z$, we conclude from (8.53) and (8.54) that

$$S_{2r} = \frac{(-1)^r}{6 \cdot 2^{2r}} s_{2r}(Z,x,Z).$$

A program was devised in MAPLE to calculate S_{2r}. This completes the proof.

We also calculated S_{2r} by using Theorem 8.1 in another way. From P. Henrici's book [1, p. 102],

$$\varphi = \sum_{n=1}^{\infty} \frac{1}{n}\operatorname{Res}\left(\Phi^{-1}\right)^{-n}\theta^n, \tag{8.55}$$

where $\operatorname{Res}\left(\Phi^{-1}\right)^{-n}$ denotes the residue of $\left(\Phi^{-1}\right)^{-n}$ at $\varphi = 0$. By using (8.2), it is easy to prove that

$$\varphi = \theta + 3\sum_{r=0}^{\infty} \frac{(-1)^r(2\theta)^{2r+1}}{(2r+1)!}S_{2r}. \tag{8.56}$$

Equating coefficients of θ^{2r+1} in (8.55) and (8.56), we find that

$$S_{2r}=\frac{(-1)^r(2r)!\,\operatorname{Res}\left(\Phi^{-1}\right)^{-2r-1}}{3\cdot 2^{2r+1}}.$$

We then used *Mathematica* to expand $\left(Z\Phi^{-1}(\varphi)\right)^{-2r-1}$ in powers of φ so that the desired residues could be calculated.

After our proof of Theorem 8.1 was completed, L.–C. Shen [1] found another proof based on the classical theory of elliptic functions.

9. The Theory for Signature 4

The theory for signature 4 is simpler than that for signature 3, primarily because of Theorem 9.3 below.

Theorem 9.1 (p. 260). *For* $0<x<1$,

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{2x}{1+x}\right)=\sqrt{1+x}\;{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;x^2\right). \tag{9.1}$$

Theorem 9.1 is precisely Entry 33(i) of Chapter 11 of Ramanujan's second notebook (Part II [2, p. 94]).

Theorem 9.2. *For* $0<x<1$,

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1-x}{1+x}\right)=\sqrt{\frac{1}{2}(1+x)}\;{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;1-x^2\right). \tag{9.2}$$

Proof. With x complex and $|x|$ sufficiently small, by Entry 33(iv) of Chapter 11 (Part II [2, p. 95]),

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right)=\frac{1}{\sqrt{1+x}}\;{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{4x}{(1+x)^2}\right). \tag{9.3}$$

Replacing x by $(1-x)/(1+x)$ in (9.3), we find that, for $|1-x|$ sufficiently small, (9.2) holds. By analytic continuation, (9.2) holds for $0<x<1$.

Theorem 9.3 (p. 260). *Let $q=:q(x)$ denote the classical base, and let $q_4=:$ $q_4(x)$ be defined by (1.8). Then, for* $0<x<1$,

$$q_4(x)=q^2\left(\frac{2\sqrt{x}}{1+\sqrt{x}}\right). \tag{9.4}$$

Proof. Dividing (9.2) by (9.1), we find that, for $0<x<1$,

$$\sqrt{2}\frac{{}_2F_1\left(\frac14,\frac34;1;1-x^2\right)}{{}_2F_1\left(\frac14,\frac34;1;x^2\right)}=2\frac{{}_2F_1\left(\frac12,\frac12;1;1-\frac{2x}{1+x}\right)}{{}_2F_1\left(\frac12,\frac12;1;\frac{2x}{1+x}\right)}.$$

Replacing x by $\sqrt{x}$ and recalling the definitions of q and q_4, we see that (9.4) follows.

Theorem 9.4 (p. 260). *For* $0 < x < 1$,

$$ {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-\left(\frac{1-x}{1+3x}\right)^2\right) = \sqrt{1+3x}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; x^2\right). \tag{9.5} $$

Proof. Applying (9.1) and then (9.3) with x replaced by $2x/(1+x)$, we find that, for x complex and $|x|$ sufficiently small,

$$ \begin{aligned} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; x^2\right) &= (1+x)^{-1/2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{2x}{1+x}\right) \\ &= (1+x)^{-1/2}\left(\frac{1+3x}{1+x}\right)^{-1/2} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{8x(1+x)}{(1+3x)^2}\right) \\ &= (1+3x)^{-1/2}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-\left(\frac{1-x}{1+3x}\right)^2\right). \end{aligned} $$

Thus, (9.5) has been established for $|x|$ sufficiently small, and by analytic continuation, (9.5) is valid for $0 < x < 1$.

We now describe a process for deducing formulas in the theory of signature 4 from corresponding formulas in the classical setting. Suppose that we have a formula

$$ \Omega(x, q^2, z) = 0. \tag{9.6} $$

Let us replace x by $2\sqrt{x}/(1+\sqrt{x})$. Then by Theorem 9.3, q^2 is replaced by q_4. By (9.1),

$$ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{2\sqrt{x}}{1+\sqrt{x}}\right) = \sqrt{1+\sqrt{x}}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; x\right), $$

i.e., z is replaced by $\sqrt{1+\sqrt{x}}z(4)$. Hence, from (9.6), we deduce the formula,

$$ \Omega\left(\frac{2\sqrt{x}}{1+\sqrt{x}}, q_4, \sqrt{1+\sqrt{x}}z(4)\right) = 0. \tag{9.7} $$

In each proof below, we apply (9.7) to results from Chapter 17 of Ramanujan's second notebook, proofs of which are given in Part III [3].

Theorem 9.5 (p. 260). *We have*

$$ M(q_4) = z^4(4)(1+3x). $$

Proof. By Entry 13(i) of Chapter 17 (Part III [3, p. 126]),

$$ M(q^2) = z^4(1-x+x^2). $$

Thus, by (9.7),

$$M(q_4) = \left\{\sqrt{1+\sqrt{x}}\right\}^4 z^4(4)\left(1 - \frac{2\sqrt{x}}{1+\sqrt{x}} + \left(\frac{2\sqrt{x}}{1+\sqrt{x}}\right)^2\right)$$
$$= z^4(4)\left\{(1+\sqrt{x})^2 - 2\sqrt{x}(1+\sqrt{x}) + 4x\right\} = z^4(4)(1+3x).$$

Theorem 9.6 (p. 260). *We have*

$$N(q_4) = z^6(4)(1-9x).$$

Proof. By Entry 13(ii) of Chapter 17 (Part III [3, p. 126]),

$$N(q^2) = z^6(1+x)(1-\tfrac{1}{2}x)(1-2x).$$

Using (9.7), we deduce that

$$N(q_4) = (1+\sqrt{x})^3 z^6(4)\left(1 + \frac{2\sqrt{x}}{1+\sqrt{x}}\right)\left(1 - \frac{\sqrt{x}}{1+\sqrt{x}}\right)\left(1 - \frac{4\sqrt{x}}{1+\sqrt{x}}\right)$$
$$= z^6(4)(1+3\sqrt{x})(1-3\sqrt{x}) = z^6(4)(1-9x).$$

Theorem 9.7 (p. 260). *We have*

$$M(q_4^2) = z^4(4)\left(1 - \tfrac{3}{4}x\right).$$

Proof. By Entry 13(v) of Chapter 17 (Part III [3, p. 127]),

$$M(q^4) = z^4\left(1 - x + \tfrac{1}{16}x^2\right).$$

Hence, by (9.7),

$$M(q_4^2) = (1+\sqrt{x})^2 z^4(4)\left(1 - \frac{2\sqrt{x}}{1+\sqrt{x}} + \frac{x}{4(1+\sqrt{x})^2}\right)$$
$$= \tfrac{1}{4}z^4(4)\left\{4(1+\sqrt{x})^2 - 8\sqrt{x}(1+\sqrt{x}) + x\right\} = \tfrac{1}{4}z^4(4)(4-3x).$$

Theorem 9.8 (p. 260). *We have*

$$N(q_4^2) = z^6(4)\left(1 - \tfrac{9}{8}x\right).$$

Proof. By Entry 13(vi) of Chapter 17 (Part III [3, p. 127]),

$$N(q^4) = z^6(1-\tfrac{1}{2}x)(1 - x - \tfrac{1}{32}x^2).$$

Thus, from (9.7),

$$N(q_4^2) = (1+\sqrt{x})^3 z^6(4)\left(1 - \frac{\sqrt{x}}{1+\sqrt{x}}\right)\left(1 - \frac{2\sqrt{x}}{1+\sqrt{x}} - \frac{x}{8(1+\sqrt{x})^2}\right)$$
$$= \tfrac{1}{8}z^6(4)\left\{8(1+\sqrt{x})^2 - 16\sqrt{x}(1+\sqrt{x}) - x\right\} = \tfrac{1}{8}z^6(4)(8-9x).$$

It is interesting that the two previous results have simpler formulations in the theory of signature 4 than in the classical theory.

Theorem 9.9 (p. 260). *We have*

$$q_4^{1/24} f(-q_4) = \sqrt{z(4)}2^{-1/4}x^{1/24}(1-x)^{1/12}.$$

Proof. From Entry 12(iii) of Chapter 17 (Part III [3, p. 124]),

$$f(-q^2) = \sqrt{z}2^{-1/3}\{x(1-x)/q\}^{1/12}.$$

Hence, by (9.7),

$$\begin{aligned} q_4^{1/24} f(-q_4) &= \left(1+\sqrt{x}\right)^{1/4}\sqrt{z(4)}2^{-1/3}\left(\frac{2\sqrt{x}}{1+\sqrt{x}}\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{1/12} \\ &= \sqrt{z(4)}2^{-1/4}x^{1/24}\left(1+\sqrt{x}\right)^{1/12}\left(1-\sqrt{x}\right)^{1/12} \\ &= \sqrt{z(4)}2^{-1/4}x^{1/24}(1-x)^{1/12}. \end{aligned}$$

Theorem 9.10 (p. 260). *We have*

$$q_4^{1/12} f(-q_4^2) = \sqrt{z(4)}2^{-1/2}x^{1/12}(1-x)^{1/24}.$$

Proof. By Entry 12(iv) of Chapter 17 (Part III [3, p. 124]),

$$q^{1/6} f(-q^4) = \sqrt{z}4^{-1/3}(1-x)^{1/24}x^{1/6}.$$

Thus, by (9.7),

$$\begin{aligned} q_4^{1/12} f(-q_4^2) &= \left(1+\sqrt{x}\right)^{1/4}\sqrt{z(4)}4^{-1/3}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{1/24}\left(\frac{2\sqrt{x}}{1+\sqrt{x}}\right)^{1/6} \\ &= \sqrt{z(4)}2^{-1/2}x^{1/12}(1+\sqrt{x})^{1/24}(1-\sqrt{x})^{1/24} \\ &= \sqrt{z(4)}2^{-1/2}x^{1/12}(1-x)^{1/24}. \end{aligned}$$

Recall the definition of $L(q)$ at the beginning of Section 4. The following intriguing formula does not appear in the second notebook but can be found in the first notebook [9].

Theorem 9.11 (p. 214, NB 1). *We have*

$${}_2F_1^2\left(\tfrac{1}{4}, \tfrac{3}{4}; 1; x\right) = 2L(q_4^2) - L(q_4).$$

Proof. For brevity, set $q = q_4$ and $z = z(4)$. From (4.2) and Theorem 9.9,

$$\begin{aligned} L(q) = q\frac{d}{dq}\log\left(qf^{24}(-q)\right) &= q\frac{d}{dq}\log\left(z^{12}2^{-6}x(1-x)^2\right) \\ &= q\frac{d}{dx}\log\left(z^{12}2^{-6}x(1-x)^2\right)\frac{dx}{dq}. \end{aligned} \tag{9.8}$$

By Entry 30 in Chapter 11 of Ramanujan's second notebook (Part II [2, p. 87]), with $q = q_4$,

$$\frac{dq}{dx} = \frac{q}{x(1-x)z^2}. \tag{9.9}$$

Using (9.9) in (9.8), we deduce that

$$\begin{aligned} L(q) &= \left(\frac{12}{z}\frac{dz}{dx} + \frac{1}{x} - \frac{2}{1-x}\right)x(1-x)z^2 \\ &= 12x(1-x)z\frac{dz}{dx} + (1-3x)z^2. \end{aligned} \tag{9.10}$$

Repeating the same argument, but with q replaced by q^2 and with an application of Theorem 9.10 instead of Theorem 9.9, we find that

$$\begin{aligned} L(q^2) &= \frac{1}{2}q\frac{d}{dq}\log\left(q^2 f^{24}(-q^2)\right) \\ &= \frac{1}{2}q\frac{d}{dx}\log\left(z^{12}2^{-12}x^2(1-x)\right)\frac{dx}{dq} \\ &= \frac{1}{2}\left(\frac{12}{z}\frac{dz}{dx} + \frac{2}{x} - \frac{1}{1-x}\right)x(1-x)z^2 \\ &= 6x(1-x)z\frac{dz}{dx} + \tfrac{1}{2}(2-3x)z^2. \end{aligned} \tag{9.11}$$

Our theorem now easily follows from (9.10) and (9.11).

We conclude this section with a new transformation formula for ${}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; x)$ found also on page 260. We need to first establish some ancillary lemmas.

Lemma 9.12. *We have*

$$a(q) + a(q^2) = 2\frac{\psi^3(q)}{\psi(q^3)} \tag{9.12}$$

and

$$2a(q^2) - a(q) = \frac{\varphi^3(-q)}{\varphi(-q^3)}. \tag{9.13}$$

Proof. By Entries 4(iii), (iv) of Chapter 19 of Ramanujan's second notebook (Part III [3, pp. 226–227, 229]),

$$\frac{\psi^3(q)}{\psi(q^3)} = 1 + 3\sum_{n=0}^{\infty}\left(\frac{q^{6n+1}}{1-q^{6n+1}} - \frac{q^{6n+5}}{1-q^{6n+5}}\right)$$

and

$$\frac{\varphi^3(-q)}{\varphi(-q^3)} = 1 - 6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1+q^{3n+1}} - \frac{q^{3n+2}}{1+q^{3n+2}}\right).$$

Using (2.6) to calculate the Lambert series for $\frac{1}{2}\{a(q)+a(q^2)\}$ and comparing it with the Lambert series above, we deduce (9.12). The proof of (9.13) is similar and follows by substituting $x = q^{3n+1}, q^{3n+2}$ in the elementary identity

$$\frac{2x^2}{1-x^2} - \frac{x}{1-x} = -\frac{x}{1+x}.$$

Lemma 9.13. *We have*

$$\frac{\psi^6(q)}{\psi^2(q^3)} + 18q\psi^2(q)\psi^2(q^3) - 27q^2\frac{\psi^6(q^3)}{\psi^2(q)} = \varphi^4(q) + 16q\psi^4(q^2) \tag{9.14}$$

and

$$\frac{\psi^6(q)}{\psi^2(q^3)} - 6q\psi^2(q)\psi^2(q^3) - 3q^2\frac{\psi^6(q^3)}{\psi^2(q)} = \varphi^4(q^3) + 16q^3\psi^4(q^6). \tag{9.15}$$

Proof. For brevity, set $a := a(q)$ and $A := a(q^2)$. Then, squaring (9.12), we find that

$$4\frac{\psi^6(q)}{\psi^2(q^3)} = (a+A)^2. \tag{9.16}$$

From (5.7), (6.3), and (9.12),

$$12q\psi^2(q)\psi^2(q^3) = a^2 - A^2 \tag{9.17}$$

and

$$36q^2\frac{\psi^6(q^3)}{\psi^2(q)} = (a-A)^2. \tag{9.18}$$

Hence, from (9.16)–(9.18),

$$\frac{\psi^6(q)}{\psi^2(q^3)} + 18q\psi^2(q)\psi^2(q^3) - 27q^2\frac{\psi^6(q^3)}{\psi^2(q)} = a^2 + 2aA - 2A^2. \tag{9.19}$$

Next, from (9.16) and (9.18),

$$48q\psi^8(q) = \left(8\frac{\psi^9(q)}{\psi^3(q^3)}\right)\left(6q\frac{\psi^3(q^3)}{\psi(q)}\right) = (a+A)^3(a-A) \tag{9.20}$$

and

$$432q^3\psi^8(q^3) = \left(216q^3\frac{\psi^9(q^3)}{\psi^3(q)}\right)\left(2\frac{\psi^3(q)}{\psi(q^3)}\right) = (a-A)^3(a+A). \tag{9.21}$$

From (5.6) and (6.5),

$$3\frac{\varphi^3(-q^3)}{\varphi(-q)} = a + 2A. \tag{9.22}$$

Thus, from (9.13) and (9.22),

$$\varphi^8(-q) = \frac{\varphi^9(-q)}{\varphi^3(-q^3)}\frac{\varphi^3(-q^3)}{\varphi(-q)} = \frac{1}{3}(2A-a)^3(a+2A) \tag{9.23}$$

and

$$\varphi^8(-q^3) = \frac{\varphi^9(-q^3)}{\varphi^3(-q)}\frac{\varphi^3(-q)}{\varphi(-q^3)} = \frac{1}{27}(a+2A)^3(2A-a). \tag{9.24}$$

Lastly, we need the elementary identity (Part III [3, p. 40, Entry 25(iv)])

$$\psi(q^4)\varphi(q^2) = \psi^2(q^2). \tag{9.25}$$

Hence, from (2.1), (9.25), (9.23), and (9.20),

$$\begin{aligned}
\left\{\varphi^4(q) + 16q\psi^4(q^2)\right\}^2 &= \left\{\varphi^4(q) - 16q\psi^4(q^2)\right\}^2 + 64q\varphi^4(q)\psi^4(q^2) \\
&= \varphi^8(-q) + 64q\psi^8(q) \\
&= \tfrac{1}{3}(2A-a)^3(a+2A) + \tfrac{4}{3}(a+A)^3(a-A) \\
&= a^4 + 4A^4 + 4aA(a^2 - 2A^2) \\
&= (a^2 - 2A^2)^2 + 4aA(a^2 - 2A^2) + 4a^2A^2 \\
&= (a^2 + 2aA - 2A^2)^2.
\end{aligned} \tag{9.26}$$

Equality (9.14) now follows from (9.19) and (9.26), upon taking the square root of both sides of (9.26) and checking agreement at $q = 0$ to ensure that the correct square root is taken.

The proof of (9.15) is similar. Thus, from (9.16)–(9.18),

$$\frac{\psi^6(q)}{\psi^2(q^3)} - 6q\psi^2(q)\psi^2(q^3) - 3q^2\frac{\psi^6(q^3)}{\psi^2(q)} = \tfrac{1}{3}(2A^2 + 2aA - a^2). \tag{9.27}$$

Proceeding as in the proof of (9.14), from (2.1), (9.25), (9.24), and (9.21), we find that

$$\begin{aligned}
\left\{\varphi^4(q^3) + 16q^3\psi^4(q^6)\right\}^2 &= \varphi^8(-q^3) + 64q^3\psi^8(q^3) \\
&= \tfrac{1}{27}(a+2A)^3(2A-a) + \tfrac{4}{27}(a-A)^3(a+A) \\
&= \tfrac{1}{9}\left(a^4 + 4A^4 + 4aA(2A^2 - a^2)\right) \\
&= \tfrac{1}{9}\left((a^2 - 2A^2)^2 - 4aA(a^2 - 2A^2) + 4a^2A^2\right) \\
&= \tfrac{1}{9}(a^2 - 2A^2 - 2aA)^2.
\end{aligned} \tag{9.28}$$

Equality (9.15) now follows from (9.27) and (9.28).

Lastly, we need the following lemma connecting ${}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; x)$ with theta-functions.

Lemma 9.14. *We have*

$${}_2F_1^2\left(\frac{1}{4}, \frac{3}{4}; 1; \left(\frac{8\sqrt{q}\psi^2(q^2)\varphi^2(q)}{\varphi^4(q) + 16q\psi^4(q^2)}\right)^2\right) = \varphi^4(q) + 16q\psi^4(q^2). \tag{9.29}$$

Proof. Let

$$x = \frac{8q\psi^2(q^4)\varphi^2(q^2)}{\varphi^4(q^2) + 16q^2\psi^4(q^4)}.$$

Then

$$\frac{2x}{1+x} = \frac{16q\psi^2(q^4)\varphi^2(q^2)}{\left\{\varphi^2(q^2) + 4q\psi^2(q^4)\right\}^2} = 16q\frac{\psi^4(q^2)}{\varphi^4(q)}, \tag{9.30}$$

where we have employed (9.25) and the elementary identity

$$\varphi^2(q^2) + 4q\psi^2(q^4) = \varphi^2(q), \tag{9.31}$$

which is achieved by adding Entries 25(v), (vi) of Chapter 16 of Ramanujan's second notebook (Part III [3, p. 40]). Hence, from Theorem 9.1, (9.30), (9.31), (2.1), and (5.19), with $-q^3$ replaced by q,

$$\begin{aligned}
{}_2F_1^2\left(\frac{1}{4}, \frac{3}{4}; 1; x^2\right) &= \frac{1}{1+x}\,{}_2F_1^2\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{2x}{1+x}\right) \\
&= \frac{\varphi^4(q^2) + 16q^2\psi^4(q^4)}{\left\{\varphi^2(q^2) + 4q\psi^2(q^4)\right\}^2}\,{}_2F_1^2\left(\frac{1}{2}, \frac{1}{2}; 1; 16q\frac{\psi^4(q^2)}{\varphi^4(q)}\right) \\
&= \frac{\varphi^4(q^2) + 16q^2\psi^4(q^4)}{\varphi^4(q)}\varphi^4(q) \\
&= \varphi^4(q^2) + 16q^2\psi^4(q^4).
\end{aligned}$$

Replacing q^2 by q, we deduce (9.29).

Lemma 9.14 was first proved by the Borwein brothers [1, p. 179, Prop. 5.7(a)], [5, Theorem 2.6(b)].

Theorem 9.15 (p. 260). *If*

$$\alpha := \frac{64p}{(3+6p-p^2)^2} \qquad \textit{and} \qquad \beta := \frac{64p^3}{(27-18p-p^2)^2}, \tag{9.32}$$

then, for $0 \le p < 1$,

$$\sqrt{27-18p-p^2}\,{}_2F_1\left(\tfrac{1}{4}, \tfrac{3}{4}; 1; \alpha\right) = 3\sqrt{3+6p-p^2}\,{}_2F_1\left(\tfrac{1}{4}, \tfrac{3}{4}; 1; \beta\right). \tag{9.33}$$

Proof. Let

$$p = 9q\frac{\psi^4(q^3)}{\psi^4(q)}. \tag{9.34}$$

Then, by (9.14) and (9.15), respectively,

$$3 + 6p - p^2 = \frac{\sqrt{p}}{\sqrt{q}\psi^4(q)}\left\{\varphi^4(q) + 16q\psi^4(q^2)\right\} \tag{9.35}$$

and

$$27 - 18p - p^2 = \frac{p^{3/2}}{q^{3/2}\psi^4(q^3)} \left\{\varphi^4(q^3) + 16q^3\psi^4(q^6)\right\}. \tag{9.36}$$

From (9.32) and (9.35),

$$\alpha[q] := \alpha = \frac{64q\psi^8(q)}{\left\{\varphi^4(q) + 16q\psi^4(q^2)\right\}^2} = \left(\frac{8\sqrt{q}\psi^2(q^2)\varphi^2(q)}{\varphi^4(q) + 16q\psi^4(q^2)}\right)^2, \tag{9.37}$$

by (9.25). Similarly, by (9.32) and (9.36),

$$\beta[q] := \beta = \left(\frac{8q^{3/2}\psi^2(q^6)\varphi^2(q^3)}{\varphi^4(q^3) + 16q^3\psi^4(q^6)}\right)^2 = \alpha[q^3], \tag{9.38}$$

by (9.37).

Hence, from (9.36), (9.37), Lemma 9.14, (9.38), (9.35), and (9.34),

$$\begin{aligned}
&\sqrt{27 - 18p - p^2}\, {}_2F_1\left(\tfrac{1}{4}, \tfrac{3}{4}; 1; \alpha\right) \\
&= \frac{p^{3/4}}{q^{3/4}\psi^2(q^3)} \left\{\varphi^4(q^3) + 16q^3\psi^4(q^6)\right\}^{1/2} \left\{\varphi^4(q) + 16q\psi^4(q^2)\right\}^{1/2} \\
&= \frac{p^{3/4}}{q^{3/4}\psi^2(q^3)}\, {}_2F_1\left(\tfrac{1}{4}, \tfrac{3}{4}; 1; \beta\right) \frac{q^{1/4}\psi^2(q)}{p^{1/4}} \sqrt{3 + 6p - p^2} \\
&= \frac{\sqrt{p}\psi^2(q)}{\sqrt{q}\psi^2(q^3)} \sqrt{3 + 6p - p^2}\, {}_2F_1\left(\tfrac{1}{4}, \tfrac{3}{4}; 1; \beta\right) \\
&= 3\sqrt{3 + 6p - p^2}\, {}_2F_1\left(\tfrac{1}{4}, \tfrac{3}{4}; 1; \beta\right).
\end{aligned}$$

Thus, (9.33) has been proved.

Lastly, it is easily checked that $\alpha =: \alpha(p)$ and $\beta =: \beta(p)$ are monotonically increasing functions of p on (0,1). Since $\alpha(0) = 0 = \beta(0)$ and $\alpha(1) = 1 = \beta(1)$, (9.33) is valid for $0 \leq p < 1$.

10. Modular Equations in the Theory of Signature 4

Page 261 in Ramanujan's second notebook is devoted to modular equations in the theory of signature 4. In each case, our proofs rely on (9.7). Thus, we will employ modular equations from Chapters 19 and 20 of the second notebook and convert them via the transformations

$$\alpha \mapsto \frac{2\sqrt{\alpha}}{1 + \sqrt{\alpha}}, \quad 1 - \alpha \mapsto \frac{1 - \sqrt{\alpha}}{1 + \sqrt{\alpha}}, \quad \beta \mapsto \frac{2\sqrt{\beta}}{1 + \sqrt{\beta}}, \quad 1 - \beta \mapsto \frac{1 - \sqrt{\beta}}{1 + \sqrt{\beta}}. \tag{10.1}$$

Theorem 10.1 (p. 261). *If β has degree 3, then*

$$(\alpha\beta)^{1/2} + \{(1 - \alpha)(1 - \beta)\}^{1/2} + 4\{\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/4} = 1. \tag{10.2}$$

Proof. From Entry 5(ii) of Chapter 19 (Part III [3, p. 230]),

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1. \tag{10.3}$$

By (10.1), (10.3) is transformed from the classical theory to

$$\frac{\sqrt{2}(\alpha\beta)^{1/8}}{\left\{(1+\sqrt{\alpha})(1+\sqrt{\beta})\right\}^{1/4}} + \left\{\frac{(1-\sqrt{\alpha})(1-\sqrt{\beta})}{(1+\sqrt{\alpha})(1+\sqrt{\beta})}\right\}^{1/4} = 1$$

in the theory of signature 4, or

$$\sqrt{2}(\alpha\beta)^{1/8} = \left\{(1+\sqrt{\alpha})(1+\sqrt{\beta})\right\}^{1/4} - \left\{(1-\sqrt{\alpha})(1-\sqrt{\beta})\right\}^{1/4}.$$

Squaring both sides yields

$$\begin{aligned}2(\alpha\beta)^{1/4} + 2\{(1-\alpha)(1-\beta)\}^{1/4} = {} & \left\{(1+\sqrt{\alpha})(1+\sqrt{\beta})\right\}^{1/2} \\ & + \left\{(1-\sqrt{\alpha})(1-\sqrt{\beta})\right\}^{1/2}.\end{aligned}$$

Squaring each side once again, we find that

$$\begin{aligned}&2(\alpha\beta)^{1/2} + 2\{(1-\alpha)(1-\beta)\}^{1/2} + 4\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4} \\ &\quad = 1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}.\end{aligned}$$

Collecting terms, we deduce (10.2).

Theorem 10.2 (p. 261). *If β has degree 5, then*

$$\begin{aligned}&(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} \\ &+ 8\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}\left((\alpha\beta)^{1/6} + \{(1-\alpha)(1-\beta)\}^{1/6}\right) = 1.\end{aligned} \tag{10.4}$$

Proof. By Entry 13(i) of Chapter 19 of Ramanujan's second notebook (Part III [3, p. 280]),

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1. \tag{10.5}$$

Transforming (10.5) by (10.1) and simplifying, we find that, in the theory of signature 4,

$$\begin{aligned}2(\alpha\beta)^{1/4} + 2\left\{64\sqrt{\alpha\beta}(1-\alpha)(1-\beta)\right\}^{1/6} = {} & \left\{(1+\sqrt{\alpha})(1+\sqrt{\beta})\right\}^{1/2} \\ & - \left\{(1-\sqrt{\alpha})(1-\sqrt{\beta})\right\}^{1/2}.\end{aligned}$$

Squaring each side and collecting terms, we deduce that

$$\begin{aligned}2(\alpha\beta)^{1/2} + 4\left\{64\sqrt{\alpha\beta}(1-\alpha)(1-\beta)\right\}^{1/3} + 8(\alpha\beta)^{1/3}\{64(1-\alpha)(1-\beta)\}^{1/6} \\ = 2 - 2\{(1-\alpha)(1-\beta)\}^{1/2}.\end{aligned}$$

Further simplification easily yields (10.4).

Theorem 10.3 (p. 261). *If β has degree 7, then*

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 20\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4} \\ + 8\sqrt{2}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\left((\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}\right) = 1. \tag{10.6}$$

Proof. From Entry 19(i) of Chapter 19 of Ramanujan's second notebook (Part III [3, p. 314]),

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1. \tag{10.7}$$

Using (10.1) to transform (10.7) into the theory of signature 4, we find that

$$(16\alpha\beta)^{1/16} = \left\{(1+\sqrt{\alpha})(1+\sqrt{\beta})\right\}^{1/8} - \left\{(1-\sqrt{\alpha})(1-\sqrt{\beta})\right\}^{1/8}.$$

Squaring both sides, we deduce that

$$(16\alpha\beta)^{1/8} + 2\{(1-\alpha)(1-\beta)\}^{1/8} = \left\{(1+\sqrt{\alpha})(1+\sqrt{\beta})\right\}^{1/4} \\ + \left\{(1-\sqrt{\alpha})(1-\sqrt{\beta})\right\}^{1/4}.$$

Squaring again and simplifying slightly, we find that

$$(16\alpha\beta)^{1/4} + 2\{(1-\alpha)(1-\beta)\}^{1/4} + 4\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \\ = \left\{(1+\sqrt{\alpha})(1+\sqrt{\beta})\right\}^{1/2} + \left\{(1-\sqrt{\alpha})(1-\sqrt{\beta})\right\}^{1/2}.$$

Squaring one more time, we finally deduce that

$$4(\alpha\beta)^{1/2} + 4\{(1-\alpha)(1-\beta)\}^{1/2} + 32\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4} \\ + 8\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4} + 16(\alpha\beta)^{1/4}\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \\ + 16\{(1-\alpha)(1-\beta)\}^{1/4}\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \\ = 2 + 2(\alpha\beta)^{1/2} + 2\{(1-\alpha)(1-\beta)\}^{1/2}.$$

Collecting terms and dividing both sides by 2, we complete the proof of (10.6).

Theorem 10.4 (p. 261). *If β has degree 11, then*

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 68\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4} \\ + 16\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}\left((\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3}\right) \\ + 48\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}\left((\alpha\beta)^{1/6} + \{(1-\alpha)(1-\beta)\}^{1/6}\right) = 1. \tag{10.8}$$

Proof. By Entry 7(i) of Chapter 20 (Part III [3, p. 363]),

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} = 1. \tag{10.9}$$

Transforming (10.9) into an equality in the theory of signature 4, we find that

$$(4\sqrt{\alpha\beta})^{1/4} + 2\left\{64\sqrt{\alpha\beta}(1-\alpha)(1-\beta)\right\}^{1/12} = \left\{(1+\sqrt{\alpha})(1+\sqrt{\beta})\right\}^{1/4} - \left\{(1-\sqrt{\alpha})(1-\sqrt{\beta})\right\}^{1/4}.$$

Squaring both sides and simplifying slightly, we deduce that

$$\begin{aligned}&2(\alpha\beta)^{1/4} + 2\{(1-\alpha)(1-\beta)\}^{1/4} + 8(\alpha\beta)^{1/12}\{(1-\alpha)(1-\beta)\}^{1/6}\\&+ 8(\alpha\beta)^{1/6}\{(1-\alpha)(1-\beta)\}^{1/12}\\&= \left\{(1+\sqrt{\alpha})(1+\sqrt{\beta})\right\}^{1/2} + \left\{(1-\sqrt{\alpha})(1-\sqrt{\beta})\right\}^{1/2}.\end{aligned}$$

Squaring both sides again, we see that

$$\begin{aligned}&4(\alpha\beta)^{1/2} + 4\{(1-\alpha)(1-\beta)\}^{1/12} + 64(\alpha\beta)^{1/6}\{(1-\alpha)(1-\beta)\}^{1/3}\\&+ 64(\alpha\beta)^{1/3}\{(1-\alpha)(1-\beta)\}^{1/6} + 8\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}\\&+ 32(\alpha\beta)^{1/3}\{(1-\alpha)(1-\beta)\}^{1/6} + 32(\alpha\beta)^{5/12}\{(1-\alpha)(1-\beta)\}^{1/12}\\&+ 32(\alpha\beta)^{1/12}\{(1-\alpha)(1-\beta)\}^{5/12} + 32(\alpha\beta)^{1/6}\{(1-\alpha)(1-\beta)\}^{1/3}\\&+ 128\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4} = 2 + 2(\alpha\beta)^{1/2} + 2\{(1-\alpha)(1-\beta)\}^{1/2}.\end{aligned}$$

Collecting terms and dividing both sides by 2, we complete the proof.

The last six entries on page 261 in Ramanujan's second notebook give formulas for multipliers. By (9.1),

$$m(4) = \frac{{}_2F_1(\frac{1}{4},\frac{3}{4};1;\alpha)}{{}_2F_1(\frac{1}{4},\frac{3}{4};1;\beta)} = \left(\frac{1+\sqrt{\beta}}{1+\sqrt{\alpha}}\right)^{1/2} \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{2\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{2\sqrt{\beta}}{1+\sqrt{\beta}}\right)}. \tag{10.10}$$

Thus, to obtain formulas for multipliers in the theory of signature 4, take formulas from the classical theory, replace m by $\sqrt{(1+\sqrt{\alpha})/(1+\sqrt{\beta})}\; m(4)$, and utilize the transformations in (10.1).

Theorem 10.5 (p. 261). *The multiplier for degree 3 is given by*

$$m^2(4) = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \frac{9}{m^2(4)}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}. \tag{10.11}$$

Proof. From Entry 5(vii) of Chapter 19 (Part III [3, p. 230]),

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} \tag{10.12}$$

and

$$\frac{9}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}. \tag{10.13}$$

Using (10.1) and (10.10), we convert (10.12) into an equality in the theory of signature 4, namely,

$$\frac{1+\sqrt{\alpha}}{1+\sqrt{\beta}}m^2(4) = \left(\frac{\sqrt{\beta}(1+\sqrt{\alpha})}{\sqrt{\alpha}(1+\sqrt{\beta})}\right)^{1/2} + \left(\frac{(1-\sqrt{\beta})(1+\sqrt{\alpha})}{(1+\sqrt{\beta})(1-\sqrt{\alpha})}\right)^{1/2} - \left(\frac{\sqrt{\beta}(1-\sqrt{\beta})(1+\sqrt{\alpha})^2}{\sqrt{\alpha}(1-\sqrt{\alpha})(1+\sqrt{\beta})^2}\right)^{1/2},$$

or, upon rearrangement,

$$m^2(4) - \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} = \left(\frac{\sqrt{\beta}(1+\sqrt{\beta})}{\sqrt{\alpha}(1+\sqrt{\alpha})}\right)^{1/2} - \left(\frac{\sqrt{\beta}(1-\sqrt{\beta})}{\sqrt{\alpha}(1-\sqrt{\alpha})}\right)^{1/2}. \tag{10.14}$$

By (10.13), (10.14), and symmetry,

$$\frac{9}{m^2(4)} - \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} = \left(\frac{\sqrt{\alpha}(1+\sqrt{\alpha})}{\sqrt{\beta}(1+\sqrt{\beta})}\right)^{1/2} - \left(\frac{\sqrt{\alpha}(1-\sqrt{\alpha})}{\sqrt{\beta}(1-\sqrt{\beta})}\right)^{1/2}. \tag{10.15}$$

Multiplying both sides of (10.15) by $\sqrt{\beta(1-\beta)/(\alpha(1-\alpha))}$, we find that

$$\frac{9}{m^2(4)}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} - \left(\frac{\beta}{\alpha}\right)^{1/2} = \left(\frac{\sqrt{\beta}(1-\sqrt{\beta})}{\sqrt{\alpha}(1-\sqrt{\alpha})}\right)^{1/2} - \left(\frac{\sqrt{\beta}(1+\sqrt{\beta})}{\sqrt{\alpha}(1+\sqrt{\alpha})}\right)^{1/2}. \tag{10.16}$$

Comparing (10.14) and (10.16), we arrive at (10.11).

Theorem 10.6 (p. 261). *If $m(4)$ is the multiplier of degree 5, then*

$$m(4) = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \frac{5}{m(4)}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}. \tag{10.17}$$

Proof. By Entry 13(xii) of Chapter 19 (Part III [3, pp. 281–282]),

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} \tag{10.18}$$

and

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}. \tag{10.19}$$

Transforming (10.18) into the theory of signature 4 via (10.1) and (10.10), we find, after a slight amount of rearrangement, that

$$m(4) = \left(\frac{\sqrt{\beta}(1+\sqrt{\beta})}{\sqrt{\alpha}(1+\sqrt{\alpha})}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\sqrt{\beta}(1-\sqrt{\beta})}{\sqrt{\alpha}(1-\sqrt{\alpha})}\right)^{1/4}. \tag{10.20}$$

From (10.19) and symmetry,

$$\frac{5}{m(4)} = \left(\frac{\sqrt{\alpha}(1+\sqrt{\alpha})}{\sqrt{\beta}(1+\sqrt{\beta})}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\sqrt{\alpha}(1-\sqrt{\alpha})}{\sqrt{\beta}(1-\sqrt{\beta})}\right)^{1/4}. \tag{10.21}$$

Multiplying both sides of (10.21) by $\sqrt[4]{\beta(1-\beta)/(\alpha(1-\alpha))}$ and comparing the result with (10.20), we readily deduce (10.17).

Theorem 10.7 (p. 261). *Let* $m(4)$ *denote the multiplier of degree 9. Then*

$$\sqrt{m(4)} = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \frac{3}{\sqrt{m(4)}}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}. \tag{10.22}$$

Proof. The proof is almost identical to the two previous proofs. By Entries 3(x), (xi), respectively, of Chapter 20 (Part III [3, p. 352]),

$$\sqrt{m} = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} \tag{10.23}$$

and

$$\frac{3}{\sqrt{m}} = \left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}. \tag{10.24}$$

The transforming of (10.23) and (10.24) via (10.1) and (10.10) yields the equalities

$$\sqrt{m(4)} = \left(\frac{\sqrt{\beta}(1+\sqrt{\beta})}{\sqrt{\alpha}(1+\sqrt{\alpha})}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\sqrt{\beta}(1-\sqrt{\beta})}{\sqrt{\alpha}(1-\sqrt{\alpha})}\right)^{1/8} \tag{10.25}$$

and

$$\frac{3}{\sqrt{m(4)}} = \left(\frac{\sqrt{\alpha}(1+\sqrt{\alpha})}{\sqrt{\beta}(1+\sqrt{\beta})}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\sqrt{\alpha}(1-\sqrt{\alpha})}{\sqrt{\beta}(1-\sqrt{\beta})}\right)^{1/8}, \tag{10.26}$$

respectively. A multiplication of (10.26) by $\sqrt[8]{\beta(1-\beta)/(\alpha(1-\alpha))}$ and a comparison of the resulting equality with (10.25) gives (10.22).

Theorem 10.8 (p. 261). *If* $m(4)$ *denotes the multiplier of degree 7, then*

$$\begin{aligned} m^2(4) = {} & \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \frac{49}{m^2(4)}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} \\ & - 8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}\left\{\left(\frac{\beta}{\alpha}\right)^{1/6} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/6}\right\}. \end{aligned} \tag{10.27}$$

Proof. By Entry 19(v) of Chapter 19 (Part III [3, p. 314]),

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} - 8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3} \tag{10.28}$$

and

$$\frac{49}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2} - 8\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/3}. \tag{10.29}$$

Transforming (10.28) and (10.29) into the theory of signature 4 via (10.1) and (10.10), we find that, after simplification,

$$m^2(4) = \left(\frac{\sqrt{\beta}(1+\sqrt{\beta})}{\sqrt{\alpha}(1+\sqrt{\alpha})}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\sqrt{\beta}(1-\sqrt{\beta})}{\sqrt{\alpha}(1-\sqrt{\alpha})}\right)^{1/2} - 8\left(\frac{\sqrt{\beta}(1-\beta)}{\sqrt{\alpha}(1-\alpha)}\right)^{1/3} \tag{10.30}$$

and

$$\frac{49}{m^2(4)} = \left(\frac{\sqrt{\alpha}(1+\sqrt{\alpha})}{\sqrt{\beta}(1+\sqrt{\beta})}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\sqrt{\alpha}(1-\sqrt{\alpha})}{\sqrt{\beta}(1-\sqrt{\beta})}\right)^{1/2} - 8\left(\frac{\sqrt{\alpha}(1-\alpha)}{\sqrt{\beta}(1-\beta)}\right)^{1/3}, \tag{10.31}$$

respectively. Multiplying both sides of (10.31) by $\sqrt{\beta(1-\beta)/(\alpha(1-\alpha))}$, we deduce that

$$\frac{49}{m^2(4)}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} = \left(\frac{\sqrt{\beta}(1-\sqrt{\beta})}{\sqrt{\alpha}(1-\sqrt{\alpha})}\right)^{1/2} + \left(\frac{\beta}{\alpha}\right)^{1/2} - \left(\frac{\sqrt{\beta}(1+\sqrt{\beta})}{\sqrt{\alpha}(1+\sqrt{\alpha})}\right)^{1/2} - 8\left(\frac{\beta\sqrt{1-\beta}}{\alpha\sqrt{1-\alpha}}\right)^{1/3}. \tag{10.32}$$

Combining (10.30) and (10.32), we complete the proof of (10.27).

Theorem 10.9 (p. 261). *If $m(4)$ denotes the multiplier of degree 13, then*

$$m(4) = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \frac{13}{m(4)}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} - 4\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12}\left\{\left(\frac{\beta}{\alpha}\right)^{1/12} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/12}\right\}. \tag{10.33}$$

Proof. By Entries 8(iii), (iv) of Chapter 20 of Ramanujan's second notebook (Part III [3, p. 376]),

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} - 4\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6} \tag{10.34}$$

and

$$\frac{13}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4} - 4\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/6}. \tag{10.35}$$

Transforming (10.34) and (10.35) into the system of signature 4 by means of (10.1) and (10.10), we find, after some simplification, that

$$m(4) = \left(\frac{\sqrt{\beta}(1+\sqrt{\beta})}{\sqrt{\alpha}(1+\sqrt{\alpha})}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\sqrt{\beta}(1-\sqrt{\beta})}{\sqrt{\alpha}(1-\sqrt{\alpha})}\right)^{1/4} - 4\left(\frac{\sqrt{\beta}(1-\beta)}{\sqrt{\alpha}(1-\alpha)}\right)^{1/6} \tag{10.36}$$

and

$$\frac{13}{m(4)} = \left(\frac{\sqrt{\alpha}(1+\sqrt{\alpha})}{\sqrt{\beta}(1+\sqrt{\beta})}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\sqrt{\alpha}(1-\sqrt{\alpha})}{\sqrt{\beta}(1-\sqrt{\beta})}\right)^{1/4} - 4\left(\frac{\sqrt{\alpha}(1-\alpha)}{\sqrt{\beta}(1-\beta)}\right)^{1/6}, \tag{10.37}$$

respectively. Multiplying both sides of (10.37) by $\sqrt[4]{\beta(1-\beta)/(\alpha(1-\alpha))}$ and combining the resulting equality with (10.36), we finish the proof of (10.33).

Theorem 10.10 (p. 261). *If $m(4)$ denotes the multiplier of degree 25, then*

$$\sqrt{m(4)} = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \frac{5}{\sqrt{m(4)}}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} - 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/24}\left\{\left(\frac{\beta}{\alpha}\right)^{1/24} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/24}\right\}. \tag{10.38}$$

Proof. By Entries 15(i), (ii) of Chapter 19 of Ramanujan's second notebook (Part III [3, p. 291]),

$$\sqrt{m} = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} - 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12} \tag{10.39}$$

and

$$\frac{5}{\sqrt{m}} = \left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} - 2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12}. \tag{10.40}$$

Transforming (10.39) and (10.40) by means of (10.1) and (10.10) into equalities in the theory of signature 4, we find that

$$\sqrt{m(4)} = \left(\frac{\sqrt{\beta}(1+\sqrt{\beta})}{\sqrt{\alpha}(1+\sqrt{\alpha})}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\sqrt{\beta}(1-\sqrt{\beta})}{\sqrt{\alpha}(1-\sqrt{\alpha})}\right)^{1/8} - 2\left(\frac{\sqrt{\beta}(1-\beta)}{\sqrt{\alpha}(1-\alpha)}\right)^{1/12} \tag{10.41}$$

and

$$\frac{5}{\sqrt{m(4)}} = \left(\frac{\sqrt{\alpha}(1+\sqrt{\alpha})}{\sqrt{\beta}(1+\sqrt{\beta})}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\sqrt{\alpha}(1-\sqrt{\alpha})}{\sqrt{\beta}(1-\sqrt{\beta})}\right)^{1/8} - 2\left(\frac{\sqrt{\alpha}(1-\alpha)}{\sqrt{\beta}(1-\beta)}\right)^{1/12}, \tag{10.42}$$

respectively. Multiplying both sides of (10.42) by $\sqrt[8]{\beta(1-\beta)/(\alpha(1-\alpha))}$ and combining the resulting equality with (10.41), we finish the proof of (10.38).

11. The Theory for Signature 6

The most important theorem in this section is Theorem 11.3 below. This result allows us to employ formulas in the classical theory to prove corresponding theorems in the theory of signature 6. To prove Theorem 11.3, we need the following two results.

Theorem 11.1 (p. 262). *If*

$$\alpha := \frac{p(2+p)}{1+2p} \quad \textit{and} \quad \beta := \frac{27p^2(1+p)^2}{4(1+p+p^2)^3}, \tag{11.1}$$

then, for $0 \le p < 1$,

$$\sqrt{1+2p}\,{}_2F_1\left(\tfrac{1}{6}, \tfrac{5}{6}; 1; \beta\right) = \sqrt{1+p+p^2}\,{}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \alpha\right). \tag{11.2}$$

Proof. From Erdélyi's treatise [1, p. 114, eq. (42)],

$${}_2F_1\left(\tfrac{1}{4}, \tfrac{1}{4}; 1; z\right) = (1-z/4)^{-1/4}\,{}_2F_1\left(\tfrac{1}{12}, \tfrac{5}{12}; 1; -27z^2(z-4)^{-3}\right), \tag{11.3}$$

for z sufficiently near the origin. By Example (ii) in Section 33 of Chapter 11 (Part II [2, p. 95]),

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) = (1-z)^{-1/2}\,{}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; -\frac{4z}{(1-z)^2}\right), \tag{11.4}$$

for $|z|$ sufficiently small. Thus, combining (11.3) and (11.4), we find that

$$\begin{aligned} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) &= (1-z)^{-1/2}\left(1+\frac{z}{(1-z)^2}\right)^{-1/4} \\ &\quad \times {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{27z^2}{4(1-z)^4}\left(\frac{z}{(1-z)^2}+1\right)^{-3}\right) \\ &= (1-z+z^2)^{-1/4}\,{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{27z^2(1-z)^2}{4(1-z+z^2)^3}\right). \end{aligned} \tag{11.5}$$

Next, recall the well-known transformation (Erdélyi [1, p. 111, eq. (2)]),

$$ {}_2F_1\left(\tfrac{1}{6},\tfrac{5}{6};1;z\right) = {}_2F_1\left(\tfrac{1}{12},\tfrac{5}{12};1;4z(1-z)\right) \tag{11.6}$$

for z in some neighborhood of the origin. Examining (11.5) and (11.6) in relation to (11.2), we see that we want to solve the equation

$$4x(1-x) = \frac{27z^2(1-z)^2}{4(1-z+z^2)^3}.$$

Solving this quadratic equation in x and choosing that root which is near the origin when z is close to 0, we find that

$$\begin{aligned} x &= \frac{1}{2}\left(1 - \left\{\frac{4(1-z+z^2)^3 - 27z^2(1-z)^2}{4(1-z+z^2)^3}\right\}^{1/2}\right) \\ &= \frac{1}{2}\left(1 - \frac{(1+z)(2-5z+2z^2)}{2(1-z+z^2)^{3/2}}\right). \end{aligned} \tag{11.7}$$

Thus, by (11.5)–(11.7), we have shown that

$$\begin{aligned} &{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{1}{2}\left(1 - \frac{(1+z)(2-5z+2z^2)}{2(1-z+z^2)^{3/2}}\right)\right) \\ &\qquad = (1-z+z^2)^{1/4}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;z\right). \end{aligned} \tag{11.8}$$

Now set

$$z = \frac{p(2+p)}{1+2p}.$$

Then elementary calculations give

$$1 - z + z^2 = \frac{(1+p+p^2)^2}{(1+2p)^2}$$

and

$$1 - \frac{(1+z)(2-5z+2z^2)}{2(1-z+z^2)^{3/2}} = \frac{27p^2(1+p)^2}{2(1+p+p^2)^3}.$$

Using these calculations in (11.8), we deduce (11.2).

Lastly, α and β are monotonically increasing functions of p on $[0, 1]$ with $\alpha(0) = 0 = \beta(0)$ and $\alpha(1) = 1 = \beta(1)$. It follows that (11.2) is valid for $0 \le p < 1$.

Corollary 11.2. *Let α and β be defined by (11.1). For $0 < p \le 1$,*

$$\sqrt{1+2p}\,{}_2F_1\left(\tfrac{1}{6},\tfrac{5}{6};1;1-\beta\right) = \sqrt{1+p+p^2}\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;1-\alpha\right). \tag{11.9}$$

Proof. From (11.1),

$$1 - \alpha = \frac{1-p^2}{1+2p} \qquad \text{and} \qquad 1 - \beta = \frac{(1-p)^2(1+2p)^2(2+p)^2}{4(1+p+p^2)^3}. \tag{11.10}$$

(Recall that $1 - \beta$ was previously calculated in (5.31).) Setting $z = (1-p^2)/(1+2p)$ in (11.8), we complete the proof.

Theorem 11.3. *Let α and β be defined by (11.1). If $0 < p < 1$, then*

$$q_6 =: q_6(\beta) = q^2(\alpha) := q^2, \tag{11.11}$$

where q denotes the classical base.

Proof. Divide (11.9) by (11.2) to obtain the equality

$$\frac{{}_2F_1(\frac{1}{6},\frac{5}{6};1;1-\beta)}{{}_2F_1(\frac{1}{6},\frac{5}{6};1;\beta)} = \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}, \tag{11.12}$$

valid for $0 < p < 1$. From (11.12) and (1.9), we immediately deduce (11.11).

From Theorem 11.1, we also can deduce that

$$\begin{aligned}\sqrt{1+2p}\, z(6;\beta) :&= \sqrt{1+2p}\, z(6)\\ &= \sqrt{1+p+p^2}\, z(2) =: \sqrt{1+p+p^2}\, z(2;\alpha).\end{aligned} \tag{11.13}$$

Hence, from (11.1), (11.11), and (11.13), we derive the following principle. Suppose that we have an equality

$$\Omega\left(q^2, z(2;\alpha(p))\right) = 0$$

in the classical theory. Then in the theory of signature 6, we may deduce the corresponding equality

$$\Omega\left(q_6, \left(\frac{1+2p}{1+p+p^2}\right)^{\frac{1}{2}} z(6;\beta(p))\right) = 0.$$

We now give some applications of this principle.

Theorem 11.4 (p. 262). *We have*

$$M(q_6) = z^4(6).$$

Proof. By Entry 13(i) of Chapter 17 of Ramanujan's second notebook (Part III [3, p. 126]), (11.11), and (11.1),

$$\begin{aligned}M(q_6) = M(q^2) = z^4(1-\alpha+\alpha^2) &= z^4\left(1 - \frac{p(2+p)}{1+2p} + \frac{p^2(2+p)^2}{(1+2p)^2}\right)\\ &= z^4\left(\frac{1+p+p^2}{1+2p}\right)^2 = z^4(6),\end{aligned}$$

by (11.13).

Theorem 11.5 (p. 262). *We have*

$$N(q_6) = z^6(6)(1-2\beta).$$

Proof. By Entry 13(ii) of Chapter 17 (Part III [3, p. 126]), (11.11), and (11.1),

$$\begin{aligned}
N(q_6) = N(q^2) &= z^6(1+\alpha)(1-\tfrac{1}{2}\alpha)(1-2\alpha)\\
&= z^6\left(1+\frac{p(2+p)}{1+2p}\right)\left(1-\frac{p(2+p)}{2(1+2p)}\right)\left(1-\frac{2p(2+p)}{1+2p}\right)\\
&= z^6\frac{(1+4p+p^2)(2+2p-p^2)(1-2p-2p^2)}{2(1+2p)^3}\\
&= z^6\frac{2(1+p+p^2)^3-27p^2(1+p)^2}{2(1+2p)^3}\\
&= z^6\frac{(1+p+p^2)^3}{(1+2p)^3}\frac{2(1+p+p^2)^3-27p^2(1+p)^2}{2(1+p+p^2)^3}\\
&= z^6(6)(1-2\beta),
\end{aligned}$$

by (11.13) and (11.1).

Theorem 11.6 (p. 262). *We have*

$$q_6^{1/24}f(-q_6) = \sqrt{z(6)}\left(\frac{\beta(1-\beta)}{432}\right)^{1/24}.$$

Proof. By Entry 12(iii) of Chapter 17 (Part III [3, p. 124]), (11.11), (11.13), and (11.1),

$$\begin{aligned}
q_6^{1/24}f(-q_6) = q^{1/12}f(-q^2) &= \sqrt{z}2^{-1/3}\{\alpha(1-\alpha)\}^{1/12}\\
&= \sqrt{z(6)}\left(\frac{1+2p}{1+p+p^2}\right)^{1/4}2^{-1/3}\left(\frac{p(2+p)}{1+2p}\frac{1-p^2}{1+2p}\right)^{1/12}\\
&= \sqrt{z(6)}\frac{\{p(1-p^2)(1+2p)(2+p)\}^{1/12}}{2^{1/3}(1+p+p^2)^{1/4}}\\
&= \sqrt{z(6)}\left(\frac{27p^2(1+p)^2}{4(1+p+p^2)^3}\frac{(1-p)^2(1+2p)^2(2+p)^2}{4(1+p+p^2)^3}\right)^{1/24}\left(\frac{1}{432}\right)^{1/24}\\
&= \sqrt{z(6)}\left(\frac{\beta(1-\beta)}{432}\right)^{1/24},
\end{aligned}$$

by (11.1) and (11.10).

12. An Identity from the First Notebook and Further Hypergeometric Transformations

On page 96 in his first notebook, Ramanujan claims that

$$\varphi(e^{-\pi y/\sin(\pi h)}) = \mu\sqrt{{}_2F_1(h, 1-h; 1; x)}, \tag{12.1}$$

where "μ can be expressed in radicals of x and h" and where

$$y = \frac{{}_2F_1(h, 1-h; 1; 1-x)}{{}_2F_1(h, 1-h; 1; x)}. \tag{12.2}$$

It is unclear what Ramanujan precisely meant by the phrase, "μ can be expressed in radicals of x and h." If Ramanujan meant that μ is contained in some radical extension of the field $\mathbb{Q}(x, h)$, the field of rational functions in x and h, then his statement is false for general h. We now sketch a proof.

By Corollary (ii) in Section 2 of Chapter 17 in Ramanujan's second notebook (Part III [3, p. 90]),

$$\begin{aligned} \exp(-\pi y/\sin(\pi h)) &= x\exp(\psi(h) + \psi(1-h) + 2\gamma) \\ &\times \left\{1 + (2h^2 - 2h + 1)x + (1 - \frac{7}{2}(h - h^2) + \frac{13}{4}(h - h^2)^2)x^2 + \cdots\right\}, \end{aligned} \tag{12.3}$$

where $\psi(x)$ is the logarithmic derivative of the gamma function and γ denotes Euler's constant. It should be noted that for $h = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ the quantity $\exp(\psi(h) + \psi(1-h) + 2\gamma)$ is a rational number. In fact, from Chapter 8 of Ramanujan's second notebook (Part I [1, p. 192]),

$$\exp(2\psi(\tfrac{1}{2}) + 2\gamma) = \tfrac{1}{16},$$

$$\exp(\psi(\tfrac{1}{3}) + \psi(\tfrac{2}{3}) + 2\gamma) = \tfrac{1}{27},$$

$$\exp(\psi(\tfrac{1}{4}) + \psi(\tfrac{3}{4}) + 2\gamma) = \tfrac{1}{64},$$

and

$$\exp(\psi(\tfrac{1}{6}) + \psi(\tfrac{5}{6}) + 2\gamma) = \tfrac{1}{432}.$$

However, for $h = \frac{1}{5}$ the quantity is transcendental, for (Part I [1, p. 192]),

$$\exp(\psi(\tfrac{1}{5}) + \psi(\tfrac{4}{5}) + 2\gamma) = 5^{-5/2}\left(\frac{\sqrt{5}+1}{2}\right)^{-\sqrt{5}},$$

which is transcendental by the Gelfond–Schneider theorem. If Ramanujan's statement were true for general h, this would imply that, for $h = \frac{1}{5}$, μ could be expressed in terms of radicals over some algebraic extension field of $\mathbb{Q}$. Now $\sqrt{{}_2F_1(h, 1-h; 1; x)}$ and, by (12.3), $\varphi(e^{-\pi y/\sin(\pi h)})$ have Taylor expansions about $x = 0$. Thus, (12.1) implies that μ has a Taylor expansion about $x = 0$. But since μ can be expressed in terms of radicals in x, it follows that the coefficients in the Taylor expansion of μ are algebraic over $\mathbb{Q}$. The function $\sqrt{{}_2F_1(h, 1-h; 1; x)}$

has the same property. Hence, by equating coefficients on both sides of (12.1), we deduce that $\exp(\psi(\frac{1}{5})+\psi(\frac{4}{5})+2\gamma)$ is algebraic, which is a contradiction. Hence, Ramanujan's statement is false for general h.

However, the statement is true if we restrict the values of h to $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$. Let $\mu = \mu(x,h)$. By (1.15),

$$\mu(x, \tfrac{1}{2}) = 1.$$

We now describe $\mu(x, \frac{1}{3})$ in terms of radicals in x. Recall that $m = z_1/z_3$.

Theorem 12.1. *For* $0 < x < 1$,

$$\mu\left(x, \tfrac{1}{3}\right) = \frac{2m^{3/4}}{\sqrt{m^2+6m-3}}, \tag{12.4}$$

where

$$m = \frac{M - \sqrt{(M+2)(M-6)}}{2}, \tag{12.5}$$

and M is the root of the cubic equation

$$x = \frac{27(M+2)^2}{(M+6)^3}$$

that is greater then 6. More explicitly, M may be given as

$$\begin{aligned} M = \frac{3}{x}\Bigg(&\sqrt[3]{8x^2-36x+27+8i\sqrt{(1-x)x^3}} \\ &+\sqrt[3]{8x^2-36x+27-8i\sqrt{(1-x)x^3}}+3-2x\Bigg). \end{aligned} \tag{12.6}$$

To make explicit which cube root is taken, we rewrite (12.6) as

$$M = \frac{3}{x}\left(2\sqrt{9-8x}\,\cos\left(\tfrac{1}{3}\tan^{-1}(8\sqrt{(1-x)x^3}, 8x^2-36x+27)\right)+3-2x\right), \tag{12.7}$$

where $\tan^{-1}(\beta,\alpha)$ *is that angle θ such that $-\pi < \theta \le \pi$ and* $\arg(\alpha+i\beta)=\theta$.

Proof. By Lemma 2.9 and Theorem 2.10,

$$\mu\left(\frac{c^3(q)}{a^3(q)}, \frac{1}{3}\right) = \frac{\varphi(q)}{\sqrt{a(q)}},$$

for $0 < q < 1$. By Lemma 2.1,

$$x := \frac{c^3(q)}{a^3(q)} = \frac{27(m+1)^3(m^2-1)}{(m^2+6m-3)^3} \tag{12.8}$$

and

$$\mu\left(x, \tfrac{1}{3}\right) = \frac{\varphi(q)}{\sqrt{a(q)}} = \frac{2m^{3/4}}{\sqrt{m^2+6m-3}}, \tag{12.9}$$

which is (12.4). From (3.12), (3.15), and the definition of m, we observe that $m = m(q)$ maps the interval (0, 1) monotonically onto the interval $(1, 3)$, and if we view (12.8) as defining x in terms of m, $x = x(m)$ maps the interval $(1, 3)$ monotonically onto the interval (0, 1). We conclude that $x = c^3(q)/a^3(q)$ maps the interval (0, 1) monotonically onto itself. We see that equation (12.8) is solvable by rewriting it as

$$x = \frac{c^3(q)}{a^3(q)} = \frac{27(M+2)^2}{(M+6)^3} =: x(M), \tag{12.10}$$

where

$$M = \frac{m^2+3}{m-1}. \tag{12.11}$$

In terms of q–series, M is the sum of two theta–products, namely,

$$M = m + \frac{m+3}{m-1} = \frac{\varphi^2(q)}{\varphi^2(q^3)} + \frac{\psi^2(q^2)}{q\psi^2(q^6)},$$

by (1.17) and (2.13)–(2.15). Since $1 < m < 3$, we see that $M > 6$. We may solve the quadratic equation (12.11) for m. The solution is given above in (12.5). We have taken the negative square root since $1 < m < 3$. Since (12.10) is a cubic equation in M, we may solve it in terms of radicals. Note that $x(-3) = x(6) = 1$, $x(-2) = 0$, and

$$\frac{dx}{dM} = -\frac{27(M+2)(M-6)}{(M+6)^4}.$$

Thus, $x(M)$ decreases from 1 to 0 on $(-3, -2)$, increases from 0 to 1 on $(-2, 6)$, and decreases from 1 to 0 on $(6, \infty)$. Therefore, if $0 < x < 1$, equation (12.10) has three real roots, and each of the intervals $(-3, -2)$, $(-2, 6)$, and $(6, \infty)$ contains exactly one root. Since $M > 6$, we must take M to be the largest of the roots. The solution is given in terms of radicals in (12.6) above and more explicitly in (12.7).

For $h = \frac{1}{4}$, μ has a particularly simple form.

Theorem 12.2. *For* $0 < x < 1$,

$$\mu\left(x, \tfrac{1}{4}\right) = \left(\frac{1+\sqrt{1-x}}{2}\right)^{1/4}. \tag{12.12}$$

Proof. With $h = 1/r$, we recall that

$$q_r = q_r(x) = \exp\left(-\pi y / \sin(\pi h)\right), \tag{12.13}$$

where y is defined by (12.2). We shall prove that

$$q_4\left(\left(\frac{8\sqrt{q}\psi^2(q^2)\varphi^2(q)}{\varphi^4(q)+16q\psi^4(q^2)}\right)^2\right) = q. \tag{12.14}$$

If $0 < q < 1$, then

$$q = \exp(-\pi t) \tag{12.15}$$

for some unique $t > 0$. Set

$$x := \left(\frac{8\sqrt{q}\psi^2(q^2)\varphi^2(q)}{\varphi^4(q) + 16q\psi^4(q^2)}\right)^2. \tag{12.16}$$

In view of (12.15), we consider $x = x(t)$ as a function of t. Define

$$z(t) = \varphi^4(q) + 16q\psi^4(q^2). \tag{12.17}$$

Then, by Lemma 9.14,

$${}_2F_1(\tfrac{1}{4}, \tfrac{3}{4}; 1; x(t)) = \sqrt{z(t)}. \tag{12.18}$$

We will prove that

$$x(2/t) = 1 - x(t) \tag{12.19}$$

and

$$z(2/t) = \tfrac{1}{2}t^2 z(t). \tag{12.20}$$

We first show that (12.19) and (12.20) imply (12.14). To that end, by (12.2), (12.19), (12.18), and (12.20),

$$\begin{aligned} y &= \frac{{}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; 1 - x(t))}{{}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; x(t))} \\ &= \frac{{}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; x(2/t))}{{}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; x(t))} = \sqrt{\frac{z(2/t)}{z(t)}} = \frac{t}{\sqrt{2}}. \end{aligned} \tag{12.21}$$

Hence, by (12.13), (12.21), and (12.15),

$$q_4(x(t)) = \exp(-\pi\sqrt{2}y) = \exp(-\pi t) = q,$$

which is (12.14). It remains to prove (12.19) and (12.20).

We will need the transformation formulas (Part III [3, p. 43, Entry 27(ii), (i)])

$$2e^{-\pi/(4t)}\psi(e^{-2\pi/t}) = \sqrt{t}\varphi(-e^{-\pi t}) \tag{12.22}$$

and

$$\varphi(e^{-\pi/t}) = \sqrt{t}\varphi(e^{-\pi t}). \tag{12.23}$$

Thus, by (12.23), (12.22), (2.1), (9.31), and (9.25),

$$\begin{aligned} z(2/t) &= \tfrac{1}{4}t^2\left(\varphi^4(\sqrt{q}) + \varphi^4(-\sqrt{q})\right) \\ &= \tfrac{1}{4}t^2\left(2\varphi^4(\sqrt{q}) - \left(\varphi^4(\sqrt{q}) - \varphi^4(-\sqrt{q})\right)\right) \\ &= \tfrac{1}{4}t^2\left(2\varphi^4(\sqrt{q}) - 16\sqrt{q}\psi^4(q)\right) \\ &= \tfrac{1}{4}t^2\left(2\left(\varphi^2(q) + 4\sqrt{q}\psi^2(q^2)\right)^2 - 16\sqrt{q}\psi^2(q^2)\varphi^2(q)\right) \end{aligned}$$

$$= \tfrac{1}{2}t^2\left(\varphi^4(q) + 16q\psi^4(q^2)\right)$$
$$= \tfrac{1}{2}t^2 z(t),$$

by (12.17). This proves (12.20).

Next, define

$$w(t) := 64q\psi^4(q^2)\varphi^4(q) = 64q\psi^8(e^{-\pi t}), \tag{12.24}$$

by (9.25). Hence, by (12.16) and (12.17),

$$x(t) = \frac{w(t)}{z^2(t)}. \tag{12.25}$$

Now, by (12.24), (12.22), and (2.1),

$$\begin{aligned} w(2/t) &= \tfrac{1}{4}t^4\varphi^8(-q) \\ &= \tfrac{1}{4}t^4\left(\varphi^4(q) - 16q\psi^4(q^2)\right)^2 \\ &= \tfrac{1}{4}t^4\left(\left(\varphi^4(q) + 16q\psi^4(q^2)\right)^2 - 64q\psi^4(q^2)\varphi^4(q)\right) \\ &= \tfrac{1}{4}t^4\left(z^2(t) - w(t)\right), \end{aligned} \tag{12.26}$$

by (12.17) and (12.24). Hence, by (12.25), (12.26), and (12.20),

$$x(2/t) = \frac{w(2/t)}{z^2(2/t)} = \frac{z^2(t) - w(t)}{z^2(t)} = 1 - x(t),$$

which is (12.19).

Hence, from (12.1), (12.14), and (12.18),

$$\mu\left(\left(\frac{8\sqrt{q}\psi^2(q^2)\varphi^2(q)}{\varphi^4(q) + 16q\psi^4(q^2)}\right)^2, \frac{1}{4}\right) = \frac{\varphi(q)}{\left(\varphi^4(q) + 16q\psi^4(q^2)\right)^{1/4}}. \tag{12.27}$$

If we let

$$\alpha = \alpha(q) = \frac{16q\psi^4(q^2)}{\varphi^4(q)},$$

then

$$x = \left(\frac{8\sqrt{q}\psi^2(q^2)\varphi^2(q)}{\varphi^4(q) + 16q\psi^4(q^2)}\right)^2 = \frac{4\alpha}{(1+\alpha)^2}, \tag{12.28}$$

and so, by (12.27),

$$\mu\left(x, \tfrac{1}{4}\right) = \frac{1}{(1+\alpha)^{1/4}}. \tag{12.29}$$

By (2.1) and (5.1),

$$\alpha = \frac{16q\psi^4(q^2)}{\varphi^4(q)} = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)} = 1 - \frac{(q;q^2)_\infty^8}{(-q;q^2)_\infty^8}. \tag{12.30}$$

Thus, $\alpha(q)$ maps the unit interval $(0, 1)$ monotonically onto itself. If we define x as a function of α or q via (12.28), then $x(\alpha)$ and hence $x(q)$ map the unit interval $(0, 1)$ monotonically onto itself. Hence, given $0 < x < 1$,

$$\alpha = \frac{2 - x - 2\sqrt{1-x}}{x},$$

since $0 < \alpha < 1$. An easy calculation gives

$$\frac{1}{1+\alpha} = \frac{1+\sqrt{1-x}}{2},$$

and putting this in (12.29), we deduce (12.12).

We now relate $\mu(x, \frac{1}{6})$ in terms of radicals.

Theorem 12.3. *If* $0 < x < 1$, *then*

$$\mu\left(x, \tfrac{1}{6}\right) = \frac{\sqrt{\sqrt{1+2p} + \sqrt{1-p^2}}}{\sqrt{2}\,(1+p+p^2)^{1/4}},$$

where

$$p = \frac{-1+\sqrt{1+4y}}{2}, \tag{12.31}$$

and y *is the root of the cubic equation*

$$x = \frac{27y^2}{4(1+y)^3},$$

which is between 0 *and* 2. *Explicitly,* y *is given by*

$$y = \frac{1}{4x}\left(6\sqrt{9-8x}\,\cos\left(\frac{1}{3}\tan^{-1}(8\sqrt{(1-x)x^3}, 8x^2 - 36x + 27) - \frac{2\pi}{3}\right)\right.$$
$$\left. + 9 - 4x\right). \tag{12.32}$$

Proof. Let $M(q)$ and $N(q)$ be the Eisenstein series defined at the beginning of Section 4. Let

$$\gamma = \gamma(q) = \frac{\varphi^4(-\sqrt{q})}{\varphi^4(\sqrt{q})}, \tag{12.33}$$

and define

$$p = p(q) = \sqrt{1 - \gamma + \gamma^2} - \gamma. \tag{12.34}$$

We note that both $\gamma(q)$ and $p(q)$ map the unit interval $(0, 1)$ monotonically onto itself. Recall from Entries 13(i), (ii) of Chapter 17 (Part III [3, p. 126]) that

$$M(q^2) = \varphi^8(q)(1 - \alpha + \alpha^2)$$

and

$$N(q^2) = \varphi^{12}(q)(1+\alpha)(1-\tfrac{1}{2}\alpha)(1-2\alpha),$$

where, by (5.18) and (5.19),

$$\alpha = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}.$$

Replace q by $\sqrt{q}$, so that α is replaced by $1-\gamma$. We then deduce that

$$M(q) = \varphi^8(\sqrt{q})(1-\gamma+\gamma^2) \tag{12.35}$$

and

$$N(q) = \frac{\varphi^{12}(\sqrt{q})}{2}(2-\gamma)(2\gamma-1)(1+\gamma). \tag{12.36}$$

From (12.34),

$$\gamma = \frac{1-p^2}{1+2p}, \tag{12.37}$$

so that

$$\sqrt{1-\gamma+\gamma^2} = \frac{1+p+p^2}{1+2p}. \tag{12.38}$$

It is interesting to note that, by (2.1),

$$\frac{16\sqrt{q}\psi^4(q)}{\varphi^4(\sqrt{q})} = 1-\gamma = \frac{p(2+p)}{1+2p}, \tag{12.39}$$

which gives an identification for α in (11.1). Thus, by (12.34)–(12.38),

$$\begin{aligned} x := \frac{M^{3/2}(q)-N(q)}{2M^{3/2}(q)} &= \frac{(\gamma-2)(2\gamma-1)(1+\gamma)}{4(1-\gamma+\gamma^2)^{3/2}} + \frac{1}{2} \\ &= \frac{27p^2(1+p)^2}{4(1+p+p^2)^3}, \end{aligned} \tag{12.40}$$

which gives an identification for β in (11.1). Hence, from Theorem 11.1, (12.40), (12.39), (12.38), (12.35), (5.18), and (5.19),

$$\begin{aligned} {}_2F_1^4\left(\frac{1}{6},\frac{5}{6};1;\frac{M^{3/2}(q)-N(q)}{2M^{3/2}(q)}\right) &= (1-\gamma+\gamma^2)\,{}_2F_1^4\left(\frac{1}{2},\frac{1}{2};1;1-\gamma\right) \\ &= \frac{M(q)}{\varphi^8(\sqrt{q})}\varphi^8(\sqrt{q}) = M(q). \end{aligned} \tag{12.41}$$

We now prove that

$$q_6\left(\frac{M^{3/2}(q)-N(q)}{2M^{3/2}(q)}\right) = q, \tag{12.42}$$

where $q_6(x)$ is defined by (12.13) or (1.9). It is well known that (Rankin [1, p. 198]) $M(q)$ and $N(q)$ (with $q = \exp(2\pi i\tau)$) are modular forms of weights 4 and 6, respectively, and multiplier systems identically equal to 1 on the full modular group. Hence, for $\operatorname{Im}(\tau) > 0$,

$$M(e^{-2\pi i/\tau}) = \tau^4 M(e^{2\pi i\tau}) \tag{12.43}$$

and

$$N(e^{-2\pi i/\tau}) = \tau^6 N(e^{2\pi i\tau}). \tag{12.44}$$

As in the previous proof, we set

$$q = \exp(-2\pi t) \tag{12.45}$$

for $t > 0$. Then (12.43) and (12.44) become

$$M(e^{-2\pi/t}) = t^4 M(e^{-2\pi t}) \tag{12.46}$$

and

$$N(e^{-2\pi/t}) = -t^6 N(e^{-2\pi t}). \tag{12.47}$$

With x defined by (12.40), we consider x as a function of t. Set $z(t) := M(q)$, so that, from (12.46),

$$z(1/t) = t^4 z(t), \tag{12.48}$$

and, from (12.41),

$${}_2F_1(\tfrac{1}{6}, \tfrac{5}{6}; 1; x(t)) = (z(t))^{1/4}. \tag{12.49}$$

By (12.40), (12.46), and (12.47),

$$x(1/t) = 1 - x(t). \tag{12.50}$$

Hence, from (12.2), (12.50), (12.49), and (12.48),

$$\begin{aligned} y &= \frac{{}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; 1 - x(t))}{{}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; x(t))} \\ &= \frac{{}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; x(1/t))}{{}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; x(t))} = \left(\frac{z(1/t)}{z(t)}\right)^{1/4} = t. \end{aligned} \tag{12.51}$$

Therefore, by (12.13), (12.51), and (12.45),

$$q_6(x(t)) = \exp(-2\pi y) = \exp(-2\pi t) = q,$$

which is (12.42).

It follows from (12.42), (12.41), and (12.1) that

$$\mu\left(\frac{M^{3/2}(q) - N(q)}{2M^{3/2}(q)}, \frac{1}{6}\right) = \frac{\varphi(q)}{M^{1/8}(q)}. \tag{12.52}$$

From Entry 25(vi) of Chapter 16 (Part III [3, p. 40]),

$$\varphi^2(q) = \tfrac{1}{2}\left(\varphi^2(\sqrt{q}) + \varphi^2(-\sqrt{q})\right),$$

and so, by (12.33),

$$\frac{\varphi^2(q)}{\varphi^2(\sqrt{q})} = \tfrac{1}{2}(1+\sqrt{\gamma}).$$

Hence, from (12.35), we find that

$$M(q) = \varphi^8(q)\left(\frac{2}{1+\sqrt{\gamma}}\right)^4 (1-\gamma+\gamma^2). \tag{12.53}$$

From (12.40), (12.52), (12.53), (12.37), and (12.38), we find that

$$\mu\left(x, \frac{1}{6}\right) = \frac{\varphi(q)}{M^{1/8}(q)} = \frac{\sqrt{\frac{1+\sqrt{\gamma}}{2}}}{(1-\gamma+\gamma^2)^{1/8}} = \frac{\sqrt{\sqrt{1+2p}+\sqrt{1-p^2}}}{\sqrt{2}\,(1+p+p^2)^{1/4}},$$

as claimed.

We now rewrite (12.40) as

$$x := x(y) := \frac{27y^2}{4(1+y)^3}, \tag{12.54}$$

where

$$y = p(1+p). \tag{12.55}$$

Since $0 < p < 1$, we have $0 < y < 2$. Solving (12.55) for p, we deduce (12.31). We have taken the positive square root since $0 < p < 1$. Since (12.54) is a cubic equation in y, we may solve it in terms of radicals. Note that $x(-\frac{1}{4}) = x(2) = 1, x(0) = 0$, and

$$\frac{dx}{dy} = -\frac{27y(y-2)}{4(1+y)^4}.$$

Thus, $x(y)$ decreases from 1 to 0 on $(-\frac{1}{4}, 0)$, increases from 0 to 1 on $(0, 2)$, and decreases from 1 to 0 on $(2, \infty)$. Therefore, if $0 < x < 1$, equation (12.54) has three real roots, and each of the intervals $(-\frac{1}{4}, 0)$, $(0, 2)$, and $(2, \infty)$ contains exactly one root. We take y to be the unique root that satisfies $0 < y < 2$. The root y is given explicitly in (12.32) above.

Some consequences of our derivations of the values of $\mu(x, h)$ are some new hypergeometric transformations. From (2.13) and (2.14), we see that

$$\alpha = \frac{16q\psi^4(q^2)}{\varphi^4(q)} = \frac{(m-1)(m+3)^3}{16m^3}. \tag{12.56}$$

Hence, from (12.56), (12.30), (5.18), (5.19), (12.9), Lemma 2.6, and (12.8),

$$\begin{aligned}{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(m-1)(m+3)^3}{16m^3}\right) &= {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\frac{\varphi^4(-q)}{\varphi^4(q)}\right) \\ = \varphi^2(q) &= \frac{4m^{3/2}}{m^2+6m-3}a(q)\end{aligned}$$

$$= \frac{4m^{3/2}}{m^2+6m-3}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{27(m+1)^3(m^2-1)}{(m^2+6m-3)^3}\right). \tag{12.57}$$

This transformation is remindful of Theorem 5.6 but is a different transformation. We have found a generalization of (12.57) via MAPLE, namely,

$$\begin{aligned}{}_2F_1&\left(3d,2d+\frac{1}{6};d+\frac{5}{6};\frac{(m-1)(m+3)^3}{16m^3}\right)\\&=\left(\frac{4m^{3/2}}{m^2+6m-3}\right)^{6d}{}_2F_1\left(2d,2d+\frac{1}{3};d+\frac{5}{6};\frac{27(m+1)^3(m^2-1)}{(m^2+6m-3)^3}\right).\end{aligned} \tag{12.58}$$

We omit the details. We have also found a generalization of Theorem 5.6 via MAPLE, viz.,

$$\begin{aligned}{}_2F_1&\left(3d,d+\frac{1}{3};2d+\frac{2}{3};\frac{p^3(2+p)}{1+2p}\right)\\&=\left(\frac{1+2p}{(1+p+p^2)^2}\right)^{3d}{}_2F_1\left(2d,2d+\frac{1}{3};3d+\frac{1}{2};\frac{27p^2(1+p)^2}{4(1+p+p^2)^3}\right).\end{aligned} \tag{12.59}$$

We have investigated the connection between (12.57) and Theorem 5.6 and found that (12.57) follows from Theorem 5.6 in the following way. Replace m by $1+2p$, so that (12.57) can be rewritten as

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{p(2+p)^3}{(1+2p)^3}\right)=\frac{(1+2p)^{3/2}}{1+4p+p^2}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{27p(1+p)^4}{2(1+4p+p^2)^3}\right). \tag{12.60}$$

Also, replace p by $2(1+p+p^2)/(2+2p-p^2)-1$ in Theorem 6.1 to obtain the identity

$$\begin{aligned}2(1+p+p^2)\,{}_2F_1&\left(\frac{1}{3},\frac{2}{3};1;\frac{27p^4(1+p)}{2(2+2p-p^2)^3}\right)\\&=(2+2p-p^2)\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{27p^2(1+p)^2}{4(1+p+p^2)^3}\right).\end{aligned} \tag{12.61}$$

We now indicate the steps that lead from Theorem 5.6 to (12.60). First, apply Entry 6(i) of Chapter 17 of the second notebook (Part III [3, p. 238]), i.e.,

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{p(2+p)^3}{(1+2p)^3}\right)=(1+2p)\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{p^3(2+p)}{1+2p}\right),$$

to the left side of (12.60). Second, apply Theorem 5.6. Third, invoke (12.61). Last, after employing Theorem 6.4, we deduce (12.60).

It is interesting to note that (12.61) and Theorem 5.6 give the transformation

$$(2+2p-p^2)\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{p^3(2+p)}{1+2p}\right) = 2\sqrt{1+2p}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{27p^4(1+p)}{2(2+2p-p^2)^3}\right). \tag{12.62}$$

We have found a generalization of (12.62) via MAPLE that is different from (12.58) and (12.59), namely,

$$\begin{aligned}&{}_2F_1\left(3d,2d+\frac{1}{6};4d+\frac{1}{3};\frac{p^3(2+p)}{1+2p}\right)\\ &=\left(\frac{4(1+2p)}{(2+2p-p^2)^2}\right)^{3d}{}_2F_1\left(2d,2d+\frac{1}{3};3d+\frac{1}{2};\frac{27p^4(1+p)}{2(2+2p-p^2)^3}\right).\end{aligned} \tag{12.63}$$

Using Theorems 4.2 and 4.3 in (12.41) and then employing Theorem 2.10, we find that

$$(1+8x)\,{}_2F_1^4\left(\frac{1}{3},\frac{2}{3};1;x\right) = {}_2F_1^4\left(\frac{1}{6},\frac{5}{6};1;\frac{1}{2}-\frac{(1-20x-8x^2)}{2(1+8x)^{\frac{3}{2}}}\right).$$

On replacing x by $(x^2-1)/8$ we find that

$${}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{(x-1)(x+3)^3}{16x^3}\right) = \sqrt{x}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{(x-1)(x+1)}{8}\right). \tag{12.64}$$

We have found a generalization via MAPLE, namely,

$$\begin{aligned}&{}_2F_1\left(d,d+\frac{2}{3};d+\frac{5}{6};\frac{(x-1)(x+3)^3}{16x^3}\right)\\ &=x^{3d}\,{}_2F_1\left(2d,2d+\frac{1}{3};d+\frac{5}{6};\frac{(x-1)(x+1)}{8}\right).\end{aligned} \tag{12.65}$$

Equation (12.64) has an elegant q–version; if we replace x by m in (12.64) and use (12.56) and Lemma 5.5, we find that

$$\varphi(q^3)\,{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{16q\psi^4(q^2)}{\varphi^4(q)}\right) = \varphi(q)\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;-\frac{c^3(q^2)}{c^3(-q)}\right). \tag{12.66}$$

13. Some Enigmatic Formulas Near the End of the Third Notebook

Near the bottom of page 392 (or possibly at the top of page 392, since the page is reproduced upside down in the published notebooks [9]), Ramanujan recorded the following claims:

$$1+5\cdot\frac{1}{2}\cdot\frac{1\cdot 3}{4^2}4x(1-x)+\cdots$$
$$=\frac{1}{1-2x}\left(1-\left[\frac{4}{y}\right]-8\sum_{n=1}^{\infty}\frac{n}{e^{ny}-1}-16\sum_{n=1}^{\infty}\frac{n}{e^{2ny}-1}\right)\tag{13.1}$$

and

$$1+4\cdot\frac{1}{2}\cdot\frac{1\cdot 2}{3^2}4x(1-x)+\cdots$$
$$=\frac{1}{1-2x}\left(1-\left[\frac{3}{y}\right]-6\sum_{n=1}^{\infty}\frac{n}{e^{ny}-1}-18\sum_{n=1}^{\infty}\frac{n}{e^{3ny}-1}\right).\tag{13.2}$$

These two claims were very difficult for us to interpret. First, Ramanujan did not provide enough terms on the left sides to determine a general term in either case, and it would appear that the series on the left sides do not converge for any x, except trivially for $x = 0, 1$. Second, although y is not defined, it is reasonable to guess (from Chapter 17) that

$$y=\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}.\tag{13.3}$$

Third, Ramanujan had never before used the notation [ ], and so we did not know if $[4/y] = 4/y$ and $[3/y] = 3/y$, or if some other meaning should be attached to the notation [ ]. (Ramanujan never used [ ] to denote the greatest integer function.) Fourth, by (13.3), the right sides of (13.1) and (13.2) have singularities at $x = 0, 1$, but if the left sides are convergent series, they must be analytic at $x = 0, 1$.

Eventually, we determined that the nth terms of the series on the left sides of (13.1) and (13.2) have $(n!)^3$ in the denominators, which is not evident from Ramanujan's formulations. The conjectured definition of y in (13.3) is incorrect, and the expressions $[4/y]$ and $[3/y]$ indicate that the results belong to Ramanujan's alternative theories of signatures 4 and 3, respectively. (See (1.8) and (1.7), respectively, for the definitions of y in these theories.) Moreover, the expressions $[4/y]$ and $[3/y]$ should be deleted from (13.1) and (13.2), respectively. For some inexplicable reason, Ramanujan indicated that his identities arose from the two alternative theories by placing the "symbols" $[4/y]$ and $[3/y]$ in the midst of the formulas.

We now offer precise renditions of (13.1) and (13.2).

Entry 13.1 (p. 392). *Let* $0 < x < 1$. *If* y *is given by (1.8) and* $z = {}_2F_1(\frac{1}{4},\frac{3}{4};1;x)$, *then*

$$\sum_{n=0}^{\infty}\frac{(4n+1)(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3}\{4x(1-x)\}^n$$

$$= \frac{1}{1-2x}\left(1 - 8\sum_{n=1}^{\infty}\frac{n}{e^{ny}-1} - 16\sum_{n=1}^{\infty}\frac{n}{e^{2ny}-1}\right). \tag{13.4}$$

If y is given by (1.7) and $z = {}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; x)$, then

$$\sum_{n=0}^{\infty}\frac{(3n+1)(\frac{1}{2})_n(\frac{1}{3})_n(\frac{2}{3})_n}{(n!)^3}\{4x(1-x)\}^n$$
$$= \frac{1}{1-2x}\left(1 - 6\sum_{n=1}^{\infty}\frac{n}{e^{ny}-1} - 18\sum_{n=1}^{\infty}\frac{n}{e^{3ny}-1}\right). \tag{13.5}$$

Proof. We first prove (13.4). Recall that

$$L(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \tag{13.6}$$

where $q = e^{-y}$. From the definition (13.6), it is easy to see that

$$1 - 8\sum_{n=1}^{\infty}\frac{n}{e^{ny}-1} - 16\sum_{n=1}^{\infty}\frac{n}{e^{2ny}-1} = \frac{1}{3}L(q) + \frac{2}{3}L(q^2). \tag{13.7}$$

From the proof of Theorem 9.11,

$$L(q) = (1-3x)z^2 + 12x(1-x)z\frac{dz}{dx} \tag{13.8}$$

and

$$L(q^2) = \tfrac{1}{2}(2-3x)z^2 + 6x(1-x)z\frac{dz}{dx}. \tag{13.9}$$

Thus, by (13.7)–(13.9), it suffices to prove that

$$\sum_{n=0}^{\infty}\frac{(4n+1)(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3}\{4x(1-x)\}^n$$
$$= \frac{1}{1-2x}\left((1-2x)z^2 + 8x(1-x)z\frac{dz}{dx}\right). \tag{13.10}$$

In Clausen's formula, Entry 13 of Chapter 11 (Part II [2, p. 58]), put $\alpha = -\frac{1}{8}$, $\beta = -\frac{3}{8}$, and $\gamma = \frac{1}{2}$ to deduce that

$${}_2F_1^2(\tfrac{1}{8}, \tfrac{3}{8}; 1; x) = {}_3F_2(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{2}; 1, 1; x). \tag{13.11}$$

In Entry 12 of Chapter 11 (Part II [2, p. 56]), set $x = -\frac{1}{8}$, $y = -\frac{3}{8}$, $z = 1$, and $p = 4x(1-x)$. Accordingly,

$${}_2F_1(\tfrac{1}{8}, \tfrac{3}{8}; 1; 4x(1-x)) = {}_2F_1(\tfrac{1}{4}, \tfrac{3}{4}; 1; x). \tag{13.12}$$

Thus, from (13.11) and (13.12),

$${}_2F_1^2(\tfrac{1}{4}, \tfrac{3}{4}; 1; x) = {}_3F_2(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{2}; 1, 1; 4x(1-x)). \tag{13.13}$$

Using (13.13), we see that (13.10) is equivalent to the identity

$$\sum_{n=1}^{\infty} \frac{n(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3} \{4x(1-x)\}^{n-1} = \frac{z}{2(1-2x)} \frac{dz}{dx}. \tag{13.14}$$

Differentiating (13.13), we find that

$$2z\frac{dz}{dx} = \sum_{n=1}^{\infty} \frac{n(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3} \{4x(1-x)\}^{n-1}(4-8x),$$

which is readily seen to be equivalent to (13.14), and so the proof of (13.4) is complete.

The proof of (13.5) is similar. First, from (13.6), we easily find that

$$1 - 6\sum_{n=1}^{\infty} \frac{n}{e^{ny}-1} - 18\sum_{n=1}^{\infty} \frac{n}{e^{3ny}-1} = \frac{1}{4}L(q) + \frac{3}{4}L(q^3). \tag{13.15}$$

From Lemma 4.1,

$$L(q) = (1-4x)z^2 + 12x(1-x)z\frac{dz}{dx}, \tag{13.16}$$

where $z = {}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; x)$. We need a similar representation for $L(q^3)$. To that end, from Corollary 3.5 and (4.4), we find that

$$\begin{aligned} L(q^3) &= \frac{1}{3}q\frac{d}{dq}\log\left(q^3 f^{24}(-q)\right) \\ &= \frac{1}{3}q\frac{d}{dx}\log\left(z^{12}3^{-9}x^3(1-x)\right)\frac{dx}{dq} \\ &= \frac{1}{3}q\left(\frac{12}{z}\frac{dz}{dx} + \frac{3}{x} - \frac{1}{1-x}\right)\frac{x(1-x)z^2}{q} \\ &= 4x(1-x)z\frac{dz}{dx} + \left(1 - \tfrac{4}{3}x\right)z^2. \end{aligned} \tag{13.17}$$

Hence, from (13.15)–(13.17),

$$\tfrac{1}{4}L(q) + \tfrac{3}{4}L(q^3) = (1-2x)z^2 + 6x(1-x)z\frac{dz}{dx}. \tag{13.18}$$

Thus, by (13.5), (13.15), and (13.18), it is sufficient to prove that

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(3n+1)(\frac{1}{2})_n(\frac{1}{3})_n(\frac{2}{3})_n}{(n!)^3} \{4x(1-x)\}^n \\ &= \frac{1}{1-2x}\left((1-2x)z^2 + 6x(1-x)z\frac{dz}{dx}\right). \end{aligned} \tag{13.19}$$

Proceeding as in the proof of (13.4), we put $\alpha = -\frac{1}{6}$, $\beta = -\frac{1}{3}$, and $\gamma = \frac{1}{2}$ in Clausen's formula, Entry 13 of Chapter 11 (Part II [2, p. 58]). Then we set

$x = -\frac{1}{6}, y = -\frac{1}{3}, z = 1$, and $p = 4x(1-x)$ in Entry 12 of Chapter 11 (Part II [2, p. 56]). Combining these two formulas together, we deduce that

$$z^2 = {}_3F_2(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{2}; 1, 1; 4x(1-x)). \tag{13.20}$$

Using (13.20) in (13.19), we find that it suffices to prove that

$$\sum_{n=1}^{\infty} \frac{n(\frac{1}{2})_n(\frac{1}{3})_n(\frac{2}{3})_n}{(n!)^3}\{4x(1-x)\}^{n-1} = \frac{z}{2(1-2x)}\frac{dz}{dx}. \tag{13.21}$$

Differentiating (13.20), we achieve (13.21) to complete the proof.

At the very bottom of page 392, Ramanujan [9] wrote

$$\begin{aligned} 1 &+ \frac{1 \cdot 2}{3^2}t + \cdots \\ &\{t(1-t)\}^{1/3} = \frac{2}{3}\frac{2-(GG')^{1/4}}{(GG')^{1/6}} = \frac{2}{3}\frac{2+(gg')^{1/4}}{(gg')^{1/6}}. \end{aligned} \tag{13.22}$$

The notations G, G', g, and g' were not defined by Ramanujan. In view of the appearance of ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; t)$, it would appear that the latter two equalities pertain to a new (unknown) type of class invariant associated with the theory of signature 3. Chapter 34 is devoted to class invariants, and in the table of class invariants in his second notebook, Ramanujan uses the notations $G := G_n$ and $g := g_n$. However, no dependence on n is indicated in (13.22). Also, the appearance of two invariants in each equality should be reflected in the appearance of two distinct moduli on the left side. Thus, we are left with the conclusion that Ramanujan is claiming two q–series identities, one for the pair G, G', and the other for the pair g, g', whatever these "invariants" might be.

Now, by Corollary 3.2 and Lemma 5.1,

$$\begin{aligned} \{t(1-t)\}^{1/3} &= \frac{b(q)}{a(q)}\frac{c(q)}{a(q)} = \frac{3q^{1/3}f^2(-q)f^2(-q^3)}{a^2(q)} \\ &= q^{1/3}\left(3 - 42q + 393q^2 - 3240q^3 + \cdots\right), \end{aligned} \tag{13.23}$$

where we employed *Mathematica* to obtain the q–expansion. Ramanujan, in his second notebook, defined G by

$$G^{1/24} = \frac{2^{1/4}q^{1/24}}{(-q;q^2)_\infty},$$

and later, in his paper [3], he gave the different definition

$$G = 2^{1/4}q^{-1/24}(-q;q^2)_\infty.$$

Using either definition, and any similar representation for G', we do not obtain an expansion for the middle expression in (13.22) that is close to that in (13.23).

More generally, suppose that

$$GG' = 2^a(1 + bq^c + \cdots).$$

Then

$$\frac{2}{3}\frac{2 - (GG')^{1/4}}{(GG')^{1/6}} = \frac{2}{3}\frac{2 - 2^{a/4}(1 + (b/4)q^c + \cdots)}{2^{a/6}(1 + (b/6)q^c + \cdots)}.$$

Thus, clearly, if (13.22) holds, we must have $a = 4$. It then further follows that $c = \frac{1}{3}$ and $b = -9 \cdot 2^{2/3}$. We are unable to use the resulting expansion

$$GG' = 16(1 - 9 \cdot 2^{2/3}q^{1/3} + \cdots)$$

to identify GG' in any meaningful way.

14. Concluding Remarks

It seems inconceivable that Ramanujan could have developed the theory of signature 3 without being aware of the cubic theta–function identity (2.5), and in Lemma 2.1 and Theorem 2.2 we showed how (2.5) follows from results of Ramanujan. H. H. Chan [1] has found a much shorter proof of (2.5) based upon results found in Ramanujan's notebooks.

H. F. Farkas and I. Kra [1, p. 124] have discovered two cubic theta–function identities different from that found by the Borweins. Let $\omega = \exp(2\pi i/6)$. Then, in Ramanujan's notation,

$$\omega^2 f^3\left(\omega q^{1/3}, \bar{\omega} q^{2/3}\right) + f^3\left(\omega q^{2/3}, \bar{\omega} q^{1/3}\right) = \omega f^3\left(-q^{1/3}, -q^{2/3}\right)$$

and

$$f^3\left(\omega q^{2/3}, \bar{\omega} q^{1/3}\right) - f^3\left(\omega q^{1/3}, \bar{\omega} q^{2/3}\right) = q^{1/3} f^3(\omega q, \bar{\omega}).$$

H. F. Farkas and Y. Kopeliovich [1] have generalized this to a pth order identity. Garvan [2] has recently found elementary proofs of the cubic identity and the pth order identities, and has found more general relations.

Almost all of the results on pages 257–262 in Ramanujan's second notebook devoted to his alternative theories are found in the first notebook, but they are scattered. In particular, they can be found on pages 96, 162, 204, 210, 212, 214, 216, 218, 220, 242, 300, 301, 310, and 328 of the first notebook. Moreover, Theorem 9.11 is only found in the first notebook.

In Section 8, we crucially used properties of $b(z, q)$, a two variable analogue of $b(q)$. The theory of two variable analogues of $a(q)$, $b(q)$, and $c(q)$ has been extensively developed by Hirschhorn, Garvan, and J. M. Borwein [1] and by S. Bhargava [1].

Some of Ramanujan's formulas for Eisenstein series in this chapter were also established by Venkatachaliengar [1].

In [3], Garvan describes how the computer algebra package MAPLE was used to understand, prove, and generalize some of the results in this chapter.

Small portions of the material in this chapter have been described in expository lectures by Berndt [10] and Garvan [1].

34

Class Invariants and Singular Moduli

1. Introduction

So that we may define Ramanujan's class invariants, set

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1,$$

and

$$\chi(q) = (-q;q^2)_\infty. \tag{1.1}$$

If

$$q = \exp(-\pi\sqrt{n}), \tag{1.2}$$

where n is a positive rational number, the two *class invariants* G_n and g_n are defined by

$$G_n := 2^{-1/4}q^{-1/24}\chi(q) \qquad \text{and} \qquad g_n := 2^{-1/4}q^{-1/24}\chi(-q). \tag{1.3}$$

In the notation of Weber [2], $G_n =: 2^{-1/4}\mathfrak{f}(\sqrt{-n})$ and $g_n =: 2^{-1/4}\mathfrak{f}_1(\sqrt{-n})$. The definition of G_n employed by Ramanujan in his paper [3], [10, pp. 23–39] is not the same as that used by him in his notebooks [9], while his definition of g_n in [3] is that used in his first notebook but not in his second notebook. More precisely, If we replace G and g in the second notebook by H and h, respectively, the relations between the definitions are given by

$$G_n^{24} = \frac{1}{H_{\sqrt{n}}} \qquad \text{and} \qquad g_n^{24} = \frac{1}{h_{\sqrt{n}}}.$$

As usual, in the theory of elliptic functions, let $k := k(q)$, $0 < k < 1$, denote the modulus. The singular modulus k_n is defined by $k_n := k(e^{-\pi\sqrt{n}})$, where n is a natural number. Following Ramanujan, set $\alpha = k^2$ and $\alpha_n = k_n^2$.

It is well known that G_n and g_n are algebraic; for example, see Cox's book [1, p. 214, Theorem 10.23; p. 257, Theorem 12.17]. However, much more is known. Weber [2, p. 540] and, more recently, H. H. Chan and S.-S. Huang [1], using a result of Deuring [1], have proved the following theorem.

Theorem 1.1.

(a) If $n \equiv 1 \pmod 4$, *then* G_n *and* $2\alpha_n$ *are units.*
(b) If $n \equiv 3 \pmod 8$, *then* $2^{-1/12}G_n$ *and* $2^2\alpha_n$ *are units.*
(c) If $n \equiv 7 \pmod 8$, *then* $2^{-1/4}G_n$ *and* $2^4\alpha_n$ *are units.*
(d) If $n \equiv 2 \pmod 4$, *then* g_n *and* α_n *are units.*

As G. N. Watson [6] remarked, "For reasons which had commended themselves to Weber and Ramanujan independently, it is customary to determine G_n for odd values of n, and g_n for even values of n."

At scattered places in his first notebook [9], Ramanujan recorded the values for 107 class invariants, or polynomials satisfied by them. On pages 294–299 in his second notebook [9], Ramanujan gave a table of values for 77 class invariants, three of which are not found in the first notebook. Since the second notebook is an enlarged revision of the first, it is unclear why Ramanujan failed to record 33 class invariants that he offered in the first notebook. By the time Ramanujan wrote his paper [3], [10, pp. 23–39], he was aware of Weber's work [2], and so his table of 46 class invariants in [3] does not contain any that are found in Weber's book [2]. Except for G_{325} and G_{363}, all of the remaining values are found in Ramanujan's notebooks; twenty-one of these class invariants are found in his second notebook. At scattered places in the second and third notebooks, Ramanujan recorded irreducible polynomials satisfied by four further invariants. In conclusion, to the best of our tallying, Ramanujan calculated a total of 116 class invariants, or monic, irreducible polynomials satisfied by them.

In two papers [6], [7], Watson proved 24 of Ramanujan's class invariants from Ramanujan's paper [3]. In the first [6], Watson devised an "empirical process" to calculate 14 of the 24 invariants, while in the second [7], he employed modular equations to prove 10 invariants. In another paper [5], Watson established Ramanujan's value for G_{1353}, communicated by Ramanujan [10, p. xxix] in his first letter to Hardy, and also stated in his paper [3]. In the introduction to [6], Watson remarked, "It is intended to publish the calculations involved in the construction of the set $N + Q$ (the invariants appearing in both Ramanujan's paper [3] and the second notebook) as part of the commentary on the notebooks by Dr. B. M. Wilson and myself." Although Watson and Wilson's efforts to edit Ramanujan's notebooks have been preserved in the library at Trinity College, Cambridge, Watson's calculations of these twenty-one invariants are not found there. If Watson actually calculated these invariants, it appears that his work has been lost. The twenty-one values of n are: 65, 69, 77, 81, 117, 141, 145, 147, 153, 205, 213, 217, 265, 289, 301, 441, 445, 505, 553, 90, and 198.

Watson wrote four further papers [9], [10], [12], [13] on the calculation of class invariants. The values of n considered by Watson depend upon the class numbers for positive definite quadratic forms of discriminant $-n$. In the course of his evaluations, he determined the class invariants for $n = 81$ [12], 147 [12], and 289 [13]. Thus, after Watson's work, 18 invariants of Ramanujan from his paper [3] and notebooks [9] remained to be verified.

Five of these invariants are established in Section 3. For each of these five values, 117, 153, 441, 90, and 198, n is a multiple of 9, and proofs are effected by formulas relating G_{9n} with G_n and g_{9n} with g_n, which we establish by using one of Ramanujan's modular equations of degree 3. The latter formula is found on page 318 of Ramanujan's first notebook, but not in his second notebook, while the former formula is not found in any of the notebooks.

Since modular equations are crucial in our work on class invariants, we now give a precise definition of a modular equation. Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, $k' := \sqrt{1-k^2}$, ℓ, and $\ell' := \sqrt{1-\ell^2}$, respectively, where $0 < k, \ell < 1$. Suppose that

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.4}$$

for some positive integer n. A relation between k and ℓ induced by (1.4) is called a *modular equation of degree n*. Following Ramanujan, set

$$\alpha = k^2 \qquad \text{and} \qquad \beta = \ell^2.$$

We often say that β has degree n over α.

As usual, in the theory of elliptic functions, set

$$q := \exp(-\pi K'/K). \tag{1.5}$$

Since $\chi(q) = 2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24}$ and $\chi(-q) = 2^{1/6}\{\alpha(1-\alpha)^{-2}/q\}^{-1/24}$ (Part III [3, p. 124]), it follows from (1.1), (1.3), and (1.5) that

$$G_n = \{4\alpha_n(1-\alpha_n)\}^{-1/24} \qquad \text{and} \qquad g_n = \{4\alpha_n(1-\alpha_n)^{-2}\}^{-1/24}. \tag{1.6}$$

This formula for G_n will be used in certain modular equations.

In Sections 4–7, we establish the remaining 13 values, each for G_n, $n = 65, 69, 77, 141, 145, 205, 213, 217, 265, 301, 445, 505$, and 553, claimed by Ramanujan. Quite remarkably, the class number for each of the 13 imaginary quadratic fields $\mathbb{Q}(\sqrt{-n})$ equals 8. Moreover, there are precisely two classes per genus in each case. Our first proofs, given in Sections 4 and 5, employ Kronecker's limit formula, which is used to find representations for certain products of Dedekind eta-functions in terms of fundamental units; see Theorems 4.1, 4.2, and 4.7. Each of the 13 values of n is a product of a small prime (3, 5, or 7) and a larger prime. Thus, our proofs also crucially employ certain modular equations of Ramanujan of degrees 3, 5, and 7. It is highly unlikely that Ramanujan was familiar with Kronecker's limit formula and the arithmetic of quadratic fields, and so our proofs certainly are not those found by Ramanujan. However, Ramanujan obviously discerned some unique arithmetical properties in these instances, and it would be fascinating to discover Ramanujan's approach. The methods in Sections 4 and 5 have been further extended by L.–C. Zhang [2], [3] who has rigorously established all the invariants in Watson's paper [6] that were calculated there with his "empirical process."

Ramanujan used modular equations to calculate only a couple of simple invariants in [3]. This fact and the sentence, "The values of G_n and g_{2n} are got from the same modular equation." [3], [10, p. 25] are the only clues to his methods that Ramanujan provided for us. It would seem that if Ramanujan had employed another type of reasoning, he would have dropped some hint about it. As mentioned earlier, Watson [7] used modular equations to establish some of Ramanujan's invariants. However, for his calculations of G_n, it is important that n be a square or a simple multiple of a square. We have been able to prove six of the remaining thirteen values for G_n, namely, for $n = 65, 69, 77, 141, 145$, and 213, by using modular equations. As will be seen in our proofs in Section 6, we need some new ideas to effect proofs of these six invariants via modular equations. To prove the remaining seven invariants by employing modular equations, we would need modular equations of degrees 31, 41, 43, 53, 79, 89, and 101. Apparently, only for degree 31 did Ramanujan derive a modular equation, for he recorded no modular equations for the other six degrees in his notebooks. Thus, Ramanujan's methods appear to be even more elusive.

Watson [6, p. 82] opined that "I believe that fourteen were obtained by Ramanujan by means of the empirical process which I described in the discussion of G_{1353}." We are not so confident that Ramanujan used this empirical process, for which Watson offered little explanation. In fact, Watson's "empirical process" is not rigorous. However, in Section 7 we shall use class field theory to make Watson's procedure rigorous for a large class of invariants including those 13 invariants mentioned above, and we use the process to calculate two new invariants as well. Chan [3] has further extended the methods of Section 7 and calculated 27 new class invariants.

Section 8 is devoted to some miscellaneous results on class invariants, including two entries to which we have not been able to attach any meaning. Here we also establish Ramanujan's more detailed assertions about the irreducible polynomials satisfied by G_{29} and G_{79}.

In Section 9, we turn to Ramanujan's singular moduli. Once G_n or g_n is known, then it is easy to calculate α_n from (1.6) by simply solving a quadratic equation. However, this expression for α_n that one trivially obtains from the quadratic formula is usually not very interesting or attractive. Thus, it is desirable to develop other algorithms for the calculation of α_n that will reflect the unit structure of α_n described in Theorem 1.1. For the calculation of α_n, when n is even, Ramanujan devised a very clever algorithm given in Theorem 9.1. For odd n, we do not have such an inclusive algorithm, and so we had to develop some lemmas to facilitate calculations.

In Section 10, a simple function of singular moduli is studied.

In the last two pages of his notebooks, Ramanujan studied the j-invariant. He seems to have quoted some results from the literature. However, Ramanujan made some remarkable discoveries, including very simple polynomials satisfied by certain algebraic functions of the j-invariant. Ramanujan's work on the j-invariant is the topic of Section 11.

2. Table of Class Invariants

Both prior to his table of class invariants in his second notebook and at the beginning of his paper [3, eqs. (5), (7)], [10, p. 23], Ramanujan recorded two simple formulas relating these invariants, and so we first state and prove these here. See also Exercise 5c on page 73 of the Borweins' treatise [1].

Entry 2.1 (p. 294, NB 2). *For $n > 0$,*

$$g_{4n} = 2^{1/4} g_n G_n.$$

Proof. This identity is an immediate consequence of the definitions of G_n and g_n in (1.1)–(1.3) and the elementary identity

$$(q^2; q^4)_\infty = (q; q^2)_\infty (-q; q^2)_\infty. \tag{2.1}$$

Entry 2.2 (p. 294, NB2). *For $n > 0$,*

$$(g_n G_n)^8 (G_n^8 - g_n^8) = \tfrac{1}{4}.$$

Proof. Jacobi's identity for fourth powers of theta–functions (Part III [3, p. 40, Entry 25(vii)]) can be written in the form (Whittaker and Watson [1, p. 470])

$$(-q; q^2)_\infty^8 - (q; q^2)_\infty^8 = 16q(-q^2; q^2)_\infty^8 = \frac{16q}{(q^2; q^4)_\infty^8}, \tag{2.2}$$

where we used Euler's identity

$$(-q; q)_\infty = \frac{1}{(q : q^2)_\infty}.$$

By (2.1), we can write (2.2) in the form

$$\begin{aligned} &2^{-2} q^{-1/3} (q; q^2)_\infty^8 2^{-2} q^{-1/3} (-q; q^2)_\infty^8 \\ &\qquad \times \left(2^{-2} q^{-1/3} (-q; q^2)_\infty^8 - 2^{-2} q^{-1/3} (q; q^2)_\infty^8\right) = \tfrac{1}{4}, \end{aligned}$$

that is, by (1.1)–(1.3), when $q = \exp(-\pi\sqrt{n})$,

$$(g_n G_n)^8 (G_n^8 - g_n^8) = \tfrac{1}{4}.$$

Recall from Part III [3, pp. 91, 102] the definition

$$F(x) = \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right), \qquad 0 < x < 1, \tag{2.3}$$

where ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$ denotes the ordinary hypergeometric function. Recall also from Part III [3, p. 36] or Chapter 33 the theta–function

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad |q| < 1, \tag{2.4}$$

and the fundamental formula (Part III [3, p. 102])

$$K(k) = \tfrac{1}{2}\pi\ {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) = \tfrac{1}{2}\pi\varphi^2(q), \tag{2.5}$$

if q is given by (1.5). (The evaluations of $\chi(q)$ and $\chi(-q)$ used in (1.6) depend upon (2.5).) If $x = \alpha_n$ in (2.3), then

$$F(\alpha_n) = \exp(-\pi\sqrt{n}). \tag{2.6}$$

Now from (1.6),

$$\alpha_n^2 - \alpha_n + \frac{1}{4G_n^{24}} = 0, \tag{2.7}$$

and so, since $\alpha_n \leq \frac{1}{2}$,

$$\alpha_n = \tfrac{1}{2}\left(1 - \sqrt{1 - 1/G_n^{24}}\right). \tag{2.8}$$

In the first notebook, Ramanujan frequently records G_n, or equivalently, α_n, in the form

$$F\left(\tfrac{1}{2}\left(1 - \sqrt{1 - 1/G_n^{24}}\right)\right) = \exp(-\pi\sqrt{n}), \tag{2.9}$$

by (2.6) and (2.8). We have indicated such a representation in the tables by placing "F" after the page number in the first notebook where the corresponding invariant (or, equivalently, singular modulus) is located.

We next give a table of all the values of G_n and g_n found by Ramanujan. We emphasize that formulas for certain values of G_n and g_n may be different in both the first and second notebooks and Ramanujan's paper [3]. In most cases, it is not difficult to verify that Ramanujan's formulations are equivalent. In particular, when Ramanujan employs (2.9), a modest amount of calculation is needed. For example, such calculations are necessary on pages 284, 287, 292, 293, 296, and 311. In all these instances, the calculations are routine, and there is no need to give them here. In other instances, for example, on page 314 in the first notebook, Ramanujan records the value for G_n^{-24} in the notation "$4\beta(1-\beta)$," that is, he is using (1.6) with α_n replaced by β.

When the invariant is a root of a cubic polynomial, Ramanujan, as well as Weber [2], normally, but not always, only gives the polynomial. There are some instances when Ramanujan calculated the appropriate root but Weber did not. In particular, we establish Ramanujan's values for G_{75} and G_{175} in Section 8. However, Ramanujan's polynomials are those satisfied by $1/(2^{1/4}G_n)$, instead of G_n. Thus, on page 345 in his first notebook, Ramanujan recorded the irreducible polynomials satisfied by $1/(2^{1/4}G_n)$, for $n = 23, 31, 11, 19, 27, 43$, and 67. These polynomials are repeated on page 351 together with monic irreducible polynomials satisfied by $1/(2^{1/4}G_3)$ and by (erroneously) $1/(2^{1/4}G_7)$. In the tables in his second notebook, Ramanujan explicitly states G_n for $n = 3, 7$, and 27.

Lastly, we remark that the tables of Weber [2] contain some errors. Corrections have been made by J. Brillhart and P. Morton [1].

For convenience, we use the following abbreviations in citing sources for the listed invariants:
Ramanujan's first notebook: N1,
Ramanujan's second notebook: N2,
Ramanujan's third notebook: N3,
Ramanujan's paper [3]: RP,
Watson's paper [5]: WXIV,
Watson's paper [6]: WI,
Watson's paper [7]: WII,
Watson's paper [9]: W3,
Watson's paper [10]: W4,
Watson's paper [12]: W5,
Watson's paper [13]: W6,
Weber's treatise [2]: We,
Brillhart and Morton's corrections [1]: BM.

Table of G_n

$n = 1$

$$1$$

Refs.: N2,RP,W5,We

$n = 3$

$$2^{1/12}$$

Refs.: N1(282F),N2,W3,We

$n = 5$

$$\left(\frac{1+\sqrt{5}}{2}\right)^{1/4}$$

Refs.: N1(287F), N2,RP,We

$n = 7$

$$2^{1/4}$$

Refs.: N1(287F),N2,W3,W4,We

$n = 9$

$$\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{1/3}$$

Refs.: N1(284F,287F),N2,RP,We

$n = 11$

$$2^{-1/4}x, \text{ where } x^3 - 2x^2 + 2x - 2 = 0$$

Refs.: N1(295F,345,351)N2,W3,We

Table of G_n (*Continued*)

$n = 13$

$$\left(\frac{3+\sqrt{13}}{2}\right)^{1/4}$$

Refs.: N1(292F),N2,RP,We

$n = 15$

$$2^{1/4}\left(\frac{1+\sqrt{5}}{2}\right)^{1/3}$$

Refs.: N1(289F),N2,W4,We

$n = 17$

$$\sqrt{\frac{5+\sqrt{17}}{8}}+\sqrt{\frac{\sqrt{17}-3}{8}}$$

Refs.: N1(296F),N2,RP,We

$n = 19$

$$2^{-1/4}x \text{ where } x^3 - 2x - 2 = 0$$

Refs.: N1(295F,345,351),N2,W3,We

$n = 21$

$$\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)^{1/4}\left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^{1/6}$$

Refs.: N1(293F),N2,We

$n = 23$

$$2^{1/4}x, \quad \text{where } x^3 - x - 1 = 0$$

Refs.: N1(295F,345,351),N2,W3,W4,We

$n = 25$

$$\frac{1+\sqrt{5}}{2}$$

Refs.: N1(287F),N2,RP,We

$n = 27$

$$2^{1/12}\left(\sqrt[3]{2}-1\right)^{-1/3}$$

Refs.: N1(305F,345,351),N2,W3,We

$n = 29$

$$G_{29}^4 = x, \quad \text{where } x^6 - 9x^5 + 5x^4 + 2x^3 - 5x^2 - 9x - 1 = 0$$

Refs.: N2(263),We

Table of G_n (*Continued*)

$n = 31$

$$2^{1/4}x, \quad \text{where } x^3 - x^2 - 1 = 0$$

Refs.: N1(296F,345,351),N2,W3,W4,We

$n = 33$

$$\left(\frac{3+\sqrt{11}}{\sqrt{2}}\right)^{1/6}\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{1/2}$$

Refs.: N1(311F),N2,We

$n = 37$

$$\left(6+\sqrt{37}\right)^{1/4}$$

Refs.: N1(305F),N2,RP,We

$n = 39$

$$2^{1/4}\left(\frac{\sqrt{13}+3}{2}\right)^{1/6}\left(\sqrt{\frac{5+\sqrt{13}}{8}}+\sqrt{\frac{\sqrt{13}-3}{8}}\right)$$

Refs.: N1(305F),N2,W4,We

$n = 41$

$$G_{41}^2 = x, \quad \text{where } \left(x+\frac{1}{x}\right)^2 - \frac{5+\sqrt{41}}{2}\left(x+\frac{1}{x}\right) + \frac{7+\sqrt{41}}{2} = 0$$

Refs.: N3(382),WE,BM

$n = 43$

$$2^{-1/4}x \quad \text{where } x^3 - 2x^2 - 2 = 0$$

Refs.: N1(313F,345,351),N2,W3,We

$n = 45$

$$\left(2+\sqrt{5}\right)^{1/4}\left(\frac{\sqrt{3}+\sqrt{5}}{\sqrt{2}}\right)^{1/3}$$

Refs.: N2,We

$n = 47$

$$2^{1/4}x, \quad \text{where } x^5 = (1+x)(1+x+x^2)$$

Refs.: N1(234),N2(263),We

$n = 49$

$$\frac{7^{1/4}+\sqrt{4+\sqrt{7}}}{2}$$

Refs.: N1(293F)N2,RP,We

Table of G_n (*Continued*)

$n = 55$

$$2^{1/4}\left(\sqrt{5}+2\right)^{1/6}\left(\sqrt{\frac{7+\sqrt{5}}{8}}+\sqrt{\frac{\sqrt{5}-1}{8}}\right)$$

Refs.: N1(315),N2,W4,We

$n = 57$

$$\left(\frac{3\sqrt{19}+13}{\sqrt{2}}\right)^{1/6}\left(2+\sqrt{3}\right)^{1/4}$$

Refs.: N1(315),N2,We

$n = 63$

$$2^{1/4}\left(\frac{5+\sqrt{21}}{2}\right)^{1/6}\left(\sqrt{\frac{5+\sqrt{21}}{8}}+\sqrt{\frac{\sqrt{21}-3}{8}}\right)$$

Refs.: N1(305F),N2,W4,We

$n = 65$

$$\left(\frac{\sqrt{13}+3}{2}\right)^{1/4}\left(\frac{\sqrt{5}+1}{2}\right)^{1/4}\left(\sqrt{\frac{9+\sqrt{65}}{8}}+\sqrt{\frac{1+\sqrt{65}}{8}}\right)^{1/2}$$

Refs.: N1(315),N2,RP

$n = 67$

$$2^{-1/4}x \quad \text{where } x^3-2x^2-2x-2=0$$

Refs.: N1(345,351),N2,W3,We

$n = 69$

$$\left(\frac{5+\sqrt{23}}{\sqrt{2}}\right)^{1/12}\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)^{1/8}\left(\sqrt{\frac{6+3\sqrt{3}}{4}}+\sqrt{\frac{2+3\sqrt{3}}{4}}\right)^{1/2}$$

Refs.: N1(314F,315),N2,RP

$n = 73$

$$\sqrt{\frac{9+\sqrt{73}}{8}}+\sqrt{\frac{1+\sqrt{73}}{8}}$$

Refs.: N1(313F),N2,RP,We

$n = 75$

$$\frac{3\cdot 2^{5/12}}{\frac{\sqrt{5}+1}{2}(10)^{1/3}+\frac{\sqrt{5}-1}{2}4^{1/3}\cdot 5^{1/6}-\sqrt{5}-1}$$

Refs.: N1(311),We

Table of G_n (*Continued*)

$n = 77$

$$\left(8+3\sqrt{7}\right)^{1/8}\left(\frac{\sqrt{11}+\sqrt{7}}{2}\right)^{1/8}\left(\sqrt{\frac{6+\sqrt{11}}{4}}+\sqrt{\frac{2+\sqrt{11}}{4}}\right)^{1/2}$$

Refs.: N1(315),N2,RP

$n = 79$

$$t = 2^{1/4}/G_{79}, \quad \text{where } t^5 - t^4 + t^3 - 2t^2 + 3t - 1 = 0$$

Refs.: N2(263,300)

$n = 81$

$$\left(\frac{\sqrt[3]{2(\sqrt{3}+1)}+1}{\sqrt[3]{2(\sqrt{3}-1)}-1}\right)^{1/3}$$

Refs.: N2,W5,RP

$n = 85$

$$\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{\sqrt{85}+9}{2}\right)^{1/4}$$

Refs.: N1(315),N2,RP,We

$n = 93$

$$\left(\frac{39+7\sqrt{31}}{\sqrt{2}}\right)^{1/6}\left(\frac{\sqrt{31}+3\sqrt{3}}{2}\right)^{1/4}$$

Refs.: N1(315), N2,We

$n = 97$

$$\sqrt{\frac{13+\sqrt{97}}{8}}+\sqrt{\frac{5+\sqrt{97}}{8}}$$

Refs.: N1(305F),N2,RP,We

$n = 105$

$$\left(\frac{5+\sqrt{21}}{2}\right)^{1/4}\left(2+\sqrt{3}\right)^{1/4}\left(\sqrt{5}+2\right)^{1/6}\left(6+\sqrt{35}\right)^{1/12}$$

Refs.: N1(317),N2,We

$n = 117$

$$\left(\frac{3+\sqrt{13}}{2}\right)^{1/4}\left(2\sqrt{3}+\sqrt{13}\right)^{1/6}\left(\frac{3^{1/4}+\sqrt{4+\sqrt{3}}}{2}\right)$$

Refs.: N1(315),N2,RP

Table of G_n (*Continued*)

$n = 121$

$$\frac{1}{3\sqrt{2}}\left[\left(11+3\sqrt{11}\right)^{1/3} \times \left(\left(3\sqrt{11}+3\sqrt{3}+4\right)^{1/3}+\left(3\sqrt{11}-3\sqrt{3}+4\right)^{1/3}\right)+2\right]$$

Refs.: N1(294F,317),RP,WII,W5

$n = 133$

$$(8+3\sqrt{7})^{1/4}\left(\frac{5\sqrt{7}+3\sqrt{19}}{2}\right)^{1/4}$$

Refs.: N1(315),N2,We

$n = 141$

$$\left(4\sqrt{3}+\sqrt{47}\right)^{1/8}\left(\frac{7+\sqrt{47}}{\sqrt{2}}\right)^{1/12}\left(\sqrt{\frac{18+9\sqrt{3}}{4}}+\sqrt{\frac{14+9\sqrt{3}}{4}}\right)^{1/2}$$

Refs.: N1(320),N2,RP

$n = 145$

$$(\sqrt{5}+2)^{1/4}\left(\frac{5+\sqrt{29}}{2}\right)^{1/4}\left(\sqrt{\frac{17+\sqrt{145}}{8}}+\sqrt{\frac{9+\sqrt{145}}{8}}\right)^{1/2}$$

Refs.: N1(315),N2,RP

$n = 147$

$$2^{1/12}\left(\frac{1}{2}+\frac{1}{\sqrt{3}}\left\{\sqrt{\frac{7}{4}}-(28)^{1/6}\right\}\right)^{-1}$$

Refs.: N2,RP,W5

$n = 153$

$$\left(\sqrt{\frac{5+\sqrt{17}}{8}}+\sqrt{\frac{\sqrt{17}-3}{8}}\right)^{2}\left(\sqrt{\frac{37+9\sqrt{17}}{4}}+\sqrt{\frac{33+9\sqrt{17}}{4}}\right)^{1/3}$$

Refs.: N1(315),N2,RP

$n = 163$

$$2^{-1/4}x \quad \text{where } x^3-6x^2+4x-2=0$$

Refs.: N2(300),We

$n = 165$

$$\left(4+\sqrt{15}\right)^{1/4}\left(3\sqrt{5}+2\sqrt{11}\right)^{1/6}\left(\frac{\sqrt{15}+\sqrt{11}}{2}\right)^{1/4}\left(\sqrt{5}+2\right)^{1/6}$$

Refs.: N1(317),N2,We

Table of G_n (*Continued*)

$n = 169$

$$\frac{1}{3}\left[(\sqrt{13}+2)+\left(\frac{13+3\sqrt{13}}{2}\right)^{1/3}\right.$$
$$\left.\times\left\{\left(\frac{11+\sqrt{13}}{2}+3\sqrt{3}\right)^{1/3}+\left(\frac{11+\sqrt{13}}{2}-3\sqrt{3}\right)^{1/3}\right\}\right]$$

Refs.: N1(294F,317),RP,WII,W5

$n = 175$

$$\frac{3\cdot 2^{1/4}}{\dfrac{\sqrt{5}-1}{2}+\left(\dfrac{5-\sqrt{5}}{4}\right)^{1/3}\left(\sqrt[3]{8-3\sqrt{5}+3\sqrt{21}}+\sqrt[3]{8-3\sqrt{5}-3\sqrt{21}}\right)}$$

Refs.: N1(316),We

$n = 177$

$$\left(\frac{3\sqrt{59}+23}{\sqrt{2}}\right)^{1/6}\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{3/2}$$

Refs.: N1(315),N2,We

$n = 205$

$$\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3\sqrt{5}+\sqrt{41}}{2}\right)^{1/4}\left(\sqrt{\frac{7+\sqrt{41}}{8}}+\sqrt{\frac{\sqrt{41}-1}{8}}\right)$$

Refs.: N1(314),N2,RP

$n = 213$

$$\left(\frac{5\sqrt{3}+\sqrt{71}}{2}\right)^{1/8}\left(\frac{59+7\sqrt{71}}{\sqrt{2}}\right)^{1/12}\left(\sqrt{\frac{21+12\sqrt{3}}{2}}+\sqrt{\frac{19+12\sqrt{3}}{2}}\right)^{1/2}$$

Refs.: N1(315),N2,RP

$n = 217$

$$\left(\sqrt{\frac{9+4\sqrt{7}}{2}}+\sqrt{\frac{11+4\sqrt{7}}{2}}\right)^{1/2}\left(\sqrt{\frac{12+5\sqrt{7}}{4}}+\sqrt{\frac{16+5\sqrt{7}}{4}}\right)^{1/2}$$

Refs.: N1(314),N2,RP

$n = 225$

$$\left(\frac{1+\sqrt{5}}{2}\right)(2+\sqrt{3})^{1/3}\left(\frac{\sqrt{4+\sqrt{15}}+(15)^{1/4}}{2}\right)$$

Refs.: N1(293F),RP,WII

Table of G_n (*Continued*)

$n = 253$

$$\left(\frac{5+\sqrt{23}}{\sqrt{2}}\right)^{1/2}\left(\frac{13\sqrt{11}+9\sqrt{23}}{2}\right)^{1/4}$$

Refs.: N1(315),N2,We

$n = 265$

$$(2+\sqrt{5})^{1/4}\left(\frac{7+\sqrt{53}}{2}\right)^{1/4}\left(\sqrt{\frac{89+5\sqrt{265}}{8}}+\sqrt{\frac{81+5\sqrt{265}}{8}}\right)^{1/2}$$

Refs.: N1(314),N2,RP

$n = 273$

$$\left(\frac{15\sqrt{7}+11\sqrt{13}}{\sqrt{2}}\right)^{1/6}\left(\frac{\sqrt{13}+3}{2}\right)^{1/2}\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^{1/2}\left(2+\sqrt{3}\right)^{1/4}$$

Refs.: N1(317),N2,We

$n = 289$

$$\left(\sqrt{\frac{17+\sqrt{17}+(17)^{1/4}(5+\sqrt{17})}{16}}+\sqrt{\frac{1+\sqrt{17}+(17)^{1/4}(5+\sqrt{17})}{16}}\right)^{2}$$

Refs.: N1(317),N2,RP,W6

$n = 301$

$$\left(8+3\sqrt{7}\right)^{1/8}\left(\frac{23\sqrt{43}+57\sqrt{7}}{2}\right)^{1/8}\left(\sqrt{\frac{46+7\sqrt{43}}{4}}+\sqrt{\frac{42+7\sqrt{43}}{4}}\right)^{1/2}$$

Refs.: N1(325),N2,RP

$n = 325$

$$\left(\frac{3+\sqrt{13}}{2}\right)^{1/4} t, \quad \text{where } t^3+t^2\left(\frac{1-\sqrt{13}}{2}\right)^2+t\left(\frac{1+\sqrt{13}}{2}\right)^2+1 = \sqrt{5}\left\{t^3-t^2\left(\frac{1+\sqrt{13}}{2}\right)+t\left(\frac{1-\sqrt{13}}{2}\right)-1\right\}$$

Refs.: RP,WII

$n = 333$

$$(6+\sqrt{37})^{1/4}(7\sqrt{3}+2\sqrt{37})^{1/6}\left(\frac{\sqrt{7+2\sqrt{3}}+\sqrt{3+2\sqrt{3}}}{2}\right)$$

Refs.: N1(314,315),RP,WI

Table of G_n (*Continued*)

$n = 345$

$$\frac{1+\sqrt{5}}{2}\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{1/2}\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)^{1/2}\left(\frac{15\sqrt{5}+7\sqrt{23}}{\sqrt{2}}\right)^{1/6}$$

Refs.: N1(317),N2,We

$n = 357$

$$\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)(8+3\sqrt{7})^{1/4}\left(\frac{\sqrt{17}+\sqrt{21}}{2}\right)^{1/4}\left(\frac{11+\sqrt{119}}{\sqrt{2}}\right)^{1/6}$$

Refs.: N1(317),N2,We,BM

$n = 363$

$$2^{5/12}t, \quad \text{where } 2t^3 - t^2\left\{(4+\sqrt{33})+\sqrt{11+2\sqrt{33}}\right\} - t\left\{1+\sqrt{11+2\sqrt{33}}\right\} - 1 = 0$$

Refs.: RP,WII

$n = 385$

$$\sqrt{\tfrac{1}{8}(3+\sqrt{11})(\sqrt{5}+\sqrt{7})(\sqrt{7}+\sqrt{11})(3+\sqrt{5})}$$

Refs.: N1(317),N2,RP,We

$n = 441$

$$\sqrt{\frac{\sqrt{3}+\sqrt{7}}{2}}(2+\sqrt{3})^{1/6}\sqrt{\frac{2+\sqrt{7}+\sqrt{7+4\sqrt{7}}}{2}}\sqrt{\frac{\sqrt{3+\sqrt{7}}+(6\sqrt{7})^{1/4}}{\sqrt{3+\sqrt{7}}-(6\sqrt{7})^{1/4}}}$$

Refs.: N1(46),N2,RP

$n = 445$

$$\sqrt{2+\sqrt{5}}\left(\frac{21+\sqrt{445}}{2}\right)^{1/4}\left(\sqrt{\frac{13+\sqrt{89}}{8}}+\sqrt{\frac{5+\sqrt{89}}{8}}\right)$$

Refs.: N1(320),N2,RP

$n = 465$

$$(2+\sqrt{3})^{1/4}\left(\frac{1+\sqrt{5}}{2}\right)^{1/4}\left(\frac{3\sqrt{3}+31}{2}\right)^{1/4}(5\sqrt{5}+2\sqrt{31})^{1/12}$$
$$\times\left(\sqrt{\frac{2+\sqrt{31}}{4}}+\sqrt{\frac{6+\sqrt{31}}{4}}\right)^{1/2}\left(\sqrt{\frac{11+2\sqrt{31}}{2}}+\sqrt{\frac{13+2\sqrt{31}}{2}}\right)^{1/2}$$

Refs.: N1(319),RP,WI

Table of G_n (*Continued*)

$n = 505$

$$\sqrt{\left(2+\sqrt{5}\right)}\left(\frac{1+\sqrt{5}}{2}\right)^{1/4}(10+\sqrt{101})^{1/4}$$
$$\times\left(\frac{113+5\sqrt{105}}{8}+\sqrt{\frac{105+5\sqrt{105}}{8}}\right)^{1/2}$$

Refs.: N1(344),N2,RP

$n = 553$

$$\left(\sqrt{\frac{96+11\sqrt{79}}{4}}+\sqrt{\frac{100+11\sqrt{79}}{4}}\right)^{1/2}$$
$$\times\left(\sqrt{\frac{141+16\sqrt{79}}{2}}+\sqrt{\frac{143+16\sqrt{79}}{2}}\right)^{1/2}$$

Refs.: N1(320),N2,RP

$n = 765$

$$\sqrt{\frac{3+\sqrt{5}}{2}}(16+\sqrt{225})^{1/12}(4+\sqrt{15})^{1/4}\left(\frac{9+\sqrt{85}}{2}\right)^{1/4}$$
$$\times\left(\sqrt{\frac{6+\sqrt{51}}{4}}+\sqrt{\frac{10+\sqrt{51}}{4}}\right)^{1/2}\left(\sqrt{\frac{18+3\sqrt{51}}{4}}+\sqrt{\frac{22+3\sqrt{51}}{4}}\right)^{1/2}$$

Refs.: N1(343),RP,WI

$n = 777$

$$(2+\sqrt{3})^{1/4}(6+\sqrt{37})^{1/4}\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)^{1/4}\left(246\sqrt{7}+107\sqrt{37}\right)^{1/12}$$
$$\times\left(\sqrt{\frac{6+3\sqrt{7}}{4}}+\sqrt{\frac{10+3\sqrt{7}}{4}}\right)^{1/2}\left(\sqrt{\frac{15+6\sqrt{7}}{2}}+\sqrt{\frac{17+6\sqrt{7}}{2}}\right)^{1/2}$$

Refs.: N1(319),RP,WI

$n = 897$

$$\sqrt{2+\sqrt{3}}\sqrt{\frac{\sqrt{13}+3}{2}}\left(4\sqrt{13}+3\sqrt{23}\right)^{1/12}\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)^{1/4}$$
$$\times\left(\sqrt{\frac{60+9\sqrt{39}}{4}}+\sqrt{\frac{56+9\sqrt{39}}{4}}\right)^{1/2}\left(\sqrt{\frac{8+\sqrt{39}}{4}}+\sqrt{\frac{4+\sqrt{39}}{4}}\right)^{1/2}$$

Refs.: N1(320),WI

Table of G_n (*Continued*)

$n = 1225$

$$\frac{1+\sqrt{5}}{2}(6+\sqrt{35})^{1/4}\left(\frac{7^{\frac{1}{4}}+\sqrt{4+\sqrt{7}}}{2}\right)^{3/2}$$
$$\times\left(\sqrt{\frac{43+15\sqrt{7}+(8+3\sqrt{7})\sqrt{10\sqrt{7}}}{8}}+\sqrt{\frac{35+15\sqrt{7}+(8+3\sqrt{7})\sqrt{10\sqrt{7}}}{8}}\right)$$

Refs.: N1(46),RP,WII

$n = 1353$

$$\left(\frac{3+\sqrt{11}}{\sqrt{2}}\right)^{1/4}\left(\frac{5+3\sqrt{3}}{\sqrt{2}}\right)^{1/4}\left(\frac{11+\sqrt{123}}{\sqrt{2}}\right)^{1/4}\left(\frac{6817+321\sqrt{451}}{\sqrt{2}}\right)^{1/12}$$
$$\times\left(\sqrt{\frac{17+3\sqrt{33}}{8}}+\sqrt{\frac{25+3\sqrt{33}}{8}}\right)^{1/2}$$
$$\times\left(\sqrt{\frac{561+99\sqrt{33}}{8}}+\sqrt{\frac{569+99\sqrt{33}}{8}}\right)^{1/2}$$

Refs.: N1(319),RP,WXIV

$n = 1645$

$$\sqrt{2+\sqrt{5}}\left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^{1/4}\left(\frac{7+\sqrt{47}}{\sqrt{2}}\right)^{1/4}\left(\frac{73\sqrt{5}+9\sqrt{329}}{2}\right)^{1/8}$$
$$\times\left(\sqrt{\frac{119+7\sqrt{329}}{8}}+\sqrt{\frac{127+7\sqrt{329}}{8}}\right)^{1/2}$$
$$\times\left(\sqrt{\frac{743+41\sqrt{329}}{8}}+\sqrt{\frac{751+41\sqrt{329}}{8}}\right)^{1/2}$$

Refs.: N1(320),RP,WI

$n = 1677$

$$(4414\sqrt{13}+2427\sqrt{43})^{1/12}\left(\frac{\sqrt{13}+3}{2}\right)^{3/4}\left(\frac{\sqrt{43}+\sqrt{39}}{2}\right)^{1/4}\times(\sqrt{13}$$
$$+2\sqrt{3})^{1/4}\left(\sqrt{\frac{355+54\sqrt{43}}{4}}+\sqrt{\frac{351+54\sqrt{43}}{4}}\right)^{1/2}$$
$$\times\left(\sqrt{\frac{17+2\sqrt{43}}{4}}+\sqrt{\frac{13+2\sqrt{43}}{4}}\right)^{1/2}$$

Refs.: N1(320),WI

Table of g_n

$n = 2$

$$1$$

Refs.: N1(316),N2,We

$n = 6$

$$(1+\sqrt{2})^{1/6}$$

Refs.: N1(316),N2,We

$n = 10$

$$\sqrt{\frac{1+\sqrt{5}}{2}}$$

Refs.: N1(316),N2,RP,We

$n = 14$

$$\sqrt{\frac{1+\sqrt{2}+\sqrt{2\sqrt{2}-1}}{2}}$$

Refs.: N1(316),N2,We

$n = 18$

$$(\sqrt{2}+\sqrt{3})^{1/3}$$

Refs.: N1(316),N2,RP,We

$n = 22$

$$\sqrt{1+\sqrt{2}}$$

Refs.: N1(316),N2,RP,We,BM

$n = 26$

$$\frac{1}{3}\Bigg(\sqrt{2+\sqrt{13}}+\sqrt[3]{(2+\sqrt{13})\sqrt{2+\sqrt{13}}+3\sqrt{3(3+\sqrt{13})}} + \sqrt[3]{(2+\sqrt{13})\sqrt{2+\sqrt{13}}-3\sqrt{3(3+\sqrt{13})}}\Bigg)$$

Refs.: N1(318,344),W5,We

$n = 30$

$$(2+\sqrt{5})^{1/6}(3+\sqrt{10})^{1/6}$$

Refs.: N1(316),N2,RP,We

$n = 34$

$$\sqrt{\frac{3+\sqrt{17}}{4}+\frac{\sqrt{10+6\sqrt{17}}}{4}}$$

Refs.: N1(316),N2,RP,We

Table of g_n (*Continued*)

$n = 38$

$$g = g_{38}, \quad \text{where} \quad g^3 + g\sqrt{2} = \sqrt{1+\sqrt{2}}\left(1 + g^2\sqrt{2}\right)$$

Refs.: N1(344),We

$n = 42$

$$(2\sqrt{2}+\sqrt{7})^{1/6}\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)^{1/2}$$

Refs.: N1(316),N2,We,BM

$n = 46$

$$\sqrt{\frac{3+\sqrt{2}+\sqrt{7+6\sqrt{2}}}{2}}$$

Refs.: N1(316),N2,We

$n = 50$

$$\frac{1}{3}\left(1+\left(\frac{5+\sqrt{5}}{4}\right)^{1/3}\left(\sqrt[3]{1+7\sqrt{5}+6\sqrt{6}}+\sqrt[3]{1+7\sqrt{5}-6\sqrt{6}}\right)\right)$$

Refs.: N1(318,344),W5,We

$n = 58$

$$\sqrt{\frac{5+\sqrt{29}}{2}}$$

Refs.: N1(316),N2,RP,We

$n = 62$

$$\left(\sqrt{\frac{4+\sqrt{1+\sqrt{2}}+\sqrt{9+5\sqrt{2}}}{8}}+\sqrt{\frac{\sqrt{1+\sqrt{2}}+\sqrt{9+5\sqrt{2}}-4}{8}}\right)^2$$

Refs.: N1(319),RP,WII,W6

$n = 66$

$$\left(\sqrt{2}+\sqrt{3}\right)^{1/4}\left(7\sqrt{2}+3\sqrt{11}\right)^{1/12}\left(\sqrt{\frac{7+\sqrt{33}}{8}}+\sqrt{\frac{\sqrt{33}-1}{8}}\right)^{1/2}$$

Refs.: N1(319),RP,WI

$n = 70$

$$\sqrt{\frac{(3+\sqrt{5})(1+\sqrt{2})}{2}}$$

Refs.: N1(316),N2,RP,We

Table of g_n (*Continued*)

$n = 78$

$$\left(\frac{3+\sqrt{13}}{2}\right)^{1/2}(5+\sqrt{26})^{1/6}$$

Refs.: N1(316),N2,We

$n = 82$

$$\sqrt{\frac{13+\sqrt{41}}{8}}+\sqrt{\frac{5+\sqrt{41}}{8}}$$

Refs.: N1(316),N2,We,BM

$n = 90$

$$\left\{(2+\sqrt{5})(\sqrt{5}+\sqrt{6})\right\}^{1/6}\left(\sqrt{\frac{3+\sqrt{6}}{4}}+\sqrt{\frac{\sqrt{6}-1}{4}}\right)$$

Refs.: N1(318),N2,RP

$n = 94$

$$\left(\sqrt{\frac{4+\sqrt{7+\sqrt{2}}+\sqrt{7+5\sqrt{2}}}{8}}+\sqrt{\frac{\sqrt{7+\sqrt{2}}+\sqrt{7+5\sqrt{2}}-4}{8}}\right)^2$$

Refs.: N1(319),RP,WII,W6

$n = 98$

$$\left(\sqrt{\frac{4+\sqrt{2}+\sqrt{14+4\sqrt{14}}}{8}}+\sqrt{\frac{\sqrt{2}+\sqrt{14+4\sqrt{14}}-4}{8}}\right)^2$$

Refs.: N1(318),RP,WII,W6

$n = 102$

$$(1+\sqrt{2})^{1/2}(3\sqrt{2}+\sqrt{17})^{1/3}$$

Refs.: N1(316),N2,We

$n = 114$

$$\left(\sqrt{2}+\sqrt{3}\right)^{1/4}\left(3\sqrt{2}+\sqrt{19}\right)^{1/12}\left(\sqrt{\frac{23+3\sqrt{57}}{8}}+\sqrt{\frac{15+3\sqrt{57}}{8}}\right)^{1/2}$$

Refs.: N1(319),RP,WI

$n = 126$

$$\sqrt{\frac{\sqrt{3}+\sqrt{7}}{2}}\left(\sqrt{6}+\sqrt{7}\right)^{1/6}\left(\sqrt{\frac{3+\sqrt{2}}{4}}+\sqrt{\frac{\sqrt{2}-1}{4}}\right)^2$$

Refs.: N1(318),RP,WI

Table of g_n (*Continued*)

$n = 130$

$$\left(\frac{1+\sqrt{5}}{2}\right)^{3/2}\left(\frac{3+\sqrt{13}}{2}\right)^{1/2}$$

Refs.: N1(316),N2,RP,We

$n = 138$

$$\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)^{1/4}\left(78\sqrt{2}+23\sqrt{23}\right)^{1/12}\left(\sqrt{\frac{5+2\sqrt{6}}{4}}+\sqrt{\frac{1+2\sqrt{6}}{4}}\right)^{1/2}$$

Refs.: N1(319),RP,WI

$n = 142$

$$\sqrt{\frac{9+5\sqrt{2}+\sqrt{127+90\sqrt{2}}}{2}}$$

Refs.: N1(316),N2,We

$n = 154$

$$\left(2\sqrt{2}+\sqrt{7}\right)^{1/4}\left(\frac{\sqrt{7}+\sqrt{11}}{2}\right)^{1/4}\left(\sqrt{\frac{13+2\sqrt{22}}{4}}+\sqrt{\frac{9+2\sqrt{22}}{4}}\right)^{1/2}$$

Refs.: N1(319),RP,WI

$n = 158$

$$\left(\sqrt{\frac{4+\sqrt{9+\sqrt{2}}+\sqrt{17+13\sqrt{2}}}{8}}+\sqrt{\frac{\sqrt{9+\sqrt{2}}+\sqrt{17+13\sqrt{2}}-4}{8}}\right)^{2}$$

Refs.: N1(319),RP,WII,W6

$n = 190$

$$\left(\frac{1+\sqrt{5}}{2}\right)^{3/2}(3+\sqrt{10})^{1/2}$$

Refs.: N1(316),N2,RP,We

$n = 198$

$$\sqrt{1+\sqrt{2}}(4\sqrt{2}+\sqrt{33})^{1/6}\left(\sqrt{\frac{9+\sqrt{33}}{8}}+\sqrt{\frac{1+\sqrt{33}}{8}}\right)$$

Refs.: N1(318),N2,RP

$n = 210$

$$\sqrt{\sqrt{3}+\sqrt{2}}(3\sqrt{14}+5\sqrt{5})^{1/6}\sqrt{\frac{\sqrt{7}+\sqrt{3}}{2}}\sqrt{\frac{\sqrt{5}+1}{2}}$$

Refs.: N1(320),We,BM

Table of g_n (*Continued*)

$n = 238$

$$\left(\sqrt{\frac{1+2\sqrt{2}}{4}}+\sqrt{\frac{5+2\sqrt{2}}{4}}\right)\left(\sqrt{\frac{1+3\sqrt{2}}{4}}+\sqrt{\frac{5+3\sqrt{2}}{4}}\right)$$

Refs.: N1(319),RP,WI

$n = 310$

$$\left(\frac{1+\sqrt{5}}{2}\right)\sqrt{1+\sqrt{2}}\left(\sqrt{\frac{7+2\sqrt{10}}{4}}+\sqrt{\frac{3+2\sqrt{10}}{4}}\right)$$

Refs.: N1(319),RP,WI

$n = 330$

$$\sqrt{\sqrt{6}+\sqrt{5}}\sqrt{\frac{\sqrt{15}+\sqrt{11}}{2}}\left(\frac{\sqrt{5}+1}{2}\right)(\sqrt{11}+\sqrt{10})^{1/6}$$

Refs.: N1(320),We

$n = 522$

$$\sqrt{\frac{5+\sqrt{29}}{2}}\left(5\sqrt{29}+11\sqrt{6}\right)^{1/6}\left(\sqrt{\frac{9+3\sqrt{6}}{4}}+\sqrt{\frac{5+3\sqrt{6}}{4}}\right)$$

Refs.: N1(318),RP,WI

$n = 630$

$$\begin{aligned}&\left(\sqrt{14}+\sqrt{15}\right)^{1/6}\sqrt{\left(\left(1+\sqrt{2}\right)\left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)\right)}\\ &\times\left(\sqrt{\frac{\sqrt{15}+\sqrt{7}+2}{4}}+\sqrt{\frac{\sqrt{15}+\sqrt{7}-2}{4}}\right)\\ &\times\left(\sqrt{\frac{\sqrt{15}+\sqrt{7}+4}{8}}+\sqrt{\frac{\sqrt{15}+\sqrt{7}-4}{8}}\right)\end{aligned}$$

Refs.: N1(318),RP,WI

3. Computation of G_n and g_n when $9|n$

In this section, we establish Ramanujan's class invariants for $n = 117, 153, 441, 90$, and 198. Note that for each such n, $9|n$. Our starting point is a relation connecting g_n and g_{9n}, found on page 318 of Ramanujan's first notebook, but not in his second notebook. K. G. Ramanathan [4] noticed this relation in the first notebook, but apparently he never gave a proof. Also unaware of its appearance in the first notebook, J. M. and P. B. Borwein [1, pp. 145, 149], although not stating the results explicitly, derived a formula connecting g_n and g_{9n}, as well as a formula relating G_n and G_{9n}. We use one of Ramanujan's modular equations of degree 3 to establish the aforementioned formulas connecting G_n and G_{9n}, and g_n and g_{9n}. The former formula is not found in the notebooks, but it can be proved along the same lines as the latter. The Borweins [1, pp. 145, 146] also derived formulas connecting G_{81n} with G_n and G_{9n}, and g_{81n} with g_n and g_{9n}.

Of course, the theorems described above can be utilized to establish other class invariants found by Ramanujan when $9|n$, in particular, for $n = 27, 45, 63, 81, 225, 333, 765, 18, 126, 522$, and 630. Undoubtedly, some of these proofs would be simpler than previous proofs, for example, for 81, 225, 333, 765, 126, 522, and 630. In particular, Watson's proofs for $n = 333, 765, 522$, and 630 [6] were based on his "empirical method." In fact, the Borweins [1, pp. 147, 149, 150] employed the aforementioned formulas to calculate the invariants G_{27}, G_{81}, G_{225}, and g_{522}. Moreover, previously undetermined class invariants, for example, for $n = 171$, 189, and 279 can be calculated. However, we shall confine ourselves here to the cases, $n = 117, 153, 441, 90$, and 98.

Theorem 3.1. *Let*

$$p = G_n^4 + G_n^{-4}. \tag{3.1}$$

Then

$$G_{9n} = G_n\left(p+\sqrt{p^2-1}\right)^{1/6}\left\{\sqrt{\frac{p^2-2+\sqrt{(p^2-1)(p^2-4)}}{2}} + \sqrt{\frac{p^2-4+\sqrt{(p^2-1)(p^2-4)}}{2}}\right\}^{1/3}. \tag{3.2}$$

Proof. For brevity, we set $G = G_n$ in most of the proof.

We shall employ Entry 15(xii) of Chapter 19 of Ramanujan's second notebook (Part III [3, p. 231]). Let

$$P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \quad \text{and} \quad Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}, \tag{3.3}$$

where β has degree 3. Then

$$Q + \frac{1}{Q} + 2\sqrt{2}\left(P - \frac{1}{P}\right) = 0. \tag{3.4}$$

Recall from (1.6) that, when $q = \exp(-\pi\sqrt{n})$,

$$G_n = \{4\alpha_n(1-\alpha_n)\}^{-1/24} \quad \text{and} \quad G_{9n} = \{4\beta_n(1-\beta_n)\}^{-1/24}.$$

Hence, by (3.3), $P = (G_nG_{9n})^{-3}$ and $Q = (G_n/G_{9n})^6$. Thus, from (3.4),

$$(G_n/G_{9n})^6 + (G_n/G_{9n})^{-6} + 2\sqrt{2}\left((G_nG_{9n})^{-3} - (G_nG_{9n})^3\right) = 0. \tag{3.5}$$

Set $x = (G_{9n}/G_n)^3$. Then (3.5) can be rewritten in the form

$$x^4 - 2\sqrt{2}G^6x^3 + 2\sqrt{2}G^{-6}x + 1 = 0. \tag{3.6}$$

Rearranging and using the notation (3.1), we find that we can recast (3.6) in the form

$$(x^2 - \sqrt{2}G^6x + p)^2 = 2(p^2 - 1)\left(G^2x - \frac{1}{\sqrt{2}}\right)^2.$$

Since $x > 1$,

$$x^2 - \sqrt{2}G^6x + p = \sqrt{2(p^2 - 1)}\left(G^2x - \frac{1}{\sqrt{2}}\right),$$

or

$$x^2 - \sqrt{2}G^2\left(G^4 + \sqrt{p^2 - 1}\right)x + p + \sqrt{p^2 - 1} = 0.$$

Remembering that $x > 1$ when solving for x, we find that

$$\begin{aligned} x = {} & \frac{1}{\sqrt{2}}G^2\left(G^4 + \sqrt{p^2 - 1}\right) \\ & + \frac{1}{\sqrt{2}}\sqrt{G^4\left(G^8 + 2G^4\sqrt{p^2 - 1} + p^2 - 1\right) - 2p - 2\sqrt{p^2 - 1}}. \end{aligned} \tag{3.7}$$

Now,

$$G^4 = \tfrac{1}{2}\left(p + \sqrt{p^2 - 4}\right) \qquad \text{and} \qquad G^8 = \tfrac{1}{2}\left(p^2 - 2 + p\sqrt{p^2 - 4}\right).$$

Thus, squaring and expanding, we find that

$$\begin{aligned} & G^4\left(G^4 + \sqrt{p^2 - 1}\right)^2 \\ & = \tfrac{1}{2}\left(p + \sqrt{p^2 - 4}\right)\tfrac{1}{2}\left(p^2 - 2 + p\sqrt{p^2 - 4}\right) \\ & \quad + \left(p^2 - 2 + p\sqrt{p^2 - 4}\right)\sqrt{p^2 - 1} + \tfrac{1}{2}\left(p + \sqrt{p^2 - 4}\right)(p^2 - 1) \\ & = \left(p + \sqrt{p^2 - 1}\right)\left(p^2 - 2 + \sqrt{(p^2 - 1)(p^2 - 4)}\right) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} & G^4\left(G^8 + 2G^4\sqrt{p^2 - 1} + p^2 - 1\right) - 2p - 2\sqrt{p^2 - 1} \\ & = \tfrac{1}{2}\left(p + \sqrt{p^2 - 4}\right)\tfrac{1}{2}\left(p^2 - 2 + p\sqrt{p^2 - 4}\right) \\ & \quad + \left(p^2 - 2 + p\sqrt{p^2 - 4}\right)\sqrt{p^2 - 1} \\ & \quad + \tfrac{1}{2}\left(p + \sqrt{p^2 - 4}\right)(p^2 - 1) - 2p - 2\sqrt{p^2 - 1} \\ & = \left(p + \sqrt{p^2 - 1}\right)\left(p^2 - 4 + \sqrt{(p^2 - 1)(p^2 - 4)}\right). \end{aligned} \tag{3.9}$$

Using (3.8) and (3.9) in (3.7), we deduce that

$$x = \frac{1}{\sqrt{2}}\sqrt{\left(p+\sqrt{p^2-1}\right)\left(p^2-2+\sqrt{(p^2-1)(p^2-4)}\right)} + \frac{1}{\sqrt{2}}\sqrt{\left(p+\sqrt{p^2-1}\right)\left(p^2-4+\sqrt{(p^2-1)(p^2-4)}\right)}. \tag{3.10}$$

Recalling that $x = (G_{9n}/G)^3$, we see that (3.10) is equivalent to (3.2), and so the proof is complete.

We next prove the aforementioned result found on page 318 in Ramanujan's first notebook.

Theorem 3.2 (p. 318). *Let*

$$p = g_n^4 - g_n^{-4}. \tag{3.11}$$

Then

$$g_{9n} = g_n\left(p+\sqrt{p^2+1}\right)^{1/6}\left\{\sqrt{\frac{p^2+4+\sqrt{(p^2+1)(p^2+4)}}{2}} + \sqrt{\frac{p^2+2+\sqrt{(p^2+1)(p^2+4)}}{2}}\right\}^{1/3}. \tag{3.12}$$

Proof. Set $g = g_n$ throughout the proof.

Using (1.3) and (1.1), we rewrite (3.5) in the form

$$\frac{\sqrt{q}(-q;q^2)_\infty^6}{(-q^3;q^6)_\infty^6} + \frac{(-q^3;q^6)_\infty^6}{\sqrt{q}(-q;q^2)_\infty^6} + \frac{8\sqrt{q}}{(-q;q^2)_\infty^3(-q^3;q^6)_\infty^3} - \frac{(-q;q^2)_\infty^3(-q^3;q^6)_\infty^3}{\sqrt{q}} = 0.$$

Multiplying both sides by $\sqrt{q}$ and then replacing q by $-q$, we find that

$$-\frac{q(q;q^2)_\infty^6}{(q^3;q^6)_\infty^6} + \frac{(q^3;q^6)_\infty^6}{(q;q^2)_\infty^6} - \frac{8q}{(q;q^2)_\infty^3(q^3;q^6)_\infty^3} - (q;q^2)_\infty^3(q^3;q^6)_\infty^3 = 0.$$

Using (1.3) and (1.1) again, we find that

$$-(g/g_{9n})^6 + (g_{9n}/g)^6 - 2\sqrt{2}(gg_{9n})^{-3} - 2\sqrt{2}(gg_{9n})^3 = 0.$$

Setting $x = (g_{9n}/g)^3$, we deduce that

$$x^4 - 2\sqrt{2}g^6x^3 - 2\sqrt{2}g^{-6}x - 1 = 0.$$

Recalling the notation (3.11), we see that the last equation can be rewritten in the form

$$(x^2 - \sqrt{2}g^6 x - p)^2 = 2(p^2 + 1)\left(g^2 x + \frac{1}{\sqrt{2}}\right)^2.$$

It should now be clear that the remainder of the proof is completely analogous to that for Theorem 3.1, and so we omit the rest of the proof.

As a bonus, the formulas connecting G_{9n} with G_n and g_{9n} with g_n led to closed form evaluations of Ramanujan's cubic continued fraction at the arguments $\pm\exp(-\pi\sqrt{n})$ by the author, H. H. Chan, and L.–C. Zhang [1].

The cube roots in (3.2) and (3.12) are not very attractive, and usually in applications Ramanujan found more appealing expressions for these cube roots. Ramanujan had an amazing ability for denesting and simplifying radicals, and we do not have the insights into radicals that Ramanujan had. However, it seems quite likely that in several instances Ramanujan used the following elementary result from Carr's book [1, p. 52]. Since Carr does not give a proof and since he adds the extraneous hypothesis that $(a^2 - b)^{1/3}$ be a perfect cube, we provide a proof here.

Lemma 3.3. *Suppose that* $c := (a^2 - b)^{1/3}$. *Then we can write*

$$(a + \sqrt{b})^{1/3} = x + \sqrt{y}, \tag{3.13}$$

where

$$4x^3 - 3cx = a \tag{3.14}$$

and

$$y = x^2 - c. \tag{3.15}$$

Proof. From (3.13), we easily see that

$$\frac{(a - \sqrt{b})^{1/3}}{c} = \frac{x - \sqrt{y}}{x^2 - y}.$$

Suppose we set $c = x^2 - y$, so that

$$(a - \sqrt{b})^{1/3} = x - \sqrt{y}. \tag{3.16}$$

Cubing both sides of (3.13) and (3.16) and solving for a, we find that

$$a = x^3 + 3xy. \tag{3.17}$$

But since $y = x^2 - c$, we deduce (3.14) from (3.17).

Usually, it is best to solve (3.14) by trial or inspection, for if, for example, Cardan's method is used, the value of x so obtained most frequently is the cube root that we originally sought to simplify.

Since Carr's book [1] was Ramanujan's primary source for learning mathematics, it seems likely that Ramanujan employed Lemma 3.3 in simplifications. However, because most of us do not possess Ramanujan's ability to discern algebraic relationships, we describe another procedure that rests upon elementary considerations in algebraic number theory and involves less guessing.

We see from Theorems 3.1 and 3.2 that it would be advantageous to find a number a such that

$$a^3 := \sqrt{b+1+c\sqrt{d}} + \sqrt{b+c\sqrt{d}}. \tag{3.18}$$

Then

$$a^{-3} = \sqrt{b+1+c\sqrt{d}} - \sqrt{b+c\sqrt{d}}$$

and

$$a^3 + a^{-3} = 2\sqrt{b+1+c\sqrt{d}}. \tag{3.19}$$

Since

$$a^3 + a^{-3} = \left(a + \frac{1}{a}\right)^3 - 3\left(a + \frac{1}{a}\right),$$

set

$$u = a + \frac{1}{a}, \tag{3.20}$$

so that, after squaring, (3.19) takes the shape

$$u^2(u^2-3)^2 = 4\left(b+1+c\sqrt{d}\right). \tag{3.21}$$

Assuming the relevant expressions above are algebraic integers, we find that, upon taking norms in (3.21),

$$N(u^2)N^2(u^2-3) = N\left(4\left(b+1+c\sqrt{d}\right)\right). \tag{3.22}$$

Using (3.22), we determine u. We then solve (3.20) for a.

Entry 3.4.

$$G_{117} = \frac{1}{2}\left(\frac{3+\sqrt{13}}{2}\right)^{1/4} (2\sqrt{3}+\sqrt{13})^{1/6}\left(3^{1/4} + \sqrt{4+\sqrt{3}}\right). \tag{3.23}$$

Proof. Let $n = 13$ in Theorem 3.1. From Weber's treatise [2, p. 721], or from the tables of the last section,

$$G_{13} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4}.$$

By (3.1), $p = \sqrt{13}$. Thus, by a direct application of (3.2),

$$G_{117} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4} (2\sqrt{3}+\sqrt{13})^{1/6} \left\{\sqrt{\frac{11+6\sqrt{3}}{2}} + \sqrt{\frac{9+6\sqrt{3}}{2}}\right\}^{1/3}.$$

(On page 314 in his first notebook [9], Ramanujan recorded G_{117} in the form given above, which is strong evidence that Ramanujan utilized Theorem 3.1 as we have done.) It therefore remains to show that

$$\left\{\sqrt{\frac{11+6\sqrt{3}}{2}} + \sqrt{\frac{9+6\sqrt{3}}{2}}\right\}^{1/3} = \frac{1}{2}\left(3^{1/4} + \sqrt{4+\sqrt{3}}\right). \tag{3.24}$$

We apply Lemma 3.3 with

$$a = \sqrt{\frac{11+6\sqrt{3}}{2}} \qquad \text{and} \qquad b = \frac{9+6\sqrt{3}}{2}.$$

Thus, $c = 1$. We therefore need to solve

$$4x^3 - 3x = \sqrt{\frac{11+6\sqrt{3}}{2}}.$$

To solve this by inspection, it perhaps is best to square both sides and set $t = x^2$. Thus,

$$t(4t-3)^2 = \frac{11+6\sqrt{3}}{2}.$$

It is not difficult to see that $t = 1 + \sqrt{3}/4$. Hence, $x = \frac{1}{2}\sqrt{4+\sqrt{3}}$, and, from (3.15), $y = \sqrt{3}/4$. Thus, (3.24) follows, and the proof is complete.

Alternatively, in the notation (3.18) and (3.20), by (3.21), we want to solve

$$u^2(u^2-3)^2 = 22 + 12\sqrt{3}. \tag{3.25}$$

Factoring in $\mathbb{Z}[\sqrt{3}]$ and using (3.22), we find that

$$N(u^2)N^2(u^2-3) = N(22+12\sqrt{3}) = 2^2 \cdot 13.$$

Then

$$\pm 2 = N(u^2-3) =: N(A+B\sqrt{3}) = A^2 - 3B^2.$$

Choose $A = 1 = B$. We now observe that

$$N(u^2) = N(4+\sqrt{3}) = 13,$$

as required. It is easily checked that, when $u = \sqrt{4+\sqrt{3}}$, (3.25) holds. We readily then find that

$$a = \frac{1}{2}\left(3^{1/4} + \sqrt{4+\sqrt{3}}\right).$$

Thus, (3.24) has been shown once again.

Entry 3.5.

$$G_{153} = \left(\sqrt{\frac{5+\sqrt{17}}{8}} + \sqrt{\frac{\sqrt{17}-3}{8}}\right)^2 \left(\sqrt{\frac{37+9\sqrt{17}}{4}} + \sqrt{\frac{33+9\sqrt{17}}{4}}\right)^{1/3}. \tag{3.26}$$

Proof. Set $n = 17$ in Theorem 3.1. From Weber's treatise [2, p. 721], or from the tables in Section 2,

$$G_{17} = \sqrt{\frac{5+\sqrt{17}}{8}} + \sqrt{\frac{\sqrt{17}-3}{8}}. \tag{3.27}$$

Since

$$G_{17}^4 = \frac{5+\sqrt{17}}{4} + \frac{(1+\sqrt{17})^{3/2}}{4\sqrt{2}},$$

from (3.1) we find that $p = (5+\sqrt{17})/2$. Also,

$$p + \sqrt{p^2-1} = \frac{5+\sqrt{17}}{2} + \sqrt{\frac{19+5\sqrt{17}}{2}}.$$

Thus, from Theorem 3.1,

$$G_{153} = \left(\sqrt{\frac{5+\sqrt{17}}{8}} + \sqrt{\frac{\sqrt{17}-3}{8}}\right)\left(\frac{5+\sqrt{17}}{2} + \sqrt{\frac{19+5\sqrt{17}}{2}}\right)^{1/6}$$
$$\times \left\{\sqrt{\frac{17+5\sqrt{17}+\sqrt{(19+5\sqrt{17})(13+5\sqrt{17})}}{4}}\right.$$
$$\left. + \sqrt{\frac{13+5\sqrt{17}+\sqrt{(19+5\sqrt{17})(13+5\sqrt{17})}}{4}}\right\}^{1/3}.$$

However, note that

$$(19+5\sqrt{17})(13+5\sqrt{17}) = 672 + 160\sqrt{17} = (20+4\sqrt{17})^2.$$

Hence,

$$G_{153} = \left(\sqrt{\frac{5+\sqrt{17}}{8}} + \sqrt{\frac{\sqrt{17}-3}{8}}\right)\left(\frac{5+\sqrt{17}}{2} + \sqrt{\frac{19+5\sqrt{17}}{2}}\right)^{1/6}$$

$$\times\left\{\sqrt{\frac{37+9\sqrt{17}}{4}}+\sqrt{\frac{33+9\sqrt{17}}{4}}\right\}^{1/3}.$$

Comparing the equality above with (3.26), by (3.27), we see that it remains to show that

$$\frac{5+\sqrt{17}}{2}+\sqrt{\frac{19+5\sqrt{17}}{2}}=G_{17}^{6}, \tag{3.28}$$

which is rather curious indeed. Note that

$$\left(\sqrt{\frac{7+\sqrt{17}}{4}}+\sqrt{\frac{3+\sqrt{17}}{4}}\right)^{2}=\frac{5+\sqrt{17}}{2}+\sqrt{\frac{19+5\sqrt{17}}{2}}.$$

Thus, by (3.28), it suffices to show that

$$\sqrt{\frac{7+\sqrt{17}}{4}}+\sqrt{\frac{3+\sqrt{17}}{4}}=G_{17}^{3}. \tag{3.29}$$

In the notation (3.18) and (3.20), we solve

$$u^2(u^2-3)^2=7+\sqrt{17}. \tag{3.30}$$

We now factor in the principal ideal domain $\mathbb{Z}\left[(1+\sqrt{17})/2\right]$. Thus,

$$N(u^2)N^2(u^2-3)=N(7+\sqrt{17})=2^5.$$

We attempt to solve

$$\pm 4=N(u^2-3)=:N\left(A+B\frac{1+\sqrt{17}}{2}\right)=A^2+AB-4B^2.$$

Take $A=-1$ and $B=1$. Then $u^2=(5+\sqrt{17})/2$ and $N(u^2)=2$. A simple calculation shows that (3.30) indeed holds. Lastly, we find that

$$a=\sqrt{\frac{5+\sqrt{17}}{8}}+\sqrt{\frac{\sqrt{17}-3}{8}}.$$

By (3.27), we conclude that (3.29) holds to complete the proof.

Entry 3.6.

$$G_{441}=\sqrt{\frac{2+\sqrt{7}+\sqrt{7+4\sqrt{7}}}{2}}\sqrt{\frac{\sqrt{3}+\sqrt{7}}{2}}(2+\sqrt{3})^{1/6}\sqrt{\frac{\sqrt{3+\sqrt{7}}+6^{1/4}7^{1/8}}{\sqrt{3+\sqrt{7}}-6^{1/4}7^{1/8}}}. \tag{3.31}$$

Proof. We apply Theorem 3.1 with $n = 49$. From Weber's treatise [2, p. 723], or from the tables of Section 2,

$$G_{49} = \frac{\sqrt{4+\sqrt{7}}+7^{1/4}}{2}. \tag{3.32}$$

It is easily checked that

$$\frac{\sqrt{4+\sqrt{7}}+7^{1/4}}{2} = \sqrt{\frac{2+\sqrt{7}+\sqrt{7+4\sqrt{7}}}{2}}. \tag{3.33}$$

After a somewhat lengthy calculation, we find that

$$G_{49}^{\pm 4} = \frac{9+4\sqrt{7}}{2} \pm \frac{(9+3\sqrt{7})7^{1/4}}{2\sqrt{2}}.$$

Thus, $p = 9+4\sqrt{7}$. After a mild calculation,

$$\begin{aligned} p+\sqrt{p^2-1} &= 9+4\sqrt{7}+2\sqrt{3}\sqrt{16+6\sqrt{7}} \\ &= 9+4\sqrt{7}+2\sqrt{3}(3+\sqrt{7}) \\ &= (2+\sqrt{3})(3\sqrt{3}+2\sqrt{7}) \\ &= (2+\sqrt{3})\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)^3. \end{aligned} \tag{3.34}$$

Using (3.32)–(3.34) in Theorem 3.1, we deduce that

$$\begin{aligned} G_{441} = {} & \sqrt{\frac{2+\sqrt{7}+\sqrt{7+4\sqrt{7}}}{2}}\sqrt{\frac{\sqrt{3}+\sqrt{7}}{2}}(2+\sqrt{3})^{1/6} \\ & \times \left\{\sqrt{\frac{191+72\sqrt{7}+A}{2}}+\sqrt{\frac{189+72\sqrt{7}+A}{2}}\right\}^{1/3}, \end{aligned} \tag{3.35}$$

where

$$\begin{aligned} A := {} & \sqrt{(192+72\sqrt{7})(189+72\sqrt{7})} \\ = {} & 6\sqrt{2016+762\sqrt{7}} = 6\cdot 7^{1/4}\sqrt{\tfrac{3}{2}}(3+\sqrt{7})^2. \end{aligned}$$

Using this calculation in (3.35), we see from (3.35) and (3.31) that it remains to prove that

$$\sqrt[3]{\sqrt{\frac{191+72\sqrt{7}+6\cdot 7^{1/4}\sqrt{\frac{3}{2}}(3+\sqrt{7})^2}{2}} + \sqrt{\frac{189+72\sqrt{7}+6\cdot 7^{1/4}\sqrt{\frac{3}{2}}(3+\sqrt{7})^2}{2}}}$$

$$= \sqrt{\frac{\sqrt{3+\sqrt{7}}+6^{1/4}7^{1/8}}{\sqrt{3+\sqrt{7}}-6^{1/4}7^{1/8}}}$$

$$= \frac{(\sqrt{3+\sqrt{7}}+6^{1/4}7^{1/8})\sqrt{3+\sqrt{7}+6^{1/2}7^{1/4}}}{4}$$

$$= \tfrac{1}{4}\sqrt{16+6\sqrt{7}+(3+\sqrt{7})6^{1/2}7^{1/4}}+\tfrac{1}{4}\sqrt{6\sqrt{7}+(3+\sqrt{7})6^{1/2}7^{1/4}}. \tag{3.36}$$

We apply Lemma 3.3 with

$$a=\sqrt{\frac{191+72\sqrt{7}+6\cdot 7^{1/4}\sqrt{\frac{3}{2}}(3+\sqrt{7})^2}{2}}$$

and

$$b=\frac{189+72\sqrt{7}+6\cdot 7^{1/4}\sqrt{\frac{3}{2}}(3+\sqrt{7})^2}{2}.$$

Again, $c=1$. Setting $x^2=t$ in (3.14), we see that we must solve

$$\begin{aligned}t(4t-3)^2 &= \frac{191+72\sqrt{7}+6\cdot 7^{1/4}\sqrt{\frac{3}{2}}(3+\sqrt{7})^2}{2}\\ &= \frac{(16+3\sqrt{7})(3+\sqrt{7})^2+6\sqrt{6}\cdot 7^{1/4}(3+\sqrt{7})^2}{4}\\ &= \frac{(3+\sqrt{7})^2}{4}\left(16+3\sqrt{7}+6\sqrt{6}\cdot 7^{1/4}\right).\end{aligned}$$

We now verify that

$$t=\frac{3+\sqrt{7}}{16}\left(3+\sqrt{7}+6^{1/2}7^{1/4}\right).$$

Thus,

$$x=\frac{\sqrt{(3+\sqrt{7})^2+(3+\sqrt{7})6^{1/2}7^{1/4}}}{4} \quad \text{and} \quad y=6\sqrt{7}+(3+\sqrt{7})6^{1/2}7^{1/4},$$

by (3.15). Thus, (3.36) follows, and the proof is complete.

Entry 3.7.

$$g_{90}=(2+\sqrt{5})^{1/6}(\sqrt{5}+\sqrt{6})^{1/6}\left(\sqrt{\frac{3+\sqrt{6}}{4}}+\sqrt{\frac{\sqrt{6}-1}{4}}\right). \tag{3.37}$$

Proof. Set $n = 10$ in Theorem 3.2. Now from Weber's book [2, p. 721], or from the tables of the preceding section,

$$g_{10} = \sqrt{\frac{1+\sqrt{5}}{2}}.$$

An easy calculation shows that $p = \sqrt{5}$. Note also that

$$g_{10} = \sqrt{\frac{1+\sqrt{5}}{2}} = \left(\left(\frac{1+\sqrt{5}}{2}\right)^3\right)^{1/6} = (2+\sqrt{5})^{1/6}. \tag{3.38}$$

Thus, from (3.12) and (3.38),

$$g_{90} = (2+\sqrt{5})^{1/6}(\sqrt{5}+\sqrt{6})^{1/6}\left\{\sqrt{\frac{9+3\sqrt{6}}{2}}+\sqrt{\frac{7+3\sqrt{6}}{2}}\right\}^{1/3}. \tag{3.39}$$

Comparing (3.39) and (3.37), we see that we must prove that

$$\left\{\sqrt{\frac{9+3\sqrt{6}}{2}}+\sqrt{\frac{7+3\sqrt{6}}{2}}\right\}^{1/3} = \sqrt{\frac{3+\sqrt{6}}{4}}+\sqrt{\frac{\sqrt{6}-1}{4}}. \tag{3.40}$$

In the notation (3.18) and (3.20), we solve

$$u^2(u^2-3)^2 = 18+6\sqrt{6}. \tag{3.41}$$

Factoring in the unique factorization domain $\mathbb{Z}[\sqrt{6}]$, we find that

$$N(u^2)N^2(u^2-3) = N(18+6\sqrt{6}) = 3\cdot 6^2.$$

We thus want to solve

$$\pm 6 = N(u^2-3) =: N(A+B\sqrt{6}) = A^2-6B^2.$$

Choose $A = 0$ and $B = 1$, so that $u^2 = 3+\sqrt{6}$ and $N(u^2) = 3$. It is trivial to see that (3.41) is satisfied. Then

$$a = \sqrt{\frac{3+\sqrt{6}}{4}}+\sqrt{\frac{\sqrt{6}-1}{4}},$$

and the verification of (3.40) is complete.

Entry 3.8.

$$g_{198} = \sqrt{1+\sqrt{2}}\left(4\sqrt{2}+\sqrt{33}\right)^{1/6}\left(\sqrt{\frac{9+\sqrt{33}}{8}}+\sqrt{\frac{1+\sqrt{33}}{8}}\right). \tag{3.42}$$

Proof. Set $n = 22$ in Theorem 3.2. From Weber's work [2, p. 722], or from our tables in the foregoing section,

$$g_{22} = \sqrt{1+\sqrt{2}}.$$

An easy calculation yields $p = 4\sqrt{2}$. Thus, from Theorem 3.2,

$$g_{198} = \sqrt{1+\sqrt{2}}\left(4\sqrt{2}+\sqrt{33}\right)^{1/6}\left\{\sqrt{18+3\sqrt{33}}+\sqrt{17+3\sqrt{33}}\right\}^{1/3}. \tag{3.43}$$

By (3.42) and (3.43), we must show that

$$\left\{\sqrt{18+3\sqrt{33}}+\sqrt{17+3\sqrt{33}}\right\}^{1/3} = \sqrt{\frac{9+\sqrt{33}}{8}}+\sqrt{\frac{1+\sqrt{33}}{8}}. \tag{3.44}$$

In the notation of (3.18) and (3.20), we seek to solve

$$u^2(u^2-3)^2 = 4(18+3\sqrt{33}). \tag{3.45}$$

Thus,

$$N(u^2)N^2(u^2-3) = N(4(18+3\sqrt{33})) = 2^4\cdot 3^3.$$

Factoring in $\mathbb{Z}\left[(1+\sqrt{33})/2\right]$, we attempt to find a solution of

$$\pm 6 = N(u^2-3) =: N\left(A+B\frac{1+\sqrt{33}}{2}\right) = A^2+AB-8B^2.$$

Let $A = 1 = B$, so that $u^2 = (9+\sqrt{33})/2$ and $N(u^2) = 12$. It is easily checked that (3.45) holds. Hence,

$$a = \sqrt{\frac{9+\sqrt{33}}{8}}+\sqrt{\frac{1+\sqrt{33}}{8}}.$$

Thus, we can deduce (3.44), and the proof is complete.

4. Kronecker's Limit Formula and General Formulas for Class Invariants

Let $Q(u, v) := y^{-1}(u+vz)(u+v\bar{z})$, where $z = x+iy$ with $y > 0$. Epstein zeta–function $\zeta_Q(s)$ is defined for $\sigma = \operatorname{Re} s > 1$ by

$$\zeta_Q(s) := \sum_{u,v}\{Q(u, v)\}^{-s}, \tag{4.1}$$

where the sum is over all pairs of integers (u, v) except $(0, 0)$. It is well known that $\zeta_Q(s)$ can be analytically continued to the entire complex $s-$plane, where

$\zeta_Q(s)$ is analytic except for a simple pole at $s = 1$. The Kronecker limit formula provides the constant term in the Laurent expansion about $s = 1$. More precisely,

$$\zeta_Q(s) = \frac{\pi}{s-1} + 2\pi\left(\gamma - \log 2 - \log\left(\sqrt{y}|\eta(z)|^2\right)\right) + O(s-1), \tag{4.2}$$

where γ denotes Euler's constant, and $\eta(z)$ is the Dedekind eta–function defined by

$$\eta(z) := q^{1/24}(q;q)_\infty =: q^{1/24}f(-q), \qquad q = e^{2\pi i z}, y > 0, \tag{4.3}$$

where $f(-q)$ is defined in (1.14) of Chapter 33.

Next, let K be an algebraic number field over the rational numbers. Let $N(\mathfrak{A})$ denote the norm of an ideal $\mathfrak{A}$. Then the Dedekind zeta–function for K is defined by

$$\zeta_K(s) := \sum_{\mathfrak{A}} (N(\mathfrak{A}))^{-s}, \qquad \sigma > 1,$$

where the sum is over all nonzero integral ideals $\mathfrak{A}$ of K. Let C_K denote the ideal class group of K. Then the Dedekind zeta–function for an ideal class A of C_K is defined by

$$\zeta(s, A) := \sum_{\mathfrak{A}\in A} (N(\mathfrak{A}))^{-s}, \qquad \sigma > 1.$$

If χ denotes an ideal class character, then the L–series for K is given, for $\sigma > 1$, by

$$L_K(s, \chi) := \sum_{\mathfrak{A}} \chi(\mathfrak{A})(N(\mathfrak{A}))^{-s} = \sum_{A} \chi(A)\zeta(s, A), \tag{4.4}$$

where the former sum is over all nonzero integral ideals $\mathfrak{A}$ of K, and the latter sum is over all ideal classes A of C_K.

In the sequel we assume that K is a quadratic field. It is well known that (C. L. Siegel [1, p. 58])

$$\lim_{s\to 1}(s-1)\zeta_K(s) = h\kappa, \tag{4.5}$$

where h is the class number of K, i.e., $h = |C_K|$, and where

$$\kappa := \begin{cases} \dfrac{2\pi}{w\sqrt{-d}}, & \text{if } K \text{ is imaginary,} \\[2ex] \dfrac{2\log\epsilon}{\sqrt{d}}, & \text{if } K \text{ is real.} \end{cases} \tag{4.6}$$

Here w is the number of roots of unity in K, d is the discriminant of K, and ϵ is the fundamental unit in K.

Let

$$L_d(s) := \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) n^{-s}, \qquad \sigma > 1,$$

where $\left(\frac{d}{n}\right)$ is the Kronecker symbol. Then (Siegel [1, p. 58])

$$\zeta_K(s) = \zeta(s)L_d(s), \tag{4.7}$$

where $\zeta(s)$ denotes the Riemann zeta-function.

Now let $d = d_1d_2$, where $d_1 > 1$ and, for $i = 1, 2$, $d_i \equiv 1 \pmod 4$ or $d_i \equiv 0 \pmod 4$. Let $\mathfrak{P}$ denote a prime ideal in K. Then a Gauss genus character χ is defined by

$$\chi(\mathfrak{P}) = \begin{cases} \left(\dfrac{d_1}{N(\mathfrak{P})}\right), & \text{if } N(\mathfrak{P}) \nmid d_1, \\ \left(\dfrac{d_2}{N(\mathfrak{P})}\right), & \text{if } N(\mathfrak{P}) | d_1, \end{cases}$$

where $\left(\frac{d_i}{N(\mathfrak{P})}\right)$ again denotes the Kronecker symbol. Note that $N(\mathfrak{P}) \nmid d_2$ if $N(\mathfrak{P})|d_1$. This definition can be extended to all ideals of K by multiplicativity. It is well known that the genus characters form an abelian group, denoted by $G(K)$, of order 2^{k-1}, where k is the number of distinct prime divisors of d.

Next define

$$G_0 := \{A \in C_K : \chi(A) = 1, \chi \in G(K)\},$$

which is named the *principal genus*. Clearly, G_0 is a subgroup of C_K, and C_K/G_0 is called the genus group. Furthermore, $C_K/G_0 \cong G(K)$. Obviously, A_1 and A_2 are in the same genus if and only if $\chi(A_1) = \chi(A_2)$ for each $\chi \in G(K)$.

Kronecker (Siegel [1, p. 62, Theorem 4]) proved that, for a genus character χ of K corresponding to the decomposition $d = d_1d_2$,

$$L_K(s, \chi) = L_{d_1}(s)L_{d_2}(s). \tag{4.8}$$

Thus, by (4.4) and (4.8),

$$L_{d_1}(s)L_{d_2}(s) = \sum_{A \in C_K} \chi(A)\zeta(s, A).$$

For a fixed nonzero integral ideal $\mathfrak{B} \in A^{-1}$,

$$\zeta(s, A) = N(\mathfrak{B})^s \sum_{\mathfrak{A} \in A} (N(\mathfrak{A}\mathfrak{B}))^{-s} = N(\mathfrak{B})^s \sum_{\lambda \in \mathfrak{B}/U} (N(\lambda))^{-s}, \qquad \sigma > 1, \tag{4.9}$$

where U is the group of units in K. Now assume that $K = \mathbb{Q}(\sqrt{-m})$ is an imaginary quadratic field, and so m is a squarefree positive integer. Recalling that w is the number of roots of unity in K, we see that, from (4.9),

$$\zeta(s, A) = \frac{N(\mathfrak{B})^s}{w} \sum_{\substack{\lambda \in \mathfrak{B} \\ \lambda \neq 0}} (N(\lambda))^{-s}, \qquad \sigma > 1. \tag{4.10}$$

Let

$$\Omega = \begin{cases} \sqrt{-m}, & \text{if } -m \equiv 2, 3 \pmod 4, \\ (1 + \sqrt{-m})/2, & \text{if } -m \equiv 1 \pmod 4. \end{cases}$$

Then

$$d = \begin{cases} -4m, & \text{if } -m \equiv 2, 3 \pmod 4, \\ -m, & \text{if } -m \equiv 1 \pmod 4. \end{cases}$$

It is known (R. Mollin and L.-C. Zhang [1]) that each ideal class contains primitive ideals which are $\mathbb{Z}$-modules of the form $\mathfrak{B} = [a, b+\Omega]$, where a and b are rational integers, $a > 0$, $a|N(b+\Omega)$, $|b| \le a/2$, a is the smallest positive integer in $\mathfrak{B}$, and $N(\mathfrak{B}) = a$.

Let $z = (b+\Omega)/a$. Then, for $\lambda = ua + v(b+\Omega)$,

$$\begin{aligned} N(\lambda) &= (ua + v(b+\Omega))(ua + v(b+\bar{\Omega})) \\ &= a^2(u+vz)(u+v\bar{z}) \\ &= \frac{a\sqrt{|d|}}{2}\left(\frac{\sqrt{|d|}}{2a}\right)^{-1}(u+vz)(u+v\bar{z}). \end{aligned} \tag{4.11}$$

Thus, for $z = (b+\Omega)/a$ and $y = \operatorname{Im} z = \sqrt{|d|}/(2a)$,

$$Q(u,v) = \left(\frac{\sqrt{|d|}}{2a}\right)^{-1}(u+vz)(u+v\bar{z}).$$

And, from (4.1), (4.10), and (4.11),

$$\zeta(s, A) = \frac{1}{w}\left(\frac{2}{\sqrt{|d|}}\right)^s \zeta_Q(s).$$

Thus, from (4.2),

$$\begin{aligned} \zeta(s, A) = &\frac{1}{w}\left(\frac{2}{\sqrt{|d|}}\right)^s \left(\frac{\pi}{s-1} + 2\pi\gamma - 2\pi\log 2 + \frac{\pi}{2}\log\sqrt{|d|}\right) \\ &- \frac{2\pi}{w}\left(\frac{2}{\sqrt{|d|}}\right)^s \left(-\tfrac{1}{2}\log(2a) + \log|\eta(z)|^2\right) + O(s-1). \end{aligned} \tag{4.12}$$

Since, for any nonprincipal genus character χ,

$$\sum_{A\in C_K} \chi(A) = 0,$$

it follows from (4.4) and (4.12) that

$$L_K(s,\chi) = -\frac{2\pi}{w}\left(\frac{2}{\sqrt{|d|}}\right)^s \sum_{A\in I_K} \chi(A)\left(-\tfrac{1}{2}\log a + \log|\eta(z)|^2\right) + O(s-1). \tag{4.13}$$

Recall that in the decomposition $d = d_1 d_2$ we assume that $d_1 > 1$ and $d_2 < 0$. Let $K_i = \mathbb{Q}(\sqrt{d_i})$, $i = 1, 2$. By (4.7),

$$\lim_{s\to 1}(s-1)\zeta_{K_i}(s) = L_{d_i}(1), \qquad i = 1, 2.$$

Then, by (4.5) and (4.6),

$$L_{d_1}(1) = \frac{2h_1 \log \epsilon_1}{\sqrt{d_1}} \tag{4.14}$$

and

$$L_{d_2}(1) = \frac{2h_2\pi}{w_2\sqrt{|d_2|}}, \tag{4.15}$$

where h_i is the class number of K_i, $i = 1, 2$, ϵ_1 is the fundamental unit of K_1, and w_2 is the number of roots of unity in K_2. Thus, setting $s = 1$ in (4.13) and using (4.8), we deduce that

$$L_{d_1}(1)L_{d_2}(1) = -\frac{4\pi}{w\sqrt{|d|}} \sum_{A \in C_K} \chi(A)\left(-\tfrac{1}{2}\log a + \log|\eta(z)|^2\right). \tag{4.16}$$

Thus, setting

$$F(A) = |\eta(z)|^2/\sqrt{a}, \tag{4.17}$$

where $z = (b + \Omega)/a$, with $[a, b + \Omega] \in A^{-1}$, we conclude from (4.14)–(4.17) that, for χ nonprincipal (Siegel [1, p. 72]),

$$\frac{wh_1h_2\log\epsilon_1}{w_2} = -\sum_{A \in C_K} \chi(A)\log F(A),$$

or

$$\epsilon_1^{wh_1h_2/w_2} = \prod_{A \in C_K} F(A)^{-\chi(A)}. \tag{4.18}$$

We remark that (4.18) was utilized by K. G. Ramanathan [1], [3], [4], [5], [7] to calculate class invariants, values of the Rogers–Ramanujan continued fraction, and certain other invariants of Ramanujan.

We next prove the three primary theorems that we need to calculate Ramanujan's class invariants G_m when $K = \mathbb{Q}(\sqrt{-m})$ has class number 8. Let $\tau = \sqrt{-m}$. Then, by (1.1), (1.3), and (4.3), it is easily seen that

$$\frac{\eta((\tau+1)/2)}{\eta(\tau)} = 2^{1/4}G_m. \tag{4.19}$$

Equalities (4.18) and (4.19) are the key ingredients for deriving formulas that will enable us to calculate G_m. We consider three different genus structures, and the first two theorems that we prove can be utilized to determine G_m for $m = 65, 69, 77, 141, 145, 205, 213, 265, 301, 445$, and 505. For $m = 217, 553$, the genus structure is of a third type. In each case, $K = \mathbb{Q}(\sqrt{-m})$ has class number 8, and the number of genera equals 4. Thus, each genus contains exactly two ideal classes. Also note that A and A^{-1} are clearly in the same genus.

Throughout the next two sections, for simplicity, we use the notation for a primitive ideal to denote the ideal class containing it; this abuse of notation should not cause difficulty.

Theorem 4.1. *Let $m \equiv 1 \pmod 4$, where m is a positive squarefree integer with prime divisor p. Let $K = \mathbb{Q}(\sqrt{-m})$ be an imaginary quadratic field such that each genus contains exactly two ideal classes and such that the principal genus*

G_0 contains the classes $[1, \Omega]$ *and* $[2p, p+\Omega]$. *Let* G_1 *be a nonprincipal genus containing the two classes* $[2, 1+\Omega]$ *and* $[p, \Omega]$. *Then*

$$\left(\frac{G_m}{G_{m/p^2}}\right)^{h/2} = \prod_{\chi(G_1)=-1} \epsilon_1^{wh_1h_2/w_2},$$

where h, h_1, *and* h_2 *are the class numbers of* K, $\mathbb{Q}(\sqrt{d_1})$, *and* $\mathbb{Q}(\sqrt{d_2})$, *respectively,* w *and* w_2 *are the numbers of roots of unity in* K *and* $\mathbb{Q}(\sqrt{d_2})$, *respectively,* ϵ_1 *is the fundamental unit in* $\mathbb{Q}(\sqrt{d_1})$, *and the product is over all characters* χ *(with* $\chi(G_1) = -1$*), associated with the decomposition* $d = d_1d_2$, *and therefore* d_1, d_2, h_1, h_2, w_2, *and* ϵ_1 *are dependent on* χ.

Proof. Each of the ideals $[1, \Omega]$, $[2p, p+\Omega]$, $[2, 1+\Omega]$, and $[p, \Omega]$ is ambiguous. If $\mathfrak{A} \in A$ is any one of these ideals, then $\mathfrak{A} \sim \mathfrak{A}^{-1}$, $A = A^{-1}$, and $\mathfrak{A} \in A^{-1}$.

For any ideal class $B \notin G_0 \cup G_1$, it is not difficult to see that (Ramanathan [5, p. 77])

$$\sum_{\chi(G_1)=-1} \chi(B) = 0,$$

which implies that

$$\prod_{\chi(G_1)=-1} F(B)^{-\chi(B)} = 1,$$

where $F(B)$ is defined by (4.17). Therefore, by (4.18),

$$\prod_{\chi(G_1)=-1} \epsilon_1^{wh_1h_2/w_2} = \prod_{\chi(G_1)=-1} \prod_{A\in G_0\cup G_1} F(A)^{-\chi(A)} = \prod_{A\in G_0\cup G_1} F(A)^{-\chi(A)h/4}, \tag{4.20}$$

since the number of genus characters equals $h/2$, and so the number of genus characters with $\chi(G_1) = -1$ is $h/4$.

Let $A_0 = [1, \Omega]$, $A_0' = [2p, p+\Omega]$, $A_1 = [2, 1+\Omega]$, and $A_1' = [p, \Omega]$. Then, by (4.20),

$$\prod_{\chi(G_1)=-1} \epsilon_1^{wh_1h_2/w_2} = \left(\frac{F(A_1)/F(A_0)}{F(A_0')/F(A_1')}\right)^{h/4}. \tag{4.21}$$

By (4.17) and (4.19),

$$\frac{F(A_1)}{F(A_0)} = \frac{\eta^2(\frac{\Omega+1}{2})/\sqrt{2}}{\eta^2(\Omega)} = G_m^2. \tag{4.22}$$

Let $\Omega' = \Omega/p = \sqrt{-m/p^2}$. Again, by (4.17) and (4.19),

$$\frac{F(A_0')}{F(A_1')} = \frac{\eta^2(\frac{\Omega+p}{2p})/\sqrt{2p}}{\eta^2(\frac{\Omega}{p})} = \frac{\eta^2(\frac{\Omega'+1}{2})/\sqrt{2}}{\eta^2(\Omega')} = G_{m/p^2}^2. \tag{4.23}$$

The theorem now follows from (4.21)–(4.23).

Theorem 4.2. *Let $m \equiv 1 \pmod 4$, where m is a positive squarefree integer with prime divisor p. Let $K = \mathbb{Q}(\sqrt{-m})$ be an imaginary quadratic field such that each genus contains exactly two ideal classes and such that the principal genus G_0 contains the classes $[1, \Omega]$ and $[p, \Omega]$. Let G_1 be a nonprincipal genus containing the two classes $[2, 1+\Omega]$ and $[2p, p+\Omega]$. Then*

$$\left(G_m G_{m/p^2}\right)^{h/2} = \prod_{\chi(G_1)=-1} \epsilon_1^{wh_1h_2/w_2},$$

where h, h_1, and h_2 are the class numbers of K, $\mathbb{Q}(\sqrt{d_1})$, and $\mathbb{Q}(\sqrt{d_2})$, respectively, w and w_2 are the numbers of roots of unity in K and $\mathbb{Q}(\sqrt{d_2})$, respectively, ϵ_1 is the fundamental unit in $\mathbb{Q}(\sqrt{d_1})$, and the product is over all characters χ (with $\chi(G_1) = -1$), associated with the decomposition $d = d_1d_2$, and therefore d_1, d_2, h_1, h_2, w_2, and ϵ_1 are dependent on χ.

The proof of Theorem 4.2 is analogous to that for Theorem 4.1, and so we omit it.

We say that m is of the *first kind* or *second kind* according as it satisfies the conditions of Theorem 4.1 or Theorem 4.2, respectively.

It is not difficult to show that $[1, \Omega]$, $[2, 1+\Omega]$, $[p, \Omega]$, and $[2p, p+\Omega]$ are representatives of different ideal classes (Mollin and Zhang [1]).

Theorems 4.1 and 4.2 need to be combined with three modular equations of Ramanujan (Part III [3, pp. 231, 282, 315]) in order to calculate Ramanujan's class invariants. We have already employed Lemma 4.3 in our proof of Theorem 3.1.

Lemma 4.3 (Modular Equation of Degree 3). *Let*

$$P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \qquad \text{and} \qquad Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}.$$

Then

$$Q + \frac{1}{Q} + 2\sqrt{2}\left(P - \frac{1}{P}\right) = 0.$$

Lemma 4.4 (Modular Equation of Degree 5). *Let*

$$P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} \qquad \text{and} \qquad Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}.$$

Then

$$Q + \frac{1}{Q} + 2\left(P - \frac{1}{P}\right) = 0.$$

Lemma 4.5 (Modular Equation of Degree 7). *Let*

$$P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \qquad \text{and} \qquad Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}.$$

Then

$$Q+\frac{1}{Q}+7=2\sqrt{2}\left(P+\frac{1}{P}\right).$$

Let $q=\exp(-\pi/\sqrt{n})$. Since $G_n=G_{1/n}$ (Ramanujan [3], [10, p. 23]), by (1.6), $G_n=\{4\alpha(1-\alpha)\}^{-1/24}$. If β has degree p over α, then $G_{n/p^2}=G_{p^2/n}=\{4\beta(1-\beta)\}^{-1/24}$. In summary, we can express the equalities of Lemmas 4.3–4.5 in terms of G_n and G_{n/p^2}, $p=3,5,7$, respectively, by employing the formulas

$$G_n=\{4\alpha(1-\alpha)\}^{-1/24} \qquad \text{and} \qquad G_{n/p^2}=\{4\beta(1-\beta)\}^{-1/24}. \tag{4.24}$$

The genus structures for $\mathbb{Q}(\sqrt{-217})$ and $\mathbb{Q}(\sqrt{-553})$ are different from those of the eleven imaginary quadratic fields to which either Theorem 4.1 or Theorem 4.2 applies, and so G_{217} and G_{553} must be calculated by another means.

Lemma 4.6. *Let m denote a positive integer with $7|m$. Let $\tau=\sqrt{-m}/7$ and $Q=(G_m/G_{m/49})^4$. Then*

$$\left(\frac{\eta(\tau)\eta\left(\frac{\tau+1}{2}\right)}{\eta(7\tau)\eta\left(\frac{7\tau+1}{2}\right)}\right)^2-49\left(\frac{\eta(\tau)\eta\left(\frac{\tau+1}{2}\right)}{\eta(7\tau)\eta\left(\frac{7\tau+1}{2}\right)}\right)^{-2}=Q^{3/2}+8Q^{1/2}-8Q^{-1/2}-Q^{-3/2}. \tag{4.25}$$

Proof. With $q=\exp(-\pi\sqrt{m}/7)$, it follows from (4.3) that

$$\left(\frac{\eta(\tau)\eta\left(\frac{\tau+1}{2}\right)}{\eta(7\tau)\eta\left(\frac{7\tau+1}{2}\right)}\right)^2=\frac{f^2(-q^2)f^2(q)}{q^{3/2}f^2(-q^{14})f^2(q^7)}. \tag{4.26}$$

Next, by an entry from Ramanujan's second notebook (Part IV [4, p. 209, Entry 55]),

$$\begin{aligned}\frac{f^2(-q)f^2(-q^2)}{q^{3/2}f^2(-q^7)f^2(-q^{14})}+49\frac{q^{3/2}f^2(-q^7)f^2(-q^{14})}{f^2(-q)f^2(-q^2)}&=\frac{f^6(-q^2)f^6(-q^7)}{q^{3/2}f^6(-q)f^6(-q^{14})}\\ -8\frac{f^2(-q^2)f^2(-q^7)}{q^{1/2}f^2(-q)f^2(-q^{14})}-8\frac{q^{1/2}f^2(-q)f^2(-q^{14})}{f^2(-q^2)f^2(-q^7)}&+\frac{q^{3/2}f^6(-q)f^6(-q^{14})}{f^6(-q^2)f^6(-q^7)}.\end{aligned}$$

Multiplying both sides by $q^{3/2}$ and then replacing q by $-q$, we find that

$$\begin{aligned}\frac{f^2(q)f^2(-q^2)}{f^2(q^7)f^2(-q^{14})}-49\frac{q^3f^2(q^7)f^2(-q^{14})}{f^2(q)f^2(-q^2)}&=\frac{f^6(-q^2)f^6(q^7)}{f^6(q)f^6(-q^{14})}\\ +8\frac{qf^2(-q^2)f^2(q^7)}{f^2(q)f^2(-q^{14})}-8\frac{q^2f^2(q)f^2(-q^{14})}{f^2(-q^2)f^2(q^7)}&-\frac{q^3f^6(q)f^6(-q^{14})}{f^6(-q^2)f^6(q^7)}.\end{aligned} \tag{4.27}$$

Recall that $q = \exp(-\pi\sqrt{m}/7)$ and recall that $G_{m/49}$ is then given by (1.3). Thus, $G_m = 2^{-1/4}q^{-7/24}(-q^7;q^{14})_\infty$. Hence,

$$\begin{aligned}\left(\frac{G_m}{G_{m/49}}\right)^2 &= \frac{(-q^7;q^{14})_\infty^2}{q^{1/2}(-q;q^2)_\infty^2}\\ &= \frac{(q^2;q^2)_\infty^2(-q^7;-q^7)_\infty^2}{q^{1/2}(-q;-q)_\infty^2(q^{14};q^{14})_\infty^2} = \frac{f^2(-q^2)f^2(q^7)}{q^{1/2}f^2(q)f^2(-q^{14})}.\end{aligned} \tag{4.28}$$

Dividing (4.27) by $q^{3/2}$ and substituting (4.26) and (4.28) into the resulting equality, we deduce (4.25) to complete the proof.

Theorem 4.7. *Let m be a squarefree positive integer with $7|m$ and $m \equiv 1 \pmod 4$. Let $K = \mathbb{Q}(\sqrt{-m})$ be an imaginary quadratic field such that each genus contains exactly two classes and such that the principal genus G_0 comprises $[1, \Omega]$ and $[2, 1+\Omega]$, while $[7, \Omega]$ and $[14, 7+\Omega]$ form a nonprincipal genus G_1. Then*

$$\left\{\frac{1}{\sqrt{7}}\left(\frac{\eta(\tau)\eta\left(\frac{\tau+1}{2}\right)}{\eta(7\tau)\eta\left(\frac{7\tau+1}{2}\right)}\right)\right\}^{h/2} = \prod_{\chi(G_1)=-1} \epsilon_1^{wh_1h_2/w_2}, \tag{4.29}$$

where h, h_1, and h_2 are the class numbers of K, $\mathbb{Q}(\sqrt{d_1})$, and $\mathbb{Q}(\sqrt{d_2})$, respectively, w and w_2 are the numbers of roots of unity in K and $\mathbb{Q}(\sqrt{d_2})$, respectively, ϵ_1 is the fundamental unit in $\mathbb{Q}(\sqrt{d_1})$, and the product is over all characters χ (with $\chi(G_1) = -1$), associated with the decomposition $d = d_1d_2$, and therefore d_1, d_2, h_1, h_2, w_2, and ϵ_1 are dependent on χ.

Proof. Let $A_0 = [1, \Omega]$, $A_0' = [2, 1+\Omega]$, $A_1 = [7, \Omega]$, and $A_1' = [14, 7+\Omega]$. Then by the same reasoning that we used in the proof of Theorem 4.1,

$$\left(\frac{F(A_1)F(A_1')}{F(A_0)F(A_0')}\right)^{h/4} = \prod_{\chi(G_1)=-1} \epsilon_1^{wh_1h_2/w_2}. \tag{4.30}$$

By (4.17),

$$\begin{aligned}F(A_0) &= \eta^2(\Omega) = \eta^2(7\tau),\\ F(A_0') &= \eta^2\left(\tfrac{\Omega+1}{2}\right)/\sqrt{2} = \eta^2\left(\tfrac{7\tau+1}{2}\right)/\sqrt{2},\\ F(A_1) &= \eta^2\left(\tfrac{\Omega}{7}\right)/\sqrt{7} = \eta^2(\tau)/\sqrt{7},\end{aligned}$$

and

$$F(A_1') = \eta^2\left(\tfrac{\Omega+7}{14}\right)/\sqrt{14} = \eta^2\left(\tfrac{\tau+1}{2}\right)/\sqrt{14}.$$

Substituting these values into (4.30) and recalling that the number of genus characters χ with $\chi(G_1) = -1$ is equal to $h/4$, we deduce (4.29) to complete the proof.

The class numbers cited below for $|d| < 500$ can be found in tables in the texts by Z. I. Borevich and I. R. Shafarevich [1, pp. 422–426], H. Cohen [1, pp.

503–509], and for $0 < d < 10,000$ in the book by D. A. Buell [1, pp. 224–234]. Lists of fundamental units can be found in the book by Borevich and Shafarevich [1] (for $d \leq 101$), the book by M. Pohst and H. Zassenhaus [1, pp. 432–435] (up to $d \leq 299$), and the tables of R. Kortum and G. McNiel [1] (up to $d = 10,000$). In Cohen's book [1, pp. 262–274], there is a table providing the ideal class structure for $\mathbb{Q}(\sqrt{-d})$, $d \leq 97$ and for $\mathbb{Q}(\sqrt{d})$, $d \leq 97$.

5. Class Invariants Via Kronecker's Limit Formula

Theorem 5.1.

$$G_{65} = \left(\frac{\sqrt{13}+3}{2}\right)^{1/4}\left(\frac{\sqrt{5}+1}{2}\right)^{1/4}\left(\sqrt{\frac{9+\sqrt{65}}{8}}+\sqrt{\frac{1+\sqrt{65}}{8}}\right)^{1/2}.$$

Proof. The following table summarizes the needed information about ideal classes and their characters.

d_1	d_2	χ	G	C	$\chi(G_0)$ $\chi(G_1)$ $\chi(G_2)$ $\chi(G_3)$	h_1	h_2	w_2	ϵ_1
1	-260	χ_0	G_0	$[1, \Omega]$ $[10, 5+\Omega]$	1 1 1 1				
5	-52	χ_1	G_1	$[2, 1+\Omega]$ $[5, \Omega]$	1 −1 −1 1	1	2	2	$\frac{\sqrt{5}+1}{2}$
13	-20	χ_2	G_2	$[3, 1+\Omega]$ $[3, -1+\Omega]$	1 −1 1 −1	1	2	2	$\frac{\sqrt{13}+3}{2}$
65	-4	χ_3	G_3	$[6, 1+\Omega]$ $[6, -1+\Omega]$	1 1 −1 −1				

Note that 65 is of the first kind. Applying Theorem 4.1 with $h = 8$ and $w = 2$, we find that

$$\left(\frac{G_{65}}{G_{13/5}}\right)^4 = \left(\frac{\sqrt{5}+1}{2}\right)^2\left(\frac{\sqrt{13}+3}{2}\right)^2. \tag{5.1}$$

Let $Q = (G_{65}/G_{13/5})^3$ and $P = (G_{65}G_{13/5})^{-2}$. Then, by (5.1),

$$Q = \left(\frac{\sqrt{5}+1}{2}\right)^{3/2}\left(\frac{\sqrt{13}+3}{2}\right)^{3/2} = (\sqrt{5}+2)^{1/2}(5\sqrt{13}+18)^{1/2}. \tag{5.2}$$

By Lemma 4.4,

$$P^{-1} = \frac{(Q+Q^{-1})+\sqrt{(Q+Q^{-1})^2+16}}{4}. \tag{5.3}$$

Now, by (5.2),

$$\begin{aligned}(Q+Q^{-1})^2+16 &= Q^2+Q^{-2}+18\\ &= (\sqrt{5}+2)(5\sqrt{13}+18)+(\sqrt{5}-2)(5\sqrt{13}-18)+18\\ &= (5+\sqrt{65})^2, \end{aligned}\tag{5.4}$$

and, by (5.4),

$$Q+Q^{-1}=\sqrt{74+10\sqrt{65}}.\tag{5.5}$$

Thus, by (5.3)–(5.5),

$$P^{-1}=\tfrac14\sqrt{74+10\sqrt{65}}+\tfrac14(5+\sqrt{65}).\tag{5.6}$$

Thus, by (5.2) and (5.6),

$$\begin{aligned}G_{65} &= Q^{1/6}P^{-1/4}\\ &= \left(\frac{\sqrt{5}+1}{2}\right)^{1/4}\left(\frac{\sqrt{13}+3}{2}\right)^{1/4}\left(\tfrac14\sqrt{74+10\sqrt{65}}+\tfrac14(5+\sqrt{65})\right)^{1/4}.\end{aligned}$$

Thus, it remains to show that

$$\tfrac14\sqrt{74+10\sqrt{65}}+\tfrac14(5+\sqrt{65})=\left(\sqrt{\frac{9+\sqrt{65}}{8}}+\sqrt{\frac{1+\sqrt{65}}{8}}\right)^2,$$

which is easily shown by a routine calculation.

Theorem 5.2.

$$G_{69}=\left(\frac{5+\sqrt{23}}{\sqrt{2}}\right)^{1/12}\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)^{1/8}\left(\sqrt{\frac{6+3\sqrt{3}}{4}}+\sqrt{\frac{2+3\sqrt{3}}{4}}\right)^{1/2}.$$

Proof. We summarize the needed information in the following table.

d_1	d_2	χ	G	C	$\chi(G_0)$ $\chi(G_1)$ $\chi(G_2)$ $\chi(G_3)$	h_1	h_2	w_2	ϵ_1
1	-276	χ_0	G_0	$[1,\Omega]$ $[6,3+\Omega]$	$1 \quad 1$ $1 \quad 1$				
92	-3	χ_1	G_1	$[2,1+\Omega]$ $[3,\Omega]$	$1 \quad -1$ $1 \quad -1$	1	1	6	$24+5\sqrt{23}$
69	-4	χ_2	G_2	$[5,1+\Omega]$ $[5,-1+\Omega]$	$1 \quad -1$ $-1 \quad 1$	1	1	4	$\frac{25+3\sqrt{69}}{2}$
12	-23	χ_3	G_3	$[7,1+\Omega]$ $[7,-1+\Omega]$	$1 \quad 1$ $-1 \quad -1$				

We apply Theorem 4.1 with $h = 8$ and $w = 2$, as 69 is of the first kind. Thus,

$$\left(\frac{G_{69}}{G_{23/3}}\right)^4 = (24+5\sqrt{23})^{1/3}\left(\frac{25+3\sqrt{69}}{2}\right)^{1/2}. \tag{5.7}$$

Let $Q = (G_{69}/G_{23/3})^6$ and $P = (G_{69}G_{23/3})^{-3}$. By (5.7),

$$\begin{aligned} Q &= (24+5\sqrt{23})^{1/2}\left(\frac{25+3\sqrt{69}}{2}\right)^{3/4} \\ &= \left(\frac{5+\sqrt{23}}{\sqrt{2}}\right)(36\sqrt{3}+13\sqrt{23})^{1/2} = \left(\frac{5+\sqrt{23}}{\sqrt{2}}\right)\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)^{3/2}. \end{aligned} \tag{5.8}$$

By Lemma 4.3,

$$P^{-1} = \frac{1}{4\sqrt{2}}(Q+Q^{-1}) + \frac{1}{4\sqrt{2}}\sqrt{(Q+Q^{-1})^2+32}. \tag{5.9}$$

From (5.8),

$$Q+Q^{-1} = \sqrt{Q^2+Q^{-2}+2} = \sqrt{16(187+108\sqrt{3})}, \tag{5.10}$$

and, from (5.10),

$$(Q+Q^{-1})^2+32 = 16(9+6\sqrt{3})^2.$$

Putting these calculations in (5.9), we find that

$$P^{-1} = \frac{1}{\sqrt{2}}\sqrt{187+108\sqrt{3}} + \frac{1}{\sqrt{2}}(9+6\sqrt{3}). \tag{5.11}$$

By (5.8),

$$G_{69} = Q^{1/12}P^{-1/6} = \left(\frac{5+\sqrt{23}}{\sqrt{2}}\right)^{1/12}\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)^{1/8}P^{-1/6},$$

and thus, by (5.11), it remains to show that

$$\left(\sqrt{\frac{6+3\sqrt{3}}{4}}+\sqrt{\frac{2+3\sqrt{3}}{4}}\right)^3 = \frac{1}{\sqrt{2}}\sqrt{187+108\sqrt{3}} + \frac{1}{\sqrt{2}}(9+6\sqrt{3}).$$

This can be achieved by a straightforward computation.

Theorem 5.3.

$$G_{77} = (8+3\sqrt{7})^{1/8}\left(\frac{\sqrt{11}+\sqrt{7}}{2}\right)^{1/8}\left(\sqrt{\frac{6+\sqrt{11}}{4}}+\sqrt{\frac{2+\sqrt{11}}{4}}\right)^{1/2}.$$

Proof. We compose the following table giving needed information about ideal classes and characters.

d_1	d_2	χ	G	C	$\chi(G_0)$ $\chi(G_1)$ $\chi(G_2)$ $\chi(G_3)$	h_1	h_2	w_2	ϵ_1
1	-308	χ_0	G_0	$[1, \Omega]$ $[14, 7+\Omega]$	$1 \quad 1$ $1 \quad 1$				
28	-11	χ_1	G_1	$[2, 1+\Omega]$ $[7, \Omega]$	$1 \quad -1$ $1 \quad -1$	1	1	2	$8+3\sqrt{7}$
77	-4	χ_2	G_2	$[3, 1+\Omega]$ $[3, -1+\Omega]$	$1 \quad -1$ $-1 \quad 1$	1	1	4	$\dfrac{9+\sqrt{77}}{2}$
44	-7	χ_3	G_3	$[6, 1+\Omega]$ $[6, -1+\Omega]$	$1 \quad 1$ $-1 \quad -1$				

We see from the table that 77 is of the first kind. Thus, by Theorem 4.1, since $h = 8$ and $w = 2$,

$$Q := \left(\frac{G_{77}}{G_{11/7}}\right)^4 = (8+3\sqrt{7})\left(\frac{9+\sqrt{77}}{2}\right)^{1/2} = (8+3\sqrt{7})\left(\frac{\sqrt{11}+\sqrt{7}}{2}\right). \tag{5.12}$$

If $P = (G_{77}G_{11/7})^{-3}$, then, from Lemma 4.5,

$$P^{-1} = \frac{Q+Q^{-1}+7+\sqrt{(Q+Q^{-1}+7)^2-32}}{4\sqrt{2}}. \tag{5.13}$$

Now

$$Q+Q^{-1}+7 = 8\sqrt{11}+28 = 2\sqrt{2}(2\sqrt{22}+7\sqrt{2}).$$

Using this in (5.13), we find that

$$P^{-1} = \tfrac{1}{2}\left(2\sqrt{22}+7\sqrt{2}+\sqrt{182+56\sqrt{11}}\right). \tag{5.14}$$

By (5.12),

$$G_{77} = Q^{1/8}P^{-1/6} = (8+3\sqrt{7})^{1/8}\left(\frac{\sqrt{11}+\sqrt{7}}{2}\right)^{1/8} P^{-1/6},$$

and thus by (5.14) it remains to show that

$$\left(\sqrt{\frac{6+\sqrt{11}}{4}}+\sqrt{\frac{2+\sqrt{11}}{4}}\right)^3 = \tfrac{1}{2}\left(2\sqrt{22}+7\sqrt{2}+\sqrt{182+56\sqrt{11}}\right),$$

which is readily shown by a straightforward calculation.

Theorem 5.4.

$$G_{141} = (4\sqrt{3}+\sqrt{47})^{1/8}\left(\frac{7+\sqrt{47}}{\sqrt{2}}\right)^{1/12}\left(\sqrt{\frac{18+9\sqrt{3}}{4}}+\sqrt{\frac{14+9\sqrt{3}}{4}}\right)^{1/2}.$$

Proof. We record the necessary information in the following table:

d_1	d_2	χ	G	C	$\chi(G_0)$ $\chi(G_1)$ $\chi(G_2)$ $\chi(G_3)$	h_1	h_2	w_2	ϵ_1
1	-564	χ_0	G_0	$[1, \Omega]$ $[6, 3+\Omega]$	1 1 1 1				
188	-3	χ_1	G_1	$[2, 1+\Omega]$ $[3, \Omega]$	1 -1 -1 1	1	1	6	$48+7\sqrt{47}$
141	-4	χ_2	G_2	$[5, 2+\Omega]$ $[5, -2+\Omega]$	1 -1 1 -1	1	1	4	$95+8\sqrt{141}$
12	-47	χ_3	G_3	$[10, 3+\Omega]$ $[10, -3+\Omega]$	1 1 -1 -1				

We see that 141 is again of the first kind. Applying Theorem 4.1, we find that, since $h = 8$ and $w = 2$,

$$\left(\frac{G_{141}}{G_{47/3}}\right)^4 = (48+7\sqrt{47})^{1/3}(95+8\sqrt{141})^{1/2}. \tag{5.15}$$

Let $Q = (G_{141}/G_{47/3})^6$. Then, by (5.15),

$$\begin{aligned} Q &= (48+7\sqrt{47})^{1/2}(95+8\sqrt{141})^{3/4} = \left(\frac{7+\sqrt{47}}{\sqrt{2}}\right)(4\sqrt{3}+\sqrt{47})^{3/2} \\ &= (48+7\sqrt{47})^{1/2}(756\sqrt{3}+191\sqrt{47})^{1/2}. \end{aligned} \tag{5.16}$$

Let $P = (G_{141}G_{47/3})^{-3}$. Then, by Lemma 4.3,

$$P^{-1} = \frac{(Q+Q^{-1})+\sqrt{(Q+Q^{-1})^2+32}}{4\sqrt{2}}. \tag{5.17}$$

From the last representation of Q in (5.16),

$$Q^2+Q^{-2}+34 = \left(36(7+4\sqrt{3})\right)^2,$$

and so

$$Q+Q^{-1}=\sqrt{Q^2+Q^{-2}+2}=4\sqrt{7855+4536\sqrt{3}}.$$

Using these calculations in (5.17), we deduce that

$$P^{-1}=\frac{1}{\sqrt{2}}\sqrt{7855+4536\sqrt{3}}+\frac{9}{\sqrt{2}}(7+4\sqrt{3}). \tag{5.18}$$

Hence, by (5.16) and (5.18),

$$\begin{aligned}G_{141}=Q^{1/12}P^{-1/6}=&\left(\frac{7+\sqrt{47}}{\sqrt{2}}\right)^{1/12}(4\sqrt{3}+\sqrt{47})^{1/8}\\&\times\left(\frac{1}{\sqrt{2}}\sqrt{7855+4536\sqrt{3}}+\frac{9}{\sqrt{2}}(7+4\sqrt{3})\right)^{1/6}.\end{aligned}$$

It thus remains to show that

$$\left(\sqrt{\frac{18+9\sqrt{3}}{4}}+\sqrt{\frac{14+9\sqrt{3}}{4}}\right)^3=\frac{1}{\sqrt{2}}\sqrt{7855+4536\sqrt{3}}+\frac{9}{\sqrt{2}}(7+4\sqrt{3}),$$

which is a straightforward, albeit laborious, task.

Theorem 5.5.

$$G_{145}=(\sqrt{5}+2)^{1/4}\left(\frac{\sqrt{29}+5}{2}\right)^{1/4}\left(\sqrt{\frac{17+\sqrt{145}}{8}}+\sqrt{\frac{9+\sqrt{145}}{8}}\right)^{1/2}.$$

Proof. We compose the following table:

d_1	d_2	χ	G	C	$\chi(G_0)$ $\chi(G_1)$ $\chi(G_2)$ $\chi(G_3)$	h_1	h_2	w_2	ϵ_1
1	-580	χ_0	G_0	$[1,\Omega]$ $[5,\Omega]$	$1\quad 1$ $1\quad 1$				
29	-20	χ_1	G_1	$[2,1+\Omega]$ $[10,5+\Omega]$	$1\quad -1$ $1\quad -1$	1	2	2	$\frac{\sqrt{29}+5}{2}$
5	-116	χ_2	G_2	$[7,3+\Omega]$ $[7,-3+\Omega]$	$1\quad -1$ $-1\quad 1$	1	6	2	$\frac{\sqrt{5}+1}{2}$
145	-4	χ_3	G_3	$[11,3+\Omega]$ $[11,-3+\Omega]$	$1\quad 1$ $-1\quad -1$				

Thus, 145 is of the second kind. Thus, by Theorem 4.2, since $h = 8$ and $w = 2$,

$$(G_{145}G_{29/5})^4 = \left(\frac{\sqrt{29}+5}{2}\right)^2 \left(\frac{\sqrt{5}+1}{2}\right)^6 = \left(\frac{\sqrt{29}+5}{2}\right)^2 (\sqrt{5}+2)^2.$$

Hence,

$$P^{-1} := (G_{145}G_{29/5})^2 = \left(\frac{\sqrt{29}+5}{2}\right)(\sqrt{5}+2). \tag{5.19}$$

By Lemma 4.4, with $Q = (G_{145}/G_{29/5})^3$,

$$Q = P^{-1} - P + \sqrt{(P^{-1}-P)^2 - 1}. \tag{5.20}$$

By (5.19), we readily find that

$$P^{-1} - P = 2\sqrt{29} + 5\sqrt{5},$$

and so, by (5.20),

$$Q = 2\sqrt{29} + 5\sqrt{5} + \sqrt{240 + 20\sqrt{145}}. \tag{5.21}$$

Thus, by (5.19) and (5.21),

$$G_{145} = P^{-1/4}Q^{1/6} = \left(\frac{\sqrt{29}+5}{2}\right)^{1/4} (\sqrt{5}+2)^{1/4} \times \left(2\sqrt{29} + 5\sqrt{5} + \sqrt{240 + 20\sqrt{145}}\right)^{1/6}.$$

Hence, it remains to show that

$$2\sqrt{29} + 5\sqrt{5} + \sqrt{240 + 20\sqrt{145}} = \left(\sqrt{\frac{17+\sqrt{145}}{8}} + \sqrt{\frac{9+\sqrt{145}}{8}}\right)^3,$$

which is readily shown.

Theorem 5.6.

$$G_{205} = \left(\frac{\sqrt{5}+1}{2}\right)\left(\frac{3\sqrt{5}+\sqrt{41}}{2}\right)^{1/4} \left(\sqrt{\frac{7+\sqrt{41}}{8}} + \sqrt{\frac{\sqrt{41}-1}{8}}\right).$$

Proof. We record the following table:

d_1	d_2	χ	G	C	$\chi(G_0)$ $\chi(G_2)$	$\chi(G_1)$ $\chi(G_3)$	h_1	h_2	w_2	ϵ_1
1	-820	χ_0	G_0	$[1, \Omega]$ $[5, \Omega]$	1 1	1 1				
5	-164	χ_1	G_1	$[2, 1+\Omega]$ $[10, 5+\Omega]$	1 1	-1 -1	1	8	2	$\dfrac{\sqrt{5}+1}{2}$
205	-4	χ_2	G_2	$[11, 2+\Omega]$ $[11, -2+\Omega]$	1 -1	-1 1	2	1	4	$\dfrac{43+3\sqrt{205}}{2}$
41	-20	χ_3	G_3	$[13, 4+\Omega]$ $[13, -4+\Omega]$	1 -1	1 -1				

Note that 205 is of the second kind. Applying Theorem 4.2 with $h = 8$ and $w = 2$, we deduce that

$$P^{-2} := (G_{205}G_{41/5})^4 = \left(\frac{\sqrt{5}+1}{2}\right)^8 \left(\frac{43+3\sqrt{205}}{2}\right)$$
$$= \left(\frac{7+3\sqrt{5}}{2}\right)^2 \left(\frac{3\sqrt{5}+\sqrt{41}}{2}\right)^2. \tag{5.22}$$

Letting $Q = (G_{205}/G_{41/5})^3$, we deduce from Lemma 4.4 that

$$Q = (P^{-1} - P) + \sqrt{(P^{-1} - P)^2 - 1}. \tag{5.23}$$

From (5.22),

$$P^{-1} - P = \frac{45 + 7\sqrt{41}}{2}.$$

Thus, from (5.23),

$$Q = \tfrac{1}{2}\left(45 + 7\sqrt{41} + \sqrt{4030 + 630\sqrt{41}}\right). \tag{5.24}$$

If follows from (5.22) and (5.24) that

$$G_{205} = P^{-1/4}Q^{1/6} = \left(\frac{\sqrt{5}+1}{2}\right)\left(\frac{3\sqrt{5}+\sqrt{41}}{2}\right)^{1/4}$$
$$\times \left(\tfrac{1}{2}\left(45 + 7\sqrt{41} + \sqrt{4030 + 630\sqrt{41}}\right)\right)^{1/6}.$$

It thus remains to show that

$$\tfrac{1}{2}\left(45 + 7\sqrt{41} + \sqrt{4030 + 630\sqrt{41}}\right) = \left(\sqrt{\frac{7+\sqrt{41}}{8}} + \sqrt{\frac{\sqrt{41}-1}{8}}\right)^6.$$

This is more readily accomplished if we first note that

$$\left(\sqrt{\frac{7+\sqrt{41}}{8}}+\sqrt{\frac{\sqrt{41}-1}{8}}\right)^2=\frac{\sqrt{41}+3}{4}+\sqrt{\frac{17+3\sqrt{41}}{8}}.$$

Theorem 5.7.

$$G_{213}=\left(\frac{5\sqrt{3}+\sqrt{71}}{2}\right)^{1/8}\left(\frac{59+7\sqrt{71}}{\sqrt{2}}\right)^{1/12}$$
$$\times\left(\sqrt{\frac{21+12\sqrt{3}}{2}}+\sqrt{\frac{19+12\sqrt{3}}{2}}\right)^{1/2}.$$

Proof. We have the following table:

d_1	d_2	χ	G	C	$\chi(G_0)$ $\chi(G_2)$	$\chi(G_1)$ $\chi(G_3)$	h_1	h_2	w_2	ϵ_1
1	-852	χ_0	G_0	$[1, \Omega]$ $[6, 3, \Omega]$	1 1	1 1				
284	-3	χ_1	G_1	$[2, 1+\Omega]$ $[3, \Omega]$	1 1	-1 -1	1	1	6	$3480+413\sqrt{71}$
213	-4	χ_2	G_2	$[7, 2+\Omega]$ $[7, -2+\Omega]$	1 -1	-1 1	1	1	4	$\frac{73+5\sqrt{213}}{2}$
12	-71	χ_3	G_3	$[14, 5+\Omega]$ $[14, -5+\Omega]$	1 -1	1 -1				

Observe that 213 is of the first kind. Applying Theorem 4.1 with $h=8$ and $w=2$, we find that

$$Q^{2/3}:=\left(\frac{G_{213}}{G_{71/3}}\right)^4=(3480+413\sqrt{71})^{1/3}\left(\frac{73+5\sqrt{213}}{2}\right)^{1/2}$$
$$=\left(\frac{59+7\sqrt{71}}{\sqrt{2}}\right)^{2/3}\left(\frac{5\sqrt{3}+\sqrt{71}}{2}\right),$$

so that

$$Q=\left(\frac{59+7\sqrt{71}}{\sqrt{2}}\right)\left(\frac{5\sqrt{3}+\sqrt{71}}{2}\right)^{3/2}=\left(\frac{59+7\sqrt{71}}{\sqrt{2}}\right)(180\sqrt{3}+37\sqrt{21})^{1/2}. \tag{5.25}$$

Let $P = (G_{213}G_{71/3})^{-3}$. Then, by Lemma 4.3,

$$P^{-1} = \frac{1}{4\sqrt{2}}\left((Q + Q^{-1}) + \sqrt{(Q + Q^{-1})^2 + 32}\right). \tag{5.26}$$

By (5.25) and moderate calculations,

$$(Q + Q^{-1})^2 + 32 = \left(12(87 + 50\sqrt{3})\right)^2$$

and

$$\frac{1}{4\sqrt{2}}(Q + Q^{-1}) = \frac{1}{4\sqrt{2}}\sqrt{Q^2 + Q^{-2} + 2} = \sqrt{\tfrac{1}{2}(135619 + 78300\sqrt{3})}.$$

Thus, by (5.26),

$$P^{-1} = \sqrt{\tfrac{1}{2}(135619 + 78300\sqrt{3})} + \frac{3}{\sqrt{2}}(87 + 50\sqrt{3}). \tag{5.27}$$

Thus, by (5.25) and (5.27),

$$\begin{aligned} G_{213} = Q^{1/12}P^{-1/6} &= \left(\frac{5\sqrt{3} + \sqrt{71}}{2}\right)^{1/8}\left(\frac{59 + 7\sqrt{71}}{\sqrt{2}}\right)^{1/12} \\ &\quad \times \left(\sqrt{\tfrac{1}{2}(135619 + 78300\sqrt{3})} + \frac{3}{\sqrt{2}}(87 + 50\sqrt{3})\right)^{1/6}. \end{aligned}$$

Hence, it remains to show that

$$\begin{aligned} &\sqrt{\tfrac{1}{2}(135619 + 78300\sqrt{3})} + \frac{3}{\sqrt{2}}(87 + 50\sqrt{3}) \\ &\qquad = \left(\sqrt{\frac{21 + 12\sqrt{3}}{2}} + \sqrt{\frac{19 + 12\sqrt{3}}{2}}\right)^3, \end{aligned}$$

which is accomplished by a direct calculation.

Theorem 5.8.

$$G_{265} = (\sqrt{5} + 2)^{1/4}\left(\frac{\sqrt{53} + 7}{2}\right)^{1/4}\left(\sqrt{\frac{89 + 5\sqrt{265}}{8}} + \sqrt{\frac{81 + 5\sqrt{265}}{8}}\right)^{1/2}.$$

Proof. The following table is easily verified:

d_1	d_2	χ	G	C	$\chi(G_0)$ $\chi(G_2)$	$\chi(G_1)$ $\chi(G_3)$	h_1	h_2	w_2	ϵ_1
1	-1060	χ_0	G_0	$[1, \Omega]$ $[10, 5, \Omega]$	1 1	1 1				
5	-212	χ_1	G_1	$[2, 1+\Omega]$ $[5, \Omega]$	1 -1	-1 1	1	6	2	$\frac{\sqrt{5}+1}{2}$
53	-20	χ_2	G_2	$[7, 1+\Omega]$ $[7, -1+\Omega]$	1 1	-1 -1	1	2	2	$\frac{\sqrt{53}+7}{2}$
265	-4	χ_3	G_3	$[14, 1+\Omega]$ $[14, -1+\Omega]$	1 -1	1 -1				

Note that 265 is of the first kind. Applying Theorem 4.1 with $h = 8$ and $w = 2$, we find that

$$Q^{4/3} := \left(\frac{G_{265}}{G_{53/5}}\right)^4 = \left(\frac{\sqrt{5}+1}{2}\right)^6 \left(\frac{\sqrt{53}+7}{2}\right)^2,$$

so that

$$Q = \left(\frac{\sqrt{5}+1}{2}\right)^{9/2} \left(\frac{\sqrt{53}+7}{2}\right)^{3/2} = (38+17\sqrt{5})^{1/2}(182+25\sqrt{53})^{1/2}. \tag{5.28}$$

Let $P = (G_{265}G_{53/5})^{-2}$. Then, by Lemma 4.4,

$$P^{-1} = \frac{(Q+Q^{-1}) + \sqrt{(Q+Q^{-1})^2+16}}{4}. \tag{5.29}$$

By using (5.28) and the identity $Q + Q^{-1} = \sqrt{Q^2+Q^{-2}+2}$ in (5.29), we find that

$$P^{-1} = \frac{1}{2\sqrt{2}}\sqrt{6917+425\sqrt{265}} + \frac{1}{4}(85+5\sqrt{265}). \tag{5.30}$$

By (5.28) and (5.30),

$$G_{265} = Q^{1/6}P^{-1/4} = (\sqrt{5}+2)^{1/4}\left(\frac{\sqrt{53}+7}{2}\right)^{1/4} \times \left(\frac{1}{2\sqrt{2}}\sqrt{6917+425\sqrt{265}} + \frac{1}{4}(85+5\sqrt{265})\right)^{1/4}.$$

Hence, it remains to show that

$$\frac{1}{2\sqrt{2}}\sqrt{6917+425\sqrt{265}} + \frac{1}{4}(85+5\sqrt{265}) = \left(\sqrt{\frac{89+5\sqrt{265}}{8}} + \sqrt{\frac{81+5\sqrt{265}}{8}}\right)^2,$$

which is easy to establish.

Theorem 5.9.

$$G_{301} = (8+3\sqrt{7})^{1/8}\left(\frac{23\sqrt{43}+57\sqrt{7}}{2}\right)^{1/8} \times \left(\sqrt{\frac{46+7\sqrt{43}}{4}}+\sqrt{\frac{42+7\sqrt{43}}{4}}\right)^{1/2}.$$

Proof. We compose the following table:

d_1	d_2	χ	G	C	$\chi(G_0)$ $\chi(G_2)$	$\chi(G_1)$ $\chi(G_3)$	h_1	h_2	w_2	ϵ_1
1	-1204	χ_0	G_0	$[1,\Omega]$ $[14,7,\Omega]$	1 1	1 1				
28	-43	χ_1	G_1	$[2,1+\Omega]$ $[7,\Omega]$	1 -1	-1 1	1	1	2	$8+3\sqrt{7}$
301	-4	χ_2	G_2	$[5,2+\Omega]$ $[5,-2+\Omega]$	1 1	-1 -1	1	1	4	$\frac{22745+1311\sqrt{301}}{2}$
172	-7	χ_3	G_3	$[10,3+\Omega]$ $[10,-3+\Omega]$	1 -1	1 -1				

Thus, 301 is of the first kind. Applying Theorem 4.1 with $h=8$ and $w=2$, we find that

$$Q := \left(\frac{G_{301}}{G_{43/7}}\right)^4 = (8+3\sqrt{7})\left(\frac{22745+1311\sqrt{301}}{2}\right)^{1/2}$$
$$= (8+3\sqrt{7})\left(\frac{23\sqrt{43}+57\sqrt{7}}{2}\right). \tag{5.31}$$

Let $P=(G_{301}G_{43/7})^{-3}$. Then, by Lemma 4.5 and (5.31),

$$P^{-1} = \frac{1}{4\sqrt{2}}(Q+Q^{-1}+7)+\frac{1}{4\sqrt{2}}\sqrt{(Q+Q^{-1}+7)^2-32}$$
$$= \frac{1}{\sqrt{2}}(301+46\sqrt{43})+\frac{1}{\sqrt{2}}\sqrt{7(25941+3956\sqrt{43})}. \tag{5.32}$$

Therefore, by (5.31) and (5.32),

$$G_{301} = Q^{1/8}P^{-1/6} = (8+3\sqrt{7})^{1/8}\left(\frac{23\sqrt{43}+57\sqrt{7}}{2}\right)^{1/8}$$

$$\times\left(\frac{1}{\sqrt{2}}(301+46\sqrt{43})+\frac{1}{\sqrt{2}}\sqrt{7(25941+3956\sqrt{43})}\right)^{1/6}.$$

It remains to show that

$$\frac{1}{\sqrt{2}}(301+46\sqrt{43})+\frac{1}{\sqrt{2}}\sqrt{7(25941+3956\sqrt{43})}$$
$$=\left(\sqrt{\frac{46+7\sqrt{43}}{4}}+\sqrt{\frac{42+7\sqrt{43}}{4}}\right)^3,$$

which is a routine task.

Theorem 5.10.

$$G_{445}=(\sqrt{5}+2)^{1/2}\left(\frac{\sqrt{445}+21}{2}\right)^{1/4}\left(\sqrt{\frac{13+\sqrt{89}}{8}}+\sqrt{\frac{5+\sqrt{89}}{8}}\right).$$

Proof. We form the following table:

d_1	d_2	χ	G	C	$\chi(G_0)$ $\chi(G_1)$ $\chi(G_2)$ $\chi(G_3)$	h_1	h_2	w_2	ϵ_1
1	-1780	χ_0	G_0	$[1,\Omega]$ $[5,\Omega]$	1 1 1 1				
5	-356	χ_1	G_1	$[2,1+\Omega]$ $[10,5+\Omega]$	1 -1 -1 1	1	12	2	$\frac{\sqrt{5}+1}{2}$
445	-4	χ_2	G_2	$[13,6+\Omega]$ $[13,-6+\Omega]$	1 -1 1 -1	4	1	4	$\frac{\sqrt{445}+21}{2}$
89	-20	χ_3	G_3	$[19,7+\Omega]$ $[19,-7+\Omega]$	1 1 -1 -1				

Thus, 445 is of the second kind. Applying Theorem 4.2 with $h=8$ and $w=2$, we deduce that

$$P^{-2}:=(G_{445}G_{89/5})^4=\left(\frac{\sqrt{5}+1}{2}\right)^{12}\left(\frac{\sqrt{445}+21}{2}\right)^2,$$

so that

$$P^{-1}=(9+4\sqrt{5})\left(\frac{\sqrt{445}+21}{2}\right). \tag{5.33}$$

Let $Q=(G_{445}/G_{89/5})^3$. Then, by Lemma 4.4 and (5.33),

$$Q=(P^{-1}-P)+\sqrt{(P^{-1}-P)^2-1}=189+20\sqrt{89}+\sqrt{71320+7560\sqrt{89}}. \tag{5.34}$$

Therefore, by (5.33) and (5.34),

$$G_{445} = P^{-1/4}Q^{1/6} = (9+4\sqrt{5})^{1/4}\left(\frac{\sqrt{445}+21}{2}\right)^{1/4} \times \left(189+20\sqrt{89}+\sqrt{71320+7560\sqrt{89}}\right)^{1/6}.$$

It thus remains to show that

$$189+20\sqrt{89}+\sqrt{71320+7560\sqrt{89}} = \left(\sqrt{\frac{13+\sqrt{89}}{8}}+\sqrt{\frac{5+\sqrt{89}}{8}}\right)^6.$$

By first squaring the binomial on the right side and then cubing the resulting expression, we can easily verify the desired equality.

Theorem 5.11.

$$G_{505} = (\sqrt{5}+2)^{1/2}\left(\frac{\sqrt{5}+1}{2}\right)^{1/4}(\sqrt{101}+10)^{1/4} \times \left(\sqrt{\frac{113+5\sqrt{505}}{8}}+\sqrt{\frac{105+5\sqrt{505}}{8}}\right)^{1/2}.$$

Proof. We compose the following table:

d_1	d_2	χ	G	C	$\chi(G_0)$ $\chi(G_1)$ $\chi(G_2)$ $\chi(G_3)$	h_1	h_2	w_2	ϵ_1
1	-2020	χ_0	G_0	$[1, \Omega]$ $[5, \Omega]$	1 1 1 1				
5	-404	χ_1	G_1	$[2, 1+\Omega]$ $[10, 5+\Omega]$	1 -1 1 -1	1	14	2	$\frac{\sqrt{5}+1}{2}$
101	-20	χ_2	G_2	$[11, 1+\Omega]$ $[11, -1+\Omega]$	1 -1 -1 1	1	2	2	$\sqrt{101}+10$
505	-4	χ_3	G_3	$[22, 1+\Omega]$ $[22, -1+\Omega]$	1 1 -1 -1				

Hence, 505 is of the second kind. Applying Theorem 4.2 with $h = 8$ and $w = 2$, we find that

$$P^{-2} := (G_{505}G_{101/5})^4 = \left(\frac{\sqrt{5}+1}{2}\right)^{14}(\sqrt{101}+10)^2,$$

so that

$$P^{-1} = \left(\frac{\sqrt{5}+1}{2}\right)^7 (\sqrt{101}+10) = (\sqrt{5}+2)\left(\frac{\sqrt{5}+1}{2}\right)^4 (\sqrt{101}+10)$$
$$= (\sqrt{5}+2)\left(\frac{7+3\sqrt{5}}{2}\right)(\sqrt{101}+10). \tag{5.35}$$

Let $Q = (G_{505}/G_{101/5})^3$. Then, by Lemma 4.4 and (5.35),

$$Q = (P^{-1}-P) + \sqrt{(P^{-1}-P)^2 - 1}$$
$$= (130\sqrt{5} + 29\sqrt{101}) + \sqrt{169440 + 7540\sqrt{505}}. \tag{5.36}$$

Therefore, by (5.35) and (5.36),

$$G_{505} = P^{-1/4}Q^{1/6} = (\sqrt{5}+2)^{1/2}\left(\frac{\sqrt{5}+1}{2}\right)^{1/4}(\sqrt{101}+10)^{1/4}$$
$$\times \left((130\sqrt{5}+29\sqrt{101}) + \sqrt{169440+7540\sqrt{505}}\right)^{1/6}.$$

Thus, it remains to show that

$$(130\sqrt{5}+29\sqrt{101}) + \sqrt{169440+7540\sqrt{505}}$$
$$= \left(\sqrt{\frac{113+5\sqrt{505}}{8}} + \sqrt{\frac{105+5\sqrt{505}}{8}}\right)^3,$$

which is straightforward.

Theorem 5.12.

$$G_{217} = \left(\sqrt{\frac{11+4\sqrt{7}}{2}} + \sqrt{\frac{9+4\sqrt{7}}{2}}\right)^{1/2} \left(\sqrt{\frac{16+5\sqrt{7}}{4}} + \sqrt{\frac{12+5\sqrt{7}}{4}}\right)^{1/2}.$$

Proof. We set up a table to summarize some information that we need.

d_1	d_2	χ	G	C	$\chi(G_0)$ $\chi(G_1)$ $\chi(G_2)$ $\chi(G_3)$	h_1	h_2	w_2	ϵ_1
1	-868	χ_0	G_0	$[1, \Omega]$ $[2, 1+\Omega]$	1 1 1 1				
124	-7	χ_1	G_1	$[7, \Omega]$ $[14, 7+\Omega]$	1 -1 -1 1	1	1	2	$1520+273\sqrt{31}$
217	-4	χ_2	G_2	$[11, 5+\Omega]$ $[11, -5+\Omega]$	1 1 -1 -1	1	1	4	$3844063+260952\sqrt{217}$
28	-31	χ_3	G_3	$[13, 2+\Omega]$ $[13, -2+\Omega]$	1 -1 1 -1				

It is clear that $\mathbb{Q}(\sqrt{-217})$ satisfies the conditions of Theorem 4.7. Thus, since $h = 8$ and $w = 2$, we deduce that

$$\frac{1}{7^2}\left(\frac{\eta(\tau)\eta\left(\frac{\tau+1}{2}\right)}{\eta(7\tau)\eta\left(\frac{7\tau+1}{2}\right)}\right)^4 = (1520+273\sqrt{31})(3844063+260952\sqrt{217})^{1/2}$$
$$= (1520+273\sqrt{31})(524\sqrt{7}+249\sqrt{31}),$$

so that

$$\left(\frac{\eta(\tau)\eta\left(\frac{\tau+1}{2}\right)}{\eta(7\tau)\eta\left(\frac{7\tau+1}{2}\right)}\right)^2 = 7\epsilon,$$

where

$$\epsilon = (1520+273\sqrt{31})^{1/2}(524\sqrt{7}+249\sqrt{31})^{1/2}. \tag{5.37}$$

It follows from (4.25) that

$$Q^{3/2} + 8Q^{1/2} - 8Q^{-1/2} - Q^{3/2} = 7(\epsilon - \epsilon^{-1}), \tag{5.38}$$

where $Q = (G_{217}/G_{31/7})^4$. By an elementary calculation,

$$(\epsilon - \epsilon^{-1})^2 = \epsilon^2 + \epsilon^{-2} - 2$$
$$= 4(1053643 + 398240\sqrt{7}) = 4(27+10\sqrt{7})^2(367+140\sqrt{7}).$$

Let $x = Q^{1/2} - Q^{-1/2}$. Then (5.38) can be recast in the form

$$x^3 + 11x = 14(27+10\sqrt{7})\sqrt{367+140\sqrt{7}}$$
$$= (367+140\sqrt{7})\sqrt{367+140\sqrt{7}} + 11\sqrt{367+140\sqrt{7}}.$$

It is now obvious that

$$x = \sqrt{367+140\sqrt{7}}. \tag{5.39}$$

Solving (5.39) for $Q^{1/2}$, we find that

$$Q^{1/2} = \tfrac{1}{2}\left(\sqrt{367+140\sqrt{7}}+\sqrt{371+140\sqrt{7}}\right)$$
$$= \tfrac{1}{2}\left(\sqrt{367+140\sqrt{7}}+(14+5\sqrt{7})\right)$$
$$= \left(\sqrt{\frac{16+5\sqrt{7}}{4}}+\sqrt{\frac{12+5\sqrt{7}}{4}}\right)^2. \tag{5.40}$$

Now let $P = (G_{217}G_{31/7})^{-3}$. Using Lemma 4.5 and (5.39), we deduce that

$$P+P^{-1} = \frac{1}{2\sqrt{2}}\left(Q+Q^{-1}+7\right)$$
$$= \frac{1}{2\sqrt{2}}(x^2+9) = \frac{1}{2\sqrt{2}}(376+140\sqrt{7}).$$

Solving for P^{-1}, we find that

$$P^{-1} = \frac{94+35\sqrt{7}}{\sqrt{2}}+\sqrt{\frac{17409+6580\sqrt{7}}{2}}$$
$$= \left(\sqrt{\frac{11+4\sqrt{7}}{2}}+\sqrt{\frac{9+4\sqrt{7}}{2}}\right)^3. \tag{5.41}$$

Thus, from (5.40) and (5.41),

$$G_{217} = P^{-1/6}Q^{1/8} = \left(\sqrt{\frac{11+4\sqrt{7}}{2}}+\sqrt{\frac{9+4\sqrt{7}}{2}}\right)^{1/2}$$
$$\times\left(\sqrt{\frac{16+5\sqrt{7}}{4}}+\sqrt{\frac{12+5\sqrt{7}}{4}}\right)^{1/2},$$

which completes the proof.

Theorem 5.13.

$$G_{553} = \left(\sqrt{\frac{100+11\sqrt{79}}{4}}+\sqrt{\frac{96+11\sqrt{79}}{4}}\right)^{1/2}$$
$$\times\left(\sqrt{\frac{143+16\sqrt{79}}{2}}+\sqrt{\frac{141+16\sqrt{79}}{2}}\right)^{1/2}.$$

Proof. We set up the following table to summarize the information that we need.

d_1	d_2	χ	G	C	$\chi(G_0)$ $\chi(G_1)$ $\chi(G_2)$ $\chi(G_3)$	h_1	h_2	w_2	ϵ_1
1	-2212	χ_0	G_0	$[1, \Omega]$ $[2, 1+\Omega]$	1 1 1 1				
28	-79	χ_1	G_1	$[7, \Omega]$ $[14, 7+\Omega]$	1 -1 -1 1	1	5	2	$8+3\sqrt{7}$
553	-4	χ_2	G_2	$[17, 5+\Omega]$ $[17, -5+\Omega]$	1 -1 1 -1	1	1	4	$624\,635\,837\,407$ $+26\,562\,217\,704\sqrt{553}$
316	-7	χ_3	G_3	$[19, 6+\Omega]$ $[19, -6+\Omega]$	1 1 -1 -1				

It is clear that $\mathbb{Q}(\sqrt{-553})$ satisfies the hypotheses of Theorem 4.7. Thus, since $h = 8$ and $w = 2$,

$$\frac{1}{7^2}\left(\frac{\eta(\tau)\eta\left(\frac{\tau+1}{2}\right)}{\eta(7\tau)\eta\left(\frac{7\tau+1}{2}\right)}\right)^4$$
$$= (8+3\sqrt{7})^5(624,635,837,407+26,562,217,704\sqrt{553})^{1/2},$$

so that

$$\left(\frac{\eta(\tau)\eta\left(\frac{\tau+1}{2}\right)}{\eta(7\tau)\eta\left(\frac{7\tau+1}{2}\right)}\right)^2 = 7\epsilon, \tag{5.42}$$

where

$$\epsilon = (514,088+194,307\sqrt{7})^{1/2}(211,227\sqrt{7}+62,876\sqrt{79})^{1/2}. \tag{5.43}$$

Then an elementary calculation gives

$$\begin{aligned}(\epsilon-\epsilon^{-1})^2 &= \epsilon^2+\epsilon^{-2}-2\\ &= 4(143,650,096,411+16,161,898,544\sqrt{79})\\ &= (391+44\sqrt{79})(19170+2156\sqrt{79})^2.\end{aligned} \tag{5.44}$$

By Lemma 4.6 and (5.42)–(5.44), with $Q = (G_{553}/G_{79/7})^4$,

$$\begin{aligned}Q^{3/2}+8Q^{1/2}-8Q^{-1/2}-Q^{-3/2} &= 7(\epsilon-\epsilon^{-1})\\ &= 7\sqrt{391+44\sqrt{79}}(19170+2156\sqrt{79}).\end{aligned}$$

If $x = Q^{1/2}-Q^{-1/2}$, then the foregoing equality may be written in the form

$$x^3+11x = 7\sqrt{391+44\sqrt{79}}\left(7^2(391+44\sqrt{79})+11\right),$$

from which it is obvious that

$$x = Q^{1/2} - Q^{-1/2} = 7\sqrt{391 + 44\sqrt{79}}. \tag{5.45}$$

Solving for $Q^{1/2}$, we readily find that

$$\begin{aligned} Q^{1/2} &= \tfrac{1}{2}\left(7\sqrt{391 + 44\sqrt{79}} + (98 + 11\sqrt{79})\right) \\ &= \left(\sqrt{\frac{100 + 11\sqrt{79}}{4}} + \sqrt{\frac{96 + 11\sqrt{79}}{4}}\right)^2. \end{aligned} \tag{5.46}$$

Now let $P = (G_{553}G_{79/7})^{-3}$. Then, by Lemma 4.5 and (5.45),

$$2\sqrt{2}(P + P^{-1}) = Q + Q^{-1} + 7 = x^2 + 9 = 19168 + 2156\sqrt{79}.$$

Solving for P^{-1}, we find that

$$\begin{aligned} P^{-1} &= \frac{1}{\sqrt{2}}\left(4792 + 539\sqrt{79} + \sqrt{45,914,421 + 5,165,776\sqrt{79}}\right) \\ &= \left(\sqrt{\frac{143 + 16\sqrt{79}}{2}} + \sqrt{\frac{141 + 16\sqrt{79}}{2}}\right)^3. \end{aligned} \tag{5.47}$$

Thus, by (5.46) and (5.47),

$$\begin{aligned} G_{553} = Q^{1/8}P^{-1/6} &= \left(\sqrt{\frac{100 + 11\sqrt{79}}{4}} + \sqrt{\frac{96 + 11\sqrt{79}}{4}}\right)^{1/2} \\ &\quad \times \left(\sqrt{\frac{143 + 16\sqrt{79}}{2}} + \sqrt{\frac{141 + 16\sqrt{79}}{2}}\right)^{1/2}, \end{aligned}$$

and the proof is complete.

6. Class Invariants Via Modular Equations

In this section we establish six of Ramanujan's class invariants by using tools well known to Ramanujan, in particular, modular equations.

Second Proof of Theorem 5.1. From (1.1) and (4.3) it is easily seen that

$$\frac{f(-q)}{f(-q^2)} = \chi(-q). \tag{6.1}$$

Using this equality, we rewrite two of Ramanujan's eta–function identities in terms of χ. Thus (Part IV [4, pp. 206, 211])

$$\frac{f(-q)f(-q^2)}{q^{3/2}f(-q^{13})f(-q^{26})}+13\frac{q^{3/2}f(-q^{13})f(-q^{26})}{f(-q)f(-q^2)}=\left(q^{-1/2}\frac{\chi(-q^{13})}{\chi(-q)}\right)^3$$
$$-4\left(q^{-1/2}\frac{\chi(-q^{13})}{\chi(-q)}\right)-4\left(q^{1/2}\frac{\chi(-q)}{\chi(-q^{13})}\right)+\left(q^{1/2}\frac{\chi(-q)}{\chi(-q^{13})}\right)^3 \tag{6.2}$$

and

$$\frac{f(-q)f(-q^2)}{q^{1/2}f(-q^5)f(-q^{10})}+5\frac{q^{1/2}f(-q^5)f(-q^{10})}{f(-q)f(-q^2)}$$
$$=\left(q^{-1/6}\frac{\chi(-q^5)}{\chi(-q)}\right)^3+\left(q^{1/6}\frac{\chi(-q)}{\chi(-q^5)}\right)^3. \tag{6.3}$$

Replace q by $-q$ in (6.2) and then set $q=\exp(-\pi\sqrt{5/13})$. If

$$A:=e^{(3\pi/2)\sqrt{5/13}}\frac{f(e^{-\pi\sqrt{5/13}})f(-e^{-2\pi\sqrt{5/13}})}{f(e^{-\pi\sqrt{65}})f(-e^{-2\pi\sqrt{65}})} \tag{6.4}$$

and

$$B:=e^{(\pi/2)\sqrt{5/13}}\frac{\chi(e^{-\pi\sqrt{65}})}{\chi(e^{-\pi\sqrt{5/13}})}, \tag{6.5}$$

then (6.2) can be recast in the form

$$A-13A^{-1}=B^3+4B-4B^{-1}-B^{-3}. \tag{6.6}$$

Next, replace q by $-q$ in (6.3) and then set $q=\exp(-\pi\sqrt{13/5})$. If

$$A':=e^{(\pi/2)\sqrt{13/5}}\frac{f(e^{-\pi\sqrt{13/5}})f(-e^{-2\pi\sqrt{13/5}})}{f(e^{-\pi\sqrt{65}})f(-e^{-2\pi\sqrt{65}})} \tag{6.7}$$

and

$$B':=e^{(\pi/6)\sqrt{13/5}}\frac{\chi(e^{-\pi\sqrt{65}})}{\chi(e^{-\pi\sqrt{13/5}})}, \tag{6.8}$$

then (6.3) takes the shape

$$A'-5A'^{-1}=B'^3-B'^{-3}. \tag{6.9}$$

We shall prove that

$$B=B' \qquad \text{and} \qquad A=\sqrt{\frac{13}{5}}A'. \tag{6.10}$$

Now, $G_n=2^{-1/4}e^{\pi\sqrt{n}/24}\chi(e^{-\pi\sqrt{n}})$ and $G_{1/n}=2^{-1/4}e^{\pi/(24\sqrt{n})}\chi(e^{-\pi/\sqrt{n}})$, by (1.3). Since $G_n=G_{1/n}$, we find that

$$\chi(e^{-\pi\sqrt{n}})=e^{(\pi/24)(1/\sqrt{n}-\sqrt{n})}\chi(e^{-\pi/\sqrt{n}}). \tag{6.11}$$

(This could also be proved by using (6.1) along with the transformation formula for f.) In particular, if $n = 5/13$, (6.11) yields the equality

$$\chi(e^{-\pi\sqrt{5/13}}) = e^{\pi/(3\sqrt{65})}\chi(e^{-\pi\sqrt{13/5}}). \tag{6.12}$$

The aforementioned transformation formula for $f(-q)$ is given by (Part III [3, p. 43, Entry 27(iii)])

$$e^{-a/12}a^{1/4}f(-e^{-2a}) = e^{-b/12}b^{1/4}f(-e^{-2b}), \tag{6.13}$$

where $a, b > 0$ with $ab = \pi^2$. If $a = \pi\sqrt{5/13}$, so that $b = \pi\sqrt{13/5}$, then we deduce from (6.13) that

$$f(-e^{-2\pi\sqrt{5/13}}) = (13/5)^{1/4}e^{-2\pi/(3\sqrt{65})}f(-e^{-2\pi\sqrt{13/5}}). \tag{6.14}$$

First, from (6.5) and (6.12),

$$B = \frac{e^{(\pi/2)\sqrt{5/13}}\chi(e^{-\pi\sqrt{65}})}{e^{\pi/(3\sqrt{65})}\chi(e^{-\pi\sqrt{13/5}})} = e^{(\pi/6)\sqrt{13/5}}\frac{\chi(e^{-\pi\sqrt{65}})}{\chi(e^{-\pi\sqrt{13/5}})} = B',$$

by (6.8). Thus, the first equality of (6.10) has been demonstrated. Second, by (6.4), (6.1) with $q = -\exp(-\pi\sqrt{5/13})$, (6.12), (6.14), and lastly (6.1) with $q = \exp(-\pi\sqrt{13/5})$,

$$\begin{aligned} A &= e^{(3\pi/2)\sqrt{5/13}}\frac{\chi(e^{-\pi\sqrt{5/13}})f^2(-e^{-2\pi\sqrt{5/13}})}{f(e^{-\pi\sqrt{65}})f(-e^{-2\pi\sqrt{65}})} \\ &= \sqrt{\tfrac{13}{5}}e^{(\pi/2)\sqrt{13/5}}\frac{\chi(e^{-\pi\sqrt{13/5}})f^2(-e^{-2\pi\sqrt{13/5}})}{f(e^{-\pi\sqrt{65}})f(-e^{-2\pi\sqrt{65}})} \\ &= \sqrt{\tfrac{13}{5}}e^{(\pi/2)\sqrt{13/5}}\frac{f(e^{-\pi\sqrt{13/5}})f(-e^{-2\pi\sqrt{13/5}})}{f(e^{-\pi\sqrt{65}})f(-e^{-2\pi\sqrt{65}})} \\ &= \sqrt{\tfrac{13}{5}}A', \end{aligned}$$

by (6.7). Thus, the second equality of (6.10) has been established.

Employing (6.10) in (6.9), we find that

$$\sqrt{\tfrac{5}{13}}A - \sqrt{65}A^{-1} = B^3 - B^{-3} = (B - B^{-1})^3 + 3(B - B^{-1}).$$

Dividing both sides by $u := B - B^{-1} (\neq 0)$, we find that

$$\frac{\sqrt{65}}{13}(u^2 + 7) = u^2 + 3.$$

Solving for u^2, we find that $u^2 = (\sqrt{65} - 1)/2$. Thus, since, clearly, $B > 1$,

$$B - B^{-1} = \sqrt{\frac{\sqrt{65} - 1}{2}}.$$

Now solving for B, we find that

$$B = \sqrt{\frac{\sqrt{65}-1}{8}} + \sqrt{\frac{\sqrt{65}+7}{8}}, \tag{6.15}$$

where in solving the quadratic equation we took the plus sign since $B > 0$.

If $q = \exp(-\pi\sqrt{13/5})$, then $q^5 = \exp(-\pi\sqrt{65})$. Hence, from (1.3) and (6.5), we readily see that $B = G_{65}/G_{13/5}$. Furthermore, from (1.6), $G_{13/5} = \{4\alpha(1-\alpha)\}^{-1/24}$. Hence, if β has degree 5 over α, then $G_{65} = \{4\beta(1-\beta)\}^{-1/24}$.

We now employ Lemma 4.4, where it is to be noted that $P = (G_{65}G_{13/5})^{-2}$ and $Q = B^{-3} = (G_{65}/G_{13/5})^{-3}$. We already know Q from (6.15). To determine P from Lemma 4.4, we first calculate

$$\begin{aligned} Q + Q^{-1} = B^3 + B^{-3} &= (B + B^{-1})\{(B+B^{-1})^2 - 3\} \\ &= \sqrt{\frac{\sqrt{65}+7}{2}}\left(\frac{\sqrt{65}+7}{2} - 3\right) = \sqrt{74+10\sqrt{65}}. \end{aligned} \tag{6.16}$$

Thus, using (6.16) in Lemma 4.4 and solving for P^{-1}, we find that

$$P^{-1} = \tfrac{1}{4}\left(\sqrt{74+10\sqrt{65}} + \sqrt{90+10\sqrt{65}}\right), \tag{6.17}$$

since $P > 0$.

Hence, by (6.15) and (6.17),

$$\begin{aligned} G_{65} = B^{1/2}P^{-1/4} &= \left(\sqrt{\frac{\sqrt{65}+7}{8}} + \sqrt{\frac{\sqrt{65}-1}{8}}\right)^{1/2} \\ &\times \left(\tfrac{1}{4}\left(\sqrt{74+10\sqrt{65}} + \sqrt{90+10\sqrt{65}}\right)\right)^{1/4}. \end{aligned} \tag{6.18}$$

We must show that (6.18) can be transformed into the form of Theorem 5.1. First,

$$\begin{aligned} &\tfrac{1}{2}\left(\sqrt{74+10\sqrt{65}} + \sqrt{90+10\sqrt{65}}\right)^{1/2} \\ &= \tfrac{1}{2}\left(\sqrt{(9+\sqrt{65})(1+\sqrt{65})} + 5 + \sqrt{65}\right)^{1/2} = \sqrt{\frac{9+\sqrt{65}}{8}} + \sqrt{\frac{1+\sqrt{65}}{8}}. \end{aligned} \tag{6.19}$$

Second,

$$\begin{aligned} \left(\sqrt{\frac{\sqrt{65}+7}{8}} + \sqrt{\frac{\sqrt{65}-1}{8}}\right)^2 &= \tfrac{1}{4}\left(3 + \sqrt{65} + \sqrt{58+6\sqrt{65}}\right) \\ &= \tfrac{1}{4}\left(3 + \sqrt{65} + 3\sqrt{5} + \sqrt{13}\right) \end{aligned}$$

$$= \left(\frac{\sqrt{13}+3}{2}\right)\left(\frac{\sqrt{5}+1}{2}\right). \tag{6.20}$$

Putting (6.19) and (6.20) in (6.18), we complete the proof.

Before commencing our second proof of Theorem 5.2, we establish a general principle. Let p and r denote coprime, positive integers. Set $q = \exp(-\pi\sqrt{p/r})$ and $q' = \exp(-\pi\sqrt{pr})$, and let β have degree r over α. Then, by (1.6),

$$G_{p/r} = \{4\alpha(1-\alpha)\}^{-1/24} \qquad \text{and} \qquad G_{pr} = \{4\beta(1-\beta)\}^{-1/24}. \tag{6.21}$$

Furthermore, from (1.2) and (1.5),

$$\frac{K(\sqrt{1-\alpha})}{K(\sqrt{\alpha})} = \sqrt{\frac{p}{r}}, \tag{6.22}$$

and from the definition (1.4) of a modular equation,

$$r\frac{K(\sqrt{1-\alpha})}{K(\sqrt{\alpha})} = \frac{K(\sqrt{1-\beta})}{K(\sqrt{\beta})}. \tag{6.23}$$

If we solve (6.22) for r and substitute this in (6.23), we find that

$$p\frac{K(\sqrt{\alpha})}{K(\sqrt{1-\alpha})} = \frac{K(\sqrt{1-\beta})}{K(\sqrt{\beta})}.$$

From the last equality we conclude:

If β has degree r over α, then β has degree p over $1-\alpha$. (6.24)

Furthermore, from (2.5) and (6.22),

$$\frac{\varphi^2(e^{-\pi\sqrt{p/r}})}{\varphi^2(e^{-\pi\sqrt{r/p}})} = \frac{K(\sqrt{\alpha})}{K(\sqrt{1-\alpha})} = \sqrt{\frac{r}{p}}. \tag{6.25}$$

Second Proof of Theorem 5.2. We need two of Ramanujan's modular equations of degree 23 (Part III [3, p. 411, Entry 15(i), (ii)]). If β has degree 23 over α, then

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1 \tag{6.26}$$

and

$$\begin{aligned} 1 &+ (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} + 2^{4/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} \\ &= \left\{2\left(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}\right)\right\}^{1/2}. \end{aligned} \tag{6.27}$$

We also need two of Ramanujan's modular equations of degree 3. The first is given by Lemma 4.3, while the second is given by (Part III [3, p. 231, Entry 5(ix)])

$$\{\alpha(1-\beta)\}^{1/2} + \{(1-\alpha)\beta\}^{1/2} = 2\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}. \tag{6.28}$$

We shall apply (6.24) with $r = 3$ and $p = 23$. Thus, β has degree 23 over $(1-\alpha)$. Thus, replacing α by $(1-\alpha)$, from (6.26) and (6.27), we find that, respectively,

$$\{(1-\alpha)\beta\}^{1/8} + \{\alpha(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1 \tag{6.29}$$

and

$$\begin{aligned} 1 + \{(1-\alpha)\beta\}^{1/4} + \{\alpha(1-\beta)\}^{1/4} + 2^{4/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} \\ = \left\{2\left(1 + \{(1-\alpha)\beta\}^{1/2} + \{\alpha(1-\beta)\}^{1/2}\right)\right\}^{1/2}. \end{aligned} \tag{6.30}$$

For brevity, in the remainder of the proof, set $G = G_{69}$ and $G' = G_{23/3}$. By (6.21), we can rewrite (6.29) in the form

$$\{(1-\alpha)\beta\}^{1/8} + \{\alpha(1-\beta)\}^{1/8} = 1 - \sqrt{2}(GG')^{-1}.$$

Setting $u = (GG')^{-1}$ and squaring both sides, we deduce that

$$\{(1-\alpha)\beta\}^{1/4} + \{\alpha(1-\beta)\}^{1/4} = 1 + 2u^2 - 2\sqrt{2}u - \sqrt{2}u^3. \tag{6.31}$$

Substituting (6.31) into (6.30), we find that

$$2 + 4u^2 - 2\sqrt{2}u - \sqrt{2}u^3 = \sqrt{2}\left(1 + \{(1-\alpha)\beta\}^{1/2} + \{\alpha(1-\beta)\}^{1/2}\right)^{1/2}. \tag{6.32}$$

Then, using (6.28) in (6.32), we deduce that

$$2 + 4u^2 - 2\sqrt{2}u - \sqrt{2}u^3 = \sqrt{2}(1 + \sqrt{2}u^3)^{1/2}.$$

Squaring both sides and simplifying, we arrive at

$$2 - 8\sqrt{2}u + 24u^2 - 22\sqrt{2}u^3 + 24u^4 - 8\sqrt{2}u^5 + 2u^6 = 0,$$

which, with $x = u + 1/u$, is equivalent to

$$\begin{aligned} 22\sqrt{2} &= 2(u^3 + u^{-3}) - 8\sqrt{2}(u^2 + u^{-2}) + 24(u + u^{-1}) \\ &= 2(x^3 - 3x) - 8\sqrt{2}(x^2 - 2) + 24x. \end{aligned}$$

Simplifying, we find that

$$x^3 - 4\sqrt{2}x^2 + 9x - 3\sqrt{2} = 0.$$

By inspection, we verify that $\sqrt{2}$ is a root. Now G_n is a monotonically increasing function of n, and it is not difficult to check numerically that the root that we seek is greater than $\sqrt{2}$. Thus,

$$x^2 - 3\sqrt{2}x + 3 = 0,$$

and so $x = (3 + \sqrt{3})/\sqrt{2}$. Since $x = u + 1/u$, we find that

$$\frac{1}{u} = \sqrt{\frac{6 + 3\sqrt{3}}{4}} + \sqrt{\frac{2 + 3\sqrt{3}}{4}}, \tag{6.33}$$

since $u < 1$.

We now apply Lemma 4.3. Noting that $P = u^3$, we see that we want to calculate

$$
\begin{aligned}
u^{-3} - u^3 &= \sqrt{(u^{-3} + u^3)^2 - 4} \\
&= \sqrt{(x^3 - 3x)^2 - 4} \\
&= \sqrt{\left(\left(\frac{3+\sqrt{3}}{\sqrt{2}}\right)^3 - 3\left(\frac{3+\sqrt{3}}{\sqrt{2}}\right)\right)^2 - 4} \\
&= \sqrt{374 + 216\sqrt{3}}.
\end{aligned}
$$

Thus, by Lemma 4.3,

$$\left(\frac{G'}{G}\right)^6 + \left(\frac{G}{G'}\right)^6 = 2\sqrt{2}(u^{-3} - u^3) = 2\sqrt{2}\sqrt{374 + 216\sqrt{3}}.$$

Solving for G/G', we deduce that

$$\frac{G}{G'} = \left(\sqrt{748 + 432\sqrt{3}} + \sqrt{747 + 432\sqrt{3}}\right)^{1/6}. \tag{6.34}$$

Thus, by (6.33) and (6.34),

$$
\begin{aligned}
G = \sqrt{\frac{G}{G'}}u^{-1/2} &= \left(\sqrt{748 + 432\sqrt{3}} + \sqrt{747 + 432\sqrt{3}}\right)^{1/12} \\
&\quad \times \left(\sqrt{\frac{6+3\sqrt{3}}{4}} + \sqrt{\frac{2+3\sqrt{3}}{4}}\right)^{1/2}.
\end{aligned}
$$

To complete the proof, it suffices to show that

$$\left(\sqrt{748 + 432\sqrt{3}} + \sqrt{747 + 432\sqrt{3}}\right)^2 = \left(\frac{5+\sqrt{23}}{\sqrt{2}}\right)^2 \left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)^3,$$

which is a straightforward task.

Second Proof of Theorem 5.3. We need two of Ramanujan's modular equations of both degrees 7 and 11. If β has degree 7 over α, then (Part III [3, pp. 314, 315, Entry 19(i), (viii)])

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1 \tag{6.35}$$

and

$$
\begin{aligned}
m - \frac{7}{m} &= 2\left((\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}\right) \\
&\quad \times \left(2 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}\right), \qquad (6.36)
\end{aligned}
$$

where $m = \varphi^2(q)/\varphi^2(q^7)$. If β has degree 11 over α, then (Part III [3, p. 363, Entry 7(i), (ii)])

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} = 1 \tag{6.37}$$

and

$$\begin{aligned} m' - \frac{11}{m'} = 2\left((\alpha\beta)^{1/4} - \{(1-\alpha)(1-\beta)\}^{1/4}\right) \\ \times \left(4 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}\right), \end{aligned} \tag{6.38}$$

where $m' = \varphi^2(q)/\varphi^2(q^{11})$.

If $q = \exp(-\pi\sqrt{11/7})$, by (6.21),

$$G_{11/7} = \{4\alpha(1-\alpha)\}^{-1/24} \qquad \text{and} \qquad G_{77} = \{4\beta(1-\beta)\}^{-1/24}.$$

Thus, setting $u = (G_{77}G_{11/7})^{-1}$, we deduce from (6.35) that

$$\begin{aligned} \left((\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}\right)^2 &= \left((\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8}\right)^2 \\ &\quad - 4\{\alpha(1-\alpha)\beta(1-\beta)\}^{1/8} \\ &= 1 - 2\sqrt{2}u^3 \end{aligned}$$

and

$$\begin{aligned} (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} &= \left((\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8}\right)^2 \\ &\quad - 2\{\alpha(1-\alpha)\beta(1-\beta)\}^{1/8} \\ &= 1 - \sqrt{2}u^3. \end{aligned}$$

Thus, from (6.36),

$$m - \frac{7}{m} = 2\left(1 - 2\sqrt{2}u^3\right)^{1/2}(3 - \sqrt{2}u^3), \tag{6.39}$$

where $m = \varphi^2(e^{-\pi\sqrt{11/7}})/\varphi^2(e^{-\pi\sqrt{77}})$.

Let $q = \exp(-\pi\sqrt{7/11})$, and note that $u = (G_{77}G_{11/7})^{-1} = (G_{77}G_{7/11})^{-1}$. Thus, by (6.37),

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1 - 2u^2$$

and

$$\begin{aligned} \left((\alpha\beta)^{1/4} - \{(1-\alpha)(1-\beta)\}^{1/4}\right)^2 &= \left((\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}\right)^2 \\ &\quad - 4\{\alpha(1-\alpha)\beta(1-\beta)\}^{1/4} \\ &= (1-2u^2)^2 - 2u^6. \end{aligned}$$

Hence, from (6.38),

$$m' - \frac{11}{m'} = 2\left((1-2u^2)^2 - 2u^6\right)^{1/2}(5 - 2u^2), \tag{6.40}$$

where $m' = \varphi^2(e^{-\pi\sqrt{7/11}})/\varphi^2(e^{-\pi\sqrt{77}})$.

From (6.25), we see that

$$m' = \sqrt{\tfrac{11}{7}}m.$$

Since

$$m - \frac{7}{m} = \sqrt{\frac{7}{11}}m' - \sqrt{\frac{11}{7}}\frac{7}{m'} = \sqrt{\frac{7}{11}}\left(m' - \frac{11}{m'}\right),$$

we deduce from (6.39) and (6.40) that

$$2\left(1 - 2\sqrt{2}u^3\right)^{1/2}(3 - \sqrt{2}u^3) = \sqrt{\tfrac{7}{11}}2\left((1 - 2u^2)^2 - 2u^6\right)^{1/2}(5 - 2u^2).$$

Squaring both sides and simplifying, we find that

$$4u^{10} - 11\sqrt{2}u^9 - 98u^8 + 327u^6 - 322u^4 - 66\sqrt{2}u^3 + 210u^2 - 19 = 0.$$

Isolating the terms involving $\sqrt{2}$ on one side of the equation, squaring both sides, simplifying, and factoring, we deduce that

$$\begin{aligned}&(u^8 - 8u^6 + 7u^4 - 8u^2 + 1)\\ &\times (196u^{12} - 1418u^{10} + 6044u^8 - 13262u^6 + 13073u^4 - 5092u^2 + 361) = 0.\end{aligned} \tag{6.41}$$

Now $x := u^2$ is an algebraic integer (see Lemma 7.2) and so must be a root of a monic irreducible polynomial. The latter polynomial in (6.41) is irreducible, and so x must be a root of the former polynomial in (6.41). Alternatively, we used *Mathematica* to check numerically that x is not a root of the latter polynomial on the left side of (6.41). Thus,

$$x^4 - 8x^3 + 7x^2 - 8x + 1 = x^2\left((x + 1/x)^2 - 8(x + 1/x) + 5\right) = 0.$$

Since $x + 1/x > 1$,

$$x + \frac{1}{x} = 4 + \sqrt{11}.$$

Thus,

$$u + \frac{1}{u} = \sqrt{x + \frac{1}{x} + 2} = \sqrt{6 + \sqrt{11}}.$$

Since $u < 1$, we find that

$$\frac{1}{u} = \sqrt{\frac{6 + \sqrt{11}}{4}} + \sqrt{\frac{2 + \sqrt{11}}{4}}. \tag{6.42}$$

Lastly, we apply Lemma 4.5. Since $P = u^{-3}$, we deduce, by (6.42), that

$$\begin{aligned}Q + Q^{-1} &= 2\sqrt{2}(u^{-3} + u^3) - 7 = 2\sqrt{2}\left((u + u^{-1})^3 - 3(u + u^{-1})\right) - 7\\ &= 2\sqrt{2}(3 + \sqrt{11})\sqrt{6 + \sqrt{11}} - 7 = 2(3 + \sqrt{11})(1 + \sqrt{11}) - 7\\ &= 21 + 8\sqrt{11}.\end{aligned}$$

Hence,

$$\frac{1}{Q}=\frac{21+8\sqrt{11}}{2}+\frac{\sqrt{1141+326\sqrt{11}}}{2}=\frac{21}{2}+4\sqrt{11}+\frac{\sqrt{7}}{2}(8+3\sqrt{11})$$
$$=(8+3\sqrt{7})\left(\frac{\sqrt{11}+\sqrt{7}}{2}\right). \tag{6.43}$$

In conclusion, by (6.42) and (6.43),

$$G_{77}=Q^{-1/8}u^{-1/2}$$
$$=(8+3\sqrt{7})^{1/8}\left(\frac{\sqrt{11}+\sqrt{7}}{2}\right)^{1/8}\left(\sqrt{\frac{6+\sqrt{11}}{4}}+\sqrt{\frac{2+\sqrt{11}}{4}}\right)^{1/2},$$

and the proof is complete.

Second Proof of Theorem 5.4. We need two of Ramanujan's modular equations, one of degree 3 and one of degree 47. If β has degree 3 over α (Part III [3, p. 231, Entry 5(ix)]),

$$\{\alpha(1-\beta)\}^{1/2}+\{\beta(1-\alpha)\}^{1/2}=2\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}. \tag{6.44}$$

If β is of degree 47 over α (Part III [3, p. 444, Entry 23(i)]),

$$2\left(\tfrac{1}{2}\left(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}\right)\right)^{1/2}$$
$$=1+(\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}$$
$$+4^{1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}\left(1+(\alpha\beta)^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}\right). \tag{6.45}$$

Let $q=\exp(-\pi\sqrt{47/3})$. Then, by (6.21),

$$G':=G_{47/3}=\{4\alpha(1-\alpha)\}^{-1/24} \quad \text{and} \quad G:=G_{141}=\{4\beta(1-\beta)\}^{-1/24}.$$

Applying (6.24) with $r=3$ and $p=47$, we find that β has degree 47 over $(1-\alpha)$ when β has degree 3 over α. Thus, by (6.45),

$$2\left(\tfrac{1}{2}\left(1+\{(1-\alpha)\beta\}^{1/2}+\{\alpha(1-\beta)\}^{1/2}\right)\right)^{1/2}$$
$$=1+\{(1-\alpha)\beta\}^{1/4}+\{\alpha(1-\beta)\}^{1/4}$$
$$+4^{1/3}\{\alpha(1-\alpha)\beta(1-\beta)\}^{1/24}\left(1+\{(1-\alpha)\beta\}^{1/8}+\{\alpha(1-\beta)\}^{1/8}\right). \tag{6.46}$$

If $u:=(GG')^{-1}$, by (6.44),

$$\{\alpha(1-\beta)\}^{1/2}+\{\beta(1-\alpha)\}^{1/2}=\sqrt{2}u^3. \tag{6.47}$$

Hence,

$$\begin{aligned}
&\left(\{\alpha(1-\beta)\}^{1/4}+\{\beta(1-\alpha)\}^{1/4}\right)^2 \\
&\qquad = \{\alpha(1-\beta)\}^{1/2}+\{\beta(1-\alpha)\}^{1/2}+2\{\alpha(1-\alpha)\beta(1-\beta)\}^{1/4} \\
&\qquad = \sqrt{2}u^3+u^6 \qquad (6.48)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\{\alpha(1-\beta)\}^{1/8}+\{\beta(1-\alpha)\}^{1/8}\right)^2 \\
&\qquad = \{\alpha(1-\beta)\}^{1/4}+\{\beta(1-\alpha)\}^{1/4}+2\{\alpha(1-\alpha)\beta(1-\beta)\}^{1/8} \\
&\qquad = \left(\sqrt{2}u^3+u^6\right)^{1/2}+\sqrt{2}u^3. \qquad (6.49)
\end{aligned}$$

Substituting (6.47)–(6.49) into (6.46), we find that

$$\begin{aligned}
2\left(\tfrac{1}{2}(1+\sqrt{2}u^3)\right)^{1/2} &= 1+(\sqrt{2}u^3+u^6)^{1/2} \\
&\quad + \sqrt{2}u\left(1+\left((\sqrt{2}u^3+u^6)^{1/2}+\sqrt{2}u^3\right)^{1/2}\right). \qquad (6.50)
\end{aligned}$$

Using Gröbner bases, A. Strzebonski denested (6.50) and obtained a polynomial of degree 48 for u. The value of u that we seek is a root of the factor $u^8-32u^6+15u^4-32u^2+1$ of this 48th degree polynomial. If $x=u^2$, then

$$x^4-32x^3+15x^2-32x+1=x^2\left((x+1/x)^2-32(x+1/x)+13\right)=0.$$

Since $x+1/x>1$, we find that

$$x+\frac{1}{x}=16+9\sqrt{3}.$$

Hence,

$$u+\frac{1}{u}=\sqrt{18+9\sqrt{3}},$$

so that

$$\frac{1}{u}=\sqrt{\frac{18+9\sqrt{3}}{4}}+\sqrt{\frac{14+9\sqrt{3}}{4}}. \qquad (6.51)$$

Lastly, we apply Lemma 4.3 with $P=u^3$ and $Q=(G'/G)^6$ to deduce, from (6.51), that

$$\begin{aligned}
Q+\frac{1}{Q} &= 2\sqrt{2}(u^{-3}-u^3)=2\sqrt{2}\left((u^{-1}-u)^3+3(u^{-1}-u)\right) \\
&= 2\sqrt{2}(14+9\sqrt{3})^{1/2}(17+9\sqrt{3}).
\end{aligned}$$

Solving for $1/Q$, we find that

$$\frac{1}{Q}=\sqrt{2}(14+9\sqrt{3})^{1/2}(17+9\sqrt{3})+\sqrt{31419+18144\sqrt{3}}. \qquad (6.52)$$

Thus, by (6.51) and (6.52),

$$\begin{aligned} G_{141} &= Q^{-1/12}u^{-1/2} \\ &= \left(\sqrt{2}(14+9\sqrt{3})^{1/2}(17+9\sqrt{3}) + \sqrt{31419+18144\sqrt{3}}\right)^{1/12} \\ &\quad \times \left(\sqrt{\frac{18+9\sqrt{3}}{4}} + \sqrt{\frac{14+9\sqrt{3}}{4}}\right)^{1/2}. \end{aligned}$$

It remains to show that

$$\begin{aligned} \sqrt{2}(14+9\sqrt{3})^{1/2}(17+9\sqrt{3}) &+ \sqrt{31419+18144\sqrt{3}} \\ &= (4\sqrt{3}+\sqrt{47})^{3/2}\left(\frac{7+\sqrt{47}}{\sqrt{2}}\right), \end{aligned}$$

which is easily accomplished via *Mathematica.*

Second Proof of Theorem 5.5. We need two modular equations, one of degree 5 and the other of degree 29. The first is found in Ramanujan's second notebook. If β has degree 5 over α, then (Part III [3, p. 281, Entry 13(x)])

$$\{\alpha(1-\beta)\}^{1/4} + \{\beta(1-\alpha)\}^{1/4} = 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}. \tag{6.53}$$

The second is found in Ramanujan's first notebook on page 304, but curiously not in his second. R. Russell [2] established this modular equation in 1890, but his formulation is imprecise; in particular, it has a sign ambiguity. We give Ramanujan's formulation as stated in Entry 65 of Chapter 36. Let

$$\begin{aligned} P &= 1 - \sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}, \\ Q &= 64\left\{\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} - \sqrt{\alpha\beta(1-\alpha)(1-\beta)}\right\}, \end{aligned}$$

and

$$R = 32\sqrt{\alpha\beta(1-\alpha)(1-\beta)}.$$

Then, if β has degree 29 over α,

$$\sqrt{P}(P^2 + 17PR^{1/3} - 9R^{2/3}) = R^{1/6}(9P^2 + Q - 13PR^{1/3} + 15R^{2/3}). \tag{6.54}$$

Let $q = \exp(-\pi\sqrt{29/5})$, so that we may apply (6.21) and (6.24) with $r = 5$ and $p = 29$. If $u = (G_{145}G_{29/5})^{-1}$, then, by (6.53),

$$\begin{aligned} \{\alpha(1-\beta)\}^{1/2} + \{\beta(1-\alpha)\}^{1/2} &= \left(\{\alpha(1-\beta)\}^{1/4} + \{\beta(1-\alpha)\}^{1/4}\right)^2 \\ &\quad - 2\{\alpha(1-\alpha)\beta(1-\beta)\}^{1/4} \\ &= 2u^2 - u^6. \end{aligned} \tag{6.55}$$

Thus, by (6.55), with α replaced by $(1-\alpha)$,

$$P = 1 - 2u^2 + u^6,$$
$$Q = 128u^2 - 64u^6 - 16u^{12},$$

and

$$R = 8u^{12}.$$

Substitute these values into (6.54), square both sides, simplify, and factor, with the help of *Mathematica*. We then find that

$$(u^2+1)(u^4-u^2-1)(u^4-u^2+1)(u^8-20u^6-43u^4-20u^2+1) \times (u^{12}-9u^{10}+181u^8-126u^6-181u^4-9u^2-1)=0.$$

In numerically checking the roots of each of these polynomials, we find that $x := u^2$ is a root of

$$x^4 - 20x^3 - 43x^2 - 20x + 1 = x^2\left((x+1/x)^2 - 20(x+1/x) - 45\right) = 0.$$

Thus, $x + 1/x = 10 + \sqrt{145}$, and so $u - 1/u = \sqrt{8+\sqrt{145}}$. Hence,

$$\frac{1}{u} = \sqrt{\frac{8+\sqrt{145}}{4}} + \sqrt{\frac{12+\sqrt{145}}{4}}. \tag{6.56}$$

Lastly, we apply Lemma 4.4 with $P = u^2$ and $Q = (G_{29/5}/G_{145})^3$. Then

$$Q + \frac{1}{Q} = 2\left(\frac{1}{P} - P\right) = 2\left(\frac{1}{u} + u\right)\left(\frac{1}{u} - u\right)$$
$$= 2\sqrt{12+\sqrt{145}}\sqrt{8+\sqrt{145}}$$
$$= 2\sqrt{241+20\sqrt{145}}.$$

Hence,

$$\frac{1}{Q} = \sqrt{241+20\sqrt{145}} + \sqrt{240+20\sqrt{145}}. \tag{6.57}$$

From (6.56) and (6.57),

$$G_{145} = Q^{-1/6}u^{-1/2} = \left(\sqrt{241+20\sqrt{145}} + \sqrt{240+20\sqrt{145}}\right)^{1/6} \times \left(\sqrt{\frac{12+\sqrt{145}}{4}} + \sqrt{\frac{8+\sqrt{145}}{4}}\right)^{1/2}.$$

To complete the proof, we must show that

$$(\sqrt{5}+2)\left(\frac{\sqrt{29}+5}{2}\right) = \left(\sqrt{\frac{12+\sqrt{145}}{4}} + \sqrt{\frac{8+\sqrt{145}}{4}}\right)^2$$

and

$$\left(\sqrt{\frac{17+\sqrt{145}}{8}}+\sqrt{\frac{9+\sqrt{145}}{8}}\right)^3=\sqrt{241+20\sqrt{145}}+\sqrt{240+20\sqrt{145}}.$$

Both equalities are easily verified.

Second Proof of Theorem 5.7. In addition to the modular equation of degree 3 given by (6.44), we need Ramanujan's modular equation of degree 71. If β has degree 71 over α, then (Part III [3, p. 444, Entry 23(ii)])

$$\begin{aligned}1&+(\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}-\left(\tfrac{1}{2}\left(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}\right)\right)^{1/2}\\&=(\alpha\beta)^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}-\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\\&\quad+2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}\left(1-(\alpha\beta)^{1/8}-\{(1-\alpha)(1-\beta)\}^{1/8}\right).\end{aligned}\tag{6.58}$$

Let $r=3$ and $p=71$ in equalities (6.21) and principle (6.24). Thus, β has degree 71 over $(1-\alpha)$. Replacing α by $1-\alpha$ in (6.58) and employing (6.47)–(6.49), but now with $u=(G_{213}G_{71/3})^{-1}$, we deduce that

$$\begin{aligned}1&+\left(\sqrt{2}u^3+u^6\right)^{1/2}-\left(\tfrac{1}{2}(1+\sqrt{2}u^3)\right)^{1/2}\\&=\left(\left(\sqrt{2}u^3+u^6\right)^{1/2}+\sqrt{2}u^3\right)^{1/2}\\&\quad-\frac{1}{\sqrt{2}}u^3+\sqrt{2}u\left(1-\left(\left(\sqrt{2}u^3+u^6\right)^{1/2}+\sqrt{2}u^3\right)^{1/2}\right)\\&=\left(\left(\sqrt{2}u^3+u^6\right)^{1/2}+\sqrt{2}u^3\right)^{1/2}(1-\sqrt{2}u)-\frac{1}{\sqrt{2}}u^3+\sqrt{2}u.\end{aligned}\tag{6.59}$$

Using resultants, A. Strzebonski and M. Trott denested (6.59) and found a polynomial that factors into several polynomials of degrees 8, 12, and 28. Numerically eliminating all factors except one, we find that u satisfies

$$u^8-80u^6-126u^4-80u^2+1=0.$$

Letting $u^2=:x$, and solving for $x+1/x$, we find that $x+1/x=40+24\sqrt{3}$. It then follows that $u+1/u=\sqrt{42+24\sqrt{3}}$. Hence,

$$\frac{1}{u}=\sqrt{\frac{21+12\sqrt{3}}{2}}+\sqrt{\frac{19+12\sqrt{3}}{2}}.\tag{6.60}$$

Lastly, apply Lemma 4.3 with $P=(G_{213}G_{71/3})^{-3}=u^3$ and $Q=(G_{71/3}/G_{213})^6$. So, by Lemma 4.3 and (6.60),

$$Q+\frac{1}{Q}=2\sqrt{2}(u^{-3}-u^3)=4(19+12\sqrt{3})^{1/2}(41+24\sqrt{3}).$$

Solving for $1/Q$, we find that

$$\frac{1}{Q} = 2(19 + 12\sqrt{3})^{1/2}(41 + 24\sqrt{3}) + \sqrt{542,475 + 313,200\sqrt{3}}. \qquad (6.61)$$

Hence, by (6.60) and (6.61),

$$\begin{aligned}G_{213} &= Q^{-1/12}u^{-1/2} \\ &= \left(2(19 + 12\sqrt{3})^{1/2}(41 + 24\sqrt{3}) + \sqrt{542,475 + 313,200\sqrt{3}}\right)^{1/12} \\ &\quad \times \left(\sqrt{\frac{21 + 12\sqrt{3}}{2}} + \sqrt{\frac{19 + 12\sqrt{3}}{2}}\right)^{1/2}.\end{aligned}$$

It thus remains to show that

$$\begin{aligned}&2(19 + 12\sqrt{3})^{1/2}(41 + 24\sqrt{3}) + \sqrt{542,475 + 313,200\sqrt{3}} \\ &\quad = \left(\frac{5\sqrt{3} + \sqrt{71}}{2}\right)^{3/2}\left(\frac{59 + 7\sqrt{71}}{\sqrt{2}}\right),\end{aligned}$$

which can be verified via *Mathematica.*

7. Class Invariants Via Class Field Theory

In [6], Watson employed an "empirical process" to evaluate 14 of Ramanujan's class invariants. Motivated by Watson's idea, we succeeded in formulating theorems which give *rigorous* evaluations of G_{pq} and $G_{p/q}$ when p and q are distinct primes satisfying $pq \equiv 1 \pmod 4$ and $h(\sqrt{-pq}) = 8$.

Let $K = \mathbb{Q}(\sqrt{-m})$ (m squarefree) be an imaginary quadratic field, and let $\mathfrak{O}_K$ be its ring of integers. By class field theory (J. Janusz [1, p. 228, Theorem 12.1]), there exists an everywhere unramified extension $K^{(1)}|K$ such that the Galois group $Gal(K^{(1)} \mid K) \simeq C_K$, where C_K is the ideal class group of K. The field $K^{(1)}$ is known as the *Hilbert class field of* K. A Hilbert class field of K is usually defined as the maximal unramified abelian extension of K.

Let $\mathfrak{a} = [\tau_1, \tau_2]$ be an $\mathfrak{O}_K$-ideal. Define

$$j(\mathfrak{a}) = j([\tau_1, \tau_2]) = 1728\frac{g_2^3([\tau_1, \tau_2])}{g_2^3([\tau_1, \tau_2]) - 27g_3^2([\tau_1, \tau_2])},$$

where

$$g_2([\tau_1, \tau_2]) = 60 \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{1}{(m\tau_1 + n\tau_2)^4}$$

and

$$g_3([\tau_1, \tau_2]) = 140 \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \frac{1}{(m\tau_1 + n\tau_2)^6}.$$

It is clear from the definitions of $g_2([\tau_1, \tau_2])$ and $g_3([\tau_1, \tau_2])$ that

$$j([\tau_1, \tau_2]) = j([1, \tau]) =: j(\tau),$$

where $\tau = \tau_2/\tau_1$. We also let

$$\gamma_2(\tau) = \sqrt[3]{j(\tau)}$$

with the cube root being real–valued when $j(\mathfrak{a})$ is real.

It is well known that $K^{(1)} = K(j(\mathfrak{O}_K))$ (D. A. Cox [1, p. 220, Theorem 11.1]). If D_K is the discriminant of K and $3 \nmid D_K$, then $K^{(1)} = K(\gamma_2(\tau_K))$ (Cox [1, p. 249, Theorem 12.2]), where

$$\tau_K = \begin{cases} \sqrt{-m}, & D_K \equiv 0 \pmod 4, \\ \dfrac{3+\sqrt{-m}}{2}, & D_K \equiv 1 \pmod 4. \end{cases}$$

Lemma 7.1. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two $\mathfrak{O}_K$-ideals. Define $\sigma_{\mathfrak{a}}(j(\mathfrak{b}))$ by*

$$\sigma_{\mathfrak{a}}(j(\mathfrak{b})) = j(\bar{\mathfrak{a}}\mathfrak{b}), \tag{7.1}$$

where $\mathfrak{a}\bar{\mathfrak{a}}$ is a principal ideal. Then $\sigma_{\mathfrak{a}}$ is a well–defined element of $Gal(K^{(1)} \mid K)$, and $\mathfrak{a} \mapsto \sigma_{\mathfrak{a}}$ induces an isomorphism

$$C_K \longrightarrow Gal(K^{(1)} \mid K).$$

Proof. See Cox's book [1, p. 240, Corollary 11.37].

Lemma 7.2. *Let $K = \mathbb{Q}(\sqrt{-pq})$, where p and q are two distinct primes satisfying $pq \equiv 1 \pmod 4$, and let*

$$\gamma = \begin{cases} 4, & \text{if } 3 \nmid pq, \\ 12, & \text{if } 3 \mid pq. \end{cases}$$

Then G_{pq}^{γ} is a real unit generating the field $K^{(1)}$.

Proof. From a paper by B. J. Birch [1, p. 290], we find that G_{pq}^{12} is a real unit of $K^{(1)}$. Since (Cox [1, p. 257, Theorem 12.17])

$$j(\mathfrak{O}_K) = j(\sqrt{-pq}) = \frac{(16G_{pq}^{24} - 4)^3}{G_{pq}^{24}}, \tag{7.2}$$

we conclude that

$$K^{(1)} = K(G_{pq}^{12}). \tag{7.3}$$

Next, suppose that $3 \nmid pq$. Then $3 \nmid D_K$ and $\gamma_2(\tau_K)$ generates $K^{(1)}$. From the equality (Cox [1, p. 257, Theorem 12.17])

$$\gamma_2(\sqrt{-pq}) = \frac{16G_{pq}^{24} - 4}{G_{pq}^{8}} \tag{7.4}$$

and (7.3), we find that $G_{pq}^8 \in K^{(1)}$. Hence, $G_{pq}^4 \in K^{(1)}$, by (7.3).

In [1, p. 290], Birch quoted Deuring's results [1, p. 43] and indicated that G_{pq} is a unit when $pq \equiv 1 \pmod 4$. A more elaborate proof of this statement was given in a paper by Chan and Huang [1, Cor. 5.2] and is contained in Theorem 1.1 in this chapter. In fact, from the treatment given in their paper, one can show that $G_{p/q}$ is also a unit. This fact will be needed in our main theorem.

From class field theory, we know that if H is a subgroup of C_K, then there exists an abelian and everywhere unramified extension $L|K$ such that

$$Gal(K^{(1)}|L) \simeq H.$$

In particular, when $H = C_K^2 :=$ the subgroup of squares in C_K, the corresponding field $M|K$ is known as the *genus field* of K. One can show that M is the maximal unramified extension of K which is abelian over $\mathbb{Q}$ (Cox [1, p. 122]).

Theorem 7.3. *Let K and γ be defined as in Lemma 7.2. If the order of C_K is 8, then*

$$\alpha_{p,q} := (G_{pq}G_{p/q})^{\gamma} + (G_{pq}G_{p/q})^{-\gamma}$$

and

$$\beta_{p,q} := \left(\frac{G_{pq}}{G_{p/q}}\right)^{\gamma} + \left(\frac{G_{pq}}{G_{p/q}}\right)^{-\gamma}$$

are algebraic integers which belong to the real quadratic field R, where $R \in \{\mathbb{Q}(\sqrt{p}), \mathbb{Q}(\sqrt{q}), \mathbb{Q}(\sqrt{pq})\}$, and where R is a field such that none of the prime ideals (2), (p), *or* (q) *are inert.*

Proof. From the hypothesis, we deduce that $\mathfrak{a}_1 = [1, \sqrt{-pq}]$, $\mathfrak{a}_2 = [q, \sqrt{-pq}]$, $\mathfrak{a}_3 = [2, 1 + \sqrt{-pq}]$, and $\mathfrak{a}_4 = [2q, q + \sqrt{-pq}]$ are $\mathfrak{O}_K$-ideals lying in distinct equivalence classes (see Section 4). This implies that C_K contains the Klein four-group generated by the ideal classes $[\mathfrak{a}_i]$ and $[\mathfrak{a}_j]$ for $i > j > 1$. Using the isomorphism described in Lemma 7.1, we conclude that $Gal(K^{(1)} \mid K)$ contains a Klein four–group V generated by $\sigma_{\mathfrak{a}_i}$ and $\sigma_{\mathfrak{a}_j}$ for $i > j > 1$. To show that $\alpha_{p,q}$ and $\beta_{p,q}$ belong to a field with degree 2 over K, it suffices to show that $\sigma_{\mathfrak{a}_i}$ and $\sigma_{\mathfrak{a}_j}$ fix $\alpha_{p,q}$ and $\beta_{p,q}$. More precisely, if $F := Fix(V)$ is the field fixed by V, then by Galois theory (J. Rotman [1, p. 49, Theorem 63]), $|F : K| = |Gal(K^{(1)} \mid K) : V|$ $= 2$ (since $|C_K| = 8$), which implies that F is of degree 2 over K. Since $\alpha_{p,q}$ and $\beta_{p,q}$ are real numbers in F, they belong to $R := F \cap \mathbb{R}$, and R is clearly a real quadratic field over $\mathbb{Q}$. The fact that they are algebraic integers follows from the fact that G_{pq}^{γ} and $G_{p/q}^{\gamma}$ are units (see Lemma 7.2).

At this stage, we will assume that $3 \mid pq$. From Cox's text [1, p. 257, Theorem 12.17],

$$j(\mathfrak{a}_2) = j(\sqrt{-p/q}) = \frac{(16G_{p/q}^{24} - 4)^3}{G_{p/q}^{24}}. \tag{7.5}$$

By Lemma 7.1, we find that

$$\sigma_{\mathfrak{a}_2}(j(\mathfrak{a}_1)) = j(\mathfrak{a}_2\mathfrak{a}_1) = j(\mathfrak{a}_2). \tag{7.6}$$

From (7.2), (7.5), and (7.6), we find that

$$\frac{(16\sigma_{\mathfrak{a}_2}^2(G_{pq}^{12}) - 4)^3}{\sigma_{\mathfrak{a}_2}^2(G_{pq}^{12})} = \frac{(16G_{p/q}^{24} - 4)^3}{G_{p/q}^{24}}. \tag{7.7}$$

Simplifying (7.7), we deduce that

$$(a-b)(a+b)\left\{64(a^2+b^2)a^2b^2 - 48a^2b^2 + 1\right\} = 0,$$

where $a = \sigma_{\mathfrak{a}_2}(G_{pq}^{12})$ and $b = G_{p/q}^{12}$. But

$$64(a^2+b^2)a^2b^2 - 48a^2b^2 + 1 \neq 0,$$

for otherwise it would contradict the fact that a and b are algebraic integers. Thus, we deduce that

$$\sigma_{\mathfrak{a}_2}(G_{pq}^{12}) = \pm G_{p/q}^{12}. \tag{7.8}$$

Similarly, corresponding to (7.8),

$$\sigma_{\mathfrak{a}_2}(G_{p/q}^{12}) = \pm G_{pq}^{12} \qquad \text{or} \qquad \sigma_{\mathfrak{a}_2}(G_{p/q}^{12}) = \mp G_{pq}^{12},$$

i.e., $\sigma_{\mathfrak{a}_2}(G_{p/q}^{12})$ may have the same or opposite sign as $\sigma_{\mathfrak{a}_2}(G_{pq}^{12})$. Since $\sigma_{\mathfrak{a}_2}^2 = 1$, the latter is inadmissible. Hence,

$$\sigma_{\mathfrak{a}_2}(G_{p/q}^{12}) = \pm G_{pq}^{12}. \tag{7.9}$$

From (7.8) and (7.9), it is now clear that

$$\sigma_{\mathfrak{a}_2}(\alpha_{p,q}) = \alpha_{p,q}$$

and

$$\sigma_{\mathfrak{a}_2}(\beta_{p,q}) = \beta_{p,q}.$$

Next, from Cox's text [1, p. 263], we find that

$$j(\mathfrak{a}_3) = j\left(\frac{3+\sqrt{-pq}}{2}\right) = G_{pq}^{24}\left(\frac{16}{G_{pq}^{24}} - 4\right)^3 \tag{7.10}$$

and

$$j(\mathfrak{a}_4) = j\left(\frac{3+\sqrt{-p/q}}{2}\right) = G_{p/q}^{24}\left(\frac{16}{G_{p/q}^{24}} - 4\right)^3. \tag{7.11}$$

Now, applying Lemma 7.1 again, we have

$$\sigma_{\mathfrak{a}_3}(j(\mathfrak{a}_1)) = j(\mathfrak{a}_3).$$

By (7.10) and (7.2), we find that

$$\frac{(16\sigma_{\mathfrak{a}_3}^2(G_{pq}^{12}) - 4)^3}{\sigma_{\mathfrak{a}_3}^2(G_{pq}^{12})} = G_{pq}^{24}\left(\frac{16}{G_{pq}^{24}} - 4\right)^3,$$

which implies that

$$\sigma_{\mathfrak{a}_3}(G_{pq}^{12}) = \pm G_{pq}^{-12}.$$

Similarly, since $\mathfrak{a}_3\mathfrak{a}_2$ is equivalent to $\mathfrak{a}_4$,

$$\sigma_{\mathfrak{a}_3}(G_{p/q}^{12}) = \pm G_{p/q}^{-12} \quad \text{or} \quad \mp G_{p/q}^{-12},$$

by (7.11) and (7.5), i.e., $\sigma_{\mathfrak{a}_3}(G_{p/q}^{12})$ may have the same or opposite sign as $\sigma_{\mathfrak{a}_3}(G_{pq}^{12})$. We will show that the latter case is inadmissible. If

$$\sigma_{\mathfrak{a}_2}(G_{p/q}^{12}) = \pm G_{pq}^{12} \quad \text{and} \quad \sigma_{\mathfrak{a}_3}(G_{p/q}^{12}) = \mp G_{p/q}^{-12},$$

then

$$\sigma_{\mathfrak{a}_2}\sigma_{\mathfrak{a}_3}(G_{pq}^{12}) = \pm G_{p/q}^{-12} \quad \text{and} \quad \sigma_{\mathfrak{a}_3}\sigma_{\mathfrak{a}_2}(G_{pq}^{12}) = \mp G_{p/q}^{-12}.$$

This is clearly a contradiction since $\sigma_{\mathfrak{a}_2}\sigma_{\mathfrak{a}_3} = \sigma_{\mathfrak{a}_3}\sigma_{\mathfrak{a}_2}$. Hence,

$$\sigma_{\mathfrak{a}_3}(\alpha_{p,q}) = \alpha_{p,q}$$

and

$$\sigma_{\mathfrak{a}_3}(\beta_{p,q}) = \beta_{p,q}.$$

Collecting our results, we see that both $\sigma_{\mathfrak{a}_2}$ and $\sigma_{\mathfrak{a}_3}$ fix $\alpha_{p,q}$ and $\beta_{p,q}$, and this implies that $\alpha_{p,q}$ and $\beta_{p,q}$ are real quadratic algebraic integers.

The proof for the case when $3 \nmid pq$ is similar. In this case, G_{pq}^4 generates $K^{(1)}$, and so $\sigma_{\mathfrak{a}_i}(G_{pq}^4)$ is well defined for $i > 1$. Hence, we may deduce from (7.7) that

$$16\left(\sigma_{\mathfrak{a}_2}(G_{pq}^8)\right)^2 - \frac{4}{\sigma_{\mathfrak{a}_2}(G_{pq}^8)} = 16G_{p/q}^{16} - \frac{4}{G_{p/q}^8}. \tag{7.12}$$

Simplifying (7.12), we have

$$(a-b)(4a^2b + 4ab^2 + 1) = 0,$$

where $a = \sigma_{\mathfrak{a}_2}(G_{pq}^8)$ and $b = G_{p/q}^8$. But

$$4a^2b + 4ab^2 + 1 \neq 0,$$

for otherwise it would contradict the fact that a and b are algebraic integers. Hence, we deduce that

$$\sigma_{\mathfrak{a}_2}(G_{pq}^8) = G_{p/q}^8.$$

Now, since $\sigma_{\mathfrak{a}_2} \in Gal(K^{(1)}|K)$ and G_{pq}^4 generates $K^{(1)}$ (see Lemma 7.2),

$$\sigma_{\mathfrak{a}_2}(G_{pq}^4) = \pm G_{p/q}^4.$$

The rest of the arguments are similar to those of the previous case, and we shall omit them.

We have already seen that $\alpha_{p,q}$ and $\beta_{p,q}$ lie in a real quadratic field R. Our next task is to give a necessary condition for R. First, we observe that $R = F \cap \mathbb{R}$, where $F = Fix(V)$ is an abelian, everywhere unramified extension of K (see the paragraph before the statement of Theorem 7.3). Hence, $R \in \{\mathbb{Q}(\sqrt{p}), \mathbb{Q}(\sqrt{q}), \mathbb{Q}(\sqrt{pq})\}$. Next, we will show that none of the prime ideals (2), (p) or (q) are inert in R. Suppose the contrary holds. Then without loss of generality, we may assume that (p) is inert in R. This implies that $\mathfrak{p}$ in K is inert in F, where $\mathfrak{p}|(p)$ and the Frobenius automorphism $\left[\dfrac{F|K}{\mathfrak{p}}\right]$ has order 2 (see the books by Cox [1, pp. 106–107] or Janusz [1, pp. 126–127]).

On the other hand, we know that the Frobenius automorphism $\sigma_{\mathfrak{p}} = \left[\dfrac{K^{(1)}|K}{\mathfrak{p}}\right]$ has order 2 and that $\left.\left[\dfrac{K^{(1)}|K}{\mathfrak{p}}\right]\right|_E = 1$, where

$$E = Fix\left(\left[\frac{K^{(1)}|K}{\mathfrak{p}}\right]\right).$$

Since (Janusz [1, p. 127, Property 2.3])

$$\left[\frac{E|K}{\mathfrak{p}}\right] = \left.\left[\frac{K^{(1)}|K}{\mathfrak{p}}\right]\right|_E,$$

we find that $\left[\dfrac{E|K}{\mathfrak{p}}\right]$ has order 1. Consequently,

$$\left[\frac{F|K}{\mathfrak{p}}\right] = \left.\left[\frac{E|K}{\mathfrak{p}}\right]\right|_F = 1.$$

This clearly contradicts the last statement of the previous paragraph. Thus, (p) is not inert in R.

Our next step is to determine $\alpha_{p,q}$ and $\beta_{p,q}$ using the numerical values of G_{pq} and $G_{p/q}$. To achieve this, we need the following result.

Theorem 7.4. *Let $R = \mathbb{Q}(\sqrt{m})$ be the field which contains $\alpha_{p,q}$ and $\beta_{p,q}$. If*

$$2\alpha_{p,q} = a_1 + a_2\sqrt{m} \tag{7.13}$$

and

$$2\beta_{p,q} = b_1 + b_2\sqrt{m}, \tag{7.14}$$

then a_1, a_2, b_1, and b_2 are positive integers.

Proof. Let $[\mathfrak{a}] \in A := \{[\mathfrak{a}_2], [\mathfrak{a}_3], [\mathfrak{a}_4]\}$, and let H be the group generated by $[\mathfrak{a}]$. From the paragraph before the statement of Theorem 7.3, we know that there exists an abelian and everywhere unramified extension $L|K$ such that

$$Gal(K^{(1)}|L) \simeq H.$$

In fact, from the isomorphism of Lemma 7.1, we find that $L = Fix(\sigma_{\mathfrak{a}})$. Since $Gal(K^{(1)}|K) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_4$, the group

$$Gal(L|K) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad \text{or} \quad \mathbb{Z}_4.$$

The first case can only happen for exactly one element in A, and the field L in this case is the genus field M of K. As for the second case, $Gal(L|\mathbb{Q}) \simeq D_8$, the dihedral group of eight elements, since L is generalized dihedral over $\mathbb{Q}$ (Cox [1, p. 191]). Hence, $Gal(L|\mathbb{Q})$ is nonabelian.

Now, rewrite (7.13) as

$$2(\eta + \eta^{-1}) = a_1 + a_2\sqrt{m}, \tag{7.15}$$

where $\eta = (G_{pq}G_{p/q})^{\gamma}$. Note that $\sigma_{\mathfrak{a}_2}$ fixes η and $\sigma_{\mathfrak{a}_3}(\eta) = \eta^{-1}$. Therefore, the field $L := K(\eta) = Fix(< \sigma_{\mathfrak{a}_2} >)$ is of degree 4 over K.

Suppose L is the genus field of K. Since $\sigma_{\mathfrak{a}_2}|_L = 1$, we conclude that the ideal $[\mathfrak{a}_2]$ lies in an ideal class belonging to the principal genus. Hence, by Theorem 4.2, we may write

$$\eta = \prod_{\chi(G_1)=-1} \epsilon_1^{e_1}, \tag{7.16}$$

where

$$e_1 = \begin{cases} wh_1h_2/w_2, & \text{if } 3 \nmid pq, \\ 3wh_1h_2/w_2, & \text{if } 3|pq. \end{cases}$$

Since $w = 2$ and

$$w_2 = \begin{cases} 2 \text{ or } 4, & \text{if } 3 \nmid pq, \\ 2,\ 4 \text{ or } 6, & \text{if } 3|pq, \end{cases}$$

we conclude that e_1 must be of the form $e_1'/2$, where $e_1' \in \mathbb{N}$. Hence, we may rewrite (7.16) as

$$\eta = \prod_{\chi(G_1)=-1} \epsilon_1^{e_1'/2}. \tag{7.17}$$

Now, it is known that a fundamental unit of a real quadratic field takes the form $u + v\sqrt{d}$ with $u, v > 0$ (Borevich and Shafarevich [1, p. 133]). Furthermore, if $\sqrt{u + v\sqrt{d}} = u'\sqrt{d_1} + v'\sqrt{d_2}$, then $u', v' \geq 0$. Collecting these observations, we deduce that η is of the form $u_1 + u_2\sqrt{p} + u_3\sqrt{q} + u_4\sqrt{pq}$, where $u_i \geq 0$ for each i. Using (7.15) and (7.17), we conclude that a_1 and a_2 are positive integers.

Next, suppose L is not the genus field. Then from the beginning of our discussion, $Gal(L|\mathbb{Q}) \simeq D_8$ is nonabelian. We claim that there exists an element σ in $Gal(L|K)$ such that $\sigma(\eta)$ is complex. Suppose the contrary holds. Then

$L \cap \mathbb{R} = \mathbb{Q}(\eta)$ would be Galois over $\mathbb{Q}$, and hence $Gal(L|\mathbb{Q}(\eta))$ is a normal subgroup of $Gal(L|\mathbb{Q})$. On the other hand, $Gal(\mathbb{Q}(\eta)|\mathbb{Q}) \simeq Gal(L|K)$, a normal subgroup of $Gal(L|\mathbb{Q})$ (Cox [1, p. 191]). Hence, $Gal(L|\mathbb{Q})$ is isomorphic to the direct sum of $Gal(L|\mathbb{Q}(\eta))$ and $Gal(\mathbb{Q}(\eta)|\mathbb{Q})$ and is therefore an abelian group, and this contradicts our initial assumption.

Next, we will show that $\sigma(\sqrt{m}) = -\sqrt{m}$. Suppose that the contrary holds. Then

$$\sigma(\eta) + \sigma(\eta)^{-1} = \eta + \eta^{-1},$$

and therefore

$$\sigma(\eta) = \eta \quad \text{or} \quad \eta^{-1}.$$

This shows that $\sigma(\eta)$ is real, which contradicts our choice of σ. Now, applying σ to (7.15), we deduce that

$$2(\sigma(\eta) + \sigma(\eta)^{-1}) = a_1 - a_2\sqrt{m}. \tag{7.18}$$

From (7.15), (7.18), and the fact that $\sigma(\eta)$ is complex, we find that

$$(a_1 + a_2\sqrt{m})^2 \geq 16$$

and

$$(a_1 - a_2\sqrt{m})^2 < 16.$$

This implies that $4a_1a_2\sqrt{m} > 0$. Since $\eta > 0$, we deduce that a_1 and a_2 are positive. The integrality of a_1 and a_2 follows easily from Theorem 7.3. In a similar way, we can show that b_1 and b_2 are positive integers in (7.14).

The argument given here for the case when $Gal(L|\mathbb{Q})$ is nonabelian is due to H. Weber; see Cox's book [1, p. 269].

Let $\mathfrak{R}_K$ be the subset of $\{\mathbb{Q}(\sqrt{p}), \mathbb{Q}(\sqrt{q}), \mathbb{Q}(\sqrt{pq})\}$ satisfying the last statement in Theorem 7.3. Note that, since $|\mathfrak{R}_K|$ is finite and $2\alpha_{p,q}$ and $2\beta_{p,q}$ lie in a discrete subset of the ring $\mathbb{Z}(\sqrt{m})$ for some $\mathbb{Q}(\sqrt{m}) \in \mathfrak{R}_K$, we can therefore determine their exact values, based on the numerical values of G_{pq} and $G_{p/q}$, in a finite number of steps. This will in turn lead to exact values of G_{pq}.

Except for $K = \mathbb{Q}(\sqrt{-217})$ and $\mathbb{Q}(\sqrt{-553})$, in all of our calculations, $|\mathfrak{R}_K| = 1$.

We illustrate our computations with two examples. Before we proceed, we let $u := G_{pq}G_{p/q}$, $v := G_{pq}/G_{p/q}$, $U_i := (u^i + u^{-i})^2$, and $V_j := (v^j + v^{-j})^2$.

Example 1. Let $p = 5$ and $q = 13$. In this case, $\gamma = 4$. By Theorem 7.3, $\alpha_{5,13}$ and $\beta_{5,13}$ are real quadratic algebraic integers. Since the primes 2, 5, and 13 are not inert in $\mathbb{Q}(\sqrt{65})$, we deduce that they are in $\mathbb{Q}(\sqrt{65})$.

Now, evaluating u and v using the product representation of G_n (see (1.3)), we find that $\alpha_{5,13} = 81.311288\ldots$ and $\beta_{5,13} = 57.186772\ldots$. We know that these numbers are of the form $a + b\sqrt{65}$, and, by Theorem 7.4, we conclude that

$$\alpha_{5,13} = \frac{41 + 5\sqrt{65}}{2} \qquad \text{and} \qquad \beta_{5,13} = \frac{33 + 3\sqrt{65}}{2}.$$

Therefore,

$$U_2 = \frac{45+5\sqrt{65}}{2} \qquad \text{and} \qquad V_2 = \frac{37+3\sqrt{65}}{2},$$

which implies that

$$U_1 = \frac{\sqrt{65}+9}{2} \qquad \text{and} \qquad V_1 = \frac{\sqrt{65}+7}{2}.$$

This further implies that

$$u = \sqrt{\frac{\sqrt{65}+9}{8}} + \sqrt{\frac{\sqrt{65}+1}{8}} \qquad \text{and} \qquad v = \sqrt{\frac{\sqrt{65}-1}{8}} + \sqrt{\frac{\sqrt{65}+7}{8}}.$$

Hence,

$$G_{65} = \left(\sqrt{\frac{\sqrt{65}+9}{8}} + \sqrt{\frac{\sqrt{65}+1}{8}}\right)^{1/2} \left(\sqrt{\frac{\sqrt{65}+7}{8}} + \sqrt{\frac{\sqrt{65}-1}{8}}\right)^{1/2}$$

and

$$G_{13/5} = \left(\sqrt{\frac{\sqrt{65}+9}{8}} + \sqrt{\frac{\sqrt{65}+1}{8}}\right)^{1/2} \left(\sqrt{\frac{\sqrt{65}+7}{8}} - \sqrt{\frac{\sqrt{65}-1}{8}}\right)^{1/2}.$$

Example 2. Let $p = 3$ and $q = 23$. In this case, $\gamma = 12$. Using the numerical values of u and v and Theorems 7.3 and 7.4, we find that

$$U_6 = 281344 + 162432\sqrt{3} \qquad \text{and} \qquad V_6 = 2992 + 1728\sqrt{3}.$$

The first equality implies that

$$u^6 + u^{-6} = 8(47 + 27\sqrt{3}).$$

Since

$$(u^2+u^{-2})^3 - 3(u^2+u^{-2}) = u^6+u^{-6} = 8(47+27\sqrt{3}) = (4+3\sqrt{3})^3 - 3(4+3\sqrt{3}),$$

we conclude that

$$u^2 + u^{-2} = 4 + 3\sqrt{3} \qquad \text{and} \qquad U_1 = 6 + 3\sqrt{3}.$$

Collecting our results and simplifying, we deduce that

$$G_{69} = \left(\sqrt{748+432\sqrt{3}} + \sqrt{747+432\sqrt{3}}\right)^{1/12} \times \left(\sqrt{\frac{6+3\sqrt{3}}{4}} + \sqrt{\frac{2+3\sqrt{3}}{4}}\right)^{1/2}.$$

The following table summarizes our further calculations. The values for G_{697} and G_{793} are new.

$n=77$	$U_1=6+\sqrt{11}$	$V_2=23+8\sqrt{11}$
$$G_n=\left(\sqrt{\frac{6+\sqrt{11}}{4}}+\sqrt{\frac{2+\sqrt{11}}{4}}\right)^{1/2}\times\left(\sqrt{\frac{23+8\sqrt{11}}{4}}\sqrt{\frac{19+8\sqrt{11}}{4}}\right)^{1/4}$$		
$n=141$	$U_1=18+9\sqrt{3}$	$V_6=125680+72576\sqrt{3}$
$$G_n=\left(\sqrt{\frac{18+9\sqrt{3}}{4}}\sqrt{\frac{14+9\sqrt{3}}{4}}\right)^{1/2}\times\left(\sqrt{31420+18144\sqrt{3}}\sqrt{31419+18144\sqrt{3}}\right)^{1/12}$$		
$n=145$	$U_1=12+\sqrt{145}$	$V_1=\frac{17+\sqrt{145}}{2}$
$$G_n=\left(\sqrt{\frac{17+\sqrt{145}}{8}}\sqrt{\frac{9+\sqrt{145}}{8}}\right)^{1/2}\times\left(\sqrt{\frac{12+\sqrt{145}}{4}}\sqrt{\frac{8+\sqrt{145}}{4}}\right)^{1/2}$$		
$n=205$	$U_2=\frac{2025+315\sqrt{41}}{2}$	$V_1=\frac{25+3\sqrt{41}}{2}$
$$G_n=\left(\sqrt{\frac{2025+315\sqrt{41}}{8}}\sqrt{\frac{2017+315\sqrt{41}}{8}}\right)^{1/4}\times\left(\sqrt{\frac{25+3\sqrt{41}}{8}}\sqrt{\frac{17+3\sqrt{41}}{8}}\right)^{1/2}$$		

$n = 213$	$U_1 = 42 + 24\sqrt{3}$	$V_6 = 2169904 + 1252800\sqrt{3}$
$G_n = \left(\sqrt{\frac{21+12\sqrt{3}}{2}}\sqrt{\frac{19+12\sqrt{3}}{2}}\right)^{1/2} \times \left(\sqrt{542476+313200\sqrt{3}}\sqrt{542475+313200\sqrt{3}}\right)^{1/12}$		
$n = 217$	$U_1 = 22 + 8\sqrt{7}$	$V_1 = 16 + 5\sqrt{7}$
$G_n = \left(\sqrt{\frac{22+8\sqrt{7}}{4}}\sqrt{\frac{18+8\sqrt{7}}{4}}\right)^{1/2} \times \left(\sqrt{\frac{16+5\sqrt{7}}{4}}\sqrt{\frac{12+5\sqrt{7}}{4}}\right)^{1/2}$		
$n = 265$	$U_1 = \frac{89+5\sqrt{265}}{2}$	$V_1 = 16 + \sqrt{265}$
$G_n = \left(\sqrt{\frac{89+5\sqrt{265}}{8}}\sqrt{\frac{81+5\sqrt{265}}{8}}\right)^{1/2} \times \left(\sqrt{\frac{16+\sqrt{265}}{4}}\sqrt{\frac{12+\sqrt{265}}{4}}\right)^{1/2}$		
$n = 301$	$U_1 = 46 + 7\sqrt{43}$	$V_2 = 1199 + 184\sqrt{43}$
$G_n = \left(\sqrt{\frac{46+7\sqrt{43}}{4}}\sqrt{\frac{42+7\sqrt{43}}{4}}\right)^{1/2} \times \left(\sqrt{\frac{1199+84\sqrt{43}}{4}}\sqrt{\frac{1195+84\sqrt{43}}{4}}\right)^{1/4}$		

$n = 445$	$U_2 = 71325 + 7560\sqrt{89}$	$V_1 = \dfrac{85 + 9\sqrt{89}}{2}$

$$G_n = \left(\sqrt{\frac{71325 + 7560\sqrt{89}}{4}}\sqrt{\frac{71321 + 7560\sqrt{89}}{4}}\right)^{1/4} \times \left(\sqrt{\frac{85 + 9\sqrt{89}}{8}}\sqrt{\frac{77 + 9\sqrt{89}}{8}}\right)^{1/2}$$

$n = 505$	$U_1 = 292 + 13\sqrt{505}$	$V_1 = \dfrac{113 + 5\sqrt{505}}{2}$

$$G_n = \left(\sqrt{\frac{292 + 13\sqrt{505}}{4}}\sqrt{\frac{288 + 13\sqrt{505}}{4}}\right)^{1/2} \times \left(\sqrt{\frac{113 + 5\sqrt{505}}{8}}\sqrt{\frac{105 + 5\sqrt{505}}{8}}\right)^{1/2}$$

$n = 553$	$U_1 = 286 + 32\sqrt{79}$	$V_2 = 19163 + 2156\sqrt{79}$

$$G_n = \left(\sqrt{\frac{286 + 32\sqrt{79}}{4}}\sqrt{\frac{282 + 32\sqrt{79}}{4}}\right)^{1/2} \times \left(\sqrt{\frac{19163 + 2156\sqrt{79}}{4}}\sqrt{\frac{19159 + 2156\sqrt{79}}{4}}\right)^{1/4}$$

$n = 697$	$U_1 = \dfrac{769 + 29\sqrt{697}}{2}$	$V_1 = \dfrac{661 + 25\sqrt{697}}{2}$

$$G_n = \left(\sqrt{\frac{769 + 29\sqrt{697}}{8}}\sqrt{\frac{761 + 29\sqrt{697}}{8}}\right)^{1/2} \times \left(\sqrt{\frac{661 + 25\sqrt{697}}{8}}\sqrt{\frac{653 + 25\sqrt{697}}{8}}\right)^{1/2}$$

$n = 793$	$U_1 = 704 + 25\sqrt{793}$	$V_1 = 452 + 16\sqrt{793}$
$G_n = \left(\sqrt{\frac{704+25\sqrt{793}}{4}}\sqrt{\frac{700+25\sqrt{793}}{4}}\right)^{1/2} \times \left(\sqrt{\frac{452+16\sqrt{793}}{4}}\sqrt{\frac{448+16\sqrt{793}}{4}}\right)^{1/2}$		

8. Miscellaneous Results

In this section we collect together some miscellaneous results of Ramanujan on class invariants. We have been unable to provide meaningful interpretations for two of these entries.

Entry 8.1 (p. 311, NB 1). *We have*

$$G_{75} = \frac{3 \cdot 2^{5/12}}{\frac{\sqrt{5}+1}{2}(10)^{1/3} + \frac{\sqrt{5}-1}{2}4^{1/3} \cdot 5^{1/6} - \sqrt{5} - 1} \tag{8.1}$$

and

$$G_{3/25} = \frac{3 \cdot 2^{5/12}}{\frac{\sqrt{5}+1}{2}4^{1/3}5^{1/6} - \frac{\sqrt{5}-1}{2}(10)^{1/3} + \sqrt{5} - 1}. \tag{8.2}$$

Proof. We apply Lemma 4.4 with $q = \exp(-\pi\sqrt{n})$, and so by (1.6),

$$P = G_n^{-2}G_{25n}^{-2}, \qquad Q = G_n^3/G_{25n}^3,$$

and

$$\frac{G_n^3}{G_{25n}^3} + \frac{G_{25n}^3}{G_n^3} + 2\left(G_n^{-2}G_{25n}^{-2} - G_n^2G_{25n}^2\right) = 0. \tag{8.3}$$

Let $n = 3$. From our table in Section 2, $G_3 = 2^{1/12}$. Thus, by (8.3), with $x = G_{75}$,

$$\frac{2^{1/4}}{x^3} + \frac{x^3}{2^{1/4}} + 2(2^{-1/6}x^{-2} - 2^{1/6}x^2) = 0. \tag{8.4}$$

This sextic polynomial has two real roots, and, via *Mathematica*, we easily checked that the expression on the right side of (8.1) satisfies (8.4) and is the correct real root.

Alternatively, from Weber's book [2, p. 724], G_{75} is a root of a certain cubic polynomial. In fact, (8.4) factors over $\mathbb{Q}(\sqrt{5})$ into a product of two cubic polynomials, one of which is Weber's cubic polynomial, and so we have given another derivation of Weber's cubic polynomial for G_{75}.

To prove (8.2), we again use (8.3), but now we set $n = \frac{3}{25}$. Thus, $G_{25n} = G_3 = 2^{1/12}$. Hence, with $y = G_{3/25}$,

$$\frac{y^3}{2^{1/4}} + \frac{2^{1/4}}{y^3} + 2(2^{-1/6}y^{-2} - 2^{1/6}y^2) = 0, \tag{8.5}$$

which is exactly the same as (8.4), but with x replaced by y. We used *Mathematica* to verify that the right side of (8.2) is a root of (8.5) and indeed is the correct one. Of course, $G_{3/25}$ is the "other" real root of (8.4) that we mentioned above.

Entry 8.2 (p. 316, NB 1). *We have*

$$G_{175} = \frac{3 \cdot 2^{1/4}}{\frac{\sqrt{5}-1}{2} + \left(\frac{5-\sqrt{5}}{4}\right)^{1/3}\left(\sqrt[3]{8-3\sqrt{5}+3\sqrt{21}} + \sqrt[3]{8-3\sqrt{5}-3\sqrt{21}}\right)} \tag{8.6}$$

and

$$G_{25/7} = \frac{3 \cdot 2^{1/4}}{\frac{\sqrt{5}+1}{2} + \left(\frac{5+\sqrt{5}}{4}\right)^{1/3}\left(\sqrt[3]{8+3\sqrt{5}+3\sqrt{21}} + \sqrt[3]{8+3\sqrt{5}-3\sqrt{21}}\right)}. \tag{8.7}$$

Proof. We employ (8.3) with $n = 7$. From our tables in Section 2, $G_7 = 2^{1/4}$. Thus, with $G = G_{175}$, we find that

$$\frac{2^{3/4}}{G^3} + \frac{G^3}{2^{3/4}} + 2(2^{-1/2}G^{-2} - 2^{1/2}G^2) = 0. \tag{8.8}$$

Now, by straightforward algebra, it is easily checked that (8.8) yields

$$2G^3 - 4 \cdot 2^{1/4}G^2 + \sqrt{2}G - 3 \cdot 2^{3/4} = \pm\sqrt{5}(2 \cdot 2^{1/4}G^2 - \sqrt{2}G + 2^{3/4}). \tag{8.9}$$

We shall show that the right side of (8.6) is a solution of the cubic equation in (8.9) where the plus sign is chosen on the right side. Thus, with the plus sign chosen in (8.9), we find that, after simplification,

$$2G^3 - 2^{1/4}(4+2\sqrt{5})G^2 + (\sqrt{2}+\sqrt{10})G - 2^{3/4}(3+\sqrt{5}) = 0. \tag{8.10}$$

The form of Ramanujan's formula (8.6) suggests that he set $G = 1/x$ and solved for x. Thus, by (8.10),

$$2^{3/4}(3+\sqrt{5})x^3 - (\sqrt{2}+\sqrt{10})x^2 + 2^{1/4}(4+2\sqrt{5})x - 2 = 0. \tag{8.11}$$

We solve (8.11) by employing Cardan's method (Hall and Knight [1, p. 480]). Thus, set

$$x = y + \frac{\sqrt{2}+\sqrt{10}}{3\cdot 2^{3/4}(3+\sqrt{5})}$$

in (8.11), and, after dividing out the common factor $47+21\sqrt{5}$, we find that

$$27\cdot 2^{3/4}y^3 + 18\cdot 2^{1/4}\sqrt{5}y + \tfrac{1}{2}(-55+23\sqrt{5}) = 0. \tag{8.12}$$

The solution of the general cubic equation $y^3+qy+r=0$ requires the calculation of $\sqrt{r^4/4+q^3/27}$. With

$$r = \frac{23\sqrt{5}-55}{27\cdot 2^{7/4}} \qquad \text{and} \qquad q = \frac{\sqrt{10}}{3},$$

we find, after much simplification, that

$$\sqrt{\frac{r^2}{4}+\frac{q^3}{27}} = \frac{\sqrt{35}(\sqrt{5}-1)}{3\sqrt{3}2^{11/4}}.$$

Thus, from Hall and Knight's text [1, p. 480], the real root of (8.12) that we seek is

$$\begin{aligned}
y &= \left\{\frac{55-23\sqrt{5}}{27\cdot 2^{11/4}} + \frac{\sqrt{35}(\sqrt{5}-1)}{3\sqrt{3}2^{11/4}}\right\}^{1/3} + \left\{\frac{55-23\sqrt{5}}{27\cdot 2^{11/4}} - \frac{\sqrt{35}(\sqrt{5}-1)}{3\sqrt{3}2^{11/4}}\right\}^{1/3} \\
&= \frac{1}{3\cdot 2^{11/12}}\left(\left\{55-23\sqrt{5}+15\sqrt{21}-3\sqrt{105}\right\}^{1/3}\right. \\
&\qquad \left. + \left\{55-23\sqrt{5}-15\sqrt{21}+3\sqrt{105}\right\}^{1/3}\right) \\
&= \frac{(5-\sqrt{5})^{1/3}}{3\cdot 2^{11/12}}\left(\left\{8-3\sqrt{5}+3\sqrt{21}\right\}^{1/3} + \left\{8-3\sqrt{5}-3\sqrt{21}\right\}^{1/3}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{G} = x = \frac{1}{3\cdot 2^{1/4}} & \left(\frac{1+\sqrt{5}}{3+\sqrt{5}} + \left(\frac{5-\sqrt{5}}{4}\right)^{1/3}\right. \\
& \times \left.\left(\left\{8-3\sqrt{5}+3\sqrt{21}\right\}^{1/3} + \left\{8-3\sqrt{5}-3\sqrt{21}\right\}^{1/3}\right)\right).
\end{aligned}$$

This is easily seen to be equivalent to (8.6).

To prove (8.7), we set $n = \frac{1}{7}$ in (8.3). Since $G_n = G_{1/n}$, it follows that $G_{1/7} = G_7 = 2^{1/4}$. Thus, with $G = G_{25/7}$, we deduce (8.8) once again. Hence, $G_{25/7}$ is the other real root of (8.8). Therefore, taking the minus sign on the right side of (8.9), we find that G satisfies the equation

$$2G^3 - 2^{1/4}(4-2\sqrt{5})G^2 + (\sqrt{2}-\sqrt{10})G - 2^{3/4}(3-\sqrt{5}) = 0.$$

Now repeat the calculations from the proof of (8.6), but with $\sqrt{5}$ replaced by $-\sqrt{5}$. We then deduce (8.7) to complete the proof.

We also calculated G_{175} from (8.10) by using Cardan's method and found that

$$G_{175} = \frac{2^{1/4}}{3} \times \left(2+\sqrt{5}+\left(\frac{5+2\sqrt{5}}{2}\right)^{1/3}\left(\left\{17+3\sqrt{21}\right\}^{1/3}+\left\{17-3\sqrt{21}\right\}^{1/3}\right)\right), \tag{8.13}$$

which is a slightly more elegant representation than (8.6). By combining (8.6) and (8.13), we deduce that

$$\begin{aligned}&\left(\frac{\sqrt{5}-1}{2}+\left(\frac{5-\sqrt{5}}{4}\right)^{1/3}\left(\left\{8-3\sqrt{5}+3\sqrt{21}\right\}^{1/3}\right.\right.\\ &\qquad\left.\left.+\left\{8-3\sqrt{5}-3\sqrt{21}\right\}^{1/3}\right)\right)\\ &\times\left(2+\sqrt{5}+\left(\frac{5+2\sqrt{5}}{2}\right)^{1/3}\right.\\ &\qquad\left.\times\left(\left\{17+3\sqrt{21}\right\}^{1/3}+\left\{17-3\sqrt{21}\right\}^{1/3}\right)\right)=9.\end{aligned} \tag{8.14}$$

We are unable to establish (8.14) directly.

At scattered places in the second notebook, Ramanujan discusses a few additional class invariants.

Entry 8.3 (p. 263, NB 2). *Let $t = 1/G_{29}^4$ and let x denote the positive real root of*

$$x^6 + 9x^5 + 5x^4 - 2x^3 - 5x^2 + 9x - 1 = 0. \tag{8.15}$$

Then $x = t^4$, where $t > 0$. Furthermore,

$$\frac{t^6+t^2}{1-t^4} = \sqrt{\frac{\sqrt{29}-5}{2}} \tag{8.16}$$

and

$$\frac{t^3+t\sqrt{\sqrt{29}-2}}{1+t^2\sqrt{\sqrt{29}+2}} = \sqrt[4]{\frac{\sqrt{29}-5}{2}}. \tag{8.17}$$

Lastly, if

$$\sqrt[4]{1-t^8} = t(1+\mu^2), \tag{8.18}$$

where $\mu > 0$, *then*

$$\mu^3 + \mu = \sqrt{2}. \tag{8.19}$$

Proof. It is not difficult to show that the first claim is equivalent to a result in Weber's book [2, p. 722].

We now prove (8.16). By straightforward algebra, it is readily verified that (8.15) is equivalent to the equation

$$\frac{1}{x}\left(\frac{1-x}{1+x}\right)^2 - x\left(\frac{1+x}{1-x}\right)^2 = 5. \tag{8.20}$$

Let

$$y = \frac{1}{x}\left(\frac{1-x}{1+x}\right)^2.$$

Then, from (8.20),

$$y^2 - 5y - 1 = 0,$$

which has the roots $\frac{1}{2}\left(5 \pm \sqrt{29}\right)$. Since $x = 0.119252\ldots$, the plus sign must be taken. Hence,

$$\frac{1}{y} = x\left(\frac{1+x}{1-x}\right)^2 = \frac{\sqrt{29}-5}{2}. \tag{8.21}$$

Taking the square root of each side and recalling that $x = t^4$, we complete the proof of (8.16).

We next prove (8.17). Employing (8.21) and (8.16) and remembering that $x = t^4$, we see that we are required to prove that

$$\left(\frac{\sqrt{x} + \sqrt{\sqrt{29}-2}}{1 + \sqrt{x}\sqrt{\sqrt{29}+2}}\right)^2 = \frac{1+x}{1-x}. \tag{8.22}$$

Again, from (8.21),

$$\begin{aligned}\left(\frac{1}{\sqrt{x}}\frac{1-x}{1+x} \pm \sqrt{x}\frac{1+x}{1-x}\right)^2 &= \left\{\sqrt{\frac{\sqrt{29}+5}{2}} \pm \sqrt{\frac{\sqrt{29}-5}{2}}\right\}^2 \\ &= \sqrt{29} \pm 2,\end{aligned}$$

i.e.,

$$\frac{1}{\sqrt{x}}\frac{1-x}{1+x} \pm \sqrt{x}\frac{1+x}{1-x} = \sqrt{\sqrt{29} \pm 2}. \tag{8.23}$$

Hence, from (8.23),

$$\sqrt{x} + \sqrt{\sqrt{29}-2} = \frac{-2x^3 - x^2 - 2x + 1}{\sqrt{x}(1-x^2)}.$$

and

$$1+\sqrt{x}\sqrt{\sqrt{29}+2}=\frac{x^3+2x^2-x+2}{1-x^2}.$$

Thus,

$$\left(\frac{\sqrt{x}+\sqrt{\sqrt{29}-2}}{1+\sqrt{x}\sqrt{\sqrt{29}+2}}\right)^2=\frac{(-2x^3-x^2-2x+1)^2}{x(x^3+2x^2-x+2)^2}.$$

By (8.22), we want to show that the right side above is equal to $(1+x)/(1-x)$. Thus, it suffices to prove that

$$(1-x)(-2x^3-x^2-2x+1)^2=x(1+x)(x^3+2x^2-x+2)^2.$$

Expanding both sides and collecting terms, we find that the foregoing equation is equivalent to the equation

$$\begin{aligned}0&=x^8+9x^7+6x^6+7x^5+7x^3-6x^2+9x-1\\&=(x^2+1)(x^6+9x^5+5x^4-2x^3-5x^2+9x-1).\end{aligned}$$

Since $x^2+1\neq 0$, it suffices to prove that the second factor above equals 0. But this is true by (8.15), and so the proof of (8.17) is complete.

Next, we prove (8.19). From (8.18),

$$\sqrt[4]{\frac{1}{x}-x}=1+\mu^2. \tag{8.24}$$

This equality and the symmetry in (8.15) suggest that we set $p=x^{-1}-x$. A brief calculation shows that

$$x^6+9x^5+5x^4-2x^3-5x^2+9x-1=-x^3\left(p^3-9p^2+8p-16\right).$$

Since $x\neq 0$, by (8.15),

$$p^3-9p^2+8p-16=0. \tag{8.25}$$

From (8.24) and the definition of p,

$$\mu=\sqrt{\sqrt[4]{\frac{1}{x}-x}-1}=\sqrt{\sqrt[4]{p}-1}.$$

Thus,

$$\mu^3+\mu=\sqrt[4]{p}\sqrt{\sqrt[4]{p}-1}.$$

Since clearly $\mu^3+\mu>0$, it therefore suffices to prove that

$$\sqrt{p}\left(\sqrt[4]{p}-1\right)=2,$$

and since $p>0$, it is sufficient to prove that

$$\sqrt{p}-\frac{4}{\sqrt{p}}=\frac{4}{p}+1.$$

Squaring both sides and simplifying, we find that it suffices to prove that

$$p^3 - 9p^2 + 8p - 16 = 0.$$

But this is precisely (8.25), and so the proof is complete.

Parts of the proof of Entry 8.3 were taken from the notes of V. R. Thiruvenkatachar and K. Venkatachaliengar [1].

Entry 8.4 (pp. 263, 300, NB 2). *Let $t = 2^{1/4}/G_{79}$. Then t is the real root of*

$$t^5 - t^4 + t^3 - 2t^2 + 3t - 1 = 0. \tag{8.26}$$

Furthermore, if

$$\sqrt{\frac{1}{t} - t} = \mu, \tag{8.27}$$

then

$$\mu^5 - 2\mu^4 + \mu^3 + 2\mu - 3 = 0. \tag{8.28}$$

Proof. The class equation for $n = 79$ was not computed by Weber [2] and is not otherwise given by Ramanujan in his paper [3] or notebooks [9]. However, Russell [2] and Watson [10] determined the class equation, which is easily shown to be equivalent to that of Ramanujan.

Now (8.28) is valid if and only if

$$\left(\mu^5 + \mu^3 + 2\mu\right)^2 = \left(3 + 2\mu^4\right)^2, \tag{8.29}$$

since the square root of each side is positive. Also, (8.29) holds if and only if, by (8.27),

$$\begin{aligned}
0 &= \mu^{10} - 2\mu^8 + 5\mu^6 - 8\mu^4 + 4\mu^2 - 9 \\
&= \left(\frac{1}{t} - t\right)^5 - 2\left(\frac{1}{t} - t\right)^4 + 5\left(\frac{1}{t} - t\right)^3 - 8\left(\frac{1}{t} - t\right)^2 + 4\left(\frac{1}{t} - t\right) - 9 \\
&= -t^5 - 2t^4 + t - 5 - t^{-1} - 2t^{-4} + t^{-5} \\
&= -\left(t^{10} + 2t^9 - t^6 + 5t^5 + t^4 + 2t - 1\right)t^{-5} \\
&= -\left(t^5 - t^4 + t^3 - 2t^2 + 3t - 1\right)\left(t^5 + 3t^4 + 2t^3 + t^2 + t + 1\right)t^{-5}.
\end{aligned}$$

Since $t > 0$, $t^5 + 3t^4 + 2t^3 + t^2 + t + 1 > 0$. Hence, (8.28) holds if and only if t satisfies (8.26), and this is what we wanted to prove.

Entry 8.5 (p. 382, NB 3). *Let $z = x + 1/x$, where $x = G_{41}^2$. Then*

$$z^2 - z\frac{5 + \sqrt{41}}{2} + \frac{7 + \sqrt{41}}{2} = 0. \tag{8.30}$$

Proof. From Weber's book [2, p. 722], as corrected by Brillhart and Morton [1], if $\mathfrak{f}^2(\sqrt{-41}) = \sqrt{2}u$, then

$$\left(u+\frac{1}{u}\right)^2 - \frac{5+\sqrt{41}}{2}\left(u+\frac{1}{u}\right) + \frac{7+\sqrt{41}}{2} = 0. \tag{8.31}$$

Since $u = G_{41}^2$, (8.30) and (8.31) are equivalent.

Ramanujan's formulation of Entry 8.5 is, in fact, slightly enigmatic. In particular, in contrast to the notation used throughout the second notebook [9], Ramanujan employed the conventional notation in the theory of elliptic functions and wrote $1/x = \sqrt[6]{2kk'}$.

We now make a few remarks about two entries possibly related to invariants.

On page 294 of the first notebook, Ramanujan claims that

$$\text{“}F\left(\frac{1-\sqrt{1-x^{24}}}{2}\right) = e^{-11\sqrt{\pi}}$$

where

$$x+\frac{1}{x} = \frac{(9\sqrt{3}+1)^{1/3}+(9\sqrt{3}-1)^{1/3}}{\sqrt{3}}\left(\frac{11}{2}\right)^{1/6}$$

$$x^{3/8}\left\{\sqrt{\frac{\sqrt{x}+1/\sqrt{x}+1}{2}} - \sqrt{\frac{\sqrt{x}+1/\sqrt{x}-1}{2}}\right\}$$

$$\times\left\{\sqrt{1+A}\pm\sqrt{A}\right\} \qquad \text{where} \qquad A = \frac{3}{2}\frac{1}{\sqrt{\dfrac{x+1/x+1}{x+1/x-2}}-1}.\text{”}$$

As intimated in Section 2 prior to the tables, it is not difficult to show that the indicated formula for G_{121} is equivalent to the one given in the tables. Parts of the last two lines of the entry above are difficult to decipher, especially the definition of A. Moreover, we are uncertain that these two lines pertain to the first two lines, that is to say, that the value of x in the first two lines is the same as in the second two lines. Even more enigmatic is that an equality sign seems to be missing in the last two lines. With the two values of x arising from the second line, we calculated the expressions in the last two lines and could not account for a missing equality sign. In conclusion, we are unable to supply any meaningful interpretation to the incomplete entry offered in the last two lines.

On page 343 in the first notebook, Ramanujan wrote

$$\text{“}y\left(\frac{\sqrt{13}-3}{2}\right)^{1/4} \quad \text{where } \sqrt{5}\left(y^3+y^2\frac{\sqrt{13}-1}{2}+y\frac{\sqrt{13}+1}{2}-1\right)$$

$$\pm\left(y^3+y^2\left(\frac{\sqrt{13}+1}{2}\right)^2+y\left(\frac{\sqrt{13}-1}{2}\right)^2+1\right) = 0.\text{”}$$

We have no explanation for this mysterious fragment. The definition of y is not given, but perhaps Ramanujan intended to write $y = \sqrt[4]{(\sqrt{13}-3)/2}$. However, this value of y is not a solution of either of the indicated polynomial equations. The top half of the page comprises results discussed in Chapter 37, and underneath the fragment is Ramanujan's (equivalent) representation for G_{765}; neither topic appears to be connected with this fragment.

9. Singular Moduli

Recall from the Introduction that the singular modulus k_n is defined by $k_n := k(e^{-\pi\sqrt{n}})$, where n is a positive integer. It is clear from (1.6) that if the value of G_n (or g_n) can be determined, then $\alpha_n := k_n^2$ can be computed by solving a quadratic equation. For example, see (2.8) or (9.1) below. However, the expression that one obtains generally is unattractive and does not evince the fact that α_n can be expressed in terms of units in certain algebraic number fields. (See Theorem 1.1.) Thus, formulas for α_n that facilitate their representations via units are desirable.

In his second letter to Hardy, Ramanujan [10, p. xxix] asserted that

$$\begin{aligned} k_{210} &= (\sqrt{2}-1)^4(2-\sqrt{3})^2(\sqrt{7}-\sqrt{6})^4(8-3\sqrt{7})^2 \\ &\quad \times (\sqrt{10}-3)^4(4-\sqrt{15})^4(\sqrt{15}-\sqrt{14})^2(6-\sqrt{35})^2. \end{aligned}$$

This was first proved by Watson [4], who used the following remarkable formula which he found in Ramanujan's first notebook [9, vol. 1, p. 320] and which enables one to calculate α_n for even n.

Theorem 9.1. *Set*

$$g_n^6 = uv,$$
$$u^2 + 1/u^2 = 2U, \qquad v^2 + 1/v^2 = 2V,$$
$$W = \sqrt{U^2 + V^2 - 1},$$

and

$$2S = U + V + W + 1.$$

Then

$$\begin{aligned} \alpha_n &= \{\sqrt{S} - \sqrt{S-1}\}^2\{\sqrt{S-U} - \sqrt{S-U-1}\}^2 \\ &\quad \times \{\sqrt{S-V} - \sqrt{S-V-1}\}^2\{\sqrt{S-W} - \sqrt{S-W-1}\}^2. \end{aligned}$$

Watson's proof of Theorem 9.1 is a *verification*; it does not shed any light on how Ramanujan might have discovered the formula. K. G. Ramanathan [1], [5] stated Ramanujan's Theorem 9.1 but did not find another proof. Heng Huat Chan has found a much more motivated proof of Theorem 9.1, and we present his proof below. Later we show that the algorithm implicit in Theorem 9.1 can be adopted to determine α_n for odd n as well.

Proof. From (1.6) and the notation above,

$$\begin{aligned}\frac{1}{\sqrt{\alpha}} - \sqrt{\alpha} = 2u^2v^2 &= 2(U + \sqrt{U^2 - 1})(V + \sqrt{V^2 - 1}) \\ &= 2(U\sqrt{V^2 - 1} + V\sqrt{U^2 - 1} + UV + \sqrt{(U^2 - 1)(V^2 - 1)}) \\ &= 2\left(\sqrt{(a+1)(b-1)} + \sqrt{ab}\right),\end{aligned}$$

where we set

$$\sqrt{ab} = UV + \sqrt{(U^2 - 1)(V^2 - 1)}$$

and

$$\sqrt{(a+1)(b-1)} = U\sqrt{V^2 - 1} + V\sqrt{U^2 - 1}.$$

Squaring each of the last two equalities, we find that, respectively,

$$ab = 2U^2V^2 - U^2 - V^2 + 1 + 2UV\sqrt{(U^2 - 1)(V^2 - 1)}$$

and

$$ab + b - a - 1 = 2U^2V^2 - U^2 - V^2 + 2UV\sqrt{(U^2 - 1)(V^2 - 1)}.$$

These two equalities imply that $a = b$. Thus,

$$\frac{1}{\sqrt{\alpha}} - \sqrt{\alpha} = 2\left(\sqrt{(a+1)(a-1)} + a\right).$$

Solving for $\sqrt{\alpha}$, we find that

$$\sqrt{\alpha} = (\sqrt{a+1} - \sqrt{a})(\sqrt{a} - \sqrt{a-1}). \tag{9.1}$$

It thus suffices to compute $\sqrt{a}$.

From the definition of a and the fact that $a = b$,

$$\begin{aligned}a &= UV + \sqrt{(U^2 - 1)(V^2 - 1)} \\ &= \tfrac{1}{2}\left(2UV + 2\sqrt{U^2V^2 - U^2 - V^2 + 1}\right) \\ &= \tfrac{1}{2}\left(2UV + 2\sqrt{U^2V^2 - W^2}\right) \\ &= \tfrac{1}{2}\left(\sqrt{UV - W} + \sqrt{UV + W}\right)^2.\end{aligned}$$

We write the last equality in two ways as follows:

$$a = \begin{cases} \dfrac{1}{2}\left(\sqrt{\dfrac{(U+V)^2 - U^2 - V^2 - 2W}{2}} \right. \\ \qquad \left. + \sqrt{\dfrac{(U+V)^2 - U^2 - V^2 + 2W}{2}}\right)^2 \\ \dfrac{1}{2}\left(\sqrt{\dfrac{-(U-V)^2 + U^2 + V^2 - 2W}{2}} \right. \\ \qquad \left. + \sqrt{\dfrac{-(U-V)^2 + U^2 + V^2 + 2W}{2}}\right)^2. \end{cases} \tag{9.2}$$

Now,

$$\begin{aligned}(S-W)(S-1) &= S(S-W-1) + W \\ &= \frac{(U+V+W+1)}{2}\frac{(U+V-W-1)}{2} + W \\ &= \frac{(U+V)^2 - (W+1)^2}{4} + W \\ &= \frac{(U+V)^2 - U^2 - V^2 - 2W}{4} + W \\ &= \frac{(U+V)^2 - U^2 - V^2 + 2W}{4}.\end{aligned}$$

Thus, from the first equality in (9.2),

$$\begin{aligned} a &= \left(\sqrt{S(S-W-1)} + \sqrt{(S-W)(S-1)}\right)^2 \\ &= S(S-W-1) + (S-W)(S-1) + 2\sqrt{S(S-W-1)(S-W)(S-1)} \\ &= S(S-W) + (S-W-1)(S-1) - S + (S-1) \\ &\quad + 2\sqrt{S(S-W-1)(S-W)(S-1)}, \end{aligned}$$

i.e.,

$$\begin{aligned} a+1 &= S(S-W) + (S-W-1)(S-1) + 2\sqrt{S(S-W)(S-W-1)(S-1)} \\ &= \left(\sqrt{S(S-W)} + \sqrt{(S-W-1)(S-1)}\right)^2. \end{aligned}$$

Hence,

$$\begin{aligned} \sqrt{a+1} - \sqrt{a} &= \sqrt{S(S-W)} + \sqrt{(S-W-1)(S-1)} \\ &\quad - \sqrt{S(S-W-1)} - \sqrt{(S-W)(S-1)} \\ &= \left(\sqrt{S} - \sqrt{S-1}\right)\left(\sqrt{S-W} - \sqrt{S-W-1}\right). \end{aligned}$$

Next,

$$\begin{aligned}(S-U-1)(S-V-1) &= (S-U)(S-V)-W\\ &= \frac{W+1+V-U}{2}\,\frac{W+1+U-V}{2}-W\\ &= \frac{(W+1)^2-(U-V)^2}{4}-W\\ &= \frac{U^2+V^2-2W-(U-V)^2}{4}.\end{aligned}$$

Thus, from the second equality of (9.2),

$$\begin{aligned}a &= \left(\sqrt{(S-U-1)(S-V-1)}+\sqrt{(S-U)(S-V)}\right)^2\\ &= (S-U-1)(S-V-1)+(S-U)(S-V)\\ &\quad +2\sqrt{(S-U-1)(S-V-1)(S-U)(S-V)}\\ &= (S-U-1)(S-V)+(S-U)(S-V-1)+1\\ &\quad +2\sqrt{(S-U-1)(S-V)(S-U)(S-V-1)},\end{aligned}$$

i.e.,

$$a-1=\left(\sqrt{(S-U-1)(S-V)}+\sqrt{(S-U)(S-V-1)}\right)^2.$$

Hence,

$$\begin{aligned}\sqrt{a}-\sqrt{a-1} &= \sqrt{(S-U-1)(S-V-1)}+\sqrt{(S-U)(S-V)}\\ &\quad -\sqrt{(S-U-1)(S-V)}-\sqrt{(S-U)(S-V-1)}\\ &= \left(\sqrt{S-V}-\sqrt{S-V-1}\right)\left(\sqrt{S-U}-\sqrt{S-U-1}\right).\end{aligned}$$

Using our just calculated formulas for $\sqrt{a+1}-\sqrt{a}$ and $\sqrt{a}-\sqrt{a-1}$ in (9.1), we complete the proof.

Furthermore, Watson [4] inexplicably claimed, "... this is the sole instance in which Ramanujan has calculated the value of k for an even integer n." In fact, 20 additional values of k_n for even n are found in the first notebook. Theorem 9.2 gives 13 of these values.

On page 82 of his first notebook, Ramanujan offers three additional theorems for calculating α_n when n is even. The first (Theorem 9.3) expresses α_{4p} as a product of units involving G_p. The second (Theorem 9.5) expresses α_{16p} as a product of units involving G_p. The third (Theorem 9.6) enables one to determine α_{8p} as a product of two fourth powers of units, provided that α_{2p} can be expressed as a product of units of a certain form. We calculate eight examples of Ramanujan as illustrations.

The calculation of α_n when n is odd is slightly more difficult. On page 80 in his first notebook, Ramanujan recorded the values of α_{21}, α_{33}, and α_{45} in terms of units. This list is repeated, with the addition of α_{15}, at the bottom of page 262 in his second notebook. On pages 345 and 346 in his first notebook, Ramanujan recorded units that appear in representations of α_n when $n = 3, 5, 7, 9, 13, 15, 17, 25$, and 55. (Inexplicably, the units for α_7 and α_{15} are recorded twice.) Ramanujan also indicated that he had intended to calculate α_{39}, but no factors are given. Of course, the result for $n = 15$ is superseded by the complete formula given on page 262 in the second notebook. It is unclear to us why Ramanujan only listed portions of α_n and not complete formulas. Initially in our investigations we employed computational "trial and error" to "guess" the complete formulas for α_n, $n = 5, 9, 13, 17, 25$, and 55. We remark that the values for α_3 and α_7 are easily determined from (2.8), i.e.,

$$\alpha_n = \tfrac{1}{2}G_n^{-12}\left(G_n^{12} - \sqrt{G_n^{24} - 1}\right). \tag{9.3}$$

For further values of n, however, (9.3) becomes unwieldy, and so better algorithms were sought.

We adopt the algorithm of Theorem 9.1 and reformulate it in Theorem 9.8 in terms of G_n to calculate some values of α_n when n is odd. Theorem 9.9 provides a list of all of Ramanujan's values for odd n. Although Theorem 9.8 yields a systematic procedure for calculating α_n when n is odd, the calculations are often cumbersome and the representations that we obtain, although expressed in terms of units, are frequently more complicated than we would like. Thus, we establish three simple lemmas, Lemmas 9.10–9.12, that provide an alternative procedure for calculating all of Ramanujan's singular moduli for odd n.

We conclude Section 9 with two further algorithms of Ramanujan for computing α_n. These were cryptically stated by Ramanujan in his first notebook and are rather different from his other algorithms. The first provides a method for determining α_{2n} from a certain type of modular equation of degree n. The second also arises from modular equations and gives a formula for α_{3n}.

Ramanujan likely learned about singular moduli from a brief discussion in A. G. Greenhill's book [3, p. 331]. It would be extremely difficult to assess the priority of each singular modulus that has been determined. Ramanathan [1] and J. M. and P. B. Borwein [1] previously calculated some of Ramanujan's values for α_n, and we shall cite their specific determinations in the sequel.

Singular Moduli for Even n

We begin with a list of 13 values for α_n found on scattered pages in the first notebook.

Theorem 9.2 (pp. 214, 288, 289, 310, 312, 313, NB 1). *We have*

$$\alpha_2 = (\sqrt{2} - 1)^2,$$

$$\alpha_6 = (2-\sqrt{3})^2(\sqrt{3}-\sqrt{2})^2 = \frac{\sqrt{6}-\sqrt{2}-1}{\sqrt{6}+\sqrt{2}+1} = \left(\frac{\sqrt{6}-\sqrt{2}-1}{\sqrt{2}-1}\right)^2,$$

$$\alpha_{10} = (\sqrt{10}-3)^2(3-2\sqrt{2})^2 = \frac{3\sqrt{2}-\sqrt{5}-2}{3\sqrt{2}+\sqrt{5}+2},$$

$$\alpha_{18} = (5\sqrt{2}-7)^2(7-4\sqrt{3})^2 = (2-\sqrt{3})^4(\sqrt{2}-1)^6 = \frac{7\sqrt{2}-2\sqrt{6}-5}{7\sqrt{2}+2\sqrt{6}+5},$$

$$\alpha_{22} = (10-3\sqrt{11})^2(3\sqrt{11}-7\sqrt{2})^2,$$

$$\alpha_{30} = (5-2\sqrt{6})^2(4-\sqrt{15})^2(\sqrt{6}-\sqrt{5})^2(2-\sqrt{3})^2,$$

$$\alpha_{42} = (8-3\sqrt{7})^2(7-4\sqrt{3})^2(3-2\sqrt{2})^2(\sqrt{7}-\sqrt{6})^2,$$

$$\alpha_{58} = (13\sqrt{58}-99)^2(99-70\sqrt{2})^2,$$

$$\alpha_{70} = (15-4\sqrt{14})^2(8-3\sqrt{7})^2(3\sqrt{14}-5\sqrt{5})^2(6-\sqrt{35})^2,$$

$$\alpha_{78} = (2-\sqrt{3})^6(3\sqrt{3}-\sqrt{26})^2(\sqrt{13}-2\sqrt{3})^4(5-2\sqrt{6})^2,$$

$$\alpha_{102} = \left(\frac{\sqrt{51}-7}{\sqrt{2}}\right)^4(5-2\sqrt{6})^4(\sqrt{51}-5\sqrt{2})^2(2-\sqrt{3})^4,$$

$$\alpha_{130} = (5\sqrt{130}-57)^2(\sqrt{10}-3)^4(\sqrt{26}-5)^4(3-2\sqrt{2})^4,$$

and

$$\alpha_{190} = \left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^4(37\sqrt{19}-51\sqrt{10})^2(2\sqrt{5}-\sqrt{19})^4(\sqrt{19}-3\sqrt{2})^4.$$

Proof. The value of α_2 was, in fact, established in Example 1, Section 2 of Chapter 17 in the second notebook (Part III [3, p. 97]). Ramanathan [1] and the Borweins [1, p. 139] also determined α_2.

All of the remaining values for α_n are easily determined from Theorem 9.1. The required values for g_n can be obtained from the tables of Weber [2] or the table in Section 2. In each instance, we list the values for u, v, U, V, W, and S in the table below. The reader can easily verify the calculations.

n	u	v	U	V	W	S
6	1	$1+\sqrt{2}$	1	3	3	4
10	1	$2+\sqrt{5}$	1	9	9	10
18	1	$5+2\sqrt{6}$	1	49	49	50
22	1	$7+5\sqrt{2}$	1	99	99	100
30	$2+\sqrt{5}$	$3+\sqrt{10}$	9	19	21	25
42	$2\sqrt{2}+\sqrt{7}$	$3\sqrt{3}+2\sqrt{7}$	15	55	57	64
58	1	$70+13\sqrt{29}$	1	9801	9801	9802
70	$9+4\sqrt{5}$	$7+5\sqrt{2}$	161	99	189	225
78	$18+5\sqrt{13}$	$5+\sqrt{26}$	649	51	651	676
102	$7+5\sqrt{2}$	$35+6\sqrt{34}$	99	2449	2451	2500
130	$38+17\sqrt{5}$	$18+5\sqrt{13}$	2889	649	2961	3250
190	$38+17\sqrt{5}$	$117+37\sqrt{10}$	2889	27,379	27,531	28,900

The second and third formulas for α_6 and α_{18}, and the second formula for α_{10} can be easily verified by direct calculations.

The Borweins [1, p. 139] calculated α_n for $1 \le n \le 9$. Ramanathan [1] also established α_{130} by using Theorem 9.1.

Recall the definition of $F(x)$ given in (2.3).

Theorem 9.3 (p. 82, NB 1). *If $p > 0$, $n \ge 1$, and*

$$e^{-\pi\sqrt{p}} = F\left(\frac{1-\sqrt{1-1/n^2}}{2}\right),$$

then

$$e^{-2\pi\sqrt{p}} = F\left(\left(\sqrt{n+1}-\sqrt{n}\right)^4\left(\sqrt{n}-\sqrt{n-1}\right)^4\right).$$

From (2.9), $n = G_p^{12}$. Hence, in Theorem 9.3 Ramanujan provides an algorithm for determining α_{4p} from the value of α_p, or from G_p, namely,

$$\alpha_{4p} = \left(\sqrt{G_p^{12}+1}-\sqrt{G_p^{12}}\right)^4\left(\sqrt{G_p^{12}}-\sqrt{G_p^{12}-1}\right)^4. \tag{9.4}$$

Before proving Theorem 9.3, we verify four examples recorded by Ramanujan.

Examples 9.4 (p. 82, NB 1). *We have*

$$\alpha_4 = (\sqrt{2}-1)^4,$$
$$\alpha_{12} = (\sqrt{3}-\sqrt{2})^4(\sqrt{2}-1)^4,$$
$$\alpha_{28} = (\sqrt{2}-1)^8(2\sqrt{2}-\sqrt{7})^4,$$

and

$$\alpha_{60} = (\sqrt{10}-3)^4(\sqrt{2}-1)^4(\sqrt{6}-\sqrt{5})^4(\sqrt{3}-\sqrt{2})^4.$$

The value of α_4 was also recorded in the second notebook (Part III [3, p. 97]). Both α_4 and α_{28} were also determined by Ramanathan [1], and the Borwein brothers have determined α_4 and α_{12} [1, pp. 139, 151].

Proof. Let $p = 1$, so that trivially $G_1 = 1$. Then, from (9.4),

$$\alpha_4 = (\sqrt{2}-1)^4(1-0)^4 = (\sqrt{2}-1)^4.$$

Let $p = 3$, so that, from the table in Section 2, $G_3 = 2^{1/12}$ and $n = 2$. Thus, from (9.4),

$$\alpha_{12} = (\sqrt{3}-\sqrt{2})^4(\sqrt{2}-1)^4.$$

Let $p = 7$, so that, from the table in Section 2, $G_7 = 2^{1/4}$ and $n = 8$. Thus, from (9.4),

$$\alpha_{28} = (3-2\sqrt{2})^4(2\sqrt{2}-\sqrt{7})^4 = (\sqrt{2}-1)^8(2\sqrt{2}-\sqrt{7})^4.$$

Let $p = 15$. From the table in Section 2, $G_{15} = 2^{-1/12}(1+\sqrt{5})^{1/3}$. Thus, $n = (1+\sqrt{5})^4/2 = 4(7+3\sqrt{5})$. Hence, from (9.4),

$$\alpha_{60} = \left(\sqrt{29+12\sqrt{5}} - 2\sqrt{7+3\sqrt{5}}\right)^4\left(2\sqrt{7+3\sqrt{5}} - \sqrt{27+12\sqrt{5}}\right)^4.$$

To denest these radicals, we employ the following denesting theorem (Landau [1]). If $a^2 - qb^2 = d^2$, a perfect square, then

$$\sqrt{a+b\sqrt{q}} = \sqrt{\frac{a+d}{2}} + (\operatorname{sgn} b)\sqrt{\frac{a-d}{2}}, \tag{9.5}$$

where we have corrected a misprint. To that end, from (9.5),

$$\begin{aligned}\alpha_{60} &= \left(\sqrt{\frac{29+11}{2}} + \sqrt{\frac{29-11}{2}} - 2\sqrt{\frac{7+2}{2}} - 2\sqrt{\frac{7-2}{2}}\right)^4 \\ &\quad\times\left(2\sqrt{\frac{7+2}{2}} + 2\sqrt{\frac{7-2}{2}} - \sqrt{\frac{27+3}{2}} - \sqrt{\frac{27-3}{2}}\right)^4 \\ &= (\sqrt{20}+3-3\sqrt{2}-\sqrt{10})^4(3\sqrt{2}+\sqrt{10}-\sqrt{15}-\sqrt{12})^4 \\ &= \{(\sqrt{10}-3)(\sqrt{2}-1)\}^4\{(\sqrt{6}-\sqrt{5})(\sqrt{3}-\sqrt{2})\}^4.\end{aligned}$$

Proof of Theorem 9.3. From Part III [3, p. 215, eq. (24.21)], we find that

$$\beta = \frac{1}{\alpha^2}\left(1 - \sqrt{1-\alpha}\right)^4, \tag{9.6}$$

where β has degree 2 over α. To prove Theorem 9.3, we must show that

$$\beta = (\sqrt{n-1} - \sqrt{n})^4(\sqrt{n+1} - \sqrt{n})^4. \tag{9.7}$$

From our hypothesis, with $n := G_p^{12}$ and $\alpha := \alpha_p$,

$$\alpha = \frac{n - \sqrt{n^2-1}}{2n},$$

which implies that

$$\alpha^{-1} = (\sqrt{n(n+1)} + \sqrt{n(n-1)})^2. \tag{9.8}$$

Since $4\alpha(1-\alpha) = n^{-2}$, we deduce that

$$\frac{1}{1-\alpha} = (\sqrt{n(n+1)} - \sqrt{n(n-1)})^2. \tag{9.9}$$

Hence, from (9.6), (9.8), and (9.9),

$$\begin{aligned}
\beta &= (\sqrt{n(n+1)} + \sqrt{n(n-1)})^4 \left(1 - \frac{1}{\sqrt{n(n+1)} - \sqrt{n(n-1)}}\right)^4 \\
&= (\sqrt{n+1} + \sqrt{n-1})^4 \left(\sqrt{n} - \frac{\sqrt{n+1} + \sqrt{n-1}}{2}\right)^4 \\
&= (\sqrt{n^2+n} + \sqrt{n^2-n} - n - \sqrt{n^2-1})^4 \\
&= (\sqrt{n-1} - \sqrt{n})^4(\sqrt{n+1} - \sqrt{n})^4,
\end{aligned}$$

and so (9.7) has been shown.

Theorem 9.5 (p. 82, NB 1). *Under the same hypotheses as Theorem 9.3,*

$$\begin{aligned}
e^{-4\pi\sqrt{p}} = F\Bigg(&(\sqrt{n+1} + \sqrt{n})^8 \left\{\sqrt{2n} + 1 - \sqrt{2\sqrt{n}(\sqrt{n+1} + \sqrt{2})}\right\}^4 \\
&\times \left\{\sqrt{2n} - 1 - \sqrt{2\sqrt{n}(\sqrt{n+1} - \sqrt{2})}\right\}^4\Bigg).
\end{aligned} \tag{9.10}$$

Thus, together Theorems 9.3 and 9.5 yield the formula

$$\begin{aligned}
\alpha_{16p} = &\left(\sqrt{G_p^{12}+1} + \sqrt{G_p^{12}}\right)^8 \left\{\sqrt{2G_p^{12}} + 1 - \sqrt{2\sqrt{G_p^{12}}\left(\sqrt{G_p^{12}+1} + \sqrt{2}\right)}\right\}^4 \\
&\times \left\{\sqrt{2G_p^{12}} - 1 - \sqrt{2\sqrt{G_p^{12}}\left(\sqrt{G_p^{12}+1} - \sqrt{2}\right)}\right\}^4.
\end{aligned}$$

For example, if $p = 1$, then $n = G_1 = 1$, and a simple calculation shows that

$$\alpha_{16} = (\sqrt{2}+1)^4(2^{1/4}-1)^8.$$

Proof. From Theorem 9.3,

$$e^{-4\pi\sqrt{p}} = F^2\left(\left(\sqrt{n+1}-\sqrt{n}\right)^4\left(\sqrt{n}-\sqrt{n-1}\right)^4\right) =: F^2\left(\frac{4x}{(1+x)^2}\right). \tag{9.11}$$

Also, by Entry 2(v) of Chapter 17 in the second notebook (Part III [3, p. 93]), for $0 < x < 1$,

$$F(x^2) = F^2\left(\frac{4x}{(1+x)^2}\right). \tag{9.12}$$

Thus, from (9.12), (9.10), and (9.11), it suffices to show that

$$\begin{aligned} x = {} & (\sqrt{n+1}+\sqrt{n})^4\left\{\sqrt{2n}+1-\sqrt{2\sqrt{n}(\sqrt{n+1}+\sqrt{2})}\right\}^2 \\ & \times\left\{\sqrt{2n}-1-\sqrt{2\sqrt{n}(\sqrt{n+1}-\sqrt{2})}\right\}^2. \end{aligned} \tag{9.13}$$

From (9.11), it follows that

$$\frac{2}{(\sqrt{n+1}-\sqrt{n})^2(\sqrt{n}-\sqrt{n-1})^2} = \frac{1}{\sqrt{x}}+\sqrt{x}. \tag{9.14}$$

Let $u = \sqrt{x}$. Since u tends to 0 as n tends to ∞, the solution of (9.14) that we seek is

$$\begin{aligned} u = {} & \left(\sqrt{\frac{1}{2(\sqrt{n+1}-\sqrt{n})^2(\sqrt{n}-\sqrt{n-1})^2}+\frac{1}{2}}\right. \\ & \left.-\sqrt{\frac{1}{2(\sqrt{n+1}-\sqrt{n})^2(\sqrt{n}-\sqrt{n-1})^2}-\frac{1}{2}}\right)^2 \\ = {} & \tfrac{1}{2}\left(\sqrt{n+1}+\sqrt{n}\right)^2\left(\sqrt{\left(\sqrt{n}+\sqrt{n-1}\right)^2+\left(\sqrt{n+1}-\sqrt{n}\right)^2}\right. \\ & \left.-\sqrt{\left(\sqrt{n}+\sqrt{n-1}\right)^2-\left(\sqrt{n+1}-\sqrt{n}\right)^2}\right)^2 \\ = {} & \left(\sqrt{n+1}+\sqrt{n}\right)^2\left(\sqrt{2n+\sqrt{n(n-1)}-\sqrt{n(n+1)}}\right. \\ & \left.-\sqrt{\sqrt{n(n-1)}+\sqrt{n(n+1)}-1}\right)^2 \\ = {} & \left(\sqrt{n+1}+\sqrt{n}\right)^2\left(2n-1+2\sqrt{n(n-1)}\right. \end{aligned}$$

$$-2\sqrt{(2n+1)\sqrt{n(n+1)}+(2n-1)\sqrt{n(n-1)}-4n}\Big). \tag{9.15}$$

Comparing the proposed value of u from (9.13) with that of (9.15) above, we see that it suffices to show that

$$\begin{aligned}&2\sqrt{(2n+1)\sqrt{n(n+1)}+(2n-1)\sqrt{n(n-1)}-4n}\\&=(\sqrt{2n}+1)\sqrt{2\sqrt{n}(\sqrt{n+1}-\sqrt{2})}+(\sqrt{2n}-1)\sqrt{2\sqrt{n}(\sqrt{n+1}+\sqrt{2})}.\end{aligned} \tag{9.16}$$

If we square both sides of (9.16), it is a routine matter to show that (9.16) indeed is a correct equality. This therefore completes the proof.

The next theorem enables one to determine α_{8p} from the value of α_{2p}.

Theorem 9.6 (p. 82, NB 1). *If* $n \geq 1$, $p > 0$, *and*

$$e^{-\pi\sqrt{2p}} = F\left((\sqrt{n+1}-\sqrt{n})^2(\sqrt{n}-\sqrt{n-1})^2\right), \tag{9.17}$$

then

$$\begin{aligned}e^{-2\pi\sqrt{2p}} = F\Bigg(&\left\{\frac{\sqrt{n}+1+\sqrt{n+1}}{\sqrt{2}}-\sqrt{(\sqrt{n}+1)(\sqrt{n}+\sqrt{n+1})}\right\}^4\\&\times\left\{\frac{\sqrt{n}-1+\sqrt{n+1}}{\sqrt{2}}-\sqrt{(\sqrt{n}-1)(\sqrt{n}+\sqrt{n+1})}\right\}^4\Bigg).\end{aligned} \tag{9.18}$$

Observe that

$$\begin{aligned}&\left\{\frac{\sqrt{n}\pm 1+\sqrt{n+1}}{\sqrt{2}}-\sqrt{(\sqrt{n}\pm 1)(\sqrt{n}+\sqrt{n+1})}\right\}\\&\times\left\{\frac{\sqrt{n}\pm 1+\sqrt{n+1}}{\sqrt{2}}+\sqrt{(\sqrt{n}\pm 1)(\sqrt{n}+\sqrt{n+1})}\right\}=1.\end{aligned} \tag{9.19}$$

Thus, if α_{2p} can be expressed as a product of units of the form $(\sqrt{n+1}-\sqrt{n})^2$ $(\sqrt{n}-\sqrt{n-1})^2$, then α_{8p} can be expressed as a product of two fourth powers of units. Before proving Theorem 9.6, we present three examples recorded by Ramanujan.

Examples 9.7 (p. 82, NB 1). *We have*

$$\alpha_8 = \left(\sqrt{3+2\sqrt{2}}-\sqrt{2+2\sqrt{2}}\right)^4,$$

$$\alpha_{24} = \left(\sqrt{6+3\sqrt{3}}-\sqrt{5+3\sqrt{3}}\right)^4\left(\sqrt{2+\sqrt{3}}-\sqrt{1+\sqrt{3}}\right)^4,$$

and

$$\alpha_{40} = \left(2\sqrt{2}+\sqrt{5}-2\sqrt{3+\sqrt{10}}\right)^4 \left(\sqrt{2}+\sqrt{5}-\sqrt{6+2\sqrt{10}}\right)^4.$$

Proof. Let $p = 1$. Then from Theorem 9.2, $\alpha_2 = (\sqrt{2}-1)^2$. Thus, $n = 1$, and from Theorem 9.6,

$$\alpha_8 = \left(\sqrt{2}+1-\sqrt{2(1+\sqrt{2})}\right)^4. \tag{9.20}$$

But from (9.5),

$$\sqrt{3+2\sqrt{2}} = \sqrt{\frac{3+1}{2}}+\sqrt{\frac{3-1}{2}} = \sqrt{2}+1.$$

Using this in (9.20), we achieve the desired representation of α_8.

Let $p = 3$. From Theorem 9.2, $\alpha_6 = (2-\sqrt{3})^2(\sqrt{3}-\sqrt{2})^2$. Thus, $n = 3$, and from Theorem 9.6,

$$\alpha_{24} = \left(\frac{3+\sqrt{3}}{\sqrt{2}}-\sqrt{5+3\sqrt{3}}\right)^4 \left(\frac{1+\sqrt{3}}{\sqrt{2}}-\sqrt{1+\sqrt{3}}\right)^4. \tag{9.21}$$

But from (9.5),

$$\sqrt{6+3\sqrt{3}} = \sqrt{\frac{6+3}{2}}+\sqrt{\frac{6-3}{2}} = \frac{3+\sqrt{3}}{\sqrt{2}}$$

and

$$\sqrt{2+\sqrt{3}} = \sqrt{\frac{2+1}{2}}+\sqrt{\frac{2-1}{2}} = \frac{\sqrt{3}+1}{\sqrt{2}}.$$

Using these calculations in (9.21), we complete the verification of Ramanujan's representation for α_{24}.

Let $p = 5$. From Theorem 9.2, $\alpha_{10} = (\sqrt{10}-3)^2(3-2\sqrt{2})^2$. Thus, $n = 9$, and from Theorem 9.6,

$$\alpha_{40} = \left(2\sqrt{2}+\sqrt{5}-\sqrt{4(3+\sqrt{10})}\right)^4 \left(\sqrt{2}+\sqrt{5}-\sqrt{2(3+\sqrt{10})}\right)^4,$$

which is what is claimed.

Proof of Theorem 9.6. From (9.17),

$$e^{-2\pi\sqrt{2p}} = F^2\left((\sqrt{n+1}-\sqrt{n})^2(\sqrt{n}-\sqrt{n-1})^2\right) =: F^2\left(\frac{4x}{(1+x)^2}\right).$$

Hence, as in (9.14),

$$\frac{2}{(\sqrt{n+1}-\sqrt{n})(\sqrt{n}-\sqrt{n-1})} = \frac{1}{\sqrt{x}}+\sqrt{x} \tag{9.22}$$

and, by (9.18) and (9.12), it suffices to prove that

$$x = \left\{\frac{\sqrt{n}+1+\sqrt{n+1}}{\sqrt{2}} - \sqrt{(\sqrt{n}+1)(\sqrt{n}+\sqrt{n+1})}\right\}^2 \times \left\{\frac{\sqrt{n}-1+\sqrt{n+1}}{\sqrt{2}} - \sqrt{(\sqrt{n}-1)(\sqrt{n}+\sqrt{n+1})}\right\}^2. \tag{9.23}$$

Let $u = \sqrt{x}$. In view of the form (9.22), it is natural to assume that

$$u = (a_1 - b_1)(a_2 - b_2), \tag{9.24}$$

where

$$a_1^2 - b_1^2 = 1 = a_2^2 - b_2^2. \tag{9.25}$$

Then, by (9.22) and (9.24),

$$(\sqrt{n+1}+\sqrt{n})(\sqrt{n}+\sqrt{n-1}) = \frac{1}{2}\left(u + \frac{1}{u}\right) = a_1a_2 + b_1b_2. \tag{9.26}$$

If $s := \sqrt{n}+\sqrt{n+1}$, the values of a_1, b_1, a_2, and b_2 that satisfy (9.25) and (9.26) are

$$a_1 = \frac{s+1}{\sqrt{2}}, \quad a_2 = \frac{s-1}{\sqrt{2}}, \quad b_1 = \sqrt{s}\sqrt{\sqrt{n}+1}, \quad \text{and} \quad b_2 = \sqrt{s}\sqrt{\sqrt{n}-1}.$$

Then, as already observed in (9.19), (9.25) is satisfied. Furthermore,

$$\begin{aligned} a_1a_2 + b_1b_2 &= n + \sqrt{n(n+1)} + (\sqrt{n}+\sqrt{n+1})\sqrt{n-1} \\ &= (\sqrt{n+1}+\sqrt{n})(\sqrt{n}+\sqrt{n-1}), \end{aligned}$$

and so (9.26) is satisfied. Hence, (9.23) has been shown, and the proof of Theorem 9.6 is complete.

Singular Moduli for Odd n

From (1.6), we find that, in the notation of Theorem 9.1,

$$2g_n^{12} = 2u^2v^2 = \frac{1}{\sqrt{\alpha_n}} - \sqrt{\alpha_n}. \tag{9.27}$$

By elementary manipulation, we find from the other equality of (1.6) that

$$2iG_n^{12} = \frac{i}{2\alpha_n G_n^{12}} - \frac{2\alpha_n G_n^{12}}{i}. \tag{9.28}$$

If we set

$$(g_n^*)^{12} := iG_n^{12} \qquad \text{and} \qquad \sqrt{\alpha_n^*} := \frac{2\alpha_n G_n^{12}}{i},$$

then (9.28) takes the form

$$2(g_n^*)^{12} = \frac{1}{\sqrt{\alpha_n^*}} - \sqrt{\alpha_n^*}. \tag{9.29}$$

Comparing (9.27) and (9.29), we deduce the following theorem from Theorem 9.1.

Theorem 9.8. *Set*

$$(g_n^*)^6 = uv,$$
$$u^2 + 1/u^2 = 2U, \qquad v^2 + 1/v^2 = 2V,$$
$$W = \sqrt{U^2 + V^2 - 1},$$

and

$$2S = U + V + W + 1.$$

Then

$$\alpha_n^* = \{\sqrt{S} - \sqrt{S-1}\}^2\{\sqrt{S-U} - \sqrt{S-U-1}\}^2$$
$$\times \{\sqrt{S-V} - \sqrt{S-V-1}\}^2\{\sqrt{S-W} - \sqrt{S-W-1}\}^2.$$

The next theorem gives the twelve values of α_n, when n is odd, that are found in Ramanujan's notebooks. In those instances when two representations are given, the former one is that which is in the notebooks, or that which contains the units provided by Ramanujan in his notebooks. The Borweins [1, pp. 139, 151] calculated α_n, for $n = 3, 5, 7, 9$, and 15, and Ramanathan [1] determined α_n, for $n = 3, 7, 9$, and 15.

Theorem 9.9 (pp. 80, 345, 346, NB 1; p. 262, NB 2). *We have*

$$\alpha_3 = \frac{2-\sqrt{3}}{4},$$

$$\alpha_5 = \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^3\left(\sqrt{\frac{3+\sqrt{5}}{4}} - \sqrt{\frac{\sqrt{5}-1}{4}}\right)^4$$

$$= \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^3\left(\frac{\sqrt{5}+1}{2} - \sqrt{\frac{\sqrt{5}+1}{2}}\right)^2,$$

$$\alpha_7 = \frac{8-3\sqrt{7}}{16},$$

$$\alpha_9 = \frac{1}{2}\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4\left(\sqrt{\frac{3+\sqrt{3}}{4}} - \sqrt{\frac{\sqrt{3}-1}{4}}\right)^8$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4 \left(\sqrt{4+2\sqrt{3}} - \sqrt{3+2\sqrt{3}}\right)^2,$$

$$\alpha_{13} = \frac{1}{2}\left(\frac{\sqrt{13}-3}{2}\right)^3 \left(\sqrt{\frac{7+\sqrt{13}}{4}} - \sqrt{\frac{3+\sqrt{13}}{4}}\right)^4$$

$$= df12\left(\frac{\sqrt{13}-3}{2}\right)^3 \left(\sqrt{\frac{19+5\sqrt{13}}{2}} - \sqrt{\frac{17+5\sqrt{13}}{2}}\right)^2,$$

$$\alpha_{15} = \frac{1}{16}\left(\frac{\sqrt{5}-1}{2}\right)^4 (2-\sqrt{3})^2(4-\sqrt{15}),$$

$$\alpha_{17} = \frac{1}{2}\left(\sqrt{\frac{7+\sqrt{17}}{4}} - \sqrt{\frac{3+\sqrt{17}}{4}}\right)^4$$

$$\times \left(\sqrt{\frac{3+\sqrt{4+\sqrt{17}}}{4}} - \sqrt{\frac{\sqrt{4+\sqrt{17}}-1}{4}}\right)^8,$$

$$\alpha_{21} = \frac{1}{2}\left(\frac{3-\sqrt{7}}{\sqrt{2}}\right)^2 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3 \left(\sqrt{\frac{5+\sqrt{7}}{4}} - \sqrt{\frac{1+\sqrt{7}}{4}}\right)^4$$

$$\times \left(\sqrt{\frac{3+\sqrt{7}}{4}} - \sqrt{\frac{\sqrt{7}-1}{4}}\right)^4,$$

$$\alpha_{25} = \tfrac{1}{2}(161-72\sqrt{5}) \left(\sqrt{\frac{5+\sqrt{5}}{4}} - \sqrt{\frac{1+\sqrt{5}}{4}}\right)^8,$$

$$\alpha_{33} = \tfrac{1}{2}(2-\sqrt{3})^3 \left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^2 \left(\sqrt{\frac{7+3\sqrt{3}}{4}} - \sqrt{\frac{3+3\sqrt{3}}{4}}\right)^4$$

$$\times \left(\sqrt{\frac{5+\sqrt{3}}{4}} - \sqrt{\frac{1+\sqrt{3}}{4}}\right)^4,$$

$$\alpha_{45} = \tfrac{1}{2}(\sqrt{5}-2)^3 \left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^4 \left(\sqrt{\frac{7+3\sqrt{5}}{4}} - \sqrt{\frac{3+3\sqrt{5}}{4}}\right)^4$$

$$\times \left(\sqrt{\frac{3+\sqrt{5}}{2}} - \sqrt{\frac{1+\sqrt{5}}{2}}\right)^4,$$

and

$$\alpha_{55} = \frac{1}{16}\left(\sqrt{5}-2\right)^2 (10-3\sqrt{11})(3\sqrt{5}-2\sqrt{11})$$
$$\times\left(\sqrt{\frac{7+\sqrt{5}}{8}}-\sqrt{\frac{\sqrt{5}-1}{8}}\right)^{12}\left(\sqrt{\frac{4+\sqrt{5}}{2}}-\sqrt{\frac{2+\sqrt{5}}{2}}\right)^4.$$

Proof of Theorem 9.9 for $n = 3, 5, 7, 9$, and 13. These five values are easily computed by using Theorem 9.8. The required values for G_n may be found in the table in Section 2. In each instance, we list the values for u, v, U, V, W, and S in the table below. The reader should easily be able to verify the calculations.

n	u	v	U	V	W	S
3	$\exp(\pi i/4)$	$\sqrt{2}$	0	$\frac{5}{4}$	$\frac{3}{4}$	$\frac{3}{2}$
5	$\exp(\pi i/4)$	$\sqrt{2+\sqrt{5}}$	0	$\sqrt{5}$	2	$\frac{1}{2}(3+\sqrt{5})$
7	$\exp(\pi i/4)$	$2^{3/2}$	0	$\frac{65}{16}$	$\frac{63}{16}$	$\frac{9}{2}$
9	$\exp(\pi i/4)$	$2+\sqrt{3}$	0	7	$4\sqrt{3}$	$4+2\sqrt{3}$
13	$\exp(\pi i/4)$	$\sqrt{18+5\sqrt{13}}$	0	$5\sqrt{13}$	18	$\frac{1}{2}(19+5\sqrt{13})$

Except for $n = 55$, we have also used Theorem 9.8 to calculate the remaining values in Theorem 9.9. However, the following lemmas lead to simpler calculations.

Lemma 9.10. *If r is any positive real number and $t = \sqrt{(r+1)/8}$, then*

$$r-\sqrt{r^2-1} = \left(\sqrt{t+\tfrac{1}{2}}-\sqrt{t-\tfrac{1}{2}}\right)^4. \tag{9.30}$$

Proof. The equality (9.30) can be readily verified by elementary algebra.

Lemma 9.11. *If r and t are as given in Lemma 9.10, then*

$$r-\sqrt{r^2-1} = \left(\sqrt{\frac{\sqrt{t+\frac{1}{2}}+1}{2}}-\sqrt{\frac{\sqrt{t+\frac{1}{2}}-1}{2}}\right)^8. \tag{9.31}$$

Proof. It is readily verified that

$$\left(\sqrt{\frac{\sqrt{t+\frac{1}{2}}+1}{2}}-\sqrt{\frac{\sqrt{t+\frac{1}{2}}-1}{2}}\right)^2=\sqrt{t+\tfrac{1}{2}}-\sqrt{t-\tfrac{1}{2}}. \tag{9.32}$$

Using Lemma 9.10 in (9.32), we deduce (9.31).

We frequently set $G = G_n$ below, when the value of n is understood.

Proof of Theorem 9.9 for n = 5, 9, 13, 15, 17, and 25. Let $n = 5$. From the table in Section 2,

$$G_5^{12}=\left(\frac{\sqrt{5}+1}{2}\right)^3=\sqrt{5}+2. \tag{9.33}$$

If $r = G_5^{12}$ in Lemma 9.10, then

$$t=\sqrt{\frac{3+\sqrt{5}}{8}}=\frac{\sqrt{5}+1}{4}$$

and

$$G^{12}-\sqrt{G^{24}-1}=\left(\sqrt{\frac{\sqrt{5}+3}{4}}-\sqrt{\frac{\sqrt{5}-1}{4}}\right)^4. \tag{9.34}$$

Thus, the given value for α_5 follows immediately from (9.3), (9.33), and (9.34).

Let $n = 9$. From the table in Section 2,

$$G_9^{12}=\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^4=7+4\sqrt{3}. \tag{9.35}$$

Applying Lemma 9.11 with $r = G_9^{12}$, we find that

$$t=\sqrt{\frac{8+4\sqrt{3}}{8}}=\frac{\sqrt{3}+1}{2},$$

$$\sqrt{t+\tfrac{1}{2}}=\sqrt{\frac{2+\sqrt{3}}{2}}=\frac{1+\sqrt{3}}{2},$$

and

$$G^{12}-\sqrt{G^{24}-1}=\left(\sqrt{\frac{3+\sqrt{3}}{4}}-\sqrt{\frac{\sqrt{3}-1}{4}}\right)^8. \tag{9.36}$$

Thus, by (9.3), (9.35), and (9.36), we deduce Ramanujan's value for α_9.

Let $n = 13$. From the table in Section 2,

$$G_{13}^{12} = \left(\frac{\sqrt{13}+3}{2}\right)^3 = 18 + 5\sqrt{13}. \tag{9.37}$$

Then in Lemma 9.10, set $r = G_{13}^{12}$ to deduce that

$$t = \sqrt{\frac{19+5\sqrt{13}}{8}} = \frac{5+\sqrt{13}}{4}$$

and

$$G^{12} - \sqrt{G^{24}-1} = \left(\sqrt{\frac{7+\sqrt{13}}{4}} - \sqrt{\frac{3+\sqrt{13}}{4}}\right)^4. \tag{9.38}$$

Hence, the given value for α_{13} follows from (9.3), (9.37), and (9.38).

Let $n = 15$. From the table in Section 2,

$$G_{15}^{12} = \tfrac{1}{2}(\sqrt{5}+1)^4 = 28 + 12\sqrt{5}. \tag{9.39}$$

Apply Lemma 9.10 with $r = G_{15}^{12}$. Then

$$t = \sqrt{\frac{29+12\sqrt{5}}{8}},$$

and so

$$\begin{aligned} G^{12} - \sqrt{G^{24}-1} &= \left(\sqrt{\sqrt{\frac{29+12\sqrt{5}}{8}} + \frac{1}{2}} - \sqrt{\sqrt{\frac{29+12\sqrt{5}}{8}} - \frac{1}{2}}\right)^4 \\ &= \left(\sqrt{\frac{29+12\sqrt{5}}{2}} - \sqrt{\frac{27+12\sqrt{5}}{2}}\right)^2 \\ &= 28 + 12\sqrt{5} - \sqrt{(29+12\sqrt{5})(27+12\sqrt{5})} \\ &= 28 + 12\sqrt{5} - 16\sqrt{3} - 7\sqrt{15} \\ &= (2-\sqrt{3})^2(4-\sqrt{15}). \end{aligned} \tag{9.40}$$

Hence, by (9.3), (9.39), and (9.40), the desired result follows.

Let $n = 17$. From the table in Section 2,

$$G_{17}^{12} = \left(\sqrt{\frac{5+\sqrt{17}}{8}} + \sqrt{\frac{\sqrt{17}-3}{8}}\right)^{12} = 20 + 5\sqrt{17} + \sqrt{(20+5\sqrt{17})^2 - 1},$$

after a lengthy calculation. We now apply Lemma 9.10 with $r = 20+5\sqrt{17}$. Then

$$t = \sqrt{\frac{21+5\sqrt{17}}{8}} = \frac{5+\sqrt{17}}{4},$$

and so

$$G_{17}^{-12} = \left(\sqrt{\frac{7+\sqrt{17}}{4}} - \sqrt{\frac{3+\sqrt{17}}{4}}\right)^4. \tag{9.41}$$

Next, set $r = 20+5\sqrt{17}+\sqrt{(20+5\sqrt{17})^2-1}$ in Lemma 9.11. Then

$$\begin{aligned} t &= \sqrt{\frac{21+5\sqrt{17}+\sqrt{(20+5\sqrt{17})^2-1}}{8}} \\ &= \sqrt{\frac{42+10\sqrt{17}+4\sqrt{206+50\sqrt{17}}}{16}} \\ &= \frac{3+\sqrt{17}+2\sqrt{4+\sqrt{17}}}{4} \end{aligned}$$

and

$$\sqrt{t+\frac{1}{2}} = \sqrt{\frac{5+\sqrt{17}+2\sqrt{4+\sqrt{17}}}{4}} = \frac{1+\sqrt{4+\sqrt{17}}}{2}.$$

Thus,

$$G^{12} - \sqrt{G^{24}-1} = \left(\sqrt{\frac{3+\sqrt{4+\sqrt{17}}}{4}} - \sqrt{\frac{\sqrt{4+\sqrt{17}}-1}{4}}\right)^8. \tag{9.42}$$

Using (9.41) and (9.42) in (9.3), we complete the proof.

Let $n = 25$. From the table in Section 2,

$$G_{25}^{12} = \left(\frac{\sqrt{5}+1}{2}\right)^{12} = 161+72\sqrt{5}. \tag{9.43}$$

With $r = G_{25}^{12}$ in Lemma 9.11,

$$t = \sqrt{\frac{162+72\sqrt{5}}{8}} = \frac{6+3\sqrt{5}}{2}$$

and

$$\sqrt{t+\frac{1}{2}} = \sqrt{\frac{7+3\sqrt{5}}{2}} = \frac{3+\sqrt{5}}{2}.$$

Hence,

$$G^{12}-\sqrt{G^{24}-1}=\left(\sqrt{\frac{5+\sqrt{5}}{4}}-\sqrt{\frac{1+\sqrt{5}}{4}}\right)^{8}. \tag{9.44}$$

Putting (9.43) and (9.44) in (9.3), we complete the proof.

The next lemma will enable us to calculate $\alpha_{21}, \alpha_{33}, \alpha_{45}$, and α_{55}.

Lemma 9.12. *Let $r = uv$, where $v > u$ and u is a unit. Set $u = u_1 + u_2$, where $u_1, u_2 > 0$ and $u_1^2 - u_2^2 = 1$. Furthermore, let*

$$a^2 = 1 + 2vu_1 + v^2 \qquad \text{and} \qquad b^2 = 1 - 2vu_1 + v^2,$$

where $a, b > 0$. Then

$$\begin{aligned} r-\sqrt{r^2-1} &= \left(\sqrt{\sqrt{\frac{a+b+2}{16}}+\frac{1}{2}}-\sqrt{\sqrt{\frac{a+b+2}{16}}-\frac{1}{2}}\right)^4 \\ &\quad\times\left(\sqrt{\sqrt{\frac{a-b+2}{16}}+\frac{1}{2}}-\sqrt{\sqrt{\frac{a-b+2}{16}}-\frac{1}{2}}\right)^4. \end{aligned} \tag{9.45}$$

Proof. The right side of (9.45) equals

$$\begin{aligned} &\left(\sqrt{\frac{a+b+2}{4}}-\sqrt{\frac{a+b-2}{4}}\right)^2\left(\sqrt{\frac{a-b+2}{4}}-\sqrt{\frac{a-b-2}{4}}\right)^2 \\ &= \left(\frac{a+b}{2}-\sqrt{\left(\frac{a+b}{2}\right)^2-1}\right)\left(\frac{a-b}{2}-\sqrt{\left(\frac{a-b}{2}\right)^2-1}\right) \\ &= \left(\frac{a^2-b^2}{4}+\sqrt{\left(\frac{a^2-b^2}{4}\right)^2-\frac{a^2+b^2}{2}+1}\right) \\ &\quad -\left(\frac{a+b}{2}\sqrt{\left(\frac{a-b}{2}\right)^2-1}+\frac{a-b}{2}\sqrt{\left(\frac{a+b}{2}\right)^2-1}\right). \end{aligned} \tag{9.46}$$

Set

$$r' = \frac{a^2-b^2}{4}+\sqrt{\left(\frac{a^2-b^2}{4}\right)^2-\frac{a^2+b^2}{2}+1}.$$

Then, by an elementary calculation,

$$\sqrt{r'^2-1}=\frac{a+b}{2}\sqrt{\left(\frac{a-b}{2}\right)^2-1}+\frac{a-b}{2}\sqrt{\left(\frac{a+b}{2}\right)^2-1}.$$

Hence, we see that the right side of (9.46) equals $r' - \sqrt{r'^2 - 1}$, and it therefore remains to show that $r = r'$.

In fact, from the definitions of a^2 and b^2,

$$\begin{aligned} r' &= vu_1 + \sqrt{v^2u_1^2 + 1 - (1 + v^2)} \\ &= vu_1 + v\sqrt{u_1^2 - 1} = vu_1 + vu_2 = vu = r, \end{aligned}$$

which completes the proof.

Proof of Theorem 9.9 for $n = 21, 33, 45$, and 55. Let $n = 21$. From the table in Section 2,

$$r := G_{21}^{12} = \left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^2 \left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^3 = (8 + 3\sqrt{7})(2\sqrt{7} + 3\sqrt{3}). \quad (9.47)$$

Set $u_1 = 2\sqrt{7}$ and $v = 8 + 3\sqrt{7}$ in Lemma 9.12. Then

$$a^2 = 1 + 2(8 + 3\sqrt{7})2\sqrt{7} + (8 + 3\sqrt{7})^2 = 212 + 80\sqrt{7} = (10 + 4\sqrt{7})^2$$

and

$$b^2 = 1 - 2(8 + 3\sqrt{7})2\sqrt{7} + (8 + 3\sqrt{7})^2 = 44 + 16\sqrt{7} = (4 + 2\sqrt{7})^2.$$

Hence, $a = 10 + 4\sqrt{7}$ and $b = 4 + 2\sqrt{7}$. Moreover,

$$\sqrt{\frac{a+b+2}{16}} = \sqrt{\frac{16+6\sqrt{7}}{16}} = \frac{3+\sqrt{7}}{4}$$

and

$$\sqrt{\frac{a-b+2}{16}} = \sqrt{\frac{8+2\sqrt{7}}{16}} = \frac{1+\sqrt{7}}{4}.$$

Thus, by Lemma 9.12,

$$G^{12} - \sqrt{G^{24} - 1} = \left(\sqrt{\frac{5+\sqrt{7}}{4}} - \sqrt{\frac{1+\sqrt{7}}{4}}\right)^4 \left(\sqrt{\frac{3+\sqrt{7}}{4}} - \sqrt{\frac{\sqrt{7}-1}{4}}\right)^4. \quad (9.48)$$

On using (9.47) and (9.48) in (9.3), we complete the proof.

Let $n = 33$. From the table in Section 2,

$$G_{33}^{12} = (2 + \sqrt{3})^3 \left(\frac{\sqrt{11}+3}{\sqrt{2}}\right)^2 = (26 + 15\sqrt{3})(10 + 3\sqrt{11}). \quad (9.49)$$

Apply Lemma 9.12 with $u_1 = 10$ and $v = 26 + 15\sqrt{3}$. Then

$$a^2 = 1 + 2(26 + 15\sqrt{3})10 + (26 + 15\sqrt{3})^2 = 1872 + 1080\sqrt{3} = (30 + 18\sqrt{3})^2$$

and

$$b^2 = 1 - 2(26 + 15\sqrt{3})10 + (26 + 15\sqrt{3})^2 = 832 + 480\sqrt{3} = (20 + 12\sqrt{3})^2.$$

Thus, $a = 30 + 18\sqrt{3}$ and $b = 20 + 12\sqrt{3}$, so that

$$\sqrt{\frac{a+b+2}{16}} = \sqrt{\frac{52+30\sqrt{3}}{16}} = \frac{5+3\sqrt{3}}{4}$$

and

$$\sqrt{\frac{a-b+2}{16}} = \sqrt{\frac{12+6\sqrt{3}}{16}} = \frac{3+\sqrt{3}}{4}.$$

Thus, by Lemma 9.12,

$$G^{12} - \sqrt{G^{24}-1} = \left(\sqrt{\frac{7+3\sqrt{3}}{4}} - \sqrt{\frac{3+3\sqrt{3}}{4}}\right)^4 \left(\sqrt{\frac{5+\sqrt{3}}{4}} - \sqrt{\frac{1+\sqrt{3}}{4}}\right)^4. \tag{9.50}$$

Upon substituting (9.49) and (9.50) in (9.3), we complete the proof.

Let $n = 45$. From the table in Section 2,

$$G_{45}^{12} = (\sqrt{5}+2)^3 \left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^4 = (38 + 17\sqrt{5})(31 + 8\sqrt{15}). \tag{9.51}$$

We apply Lemma 9.12 with $u_1 = 31$ and $v = 38 + 17\sqrt{5}$. Thus,

$$a^2 = 1 + 2(38 + 17\sqrt{5})31 + (38 + 17\sqrt{5})^2 = 5246 + 2346\sqrt{5} = (51 + 23\sqrt{5})^2$$

and

$$b^2 = 1 - 2(38 + 17\sqrt{5})31 + (38 + 17\sqrt{5})^2 = 534 + 238\sqrt{5} = (17 + 7\sqrt{5})^2.$$

Hence, $a = 51 + 23\sqrt{5}$ and $b = 17 + 7\sqrt{5}$, so that

$$\sqrt{\frac{a+b+2}{16}} = \sqrt{\frac{70+30\sqrt{5}}{16}} = \frac{5+3\sqrt{5}}{4}$$

and

$$\sqrt{\frac{a-b+2}{16}} = \sqrt{\frac{36+16\sqrt{5}}{16}} = \frac{2+\sqrt{5}}{2}.$$

So, by Lemma 9.12,

$$G^{12} - \sqrt{G^{24}-1} = \left(\sqrt{\frac{7+3\sqrt{5}}{4}} - \sqrt{\frac{3+3\sqrt{5}}{4}}\right)^4 \left(\sqrt{\frac{3+\sqrt{5}}{2}} - \sqrt{\frac{1+\sqrt{5}}{2}}\right)^4. \tag{9.52}$$

The desired evaluation now follows immediately from (9.3), (9.51), and (9.52).

Let $n = 55$. From the table in Section 2,

$$G_{55}^{12} = 8(\sqrt{5}+2)^2\left(\sqrt{\frac{7+\sqrt{5}}{8}}+\sqrt{\frac{\sqrt{5}-1}{8}}\right)^{12}$$

$$= 8(\sqrt{5}+2)^2\left(\frac{99+45\sqrt{5}}{4}+\sqrt{\left(\frac{99+45\sqrt{5}}{4}\right)^2-1}\right). \tag{9.53}$$

Apply Lemma 9.12 with $u_1 = (99+45\sqrt{5})/4$ and $v = 8(\sqrt{5}+2)^2$. Thus,

$$a^2 = 1 + 2\cdot 8(\sqrt{5}+2)^2\tfrac{1}{4}(99+45\sqrt{5}) + 64(\sqrt{5}+2)^4$$
$$= 17469 + 7812\sqrt{5} = (93+42\sqrt{5})^2$$

and

$$b^2 = 1 - 2\cdot 8(\sqrt{5}+2)^2\tfrac{1}{4}(99+45\sqrt{5}) + 64(\sqrt{5}+2)^4$$
$$= 3141 + 1404\sqrt{5} = (39+18\sqrt{5})^2.$$

Thus, $a = 93+42\sqrt{5}$ and $b = 39+18\sqrt{5}$, so that

$$\sqrt{\frac{a+b+2}{16}} = \sqrt{\frac{67+30\sqrt{5}}{8}}$$

and

$$\sqrt{\frac{a-b+2}{16}} = \sqrt{\frac{7+3\sqrt{5}}{2}} = \frac{3+\sqrt{5}}{2}.$$

Thus, by Lemma 9.12,

$$G^{12} - \sqrt{G^{24}-1} = \left(\sqrt{\sqrt{\frac{67+30\sqrt{5}}{8}}+\frac{1}{2}}-\sqrt{\sqrt{\frac{67+30\sqrt{5}}{8}}-\frac{1}{2}}\right)^4$$
$$\times\left(\sqrt{\frac{4+\sqrt{5}}{2}}-\sqrt{\frac{2+\sqrt{5}}{2}}\right)^4. \tag{9.54}$$

Now, by Lemma 9.10,

$$\left(\sqrt{\sqrt{\frac{67+30\sqrt{5}}{8}}+\frac{1}{2}}-\sqrt{\sqrt{\frac{67+30\sqrt{5}}{8}}-\frac{1}{2}}\right)^4$$
$$= 66+30\sqrt{5}-\sqrt{(66+30\sqrt{5})^2-1}$$
$$= 66+30\sqrt{5}-9\sqrt{55}-20\sqrt{11}$$
$$= (10-3\sqrt{11})(3\sqrt{5}-2\sqrt{11}). \tag{9.55}$$

Using (9.55) in (9.54) and then (9.53) and (9.54) in (9.3), we complete the proof.

We close this subsection by showing how different modes of calculating α_n can lead to interesting identities between radicals.

Entry 9.13 (p. 311, NB 1). *We have*

(a)
$$\sqrt[8]{1 \pm \sqrt{1-(\sqrt{5}-2)^8}} = \frac{\sqrt{5}-1}{2}\frac{\sqrt[4]{5}\pm 1}{\sqrt{2}}$$

and

(b)
$$\sqrt[8]{1 \pm \sqrt{1-(2-\sqrt{3})^4}} = \frac{\sqrt{2}\sqrt[4]{3}\pm(\sqrt{3}-1)}{2\sqrt[4]{2}}.$$

Proof. From the table in Section 2, $G_{25} = (\sqrt{5}+1)/2$, and using this value in (9.3), we easily find that

$$(2\alpha_{25})^{1/8} = \sqrt[8]{1-\sqrt{1-(\sqrt{5}-2)^8}}.$$

On the other hand, from Theorem 9.9,

$$\begin{aligned}(2\alpha_{25})^{1/8} &= \sqrt[8]{161-72\sqrt{5}}\left(\sqrt{\frac{5+\sqrt{5}}{4}}-\sqrt{\frac{1+\sqrt{5}}{4}}\right)\\ &= \sqrt{\sqrt{5}-2}\left(\sqrt{\frac{5+\sqrt{5}}{4}}-\sqrt{\frac{1+\sqrt{5}}{4}}\right)\\ &= \left(\frac{\sqrt{5}-1}{2}\right)^{3/2}\left(\sqrt{\frac{5+\sqrt{5}}{4}}-\sqrt{\frac{1+\sqrt{5}}{4}}\right)\\ &= \frac{\sqrt{5}-1}{2}\left(\frac{5^{1/4}}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right).\end{aligned}$$

Combining these two calculations, we deduce (a) with the minus signs chosen on each side.

From the table in Section 2, $G_9 = \left((1+\sqrt{3})/\sqrt{2}\right)^{1/3}$, and with this value in (9.3), we readily find that

$$(2\alpha_9)^{1/8} = \sqrt[8]{1-\sqrt{1-(2-\sqrt{3})^4}}.$$

On the other hand, from Theorem 9.9,

$$(2\alpha_9)^{1/8} = \sqrt{\frac{\sqrt{3}-1}{\sqrt{2}}}\left(\sqrt{\frac{3+\sqrt{3}}{4}} - \sqrt{\frac{\sqrt{3}-1}{4}}\right)$$
$$= \frac{1}{2^{5/4}}\left(\sqrt{2\sqrt{3}} - \left(\sqrt{3}-1\right)\right).$$

Combining the last two calculations, we obtain (b) with the minus signs chosen on both sides.

We verified via *Mathematica* that both (a) and (b) also hold with the plus signs chosen on each side.

Calculating α_{2n} and α_{3n} with the Help of Modular Equations

Ramanujan obscurely described two further methods for calculating α_n in his first notebook [9].

In the first, Ramanujan indicated that α_{2n} may be calculated by solving a certain type of modular equation of degree n. For several prime values of n, the desired form of modular equation exists; many of these modular equations can be found in Ramanujan's notebooks and are proved in Part III [3]. This very novel method is the only known method that does not require *a priori* the value of g_{2n}. Thus, the method is a new, valuable tool in the computation of α_{2n}.

In the second, Ramanujan disclosed a method for determining α_{3n} arising from the definition of a modular equation of degree n. We give a rigorous formulation of this formula and prove it by using a device introduced in Section 6 to calculate certain class invariants.

The results in this subsection are due to the author and Chan [3]. In this paper we also devise a method for calculating α_{5n} that is similar to Ramanujan's method for determining α_{3n}.

In the middle of page 292 in his first notebook, Ramanujan claims that, "Changing β to $4B/(1+B)^2$ and α to $1-B^2$ we get an equation in $4B(1-B)/(1+B)$ and the value of B^2 is for $e^{-\pi\sqrt{2n}}$." We now state and prove a rigorous formulation of this assertion.

Theorem 9.14. *Let β have degree n over α, and suppose α and β are related by a modular equation of the form*

$$F\left((\alpha\beta)^r, \{(1-\alpha)(1-\beta)\}^r\right) = 0, \tag{9.56}$$

for some polynomial F and for some positive rational number r. If we replace α by $1-x^2$ and β by $4x/(1+x)^2$, then (2.1) becomes an equation of the form

$$G(x) := g(\{4x(1-x)/(1+x)\}^r) = 0,$$

for some polynomial g. Furthermore, $x = \sqrt{\alpha_{2n}}$. is a root of this equation.

Proof. Under the designated substitutions,

$$\begin{aligned}
&F\big((\alpha\beta)^r, \{(1-\alpha)(1-\beta)\}^r\big) \\
&= F\left(\left((1-x^2)\frac{4x}{(1+x)^2}\right)^r, \left(x^2\left(1-\frac{4x}{(1+x)^2}\right)\right)^r\right) \\
&= F\left(\left(\frac{4x(1-x)}{1+x}\right)^r, \frac{1}{4^{2r}}\left(\frac{4x(1-x)}{1+x}\right)^{2r}\right) \\
&= 0.
\end{aligned}$$

Hence, the first part of Theorem 9.14 follows.

For brevity, let ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x) = {}_2F_1(x)$. Setting $\beta = 4x/(1+x)^2$, we find that

$$\frac{{}_2F_1(1-\beta)}{{}_2F_1(\beta)} = \frac{{}_2F_1\left(1-\dfrac{4x}{(1+x)^2}\right)}{{}_2F_1\left(\dfrac{4x}{(1+x)^2}\right)} = \frac{1}{2}\frac{{}_2F_1(1-x^2)}{{}_2F_1(x^2)}, \tag{9.57}$$

by a fundamental transformation for $F(x)$ (Part III [3, p. 93]), which actually arises from a special case of Pfaff's transformation. With α replaced by $1-x^2$, we find from (1.4), (2.5), and (9.57) that

$$n\frac{{}_2F_1(x^2)}{{}_2F_1(1-x^2)} = \frac{1}{2}\frac{{}_2F_1(1-x^2)}{{}_2F_1(x^2)}.$$

Therefore,

$$\frac{{}_2F_1(1-x^2)}{{}_2F_1(x^2)} = \sqrt{2n}.$$

Now recalling the definition of a singular modulus and (1.2), we deduce that $x^2 = \alpha_{2n}$, and the proof is complete.

Example 9.15(a). Let $n = 3$. Then

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1, \tag{9.58}$$

which is originally due to Legendre and was rediscovered by Ramanujan (Part III [3, pp. 230, 232]). With Ramanujan's substitutions, (9.58) takes the form

$$u + \tfrac{1}{2}u^2 = 1, \tag{9.59}$$

where

$$u = \left(4x\frac{1-x}{1+x}\right)^{1/4}. \tag{9.60}$$

Solving (9.59), we find that $u = \sqrt{3} - 1$. Then solving (9.60), we find that $x = 2\sqrt{3} - 3 - 2\sqrt{2} + \sqrt{6}$. Using two different modes of calculation, we find that

$$\alpha_6 = x^2 = (2-\sqrt{3})^2(\sqrt{3}-\sqrt{2})^2 = \frac{2\sqrt{3}+\sqrt{6}-3-2\sqrt{2}}{2\sqrt{3}+\sqrt{6}+3+2\sqrt{2}}.$$

The former representation for α_6 can be found in Theorem 9.2.

Example 9.15(b). Let $n = 5$. Then we have Ramanujan's modular equation of degree 5 (Part III [3, p. 280]),

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1. \tag{9.61}$$

Using Ramanujan's substitutions, we find that (9.61) can be put in the shape

$$u + \tfrac{1}{4}u^2 + 2u = \tfrac{1}{4}u^2 + 3u = 1, \tag{9.62}$$

where

$$u = \left(4x\frac{1-x}{1+x}\right)^{1/2}. \tag{9.63}$$

Solving (9.62), we deduce that $u = -6 + 2\sqrt{10}$. Next, solving (9.63), we find that $x = 3\sqrt{10} - 9 - 4\sqrt{5} + 6\sqrt{2}$. Lastly, by two distinct routes for calculation,

$$\alpha_{10} = x^2 = (\sqrt{10}-3)^2(3-2\sqrt{2})^2 = \frac{3\sqrt{10}+6\sqrt{2}-9-4\sqrt{5}}{3\sqrt{10}+6\sqrt{2}+9+4\sqrt{5}}.$$

The former representation is given in Theorem 9.2.

We next derive Ramanujan's formula for α_{3n}.

Let q be given by (1.5), and suppose that β has degree n over α. Thus, (1.4) holds. Now suppose also that β has degree 3 over $1-\alpha =: \alpha'$. Then, by (1.4),

$$3\frac{{}_2F_1(\alpha)}{{}_2F_1(1-\alpha)} = \frac{{}_2F_1(1-\beta)}{{}_2F_1(\beta)} = n\frac{{}_2F_1(1-\alpha)}{{}_2F_1(\alpha)}. \tag{9.64}$$

Hence,

$$\frac{{}_2F_1(\alpha)}{{}_2F_1(1-\alpha)} = \sqrt{\frac{n}{3}}, \tag{9.65}$$

and, from (9.64) and (9.65),

$$\frac{{}_2F_1(1-\beta)}{{}_2F_1(\beta)} = 3\sqrt{\frac{n}{3}} = \sqrt{3n}. \tag{9.66}$$

Next, from Part III [3, p. 237], since β has degree 3 over α', we have the parametrizations

$$\alpha' = p\left(\frac{2+p}{1+2p}\right)^3 \qquad \text{and} \qquad \beta = p^3\left(\frac{2+p}{1+2p}\right), \tag{9.67}$$

where $0 < p < 1$. It follows from (9.67) that

$$(1-\alpha')\beta = \frac{(1-p)^3(1+p)}{(1+2p)^3}p^3\left(\frac{2+p}{1+2p}\right) = \left(p\frac{1-p}{1+2p}\right)^3\frac{(1+p)(2+p)}{1+2p} \tag{9.68}$$

and

$$(1-\beta)\alpha' = \frac{(1-p)(1+p)^3}{(1+2p)} p \left(\frac{2+p}{1+2p}\right)^3 = p\frac{1-p}{1+2p}\left(\frac{(1+p)(2+p)}{1+2p}\right)^3. \tag{9.69}$$

Next, set

$$2t = p\frac{1-p}{1+2p} \tag{9.70}$$

and observe that

$$2(1-t) = \frac{(1+p)(2+p)}{1+2p}. \tag{9.71}$$

It follows from (9.68)–(9.71) that

$$\alpha\beta = (1-\alpha')\beta = 16t^3(1-t) \tag{9.72}$$

and

$$(1-\alpha)(1-\beta) = (1-\beta)\alpha' = 16t(1-t)^3. \tag{9.73}$$

Now, set

$$k = 4t(1-t). \tag{9.74}$$

Observe from (9.72), (9.73), (1.6), (9.65), and (9.66) that

$$k = \left(2G_{n/3}^6 G_{3n}^6\right)^{-1}. \tag{9.75}$$

We determine β (α_{3n}) as a function of k. From (9.74), we find that, with no loss of generality in choosing the minus sign in the first equality below,

$$t = \frac{1-\sqrt{1-k}}{2} \qquad \text{and} \qquad 1-t = \frac{1+\sqrt{1-k}}{2}.$$

Thus, by (9.72) and (9.73), we find that

$$(1-\alpha')\beta = (1-\sqrt{1-k})^3(1+\sqrt{1-k}) = k(1-\sqrt{1-k})^2 \tag{9.76}$$

and

$$(1-\beta)\alpha' = (1-\sqrt{1-k})(1+\sqrt{1-k})^3 = k(1+\sqrt{1-k})^2. \tag{9.77}$$

Subtracting (9.76) from (9.77), we find that

$$\alpha' = 4k\sqrt{1-k} + \beta. \tag{9.78}$$

Substituting (9.78) into (9.77), we deduce that

$$\beta^2 + \beta(4k\sqrt{1-k} - 1) + k(1-\sqrt{1-k})^2 = 0.$$

Thus,

$$\alpha_{3n} = \beta = \frac{1 - 4k\sqrt{1-k} \pm \sqrt{(4k\sqrt{1-k} - 1)^2 - 4k(1 - \sqrt{1-k})^2}}{2}$$
$$= \frac{1 - 4k\sqrt{1-k} \pm (1-2k)\sqrt{1-4k}}{2}$$
$$= \frac{1 - \sqrt{1 - 4k\left(1 - 2k \pm \sqrt{(1-k)(1-4k)}\right)^2}}{2}. \qquad (9.79)$$

This last formulation was that given by Ramanujan.

We now resolve the sign ambiguity in (9.79). It is clear that both α' and β satisfy the equation

$$y^2 + y(4k\sqrt{1-k} - 1) + k(1 - \sqrt{1-k})^2 = 0. \qquad (9.80)$$

Since $\alpha' = 1 - \alpha$, it follows from (9.65) and (9.66) that the solutions of (9.80) are α_{3n} and $\alpha_{3/n}$. Thus, it suffices to show that

$$\alpha_{3/n} > \alpha_{3n}, \qquad (9.81)$$

for $n > 1$. From the definition of φ in (2.4), it is obvious that

$$\varphi^2(e^{-\pi\sqrt{3/n}}) > \varphi^2(e^{-\pi\sqrt{3n}}).$$

From (2.5), it follows that

$${}_2F_1(\alpha_{3/n}) > {}_2F_1(\alpha_{3n}).$$

Since ${}_2F_1(x)$ is increasing on $(0, 1)$, (9.81) follows.

Thus, we have proved the following theorem.

Theorem 9.16. *Let q be given by (1.5), suppose that β has degree n over α, let β have the parametrization (9.67), and define t by (9.70). Then, if k is defined by (9.74), α_{3n} has the representations given in (9.79), where the minus sign must be chosen.*

Ramanujan's formulation of Theorem 9.16 at the bottom of page 310 in his first notebook is a bit different. He first gives (9.72) and (9.73), but with the left sides switched. He then states (9.70), followed by the equality

$$F\left(p^3 \frac{2+p}{1+2p}\right) = e^{-\pi\sqrt{3n}},$$

which is a consequence of (9.66) and (9.67), where F is defined in (2.3). He concludes by defining k in (9.74) and by claiming that a more complicated, somewhat ambiguous version of the right side of (9.79) equals $e^{-\pi\sqrt{3n}}$, i.e., he forgot to write "F" in front of the right side of (9.79). (In recording specific values of $F(\alpha_n)$, Ramanujan frequently omitted parentheses about the arguments.)

Example 9.17(a). Let $n = 1$. Then, from the table in Section 2, $G_3 = 2^{1/12} = G_{1/3}$, since $G_n = G_{1/n}$ (Ramanujan [3], [10, p. 23]). Hence, by (9.75), $k = \frac{1}{4}$, and by (9.79),

$$\alpha_3 = \frac{2-\sqrt{3}}{4},$$

which is given in Theorem 9.9.

Example 9.17(b). Let $n = 5$. Then, from the table in Section 2, $G_{15} = 2^{-1/12}(\sqrt{5}+1)^{1/3}$. It can also be verified that $G_{5/3} = 2^{-1/12}(\sqrt{5}-1)^{1/3}$. Hence, it is easily seen from (9.75) that $k = \frac{1}{16}$. Therefore, from (9.79),

$$\alpha_{15} = \frac{16 - 7\sqrt{3} - \sqrt{15}}{32},$$

which is simpler than the formula given in Theorem 9.9.

10. A Certain Rational Function of Singular Moduli

Set $b^2 := b_n^2 := \beta$, where β has degree $2n$ and $b > 0$. On page 312 in his first notebook, Ramanujan defines

$$u := u_n := \frac{b(1-b)}{1+b} \tag{10.1}$$

and

$$U := U_n := \frac{1}{4}\left(u + \frac{1}{u} - 2\right). \tag{10.2}$$

(In fact, Ramanujan uses the notation β, instead of b above, and he has no notation for the right side of (10.2).) He then provides the following table.

n	U_n
1	1
3	3
5	9
7	$(\sqrt{2}+1)^2(1+2\sqrt{2})$
9	49
11	99
15	$3(5+4\sqrt{2})^2$
23	$9(1+\sqrt{2})^4(3+4\sqrt{2})$
29	99^2
35	$63(8\sqrt{2}+5\sqrt{5})^2$
71	$9(1+\sqrt{2})^{10}(2\sqrt{2}+1)^2(6\sqrt{2}+1)$

In this section, these eleven values are established. It is unclear to us why Ramanujan studied this particular function U_n.

Recall from (1.6) that

$$g := g_{2n} = \left(\frac{(1-\beta)^2}{4\beta}\right)^{1/24}. \tag{10.3}$$

It follows that

$$\frac{1}{b} - b = 2g^{12}. \tag{10.4}$$

Hence, from (10.1)–(10.4),

$$\begin{aligned} U &= \frac{1}{4}\left(\frac{b(1-b)}{1+b} + \frac{1+b}{b(1-b)} - 2\right) = \frac{(1+b^2)^2}{4b(1-b^2)} = \frac{2b}{1-b^2}\frac{(1+b^2)^2}{8b^2} \\ &= \frac{2\sqrt{\beta}}{1-\beta}\frac{(1/b-b)^2+4}{8} = \tfrac{1}{8}g^{-12}\left(4g^{24}+4\right) = \tfrac{1}{2}\left(g^{12}+g^{-12}\right). \end{aligned} \tag{10.5}$$

Equality (10.5) will now be utilized to calculate U_n for eleven values of n. All values for g_n below can be found in the table in Section 2.

First, $g_2 = 1$, and so, by (10.5),

$$U_1 = \tfrac{1}{2}(1+1) = 1.$$

Second, since $g_6 = 2^{-1/4}(4+2\sqrt{2})^{1/6}$, we find that

$$g^{\pm 12} = 3 \pm 2\sqrt{2}.$$

Thus, from (10.5),

$$U_3 = \tfrac{1}{2}\left((3+2\sqrt{2}) + (3-2\sqrt{2})\right) = 3.$$

Third, $g_{10} = \sqrt{(\sqrt{5}+1)/2}$, and so

$$g^{\pm 12} = 9 \pm 4\sqrt{5}.$$

Thus, from (10.5),

$$U_5 = \tfrac{1}{2}\left((9+4\sqrt{5}) + (9-4\sqrt{5})\right) = 9.$$

Fourth, $g_{14} = \sqrt{\left(1+\sqrt{2}+\sqrt{2\sqrt{2}-1}\right)/2}$, and so

$$g_{14}^{\pm 12} = 11 + 8\sqrt{2} \pm (8+6\sqrt{2})\sqrt{2\sqrt{2}-1}.$$

Thus, from (10.5),

$$U_7 = 11 + 8\sqrt{2} = (\sqrt{2}+1)^2(1+2\sqrt{2}).$$

Fifth, $g_{18} = (\sqrt{2}+\sqrt{3})^{1/3}$. Thus,

$$g^{\pm 12} = 49 \pm 20\sqrt{6},$$

and so, from (10.5),

$$U_9 = 49.$$

Sixth, $g_{22} = \sqrt{\sqrt{2}+1}$, and so

$$g^{\pm 12} = 99 \pm 70\sqrt{2}.$$

Hence, from (10.5),

$$U_{11} = 99.$$

Seventh, $g_{30} = (2+\sqrt{5})^{1/6}(3+\sqrt{10})^{1/6}$. It follows that

$$g^{\pm 12} = (9 \pm 4\sqrt{5})(19 \pm 6\sqrt{10}).$$

Hence,

$$U_{15} = 171 + 120\sqrt{2} = 3(5+4\sqrt{2})^2.$$

Eighth, $g_{46} = \sqrt{(3+\sqrt{2}+\sqrt{7+6\sqrt{2}})/2}$. Thus,

$$g^{\pm 12} = \left\{18 + 13\sqrt{2} \pm (5+3\sqrt{2})\sqrt{7+6\sqrt{2}}\right\}^2.$$

It follows that

$$\begin{aligned} U_{23} &= (18+13\sqrt{2})^2 + (5+3\sqrt{2})^2(7+6\sqrt{2}) = 9(147+104\sqrt{2}) \\ &= 9(17+12\sqrt{2})(3+4\sqrt{2}) = 9(1+\sqrt{2})^4(3+4\sqrt{2}). \end{aligned}$$

Ninth, $g_{58} = \sqrt{(5+\sqrt{29})/2}$. Thus,

$$g^{\pm 12} = (70 \pm 13\sqrt{29})^2,$$

and so

$$U_{29} = 70^2 + 13^2 \cdot 29 = 9801 = 99^2.$$

Tenth, $g_{70} = \sqrt{(3+\sqrt{5})(1+\sqrt{2})/2}$. It follows that

$$g^{\pm 12} = (9 \pm 4\sqrt{5})^2(7 \pm 5\sqrt{2})^2.$$

Thus, from (10.5),

$$U_{35} = 63(253 + 80\sqrt{10}) = 63(8\sqrt{2}+5\sqrt{5})^2.$$

Eleventh, $g_{142} = \sqrt{(9+5\sqrt{2}+\sqrt{127+90\sqrt{2}})/2}$. So,

$$g^{\pm 12} = \left\{1026 + 725\sqrt{2} \pm (65+45\sqrt{2})\sqrt{127+90\sqrt{2}}\right\}^2.$$

Thus, from (10.5),

$$\begin{aligned}U_{71} &= (1026 + 725\sqrt{2})^2 + (65 + 45\sqrt{2})^2(127 + 90\sqrt{2})\\ &= 9(467539 + 330600\sqrt{2})\\ &= 9(3363 + 2378\sqrt{2})(57 + 58\sqrt{2})\\ &= 9(1 + \sqrt{2})^{10}(2\sqrt{2} + 1)^2(6\sqrt{2} + 1).\end{aligned}$$

11. The Modular j–invariant

Except for four entries discussed in Section 13 of Chapter 33, the last two pages, 392–393, in Ramanujan's second notebook are devoted to values of the modular j–invariant. Recall (K. Chandrasekharan [1, p. 81], Cox [1, p. 224]) that the invariants $J(\tau)$ and $j(\tau)$, for $\tau \in \mathbb{H}$, are defined by

$$J(\tau) = \frac{g_2^3(\tau)}{\Delta(\tau)} \qquad \text{and} \qquad j(\tau) = 1728J(\tau),$$

where, for $q = \exp(2\pi i\tau)$,

$$g_2(\tau) = 60 \sum_{\substack{m,n=-\infty\\(m,n)\neq(0,0)}}^{\infty} (m\tau + n)^{-4} = \frac{4\pi^4}{3}M(q),$$

$$g_3(\tau) = 140 \sum_{\substack{m,n=-\infty\\(m,n)\neq(0,0)}}^{\infty} (m\tau + n)^{-6} = \frac{8\pi^6}{27}N(q),$$

and

$$\Delta(\tau) = g_2^3(\tau) - 27g_3^2(\tau) = 1728\frac{M^3(q)}{M^3(q) - N^2(q)}. \tag{11.1}$$

Here $M(q)$ and $N(q)$ are the Eisenstein series defined at the beginning of Section 4 of Chapter 33. (See also the beginning of Section 7 in this chapter.) Furthermore, the function $\gamma_2(\tau)$ is defined by (Cox [1, p. 249])

$$\gamma_2(\tau) = \sqrt[3]{j(\tau)}, \tag{11.2}$$

where that branch which is real when τ is purely imaginary is chosen.

At the top of page 392, which inexplicably is printed upside down, Ramanujan defines $J := J_n$ and $u := u_n$ by

$$J_n = \frac{1 - 16\alpha_n(1 - \alpha_n)}{8(4\alpha_n(1 - \alpha_n))^{1/3}} \qquad \text{and} \qquad J_n = \frac{\sqrt[3]{4u_n}}{2^3}, \tag{11.3}$$

where n is a natural number. (To avoid a conflict of notation later, we have replaced Ramanujan's t_n by u_n.) For 15 values of n, $n \equiv 3 \pmod 4$, Ramanujan indicates the corresponding values for J_n, although not all values are explicitly given. In each case, the value had been given in the literature or can be readily deduced from results in the literature. Now, from (1.6) and (11.3), we easily find that

$$J_n = \tfrac{1}{8}G_n^8(1 - 4G_n^{-24}). \tag{11.4}$$

We now identify J_n with γ_2. From Cox's text [1, p. 257, Theorem 12.17], for $q = \exp(2\pi i \tau)$,

$$\gamma_2(\tau) = 2^8 \frac{q^{2/3} f^{16}(-q^2)}{f^{16}(-q)} + \frac{f^8(-q)}{q^{1/3} f^8(-q^2)}. \tag{11.5}$$

Setting $\tau = (3 + \sqrt{-n})/2$, we deduce from (11.5), (6.1), and (1.3) that

$$\gamma_2\left(\frac{3+\sqrt{-n}}{2}\right) = \frac{2^8 - 2^6 G_n^{24}}{2^4 G_n^{16}} = -4G_n^8(1 - 4G_n^{-24}).$$

Hence, from (11.4) and (11.2),

$$J_n = -\frac{1}{32}\gamma_2\left(\frac{3+\sqrt{-n}}{2}\right) = -\frac{1}{32}\sqrt[3]{j\left(\frac{3+\sqrt{-n}}{2}\right)}. \tag{11.6}$$

There are 13 cases when the class number of the order in an imaginary quadratic field equals 1 (Cox [1, p. 260]). In such an instance, the value of the j–invariant is known to be a rational integer (Cox [1, p. 261]), and we first turn to Ramanujan's seven values in these cases. Formula (11.3) can be used to calculate some values, but in most instances the value of G_n is unavailable.

Entry 11.1. $J_3 = 0$.

Proof. From the table in Section 2, $G_3 = 2^{1/12}$. Thus, by (11.4),

$$J_3 = \tfrac{1}{8}(G_3^8 - 4G_3^{-16}) = \tfrac{1}{8}(2^{2/3} - 4 \cdot 2^{-4/3}) = 0.$$

Entry 11.2. $J_{27} = 5 \cdot 3^{1/3}$.

Proof. From the table in Section 2, $G_{27} = 2^{1/12}(2^{1/3} - 1)^{-1/3}$. Thus, from (11.4),

$$J_{27}^3 = \frac{1}{8^3}\left(4(2^{1/3} - 1)^{-8}\right)\left(1 - (2^{1/3} - 1)^8\right)^3 = 375,$$

with the help of *Mathematica*.

For the last five cases of degree 1, we simply use the relation (11.6) in conjunction with the table in Cox's book [1, p. 261].

Entry 11.3. $J_{11} = 1$.

Entry 11.4. $J_{19} = 3$.

Entry 11.5. $J_{43} = 30$.

Entry 11.6. $J_{67} = 165.$

Entry 11.7. $J_{163} = 20,010.$

There are a total of 29 cases when the degree of $j\left((3+\sqrt{-n})/2\right)$ equals 2 (W. E. Berwick [1]). Ramanujan gives the values of J_n in six of these cases. We will verify Ramanujan's values by simply referring to the tables in Berwick's paper [1, pp. 55, 57–59].

Entry 11.8. $J_{35} = \sqrt{5}\left(\dfrac{\sqrt{5}+1}{2}\right)^4.$

Entry 11.9. $J_{51} = 3\left(\sqrt{17}+4\right)^{2/3}\left(\dfrac{5+\sqrt{17}}{2}\right).$

Entry 11.10. $J_{75} = 3\cdot 5^{1/6}\left(\dfrac{69+31\sqrt{5}}{2}\right).$

Proof. In Berwick's table on page 58, we need the values of ϵ (given on page 55) and γ (given at the top of page 57). Thus,

$$\begin{aligned} J_{75} &= 3\cdot 5^{1/6}\left(\frac{\sqrt{5}+1}{2}\right)^6\left(3\frac{\sqrt{5}+1}{2}-1\right) \\ &= 3\cdot 5^{1/6}(9+4\sqrt{5})\left(\frac{1+3\sqrt{5}}{2}\right) = 3\cdot 5^{1/6}\left(\frac{69+31\sqrt{5}}{2}\right). \end{aligned}$$

Entry 11.11. $J_{91} = \frac{3}{2}(227+63\sqrt{13}).$

Proof. Proceeding with the same tables as in the preceding proof, we find that

$$J_{91} = 3\left(\frac{\sqrt{13}+3}{2}\right)^4\left(\frac{3\sqrt{13}-7}{2}\right) = \tfrac{3}{2}(227+63\sqrt{13}).$$

Entry 11.12. $J_{99} = \frac{1}{2}(23+4\sqrt{33})^{2/3}(77+15\sqrt{33}).$

Proof. From the work of Berwick [1, pp. 55, 58],

$$\begin{aligned} J_{99} &= (23+4\sqrt{33})^{2/3}\frac{\sqrt{33}-5}{2}(2\sqrt{33}+11)(\sqrt{33}+4) \\ &= \tfrac{1}{2}(23+4\sqrt{33})^{2/3}(77+15\sqrt{33}). \end{aligned}$$

The value of J_{99} was not given by Ramanujan; he wrote "$J_{99} = \cdots$."

Entry 11.13. $J_{115} = \frac{3}{2}(785 + 351\sqrt{5})$.

Proof. From pages 58, 55, and 57 in Berwick's paper [1],

$$J_{115} = 3\sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^{10}\left(\frac{3\sqrt{5}-1}{2}\right) = \tfrac{3}{2}(785 + 351\sqrt{5}).$$

Ramanujan misrecorded the value of J_{115} as

$$\tfrac{3}{2}(785 + 341\sqrt{5}).$$

Lastly, Ramanujan discusses two values of J_n when $j\left((3 + \sqrt{-n})/2\right)$ has degree 3.

Entry 11.14. *Let u_n be defined by (11.3) and define $p > 0$ by*

$$u_{59} = \frac{(1-p^8)^3}{p^8}.$$

Then

$$p^9 - 7p^8 + 22p^7 - 34p^6 + 40p^5 - 28p^4 + 22p^3 - 10p^2 + 11p - 1 = 0. \quad (11.7)$$

Furthermore, $u_{59}^{1/3}$ satisfies an irreducible cubic polynomial over $\mathbb{Q}$.

Proof. We turn to the work of A. G. Greenhill [1] who sets (with notation altered so as not to conflict with that of Ramanujan)

$$a_n = \frac{(1 - 16\alpha_n(1-\alpha_n))^3}{16\alpha_n(1-\alpha_n)}. \tag{11.8}$$

Comparing (11.8) and (11.3), we see that $a_n = 128J_n^3$. Thus, from (11.3), $u_n = a_n$. Then Greenhill sets (p. 311)

$$a_{59} = \frac{(1-p^8)^3}{p^8}$$

and shows that p is a root of (11.7). Greenhill [1] claims that $a := a_{59}$ satisfies a cubic equation but does not give it. However, in a subsequent paper [2, p. 404], Greenhill gives the equation

$$a - 392 \cdot 2^{1/3}a^{2/3} + 1072 \cdot 4^{1/3}a^{1/3} - 2816 = 0. \tag{11.9}$$

Entry 11.15. *Let u_n be defined by (11.3). Define $s > 0$ by*

$$u_{83} = \frac{(1-256s^{24})^3}{256s^{24}}$$

and set

$$\beta = \frac{1 - 2s - 2s^2 - 2s^3}{2s^3}.$$

Then

$$\beta^3 + 4\beta^2 + 2\beta - 5 = 0.$$

Furthermore, $u_{83}^{1/3}$ *satisfies an irreducible cubic polynomial over* $\mathbb{Q}$.

Proof. Ramanujan's claim, which is incomplete as stated on page 392, is a quotation from Greenhill's paper [1, pp. 312, 313]. In fact, except for u_{83}, the notation is the same. Greenhill [1] indicates that $a^{1/3} := u_{83}^{1/3}$ satisfies a cubic polynomial, which he gives in his second paper [2, p. 405], namely,

$$a - 1740 \cdot 2^{1/3} a^{2/3} + 2000 \cdot 4^{1/3} a^{1/3} - 32000 = 0, \tag{11.10}$$

which he claims is due to Russell.

On page 393, Ramanujan considers various polynomials satisfied by J_n. The work is divided into three parts. In the first part, Ramanujan gives the following table.

Entry 11.16.

n	J_n	$8J_n + 3$	$64J_n^2 - 24J_n + 9$
11	1	11	$49 = 7^2$
19	3	27	$513 = 27 \cdot 19$
43	30	$243 = 27 \cdot 3^2$	$56,889 = 27 \cdot 43 \cdot 7^2$
67	165	$1323 = 27 \cdot 7^2$	$1,738,449 = 27 \cdot 31^2 \cdot 67$
163	20010	$160,083 = 27 \cdot 77^2$	$25,625,126,169 = 27 \cdot 163 \cdot 2413^2$

Table 11.1

Note that

$$(8J_n + 3)^2 = (64J_n^2 - 24J_n + 9) + 72J_n.$$

Observe that, except for $n = 11$, $64J_n^2 - 24J_n + 9$ contains n as a factor. Also note that, except for $n = 11$, both $8J_n + 3$ and $64J_n^2 - 24J_n + 9$ contain 27 as a factor. The factorizations in the last column are presented as Ramanujan recorded them. Perhaps he failed to observe that $2413 = 19 \cdot 127$.

In the second part of the page, Ramanujan recorded the following remarkable theorem.

Entry 11.17. *For* $q = \exp(-\pi\sqrt{n})$, *define*

$$t := t_n := \sqrt{3}q^{1/18} \frac{f(q^{1/3}) f(q^3)}{f^2(q)}. \tag{11.11}$$

Then

$$t_n = \left(2\sqrt{64J_n^2 - 24J_n + 9} - (16J_n - 3)\right)^{1/6}. \tag{11.12}$$

Entry 11.18. *For the values of n given below, we have the following table of polynomials $p_n(t)$ satisfied by t_n:*

n	$p_n(t)$
11	$t - 1$
35	$t^2 + t - 1$
59	$t^3 + 2t - 1$
83	$t^3 + 2t^2 + 2t - 1$
107	$t^3 - 2t^2 + 4t - 1$

Table 11.2

The simplicity of these polynomials is remarkable, since the corresponding well–known polynomials of the same degrees satisfied by J_n are considerably more complicated, especially in the latter three instances. If n is squarefree, $n \equiv 11 \pmod{24}$, and the class number of the Hilbert class field $K^{(1)}$ is odd, then t_n and J_n satisfy irreducible polynomials of the same degree; for a proof, see a paper by the author and Chan [4].

The form of Entry 11.17 suggests hitherto unknown connections between the j–invariant and Ramanujan's cubic theory of elliptic functions to alternative bases developed in Chapter 33. We defer a proof of Entry 11.17, as a similar result commences the third part of page 393. A proof of the latter claim also depends upon Ramanujan's cubic theory.

Proof of Entry 11.18. Using Entry 11.3 and Table 11.1, we find that

$$t_{11} = (2 \cdot 7 - 13)^{1/6} = 1,$$

as desired.

Second, from Entry 11.8,

$$\begin{aligned} t_{35} = &\left(2\sqrt{64 \cdot 5\left(\frac{\sqrt{5}+1}{2}\right)^8 - 24\sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^4 + 9}\right. \\ &\left.- \left(16\sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^4 - 3\right)\right)^{1/6} \\ = &\left(2\sqrt{7349 + 3276\sqrt{5}} - 117 - 56\sqrt{5}\right)^{1/6}. \end{aligned}$$

Now,

$$7349^2 - 5 \cdot 3276^2 = 589^2.$$

Thus, by the denesting theorem (9.5),

$$\begin{aligned}\sqrt{7349 + 3276\sqrt{5}} &= \sqrt{\frac{7349 + 589}{2}} + \sqrt{\frac{7349 - 589}{2}} \\ &= \sqrt{3969} + \sqrt{3380} = 63 + 26\sqrt{5}.\end{aligned}$$

Hence,

$$t_{35} = \left(2\left(63 + 26\sqrt{5}\right) - 117 - 56\sqrt{5}\right)^{1/6} = (9 - 4\sqrt{5})^{1/6} = \frac{\sqrt{5}-1}{2}.$$

Hence, t_{35} is a root of $t^2 + t - 1$, and the second result is established.

For $n = 59$, recall that $a_{59} = -\gamma_2^3/256$ is a root of (11.9), where $\gamma_2 = \gamma_2\left((3 + \sqrt{-59})/2\right)$. Thus, γ_2 satisfies

$$\gamma_2^3 + 3136\gamma_2^2 + 68608\gamma_2 + 720896 = 0. \tag{11.13}$$

Set $x := t^6 := t_n^6$ and $J := J_n$. From (11.12), we see that x is a root of a quadratic polynomial x^2+bx+c, with $b = 2(16J-3)$. Since $b^2-4c = 16(64J^2-24J+9)$, we easily calculate that $c = -27$. Thus,

$$x^2 + 2(16J - 3)x - 27 = 0. \tag{11.14}$$

This suggests that we make the substitution

$$\gamma_2 = x - 6 - 27x^{-1} \tag{11.15}$$

in (11.13). Upon turning to *Mathematica*, we find that

$$\begin{aligned}&x^6 + 3118x^5 + 31003x^4 + 25355x^3 - 837081x^2 + 2273022x - 19683 \\ &\quad = (x^3 + 13x^2 + 115x - 1)(x^3 + 3105x^2 - 9477x + 19683) = 0.\end{aligned}$$

Numerically checking the roots, we see that x is a root of the first factor above, i.e.,

$$t^{18} + 13t^{12} + 115t^6 - 1 = 0.$$

We use *Mathematica* to factor this polynomial and find that

$$\begin{aligned}t^{18} + 13t^{12} + 115t^6 - 1 &= (t^3 + 2t - 1)(t^3 + 2t + 1) \\ &\quad \times (t^6 - 2t^4 + 2t^3 + 4t^2 - 2t + 1)(t^6 - 2t^4 - 2t^3 + 4t^2 + 2t + 1).\end{aligned}$$

Again, numerically calculating the roots, we find that

$$t^3 + 2t - 1 = 0,$$

as claimed by Ramanujan.

For $n = 83$, recall that $a_{83} = -\gamma_2^3/256$ is a root of (11.10), where $\gamma_2 = \gamma_2\left((3 + \sqrt{-83})/2\right)$. Thus, γ_2 satisfies

$$\gamma_2^3 + 13920\gamma_2^2 + 128000\gamma_2 + 8192000 = 0.$$

As in the last example, set $x = t^6$ and $\gamma_2 = x - 6 - 27x^{-1}$. Using *Mathematica*, we find that

$$\begin{aligned}x^6 &+ 13902x^5 + 39013x^4 + 7174196x^3 + 1053351x^2 + 10134558x - 19683\\ &= (x^3 - 3x^2 + 515x - 1)(x^3 + 13905x^2 + 2187x + 19683) = 0.\end{aligned}$$

A numerical examination of the roots shows that t is a root of the first factor, i.e.,

$$\begin{aligned}&t^{18} - 3t^{12} + 515t^6 - 1\\ &= (t^3 + 2t^2 + 2t - 1)(t^3 - 2t^2 + 2t + 1)(t^6 + 2t^5 + 2t^4 + 6t^3 + 6t^2 - 2t + 1)\\ &\quad \times (t^6 - 2t^5 + 2t^4 - 6t^3 + 6t^2 + 2t + 1) = 0.\end{aligned}$$

Numerically inspecting the roots, we conclude that

$$t^3 + 2t^2 + 2t - 1 = 0,$$

which is what Ramanujan claimed.

For $n = 107$, Greenhill [2, p. 405] proved that $a := a_{107}$ satisfies

$$a - 79 \cdot 80 \cdot 2^{1/3}a^{2/3} - 69 \cdot 800 \cdot 4^{1/3}a^{1/3} - 17 \cdot 16000 = 0.$$

As in the previous two examples, setting $a = -\gamma_2^3/256$, we find that

$$\gamma_2^3 + 50560\gamma_2^2 - 3532800\gamma_2 + 69632000 = 0.$$

As before, set $x = t^6$ and $\gamma_2 = x - 6 - 27x^{-1}$. Using *Mathematica*, we find that

$$\begin{aligned}x^6 &+ 50542x^5 - 4139493x^4 + 89919476x^3\\ &\quad + 111766311x^2 + 36845118x - 19683\\ &= (x^3 - 83x^2 + 1875x - 1)(x^3 + 50625x^2 + 60507x + 19683) = 0.\end{aligned}$$

A numerical examination of the roots indicates that the first factor equals 0, i.e.,

$$\begin{aligned}t^{18} &- 83t^{12} + 1875t^6 - 1\\ &= (t^3 - 2t^2 + 4t - 1)(t^3 + 2t^2 + 4t + 1)(t^6 + 2t^5 + 6t^3 + 14t^2 + 4t + 1)\\ &\quad \times (t^6 - 2t^5 - 6t^3 + 14t^2 - 4t + 1) = 0.\end{aligned}$$

Checking the roots of each polynomial, we conclude that

$$t^3 - 2t^2 + 4t - 1 = 0,$$

which is what Ramanujan asserted.

In the third portion of page 393, Ramanujan first sets

$$t_n = \frac{1}{3}\sqrt{1 + \frac{8}{3}J_n} \tag{11.16}$$

and then gives the following table of values for t_n.

Entry 11.19.

n	t_n
19	1
43	3
67	7
91	$7+2\sqrt{13}$ $(t^2-14t-3=0)$
115	$13+6\sqrt{5}$ $(t^2-26t-11=0)$
163	77

Table 11.3

Proof. The values of t_n for $n = 19, 43$, and 67 follow trivially from Entries 11.4–11.6, respectively.

From Entry 11.11,

$$t_{91} = \tfrac{1}{3}\sqrt{909 + 252\sqrt{13}}.$$

Now

$$909^2 - 13 \cdot 252^2 = 27^2.$$

Thus, by the denesting equality (9.5),

$$t_{91} = \tfrac{1}{3}\left(\sqrt{468} + \sqrt{441}\right) = \tfrac{1}{3}\left(6\sqrt{13} + 21\right) = 7 + 2\sqrt{13},$$

which proves Ramanujan's assertion.

Next, by Entry 11.13 and another application of the denesting theorem (9.5),

$$t_{115} = \tfrac{1}{3}\sqrt{3141 + 1404\sqrt{5}} = \tfrac{1}{3}(39 + 18\sqrt{5}) = 13 + 6\sqrt{5},$$

which establishes the desired result.

Lastly, the value for t_{163} follows very readily from Entry 11.7.

Return to the definition of t_n given in (11.11) and set $H(n) = 27t_n^{-12}$. For the four values 19, 43, 67, and 163 in Entry 11.19, H. H. Chan found that

$$\begin{aligned} H(19) &= 151 + 20\sqrt{57}, \\ H(43) &= 16855 + 1484\sqrt{129}, \\ H(67) &= 515095 + 36332\sqrt{201}, \end{aligned}$$

and

$$H(163) = 7592629975 + 343350596\sqrt{489}.$$

Amazingly, these numbers are fundamental units of their respective real quadratic fields. Essentially, these same observations were also made by H. M. Stark [1].

Entry 11.20. *Let t_n be defined by (11.16). Then, as n tends to ∞,*

$$t_n \sim \frac{e^{\pi\sqrt{n}/6} + 6e^{-\pi\sqrt{n}/6}}{6\sqrt{3}}.$$

Proof. From the definition (1.3), as n tends to ∞,

$$G_n \sim 2^{-1/4} e^{\pi\sqrt{n}/24}(1 + e^{-\pi\sqrt{n}}).$$

Hence, from (11.4),

$$J_n \sim \tfrac{1}{8} 2^{-2} e^{\pi\sqrt{n}/3}(1 + e^{-\pi\sqrt{n}})^8 \left(1 - 4 \cdot 2^6 e^{-\pi\sqrt{n}}(1 + e^{-\pi\sqrt{n}})^{-24}\right),$$

which implies that

$$J_n = \tfrac{1}{32} e^{\pi\sqrt{n}/3} + O\left(e^{-2\pi\sqrt{n}/3}\right).$$

Thus,

$$\begin{aligned} t_n = \tfrac{1}{3}\sqrt{1 + \tfrac{8}{3}J_n} &= \tfrac{1}{3}\sqrt{1 + \tfrac{1}{12}e^{\pi\sqrt{n}/3} + O\left(e^{-2\pi\sqrt{n}/3}\right)} \\ &= \frac{1}{6\sqrt{3}} e^{\pi\sqrt{n}/6} \sqrt{1 + 12e^{-\pi\sqrt{n}/3} + O\left(e^{-\pi\sqrt{n}}\right)} \\ &= \frac{1}{6\sqrt{3}} e^{\pi\sqrt{n}/6} \left(1 + 6e^{-\pi\sqrt{n}/3} + O\left(e^{-2\pi\sqrt{n}/3}\right)\right), \end{aligned}$$

from which Ramanujan's asymptotic formula for t_n follows.

Entry 11.21. *Let $q = \exp(-\pi\sqrt{n})$ and put*

$$R := R_n := 3^{1/4} q^{1/36} \frac{f(q)}{f(q^{1/3})}. \tag{11.17}$$

Then

$$\frac{3\sqrt{3}}{R_n^6} = \sqrt{8J_n + 3} + \sqrt{2\sqrt{64J_n^2 - 24J_n + 9} - 8J_n + 6}. \tag{11.18}$$

We now use Ramanujan's cubic theory from Chapter 33 to prove both Entries 11.17 and 11.21.

Proof of Entry 11.17. We shall prove a stronger version of Entry 11.17 by removing the condition $q = \exp(-\pi\sqrt{n})$, i.e., we interpret (11.12) as a q–identity in the following way. The more general definition of t is clear from (11.11). To extend the definition of J_n, note that, from (1.3), $G_n = 2^{-1/4} q^{-1/24} f(q)/f(-q^2)$, where $q = \exp(-\pi\sqrt{n})$. Removing this stipulation on q, in view of (11.4), it is natural to define J by

$$J = \tfrac{1}{32} q^{-1/3} \frac{f^8(q)}{f^8(-q^2)} - 8q^{2/3} \frac{f^{16}(-q^2)}{f^{16}(q)}. \tag{11.19}$$

To prove (11.12), recall that in proving Entry 11.18 we showed that it suffices to establish (11.14). Replacing q by $-q^3$ in (11.14) and using (11.11) and (11.19), we find that it suffices to prove that

$$3^6 q^2 \frac{f^{12}(-q)f^{12}(-q^9)}{f^{24}(-q^3)} + 2 \cdot 3^3 q \frac{f^6(-q)f^6(-q^9)}{f^{12}(-q^3)} \times \left(\frac{1}{2q} \frac{f^8(-q^3)}{f^8(-q^6)} + 128q^2 \frac{f^{16}(-q^6)}{f^{16}(-q^3)} + 3 \right) - 27 = 0. \tag{11.20}$$

Setting

$$h(q) = \frac{f^{12}(-q^3)}{qf^6(-q)f^6(-q^9)} \tag{11.21}$$

and using (11.5) with q replaced by q^3, we find that (11.20) assumes the more palatable form

$$27h^{-2}(q) + 2h^{-1}(q)\left(\tfrac{1}{2}\gamma_2(3\tau) + 3\right) - 1 = 0,$$

or

$$\gamma_2(3\tau) = h(q) - 6 - 27h^{-1}(q). \tag{11.22}$$

Set

$$s = \frac{f^3(-q)}{qf^3(-q^9)}. \tag{11.23}$$

Then upon cubing Entry 1(iv) of Chapter 20 of Ramanujan's second notebook with q replaced by q^3 (Part III [3, p. 345]), we find that

$$h(q) = s + 9 + \frac{27}{s}. \tag{11.24}$$

Substituting (11.24) into (11.22) and utilizing elementary algebra, we now find that we are required to prove that

$$\gamma_2(3\tau) = \frac{(s+9)(s+3)(s^2+27)}{s(s^2+9s+27)}. \tag{11.25}$$

We now recall some basic facts from Ramanujan's cubic theory. From (1.7) of Chapter 33,

$$q := q_3 := \exp\left(-\frac{2\pi}{\sqrt{3}} \frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-\alpha)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; \alpha)} \right), \qquad 0 < \alpha < 1.$$

Returning to (11.1), if we employ Theorems 4.2 and 4.3 of Chapter 33, we deduce that

$$j(\tau) = 27\frac{(1+8\alpha)^3}{\alpha(1-\alpha)^3}. \tag{11.26}$$

By Lemma 2.9 of Chapter 33, α has the representation

$$\alpha := \alpha(q) = \frac{c^3(q)}{a^3(q)}, \tag{11.27}$$

where $a(q)$ and $c(q)$ are defined by (2.2) and (2.4), respectively, in Chapter 33. From Lemma 5.1 of Chapter 33,

$$c(q) = 3q^{1/3}\frac{f^3(-q^3)}{f(-q)}. \tag{11.28}$$

By combining Entry 1(v) of Chapter 20 of Ramanujan's second notebook (Part III [3, p. 346]) with (2.6) of Chapter 33, we find that

$$a(q^3) = \frac{f^3(-q) + 3qf^3(-q^9)}{f(-q^3)}. \tag{11.29}$$

Hence, by (11.27)–(11.29) and (11.23),

$$\frac{1}{\alpha^{1/3}(q^3)} = \frac{f^3(-q)}{3qf^3(-q^9)} + 1 = \frac{1}{3s} + 1,$$

or, with the argument q^3 suppressed,

$$s = 3\frac{1-\alpha^{1/3}}{\alpha^{1/3}}. \tag{11.30}$$

Upon substituting (11.30) into (11.25) and simplifying, we find that we are required to prove that

$$\gamma_2(3\tau) = 3\frac{(1+2\alpha^{1/3})(1-2\alpha^{1/3}+4\alpha^{2/3})}{\alpha^{1/3}(1-\alpha^{1/3})(1+\alpha^{1/3}+\alpha^{2/3})} = 3\frac{1+8\alpha}{\alpha^{1/3}(1-\alpha)}. \tag{11.31}$$

But, by (11.2) and (11.26), since the argument of α is q^3,

$$\gamma_2(3\tau) = \sqrt[3]{j(3\tau)} = 3\frac{1+8\alpha}{\alpha^{1/3}(1-\alpha)}.$$

Thus, (11.31) has been shown, and so the proof of Entry 11.17 is complete.

Proof of Entry 11.21. By (11.17),

$$x := \frac{3\sqrt{3}}{R^6} = \frac{f^6(q^{1/3})}{q^{1/6}f^6(q)}.$$

From (11.18), we want to show that x is a root of a certain quadratic polynomial $x^2 + bx + c$, where $b = -2\sqrt{8J+3}$ and

$$b^2 - 4c = 4\left(2\sqrt{64J^2 - 24J + 9} - 8J + 6\right).$$

A simple calculation shows that

$$c = 16J - 3 - 2\sqrt{64J^2 - 24J + 9} = -27q^{1/3}\frac{f^6(q^{1/3})f^6(q^3)}{f^{12}(q)},$$

by Entry 11.17. Hence, we want to prove that

$$\frac{f^{12}(q^{1/3})}{q^{1/3}f^{12}(q)} - 2\sqrt{8J+3}\frac{f^6(q^{1/3})}{q^{1/6}f^6(q)} - 27q^{1/3}\frac{f^6(q^{1/3})f^6(q^3)}{f^{12}(q)} = 0. \quad (11.32)$$

Cancelling $f^6(q^{1/3})/(q^{1/6}f^6(q))$ in (11.32), we find that

$$2\sqrt{8J+3} = \frac{f^6(q^{1/3})}{q^{1/6}f^6(q)} - 27\sqrt{q}\frac{f^6(q^3)}{f^6(q)}. \quad (11.33)$$

By (11.19),

$$8J+3 = \tfrac{1}{4}q^{-1/3}\frac{f^8(q)}{f^8(-q^2)} - 64q^{2/3}\frac{f^{16}(-q^2)}{f^{16}(q)} + 3. \quad (11.34)$$

Squaring both sides of (11.33), substituting (11.34) into the left side, replacing q by $-q^3$, and simplifying slightly, we find that

$$\begin{aligned}&\frac{f^8(-q^3)}{qf^8(-q^6)} + 256q^2\frac{f^{16}(-q^6)}{f^{16}(-q^3)} - 12\\ &= \frac{f^{12}(-q)}{qf^{12}(-q^3)} + 3^6q^3\frac{f^{12}(-q^9)}{f^{12}(-q^3)} - 54q\frac{f^6(-q)f^6(-q^9)}{f^{12}(-q^3)}.\end{aligned} \quad (11.35)$$

By using (11.5), (11.21), and (11.23), we can rewrite (11.35) in the form

$$\gamma_2(3\tau) - 12 = \frac{s^2}{h(q)} + \frac{3^6}{s^2h(q)} - \frac{54}{h(q)},$$

or, by (11.24),

$$\begin{aligned}\gamma_2(3\tau) &= \left(s^2 + \frac{3^6}{s^2} - 54\right)\frac{1}{s+9+27/s} + 12\\ &= \frac{(s+9)(s+3)(s^2+27)}{s(s^2+9s+27)},\end{aligned} \quad (11.36)$$

after elementary simplification. However, (11.36) is the same as (11.25), established in the proof of Entry 11.17. This completes the proof of Entry 11.21.

It has been conjectured that Ramanujan had seen Greenhill's book [3] on elliptic functions. For example, see a letter by K. Ananda Rau and the following commentary in the book by R. A. Rankin and the author [1, pp. 289, 290]. On pages 327–329, Greenhill [3] briefly summarizes much of Russell's work on modular equations. This is further evidence that Ramanujan had seen both of Russell's papers [1], [2], and also both of Greenhill's papers [1], [2], since all four of these papers appear in volumes 19 and 21 of the *Proceedings of the London Mathematical Society*.

We have, for convenience, quoted Berwick's paper [1], published in 1923, for some of Ramanujan's results. However, the quoted results were originally discovered much earlier, as Berwick cites Chapter 19 of the third volume of Weber's *Lehrbuch der Algebra* [2]. It is interesting that in 1916 Ramanujan wrote Berwick inquiring about references for singular moduli and modular equations. Ramanujan's letter is evidently not extant, but Berwick's reply has been preserved (Berndt and Rankin [1, pp. 138–141]). Among the several references that Berwick provides are the four aforementioned papers by Greenhill and Russell.

There is evidence that several entries in the third notebook were recorded while Ramanujan was at Cambridge. From the remarks above, we conjecture that the entries on pages 392 and 393, which are the last two pages in the third notebook, were also recorded during Ramanujan's stay in England.

Following a suggestion by A. O. L. Atkin, Birch [1, p. 291] briefly considered the function g defined by

$$g^6 - 27g^{-6} = \gamma_2 + 6.$$

This is exactly the function studied by Ramanujan in Entry 11.17. In particular, see (11.15). Atkin's suggestion was motivated by certain relations between modular forms that would have been unknown to Ramanujan.

For further recent work on invariants, readers might examine papers by B. H. Gross and D. B. Zagier [1], E. Kaltofen and N. Yui [1], N. Yui and D. Zagier [1], and I. Chen and N. Yui [1].

35

Values of Theta–Functions

0. Introduction

For the convenience of the reader we briefly review here some definitions from earlier chapters.

After Ramanujan, define the theta–functions $\varphi(q)$ and $\psi(q)$ by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty} \tag{0.1}$$

and

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{0.2}$$

where the infinite product representations arise from the Jacobi triple product identity (Part III [3, pp. 35–37]). The Dedekind eta–function $\eta(z)$ and Ramanujan's function $f(-q)$ are defined by

$$\eta(z) := q^{1/24}(q;q)_\infty =: q^{1/24} f(-q), \qquad q = e^{2\pi i z}, \ \mathrm{Im}\, z > 0. \tag{0.3}$$

It is well known that (Part III [3, p. 102 (with a misprint corrected)])

$$\varphi^2(q) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} =: \frac{2}{\pi} K(k), \tag{0.4}$$

where ${}_2F_1$ denotes the ordinary or Gaussian hypergeometric function; k, $0 < k < 1$, is the modulus; K is the complete elliptic integral of the first kind; and

$$q = \exp(-\pi K'/K), \tag{0.5}$$

where $K' = K(k')$, and $k' = \sqrt{1-k^2}$ is the complementary modulus. Thus, an evaluation of any one of the functions φ, ${}_2F_1$, or K yields an evaluation of the other two functions. However, such evaluations may not be very explicit. For example, if $K(k)$ is known for a certain value of k, it may be difficult or impossible to determine explicitly K', and so q cannot be explicitly determined. Conversely, it may be possible to evaluate $\varphi(q)$ for a certain value of q, but it may be impossible to determine the corresponding value of k. (Recall that $k = \sqrt{1 - \varphi^4(-q)/\varphi^4(q)}$ (Part III [3, p. 102]).)

In the literature more attention has been devoted to determining ${}_2F_1$ and $K(k)$. In particular, using the Selberg–Chowla formula, I. J. Zucker [1] evaluated K when $K'/K = \sqrt{\lambda}$, where λ is a positive integer such that $1 \leq \lambda \leq 16$, $\lambda \neq 14$. See also papers of G. S. Joyce and Zucker [1], J. M. Borwein and Zucker [1], and J. G. Huard, P. Kaplan, and K. S. Williams [1].

In his second notebook, Ramanujan recorded the values of $\varphi(e^{-\pi})$, $\varphi(e^{-\pi\sqrt{2}})$, $\varphi(e^{-2\pi})$, and $\varphi(e^{-5\pi})$ (Part III [3, pp. 103, 104]); no other values of φ are recorded. The first three values are classical (Whittaker and Watson [1, p. 525]), while the value of $\varphi(e^{-5\pi})$ is new. However, in his first notebook, Ramanujan recorded many values of φ, as well as values of ψ and f.

Ramanujan's modular equations are central to most of our proofs of his evaluations. Recall that in the theory of modular equations the multiplier m is defined by

$$m := \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \alpha)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \beta)} = \frac{\varphi^2(q)}{\varphi^2(q^n)}, \tag{0.6}$$

by (0.4). This fact is the key to our evaluations below.

For some evaluations of φ in Section 2, we need to recall a few basic facts about class invariants from Chapter 34. Define, for $|q| < 1$,

$$\chi(q) = (-q; q^2)_\infty.$$

The *class invariant* G_n is defined by

$$G_n := 2^{-1/4} q^{-1/24} \chi(q), \tag{0.7}$$

where $q = \exp(-\pi\sqrt{n})$. Since $\chi(q) = 2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24}$ (Part III [3, p. 124]), it follows from (0.7) that

$$G_n = \{4\alpha(1-\alpha)\}^{-1/24}. \tag{0.8}$$

If β has degree n over α, it follows from (0.8) that

$$G_{n^2} = \{4\beta(1-\beta)\}^{-1/24}, \tag{0.9}$$

where now $q = \exp(-\pi)$.

This chapter is divided into three parts. In Section 1, elementary values of $\varphi(q)$, $\psi(q)$, $f(-q)$, and $\chi(q)$ are given. These are easy to derive and probably most of these values can be found somewhere in the classical literature. Section 2 is devoted to values of $\varphi(e^{-n\pi})$, for certain positive integers n, and these values are mostly new. Section 3 focuses on values for a certain remarkable product of the functions φ and ψ.

1. Elementary Values

The values given in Entries 1 and 2 are easy consequences of the "catalogue" of evaluations given by Ramanujan in Chapter 17 (Part III [3, pp. 122–124]). Since the more general evaluations are given in the second notebook and not in the first, it appears that Ramanujan first derived the special values and then observed that his arguments held more generally. Since proofs of the more general results have already been given in Part III, we follow the reverse tack.

Entry 1 (p. 248). *Let*

$$a = \frac{\pi^{1/4}}{\Gamma(3/4)}. \tag{1.1}$$

Then

(i) $\varphi(e^{-\pi}) = a,$

(ii) $\varphi(-e^{-\pi}) = a2^{-1/4},$

(iii) $\varphi(e^{-2\pi}) = a2^{-1}(2+\sqrt{2})^{1/2},$

(iv) $\varphi(-e^{-2\pi}) = a2^{-1/8},$

(v) $\varphi(e^{-4\pi}) = a2^{-1}(1+2^{-1/4}),$

(vi) $\varphi(-e^{-4\pi}) = a2^{-1/2}(2^{3/4}+2^{1/4})^{1/4},$

(vii) $\varphi(e^{-\pi/2}) = a2^{-1/4}(\sqrt{2}+1)^{1/2},$

(viii) $\varphi(-e^{-\pi/2}) = a2^{-1/4}(\sqrt{2}-1)^{1/2},$

(ix) $\varphi(e^{-\pi/4}) = a(1+2^{-1/4}),$

(x) $\varphi(-e^{-\pi/4}) = a(1-2^{-1/4}),$

(xi) $\psi(e^{-\pi}) = a2^{-5/8}e^{\pi/8},$

(xii) $\psi(e^{-2\pi}) = a2^{-5/4}e^{\pi/4},$

(xiii) $\psi(e^{-4\pi}) = a2^{-2}(2-\sqrt{2})^{1/2}e^{\pi/2},$

(xiv) $\psi(e^{-8\pi}) = a2^{-2}(1-2^{-1/4})e^{\pi},$

(xv) $\psi(e^{-\pi/2}) = a2^{-7/16}(1+\sqrt{2})^{1/4}e^{\pi/16},$

(xvi) $\psi(e^{-\pi/4}) = a2^{-11/32}(2^{1/4}+1)^{1/2}(\sqrt{2}+1)^{1/8}e^{\pi/32}.$

The evaluations in (xv) and (xvi) give an extra factor of $2^{1/16}$ in the denominators that Ramanujan does not have. In our proofs below, all references are to Entries 10 and 11 of Chapter 17 (Part III [3, pp. 122, 123]). Note that the evaluations arise from taking $x = \frac{1}{2}$ in these formulas, and so $q = \exp(-\pi)$.

Proof. Part (i) is the same as Example (i) in Section 6 of Chapter 17 (Part III [3, p. 103]).

Part (ii) follows from Entry 10(ii).

Part (iii) is the same as Example (iii) in Section 6 of Chapter 17 (Part III [3, p. 104]).

Part (iv) follows from Entry 10(iii).

Use Entry 10(v) to prove part (v).

From (10.1) of Chapter 17 (Part III [3, p. 123]),

$$\varphi(-e^{-4\pi}) = \sqrt{\varphi(e^{-2\pi})\varphi(-e^{-2\pi})}.$$

Now use parts (iii) and (iv) to complete the proof of (vi).

Parts (vii)–(x) follow from Entries 10(vi)–(ix), respectively.

Parts (xi)–(xvi) follow from Entries 11(i), (iii), (iv), (v), (vi), and (viii), respectively.

Entry 2 (p. 250). *Let a be given by (1.1). Then*

(i) $$f(-e^{-\pi}) = a2^{-3/8}e^{\pi/24},$$

(ii) $$f(-e^{-2\pi}) = a2^{-1/2}e^{\pi/12},$$

(iii) $$f(-e^{-4\pi}) = a2^{-7/8}e^{\pi/6},$$

(iv) $$f(-e^{-8\pi}) = a2^{-21/16}(\sqrt{2}-1)^{1/4}e^{\pi/3},$$

(v) $$\chi(-e^{-\pi}) = 2^{1/8}e^{-\pi/24},$$

(vi) $$\chi(-e^{-2\pi}) = 2^{3/8}e^{-\pi/12},$$

(vii) $$\chi(-e^{-4\pi}) = 2^{5/16}(2+\sqrt{2})^{1/4}e^{-\pi/6},$$

(viii) $$\chi(e^{-\pi}) = 2^{1/4}e^{-\pi/24},$$

(ix) $$\chi(e^{-2\pi}) = 2^{1/16}(\sqrt{2}+1)^{1/4}e^{-\pi/12}.$$

Proof. We shall make several references to Entry 12 of Chapter 17 (Part III [3, p. 124]).

Parts (i)–(iii) follow readily from Entries 12(ii)–(iv), respectively, and Entry 1(i) above.

From Part III [3, p. 124, eq. (12.3)],

$$f(-e^{-8\pi}) = \{\varphi(-e^{-4\pi})\psi^2(e^{-4\pi})\}^{1/3}.$$

If we now employ Entries 1(vi), (xiii) above, we readily deduce (iv).

Parts (v) and (vi) are easy consequences of Entries 12(vi), (vii), respectively.

From Part III [3, p. 124, eq. (12.4)],

$$\chi(-e^{-4\pi}) = \frac{\varphi(-e^{-4\pi})}{f(-e^{-4\pi})}.$$

If we now employ Entry 1(vi) and Entry 2(iii), we easily complete the proof of (vii).

Part (viii) follows readily from Entry 12(v).

From Entry 24(i) of Chapter 16 (Part III [3, p. 39]),

$$\chi(e^{-2\pi}) = \chi(-e^{-2\pi})\sqrt{\frac{\varphi(e^{-2\pi})}{\varphi(-e^{-2\pi})}}.$$

Substituting the values from Entries 1(iii), (iv) and Entry 2(vi), we readily deduce (ix).

By repeated applications of Entries 10–12 of Chapter 17 along with identities from Entry 24 of Chapter 16, it is clear that we can explicitly determine $F(\pm e^{-2^n\pi})$, where n is any integer and F is any of the functions φ, ψ, f, or χ.

2. Nonelementary Values of $\varphi(e^{-n\pi})$

Ramanujan recorded the values of $\varphi(e^{-n\pi})$ when $n = 3, 5, 9, 45$, and 7 in his first notebook. We give here the proofs that appeared in our paper with Chan [2]. These proofs are very natural and employ some of Ramanujan's modular equations and class invariants. As corollaries, we are able to obtain three new explicit determinations of ${}_2F_1$. We also explicitly determine $a(e^{-2\pi})$, where $a(q)$ is the Borweins' cubic theta–function studied in Chapter 33; no other value of $a(q)$ had been previously determined. Other new values for $\varphi(e^{-n\pi})$ are also obtained in our paper with Chan [2]. Different proofs for the cases $n = 3, 5$ have been given elsewhere in the literature; see Zucker's paper [1] for $n = 3$, and the paper of Joyce and Zucker [1] and our book [3, p. 210] for $n = 5$.

Ramanujan recorded most of his values for $\varphi(e^{-n\pi})$ in terms of $\varphi(e^{-\pi})$, but in view of (1.1), $\varphi(e^{-n\pi})$ is therefore determined explicitly.

Entry 3 (p. 285).

$$\frac{\varphi(e^{-5\pi})}{\varphi(e^{-\pi})} = \frac{1}{\sqrt{5\sqrt{5}-10}}.$$

Entry 4 (p. 284).

$$\frac{\varphi(e^{-3\pi})}{\varphi(e^{-\pi})} = \frac{1}{\sqrt[4]{6\sqrt{3}-9}}.$$

Corollary 1. *If*

$$p = \frac{\sqrt{6\sqrt{3}-9}-1}{2},$$

then

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; p^3\frac{2+p}{1+2p}\right) = \frac{\sqrt{\pi}}{\sqrt{6\sqrt{3}-9}\,\Gamma^2(\frac{3}{4})}.$$

Corollary 2.

$$ {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{3\sqrt{3}-5}{4}\right) = \frac{\sqrt{\pi}}{(12)^{1/8}\sqrt{\sqrt{3}-1}\,\Gamma^2(\frac{3}{4})}. $$

Corollary 3. *Let $a(q)$ be defined by (2.2) in Chapter 33. Then*

$$ \frac{a(e^{-2\pi})}{\varphi^2(e^{-\pi})} = \frac{1}{(12)^{1/8}\sqrt{\sqrt{3}-1}}. $$

Entry 5 (p. 287).

$$ \frac{\varphi(e^{-9\pi})}{\varphi(e^{-\pi})} = \frac{1+\sqrt[3]{2(\sqrt{3}+1)}}{3}. $$

Corollary 4. *If*

$$ t = \frac{\left(2(\sqrt{3}-1)\right)^{1/3} - 1}{\left(2(\sqrt{3}-1)\right)^{1/3} + 2}, $$

then

$$ {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; 64t^9\frac{1-t^3}{1+8t^3}\right) = \frac{(1+(2(\sqrt{3}+1))^{1/3})^2\sqrt{\pi}}{9\Gamma^2(\frac{3}{4})}. $$

Entry 6 (p. 312).

$$ \frac{\varphi(e^{-45\pi})}{\varphi(e^{-\pi})} = \frac{3+\sqrt{5}+\left(\sqrt{3}+\sqrt{5}+(60)^{1/4}\right)\sqrt[3]{2+\sqrt{3}}}{3\sqrt{10+10\sqrt{5}}}. $$

Entry 7 (p. 297).

$$ \frac{\varphi^2(e^{-7\pi})}{\varphi^2(e^{-\pi})} = \frac{\sqrt{13+\sqrt{7}}+\sqrt{7+3\sqrt{7}}}{14}(28)^{1/8}. $$

We now provide proofs for the entries and corollaries given above.

Proof of Entry 3. If β has degree 5 and m is the multiplier for degree 5, then from Chapter 19 of Ramanujan's second notebook (Part III [3, Entry 13(xii), pp. 281, 282])

$$ m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} \tag{3.1} $$

and

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}. \tag{3.2}$$

Set $\alpha = \frac{1}{2}$, so that, by (0.5), $q = e^{-\pi}$. From (3.1) we find that

$$(2\beta)^{1/4} + (2(1-\beta))^{1/4} = m + (4\beta(1-\beta))^{1/4}, \tag{3.3}$$

and, from (3.2) and (3.3), we find that

$$\begin{aligned}\frac{5}{m} &= \frac{(2(1-\beta))^{1/4} + (2\beta)^{1/4} - 1}{(4\beta(1-\beta))^{1/4}} \\ &= \frac{m + (4\beta(1-\beta))^{1/4} - 1}{(4\beta(1-\beta))^{1/4}} = \frac{m + G_{25}^{-6} - 1}{G_{25}^{-6}},\end{aligned} \tag{3.4}$$

by (0.9), with $n = 5$. From Lemma 7.1 of Chapter 32, or the table in Section 2 of Chapter 34,

$$G := G_{25} = \frac{1+\sqrt{5}}{2}. \tag{3.5}$$

Hence, from (3.4), since $G^3 = 2 + \sqrt{5}$,

$$G^3 m - \frac{5}{G^3 m} = G^3 - G^{-3} = 4,$$

from which we deduce that $G^3 m = 5$, or $m = 5(\sqrt{5} - 2)$. Entry 3 now follows from (0.6).

First Proof of Entry 4. Our first proof is similar to that for Entry 3.

From Entry 5(vii) of Chapter 19 of Ramanujan's second notebook (Part III [3, p. 230]),

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} \tag{4.1}$$

and

$$\frac{9}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}, \tag{4.2}$$

where β has degree 3. Setting $\alpha = \frac{1}{2}$ in (4.1) and (4.2) and eliminating the terms $(2\beta)^{1/2} + (2(1-\beta))^{1/2}$, we deduce that

$$m^2 - \frac{9}{m^2} G^{-12} = 1 - G^{-12}, \tag{4.3}$$

by (0.9) with $n = 3$, where, from the table in Section 2 of Chapter 34,

$$G := G_9 = \left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{1/3}. \tag{4.4}$$

Rewriting (4.3) and employing (4.4), we arrive at

$$(G^3m)^2 - \frac{9}{(G^3m)^2} = G^6 - G^{-6} = 2\sqrt{3}.$$

Hence, $(G^3m)^2 = 3\sqrt{3}$, or, by (4.4), $m^2 = 6\sqrt{3} - 9$. Appealing to (0.6), we complete the first proof of Entry 4.

Second Proof of Entry 4. From the Borwein brothers' book [1, p. 145],

$$9\frac{\varphi^4(e^{-3n\pi})}{\varphi^4(e^{-n\pi})} = 1 + 2\sqrt{2}\frac{G_{9n^2}^3}{G_{n^2}^9}, \tag{4.5}$$

where n is any positive rational number. We provide here a proof somewhat different from that in the Borweins' book [1].

From Part III [3, p. 347],

$$\frac{\varphi^4(q)}{\varphi^4(q^3)} = 1 + 8q\frac{\chi^3(q)}{\chi^9(q^3)}. \tag{4.6}$$

Recall the transformation formula for φ (Part III [3, p. 43]). If $a, b > 0$ with $ab = \pi$, then

$$\sqrt{a}\varphi(e^{-a^2}) = \sqrt{b}\varphi(e^{-b^2}). \tag{4.7}$$

Using (4.7) twice, we easily find that, for $\text{Re}(z) > 0$,

$$\frac{\varphi^4(e^{-\pi z})}{\varphi^4(e^{-3\pi z})} = 9\frac{\varphi^4(e^{-\pi/z})}{\varphi^4(e^{-\pi/(3z)})}. \tag{4.8}$$

Recall also the transformation formula for χ (Part III [3, p. 43]). If $a, b > 0$ with $ab = \pi^2$, then

$$e^{a/24}\chi(e^{-a}) = e^{b/24}\chi(e^{-b}). \tag{4.9}$$

Hence, from (4.9), we deduce that

$$e^{-\pi z}\frac{\chi^3(e^{-\pi z})}{\chi^9(e^{-3\pi z})} = \frac{e^{3\pi z/24}\chi^3(e^{-\pi z})}{e^{27\pi z/24}\chi^9(e^{-3\pi z})} = \frac{(e^{\pi/(24z)}\chi(e^{-\pi/z}))^3}{(e^{\pi/(24(3z))}\chi(e^{-\pi/(3z)}))^9} = \frac{\chi^3(e^{-\pi/z})}{\chi^9(e^{-\pi/(3z)})}. \tag{4.10}$$

Utilizing (4.8) and (4.10) in (4.6), we deduce that

$$9\frac{\varphi^4(q_1^3)}{\varphi^4(q_1)} = 1 + 8\frac{\chi^3(q_1^3)}{\chi^9(q_1)}, \tag{4.11}$$

where $q_1 = e^{-\pi/(3z)}$. If we now set $q_1 = e^{-\pi n}$ in (4.11), we deduce (4.5).

Setting $n = 1$ in (4.5), employing (4.4), and noting that $G_1 = 1$, we find that

$$9\frac{\varphi^4(e^{-3\pi})}{\varphi^4(e^{-\pi})} = 1 + 2\sqrt{2}\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right) = 3 + 2\sqrt{3}.$$

The desired formula now follows by elementary algebra.

Third Proof of Entry 4. From Entry 6a of Chapter 19 in Ramanujan's second notebook (Part III [3, p. 238]),

$$\frac{\varphi^2(q)}{\varphi^2(q^3)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; p\left(\frac{2+p}{1+2p}\right)^3\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; p^3\left(\frac{2+p}{1+2p}\right)\right)} = 1 + 2p, \tag{4.12}$$

where $0 < p < 1$. Set $\alpha = \frac{1}{2}$, so that, by (0.6) and (4.12),

$$p\left(\frac{2+p}{1+2p}\right)^3 = \frac{1}{2}.$$

Hence,

$$p^4 + 2p^3 + 6p^2 + 5p - \tfrac{1}{2} = 0.$$

To solve this quartic equation, we use Ferrari's method, as found, for example, in H. S. Hall and S. R. Knight's text [1, pp. 483, 484]. Thus, adding $(ap + b)^2$ to both sides above, writing the left side as $(p^2 + p + k)^2$, and equating coefficients of like powers of p, we are led to the equations

$$\begin{aligned} 1 + 2k &= 6 + a^2, \\ k &= \tfrac{5}{2} + ab, \\ k^2 &= -\tfrac{1}{2} + b^2. \end{aligned}$$

Hence,

$$(k - \tfrac{5}{2})^2 = a^2b^2 = (2k - 5)(k^2 + \tfrac{1}{2}).$$

Obviously, $k = \frac{5}{2}$ is the real root. It follows that $a = 0$ and $b = \frac{3}{2}\sqrt{3}$. Hence,

$$(p^2 + p + \tfrac{5}{2})^2 = \tfrac{27}{4}.$$

Since $p > 0$,

$$p = \frac{-1 + \sqrt{6\sqrt{3} - 9}}{2}. \tag{4.13}$$

Using (4.13) in (4.12), we complete the proof.

Proof of Corollary 1. From (0.4) and (1.1),

$${}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{\Gamma^2\left(\frac{3}{4}\right)}. \tag{4.14}$$

Hence, the desired result follows from (4.12) and (4.13).

Proof of Corollary 2. By Theorem 5.6 in Chapter 33, if

$$\alpha = p^3\frac{2+p}{1+2p} \qquad \text{and} \qquad \beta = \frac{27p^2(1+p)^2}{4(1+p+p^2)^3}, \tag{4.15}$$

where $0 \leq p < 1$, then

$$(1+p+p^2)\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\alpha) = \sqrt{1+2p}\,{}_2F_1(\tfrac{1}{3},\tfrac{2}{3};1;\beta). \tag{4.16}$$

If p is given by (4.13), then, by (4.15) and a straightforward calculation, we find that

$$\beta = \frac{3\sqrt{3}-5}{4}. \tag{4.17}$$

Another elementary calculation shows that

$$\frac{1+p+p^2}{\sqrt{6\sqrt{3}-9}\sqrt{1+2p}} = \frac{1}{(12)^{1/8}\sqrt{\sqrt{3}-1}}. \tag{4.18}$$

Using Corollary 1, (4.17), and (4.18) in (4.16), we readily complete the proof.

Proof of Corollary 3. From Theorem 2.12 of Chapter 33, if

$$q = \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-x\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x\right)}\right), \qquad |x|<1, \tag{4.19}$$

then

$$a(q) = {}_2F_1\left(\tfrac{1}{3},\tfrac{2}{3};1;x\right). \tag{4.20}$$

We also need Ramanujan's cubic transformation, Corollary 2.4 of Chapter 33,

$${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\left(\frac{1-x}{1+2x}\right)^3\right) = (1+2x)\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x^3\right). \tag{4.21}$$

Now let $x = (\sqrt{3}-1)/2$. Then a simple calculation shows that $(1-x)/(1+2x) = (\sqrt{3}-1)/2$. Using these values in (4.21), we find that

$$\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\left(\frac{\sqrt{3}-1}{2}\right)^3\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\left(\frac{\sqrt{3}-1}{2}\right)^3\right)} = \sqrt{3}. \tag{4.22}$$

Substituting (4.22) into (4.19), we see that $q = e^{-2\pi}$. Using this value of q in (4.20) and noting that $x^3 = (3\sqrt{3}-5)/4$, we then find that

$$a(e^{-2\pi}) = {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{3\sqrt{3}-5}{4}\right) = \frac{\varphi^2(e^{-\pi})}{(12)^{1/8}\sqrt{\sqrt{3}-1}},$$

by Corollary 2 and (1.1). This completes the proof.

Another short proof of Corollary 3 can be effected by employing a formula of Ramanujan for $a(q^2)$ (Part III [3, p. 460]) and Entry 4.

First Proof of Entry 5. If β and γ have degrees 3 and 9, respectively, over α, then from Entries 3(vi), (x) of Chapter 20 in Ramanujan's second notebook (Part III [3, p. 352])

$$(\alpha(1-\gamma))^{1/8} + (\gamma(1-\alpha))^{1/8} = 2^{1/3}(\beta(1-\beta))^{1/24} \tag{5.1}$$

and

$$\left(\frac{\gamma}{\alpha}\right)^{1/8} + \left(\frac{1-\gamma}{1-\alpha}\right)^{1/8} - \left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8} = \sqrt{mm'}, \tag{5.2}$$

respectively, where m is the multiplier connecting α and β and m' is the multiplier relating β and γ. Setting $\alpha = \frac{1}{2}$ in (5.1) and (5.2), we find that, respectively,

$$(2(1-\gamma))^{1/8} + (2\gamma)^{1/8} = \sqrt{2}G_9^{-1} \tag{5.3}$$

and

$$(2\gamma)^{1/8} + (2(1-\gamma))^{1/8} - G_{81}^{-3} = \frac{\varphi(e^{-\pi})}{\varphi(e^{-9\pi})}. \tag{5.4}$$

From the table in Section 2 of Chapter 34,

$$G_{81} = \left(\frac{\sqrt[3]{2(\sqrt{3}+1)}+1}{\sqrt[3]{2(\sqrt{3}-1)}-1}\right)^{1/3}, \tag{5.5}$$

which was first proved in print by Watson [12]. Thus, from (5.3), (5.4), (4.4), and (5.5), we conclude that

$$\begin{aligned}
\frac{\varphi(e^{-\pi})}{\varphi(e^{-9\pi})} &= \sqrt{2}\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{-1/3} - \frac{\sqrt[3]{2(\sqrt{3}-1)}-1}{\sqrt[3]{2(\sqrt{3}+1)}+1} \\
&= \frac{2}{2^{1/3}(\sqrt{3}+1)^{1/3}} - \frac{2-(2(\sqrt{3}+1))^{1/3}}{(2(\sqrt{3}+1))^{1/3}\left(\sqrt[3]{2(\sqrt{3}+1)}+1\right)} \\
&= \frac{3}{\sqrt[3]{2(\sqrt{3}+1)}+1},
\end{aligned} \tag{5.6}$$

and so the proof is complete.

Another proof can be constructed by combining (5.2) and the "reciprocal" modular equation (Part III [3, p. 352, Entry 3(xi)])

$$\left(\frac{\alpha}{\gamma}\right)^{1/8} + \left(\frac{1-\alpha}{1-\gamma}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/8} = \frac{3}{\sqrt{mm'}},$$

when $\alpha = \frac{1}{2}$. However, the resulting radicals are more difficult to simplify.

Second Proof of Entry 5. From the Borweins' book [1, p. 145], for any positive integer n,

$$3\frac{\varphi(e^{-9n\pi})}{\varphi(e^{-n\pi})} = 1 + \sqrt{2}\frac{G_{9n^2}}{G_{n^2}^3}. \tag{5.7}$$

The proof of (5.7) is very similar to that for (4.5). We begin with Ramanujan's identity (Part III [3, p. 345, Entry 1(ii)])

$$\frac{\varphi(q^{1/3})}{\varphi(q^3)} = 1 + 2q^{1/3}\frac{\chi(q)}{\chi^3(q^3)} \tag{5.8}$$

and set $q = e^{-\pi z}$. After applying the transformation formulas (4.7) and (4.9) to (5.8), we obtain the identity

$$3\frac{\varphi(q_1^3)}{\varphi(q_1^{1/3})} = 1 + 2\frac{\chi(q_1)}{\chi^3(q_1^{1/3})}, \tag{5.9}$$

where $q_1 = e^{-\pi/z}$. If we now set $q_1 = e^{-3n\pi}$ in (5.9), we easily deduce (5.7).

Setting $n = 1$ in (5.7), we deduce that

$$3\frac{\varphi(e^{-9\pi})}{\varphi(e^{-\pi})} = 1 + \sqrt{2}G_9 = 1 + (2(1+\sqrt{3}))^{1/3},$$

by (4.4). This completes the second proof.

Third Proof of Entry 5. From Part III [3, p. 354, eqs. (3.10), (3.11)],

$$\frac{\varphi^4(q)}{\varphi^4(q^9)} = m^2m'^2 = \frac{(1+2t)^4}{1+8t^3}(1+8t^3) = (1+2t)^4. \tag{5.10}$$

Setting $q = e^{-\pi}$ and comparing (5.10) with Entry 5, we see that it remains to show that

$$1 + 2t = \frac{3}{1 + (2(\sqrt{3}+1))^{1/3}}. \tag{5.11}$$

But from (5.10) and Entry 4, we know that

$$m^2 = \frac{(1+2t)^4}{1+8t^3} = 6\sqrt{3} - 9.$$

After some elementary algebra, we find that

$$t^4 + \left(\tfrac{3}{2} + \tfrac{1}{2}(1-\sqrt{3})^3\right)t^3 + \tfrac{3}{2}t^2 + \tfrac{1}{2}t + \tfrac{1}{16}(1-\sqrt{3})^3 = 0.$$

It is easily checked that $t = -\frac{1}{2}$ is a root, and so upon dividing by $t + \frac{1}{2}$, we deduce that

$$t^3 + (6 - 3\sqrt{3})t^2 - \tfrac{3}{2}(1-\sqrt{3})t + \tfrac{1}{8}(1-\sqrt{3})^3 = 0.$$

This equation has two complex roots and a real root given by

$$t = \frac{(2(\sqrt{3}-1))^{1/3} - 1}{(2(\sqrt{3}+1))^{1/3} + 2}, \tag{5.12}$$

a fact easily verified via *Mathematica*. By elementary algebra, similar to that used in (5.6), it may be verified that (5.11) and (5.12) are equivalent, and so the third proof is complete.

Proof of Corollary 4. From Entry 41(iii) and (41.3) in Chapter 25 of Part IV [4, pp. 193, 194],

$$\begin{aligned} {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; 64t^9\frac{1-t^3}{1+8t^3}\right) &= \frac{1}{(1+2t)^2}\, {}_2F_1\left(\tfrac{1}{4}, \tfrac{1}{4}; 1; 4\alpha(1-\alpha)\right) \\ &= \frac{1}{(1+2t)^2}\, {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \alpha\right). \end{aligned}$$

Set $\alpha = \frac{1}{2}$ and use (4.14), (5.11), and (5.12) to complete the proof.

Proof of Entry 6. Setting $n = 5$ in (5.7), we find that

$$3\frac{\varphi(e^{-45\pi})}{\varphi(e^{-5\pi})} = 1 + \sqrt{2}\frac{G_{225}}{G_{25}^3}. \tag{6.1}$$

Now from the table in Section 2 of Chapter 34,

$$G_{225} = \left(\frac{1+\sqrt{5}}{4}\right)(2+\sqrt{3})^{1/3}\left(\sqrt{4+\sqrt{15}} + (15)^{1/4}\right), \tag{6.2}$$

which was first proved in print by Watson [7]. Hence, by (6.1), (6.2), (3.5), and Entry 3,

$$\begin{aligned} \frac{\varphi(e^{-45\pi})}{\varphi(e^{-\pi})} &= \frac{\varphi(e^{-45\pi})}{\varphi(e^{-5\pi})}\frac{\varphi(e^{-5\pi})}{\varphi(e^{-\pi})} = \frac{1}{3\sqrt{5\sqrt{5}-10}}\left(\sqrt{2}\frac{G_{225}}{G_{25}^3} + 1\right) \\ &= \frac{1}{3\sqrt{5\sqrt{5}-10}}\left(\sqrt{2}\left(\frac{1+\sqrt{5}}{4}\right)\left(\frac{\sqrt{5}-1}{2}\right)^3\right. \\ &\quad \left.\times\, (2+\sqrt{3})^{1/3}\left(\sqrt{4+\sqrt{15}} + (15)^{1/4}\right) + 1\right) \\ &= \frac{1}{3(3+\sqrt{5})\sqrt{5\sqrt{5}-10}} \\ &\quad \times \left(\sqrt{2}(2+\sqrt{3})^{1/3}\left(\sqrt{4+\sqrt{15}} + (15)^{1/4}\right) + 3 + \sqrt{5}\right). \end{aligned} \tag{6.3}$$

By (9.5) of Chapter 34,

$$\sqrt{4+\sqrt{15}} = \sqrt{\frac{5}{2}} + \sqrt{\frac{3}{2}}.$$

Using this in (6.3), we find that

$$\frac{\varphi(e^{-45\pi})}{\varphi(e^{-\pi})} = \frac{1}{3\sqrt{10\sqrt{5}+10}}\left((2+\sqrt{3})^{1/3}\left(\sqrt{5}+\sqrt{3}+(60)^{1/4}\right) + 3 + \sqrt{5}\right),$$

and so the proof of Entry 6 is complete.

The following proof of Entry 7 due to H. H. Chan supplants the more lengthy proof given in the author's paper with Chan [2].

Proof of Entry 7. We employ the two modular equations of degree 7,

$$\left(\frac{1}{2}\left(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}\right)\right)^{1/2}=1-\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \quad (7.1)$$

and

$$\frac{49}{m^2}=\left(\frac{\alpha}{\beta}\right)^{1/2}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/2}-\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}-8\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/3}, \quad (7.2)$$

recorded as Entries 19(i), (v), respectively, in Ramanujan's second notebook (Part III [3, p. 314]). Setting $\alpha=\frac{1}{2}$ in (7.1) and using (0.9), we find that

$$\sqrt{\beta}+\sqrt{1-\beta}=\sqrt{2}\left(2\left(1-\frac{1}{\sqrt{2}G_{49}^{3}}\right)^{2}-1\right). \quad (7.3)$$

Next, set $\alpha=\frac{1}{2}$ in (7.2), use (0.9) again, and substitute (7.3) into (7.2). Accordingly,

$$\frac{49}{m^2}=2G_{49}^{12}\left(2\left(1-\frac{1}{\sqrt{2}G_{49}^{3}}\right)^{2}-1\right)-G_{49}^{12}-8G_{49}^{8}. \quad (7.4)$$

Since, by (9.5) of Chapter 34,

$$\sqrt{4+\sqrt{7}}=\sqrt{7/2}+\sqrt{1/2},$$

from the table in Section 2 of Chapter 34,

$$G_{49}^{\pm 1}=\frac{\sqrt{14}}{4}+\frac{\sqrt{2}}{4}\pm\frac{7^{1/4}}{2}. \quad (7.5)$$

Substituting (7.5) into (7.4) and turning to *Mathematica*, we deduce that

$$\frac{49}{m^2}=7+5\sqrt{2}\cdot 7^{1/4}+4\sqrt{7}+\sqrt{2}\cdot 7^{3/4}, \quad (7.6)$$

or, upon taking the square root of each side,

$$\frac{1}{m}=\frac{(28)^{1/8}}{14}\left(2\sqrt{2}\cdot 7^{3/4}+20+8\sqrt{2}\cdot 7^{1/4}+4\sqrt{7}\right)^{1/2}. \quad (7.7)$$

Since

$$2\sqrt{2}\cdot 7^{3/4}+8\sqrt{2}\cdot 7^{1/4}=2\sqrt{(13+\sqrt{7})(7+3\sqrt{7})}$$

and

$$20+4\sqrt{7}=(13+\sqrt{7})+(7+3\sqrt{7}),$$

we find that (7.7) can be put in the form given in Entry 7.

Values for $\varphi(e^{-13\pi})$, $\varphi(e^{-27\pi})$, and $\varphi(e^{-63\pi})$ can be found in the paper by the author and Chan [2].

It is easily noticed that all of the values for $\varphi(e^{-n\pi})/\varphi(e^{-\pi})$ that we have calculated are algebraic. Indeed, this holds in general, and in a paper with Chan and Zhang [4], a stronger theorem is proved: If n is an odd positive integer, then $\sqrt{2n}\varphi(e^{-n\pi})/\varphi(e^{-\pi})$ is an algebraic integer dividing $2\sqrt{n}$, while if n is an even positive integer, then $2\sqrt{n}\varphi(e^{-n\pi})/\varphi(e^{-\pi})$ is an algebraic integer dividing $4\sqrt{n}$.

3. A Remarkable Product of Theta-Functions

On page 338 of his first notebook Ramanujan defines

$$a_{m,n} := ne^{-(\pi/4)(n-1)\sqrt{m/n}}\frac{\psi^2(e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{m/n}})\varphi^2(-e^{-2\pi\sqrt{m/n}})}, \tag{8.1}$$

where evidently m and n are positive integers. He then, on pages 338 and 339, offers a list of 18 particular values, which we present in the following table.

Entry 8 (pp. 338, 339).

m,n	$a_{m,n}$	m,n	$a_{m,n}$
3,3	$\frac{1}{\sqrt{3}}$	3, 13	$\left(\sqrt{\frac{5+\sqrt{13}}{8}}-\sqrt{\frac{\sqrt{13}-3}{8}}\right)^8$
3,9	$\frac{1}{(2^{1/3}+1)^2}$	3, 23	$\left(\sqrt{\frac{7+4\sqrt{3}}{2}}-\sqrt{\frac{5+4\sqrt{3}}{2}}\right)^2$
3,15	$\frac{2-\sqrt{3}}{3}$	3, 71	$\left(\sqrt{\frac{175+100\sqrt{3}}{2}}-\sqrt{\frac{173+100\sqrt{3}}{2}}\right)^2$
3,5	$\frac{3-\sqrt{5}}{2}$	5, 9	$(2-\sqrt{3})^2$
3,7	$2-\sqrt{3}$	5, 11	$\left(\sqrt{\frac{7+\sqrt{5}}{8}}-\sqrt{\frac{\sqrt{5}-1}{8}}\right)^8$
3,11	$2\sqrt{3}-\sqrt{11}$	5, 13	$\left(\sqrt{\frac{9+\sqrt{65}}{2}}-\sqrt{\frac{7+\sqrt{65}}{2}}\right)^2$

3,19	$2\sqrt{19}-5\sqrt{3}$	5, 17	$(\sqrt{17}-4)^2$
3,31	$(2-\sqrt{3})^3$	5, 29	$\left(\sqrt{49+4\sqrt{145}}-\sqrt{48+4\sqrt{145}}\right)^2$
3,59	$102\sqrt{3}-23\sqrt{59}$	7, 9	$\left(\sqrt{\frac{5+\sqrt{21}}{8}}-\sqrt{\frac{\sqrt{21}-5}{8}}\right)^8$

Our goal in this section is to establish each of these 18 values, and the tools we shall use were entirely known to Ramanujan. The methods we develop can be easily applied to determine further values of $a_{m,n}$, and the reader is referred to our paper with Chan and Zhang [4] for these. Upon examining the table, we see that when $(m,n)=1$, each value is a unit in some algebraic number field. In the paper mentioned above [4] we prove that $a_{m,n}$ is a unit for large classes of pairs m, n. We also establish in [4] some general formulas for evaluating $a_{m,n}$, but we use ideal class theory, with which Ramanujan was likely unfamiliar.

Ramanujan's class invariants are at the heart of our evaluations. Replacing n by m/n in (0.8) and (0.9), we find that if $q=\exp(-\pi\sqrt{m/n})$ and β has degree n over α, then

$$G_{m/n}=\{4\alpha(1-\alpha)\}^{-1/24} \qquad \text{and} \qquad G_{mn}=\{4\beta(1-\beta)\}^{-1/24}. \tag{8.2}$$

Next, we derive some formulas for $a_{m,n}$ that will be useful in our calculations. Set $q=\exp(-\pi\sqrt{m/n})$. Then, by (0.1), (0.2), and (8.1),

$$a_{m,n}=nq^{(n-1)/4}\frac{(q^{2n};q^{2n})_\infty^2(q^{2n};q^{2n})_\infty^2(q;q^2)_\infty^2(-q^2;q^2)_\infty^2}{(q^n;q^{2n})_\infty^2(-q^{2n};q^{2n})_\infty^2(q^2;q^2)_\infty^2(q^2;q^2)_\infty^2}. \tag{8.3}$$

Since $(-q;-q)_\infty=(q^2;q^2)_\infty(-q;q^2)_\infty$,

$$\begin{aligned}\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty(-q^2;q^2)_\infty}&=\frac{(-q;-q)_\infty^2}{(q;q^2)_\infty(-q^2;q^2)_\infty(-q;q^2)_\infty^2}\\&=\frac{(-q;-q)_\infty^2}{(q;-q)_\infty(-q;q^2)_\infty^2}.\end{aligned} \tag{8.4}$$

Using Euler's identity,

$$1=(q;-q)_\infty(-q;q^2)_\infty, \tag{8.5}$$

in (8.4) and then using (8.4) in (8.3) twice (once with q replaced by q^n), we find that

$$\begin{aligned}a_{m,n}&=nq^{(n-1)/4}\frac{(-q^n;-q^n)_\infty^4(q;-q)_\infty^4(-q;q^2)_\infty^6}{(q^n;-q^n)_\infty^4(-q;-q)_\infty^4(-q^n;q^{2n})_\infty^6}\\&=n\frac{\varphi^4(e^{-\pi\sqrt{mn}})}{\varphi^4(e^{-\pi\sqrt{m/n}})}\frac{G_{m/n}^6}{G_{mn}^6},\end{aligned} \tag{8.6}$$

by (0.1) and (0.7). Using (8.5) once again in (8.4), but with the reverse substitution, and then using (8.4) in (8.3) twice, we also find that

$$a_{m,n} = nq^{(n-1)/4}\frac{(-q^n;-q^n)_\infty^4(-q;q^2)_\infty^2}{(-q;-q)_\infty^4(-q^n;q^{2n})_\infty^2}$$

$$= n\frac{\eta^4\left(\frac{1+\sqrt{-mn}}{2}\right)}{\eta^4\left(\frac{1+\sqrt{-m/n}}{2}\right)}\frac{G_{m/n}^2}{G_{mn}^2}, \tag{8.7}$$

by (0.3) and (0.7). From (0.3) and (0.7), it is easy to see that, if $\tau = \sqrt{-r}$,

$$G_r = 2^{-1/4}\frac{\eta\left(\frac{1+\tau}{2}\right)}{\eta(\tau)}, \tag{8.8}$$

and so, using (8.8) in (8.7), we find that

$$a_{m,n} = n\frac{\eta^2\left(\frac{1+\sqrt{-mn}}{2}\right)\eta^2(\sqrt{-mn})}{\eta^2\left(\frac{1+\sqrt{-m/n}}{2}\right)\eta^2(\sqrt{-m/n})}. \tag{8.9}$$

K. G. Ramanathan [5, p. 88] very briefly discussed the numbers $a_{m,n}$ and derived an equivalent formulation of (8.7) (with a misprint). He wrote that he planned to return to $a_{m,n}$ in a future paper, but unfortunately he died before he was able to do so.

To help determine $G_{m/n}$, we shall employ three of Ramanujan's modular equations, Lemmas 4.3–4.5 of Chapter 34.

We also need three additional facts about $\varphi(q)$ and invariants. The first is a restatement of (4.5), but with n replaced by $\sqrt{n}$. If n is any positive rational number,

$$9\frac{\varphi^4(e^{-3\sqrt{n}\pi})}{\varphi^4(e^{-\sqrt{n}\pi})} = 1 + 2\sqrt{2}\frac{G_{9n}^3}{G_n^9}. \tag{8.10}$$

Second, if β has degree 5 over α (Part III [3, p. 281, Entry 13(iv)]),

$$5\frac{\varphi^2(q^5)}{\varphi^2(q)} = 1 + 2^{4/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24}$$

$$= 1 + 2\frac{G_{5n}}{G_{n/5}^5} \qquad (\text{if } q = \exp(-\pi\sqrt{n/5})), \tag{8.11}$$

by (8.2). Third, we need the standard theta-transformation formula, (4.7). In particular, if $\alpha^2 = \pi/\sqrt{r}$, then, from (4.7),

$$\varphi(e^{-\pi/\sqrt{r}}) = r^{1/4}\varphi(e^{-\pi\sqrt{r}}). \tag{8.12}$$

Calculation of $a_{m,n}$

$m = 3$

We first calculate $a_{3,3}$. By (8.6),

$$a_{3,3} = 3\frac{\varphi^4(e^{-3\pi})}{\varphi^4(e^{-\pi})}\frac{G_1^6}{G_9^6}. \tag{8.13}$$

The value for $\varphi^4(e^{-3\pi})/\varphi^4(e^{-\pi})$ is given in Entry 4 of this chapter, and the values

$$G_1 = 1 \qquad \text{and} \qquad G_9 = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^{1/3}$$

are found in the table in Section 2 of Chapter 34. Thus, putting these values in (8.13), we find that

$$a_{3,3} = 3\frac{1}{6\sqrt{3}-9}\left(\frac{\sqrt{2}}{\sqrt{3}+1}\right)^2 = \frac{1}{\sqrt{3}}.$$

By (8.6) and (8.12),

$$a_{3,n} = n\frac{\varphi^4(e^{-\pi\sqrt{3n}})}{\varphi^4(e^{-\pi\sqrt{3/n}})}\frac{G_{3/n}^6}{G_{3n}^6} = 3\frac{\varphi^4(e^{-\pi\sqrt{3n}})}{\varphi^4(e^{-\pi\sqrt{n/3}})}\frac{G_{n/3}^6}{G_{3n}^6}, \tag{8.14}$$

since $G_n = G_{1/n}$ (Ramanujan [3], [10, p. 23]). Using (8.10) in (8.14), we find that

$$a_{3,n} = \frac{1}{3}\left(\frac{G_{n/3}^6}{G_{3n}^6} + \frac{2\sqrt{2}}{G_{n/3}^3G_{3n}^3}\right). \tag{8.15}$$

Provided that the invariants $G_{n/3}$ and G_{3n} are known, (8.15) can be utilized to compute several values of $a_{3,n}$.

First, let $n = 9$. Then from the table in Chapter 34,

$$G_3 = 2^{1/12} \qquad \text{and} \qquad G_{27} = 2^{1/12}(2^{1/3}-1)^{-1/3}.$$

Putting these values in (8.15), we find that, upon simplification,

$$a_{3,9} = \tfrac{1}{3}(2^{2/3}-1) = \frac{1}{(2^{1/3}+1)^2}.$$

Second, let $n = 15$. Then, from the table in Chapter 34,

$$G_5 = \left(\frac{\sqrt{5}+1}{2}\right)^{1/4} \qquad \text{and} \qquad G_{45} = (\sqrt{5}+2)^{1/4}\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^{1/3}. \tag{8.16}$$

Using these values in (8.15), we find, after a straightforward, lengthy calculation, that

$$a_{3,15} = \frac{2^{5/2}(1+2\sqrt{3}+2\sqrt{5})}{3(\sqrt{5}+\sqrt{3})^2(3+\sqrt{5})^{3/2}}.$$

But $\sqrt{3+\sqrt{5}} = \sqrt{5/2} + \sqrt{1/2}$. Thus, after further simplification, we find that

$$a_{3,15} = \frac{2(1+2\sqrt{3}+2\sqrt{5})}{3(\sqrt{5}+\sqrt{3})^2(2+\sqrt{5})} = \frac{2-\sqrt{3}}{3},$$

by a direct calculation. (It is curious that the middle expression above is in $\mathbb{Q}(\sqrt{3})$ and not in $\mathbb{Q}(\sqrt{3}, \sqrt{5})$, as we would expect.)

In the remaining calculations, we need to calculate $G_{n/3}$ when $n/3$ is nonintegral. First, we could use the methods developed in our paper with Chan and Zhang [2], but these are nonelementary and depend upon the Kronecker limit formula. Second, with the use of (8.2), we can employ Lemma 4.3 of Chapter 34 with $P = (G_{n/3}G_{3n})^{-3}$ and $Q = (G_{n/3}/G_{3n})^6$ to deduce that

$$\left(\frac{G_{n/3}}{G_{3n}}\right)^6 + \left(\frac{G_{3n}}{G_{n/3}}\right)^6 + 2\sqrt{2}\left(\frac{1}{G_{n/3}^3 G_{3n}^3} - G_{n/3}^3 G_{3n}^3\right) = 0. \tag{8.17}$$

For specific values of n and G_{3n}, we could solve (8.17) for $G_{n/3}$, but this procedure is normally very laborious. It is better to "guess" the solution and then to verify that our "guess" is indeed correct. That we have guessed the correct solution and not another solution can be simply verified by numerically checking all the roots of the polynomial equation.

Now set $n = 5$ in (8.15). From the table in Chapter 34,

$$G_{15} = 2^{-1/12}(\sqrt{5}+1)^{1/3}. \tag{8.18}$$

From (8.17) we easily verify that

$$G_{5/3} = 2^{-1/12}(\sqrt{5}-1)^{1/3}.$$

Using this value and (8.18) in (8.15), with $n = 5$, we easily find that

$$a_{3,5} = \frac{3-\sqrt{5}}{2}.$$

Next, let $n = 7$. From the table in Chapter 34,

$$G_{21} = 2^{-1/3}(\sqrt{7}+\sqrt{3})^{1/4}(3+\sqrt{7})^{1/6}.$$

Just as in the previous proof, we substitute this value in (8.17), when $n = 7$, and verify that

$$G_{7/3} = 2^{-1/3}(\sqrt{7}-\sqrt{3})^{1/4}(3+\sqrt{7})^{1/6}.$$

Using the foregoing values in (8.15) when $n = 7$, we readily find that

$$a_{3,7} = 2-\sqrt{3}.$$

Next, let $n = 11$. From the table in Chapter 34,

$$G_{33} = 2^{-1/3}(\sqrt{11}+3)^{1/6}(\sqrt{3}+1)^{1/2}.$$

Letting $n = 11$ in (8.17), we check that

$$G_{11/3} = 2^{-1/3}(\sqrt{11}-3)^{1/6}(\sqrt{3}+1)^{1/2}.$$

Upon putting these values in (8.15) when $n = 11$, we readily deduce that

$$a_{3,11} = 2\sqrt{3}-\sqrt{11}.$$

Let $n = 19$. From the table in Chapter 34,

$$G_{57} = \left(\frac{3\sqrt{19}+13}{\sqrt{2}}\right)^{1/6} (2+\sqrt{3})^{1/4}.$$

Putting $n = 19$ in (8.17), we verify that

$$G_{19/3} = \left(\frac{3\sqrt{19}+13}{\sqrt{2}}\right)^{1/6} (2-\sqrt{3})^{1/4}.$$

Employing these values in (8.15) with $n = 19$, we easily deduce that

$$a_{3,19} = 2\sqrt{19} - 5\sqrt{3}.$$

Let $n = 31$. From the table in Chapter 34,

$$G_{93} = \left(\frac{39+7\sqrt{31}}{\sqrt{2}}\right)^{1/6} \left(\frac{\sqrt{31}+3\sqrt{3}}{2}\right)^{1/4}.$$

Using (8.17) with $n = 31$, we check that

$$G_{31/3} = \left(\frac{39+7\sqrt{31}}{\sqrt{2}}\right)^{1/6} \left(\frac{\sqrt{31}-3\sqrt{3}}{2}\right)^{1/4}.$$

Using these values in (8.15) when $n = 31$, we easily find that

$$a_{3,31} = (2-\sqrt{3})^3.$$

Let $n = 59$. From the table in Chapter 34,

$$G_{177} = \left(\frac{3\sqrt{59}+23}{\sqrt{2}}\right)^{1/6} \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^{3/2}.$$

By using (8.17) with $n = 59$, we find that

$$G_{59/3} = \left(\frac{3\sqrt{59}-23}{\sqrt{2}}\right)^{1/6} \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^{3/2}.$$

With these values in (8.15) with $n = 59$, it is now an easy task to show that

$$a_{3,59} = 102\sqrt{3} - 23\sqrt{59}.$$

Let $n = 39$. From the table in Chapter 34,

$$G_{39} = 2^{1/4}\left(\frac{\sqrt{13}+3}{2}\right)^{1/6} \left(\sqrt{\frac{5+\sqrt{13}}{8}} + \sqrt{\frac{\sqrt{13}-3}{8}}\right).$$

From (8.17), we can verify that

$$G_{13/3} = 2^{1/4}\left(\frac{\sqrt{13}+3}{2}\right)^{1/6} \left(\sqrt{\frac{5+\sqrt{13}}{8}} - \sqrt{\frac{\sqrt{13}-3}{8}}\right).$$

Using these values in (8.15) with $n = 39$, after a calculation with the help of *Mathematica*, we find that

$$a_{3,13} = \frac{1}{4}\left(7 + 3\sqrt{13} - (4 + \sqrt{13})\sqrt{2\sqrt{13} - 2}\right)$$
$$= \left(\sqrt{\frac{5 + \sqrt{13}}{8}} - \sqrt{\frac{\sqrt{13} - 3}{8}}\right)^8,$$

where we established the last equality by applying the binomial theorem on the right side.

Let $n = 23$. From the table in Chapter 34,

$$G_{69} = \left(\frac{5 + \sqrt{23}}{\sqrt{2}}\right)^{1/12} \left(\frac{3\sqrt{3} + \sqrt{23}}{2}\right)^{1/8} \left(\sqrt{\frac{6 + 3\sqrt{3}}{4}} + \sqrt{\frac{2 + 3\sqrt{3}}{4}}\right)^{1/2}.$$

Next,

$$G_{23/3} = \left(\frac{5 - \sqrt{23}}{\sqrt{2}}\right)^{1/12} \left(\frac{3\sqrt{3} - \sqrt{23}}{2}\right)^{1/8} \left(\sqrt{\frac{6 + 3\sqrt{3}}{4}} + \sqrt{\frac{2 + 3\sqrt{3}}{4}}\right)^{1/2}.$$

Once again, we can use (8.17) to verify this value, but the calculation is quite laborious. Alternatively, and preferably, we can use the method that we used in our joint paper with Chan and Zhang [2] to calculate G_{69} to also calculate $G_{23/3}$. We use the values above in (8.15), when $n = 23$, and find that, after a very lengthy calculation,

$$a_{3,23} = 6 + 4\sqrt{3} - \tfrac{1}{2}(5\sqrt{2} + 3\sqrt{6})\sqrt{3\sqrt{3} - 2}$$
$$= 6 + 4\sqrt{3} - \sqrt{(15\sqrt{3} + 26)(3\sqrt{3} - 2)}$$
$$= \left(\sqrt{\frac{7 + 4\sqrt{3}}{2}} - \sqrt{\frac{5 + 4\sqrt{3}}{2}}\right)^2.$$

Let $n = 71$. From the table in Chapter 34,

$$G_{213} = \left(\frac{5\sqrt{3} + \sqrt{71}}{2}\right)^{1/8} \left(\frac{59 + 7\sqrt{71}}{\sqrt{2}}\right)^{1/12}$$
$$\times \left(\sqrt{\frac{21 + 12\sqrt{3}}{2}} + \sqrt{\frac{19 + 12\sqrt{3}}{2}}\right)^{1/2}.$$

By using the same method that we employed in the aforementioned paper [2], which also utilized Lemma 4.3 of Chapter 34, or (8.17), we can deduce that

$$G_{71/3} = \left(\frac{5\sqrt{3}-\sqrt{71}}{2}\right)^{1/8}\left(\frac{59-7\sqrt{71}}{\sqrt{2}}\right)^{1/12}$$
$$\times\left(\sqrt{\frac{21+12\sqrt{3}}{2}}+\sqrt{\frac{19+12\sqrt{3}}{2}}\right)^{1/2}.$$

Putting these two values in (8.15) when $n = 71$, we find that, after a lengthy calculation,

$$\begin{aligned}
a_{3,71} &= 2(87+50\sqrt{3}) - \frac{1}{\sqrt{2}}(95+55\sqrt{3})\sqrt{5\sqrt{3}-2} \\
&= 2(87+50\sqrt{3}) - 5\sqrt{(362+209\sqrt{3})(5\sqrt{3}-2)} \\
&= \left(\sqrt{\frac{175+100\sqrt{3}}{2}} - \sqrt{\frac{173+100\sqrt{3}}{2}}\right)^2.
\end{aligned}$$

We close this subsection by proving two additional formulas for $a_{3,n}$. Theorem 8.1 below is an analogue of Corollary 8.4 (for $m = 5$) and Theorem 8.5 (for $m = 7$), and provides an optional method for calculating $a_{3,n}$. For calculational purposes, the formulas for $m = 5, 7$ are more advantageous than the ones below for $m = 3$.

Theorem 8.1. *If n is a positive integer and*

$$V_n = \frac{G_{n/3}}{G_{3n}},$$

then

$$a_{3,n} - \frac{1}{a_{3,n}} = \frac{1}{3}\left(V_n^6 - V_n^{-6}\right). \tag{8.19}$$

Proof. If β has degree 3 over α, then, from Part III [3, p. 230, Entry 5(vii)],

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} \tag{8.20}$$

and

$$\frac{9}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}, \tag{8.21}$$

where $m = \varphi^2(q)/\varphi^2(q^3)$. By combining (8.20) and (8.21), we deduce that

$$\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}\left(m^2 + \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}\right) = \frac{9}{m^2} + \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}. \tag{8.22}$$

Let $q = \exp(-\pi\sqrt{n/3})$, so that $q^3 = \exp(-\pi\sqrt{3n})$. Then, by (8.2), $G_{n/3} = \{4\alpha(1-\alpha)\}^{-1/24}$ and $G_{3n} = \{4\beta(1-\beta)\}^{-1/24}$. Thus, (8.22) may be rewritten in the form

$$\left(\frac{G_{3n}}{G_{n/3}}\right)^{12}\left(m^2+\left(\frac{G_{n/3}}{G_{3n}}\right)^{12}\right)=\frac{9}{m^2}+\left(\frac{G_{3n}}{G_{n/3}}\right)^{12}. \tag{8.23}$$

Rearranging (8.23) and using (8.6), we readily deduce (8.19).

By combining Theorem 8.4 with the modular equation (8.17), it is not difficult to deduce the following corollary.

Corollary 8.2. *If n is a positive integer and $U_n = G_{n/3}G_{3n}$, then*

$$a_{3,n}-\frac{1}{a_{3,n}}=\tfrac{1}{3}\left(8U_n^6+8U_n^{-6}-20\right)^{1/2}.$$

$m = 5$

First, let $n = 9$. By (8.6) and two applications of (8.10),

$$\begin{aligned}
a_{5,9} &= 9\frac{\varphi^4(e^{-\pi\sqrt{45}})}{\varphi^4(e^{-\pi\sqrt{5/9}})}\frac{G_{5/9}^6}{G_{45}^6}\\
&= 9\frac{\varphi^4(e^{-\pi\sqrt{45}})}{\varphi^4(e^{-\pi\sqrt{5}})}\frac{\varphi^4(e^{-\pi\sqrt{5}})}{\varphi^4(e^{-\pi\sqrt{5}/3})}\frac{G_{9/5}^6}{G_{45}^6}\\
&= \frac{1}{9}\left(1+2\sqrt{2}\frac{G_{45}^3}{G_5^9}\right)\left(1+2\sqrt{2}\frac{G_5^3}{G_{5/9}^9}\right)\frac{G_{9/5}^6}{G_{45}^6}.
\end{aligned} \tag{8.24}$$

Next, by (8.2) and Lemma 4.4 of Chapter 34 with $P = (G_{n/5}G_{5n})^{-2}$ and $Q = (G_{n/5}/G_{5n})^3$,

$$\left(\frac{G_{n/5}}{G_{5n}}\right)^3+\left(\frac{G_{5n}}{G_{n/5}}\right)^3+2\left(\frac{1}{G_{n/5}^2G_{5n}^2}-G_{n/5}^2G_{5n}^2\right)=0. \tag{8.25}$$

Set $n = 9$ in (8.25) and substitute the value of G_{45} given in (8.16). We then verify that, as in our applications of (8.17),

$$G_{9/5}=(\sqrt{5}+2)^{1/4}\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^{1/3}.$$

Putting this value and the values for G_5 and G_{45}, given by (8.16), into (8.24), we find, after a long, but straightforward, calculation, that

$$a_{5,9}=(2-\sqrt{3})^2.$$

From (8.6) and (8.11),

$$a_{5,n} = 5\frac{\varphi^4(e^{-\pi\sqrt{5n}})}{\varphi^4(e^{-\pi\sqrt{n/5}})}\frac{G_{n/5}^6}{G_{5n}^6} = \frac{1}{5}\left(1+2\frac{G_{5n}}{G_{n/5}^5}\right)^2\frac{G_{n/5}^6}{G_{5n}^6}. \tag{8.26}$$

We shall use (8.26) in further computations and in our proof of Corollary 8.4 below.

We next derive a formula by which the computation of $a_{5,n}$ generally will be simpler than that by using an analogue of (8.15). The following theorem is proved in a paper with Chan and Zhang [3].

Theorem 8.3. *Let $f(-q)$ be defined by (0.3), and let k be a positive rational number. Put*

$$A_1' = e^{\pi\sqrt{k}/6}\frac{f(e^{-\pi\sqrt{k}})}{f(e^{-5\pi\sqrt{k}})} \qquad \textit{and} \qquad V' = \frac{G_{25k}}{G_k}.$$

Then

$$\frac{A_1'^2V'}{\sqrt{5}} - \frac{\sqrt{5}}{A_1'^2V'} = \frac{1}{\sqrt{5}}\left(V'^3 - V'^{-3}\right).$$

Corollary 8.4. *If n is a positive integer and*

$$V_n = \frac{G_{n/5}}{G_{5n}},$$

then

$$\sqrt{a_{5,n}} - \frac{1}{\sqrt{a_{5,n}}} = \frac{1}{\sqrt{5}}\left(V_n^3 - V_n^{-3}\right) = \frac{1}{\sqrt{5}}\left(V_n - V_n^{-1}\right)\left(\left(V_n - V_n^{-1}\right)^2 + 3\right).$$

Proof. From (8.26) and (8.12),

$$a_{5,n} = \frac{1}{5}\frac{\varphi^4(e^{-\pi/\sqrt{5n}})}{\varphi^4(e^{-\pi\sqrt{5/n}})}\frac{G_{5/n}^6}{G_{1/(5n)}^6},$$

and from (0.1), (0.3), and (0.7), with $k = 1/(5n)$ in the definition of A_1',

$$A_1' = e^{\pi/(6\sqrt{5n})}\frac{\varphi(e^{-\pi/\sqrt{5n}})}{\varphi(e^{-\pi\sqrt{5/n}})}\frac{\chi(e^{-\pi\sqrt{5/n}})}{\chi(e^{-\pi/\sqrt{5n}})}.$$

Therefore,

$$\frac{A_1'^2V'}{\sqrt{5}} = \frac{1}{\sqrt{5}}\frac{\varphi^2(e^{-\pi/\sqrt{5n}})}{\varphi^2(e^{-\pi\sqrt{5/n}})}\frac{G_{5/n}^2}{G_{1/(5n)}^2}\frac{G_{5/n}}{G_{1/(5n)}} = \sqrt{a_{5,n}}.$$

Thus, by Theorem 8.3,

$$\sqrt{a_{5,n}} - \frac{1}{\sqrt{a_{5,n}}} = \frac{1}{\sqrt{5}}\left(V_n^3 - V_n^{-3}\right),$$

since $V' = V_n$. This completes the proof.

Now let $n = 11$. From the table in Chapter 34,

$$G_{55} = 2^{1/4}(\sqrt{5}+2)^{1/6}\left(\sqrt{\frac{7+\sqrt{5}}{8}}+\sqrt{\frac{\sqrt{5}-1}{8}}\right).$$

Setting $n = 11$ in (8.25), we verify that

$$G_{11/5} = 2^{1/4}(\sqrt{5}+2)^{1/6}\left(\sqrt{\frac{7+\sqrt{5}}{8}}-\sqrt{\frac{\sqrt{5}-1}{8}}\right).$$

Thus,

$$V_{11} - V_{11}^{-1} = -\sqrt{\frac{3\sqrt{5}-1}{2}},$$

and so from Corollary 8.4,

$$\sqrt{a_{5,11}} - \frac{1}{\sqrt{a_{5,11}}} = -\frac{1}{\sqrt{5}}\sqrt{\frac{3\sqrt{5}-1}{2}}\left(\frac{3\sqrt{5}+5}{2}\right) = -\sqrt{\frac{19+9\sqrt{5}}{2}}.$$

Solving for $\sqrt{a_{5,11}}$, we find that

$$a_{5,11} = \left(\sqrt{\frac{27+9\sqrt{5}}{8}}-\sqrt{\frac{19+9\sqrt{5}}{8}}\right)^2 = \left(\sqrt{\frac{7+\sqrt{5}}{8}}-\sqrt{\frac{\sqrt{5}-1}{8}}\right)^8.$$

Let $n = 13$. From (6.15) and the paragraph following (6.15) of Chapter 34,

$$V_{13} = \sqrt{\frac{\sqrt{65}+7}{8}}-\sqrt{\frac{\sqrt{65}-1}{8}}.$$

Thus,

$$V_{13} - V_{13}^{-1} = -\sqrt{\frac{\sqrt{65}-1}{2}}.$$

Therefore, by Corollary 8.4,

$$\sqrt{a_{5,13}} - \frac{1}{\sqrt{a_{5,13}}} = -\frac{5+\sqrt{65}}{2\sqrt{5}}\sqrt{\frac{\sqrt{65}-1}{2}} = -\sqrt{14+2\sqrt{65}}.$$

Hence,

$$\sqrt{a_{5,13}} + \frac{1}{\sqrt{a_{5,13}}} = \sqrt{18+2\sqrt{65}}.$$

Solving for $a_{5,13}$, we deduce that

$$a_{5,13} = \left(\sqrt{\frac{9+\sqrt{65}}{2}} - \sqrt{\frac{7+\sqrt{65}}{2}}\right)^2.$$

Let $n = 17$. From the table in Chapter 34,

$$G_{85} = \left(\frac{\sqrt{5}+1}{2}\right)\left(\frac{\sqrt{85}+9}{2}\right)^{1/4}.$$

With $n = 17$ in (8.25), it is quite easy to verify that

$$G_{17/5} = \left(\frac{\sqrt{5}-1}{2}\right)\left(\frac{\sqrt{85}+9}{2}\right)^{1/4}.$$

Putting these values in (8.26), with $n = 17$, we readily find that

$$a_{5,17} = (\sqrt{17}-4)^2.$$

Next, let $n = 29$. In the course of establishing the value of G_{145} in the proof of Theorem 5.5, either in Section 5 or Section 6 of Chapter 34, we proved that

$$V_{29} = \sqrt{\frac{17+\sqrt{145}}{8}} - \sqrt{\frac{9+\sqrt{145}}{8}}.$$

Thus,

$$V_{29} - V_{29}^{-1} = -\sqrt{\frac{9+\sqrt{145}}{2}}$$

and so, by Corollary 8.4,

$$\sqrt{a_{5,29}} - \frac{1}{\sqrt{a_{5,29}}} = -\frac{15+\sqrt{145}}{2\sqrt{5}}\sqrt{\frac{9+\sqrt{145}}{2}} = -\sqrt{192+16\sqrt{145}}.$$

Hence,

$$\sqrt{a_{5,29}} + \frac{1}{\sqrt{a_{5,29}}} = \sqrt{196+16\sqrt{145}}.$$

Solving for $a_{5,29}$, we conclude that

$$a_{5,29} = \left(\sqrt{49+4\sqrt{145}} - \sqrt{48+4\sqrt{145}}\right)^2.$$

Similarly, by using Theorems 5.6, 5.8, 5.10, and 5.11 of Chapter 34, we can deduce the values

$$a_{5,41} = \left(\sqrt{\frac{23+3\sqrt{41}}{4}} - \sqrt{\frac{19+3\sqrt{41}}{4}}\right)^4,$$

$$a_{5,53} = \left(\sqrt{\frac{5\sqrt{53} + 17\sqrt{5} + 2}{4}} - \sqrt{\frac{5\sqrt{53} + 17\sqrt{5} - 2}{4}} \right)^4,$$

$$a_{5,89} = \left(\sqrt{\frac{85 + 9\sqrt{89}}{2}} - \sqrt{\frac{83 + 9\sqrt{89}}{2}} \right)^4,$$

and

$$a_{5,101} = \left(\sqrt{\frac{13\sqrt{101} + 58\sqrt{5} + 1}{2}} - \sqrt{\frac{13\sqrt{101} + 58\sqrt{5} - 1}{2}} \right)^4.$$

$m = 7$

First, let $n = 9$. By (8.6) and two applications of (8.10),

$$\begin{aligned} a_{7,9} &= 9\frac{\varphi^4(e^{-3\pi\sqrt{7}})}{\varphi^4(e^{-\pi\sqrt{7}})} \frac{\varphi^4(e^{-\pi\sqrt{7}})}{\varphi^4(e^{-\pi\sqrt{7}/3})} \frac{G_{9/7}^6}{G_{63}^6} \\ &= \frac{1}{9}\left(1 + 2\sqrt{2}\frac{G_{63}^3}{G_7^9}\right)\left(1 + 2\sqrt{2}\frac{G_7^3}{G_{7/9}^9}\right)\frac{G_{9/7}^6}{G_{63}^6}. \end{aligned} \tag{8.27}$$

Using (8.2) and setting $P = (G_{n/7}G_{7n})^{-3}$ and $Q = (G_{n/7}/G_{7n})^4$, we find that Lemma 4.5 of Chapter 34 assumes the form

$$\left(\frac{G_{n/7}}{G_{7n}}\right)^4 + \left(\frac{G_{7n}}{G_{n/7}}\right)^4 + 7 = 2\sqrt{2}\left(\frac{1}{G_{n/7}^3 G_{7n}^3} - G_{n/7}^3 G_{7n}^3\right). \tag{8.28}$$

From the table in Chapter 34,

$$G_{63} = 2^{1/4}\left(\frac{5 + \sqrt{21}}{2}\right)^{1/6}\left(\sqrt{\frac{5 + \sqrt{21}}{8}} + \sqrt{\frac{\sqrt{21} - 3}{8}}\right).$$

Using (8.28) with $n = 9$, we easily verify that

$$G_{9/7} = 2^{1/4}\left(\frac{5 + \sqrt{21}}{2}\right)^{1/6}\left(\sqrt{\frac{5 + \sqrt{21}}{8}} - \sqrt{\frac{\sqrt{21} - 3}{8}}\right).$$

Substituting in (8.27) and using the denestings

$$\sqrt{5 \pm \sqrt{21}} = \sqrt{7/2} \pm \sqrt{3/2},$$

we find that

$$a_{7,9} = \tfrac{1}{8}\left(62 + 14\sqrt{21} - (21\sqrt{6} + 13\sqrt{14})\sqrt{\sqrt{21} - 3}\right)$$
$$= \left(\sqrt{\frac{5 + \sqrt{21}}{8}} - \sqrt{\frac{\sqrt{21} - 3}{8}}\right)^8,$$

upon expanding the right side and simplifying.

Of course, the method of calculation used above cannot be generalized to $n \neq 9$. Thus, generally, it seems best to prove an analogue of Theorem 8.1 and Corollary 8.4.

Theorem 8.5. *If n is any positive integer and*

$$V_n = \frac{G_{n/7}}{G_{7n}},$$

then

$$a_{7,n} - \frac{1}{a_{7,n}} = \tfrac{1}{7}\left(\left(V_n^6 - V_n^{-6}\right) + 8\left(V_n^2 - V_n^{-2}\right)\right)$$
$$= \tfrac{1}{7}\left(V_n^2 - V_n^{-2}\right)\left(\left(V_n^2 - V_n^{-2}\right)^2 + 11\right).$$

The proof of Theorem 8.5 follows along exactly the same lines as that for Theorem 8.1. The relevant modular equations of degree 7 are given by (Part III [3, p. 314, Entry 9(v)])

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} - 8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3}$$

and

$$\frac{49}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2} - 8\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/3},$$

where $m = \varphi^2(q)/\varphi^2(q^7)$.

As an example, let $n = 11$. Then, from (5.12) and (9.5) of Chapter 34, we find that

$$V_{11}^2 = \sqrt{\frac{23 + 8\sqrt{11}}{4}} - \sqrt{\frac{19 + 8\sqrt{11}}{4}}.$$

Hence,

$$V_{11}^2 - V_{11}^{-2} = -\sqrt{19 + 8\sqrt{11}},$$

and, from Theorem 8.5,

$$a_{7,11} - \frac{1}{a_{7,11}} = -\frac{30 + 8\sqrt{11}}{7}\sqrt{19 + 8\sqrt{11}} = -\sqrt{1484 + 448\sqrt{11}}.$$

Solving for $a_{7,11}$, we deduce that

$$a_{7,11} = \sqrt{372 + 112\sqrt{11}} - \sqrt{371 + 112\sqrt{11}}.$$

In the author's paper with Chan and Zhang [4], the following two theorems, guaranteeing that $a_{m,n}$ is a unit, are proved.

Theorem 8.6. *Let m and n be odd positive integers such that mn is squarefree and $-mn \equiv 3 \pmod 4$. Then $a_{m,n}$ is a unit.*

Theorem 8.7. *Let m and n be odd positive integers such that mn is squarefree and $-mn \equiv 1 \pmod 8$. Then $a_{m,n}$ is a unit.*

In this same paper, we also prove two theorems (Theorems 3.3 and 3.4) which give formulas for $a_{m,n}$ in terms of various parameters connected with the theory of quadratic number fields.

36

Modular Equations and Theta–Function Identities in Notebook 1

Chapters 19–21 in Ramanujan's second notebook are devoted almost exclusively to modular equations (Part III [3, pp. 220–488]). Ramanujan clearly loved modular equations, and, as the content of Chapters 34 and 35 makes manifest, he found many applications of these equations. Thus, it is surprising that the first notebook contains several dozen modular equations that he failed to record in his second notebook. Some of these are easy to prove with the help of modular equations in the second notebook, and so Ramanujan might have considered them less important and not worthy of repeating in his second notebook. However, many of them are apparently not so easy to prove. Some have degrees not examined in the second notebook. For example, on page 298 Ramanujan records a modular equation of degree 49. Not only does Ramanujan not consider modular equations of this degree in his second notebook, but apparently no one else had found a modular equation of degree 49 up until that time. Even at this writing, we know of no other modular equation of degree 49.

It is regrettable that we are unable to prove some of Ramanujan's modular equations by methods familiar to Ramanujan. As in Part III [3], we have had to use the theory of modular forms in some instances. In particular, for several modular equations on pages 86 and 88 in the spirit of L. Schläfli [1], we have had to resort to modular forms.

As with much of Ramanujan's work on modular equations and class invariants, one continually wonders about Ramanujan's methods. Several modular equations are in the forms of Schläfli [1], H. Weber [1], or R. Russell [1], [2]. Ramanujan employs the same functions of the moduli k and ℓ that these authors considered. He uses similar, but not exactly the same, notation as these predecessors. It would then seem that Ramanujan had read these papers, but it is doubtful that Ramanujan had access to them in Madras. As we pointed out at the close of Section 11 in Chapter 34, Ramanujan most likely had studied Greenhill's book [3], and, if so, he would have learned about Russell's discoveries on pages 327 and 328 of this book, but not his methods. For those modular equations of Schläfli–type, Weber–type, and Russell–type, we could have employed, respectively, the methods of these three men. However, since it seems unlikely that Ramanujan would have been acquainted with these methods, we have not proceeded by these means. Nonetheless, we feel that a study of Ramanujan's work along with that of Schläfli and Weber would be profitable.

Many deep modular equations are found on page 309, but two of them, of degrees 25 and 27, are incorrect. As we show in the sequel, if these modular equations are set in the form $F(q) = 0$, then the q–expansions of F have coefficients equal to 0 up through powers of 7 and 18, respectively. This is evidence that Ramanujan possibly used a method of comparing coefficients. Of course, this is also the crux of the method employing modular forms. It is also the underlying principle in Russell's method. Thus, one might conjecture that Ramanujan was acquainted with Russell's ideas, which were developed for certain modular equations of prime degree. Possibly Ramanujan tried to adopt such methods in these cases and either miscalculated some coefficients or did not calculate a sufficiently large number of them. From the point of view of modular forms, the subgroups $\Gamma_0(n)$ and $\Gamma(2) \cap \Gamma_0(n)$, on which the theta–functions act, have more cusps when n is composite than when n is prime. This generally requires the calculation of more coefficients of the q–series in order to rigorously establish modular equations via the theory of modular forms. Also, if one examines the additive order of terms in Ramanujan's formulations of his modular equations of this type, one sees that the q–expansions begin with successively higher powers of q. This gives further evidence that Ramanujan's methods hinged upon comparing coefficients in q–series.

K. G. Ramanathan [8] illuminatingly examined Ramanujan's work on modular equations pertaining to those of Schläfli, Weber, and Russell. Near the end of the paper, Ramanathan [8, p. 419] remarked, "We will defer a detailed consideration of these, in the light of Weber's method, to a succeeding paper." Unfortunately, Ramanathan died shortly thereafter, and no manuscript of the promised paper is extant.

This chapter contains other material besides modular equations, but all of the entries pertain to theta–functions in some way.

1. Modular Equations of Degree 3 and Related Theta–Function Identities

Ramanujan states that Entry 1 is valid provided that

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1, \tag{1.1}$$

but the degree of this modular equation is not given. Although (1.1) could be a modular equation of another degree, we are confident that it is a modular equation of degree 3 (Part III [3, p. 230, Entry 5(ii)]).

Entry 1 (p. 176). *If β has degree 3 over α, then*

$$\int_0^{\sin^{-1}(\beta/\alpha^3)^{1/8}} \frac{d\theta}{\sqrt{1-\alpha\sin^2\theta}} = \frac{\pi}{6}\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;\alpha\right).$$

Proof. We shall apply Entry 6(v) of Chapter 19 of the second notebook (Part III [3, pp. 238–239]) with $B = \sin^{-1}(\beta/\alpha^3)^{1/8}$ and $C = \pi/2$. Hence, if

$$\tan\frac{1}{2}\left(\frac{\pi}{2} + \sin^{-1}\left(\frac{\beta}{\alpha^3}\right)^{1/8}\right)$$
$$= \frac{2\tan\left(\sin^{-1}(\beta/\alpha^3)^{1/8}\right) + 2(1-\alpha)\tan^3\left(\sin^{-1}(\beta/\alpha^3)^{1/8}\right)}{1-(1-\alpha)\tan^4\left(\sin^{-1}(\beta/\alpha^3)^{1/8}\right)}, \tag{1.2}$$

then (Part III [3, p. 102], with a misprint corrected)

$$\int_0^{\sin^{-1}(\beta/\alpha^3)^{1/8}} \frac{d\theta}{\sqrt{1-\alpha\sin^2\theta}} = \frac{1}{3}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\alpha\sin^2\theta}} = \frac{\pi}{6}\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;\alpha\right).$$

It remains to show that (1.2) is a modular equation of degree 3.

By the addition and half-angle formulas for the tangent function, (1.2) is equivalent to the formula

$$\tan\frac{1}{2}\left(\frac{\pi}{2} + \sin^{-1}\left(\frac{\beta}{\alpha^3}\right)^{1/8}\right) = \frac{1 + \dfrac{(\beta/\alpha^3)^{1/8}}{1+\sqrt{1-(\beta/\alpha^3)^{1/4}}}}{1 - \dfrac{(\beta/\alpha^3)^{1/8}}{1+\sqrt{1-(\beta/\alpha^3)^{1/4}}}}$$
$$= \frac{\dfrac{2(\beta/\alpha^3)^{1/8}}{\sqrt{1-(\beta/\alpha^3)^{1/4}}} + \dfrac{2(1-\alpha)(\beta/\alpha^3)^{3/8}}{\{1-(\beta/\alpha^3)^{1/4}\}^{3/2}}}{1 - \dfrac{(1-\alpha)(\beta/\alpha^3)^{1/2}}{\{1-(\beta/\alpha^3)^{1/4}\}^2}}. \tag{1.3}$$

In proving some of Ramanujan's modular equations of degree 3 (Part III [3, p. 237]), we found it convenient to use the parametrizations

$$\alpha = p\left(\frac{2+p}{1+2p}\right)^3 \quad \text{and} \quad \beta = p^3\left(\frac{2+p}{1+2p}\right), \tag{1.4}$$

where $0 < p < 1$. (It is easy to verify that (1.4) is compatible with (1.1).) Hence,

$$\frac{\beta}{\alpha^3} = \left(\frac{1+2p}{2+p}\right)^8 \quad \text{and} \quad 1-\alpha = (1+p)\left(\frac{1-p}{1+2p}\right)^3. \tag{1.5}$$

Substituting (1.5) into (1.3) and simplifying each side, we see that we are now required to show that

$$\frac{3+3p+\sqrt{3-3p^2}}{1-p+\sqrt{3-3p^2}} = \frac{\sqrt{3-3p^2}\left\{2(1+2p)(3-3p^2)+2(1+p)(1-p)^3\right\}}{(3-3p^2)^2-(1+p)(1-p)^3(1+2p)}. \tag{1.6}$$

After much simplification, each side of (1.6) reduces to $\sqrt{3-3p^2}/(1-p)$, and so the proof is complete.

On page 172 Ramanujan records two further results on elliptic integrals. We have placed these results at the beginning of Section 10 in this chapter, because their statements require no information about modular equations, although one of the entries has a connection with modular equations of degree 3.

Entry 2 (p. 230). *We have*

$$\psi(q)\psi(q^3) - \psi(-q)\psi(-q^3) = 2q\varphi(q^2)\psi(q^{12}). \tag{2.1}$$

Proof. In (36.8) of Chapter 16 of our book [3, p. 69], set $\mu = 2$ and $\nu = 1$ to deduce that

$$\psi(q)\psi(q^3) = \varphi(q^6)\psi(q^4) + q\psi(q^{12})\varphi(q^2).$$

Replacing q by $-q$, we find that

$$\psi(-q)\psi(-q^3) = \varphi(q^6)\psi(q^4) - q\psi(q^{12})\varphi(q^2).$$

Subtracting the latter equality from the former, we deduce (2.1).

Entry 3 (p. 254). *We have*

$$\psi(q) - q\psi(q^9) = \varphi(-q^9)\left(\frac{\psi(q^3)}{\varphi(-q^3)}\right)^{1/3}.$$

Proof. From (22.4) and Entry 24(iii) of Chapter 16 (Part III [3, pp. 37, 39]),

$$\varphi(-q^9)\left(\frac{\psi(q^3)}{\varphi(-q^3)}\right)^{1/3} = \frac{(q^9;q^9)_\infty(q^9;q^{18})_\infty}{\chi(-q^3)} = \frac{(q^9;q^9)_\infty(q^9;q^{18})_\infty}{(q^3;q^6)_\infty}. \tag{3.1}$$

On the other hand, from Corollary (ii) in Section 31 of Chapter 16 and the Jacobi triple product identity (Part III [3, pp. 35, 49]),

$$\begin{aligned}
\psi(q) - q\psi(q^9) &= f(q^3,q^6) \\
&= (-q^3;q^9)_\infty(-q^6;q^9)_\infty(q^9;q^9)_\infty \\
&= \frac{(q^6;q^{18})_\infty(q^{12};q^{18})_\infty(q^9;q^9)_\infty}{(q^3;q^9)_\infty(q^6;q^9)_\infty} \\
&= \frac{(q^6;q^6)_\infty(q^9;q^9)_\infty^2}{(q^3;q^3)_\infty(q^{18};q^{18})_\infty} \\
&= \frac{(q^9;q^9)_\infty(q^9;q^{18})_\infty}{(q^3;q^6)_\infty}.
\end{aligned} \tag{3.2}$$

Comparing (3.1) and (3.2), we see that we have completed the proof.

Entry 4 (p. 254). *We have*

$$\{3\varphi(-q^9) - \varphi(-q)\}^3 = 8\frac{\psi^3(q)}{\psi(q^3)}\varphi(-q^3). \tag{4.1}$$

Proof. By Entry 1(iii) of Chapter 20 (Part III [3, p. 345]),

$$\{3\varphi(-q^9) - \varphi(-q)\}^3 = \varphi(-q^3)\left(9\frac{\varphi^3(-q^3)}{\varphi(-q)} - \frac{\varphi^3(-q)}{\varphi(-q^3)}\right). \tag{4.2}$$

Comparing (4.1) and (4.2), we see that we must show that

$$9\frac{\varphi^3(-q^3)}{\varphi(-q)} - \frac{\varphi^3(-q)}{\varphi(-q^3)} = 8\frac{\psi^3(q)}{\psi(q^3)}. \tag{4.3}$$

We now translate (4.3) into a modular equation of degree 3 by means of Entries 10(ii) and 11(i) of Chapter 17 (Part III [3, pp. 122–123]). Thus, if β is of degree 3 over α and $m = z_1/z_3$, (4.3) is equivalent to the equality

$$9\frac{z_3^{3/2}(1-\beta)^{3/4}}{z_1^{1/2}(1-\alpha)^{1/4}} - \frac{z_1^{3/2}(1-\alpha)^{3/4}}{z_3^{1/2}(1-\beta)^{1/4}} = 4\frac{z_1^{3/2}\alpha^{3/8}}{z_3^{1/2}\beta^{1/8}},$$

or

$$\frac{9}{m}\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/4} - m\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/4} = 4m\left(\frac{\alpha^3}{\beta}\right)^{1/8}. \tag{4.4}$$

This modular equation is not found among Ramanujan's modular equations of degree 3 given in Entry 5 of Chapter 19 (Part III [3, pp. 230–231]). However, it can easily be verified by using the parametrizations (Part III [3, p. 232, eq. (5.1)])

$$\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} = \frac{m+1}{2}, \qquad \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} = \frac{3-m}{2m}, \tag{4.5}$$

$$\left(\frac{\beta^3}{\alpha}\right)^{1/8} = \frac{m-1}{2}, \qquad \text{and} \qquad \left(\frac{\alpha^3}{\beta}\right)^{1/8} = \frac{3+m}{2m}. \tag{4.6}$$

Thus, by (4.5) and (4.6), (4.4) is equivalent to the equality

$$\frac{9}{m}\left(\frac{m+1}{2}\right)^2 - m\left(\frac{3-m}{2m}\right)^2 = 4m\frac{3+m}{2m},$$

which is easy to verify, and so the proof is complete.

Entry 5 (p. 266). *We have*

$$\varphi^2(q)\psi(-q) + 3q\varphi^2(q^9)\psi(-q^9) = 1 - 5q^3 - 7q^6 + 11q^{15} + 13q^{21} - \cdots. \tag{5.1}$$

This result is not correct. However, by Entry 24(ii) of Chapter 16 (Part III [3, p. 39]),

$$\begin{aligned}\varphi^2(q)\psi(-q) + 3q\varphi^2(q^9)\psi(-q^9) &= f^3(q) + 3qf^3(q^9)\\ &= \sum_{k=0}^{\infty}(-1)^{k(k-1)/2}(2k+1)q^{k(k+1)/2}\\ &\quad + 3q\sum_{k=0}^{\infty}(-1)^{k(k-1)/2}(2k+1)q^{9k(k+1)/2}.\end{aligned} \tag{5.2}$$

The right side of (5.1) comprises those terms on the right side of (5.2) when the power of q is a multiple of 3, except there is a discrepancy in sign in the last displayed term. Of course, the second series on the right side of (5.2) provides no contributions to the right side of (5.1), and so this entry is very puzzling indeed.

Entry 6 (p. 282). *If β has degree 3 over α, then*

$$\left(\frac{\alpha^3}{\beta}\right)^{1/8} + \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} = 3\left(\frac{(\alpha^3/\beta)^{1/8}-\alpha}{(\alpha^3/\beta)^{1/8}-\beta}\right)^{1/2}. \tag{6.1}$$

Proof. By (4.5) and (4.6), the left side of (6.1) equals

$$\frac{3+m}{2m} + \frac{3-m}{2m} = \frac{3}{m}.$$

Thus, it suffices to prove that

$$m = \left(\frac{(\alpha^3/\beta)^{1/8}-\beta}{(\alpha^3/\beta)^{1/8}-\alpha}\right)^{1/2}.$$

However, this last equality is Entry 5(iv) of Chapter 19 (Part III [3, p. 230]), and so the proof is complete.

Entry 7 (p. 283). *Let m and n be positive numbers such that $m - n = 1$. Define θ and φ, $0 \le \theta, \varphi \le \pi/2$, by*

$$\sin^3\theta = m^4 \sin\varphi \tag{7.1}$$

and

$$\cos^3\theta = n^4\cos\varphi. \tag{7.2}$$

Then

$$\sqrt{\frac{m-\sin^2\theta}{m-\sin^2\varphi}} = \frac{m+n}{3} = \frac{{}_2F_1(\frac12,\frac12;1;\sin^2\varphi)}{{}_2F_1(\frac12,\frac12;1;\sin^2\theta)}. \tag{7.3}$$

Proof. The far right side of (7.3) suggests that we set $\alpha = \sin^2\theta$ and $\beta = \sin^2\varphi$, so that $1-\alpha = \cos^2\theta$ and $1-\beta = \cos^2\varphi$. Thus, from (7.1) and (7.2), respectively,

$$m = \left(\frac{\alpha^3}{\beta}\right)^{1/8} \quad \text{and} \quad n = \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8}. \tag{7.4}$$

Thus,

$$1 = m - n = \left(\frac{\alpha^3}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8}. \tag{7.5}$$

In fact, by Entry 5(i) of Chapter 19 (Part III [3, p. 230]), (7.5) is a modular equation of degree 3. This confirms our definitions of α and β. Also, the far right side of (7.3) is then the reciprocal of the multiplier of degree 3, which, to avoid a conflict of notation, we denote here by M. By (7.4), the last equality in (7.3) then takes the form

$$\left(\frac{\alpha^3}{\beta}\right)^{1/8} + \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} = \frac{3}{M}. \tag{7.6}$$

However, (7.6) is a modular equ ation of degree 3 that can be deduced by combining parts of Entries 5(i), (iii) of Chapter 19 (Part III [3, p. 230]).

Lastly, by (7.4), the extremal equality proposed in (7.3) takes the shape

$$\sqrt{\frac{m-\sin^2\theta}{m-\sin^2\varphi}} = \sqrt{\frac{(\alpha^3/\beta)^{1/8}-\alpha}{(\alpha^3/\beta)^{1/8}-\beta}} = \frac{1}{M}.$$

However, this equality is precisely Entry 5(iv) of Chapter 19 (Part III [3, p. 230]). This completes the proof.

Entry 8 (p. 283). *Let* $0 < n < 1$ *and set*

$$p^3\frac{2+p}{1+2p} = \frac{1-\sqrt{1-n^3}}{2}. \tag{8.1}$$

Then, if $p > 0$,

$$p = \frac{-(1+\sqrt{1-n}) + \sqrt{2+n+2\sqrt{1+n+n^2}}}{2}. \tag{8.2}$$

Proof. Let $f(n)$ denote the right side of (8.1). Thus,

$$p^4 + 2p^3 - 2pf(n) - f(n) = 0. \tag{8.3}$$

To solve this quartic equation, we follow a standard procedure, as set forth, for example, in Hall and Knight's text [1, pp. 483–484]. Write (8.3) in the form

$$p^4 + 2p^3 + a^2p^2 + 2(ab - f(n))p - f(n) + b^2 = (ap+b)^2.$$

We seek constants a, b, and k such that

$$(p^2 + p + k)^2 = (ap+b)^2. \tag{8.4}$$

By equating coefficients, we deduce that

$$1 + 2k = a^2, \tag{8.5}$$

$$k = -f(n) + ab, \tag{8.6}$$

and

$$k^2 = -f(n) + b^2. \tag{8.7}$$

Thus,

$$(k + f(n))^2 = (2k + 1)(k^2 + f(n)),$$

from which it follows that

$$2k^3 = f^2(n) - f(n) = -\tfrac{1}{4}n^3.$$

Let k be the real root $-n/2$. So, from (8.5),

$$a = \sqrt{1-n}.$$

Then, from (8.7),

$$b = -\tfrac{1}{2}\sqrt{n^2 + 2\left(1 - \sqrt{1-n^3}\right)}.$$

That we have taken the correct square root can be verified by letting n tend to 1 above and using (8.6). Thus, from (8.4),

$$(p^2 + p - \tfrac{1}{2}n)^2 = \left(\sqrt{1-n}\,p - \tfrac{1}{2}\sqrt{n^2 + 2\left(1 - \sqrt{1-n^3}\right)}\right)^2.$$

Letting n tend to 1 and recalling that $p > 0$, we determine the proper square root on the right side above to be

$$p^2 + p - \tfrac{1}{2}n = -\left(\sqrt{1-n}\,p - \tfrac{1}{2}\sqrt{n^2 + 2\left(1 - \sqrt{1-n^3}\right)}\right).$$

In solving this quadratic equation, we again let n tend to 1 and use the fact that $p > 0$ to determine the correct square root. To that end,

$$p = \frac{-(1 + \sqrt{1-n}) + \sqrt{2 + n + 2\sqrt{1-n} + 2\sqrt{n^2 + 2(1 - \sqrt{1-n^3})}}}{2}. \tag{8.8}$$

We shall use (9.5) of Chapter 34 to denest the second rightmost inner radical above. Here, $d^2 = n^2(n+2)^2$, and so

$$\begin{aligned}\sqrt{n^2 + 2(1 - \sqrt{1-n^3})} &= \sqrt{\frac{n^2 + 2 + n(n+2)}{2}} - \sqrt{\frac{n^2 + 2 - n(n+2)}{2}}\\ &= \sqrt{1 + n + n^2} - \sqrt{1-n}.\end{aligned} \tag{8.9}$$

Using (8.9) in (8.8) and then simplifying, we deduce (8.2) to complete the proof.

The quotient $p^3(2+p)/(1+2p)$ occurs in the theory of modular equations of degree 3 (Part III [3, pp. 237, 238]). Taking the parametrizations of β and $1-\beta$ in terms of p on page 237, by an elementary calculation, we can show that $n^2 = 4\beta(1-\beta)$, where β has degree 3. Thus, by (1.6) of Chapter 34, n^2 is related to class invariants.

Entry 9 (p. 283). *Let β have degree 3 over α. If $m = \alpha^{1/8}$ and $n = \beta^{1/8}$, then*

$$m^4 + 2m^3n^3 - 2mn - n^4 = 0. \tag{9.1}$$

In fact, Ramanujan does not indicate that β has degree 3, i.e., that (9.1) is a modular equation of degree 3.

Proof. Ramanujan's equation (9.1) is equivalent to the equation

$$1 + 2\left(\frac{\beta^3}{\alpha}\right)^{1/8} - 2\left(\frac{\beta}{\alpha^3}\right)^{1/8} - \sqrt{\frac{\beta}{\alpha}} = 0. \tag{9.2}$$

Using (4.5) and (4.6), we find that (9.2) takes the form

$$1 + (m-1) - \frac{4m}{3+m} - \frac{m-1}{3+m}m = 0,$$

which is trivial to verify.

The modular equations in Entry 10 do not seem to be easily deducible from Ramanujan's modular equations of degree 3 listed in Entry 5 of Chapter 19 (Part III [3, pp. 230–231]). However, they can be easily verified by using the parametrizations given in (4.5) and (4.6). Since the details are simple and straightforward, we do not provide them.

Entry 10 (p. 290). *We have*

$$m = \frac{-1 + \left(\dfrac{(1-\beta)^3}{1-\alpha}\right)^{1/4}}{1 - \{(1-\alpha)(1-\beta)\}^{1/4}} = \frac{1 - \left(\dfrac{\beta^3}{\alpha}\right)^{1/4}}{1 - (\alpha\beta)^{1/4}}$$

and

$$\frac{m^2}{3} = \frac{1 + \left(\dfrac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}}{1 + \left(\dfrac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8}}.$$

Entry 11 (p. 295). *Let β have degree 3 over α, and define x and y by*

$$\alpha = \frac{1 \pm \sqrt{1-x^3}}{2} \qquad \text{and} \qquad \beta = \frac{1-\sqrt{1-y^8}}{2}. \tag{11.1}$$

Then

$$\sqrt{2}y = x^{1/8}\left(\sqrt{1+x+x^2} - x - \sqrt{(1-x)(2\sqrt{1+x+x^2}-1-2x)}\right). \tag{11.2}$$

The following proof by H. H. Chan supplants the author's more *ad hoc* proof.

Proof. Routinely solving (11.1) and (11.2) for x and y, we find that

$$x = (4\alpha(1-\alpha))^{1/3} \qquad \text{and} \qquad y = (4\beta(1-\beta))^{1/8}.$$

Replacing G_n by $(4\alpha(1-\alpha))^{-1/24} = x^{-1/8}$ and G_{9n} by $(4\beta(1-\beta))^{-1/24} = y^{-1/3}$ in Theorem 3.1 of Chapter 34, we find that

$$y^{-1/3} = x^{-1/8}\left(p+\sqrt{p^2-1}\right)^{1/6}\left(\sqrt{u}+\sqrt{u-1}\right)^{1/3},$$

where

$$p = \sqrt{x} + \frac{1}{\sqrt{x}} \quad \text{and} \quad u = \tfrac{1}{2}\left(p^2 - 2 + \sqrt{(p^2-1)(p^2-4)}\right).$$

After straightforward elementary algebra, we find that

$$\begin{aligned}
\sqrt{2}y &= x^{-3/8}\left(x+1-\sqrt{1+x+x^2}\right)^{1/2}\left(\sqrt{1+x^2+(1-x)\sqrt{1+x+x^2}}\right. \\
&\qquad \left. -\sqrt{(1-x)(1-x+\sqrt{1+x+x^2})}\right) \\
&= x^{1/8}\left(\sqrt{1+x+2x^2-2x\sqrt{1+x+x^2}}\right. \\
&\qquad \left. -\sqrt{(1-x)\left(-1-2x+2\sqrt{1+x+x^2}\right)}\right) \\
&= x^{1/8}\left(\sqrt{1+x+x^2} - x - \sqrt{(1-x)\left(2\sqrt{1+x+x^2}-1-2x\right)}\right),
\end{aligned}$$

since

$$\sqrt{1+x+2x^2-2x\sqrt{1+x+x^2}} = \sqrt{1+x+x^2} - x.$$

Entry 12 (p. 297). *If β has degree 3, then*

$$(\alpha\beta^5)^{1/8} + \{(1-\alpha)(1-\beta)^5\}^{1/8} = \sqrt{1-\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}}. \tag{12.1}$$

Proof. Comparing (12.1) with Entry 5(viii) of Chapter 19 (Part III [3, p. 231]), we find that we must show that

$$1-\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}=\tfrac{1}{2}\left(1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\right),$$

or

$$1=(\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}. \tag{12.2}$$

But (12.2) is identical to Entry 5(ii) of Chapter 19 (Part III [3, p. 230]), and so the proof is complete.

Entry 13 (p. 297). *If β has degree 3, then*

$$1+\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}=m\sqrt{1-\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}}. \tag{13.1}$$

Proof. We are unable to simply deduce (13.1) from Ramanujan's modular equations of degree 3 found in Entry 5 of Chapter 19 (Part III [3, pp. 230–231]). Thus, we use (4.5) and (4.6) to verify (13.1). Thus, (13.1) is equivalent to the equation

$$1+\frac{m^2-1}{4}=m\sqrt{1-\frac{(m^2-1)(9-m^2)}{16m^2}},$$

which is easily verified.

2. Modular Equations of Degree 5 and Related Theta–Function Identities

Entry 14 (p. 222). *We have*

$$\frac{\dfrac{\varphi^5(q)}{\varphi(q^5)}+4\dfrac{\psi^5(q)}{\psi(q^5)}}{\varphi(q)\varphi^3(q^5)+4q^2\psi(q)\psi^3(q^5)}=5\frac{\varphi^2(q)}{\varphi^2(q^5)}. \tag{14.1}$$

Proof. Let β have degree 5 over α, and let m denote the multiplier of degree 5. By Entries 10(i) and 11(i) of Chapter 17 (Part III [3, pp. 122–123]), (14.1) is equivalent to the modular equation

$$\frac{\dfrac{z_1^{5/2}}{z_5^{1/2}}+\dfrac{z_1^{5/2}}{z_5^{1/2}}\left(\dfrac{\alpha^5}{\beta}\right)^{1/8}}{z_1^{1/2}z_5^{3/2}+z_1^{1/2}z_5^{3/2}(\alpha\beta^3)^{1/8}}=5\frac{z_1}{z_5},$$

or

$$\frac{1+(\alpha^5/\beta)^{1/8}}{1+(\alpha\beta^3)^{1/8}}=\frac{5}{m}. \tag{14.2}$$

However, (14.2) is one of Ramanujan's modular equations of degree 5 (Part III [3, p. 281, Entry 13(vi)]), and so the proof is complete.

Entry 15 (p. 284). *We have*

$$\frac{\psi^5(-q^5)}{\psi(-q)} - \frac{\psi^5(q^5)}{\psi(q)} = 4q^3\frac{\psi^5(q^{10})}{\psi(q^2)} + 2q\frac{f^5(-q^{20})}{f(-q^4)}. \tag{15.1}$$

Proof. By using Entries 11(i)–(iii) and 12(iv) of Chapter 17 (Part III [3, pp. 123–124]), we can easily show that (15.1) can be translated into the modular equation of degree 5,

$$\left(\frac{(1-\beta)^5}{1-\alpha}\right)^{1/8} - 1 = \left(\frac{\beta^5}{\alpha}\right)^{1/8} + 2^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24},$$

which is Entry 13(iii) of Chapter 19 (Part III [3, p. 280]). This completes the proof.

Entry 16 (p. 285). *We have*

$$\frac{\psi^5(q)}{\psi(q^5)} + \frac{\psi^5(-q)}{\psi(-q^5)} + 2\frac{f^5(-q^4)}{f(-q^{20})} = 4\frac{\psi^5(q^2)}{\psi(q^{10})}. \tag{16.1}$$

Proof. With the use of Entries 11(i)–(iii) and 12(iv) of Chapter 17 (Part III [3, pp. 123–124]), it is easily shown that (16.1) is equivalent to the modular equation of degree 5,

$$1 + \left(\frac{(1-\alpha)^5}{1-\beta}\right)^{1/8} + 2^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24} = \left(\frac{\alpha^5}{\beta}\right)^{1/8},$$

which is Entry 13(ii) of Chapter 19 (Part III [3, p. 280]), and so the proof is complete.

Entry 17 (p. 286). *We have*

$$\frac{\varphi^5(-q)}{\varphi(-q^5)} + 4\frac{f^5(-q)}{f(-q^5)} = 5\varphi^3(-q)\varphi(-q^5).$$

Proof. Replacing q by $-q$, we are required to show that

$$\frac{\varphi^5(q)}{\varphi(q^5)} + 4\frac{f^5(q)}{f(q^5)} = 5\varphi^3(q)\varphi(q^5). \tag{17.1}$$

Now by Entry 9(ii) of Chapter 19 (Part III [3, p. 258]),

$$4q\frac{f^5(q^5)}{f(q)} + \frac{\varphi^5(q^5)}{\varphi(q)} = \varphi(q)\varphi^3(q^5). \tag{17.2}$$

We shall apply transformation formulas to the functions in (17.2) to deduce (17.1). If α, $\beta > 0$ and $\alpha\beta = \pi^2$, then, by Entry 27(iv) of Chapter 16 (Part III [3, p. 43]),

$$e^{-\alpha/24}\alpha^{1/4}f(e^{-\alpha}) = e^{-\beta/24}\beta^{1/4}f(e^{-\beta})$$

and

$$e^{-5\alpha/24}(5\alpha)^{1/4}f(e^{-5\alpha}) = e^{-\beta/120}(\beta/5)^{1/4}f(e^{-\beta/5}).$$

Thus,

$$e^{-\alpha}\frac{f^5(e^{-5\alpha})}{f(e^{-\alpha})} = \frac{\left(e^{-5\alpha/24}f(e^{-5\alpha})\right)^5}{e^{-\alpha/24}f(e^{-\alpha})} = \frac{1}{5^{5/2}}\left(\frac{\beta}{\alpha}\right)\frac{f^5(e^{-\beta/5})}{f(e^{-\beta})}. \tag{17.3}$$

Next, from Entry 27(i) of Chapter 16 (Part III [3, p. 43]), if α, $\beta > 0$ and $\alpha\beta = \pi^2$,

$$\alpha^{1/4}\varphi(e^{-\alpha}) = \beta^{1/4}\varphi(e^{-\beta})$$

and

$$(5\alpha)^{1/4}\varphi(e^{-5\alpha}) = (\beta/5)^{1/4}\varphi(e^{-\beta/5}).$$

Hence,

$$\frac{\varphi^5(e^{-5\alpha})}{\varphi(e^{-\alpha})} = \frac{1}{5^{5/2}}\left(\frac{\beta}{\alpha}\right)\frac{\varphi^5(e^{-\beta/5})}{\varphi(e^{-\beta})} \tag{17.4}$$

and

$$\varphi(e^{-\alpha})\varphi^3(e^{-5\alpha}) = \frac{1}{5^{3/2}}\left(\frac{\beta}{\alpha}\right)\varphi(e^{-\beta})\varphi^3(e^{-\beta/5}). \tag{17.5}$$

Therefore, using (17.3)–(17.5) in (17.2), we deduce that

$$4\frac{1}{5^{5/2}}\left(\frac{\beta}{\alpha}\right)\frac{f^5(e^{-\beta/5})}{f(e^{-\beta})} + \frac{1}{5^{5/2}}\left(\frac{\beta}{\alpha}\right)\frac{\varphi^5(e^{-\beta/5})}{\varphi(e^{-\beta})} = \frac{1}{5^{3/2}}\left(\frac{\beta}{\alpha}\right)\varphi(e^{-\beta})\varphi^3(e^{-\beta/5}). \tag{17.6}$$

Cancel the common expression $5^{-5/2}\beta/\alpha$ and set $e^{-\beta/5} = q_1$. Hence, (17.6) simplifies to the equality

$$4\frac{f^5(q_1)}{f(q_1^5)} + \frac{\varphi^5(q_1)}{\varphi(q_1^5)} = 5\varphi(q_1^5)\varphi^3(q_1),$$

which is (17.1) with q replaced by q_1.

Entry 18 (p. 295). *We have*

$$\psi^2(-q) + q\psi^2(-q^5) = \frac{f(q^5)\varphi(q^5)}{\chi(q)}.$$

Proof. Replacing q by $-q$, we find that it suffices to prove that

$$\psi^2(q) - q\psi^2(q^5) = \frac{f(-q^5)\varphi(-q^5)}{\chi(-q)} = \frac{(q^5;q^5)_\infty^2(q^5;q^{10})_\infty}{(q;q^2)_\infty}, \tag{18.1}$$

by (22.4) and Entries 22(iii), (iv) of Chapter 16 of Part III [3, pp. 36, 37].

Now by Entry 10(v) of Chapter 19 and the Jacobi triple product identity (Part III [3, pp. 262, 35]),

$$\begin{aligned}\psi^2(q) - q\psi^2(q^5) &= f(q, q^4)f(q^2, q^3)\\ &= (-q; q^5)_\infty(-q^4; q^5)_\infty(-q^2; q^5)_\infty(-q^3; q^5)_\infty(q^5; q^5)^2_\infty\\ &= \frac{(-q; q)_\infty(q^5; q^5)^2_\infty}{(-q^5; q^5)_\infty} = \frac{(q^5; q^5)^2_\infty(q^5; q^{10})_\infty}{(q; q^2)_\infty},\end{aligned} \tag{18.2}$$

by Euler's identity, (22.3) of Chapter 16 (Part III [3, p. 37]). The desired result now follows from (18.1) and (18.2).

Entry 19 (p. 295). *We have*

$$\psi^2(-q) + 5q\psi^2(-q^5) = \frac{\varphi^2(q)}{\chi(q)\chi(q^5)}. \tag{19.1}$$

Proof. After replacing q by $-q$ and employing Entries 11(i), 10(ii), and 12(vi) in Chapter 17 (Part III [3, pp. 122–124]), we can translate (19.1) into the modular equation of degree 5,

$$\left(\frac{\alpha^5}{\beta}\right)^{1/8} - \frac{5}{m}(\alpha^3\beta)^{1/8} = 4^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/12}. \tag{19.2}$$

On the other hand, by Entry 13(iv) of Chapter 19 (Part III [3, p. 281]),

$$\frac{1}{2}\left(\frac{5}{m} - 1\right) = 2^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24}. \tag{19.3}$$

Squaring (19.3) and subtracting the result from (19.2), we see that it suffices to prove that

$$\left(\frac{\alpha^5}{\beta}\right)^{1/8} - \frac{5}{m}(\alpha^3\beta)^{1/8} = \frac{1}{4}\left(\frac{5}{m} - 1\right)^2. \tag{19.4}$$

Recall from Part III [3, p. 284, eq. (13.3)] the notation

$$\rho = (m^3 - 2m^2 + 5m)^{1/2}.$$

By Part III [3, p. 284, eq. (13.4); p. 285, eq. (13.10)],

$$\left(\frac{\alpha^5}{\beta}\right)^{1/8} - \frac{5}{m}(\alpha^3\beta)^{1/8} = \frac{5\rho + m^2 + 5m}{4m^2} - \frac{5(\rho + 3m - 5)}{4m^2} = \frac{1}{4}\left(\frac{5}{m} - 1\right)^2.$$

Thus, (19.4) has been proved, and the proof of Entry 19 is complete.

Entry 20 (p. 297). *If β has degree 5 over α, then*

$$(\alpha\beta^3)^{1/8} + \{(1-\alpha)(1-\beta)^3\}^{1/8} = \sqrt{1 - \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}}. \tag{20.1}$$

Proof. Comparing (20.1) with Entry 13(vii) of Chapter 19 (Part III [3, p. 281]), we find that it suffices to show that

$$1-\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}=\tfrac{1}{2}\left(1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\right),$$

or

$$1-2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}=\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}. \tag{20.2}$$

But (20.2) is the same as Entry 13(i) of Chapter 19 (Part III [3, p. 280]), and so the proof is complete.

Entry 21 (p. 325). *We have*

$$f(-q,-q^{14})f(-q^4,-q^{11})f(-q^6,-q^9)f(-q^5)=f(-q,-q^4)f^3(-q^{15})$$

and

$$f(-q^7,-q^8)f(-q^2,-q^{13})f(-q^3,-q^{12})f(-q^5)=f(-q^2,-q^3)f^3(-q^{15}).$$

Proof. Both of these identities are readily established by employing the Jacobi triple product identity.

3. Other Modular Equations and Related Theta–Function Identities

Entry 22 (p. 246). *We have*

$$2\frac{\psi'(q)}{\psi(q)}-2q\frac{\psi'(q^2)}{\psi(q^2)}=\frac{\varphi'(q)}{\varphi(q)} \tag{22.1}$$

and

$$\frac{\varphi'(-q)}{\varphi(-q)}-4q\frac{\varphi'(-q^2)}{\varphi(-q^2)}=\frac{\varphi'(q)}{\varphi(q)}. \tag{22.2}$$

Proof. By using the product representations for $\psi(q)$, $\psi(q^2)$, $\varphi(q)$, $\varphi(-q)$, and $\varphi(-q^2)$ (Part III [3, p. 36]), we can easily show that

$$\frac{\psi^2(q)}{\psi(q^2)}=\varphi(q)=\frac{\varphi^2(-q^2)}{\varphi(-q)}. \tag{22.3}$$

Taking the logarithmic derivatives of both equalities in (22.3), we deduce (22.1) and (22.2) at once.

The following two entries can be found in Section 24 of Chapter 18. Regrettably, in Part III [3, p. 216], we claimed that the two results are false.

Entry 23 (p. 300). *The equation*

$$\sqrt{m}(1-\alpha)^{1/8}+\beta^{1/4}=1 \tag{23.1}$$

is a modular equation of degree 8.

Proof. By using Entries 10(iii) and 11(iii) of Chapter 17 (Part III [3, pp. 122–123]), we find that (23.1) is equivalent to the theta-function identity

$$\varphi(-q^2)+2q^2\psi(q^{16})=\varphi(q^8).$$

By Entries 25(ii), (i) of Chapter 16 (Part III [3, p. 40]),

$$\begin{aligned}\varphi(-q^2)+2q^2\psi(q^{16}) &= \varphi(-q^2)+\tfrac{1}{2}\{\varphi(q^2)-\varphi(-q^2)\}\\ &= \tfrac{1}{2}\{\varphi(-q^2)+\varphi(q^2)\}=\varphi(q^8),\end{aligned}$$

and the proof is complete.

Entry 24 (p. 300). *If β has degree 16 over α, then*

$$\sqrt{m}=2\frac{1+\beta^{1/4}}{1+(1-\alpha)^{1/4}}. \tag{24.1}$$

Proof. By Entries 10(i), (ii), and 11(iii) of Chapter 17 (Part III [3, pp. 122–123]), (24.1) is equivalent to the theta-function identity

$$\varphi(q)+\varphi(-q)=2\varphi(q^{16})+4q^4\psi(q^{32}).$$

By Entries 25(i), (ii) of Chapter 16 (Part III [3, p. 40]),

$$\begin{aligned}2\varphi(q^{16})+4q^4\psi(q^{32}) &= \{\varphi(q^4)+\varphi(-q^4)\}+\{\varphi(q^4)-\varphi(-q^4)\}\\ &= 2\varphi(q^4)=\varphi(q)+\varphi(-q),\end{aligned}$$

and so the proof is complete.

Entry 25 (p. 297). *If β has degree 7 over α, then*

$$\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8}-\left(\frac{\beta^7}{\alpha}\right)^{1/8}=m\left(1-\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\right).$$

Proof. The result follows by combining the first part of Entry 19(iii) with the second part of Entry 19(i) of Chapter 19 (Part III [3, p. 314]).

Entry 26 (p. 296). *If β has degree 7 over α, then*

$$\left\{\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/4}+\left(\frac{\beta^7}{\alpha}\right)^{1/4}+1\right\}-2\left\{\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/8}+\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8}\right.$$
$$\left.+\left(\frac{\beta^7}{\alpha}\right)^{1/8}\right\}-\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24}\left\{\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8}+\left(\frac{\beta^7}{\alpha}\right)^{1/8}+1\right\}$$
$$-3\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/12}=0. \tag{26.1}$$

Proof. Let

$$A:=\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24} \tag{26.2}$$

and

$$R:=\left\{(2-3t+2t^2)(2-t+t^2)(1-t+2t^2)\right\}^{1/2},$$

where $\alpha\beta=t^8$, $t>0$. From Part III [3, pp. 316, 318, eqs. (19.2), (19.3), (19.15)],

$$A=\tfrac{1}{2}\left\{2-7t+11t^2-8t^3+4t^4-(1-2t)R\right\}. \tag{26.3}$$

From Entry 19(vii) of Chapter 19 in Ramanujan's second notebook (Part III [3, p. 314]),

$$\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8}+\left(\frac{\beta^7}{\alpha}\right)^{1/8}+2\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24}=\frac{3+m^2}{4}. \tag{26.4}$$

Thus, using (26.2) and (26.4), we may recast (26.1) in the form

$$1+\left(\frac{3+m^2}{4}-2A\right)^2-2A^3-2\left(A^3+\frac{3+m^2}{4}-2A\right)$$
$$-A\left(\frac{7+m^2}{4}-2A\right)-3A^2=0,$$

or

$$(m^2-1)^2-20Am^2-12A+48A^2-64A^3=0. \tag{26.5}$$

We also know that (Part III [3, p. 319, eq. (19.20)])

$$m=-3+8t-6t^2+4t^3+2R. \tag{26.6}$$

Putting (26.3) and (26.6) in the left side of (26.5) and simplifying with the aid of *Mathematica*, we verify that indeed (26.5) holds to complete the proof.

We have altered Ramanujan's notation in the next entry so that it is consistent with that in Entry 3 of Chapter 20 (Part III [3, pp. 352–353]).

Entry 27 (p. 286). *Let γ have degree 9, and put $m = z_1/z_3$ and $m' = z_3/z_9$. Then*

$$\left(\frac{\alpha}{\gamma}\right)^{1/8}\left(\frac{1-\alpha}{1-\gamma}\right)^{1/4}\sqrt{mm'}+\frac{3}{\sqrt{mm'}}=\left(\frac{\alpha}{\gamma}\right)^{1/8}+\left(\frac{1-\alpha}{1-\gamma}\right)^{1/4}.$$

Proof. Using Entries 3(x), (xi) of Chapter 20 (Part III [3, p. 352]), we find that

$$\begin{aligned}
&\left(\frac{\alpha}{\gamma}\right)^{1/8}\left(\frac{1-\alpha}{1-\gamma}\right)^{1/4}\sqrt{mm'}+\frac{3}{\sqrt{mm'}}\\
&=\left(\frac{\alpha}{\gamma}\right)^{1/8}\left(\frac{1-\alpha}{1-\gamma}\right)^{1/4}\left(\left(\frac{\gamma}{\alpha}\right)^{1/8}+\left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}-\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8}\right)\\
&\quad+\left(\frac{\alpha}{\gamma}\right)^{1/8}+\left(\frac{1-\alpha}{1-\gamma}\right)^{1/8}-\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/8}\\
&=\left(\frac{1-\alpha}{1-\gamma}\right)^{1/4}+\left(\frac{\alpha}{\gamma}\right)^{1/8},
\end{aligned}$$

which completes the proof.

Entry 28 (p. 296). *If m denotes the multiplier of degree 9 and γ has degree 9, then*

$$\begin{aligned}
&\left(\frac{\gamma}{\alpha}\right)^{1/2}+\left(\frac{1-\gamma}{1-\alpha}\right)^{1/2}+\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/2}\\
&-4\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/4}\left\{1+\left(\frac{\gamma}{\alpha}\right)^{1/4}+\left(\frac{1-\gamma}{1-\alpha}\right)^{1/4}\right\}=m^2.
\end{aligned}$$

Disproof. Write

$$\begin{aligned}
S:=&\left(\frac{\gamma}{\alpha}\right)^{1/2}+\left(\frac{1-\gamma}{1-\alpha}\right)^{1/2}+\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/2}\\
&-4\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/4}\left\{1+\left(\frac{\gamma}{\alpha}\right)^{1/4}+\left(\frac{1-\gamma}{1-\alpha}\right)^{1/4}\right\}\\
=&\left\{\left(\frac{\gamma}{\alpha}\right)^{1/8}+\left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}\right\}^4-4\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8}\\
&\times\left\{\left(\frac{\gamma}{\alpha}\right)^{1/4}+\left(\frac{1-\gamma}{1-\alpha}\right)^{1/4}\right\}-6\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/4}\\
&+\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/2}-4\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/4}\left\{1+\left(\frac{\gamma}{\alpha}\right)^{1/4}+\left(\frac{1-\gamma}{1-\alpha}\right)^{1/4}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{\left(\frac{\gamma}{\alpha}\right)^{1/8} + \left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}\right\}^4 - 10\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/4} + \left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/2} \\
&\quad - 4\left\{\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8} + \left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/4}\right\} \\
&\quad \textit{times}\left\{\left\{\left(\frac{\gamma}{\alpha}\right)^{1/8} + \left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}\right\}^2 - 2\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8}\right\} \\
&= \left\{\left(\frac{\gamma}{\alpha}\right)^{1/8} + \left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}\right\}^4 - 2\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/4} \\
&\quad + \left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/2} + 8\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{3/8} \\
&\quad - 4\left\{\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8} + \left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/4}\right\}\left\{\left(\frac{\gamma}{\alpha}\right)^{1/8} + \left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}\right\}^2.
\end{aligned} \tag{28.1}
$$

From our study of Ramanujan's modular equations of degree 9 in Part III [3, pp. 352–356], each of the expressions above can be expressed in terms of a parameter t. In particular, from page 356,

$$\left(\frac{\gamma}{\alpha}\right)^{1/8} + \left(\frac{1-\gamma}{1-\alpha}\right)^{1/8} = \frac{1+2t}{1-t},$$

and from equations (3.7) and (3.9) on page 354,

$$\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8} = t\frac{1+2t}{1-t}.$$

Thus, from (28.1),

$$
\begin{aligned}
S &= \left(\frac{1+2t}{1-t}\right)^4 - 2t^2\left(\frac{1+2t}{1-t}\right)^2 + t^4\left(\frac{1+2t}{1-t}\right)^4 + 8t^3\left(\frac{1+2t}{1-t}\right)^3 \\
&\quad - 4\left\{t\frac{1+2t}{1-t} + t^2\left(\frac{1+2t}{1-t}\right)^2\right\}\left(\frac{1+2t}{1-t}\right)^2 \\
&= \frac{(1+2t)^2}{(1-t)^4}\left\{(1+2t)^2 - 2t^2(1-t)^2 + t^4(1+2t)^2\right. \\
&\quad \left. +8t^3(1+2t)(1-t) - 4\left(t(1+2t)(1-t) + t^2(1+2t)^2\right)\right\} \\
&= \frac{(1+2t)^2}{(1-t)^4}\left\{1 - 6t^2 + 4t^3 - 9t^4 - 12t^5 + 4t^6\right\}.
\end{aligned} \tag{28.2}
$$

Now from equations (3.10) and (3.11) on page 354 of Part III [3],

$$m^2 = (1+2t)^4.$$

Thus, if Ramanujan were correct, the expression in curly braces on the far right side of (28.2) should equal $(1+2t)^2(1-t)^4$. But

$$(1+2t)^2(1-t)^4 = 1 - 6t^2 + 4t^3 + 9t^4 - 12t^5 + 4t^6. \tag{28.3}$$

Thus, there is a discrepancy between (28.2) and (28.3) in the terms $-9t^4$ and $+9t^4$, respectively. It seems therefore that Ramanujan made a sign error in his calculations. This analysis is evidence that Ramanujan also used parametric representations.

Entry 29 (p. 230). *We have*

$$\psi(q)\psi(q^{11}) - \psi(-q)\psi(-q^{11}) = 2qf(q^2, q^{10})f(q^{44}, q^{88}) + 2q^{15}\varphi(q^6)\psi(q^{132}). \tag{29.1}$$

Proof. In (36.8) of Chapter 16 (Part III [3, p. 69]), we set $\mu = 6$ and $\nu = 5$ to find that

$$\begin{aligned}\psi(q^{11})\psi(q) &= \varphi(q^{66})\psi(q^{12}) + qf(q^{44}, q^{88})f(q^2, q^{10}) \\ &\quad + q^{14}f(q^{22}, q^{110})f(q^{20}, q^{-8}) + q^{39}\psi(q^{132})f(q^{30}, q^{-18}).\end{aligned}$$

Replacing q by $-q$ and subtracting the resulting equality from that above, we find that

$$\begin{aligned}&\psi(q)\psi(q^{11}) - \psi(-q)\psi(-q^{11}) \\ &\qquad = 2qf(q^{44}, q^{88})f(q^2, q^{10}) + 2q^{39}\psi(q^{132})f(q^{30}, q^{-18}).\end{aligned}$$

But by Entry 18(iv) of Chapter 16 with $n = 2$ (Part III [3, p. 34]),

$$f(q^{30}, q^{-18}) = q^{-24}\varphi(q^6).$$

Using this above, we deduce (29.1).

Entry 30 (p. 298). *If β has degree 11 over α, then*

$$\begin{aligned}1 + 2^{10/3}\left(\frac{\beta^{11}(1-\beta)^{11}}{\alpha(1-\alpha)}\right)^{1/24} &= m\left\{(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}\right\} \\ \times\left\{(\alpha\beta)^{1/4} - \{(1-\alpha)(1-\beta)\}^{1/4} + \left(2 + 2\sqrt{\alpha\beta} + 2\sqrt{(1-\alpha)(1-\beta)}\right)^{1/2}\right\}&.\end{aligned} \tag{30.1}$$

Proof. By adding Entries 7(vi), (vii) of Chapter 20 (Part III [3, p. 364]), we find that

$$\begin{aligned}\frac{1}{m}\left(1 + 2^{10/3}\left(\frac{\beta^{11}(1-\beta)^{11}}{\alpha(1-\alpha)}\right)^{1/24}\right) &= \sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)} \\ + \sqrt{2}\left\{(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}\right\}&\left(1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}\right)^{1/2}.\end{aligned} \tag{30.2}$$

Comparing (30.1) and (30.2), we find that the proof is complete.

4. Identities Involving Lambert Series

Entry 31 (p. 266).

$$\tfrac{1}{4}\varphi^2(q)\varphi^2(q^2) = \frac{1}{4} + \sum_{k=1}^{\infty} \frac{(2k-1)q^{2k-1}}{1-q^{2k-1}} + \sum_{k=1}^{\infty} \frac{(2+(-1)^k)kq^{2k}}{1+q^{2k}}.$$

Proof. Using successively Entries 25(vi) and 25(iii) of Chapter 16 and Entry 8(ii) of Chapter 17 (Part III [3, pp. 40, 114]), we find that

$$\begin{aligned}\tfrac{1}{4}\varphi^2(q)\varphi^2(q^2) &= \tfrac{1}{8}\varphi^2(q)\left(\varphi^2(q)+\varphi^2(-q)\right) = \tfrac{1}{8}\left(\varphi^4(q)+\varphi^4(-q^2)\right)\\ &= \frac{1}{8} + \sum_{k=1}^{\infty} \frac{kq^k}{1+(-q)^k} + \frac{1}{8} + \sum_{k=1}^{\infty} \frac{k(-q^2)^k}{1+q^{2k}}\\ &= \frac{1}{4} + \sum_{k=1}^{\infty} \frac{(2k-1)q^{2k-1}}{1-q^{2k-1}} + \sum_{k=1}^{\infty} \frac{(2+(-1)^k)kq^{2k}}{1+q^{2k}},\end{aligned}$$

and the proof is complete.

Entry 32 (p. 267).

$$\varphi(q)\varphi(q^4) = 1 + 2\sum_{k=0}^{\infty} \frac{(-1)^k q^{2k+1}}{1-q^{2k+1}} - 2\sum_{k=0}^{\infty} \frac{(-1)^k q^{4k+2}}{1+q^{4k+2}}. \tag{32.1}$$

Proof. By Entries 25(i), (iii) of Chapter 16 (Part III [3, p. 40]),

$$\varphi(q)\varphi(q^4) = \tfrac{1}{2}\varphi(q)\left(\varphi(q)+\varphi(-q)\right) = \tfrac{1}{2}\left(\varphi^2(q)+\varphi^2(-q^2)\right). \tag{32.2}$$

Recall from Entry 8(i) of Chapter 17 (Part III [3, p. 114]) that

$$\varphi^2(q) = 1 + 4\sum_{k=0}^{\infty} \frac{(-1)^k q^{2k+1}}{1-q^{2k+1}}.$$

Using this twice in (32.2), we deduce (32.1) at once.

Note that Entry 32 is a companion to Entries 8(iii), (iv) of Chapter 17 (Part III [3, p. 114]), and, in fact, is placed after these entries in the first notebook.

Entry 33 (p. 274). *If n is real, then*

$$1 + 4\sum_{k=1}^{\infty} \frac{q^k \cos(nk)}{1+q^{2k}} = \varphi^3(-q^2)\frac{\sum_{k=-\infty}^{\infty}(-1)^k q^{2k^2}\cos(2kn)}{\left(\sum_{k=-\infty}^{\infty}(-1)^k q^{k^2}\cos(kn)\right)^2}. \tag{33.1}$$

Proof. Comparing (33.1) with Entry 33(iii) of Chapter 16 (Part III [3, p. 53]), we find that it suffices to prove that, with $z = e^{in}$,

$$\varphi^2(-q^2)\frac{f(zq, q/z)}{f(-zq, -q/z)} = \frac{\varphi^3(-q^2)f(-z^2q^2, -q^2/z^2)}{f^2(-zq, -q/z)},$$

or

$$f(zq, q/z) = \frac{\varphi(-q^2)f(-z^2q^2, -q^2/z^2)}{f(-zq, -q/z)}. \tag{33.2}$$

By the Jacobi triple product identity and (22.4) of Chapter 16 (Part III [3, pp. 35, 37]), the right side of (33.2) equals

$$\begin{aligned}&\frac{(q^2;q^2)_\infty(q^2;q^4)_\infty(z^2q^2;q^4)_\infty(q^2/z^2;q^4)_\infty(q^4;q^4)_\infty}{(zq;q^2)_\infty(q/z;q^2)_\infty(q^2;q^2)_\infty}\\ &\quad= (q^2;q^2)_\infty(-zq;q^2)_\infty(-q/z;q^2)_\infty = f(zq, q/z),\end{aligned}$$

again, by the Jacobi triple product identity. This proves (33.2), and so the proof is complete.

Entry 34 (p. 284). *If* $\left(\frac{n}{3}\right)$*denotes the Legendre symbol, then*

$$q\frac{\psi^3(q^3)}{\psi(q)} = \sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n}{1-q^{2n}}.$$

Proof. By Entry 4(iii) of Chapter 19 (Part III [3, p. 226]),

$$\frac{\psi^3(q)}{\psi(q^3)} = 1 + 3\sum_{n=0}^{\infty}\left(\frac{q^{6n+1}}{1-q^{6n+1}} - \frac{q^{6n+5}}{1-q^{6n+5}}\right), \tag{34.1}$$

and by Entry 3(i) of Chapter 21 (Part III [3, p. 460]),

$$\frac{\psi^3(q)}{\psi(q^3)} + 3q\frac{\psi^3(q^3)}{\psi(q)} = 1 + 6\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n}{1-q^n}. \tag{34.2}$$

Thus, from (34.1) and (34.2),

$$q\frac{\psi^3(q^3)}{\psi(q)} = \sum_{n=0}^{\infty}\left(\frac{q^{6n+1}}{1-q^{6n+1}} - 2\frac{q^{6n+2}}{1-q^{6n+2}} + 2\frac{q^{6n+4}}{1-q^{6n+4}} - \frac{q^{6n+5}}{1-q^{6n+5}}\right).$$

We now use the elementary equality

$$\frac{2q^{2m}}{1-q^{2m}} = \frac{q^m}{1-q^m} - \frac{q^m}{1+q^m} \tag{34.3}$$

with $m = 3n + 1, 3n + 2$. Hence, after some cancellation,

$$q\frac{\psi^3(q^3)}{\psi(q)} = \sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1+q^{3n+1}} + \frac{q^{6n+2}}{1-q^{6n+2}} - \frac{q^{3n+2}}{1+q^{3n+2}} - \frac{q^{6n+4}}{1-q^{6n+4}}\right)$$
$$= \sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{6n+2}} - \frac{q^{3n+2}}{1-q^{6n+4}}\right),$$

where we used the elementary equality

$$\frac{q^m}{1+q^m} + \frac{q^{2m}}{1-q^{2m}} = \frac{q^m}{1-q^{2m}}$$

with $m = 3n + 1, 3n + 2$. This completes the proof.

Entry 35 (p. 284). *If* $(\frac{n}{3})$ *denotes the Legendre symbol, then*

$$\frac{\varphi^3(q^3)}{\varphi(q)} = 1 - 2\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n}{1-(-q)^n}. \tag{35.1}$$

Proof. From Entries 3(i), (ii) of Chapter 21 (Part III [3, p. 460]),

$$\frac{\varphi^3(q)}{\varphi(q^3)} + 3\frac{\varphi^3(q^3)}{\varphi(q)} = 4\left(1 + 6\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^{2n}}{1-q^{2n}}\right),$$

and from Entry 4(iv) of Chapter 19 (Part III [3, p. 227]),

$$\frac{\varphi^3(q)}{\varphi(q^3)} = 1 + 6\sum_{n=0}^{\infty}\left(\frac{q^{6n+1}}{1-q^{6n+1}} + \frac{q^{6n+2}}{1+q^{6n+2}} - \frac{q^{6n+4}}{1+q^{6n+4}} - \frac{q^{6n+5}}{1-q^{6n+5}}\right).$$

Hence, using (34.3), we find that

$$\begin{aligned}\frac{\varphi^3(q^3)}{\varphi(q)} &= 1 + 8\sum_{n=0}^{\infty}\left(\frac{q^{6n+2}}{1-q^{6n+2}} - \frac{q^{6n+4}}{1-q^{6n+4}}\right)\\ &\quad - 2\sum_{n=0}^{\infty}\left(\frac{q^{6n+1}}{1-q^{6n+1}} + \frac{q^{6n+2}}{1-q^{6n+2}} - 2\frac{q^{12n+4}}{1-q^{12n+4}}\right.\\ &\quad \left. - \frac{q^{6n+4}}{1-q^{6n+4}} + 2\frac{q^{12n+8}}{1-q^{12n+8}} - \frac{q^{6n+5}}{1-q^{6n+5}}\right)\\ &= 1 + 2\sum_{n=0}^{\infty}\left(-\frac{q^{12n+1}}{1-q^{12n+1}} + 3\frac{q^{12n+2}}{1-q^{12n+2}} - \frac{q^{12n+4}}{1-q^{12n+4}} + \frac{q^{12n+5}}{1-q^{12n+5}}\right.\\ &\quad \left. - \frac{q^{12n+7}}{1-q^{12n+7}} + \frac{q^{12n+8}}{1-q^{12n+8}} - 3\frac{q^{12n+10}}{1-q^{12n+10}} + \frac{q^{12n+11}}{1-q^{12n+11}}\right).\end{aligned} \tag{35.2}$$

On the other hand, the right side of (35.1) equals, by (34.3),

$$1-2\sum_{n=0}^{\infty}\left(\frac{q^{6n+1}}{1+q^{6n+1}}-\frac{q^{6n+2}}{1-q^{6n+2}}+\frac{q^{6n+4}}{1-q^{6n+4}}-\frac{q^{6n+5}}{1+q^{6n+5}}\right)$$
$$=1-2\sum_{n=0}^{\infty}\left(\frac{q^{6n+1}}{1-q^{6n+1}}-2\frac{q^{12n+2}}{1-q^{12n+2}}-\frac{q^{6n+2}}{1-q^{6n+2}}\right.$$
$$\left.+\frac{q^{6n+4}}{1-q^{6n+4}}-\frac{q^{6n+5}}{1-q^{6n+5}}+2\frac{q^{12n+10}}{1-q^{12n+10}}\right). \tag{35.3}$$

If we now compare (35.2) and (35.3), we find that they are equal, and so the proof is complete.

Entry 36 (p. 284). *We have*

$$\frac{\psi^3(-q^3)}{\psi(-q)}-\frac{\psi^3(q^3)}{\psi(q)}=2q\frac{\psi^3(q^6)}{\psi(q^2)}.$$

Proof. From Entry 34,

$$\frac{\psi^3(-q^3)}{\psi(-q)}-\frac{\psi^3(q^3)}{\psi(q)}=-\frac{1}{q}\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\left(\frac{(-q)^n}{1-q^{2n}}+\frac{q^n}{1-q^{2n}}\right)$$
$$=-\frac{2}{q}\sum_{n=1}^{\infty}\left(\frac{2n}{3}\right)\frac{q^{2n}}{1-q^{4n}}=2q\frac{\psi^3(q^6)}{\psi(q^2)},$$

since $(\frac{2}{3})=-1$.

Entry 37 (p. 285). *We have*

$$\frac{\psi^3(q)}{\psi(q^3)}+\frac{\psi^3(-q)}{\psi(-q^3)}=2\frac{\psi^3(q^2)}{\psi(q^6)}.$$

Proof. This result easily follows from (34.1) and (34.3).

5. Identities Involving Eisenstein Series

Recall that

$$L:=L(q):=1-24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}, \qquad |q|<1. \tag{38.1}$$

From Entry 12(ix) of Chapter 15 (Part II [2, p. 326]), or from the fourth entry on page 264 of the first notebook,

$$L(q)=\frac{\sum_{k=0}^{\infty}(-1)^k(2k+1)^3q^{k(k+1)/2}}{\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2}}. \tag{38.2}$$

Ramanujan, in fact, expresses the next entry in terms of the right side of (38.2).

Entry 38 (p. 264). *We have*

$$4L(q^4) - L(q) = 3\varphi^4(q). \tag{38.3}$$

Proof. From Entries 13(viii), (ix) of Chapter 17 (Part III [3, p. 127]),

$$\begin{aligned} 4L(q^4) - L(q) &= \left(4L(q^4) - 2L(q^2)\right) + \left(2L(q^2) - L(q)\right) \\ &= 2z^2(1 - \tfrac{1}{2}x) + z^2(1 + x) = 3z^2 = 3\varphi^4(q), \end{aligned}$$

by Entry 6 of Chapter 17 (Part III [3, p. 101]).

Suppose that, in the summands on the left side of (38.3), we expand $1/(1-q^k)$ and $1/(1-q^{4k})$ into geometric series. Collecting coefficients of like powers of q^n, $n \geq 0$, we find that

$$4L(q^4) - L(q) = 3 - 96\sum_{n=1}^{\infty} \sigma(n) q^{4n} + 24\sum_{n=1}^{\infty} \sigma(n) q^n,$$

where $\sigma(n)$ is the sum of the positive divisors of n. Thus, equating coefficients of q^n, $n \geq 1$, on both sides of (38.3), we find that, if $r_4(n)$ denotes the number of representations of n as a sum of four squares,

$$\begin{aligned} r_4(n) &= \begin{cases} 8\sigma(n), & \text{if } n \not\equiv 0 \pmod 4, \\ 8\sigma(n) - 32\sigma(n/4), & \text{if } n \equiv 0 \pmod 4, \end{cases} \\ &= 8\sum_{\substack{d \mid n \\ 4 \nmid n}} d. \end{aligned}$$

Thus, Entry 38 yields a very short proof of a famous result of C. G. J. Jacobi [1], [2].

Recall that

$$M := M(q) := 1 + 240\sum_{k=1}^{\infty} \frac{k^3 q^k}{1 - q^k}, \qquad |q| < 1. \tag{39.1}$$

Entry 39 (p. 264). *We have*

$$\frac{\sum_{k=0}^{\infty}(-1)^k (2k+1)^5 q^{k(k+1)/2}}{\sum_{k=0}^{\infty}(-1)^k (2k+1) q^{k(k+1)/2}} = M - 480\sum_{k=1}^{\infty} \frac{k^2 q^k}{(1 - q^k)^2}.$$

Proof. By Entry 12(v) of Chapter 15 (Part II [2, p. 326]),

$$\sum_{k=1}^{\infty} \frac{k^2 q^k}{(1 - q^k)^2} = \frac{M - L^2}{288},$$

where L is defined in (38.1). By Example (ii) in Section 35 of Chapter 16 (Part III [3, p. 65]),

$$\frac{\sum_{k=0}^{\infty}(-1)^k(2k+1)^5 q^{k(k+1)/2}}{\sum_{k=0}^{\infty}(-1)^k(2k+1) q^{k(k+1)/2}} = \frac{5L^2 - 2M}{3}.$$

Entry 39 thus easily follows from the last two equalities.

Entry 40 (p. 271). *If $M(q)$ is defined by (39.1), then*

$$M(q) = \varphi^8(-q) + 256q\psi^8(q).$$

Proof. From Entry 13(iii) of Chapter 17 (Part III [3, p. 127]),

$$M(q) = z^4(1 + 14x + x^2). \tag{40.1}$$

On the other hand, by Entries 10(ii) and 11(i) of Chapter 17 (Part III [3, pp. 122–123]),

$$\varphi^8(-q) + 256q\psi^8(q) = z^4(1-x)^2 + 16z^4x = z^4(1 + 14x + x^2). \tag{40.2}$$

Combining (40.1) and (40.2), we complete the proof.

6. Modular Equations in the Form of Schläfli

This section contains some of the deepest results in the chapter. As indicated at the beginning of the chapter, Ramanujan's methods for some entries have remained hidden from us.

Entry 41 (p. 90). *Let*

$$P := 2^{1/6}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} \qquad \textit{and} \qquad Q := \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/24}.$$

Then, if β has degrees 11, 13, 17, and 19, respectively, over α,

$$Q^6 + \frac{1}{Q^6} - 2\sqrt{2}\left(\frac{2}{P^5} - \frac{11}{P^3} + \frac{22}{P} - 22P + 11P^3 - 2P^5\right) = 0,$$

$$\begin{aligned} &Q^7 + \frac{1}{Q^7} + 13\left(Q^5 + \frac{1}{Q^5}\right) + 52\left(Q^3 + \frac{1}{Q^3}\right) \\ &\quad + 78\left(Q + \frac{1}{Q}\right) - 8\left(\frac{1}{P^6} - P^6\right) = 0, \end{aligned}$$

$$\begin{aligned} &Q^9 + \frac{1}{Q^9} - 34\left(Q^6 + \frac{1}{Q^6}\right) + 17\left(Q^3 + \frac{1}{Q^3}\right)\left(\frac{4}{P^4} + 7 + 4P^4\right) \\ &\quad - \left(\frac{16}{P^8} - \frac{136}{P^4} - 340 - 136P^4 + 16P^8\right) = 0, \end{aligned}$$

and

$$Q^{10}+\frac{1}{Q^{10}}+114\left(Q^6+\frac{1}{Q^6}\right)-190\sqrt{2}\left(Q^4+\frac{1}{Q^4}\right)\left(\frac{1}{P^3}-P^3\right)$$
$$+19\left(Q^2+\frac{1}{Q^2}\right)\left(\frac{8}{P^6}-5+8P^6\right)-4\sqrt{2}\left(\frac{4}{P^9}+\frac{19}{P^3}-19P^3-4P^9\right)=0.$$

These modular equations were established by Schläfli [1] in 1870. Schläfli also discovered similar modular equations of degrees 3, 5, and 7, which also appear on page 90 of the first notebook but are not given here, since they also appear in the second notebook and so were proved by us in Part III [3, pp. 231, 282, 315]. In one sentence, Ramanathan [8, p. 411] indicated a proof of the equation of degree 11 above, but we have been unable to complete the proof along the lines that he indicated. Watson [7] gave a proof of Schläfli's modular equation of degree 13 but erroneously remarked that Ramanujan did not discover it. Schläfli's modular equations were also examined in another paper by Watson [8].

In Entries 42–49 we set

$$P := (256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/48} \tag{42.1}$$

and

$$Q := \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/48}, \tag{42.2}$$

where, say, α, β, γ, and δ have degrees a, b, c, and d, respectively. By using Entries 10(i) and 12(i) of Chapter 17 (Part III [3, pp. 122, 124]), we may transform (42.1) and (42.2) into the representations,

$$P = \sqrt{2}q^{(a+b+c+d)/48}\sqrt{\frac{f(q^a)f(q^b)f(q^c)f(q^d)}{\varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d)}} \tag{42.3}$$

and

$$Q = q^{(a+d-b-c)/48}\sqrt{\frac{f(q^a)f(q^d)\varphi(q^b)\varphi(q^c)}{f(q^b)f(q^c)\varphi(q^a)\varphi(q^d)}} \tag{42.4}$$

in terms of theta-functions.

Entry 42 (p. 86). *Let P and Q be defined by (42.1) and (42.2), respectively. Suppose that α, β, γ, and δ have degrees 1, 5, 7, and 35, respectively. Then*

$$Q^4+\frac{1}{Q^4}-\left(Q^2+\frac{1}{Q^2}\right)-2\left(P^2+\frac{1}{P^2}\right)=0.$$

Proof. The following proof was given by Ramanathan [8]. Observe that

$$PQ^{-1} = 2^{1/6}(\beta\gamma(1-\beta)(1-\gamma))^{1/24} \tag{42.5}$$

and

$$PQ = 2^{1/6}(\alpha\delta(1-\alpha)(1-\delta))^{1/24}. \tag{42.6}$$

Now from Entry 18(v) of Chapter 20 of Ramanujan's second notebook (Part III [3, p. 423]),

$$\begin{aligned}&\frac{(16\beta\gamma(1-\beta)(1-\gamma))^{1/24} - (16\alpha\delta(1-\alpha)(1-\delta))^{1/8}}{(16\beta\gamma(1-\beta)(1-\gamma))^{1/24} + (16\beta\gamma(1-\beta)(1-\gamma))^{1/8}}\\ &= \frac{(16\alpha\delta(1-\alpha)(1-\delta))^{1/8} + (16\alpha\delta(1-\alpha)(1-\delta))^{1/24}}{(16\beta\gamma(1-\beta)(1-\gamma))^{1/8} - (16\alpha\delta(1-\alpha)(1-\delta))^{1/24}}.\end{aligned}$$

Using (42.5) and (42.6), we can rewrite the latter equation in the form,

$$\frac{PQ^{-1} - P^3Q^3}{PQ^{-1} + P^3Q^{-3}} = \frac{P^3Q^3 + PQ}{P^3Q^{-3} - PQ}.$$

Clearing fractions above, we easily deduce the desired result.

Entry 43 (p. 86). *If α, β, γ, and δ have degrees 3, 1, 5, and 15, respectively, then*

$$Q^4 + \frac{1}{Q^4} - 2\left(P^2 + \frac{1}{P^2}\right) + 3 = 0. \tag{43.1}$$

Entry 44 (p. 86). *If α, β, γ, and δ have degrees 5, 1, 3, and 15, respectively, then*

$$Q^6 + \frac{1}{Q^6} - 4\left(P^4 + \frac{1}{P^4}\right) + 10\left(P^2 + \frac{1}{P^2} - 1\right) = 0. \tag{44.1}$$

Entry 45 (p. 86). *If α, β, γ, and δ have degrees 1, 3, 7, and 21, respectively, then*

$$\begin{aligned}Q^{16} + \frac{1}{Q^{16}} - 5\left(Q^{12} + \frac{1}{Q^{12}}\right) + 5\left(Q^8 + \frac{1}{Q^8}\right) + 6\left(Q^4 + \frac{1}{Q^4}\right)&\\ - 8\left(P^6 + \frac{1}{P^6}\right) + 6 = 0.&\end{aligned} \tag{45.1}$$

Entry 46 (p. 86). *If α, β, γ, and δ have degrees 7, 1, 3, and 21, respectively, then*

$$\begin{aligned}Q^8 + \frac{1}{Q^8} + 7\left(Q^6 + \frac{1}{Q^6}\right) + 14\left(Q^4 + \frac{1}{Q^4}\right) + 21\left(Q^2 + \frac{1}{Q^2}\right)&\\ - 8\left(P^6 + \frac{1}{P^6}\right) + 42 = 0.&\end{aligned} \tag{46.1}$$

Entry 47 (p. 86). *If α, β, γ, and δ have degrees 3, 1, 11, and 33, respectively, then*

$$Q^4 + \frac{1}{Q^4} + 3\left(Q^2 + \frac{1}{Q^2}\right) - 2\left(P^2 + \frac{1}{P^2}\right) = 0. \tag{47.1}$$

Entry 48 (p. 86). *If α, β, γ, and δ have degrees 5, 1, 7, and 35, respectively, then*

$$Q^6+\frac{1}{Q^6}+5\sqrt{2}\left(Q^3+\frac{1}{Q^3}\right)\left(P+\frac{1}{P}\right)-4\left(P^4+\frac{1}{P^4}\right)+10=0. \quad (48.1)$$

Entry 49 (p. 86). *If α, β, γ, and δ have degrees 5, 1, 11, and 55, respectively, then*

$$\begin{aligned} Q^6+\frac{1}{Q^6}-5\left(Q^4+\frac{1}{Q^4}\right)+10\left(Q^2+\frac{1}{Q^2}\right)\left(P^2+\frac{1}{P^2}-1\right)& \\ -4\left(P^4+\frac{1}{P^4}\right)+10\left(P^2+\frac{1}{P^2}\right)-25=0.& \quad (49.1)\end{aligned}$$

For the next three entries, we need the definitions

$$R:=\left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/48} \quad (50.1)$$

and

$$T:=\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/48}. \quad (50.2)$$

By Entries 10(i) and 12(i) of Chapter 17 (Part III [3, pp. 122, 124]), R and T may be expressed in the forms

$$R=q^{(c+d-a-b)/48}\sqrt{\frac{f(q^c)f(q^d)\varphi(q^a)\varphi(q^b)}{f(q^a)f(q^b)\varphi(q^c)\varphi(q^d)}} \quad (50.3)$$

and

$$T=q^{(b+d-a-c)/48}\sqrt{\frac{f(q^b)f(q^d)\varphi(q^a)\varphi(q^c)}{f(q^a)f(q^c)\varphi(q^b)\varphi(q^d)}}. \quad (50.4)$$

Entry 50 (p. 86). *If α, β, γ, and δ have degrees 1, 5, 7, and 35, respectively, then*

$$R^4+\frac{1}{R^4}-\left(Q^6+\frac{1}{Q^6}\right)+5\left(Q^4+\frac{1}{Q^4}\right)-10\left(Q^2+\frac{1}{Q^2}\right)+15=0. \quad (50.5)$$

Entry 51 (p. 88). *If α, β, γ, and δ have degrees 1, 13, 3, and 39, respectively, then*

$$Q^4+\frac{1}{Q^4}-3\left(Q^2+\frac{1}{Q^2}\right)-\left(T^2+\frac{1}{T^2}\right)+3=0. \quad (51.1)$$

Entry 52 (p. 88). *If α, β, γ, and δ have degrees 1, 13, 5, and 65, respectively, then*

$$Q^6+\frac{1}{Q^6}-5\left(Q+\frac{1}{Q}\right)^2\left(T+\frac{1}{T}\right)^2-\left(T^4+\frac{1}{T^4}\right)=0. \quad (52.1)$$

Each of the previous eleven modular equations is of Schläfli–type. They are equations of composite degree that are analogues of the Schläfli modular equations of degrees 11, 13, 17, and 19 given in Entry 41. We have been unable to prove Entries 43–52 by employing ideas known to Ramanujan and so we have resorted to the theory of modular forms. By using (42.3), (42.4), (50.3), and (50.4), we may convert each of Entries 43-52 into a proposed identity involving modular forms.

Let $\Gamma(1)$ denote the full modular group, and let $\Gamma(2)$ and $\Gamma_0(n)$ be the subgroups usually so denoted (Part III [3, p. 327]). If $q = \exp(\pi i \tau)$, where $\tau \in \mathbb{H} = \{\tau : \operatorname{Im} \tau > 0\}$, $f_1(\tau) := q^{1/24} f(q)$, and $g_1(\tau) := \varphi(q)$, then f_1 and g_1 are modular forms of weight $\frac{1}{2}$ on $\Gamma(2)$ (Part III [3, pp. 330, 331]). Their multiplier systems may be found on page 331 of Part III [3]. In each of Entries 43–52, the degree $n = p_1 p_2$, where p_1 and p_2 are distinct odd primes. Each of the modular forms $f_1(m\tau)$ and $g_1(m\tau), m = 1, p_1, p_2, n$, appearing in these entries is a modular form of weight $\frac{1}{2}$ on $\Gamma := \Gamma(2) \cap \Gamma_0(n)$ (Part III [3, p. 332]). If n is any odd positive integer, then (Part III [3, p. 332])

$$(\Gamma(1) : \Gamma(2) \cap \Gamma_0(n)) = 6n \prod_{p|n} \left(1 + \frac{1}{p}\right), \tag{52.2}$$

where the product is over all odd primes p dividing n. It is easily checked that each of the products of powers of modular forms appearing in the translations of Entries 43–52 into modular forms is a modular form of weight 0 and multiplier system identically equal to 1.

As in Part III, the principal theorem to be utilized is the valence formula (Rankin [1, p. 98, Theorem 4.1.4], Part III [3, p. 329]). If F is a modular form of weight r and multiplier system v on Γ, a subgroup of finite index in $\Gamma(1)$, and if F is any fundamental set for Γ, then, provided that F is not constant,

$$\sum_{z \in \mathrm{F}} \operatorname{Ord}_\Gamma(F; z) = r\rho_\Gamma, \tag{52.3}$$

where $\operatorname{Ord}_\Gamma(F; z)$ denotes the order of F at z with respect to Γ and

$$\rho_\Gamma = \tfrac{1}{12}(\Gamma(1) : \Gamma). \tag{52.4}$$

In Entries 43–52, as already observed, $r = 0$, and so (52.3) reduces to the equality

$$\sum_{z \in \mathrm{F}} \operatorname{Ord}_\Gamma(F; z) = 0. \tag{52.5}$$

Since, in our applications, F is analytic on $\mathbb{H}$, then, from (52.5),

$$0 \geq \sum_{z \in \mathbb{C}} \operatorname{Ord}_\Gamma(F; z), \tag{52.6}$$

where $\mathbb{C}$ is a complete set of inequivalent cusps for Γ.

When $\Gamma = \Gamma(2) \cap \Gamma_0(n)$, and $n = p_1 p_2$, where p_1 and p_2 are distinct odd primes, then a complete set of inequivalent cusps is given by (Part III [3, p. 403])

$$\frac{1}{0}, \frac{2}{n}, \frac{1}{n}, \frac{1}{2p_1}, \frac{2}{p_1}, \frac{1}{p_1}, \frac{1}{2p_2}, \frac{2}{p_2}, \frac{1}{p_2}, \frac{1}{2}, \frac{2}{1}, \frac{1}{1}. \tag{52.7}$$

If $\mathbb{C}^* := \mathbb{C} - 1/0$, we rewrite (52.6) as

$$0 \geq \operatorname{Ord}_\Gamma(F;\infty) + \sum_{z\in\mathbb{C}^*} \operatorname{Ord}_\Gamma(F;z). \tag{52.8}$$

For a proposed identity

$$F(\tau) := \sum_{j=1}^{m} F_j(\tau) = 0 \tag{52.9}$$

involving modular forms $F_j(\tau)$, $1 \leq j \leq m$, of weight 0, we calculate a lower bound for the order of $F(\tau)$ at each cusp in $\mathbb{C}^*$. Thus, from (52.8), we obtain an upper bound for $\operatorname{Ord}_\Gamma(F;\infty)$. If we can show by a direct calculation of the Fourier expansion of F about $\tau = i\infty$, i.e., the power series of F about $q = 0$, that $\operatorname{Ord}_\Gamma(F;\infty)$ exceeds this bound, then we will obtain a contradiction to (52.8), unless $F(\tau)$ is a constant. Letting τ approach $i\infty$ (or $q \to 0$), we see that this constant must be 0. This then proves the proposed identity (52.9).

To calculate this bound, we need formulas for the orders of $f_1(m\tau)$ and $g_1(m\tau)$ at cusps. From Part III [3, p. 333, Lemma 0.1 and Table 1; p. 404, eq. (13.11)], we deduce the following formulas for orders at the cusps r/s, where $(r,s) = 1$:

$$48\operatorname{Ord}(f_1(m\tau); r/s) = \frac{(mr,s)^2}{m}(mr+s,2)^2 \tag{52.10}$$

and

$$24\operatorname{Ord}(g_1(m\tau); r/s) = \frac{(mr,s)^2}{m}\left((mr+s,2)^2 - 1\right), \tag{52.11}$$

where (a,b) denotes the greatest common divisor of a and b. We thus use (52.10) and (52.11) to calculate, for each of the eleven finite cusps in (52.7), the order of each modular form $f_1(m\tau)$ and $g_1(m\tau)$, $m = a,b,c,d$, appearing in Entries 43-52. Thus, a total of 88 orders for each entry are calculated. We then calculate the order of each term Q^j, P^j, R^j, T^j, P^iQ^j, T^iQ^j appearing in the proposed modular equations of Entries 43–52. This provides a lower bound for the order of each finite cusp on the left side of each equation. All of this work was programmed by Jeff Meyer on *Mathematica*. For each of Entries 43–52 we provide a table giving the eleven finite cusps and a lower bound for the order of each cusp on the left side of each modular equation.

cusp	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{2}$	1	2	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{3}$
order	$-\frac{1}{2}$	-1	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{30}$	$-\frac{1}{15}$	$-\frac{1}{30}$	$-\frac{1}{10}$	$-\frac{1}{10}$	$-\frac{1}{5}$

cusp	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{2}$	1	2	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{3}$
order	-1	-2	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{15}$	$-\frac{2}{15}$	$-\frac{1}{15}$	$-\frac{1}{5}$	$-\frac{1}{5}$	$-\frac{2}{5}$

cusp	$\frac{2}{21}$	$\frac{1}{21}$	$\frac{1}{14}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{1}{2}$	1	2	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{3}$
order	-2	-4	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{2}{21}$	$-\frac{4}{21}$	$-\frac{2}{21}$	$-\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{4}{7}$

cusp	$\frac{2}{21}$	$\frac{1}{21}$	$\frac{1}{14}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{1}{2}$	1	2	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{3}$
order	-2	-4	$-\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{4}{3}$	$-\frac{2}{21}$	$-\frac{4}{21}$	$-\frac{2}{21}$	$-\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{4}{7}$

cusp	$\frac{2}{33}$	$\frac{1}{33}$	$\frac{1}{22}$	$\frac{2}{11}$	$\frac{1}{11}$	$\frac{1}{2}$	1	2	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{3}$
order	-1	-2	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{33}$	$-\frac{2}{33}$	$-\frac{1}{33}$	$-\frac{1}{11}$	$-\frac{1}{11}$	$-\frac{2}{11}$

cusp	$\frac{2}{35}$	$\frac{1}{35}$	$\frac{1}{14}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{1}{2}$	1	2	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{5}$
order	-2	-4	$-\frac{2}{5}$	$-\frac{2}{5}$	$-\frac{4}{5}$	$-\frac{2}{35}$	$-\frac{4}{35}$	$-\frac{2}{35}$	$-\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{4}{7}$

cusp	$\frac{2}{55}$	$\frac{1}{55}$	$\frac{1}{22}$	$\frac{2}{11}$	$\frac{1}{11}$	$\frac{1}{2}$	1	2	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{5}$
order	-3	-6	$-\frac{3}{5}$	$-\frac{3}{5}$	$-\frac{6}{5}$	$-\frac{3}{55}$	$-\frac{6}{55}$	$-\frac{3}{55}$	$-\frac{3}{11}$	$-\frac{3}{11}$	$-\frac{6}{11}$

cusp	$\frac{2}{35}$	$\frac{1}{35}$	$\frac{1}{14}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{1}{2}$	1	2	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{5}$
order	$-\frac{3}{2}$	-3	$-\frac{3}{10}$	$-\frac{3}{10}$	$-\frac{3}{5}$	$-\frac{3}{70}$	$-\frac{3}{35}$	$-\frac{3}{70}$	$-\frac{3}{14}$	$-\frac{3}{14}$	$-\frac{3}{7}$

cusp	$\frac{2}{39}$	$\frac{1}{39}$	$\frac{1}{26}$	$\frac{2}{13}$	$\frac{1}{13}$	$\frac{1}{2}$	1	2	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{3}$
order	-1	-2	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{39}$	$-\frac{2}{39}$	$-\frac{1}{39}$	$-\frac{1}{13}$	$-\frac{1}{13}$	$-\frac{2}{13}$

cusp	$\frac{2}{65}$	$\frac{1}{65}$	$\frac{1}{26}$	$\frac{2}{13}$	$\frac{1}{13}$	$\frac{1}{2}$	1	2	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{5}$
order	-3	-6	$-\frac{3}{5}$	$-\frac{3}{5}$	$-\frac{6}{5}$	$-\frac{3}{65}$	$-\frac{6}{65}$	$-\frac{3}{65}$	$-\frac{3}{13}$	$-\frac{3}{13}$	$-\frac{6}{13}$

In the next table we sum the orders of the eleven cusps and then use (52.8) to determine the minimal number of coefficients in the q-expansion of the left side that we must show are equal to 0 in order to prove the given modular equation.

Entry	Order	Number of Terms
43	$-\frac{27}{10}$	3
44	$-\frac{27}{5}$	6
45	$-\frac{214}{21}$	11
46	$-\frac{214}{21}$	11
47	$-\frac{53}{11}$	5
48	$-\frac{314}{35}$	9
49	$-\frac{699}{55}$	13
50	$-\frac{471}{70}$	7
51	$-\frac{185}{39}$	5
52	$-\frac{813}{65}$	13

In each instance we employed *Mathematica* to compute the q-expansion of the left side of each proposed theta-function identity. In each case, we determined that each of the required coefficients is equal to zero.

7. Modular Equations in the Form of Russell

For Entries 53–57 we define

$$P = 1 \pm (\alpha\beta)^{1/8} \pm \{(1-\alpha)(1-\beta)\}^{1/8}, \tag{53.1}$$

$$Q = 4\left((\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} \pm \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\right), \tag{53.2}$$

and

$$R = 4\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}. \tag{53.3}$$

We shall give four definitions for the triples P, Q, and R in this section. Russell [1], [2] used the same notation, but his definitions of P, Q, and R are different. In Ramanujan's definitions of Q and R, powers of 2 occur which to do not appear in Russell's definitions. Moreover, the signs of Russell's P, Q, and R are generally different from those of Ramanujan.

Entry 53 (p. 302). *Let P, Q, and R be given by (53.1)–(53.3), with the plus signs taken. Then, if β has degree 15 over α,*

$$P(P^2 - Q) + R = 0. \tag{53.4}$$

An equivalent formulation of (53.4) was first proved by Russell [2, p. 388].

Entry 54 (p. 302). *Let P, Q, and R be given by (53.1)–(53.3), with the plus signs taken. Suppose that β has degree 31 over α. Then*

$$P^2 - Q = \sqrt{PR}.$$

Although not immediately obvious, Entry 54 is equivalent to Entry 22(ii) of Chapter 20 (Part III [3, p. 439]) after elementary manipulation.

Entry 55 (p. 302). *Let P, Q, and R be given by (53.1)–(53.3), with the plus signs taken. Suppose that β has degree 47 over α. Then*

$$P^2 - Q - PR^{1/3} - 2R^{2/3} = 0.$$

Entry 55 is identical to Russell's modular equation of degree 47 quoted in our book, Part III [3, p. 445, eq. (23.1)]. Note, however, that the notations in Russell's paper [2], Part III [3], and here are all different. The equivalence of Ramanujan's and Russell's formulations is provided in detail in the proof of Entry 56 below.

Entry 56 (p. 302). *Let P, Q, and R be given by (53.1)–(53.3), with the minus signs taken. Suppose that β has degree 71 over α. Then*

$$P^3 - R^{1/3}(4P^2 + Q) + 2PR^{2/3} - R = 0. \tag{56.1}$$

Proof. In the notation above, Russell's modular equation of degree 71 takes the form (Part III [3, Chap. 20, p. 445, eq. (23.2)])

$$P^3 - 4R^{1/3}(P^2 - P + 1) + 2PR^{2/3} - R + R^{4/3} = 0. \tag{56.2}$$

Comparing (56.1) and (56.2), we find that it suffices to prove that

$$4R^{1/3}(P-1) + R^{4/3} = -R^{1/3}Q,$$

or

$$4P - 4 + R = -Q. \tag{56.3}$$

Using the definitions of P, Q, and R, we see that (56.3) is a triviality, and so the proof is complete.

Entry 57 (p. 302). *Let P, Q, and R be given by (53.1)–(53.3), with the plus signs taken. Then, if β has degree 95 over α,*

$$(P^2-Q)^2P - R^{1/3}(5P^4-4QP^2) + PQR^{2/3} - R(P^2+Q) - 5PR^{4/3} - 2R^{5/3} = 0. \tag{57.1}$$

An equivalent version of (57.1) was proved by Russell [2, p. 389]. In fact, the term $-2R^{5/3}$ does not appear in the notebooks, a fact also observed by Ramanathan [8].

For Entries 58–61 we define

$$P = 1 - (\alpha\beta)^{1/4} - \{(1-\alpha)(1-\beta)\}^{1/4}, \tag{58.1}$$

$$Q = 16\left((\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}\right), \tag{58.2}$$

and

$$R = 16\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}. \tag{58.3}$$

Entry 58 (p. 302). *Let P, Q, and R be given by (58.1)–(58.3), respectively. Then, if β has degree 19 over α,*

$$P^5 - 7P^2R - QR = 0. \tag{58.4}$$

An equivalent form of (58.4) was first proved by Russell [1, p. 99], [2, p. 389].

Entry 59 (p. 302). *Let P, Q, and R be given by (58.1)–(58.3), respectively. Then, if β has degree 27 over α,*

$$P^9 - RP^2(29P^4 + 11P^2Q + Q^2) - 17R^2P^3 - 3R^2(PQ + R) = 0.$$

This modular equation was first proved by E. Fiedler [1, p. 226] in 1885, although his formulation is much different. M. Hanna [1, p. 49] translated Fiedler's equation into the form above, but with different signs on P, Q, and R. Hanna's formulation unfortunately contains three errors; the signs of the terms $29P^4$ and $11P^2Q$ are incorrect, and the coefficient 3 of R^3 was omitted.

Entry 60 (p. 302). *Let P, Q, and R be given by (58.1)–(58.3), respectively. Then, if β has degree 35 over α,*

$$P^4 - R^{1/3}(5P^3 + PQ) + 2P^2R^{2/3} - RP - R^{4/3} = 0. \tag{60.1}$$

An equivalent version of (60.1) was established by Russell [2, p. 389].

Entry 61 (p. 304). *Let P, Q, and R be given by (58.1)–(58.3), respectively. Then, if β has degree 59 over α,*

$$P^5 - R^{1/3}(14P^4 + 9P^2Q + Q^2) + R^{2/3}P(19P^2 + Q) \\ + 6R(7P^2 + Q) + 4PR^{4/3} - 8R^{5/3} = 0.$$

This result is due to Russell [2, pp. 367, 389]. The sign of $4PR^{4/3}$ is different in Russell's two formulations, with that on page 389 being correct. In both formulations, Russell wrote $-42P^2R$ instead of the correct expression $42P^2R$. Ramanujan also made a mistake; he wrote $-3R^{5/3}$ instead of $-8R^{5/3}$.

For Entries 62–66 we define

$$P = 1 - \sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}, \tag{62.1}$$

$$Q = 64\left(\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} - \sqrt{\alpha\beta(1-\alpha)(1-\beta)}\right), \tag{62.2}$$

and

$$R = 32\sqrt{\alpha\beta(1-\alpha)(1-\beta)}. \tag{62.3}$$

Entry 62 (p. 304). *If P, Q, and R are defined by (62.1)–(62.3), respectively, then, if β has degree 9 over α,*

$$P^6 - R(14P^3 + PQ) - 3R^2 = 0.$$

An equivalent form of this modular equation was first established by Russell [2, p. 390].

Entry 63 (p. 304). *If P, Q, and R are defined by (62.1)–(62.3), respectively, then, if β has degree 13 over α,*

$$\sqrt{P}(P^3 + 8R) - \sqrt{R}(11P^2 + Q) = 0. \tag{63.1}$$

Proof. With

$$P = -P_0, \qquad Q = -64Q_0, \qquad \text{and} \qquad R = -32R_0, \tag{63.2}$$

Russell [1, p. 105] proved that

$$P_0^7 - 2^5 R_0(105P_0^4 - 2^7 \cdot 11P_0^2Q_0 + 2^{12}Q_0^2) + 2^{16}R_0^2P_0 = 0,$$

which in Ramanujan's notation becomes

$$P^7 - R(105P^4 + 22P^2Q + Q^2) + 64R^2P = 0. \tag{63.3}$$

For simplicity, set $P = p^2$, $Q = q$, and $R = r^2$. Then, from (63.3),

$$\begin{aligned} p^{14} &- 105r^2p^8 - 22r^2qp^4 + 64r^4p^2 - r^2q^2 \\ &= (p^7 - 11rp^4 + 8r^2p - rq)(p^7 + 11rp^4 + 8r^2p + rq) = 0. \end{aligned}$$

Clearly,

$$p^7 + 11rp^4 + 8r^2p + rq \neq 0.$$

Thus,

$$p^7 - 11rp^4 + 8r^2p - rq = 0,$$

which is equivalent to (63.1).

Entry 64 (p. 304). *If P, Q, and R are defined by (62.1)–(62.3), respectively, then, if β has degree 17 over α,*

$$P^3 - R^{1/3}(10P^2 + Q) + 13R^{2/3}P + 12R = 0.$$

Proof. With the same notation as in the previous proof, Russell [2, p. 390] proved that

$$P_0^3 + (4R_0)^{1/3}(2^7Q_0 - 20P_0^2) + 52(4R_0)^{2/3}P_0 + 384R_0 = 0.$$

It is now an easy task to show that this equation is equivalent to that of Ramanujan.

Entry 65 (p. 304). *If P, Q, and R are defined by (62.1)–(62.3), respectively, then, if β has degree 29 over α,*

$$\sqrt{P}(P^2 + 17PR^{1/3} - 9R^{2/3}) - R^{1/6}(9P^2 + Q - 13PR^{1/3} + 15R^{2/3}) = 0. \tag{65.1}$$

This modular equation was essentially proved by Russell [2, p. 390]. As in the foregoing entries, Russell used the altered notation (63.2). Russell has a sign ambiguity in his formulation of (65.1); corresponding to the last expression in (65.1), Russell writes (in our notation) $\pm R^{1/6}(9P^2 + Q - 13PR^{1/3} + 15R^{2/3})$.

Entry 66 (p. 310). *If P, Q, and R are defined by (62.1)–(62.3), respectively, then, if β has degree 37 over α,*

$$P^3 - R^{1/3}(7P^2 + Q) - 3PR^{2/3} - 25R - M(19P^2 - \&) + \&. \tag{66.1}$$

As evinced by the two "&'s," Ramanujan evidently failed to determine an exact modular equation of degree 37. In fact, Ramanujan does not define P, Q, R, and M, but since $37 \equiv 5 \pmod 8$, the definitions of P, Q, and R are probably given by (62.1)–(62.3).

Discussing a probable modular equation of degree 37, Russell [2, p. 363] remarks, "This is of the 19th degree in $\kappa\lambda$, $\kappa'\lambda'$, and is not reducible to a simplified form. I have consequently made no attempt to form it." Contradicting Russell, Hanna [1, p. 48] derived a modular equation of degree 37 in the 19/2 degree in $\kappa\lambda$, $\kappa'\lambda'$. His modular equation is given in terms of L, M, and N, where $L = -P$, $M = -Q$, and $N = -R$, where P, Q, and R are defined in (62.1)–(62.3). Moreover, his equation has 18 terms with coefficients ranging up to 923722. In Ramanujan's notation, Hanna's "leading term" is $P^{19/2}$.

For none of the fourteen preceding modular equations, did Ramanujan record the definitions of P, Q, and R. We were only able to discern the correct definitions with the help of Russell's work.

Page 303 is an enigma. For the material on the top two–thirds of the page, we quote Ramanujan below.

$$7, 23\& \begin{cases} P = qf(q^2, q^{14}) - q^{(n+1)/8}\psi(q^2) \\ Q = \left\{\varphi(-q^2) + 4q^{(n+9)/8} f(q^2, q^{14})\right\} \psi(q^2) \\ R = q^{(n+1)/8} f^3(-q^4) \end{cases}$$

$$3, 11\& \begin{cases} P = q\psi(q^8) - q^{(n+1)/4}\psi(q^2) \\ Q = \varphi^2(-q^2) + 16q^{(n+5)/4}\psi(q^2)\psi(q^8) \\ R = q^{(n+1)/4} f^3(-q^2) \end{cases}$$

$$1, 5\& \begin{cases} P = \left\{q\psi(q^4) - 2q^{(n+1)/2}\varphi(q^2)\right\} \psi(q^4) \\ Q = \varphi^4(-q^2) + 128q^{(n+3)/2}\psi^2(q^2)\psi^2(q^4) \\ R = q^{(n+1)/2} f^6(-q^2) \end{cases}$$

$$15, 31\& \begin{cases} P = f(q^6, q^{10}) + q^{(n+1)/8}\psi(q^2) \\ Q = \left\{\varphi(-q^2) + 4q^{(n+1)/8} f(q^6, q^{10})\right\} \psi(q^2) \\ R = q^{(n+1)/8} f^3(-q^4) \\ P_2 = P^2 - Q = q^2 f^2(q^2, q^{14}) - 2q^{(n+1)/8} f(q^6, q^{10})\psi(q^2) \\ \qquad + q^{(n+1)/4}\psi^2(q^2) \end{cases}$$

The conditions at the left margin evidently indicate that the definitions to the right hold for $n \equiv 7 \pmod{16}$, $n \equiv 3 \pmod 8$, $n \equiv 1 \pmod 4$, and $n \equiv 15 \pmod{16}$, respectively. These definitions of P, Q, and R do not correspond to any other definitions of P, Q, and R that might arise in modular equations. In particular, some of the terms comprising P and Q do not depend upon n, which evidently denotes the degree of the modular equation, and the dependence of R on n is too simple.

At the bottom of the page, Ramanujan cryptically lists four triples of formulas for P, Q, and R, which we think are (53.1)–(53.3), (58.1)–(58.3), and (62.1)–(62.3), i.e., in more detail,

$$\begin{cases} P = 1 + (\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} \\ Q = 4\left((\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\right) \\ R = 4\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \end{cases}$$

$$\begin{cases} P = 1 - (\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8} \\ Q = 4\left((\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\right) \\ R = 4\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \end{cases}$$

$$\begin{cases} P = 1 - (\alpha\beta)^{1/4} - \{(1-\alpha)(1-\beta)\}^{1/4} \\ Q = 16\left((\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}\right) \\ R = 16\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4} \end{cases}$$

$$\begin{cases} P = 1 - \sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)} \\ Q = 64\left(\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} - \sqrt{\alpha\beta(1-\alpha)(1-\beta)}\right) \\ R = 32\sqrt{\alpha\beta(1-\alpha)(1-\beta)} \end{cases}$$

Only for the first triple, does Ramanujan give complete definitions. The definitions of Q and R in the remaining three triples are abbreviated by 4, 4; 16, 16; and 64, 32, respectively. One might surmise that the four triples of P, Q, and R at the bottom of the page correspond to the four sets of definitions in the top two–thirds of the page. However, there appears to be very little resemblance between the two sets of formulas. For example, consider the third triple of formulas at the bottom of the page. Transcribing P and Q via Entries 10(i), (ii) and 11(iii) of Chapter 17 (Part III [3, pp. 122–124]), and also employing Entries 24(iv) and 25(iv) of Chapter 16 (Part III [3, pp. 39–40]), we find that

$$P = 1 - 4q^{(n+1)/4}\frac{\psi(q^2)\psi(q^{2n})}{\varphi(q)\varphi(q^n)} - \frac{\varphi(-q)\varphi(-q^n)}{\varphi(q)\varphi(q^n)}$$

and

$$R = 64q^{(n+1)/4}\frac{f^3(-q^2)f^3(-q^{2n})}{\varphi^3(q)\varphi^3(q^n)}.$$

Any connections between these two formulas and those for P and Q in the third (or any) set at the top of the page appear, at best, artificial.

Observe that in Entries 53–57, the degree $n \equiv 7 \pmod 8$, and that the definitions of P, Q, and R are given in the first set of definitions at the bottom of page 303. In Entries 58–61, each degree $n \equiv 3 \pmod 8$, and the definitions of P, Q, and R are given by the third set of formulas at the bottom of page 303, but that in the third set of formulas at the top of the page, $n \equiv 1 \pmod 4$. Lastly, in Entries

62–66, $n \equiv 1 \pmod 4$, and the formulas for P, Q, and R are given by the last set of formulas at the bottom of the page, but at the top of the page in the last set of formulas, $n \equiv 15 \pmod{16}$.

Finally, we provide a proof of the second equality for P_2 given after the last triple for P, Q, and R at the top of page 303. A brief calculation shows that this equality is equivalent to the following identity.

Lemma 67 (p. 303).

$$\varphi(-q^2)\psi(q^2) = f^2(q^6, q^{10}) - q^2 f^2(q^2, q^{14}).$$

Proof. From the corollary to Entry 31 of Chapter 16 (Part III [3, p. 49]),

$$\psi(q) = f(q^6, q^{10}) + q f(q^2, q^{14}).$$

Replacing q by $-q$, we find that

$$\psi(-q) = f(q^6, q^{10}) - q f(q^2, q^{14}).$$

Upon multiplication,

$$\psi(q)\psi(-q) = f^2(q^6, q^{10}) - q^2 f^2(q^2, q^{14}). \tag{67.1}$$

But by Entry 25(iii) of Chapter 16 (Part III [3, p. 40]),

$$\psi(q)\psi(-q) = \varphi(-q^2)\psi(q^2). \tag{67.2}$$

Combining (67.1) and (67.2), we complete the proof.

8. Modular Equations in the Form of Weber

The next six modular equations are of Weber–type [1]. The definitions of P, Q, and R given below were not recorded by Ramanujan. We have deduced these definitions from Weber's modular equations, but we have replaced Ramanujan's R by R^3. Let $n = n_1 n_2$, where n_1 and n_2 are primes, which are not necessarily distinct. Suppose that μ is defined by

$$(n_1 + 1)(n_2 + 1) = 8\mu \equiv 0 \pmod 8. \tag{68.1}$$

Define

$$P = 1 + (-1)^{\mu}\left((\alpha\beta\gamma\delta)^{1/8} + ((1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/8}\right), \tag{68.2}$$

$$\begin{aligned} Q = 4\big((\alpha\beta\gamma\delta)^{1/8} &+ ((1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/8} \\ &+ (-1)^{\mu}(\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/8}\big), \end{aligned} \tag{68.3}$$

and

$$R = (256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/24}. \tag{68.4}$$

Suppose that α, β, γ, and δ have degrees 1, b, c, and d, respectively. Then using Entries 10(i), (iii), 11(i), and 12(i) of Chapter 17 (Part III [3, pp. 122–124]), we can express (68.2)–(68.4) in the forms

$$P = 1 + (-1)^{\mu}\left(\frac{4q^{(1+b+c+d)/8}\psi(q)\psi(q^b)\psi(q^c)\psi(q^d)}{\varphi(q)\varphi(q^b)\varphi(q^c)\varphi(q^d)} + \frac{\varphi(-q^2)\varphi(-q^{2b})\varphi(-q^{2c})\varphi(-q^{2d})}{\varphi(q)\varphi(q^b)\varphi(q^c)\varphi(q^d)}\right), \tag{68.5}$$

$$\begin{aligned} Q = 4\Bigg(&\frac{4q^{(1+b+c+d)/8}\psi(q)\psi(q^b)\psi(q^c)\psi(q^d)}{\varphi(q)\varphi(q^b)\varphi(q^c)\varphi(q^d)} + \frac{\varphi(-q^2)\varphi(-q^{2b})\varphi(-q^{2c})\varphi(-q^{2d})}{\varphi(q)\varphi(q^b)\varphi(q^c)\varphi(q^d)} + (-1)^{\mu} \\ &\times \frac{4q^{(1+b+c+d)/8}\psi(q)\psi(q^b)\psi(q^c)\psi(q^d)\varphi(-q^2)\varphi(-q^{2b})\varphi(-q^{2c})\varphi(-q^{2d})}{\varphi^2(q)\varphi^2(q^b)\varphi^2(q^c)\varphi^2(q^d)}\Bigg), \end{aligned} \tag{68.6}$$

and

$$R = 2q^{(1+b+c+d)/24}\frac{f(q)f(q^b)f(q^c)f(q^d)}{\varphi(q)\varphi(q^b)\varphi(q^c)\varphi(q^d)}. \tag{68.7}$$

Entry 68 (p. 309). *Let α, β, γ, and δ have degrees 1, 3, 11, and 33, respectively, and let P, Q, and R be given by (68.2)–(68.4), respectively. Then*

$$P^2 - Q - PR - 4R^2 = 0.$$

Entry 69 (p. 309). *Let α, β, γ, and δ have degrees 1, 5, 7, and 35, respectively, and let P, Q, and R be given by (68.2)–(68.4), respectively. Then*

$$P^2 - Q - PR - 2R^2 = 0.$$

Entry 70 (p. 309). *Let α, β, γ, and δ have degrees 1, 5, 11, and 55, respectively, and let P, Q, and R be given by (68.2)–(68.4), respectively. Then*

$$P^3 - R(4P^2 + Q) - PR^2 + 4R^3 = 0.$$

Each of the three preceding modular equations was first established by Weber [1]. The first two were not explicitly recorded by Ramanujan, but he must have found them, for he wrote, "N.B. For 1, 5, 7, 35 same as for 1, 3, 11, 33, $4R^{2/3}$ instead of $2R^{2/3}$."

The next three equations were not discovered by Weber, and we shall use the theory of modular forms, as described in Section 6, to prove them.

Entry 71 (p. 309). *Let α, β, γ, and δ have degrees 1, 3, 17, and 51, respectively, and let P, Q, and R be given by (68.2)–(68.4), respectively. Then*

$$P^3 - R(7P^2 + Q) + 13R^2P - 12R^3 = 0. \tag{71.1}$$

Proof. We proceed as in the proofs of Entries 43–52. For $q = \exp(\pi i\tau)$, $\tau \in \mathbb{H}$, the functions $h_2(\tau) := \varphi(-q^2)$ and $h_0(\tau) := q^{1/8}\psi(q)$ are modular forms of weight $\frac{1}{2}$ on $\Gamma(2)$. Thus, (68.5)–(68.7) can be expressed in terms of the modular forms f_1, g_1, h_0, and h_2, where f_1 and g_1 were defined prior to (52.2). From Part III [3, p. 333], for each positive integer n and cusp r/s, where $(r, s) = 1$, we deduce that

$$48\mathrm{Ord}(h_0(n\tau); r/s) = \frac{(nr, s)^2}{n}\left(4 - (nr, 2)^2\right) \tag{71.2}$$

and

$$48\mathrm{Ord}(h_2(n\tau); r/s) = \frac{(nr, s)^2}{n}\left(4 - (s, 2)^2\right). \tag{71.3}$$

Using (52.10), (52.11), (71.2), and (71.3), we calculate, for each finite cusp, the order of each function $f_1(m\tau), g_1(m\tau), h_0(m\tau)$, and $h_2(m\tau), m = 1, 3, 17, 51$, appearing in the translation of (71.1) into modular forms. We then calculate the order of each term in (71.1) at each finite cusp, so that we can obtain a lower bound for the order of the left side of (71.1) at each finite cusp. This was programmed by J. Meyer on *Mathematica*, and the following table summarizes the calculations:

cusp	$\frac{2}{51}$	$\frac{1}{51}$	$\frac{1}{34}$	$\frac{2}{17}$	$\frac{1}{17}$	$\frac{1}{2}$	1	2	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{3}$
order	0	$-\frac{27}{2}$	0	0	$-\frac{9}{2}$	0	$-\frac{9}{34}$	0	0	0	$-\frac{27}{34}$

Hence, if $F(\tau)$ denotes the left side of (71.1), after it is converted into modular forms,

$$\sum_{z \in \mathbb{C}^*} \mathrm{Ord}(F; z) = -\frac{324}{17}.$$

We thus need to show that the first 20 coefficients in the q-expansion of $F(\tau)$ are equal to 0. With the use of (68.5)–(68.7) and *Mathematica*, this can be shown, and so the proof of Entry 71 is complete.

For the next two modular equations, the degree n is not the product of two odd primes. More precisely, $n = 9$ and 27, respectively. We were unable to find a complete set of inequivalent cusps for $\Gamma(2) \cap \Gamma_0(9)$ and $\Gamma(2) \cap \Gamma_0(27)$ in the literature, nor were we able to determine complete sets. Thus, we employed the valence formula (52.3) in a way different from that in Section 6. We take the two theta-function identities equivalent to the two proposed modular equations and clear denominators. This then yields identities involving products of modular forms, with each product having a positive weight and nonnegative order at each cusp. Let $F(\tau), q = \exp(\pi i \tau)$, denote the left side of one of the two proposed theta–function identities (set equal to 0). We show that F is a modular form of weight r, say, on Γ, where Γ is one of the two modular subgroups mentioned above. If the coefficients of $q^0, q^1, q^2, \dots, q^\nu$ in the expansion of F about $q = 0$ are equal to 0, it follows that $\mathrm{Ord}_\Gamma(F; \infty) \geq \nu + 1$. Suppose further that $\nu + 1 > r\rho_\Gamma$. Then

$$\sum_{z \in \mathbb{F}} \mathrm{Ord}_\Gamma(F; z) \geq \mathrm{Ord}_\Gamma(F; \infty) \geq \nu + 1 > r\rho_\Gamma. \tag{72.1}$$

This then yields a contradiction to (52.3) unless F is constant, which must be the case. This constant trivially is equal to 0, and so the modular equation is proved.

Entry 72 (p. 309). *Let α, β, γ, and δ have degrees 1, 3, 3, and 9, respectively, and let P, Q, and R be defined by (68.2)–(68.4). Then*

$$P^3 - PQ + 3R^3 = 0. \tag{72.2}$$

Proof. Substituting (68.5)–(68.7) into (72.2) and multiplying both sides by $w^3(q)$, where

$$w(q) = \varphi(q)\varphi^2(q^3)\varphi(q^9),$$

to clear denominators, we find that

$$p^3 - ps + 3r^3 = 0, \tag{72.3}$$

where

$$p := p(q) := w(q)P, \quad s := s(q) := w^2(q)Q, \quad \text{and} \quad r := r(q) := w(q)R. \tag{72.4}$$

Each of the three terms in (72.3) has weight 6 and multiplier system identically equal to 1. By (52.2), if $\Gamma = \Gamma(2) \cap \Gamma_0(9)$, then $(\Gamma(1) : \Gamma) = 72$. Hence, by (52.4), $r\rho_\Gamma = 36$. Thus, expanding $p^3 - ps + 3r^3$ in a q–series with the help of *Mathematica*, we find that the coefficients of q^n, $0 \leq n \leq 36$, are equal to 0. Thus, (72.3), and hence (72.2), are established to complete the proof.

Entry 73 (p. 309). *Let α, β, γ, and δ have degrees 1, 3, 9, and 27, respectively, and let P, Q, and R be defined by (68.2)–(68.4). Then*

$$P^9 - R^3P^4(11P^2 + Q) + 9R^6P^3 + 6R^9 = 0. \tag{73.1}$$

Disproof. Substituting (68.5)–(68.7) into (73.1) and multiplying both sides by $w^9(q)$, where

$$w(q) = \varphi(q)\varphi(q^3)\varphi(q^9)\varphi(q^{27}),$$

we obtain the proposed equation

$$p^9 - r^3p^4(11p^2 + s) + 9r^6p^3 + 6r^9 = 0, \tag{73.2}$$

where p, s, and r are defined by the same equations as (72.4). Each of the terms in (73.2) has weight 18 and multiplier system identically equal to 1. By (52.2), if $\Gamma = \Gamma(2) \cap \Gamma_0(27)$, then $(\Gamma(1) : \Gamma) = 216$, and so, by (52.4), $r\rho_\Gamma = 324$. Expanding the left side of (73.2) in a q-series with the aid of *Mathematica*, we find that

$$p^9 - r^3p^4(11p^2+s)+9r^6p^3+6r^9 = 46080q^{19}+414720q^{20}+1175040q^{21}+\cdots.$$

Thus, the proposed modular equation is false. That the first 19 coefficients in the q-series are equal to 0 is evidence that Ramanujan used a method of comparing coefficients and did not compute enough of them, or made an error in computing them.

As in the previous entries, Ramanujan does not provide the definitions of P, Q, and R in the next entry. In the notation (68.1), $n_1 = n_2 = 5$, and so (68.1) is not satisfied by any integer μ. Thus, it might be that P, Q, and R are not defined by (68.2)–(68.4). In fact, we employed (68.2)–(68.4) twice, once with the plus signs and once with the minus signs, in an attempt to determine if these are the correct definitions for Entry 74, and indeed they are not. We tried several possible definitions for P, Q, and R, but none were viable. The definitions that came the "closest" are given by

$$P = 1 - (\alpha\beta\gamma\delta)^{1/4} - ((1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/4}, \tag{74.1}$$

$$\begin{aligned} Q = 16\big(&(\alpha\beta\gamma\delta)^{1/4} + ((1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/4} \\ &- (\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/4}\big), \end{aligned} \tag{74.2}$$

and

$$R = 2^{2/3}(\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/12}. \tag{74.3}$$

Translating (74.1)–(74.3) via Entries 10(i), (ii), 11(iii), and 12(iii) of Chapter 17 (Part III [3, pp. 122–124]), we find that

$$P = 1 - \frac{16q^9\psi(q^2)\psi^2(q^{10})\psi(q^{50})}{\varphi(q)\varphi^2(q^5)\varphi(q^{25})} - \frac{\varphi(-q)\varphi^2(-q^5)\varphi(-q^{25})}{\varphi(q)\varphi^2(q^5)\varphi(q^{25})}, \tag{74.4}$$

$$\begin{aligned} Q = 16\Bigg(&\frac{16q^9\psi(q^2)\psi^2(q^{10})\psi(q^{50})}{\varphi(q)\varphi^2(q^5)\varphi(q^{25})} + \frac{\varphi(-q)\varphi^2(-q^5)\varphi(-q^{25})}{\varphi(q)\varphi^2(q^5)\varphi(q^{25})} \\ &- \frac{16q^9\psi(q^2)\psi^2(q^{10})\psi(q^{50})\varphi(-q)\varphi^2(-q^5)\varphi(-q^{25})}{\varphi^2(q)\varphi^4(q^5)\varphi^2(q^{25})}\Bigg), \end{aligned} \tag{74.5}$$

and

$$R = 4q^3\frac{f(-q^2)f^2(-q^{10})f(-q^{50})}{\varphi(q)\varphi^2(q^5)\varphi(q^{25})}. \tag{74.6}$$

Entry 74 (p. 309). *Let α, β, γ, and δ have degrees 1, 5, 5, and 25, respectively. Then (possibly) for P, Q, and R defined by (74.1)–(74.3), Ramanujan claims that*

$$P^4 - PR(5P^2 + Q) - P^2R^2 + 3PR^3 - R^4 = 0. \tag{74.7}$$

Disproof. Transcribe (74.7) into a proposed theta–function identity by (74.4)–(74.6). Then clear fractions by multiplying both sides by $t^4(q)$, where $t(q) = \varphi(q)\varphi^2(q^5)\varphi(q^{10})$. Setting $p := p(q) := t(q)P$, $s := s(q) := t^2(q)Q$, and $r := r(q) := t(q)R$, we find that (74.7) is equivalent to the proposed identity

$$l(q) := p^4 - pr(5p^2 + s) - p^2r^2 + 3pr^3 - r^4 = 0. \tag{74.8}$$

Expanding $l(q)$ in a q–series, we find that

$$l(q) = 768q^8 - 2560q^{10} + 7680q^{12} - \cdots .$$

Thus, Ramanujan's claim (74.7), if we have interpreted it properly, is false.

By successively changing the coefficients of p^2r^2, pr^3, and r^4 in (74.8), we can increase the power of the leading term of $l(q)$ by 2. For example, if we replace $l(q)$ by

$$l^*(q) := p^4 - pr(5p^2 + s) - 4p^2r^2 + 7pr^3 - 10r^4,$$

we find that

$$l^*(q) = -1280q^{14} + 10240q^{16} + \cdots .$$

Thus, we were never able to alter sufficiently the definition of $l(q)$ to produce a function that was identically equal to 0.

We reluctantly conclude that Ramanujan either miscalculated some coefficients in the q–series for $l(q)$, or did not calculate enough of them.

The next entry does not properly fit in this section, but we place it here simply because we must again resort to the theory of modular forms to effect a proof.

Entry 75 (p. 298). *Let β have degree 49 over α. Put*

$$P = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} + \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8},$$

$$Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}\left(\left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} + 1\right),$$

and

$$R = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12}.$$

Then

$$\left(P + R - \sqrt{m}\right)^2 - (4Q + 4PR + 13R^2) = 0, \tag{75.1}$$

where m denotes the multiplier of degree 49.

Proof. By using Entries 10(i), (ii), 11(iii), and 12(iii) of Chapter 17 (Part III [3, pp. 122–124]), we can transform (75.1) into a theta–function identity. Clearing denominators, we then deduce the proposed identity

$$(p+r-t)^2 - 4s - 4pr - 13r^2 = 0, \tag{75.2}$$

where

$$\begin{aligned} p &= q^6 f(-q^2)\varphi(q)\varphi(q^{49})\varphi(-q^2)\psi(q^{49}) + f(-q^2)\varphi(q)\varphi(q^{49})\varphi(-q^{98})\psi(q) \\ &\quad + q^6 f(-q^2)\varphi^2(q)\varphi(-q^{98})\psi(q^{49}), \\ s &= q^6 f^2(-q^2)\varphi^2(q)\varphi(-q^{98})\psi(q^{49})\left(q^6\varphi(q)\varphi(q^{49})\varphi(-q^2)\psi(q^{49})\right. \\ &\quad \left.+\varphi(q)\varphi(q^{49})\varphi(-q^{98})\psi(q) + \varphi^2(q^{49})\varphi(-q^2)\psi(q)\right), \\ r &= q^4 f(-q^{98})\varphi(q)\varphi(q^{49})\varphi(-q^2)\psi(q), \end{aligned}$$

and

$$t = f(-q^2)\varphi(q)\varphi(q^{49})\varphi(-q^2)\psi(q).$$

If $q = \exp(\pi i \tau)$, where $\tau \in \mathbb{H}$, then $\eta(\tau) = q^{1/12} f(-q^2)$, $g_1(\tau) = \varphi(q)$, $g_2(\tau) = q^{1/4}\psi(q^2)$, and $h_2(\tau) = \varphi(-q^2)$ are modular forms of weight $\frac{1}{2}$ on $\Gamma(2)$ (Part III [3, pp. 330, 331]). The multiplier system for each term in (75.2) is the same, and each term has weight 5. By (52.2), if $\Gamma = \Gamma(2) \cap \Gamma_0(49)$, then $(\Gamma(1) : \Gamma) = 336$, and, by (52.4), $\rho_\Gamma = 140$. Thus, by (72.1), to prove (75.2), we must show that each of the coefficients of q^n, $0 \le n \le 140$, on the left side of (75.2) is equal to 0. Using *Mathematica*, we have indeed shown this, and so the proof of Entry 75 is complete.

To the best of our knowledge, Entry 75 is the first and only modular equation of degree 49 that can be found in the literature.

9. Series Transformations Associated with Theta–Functions

Recall from Section 6 of Chapter 17 (Part III [3, p. 101]) the fundamental notation

$$e^{-y} := F(x) := \exp\left(-\pi \frac{z(1-x)}{z(x)}\right), \tag{76.1}$$

where

$$z := z(x) := {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; x\right). \tag{76.2}$$

Entries 76 and 77 are beautiful series transformation formulas involving the variables x, y, and z above and should be compared with Entry 6 of Chapter 18 (Part III [3, p. 153]), which is also given in the first notebook along with Entries 76 and 77. Ramanujan gives still another formula in the same vein, but it appears that a faint line has been drawn through the entry to indicate that the entry is incorrect, or that Ramanujan could not prove his claim. We record in Entry 78 the entry as Ramanujan gives it. We are unable to discern a general formula for the nth term in the series on the right side. Then we sketch an attempt at proving (or disproving) the formula. As we shall see, serious technical difficulties prevent us from utilizing the ideas that we used in proving Entries 76 and 77 and Entry 6 of Chapter 18 in Part III.

Entry 76 (p. 281). *With the notation (76.1), (76.2),*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\cosh\{(2n+1)y/2\}} = \frac{\sqrt{x}}{2} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n x^n}{n!\,(2n+1)}.$$

Proof. Now,

$$\frac{d}{dy}\left(\frac{1}{e^{cy}+e^{-cy}}\right) = -c\frac{e^{cy}-e^{-cy}}{(e^{cy}+e^{-cy})^2}, \tag{76.3}$$

for any constant c, and

$$\frac{1}{2\cosh(cy)} = \frac{1}{e^{cy}+e^{-cy}} = -\int_y^{\infty} \frac{d}{dt}\left(\frac{1}{e^{ct}+e^{-ct}}\right) dt.$$

Let $c = (2n+1)/2$, multiply both sides above by $(-1)^n/c$, and sum on n, $0 \leq n < \infty$. Using also (76.3), we find that

$$\begin{aligned} S &:= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\cosh\{(2n+1)y/2\}} \\ &= -2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_y^{\infty} \frac{d}{dt}\left(\frac{1}{e^{(2n+1)t/2}+e^{-(2n+1)t/2}}\right) dt \\ &= \int_y^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{e^{(2n+1)t/2}-e^{-(2n+1)t/2}}{(e^{(2n+1)t/2}+e^{-(2n+1)t/2})^2} dt. \end{aligned} \tag{76.4}$$

We now evaluate the integral on the right side of (76.4) in another way. Replace the variable t by y_0. For brevity, set $u = \exp(-y_0/2)$. Now,

$$\begin{aligned} &\sum_{n=0}^{\infty} (-1)^n \frac{u^{-(2n+1)}-u^{2n+1}}{(u^{-(2n+1)}+u^{2n+1})^2} \\ &\quad= \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}-u^{3(2n+1)}}{(1+u^{2(2n+1)})^2} \\ &\quad= \sum_{n=0}^{\infty} (-1)^n \left(u^{2n+1}-u^{3(2n+1)}\right) \sum_{m=0}^{\infty} (-1)^m (m+1) u^{2(2n+1)m} \\ &\quad= \sum_{m=0}^{\infty} (-1)^m (m+1) \left(\frac{u^{2m+1}}{1+u^{2(2m+1)}} - \frac{u^{2m+3}}{1+u^{2(2m+3)}}\right) \end{aligned}$$

$$= \sum_{m=0}^{\infty}(-1)^m(2m+1)\frac{u^{2m+1}}{1+u^{2(2m+1)}}$$

$$= \sum_{m=0}^{\infty}(-1)^m\frac{2m+1}{2\cosh\{(2m+1)y_0/2\}}$$

$$= \tfrac{1}{4}z^2(t)\sqrt{t(1-t)}, \tag{76.5}$$

by Entry 16(i) of Chapter 17 (Part III [3, p. 134]), with x there replaced by t here.

We substitute (76.5) into the right side of (76.4) and make the change of variable $y_0 = \pi z(1-t)/z(t)$. By Entry 9(i) of Chapter 17 (Part III [3, p. 120]),

$$\frac{dy_0}{dt} = -\frac{1}{t(1-t)z^2(t)}. \tag{76.6}$$

When $y_0 = y$, then $t = x$, say. We therefore find from (76.4) that

$$S = \frac{1}{4}\int_y^{\infty} z^2(t)\sqrt{t(1-t)}\,dy_0 = \frac{1}{4}\int_0^x \frac{dt}{\sqrt{t(1-t)}}$$

$$= \frac{1}{4}\int_0^x t^{-1/2}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{n!}t^n\,dt = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n x^{n+1/2}}{n!\,(n+1/2)}. \tag{76.7}$$

It is easily seen that (76.7) is equivalent to the claim made in Entry 76.

Entry 77 (p. 280). *In the notation (76.1), (76.2),*

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2\sinh\{(2n+1)y/2\}} = \frac{\sqrt{x}}{2z}\sum_{n=0}^{\infty}\frac{(n!)^2}{\left(\frac{3}{2}\right)_n^2}\sum_{k=0}^{n}\frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2}x^n.$$

Proof. For any complex number c,

$$\frac{d}{dy}\left(\frac{1}{e^{cy}-e^{-cy}}\right) = -c\frac{e^{cy}+e^{-cy}}{(e^{cy}-e^{-cy})^2} \tag{77.1}$$

and

$$\frac{d^2}{dy^2}\left(\frac{1}{e^{cy}-e^{-cy}}\right) = c^2\frac{e^{2cy}+e^{-2cy}+6}{(e^{cy}-e^{-cy})^3}. \tag{77.2}$$

Hence,

$$\frac{1}{2\sinh(cy)} = \frac{1}{e^{cy}-e^{-cy}} = -\int_y^{\infty}\frac{d}{du}\left(\frac{1}{e^{cu}-e^{-cu}}\right)du$$

$$= \int_y^{\infty}\int_u^{\infty}\frac{d^2}{dt^2}\left(\frac{1}{e^{ct}-e^{-ct}}\right)dt\,du$$

$$= c^2\int_y^{\infty}\int_u^{\infty}\frac{e^{2ct}+e^{-2ct}+6}{(e^{ct}-e^{-ct})^3}dt\,du. \tag{77.3}$$

Now let $c = (2n+1)/2$, multiply both sides of (77.3) by $(-1)^n/c^2$, and sum on $n, 0 \le n < \infty$. Accordingly, we find that

$$\begin{aligned} S &:= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2 \sinh\{(2n+1)y/2\}} \\ &= \frac{1}{2} \int_y^{\infty} \int_u^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{e^{(2n+1)t} + e^{-(2n+1)t} + 6}{(e^{(2n+1)t/2} - e^{-(2n+1)t/2})^3}\, dt\, du. \end{aligned} \tag{77.4}$$

For brevity, set $s = \exp(-t/2)$. The integrand in (77.4) then becomes

$$\begin{aligned} &\sum_{n=0}^{\infty} (-1)^n \frac{s^{-(4n+2)} + s^{4n+2} + 6}{\left(s^{-(2n+1)} - s^{2n+1}\right)^3} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{s^{2n+1} + s^{5(2n+1)} + 6s^{3(2n+1)}}{(1 - s^{4n+2})^3} \\ &= \sum_{n=0}^{\infty} (-1)^n \left(s^{2n+1} + s^{5(2n+1)} + 6s^{3(2n+1)}\right) \sum_{m=0}^{\infty} \tfrac{1}{2}(m+2)(m+1)s^{m(4n+2)} \\ &= \sum_{m=0}^{\infty} \tfrac{1}{2}(m+2)(m+1) \sum_{n=0}^{\infty} (-1)^n \left(s^{(2m+1)(2n+1)} + s^{(2m+5)(2n+1)} \right. \\ &\qquad\qquad \left. + 6s^{(2m+3)(2n+1)}\right) \\ &= \frac{1}{2} \sum_{m=0}^{\infty} \frac{(m+2)(m+1)s^{2m+1}}{1 + s^{2(2m+1)}} + \frac{1}{2} \sum_{m=0}^{\infty} \frac{m(m-1)s^{2m+1}}{1 + s^{2(2m+1)}} \\ &\quad + 3 \sum_{m=0}^{\infty} \frac{(m+1)ms^{2m+1}}{1 + s^{2(2m+1)}} \\ &= \sum_{m=0}^{\infty} \frac{(2m+1)^2 s^{2m+1}}{1 + s^{2(2m+1)}}. \end{aligned} \tag{77.5}$$

From (77.4) and (77.5), so far, we have shown that

$$S = \frac{1}{4} \int_y^{\infty} \int_u^{\infty} \sum_{m=0}^{\infty} \frac{(2m+1)^2}{\cosh\{(2m+1)y_0/2\}}\, dy_0\, du, \tag{77.6}$$

where, for the sequel, it is more suggestive to set $t = y_0$.

We now apply Entry 16(x) of Chapter 17 (Part III [3, p. 134]) in (77.6) to deduce that

$$S = \frac{1}{4} \int_y^{\infty} \int_u^{\infty} \tfrac{1}{2} z^3(t)\sqrt{t}\, dy_0\, du.$$

We make the change of variable $y_0 = \pi z(1-t)/z(t)$. If $y_0 = u$, then $t = x_0$, say. Thus, by (76.6),

$$S = \frac{1}{8} \int_y^{\infty} \int_0^{x_0} \frac{z(t)}{\sqrt{t}(1-t)}\, dt\, du. \tag{77.7}$$

Replacing u by y_0 above, we find, from (76.6), that

$$\frac{dy_0}{dx_0} = -\frac{1}{x_0(1-x_0)z^2(x_0)}.$$

When $y_0 = y$, then $x_0 = x$, say. Hence, appealing to (77.7) and then replacing x_0 by u, we find that

$$\begin{aligned}
S &= \frac{1}{8}\int_0^x \left(\int_0^{x_0} \frac{z(t)}{\sqrt{t}(1-t)}dt\right)\frac{1}{x_0(1-x_0)z^2(x_0)}dx_0 \\
&= \frac{1}{8}\sum_{k=0}^{\infty}\int_0^x\int_0^u t^{k-1/2}z(t)\,dt\frac{du}{u(1-u)z^2(u)} \\
&= \frac{1}{8z}\sum_{k=0}^{\infty}\frac{x^{k+1/2}}{(k+1/2)^2}\,{}_3F_2\left[\begin{matrix} k+1,\ k+1,\ 1 \\ k+\frac{3}{2},\ k+\frac{3}{2} \end{matrix};x\right],
\end{aligned} \tag{77.8}$$

by Entry 9(iii) of Chapter 17 (Part III [3, p. 120]). Using the definition of ${}_3F_2$ in (77.8), setting $k+j=n$, and inverting the order of summation, we find that

$$\begin{aligned}
S &= \frac{1}{2z}\sum_{k=0}^{\infty}\frac{x^{k+1/2}}{(2k+1)^2}\sum_{j=0}^{\infty}\frac{(k+1)_j^2}{(k+\frac{3}{2})_j^2}x^j \\
&= \frac{1}{2z}\sum_{k=0}^{\infty}\frac{x^{k+1/2}}{(2k+1)^2}\frac{(\frac{3}{2})_k^2}{(1)_k^2}\sum_{j=0}^{\infty}\frac{(1)_{k+j}^2}{(\frac{3}{2})_{k+j}^2}x^j \\
&= \frac{\sqrt{x}}{2z}\sum_{k=0}^{\infty}\frac{(\frac{3}{2})_k^2}{(2k+1)^2(k!)^2}\sum_{n=k}^{\infty}\frac{(1)_n^2}{(\frac{3}{2})_n^2}x^n \\
&= \frac{\sqrt{x}}{2z}\sum_{n=0}^{\infty}\frac{(1)_n^2}{(\frac{3}{2})_n^2}\sum_{k=0}^{n}\frac{(\frac{3}{2})_k^2}{(2k+1)^2(k!)^2}x^n \\
&= \frac{\sqrt{x}}{2z}\sum_{n=0}^{\infty}\frac{(1)_n^2}{(\frac{3}{2})_n^2}\sum_{k=0}^{n}\frac{(\frac{1}{2})_k^2}{(k!)^2}x^n,
\end{aligned}$$

which completes the proof.

By using Entry 29(b) of Chapter 10 (Part II [2, p. 39]), namely,

$$\sum_{k=0}^{n}\frac{(\frac{1}{2})_k^2}{(k!)^2} = \frac{\Gamma^2(n+\frac{3}{2})}{(n+1)\Gamma^2(n+1)}\,{}_3F_2\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ n+1 \\ 1,\ n+2\end{matrix}\right],$$

we easily find that the right side of Entry 77 may be rewritten in the form

$$\frac{\sqrt{x}\pi}{8z}\sum_{n=0}^{\infty}\frac{1}{n+1}\,{}_3F_2\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ n+1 \\ 1,\ n+2\end{matrix}\right]x^n.$$

Entry 78 (p. 280). *With x, y, and z as in (76.1) and (76.2),*

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4 \cosh\{(2n+1)y/2\}}$$
$$= \frac{\sqrt{x}}{2z^3}\left(1+\left(\frac{2}{3}\right)^2\left[1+\left(\frac{2}{3}\right)^2\left\{1+\left(\frac{1}{2}\right)^2\right\}\right]x+\cdots\right).$$

We are unable to discern a general formula for the nth term of the series on the right side above.

Sketch of Attempted Proof. Differentiating (76.3), or from the proof of Entry 6 of Chapter 18 (Part III [3, p. 153]), for any complex number c, we find that

$$\frac{d^2}{dy^2}\left(\frac{1}{e^{cy}+e^{-cy}}\right) = c^2 \frac{e^{2cy}+e^{-2cy}-6}{(e^{cy}+e^{-cy})^3},$$

which is an analogue of (77.2). After two further differentiations,

$$\frac{d^4}{dy^4}\left(\frac{1}{e^{cy}+e^{-cy}}\right) = c^4 \frac{e^{4cy}-76e^{2cy}+230-76e^{-2cy}+e^{-4cy}}{(e^{cy}+e^{-cy})^5}. \tag{78.1}$$

Letting $c=(2n+1)/2$, dividing both sides of (78.1) by $(2n+1)^4$, summing both sides on n, $0 \le n < \infty$, and integrating, we find that

$$S := \sum_{n=0}^{\infty} \frac{1}{(2n+1)^4 \cosh\{(2n+1)y/2\}} = \frac{1}{8}\int_y^{\infty}\int_{y_3}^{\infty}\int_{y_2}^{\infty}\int_{y_1}^{\infty}$$
$$\times \sum_{n=0}^{\infty} \frac{e^{2(2n+1)y_0}-76e^{(2n+1)y_0}+230-76e^{-(2n+1)y_0}+e^{-2(2n+1)y_0}}{(e^{(2n+1)y_0/2}+e^{-(2n+1)y_0/2})^5} dy_0\, dy_1\, dy_2\, dy_3. \tag{78.2}$$

As before, set $u=\exp(-y_0/2)$. The integrand in (78.2) may then be written in the form

$$\sum_{n=0}^{\infty} \frac{u^{2n+1}-76u^{3(2n+1)}+230u^{5(2n+1)}-76u^{7(2n+1)}+u^{9(2n+1)}}{(1+u^{4n+2})^5}$$
$$= \sum_{m=0}^{\infty}(-1)^m \frac{(m+4)(m+3)(m+2)(m+1)}{4!}\left(\frac{u^{2m+1}}{1-u^{2(2m+1)}} - 76\frac{u^{2m+3}}{1-u^{2(2m+3)}}\right.$$
$$\left. +230\frac{u^{2m+5}}{1-u^{2(2m+5)}} - 76\frac{u^{2m+7}}{1-u^{2(2m+7)}} + \frac{u^{2m+9}}{1-u^{2(2m+9)}}\right)$$
$$= \sum_{m=0}^{\infty} \frac{(-1)^m(2m+1)^4 u^{2m+1}}{1-u^{2(2m+1)}}. \tag{78.3}$$

Employing (78.3) in (78.2), we find that

$$
\begin{aligned}
S &= \frac{1}{8}\int_y^\infty \int_{y_3}^\infty \int_{y_2}^\infty \int_{y_1}^\infty \sum_{m=0}^\infty \frac{(-1)^m(2m+1)^4}{e^{(2m+1)y_0/2}-e^{-(2m+1)y_0/2}}\,dy_0\,dy_1\,dy_2\,dy_3 \\
&= \frac{1}{8}\int_y^\infty \int_{y_3}^\infty \int_{y_2}^\infty \int_{y_1}^\infty \sum_{m=0}^\infty (-1)^m(2m+1)^4\left(\frac{1}{e^{(2m+1)y_0/2}-1}\right. \\
&\qquad \left. -\frac{1}{e^{(2m+1)y_0}-1}\right)dy_0\,dy_1\,dy_2\,dy_3. \qquad (78.4)
\end{aligned}
$$

The value of the "second" sum in the integrand on the far right side of (78.4) can be obtained from Entry 17(viii) of Chapter 17 (Part III [3, p. 138]). Thus,

$$
\sum_{m=0}^\infty \frac{(-1)^m(2m+1)^4}{e^{(2m+1)y_0}-1} = \tfrac{1}{4}\left(z^5(t)(5-t)(1-t)-5\right). \qquad (78.5)
$$

The value of the "first" sum in the integrand on the far right side of (78.4) can be obtained from that of the "second" sum by the process of dimidiation (Part III [3, p. 126]). Thus,

$$
\sum_{m=0}^\infty \frac{(-1)^m(2m+1)^4}{e^{(2m+1)y_0/2}-1} = \tfrac{1}{4}\left(z^5(t)(1-t)(1-\sqrt{t})(5+6\sqrt{t}+5t)-5\right). \qquad (78.6)
$$

Using (78.5) and (78.6) in (78.4), we find that

$$
\begin{aligned}
S &= \frac{1}{32}\int_y^\infty \int_{y_3}^\infty \int_{y_2}^\infty \int_{y_1}^\infty z^5(t)(1-t)(1-5t)\sqrt{t}\,dy_0\,dy_1\,dy_2\,dy_3 \\
&= \frac{1}{32}\int_y^\infty \int_{y_3}^\infty \int_{y_2}^\infty \int_0^{x_0} z^3(t)(1-5t)t^{-1/2}\,dt\,dy_1\,dy_2\,dy_3, \qquad (78.7)
\end{aligned}
$$

by (76.6). We can make similar changes of variables for y_1, y_2, and y_3, and we can employ (76.6) (with y_0 replaced by y_1, y_2, and y_3,) as we did above. However, these steps do not enable us to completely evaluate the quadruple integral in (78.7). Thus, at this point, we terminate our discussion of this apparently deleted entry.

10. Miscellaneous Results

Entry 79 (p. 172). *For* $0 < x < 1$, *suppose that*

$$
\frac{1+\sin\beta}{1-\sin\beta} = \frac{1+\sin\alpha}{1-\sin\alpha}\left(\frac{1+x\sin\alpha}{1-x\sin\alpha}\right)^2, \qquad (79.1)
$$

where $0 \le \alpha, \beta \le \pi/2$. *Then*

$$
(1+2x)\int_0^\alpha \frac{d\theta}{\sqrt{1-x^3\frac{2+x}{1+2x}\sin^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-x\left(\frac{2+x}{1+2x}\right)^3\sin^2\theta}}. \qquad (79.2)
$$

Proof. By Entry 6(iii) of Chapter 19 (Part III [3, p. 238]), if

$$\tan \tfrac{1}{2}(\alpha+\beta) = (1+p)\tan\alpha, \tag{79.3}$$

then (79.2) holds, with x replaced by p. Thus, it only remains to show that (79.1) and (79.3) are equivalent. In Part III [3, pp. 239–240, eq. (6.2)], we solved (79.3) for p in terms of $\sin\alpha$ and $\sin\beta$ and found that

$$p = \frac{1-\sin\alpha\sin\beta-\cos\alpha\cos\beta}{\sin\alpha(\sin\beta-\sin\alpha)}. \tag{79.4}$$

Put $\sqrt{u} = (1+x\sin\alpha)/(1-x\sin\alpha)$. Thus, from (79.1),

$$x\sin\alpha = \frac{\sqrt{u}-1}{\sqrt{u}+1} = \frac{\sqrt{\dfrac{(1+\sin\beta)(1-\sin\alpha)}{(1-\sin\beta)(1+\sin\alpha)}}-1}{\sqrt{\dfrac{(1+\sin\beta)(1-\sin\alpha)}{(1-\sin\beta)(1+\sin\alpha)}}+1}.$$

After straightforward elementary algebra and trigonometry, we find that

$$x = \frac{1-\sin\alpha\sin\beta-\cos\alpha\cos\beta}{\sin\alpha(\sin\beta-\sin\alpha)}.$$

This agrees with (79.4), and so the proof is complete.

Entry 80 (p. 172). *For* $0<x<1$, *suppose that*

$$\frac{1+\sin\beta}{1-\sin\beta} = \frac{1+\sin\alpha}{1-\sin\alpha}\left(\frac{1+x\sin\alpha}{1-x\sin\alpha}\right)^2\frac{1+x^2\sin\alpha}{1-x^2\sin\alpha}, \tag{80.1}$$

where $0\le\alpha,\beta\le\pi/2$. *Then*

$$(1+x)^2\int_0^\alpha \frac{d\theta}{\sqrt{1-x^4\sin^2\theta}} = \int_0^\beta \frac{d\theta}{\sqrt{1-\left\{1-\left(\frac{1-x}{1+x}\right)^4\right\}\sin^2\theta}}. \tag{80.2}$$

Proof. Set

$$1-\left(\frac{1-x}{1+x}\right)^4 = \frac{8x(1+x^2)}{(1+x)^4} = \frac{4y}{(1+y)^2}. \tag{80.3}$$

Solving for y in terms of x, we find that

$$y = \frac{1+6x^2+x^4\pm(1-x^2)^2}{4x(1+x^2)}.$$

We want y to approach 0 as x tends to 0, and so we need to choose the minus sign above. Hence,

$$y = \frac{2x}{1+x^2}. \tag{80.4}$$

We now apply Entry 7(xii) of Chapter 17 (Part III [3, p. 112]). If

$$(1 + y\sin^2\gamma)\sin\beta = (1 + y)\sin\gamma, \tag{80.5}$$

and we use (80.3) and (80.4), then

$$\begin{aligned}\int_0^\beta \frac{d\theta}{\sqrt{1 - \left\{1 - \left(\frac{1-x}{1+x}\right)^4\right\}\sin^2\theta}} &= \int_0^\beta \frac{d\theta}{\sqrt{1 - \frac{4y}{(1+y)^2}\sin^2\theta}} \\ &= (1 + y)\int_0^\gamma \frac{d\theta}{\sqrt{1 - y^2\sin^2\theta}} \\ &= \left(1 + \frac{2x}{1 + x^2}\right)\int_0^\gamma \frac{d\theta}{\sqrt{1 - \frac{4x^2}{(1+x^2)^2}\sin^2\theta}} \\ &= (1 + x)^2 \int_0^\alpha \frac{d\theta}{\sqrt{1 - x^4\sin^2\theta}},\end{aligned}$$

where we have applied Entry 17(xii) of Chapter 17 once again and thus need the condition

$$(1 + x^2\sin^2\alpha)\sin\gamma = (1 + x^2)\sin\alpha. \tag{80.6}$$

Combining (80.5) and (80.6), we find that

$$\frac{\sin\beta}{\sin\alpha} = \frac{(1 + x)^2(1 + x^2\sin^2\alpha)}{(1 + x^2\sin^2\alpha)^2 + 2x(1 + x^2)\sin^2\alpha}. \tag{80.7}$$

It remains to show that (80.7) is equivalent to Ramanujan's hypothesis (80.1). If we substitute in (80.1) the formula for $\sin\beta$ obtained from (80.7), we find that (80.1) and (80.7) are compatible, and so the proof of (80.2) is complete.

Entry 81 is one of many entries where Ramanujan writes "nearly" to indicate an approximation.

Entry 81 (p. 244). *Let, $F(x)$, $0 < x < 1$, be defined by (76.1). Put*

$$F(x^2) = t.$$

Then

$$F\left(\frac{2x}{1 + x}\right) = \frac{\sqrt{t}}{1 + (1 - t)(1 - t + \frac{3}{4}t^2)} \tag{81.1}$$

"nearly."

Proof. From Entry 2(vi) of Chapter 17 (Part III [3, p. 95]),

$$F\left(\frac{2x}{1 + x}\right) = \frac{1}{2^3}x + \frac{5}{2^7}x^3 + \frac{369}{2^{14}}x^5 + \frac{4097}{2^{18}}x^7 + \frac{1594895}{2^{27}}x^9 + \cdots. \tag{81.2}$$

On the other hand, using *Mathematica* to expand the right side of (81.1), we find that

$$\frac{\sqrt{t}}{1+(1-t)(1-t+\frac{3}{4}t^2)}$$
$$=\frac{1}{2^3}x+\frac{5}{2^7}x^3+\frac{369}{2^{14}}x^5+\frac{4097}{2^{18}}x^7+\frac{398697}{2^{25}}x^9+\cdots. \tag{81.3}$$

The first four coefficients in (81.2) and (81.3) agree, while

$$\frac{1594895}{2^{27}}=0.0118829\ldots$$

and

$$\frac{398697}{2^{25}}=0.0118821\ldots.$$

Thus, the fifth coefficients in (81.2) and (81.3) are remarkably close. Therefore, Ramanujan's claim has been justified.

Entry 82 (p. 244). *Let $F(x)$ be defined by (76.1). Then*

$$F(1-e^{-x})\approx\frac{x}{10+\sqrt{36+x^2}}-\frac{1}{2160}\left(\frac{x}{8+\frac{7}{50}x^2}\right)^5. \tag{82.1}$$

Proof. From Entry 2(vii) of Chapter 17 (Part III [3, p. 96]),

$$F(1-e^{-x})=\frac{1}{2^4}x-\frac{1}{3\cdot 2^{10}}x^3+\frac{31}{15\cdot 2^{19}}x^5-\frac{661}{315\cdot 2^{25}}x^7+\cdots. \tag{82.2}$$

On the other hand, expanding the right side of (82.1) by means of *Mathematica*, we arrive at

$$\frac{x}{10+\sqrt{36+x^2}}-\frac{1}{2160}\left(\frac{x}{8+\frac{7}{50}x^2}\right)^5$$
$$=\frac{1}{2^4}x-\frac{1}{3\cdot 2^{10}}x^3+\frac{31}{15\cdot 2^{19}}x^5-\frac{3187}{3^5\cdot 5^2\cdot 2^{23}}x^7+\cdots. \tag{82.3}$$

Thus, the first three coefficients in (82.2) and (82.3) agree, while

$$\frac{661}{315\cdot 2^{25}}=6.25376\cdots\times 10^{-8}$$

and

$$\frac{3187}{3^5\cdot 5^2\cdot 2^{23}}=6.25383\cdots\times 10^{-8}.$$

Thus, the fourth coefficients in (82.2) and (82.3) are amazingly close, and Ramanujan's approximation is indeed justified.

Entry 83 (p. 271). *Let*

$$x = 1 - \frac{\varphi^4(-u)}{\varphi^4(u)},$$

where

$$u := e^{-y} := \exp\left(-\pi \frac{z(1-x)}{z(x)}\right),$$

where z is defined by (76.2). Then, for any real number n,

$$\int \frac{\varphi^{2n+4}(u)}{u} du = \int \frac{z^n(x)}{x(1-x)} dx.$$

Proof. From Entries 6 and 9(i) of Chapter 17 (Part III [3, pp. 101, 120]),

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{u}{x(1-x)z^2(x)}.$$

Thus, by a change of variable,

$$\int \frac{\varphi^{2n+4}(u)}{u} du = \int \frac{z^{n+2}(x)}{u} \frac{u}{x(1-x)z^2(x)} dx = \int \frac{z^n(x)}{x(1-x)} dx.$$

37

Infinite Series

In the last two chapters devoted to the second and third notebooks, we gather together most of Ramanujan's results on series in the 133 pages of unorganized material found in these two notebooks. In this chapter, we primarily focus on exact formulas, while in Chapter 38 our attention is given to approximations and asymptotic formulas. In Part IV [4], we had disengaged Ramanujan's results on special functions, partial fraction decompositions, and elementary and miscellaneous analysis from the material on infinite series, and devoted individual chapters to these three topics. Although those three chapters contain a couple of gems, Chapters 37 and 38 have many more jewels.

We now briefly describe some of the interesting theorems proved in this chapter.

Entries 2–10 are new summation formulas, or applications thereof, akin to the Abel–Plana summation formula. Our proofs by contour integration employ the same idea that is used in the most well–known classical proof of the Abel–Plana formula.

Entries 15–19 arise from Eisenstein series and provide analogues of more well–known series identities, some of which Ramanujan had derived elsewhere and some of which the author [6], [7] had previously proved from different considerations of Eisenstein series.

Entry 21 is particularly fascinating. Ramanujan offers two transformations for doubly exponential series that are analogous to the famous theta transformation formula. That such transformations exist is surprising.

Entry 35 and Entries 36 and 37, which follow from Entry 35, are incorrect. Entry 35 gives a series transformation, reminiscent of the Poisson summation formula. However, Entry 35 involves the Möbius function, and, indeed, it is extremely unlikely that such a general transformation would exist. Numerical calculations demonstrate that Entries 36 and 37 are "close to being true." We conjecture that Ramanujan employed approximate formulas from prime number theory that he considered to be more accurate than warranted (see Chapter 24 of Part IV [4]). It would be interesting to reconstruct Ramanujan's thinking.

On pages 335, 340, and 341 Ramanujan makes successively more general claims about the behavior of partial sums of certain oscillating series. Ramanujan's claims are very remarkable in their explicit descriptions of the oscillations. We are not aware of any theorems in the literature similar to Entry 42, which is a rigorous formulation of Ramanujan's most general claim.

Beginning with Euler, many authors have written about the convergence of infinite exponentials

$$a_1^{a_2^{\cdot^{\cdot^{\cdot}}}}.$$

W. J. Thron [1] in 1957 proved a theorem which, it would seem, gives the best possible general upper bound for $|a_n|$ that suffices for the convergence of iterated exponentials. However, in his third notebook, Ramanujan offers a slightly better upper bound for $|a_n|$ and also claims, at least in one sense, that his bound is the best possible. These claims were recently proved by G. Bachman [1], and we present his elegant, difficult proofs in Section 50.

We conclude our introduction by stating some definitions, notation, and well-known results that are needed in the sequel.

Recall that the Bernoulli numbers B_n, $n \geq 0$, can be defined by the generating function

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \qquad |z| < 2\pi. \tag{0.1}$$

As usual, let $\zeta(z)$ denote the Riemann zeta–function. For each positive integer n,

$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n} B_{2n}}{2(2n)!}, \tag{0.2}$$

which is a famous formula of Euler. Also, for each positive integer n (Titchmarsh [3, p. 19]),

$$\zeta(1-2n) = -\frac{B_{2n}}{2n}. \tag{0.3}$$

For any complex number z, the functional equation of $\zeta(z)$ is given by (Titchmarsh [3, p. 22])

$$\pi^{-z/2}\Gamma(z/2)\zeta(z) = \pi^{-(1-z)/2}\Gamma((1-z)/2)\zeta(1-z),$$

which can be written in the equivalent form (Titchmarsh [3, p. 25])

$$\zeta(z) = 2\sin(\tfrac{1}{2}\pi z)(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z). \tag{0.4}$$

We shall need the Euler–Maclaurin summation formula (Olver [1, p. 285]). If f has $2n+1$ continuous derivatives on $[a, b]$, where a and b are integers, then

$$\sum_{k=a}^{b} f(k) = \int_a^b f(t)\,dt + \tfrac{1}{2}\{f(a) + f(b)\} + \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!}\left\{f^{(2k-1)}(b) - f^{(2k-1)}(a)\right\} + R_n, \tag{0.5}$$

where, for $n \geq 0$,

$$R_n = \frac{1}{(2n+1)!} \int_a^b B_{2n+1}(t - [t]) f^{(2n+1)}(t)\, dt, \tag{0.6}$$

where $B_n(x)$, $0 \leq n < \infty$, denotes the nth Bernoulli polynomial.

For each nonnegative integer n, the rising factorial $(a)_n$ is defined by

$$(a)_n := a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

If C is a simple closed contour, $I(C)$ denotes its interior. The residue of a function $f(z)$ at a pole α is denoted by R_α. (Usually, the function $f(z)$ is understood, and so there is no ambiguity by not specifying f in the notation for a residue.)

In Sections 17 and 18 of Chapter 13, Ramanujan states a version of the Abel–Plana summation formula and some variations (Part II [2, pp. 220–222]). At the top of page 335, Ramanujan offers another version of the Abel–Plana formula. A rigorous proof of this formulation can be found in Henrici's book [1, p. 274].

Entry 1 (p. 335). *Let $\varphi(z)$ be analytic for* $\operatorname{Re} z \geq 0$ *and suppose that either*

$$\sum_{n=0}^{\infty} \varphi(n) \qquad \text{or} \qquad \int_0^{\infty} \varphi(x)\, dx$$

is convergent. Assume furthermore that

$$\lim_{y\to\infty} |\varphi(x \pm iy)| e^{-2\pi y} = 0,$$

uniformly in x on every finite interval, and that

$$\int_0^{\infty} |\varphi(x \pm iy)| e^{-2\pi y} dy$$

exists for every $x \geq 0$ and tends to 0 as x tends to ∞. Then

$$\sum_{n=0}^{\infty} \varphi(n) = \tfrac{1}{2}\varphi(0) + \int_0^{\infty} \varphi(x)\, dx + i \int_0^{\infty} \frac{\varphi(iy) - \varphi(-iy)}{e^{2\pi y} - 1} dy. \tag{1.1}$$

Entry 2 (p. 335). *Let n be any complex number such that* $\operatorname{Re} n > 1$. *Let z^n have its principal value. Then, if* $\operatorname{Re} x > 0$,

$$\sum_{k=1}^{\infty} k^{n-1} e^{-kx} = \Gamma(n) x^{-n} + 2 \int_0^{\infty} \frac{y^{n-1} \cos(\frac{1}{2}\pi n - xy)}{e^{2\pi y} - 1} dy.$$

Proof. Let $f(t) = t^{n-1} e^{-tx}$. Although $f(t)$ is not necessarily analytic at $t = 0$, the proof of the Abel–Plana summation formula may be easily extended to include this function. We therefore find from Entry 1 that

$$\begin{aligned}
\sum_{k=1}^{\infty} k^{n-1}e^{-kx} &= \int_0^{\infty} t^{n-1}e^{-tx}dt + i\int_0^{\infty} \frac{(iy)^{n-1}e^{-iyx} - (-iy)^{n-1}e^{iyx}}{e^{2\pi y}-1}dy \\
&= \Gamma(n)x^{-n} + i\int_0^{\infty} \frac{y^{n-1}\left(e^{\pi i(n-1)/2-iyx} - e^{-\pi i(n-1)/2+iyx}\right)}{e^{2\pi y}-1}dy \\
&= \Gamma(n)x^{-n} - 2\int_0^{\infty} y^{n-1}\frac{\sin\left(\frac{1}{2}\pi(n-1)-yx\right)}{e^{2\pi y}-1}dy \\
&= \Gamma(n)x^{-n} + 2\int_0^{\infty} y^{n-1}\frac{\cos\left(\frac{1}{2}\pi n-yx\right)}{e^{2\pi y}-1}dy,
\end{aligned}$$

which completes the proof.

Entry 3 (p. 334). *Assume that $\varphi(z)$ is analytic for* $\operatorname{Re} z \geq 0$. *Let α be real with $0 < |\alpha| < 1$. Suppose that*

$$\lim_{y\to\infty} |\varphi(x \pm iy)|e^{-\pi y} = 0,$$

uniformly on any finite interval in $x \geq 0$. Suppose also that

$$\int_{-\infty}^{\infty} \frac{\varphi(x+iy)}{\cos(\pi(x+iy)) + \cos(\pi\alpha)}dy$$

exists for each nonnegative number x and tends to 0 as x tends to ∞. Assume also that the integral below converges. Then

$$\frac{2}{\sin(\pi\alpha)}\sum_{n=0}^{\infty}\{\varphi(2n+1-\alpha) - \varphi(2n+1+\alpha)\} = \int_0^{\infty}\frac{\varphi(ix)+\varphi(-ix)}{\cosh(\pi x)+\cos(\pi\alpha)}dx.$$

Proof. Consider, for positive integers m and N,

$$I(m,N) := \int_{C_{N,m}} \frac{\varphi(z)\,dz}{\cos(\pi z)+\cos(\pi\alpha)},$$

where $C_{N,m}$ is a positively oriented rectangle with horizontal sides passing through $\pm iN$ and vertical sides passing through 0 and m. We apply the residue theorem. Since

$$\cos(\pi z) + \cos(\pi\alpha) = 2\cos\left\{\tfrac{1}{2}\pi(z+\alpha)\right\}\cos\left\{\tfrac{1}{2}\pi(z-\alpha)\right\},$$

the integrand has simple poles at $z = 2n+1\pm\alpha$ for each nonnegative integer n. Straightforward calculations give

$$R_{2n+1\pm\alpha} = \pm\frac{\varphi(2n+1\pm\alpha)}{\pi\sin(\pi\alpha)}.$$

Hence, by the residue theorem,

$$I(m,N) = \frac{2i}{\sin(\pi\alpha)}\sum_{0<2n+1\pm\alpha<m}\{\varphi(2n+1+\alpha) - \varphi(2n+1-\alpha)\}.$$

Letting N tend to ∞, then letting m tend to ∞, and using our hypotheses, we find that

$$\left(\int_{i\infty}^{0}+\int_{0}^{-i\infty}\right)\frac{\varphi(z)\,dz}{\cos(\pi z)+\cos(\pi\alpha)}$$
$$=\frac{2i}{\sin(\pi\alpha)}\sum_{n=0}^{\infty}\{\varphi(2n+1+\alpha)-\varphi(2n+1-\alpha)\}.$$

Upon parametrizing the two integrals above, we complete the proof.

The next result is a limiting case of a lemma in Section 18 of Chapter 13 (Part II [2, pp. 221–222]), and so it is unnecessary to give a proof here.

Entry 4 (p. 335). *Let $\varphi(z)$ be analytic for* $\operatorname{Re} z \geq 0$. *Assume that*

$$\lim_{y\to\infty}|\varphi(x\pm iy)|e^{-\pi y/2}=0,$$

uniformly for x in any compact interval in $[0,\infty)$. *Suppose also that*

$$\int_0^{\infty}|\varphi(x\pm iy)|e^{-\pi y/2}\,dy$$

exists for all $x \geq 0$ and tends to 0 as x tends to ∞. Assume that the integral below exists. Then

$$4\sum_{n=0}^{\infty}(-1)^n\varphi(2n+1)=\int_0^{\infty}\frac{\varphi(ix)+\varphi(-ix)}{\cosh(\pi x/2)}dx.$$

Entry 5 (p. 335). *Let $\varphi(z)$ be analytic for* $\operatorname{Re} z \geq 0$. *Assume that*

$$\lim_{y\to\infty}|\varphi(x\pm iy)|e^{-\pi y}=0,$$

uniformly for x in any compact interval on $[0,\infty)$. *Assume that*

$$\int_0^{\infty}|\varphi(x\pm iy)|e^{-\pi y}dy$$

exists for every $x \geq 0$ and tends to 0 as x tends to ∞. Then, provided that the integral below exists,

$$\tfrac{1}{2}\varphi(0)+\sum_{n=1}^{\infty}(-1)^n\varphi(n)=\tfrac{1}{2}i\int_0^{\infty}\frac{\varphi(ix)-\varphi(-ix)}{\sinh(\pi x)}dx.$$

Proof. For positive integers N and m, let $C_{N,m}$ denote a positively oriented, indented rectangle with horizontal sides passing through $\pm iN$ and vertical sides passing through 0 and $m+\frac{1}{2}$. The indentation is a semicircle C_ϵ of radius $\epsilon > 0$

centered at the origin and lying in the left half-plane. Applying the residue theorem to

$$f(z) := \frac{\pi\varphi(z)}{\sin(\pi z)},$$

we find that

$$\frac{1}{2\pi i}\int_{C_{N,m}} f(z)\,dz = \sum_{n=0}^{m}(-1)^n\varphi(n).$$

Letting N tend to ∞, then letting m tend to ∞, and invoking our hypotheses, we deduce that

$$\frac{1}{2\pi i}\left(\int_{i\infty}^{i\epsilon} + \int_{C_\epsilon} + \int_{-i\epsilon}^{-i\infty}\right) f(z)\,dz = \sum_{n=0}^{\infty}(-1)^n\varphi(n). \tag{5.1}$$

Setting $z = \epsilon e^{i\theta}$, $\pi/2 \le \theta \le 3\pi/2$, we find that

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{C_\epsilon} f(z)\,dz &= \lim_{\epsilon\to 0}\frac{1}{2i}\int_{\pi/2}^{3\pi/2}\frac{\varphi(\epsilon e^{i\theta})\epsilon e^{i\theta} i\,d\theta}{\sin(\pi\epsilon e^{i\theta})}\\ &= \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}\varphi(0)\,d\theta = \frac{1}{2}\varphi(0).\end{aligned} \tag{5.2}$$

Hence, using (5.2) in (5.1), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\varphi(n) &= -\frac{1}{2}\int_{-\infty}^{\infty}\frac{\varphi(iy)\,dy}{\sin(i\pi y)} + \frac{1}{2}\varphi(0)\\ &= -\frac{1}{2i}\int_0^{\infty}\frac{\varphi(iy)\,dy}{\sinh(\pi y)} + \frac{1}{2i}\int_0^{\infty}\frac{\varphi(-iy)\,dy}{\sinh(\pi y)} + \frac{1}{2}\varphi(0).\end{aligned}$$

This completes the proof.

Entry 6 (p. 335). *Let $\varphi(z)$ be analytic for* $\operatorname{Re} z \ge 0$. *Suppose that*

$$\lim_{y\to\infty}|\varphi(x \pm iy)|e^{-\pi y} = 0,$$

uniformly for x in any compact interval on $[0, \infty)$. *Assume that*

$$\int_0^{\infty}|\varphi(x \pm iy)|e^{-\pi y}dy$$

exists for every $x \ge$ 0 and tends to 0 as x tends to ∞. Then, provided that the integrals below exist,

$$\sum_{n=0}^{\infty}\varphi(2n+1) = \frac{1}{2}\int_0^{\infty}\varphi(x)dx - \frac{1}{2}i\int_0^{\infty}\frac{\varphi(ix)-\varphi(-ix)}{e^{\pi x}+1}dx.$$

Proof. For positive integers N and m, let $C^{\pm}_{N,2m}$ denote the positively oriented indented rectangles with horizontal sides passing through 0 and $\pm iN$, respectively, and with vertical sides passing through 0 and $2m$. The indentations are semicircles $C_{j,\epsilon}^+$ and $C_{j,\epsilon}^-$, of radius ϵ, $0 < \epsilon < 1$, centered at $2j-1$, $1 \le j \le m$, and lying in the upper and lower half planes, respectively. Thus, observe that

$$\begin{aligned} C_{N,2m}^+ \cup C_{N,2m}^- &= [iN, -iN] \cup [-iN, -iN+2m] \\ &\quad \cup [-iN+2m, iN+2m] \cup [iN+2m, iN] \cup_{j=1}^m C_{j,\epsilon}, \end{aligned}$$

where $C_{j,\epsilon} = C_{j,\epsilon}^+ \cup C_{j,\epsilon}^-$ is a negatively oriented circle of radius ϵ and center $2j-1$, $1 \le j \le m$.

Applying Cauchy's theorem, we find that

$$\int_{C_{N,2m}^{\pm}} \frac{\varphi(z)\,dz}{e^{\mp\pi i z}+1} = 0. \tag{6.1}$$

Now,

$$\begin{aligned} \lim_{\epsilon\to0}\int_{C_{j,\epsilon}^+} \frac{\varphi(z)\,dz}{e^{-\pi i z}+1} &= \lim_{\epsilon\to0}\int_{\pi}^{0} \frac{\varphi(2j-1+\epsilon e^{i\theta})\epsilon e^{i\theta} i\,d\theta}{e^{-\pi i\{(2j-1)+\epsilon\exp(i\theta)\}}+1} \\ &= \frac{1}{\pi}\int_{\pi}^{0} \varphi(2j-1)\,d\theta = -\varphi(2j-1) \end{aligned}$$

and

$$\begin{aligned} \lim_{\epsilon\to0}\int_{C_{j,\epsilon}^-} \frac{\varphi(z)\,dz}{e^{\pi i z}+1} &= \lim_{\epsilon\to0}\int_{2\pi}^{\pi} \frac{\varphi(2j-1+\epsilon e^{i\theta})\epsilon e^{i\theta} i\,d\theta}{e^{\pi i\{(2j-1)+\epsilon\exp(i\theta)\}}+1} \\ &= -\frac{1}{\pi}\int_{\pi}^{0} \varphi(2j-1)\,d\theta = \varphi(2j-1). \end{aligned}$$

Thus, from (6.1),

$$2\sum_{j=1}^{m}\varphi(2j-1) = \lim_{\epsilon\to0}\left(\int_{C_{N,2m}^+}\frac{\varphi(z)\,dz}{e^{-\pi i z}+1} - \int_{C_{N,2m}^-}\frac{\varphi(z)\,dz}{e^{\pi i z}+1}\right).$$

Now let N tend to ∞, then let m tend to ∞, and use the given hypotheses. Hence,

$$\begin{aligned} 2\sum_{j=1}^{\infty}\varphi(2j-1) &= \int_{i\infty}^{0}\frac{\varphi(z)\,dz}{e^{-\pi i z}+1} - \int_{0}^{-i\infty}\frac{\varphi(z)\,dz}{e^{\pi i z}+1} \\ &\quad + \int_0^\infty \left(\frac{1}{e^{-\pi i z}+1} + \frac{1}{e^{\pi i z}+1}\right)\varphi(z)\,dz. \end{aligned} \tag{6.2}$$

Since

$$-1 + \frac{2}{e^{-\pi i z}+1} = i\tan\left(\tfrac12\pi x\right) = 1 - \frac{2}{e^{\pi i z}+1},$$

we find that

$$\begin{aligned}
&\int_0^\infty \left(\frac{1}{e^{-\pi i z}+1} + \frac{1}{e^{\pi i z}+1} \right) \varphi(z)\,dz \\
&\qquad = \frac{1}{2} \int_0^\infty \varphi(x) \left\{ i \tan\left(\tfrac{1}{2}\pi x\right) + 1 + 1 - i \tan\left(\tfrac{1}{2}\pi x\right) \right\} dx \\
&\qquad = \int_0^\infty \varphi(x)\,dx.
\end{aligned} \tag{6.3}$$

Also,

$$\int_{i\infty}^0 \frac{\varphi(z)\,dz}{e^{-\pi i z}+1} - \int_0^{-i\infty} \frac{\varphi(z)\,dz}{e^{\pi i z}+1} = -i \int_0^\infty \frac{\varphi(iy)\,dy}{e^{\pi y}+1} + i \int_0^\infty \frac{\varphi(-iy)\,dy}{e^{\pi y}+1}. \tag{6.4}$$

Putting (6.3) and (6.4) in (6.2), we complete the proof.

The first two formulas on page 269 are remindful of the Abel–Plana summation formula, but, in fact, a stronger version of this formula is needed, because the functions to which we would like to apply the formula have poles on the imaginary axis or in the right half-plane.

Entry 7 (Formula (1), p. 269). *Let* $\alpha, \beta > 0$ *with* $\alpha\beta = 4\pi^2$, *and let* $\operatorname{Re} n > 2$. *Then*

$$\begin{aligned}
&\sqrt{\alpha^n} \left(\frac{\Gamma(n)\zeta(n)}{(2\pi)^n} + \cos\left(\tfrac{1}{2}\pi n\right) \sum_{k=1}^\infty \frac{k^{n-1}}{e^{\alpha k}-1} \right) \\
&\quad = \sqrt{\beta^n} \left(\cos\left(\tfrac{1}{2}\pi n\right) \frac{\Gamma(n)\zeta(n)}{(2\pi)^n} + \sum_{k=1}^\infty \frac{k^{n-1}}{e^{\beta k}-1} \right. \\
&\qquad \left. - \sin\left(\tfrac{1}{2}\pi n\right) \mathrm{PV} \int_0^\infty \frac{x^{n-1}}{e^{2\pi x}-1} \cot\left(\tfrac{1}{2}\beta x\right) dx \right),
\end{aligned}$$

where PV *denotes the principal value of the integral.*

Proof. In proving the Abel–Plana summation formula of Entry 1, one proceeds as in the proof of Entry 6, except that now

$$\frac{\varphi(z)}{e^{\mp 2\pi i z}-1}$$

is integrated over $C^{\pm}_{N,m+1/2}$, where now the right vertical sides pass through $m + \frac{1}{2}$, where m is a positive integer. Furthermore, the semicircular indentations of radius ϵ, $0 < \epsilon < \frac{1}{2}$, are centered at each positive integer j, $1 \le j \le m$. On the left vertical side, we need a semicircular indentation at the origin, with the upper quartercircle being part of $C^{+}_{N,m+1/2}$ and the lower quartercircle belonging to $C^{-}_{N,m+1/2}$.

We now set

$$\varphi(z) = \frac{z^{n-1}}{e^{\beta z} - 1}, \tag{7.1}$$

where the principal branch of z^n is taken. Observe that $\varphi(z)$ has a simple pole at $z = 2\pi ik/\beta$, for each nonzero integer k, and also has a singularity at $z = 0$. Except for the singularities on the imaginary axis, $\varphi(z)$ satisfies all the remaining hypotheses of the Abel–Plana summation theorem, Entry 1. However, because of the singularities, principal values need to be taken for the integrals appearing in the Abel–Plana formula. We thus will proceed with the necessary calculations in applying Entry 1 and make the necessary modifications to accommodate our function $\varphi(z)$, defined by (7.1).

First,

$$\begin{aligned}
\varphi(ix) - \varphi(-ix) &= x^{n-1}\left(\frac{e^{\pi i(n-1)/2}}{e^{i\beta x} - 1} - \frac{e^{-\pi i(n-1)/2}}{e^{-i\beta x} - 1}\right) \\
&= -\frac{x^{n-1}}{(e^{i\beta x} - 1)(e^{-i\beta x} - 1)}\left(2i\sin\left(\tfrac{1}{2}\pi(n-1)\right) + 2i\sin\left(\beta x - \tfrac{1}{2}\pi(n-1)\right)\right) \\
&= \frac{x^{n-1}}{(e^{i\beta x} - 1)(e^{-i\beta x} - 1)}\left(2i\cos(\tfrac{1}{2}\pi n) - 2i\cos(\beta x - \tfrac{1}{2}\pi n)\right) \\
&= \frac{x^{n-1}i}{1 - \cos(\beta x)}\left(\cos(\tfrac{1}{2}\pi n)(1 - \cos(\beta x)) - \sin(\tfrac{1}{2}\pi n)\sin(\beta x)\right) \\
&= x^{n-1}i\left(\cos(\tfrac{1}{2}\pi n) - \sin(\tfrac{1}{2}\pi n)\cot(\tfrac{1}{2}\beta x)\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
i\mathrm{PV}\int_0^\infty \frac{\varphi(ix) - \varphi(-ix)}{e^{2\pi x} - 1}dx = &-\cos(\tfrac{1}{2}\pi n)\int_0^\infty \frac{x^{n-1}}{e^{2\pi x} - 1}dx \\
&+ \sin(\tfrac{1}{2}\pi n)\mathrm{PV}\int_0^\infty \frac{x^{n-1}}{e^{2\pi x} - 1}\cot(\tfrac{1}{2}\beta x)\,dx.
\end{aligned} \tag{7.2}$$

Recall that (Gradshteyn and Ryzhik [1, p. 370, formula 3.411, no. 1])

$$\int_0^\infty \frac{x^{a-1}}{e^{bx} - 1}dx = \frac{1}{b^a}\Gamma(a)\zeta(a), \qquad \mathrm{Re}\, a > 1, \quad \mathrm{Re}\, b > 0. \tag{7.3}$$

Hence, from (7.2) and (7.3),

$$\begin{aligned}
\int_0^\infty \varphi(x)\,dx + i\mathrm{PV}\int_0^\infty \frac{\varphi(ix) - \varphi(-ix)}{e^{2\pi x} - 1}dx = \frac{\Gamma(n)\zeta(n)}{\beta^n} \\
- \cos(\tfrac{1}{2}\pi n)\frac{\Gamma(n)\zeta(n)}{(2\pi)^n} + \sin(\tfrac{1}{2}\pi n)\mathrm{PV}\int_0^\infty \frac{x^{n-1}}{e^{2\pi x} - 1}\cot(\tfrac{1}{2}\beta x)\,dx.
\end{aligned} \tag{7.4}$$

We now calculate the contributions from the poles. First consider a pole $2\pi ik/\beta$, where $k \neq 0$. On $C^{\pm}_{N,m+1/2}$, we make a semicircular indentation of radius $\epsilon > 0$

in the right half-plane. For $k > 0$, we thus need to calculate

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{2\pi ik/\beta-i\epsilon}^{2\pi ik/\beta+i\epsilon} \frac{z^{n-1}dz}{(e^{\beta z}-1)(e^{-2\pi iz}-1)} \\
&= \lim_{\epsilon\to 0}\int_0^{-\pi} \frac{(2\pi ik/\beta+\epsilon e^{i\theta})^{n-1}\epsilon ie^{i\theta}d\theta}{(e^{2\pi ik+\beta\epsilon\exp(i\theta)}-1)(e^{-2\pi i(2\pi ik/\beta+\epsilon\exp(i\theta))}-1)} \\
&= -\frac{\pi(2\pi ik/\beta)^{n-1}i}{\beta(e^{4\pi^2k/\beta}-1)} \\
&= -\frac{\alpha^{n/2}k^{n-1}i^n}{2\beta^{n/2}(e^{\alpha k}-1)},
\end{aligned} \tag{7.5}$$

since $\alpha\beta = 4\pi^2$. Second, we examine the contribution about $-2\pi ik/\beta$ for $k > 0$. We thus need to calculate

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{-2\pi ik/\beta-i\epsilon}^{-2\pi ik/\beta+i\epsilon} \frac{z^{n-1}dz}{(e^{\beta z}-1)(e^{2\pi iz}-1)} \\
&= \lim_{\epsilon\to 0}\int_0^{-\pi} \frac{(-2\pi ik/\beta+\epsilon e^{i\theta})^{n-1}\epsilon ie^{i\theta}d\theta}{(e^{-2\pi ik+\beta\epsilon\exp(i\theta)}-1)(e^{2\pi i(-2\pi ik/\beta+\epsilon\exp(i\theta))}-1)} \\
&= -\frac{\pi(-2\pi ik/\beta)^{n-1}i}{\beta(e^{4\pi^2k/\beta}-1)} \\
&= \frac{\alpha^{n/2}k^{n-1}(-i)^n}{2\beta^{n/2}(e^{\alpha k}-1)}.
\end{aligned} \tag{7.6}$$

Recall that, in the proof of Entry 6, we needed to take the difference of the integrals over $C^+_{N,2m}$ and $C^-_{N,2m}$. In modifying the proof of Entry 1, we need to calculate the difference of (7.6) and (7.5). To that end,

$$\frac{\alpha^{n/2}k^{n-1}(-i)^n}{2\beta^{n/2}(e^{\alpha k}-1)} + \frac{\alpha^{n/2}k^{n-1}i^n}{2\beta^{n/2}(e^{\alpha k}-1)} = \frac{\alpha^{n/2}k^{n-1}\cos(\frac{1}{2}\pi n)}{\beta^{n/2}(e^{\alpha k}-1)}. \tag{7.7}$$

Lastly, we examine the singularity at $z = 0$. Since $\operatorname{Re} n > 2$, it is easy to see that the contributions of the two quartercircular indentations tend to 0 as their radii tend to 0.

Hence, on the right side of (1.1), we must add, by (7.7), the additional expression

$$\cos(\tfrac{1}{2}\pi n)\sqrt{\frac{\alpha^n}{\beta^n}}\sum_{k=1}^{\infty}\frac{k^{n-1}}{e^{\alpha k}-1}. \tag{7.8}$$

In conclusion, by the modified form of (1.1), (7.4), and (7.8),

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{k^{n-1}}{e^{\beta k}-1} &= \frac{\Gamma(n)\zeta(n)}{\beta^n} - \cos(\tfrac{1}{2}\pi n)\frac{\Gamma(n)\zeta(n)}{(2\pi)^n} \\
&\quad + \sin(\tfrac{1}{2}\pi n)\mathrm{PV}\int_0^{\infty}\frac{x^{n-1}}{e^{2\pi x}-1}\cot(\tfrac{1}{2}\beta x)\,dx
\end{aligned}$$

$$+\cos(\tfrac{1}{2}\pi n)\sqrt{\frac{\alpha^n}{\beta^n}}\sum_{k=1}^{\infty}\frac{k^{n-1}}{e^{\alpha k}-1}. \tag{7.9}$$

Multiplying both sides of (7.9) by $\sqrt{\beta^n}$ and rearranging, we complete the proof.

Entry 8 (Formula (2), p. 269). *Let $x > 0$ and $-1 < n < 2$. Then*

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{k^{n+1}}{k^4+4x^4} &= \tfrac{1}{4}\pi(x\sqrt{2})^{n-2}\sec(\tfrac{1}{4}\pi n)\\ &\quad -2\cos(\tfrac{1}{2}\pi n)\int_0^{\infty}\frac{t^{n+1}dt}{(e^{2\pi t}-1)(t^4+4x^4)}\\ &\quad +\tfrac{1}{2}\pi(x\sqrt{2})^{n-2}\frac{\cos(\tfrac{1}{4}\pi n+2\pi x)-e^{-2\pi x}\cos(\tfrac{1}{4}\pi n)}{\cosh(2\pi x)-\cos(2\pi x)}.\end{aligned} \tag{8.1}$$

Proof. We apply the Abel–Plana summation formula of Entry 1, but modifications are necessary because of poles. Let

$$\varphi(z) = \frac{z^{n+1}}{z^4+4x^4}.$$

Observe that $\varphi(z)$ has simple poles at $z = x\sqrt{2}\exp((\pi i + 2\pi i k)/4)$, $0 \le k \le 3$. When $k = 0, 3$, the poles lie in the right half-plane.

First, a straightforward calculation shows that

$$\varphi(it)-\varphi(-it) = \frac{2it^{n+1}\cos(\tfrac{1}{2}\pi n)}{t^4+4x^4}. \tag{8.2}$$

Second, setting $t = xu\sqrt{2}$, we find that

$$\begin{aligned}\int_0^{\infty}\varphi(t)\,dt &= \int_0^{\infty}\frac{t^{n+1}dt}{t^4+4x^4}\\ &= (x\sqrt{2})^{n-2}\int_0^{\infty}\frac{u^{n+1}du}{u^4+1} = (x\sqrt{2})^{n-2}\tfrac{1}{4}\pi\sec(\tfrac{1}{4}\pi n),\end{aligned} \tag{8.3}$$

by, for example, the calculus of residues or tables (Gradshteyn and Ryzhik [1, p. 340, formula 3.241, no. 2]). Therefore, by (8.2) and (8.3),

$$\begin{aligned}&\int_0^{\infty}\varphi(t)\,dt + i\int_0^{\infty}\frac{\varphi(it)-\varphi(-it)}{e^{2\pi t}-1}dt\\ &= \tfrac{1}{4}\pi(x\sqrt{2})^{n-2}\sec(\tfrac{1}{4}\pi n) - 2\cos(\tfrac{1}{2}\pi n)\int_0^{\infty}\frac{t^{n+1}dt}{(e^{2\pi t}-1)(t^4+4x^4)}.\end{aligned} \tag{8.4}$$

Returning to the proof of the Abel–Plana summation formula, or to the proof of Entry 6, we see that we must modify the proof by accounting for the contributions of the poles of

$$\frac{\varphi(z)}{e^{-2\pi iz}-1}$$

at $z = x\sqrt{2}\exp(\pi i/4)$ and of

$$\frac{\varphi(z)}{e^{2\pi iz}-1}$$

at $z = x\sqrt{2}\exp(-\pi i/4)$. Denoting these residues by R' and R'', respectively, we find that

$$R' = \frac{(x\sqrt{2}e^{\pi i/4})^{n+1}}{4(x\sqrt{2}e^{\pi i/4})^3(e^{-2\pi ix\sqrt{2}\exp(\pi i/4)}-1)} = -\frac{ie^{\pi in/4}(x\sqrt{2})^{n-2}}{4(e^{2\pi x(1-i)}-1)},$$

and, by a similar calculation,

$$R'' = \frac{ie^{-\pi in/4}(x\sqrt{2})^{n-2}}{4(e^{2\pi x(1+i)}-1)}.$$

Thus, by the residue theorem, we obtain on the right side of (1.1) an additional contribution of

$$\begin{aligned}
2\pi i(R'-R'') &= \frac{\pi}{2}(x\sqrt{2})^{n-2}\left(\frac{e^{\pi in/4}}{e^{2\pi x(1-i)}-1}+\frac{e^{-\pi in/4}}{e^{2\pi x(1+i)}-1}\right)\\
&= \frac{\pi}{2}(x\sqrt{2})^{n-2}\left(\frac{2e^{2\pi x}\cos(2\pi x+\frac{1}{4}\pi n)-2\cos(\frac{1}{4}\pi n)}{1+e^{4\pi x}-2e^{2\pi x}\cos(2\pi x)}\right)\\
&= \frac{\pi}{2}(x\sqrt{2})^{n-2}\left(\frac{\cos(2\pi x+\frac{1}{4}\pi n)-e^{-2\pi x}\cos(\frac{1}{4}\pi n)}{\cosh(2\pi x)-\cos(2\pi x)}\right).
\end{aligned} \tag{8.5}$$

Hence, using (8.4) and (8.5) in a modified form of (1.1), we conclude (8.1) to complete the proof.

Entry 9 (Formula (5), p. 283). *Let x be a positive, nonintegral number, and let $0 < n < 1$. Then*

$$\sum_{k=1}^{\infty}\frac{k^n}{k^2-x^2} = \frac{\pi x^{n-1}}{2}\left(\tan(\tfrac{1}{2}\pi n)-\cot(\pi x)\right) + 2\sin(\tfrac{1}{2}\pi n)\int_0^{\infty}\frac{z^n\,dz}{(e^{2\pi iz}-1)(z^2+x^2)}, \tag{9.1}$$

where principal values are taken.

Proof. We shall apply a modified form of the Abel–Plana summation formula from Entry 1. This modification is necessary because the function in our application has a singularity at the origin and a simple pole on the real axis. We thus will indicate the modifications in the proof of the Abel–Plana formula that need to be made.

Let

$$\varphi(z) = \frac{z^{n-1}}{z+x} + \frac{z^{n-1}}{z-x}.$$

By straightforward, elementary calculations,

$$\varphi(iz) - \varphi(-iz) = -\frac{2z^n(i^n - (-i)^n)}{z^2 + x^2} \tag{9.2}$$

and

$$\varphi(k) = \frac{2k^n}{k^2 - x^2}, \tag{9.3}$$

for each positive integer k.

We now indicate the alterations that we mentioned above. For positive integers N and m, let $C^{\pm}_{N,m}$ denote the positively oriented indented rectangles with horizontal sides passing through 0 and $\pm iN$, respectively, and with vertical sides passing through 0 and $m + \frac{1}{2}$. The indentations are quartercircles of radius ϵ about the origin in the upper and lower half-planes, respectively, and semicircles C^+_ϵ and C^-_ϵ, of radius ϵ about x in the upper and lower half-planes, respectively. Because $n > 0$, the limits, as ϵ tends to 0, of the integrals around these quartercircles equal 0. The union of C^+_ϵ and C^-_ϵ is a negatively oriented circle of radius ϵ centered at x. We therefore obtain contributions on the "right side" of the Abel–Plana summation formula equal to

$$\begin{aligned}
&-\lim_{\epsilon\to 0}\int_{C^+_\epsilon} \frac{z^{n-1}dz}{(e^{-2\pi iz} - 1)(z-x)} + \lim_{\epsilon\to 0}\int_{C^-_\epsilon} \frac{z^{n-1}dz}{(e^{2\pi iz} - 1)(z-x)} \\
&= -\lim_{\epsilon\to 0}\int_{\pi}^{0} \frac{(x+\epsilon e^{i\theta})^{n-1}\epsilon i e^{i\theta}d\theta}{(e^{-2\pi i(x+\epsilon\exp(i\theta))} - 1)\epsilon e^{i\theta}} + \lim_{\epsilon\to 0}\int_{2\pi}^{\pi} \frac{(x+\epsilon e^{i\theta})^{n-1}\epsilon i e^{i\theta}d\theta}{(e^{2\pi i(x+\epsilon\exp(i\theta))} - 1)\epsilon e^{i\theta}} \\
&= \frac{i\pi x^{n-1}}{e^{-2\pi ix} - 1} - \frac{i\pi x^{n-1}}{e^{2\pi ix} - 1} \\
&= \tfrac{1}{2}\pi i x^{n-1}\left((i\cot(\pi x) - 1) + (i\cot(\pi x) + 1)\right) \\
&= -\pi x^{n-1}\cot(\pi x).
\end{aligned} \tag{9.4}$$

Hence, using (9.2)–(9.4) in our modified Abel–Plana summation formula, we find that

$$\begin{aligned}
2\sum_{k=1}^{\infty} \frac{k^n}{k^2 - x^2} = \int_0^\infty \frac{z^{n-1}}{z+x}dz + \mathrm{PV}\int_0^\infty \frac{z^{n-1}}{z-x}dz - \pi x^{n-1}\cot(\pi x) \\
+ 4\sin(\tfrac{1}{2}\pi n)\int_0^\infty \frac{z^n dz}{(e^{2\pi iz} - 1)(z^2 + x^2)},
\end{aligned} \tag{9.5}$$

where PV designates the principal value of the integral. However, by (8.3) and a result in Sansone and Gerretsen's book [1, p. 133],

$$\int_0^\infty \frac{z^{n-1}}{z+x}dz + \mathrm{PV}\int_0^\infty \frac{z^{n-1}}{z-x}dz = \pi\csc(\pi n)x^{n-1} - \pi\cot(\pi n)x^{n-1}$$

$$= \pi x^{n-1} \tan(\tfrac{1}{2}\pi n). \tag{9.6}$$

Putting (9.6) in (9.5), we readily deduce (9.1) to complete the proof.

Entry 10 (Formula (6), p. 283). *Let x be positive, and let $0 < n < 1$. Then*

$$\sum_{k=1}^{\infty} \frac{k^n}{k^2 + x^2} = \tfrac{1}{2}\pi x^{n-1} \sec(\tfrac{1}{2}\pi n) + \frac{\pi x^{n-1}\cos(\tfrac{1}{2}\pi n)}{e^{2\pi x} - 1}$$
$$+ 2\sin(\tfrac{1}{2}\pi n)\mathrm{PV}\int_0^{\infty} \frac{z^n dz}{(e^{2\pi x} - 1)(z^2 - x^2)}. \tag{10.1}$$

Proof. As in the previous proof, we apply the Abel–Plana summation formula under appropriate modifications.

Let

$$\varphi(z) = \frac{z^{n-1}}{z + ix} + \frac{z^{n-1}}{z - ix}.$$

Then, by elementary calculations,

$$\varphi(iz) - \varphi(-iz) = -\frac{2z^n(i^n - (-i)^n)}{z^2 - x^2} = -\frac{4i\sin(\tfrac{1}{2}\pi n)z^n}{z^2 - x^2} \tag{10.2}$$

and

$$\varphi(k) = \frac{2k^n}{k^2 + x^2}, \tag{10.3}$$

for each positive integer k. Also (Gradshteyn and Ryzhik [1, p. 340, formula 3.241, no. 2]), for $0 < n < 1$,

$$\int_0^{\infty} \varphi(z)\, dz = 2\int_0^{\infty} \frac{z^n\, dz}{z^2 + x^2} = \pi x^{n-1}\sec(\tfrac{1}{2}\pi n). \tag{10.4}$$

In modifying the proof of the Abel–Plana summation formula for the present application, we take quartercircular indentations around the origin in the two rectangular contours. By the same argument as in the proof of Entry 9, we get contributions of 0 when we let the radii tend to 0. We also take semicircular indentations of radius ϵ, C_ϵ^+ and C_ϵ^-, in the right half-plane about the poles $z = \pm ix$, respectively. On the "right side" of the Abel–Plana summation formula, we then obtain contributions of

$$-\lim_{\epsilon\to 0}\int_{C_\epsilon^+} \frac{z^{n-1}dz}{(e^{-2\pi iz} - 1)(z - ix)} + \lim_{\epsilon\to 0}\int_{C_\epsilon^-} \frac{z^{n-1}dz}{(e^{2\pi iz} - 1)(z + ix)}$$
$$= -\lim_{\epsilon\to 0}\int_0^{-\pi} \frac{(ix + \epsilon e^{i\theta})^{n-1}\epsilon i e^{i\theta}d\theta}{(e^{-2\pi i(ix+\epsilon\exp(i\theta))} - 1)\epsilon e^{i\theta}}$$
$$+ \lim_{\epsilon\to 0}\int_0^{-\pi} \frac{(-ix + \epsilon e^{i\theta})^{n-1}\epsilon i e^{i\theta}d\theta}{(e^{2\pi i(-ix+\epsilon\exp(i\theta))} - 1)\epsilon e^{i\theta}}$$

$$= \frac{\pi x^{n-1} i^n}{e^{2\pi x}-1} + \frac{\pi x^{n-1}(-i)^n}{e^{2\pi x}-1}$$

$$= \frac{2\pi x^{n-1}\cos(\frac{1}{2}\pi n)}{e^{2\pi x}-1}. \tag{10.5}$$

Using (10.2)–(10.5) in our modified Abel–Plana summation formula and dividing both sides by 2, we arrive at (10.1) to complete the proof.

Entry 11 (Formula (2), p. 268). *Let $\varphi(z)$ be an entire function. Let C_N denote a positively oriented rectangle with its horizontal sides passing through $\pm iN$ and its vertical sides passing through $2N+1$ and $-2N$, where N is a positive integer. Assume that, for $\beta > 0$,*

$$\lim_{N\to\infty} \int_{CN} \frac{\pi\Gamma(z+1)\zeta(z)\varphi(z)(2\sqrt{\beta})^{-z}}{2\Gamma(\frac{1}{2}z+1)\sin(\frac{1}{2}\pi z)} dz = 0, \tag{11.1}$$

where $\zeta(z)$ denotes the Riemann zeta-function. Then, if $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$,

$$\alpha^{-1/4}\left(\varphi(0) + \sum_{n=1}^{\infty} \frac{B_{2n}\varphi(2n)\alpha^n}{n!}\right) = \beta^{-1/4}\left(\varphi(1) + \sum_{n=1}^{\infty} \frac{B_{2n}\varphi(-2n+1)\beta^n}{n!}\right), \tag{11.2}$$

where B_j, $j \ge 0$, denotes the jth Bernoulli number.

Proof. We integrate

$$f(z) := \frac{\pi\Gamma(z+1)\zeta(z)\varphi(z)(2\sqrt{\beta})^{-z}}{2\Gamma(\frac{1}{2}z+1)\sin(\frac{1}{2}\pi z)}$$

around the contour C_N. Observe that $f(z)$ has simple poles at $z = 1$, because $\zeta(z)$ has a simple pole at $z = 1$, at $z = 2n$, for each nonnegative integer n, and at $z = -2j-1$, for each nonnegative integer j. Note that $f(z)$ is analytic at $z = 2n$, when n is a negative integer, because $\zeta(z)$ has a simple zero at $z = 2n$ (Titchmarsh [3, p. 19]). We next calculate the residues.

First,

$$R_1 = \frac{\sqrt{\pi}\varphi(1)}{2\sqrt{\beta}}, \tag{11.3}$$

since $\Gamma(3/2) = \sqrt{\pi}/2$. Second,

$$R_0 = -\tfrac{1}{2}\varphi(0), \tag{11.4}$$

since $\zeta(0) = -\frac{1}{2}$ (Titchmarsh [3, p. 19]). Third, for each positive integer n,

$$R_{2n} = \frac{(-1)^n(2n)!\,\zeta(2n)\varphi(2n)}{n!\,(4\beta)^n} = -\frac{\pi^{2n}B_{2n}\varphi(2n)}{2n!\,\beta^n}, \tag{11.5}$$

by (0.2). Fourth, using (0.3), we find that

$$
\begin{aligned}
R_{-2j-1} &= \frac{(-1)^{j+1}\pi\zeta(-2j-1)\varphi(-2j-1)(2\sqrt{\beta})^{2j+1}}{2(2j)!\,\Gamma(-j+\frac{1}{2})} \\
&= \frac{(-1)^{j}\sqrt{\pi}B_{2j+2}\varphi(-2j-1)(-j+\frac{1}{2})(-j+\frac{3}{2})\cdots(-\frac{1}{2})(2\sqrt{\beta})^{2j+1}}{2(2j)!\,(2j+2)} \\
&= \frac{\sqrt{\pi}B_{2j+2}\varphi(-2j-1)\beta^{j+1/2}}{2(j+1)!}. \qquad (11.6)
\end{aligned}
$$

Thus, applying the residue theorem, using our calculations (11.3)–(11.6), letting N tend to ∞, and employing (11.1), we find that

$$
-\tfrac{1}{2}\varphi(0)+\frac{\sqrt{\pi}\varphi(1)}{2\sqrt{\beta}}-\sum_{n=1}^{\infty}\frac{\pi^{2n}B_{2n}\varphi(2n)}{2n!\,\beta^{n}}+\sqrt{\pi}\sum_{j=0}^{\infty}\frac{B_{2j+2}\varphi(-2j-1)\beta^{j+1/2}}{2(j+1)!}=0.
$$

Letting $n=j+1$ in the latter sum, multiplying both sides by $-2\alpha^{-1/4}$, using the hypothesis $\alpha\beta=\pi^2$, and rearranging, we complete the proof of (11.2).

Entry 12 (Formula (5), p. 269). *If n is any positive integer, then*

$$
\sum_{k=1}^{\infty}\frac{k^{4n}}{\sinh^2(\pi k)}=\frac{4n}{\pi}\left(-\frac{B_{2n}}{8n}+\sum_{k=1}^{\infty}\frac{k^{4n-1}}{e^{2\pi k}-1}\right).
$$

Entry 12 was communicated in Ramanujan's [10, p. xxvi] first letter to Hardy and was later proved by C. T. Preece [1] in 1928. The first proof in print, however, appears to be by M. B. Rao and M. V. Aiyar [1] about four or five years earlier. E. Grosswald [2] and the author [6] have also found proofs.

Entry 13 (p. 273). *If n is an odd positive integer, then*

$$
\sum_{k=1}^{\infty}\frac{\coth(\pi k)}{k^{2n+1}}=\frac{1}{2}\left(\frac{3}{\pi}\zeta(2n+2)+\frac{\pi}{3}\zeta(2n)\right)+\frac{2^{2n-2}}{(2n-1)!}\pi^{2n+1}\frac{v_{2n+2}}{270},
$$

where

$$
v_4=-\frac{3}{2},\qquad v_8=0,\qquad v_{12}=\frac{1}{2730},\qquad v_{16}=\frac{1}{340},
$$
$$
v_{20}=\frac{191}{2310},\qquad v_{24}=\frac{907}{147},\ldots.
$$

Proof. This entry is a sequel to Entries 25(i), (ii) of Chapter 14 (Part II [2, p. 293]), where only the first two cases are presented. All six evaluations above can be deduced from the general formula (Part II [2, p. 293, eq. (25.3)])

$$
\sum_{k=1}^{\infty}\frac{\coth(\pi k)}{k^{2n+1}}=2^{2n}\pi^{2n+1}\sum_{k=0}^{n+1}(-1)^{k+1}\frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!}, \qquad (13.1)
$$

where n is a positive odd integer, and B_j, $j \geq 0$, denotes the jth Bernoulli number. It is now routine to check that for $n = 1, 3, 5, 7, 9$, and 11 Ramanujan's claims agree with (13.1).

For references to proofs of (13.1) and special cases thereof, see Part II [2, p. 293].

Ramanujan writes the next entry in terms of a function φ "defined" by

$$\varphi(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)_n}{z^n}. \tag{14.1}$$

This series does not converge for any finite value of z, and so φ is not well defined. We will therefore find a function φ which has an asymptotic expansion, as z tends to ∞, given by (14.1). Several functions may have the same asymptotic expansion, but the function φ defined below will be shown to satisfy Ramanujan's claim.

Define, for $z > 0$,

$$\varphi(z) := \varphi(z, x) := z \int_0^{\infty} \frac{e^{-tz}\, dt}{(1+t)^x}, \tag{14.2}$$

where x is any complex number. Now,

$$(1+t)^{-x} = \sum_{n=0}^{\infty} \frac{(x)_n (-t)^n}{n!}, \qquad |t| < 1.$$

Applying Watson's Lemma (Olver [1, p. 71]), we find that, as z tends to ∞,

$$\varphi(z) \sim \sum_{n=0}^{\infty} \frac{(-1)^n (x)_n}{z^n},$$

which agrees with (14.1).

Entry 14 (Formula (1), p. 276). *Let $\varphi(z, x) = \varphi(z)$ be defined by (14.2). Then for $0 < \operatorname{Re} a < 1$, and any complex number x,*

$$\frac{\pi a^z \zeta(x)}{2\sin(\frac{1}{2}\pi x)} = \frac{\pi a}{2(x-1)} - \frac{1}{2x} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} a^{2n} \zeta(2n)}{2n - x} + \sum_{n=1}^{\infty} \frac{e^{-2\pi n a} \varphi(2\pi n a)}{2n}. \tag{14.3}$$

Proof. Using (14.2) and inverting the order of summation and integration by absolute convergence, we find that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{e^{-2\pi n a} \varphi(2\pi n a)}{2n} &= \pi a \sum_{n=1}^{\infty} e^{-2\pi n a} \int_0^{\infty} \frac{e^{-2\pi n a t}}{(1+t)^x} dt \\
&= \pi a \int_0^{\infty} (1+t)^{-x} \sum_{n=1}^{\infty} e^{-2\pi n a (t+1)} dt \\
&= \pi a \int_0^{\infty} \frac{dt}{(1+t)^x (e^{2\pi a (t+1)} - 1)}.
\end{aligned}$$

Thus, (14.3) is equivalent to the formula

$$\frac{\pi a^z \zeta(x)}{2\sin(\frac{1}{2}\pi x)} = \frac{\pi a}{2(x-1)} - \frac{1}{2x} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}a^{2n}\zeta(2n)}{2n-x} + \pi a \int_0^{\infty} \frac{dt}{(1+t)^x(e^{2\pi a(t+1)}-1)}. \tag{14.4}$$

We now temporarily add the restriction, Re $x < 0$. Then

$$\begin{aligned}\int_0^{\infty} \frac{dt}{(1+t)^x(e^{2\pi a(t+1)}-1)} &= \int_1^{\infty} \frac{dt}{t^x(e^{2\pi a t}-1)} \\ &= \int_0^{\infty} \frac{dt}{t^x(e^{2\pi a t}-1)} - \int_0^1 \frac{dt}{t^x(e^{2\pi a t}-1)} \\ &= (2\pi a)^{x-1}\Gamma(1-x)\zeta(1-x) - \int_0^1 \frac{dt}{t^x(e^{2\pi a t}-1)},\end{aligned} \tag{14.5}$$

by a well-known representation for $\zeta(s)$ (Titchmarsh [3, p. 18]). Employing the functional equation (0.4) of $\zeta(z)$ in (14.5), and then using (14.5) in (14.4), we see that we are required to prove that

$$\frac{\pi a}{2(x-1)} - \frac{1}{2x} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}a^{2n}\zeta(2n)}{2n-x} - \pi a \int_0^1 \frac{dt}{t^x(e^{2\pi a t}-1)} = 0, \tag{14.6}$$

where Re $x < 0$.

Using the generating function (0.1) for the Bernoulli numbers and inverting the order of integration and summation by absolute convergence, we find that, for Re $x < 0$ and $0 <$ Re $a < 1$,

$$\begin{aligned}\pi a \int_0^1 \frac{dt}{t^x(e^{2\pi a t}-1)} &= \frac{1}{2}\int_0^1 t^{-1-x} \sum_{n=0}^{\infty} \frac{B_n (2\pi a t)^n}{n!} dt \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{B_n(2\pi a)^n}{n!}\frac{1}{n-x} \\ &= -\frac{1}{2x} + \frac{\pi a}{2(x-1)} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{B_{2n}(2\pi a)^{2n}}{(2n)!(2n-x)} \\ &= -\frac{1}{2x} + \frac{\pi a}{2(x-1)} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\zeta(2n)a^{2n}}{2n-x},\end{aligned}$$

by Euler's formula (0.2). Hence, we have proved (14.6) for $0 <$ Re $a < 1$ and Re $x < 0$. Using analytic continuation, we find that (14.3) holds for all complex x.

On pages 277–278, Ramanujan records five enigmatic formulas, all of the same type and each apparently arising from Eisenstein series. These formulas are numbered (9), (10), (11), (12), and (14) in a long list of results, but the remaining formulas in this series have no relation to these five formulas. All five formulas involve the Bernoulli numbers B_n. For each of the next five entries, we precisely quote Ramanujan, even though his convention for Bernoulli numbers is different from ours in (0.1). We then determine those values of n for which the proposed claim might have validity. For each formula, there is a sequence of values of n for which the result is classical; in most cases, the theorem can be found elsewhere in Ramanujan's notebooks. After discussing the classical case, we reformulate Ramanujan's claim for those values of n for which Ramanujan's claim is completely new. We lastly, in each case, prove the new theorem.

Proofs of Entries 15–19 were first proved in a paper with P. Bialek [1].

Entry 15 (Formula (9), p. 277).

$$\sum_{k=1}^{\infty}\frac{k^{n-1}}{e^{2\pi k}-1}=\frac{|B_n|}{2n}+\frac{|B_n|}{n}\cos\left(\frac{\pi n}{4}\right)\left\{\frac{1}{2^{n/2}}+\frac{2\cos(n\tan^{-1}\frac{1}{3})}{5^{n/2}}\right.$$
$$\left.+\frac{2\cos(n\tan^{-1}\frac{1}{2})}{10^{n/2}}+\frac{2\cos(n\tan^{-1}\frac{1}{5})}{13^{n/2}}+\cdots\right\}, \tag{15.1}$$

"where 2, 5, 10, 13, ... are sum of squares of numbers that are prime to each other."

In Ramanujan's convention all the even indexed Bernoulli numbers are positive, in contrast to the usual definition in (0.1), and so we have inserted absolute value signs around his B_n's.

It appears to be difficult to discern the pattern in the numerators on the right side of (15.1), but a natural pattern will emerge in the proof below. If n is a positive odd integer exceeding 1, $B_n = 0$, and so (15.1) cannot be valid, as the left side of (15.1) is positive. Numerical calculations indicate that (15.1) is apparently false if $n = 1$, even if we assume that $B_1 = \frac{1}{2}$, instead of $-\frac{1}{2}$ from (0.1). We thus conclude that Ramanujan apparently intends n to be an even positive integer.

Let $n = 4m+2$, where m is a positive integer. Then (15.1) reduces to the claim

$$\sum_{k=1}^{\infty}\frac{k^{4m+1}}{e^{2\pi k}-1}=\frac{B_{4m+2}}{8m+4}. \tag{15.2}$$

Indeed, (15.2) is correct. To the best of our knowledge, (15.2) was first proved by J. W. L. Glaisher [1] in 1889, although an equivalent formulation was established in 1881 by A. Hurwitz [1], [2] in his thesis. Moreover, (15.2) is found in Section 13 of Chapter 14 in Ramanujan's second notebook [9, p. 171]. For proofs of other generalizations of (15.2) and for references to the many proofs of (15.2) that can be found in the literature, see our paper [6] and book [2, pp. 261–262].

In concluding our discussion of (15.1), we remark that the instances when $n \equiv 0 (\mathrm{mod} 4)$ appear to be the only ones remaining for which (15.1) may be correct and new. Indeed, (15.1) is then of great interest, for in these cases a curious infinite series appears on the right side of (15.1), and there are no comparable results in the literature.

Before we state Theorem 15.1 it is necessary to provide a discussion about solutions to

$$\ell = c^2 + d^2, \qquad \gcd(c, d) = 1, \tag{15.3}$$

where $\gcd(c, d)$ denotes the greatest common divisor of c and d. We partition the solutions of (15.3) into three classes. First, suppose that $c \neq d$ and $cd \neq 0$. Then each solution (c, d) of (15.3) generates eight solutions, namely,

$$\pm(c, d), \qquad \pm(c, -d), \qquad \pm(d, c), \qquad \pm(d, -c). \tag{i}$$

The case $c = d = 1$ generates four solutions, namely,

$$\pm(1, 1), \qquad \pm(1, -1). \tag{ii}$$

There is one further case, namely, $c = 1, d = 0$, which generates the four solutions

$$\pm(1, 0), \qquad \pm(0, 1). \tag{iii}$$

We shall say that the solutions (c_1, d_1) and (c_2, d_2) of (15.3) are *distinct* if they do not simultaneously belong to the same set of eight solutions in (i), or the same set of four solutions in (ii), or the same set of four solutions in (iii).

Recall that solutions (c, d) of (15.3) exist if and only if the prime factors of ℓ are all of the form $4k + 1$, except for the prime 2, which may occur to at most the first power (I. Niven, H. S. Zuckerman, and H. L. Montgomery [1, p. 164]). Although not needed in the sequel, we also recall (G. H. Hardy and E. M. Wright [1, pp. 241–242]; Niven, Zuckerman, and Montgomery [1, p. 167]) that if $r(\ell)$ denotes the number of representations of ℓ as a sum of two squares, then

$$r(\ell) = 4\sum_{d|\ell}(-1)^{(d-1)/2}.$$

Of course, in (15.3), we have imposed the restriction $\gcd(c, d) = 1$.

Theorem 15.1. *Let m be a positive integer. Then*

$$\sum_{k=1}^{\infty} \frac{k^{4m-1}}{e^{2\pi k} - 1} = -\frac{B_{4m}}{8m} - \frac{(-1)^m B_{4m}}{4m}\left(\frac{1}{2^{2m}} + \sum_{\ell} \frac{2\cos\left(4m\tan^{-1}\frac{c-d}{c+d}\right)}{\ell^{2m}}\right), \tag{15.4}$$

where the sum on the right side of (15.4) is over all integers $\ell > 2$ representable by (15.3), and where for each ℓ, the sum is also over all distinct solutions (c, d) of (15.3).

Observe that the four terms in curly brackets on the right side of (15.1) arise from (15.4) when $c, d = 1, 1; 2, 1; 3, 1; 3, 2$, respectively.

Proof. First,

$$\sum_{k=1}^{\infty} \frac{k^{4m-1}}{e^{2\pi k}-1} = \sum_{k=1}^{\infty}\sum_{r=1}^{\infty} k^{4m-1}e^{-2\pi kr} = \sum_{n=1}^{\infty} \sigma_{4m-1}(n)e^{-2\pi n}, \tag{15.5}$$

where $\sigma_k(n) = \sum_{d|n} d^k$. Now if n is an integer with $n \geq 2$, and if $\mathrm{Im}(\tau) > 0$ (Rankin [1, p. 194, eq. (6.1.4)]),

$$\sum_{k=1}^{\infty} \sigma_{2n-1}(k)e^{2\pi ik\tau} = \frac{B_{2n}}{4n} - \frac{B_{2n}}{8n} \sum_{\substack{c,d=-\infty \\ (c,d)=1}}^{\infty}{}' \frac{1}{(c\tau+d)^{2n}}, \tag{15.6}$$

where the prime $'$ on the summation sign indicates that the term with $c = d = 0$ is omitted from the summation. Thus, by (15.5) and (15.6),

$$\sum_{k=1}^{\infty} \frac{k^{4m-1}}{e^{2\pi k}-1} = \frac{B_{4m}}{8m} - \frac{B_{4m}}{16m} \sum_{\substack{c,d=-\infty \\ (c,d)=1}}^{\infty}{}' \frac{1}{(ci+d)^{4m}}. \tag{15.7}$$

Let $\tan^{-1} z$ denote the principal branch of the inverse tangent relation, i.e., $|\arg w| < \pi/2$. Then

$$(ci+d)^{-4m} = \ell^{-2m} \exp(-4mi \tan^{-1}(c/d)), \tag{15.8}$$

where ℓ is defined by (15.3). Since the sum on the right side of (15.7) converges absolutely, we can rearrange the terms in any order. So, we group terms according to increasing values of ℓ.

We now sum the terms in each of the cases (i)–(iii), described prior to the statement of Theorem 15.1. For fixed c, d, with $c, d > 0$, the eight terms in (15.7) for case (i) equal, by (15.8),

$$\begin{aligned}
&\frac{2}{(ci+d)^{4m}} + \frac{2}{(-ci+d)^{4m}} + \frac{2}{(di+c)^{4m}} + \frac{2}{(-di+c)^{4m}} \\
&= 4\mathrm{Re}\left\{\frac{1}{(ci+d)^{4m}} + \frac{1}{(di+c)^{4m}}\right\} \\
&= 4\ell^{-2m}\mathrm{Re}\left\{\exp\left(-4mi\tan^{-1}(c/d)\right) + \exp\left(-4mi\tan^{-1}(d/c)\right)\right\} \\
&= 4\ell^{-2m}\left\{\cos\left(4m\tan^{-1}(c/d)\right) + \cos\left(4m\tan^{-1}(d/c)\right)\right\} \\
&= 8\ell^{-2m}\cos\left(4m\tan^{-1}(c/d)\right),
\end{aligned} \tag{15.9}$$

since

$$\tan^{-1}(c/d) + \tan^{-1}(d/c) = \pi/2. \tag{15.10}$$

We now show that

$$\cos\left(4m\tan^{-1}\left(\frac{c}{d}\right)\right) = (-1)^m \cos\left(4m\tan^{-1}\left(\frac{c-d}{c+d}\right)\right). \tag{15.11}$$

Observe that

$$\tan\left(\tan^{-1}\left(\frac{d}{c}\right)+\tan^{-1}\left(\frac{c-d}{c+d}\right)\right)=\frac{\dfrac{c}{d}+\dfrac{c-d}{c+d}}{1-\dfrac{c}{d}\left(\dfrac{c-d}{c+d}\right)}=1.$$

Hence,

$$\tan^{-1}\left(\frac{d}{c}\right)+\tan^{-1}\left(\frac{c-d}{c+d}\right)=\frac{\pi}{4}+k\pi, \tag{15.12}$$

for some integer k. Thus, (15.11) follows easily. Hence, by (15.9), the sum of the eight terms in case (i) equals

$$8(-1)^m\ell^{-2m}\cos\left(4m\tan^{-1}\left(\frac{c-d}{c+d}\right)\right). \tag{15.13}$$

In case (ii), the sum of the four terms equals

$$\frac{2}{(1+i)^{4m}}+\frac{2}{(1-i)^{4m}}=4\,\mathrm{Re}\left(\frac{1}{(1+i)^{4m}}\right)=\frac{4(-1)^m}{2^{2m}}. \tag{15.14}$$

In case (iii), the sum of the four terms equals

$$4. \tag{15.15}$$

Using (15.13)–(15.15) in (15.7), we deduce (15.4).

Entry 16 (Formula (11), p. 278).

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k^{n-1}}{e^{k\pi}-e^{-k\pi}}=-(2^n-1)\frac{|B_n|}{n}\cos\left(\frac{\pi n}{4}\right)\left(\frac{1}{2^{n/2}}\right.$$
$$\left.+\frac{2\cos\left(n\tan^{-1}\frac{1}{2}\right)}{10^{n/2}}+\frac{2\cos\left(n\tan^{-1}\frac{2}{3}\right)}{26^{n/2}}+\cdots\right). \tag{16.1}$$

As before, it appears that Ramanujan intends n to be an even positive integer. If $n=4m+2$, where m is a positive integer, (16.1) reduces to the evaluation

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k^{4m+1}}{\sinh(k\pi)}=0. \tag{16.2}$$

This result is due to A. Cauchy [1, p. 362] and is also found in Glaisher's paper [1]. Many proofs of (16.2) can be found in the literature; see the author's paper [7, p. 337] for a list of several authors. If $n=2$, (16.1) and (16.2) are false. In fact,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k}{\sinh(k\pi)}=\frac{1}{4\pi},$$

a result also due to Cauchy [1, p. 361]. See the author's paper [7, p. 337] for another proof and references to further proofs.

For other generalizations of (16.2), see our paper [7] and book [2, pp. 294–295].

If $n \equiv 0 \pmod 4$, (16.1) is new, and we state a precise version of this in the next theorem.

Theorem 16.1. *Let m be a positive integer. Then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}k^{4m-1}}{e^{k\pi} - e^{-k\pi}} = (-1)^m(2^{4m}-1)\frac{B_{4m}}{4m} \times \left(\frac{1}{2^{2m}} + \sum_{\ell \text{ even}} \frac{2\cos\left(4m\tan^{-1}\left(\frac{c-d}{c+d}\right)\right)}{\ell^{2m}}\right), \tag{16.3}$$

where the sum is over all even positive integers $\ell > 2$ that can be represented by (15.3), and, for each fixed ℓ, the sum is also over all distinct odd solutions (c, d) of (15.3).

Note that the second and third summands on the right side of (16.1) arise from the terms with $c = 3, d = 1$ and $c = 5, d = 1$, respectively, in (16.3).

Proof. First,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}k^{4m-1}}{e^{k\pi} - e^{-k\pi}} = \sum_{k=1}^{\infty}\sum_{r=0}^{\infty}(-1)^{k-1}k^{4m-1}e^{-k(2r+1)\pi} = \sum_{n=1}^{\infty} f_{4m-1}(n)e^{-n\pi}, \tag{16.4}$$

where

$$f_k(n) = \sum_{\substack{d|n \\ n/d \text{ odd}}} (-1)^{d-1}d^k. \tag{16.5}$$

Second, we repeat the argument made in the proof of Theorem 15.1 with the added stipulation that the indices in the Eisenstein series be odd. We thus find that

$$\sum_{\substack{c,d=-\infty \\ (c,d)=1 \\ c,d \text{ odd}}}^{\infty} \frac{1}{(ci+d)^{4m}} = 4(-1)^m\left(\frac{1}{2^{2m}} + \sum_{\ell \text{ even}} \frac{2\cos\left(4m\tan^{-1}\left(\frac{c-d}{c+d}\right)\right)}{\ell^{2m}}\right), \tag{16.6}$$

where the summation on the right side of (16.6) is over all even $\ell > 2$ that are representable by (15.3), and where, for each fixed ℓ, the sum is also over all distinct solutions (c, d) satisfying (15.3).

Third, we need an analogue of (15.6), where in the Eisenstein series on the right side of (15.6) we sum only over odd c and d. To that end,

$$\sum_{\substack{c,d=-\infty \\ c,d \text{ odd}}}^{\infty} \frac{1}{(c\tau+d)^{2n}} = \sum_{\substack{r=1 \\ r \text{ odd}}}^{\infty} \sum_{\substack{c,d=-\infty \\ (c,d)=1 \\ c,d \text{ odd}}}^{\infty} \frac{1}{(cr\tau+dr)^{2n}}$$

$$= \zeta(2n)(1-2^{-2n}) \sum_{\substack{c,d=-\infty \\ (c,d)=1 \\ c,d \text{ odd}}}^{\infty} \frac{1}{(c\tau+d)^{2n}}. \tag{16.7}$$

Put

$$G_{2n}(\tau) = {\sum_{c,d=-\infty}^{\infty}}' \frac{1}{(c\tau+d)^{2n}}.$$

Then

$$\sum_{\substack{c,d=-\infty \\ c,d \text{ odd}}}^{\infty} \frac{1}{(c\tau+d)^{2n}} = \left(\sum_{c,d=-\infty}^{\infty} - {\sum_{\substack{c,d=-\infty \\ c \text{ even}}}^{\infty}}' - \sum_{\substack{c,d=-\infty \\ d \text{ even}}}^{\infty} + {\sum_{\substack{c,d=-\infty \\ c,d \text{ even}}}^{\infty}}' \right) \frac{1}{(c\tau+d)^{2n}}$$

$$= G_{2n}(\tau) - G_{2n}(2\tau) - 2^{-2n}G_{2n}(\tau/2) + 2^{-2n}G_{2n}(\tau). \tag{16.8}$$

Writing (15.6) in the form

$$G_{2n}(\tau) = 2\zeta(2n) + D_{2n} \sum_{k=1}^{\infty} \sigma_{2n-1}(k) e^{2\pi i k\tau},$$

where

$$D_{2n} = \frac{2(2\pi i)^{2n}}{(2n-1)!}, \tag{16.9}$$

we deduce from (16.8) that

$$\sum_{\substack{c,d=-\infty \\ c,d \text{ odd}}}^{\infty} \frac{1}{(c\tau+d)^{2n}} = (1+2^{-2n}) \left(2\zeta(2n) + D_{2n} \sum_{k=1}^{\infty} \sigma_{2n-1}(k) e^{2\pi i k\tau} \right)$$

$$- \left(2\zeta(2n) + D_{2n} \sum_{k=1}^{\infty} \sigma_{2n-1}(k) e^{4\pi i k\tau} \right)$$

$$- 2^{-2n} \left(2\zeta(2n) + D_{2n} \sum_{k=1}^{\infty} \sigma_{2n-1}(k) e^{\pi i k\tau} \right)$$

$$= D_{2n} \sum_{k=1}^{\infty} \sigma_{2n-1}(k) \left\{ (1+2^{-2n}) e^{2\pi i k\tau} - e^{4\pi i k\tau} - 2^{-2n} e^{\pi i k\tau} \right\}$$

$$= D_{2n} \sum_{k=1}^{\infty} \left\{ \chi_2(k)\sigma_{2n-1}(k/2)(1+2^{-2n}) \right.$$

$$\left. - \chi_4(k)\sigma_{2n-1}(k/4) - \sigma_{2n-1}(k) 2^{-2n} \right\} e^{\pi i k\tau}, \tag{16.10}$$

where

$$\chi_2(k) = \begin{cases} 1, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd,} \end{cases}$$

and

$$\chi_4(k) = \begin{cases} 1, & \text{if } 4|k, \\ 0, & \text{otherwise.} \end{cases}$$

Since, from (0.2) and (16.9),

$$\frac{\zeta(2n)}{D_{2n}} = -\frac{B_{2n}}{8n},$$

we conclude from (16.7) and (16.10) that

$$\sum_{k=1}^{\infty} h_{2n-1}(k) e^{\pi i k \tau} = (2^{2n} - 1)\frac{B_{2n}}{8n} \sum_{\substack{c,d=-\infty \\ (c,d)=1 \\ c,d \text{ odd}}}^{\infty} \frac{1}{(c\tau + d)^{2n}}, \tag{16.11}$$

where

$$\begin{aligned} h_{2n-1}(k) &= -2^{2n}\left\{\chi_2(k)\sigma_{2n-1}(k/2)(1 + 2^{-2n}) \right. \\ &\qquad \left. - \chi_4(k)\sigma_{2n-1}(k/4) - \sigma_{2n-1}(k)2^{-2n}\right\} \\ &= \begin{cases} \sigma_{2n-1}(k), & \text{if } k \equiv 1, 3 \pmod 4, \\ -(2^{2n}+1)\sigma_{2n-1}(k/2) + \sigma_{2n-1}(k), & \text{if } k \equiv 2 \pmod 4, \\ -(2^{2n}+1)\sigma_{2n-1}(k/2) & \\ \quad + 2^{2n}\sigma_{2n-1}(k/4) + \sigma_{2n-1}(k), & \text{if } k \equiv 0 \pmod 4. \end{cases} \end{aligned} \tag{16.12}$$

We now return to the definition of $f_{2n-1}(k)$ given in (16.5) and relate it to the definition of $h_{2n-1}(k)$ above.

First suppose that k is odd. Then k/d is odd for all divisors d of k. Hence,

$$f_{2n-1}(k) = \sum_{d|k} (-1)^{d-1} d^{2n-1} = \sum_{d|k} d^{2n-1} = \sigma_{2n-1}(k) = h_{2n-1}(k),$$

by (16.12).

Second, suppose that $k \equiv 2 \pmod 4$. If k/d is odd, then d is even. Thus,

$$f_{2n-1}(k) = -\sum_{\substack{d|k \\ d \text{ even}}} d^{2n-1}.$$

Write $d = 2d_1$. Then

$$f_{2n-1}(k) = -\sum_{2d_1|k} (2d_1)^{2n-1} = -2^{2n-1} \sum_{d_1|k/2} d_1^{2n-1} = -2^{2n-1}\sigma_{2n-1}(k/2). \tag{16.13}$$

Since $k \equiv 2 \pmod 4$, it follows from the standard product formula for $\sigma_n(k)$ (Hardy and Wright [1, p. 239]) that

$$\sigma_{2n-1}(k) = (2^{2n-1} + 1)\sigma_{2n-1}(k/2).$$

Thus, by (16.12),

$$h_{2n-1}(k) = (-2^{2n} - 1 + 2^{2n-1} + 1)\sigma_{2n-1}(k/2) = -2^{2n-1}\sigma_{2n-1}(k/2) = f_{2n-1}(k),$$

by (16.13).

Third, suppose that $k \equiv 0 \pmod 4$. Then,

$$\begin{aligned} f_{2n-1}(k) &= -\sum_{\substack{d|k \\ k/d \text{ odd}}} d^{2n-1} = -\sum_{d|k} d^{2n-1} + \sum_{\substack{d|k \\ k/d \text{ even}}} d^{2n-1} \\ &= -\sum_{d|k} d^{2n-1} + \sum_{d|\frac{1}{2}k} d^{2n-1} = -\sigma_{2n-1}(k) + \sigma_{2n-1}(k/2). \end{aligned}$$

Define the integer $a \geq 2$ by $2^a \| k$. Then by the aforementioned product formula for $\sigma_{2n-1}(k)$,

$$\begin{aligned} f_{2n-1}(k) &= -\frac{2^{(a+1)(2n-1)} - 1}{2^{a(2n-1)} - 1}\sigma_{2n-1}(k/2) + \sigma_{2n-1}(k/2) \\ &= \frac{2^{a(2n-1)}(1 - 2^{2n-1})}{2^{a(2n-1)} - 1}\sigma_{2n-1}(k/2). \end{aligned} \tag{16.14}$$

On the other hand, from (16.12), when $k \equiv 0 \pmod 4$,

$$\begin{aligned} h_{2n-1}(k) = &- (2^{2n} + 1)\sigma_{2n-1}(k/2) + 2^{2n}\left(\frac{2^{(a-1)(2n-1)} - 1}{2^{a(2n-1)} - 1}\right)\sigma_{2n-1}(k/2) \\ &+ \frac{2^{(a+1)(2n-1)} - 1}{2^{a(2n-1)} - 1}\sigma_{2n-1}(k/2) \\ = &\ \frac{2^{a(2n-1)}(1 - 2^{2n-1})}{2^{a(2n-1)} - 1}\sigma_{2n-1}(k/2) \\ = &\ f_{2n-1}(k), \end{aligned}$$

by (16.14).

In conclusion, for all k we have shown that $h_{2n-1}(k) = f_{2n-1}(k)$. Using this fact in (16.11), setting $\tau = i$ and $n = 2m$, and combining (16.11) with (16.4), we find that

$$\sum_{k=1}^{\infty} \frac{(-1)^{(k-1)}k^{4m-1}}{e^{k\pi} - e^{-k\pi}} = (2^{4m} - 1)\frac{B_{4m}}{16m} \sum_{\substack{c,d=-\infty \\ (c,d)=1 \\ c,d \text{ odd}}}^{\infty} \frac{1}{(ci + d)^{4m}}. \tag{16.15}$$

Combining (16.6) and (16.15), we complete the proof.

Entry 17 (Formula (14), p. 278).

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k^{n-1}}{e^{k\pi\sqrt{3}}-(-1)^k} = -\frac{|B_n|}{n}\cos\left(\frac{\pi n}{6}\right)-\frac{|B_n|}{n}\left(\frac{1}{2}+\cos\left(\frac{\pi n}{3}\right)\right)$$
$$\times\left\{\frac{1}{3^{n/2}}+\frac{2\cos\left(n\tan^{-1}\frac{1}{3\sqrt{3}}\right)}{7^{n/2}}+\cdots\right\}. \tag{17.1}$$

As before, Ramanujan evidently intends n to be even. Thus, set $n = 2m$. If $m \not\equiv 0 \pmod 3$ and $m > 1$, then

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k^{2m-1}}{e^{k\pi\sqrt{3}}-(-1)^k} = -\frac{|B_{2m}|}{2m}\cos\left(\frac{\pi m}{3}\right) = -\frac{B_{2m}}{4m}. \tag{17.2}$$

This result was apparently first established in print by M. B. Rao and M. V. Aiyar in papers published in 1923–1924 [1], [2]. Thus, even the special case (17.2) was first proved by Ramanujan, although he never published a proof. See also Berndt's paper [6, p. 157, Prop. 2.8], which also contains some generalizations of (17.2).

If $m = 1$, (17.1) and (17.2) reduce to the claim

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k}{e^{k\pi\sqrt{3}}-(-1)^k} = -\frac{B_2}{4} = -\frac{1}{24}. \tag{17.3}$$

Now (17.3) is false. In fact (Berndt [6, p. 159, Prop. 2.13]),

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k}{e^{k\pi\sqrt{3}}-(-1)^k} = -\frac{1}{24}+\frac{1}{4\pi\sqrt{3}},$$

which was also first established by Rao and Aiyar [1].

Thus, it remains to prove (17.1) for $n \equiv 0 \pmod 6$.

Before stating Theorem 17.1, we need to offer some remarks about the solutions of

$$\ell = c^2 - cd + d^2, \qquad \gcd(c,d) = 1. \tag{17.4}$$

We consider three cases.

First, suppose that $c \neq d, cd \neq 0$, and that (2,1) does not appear in the list immediately below. Then each solution (c, d) generates 12 solutions, namely,

(i)

$$\pm(c,d),\quad \pm(d,c),\quad \pm(c-d,c),\quad \pm(c,c-d),\quad \pm(d,d-c),\quad \pm(d-c,d).$$

Second, for $\ell = 3$, there are only six solutions, because, for example, if $c = 2, d = 1$, then $c - d = d$, and so (c, d) and $(c, c - d)$ are not distinct. The six solutions are

(ii) $\qquad \pm(2,1), \quad \pm(1,2), \quad \pm(1,-1).$

Third, for $\ell = 1$, there are again only six solutions, namely,

(iii) $\qquad \pm(1,0), \quad \pm(0,1), \quad \pm(1,1).$

We shall say that two solutions (c_1, d_1), (c_2, d_2) of (17.4) are *distinct* if they are not both simultaneously in any of the three solution sets given in cases (i)–(iii) above.

Those integers ℓ that can be represented by (17.4) have the representation

$$\ell = 3^a \prod_{j=1}^{r} p_j^{a_j},$$

where $a = 0$ or 1, the primes p_j are distinct and have the form $3k_j + 1$, and a_j is a positive integer, $1 \le j \le r$ (Niven, Zuckerman, and Montgomery [1, p. 176]).

Although not needed in the sequel, we note in passing that if $r(\ell)$ denotes the number of integral solutions to $\ell = c^2 - cd + d^2$, then by a theorem of Dirichlet [1],

$$r(\ell) = 6\left(d_{1,3}(\ell) - d_{2,3}(\ell)\right),$$

where $d_{j,3}(\ell)$, $j = 1, 2$, denotes the number of divisors of ℓ of the form $3k + j$.

Theorem 17.1. *Let m be a positive integer. Then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} k^{6m-1}}{e^{k\pi\sqrt{3}} - (-1)^k} = \frac{B_{6m}}{6m} + (-1)^m \frac{B_{6m}}{4m} \times \left\{ \frac{1}{3^{3m}} + 2 \sum_{\ell} \frac{\cos\left(6m \tan^{-1}\left(\frac{d+c}{\sqrt{3}(d-c)}\right)\right)}{\ell^{3m}} \right\}, \tag{17.5}$$

where the sum is over all integers $\ell \ge 7$ that are representable by (17.4), and where, for each fixed ℓ, the sum is also over all distinct solutions (c, d) of (17.4).

Proof. First, if $\omega = \exp(2\pi i/3)$,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} k^{6m-1}}{e^{k\pi\sqrt{3}} - (-1)^k} &= -\sum_{k=1}^{\infty} \frac{k^{6m-1} e^{2\pi i \omega k}}{1 - e^{2\pi i \omega k}} = -\sum_{k=1}^{\infty} \sum_{r=1}^{\infty} k^{6m-1} e^{2\pi i \omega k r} \\ &= -\sum_{n=1}^{\infty} \sigma_{6m-1}(n) e^{2\pi i \omega n} = -\frac{B_{6m}}{12m} + \frac{B_{6m}}{24m} \sum_{\substack{c,d=-\infty \\ (c,d)=1}}^{\infty}{}' \frac{1}{(c\omega + d)^{6m}}, \end{aligned} \tag{17.6}$$

by (15.6).

Next,

$$(c\omega + d)^{6m} = \ell^{3m} \exp\left(6mi \tan^{-1}\left(\frac{c\sqrt{3}}{2d - c}\right)\right), \tag{17.7}$$

where ℓ is given by (17.4). We shall group terms in (17.6) according to increasing values of ℓ. This rearrangement is justified by the absolute convergence of the double series in (17.6).

We consider first the 12 terms on the right side of (17.6) that arise from a typical value in case (i). This sum of 12 terms equals

$$\frac{2}{(c\omega+d)^{6m}}+\frac{2}{(d\omega+c)^{6m}}+\frac{2}{((c-d)\omega+c)^{6m}}$$
$$+\frac{2}{(c\omega+(c-d))^{6m}}+\frac{2}{(d\omega+(d-c))^{6m}}+\frac{2}{((d-c)\omega+d)^{6m}}. \tag{17.8}$$

Observe that

$$\tan\left(\tan^{-1}\left(\frac{c\sqrt{3}}{2d-c}\right)+\tan^{-1}\left(\frac{d\sqrt{3}}{2c-d}\right)\right)=\frac{\dfrac{c\sqrt{3}}{2d-c}+\dfrac{d\sqrt{3}}{2c-d}}{1-\dfrac{3cd}{(2d-c)(2c-d)}}=-\sqrt{3}.$$

Thus,

$$\tan^{-1}\left(\frac{c\sqrt{3}}{2d-c}\right)+\tan^{-1}\left(\frac{d\sqrt{3}}{2c-d}\right)=-\frac{\pi}{3}+j\pi,$$

for some integer j. Hence,

$$\arg(c\omega+d)^{6m}+\arg(d\omega+c)^{6m}=2k\pi,$$

for some integer k. Therefore, $(c\omega+d)^{-6m}$ and $(d\omega+c)^{-6m}$ are conjugates. Using (17.4) and (17.7), we find that the sum (17.8) equals

$$4\operatorname{Re}\left(\frac{1}{(c\omega+d)^{6m}}\right)+4\operatorname{Re}\left(\frac{1}{((c-d)\omega+c)^{6m}}\right)+4\operatorname{Re}\left(\frac{1}{((d-c)\omega+d)^{6m}}\right)$$
$$=\frac{4}{l^{3m}}\operatorname{Re}\left\{\exp\left(-6mi\tan^{-1}\left(\frac{c\sqrt{3}}{2d-c}\right)\right)+\exp\left(-6mi\tan^{-1}\left(\frac{(c-d)\sqrt{3}}{c+d}\right)\right)\right.$$
$$\left.+\exp\left(-6mi\tan^{-1}\left(\frac{(d-c)\sqrt{3}}{c+d}\right)\right)\right\}$$
$$=\frac{4}{l^{3m}}\left\{\cos\left(6m\tan^{-1}\left(\frac{c\sqrt{3}}{2d-c}\right)\right)+\cos\left(6m\tan^{-1}\left(\frac{(c-d)\sqrt{3}}{c+d}\right)\right)\right.$$
$$\left.+\cos\left(6m\tan^{-1}\left(\frac{(d-c)\sqrt{3}}{c+d}\right)\right)\right\}. \tag{17.9}$$

By a calculation similar to one above, we find that

$$\tan^{-1}\left(\frac{c\sqrt{3}}{2d-c}\right)-\tan^{-1}\left(\frac{(c-d)\sqrt{3}}{c+d}\right)=\frac{\pi}{3}+j\pi,$$

for some integer j. Hence, the sum in (17.9) equals

$$12\ell^{-3m}\cos\left(6m\tan^{-1}\left(\frac{c\sqrt{3}}{2d-c}\right)\right). \tag{17.10}$$

Another elementary calculation shows that

$$\tan^{-1}\left(\frac{c\sqrt{3}}{2d-c}\right)-\tan^{-1}\left(\frac{d+c}{\sqrt{3}(d-c)}\right)=-\frac{\pi}{6}+j\pi,$$

for some integer j. Thus, from (17.10), the sum of our 12 terms equals

$$12(-1)^m\ell^{-3m}\cos\left(6m\tan^{-1}\left(\frac{d+c}{\sqrt{3}(d-c)}\right)\right). \tag{17.11}$$

For case (ii), a similar argument shows that the six terms total

$$6(-1)^m3^{-3m}. \tag{17.12}$$

For case (iii), an easy argument shows that the six terms total

$$6. \tag{17.13}$$

Using (17.11)–(17.13) in (17.6), we find that

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k^{6m-1}}{e^{k\pi\sqrt{3}}-(-1)^k}=&-\frac{B_{6m}}{12m}+\frac{B_{6m}}{24m}\left\{6+\frac{6(-1)^m}{3^{3m}}\right.\\&\left.+12(-1)^m\sum_{\ell}\frac{\cos\left(6m\tan^{-1}\left(\frac{d+c}{\sqrt{3}(d-c)}\right)\right)}{\ell^{3m}}\right\},\end{aligned} \tag{17.14}$$

where the sum is over all $\ell\geq 7$ that can be represented by (17.4), and where, for fixed ℓ, the sum is also over all distinct solutions of (17.4). The desired result (17.5) now follows upon simplifying (17.14).

Formulas (10) and (12) arise from a different Eisenstein series. We thus begin by deriving an analogue of (15.6). An alternate proof may be obtained by using Fourier series of Eisenstein series of level 4 (Schoeneberg [1, pp. 154–157]).

Lemma 18.1. *Let* $\operatorname{Im}\tau>0$, *and let n be a positive integer exceeding 1. Then*

$$\begin{aligned}&\sum_{k=0}^{\infty}(-1)^k\sigma_{2n-1}(2k+1)e^{\pi i(2k+1)\tau/2}\\&=i2^{2n-2}(2^{2n}-1)\frac{B_{2n}}{4n}\sum_{\substack{j,k=-\infty\\(2j+1,2k+1)=1}}^{\infty}\frac{(-1)^{j+k}}{((2j+1)\tau+2k+1)^{2n}}.\end{aligned}$$

Proof. Recall the partial fraction decomposition

$$\sum_{k=-\infty}^{\infty}\frac{(-1)^{k-1}}{2k-1+\tau}=\tfrac{1}{2}\pi\sec\left(\tfrac{1}{2}\pi\tau\right)=\frac{\pi e^{\pi i\tau/2}}{1+e^{\pi i\tau}}=\pi\sum_{k=0}^{\infty}(-1)^ke^{\pi i\tau(2k+1)/2}.$$

Differentiate both sides with respect to τ a total of $2n-1$ times to obtain, after some rearrangement,

$$\sum_{k=0}^{\infty}(-1)^k(2k+1)^{2n-1}e^{\pi i\tau(2k+1)/2}$$
$$= \tfrac{1}{2}i(-1)^{n-1}(2n-1)!\left(\frac{2}{\pi}\right)^{2n}\sum_{k=-\infty}^{\infty}\frac{(-1)^{k-1}}{(2k-1+\tau)^{2n}}.$$

Replace τ by $(2j+1)\tau$, multiply both sides by $(-1)^j$ and sum on j, $0 \le j < \infty$. We then deduce that

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}(2k+1)^{2n-1}e^{\pi i(2j+1)(2k+1)\tau/2}$$
$$= \tfrac{1}{2}i(-1)^{n-1}(2n-1)!\left(\frac{2}{\pi}\right)^{2n}\sum_{j=0}^{\infty}\sum_{k=-\infty}^{\infty}\frac{(-1)^{j+k-1}}{((2j+1)\tau+2k-1)^{2n}}. \tag{18.1}$$

Note that the summands $(-1)^{j+k-1}/((2j+1)\tau+2k-1)^{2n}$ in (18.1) are invariant upon the replacement of j by $-j-1$ and k by $-k+1$. We also observe that the double sum on the left side of (18.1) can be rewritten in terms of σ_{2n-1}. Thus, we may rewrite (18.1) in the form

$$\sum_{r=0}^{\infty}(-1)^r\sigma_{2n-1}(2r+1)e^{\pi i(2r+1)\tau/2}$$
$$= \tfrac{1}{4}i(-1)^{n-1}(2n-1)!\left(\frac{2}{\pi}\right)^{2n}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\frac{(-1)^{j+k}}{((2j+1)\tau+2k+1)^{2n}}$$
$$= \tfrac{1}{4}i(-1)^{n-1}(2n-1)!\left(\frac{2}{\pi}\right)^{2n}(1-2^{-2n})\zeta(2n)$$
$$\times \sum_{\substack{j,k=-\infty\\(2j+1,2k+1)=1}}^{\infty}\frac{(-1)^{j+k}}{((2j+1)\tau+2k+1)^{2n}}.$$

Upon using Euler's formula (0.2), we complete the proof.

Entry 18 (Formula (10), p. 278).

$$\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)^{n-1}}{\cosh\{(2k+1)\pi/2\}} = 2^n(2^n-1)\frac{|B_n|}{n}\sin\left(\frac{\pi n}{4}\right)$$
$$\times\left\{\frac{1}{2^{n/2}}-\frac{2\cos\left(n\tan^{-1}\frac{1}{2}\right)}{10^{n/2}}+\frac{2\cos\left(n\tan^{-1}\frac{2}{3}\right)}{26^{n/2}}-\cdots\right\}. \tag{18.2}$$

As with the previous formulas, most likely, Ramanujan intended n to be an even positive integer. If we set $n=4m$, where m is a positive integer, in (18.2), we find that

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^{4m-1}}{\cosh\{(2k+1)\pi/2\}} = 0. \tag{18.3}$$

This result was first proved by Cauchy [1, pp. 313, 362]. Ramanujan [2], [10, p. 326] offered (18.3) as a problem to the *Journal of the Indian Mathematical Society*. In Section 14 of Chapter 14 in his second notebook [9], Ramanujan recorded (18.3) as a corollary of a beautiful, more general theorem (Part II [2, p. 262]). For references to the many proofs of (18.3) and statements and proofs of more general theorems, see the author's paper [6, pp. 176–178] and book [2, pp. 261–262].

The next theorem gives a precise version of (18.2) when $n \equiv 2 \pmod 4$.

Theorem 18.1. *If m is a positive integer, then*

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^{4m+1}}{\cosh\{(2k+1)\pi/2\}} = (-1)^m 2^{4m+2}(2^{4m+2}-1)\frac{B_{4m+2}}{4m+2} \times \left\{ \frac{1}{2^{2m+1}} - 2\sum_{\ell} \frac{(-1)^{(c+d)/2} \cos\left((4m+2)\tan^{-1}\left(\frac{c-d}{c+d}\right)\right)}{\ell^{2m+1}} \right\}, \tag{18.4}$$

where the summation on the right side of (18.4) is over all even positive integers $\ell > 2$ that are representable by (15.3), and where, for fixed ℓ, the sum is also over all distinct pairs (c, d) satisfying (15.3).

It is easily checked that the second and third displayed terms in Ramanujan's formulation (18.2) arise from the terms when $c = 3, d = 1$ and $c = 5, d = 1$, respectively, in (18.4).

Proof. We first transform the left side of (18.4). To that end,

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^{4m+1}}{\cosh\{(2k+1)\pi/2\}} &= 2\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^{4m+1} e^{-(2k+1)\pi/2}}{1+e^{-(2k+1)\pi}} \\
&= 2\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} (-1)^{k+n}(2k+1)^{4m+1} e^{-(2n+1)(2k+1)\pi/2} \\
&= 2\sum_{r=0}^{\infty} (-1)^r \sigma_{4m+1}(2r+1) e^{-(2r+1)\pi/2} \\
&= i2^{4m+2}(2^{4m+2}-1)\frac{B_{4m+2}}{4(4m+2)} \sum_{\substack{j,k=-\infty \\ (2j+1,2k+1)=1}}^{\infty} \frac{(-1)^{j+k}}{((2j+1)i+2k+1)^{4m+2}} \\
&= -i2^{4m+2}(2^{4m+2}-1)\frac{B_{4m+2}}{4(4m+2)} \sum_{\substack{c,d=-\infty \\ (c,d)=1 \\ c,d \text{ odd}}}^{\infty} \frac{(-1)^{(c+d)/2}}{(ci+d)^{4m+2}},
\end{aligned} \tag{18.5}$$

by Lemma 18.1 with $\tau = i$ and $n = 2m + 1$. The sum on the right side of (18.5) is similar to the Eisenstein series in (16.6). The only differences are that the power of $(ci + d)$ is $4m + 2$ instead of $4m$, and that the series above contains the extra factor of $(-1)^{(c+d)/2}$ in its summands.

We now consider cases (i)–(ii), as we did in the proof of Theorem 15.1.

The sum of the eight terms in case (i) equals

$$\begin{aligned}
&\frac{2(-1)^{(c+d)/2}}{(ci+d)^{4m+2}} + \frac{2(-1)^{(-c+d)/2}}{(-ci+d)^{4m+2}} + \frac{2(-1)^{(c+d)/2}}{(di+c)^{4m+2}} + \frac{2(-1)^{(-c+d)/2}}{(-di+c)^{4m+2}} \\
&\quad = 4i \operatorname{Im}\left(\frac{(-1)^{(c+d)/2}}{(ci+d)^{4m+2}} + \frac{(-1)^{(c+d)/2}}{(di+c)^{4m+2}}\right) \\
&\quad = 4i(-1)^{(c+d)/2}\ell^{-2m-1}\operatorname{Im}\left\{\exp\left(-i(4m+2)\tan^{-1}(c/d)\right)\right. \\
&\qquad \left. + \exp\left(-i(4m+2)\tan^{-1}(d/c)\right)\right\} \\
&\quad = -4i(-1)^{(c+d)/2}\ell^{-2m-1}\left\{\sin\left((4m+2)\tan^{-1}(c/d)\right)\right. \\
&\qquad \left. + \sin\left((4m+2)\tan^{-1}(d/c)\right)\right\} \\
&\quad = -8i(-1)^{(c+d)/2}\ell^{-2m-1}\sin\left((4m+2)\tan^{-1}(c/d)\right),
\end{aligned}$$

by (15.10). Using (15.12), we find that

$$\sin\left((4m+2)\tan^{-1}(c/d)\right) = (-1)^m \cos\left((4m+2)\tan^{-1}\left(\frac{c-d}{c+d}\right)\right).$$

Thus, the eight terms in case (i) have the sum

$$-8i(-1)^{m+(c+d)/2}\ell^{-2m-1}\cos\left((4m+2)\tan^{-1}\left(\frac{c-d}{c+d}\right)\right). \tag{18.6}$$

Next, by the same type of reasoning, we find that the four terms in case (ii) sum to

$$4i2^{-2m-1}\sin\{(4m+2)\pi/4\} = 4i(-1)^m 2^{-2m-1}. \tag{18.7}$$

Putting (18.6) and (18.7) in (18.5), we conclude that

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^{4m+1}}{\cosh\{(2k+1)\pi/2\}} &= -i2^{4m+2}(2^{4m+2}-1)\frac{B_{4m+2}}{4(4m+2)} \\
&\times \left\{\frac{4i(-1)^m}{2^{2m+1}} - 8i(-1)^m \sum_{\ell} \frac{(-1)^{(c+d)/2}\cos\left((4m+2)\tan^{-1}\left(\frac{c-d}{c+d}\right)\right)}{\ell^{2m+1}}\right\},
\end{aligned}$$

where the summation on the right side is over all even positive integers $\ell > 2$ that can be represented by (15.3), and where, for each fixed ℓ, the sum is also over all distinct solutions (c, d) of (15.3). The theorem now follows after a small amount of simplification.

Entry 19 (Formula (12), p. 278).

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^{n-1}}{\cosh\left\{(2k+1)\pi\sqrt{3}/2\right\}} = (2^n-1)\frac{|B_n|}{n}\sin\left(\frac{\pi n}{6}\right)\left\{1-\frac{2\cos(\pi n/6)}{3^{n/2}}\right.$$
$$\left.+\frac{2\cos\left(n\tan^{-1}(\sqrt{3}/2)\right)}{7^{n/2}}-\cdots\right\}. \tag{19.1}$$

As before, we assume that Ramanujan intended n to be an even positive integer. If $n = 6m$, where m is a positive integer, then we deduce from (19.1) that

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^{6m-1}}{\cosh\left\{(2k+1)\pi\sqrt{3}/2\right\}} = 0. \tag{19.2}$$

The evaluation (19.2) was first achieved by Cauchy [1, p. 317]. Ramanujan recorded (19.2) as part of Entry 18(iii) of Chapter 17 of his second notebook. For two proofs of (19.2), see the author's paper [7, Corollary 7.6] and book [3, pp. 140–141]. For references to other proofs, generalizations of (19.2), and further results of this sort, see the last two cited references.

Before stating Theorem 19.1, we need to say a few words about solutions to

$$\ell = 3c^2 + d^2, \qquad \gcd(c,d) = 1. \tag{19.3}$$

Each solution (c, d) of (19.3) generates four solutions, namely,

$$\pm(c,d), \qquad \pm(-c,d). \tag{19.4}$$

We shall say that (c_1, d_1) and (c_2, d_2) are *distinct* solutions to (19.3) if they belong to different sets of four solutions given by (19.4). We remark that the number of positive odd solutions (c, d) to $4n = 3c^2 + d^2$, where n is odd, equals $d_{1,3}(n) - d_{2,3}(n)$ (L. K. Hua [1, p. 309]).

Theorem 19.1. *Let m be a positive integer such that $m \not\equiv 0 \pmod 3$. Then*

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^{2m-1}}{\cosh\left\{(2k+1)\pi\sqrt{3}/2\right\}}$$
$$= -(2^{2m}-1)\frac{B_{2m}}{2m}\sum_{\ell \text{ even}} \frac{(-1)^{(c+d)/2}\sin\left(2m\tan^{-1}(c\sqrt{3}/d)\right)}{(\ell/4)^m}, \tag{19.5}$$

where the sum on the right side of (19.5) is over all even positive integers ℓ which can be represented by (19.3), and where, for each fixed ℓ, the sum is also over all distinct solutions (c,d) of (19.3).

Proof. By a calculation like that in (18.5) and by Lemma 18.1 with $\tau = i\sqrt{3}$, we deduce that

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^{2m-1}}{\cosh\left\{(2k+1)\pi\sqrt{3}/2\right\}} = 2\sum_{r=0}^{\infty}(-1)^r \sigma_{2m-1}(2r+1)e^{-(2r+1)\pi\sqrt{3}/2}$$

$$= i2^{2m}(2^{2m}-1)\frac{B_{2m}}{8m} \sum_{\substack{j,k=-\infty \\ (2j+1,2k+1)=1}}^{\infty} \frac{(-1)^{j+k}}{((2j+1)i\sqrt{3}+2k+1)^{2m}}$$

$$= -i2^{2m}(2^{2m}-1)\frac{B_{2m}}{8m} \sum_{\substack{c,d=-\infty \\ (c,d)=1 \\ c,d \text{ odd}}}^{\infty} \frac{(-1)^{(c+d)/2}}{(ci\sqrt{3}+d)^{2m}}. \tag{19.6}$$

Now,

$$(ci\sqrt{3}+d)^{-2m} = \ell^{-m}\exp\left(-2mi\tan^{-1}(c\sqrt{3}/d)\right), \tag{19.7}$$

where ℓ is given by (19.3). We group terms according to increasing values of ℓ. The sum of the four terms arising from (19.4) equals, by (19.7),

$$\frac{2(-1)^{(c+d)/2}}{(ci\sqrt{3}+d)^{2m}} + \frac{2(-1)^{(-c+d)/2}}{(-ci\sqrt{3}+d)^{2m}}$$
$$= 4i \operatorname{Im}\left(\frac{(-1)^{(c+d)/2}}{(ci\sqrt{3}+d)^{2m}}\right)$$
$$= -4i(-1)^{(c+d)/2}\ell^{-m}\sin\left(2m\tan^{-1}(c\sqrt{3}/d)\right). \tag{19.8}$$

Thus, from (19.6) and (19.8), we complete the proof of Theorem 19.1.

Comparing (19.5) with (19.1), we see that the trigonometric sums on the right sides have quite different shapes. The first three terms in (19.1) evidently arise from the values $c, d = 1, 1; 1, 3; 1, 5$, respectively. However, we note that 28 has two representations, $3 \cdot 1^2 + 5^2$ and $3 \cdot 3^2 + 1^2$. Easy calculations show that the first two terms on the right side of (19.1) agree with the first two terms on the right side of (19.5). However, there is a discrepancy between the third terms. This discrepancy exists if we take either term arising from the two representations of 28, or if we take the sum of the two terms, from our sum (19.5).

We shall now establish an alternative representation for the sum on the right side of (19.5). This will give a result which is "close" to that of Ramanujan and perhaps indicate where Ramanujan erred. An elementary calculation shows that

$$\tan\left(\tan^{-1}\left(\frac{c\sqrt{3}}{d}\right) + \tan^{-1}\left(\frac{\sqrt{3}(d-c)}{3c+d}\right)\right) = \sqrt{3},$$

and so

$$\tan^{-1}\left(\frac{c\sqrt{3}}{d}\right)+\tan^{-1}\left(\frac{\sqrt{3}(d-c)}{3c+d}\right)=\frac{\pi}{3}+k\pi,$$

for some integer k. Thus,

$$\begin{aligned}\sin\left(2m\tan^{-1}\left(\frac{c\sqrt{3}}{d}\right)\right)&=\sin\left(2m\left\{\frac{\pi}{3}-\tan^{-1}\left(\frac{\sqrt{3}(d-c)}{3c+d}\right)\right\}\right)\\&=\sin\left(\frac{2m\pi}{3}\right)\cos\left(2m\tan^{-1}\left(\frac{\sqrt{3}(d-c)}{3c+d}\right)\right)\\&\quad-\cos\left(\frac{2m\pi}{3}\right)\sin\left(2m\tan^{-1}\left(\frac{\sqrt{3}(d-c)}{3c+d}\right)\right)\\&=(-1)^{m-1}\sin\left(\frac{m\pi}{3}\right)\cos\left(2m\tan^{-1}\left(\frac{\sqrt{3}(d-c)}{3c+d}\right)\right)\\&\quad+\tfrac{1}{2}\sin\left(2m\tan^{-1}\left(\frac{\sqrt{3}(d-c)}{3c+d}\right)\right).\end{aligned}\tag{19.9}$$

Thus, we obtain "half" of what Ramanujan probably found, because for $c, d = 1, 1; 1, 3; 1, 5$, the first term on the far right side of (19.9) yields precisely the trigonometric functions given by Ramanujan in (19.1), except for an additional factor of 2 in the second and third terms in (19.1). For $c = d = 1$, the second term on the far right side of (19.9) vanishes. For $c, d = 1, 3$, a simple calculation shows that the first and second terms on the far right side of (19.9) are equal, while for $c, d = 1, 5$, the second term does not equal the first term.

Entry 20 (Formula (15), p. 278). *If n is a positive integer, then*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k^{6n}}{\cosh(k\pi\sqrt{3})-(-1)^k}=\frac{2n\sqrt{3}}{\pi}\left(\frac{B_{6n}}{12n}+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k^{6n-1}}{e^{k\pi\sqrt{3}}-(-1)^k}\right),$$

where B_j, $j \geq 0$, denotes the jth Bernoulli number.

Proof. We shall easily show that Entry 20 is equivalent to a result of the author [6, p. 163, Cor. 2.22], namely,

$$\sum_{k=1}^{\infty}\frac{k^{6n-1}}{(-1)^k e^{k\pi\sqrt{3}}-1}+\frac{\pi\sqrt{3}}{12n}\sum_{k=1}^{\infty}\frac{k^{6n}}{\sin^2(k\pi\rho)}=\frac{B_{6n}}{12n},\tag{20.1}$$

where n is a positive integer and $\rho = \exp(2\pi i/3)$. By a straightforward calculation,

$$\sin^2(k\pi\rho)=\tfrac{1}{2}(1-\cos(2\pi k\rho))=\tfrac{1}{2}(1-(-1)^k\cosh(k\pi\sqrt{3})).$$

Using this and elementary manipulation in (20.1), we complete the proof.

Ramanujan's statement of Entry 20 does not contain an equality sign.

Ramanujan next offers two puzzling transformations for doubly exponential series. We shall state them exactly as Ramanujan wrote them and then reformulate them.

If $\alpha\beta = 2\pi$, then

$$\alpha \sum_{k=0}^{\infty} e^{-ne^{k\alpha}} = \alpha \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} n^k}{k!\,(e^{k\alpha} - 1)} \right\} - \gamma - \log n + 2 \sum_{k=1}^{\infty} \varphi(k\beta), \quad (21.1)$$

where

$$\varphi(\beta) = \sqrt{\frac{\pi}{\beta \sinh(\pi\beta)}} \cos\left(\beta \log \frac{\beta}{n} - \beta - \frac{\pi}{4} - \frac{B_2}{1 \cdot 2\beta} + \cdots \right). \quad (21.2)$$

Second, if $\alpha\beta = \pi/2$, then

$$\alpha \sum_{k=0}^{\infty} (-1)^k e^{-ne^{(2k+1)\alpha}} = \alpha \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k n^k}{k!\,(e^{k\alpha} + e^{-k\alpha})} \right\} + \sum_{k=0}^{\infty} \psi((2k+1)\beta), \quad (21.3)$$

where

$$\psi(\beta) = \sqrt{\frac{\pi}{\beta \sinh(\pi\beta)}} \sin\left(\beta \log \frac{\beta}{n} - \beta - \frac{\pi}{4} - \frac{B_2}{1 \cdot 2\beta} + \frac{B_4}{3 \cdot 4\beta^3} - \cdots \right). \quad (21.4)$$

In (21.1), γ denotes Euler's constant, and, in (21.2) and (21.4), B_j, $j \geq 0$, denotes the jth Bernoulli number. We emphasize that in Ramanujan's notation, all even indexed Bernoulli numbers are positive.

The definitions of $\varphi(\beta)$ and $\psi(\beta)$ given in (21.2) and (21.4), respectively, are certainly enigmatic. Appearing in the arguments of the trigonometric functions are apparently asymptotic series as β tends to ∞. Thus, the definitions of $\varphi(\beta)$ and $\psi(\beta)$ are imprecise, and so Ramanujan's claims are unclear. Nonetheless, we shall show that (21.1) and (21.3) are correct, if (21.2) and (21.4) are properly interpreted.

The work on Entry 21 which follows was done jointly with J. L. Hafner [2].

We begin by defining functions $G(\beta)$ and $B(\beta)$ by

$$\Gamma(i\beta + 1) = (i\beta)^{i\beta+1/2} e^{-i\beta} \sqrt{2\pi}\, G(\beta) = (i\beta)^{i\beta+1/2} e^{-i\beta} \sqrt{2\pi}\, e^{-iB(\beta)}. \quad (21.5)$$

Then (Whittaker and Watson [1, pp. 252–253], as β tends to ∞,

$$B(\beta) \sim \sum_{k=1}^{\infty} \frac{(-1)^{k-1} B_{2k}}{(2k-1)(2k)\beta^{2k-1}}$$

and

$$G(\beta) \sim 1 + \frac{1}{12 i\beta} - \frac{1}{288\beta^2} - \frac{139}{51840 (i\beta)^3} - \frac{571}{2488320\beta^4} + \cdots. \quad (21.6)$$

Ramanujan less explicitly gives the asymptotic expansion for $B(\beta)$ in the arguments of the trigonometric functions in (21.2) and (21.4).

We now state rigorous formulations of (21.1) and (21.3), and then immediately show that Ramanujan's aforementioned claims are consequences.

Entry 21 (Formulas (4), (5), p. 279). *Let n, α, and β be positive with $\alpha\beta = 2\pi$. Then (21.1) holds, where*

$$\begin{aligned}
\varphi(\beta) &= \frac{1}{\beta}\operatorname{Im}\left\{n^{-i\beta}\Gamma(i\beta+1)\right\} \\
&= \sqrt{\frac{2\pi}{\beta}}e^{-\pi\beta/2}\left\{\sin\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}\right)\operatorname{Re}\{G(\beta)\}\right. \\
&\quad \left.+\cos\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}\right)\operatorname{Im}\{G(\beta)\}\right\} \\
&\sim\sqrt{\frac{2\pi}{\beta}}e^{-\pi\beta/2}\left\{\sin\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}\right)\left\{1-\frac{1}{288\beta^2}+\cdots\right\}\right. \\
&\quad \left.-\cos\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}\right)\left\{\frac{1}{12\beta}+\cdots\right\}\right\},
\end{aligned} \tag{21.7}$$

as β tends to ∞.

Let n, α, and β be positive with $\alpha\beta = \pi/2$. Then (21.3) holds, where

$$\begin{aligned}
\psi(\beta) &= -\frac{1}{\beta}\operatorname{Re}\left\{n^{-i\beta}\Gamma(i\beta+1)\right\} \\
&= -\sqrt{\frac{2\pi}{\beta}}e^{-\pi\beta/2}\left\{\cos\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}\right)\operatorname{Re}\{G(\beta)\}\right. \\
&\quad \left.-\sin\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}\right)\operatorname{Im}\{G(\beta)\}\right\} \\
&\sim-\sqrt{\frac{2\pi}{\beta}}e^{-\pi\beta/2}\left\{\cos\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}\right)\left\{1-\frac{1}{288\beta^2}+\cdots\right\}\right. \\
&\quad \left.+\sin\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}\right)\left\{\frac{1}{12\beta}+\cdots\right\}\right\},
\end{aligned} \tag{21.8}$$

as β tends to ∞.

We first show that Ramanujan's definitions (21.2) and (21.4) are compatible with the far right sides of (21.7) and (21.8), respectively. As β tends to ∞,

$$\begin{aligned}
&\sqrt{\frac{\pi}{\beta\sinh(\pi\beta)}}\cos\left(\beta\log\frac{\beta}{n}-\beta-\frac{\pi}{4}-B(\beta)\right) \\
&\qquad=\sqrt{\frac{2\pi}{\beta}}e^{-\pi\beta/2}(1-e^{-2\pi\beta})^{-1/2}\sin\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}-B(\beta)\right)
\end{aligned}$$

$$\sim\sqrt{\frac{2\pi}{\beta}}e^{-\pi\beta/2}\left\{\sin\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}\right)\cos\ B(\beta)\right.$$
$$\left.-\cos\left(\beta\log\frac{\beta}{n}-\beta+\frac{\pi}{4}\right)\sin\ B(\beta)\right\}.$$

Thus, (21.2) and (21.7) are in agreement. The argument showing that (21.4) and (21.8) agree is similar. We now proceed to prove Entry 21.

Proofs of (21.1) and (21.7). First,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}n^k}{k!\,(e^{k\alpha}-1)}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}n^k}{k!\,e^{k\alpha}}\sum_{j=0}^{\infty}e^{-kj\alpha}$$
$$=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}n^k e^{-kj\alpha}}{k!}=\sum_{j=1}^{\infty}\left(1-e^{-ne^{-j\alpha}}\right).$$

Thus, the proposed identity may be written in the equivalent form

$$\alpha\sum_{k=1}^{\infty}\left(e^{-ne^{k\alpha}}+e^{-ne^{-k\alpha}}-1\right)-\tfrac{1}{2}\alpha+\alpha e^{-n}=-\gamma-\log n+2\sum_{k=1}^{\infty}\varphi(k\beta). \quad (21.9)$$

Second, we apply the Poisson summation formula (Titchmarsh [2, p. 60]) to the function

$$f(x):=e^{-ne^{x}}+e^{-ne^{-x}}-1. \quad (21.10)$$

Observing that $f(0)=2e^{-n}-1$, we find that, for $\alpha,\beta>0$ with $\alpha\beta=2\pi$,

$$\alpha\left(\tfrac{1}{2}(2e^{-n}-1)+\sum_{k=1}^{\infty}\left(e^{-ne^{k\alpha}}+e^{-ne^{-k\alpha}}-1\right)\right)$$
$$=\int_0^{\infty}f(x)\,dx+2\sum_{k=1}^{\infty}\int_0^{\infty}f(x)\cos(k\beta x)\,dx. \quad (21.11)$$

Comparing (21.9) and (21.11), we see that it remains to prove that

$$-\gamma-\log n+2\sum_{k=1}^{\infty}\varphi(k\beta)=\int_0^{\infty}f(x)\,dx+2\sum_{k=1}^{\infty}\int_0^{\infty}f(x)\cos(k\beta x)\,dx. \quad (21.12)$$

Observe from (21.10) that $f(x)$ is even. Setting $u=e^x$, we find that

$$\int_0^{\infty}f(x)dx=\frac{1}{2}\int_{-\infty}^{\infty}f(x)\,dx$$
$$=\frac{1}{2}\int_0^{\infty}(e^{-nu}+e^{-n/u}-1)\frac{du}{u}$$
$$=\frac{1}{2}\left(-\int_0^{1/n}\frac{1-e^{-nu}}{u}du+\int_{1/n}^{\infty}\frac{e^{-nu}}{u}du\right.$$

$$+\int_0^{1/n} \frac{e^{-n/u}}{u}du - \int_{1/n}^{\infty} \frac{1-e^{-n/u}}{u}du\Bigg)$$

$$= \frac{1}{2}\Bigg(-\int_0^1 \frac{1-e^{-x}}{x}dx + \int_1^{\infty} \frac{e^{-x}}{x}dx$$

$$+\int_{n^2}^{\infty} \frac{e^{-x}}{x}dx - \int_0^{n^2} \frac{1-e^{-x}}{x}dx\Bigg).$$

Since (Part I [1, p. 103])

$$\gamma = \int_0^1 \frac{1-e^{-x}}{x}dx - \int_1^{\infty} \frac{e^{-x}}{x}dx,$$

we find that

$$\begin{aligned}\int_0^{\infty} f(x)\,dx &= \frac{1}{2}\left(-\gamma + \int_1^{\infty} \frac{e^{-x}}{x}dx - \int_0^1 \frac{1-e^{-x}}{x}dx - \int_0^{n^2} \frac{dx}{x}\right)\\ &= \tfrac{1}{2}\left(-\gamma - \gamma - \log n^2\right)\\ &= -\gamma - \log n. \end{aligned} \tag{21.13}$$

Using (21.13) in (21.12), we find that it suffices to prove that

$$\sum_{k=1}^{\infty} \varphi(k\beta) = \sum_{k=1}^{\infty} \int_0^{\infty} f(x)\cos(k\beta x)\,dx. \tag{21.14}$$

Set

$$I := I(\beta) := \int_0^{\infty} f(x)\cos(\beta x)\,dx.$$

By (21.14), it now suffices to prove that $I(\beta) = \varphi(\beta)$, where $\varphi(\beta)$ is defined by (21.7). Letting $u = e^x$, we find that

$$I = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\cos(\beta x)\,dx = \frac{1}{2}\int_0^{\infty} \left(e^{-nu} + e^{-n/u} - 1\right)\cos(\beta \log u)\frac{du}{u}.$$

Integrating by parts, we find that

$$\begin{aligned} I &= \frac{n}{2\beta}\int_0^{\infty} \left(e^{-nu} - \frac{1}{u^2}e^{-n/u}\right)\sin(\beta \log u)\,du\\ &= \frac{n}{2\beta}\left(\int_0^{\infty} e^{-nu}\sin(\beta\log u)\,du - \int_0^{\infty} \frac{e^{-n/u}}{u^2}\sin(\beta\log u)\,du\right)\\ &= \frac{n}{2\beta}(I_1 - I_2), \end{aligned}$$

say. Setting $t = 1/u$ in I_2, we deduce that $I_2 = -I_1$. Hence,

$$
\begin{aligned}
I = \frac{n}{\beta} I_1 &= \frac{n}{\beta} \int_0^\infty e^{-nu} \sin(\beta \log u)\, du \\
&= \frac{n}{2\beta i} \int_0^\infty \left(e^{-nu} u^{i\beta} - e^{-nu} u^{-i\beta}\right) du \\
&= \frac{1}{2\beta i} \left(n^{-i\beta}\Gamma(i\beta + 1) - n^{i\beta}\Gamma(-i\beta + 1)\right) \\
&= \frac{1}{\beta} \operatorname{Im}\left\{n^{-i\beta}\Gamma(i\beta + 1)\right\}.
\end{aligned}
$$

Hence, $I = I(\beta) = \varphi(\beta)$, by (21.7). This completes the proof of (21.1).

Lastly, from (21.5),

$$
\begin{aligned}
\varphi(\beta) &= \frac{1}{\beta} \operatorname{Im}\left((i\beta)^{1/2} \left(\frac{i\beta}{ne}\right)^{i\beta} \sqrt{2\pi}\, G(\beta)\right) \\
&= \sqrt{\frac{2\pi}{\beta}} \operatorname{Im}\left(e^{i(\pi/4 + \beta \log(i\beta/(ne)))} G(\beta)\right) \\
&= \sqrt{\frac{2\pi}{\beta}} e^{-\pi\beta/2} \operatorname{Im}\left(e^{i(\pi/4 + \beta \log(\beta/n) - \beta)} G(\beta)\right).
\end{aligned}
$$

Hence, the second equality in (21.7) follows, and the asymptotic formula follows by using (21.6).

Proofs of (21.3) and (21.8). First,

$$
\begin{aligned}
\sum_{k=1}^\infty \frac{(-1)^k n^k}{k!\,(e^{k\alpha} + e^{-k\alpha})} &= \sum_{k=1}^\infty \frac{(-1)^k n^k}{k!\, e^{k\alpha}} \sum_{j=0}^\infty (-1)^j e^{-2kj\alpha} \\
&= \sum_{j=0}^\infty (-1)^j \left(e^{-ne^{-(2j+1)\alpha}} - 1\right).
\end{aligned}
$$

Thus, the proposed identity (21.3) can be recast in the form

$$
\alpha \sum_{k=0}^\infty (-1)^k \left(e^{-ne^{(2k+1)\alpha}} - e^{-ne^{-(2k+1)\alpha}} + 1\right) = \tfrac{1}{2}\alpha + \sum_{k=0}^\infty (-1)^k \psi((2k+1)\beta). \tag{21.15}
$$

Next, we apply the Poisson summation formula for Fourier sine transforms (Titchmarsh [2, p. 66]) to the function

$$
f(x) := e^{-ne^x} - e^{-ne^{-x}} + 1.
$$

Thus, for $\alpha, \beta > 0$ and $\alpha\beta = \pi/2$,

$$\alpha\sum_{k=0}^{\infty}(-1)^k\left(e^{-ne^{(2k+1)\alpha}} - e^{-ne^{-(2k+1)\alpha}} + 1\right)$$
$$= \sum_{k=0}^{\infty}(-1)^k\int_0^{\infty} f(x)\sin((2k+1)\beta x)\,dx. \tag{21.16}$$

We recall that

$$\frac{\alpha}{2} = \frac{\pi}{4\beta} = \frac{1}{\beta}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}. \tag{21.17}$$

Using (21.16) and (21.17) in (21.15), we find that it suffices to prove that

$$\sum_{k=0}^{\infty}(-1)^k\psi((2k+1)\beta)$$
$$= \sum_{k=0}^{\infty}(-1)^k\left(\int_0^{\infty} f(x)\sin((2k+1)\beta x)\,dx - \frac{1}{(2k+1)\beta}\right).$$

Set

$$I := I(\beta) := \int_0^{\infty} f(x)\sin(\beta x)\,dx - \frac{1}{\beta}.$$

By (21.18), we now see that it suffices to prove that $I(\beta) = \psi(\beta)$, where $\psi(\beta)$ is defined by (21.8).

Setting $x = e^u$ and integrating by parts, we find that

$$\begin{aligned}
I &= \int_0^{\infty}\left(e^{-ne^x} - e^{-ne^{-x}} + 1\right)\sin(\beta x)\,dx - \frac{1}{\beta} \\
&= \int_0^{\infty}\left(e^{-nu} - e^{-n/u} + 1\right)\sin(\beta\log u)\frac{du}{u} - \frac{1}{\beta} \\
&= -\frac{n}{\beta}\int_1^{\infty}\left(e^{-nu} + \frac{1}{u^2}e^{-n/u}\right)\cos(\beta\log u)\,du \\
&= -\frac{n}{\beta}\int_0^{1}\left(e^{-nt} + \frac{1}{t^2}e^{-n/t}\right)\cos(\beta\log t)\,dt,
\end{aligned}$$

where we set $u = 1/t$. Hence,

$$I = -\frac{n}{2\beta}\int_0^{\infty}\left(e^{-nt} + \frac{1}{t^2}e^{-n/t}\right)\cos(\beta\log t)\,dt = -\frac{n}{2\beta}(I_1 + I_2),$$

say. Letting $t = 1/u$ in I_2, we easily find that $I_2 = I_1$. Consequently,

$$\begin{aligned}
I = -\frac{n}{\beta}I_1 &= -\frac{n}{\beta}\int_0^{\infty} e^{-nt}\cos(\beta\log t)\,dt \\
&= -\frac{n}{2\beta}\int_0^{\infty}\left(e^{-nt}t^{i\beta} + e^{-nt}t^{-i\beta}\right)dt
\end{aligned}$$

$$= -\frac{1}{2\beta}\left(n^{-i\beta}\Gamma(i\beta+1)+n^{i\beta}\Gamma(-i\beta+1)\right)$$

$$= -\frac{1}{\beta}\text{Re}\left\{n^{-i\beta}\Gamma(i\beta+1)\right\}.$$

Thus, we have shown that $I(\beta) = \psi(\beta)$, by (21.8). This completes the proof of (21.3).

The remaining two claims in (21.8) follow as in the proof of (21.7).

Entry 22 (Formula (2), p. 280). *Let, as usual,* $\psi(x) = \Gamma'(x)/\Gamma(x)$. *Then, if* $0 < x$,

$$\begin{aligned}\psi(x+1) = {}& \frac{\pi}{3}\log x + \frac{1}{2x} - \frac{1}{4\pi x^2} + \frac{\pi\cot(\pi x)}{e^{2\pi x}-1} + \frac{\pi\log|2\sin(\pi x)|}{2\sinh^2(\pi x)} \\ & + 2\sum_{n=1}^{\infty}\frac{n}{(e^{2\pi n}-1)(n^2-x^2)} - \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{\log|n^4-x^4|}{\sinh^2(\pi n)} \\ & - 2\pi\sum_{n=1}^{\infty}e^{-2\pi nx}\left(n^2\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k^2+n^2} - n^3\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k(k^2+n^2)}\right).\end{aligned} \tag{22.1}$$

Using Entry 8 of Chapter 30 (Part IV [4, p. 374]), we obtain a formula for Euler's constant γ, which is equivalent to (22.1), namely,

$$\begin{aligned}-\gamma = {}& \frac{\pi}{3}\log x + \frac{1}{4\pi x^2} + \frac{\pi\log|2\sin(\pi x)|}{2\sinh^2(\pi x)} - \sum_{k=1}^{\infty}\frac{x^2}{k(k^2+x^2)} \\ & + 2\sum_{n=1}^{\infty}\frac{n}{(n^2+x^2)(e^{2\pi n}-1)} - \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{\log|n^4-x^4|}{\sinh^2(\pi n)} \\ & - 2\pi\sum_{n=1}^{\infty}e^{-2\pi nx}\left(n^2\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k^2+n^2} - n^3\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k(k^2+n^2)}\right).\end{aligned} \tag{22.2}$$

In collaboration with J. M. Borwein and W. Galway, the author originally proved (22.2) by showing that the derivatives of both sides of (22.2) equal 0 and then letting x tend to ∞ to show that both sides of (22.2) are equal to $-\gamma$. Shortly thereafter, D. Bradley [1] found a more natural proof by working directly with the double series on the right side of (22.1), and therefore we shall give Bradley's elegant proof. Bradley has shown that Entry 22 has several interesting consequences. In particular, Ramanujan's famous formula for $\zeta(2n+1)$ (Part II [2, pp. 275–276, Entry 21(i)]) follows as a corollary.

Proof of Entry 22. We begin with the partial fraction expansion from Entry 3 of Chapter 30 (Part IV [4, p. 359]),

$$\frac{\pi^2\csc^2(\pi x)}{e^{2\pi x}-1}=\frac{\pi}{3x}-\frac{1}{2x^2}+\frac{1}{2\pi x^3}-\psi'(x+1)$$
$$+\sum_{k=1}^{\infty}\frac{4kx}{(e^{2\pi k}-1)(x^2-k^2)^2}-\sum_{k=1}^{\infty}\frac{2\pi x^3}{\sinh^2(\pi k)(x^4-k^4)}. \tag{22.3}$$

By the product rule for differentiation,

$$\frac{d}{dx}\frac{\pi\cot(\pi x)}{e^{2\pi x}-1}=-\frac{\pi^2\csc^2(\pi x)}{e^{2\pi x}-1}-\frac{\pi^2\cot(\pi x)}{2\sinh^2(\pi x)}$$

and

$$\frac{d}{dx}\frac{\pi\log|2\sin(\pi x)|}{2\sinh^2(\pi x)}=\frac{\pi^2\cot(\pi x)}{2\sinh^2(\pi x)}+\frac{\pi}{2}\log|2\sin(\pi x)|\frac{d}{dx}\operatorname{csch}^2(\pi x).$$

Hence, we can rewrite (22.3) in the form

$$\psi'(x+1)=\frac{\pi}{3x}-\frac{1}{2x^2}+\frac{1}{2\pi x^3}+\frac{d}{dx}\frac{\pi\cot(\pi x)}{e^{2\pi x}-1}+\frac{d}{dx}\frac{\pi\log|2\sin(\pi x)|}{2\sinh^2(\pi x)}$$
$$+\sum_{k=1}^{\infty}\frac{4kx}{(e^{2\pi k}-1)(x^2-k^2)^2}-\sum_{k=1}^{\infty}\frac{2\pi x^3}{\sinh^2(\pi k)(x^4-k^4)}$$
$$-\frac{\pi}{2}\log|2\sin(\pi x)|\frac{d}{dx}\operatorname{csch}^2(\pi x). \tag{22.4}$$

But,

$$-\frac{\pi}{2}\frac{d}{dx}\operatorname{csch}^2(\pi x)=-\frac{d}{dx}\frac{2\pi e^{2\pi x}}{\left(e^{2\pi x}-1\right)^2}=\left(\frac{d}{dx}\right)^2\frac{1}{e^{2\pi x}-1}$$
$$=\left(\frac{d}{dx}\right)^2\sum_{k=1}^{\infty}e^{-2\pi kx}=\sum_{k=1}^{\infty}(-2\pi k)^2e^{-2\pi kx}.$$

Using this in (22.4) and integrating both sides with respect to x, we deduce that

$$\psi(x+1)=\frac{\pi}{3}\log x+\frac{1}{2x}-\frac{1}{4\pi x^2}+\frac{\pi\cot(\pi x)}{e^{2\pi x}-1}+\frac{\pi\log|2\sin(\pi x)|}{2\sinh^2(\pi x)}$$
$$+\sum_{k=1}^{\infty}\frac{2k}{(e^{2\pi k}-1)(k^2-x^2)}-\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{\log|k^4-x^4|}{\sinh^2(\pi k)}-S(x)+C, \tag{22.5}$$

where C is a constant of integration to be determined and

$$S(x):=\int_x^{\infty}\log|2\sin(\pi v)|\sum_{k=1}^{\infty}(-2\pi k)^2e^{-2\pi kv}\,dv$$
$$=\int_0^{\infty}\log|2\sin(\pi(x+u))|\sum_{k=1}^{\infty}(-2\pi k)^2e^{-2\pi k(x+u)}\,du. \tag{22.6}$$

Using the Laplace transforms

$$\int_0^\infty e^{-kt}\sin(nt)\,dt = \frac{n}{k^2+n^2} \qquad \text{and} \qquad \int_0^\infty e^{-kt}\cos(nt)\,dt = \frac{k}{k^2+n^2},$$

and the well–known Fourier series (Gradshteyn and Ryzhik [1, p. 46, formula 1.441, no. 2])

$$\sum_{k=1}^\infty \frac{\cos(2\pi ky)}{k} = -\log|2\sin(\pi y)|, \qquad y \text{ real},$$

in (22.6), we find that

$$\begin{aligned}
S(x) &= (2\pi)^2\sum_{k=1}^\infty k^2 e^{-2\pi kx}\int_0^\infty e^{-2\pi ku}\log|2\sin(\pi(x+u))|\,du,\\
&= -(2\pi)^2\sum_{k=1}^\infty k^2 e^{-2\pi kx}\int_0^\infty e^{-2\pi ku}\sum_{n=1}^\infty \frac{\cos(2\pi nx+2\pi nu)}{n}\,du\\
&= -2\pi\sum_{k=1}^\infty k^2 e^{-2\pi kx}\int_0^\infty e^{-kt}\sum_{n=1}^\infty \frac{\cos(2\pi nx+nt)}{n}\,dt\\
&= 2\pi\sum_{k=1}^\infty k^2 e^{-2\pi kx}\int_0^\infty e^{-kt}\sum_{n=1}^\infty \frac{\sin(2\pi nx)\sin(nt)-\cos(2\pi nx)\cos(nt)}{n}\,dt\\
&= 2\pi\sum_{k=1}^\infty k^2 e^{-2\pi kx}\left(\sum_{n=1}^\infty \frac{\sin(2\pi nx)}{n}\int_0^\infty e^{-kt}\sin(nt)\,dt\right.\\
&\qquad\qquad\left.-\sum_{n=1}^\infty \frac{\cos(2\pi nx)}{n}\int_0^\infty e^{-kt}\cos(nt)\,dt\right)\\
&= 2\pi\sum_{k=1}^\infty e^{-2\pi nx}\left(k^2\sum_{n=1}^\infty \frac{\sin(2\pi nx)}{k^2+n^2} - k^3\sum_{n=1}^\infty \frac{\cos(2\pi nx)}{n(k^2+n^2)}\right),
\end{aligned} \tag{22.7}$$

where the inversions in the order of summation and integration are justified by absolute convergence. Substituting (22.7) in (22.5), we find that

$$\begin{aligned}
\psi(x+1) &= C + \frac{\pi}{3}\log x + \frac{1}{2x} - \frac{1}{4\pi x^2} + \frac{\pi\cot(\pi x)}{e^{2\pi x}-1} + \frac{\pi\log|2\sin(\pi x)|}{2\sinh^2(\pi x)}\\
&\quad + \sum_{k=1}^\infty \frac{2k}{(e^{2\pi k}-1)(k^2-x^2)} - \frac{\pi}{2}\sum_{k=1}^\infty \frac{\log|k^4-x^4|}{\sinh^2(\pi k)}\\
&\quad - 2\pi\sum_{k=1}^\infty e^{-2\pi nx}\left(k^2\sum_{n=1}^\infty \frac{\sin(2\pi nx)}{k^2+n^2} - k^3\sum_{n=1}^\infty \frac{\cos(2\pi nx)}{n(k^2+n^2)}\right).
\end{aligned} \tag{22.8}$$

By letting $x \to +\infty$ in (22.8), we can evaluate the integration constant, C. In fact, C must equal zero, as a comparison of (22.8) and (22.1) reveals. Since several terms in (22.8) vanish as $x \to +\infty$, it suffices to show that

$$\lim_{x\to\infty}\left(\psi(x+1)-\frac{\pi}{3}\log x+\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{\log|x^4-k^4|}{\sinh^2(\pi k)}\right)=0. \tag{22.9}$$

Set $x = N + \frac{1}{2}$, where N is a positive integer, and write

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\log|x^4-k^4|}{\sinh^2(\pi k)} &= \left(\sum_{k\le\sqrt{x}}+\sum_{\sqrt{x}<k<x}+\sum_{k>x}\right)\frac{\log|x^4-k^4|}{\sinh^2(\pi k)}\\
&= \sum_{k\le\sqrt{x}}\frac{4\log x}{\sinh^2(\pi k)}+\sum_{k\le\sqrt{x}}\frac{\log(1-k^4/x^4)}{\sinh^2(\pi k)}\\
&\quad+(4x\log x)\,O\left(e^{-2\pi\sqrt{x}}\right)+O\left(e^{-3\pi x/2}\right)\\
&= (4\log x)\left(\sum_{k=1}^{\infty}\frac{1}{\sinh^2(\pi k)}+O\left(e^{-\pi\sqrt{x}}\right)\right)\\
&\quad+O(x^{-2})+O\left(e^{-\pi\sqrt{x}}\right)\\
&= (4\log x)\sum_{k=1}^{\infty}\frac{1}{\sinh^2(\pi k)}+O(x^{-2}).\end{aligned} \tag{22.10}$$

Now, by Stirling's formula, as $x \to \infty$,

$$\psi(x+1)=\log x+O(1/x). \tag{22.11}$$

Employing (22.10) and (22.11) in (22.8), we see that

$$\begin{aligned}&\psi(x+1)-\frac{\pi}{3}\log x+\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{\log|x^4-k^4|}{\sinh^2(\pi k)}\\
&=\left(1-\frac{\pi}{3}+2\pi\sum_{k=1}^{\infty}\frac{1}{\sinh^2(\pi k)}\right)\log x+O(1/x),\end{aligned} \tag{22.12}$$

as x tends to $+\infty$. However (Berndt [2, Prop. 2.26]),

$$1-\frac{\pi}{3}+2\pi\sum_{k=1}^{\infty}\frac{1}{\sinh^2(\pi k)}=1-\frac{\pi}{3}+2\pi\left(\frac{1}{6}-\frac{1}{2\pi}\right)=0. \tag{22.13}$$

Using (22.13) in (22.12), we find that

$$\psi(x+1)-\frac{\pi}{3}\log x+\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{\log|x^4-k^4|}{\sinh^2(\pi k)}=O(1/x),$$

as x tends to $+\infty$. Thus, the limit (22.9) holds, and the proof of Entry 22 is complete.

Entry 23 (Formula (4), p. 281). *For each positive integer n, let*

$$S_{2n} := S_{2n}(\tau) := -\frac{B_{2n}}{4n} + \sum_{k=1}^{\infty} \frac{k^{2n-1}q^k}{1-q^k}, \qquad q := e^{2\pi i\tau},$$

where $\operatorname{Im}\tau > 0$. *Then, if m is a positive integer,*

$$\frac{(4m+5)(4m-2)}{24(4m+2)(4m+1)} S_{4m+4}(i) = \sum_{k=1}^{m} \binom{4m}{4k-2} S_{4k}(i) S_{4m-4k+4}(i).$$

Proof. In [7], [10, p. 140, eq. (22)], Ramanujan proved that, for $\operatorname{Im}\tau > 0$ and any integer n greater than 1,

$$\frac{(2n+5)(2n-2)}{24(2n+2)(2n+1)} S_{2n+4}(\tau) = \sum_{k=1}^{n-1} \binom{2n}{2k} S_{2+2k}(\tau) S_{2n-2k+2}(\tau). \qquad (23.1)$$

(We remark that the notation S_n has a meaning in Ramanujan's paper [7] different from that in the notebooks.) By a theorem of A. O. L. Atkin [1], $E_{4n+2}(i) = 0$, where $E_{2n}(\tau)$ is the Eisenstein series defined by (Rankin [1, p. 194])

$$E_{2n}(\tau) := -\frac{4n}{B_{2n}} S_{2n}(\tau).$$

Thus, when $\tau = i$, the desired result follows.

It is interesting that Ramanujan undoubtedly discovered the theorem, $S_{4n+2}(i) = 0$, more than 50 years before a proof was published.

The next three entries are found in Ramanujan's paper [6], [10, pp. 47–49]. Since the details of the proofs were not completely given by Ramanujan, we do so below. The constants c_1, c_2, and c_3 below were not explicitly given by Ramanujan in the notebooks.

Entry 24 (Formula (5), p. 281). *For each positive integer n,*

$$\sum_{k=1}^{n} \sqrt{k} = -\frac{1}{4\pi}\zeta(3/2) + \tfrac{2}{3}n^{3/2} + \tfrac{1}{2}n^{1/2} + \tfrac{1}{6}\sum_{k=0}^{\infty}\left\{\sqrt{n+k} + \sqrt{n+k+1}\right\}^{-3}.$$

Proof. Let

$$\varphi_1(n) := \sum_{k=1}^{n} \sqrt{k} - c_1 - \tfrac{2}{3}n^{3/2} - \tfrac{1}{2}n^{1/2} - \tfrac{1}{6}\sum_{k=0}^{\infty}\left\{\sqrt{n+k} + \sqrt{n+k+1}\right\}^{-3},$$

where c_1 is a constant such that $\varphi_1(1) = 0$. Thus,

$$\begin{aligned}\varphi_1(n) - \varphi_1(n+1) &= -(n+1)^{1/2} - \tfrac{2}{3}n^{3/2} - \tfrac{1}{2}n^{1/2} + \tfrac{2}{3}(n+1)^{3/2} + \tfrac{1}{2}(n+1)^{1/2} \\ &\quad - \tfrac{1}{6}\left\{\sqrt{n} + \sqrt{n+1}\right\}^{-3} \\ &= -\tfrac{1}{2}(n+1)^{1/2} - \tfrac{2}{3}n^{3/2} - \tfrac{1}{2}n^{1/2} + \tfrac{2}{3}(n+1)^{3/2}\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1}{6}\left\{\sqrt{n}-\sqrt{n+1}\right\}^3 \\
&= -\tfrac{1}{2}(n+1)^{1/2} - \tfrac{1}{2}n^{1/2} - \tfrac{1}{2}n^{3/2} + \tfrac{1}{2}(n+1)^{3/2} \\
&\quad -\tfrac{1}{2}n(n+1)^{1/2} + \tfrac{1}{2}n^{1/2}(n+1) \\
&= 0.
\end{aligned}
$$

Since $\varphi_1(1) = 0$, by induction, $\varphi_1(n) = 0$ for every positive integer n. From Part I [1, p. 156], $c_1 = -\zeta(3/2)/(4\pi)$, and so the proof is complete.

Entry 25 (Formula (6), p. 281). *For each positive integer n,*

$$
\begin{aligned}
\sum_{k=1}^{n} k^{3/2} = &-\frac{3}{16\pi^2}\zeta(5/2) + \tfrac{2}{5}n^{5/2} + \tfrac{1}{2}n^{3/2} + \tfrac{1}{8}n^{1/2} \\
&+ \frac{1}{40}\sum_{k=0}^{\infty}\left\{\sqrt{n+k}+\sqrt{n+k+1}\right\}^{-5}.
\end{aligned}
$$

Proof. Let

$$
\begin{aligned}
\varphi_2(n) = &\sum_{k=1}^{n} k^{3/2} - c_2 - \tfrac{2}{5}n^{5/2} - \tfrac{1}{2}n^{3/2} - \tfrac{1}{8}n^{1/2} \\
&- \frac{1}{40}\sum_{k=0}^{\infty}\left\{\sqrt{n+k}+\sqrt{n+k+1}\right\}^{-5},
\end{aligned}
$$

where c_2 is a constant such that $\varphi_2(1) = 0$. By the same sort of calculation as in the previous proof, $\varphi_2(n) - \varphi_2(n+1) = 0$. Thus, by induction, $\varphi_2(n) = 0$ for every positive integer n. From Part I [1, p. 156], $c_2 = -3\zeta(5/2)/(16\pi^2)$, and so the proof is complete.

Entry 26 (Formula (7), p. 281). *For each positive integer n,*

$$
\begin{aligned}
\sum_{k=1}^{n} k^{5/2} = &\frac{15}{64\pi^3}\zeta(7/2) - \frac{1}{64\pi}\zeta(3/2) + \tfrac{2}{7}n^{7/2} + \tfrac{1}{2}n^{5/2} + \tfrac{1}{4}n^{3/2} \\
&+ \tfrac{1}{32}n^{1/2} - \frac{1}{16}\sum_{k=1}^{n}\sqrt{k} + \frac{1}{224}\sum_{k=0}^{\infty}\left\{\sqrt{n+k}+\sqrt{n+k+1}\right\}^{-7}.
\end{aligned}
$$

Proof. Set

$$
\begin{aligned}
\varphi_3(n) = &\sum_{k=1}^{n} k^{5/2} - c_3 - \sqrt{n}\left(\tfrac{2}{7}n^3 + \tfrac{1}{2}n^2 + \tfrac{5}{24}n\right) \\
&+ \frac{1}{96}\sum_{k=0}^{\infty}\left\{\sqrt{n+k}+\sqrt{n+k+1}\right\}^{-3} \\
&- \frac{1}{224}\sum_{k=0}^{\infty}\left\{\sqrt{n+k}+\sqrt{n+k+1}\right\}^{-7},
\end{aligned}
$$

where c_3 is a constant chosen so that $\varphi_3(1) = 0$. A somewhat laborious calculation shows that $\varphi_3(n) - \varphi_3(n+1) = 0$ for each positive integer n. Thus, by induction, $\varphi_3(n) = 0$ for every positive integer n. Hence,

$$\sum_{k=1}^{n} k^{5/2} = c_3 + \sqrt{n}\left(\tfrac{2}{7}n^3 + \tfrac{1}{2}n^2 + \tfrac{5}{24}n\right)$$
$$-\frac{1}{96}\sum_{k=0}^{\infty}\left\{\sqrt{n+k} + \sqrt{n+k+1}\right\}^{-3} + \frac{1}{224}\sum_{k=0}^{\infty}\left\{\sqrt{n+k} + \sqrt{n+k+1}\right\}^{-7},$$

which is the formulation given by Ramanujan in [6]. Using Entry 24 above, we deduce that

$$\sum_{k=1}^{n} k^{5/2} = c_3 - \frac{1}{64\pi}\zeta(3/2) + \sqrt{n}\left(\tfrac{2}{7}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{4}n + \tfrac{1}{32}\right)$$
$$-\frac{1}{16}\sum_{k=1}^{n}\sqrt{k} + \frac{1}{224}\sum_{k=0}^{\infty}\left\{\sqrt{n+k} + \sqrt{n+k+1}\right\}^{-7}.$$

By Entry 1 of Chapter 7 (Part I [1, p. 150]), $c_3 = \zeta(-5/2)$. By the functional equation (0.4) of $\zeta(z)$, $c_3 = \zeta(-5/2) = 15\zeta(7/2)/(64\pi^3)$. Using this in the equality above, we complete the proof.

Entry 27 (Formula (3), p. 282). *Let*

$$\varphi(x) = \int_0^x \frac{\tan^{-1} t}{t}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)^2}, \tag{27.1}$$

where, in the latter representation, $|x| < 1$. Then, for every real number x,

$$\frac{4}{\pi}\left\{\varphi(1) - \varphi(e^{-\pi x/2})\right\} - 2x\tan^{-1}(e^{-\pi x/2})$$
$$= \sum_{n=0}^{\infty}(-1)^n(2n+1)\log\left(1 + \frac{x^2}{(2n+1)^2}\right). \tag{27.2}$$

An equivalent formulation of Entry 27 is given by Ramanujan in his paper [5, eq. (12)], [10, p. 41], where he writes that (27.2) is "very easily proved by differentiating both sides with respect to x." It seems unlikely that Ramanujan discovered (27.2) by this means, but we do not have a better proof and proceed accordingly.

Proof. Let $f(x)$ and $g(x)$ denote, respectively, the left and right sides of (27.2). Then, by straightforward differentiation,

$$f'(x) = \frac{\pi x e^{-\pi x/2}}{1 + e^{-\pi x}} = \frac{\pi x}{2\cosh(\frac{1}{2}\pi x)}$$

and

$$g'(x) = 2x \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{(2n+1)^2+x^2}.$$

The differentiation under the summation sign is justified because the differentiated series converges uniformly for all real x. But (Whittaker and Watson [1, p. 136])

$$\frac{\pi}{4\cosh(\frac{1}{2}\pi x)} = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{(2n+1)^2+x^2}. \tag{27.3}$$

Hence, $f'(x) \equiv g'(x)$. Since $f(0) = 0 = g(0)$, the proof of Entry 27 is complete.

Entry 28 (Formula (4), p. 282). *We have*

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)\log\left(1+\left(\frac{2}{(2n+1)\pi}\log\left(2+\sqrt{3}\right)\right)^2\right) = \frac{4}{3\pi}C, \tag{28.1}$$

where

$$C = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2},$$

which is Catalan's constant.

Proof. Set $x = (2/\pi)\log(2+\sqrt{3})$ in Entry 27. Then if S denotes the left side of (28.1), we find that

$$S = \frac{4}{\pi}\left\{\varphi(1) - \varphi\left(\frac{1}{2+\sqrt{3}}\right)\right\} - \frac{4}{\pi}\log(2+\sqrt{3})\tan^{-1}\left(\frac{1}{2+\sqrt{3}}\right). \tag{28.2}$$

Setting $u = \pi/12$ in the double angle formula $\cos(2u) = 2\cos^2 u - 1$, we find that $\cos(\pi/12) = \frac{1}{2}\sqrt{2+\sqrt{3}}$. Thus, $\sin(\pi/12) = \frac{1}{2}\sqrt{2-\sqrt{3}}$ and $\tan(\pi/12) = 2-\sqrt{3}$. Hence, (28.2) becomes

$$S = \frac{4}{\pi}\left\{\varphi(1) - \varphi\left(2-\sqrt{3}\right)\right\} - \tfrac{1}{3}\log(2+\sqrt{3}). \tag{28.3}$$

Now in [5, eq. (7)], [10, p. 40], Ramanujan briefly sketched a proof of the equality

$$2\varphi(1) = 3\varphi(2-\sqrt{3}) + \tfrac{1}{4}\pi\log(2+\sqrt{3}). \tag{28.4}$$

Using (28.4) in (28.3), we deduce that

$$S = \frac{4}{3\pi}\varphi(1).$$

Since $\varphi(1) = C$, the proof of (28.1) is complete.

Lastly, we provide details for the proof of (28.4). Ramanujan claims that (28.4) follows from the equality

$$\sum_{n=0}^{\infty} \frac{\sin(4n+2)x}{(2n+1)^2} = \varphi(\tan x) - x\log(\tan x), \qquad 0 < x < \pi/2, \tag{28.5}$$

by setting $x = \pi/12$. Accordingly, we find that

$$\begin{aligned}
&\varphi(2-\sqrt{3}) + \frac{\pi}{12}\log(2+\sqrt{3}) \\
&= \frac{1}{2\cdot 1^2} + \frac{1}{3^2} + \frac{1}{2\cdot 5^2} - \frac{1}{2\cdot 7^2} - \frac{1}{9^2} - \frac{1}{2\cdot 11^2} + \cdots \\
&= \frac{1}{2}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \cdots\right) + \frac{3}{2}\left(\frac{1}{3^2} - \frac{1}{9^2} + \cdots\right) \\
&= \tfrac{1}{2}\varphi(1) + \tfrac{1}{6}\varphi(1) = \tfrac{2}{3}\varphi(1).
\end{aligned}$$

Thus, (28.4) has been proved, and it remains to prove (28.5).

To prove (28.5), let $f(x)$ and $g(x)$ denote the left and right sides, respectively, of (28.5). Then

$$f'(x) = 2\sum_{n=0}^{\infty} \frac{\cos(4n+2)x}{2n+1},$$

since the differentiated series converges uniformly on any compact subinterval of $(0, \pi/2)$. Also,

$$g'(x) = -\log(\tan x) = 2\sum_{n=0}^{\infty} \frac{\cos(4n+2)x}{2n+1}, \qquad 0 < x < \pi/2,$$

by a familiar Fourier series development (Gradshteyn and Ryzhik [1, p. 46, formula 1.442, no. 2]). Thus, $f'(x) = g'(x)$, when $0 < x < \pi/2$, and so $f(x) = g(x) + c$, where c is some constant. Letting $x = \pi/4$, we find that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \varphi(1) + c.$$

By (27.1), $c = 0$. Hence, (28.5) has been proved.

Entry 29 (Formula (10), p. 286). *If* $-1 < x < 1$, *then*

$$\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^n(2n+1)\log\left(1 - \frac{x^2}{(2n+1)^2}\right) \\
&= \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}\left(1 - \cos\left\{\tfrac{1}{2}(2n+1)\pi x\right\}\right) + x\log\left(\tan\left(\frac{\pi - \pi x}{4}\right)\right) \qquad (29.1) \\
&= \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}\left(1 - \tan^{2n+1}\left(\frac{\pi - \pi x}{4}\right)\right) + \log\left(\tan\left(\frac{\pi - \pi x}{4}\right)\right). \qquad (29.2)
\end{aligned}$$

Proof. By analytic continuation, Entry 27 is valid for all complex x with $|x| < 1$. Replacing x by ix in Entry 27, we find that

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)\log\left(1-\frac{x^2}{(2n+1)^2}\right)$$
$$=\frac{4}{\pi}\left\{\varphi(1)-\varphi(e^{-\pi ix/2})\right\}-2ix\tan^{-1}(e^{-\pi ix/2}).$$

Taking the real part of each side, we deduce that

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)\log\left(1-\frac{x^2}{(2n+1)^2}\right)$$
$$=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\left(1-\cos\left\{\tfrac{1}{2}(2n+1)\pi x\right\}\right)-2\operatorname{Re}\left(ix\tan^{-1}\left(e^{-\pi ix/2}\right)\right). \tag{29.3}$$

Now

$$\begin{aligned}-2\operatorname{Re}\left(ix\tan^{-1}\left(e^{-\pi ix/2}\right)\right)&=x\operatorname{Re}\log\left(\frac{i+e^{-\pi ix/2}}{i-e^{-\pi ix/2}}\right)\\&=x\operatorname{Re}\log\left(\frac{-i\cos(\frac{1}{2}\pi x)}{1+\sin(\frac{1}{2}\pi x)}\right)\\&=x\log\left(\frac{\cos(\frac{1}{2}\pi x)}{1+\sin(\frac{1}{2}\pi x)}\right)\\&=x\log\left(\tan\left(\frac{\pi-\pi x}{4}\right)\right).\end{aligned} \tag{29.4}$$

Using (29.4) in (29.3), we complete the proof of (29.1).

To prove (29.2), which is eq. (12) in Ramanujan's paper [5], [10, p. 41], we first note that the right side of (29.2) equals

$$g(x):=\frac{4}{\pi}\left\{\varphi(1)-\varphi\left(\tan\left(\frac{\pi-\pi x}{4}\right)\right)\right\}+\log\left(\tan\left(\frac{\pi-\pi x}{4}\right)\right).$$

With $f(x)$ denoting the left side of (29.2), we see that $f(0)=0=g(0)$. Thus, it suffices to prove that $f'(x)=g'(x)$. Now, by a straightforward calculation with the use of (27.1) and (27.3),

$$\begin{aligned}g'(x)&=-\frac{\pi x\sec^2\left(\dfrac{\pi-\pi x}{4}\right)}{4\tan\left(\dfrac{\pi-\pi x}{4}\right)}=-\frac{\pi x}{2\sin\left\{\frac{1}{2}\pi(1-x)\right\}}\\&=-\frac{\pi x}{2\cos(\frac{1}{2}\pi x)}=-2x\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(2n+1)^2-x^2}=f'(x),\end{aligned}$$

and the proof of (29.2) is complete.

Entry 30 (Formula (11), p. 286). *Let* $\alpha > 0$ *and* $0 < \beta < 1$, *with*

$$\log\left(\tan\left(\tfrac{1}{4}\pi(1+\beta)\right)\right) = \tfrac{1}{2}\pi\alpha.$$

Then

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)\log\left(1+\frac{\alpha^2}{(2n+1)^2}\right)$$
$$= \frac{\pi\alpha\beta}{2} + \sum_{n=0}^{\infty}(-1)^n(2n+1)\log\left(1-\frac{\beta^2}{(2n+1)^2}\right). \tag{30.1}$$

Proof. Observe that both sides of (29.1) are even functions of x, so that we can replace x by $-x$. Thus, by Entries 27 and 29,

$$\begin{aligned} S :&= \sum_{n=0}^{\infty}(-1)^n(2n+1)\log\left(1+\frac{\alpha^2}{(2n+1)^2}\right) \\ &\quad - \sum_{n=0}^{\infty}(-1)^n(2n+1)\log\left(1-\frac{\beta^2}{(2n+1)^2}\right) \\ &= \frac{4}{\pi}\varphi\left(\tan\left(\frac{1}{4}\pi(1+\beta)\right)\right) - \frac{4}{\pi}\varphi\left(e^{-\pi\alpha/2}\right) - 2\alpha\tan^{-1}\left(e^{-\pi\alpha/2}\right) \\ &\quad - \log\left(\tan\left(\frac{1}{4}\pi(1+\beta)\right)\right) \\ &= \frac{4}{\pi}\left\{\varphi(e^{\pi\alpha/2}) - \varphi(e^{-\pi\alpha/2})\right\} - 2\alpha\tan^{-1}\left(\cot\left(\frac{1}{4}\pi(1+\beta)\right)\right) - \log(e^{\pi\alpha/2}). \end{aligned}$$

Now from Ramanujan's paper [5, eq. (4)], [10, p. 40],

$$\varphi(x) - \varphi(-1/x) = \tfrac{1}{2}\pi\log x. \tag{30.2}$$

Hence,

$$S = \pi\alpha - 2\alpha\left(\frac{\pi}{2} - \left(\frac{\pi}{4}(1+\beta)\right)\right) - \frac{\pi\alpha}{2} = \frac{\pi\alpha\beta}{2}.$$

This completes the proof of (30.1).

Entry 30 is equivalent to eq. (17) of [5], [10, p. 41], which Ramanujan stated without proof. He also [4], [10, p. 329] submitted this formula as a problem to the *Journal of the Indian Mathematical Society*. The proof of (30.2) was only briefly sketched by Ramanujan in [5]. The editors of his *Collected Papers* [10, pp. 336–337] supplied a proof in more detail.

Entry 31 (Formula (1), p. 288). *For* $x > 0$, *let*

$$F(x) = x^2\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^3\operatorname{sech}\left\{\frac{1}{2}\pi(2n+1)\right\}}{\cosh\{(2n+1)x\}+\cos\{(2n+1)x\}}.$$

Then, if $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2/2$,

$$F(\alpha) = F(\beta). \tag{31.1}$$

This is a particularly beautiful result. Ramanujan evidently did not possess a complete proof, for he recorded the result (in abbreviated notation) as "The difference between the two series ($\alpha\beta = \pi^2/2$) $F(\alpha)$ and $F(\beta)$ is 0?" As we shall see, upon examining our proof below, $(2n+1)^3$ can be replaced by $(2n+1)^{4m+3}$, for any nonnegative integer m.

Proof. Let

$$f(z) := \frac{z^3}{(\cosh(\alpha z) + \cos(\alpha z))\cos(\frac{1}{2}\pi z)\cosh(\frac{1}{2}\pi z)}.$$

Observe that $f(z)$ has simple poles at $z = 2n+1$, $(2n+1)i$, $(2n+1)\pi(1\pm i)/(2\alpha)$, for each integer n.

Let $\{C_N\}$, $N \geq 1$, denote a sequence of rectangles having vertical and horizontal sides and centers at the origin. We shall choose the rectangles so that, as N tends to ∞, the sides tend to ∞ but remain at a bounded distance away from the poles of $f(z)$. It is then easy to see that

$$\lim_{N\to\infty} \int_{C_N} f(z)\, dz = 0. \tag{31.2}$$

We apply the residue theorem. By straightforward calculations, for each integer n,

$$R_{2n+1} = -\frac{2(-1)^n(2n+1)^3 \operatorname{sech}\left\{\frac{1}{2}\pi(2n+1)\right\}}{\pi\left(\cosh\{(2n+1)\alpha\} + \cos\{(2n+1)\alpha\}\right)} = R_{-(2n+1)}, \tag{31.3}$$

$$R_{(2n+1)i} = -\frac{2(-1)^n(2n+1)^3 \operatorname{sech}\left\{\frac{1}{2}\pi(2n+1)\right\}}{\pi\left(\cosh\{(2n+1)\alpha\} + \cos\{(2n+1)\alpha\}\right)} = R_{-(2n+1)i}, \tag{31.4}$$

and

$$\begin{aligned} &R_{(2n+1)\pi(1+i)/(2\alpha)} \\ &= \frac{((2n+1)\pi(1+i))^3 \sec\left\{(2n+1)\pi^2(1+i)/(4\alpha)\right\} \operatorname{sech}\left\{(2n+1)\pi^2(1+i)/(4\alpha)\right\}}{8\alpha^4\left(\sinh\left\{\frac{1}{2}(2n+1)\pi(1+i)\right\} - \sin\left\{\frac{1}{2}(2n+1)\pi(1+i)\right\}\right)} \\ &= \frac{(-1)^n\beta^2 i(1+i)(2n+1)^3 \sec\left\{\frac{1}{2}(2n+1)\beta(1+i)\right\} \operatorname{sech}\left\{\frac{1}{2}(2n+1)\beta(1+i)\right\}}{\pi\alpha^2(i-1)\cosh\left\{\frac{1}{2}\pi(2n+1)\right\}} \\ &= \frac{2(-1)^n\beta^2(2n+1)^3}{\pi\alpha^2\cosh\left\{\frac{1}{2}\pi(2n+1)\right\}\left(\cosh\{(2n+1)\beta\} + \cos\{(2n+1)\beta\}\right)}. \end{aligned} \tag{31.5}$$

Furthermore, we see that

$$\begin{aligned} R_{(2n+1)\pi(1+i)/(2\alpha)} &= R_{-(2n+1)\pi(1+i)/(2\alpha)} \\ &= R_{(2n+1)\pi(1-i)/(2\alpha)} = R_{-(2n+1)\pi(1-i)/(2\alpha)}. \end{aligned} \tag{31.6}$$

Hence, applying the residue theorem to the integral of $f(z)$ over C_N, employing the calculations (31.3)–(31.6), letting N tend to ∞, and using (31.2), we deduce that

$$-\frac{8}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^3\operatorname{sech}\left\{\frac{1}{2}\pi(2n+1)\right\}}{\cosh\{(2n+1)\alpha\}+\cos\{(2n+1)\alpha\}}$$
$$+\frac{8\beta^2}{\pi\alpha^2}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^3\operatorname{sech}\left\{\frac{1}{2}\pi(2n+1)\right\}}{\cosh\{(2n+1)\beta\}+\cos\{(2n+1)\beta\}}=0.$$

Multiplying both sides by $\pi\alpha^2/8$, we arrive at (31.1) to complete the proof.

If we divide both sides of (31.1) by β^2 and let α tend to ∞, and therefore let β tend to 0, and replace $(2n+1)^3$ by $(2n+1)^{4m+3}$, as we may, for any nonnegative integer m, we deduce that

$$\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^{4m+3}}{\cosh\left\{\frac{1}{2}\pi(2n+1)\right\}}=0.$$

This result is due to Cauchy [1, pp. 313, 362] and is a special case of another theorem of Ramanujan (Part II [2, p. 262]).

Entry 32 (Formula (3), p. 288). *If* $\alpha, \beta > 0$ *with* $\alpha\beta = \pi^2/4$, *then*

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\left(\cosh\{(2n+1)\alpha\}+\cos\{(2n+1)\alpha\}\right)}=\frac{\pi}{8}$$
$$-2\sum_{n=0}^{\infty}\frac{(-1)^n\cosh\{(2n+1)\beta\}\cos\{(2n+1)\beta\}}{(2n+1)\cosh\left\{\frac{1}{2}\pi(2n+1)\right\}\left(\cosh\{(4n+2)\beta\}+\cos\{(4n+2)\beta\}\right)}. \tag{32.1}$$

Proof. Let

$$f(z):=\frac{1}{z\left(\cosh(\alpha z)+\cos(\alpha z)\right)\cos(\frac{1}{2}\pi z)}.$$

Note that $f(z)$ has simple poles at $z=0, 2n+1$, and $(2n+1)\pi i(1\pm i)/(2\alpha)$, for each integer n. Let $\{C_N\}$ denote the same sequence of rectangles as described in the proof of Entry 31.

We now calculate the residues of $f(z)$. First,

$$R_0=\tfrac{1}{2}. \tag{32.2}$$

Second, by an easy calculation,

$$R_{2n+1}=-\frac{2(-1)^n}{\pi(2n+1)\left(\cosh\{(2n+1)\alpha\}+\cos\{(2n+1)\alpha\}\right)}=R_{-(2n+1)}. \tag{32.3}$$

Third, by straightforward calculations,

$$R_{(2n+1)\pi i(1\pm i)/(2\alpha)} = -\frac{(-1)^n}{\pi(2n+1)\cos\{(2n+1)(\mp 1+i)\beta\}\cosh\left\{\frac{1}{2}\pi(2n+1)\right\}}$$
$$= R_{-(2n+1)\pi i(1\pm i)/(2\alpha)}. \tag{32.4}$$

By (32.4) and an elementary calculation,

$$R_{(2n+1)\pi i(1+i)/(2\alpha)} + R_{(2n+1)\pi i(1-i)/(2\alpha)}$$
$$= -\frac{4(-1)^n\cosh\{(2n+1)\beta\}\cos\{(2n+1)\beta\}}{\pi(2n+1)\cosh\left\{\frac{1}{2}\pi(2n+1)\right\}(\cosh\{(4n+2)\beta\}+\cos\{(4n+2)\beta\})}. \tag{32.5}$$

We now apply the residue theorem to the integral of $f(z)$ over C_N. It is easy to show that

$$\lim_{N\to\infty}\int_{C_N} f(z)\,dz = 0. \tag{32.6}$$

Hence, from (32.2)–(32.6),

$$0 = \frac{1}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(\cosh\{(2n+1)\alpha\}+\cos\{(2n+1)\alpha\})}$$
$$-\frac{8}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\cosh\{(2n+1)\beta\}\cos\{(2n+1)\beta\}}{(2n+1)\cosh\left\{\frac{1}{2}\pi(2n+1)\right\}(\cosh\{(4n+2)\beta\}+\cos\{(4n+2)\beta\})}.$$

Equality (32.1) now readily follows.

If we let α tend to ∞, and hence β tend to 0, in (32.1), we find that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\cosh\left\{\frac{1}{2}\pi(2n+1)\right\}} = \frac{\pi}{8},$$

which is a special case of another theorem of Ramanujan, Entry 15 of Chapter 14 (Part II [2, p. 262]).

If we let β tend to ∞, and therefore α tend to 0, we find that (32.1) yields the well-known evaluation

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \frac{\pi}{4}. \tag{32.7}$$

Entry 33 (Formula (4), p. 288). *If* $\alpha\beta = \pi^2/2$, *where* $\alpha, \beta > 0$, *then*

$$\sum_{n=0}^{\infty}\frac{(-1)^n\cos\{(2n+1)\alpha\}}{(2n+1)(\cosh\{(2n+1)\alpha\}-\cos\{(2n+1)\alpha\})} + \frac{\pi}{8} - \frac{\pi^3}{32\alpha^2}$$
$$= \sum_{n=1}^{\infty}\frac{\sin(n\beta)\sinh(n\beta)\coth(n\pi)}{n(\cosh(2n\beta)+\cos(2n\beta))}. \tag{33.1}$$

Proof. Let

$$f(z) := \frac{\cos(\alpha z)}{z\,(\cosh(\alpha z) - \cos(\alpha z))\cos(\frac{1}{2}\pi z)}.$$

We observe that $f(z)$ has a triple pole at $z = 0$ and simple poles at $z = 2n + 1$, for each integer n, and at $z = n\pi(\pm 1 + i)/\alpha$, for each nonzero integer n. Let $\{C_N\}$, $N \geq 1$, denote a sequence of rectangles having centers at the origin and horizontal and vertical sides approaching ∞ as N tends to ∞. The rectangles are also chosen so that the sides remain at a bounded distance from the poles of $f(z)$ as N approaches ∞.

We now calculate the residues of $f(z)$. First, after a modest calculation,

$$R_0 = \frac{\pi^2}{8\alpha^2} - \frac{1}{2}. \tag{33.2}$$

Second, for each integer n,

$$R_{2n+1} = -\frac{2(-1)^n \cos\{(2n+1)\alpha\}}{\pi(2n+1)\,(\cosh\{(2n+1)\alpha\} - \cos\{(2n+1)\alpha\})} = R_{-(2n+1)}. \tag{33.3}$$

Third, straightforward calculations yield, for each nonzero integer n,

$$R_{n\pi(\pm 1+i)/\alpha} = \frac{\pm(-1)^n \cos\{n\pi(\pm 1 + i)\}}{2\pi i n \sinh(n\pi)\cos\{n\beta(\pm 1 + i)\}} = R_{-n\pi(\pm 1+i)/\alpha}. \tag{33.4}$$

Using (33.4), we find that

$$R_{\pm n\pi(1+i)/\alpha} + R_{\pm n\pi(-1+i)/\alpha} = \frac{2\coth(n\pi)\sin(n\beta)\sinh(n\beta)}{\pi n\,(\cosh(2n\beta) + \cos(2n\beta))}, \tag{33.5}$$

for every positive integer n.

Lastly, we apply the residue theorem. It is easy to show that

$$\lim_{N\to\infty} \int_{C_N} f(z)\,dz = 0. \tag{33.6}$$

Thus, by (33.2), (33.3), (33.5), and (33.6),

$$\begin{aligned} 0 = \frac{\pi^2}{8\alpha^2} - \frac{1}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n \cos\{(2n+1)\alpha\}}{(2n+1)\,(\cosh\{(2n+1)\alpha\} - \cos\{(2n+1)\alpha\})} \\ + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(n\beta)\sinh(n\beta)\coth(n\pi)}{n\,(\cosh(2n\beta) + \cos(2n\beta))}. \end{aligned}$$

After a slight amount of rearrangement, we deduce (33.1).

If we let α tend to 0, and therefore β tend to ∞, in (33.1), we deduce the well-known result

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.$$

At the top of page 289, Ramanujan writes, "The difference between the series

$$\frac{\theta}{8\pi}+\sum_{n=1}^{\infty}\frac{\sin(n^2\theta)}{n(e^{2n\pi}-1)} \quad \text{and} \quad \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n B_{4n+2}\theta^{2n+1}}{(2n+1)!\,(2n+1)}."$$

The sentence is not completed. Observe that, by Stirling's formula, the series on the right diverges for all $\theta \neq 0$. Most likely, Ramanujan realized that the series diverges and stopped here, or that he proceeded formally and could not evaluate the requisite integrals. These integrals do not appear to have evaluations in closed form. Note below that Ramanujan has a (possible) misprint in the first expression of the quote above. This entry should be compared with formula (3) on page 274 (Part IV [4, p. 298]).

Entry 34 (Formula (1), p. 289). *Formally,*

$$\frac{\theta}{4\pi}+\sum_{n=1}^{\infty}\frac{\sin(n^2\theta)}{n(e^{2n\pi}-1)}=\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n B_{4n+2}\theta^{2n+1}}{(2n+1)!\,(2n+1)} +2\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\sin(x^2\theta)\cos(2\pi nx)}{x(e^{2\pi x}-1)}dx, \tag{34.1}$$

where B_j, $j \geq 0$, denotes the jth Bernoulli number.

"Formal Proof". Apply the Poisson summation formula (Titchmarsh [2, p. 60]) to

$$f(x) := \frac{\sin(x^2\theta)}{x(e^{2\pi x}-1)}.$$

Note that $f(0) = \theta/(2\pi)$. Hence,

$$\frac{\theta}{4\pi}+\sum_{n=1}^{\infty}\frac{\sin(n^2\theta)}{n(e^{2n\pi}-1)}=\int_0^{\infty}\frac{\sin(x^2\theta)}{x(e^{2\pi x}-1)}dx +2\sum_{n=1}^{\infty}\int_0^{\infty}\frac{\sin(x^2\theta)\cos(2\pi nx)}{x(e^{2\pi x}-1)}dx. \tag{34.2}$$

Now, by Entry 16(iv) of Chapter 13 (Part II [2, p. 220]),

$$\begin{aligned}\int_0^{\infty}\frac{\sin(x^2\theta)}{x(e^{2\pi x}-1)}dx &= \sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n+1}}{(2n+1)!}\int_0^{\infty}\frac{x^{4n+1}}{e^{2\pi x}-1}dx\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n+1}}{(2n+1)!}\frac{B_{4n+2}}{8n+4}.\end{aligned} \tag{34.3}$$

Note that the inversion in order of summation and integration has not been justified. Indeed, the series on the far right side of (34.3) diverges. Putting (34.3) in (34.2), we complete the "proof."

Ramanujan begins page 312 by stating two series identities involving the Möbius function $\mu(n)$. He then offers a general claim, which contains the previous two results as special cases, and which is an analogue of the Poisson summation formula, with $\mu(n)/n$ as coefficients. We first formally state Ramanujan's general claim. We then show that the two examples follow from the general claim. Next, numerical calculations show that Ramanujan's two examples are false but that the errors are numerically very small. Lastly, we indicate how G. H. Hardy and J. E. Littlewood corrected Ramanujan's claims.

In our calculations below we employ the prime number theorem in the form $\sum_{n=1}^{\infty} \mu(n)/n = 0$. Perhaps Ramanujan used a stronger, but incorrect, version of the prime number theorem, which would account for the excellent numerical agreement in the two applications. As we saw in Chapter 24 (Part IV [4]), Ramanujan made several errors in deriving approximations to $\pi(x)$ that involve $\mu(n)$.

The author [8] has established a general arithmetical Poisson summation formula which can be applied to series with $\mu(n)$ as the coefficients. However, it is considerably more complicated than Ramanujan's claim. Similar arithmetical summation formulas were derived via contour integration by the author [5] and by P. V. Krishnaiah and R. Sita Rama Chandra Rao [1].

Entry 35 (p. 312). *Let* $\alpha\beta = 2\pi$, *where* $\alpha, \beta > 0$. *Let*

$$\psi(n) = \int_0^{\infty} \varphi(x)\cos(nx)\,dx.$$

Then

$$\frac{\alpha}{2}\sum_{n=1}^{\infty} \frac{\mu(n)\varphi(\alpha/n)}{n} = \sum_{n=1}^{\infty} \frac{\mu(n)\psi(\beta/n)}{n}.$$

Entry 36 (p. 312). *If* $p > 0$,

$$\sum_{n=1}^{\infty} \frac{\mu(n)n}{p^2+n^2} = \frac{\pi}{p}\sum_{n=1}^{\infty} \frac{\mu(n)e^{-2\pi/(np)}}{n}.$$

Proof based on Entry 35. Apply Entry 35 with $\varphi(x) = 1/(x^2+1)$. Now, by contour integration or tables (Gradshteyn and Ryzhik [1, p. 445, formula 3.723, no. 2]),

$$\psi(n) = \int_0^{\infty} \frac{\cos(nx)}{x^2+1}dx = \frac{\pi}{2}e^{-n}.$$

Thus, by Entry 35,

$$\frac{\alpha}{2}\sum_{n=1}^{\infty} \frac{\mu(n)n}{\alpha^2+n^2} = \frac{\alpha}{2}\sum_{n=1}^{\infty} \frac{\mu(n)}{n\left((\alpha/n)^2+1\right)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n}\frac{\pi}{2}e^{-\beta/n},$$

or

$$\sum_{n=1}^{\infty} \frac{\mu(n)n}{\alpha^2+n^2} = \frac{\pi}{\alpha} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-2\pi/(n\alpha)}.$$

Entry 37 (p. 312). *If* $p > 0$,

$$\sum_{n=1}^{\infty} \frac{\mu(n)e^{-p/n^2}}{n} = \sqrt{\frac{\pi}{p}} \sum_{n=1}^{\infty} \frac{\mu(n)e^{-\pi^2/(n^2 p)}}{n}.$$

Proof based on Entry 35. Replace α and β in Entry 35 by $\sqrt{\alpha}$ and $\sqrt{\beta}$, respectively, so that

$$\frac{\sqrt{\alpha}}{2} \sum_{n=1}^{\infty} \frac{\mu(n)\varphi(\sqrt{\alpha}/n)}{n} = \sum_{n=1}^{\infty} \frac{\mu(n)\psi(\sqrt{\beta}/n)}{n}, \tag{37.1}$$

where $\alpha\beta = 4\pi^2$. Let $\varphi(x) = \exp(-x^2)$. Then

$$\psi(n) = \int_0^{\infty} e^{-x^2} \cos(nx)\, dx = \frac{\sqrt{\pi}}{2} e^{-n^2/4}.$$

Thus, by (37.1),

$$\frac{\sqrt{\alpha}}{2} \sum_{n=1}^{\infty} \frac{\mu(n)e^{-\alpha/n^2}}{n} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \frac{\sqrt{\pi}}{2} e^{-\beta/(4n^2)},$$

or

$$\sum_{n=1}^{\infty} \frac{\mu(n)e^{-\alpha/n^2}}{n} = \sqrt{\frac{\pi}{\alpha}} \sum_{n=1}^{\infty} \frac{\mu(n)e^{-\pi^2/(\alpha n^2)}}{n}.$$

We now numerically examine Entries 36 and 37. If $0 < p < 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\mu(n)n}{p^2+n^2} &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{k=0}^{\infty} (-p^2/n^2)^k \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{k=1}^{\infty} (-p^2/n^2)^k \\
&= \sum_{k=1}^{\infty} (-p^2)^k \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2k+1}} = \sum_{k=1}^{\infty} \frac{(-1)^k p^{2k}}{\zeta(2k+1)},
\end{aligned}$$

where we have used the fact that $\sum_{n=1}^{\infty} \mu(n)/n = 0$, which is equivalent to the prime number theorem.

Next, using this same fact, we find that

$$\begin{aligned}\frac{\pi}{p}\sum_{n=1}^{\infty}\frac{\mu(n)e^{-2\pi/(np)}}{n} &= \frac{\pi}{p}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{2\pi}{np}\right)^k \\ &= \frac{\pi}{p}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\sum_{k=1}^{\infty}\frac{1}{k!}\left(-\frac{2\pi}{np}\right)^k \\ &= \frac{\pi}{p}\sum_{k=1}^{\infty}\left(-\frac{2\pi}{p}\right)^k\frac{1}{k!\,\zeta(k+1)}.\end{aligned}$$

Thus, Entry 36 may be rewritten in the form

$$\sum_{k=1}^{\infty}\frac{(-1)^k p^{2k}}{\zeta(2k+1)} = \frac{\pi}{p}\sum_{k=1}^{\infty}\left(-\frac{2\pi}{p}\right)^k\frac{1}{k!\,\zeta(k+1)}. \tag{36.1}$$

Setting $p = \frac{1}{2}$ and summing the first 50 terms of each series via *Mathematica*, we find that

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{2^{2k}\zeta(2k+1)} = -0.1600806325\ldots$$

and

$$2\pi\sum_{k=1}^{\infty}(-4\pi)^k\frac{1}{k!\,\zeta(k+1)} = -0.1600806298\ldots.$$

Since the series are alternating, and since

$$\frac{1}{2^{102}\zeta(103)} < 1.98\times 10^{-31} \quad\text{and}\quad \frac{2\pi(4\pi)^{51}}{(51)!\zeta(52)} < 4.65\times 10^{-10},$$

these calculations show that (36.1) is false for $p = \frac{1}{2}$.

If $p > 0$, since $\sum_{n=1}^{\infty}\mu(n)/n = 0$,

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-p/n^2} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\left(\frac{p}{n^2}\right)^k = \sum_{k=1}^{\infty}\frac{(-1)^k p^k}{k!\,\zeta(2k+1)}.$$

Also, by a similar calculation,

$$\sqrt{\frac{\pi}{p}}\sum_{n=1}^{\infty}\frac{\mu(n)e^{-\pi^2/(n^2p)}}{n} = \sqrt{\frac{\pi}{p}}\sum_{k=1}^{\infty}\frac{(-1)^k\pi^{2k}}{k!\,p^k\zeta(2k+1)}.$$

Thus, Entry 37 may be rewritten in the form

$$\sum_{k=1}^{\infty}\frac{(-1)^k p^k}{k!\,\zeta(2k+1)} = \sqrt{\frac{\pi}{p}}\sum_{k=1}^{\infty}\frac{(-1)^k\pi^{2k}}{k!\,p^k\zeta(2k+1)}. \tag{37.2}$$

Setting $p = 1$ and summing the first 50 terms of each series above by *Mathematica*, we find that

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k!\,\zeta(2k+1)} = -0.4805338008\ldots$$

and

$$\sqrt{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k\pi^{2k}}{k!\,\zeta(2k+1)} = -0.4805622889\ldots.$$

Since the series are alternating, and since

$$\frac{1}{(51)!\,\zeta(103)} < 6.45\times 10^{-67} \quad\text{and}\quad \frac{\sqrt{\pi}\pi^{102}}{(51)!\,\zeta(103)} < 5.85\times 10^{-16},$$

we conclude that (37.2) is false for $p = 1$.

Now, in fact, during his stay in Cambridge, Ramanujan told G. H. Hardy and J. E. Littlewood about Entry 37, and they discuss the formula and the more general claim, Entry 35, in their paper [1] (Hardy [3, pp. 20–97, especially pp. 57–63]). Assuming that all of the zeros of $\zeta(s)$ are simple and that the series on the far right side below converges, Hardy and Littlewood proved that

$$\sum_{n=1}^{\infty}\frac{\mu(n)e^{-p/n^2}}{n} = \sqrt{\frac{\pi}{p}}\sum_{n=1}^{\infty}\frac{\mu(n)e^{-\pi^2/(n^2p)}}{n} - \frac{1}{2\sqrt{\pi}}\sum_{\rho}\left(\frac{\pi}{p}\right)^{\rho}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\rho)}{\zeta'(\rho)}, \tag{37.3}$$

where the latter sum is over all complex zeros ρ of $\zeta(s)$, arranged according to increasing moduli. Thus, Ramanujan's claim in Entry 37 must be altered by the latter expression on the right side of (37.3). This is then another instance where Ramanujan's ignorance of the complex zeros of $\zeta(s)$ led him astray.

Hardy and Littlewood also briefly address the more general formula in Entry 35. Although they do not give a complete proof, they clearly demonstrate that Ramanujan's claim must be modified by a similar sum over ρ.

Returning to (37.3), Hardy and Littlewood [1, p. 161] showed that the estimate

$$\sum_{k=1}^{\infty}\frac{(-p)^k}{k!\,\zeta(2k+1)} = O(p^{-1/4+\epsilon}), \tag{37.4}$$

as p tends to ∞, where ϵ is any positive number, is equivalent to the Riemann hypothesis.

Titchmarsh [3, pp. 186–187] also provides a proof of the corrected version (37.3) of Ramanujan's Entry 37 and briefly mentions (37.4) (with the exponent ϵ unfortunately missing) as well [3, p. 328]. We are very grateful to Richard Brent for correcting some inaccuracies in our discussions of the past two entries in an earlier version of this chapter.

Although Entries 35–37 are false, it would be exceedingly interesting to discover Ramanujan's heuristic arguments, since Hardy and Littlewood's approach is through contour integration.

The next result generalizes Entry 35 and is stated in the notebooks without the latter series on the right side of (38.2).

If $\varphi(z) \equiv 1$ and we omit the latter series on the right side of (38.2), we obtain (36.1).

If we set $\varphi(z) = 1/\Gamma(\frac{1}{2}\{z+1\})$, replace p by $\sqrt{p}$, and again ignore the second series on the right side of (38.2), we deduce (37.2). To see this, we first note that if n is odd, $\varphi(-n) = 0$. If $n = 2m$ is even, we observe that

$$\frac{2^{2m}}{(2m)!\,\Gamma(\frac{1}{2}\{-2m+1\})} = \frac{2^{2m}\Gamma(\frac{1}{2}\{2m+1\})}{(2m)!\,\pi(-1)^m}$$
$$= \frac{(-2)^m(2m-1)(2m-3)\cdots 1}{(2m)!\,\sqrt{\pi}} = \frac{(-1)^m}{m!\,\sqrt{\pi}}.$$

Hence, (37.2) readily follows.

Entry 38 (p. 312). *Let $\varphi(z)$ denote an entire function, and put*

$$f(z) := \frac{\varphi(z)p^z}{\cos(\frac{1}{2}\pi z)\zeta(z)},$$

where p is any fixed nonzero complex number, and where $\zeta(z)$ denotes the Riemann zeta-function. For simplicity, assume that each nonreal zero of $\zeta(z)$ is simple. Let C_N denote the positively oriented circle of radius $N+\frac{1}{2}$ centered at the origin, where N is a positive integer. Suppose that

$$\lim_{N\to\infty}\int_{C_N} f(z)\,dz = 0. \tag{38.1}$$

Then

$$\sum_{n=1}^{\infty}\frac{(-1)^n\varphi(2n+1)p^{2n+1}}{\zeta(2n+1)} = \pi\sum_{n=1}^{\infty}\frac{(-1)^n\varphi(-n)}{n!\,\zeta(n+1)}\left(\frac{2\pi}{p}\right)^n$$
$$+\frac{\pi}{2}\sum_{\rho}\frac{\varphi(\rho)p^{\rho}}{\cos(\frac{1}{2}\pi\rho)\zeta'(\rho)}, \tag{38.2}$$

where the sum on ρ is over all nonreal zeros of $\zeta(z)$ arranged according to increasing moduli.

Proof. We apply the residue theorem to the integral in (38.1). Observe that $f(z)$ has simple poles at $z = 2n+1$, for each integer n, at $z = -2n$, for each positive integer n, since $\zeta(-2n) = 0$ (Titchmarsh [3, p. 19]), and at each nonreal zero ρ of $\zeta(z)$. By a straightforward calculation, for each integer n,

$$R_{2n+1} = -\frac{2(-1)^n\varphi(2n+1)p^{2n+1}}{\pi\zeta(2n+1)}. \tag{38.3}$$

From the functional equation of $\zeta(z)$, (0.4), for each positive integer n,

$$\lim_{z\to-2n}\frac{z+2n}{\zeta(z)} = \frac{(-1)^n(2\pi)^{2n+1}}{\pi\Gamma(2n+1)\zeta(2n+1)}. \tag{38.4}$$

Hence,

$$R_{-2n} = \frac{(2\pi)^{2n+1}\varphi(-2n)p^{-2n}}{\pi(2n)!\zeta(2n+1)}. \tag{38.5}$$

Lastly, for each nonreal zero ρ of $\zeta(z)$, a simple calculation gives

$$R_\rho = \frac{\varphi(\rho)p^\rho}{\cos(\frac{1}{2}\pi\rho)\zeta'(\rho)}. \tag{38.6}$$

Thus, applying the residue theorem, using (38.3), (38.5), and (38.6), letting N tend to ∞, and employing (38.1), we find that

$$\begin{aligned} &-\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\varphi(2n+1)p^{2n+1}}{\zeta(2n+1)} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\varphi(-2n-1)p^{-2n-1}}{\zeta(-2n-1)} \\ &+2\sum_{n=1}^{\infty}\frac{\varphi(-2n)}{\zeta(2n+1)(2n)!}\left(\frac{2\pi}{p}\right)^{2n} + \sum_\rho \frac{\varphi(\rho)p^\rho}{\cos(\frac{1}{2}\pi\rho)\zeta'(\rho)} = 0, \end{aligned} \tag{38.7}$$

where the sum on ρ is over all nonreal zeros ρ of $\zeta(z)$ arranged according to increasing moduli. Now, by the functional equation (0.4) of $\zeta(z)$,

$$\zeta(-2n-1) = 2(-1)^{n+1}(2\pi)^{-2n-2}\Gamma(2n+2)\zeta(2n+2).$$

Using this in the second sum in (38.7), we deduce that

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{(-1)^n\varphi(2n+1)p^{2n+1}}{\zeta(2n+1)} &= -\pi\sum_{n=0}^{\infty}\frac{\varphi(-2n-1)}{\zeta(2n+2)(2n+1)!}\left(\frac{2\pi}{p}\right)^{2n+1} \\ &\quad +\pi\sum_{n=1}^{\infty}\frac{\varphi(-2n)}{\zeta(2n+1)(2n)!}\left(\frac{2\pi}{p}\right)^{2n} \\ &\quad +\frac{\pi}{2}\sum_\rho\frac{\varphi(\rho)p^\rho}{\cos(\frac{1}{2}\pi\rho)\zeta'(\rho)}. \end{aligned} \tag{38.8}$$

The first two series on the right side of (38.8) can be combined into one series, and since $1/\zeta(1) = 0$, the proof of (38.2) is complete.

In Ramanujan's formulation of Entry 39, the latter series on the right side of (39.2) does not appear.

Entry 39 (p. 312). *Let $\varphi(z)$ be an entire function, and put*

$$f(z) := \frac{\varphi(z)p^z}{\Gamma(\frac{1}{2}(z+1))\cos(\frac{1}{2}\pi z)\zeta(z)},$$

where p is any fixed nonzero complex number, and where $\zeta(z)$ denotes the Riemann zeta-function. Assume, for simplicity, that all nonreal zeros ρ of $\zeta(z)$ are simple. Let C_N be the same circle as in Entry 38, and assume that

$$\lim_{N\to\infty}\int_{C_N} f(z)\,dz = 0. \tag{39.1}$$

Then

$$\sum_{n=1}^{\infty} \frac{(-1)^n \varphi(2n+1) p^{2n+1}}{\zeta(2n+1) n!} = \sqrt{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \varphi(-2n)}{\zeta(2n+1) n!} \left(\frac{\pi}{p}\right)^{2n}$$
$$+ \frac{\pi}{2} \sum_{\rho} \frac{\varphi(\rho) p^{\rho}}{\Gamma(\frac{1}{2}\{\rho+1\}) \cos(\frac{1}{2}\pi\rho) \zeta'(\rho)}, \tag{39.2}$$

where the sum on ρ is over all nonreal zeros ρ of $\zeta(z)$ arranged according to increasing moduli.

Proof. The proof follows along the same lines as the proof of Entry 38. The function $f(z)$ has simple poles at $z = 2n+1$, for each nonnegative integer n, at $z = -2n$, for each positive integer n, and at $z = \rho$, for each nonreal zero ρ of $\zeta(z)$. Note that the zero of $\cos(\frac{1}{2}\pi z)$ at $z = 2n+1$ is cancelled by the zero of $1/\Gamma(\frac{1}{2}(z+1))$ at $z = 2n+1$ when n is a negative integer. A simple calculation yields

$$R_{2n+1} = -\frac{2(-1)^n \varphi(2n+1) p^{2n+1}}{\pi \zeta(2n+1) n!}, \qquad n \geq 0. \tag{39.3}$$

By (38.4) and the duplication formula (or the functional equation) of the gamma function, for each positive integer n,

$$R_{-2n} = \frac{(2\pi)^{2n+1} \varphi(-2n) p^{-2n}}{\pi \Gamma(-n+\frac{1}{2}) \Gamma(2n+1) \zeta(2n+1)} = \frac{2(-1)^n \varphi(-2n)}{\sqrt{\pi} n! \, \zeta(2n+1)} \left(\frac{\pi}{p}\right)^{2n}. \tag{39.4}$$

For each nonreal zero ρ of $\zeta(z)$, we easily see that

$$R_{\rho} = \frac{\varphi(\rho) p^{\rho}}{\Gamma(\frac{1}{2}\{\rho+1\}) \cos(\frac{1}{2}\pi\rho) \zeta'(\rho)}. \tag{39.5}$$

Invoking the residue theorem, using (39.3)–(39.5), letting N tend to ∞, and employing (39.1), we conclude that

$$-\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \varphi(2n+1) p^{2n+1}}{\zeta(2n+1) n!} + \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^n \varphi(-2n)}{\zeta(2n+1) n!} \left(\frac{\pi}{p}\right)^{2n}$$
$$+ \sum_{\rho} \frac{\varphi(\rho) p^{\rho}}{\Gamma(\frac{1}{2}\{\rho+1\}) \cos(\frac{1}{2}\pi\rho) \zeta'(\rho)} = 0,$$

where the sum on ρ is over all nonreal zeros ρ of $\zeta(z)$ arranged according to increasing moduli. The last equality is equivalent to (39.2).

Entry 40 (p. 324). *For each positive integer n,*

$$\sum_{k=1}^{n} \frac{1}{(4k-2)^3 - (4k-2)} = \frac{1}{2} \sum_{k=1}^{n} \frac{1}{2n+2k-1}. \tag{40.1}$$

Proof. The following proof is due to R. Sitaramachandrarao [1].

For any nonzero number a and positive integer n, define

$$\varphi(a, n) = 1 + 2\sum_{k=1}^{n} \frac{1}{(ak)^3 - ak}.$$

Then it is easy to see that

$$\sum_{k=1}^{n} \frac{1}{(4k-2)^3 - (4k-2)} = \tfrac{1}{2}\{\varphi(2, 2n) - \varphi(4, n)\}. \tag{40.2}$$

By Entry 1 of Chapter 2 and Example 4 in Section 5 of Chapter 2 (Part I [1, pp. 25, 31]),

$$\begin{aligned}\varphi(2, 2n) - \varphi(4, n) &= 1 + 2\sum_{k=1}^{2n} \frac{1}{2n+k} - \frac{4n}{4n+1} - \frac{1}{2}\sum_{k=n+1}^{2n} \frac{1}{k} - \sum_{k=2n+1}^{4n+1} \frac{1}{k}\\ &= \frac{1}{4n+1} + 2\sum_{k=2n+1}^{4n} \frac{1}{k} - \frac{1}{2}\sum_{k=n+1}^{2n} \frac{1}{k} - \sum_{k=2n+1}^{4n+1} \frac{1}{k}\\ &= \sum_{k=2n+1}^{4n} \frac{1}{k} - \frac{1}{2}\sum_{k=n+1}^{2n} \frac{1}{k} = \sum_{k=n}^{2n-1} \frac{1}{2k+1}.\end{aligned} \tag{40.3}$$

Putting (40.3) in (40.2), we obtain an equality that is easily seen to be equivalent to (40.1).

In the entry immediately following Entry 40, Ramanujan proposes an equality between two finite sums of inverse tangent functions,

$$\sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{2n+2k-1}\right) = \sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{2k^3+1}\right).$$

However, a line has been drawn through the right side. Indeed, it is easily checked that this claim is false, in general.

Entry 41 (p. 324). *For each positive integer n,*

$$\sum_{k=0}^{n-1} \tan^{-1}\left(\frac{1}{(2n+2k+1)\sqrt{3}}\right) = \sum_{k=0}^{n-1} \tan^{-1}\left(\frac{1}{(2k+1)\sqrt{3}}\right)^3. \tag{41.1}$$

Proof. We induct on n. For $n = 1$, (41.1) is trivial.

Assume that (41.1) is valid. We thus will prove that (41.1) holds with n replaced by $n + 1$. Recall that, for $0 \le xy < 1$,

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right). \tag{41.2}$$

Thus, by induction,

$$\begin{aligned}
&\sum_{k=0}^{n} \tan^{-1}\left(\frac{1}{(2k+1)\sqrt{3}}\right)^3 \\
&= \sum_{k=0}^{n-1} \tan^{-1}\left(\frac{1}{(2n+2k+1)\sqrt{3}}\right) + \tan^{-1}\left(\frac{1}{(2n+1)\sqrt{3}}\right)^3 \\
&= \sum_{k=0}^{n-2} \tan^{-1}\left(\frac{1}{(2n+2+2k+1)\sqrt{3}}\right) + \tan^{-1}\left(\frac{1}{(2n+1)\sqrt{3}}\right) \\
&\quad + \tan^{-1}\left(\frac{1}{(2n+1)\sqrt{3}}\right)^3.
\end{aligned}$$

Thus, it remains to show that

$$\begin{aligned}
&\tan^{-1}\left(\frac{1}{(2n+1)\sqrt{3}}\right) + \tan^{-1}\left(\frac{1}{(2n+1)\sqrt{3}}\right)^3 \\
&\qquad = \tan^{-1}\left(\frac{1}{(4n+1)\sqrt{3}}\right) + \tan^{-1}\left(\frac{1}{(4n+3)\sqrt{3}}\right).
\end{aligned} \tag{41.3}$$

By (41.2),

$$\begin{aligned}
&\tan^{-1}\left(\frac{1}{(4n+1)\sqrt{3}}\right) + \tan^{-1}\left(\frac{1}{(4n+3)\sqrt{3}}\right) \\
&\qquad = \tan^{-1}\left(\frac{\sqrt{3}(8n+4)}{3(4n+1)(4n+3)-1}\right) \\
&\qquad = \tan^{-1}\left(\frac{\sqrt{3}(2n+1)}{3(2n+1)^2-1}\right) \\
&\qquad = \tan^{-1}\left(\frac{\sqrt{3}(2n+1)\left(1+3(2n+1)^2\right)}{9(2n+1)^4-1}\right) \\
&\qquad = \tan^{-1}\left(\frac{1}{(2n+1)\sqrt{3}}\right) + \tan^{-1}\left(\frac{1}{(2n+1)\sqrt{3}}\right)^3,
\end{aligned}$$

by another application of (41.2). Thus, (41.3) has been proved, and the proof is complete.

On pages 334, 335, 340, and 341, Ramanujan offers four related claims about products of certain alternating series. In particular, on page 335, he asserts that "$a_1^2 - 2a_1a_2 + (2a_1a_3 + a_2^2) - (2a_1a_4 + 2a_2a_3) + \cdots$ oscillates between $(a_1 - a_2 + a_3 - \cdots)^2 \pm (\pi/2)\lim_{n\to\infty} na_n^2$. For example,

$$1 - \frac{2}{\sqrt{2}} + \left(\frac{2}{\sqrt{3}} + \frac{1}{2}\right) - \left(\frac{2}{\sqrt{4}} + \frac{2}{\sqrt{6}}\right) + \left(\frac{2}{\sqrt{5}} + \frac{2}{\sqrt{8}} + \frac{1}{3}\right) - \cdots$$

oscillates between

$$\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\cdots\right)^2+\frac{\pi}{2}$$

and

$$\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\cdots\right)^2-\frac{\pi}{2}."$$

We state this claim more precisely. Suppose that, for $0 \le x \le 1$, $f(x)$ has the power series representation $\sum_{n=1}^{\infty}(-1)^{n-1}a_n x^n$, where $a_n > 0$, $1 \le n < \infty$. Let $g(x)$ denote the power series obtained by forming the Cauchy product of $f(x)$ with itself. If $L := \lim_{n\to\infty} na_n^2$ exists and is finite, then the even and odd indexed partial sums of $g(1)$ tend to $f^2(1)+(\pi/2)L$ and $f^2(1)-(\pi/2)L$, respectively. In particular, if $a_n = 1/\sqrt{n}$, then the even and odd indexed partial sums of $g(1)$ tend to $\left\{(1-\sqrt{2})\zeta(\frac{1}{2})\right\}^2+\pi/2$ and $\left\{(1-\sqrt{2})\zeta(\frac{1}{2})\right\}^2-\pi/2$, respectively, where ζ denotes the Riemann zeta-function. The case when $L=0$ implies that the series $g(1)$ converges. In the case when $L=+\infty$, the result states that the even and odd indexed partial sums diverge to $+\infty$ and $-\infty$, respectively.

On page 340, Ramanujan states a generalization of the foregoing result for the kth power of $f(x)$, where k is any positive integer exceeding 1. Finally, on page 341, Ramanujan offers a similar theorem for the product of k (possibly distinct) alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_{n1}, \sum_{n=1}^{\infty}(-1)^{n-1}a_{n2}, \dots, \sum_{n=1}^{\infty}(-1)^{n-1}a_{nk}$ under the hypothesis that $\lim_{n\to\infty} n^{k-1}a_{n1}a_{n2}\cdots a_{nk}$ exists or is $+\infty$. The statement on page 334 is the special case $k=2$ of the last claim.

We shall prove Ramanujan's assertions under appropriate assumptions. Our results are possibly not as general as Ramanujan intended. Slightly stronger theorems can undoubtedly be established at the cost of additional technical details in the proofs. (See the remarks following our proof of Entry 42.)

Ramanujan's results are quite remarkable for their explicit description of the behavior of the partial sums of certain alternating divergent series. We know of no other comparable results in the literature.

The results in this section first appeared in a paper by the author and J. L. Hafner [1].

We begin with a simple lemma concerning the asymptotic behavior of a certain finite sum.

Lemma 42.1. *Let α and β denote constants with $0 < \alpha, \beta < 1$,*

$$f_n(x) = \frac{1}{x^{\alpha}(n-x)^{\beta}}, \qquad n \ge 1,$$

and

$$c(n) = \sum_{k=1}^{n-1} f_n(k), \qquad n \ge 2. \tag{42.1}$$

Then, as n tends to ∞,

$$c(n) = \frac{\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)n^{\alpha+\beta-1}} + \frac{\zeta(\alpha)}{n^\beta} + \frac{\zeta(\beta)}{n^\alpha} + O\left(\frac{1}{n^{\alpha+1}}\right) + O\left(\frac{1}{n^{\beta+1}}\right),$$

where ζ denotes the Riemann zeta–function.

Proof. Define, for $x > 0$ and $n \geq 1$,

$$\varphi(x) = \frac{1}{x^\alpha(1-x)^\beta} - \frac{1}{x^\alpha} - \frac{1}{(1-x)^\beta} + 1 \tag{42.2}$$

and

$$g_n(x) = \frac{1}{x^\alpha(n-x)^\beta} - \frac{1}{n^\beta x^\alpha} - \frac{1}{n^\alpha(n-x)^\beta} + \frac{1}{n^{\alpha+\beta}}. \tag{42.3}$$

Note that

$$g_n^{(j)}(x) = \frac{1}{n^{\alpha+\beta+j}}\varphi^{(j)}\left(\frac{x}{n}\right), \qquad j \geq 0.$$

Since, for $j \geq 0$,

$$\varphi^{(j)}(x) = \begin{cases} O(x^{-j+1-\alpha}), & 0 < x \leq \frac{1}{2}, \\ O((1-x)^{-j+1-\beta}), & \frac{1}{2} \leq x < 1, \end{cases}$$

uniformly for x in the given ranges, it follows that

$$g^{(j)}(x) = \begin{cases} O\left(\dfrac{x^{-j+1-\alpha}}{n^{\beta+1}}\right), & 0 < x \leq n/2, \\ O\left(\dfrac{(n-x)^{-j+1-\beta}}{n^{\alpha+1}}\right), & n/2 \leq x < n. \end{cases} \tag{42.4}$$

We now apply the Euler–Maclaurin summation formula (0.5) to $g_n(x)$. Recalling that $B_2(x)$ denotes the second Bernoulli polynomial, we find that

$$\sum_{k=1}^{n-1} g_n(k) = \int_1^{n-1} g_n(x)\,dx + \tfrac{1}{2}g_n(1) + \tfrac{1}{2}g_n(n-1) + R_n, \tag{42.5}$$

where

$$\begin{aligned} R_n &= \tfrac{1}{12}\left\{g_n'(n-1) - g_n'(1)\right\} - \tfrac{1}{2}\int_1^{n-1} B_2(x-[x])g_n''(x)\,dx \\ &= O\left(\frac{1}{n^{\alpha+1}}\right) + O\left(\frac{1}{n^{\beta+1}}\right) + O\left(\frac{1}{n^{\alpha+\beta+1}}\right), \end{aligned} \tag{42.6}$$

as n tends to ∞, by (42.4). (Note that the notation R_n has a meaning here different from that in (0.6).) Using (42.4) and (42.6) in (42.5), we deduce that

$$\sum_{k=1}^{n-1} g_n(k) = \int_0^n g_n(x)\,dx + O\left(\frac{1}{n^{\alpha+1}}\right) + O\left(\frac{1}{n^{\beta+1}}\right), \tag{42.7}$$

as n tends to ∞.

Recalling the definitions (42.2) and (42.3) of φ and g_n, respectively, and setting $x = nt$ in the integral above, we find that

$$\begin{aligned}\int_0^n g_n(x)\,dx &= \frac{1}{n^{\alpha+\beta-1}}\int_0^1 \varphi(t)\,dt\\ &= \frac{1}{n^{\alpha+\beta-1}}\left\{\frac{\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)} - \frac{1}{1-\alpha} - \frac{1}{1-\beta} + 1\right\},\end{aligned}$$

where we have used the classical integral representation for the beta function. Hence, by (42.7), as n tends to ∞,

$$\begin{aligned}\sum_{k=1}^{n-1} g_n(k) = \frac{1}{n^{\alpha+\beta-1}}&\left\{\frac{\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)} - \frac{1}{1-\alpha} - \frac{1}{1-\beta} + 1\right\}\\ &+ O\left(\frac{1}{n^{\alpha+1}}\right) + O\left(\frac{1}{n^{\beta+1}}\right).\end{aligned} \tag{42.8}$$

Now, from the definitions (42.1) and (42.3), we deduce immediately that

$$\begin{aligned}c(n) &= \sum_{k=1}^{n-1} g_n(k) + \frac{1}{n^\beta}\sum_{k=1}^{n-1}\frac{1}{k^\alpha} + \frac{1}{n^\alpha}\sum_{k=1}^{n-1}\frac{1}{(n-k)^\beta} - \frac{n-1}{n^{\alpha+\beta}}\\ &= \sum_{k=1}^{n-1} g_n(k) + \frac{1}{n^\beta}\sum_{k=1}^{n}\frac{1}{k^\alpha} + \frac{1}{n^\alpha}\sum_{k=1}^{n}\frac{1}{k^\beta} - \frac{n+1}{n^{\alpha+\beta}}.\end{aligned} \tag{42.9}$$

Recall (Part I [1, p. 150]) that for any complex number $r \neq -1$, as n tends to ∞,

$$\sum_{k=1}^{n} k^r \sim \zeta(-r) + \frac{n^{r+1}}{r+1} + \frac{n^r}{2} + \sum_{k=1}^{\infty}\frac{B_{2k}\Gamma(r+1)n^{r-2k+1}}{(2k)!\,\Gamma(r-2k+2)}, \tag{42.10}$$

where B_j, $j \geq 2$, denotes the jth Bernoulli number. Employing (42.8) and (42.10) in (42.9), we conclude that

$$\begin{aligned}c(n) = &\frac{\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)n^{\alpha+\beta-1}} - \frac{1}{n^{\alpha+\beta-1}}\left(\frac{1}{1-\alpha} + \frac{1}{1-\beta}\right) - \frac{1}{n^{\alpha+\beta}}\\ &+ \frac{1}{n^\beta}\left\{\zeta(\alpha) + \frac{n^{1-\alpha}}{1-\alpha} + \frac{n^{-\alpha}}{2} + O\left(\frac{1}{n^{\alpha+1}}\right)\right\}\\ &+ \frac{1}{n^\alpha}\left\{\zeta(\beta) + \frac{n^{1-\beta}}{1-\beta} + \frac{n^{-\beta}}{2} + O\left(\frac{1}{n^{\beta+1}}\right)\right\}\\ &+ O\left(\frac{1}{n^{\alpha+1}}\right) + O\left(\frac{1}{n^{\beta+1}}\right)\\ = &\frac{\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(2-\alpha-\beta)n^{\alpha+\beta-1}} + \frac{\zeta(\alpha)}{n^\beta} + \frac{\zeta(\beta)}{n^\alpha} + O\left(\frac{1}{n^{\alpha+1}}\right) + O\left(\frac{1}{n^{\beta+1}}\right),\end{aligned}$$

which completes the proof of Lemma 42.1.

Our next lemma extends the result to multiple sums.

Lemma 42.2. *Let $k \geq 2$, let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be constants such that $0 < \alpha_k \leq \alpha_{k-1} \leq \cdots \leq \alpha_1 < 1$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_k \geq k-1$, and let $\gamma_k = \alpha_2 + \alpha_3 + \cdots + \alpha_k - k + 3$. Then $\gamma_k > 1$ and there exist constants b_{kj} and β_{kj} such that for each j, $1 \leq j \leq 3^{k-1} - 1$, $\alpha_1 + \alpha_2 + \cdots + \alpha_k - k + 1 < \beta_{kj} < 1$ and*

$$\begin{aligned} c_k(n) &:= \sum_{n_k=1}^{n-1} \sum_{n_{k-1}=1}^{n-n_k-1} \cdots \sum_{n_2=1}^{n-n_k-\cdots-n_3-1} \frac{1}{n_k^{\alpha_k} n_{k-1}^{\alpha_{k-1}} \cdots n_2^{\alpha_2} (n - n_k - \cdots - n_2)^{\alpha_1}} \\ &= \frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\cdots\Gamma(1-\alpha_k)}{\Gamma(k - \alpha_1 - \alpha_2 - \cdots - \alpha_k) n^{\alpha_1+\alpha_2+\cdots+\alpha_k-k+1}} \\ &\quad + \sum_{j=1}^{3^{k-1}-1} \frac{b_{kj}}{n^{\beta_{kj}}} + O\left(\frac{1}{n^{\gamma_k}}\right), \end{aligned} \tag{42.11}$$

as n tends to ∞.

Proof. Naturally, we induct on k. The case $k = 2$ is just a restatement of Lemma 42.1 with $\alpha = \alpha_1$ and $\beta = \alpha_2$ (assuming $\beta \leq \alpha$). In this case, we have $b_{21} = \zeta(\alpha_2)$, $b_{22} = \zeta(\alpha_1)$, $\beta_{21} = \alpha_1$, $\beta_{22} = \alpha_2$, and $\gamma_2 = \alpha_2 + 1$.

Suppose that Lemma 42.2 is valid with k replaced by $k-1$, where $k - 1 \geq 2$. Then for some constants $b_{k-1,j}, \beta_{k-1,j}$, with $1 \leq j \leq 3^{k-2} - 1$ and $0 < \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1} - k + 2 < \beta_{k-1,j} < 1$ and $\gamma_{k-1} = \alpha_2 + \alpha_3 + \cdots + \alpha_{k-1} - k + 4$, it follows that

$$\begin{aligned} c_k(n) = \sum_{n_k=1}^{n-1} \Bigg\{ & \frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\cdots\Gamma(1-\alpha_{k-1})}{\Gamma(k-1-\alpha_1-\alpha_2-\cdots-\alpha_{k-1})(n-n_k)^{\alpha_1+\alpha_2+\cdots+\alpha_{k-1}-k+2}} \\ & + \sum_{j=1}^{3^{k-2}-1} \frac{b_{k-1,j}}{(n-n_k)^{\beta_{k-1,j}}} + O\left(\frac{1}{(n-n_k)^{\gamma_{k-1}}}\right) \Bigg\} \frac{1}{n_k^{\alpha_k}}. \end{aligned}$$

Applying Lemma 42.1 a total of 3^{k-2} times, we deduce that

$$\begin{aligned} c_k(n) = & \frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\cdots\Gamma(1-\alpha_{k-1})}{\Gamma(k-1-\alpha_1-\alpha_2-\cdots-\alpha_{k-1})} \\ & \times \Bigg\{ \frac{\Gamma(1-\alpha_k)\Gamma(1-\alpha_1-\alpha_2-\cdots-\alpha_{k-1}+k-2)}{\Gamma(k-\alpha_1-\alpha_2-\cdots-\alpha_k)n^{\alpha_1+\alpha_2+\cdots+\alpha_k-k+1}} \\ & + \frac{\zeta(\alpha_k)}{n^{\alpha_1+\alpha_2+\cdots+\alpha_{k-1}-k+2}} + \frac{\zeta(\alpha_1+\alpha_2+\cdots+\alpha_{k-1}-k+2)}{n^{\alpha_k}} \\ & + O\left(\frac{1}{n^{\alpha_1+\alpha_2+\cdots+\alpha_{k-1}-k+3}}\right) + O\left(\frac{1}{n^{\alpha_k+1}}\right) \Bigg\} \\ & + \sum_{j=1}^{3^{k-2}-1} b_{k-1,j} \Bigg\{ \frac{\Gamma(1-\beta_{k-1,j})\Gamma(1-\alpha_k)}{\Gamma(2-\beta_{k-1,j}-\alpha_k)n^{\beta_{k-1,j}+\alpha_k-1}} \\ & + \frac{\zeta(\beta_{k-1,j})}{n^{\alpha_k}} + \frac{\zeta(\alpha_k)}{n^{\beta_{k-1,j}}} + O\left(\frac{1}{n^{\alpha_k+1}}\right) + O\left(\frac{1}{n^{\beta_{k-1,j}+1}}\right) \Bigg\} \end{aligned}$$

$$+ O\left(\frac{1}{n^{\gamma_{k-1}+\alpha_k-1}}\right)$$

$$= \frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\cdots\Gamma(1-\alpha_k)}{\Gamma(k-\alpha_1-\alpha_2-\cdots-\alpha_k)n^{\alpha_1+\alpha_2+\cdots+\alpha_k-k+1}} + \sum_{j=1}^{3^{k-1}-1} \frac{b_{kj}}{n^{\beta_{kj}}} + O\left(\frac{1}{n^{\gamma_k}}\right).$$

Here the set $\{b_{kj} : 1 \le j \le 3^{k-1}-1\}$ comprises the numbers

$$\frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\cdots\Gamma(1-\alpha_{k-1})}{\Gamma(k-1-\alpha_1-\alpha_2-\cdots-\alpha_{k-1})}\zeta(\alpha_k),$$

$$\frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\cdots\Gamma(1-\alpha_{k-1})}{\Gamma(k-1-\alpha_1-\alpha_2-\cdots-\alpha_{k-1})}\zeta(\alpha_1+\alpha_2+\cdots+\alpha_{k-1}-k+2),$$

and, for $1 \le j \le 3^{k-2}-1$,

$$b_{k-1,j}\frac{\Gamma(1-\beta_{k-1,j})\Gamma(1-\alpha_k)}{\Gamma(2-\beta_{k-1,j}-\alpha_k)}, \qquad b_{k-1,j}\zeta(\beta_{k-1,j}), \qquad b_{k-1,j}\zeta(\alpha_k).$$

Furthermore, the set $\{\beta_{kj} : 1 \le j \le 3^{k-1}-1\}$ is composed of the numbers $\alpha_1+\alpha_2+\cdots+\alpha_k-k+2, \alpha_k$ (with multiplicity 3^{k-2}), $\beta_{k-1,j}+\alpha_k-1$, and $\beta_{k-1,j}$, where $1 \le j \le 3^{k-2}-1$. Lastly, we observe that

$$\begin{aligned}\gamma_k &= \alpha_2+\alpha_3+\cdots+\alpha_k-k+3\\ &= \inf_{1\le j\le 3^{k-2}-1}\{\alpha_1+\alpha_2+\cdots+\alpha_{k-1}-k+3, \alpha_k+1, \beta_{k-1,j}+1, \gamma_{k-1}+\alpha_k-1\},\end{aligned}$$

which justifies the exponent in the final O–term.

We are now in a position to state and prove one form of Ramanujan's assertion on page 341 in his second notebook [9]. Our result is the special case of the general claim to which we referred in the introductory paragraphs of this section, when $a_{ni} = n^{-\alpha_i}$ and $\lim_{n\to\infty} n^{k-1}a_{n1}a_{n2}\cdots a_{nk} = 1$.

Entry 42. *Let $k \ge 2$. Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants such that $0 < \alpha_k \le \alpha_{k-1} \le \cdots \le \alpha_1 < 1$ and $\alpha_1+\alpha_2+\cdots+\alpha_k = k-1$. Let $c_k(n)$, $n \ge k$, be defined by (42.11). Then the even and odd indexed partial sums of*

$$\sum_{n=k}^{\infty}(-1)^n c_k(n) \tag{42.12}$$

tend to $S_k + \frac{1}{2}\Gamma_k$ and $S_k - \frac{1}{2}\Gamma_k$, respectively, where

$$S_k = (-1)^k(1-2^{1-\alpha_1})\zeta(\alpha_1)(1-2^{1-\alpha_2})\zeta(\alpha_2)\cdots(1-2^{1-\alpha_k})\zeta(\alpha_k)$$

and

$$\Gamma_k = \Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\cdots\Gamma(1-\alpha_k).$$

Proof. Let $0 < z < 1$ and suppose that N is a positive integer. Then by Lemma 42.2,

$$\begin{aligned}
&\sum_{n_1=1}^{\infty} \frac{(-1)^{n_1} z^{n_1}}{n_1^{\alpha_1}} \sum_{n_2=1}^{\infty} \frac{(-1)^{n_2} z^{n_2}}{n_2^{\alpha_2}} \cdots \sum_{n_k=1}^{\infty} \frac{(-1)^{n_k} z^{n_k}}{n_k^{\alpha_k}} \\
&\quad = \sum_{n=k}^{\infty} (-1)^n c_k(n) z^n \\
&\quad = \sum_{n=k}^{N-1} (-1)^n c_k(n) z^n + \sum_{n=N}^{\infty} (-1)^n c_k(n) z^n \\
&\quad = \sum_{n=k}^{N-1} (-1)^n c_k(n) z^n + \Gamma_k \sum_{n=N}^{\infty} (-1)^n z^n \\
&\qquad + \sum_{j=1}^{3^{k-1}-1} b_{kj} \sum_{n=N}^{\infty} \frac{(-1)^n z^n}{n^{\beta_{kj}}} + O\left(\sum_{n=N}^{\infty} n^{-\gamma_k} \right) \\
&\quad = \sum_{n=k}^{N-1} (-1)^n c_k(n) z^n + \frac{\Gamma_k (-1)^N z^N}{1+z} + o(1) + O(N^{1-\gamma_k}),
\end{aligned} \tag{42.13}$$

as N tends to ∞, uniformly for $0 < z \leq 1$. The term $o(1)$ arises from the fact that the series $\sum_{n=k}^{\infty} (-z)^n / n^{\beta_{kj}}$ converges uniformly on $0 \leq z \leq 1$. Letting z tend to $1-$ in (42.13), we see that the left side approaches S_k, while the right side alternates like

$$\sum_{n=k}^{N-1} (-1)^n c_k(n) + \tfrac{1}{2}(-1)^N \Gamma_k + o(1),$$

as N tends to ∞. The proof is now complete.

We next state a simple corollary of Entry 42.

Corollary. *Let $c_k(n)$ be given by (42.11), with $\alpha_j = 1 - 1/k$, $1 \leq j \leq k$. Then the even and odd indexed partial sums of (42.12) tend to*

$$S_k + \tfrac{1}{2}\Gamma^k(1/k) \qquad \textit{and} \qquad S_k - \tfrac{1}{2}\Gamma^k(1/k),$$

respectively, where

$$S_k = (-1)^k \left\{ (1 - 2^{1/k}) \zeta(1 - 1/k) \right\}^k .$$

In particular, if $k = 2$, the even and odd indexed partial sums of (42.12) tend to

$$S_2 + \pi/2 \qquad \textit{and} \qquad S_2 - \pi/2,$$

respectively.

This corollary is an immediate consequence of Entry 42, when, for $k = 2$, we recall that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Note that this last case is the example that we mentioned in the opening paragraph of this section.

We now offer several remarks.

From the proofs of Lemma 42.2 and Entry 42, we can furthermore conclude that if $\alpha_1+\alpha_2+\cdots+\alpha_k > k-1$, then the series in (42.12) converges to S_k, while if $\alpha_1+\alpha_2+\cdots+\alpha_k < k-1$, then the even and odd indexed partial sums of this series tend to $+\infty$ and $-\infty$, respectively. This observation partially explains the meaning of Ramanujan's assertion on page 334, "The product of the two series $(a_1-a_2+a_3-\cdots)(b_1-b_2+b_3-\cdots)$ is convergent, divergent, or oscillating as $\lim_{n\to\infty} na_nb_n$ is zero, infinite, or finite, when a_n and b_n do not contain any logarithmic functions."

It seems that Ramanujan is tacitly assuming that $a_n = (a+\epsilon_{n1})n_1^{-\alpha_1}$, and $b_n = (b+\epsilon_{n2})n^{-\alpha_2}$, where a, b, α_1, and α_2 are constants with $\alpha_1, \alpha_2 > 0$, and where ϵ_{n1} and ϵ_{n2} tend to 0 as n tends to ∞. Ramanujan evidently assumes similar hypotheses for the aforementioned claims on pages 335, 340, and 341. Indeed, our theorem can undoubtedly be generalized by replacing $1/n_j^{\alpha_j}$ by $(a_j+\epsilon_{n,j})n_j^{-\alpha_j}$, where, for each j, $1 \le j \le k, a_j$ is a constant and $\epsilon_{n,j}$ is a suitable function approaching 0 as n_j tends to ∞. We have been able to show that this claim is correct under the assumption that $\epsilon_{n,j}$, $1 \le j \le k$, is monotonic, though this is probably not the weakest assumption under which Ramanujan's assertion would hold. We forego giving the details of the proof of this more general result.

Adolf Hildebrand has kindly pointed out to us that Ramanujan's weak assumption is *not* sufficient for the conclusion of Entry 42 to hold, even in the case $k=2$. He observes that the series in (42.12) cannot oscillate if $c_k(2n)-c_k(2n-1) > \delta_n$ for sufficiently large n, with $\delta_n > 0$ and $\sum_{n=1}^{\infty}\delta_n = \infty$. We reconstruct his example here. In the notation of the previous two paragraphs, set

$$a_n = b_n = \begin{cases} \dfrac{1}{\sqrt{n}}(1+\epsilon_n), & \text{if } n \text{ is even,} \\ \dfrac{1}{\sqrt{n}}, & \text{if } n \text{ is odd,} \end{cases}$$

where ϵ_n tends to zero, $\sum_{n=1}^{\infty}\epsilon_n^2 = \infty$, and ϵ_n satisfies the relation

$$\epsilon_k = \epsilon_n + o(\epsilon_n^2), \tag{42.14}$$

uniformly for $\sqrt{n} \le k \le n$, as n tends to ∞. For example, we can take $\epsilon_n = (\log\log n)^{-1/2}$.

Define, in analogy with (42.11), $c_2(n) = c(n) = \sum_{k=1}^{n-1} a_ka_{n-k}$. If n is even, then

$$\begin{aligned} c(n) &= \sum_{\substack{k=1\\ k \text{ even}}}^{n-1} \frac{(1+\epsilon_k)(1+\epsilon_{n-k})}{\sqrt{k(n-k)}} + \sum_{\substack{k=1\\ k \text{ odd}}}^{n-1} \frac{1}{\sqrt{k(n-k)}} \\ &= c'(n) + 2\sum_{\substack{k=1\\ k \text{ even}}}^{n-1} \frac{\epsilon_k}{\sqrt{k(n-k)}} + \sum_{\substack{k=1\\ k \text{ even}}}^{n-1} \frac{\epsilon_k\epsilon_{n-k}}{\sqrt{k(n-k)}} \\ &= c'(n) + 2d(n) + e(n), \end{aligned}$$

say, where

$$c'(n) = \sum_{k=1}^{n-1} \frac{1}{\sqrt{k(n-k)}} = \pi + O(n^{-1/2}),$$

according to Lemma 42.1 and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. To estimate $e(n)$ we proceed as follows:

$$e(n) = \left\{ \sum_{\substack{1 \le k \le \sqrt{n} \\ k \text{ even}}} + \sum_{\substack{\sqrt{n} < k \le n - \sqrt{n} \\ k \text{ even}}} + \sum_{\substack{n-\sqrt{n} < k \le n-1 \\ k \text{ even}}} \right\} \frac{\epsilon_k \epsilon_{n-k}}{\sqrt{k(n-k)}}.$$

The first and last sums can be estimated trivially as $O(n^{-1/4}) = o(\epsilon_n^2)$. Now we use relation (42.14) to express the second sum as

$$(\epsilon_n^2 + o(\epsilon_n^2)) \sum_{\substack{\sqrt{n} < k \le n - \sqrt{n} \\ k \text{ even}}} \frac{1}{\sqrt{k(n-k)}} = \tfrac{1}{2} c'(n/2)(\epsilon_n^2 + o(\epsilon_n^2)).$$

Consequently, $e(n) = \pi \epsilon_n^2/2 + o(\epsilon_n^2)$.

A similar and simpler argument shows that

$$d(n) = \pi \epsilon_n /2 + o(\epsilon_n^2). \tag{42.15}$$

We then deduce that for n even

$$c(n) = \pi(1 + \epsilon_n + \epsilon_n^2/2) + o(\epsilon_n^2). \tag{42.16}$$

On the other hand, if n is odd, then

$$\begin{aligned} c(n) &= 2 \sum_{\substack{k=1 \\ k \text{ even}}}^{n-1} \frac{(1+\epsilon_k)}{\sqrt{k(n-k)}} \\ &= 2 \sum_{\substack{k=1 \\ k \text{ even}}}^{n-1} \frac{1}{\sqrt{k(n-k)}} + 2 \sum_{\substack{k=1 \\ k \text{ even}}}^{n-1} \frac{\epsilon_k}{\sqrt{k(n-k)}} \\ &= \pi + O(n^{-1/2}) + 2d(n) \\ &= \pi(1 + \epsilon_n) + o(\epsilon_n^2), \end{aligned} \tag{42.17}$$

by (42.15).

Thus, by (42.16) and (42.17),

$$c(2n) - c(2n-1) = \tfrac{1}{2}\pi \epsilon_{2n}^2 + o(\epsilon_{2n}^2),$$

and hence

$$\sum_{n=1}^{\infty} (c(2n) - c(2n-1)) \gg \sum_{n=1}^{\infty} \epsilon_{2n}^2 = \infty.$$

In light of the remarks above, it seems to be very difficult to determine the most general conditions under which Ramanujan's claim is valid.

We quote Ramanujan in the next entry.

Entry 43 (p. 348). *e^{ax} can be expanded in ascending powers of $e^{bx} - e^{cx}$ and consequently e^{ax} can be expanded in ascending powers of $e^{bx}\sin x$ and hence many transcendental equations can be solved.*

The content of Entry 43 has been thoroughly discussed in Part I [1, pp. 308–312].

Entry 44 (p. 350). *Let n be complex, c be real, and $b \geq 0$. Then, if $0 \leq x \leq (1/c)\tan^{-1}(c/b)$,*

$$e^{nx} = 1 + \frac{ne^{-bx}\sin(cx)}{c} + \sum_{k=2}^{\infty} \frac{d_k}{k!}\left(\frac{e^{-bx}\sin(cx)}{c}\right)^k,$$

where

$$d_k = \begin{cases} n(n+kb)\left\{(n+kb)^2+(2c)^2\right\}\left\{(n+kb)^2+(4c)^2\right\}\cdots \\ \qquad \times\left\{(n+kb)^2+(k-2)^2c^2\right\}, \text{ if } k \text{ is even,} \\ n\left\{(n+kb)^2+c^2\right\}\left\{(n+kb)^2+(3c)^2\right\}\cdots\left\{(n+kb)^2+(k-2)^2c^2\right\}, \\ \qquad \text{if } k \text{ is odd.} \end{cases}$$

See Part I [1, pp. 309–310] for a proof.

At the top of page 352 Ramanujan writes, "If an nth degree series can be expressed in terms of M and N only, then

$$x\frac{du}{dx} - \frac{nLu}{12}$$

can be expressed in terms of M and N only." The degree of a series is vaguely defined in Chapter 15 (Part II [2, pp. 320–321]). The identity of the function u is not divulged. However, L, M, and N are undoubtedly the Eisenstein series $L(q)$, $M(q)$, and $N(q)$ defined at the beginning of Section 4 of Chapter 33. Although the meaning of Ramanujan's claim is unclear, he gives a two line "proof" of his assertion. But, Ramanujan's "proof" appears to have only a shadowy connection with his claim, and so we shall let the next entry encompass what Ramanujan sketchily proves.

Entry 45 (p. 352). *Let f be any differentiable function and set $u(q) = M^{n/4}f(M^3/N^2)$, where M and N are the Eisenstein series mentioned above. Then*

$$q\frac{du}{dq} - \frac{nL(q)u(q)}{12}$$

is a function of only M and N.

Proof. We shall employ Ramanujan's differentiation formulas ([7, eq. (30)], [10, eq. p. 142], Part II [2, p. 330])

$$q\frac{dL}{dq}=\frac{L^2-M}{12},\qquad q\frac{dM}{dq}=\frac{LM-N}{3},\qquad \text{and}\qquad q\frac{dN}{dq}=\frac{LN-M^2}{2}. \tag{45.1}$$

Thus, using (45.1), we first observe that

$$\begin{aligned} q\frac{d(M^3/N^2)}{dq} &= 3q\frac{M^2}{N^2}\frac{dM}{dq}-2q\frac{M^3}{N^3}\frac{dN}{dq} \\ &= 3\frac{M^2}{N^2}\left(\frac{LM-N}{3}\right)-2\frac{M^3}{N^3}\left(\frac{LN-M^2}{2}\right) \\ &= \frac{M^2}{N^3}\left(M^3-N^2\right). \end{aligned} \tag{45.2}$$

Hence, by (45.1) and (45.2),

$$\begin{aligned} q\frac{du}{dq}-\frac{nLu}{12} &= q\frac{n}{4}M^{n/4-1}\frac{dM}{dq}f\left(\frac{M^3}{N^2}\right)+qM^{n/4}f'\left(\frac{M^3}{N^2}\right)\frac{d(M^3/N^2)}{dq} \\ &\quad -\frac{nL}{12}M^{n/4}f\left(\frac{M^3}{N^2}\right) \\ &= \frac{n}{4}M^{n/4-1}\frac{LM-N}{3}f\left(\frac{M^3}{N^2}\right)+M^{n/4}f'\left(\frac{M^3}{N^2}\right)\frac{M^2}{N^3}(M^3-N^2) \\ &\quad -\frac{nL}{12}M^{n/4}f\left(\frac{M^3}{N^2}\right) \\ &= M^{n/4}f'\left(\frac{M^3}{N^2}\right)\frac{M^2}{N^3}(M^3-N^2)-\frac{n}{12}M^{n/4-1}Nf\left(\frac{M^3}{N^2}\right), \end{aligned}$$

which indeed is only a function of M and N (and not of L), as claimed by Ramanujan.

Corollary (p. 352). *We have*

$$\frac{d(L^4/M)}{dN}=\frac{2L^3}{3M^2}\qquad \textit{and}\qquad \frac{d(L^6/N)}{dM}=\frac{3L^5M}{2N^2}.$$

There is a misprint in the notebooks; Ramanujan wrote $3M$ instead of $3M^2$ in the first equality. It is not clear why Ramanujan used the appellation, "Corollary," here.

Proof. By (45.1),

$$\frac{d(L^4/M)}{dN}=\frac{d(L^4/M)/dq}{dN/dq}=\frac{\dfrac{4L^3}{M}\dfrac{dL}{dq}-\dfrac{L^4}{M^2}\dfrac{dM}{dq}}{dN/dq}$$

$$= \frac{\frac{4L^3}{M}\left(\frac{L^2-M}{12}\right) - \frac{L^4}{M^2}\left(\frac{LM-N}{3}\right)}{\frac{LN-M^2}{2}}$$

$$= \frac{\frac{2}{3}\left(-L^3+\frac{L^4N}{M^2}\right)}{LN-M^2} = \frac{2L^3}{3M^2}.$$

The proof of the second equality in the corollary follows along the same lines.

Entry 46 (p. 353). *For* $\operatorname{Re} a > 0$,

$$2\sum_{n=0}^{\infty} \frac{(-1)^n}{x^2+(a+n)^2} = \sum_{n=0}^{\infty} \frac{n!\,(a)_n}{\prod_{k=0}^{n}(x^2+(a+k)^2)}. \tag{46.1}$$

Proof. Observe that each side of (46.1) has simple poles at $x = \pm i(a+n)$ for each nonnegative integer n. It then suffices to show that the residues of each pole are equal. For if $F(x)$ denotes the difference of the left and right sides, $F(x)$ is then an entire function which tends to 0 as x tends to ∞. By Liouville's theorem, $F(x)$ is a constant, which obviously equals 0. Thus, (46.1) is established.

If R_α denotes the residue of a pole α on the left side of (46.1), then it is easy to see that

$$R_{i(a+n)} = \frac{(-1)^n}{i(a+n)} = -R_{-i(a+n)}. \tag{46.2}$$

The calculation of the residues on the right side of (46.1) is more difficult. Now R_α denotes the residue of a pole α on the right side of (46.1). We have

$$\begin{aligned}
R_{i(a+n)} &= \sum_{j=n}^{\infty} \frac{j!\,(a)_j}{\prod_{k=0}^{n-1}(x^2+(a+k)^2)2i(a+n)\prod_{k=n+1}^{j}(x^2+(a+k)^2)}\Bigg|_{x=i(a+n)} \\
&= \sum_{k=0}^{\infty} \frac{(k+n)!\,(a)_{k+n}(x^2+(a+n)^2)}{(a+xi)_{n+k+1}(a-xi)_{n+k+1}2i(a+n)}\Bigg|_{x=i(a+n)} \\
&= \frac{(a)_n n!\,(x^2+(a+n)^2)}{(a+xi)_{n+1}(a-xi)_{n+1}2i(a+n)} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(n+1)_k(a+n)_k}{(a+xi+n+1)_k(a-xi+n+1)_k}\Bigg|_{x=i(a+n)} \\
&= \frac{(a)_n n!}{(a+xi)_n(a-xi)_n 2i(a+n)} \\
&\quad \times {}_3F_2\left[\begin{matrix} 1, n+1, a+n \\ a+xi+n+1, a-xi+n+1 \end{matrix}; 1\right]\Bigg|_{x=i(a+n)}
\end{aligned}$$

$$= \frac{(a)_n n!}{(-n)_n(2a+n)_n 2i(a+n)} \, {}_3F_2\left[\begin{matrix}1, n+1, a+n\\ 1, 2a+2n+1\end{matrix}; 1\right]$$

$$= \frac{(a)_n(-1)^n}{(2a+n)_n 2i(a+n)} \, {}_2F_1\left[\begin{matrix}n+1, a+n\\ 2a+2n+1\end{matrix}; 1\right]$$

$$= \frac{(a)_n(-1)^n}{(2a+n)_n 2i(a+n)} \frac{\Gamma(2a+2n+1)\Gamma(a)}{\Gamma(2a+n)\Gamma(a+n+1)}$$

$$= \frac{\Gamma(a+n)(-1)^n\Gamma(2a+n)\Gamma(2a+2n+1)\Gamma(a)}{\Gamma(a)\Gamma(2a+2n)2i(a+n)\Gamma(2a+n)\Gamma(a+n+1)}$$

$$= \frac{(-1)^n(2a+2n)}{2i(a+n)(a+n)} = \frac{(-1)^n}{i(a+n)}, \tag{46.3}$$

where we applied Gauss's theorem (Bailey [1, p. 2], Part II [2, p. 25]). Thus, (46.3) agrees with (46.2). The calculation of $R_{-i(a+n)}$ is similar. Thus, the proof is complete.

Observe that the right side of (46.1) equals

$$\frac{1}{a^2+x^2} \, {}_3F_2\left[\begin{matrix}1, 1, a\\ a+xi+1, a-xi+1\end{matrix}\right].$$

Entry 47 (p. 355). *For* $|x| < 1$,

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{1-x^{2n+1}} = \sum_{n=1}^{\infty} \frac{x^{n(n+1)/2}}{1-x^n}. \tag{47.1}$$

Proof. The left side of (47.1) may be written in the form

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} x^{(2n+1)m}.$$

Arrange the terms in the array

$$\begin{array}{cccccc}
x & x^3 & x^5 & x^7 & x^9 & \dots \\
x^2 & x^6 & x^{10} & x^{14} & x^{18} & \dots \\
x^3 & x^9 & x^{15} & x^{21} & x^{27} & \dots \\
x^4 & x^{12} & x^{20} & x^{28} & x^{36} & \dots \\
x^5 & x^{15} & x^{25} & x^{35} & x^{45} & \dots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \dots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \dots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \dots .
\end{array}$$

Summing these terms by the column–row method (Part III [3, p. 114]), we arrive at the right side of (47.1).

Entry 48 (p. 364). *For* $|x| < 1$ *and positive integers* $n \geq 2$, *the polylogarithm* $\mathrm{Li}_n(x)$ *is defined by*

$$\mathrm{Li}_n(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^n}.$$

Then, for $x \geq 0$,

$$\sum_{k=1}^{\infty} \left\{ k^2 \log\left(1 + \frac{x^2}{k^2}\right) - x^2 \right\} = \frac{x^2}{2} - \frac{\pi x^3}{3} + \frac{x}{\pi} \mathrm{Li}_2(e^{-2\pi x})$$
$$- x^2 \log(1 - e^{-2\pi x})$$
$$- \frac{1}{2\pi^2} \{\zeta(3) - \mathrm{Li}_3(e^{-2\pi x})\}. \tag{48.1}$$

Proof. Trivially, (48.1) is valid for $x = 0$. It therefore suffices to show that the derivatives of both sides of (48.1) are equal.

Differentiating both sides of (48.1), simplifying, and dividing both sides by $2x$, we find that we must prove that

$$\sum_{k=1}^{\infty} \left\{ \frac{1}{1 + x^2/k^2} - 1 \right\} = \frac{1}{2} - \frac{\pi x}{2} - \frac{\pi x e^{-2\pi x}}{1 - e^{-2\pi x}}. \tag{48.2}$$

For $|x| < 1$, the left side of (48.2) equals

$$\sum_{k=1}^{\infty} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{k^{2n}} - 1 \right\} = \sum_{n=1}^{\infty} (-1)^n x^{2n} \zeta(2n).$$

On the other hand, if B_n, $0 \leq n < \infty$, denotes the nth Bernoulli number, and $|x| < 1$, the right side of (48.2) equals, by (0.1),

$$\frac{1}{2} - \frac{\pi x}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{B_n}{n!} (2\pi x)^n = -\frac{1}{2} \sum_{n=2}^{\infty} \frac{B_n}{n!} (2\pi x)^n$$
$$= -\frac{1}{2} \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} (2\pi x)^{2n} = \sum_{n=1}^{\infty} (-1)^n x^{2n} \zeta(2n),$$

by Euler's formula (0.2). Thus, (48.2) has been verified for $0 \leq x < 1$. Thus, (48.1) has been established for $0 \leq x < 1$. But both sides of (48.1) are analytic for all complex x with $\mathrm{Re}\, x > 0$. Hence, by analytic continuation, (48.1) is valid for all complex x with $\mathrm{Re}\, x > 0$.

In notation slightly different from that of Ramanujan, he claims on page 365 that if

$$x = \left(\frac{1}{\pi} \log\left(\frac{1 + \sqrt{5}}{2} \right) \right)^2, \tag{49.1}$$

then

$$e^{x/2} = \lim_{n\to\infty} e^{-nx} \prod_{k=1}^{n} \left(1 + \frac{x}{k^2}\right)^{k^2}. \tag{49.2}$$

This result is also found on page 370 of Ramanujan's lost notebook [11]. As we shall see, Ramanujan's assertion is incorrect. In our corrected version below, the proposed equality is independent of the value of x.

Entry 49 (p. 365). *If* $|x| < 1$, *then*

$$\frac{x^2}{2} - \frac{1}{\pi^2}\int_0^{\pi x} t^2 \coth t\, dt = \sum_{k=1}^{\infty} \frac{(-1)^k \zeta(2k) x^{2k+2}}{k+1}. \tag{49.3}$$

Proof. Recall that, for $|x| < \pi$ (Gradshteyn and Ryzhik [1, p. 42]),

$$\coth x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} x^{2k-1},$$

where B_n, $0 \le n < \infty$, denotes the nth Bernoulli number. It follows from Euler's formula (0.2) that, for $|x| < 1$,

$$\pi x^2 \coth(\pi x) = x + 2\sum_{k=1}^{\infty} (-1)^{k-1} \zeta(2k) x^{2k+1}.$$

Upon replacing x by t and integrating over $(0, x)$ for $|x| < 1$, we find that

$$\int_0^x \pi t^2 \coth(\pi t)\, dt = \frac{x^2}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \zeta(2k) x^{2k+2}}{k+1},$$

which is easily seen to be equivalent to (49.3).

Now suppose that (49.2) were true for $|x| < 1$. Then

$$\begin{aligned}
\frac{x}{2} &= \lim_{n\to\infty} \sum_{k=1}^{n} \left\{ k^2 \log\left(1 + \frac{x}{k^2}\right) - x \right\} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left\{ k^2 \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \left(\frac{x}{k^2}\right)^j - x \right\} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} k^2 \sum_{j=2}^{\infty} \frac{(-1)^{j-1} x^j}{j k^{2j}} \\
&= \sum_{j=1}^{\infty} \frac{(-1)^j \zeta(2j) x^{j+1}}{j+1},
\end{aligned}$$

where the inversion in order of summation is justified by the absolute convergence of the latter double sum. Replacing x by x^2, we deduce that

$$\frac{x^2}{2} = \sum_{k=1}^{\infty} \frac{(-1)^k \zeta(2k) x^{2k+2}}{k+1}. \tag{49.4}$$

Comparing (49.4) with (49.3), we see that Ramanujan's claim is equivalent to asserting that, by (49.1),

$$\int_0^{\log\left((1+\sqrt{5})\right)/2} t^2 \coth t \, dt = 0,$$

which is obviously false.

In connection with the Lagrange inversion formula, in Chapter 3 Ramanujan studied infinite exponentials

$$a^{a^{\cdot^{\cdot^{\cdot}}}} \tag{50.1}$$

(Part I [1, p. 77]). L. Euler [1], [2] was evidently the first person to seriously examine (50.1), and he showed that (50.1) is convergent if and only if $e^{-e} \le a \le e^{1/e}$.

Upside down, on page 390, Ramanujan offers a theorem about the convergence of the more general infinite exponential

$$a_1^{a_2^{\cdot^{\cdot^{\cdot}}}}. \tag{50.2}$$

Before stating his result, we mention some further relevant papers. As we indicated in [1, p. 77], many authors have written about the convergence of (50.2), and an extensive bibliography on such results is contained in A. Knoebel's comprehensive survey paper [1]. Most authors assume that $\{a_n\}$ is a real, positive sequence and establish convergence when $e^{-e} \le a_n \le e^{1/e}$, for n sufficiently large. D. F. Barrow [1] appears to have been the only one to venture outside this interval. Writing $a_n = e^{1/e} + \epsilon_n$, where $\epsilon_n \ge 0$, he showed that (50.2) converges if

$$\lim_{n\to\infty} \epsilon_n n^2 < \frac{e^{1/e}}{2e} \tag{50.3}$$

and diverges if

$$\lim_{n\to\infty} \epsilon_n n^2 > \frac{e^{1/e}}{2e}. \tag{50.4}$$

Furthermore, writing $a_n = e^{-e} - \epsilon_n$, where $\epsilon_n \ge 0$, he proved that necessarily $\lim_{n\to\infty} \epsilon_n = 0$ and that $\lim_{n\to\infty} n^q \epsilon_n = 0$, for some $q > 1$, is a sufficient condition for convergence. For complex a_n, the most general result is due to W. J. Thron [1] who proved that (50.2) converges if $|a_n| \le e^{1/e}$ for n sufficiently large.

We now state Ramanujan's claim on page 390. He asserts that (50.2) is convergent when

$$1 + \log\log a_n \le \frac{1}{2}\left\{\frac{1}{n^2} + \frac{1}{(n\log n)^2} + \frac{1}{(n\log n \log\log n)^2} + \cdots\right\}, \tag{50.5}$$

and is divergent when the left-hand side is greater than the right side when any 1 is replaced by $1+\epsilon$. This statement needs some clarification. First, the series on the right side of (50.5) is finite for each n and persists as long as the iterated logarithms remain positive. Ramanujan evidently had in mind a test for the convergence of (50.2) when $a_n \geq 1$. If for n sufficiently large, (50.5) holds, then Ramanujan claims that (50.2) converges. On the other hand, if one of the numerators 1 in the series on the right side of (50.5) is replaced by $1+\epsilon$, for some fixed number $\epsilon > 0$, and if there exists a subsequence a_{n_k} tending to ∞ for which the left side of (50.5) is greater than or equal to the right side with one of the numerators 1 replaced by $1+\epsilon$, then Ramanujan claims that (50.2) diverges.

An easy calculation shows that Barrow's theorems (50.3) and (50.4), even in the stronger form when the inequality $<$ in (50.3) is replaced by $\leq$, are contained in Ramanujan's assertion (50.5) with the right side of (50.5) truncated after the first term.

In the remainder of this section we follow the analysis in Bachman's paper [1].

We shall establish a version of Ramanujan's claim for complex a_n. So that the exponentiation is unambiguous, we assume that the sequence of complex numbers $\{b_n\}$, $1 \leq n < \infty$, is given and set

$$a_n := e^{b_n}, \qquad n \geq 1. \tag{50.6}$$

With this definition,

$$E_n := a_1^{a_2^{\cdot^{\cdot^{a_n}}}}, \qquad n \geq 1, \tag{50.7}$$

is well defined. We first give the following test for the convergence of $\{E_n\}$ for complex exponents.

Theorem 50.1. *Let $\{a_n\}$ and $\{E_n\}$ be given by (50.6) and (50.7), respectively. Set*

$$\hat{a}_n = e^{|b_n|}, \qquad n \geq 1, \tag{50.8}$$

and define $\{\hat{E}_n\}$, $n \geq 1$, by (50.7) with $\hat{a}_n$ in place of a_n. Then, if $\{\hat{E}_n\}$ converges, $\{E_n\}$ must converge as well.

The test above is of independent interest. In particular, Thron's result [1] follows from Barrow's theorem for real exponents a_n, $1 \leq a_n \leq e^{1/e}$, and Theorem 50.1.

To state Bachman's results concerning Ramanujan's test for convergence, we introduce the following notation for iterated logarithms. Setting $x_1 = e$ and

$$L_1(x) := L(x) := \log x, \qquad x \geq e,$$

we recursively define x_k and L_k, for $k \geq 2$, by $x_k := e^{x_{k-1}}$, and

$$L_k := L_{k-1}(L(x)), \qquad x \geq x_k.$$

Entry 50a (p. 390). *Let $\{a_n\}$ and $\{E_n\}$ be defined by (50.6) and (50.7), respectively. Then $\{E_n\}$ converges if there exist positive integers k_0 and n_0, such that for all*

$n \geq n_0$,

$$\begin{aligned} 1+\log|\log a_n| &= 1+\log|b_n| \\ &\leq \frac{1}{2}\left\{\frac{1}{n^2}+\frac{1}{(nL_1(n))^2}+\frac{1}{(nL_1(n)L_2(n))^2}\right. \\ &\qquad \left. +\cdots+\frac{1}{(nL_1(n)L_2(n)\cdots L_{k_0}(n))^2}\right\}. \end{aligned} \tag{50.9}$$

Entry 50b (p. 390). *Let $\{E_n\}$ be defined by (50.7), where the sequence $\{a_n\}$ is real, $a_n > 1$, for every integer n, and*

$$\begin{aligned} 1+\log\log a_n \geq \frac{1}{2}&\left\{\frac{1}{n^2}+\frac{1}{(nL_1(n))^2}+\frac{1}{(nL_1(n)L_2(n))^2}+\cdots\right. \\ &\left. +\frac{1}{(nL_1(n)L_2(n)\cdots L_{k_0-1}(n))^2}+\frac{1+\epsilon}{(nL_1(n)L_2(n)\cdots L_{k_0}(n))^2}\right\}, \end{aligned} \tag{50.10}$$

for $n \geq n_0$, for some positive integers k_0 and n_0, and for some $\epsilon > 0$. Then the infinite exponential $\{E_n\}$ diverges.

We first set some convenient notation and establish three useful lemmas. The first lemma reduces the principal case of our problem to an equivalent problem that is easier to attack. Set

$$[x_1, x_2, \ldots, x_n] := x_1^{x_2^{\cdot^{\cdot^{x_n}}}} \qquad \text{and} \qquad [x_1, x_2, \ldots] := x_1^{x_2^{\cdot^{\cdot^{\cdot}}}}.$$

Also set

$$\ell_0(x) := \frac{1}{x} \qquad \text{and} \qquad \ell_n(x) = \frac{1}{xL_1(x)L_2(x)\cdots L_n(x)}, \qquad n \geq 1. \tag{50.11}$$

Lemma 50.2. *Let $\{x_n\}$, $n \geq 1$, be a sequence of real numbers such that $x_n > 1$. Define another sequence $\{X_n\}$, $n \geq 1$, by*

$$x_n = \exp\left(\frac{1+X_n}{e}\right). \tag{50.12}$$

Then $[x_1, x_2, \ldots]$ converges if and only if there exists a sequence $\{Y_n\}$, $n \geq 1$, such that $Y_n \geq -1$ and such that the inequality

$$1+Y_n \geq (1+X_n)e^{Y_{n+1}} \tag{50.13}$$

holds.

Proof. Since $x_n > 1$, the sequence $[x_1, x_2, \ldots]$ is monotonically increasing. Hence, to prove that it converges, it suffices to show that it is bounded. By (50.12) and (50.13),

$$[x_1, x_2, \ldots, x_n] \le [x_1, x_2, \ldots, x_n, e^{1+Y_{n+1}}] \le e^{1+Y_1}.$$

In the opposite direction, suppose that $[x_1, x_2, \ldots]$ converges. Since, $x_n > 1$, then $[x_n, x_{n+1}, \ldots]$ also converges for each $n \ge 1$. Denoting the limit of $[x_n, x_{n+1}, \ldots]$ by e^{1+Y_n}, we observe that $Y_n \ge -1$ and that

$$e^{1+Y_n} = [x_n, e^{1+Y_{n+1}}] = e^{(1+X_n)e^{Y_{n+1}}}.$$

Thus, we deduce (50.13) with equality, and this completes the proof of the lemma.

The next two lemmas are the primary ingredients in the proofs of Entries 50a and 50b.

Lemma 50.3. *Let T_n^k, C_n^k, and X_n^k be defined by*

$$T_n^k := \sum_{j=0}^{k} \ell_j(n-1), \tag{50.14}$$

$$C_n^k := \frac{1}{2}\sum_{j=0}^{k} \ell_j^2(n), \tag{50.15}$$

and

$$1 + X_n^k = (1 + T_n^k)e^{-T_{n+1}^k}, \tag{50.16}$$

where $\ell_j(n)$ is defined in (50.11), and where $k \ge 0$ and $n \ge 2$ are any integers for which the right sides of (50.14) and (50.15) are defined. Then there exists a sequence of integers $\{n_k\}$ such that, for $n \ge n_k$,

$$C_n^k < X_n^k < C_n^{k+1}. \tag{50.17}$$

Proof. Let $k \ge 0$ be fixed. For brevity, set $T_n = T_n^k$ and $X_n = X_n^k$. By (50.14) and (50.11), $T_n = O_k(1/n) < 1$, for $n \ge n_k$, say, and where the notation O_k indicates that the implied constant is dependent on k. For such n, we can expand the right side of (50.16) in a Taylor series about 0 and so find that

$$\begin{aligned}
1 + X_n &= (1 + T_n)e^{-T_{n+1}} \\
&= (1 + T_n)\left(1 - T_{n+1} + \tfrac{1}{2}(T_{n+1})^2 - \tfrac{1}{6}(T_{n+1})^3 + O_k(n^{-4})\right) \\
&= 1 + T_n - T_{n+1} + \tfrac{1}{2}(T_{n+1})^2 - T_nT_{n+1} + \tfrac{1}{2}T_n(T_{n+1})^2 - \tfrac{1}{6}(T_{n+1})^3 + O_k(n^{-4}).
\end{aligned} \tag{50.18}$$

Now, by (50.14) and (50.11), expanding T_n about $n+1$, we find that

$$T_n = T_{n+1} - T'_{n+1} + \tfrac{1}{2}T''_{n+1} - \tfrac{1}{6}T'''_{\xi}, \tag{50.19}$$

for some number ξ, such that $n < \xi < n+1$. Note that

$$T'_{n+1} = \sum_{j=0}^{k} \ell'_j(n) = -\sum_{j=0}^{k} \ell_j(n) \sum_{i=0}^{j} \ell_i(n), \tag{50.20}$$

$$\begin{aligned} T''_{n+1} &= \sum_{j=0}^{k} \ell''_j(n) = -\left((\ell_0^2(n))' + \sum_{j=1}^{k} \sum_{i=0}^{j} \left(\ell_j(n)\ell_i(n)\right)' \right) \\ &= \frac{2}{n^3} + O_k\left(\frac{1}{n^3 \log n}\right), \end{aligned} \tag{50.21}$$

and

$$T'''_{\xi} = \sum_{j=0}^{k} \ell'''_j(\xi - 1) = O_k(n^{-4}). \tag{50.22}$$

Substituting (50.19)–(50.22) into (50.18) and simplifying the resulting expression, we find that

$$1 + X_n = 1 - T'_{n+1} - \tfrac{1}{2}(T_{n+1})^2 + \frac{1}{3n^3} + O_k\left(\frac{1}{n^3 \log n}\right).$$

Hence, by (50.20), (50.14), and (50.15), we deduce that

$$\begin{aligned} X_n &= -T'_{n+1} - \tfrac{1}{2}(T_{n+1})^2 + \frac{1}{3n^3} + O_k\left(\frac{1}{n^3 \log n}\right) \\ &= \frac{1}{2} \sum_{j=0}^{k} \ell_j^2(n) + \frac{1}{3n^3} + O_k\left(\frac{1}{n^3 \log n}\right) \\ &= C_n^k + \frac{1}{3n^3} + O_k\left(\frac{1}{n^3 \log n}\right). \end{aligned} \tag{50.23}$$

Thus, for $n \geq n_k$, (50.23) implies (50.17), and this completes the proof of Lemma 50.2.

Lemma 50.4. *Let T_n^k and X_n^k be defined by (50.14) and (50.16), respectively. Moreover, let x_n^k be defined by*

$$x_n^k = \exp\left(\frac{1 + X_n^k}{e}\right). \tag{50.24}$$

Then

$$\lim_{m \to \infty} [x_n^k, x_{n+1}^k, \ldots, x_m^k] = e^{1+T_n^k}. \tag{50.25}$$

Proof. We begin with the observation that it suffices to show that there exists a sequence of integers $\{n'_k\}$ such that (50.25) is valid for each $n \geq n'_k$. Indeed, assuming this, we have, for each $\ell \geq n'_k$,

$$\begin{aligned}\lim_{m\to\infty}[x_n^k, x_{n+1}^k, \dots, x_m^k] &= \left[x_n^k, x_{n+1}^k, \dots, x_\ell^k, \lim_{m\to\infty}[x_{\ell+1}^k, x_{\ell+2}^k, \dots, x_m^k]\right] \\ &= \left[x_n^k, x_{n+1}^k, \dots, x_\ell^k, e^{1+T_{\ell+1}^k}\right] \\ &= e^{1+T_n^k},\end{aligned}$$

by (50.24) and (50.16). To exhibit the existence of such a sequence $\{n'_k\}$, we first observe that, by (50.24), Lemma 50.2, and Lemma 50.3, any infinite exponential $[x_n^k, x_{n+1}^k, \dots]$, with $n \geq n_k$, is convergent, where $\{n_k\}$ is a sequence defined in the statement of Lemma 50.3. Denote the limit of such an infinite exponential by $e^{1+S_n^k}$. Then (50.25) will follow if we can show that

$$S_n^k = T_n^k, \tag{50.26}$$

for all $n \geq n'_k \geq n_k$.

To this end, we define, for integers $k \geq 0$ and $n \geq n_k$, the numbers t_n^k by

$$t_n^k := T_n^k - S_n^k. \tag{50.27}$$

We will deduce (50.26) from the three inequalities,

$$S_n^k > S_n^\ell > 0, \qquad k > \ell;\, n \geq \sup(n_k, n_\ell), \tag{50.28}$$

$$t_n^k \geq 0, \tag{50.29}$$

and

$$t_m^k \geq t_n^k \left(\frac{L_k(m-1)}{L_k(n)}\right)^{t_n^k/2} \ell_k(m-1), \qquad m > n \geq n'_k, \tag{50.30}$$

where n'_k is sufficiently large, where in the case $k = 0$, $L_0(x) =: x$. Indeed, assume that (50.26) fails for $k = 0$ and some $n \geq n'_0$. Then, by (50.29), $t_n^0 > 0$, and so, by (50.30) and (50.14), we find that

$$t_m^0 \geq t_n^0 \left(\frac{m-1}{n}\right)^{t_n^0/2} \ell_0(m-1) > \ell_0(m-1) =: T_m^0,$$

for some $m > n$, where m is sufficiently large in terms of t_n^0. But, by (50.27), this implies that $S_m^0 < 0$, which contradicts (50.28). Thus, (50.26) is valid with $k = 0$ for all $n \geq n'_0$. Proceeding by induction on k, assume that (50.26) holds up to $k - 1$. Assume, to the contrary, that (50.26) fails to hold for some $k > 0$ and $n \geq n'_k$. By the same argument as used above, we find that

$$t_m^k > \ell_k(m-1),$$

for some $m > n$ that is sufficiently large in terms of t_n^k. This, together with (50.27) and (50.14) shows that

$$S_m^k = T_m^k - t_m^k < T_m^{k-1} = S_m^{k-1}, \tag{50.31}$$

by the inductive hypothesis, provided that $m \geq n'_{k-1}$, which we may assume. But since (50.31) contradicts (50.28), we conclude that (50.26) and therefore (50.25) hold. Thus, it remains to prove (50.28)–(50.30).

To that end, assuming that $\{n_k\}$ is increasing, as we may, we find that, for $k > \ell$ and $n \geq n_k \geq n_\ell$,

$$X_n^k > X_n^\ell > 0,$$

and so

$$x_n^k > x_n^\ell > e^{1/e}, \tag{50.32}$$

by (50.17), (50.15), and (50.24). Recall that Euler [1], [2] showed that the infinite exponential with constant exponents $e^{1/e}$ converges to e. This fact, together with (50.32), yields (50.28).

To prove (50.29), we first observe that, for $m > n \geq n_k$,

$$[x_n^k, x_{n+1}^k, \dots, x_m^k] < [x_n^k, x_{n+1}^k, \dots, x_m^k, e^{1+T_{m+1}^k}] = e^{1+T_n^k},$$

by (50.24) and (50.16). Hence, $S_n^k \leq T_n^k$, and so (50.29) holds by the definition (50.26) of t_n^k.

For the proof of (50.30), we first observe that, by the definition of S_n^k and (50.24),

$$e^{1+S_n^k} = \left[x_n^k, e^{1+S_{n+1}^k}\right] = e^{(1+X_n^k)e^{S_{n+1}^k}}.$$

Hence, S_n^k satisfies (50.16) with T_n^k replaced by S_n^k. We now fix k and write S_n, T_n, and t_n for S_n^k, T_n^k, and t_n^k, respectively. From our last observation it follows that

$$(1 + S_n)e^{-S_{n+1}} = (1 + T_n)e^{-T_{n+1}}.$$

Substituting $S_m = T_m - t_m$ and $m = n, n+1$ into the last identity, we find that

$$\frac{t_n}{1 + T_n} = 1 - e^{-t_{n+1}}. \tag{50.33}$$

By (50.29), (50.28), (50.14), and (50.11),

$$0 \leq t_n \leq T_n \ll_k \frac{1}{n}. \tag{50.34}$$

Hence,

$$1 - e^{-t_{n+1}} = t_{n+1} \sum_{i=1}^\infty \frac{1}{i!}(-t_{n+1})^{i-1} < t_{n+1} \sum_{i=1}^\infty \left(-\frac{t_{n+1}}{2}\right)^{i-1} = \frac{t_{n+1}}{1 + t_{n+1}/2},$$

provided that $n \geq n'_k$, where n'_k is sufficiently large. Using this bound on the right side of (50.33), we deduce that

$$t_{n+1} > t_n \frac{1 + t_{n+1}/2}{1 + T_n}.$$

Hence, for any integers $m > n \geq n'_k$,

$$t_m > t_n \prod_{i=n}^{m-1} \frac{1 + t_{i+1}/2}{1 + T_i}. \tag{50.35}$$

We use (50.35) twice. First, by (50.35) and (50.34),

$$t_m > t_n \prod_{i=n}^{m-1} \frac{1}{1+T_i} = t_n \exp\left(\sum_{i=n}^{m-1} \log \frac{1}{1+T_i}\right) > t_n \exp\left(-\sum_{i=n}^{m-1} T_i\right),$$

for $m > n \geq n'_k$, where n'_k is sufficiently large. Now, by (50.14) and (50.11),

$$\begin{aligned} \sum_{i=n}^{m-1} T_i &= \sum_{i=n}^{m-1} \sum_{j=0}^{k} \ell_j(i-1) < \sum_{j=0}^{k} \int_{n-2}^{m-2} \ell_j(x)\,dx \\ &= \sum_{j=0}^{k} \big(L_{j+1}(m-2) - L_{j+1}(n-2)\big) < \sum_{j=0}^{k} L_{j+1}(m-1). \end{aligned} \tag{50.36}$$

Hence, by (50.11),

$$\begin{aligned} t_m > t_n \exp\left(-\sum_{j=0}^{k} L_{j+1}(m-1)\right) &= t_n \frac{1}{(m-1)L_1(m-1)\cdots L_k(m-1)} \\ &= t_n \ell_k(m-1), \end{aligned} \tag{50.37}$$

for any integers $m > n \geq n'_k$. We now reiterate the argument above but this time using (50.37) instead of (50.29) on the right side of (50.35). Employing also (50.36), (50.34), and the inequalities $t_n \ell_k(i)/2 < T_i/2 \ll_k 1/i$, we find that

$$\begin{aligned} t_m &> t_n \prod_{i=n}^{m-1} \frac{1+t_n\ell_k(i)/2}{1+T_i} = t_n \exp\left(\sum_{i=n}^{m-1} \log\left(\frac{1+t_n\ell_k(i)/2}{1+T_i}\right)\right) \\ &> t_n \exp\left(\sum_{i=n}^{m-1} \left(\tfrac{1}{2} t_n \ell_k(i) - T_i\right)\right) \\ &> t_n \exp\left(\tfrac{1}{2} t_n (L_{k+1}(m-1) - L_{k+1}(n)) - \sum_{j=0}^{k} L_{j+1}(m-1)\right) \\ &= t_n \left(\frac{L_k(m-1)}{L_k(n)}\right)^{t_n/2} \ell_k(m-1), \end{aligned}$$

where the second inequality above holds for $m > n \geq n'_k$, where n'_k is sufficiently large. Thus, (50.30) is established, and the proof of Lemma 50.4 is complete.

Proof of Theorem 50.1. We assume at the outset that $a_n \neq 1$, for each $n \geq 1$, for otherwise both $[a_1, a_2, \ldots]$ and $[\hat{a}_1, \hat{a}_2, \ldots]$ converge trivially. Fix a positive integer n and, for any complex number z, set

$$f(z) := \frac{d}{dz}[a_1, a_2, \ldots, a_n, z] \tag{50.38}$$

and

$$g(z) := \frac{d}{dz}[\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n, z]. \tag{50.39}$$

For each $m > n$,

$$\begin{aligned}[a_1, a_2, \ldots, a_m] - [a_1, a_2, \ldots, a_n] &= [a_1, a_2, \ldots, a_n, [a_{n+1}, a_{n+2}, \ldots, a_m]] \\ &\quad - [a_1, a_2, \ldots, a_n, 1] \\ &= \int_1^{[a_{n+1}, a_{n+2}, \ldots, a_m]} f(z)\, dz. \end{aligned} \tag{50.40}$$

Setting

$$u := [a_{n+1}, a_{n+2}, \ldots, a_m] \tag{50.41}$$

and

$$w := [\hat{a}_{n+1}, \hat{a}_{n+2}, \ldots, \hat{a}_m], \tag{50.42}$$

we find, upon estimating the right side of (50.40), that

$$\begin{aligned}|[a_1, a_2, \ldots, a_m] - [a_1, a_2, \ldots, a_n]| &= \left|\int_1^u f(z)\, dz\right| \\ &= \left|\int_0^1 f(1 + (u-1)t) d(1 + (u-1)t)\right| \\ &\leq |u - 1| \int_0^1 |f(1 + (u-1)t)|\, dt. \end{aligned} \tag{50.43}$$

Thus, by (50.38), (50.6), (50.39), and (50.8),

$$f(z) = b_1[a_1, a_2, \ldots, a_n, z] \frac{d}{dz}[a_1, a_2, \ldots, a_n, z] = \prod_{k=1}^{n} b_k[a_k, a_{k+1}, \ldots, a_n, z]$$

and

$$g(z) = \prod_{k=1}^{n} |b_k| [\hat{a}_k, \hat{a}_{k+1}, \ldots, \hat{a}_n, z].$$

Hence, by (50.6), (50.8), (50.41), and (50.42), we obtain the inequalities

$$\begin{aligned}|f(1 - t + ut)| &= \prod_{k=1}^{n} |b_k[a_k, a_{k+1}, \ldots, a_n, (1 - t + ut)]| \\ &\leq \prod_{k=1}^{n} |b_k| [\hat{a}_k, \hat{a}_{k+1}, \ldots, \hat{a}_n, (1 - t + |u|t)] \\ &\leq g(1 - t + wt), \end{aligned} \tag{50.44}$$

which is valid for $0 \le t \le 1$. Applying (50.44) to the right side of (50.43), we find that

$$\begin{aligned}
|[a_1, a_2, \ldots, a_m] - [a_1, a_2, \ldots, a_n]| &\le |u-1| \int_0^1 g(1+(w-1)t)\,dt \\
&= \frac{|u-1|}{w-1} \int_0^1 g(1+(w-1)t)d(1+(w-1)t) \\
&= \frac{|u-1|}{w-1} \int_1^w g(z)\,dz \\
&= \frac{|u-1|}{w-1}([\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_m] - [\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n]),
\end{aligned} \tag{50.45}$$

by (50.39) and (50.42). Observe that $w > 1$, since $a_n \ne 1$, and so $\hat{a}_n > 1$. Moreover,

$$\begin{aligned}
|u-1| &= \left| e^{b_{n+1}[a_{n+2}, a_{n+3}, \ldots, a_m]} - 1 \right| \\
&= \left| \sum_{k=1}^{\infty} \frac{1}{k!}(b_{n+1}[a_{n+2}, a_{n+3}, \ldots, a_m])^k \right| \\
&\le \sum_{k=1}^{\infty} \frac{1}{k!}(|b_{n+1}|[\hat{a}_{n+2}, \hat{a}_{n+3}, \ldots, \hat{a}_m])^k = w-1.
\end{aligned} \tag{50.46}$$

Theorem 50.1 now follows from (50.45), (50.46), and the Cauchy criterion for convergence.

Proof of Entry 50a. By Theorem 50.1, it suffices to consider real exponents $a_n \ge 1$. In such a case, $[a_1, a_2, \ldots, a_n]$ is monotonically increasing, and so it suffices to show that it is bounded. Define the sequence $\{c_n\} = \{c_n^{k_0+1}\}$ by

$$c_n = \exp\left(\frac{1 + C_n^{k_0+1}}{e}\right), \qquad n \ge n_{k_0+1},$$

where $C_n^{k_0+1}$ and n_{k_0+1} are defined in the statement of Lemma 50.3. Setting $C_n = C_n^{k_0+1}$, we find that, by (50.15), (50.11), and (50.9),

$$\begin{aligned}
1 + \log\log c_n = \log(1 + C_n) &= C_n - \frac{1}{2}C_n^2 + O(C_n^3) \\
&> \tfrac{1}{2}\sum_{j=0}^{k_0} \ell_j^2(n) \ge 1 + \log\log a_n,
\end{aligned}$$

for $n \ge n_0$, sufficiently large in terms of k_0, as we may assume. Therefore, for $n \ge n_0$, we have $a_n \le c_n$, and so

$$[a_{n_0}, a_{n_0+1}, \ldots, a_n] \le [c_{n_0}, c_{n_0+1}, \ldots, c_n].$$

Thus, it suffices to show that the infinite exponential $[c_{n_0}, c_{n_0+1}, \ldots]$ converges. By Lemma 50.2, this, in turn, is equivalent to the existence of a sequence S_n, $n =$

$n_0, n_0+1, \ldots$, such that $S_n \geq -1$ and

$$1 + S_n \geq (1 + C_n)e^{S_{n+1}}. \tag{50.47}$$

But, by Lemma 50.3,

$$1 + C_n = 1 + C_n^{k_0+1} < 1 + X_n^{k_0+1} = (1 + T_n^{k_0+1})e^{-T_{n+1}^{k_0+1}}.$$

Hence, (50.47) is satisfied with $S_n = T_n^{k_0+1}$, $n \geq n_0$. This completes the proof of Entry 50a.

Proof of Entry 50b. We proceed by contradiction. Suppose then that the infinite exponential $[a_1, a_2, \ldots]$ is convergent. Then, since $a_n > 1$, so is $[a_n, a_{n+1}, \ldots]$ convergent for any $n \geq 1$. Denote the limit of the latter infinite exponential by e^{1+S_n}. Define also a sequence $\{A_n\}$ by

$$a_n = \exp\left(\frac{1 + A_n}{e}\right). \tag{50.48}$$

Then

$$e^{1+S_n} = \left[a_n, e^{1+S_{n+1}}\right] = e^{(1+A_n)e^{S_{n+1}}}. \tag{50.49}$$

In the remainder of the proof, n always denotes an integer such that $n \geq n_0$. For such n, it follows immediately from (50.10) that $A_n > 0$, since $a_n > e^{1/e}$. Moreover, by (50.48), (50.10), (50.11), (50.15), and (50.17),

$$A_n \geq \log(1 + A_n) = 1 + \log\log a_n \geq C_n^{k_0} + \tfrac{1}{2}\epsilon\ell_{k_0}^2(n) > C_n^{k_0+1} > X_n^{k_0}, \tag{50.50}$$

for $n \geq n_0$, where n_0, which depends on k and ϵ, is sufficiently large. Thus,

$$a_n > x_n^{k_0},$$

where $x_n^{k_0}$ is defined by (50.24). Therefore, by the definition of S_n and Lemma 50.4, we find that

$$S_n > T_n^{k_0}. \tag{50.51}$$

For brevity, set $T_n = T_n^{k_0}$, $X_n = X_n^{k_0}$,

$$B_n = A_n - X_n, \qquad \text{and} \qquad R_n = S_n - T_n > 0, \tag{50.52}$$

by (50.51). By (50.50), (50.23), (50.15), and (50.11),

$$B_n \geq C_n^{k_0} + \tfrac{1}{2}\epsilon\ell_{k_0}^2(n) - X_n^{k_0} = \tfrac{1}{2}\epsilon\ell_{k_0}^2(n) - \frac{1}{3n^3} + O_k\left(\frac{1}{n^3 \log n}\right) > \tfrac{1}{2}\epsilon\ell_{k_0}^2(n), \tag{50.53}$$

where $n \geq n_0$, and where n_0 is sufficiently large.

The remainder of the argument in Bachman's paper [1] is incorrect. We are very grateful to A. Hildebrand for supplying the following elegant argument to complete the proof. We begin with a lemma.

Lemma 50.5. *For $M, N \geq n_0$,*

$$\sum_{n=N+1}^{M} T_n^k \geq \log\left(\frac{\ell_k(N)}{\ell_k(M)}\right).$$

Proof. From (50.11), we observe that

$$\ell_j(x) = \frac{L'_j(x)}{L_j(x)}.$$

Thus,

$$S := \sum_{n=N+1}^{M} T_n^k = \sum_{n=N+1}^{M} \sum_{j=0}^{k} \ell_j(n-1) = \sum_{j=0}^{k} \sum_{n=N+1}^{M} \frac{L'_j(n-1)}{L_j(n-1)}.$$

Since $\ell_j(x) = L'_j(x)/L_j(x)$ is decreasing for $x \geq n_0$,

$$\frac{L'_j(n-1)}{L_j(n-1)} \geq \int_{n-1}^{n} \frac{L'_j(x)}{L_j(x)} dx.$$

Thus,

$$S \geq \sum_{j=0}^{k} \int_{N}^{M} \frac{L'_j(x)}{L_j(x)} dx = \sum_{j=0}^{k} \log\left(\frac{L_j(M)}{L_j(N)}\right) = \log\left(\frac{\ell_k(N)}{\ell_k(M)}\right),$$

and so the proof is complete.

Recall from (50.18) and (50.49) that

$$1 + X_n = (1 + T_n)e^{-T_{n+1}} \qquad \text{and} \qquad 1 + A_n = (1 + S_n)e^{-S_{n+1}}. \tag{50.54}$$

Thus, from (50.52) and (50.54),

$$1 + \frac{B_n}{1 + X_n} = \frac{1 + A_n}{1 + X_n} = \frac{1 + S_n}{1 + T_n} e^{-R_{n+1}} = \left(1 + \frac{R_n}{1 + T_n}\right) e^{-R_{n+1}}.$$

Hence,

$$R_{n+1} = \log\left(1 + \frac{R_n}{1 + T_n}\right) - \log\left(1 + \frac{B_n}{1 + X_n}\right). \tag{50.55}$$

Since $R_{n+1} \geq 0$ by (50.52), equality (50.55) first implies that

$$\frac{R_n}{1 + T_n} \geq \frac{B_n}{1 + X_n},$$

and then, with the use of the inequality,

$$\log(1 + x + h) - \log(1 + x) \leq \frac{h}{1 + x}, \qquad x, h \geq 0,$$

secondly implies that

$$\begin{aligned} R_{n+1} &\leq \left(\frac{R_n}{1+T_n} - \frac{B_n}{1+X_n} \right) \left(1 + \frac{B_n}{1+X_n} \right)^{-1} \\ &= \frac{1}{1+X_n+B_n} \left(\frac{1+X_n}{1+T_n} R_n - B_n \right) \\ &\leq \frac{1+X_n}{1+T_n} R_n - B_n = e^{-T_{n+1}} R_n - B_n, \end{aligned}$$

where, in the last inequality, we used the fact that $R_{n+1} \geq 0$ from (50.52), and where, in the last equality, we used (50.54). Thus,

$$R_n \geq R_{n+1} e^{T_{n+1}} + B_n e^{T_{n+1}}.$$

Iterating this inequality, we find that

$$\begin{aligned} R_n &\geq R_{n+2} e^{T_{n+1}+T_{n+2}} + B_{n+1} e^{T_{n+1}+T_{n+2}} + B_n e^{T_{n+1}} \\ &\geq \cdots \\ &\geq R_{n+m} e^{T_{n+1}+\cdots+T_{n+m}} + \sum_{j=1}^{m} B_{n+j-1} e^{T_{n+1}+\cdots+T_{n+j}} \\ &\geq \sum_{j=1}^{m} B_{n+j-1} e^{T_{n+1}+\cdots+T_{n+j}}. \end{aligned} \tag{50.56}$$

By Lemma 50.5, for $n \geq n_0$,

$$e^{T_{n+1}+\cdots+T_{n+j}} \geq \frac{\ell_k(n)}{\ell_k(n+j)} \gg \frac{1}{\ell_k(n+j-1)}. \tag{50.57}$$

Hence, using (50.53) and (50.57) in (50.56), we conclude that

$$R_n \gg \sum_{j=1}^{m} \ell_k(n+j-1).$$

Since $m > 0$ is arbitrary and since $\sum_{j=1}^{\infty} \ell_k(n+j-1)$ diverges, we have reached the desired contradiction, and so the proof is complete.

38

Approximations and Asymptotic Expansions

One of the primary areas to which Ramanujan made fundamental contributions, but for which he received no recognition until recent times, is asymptotic analysis. Asymptotic formulas, both general and specific, can be found in several places in his second notebook, but perhaps the largest concentration lies in Chapter 13. Several contributions pertain to hypergeometric functions, and an excellent survey of several of these results has been made by R. J. Evans [1]. The unorganized pages in the second and third notebooks also contain many beautiful theorems in asymptotic analysis. This chapter is devoted to proving these theorems and a few approximations as well.

On pages 270–273 of the second notebook, Ramanujan examines some related functions that can be considered as hybrids of the Riemann zeta-function and hypergeometric functions. Some of these results were established in a paper with Evans [2]. Entries 2–8 contain accounts of Ramanujan's findings described on these pages.

In Entries 12 and 13, Ramanujan determines the asymptotic behavior of some multivariate exponential series. These results are in the spirit of several theorems that can be found in Chapter 15 (Part II [2, pp. 303–314]).

In Entry 16, Ramanujan derives the asymptotic expansion of

$$2\sum_{n=0}^{\infty}(-1)^n\left(\frac{1-t}{1+t}\right)^{n(n+1)},$$

as t tends to $0+$. A complete description of the asymptotic expansion of this false theta-function involves Euler numbers. All of the coefficients in Ramanujan's asymptotic expansion appear to be integers, and this was recently proved by W. Galway [1] using a formula from Ramanujan's lost notebook [11].

Entry 17, which can probably be generalized, gives the asymptotic expansion of a function which is a hybrid of a theta-function and a hypergeometric function. This is also related to the aforementioned material in Chapter 15.

Entries 18 and 19 present families of approximations to certain finite sums and certain infinite series, respectively. These approximations arise from the orthogonality, respectively, of the discrete Hahn and discrete Charlier polynomials. The results are quite remarkable, for no one had previously realized that Ramanujan must have, in essence, discovered these orthogonal polynomials.

Entry 23, which can also undoubtedly be generalized, provides an asymptotic formula for a certain Lambert series as x tends to 0.

We quote Ramanujan in the first entry of this chapter.

Entry 1 (p. 265).

$$\sum_{k=2}^{x} \frac{1}{k \log k} = 0.1015314 + \log\log(x^2 + x + \theta); \tag{1.1}$$

$$x = \infty, \qquad \theta = \tfrac{1}{3}; \tag{1.2}$$

$$x = 1, \qquad \theta = 0.46811; \tag{1.3}$$

130489 quadrillion terms to get the value of 5. (1.4)

Proof. Applying the Euler–Maclaurin summation formula, (0.5) of Chapter 37, to $f(x) = 1/(x \log x)$, we find that, for some constant c,

$$\sum_{k=2}^{x} \frac{1}{k \log k} = c + \log\log x + \frac{1}{2x \log x} - \frac{1}{12x^2 \log x} + O\left(\frac{1}{x^2 \log^2 x}\right), \tag{1.5}$$

as x tends to ∞. From Entry 14 of Chapter 7 (Part I [1, p. 166]),

$$\lim_{x\to\infty} \left(\sum_{k=2}^{x} \frac{1}{k \log k} - \log\log x\right) = 0.7946786, \tag{1.6}$$

i.e., the constant c in (1.5) equals 0.7946786. On the other hand, observe that

$$\begin{aligned}
\log\left(x^2 + x + \theta\right) &= 2\log x + \log\left(1 + \frac{1}{x} + \frac{\theta}{x^2}\right) \\
&= 2\log x + \frac{1}{x} + \frac{\theta - \frac{1}{2}}{x^2} + O\left(\frac{1}{x^3}\right) \\
&= (2\log x)\left(1 + \frac{1}{2x\log x} + \frac{\theta - \frac{1}{2}}{2x^2 \log x} + O\left(\frac{1}{x^3 \log x}\right)\right),
\end{aligned}$$

as x tends to ∞. Thus,

$$\log\log(x^2 + x + \theta) = \log 2 + \log\log x + \frac{1}{2x\log x} + \frac{\theta - \frac{1}{2}}{2x^2 \log x} + O\left(\frac{1}{x^2 \log^2 x}\right). \tag{1.7}$$

Now $\log 2 = 0.6931472$, and $0.1015314 + 0.6931472 = 0.7946786$. Hence, by (1.6) and (1.7), the constant terms in (1.1) and (1.5) are in agreement. We also see that if we set $\theta = \frac{1}{3}$ in (1.7), then the first three nonconstant terms in (1.5) and (1.7) agree. This then proves Ramanujan's assertion (1.1).

When $x = 1$ and $\theta = 0.46811$, the right side of (1.1) should approximately equal 0, if Ramanujan's claim is correct. Since

$$\log\log(2.46811) = -0.1015315,$$

the right side of (1.1) to seven decimal places is -0.0000001, which justifies Ramanujan's claim in (1.3).

The assertion (1.4) is recorded in a different color or different shade of ink in some "empty space" further down the page. With an American interpretation, 130489 quadrillion $= 1.30489 \times 10^{20}$. However, in the United Kingdom and areas under its former dominion, one quadrillion equals 10^{24} instead of 10^{15}. Therefore, Ramanujan claimed that 1.30489×10^{29} terms will give a sum exceeding 5. This is in agreement with work of Hardy [2, p. 61] and R. P. Boas Jr. [1, p. 244], [2, p. 156] who showed that 1.3×10^{29} terms are required to exceed a sum of 5. See also Part I [1, p. 328].

We are grateful to R. P. Brent for showing us the advantage of $\log\log(x^2+x+\theta)$ over $\log\log(x^2)$ through his analysis in (1.5) and (1.7).

Some of Entries 2–8 below, recorded on pages 270–273, are not approximations or asymptotic estimates, but since all the results are connected and asymptotic expansions are the focus, we prove and discuss all of them here.

Entry 2 (formula (1), p. 270). *For $p > 0$,*

$$\sum_{k=1}^{\infty} \frac{k^{k-2}}{(p+k)^k} = \frac{1-e^{-p}}{p} + \sum_{k=1}^{\infty} \frac{k^{k-2}e^{-(p+k)}}{(k-1)!} \sum_{j=1}^{k} \frac{(-1)^{j-1}(-k+1)_{j-1}}{(p+k)^j}. \tag{2.1}$$

Proof. Let S denote the double sum on the right side of (2.1). Then

$$\begin{aligned}
S &= \sum_{k=1}^{\infty} \frac{k^{k-2}}{(k-1)!} \sum_{n=0}^{\infty} \frac{(-1)^n (p+k)^n}{n!} \sum_{j=1}^{k} \frac{(k-1)!}{(k-j)!\,(p+k)^j} \\
&= \sum_{k=1}^{\infty} k^{k-2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{j=1}^{k} \frac{(p+k)^{n-j}}{(k-j)!} \\
&= \sum_{k=1}^{\infty} k^{k-2} \sum_{j=1}^{k} \frac{1}{(k-j)!} \sum_{m=-j}^{\infty} \frac{(-1)^{m+j}(p+k)^m}{(m+j)!} \\
&= \sum_{k=1}^{\infty} k^{k-2} \sum_{j=1}^{k} \frac{1}{(k-j)!} \left(\sum_{m=1}^{j} \frac{(-1)^{m+j}(p+k)^{-m}}{(j-m)!} + \sum_{m=0}^{\infty} \frac{(-1)^{m+j}(p+k)^m}{(m+j)!} \right).
\end{aligned} \tag{2.2}$$

We examine the contribution of the double finite sum on j and m. Inverting the order of summation, we find that

$$\begin{aligned}
\sum_{j=1}^{k}\frac{1}{(k-j)!}\sum_{m=1}^{j}\frac{(-1)^{m+j}(p+k)^{-m}}{(j-m)!} &= \sum_{m=1}^{k}(p+k)^{-m}\sum_{j=m}^{k}\frac{(-1)^{m+j}}{(k-j)!\,(j-m)!}\\
&= \sum_{m=1}^{k}(p+k)^{-m}\sum_{n=0}^{k-m}\frac{(-1)^n}{(k-m-n)!\,n!}\\
&= (p+k)^{-k}+\sum_{m=1}^{k-1}\frac{(p+k)^{-m}}{(k-m)!}\sum_{n=0}^{k-m}(-1)^n\binom{k-m}{n}\\
&= (p+k)^{-k}. \qquad (2.3)
\end{aligned}$$

Hence, substituting (2.3) into (2.2), we have shown that

$$S=\sum_{k=1}^{\infty}\frac{k^{k-2}}{(p+k)^k}+\sum_{k=1}^{\infty}k^{k-2}\sum_{j=1}^{k}\frac{1}{(k-j)!}\sum_{m=0}^{\infty}\frac{(-1)^{m+j}(p+k)^m}{(m+j)!}. \qquad (2.4)$$

Let A_k denote the inner double sum in (2.4). Inverting the order of summation and employing Vandermonde's theorem (Bailey [1, p. 3]), we deduce that

$$\begin{aligned}
A_k &= \sum_{m=0}^{\infty}\sum_{j=1}^{k}\frac{(-1)^{m+j}(p+k)^m}{(m+j)!\,(k-j)!}\\
&= \sum_{m=0}^{\infty}\frac{(-1)^m(p+k)^m}{k!}\sum_{j=1}^{k}\frac{(-k)_j}{(m+j)!}\\
&= \sum_{m=0}^{\infty}\frac{(-1)^m(p+k)^m}{k!\,m!}\left(\sum_{j=0}^{k}\frac{(-k)_j}{(m+1)_j}-1\right)\\
&= \sum_{m=0}^{\infty}\frac{(-1)^m(p+k)^m}{k!\,m!}\left(\frac{(m)_k}{(m+1)_k}-1\right)\\
&= -\frac{k}{k!}\sum_{m=0}^{\infty}\frac{(-1)^m(p+k)^m}{m!\,(m+k)}.
\end{aligned}$$

Thus, in view of (2.1) and (2.4), it remains to show that

$$\begin{aligned}
\frac{1-e^{-p}}{p} &= \sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{k^{k-1}(-1)^m(p+k)^m}{k!\,m!\,(m+k)}\\
&= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\sum_{m=n}^{\infty}\frac{k^{k-1+m-n}(-1)^m p^n}{k!\,n!\,(m-n)!\,(m+k)}, \qquad (2.5)
\end{aligned}$$

where we have employed the binomial theorem and inverted the order of summation. Comparing the coefficients of p^n on both sides of (2.5), we see that it suffices to show that

$$\frac{(-1)^n}{(n+1)!} = \sum_{k=1}^{\infty}\sum_{m=n}^{\infty} \frac{k^{k-1+m-n}(-1)^m}{k!\,n!\,(m-n)!\,(m+k)}, \qquad n \geq 0. \tag{2.6}$$

Multiplying both sides of (2.6) by $(-1)^n n!$, we see that (2.6) is equivalent to

$$\begin{aligned}\frac{1}{n+1} &= \sum_{k=1}^{\infty}\sum_{m=n}^{\infty} \frac{k^{k-1+m-n}(-1)^{m-n}}{k!\,(m-n)!\,(m+k)}\\ &= \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \frac{k^{k-1+j}(-1)^j}{k!\,j!\,(j+k+n)}\\ &= \sum_{k=1}^{\infty}\sum_{j=k}^{\infty} \frac{k^{j-1}(-1)^{k-j}}{k!\,(j-k)!\,(j+n)}\\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{j} \frac{k^{j-1}(-1)^{k-j}}{k!\,(j-k)!\,(j+n)}\\ &= \sum_{j=1}^{\infty} \frac{(-1)^j}{j!(j+n)} \sum_{k=1}^{j} \binom{j}{k}(-1)^k k^{j-1}\\ &= \frac{1}{n+1} + \sum_{j=2}^{\infty} \frac{(-1)^j}{j!\,(j+n)} \sum_{k=1}^{j} \binom{j}{k}(-1)^k k^{j-1}.\end{aligned} \tag{2.7}$$

It is an easy and well-known consequence of the binomial theorem that, for each $j \geq 2$, the inner sum on the far right side of (2.7) is equal to 0 (Gradshteyn and Ryzhik [1, p. 5, formula 0.154, no. 3]). Hence, the right side of (2.7) reduces to $1/(n+1)$, as desired. Thus, the proof of (2.6) and, consequently, of (2.1) is complete.

Ramanujan's formulation of Entry 2 is slightly imprecise, because he only offers the first three terms of the sum on k on the right side of (2.1). Note that the first three values of k^{k-2} are 1, 1, and 3.

Ramanujan next expands the inner sum on the right side of (2.1) in powers of k with coefficients that are powers of $1/p$. Perhaps Ramanujan was striving to obtain an asymptotic expansion for the left side as p tends to ∞. This procedure does not lead to the desired end, because the contribution of k^m, $m \geq 1$, to the sum on m yields a divergent series. At any rate, we next establish Ramanujan's expansion of the inner sum.

Entry 3 (p. 270). *Let $1 \leq k < |p|$, where k is an integer and p is any complex number. If m and n denote integers with $m \geq 0$ and $n \geq 1$, define*

$$\alpha_n(m,k) := \sum_{r=0}^{k+m-n} (-1)^{m-r}\binom{n-1}{m-r} s(r+n-m, r+1), \tag{3.1}$$

where $s(a,b)$, $a,b \geq 1$, denote the Stirling numbers of the first kind. Then

$$T := T(p,k) := \sum_{j=1}^{k} \frac{(-1)^{j-1}(1-k)_{j-1}}{(p+k)^j} = \sum_{m=0}^{\infty} k^m \sum_{n=m+1}^{m+k} \frac{\alpha_n(m,k)}{p^n}. \tag{3.2}$$

Moreover, if $n \leq k$ and $m+1 \leq n \leq 2m$, then $\alpha_n(m,k) = 0$.

Proof. The Stirling numbers $s(j,r)$, $j,r \geq 1$, of the first kind may be defined by (L. Comtet [1, p. 213, eq. [5e]])

$$(k-1)(k-2)\cdots(k-j+1) = \sum_{r=1}^{j} s(j,r)k^{r-1}. \tag{3.3}$$

Thus, using (3.3) and inverting the order of summation, we find that

$$T = \sum_{r=1}^{k}\sum_{j=r}^{k} \frac{s(j,r)k^{r-1}}{(p+k)^j}.$$

By the generalized binomial theorem, for $k < |p|$,

$$(p+k)^{-j} = \frac{1}{p^j}\sum_{\ell=0}^{\infty}\binom{\ell+j-1}{\ell}\left(-\frac{k}{p}\right)^{\ell}.$$

Thus, letting $\ell = m+1-r$ and $i = j-r$, we find that

$$\begin{aligned}
T &= \sum_{r=1}^{k}\sum_{j=r}^{k}\sum_{\ell=0}^{\infty}(-1)^{\ell}s(j,r)\frac{k^{r+\ell-1}}{p^{\ell+j}}\binom{\ell+j-1}{\ell} \\
&= \sum_{m=0}^{\infty}k^m\sum_{r=1}^{k}\sum_{j=r}^{k}\frac{(-1)^{m+1-r}s(j,r)}{p^{m+1+j-r}}\binom{m+j-r}{m+1-r} \\
&= \sum_{m=0}^{\infty}k^m\sum_{r=1}^{k}\sum_{i=0}^{k-r}\frac{(-1)^{m+1-r}s(r+i,r)}{p^{m+1+i}}\binom{m+i}{m+1-r} \\
&= \sum_{m=0}^{\infty}k^m\sum_{i=0}^{k-1}\frac{1}{p^{m+i+1}}\sum_{r=1}^{k-i}(-1)^{m+1-r}s(r+i,r)\binom{m+i}{m+1-r} \\
&= \sum_{m=0}^{\infty}k^m\sum_{i=0}^{k-1}\frac{1}{p^{m+i+1}}\sum_{r=0}^{k-i-1}(-1)^{m-r}s(r+i+1,r+1)\binom{m+i}{m-r} && (3.4) \\
&= \sum_{m=0}^{\infty}k^m\sum_{n=m+1}^{m+k}\frac{1}{p^n}\sum_{r=0}^{k+m-n}(-1)^{m-r}s(r+n-m,r+1)\binom{n-1}{m-r}, && (3.5)
\end{aligned}$$

where we have set $n = m+i+1$ in the last step. Hence, the proof of (3.2) is complete.

Lastly, from (3.5) and (3.1), we see that it remains to show that the inner sum on r vanishes if $n \leq k$ and $m+1 \leq n \leq 2m$. Alternatively, from (3.4), we will show that

$$\sum_{r=0}^{m}(-1)^{m-r}s(r+i+1,r+1)\binom{m+i}{m-r}=0, \tag{3.6}$$

for each i such that $0 \le i \le m-1$ and $m \le k-i-1$.

From Comtet's book [1, pp. 227–228], $s(r+i+1,r+1)$ is the coefficient of t^i in the power series expansion of

$$\left(\frac{t}{e^t-1}\right)^{r+i+1}\frac{(r+i)!}{r!}.$$

Therefore,

$$(-1)^{m-r}s(r+i+1,r+1)\binom{m+i}{m-r}$$

is the coefficient of t^i in the expansion of

$$\left(\frac{t}{e^t-1}\right)^{r+i+1}(-1)^{m-r}\binom{m}{r}\frac{(m+i)!}{m!}.$$

Thus, the sum in (3.6) is the coefficient of t^i in the expansion of

$$\begin{aligned}
&\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}\left(\frac{t}{e^t-1}\right)^{r}\left\{\frac{(m+i)!}{m!}\left(\frac{t}{e^t-1}\right)^{i+1}\right\}\\
&=\frac{(m+i)!}{m!}\left(\frac{t}{e^t-1}\right)^{i+1}\left(\frac{t}{e^t-1}-1\right)^{m}\\
&=\sum_{i=m}^{\infty}a_i t^i,
\end{aligned}$$

for certain numbers a_i, $i \ge m$. Thus, for $0 \le i \le m-1$, the coefficient of t^i is indeed equal to 0. Thus, (3.6) has been established, and the proof of Entry 3 is complete.

Entry 4 (Formula (2), p. 271). *For each $p \ge -1$, define $\theta = \theta_p$ by*

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{k^{k-2}}{(p+k)^k} &= \frac{1-e^{-p}}{p}+e^{-p}\left(\frac{1}{p+1}-\frac{1}{(p+1)(p+2)}\right.\\
&\quad+\frac{4}{3(p+1)(p+2)(p+4)}\\
&\quad\left.-\frac{4}{(p+1)(p+2)(p+4)(3p+23+\theta)}\right).
\end{aligned} \tag{4.1}$$

Then $\theta_{-1} = -2.5856$, $\theta_0 = 0.0069$, $\theta_1 = 0.4137$, and $\theta_\infty = \frac{3}{5}$.

Proof. Multiplying both sides of (4.1) by $(p+1)$ and then setting $p=-1$, we find that

$$1 = 0 + e\left(1 - 1 + \frac{4}{3\cdot 3} - \frac{4}{3(20+\theta_{-1})}\right).$$

Thus,

$$\theta_{-1} = \frac{12e}{4e-9} - 20 = -2.585603,$$

which is in agreement with Ramanujan's claim.

Second, letting p tend to 0 in (4.1), we find that

$$\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \left(1 - \frac{1}{2} + \frac{4}{3\cdot 2\cdot 4} - \frac{4}{2\cdot 4(23+\theta_0)}\right).$$

Solving for θ_0, we find that

$$\theta_0 = \frac{3}{10-\pi^2} - 23 = 0.006912,$$

which again is in agreement with Ramanujan.

The case $p=1$ is more challenging. Letting $p=1$ in (4.1), we see that

$$S := \sum_{k=1}^{\infty} \frac{k^{k-2}}{(k+1)^k} = 1 - \frac{1}{e} + \frac{1}{e}\left(\frac{1}{2} - \frac{1}{6} + \frac{2}{45} - \frac{2}{15(26+\theta_1)}\right).$$

Solving for θ_1, we find that

$$\theta_1 = -\frac{6}{45e(S-1)+28} - 26. \tag{4.2}$$

To determine θ_1 to the accuracy demanded by Ramanujan, we need a very precise determination of S.

Write

$$M_k = \frac{1}{k^2}\left(\left(\frac{k}{k+1}\right)^k - \frac{1}{e}\right), \qquad k \geq 1.$$

Then

$$S = \sum_{k=1}^{\infty} M_k + \frac{\pi^2}{6e} = A + B + \frac{\pi^2}{6e}, \tag{4.3}$$

where

$$A = \sum_{k=1}^{10^6} M_k \qquad \text{and} \qquad B = \sum_{k=10^6+1}^{\infty} M_k.$$

We are grateful to W. Root who calculated A and found that

$$A = 0.16410279790586.$$

Hence, from (4.3),

$$S = 0.7692402231818 + B.$$

Since $(k/(k+1))^k - 1/e$ monotonically decreases to 0 and has a value at $k = 10^6$ that is less than $2 \cdot 10^{-7}$, we deduce that

$$B < \sum_{k=10^6+1}^{\infty} \frac{2 \cdot 10^{-7}}{k^2} < \int_{10^6}^{\infty} \frac{2 \cdot 10^{-7}}{x^2} dx = 2 \cdot 10^{-13}.$$

Hence,

$$S = 0.769240223182.$$

Returning to (4.2) and using the value of S calculated above, we find that

$$\theta_1 = 0.413696,$$

which agrees with Ramanujan's calculation.

We have calculated S to more accuracy than needed to establish Ramanujan's claim. However, it seems to us that a calculator or computer is necessary to calculate S to the precision needed to determine θ_1 to the accuracy indicated by Ramanujan. So, we wonder how Ramanujan computed S.

The proof that $\theta_\infty = \frac{3}{5}$ is considerably deeper than the previous calculations. We show later that this result follows from Entry 7.

Entry 5 (Formula (3), p. 271). *Let* $0 < p < a$. *Then*

$$\sum_{n=0}^{\infty} \frac{(a+n)^{n-1}}{(p+a+n)^{n+1}} = \sum_{n=0}^{\infty} \frac{(-p)^n u_n(a)}{n!}, \tag{5.1}$$

where, for $n \geq 0$,

$$u_n(a) = \sum_{k=0}^{\infty} \frac{\Gamma(n+k+1)}{(a+k)^{n+2}\Gamma(k+1)}. \tag{5.2}$$

Furthermore, for $n \geq 1$,

$$\frac{u_{n-1}(a) - u_n(a+1)}{u_n(a) - u_n(a+1)} = \frac{a}{n}. \tag{5.3}$$

Proof. By the generalized binomial theorem, for $|p| < a$,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(a+n)^{n-1}}{(p+a+n)^{n+1}} &= \sum_{n=0}^{\infty} \frac{1}{(a+n)^2\left(1+\dfrac{p}{a+n}\right)^{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{1}{(a+n)^2} \sum_{k=0}^{\infty} \frac{(-1)^k (n+1)_k}{k!} \left(\frac{p}{a+n}\right)^k\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{(-p)^k}{k!} \sum_{n=0}^{\infty} \frac{(n+1)_k}{(a+n)^{k+2}},$$

from which (5.1) follows.

Next,

$$\frac{u_{n-1}(a) - u_n(a+1)}{u_n(a) - u_n(a+1)} = \frac{\displaystyle\sum_{k=0}^{\infty} \frac{(k+1)_{n-1}}{(a+k)^{n+1}} - \sum_{k=1}^{\infty} \frac{(k)_n}{(a+k)^{n+2}}}{\displaystyle\sum_{k=0}^{\infty} \frac{(k+1)_n}{(a+k)^{n+2}} - \sum_{k=1}^{\infty} \frac{(k)_n}{(a+k)^{n+2}}}$$

$$= \frac{\dfrac{(n-1)!}{a^{n+1}} + \displaystyle\sum_{k=1}^{\infty} \frac{(k+1)_{n-1}}{(a+k)^{n+2}} \{(a+k) - k\}}{\dfrac{n!}{a^{n+2}} + \displaystyle\sum_{k=1}^{\infty} \frac{(k+1)_{n-1}}{(a+k)^{n+2}} \{(k+n) - k\}}$$

$$= \frac{\dfrac{(n-1)!}{a^{n+1}} + a\displaystyle\sum_{k=1}^{\infty} \frac{(k+1)_{n-1}}{(a+k)^{n+2}}}{\dfrac{n!}{a^{n+2}} + n\displaystyle\sum_{k=1}^{\infty} \frac{(k+1)_{n-1}}{(a+k)^{n+2}}}$$

$$= \frac{a}{n}.$$

Thus, (5.3) has been established.

Entry 6 (Formula (4), p. 271). *Let $u_n(a)$ be defined by (5.2). Then as a tends to ∞,*

$$\begin{aligned} u_n(a) \sim{} & \frac{1}{(n+1)a} + \frac{1}{2a^2} + \left(\frac{1}{6} + \frac{n}{4}\right)\frac{1}{a^3} + \left(\frac{n}{4} + \frac{n(n-1)}{8}\right)\frac{1}{a^4} \\ & + \left(-\frac{1}{30} + \frac{n}{12} + \frac{n(n-1)}{4} + \frac{n(n-1)(n-2)}{16}\right)\frac{1}{a^5} \\ & + \left(-\frac{n}{12} + \frac{5n(n-1)}{24} + \frac{5n(n-1)(n-2)}{24} + \frac{n(n-1)(n-2)(n-3)}{32}\right)\frac{1}{a^6} \\ & + \left(\frac{1}{42} - \frac{n}{12} - \frac{n(n-1)}{18} + \frac{5n(n-1)(n-2)}{16} + \frac{5n(n-1)(n-2)(n-3)}{32}\right. \\ & \quad \left. + \frac{n(n-1)(n-2)(n-3)(n-4)}{64}\right)\frac{1}{a^7} \\ & + \left(\frac{n}{12} - \frac{7n(n-1)}{24} + \frac{7n(n-1)(n-2)}{72} + \frac{35n(n-1)(n-2)(n-3)}{96}\right. \\ & \quad + \frac{7n(n-1)(n-2)(n-3)(n-4)}{64} \end{aligned}$$

$$+\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{128}\Bigg)\frac{1}{a^8}+\cdots. \tag{6.1}$$

First Proof. Recalling the definition (3.3) of the Stirling numbers $s(n,r)$, we find that

$$\begin{aligned}(k+1)_n &= (-1)^n \sum_{r=1}^{n+1} s(n+1,r)(-k)^{r-1}\\ &= (-1)^n \sum_{r=0}^{n} s(n+1,r+1)(a-(k+a))^r\\ &= (-1)^n \sum_{r=0}^{n} s(n+1,r+1)\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}(k+a)^{r-j}a^j.\end{aligned} \tag{6.2}$$

Recall that the Hurwitz zeta–function $\zeta(s,a)$ is defined for $\operatorname{Re} s>1$ and $a>0$ by

$$\zeta(s,a)=\sum_{k=0}^{\infty}(k+a)^{-s}.$$

Thus, from (5.2) and (6.2),

$$\begin{aligned}u_n(a) &= (-1)^n \sum_{k=0}^{\infty}(a+k)^{-n-2}\sum_{r=0}^{n} s(n+1,r+1)\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}(k+a)^{r-j}a^j\\ &= (-1)^n \sum_{r=0}^{n} s(n+1,r+1)\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}a^j\zeta(n+2+j-r,a).\end{aligned} \tag{6.3}$$

By the Euler–Maclaurin summation formula, (0.5) of Chapter 37, as a tends to ∞, for $\operatorname{Re} s>1$,

$$\zeta(s,a)\sim\sum_{m=0}^{\infty}(-1)^m\frac{B_m(m+s-2)!}{m!\,(s-1)!}a^{1-s-m},$$

where B_m, $m\geq 0$, denotes the mth Bernoulli number. Employing this asymptotic expansion in (6.3), we deduce that, as a tends to ∞,

$$\begin{aligned}u_n(a) &\sim \sum_{r=0}^{n}\sum_{j=0}^{r}(-1)^{n+r+j}s(n+1,r+1)\binom{r}{j}\\ &\quad\times\sum_{m=0}^{\infty}(-1)^m\frac{B_m(m+n+j-r)!}{m!\,(1+n+j-r)!}a^{-n+r-m-1}.\end{aligned} \tag{6.4}$$

Inverting the order of summation on j and m in (6.4), we are led to the inner sum

$$\begin{aligned}\sum_{j=0}^{r}(-1)^j\binom{r}{j}\frac{(m+n+j-r)!}{(1+n+j-r)!} &= \frac{(m+n-r)!}{(1+n-r)!}\sum_{j=0}^{r}\frac{(-r)_j(m+n-r+1)_j}{j!\,(n-r+2)_j}\\ &= \frac{(m+n-r)!(1-m)_r}{(n-r+1)!\,(n-r+2)_r},\end{aligned}$$

by Vandermonde's theorem (Bailey [1, p. 3]). Observe that if $0 < m \le r$, where $r \ge 1$, then the sum above equals 0. Extracting the term for $m = 0$, we find that, from (6.4), as a tends to ∞,

$$\begin{aligned} u_n(a) \sim & \sum_{r=0}^{n} (-1)^{n+r} s(n+1, r+1) \frac{(n-r)!\, r!}{(n+1)!} a^{-n+r-1} \\ & + \sum_{r=0}^{n} (-1)^{n+r} s(n+1, r+1) \\ & \times \sum_{m=r+1}^{\infty} (-1)^m \frac{B_m}{m!} \frac{(m+n-r)!\, (1-m)_r}{(n-r+1)!\, (n-r+2)_r} a^{-n+r-m-1}. \end{aligned} \tag{6.5}$$

We now calculate the coefficients of a^{-N}, $1 \le N \le 8$.

First, the coefficient of $1/a$ equals

$$s(n+1, n+1) \frac{n!}{(n+1)!} = \frac{1}{n+1},$$

in agreement with (6.1).

For the remainder of the calculations, it will be convenient to use the following formula (C. Jordan [1, p. 150, formula (3)]):

$$s(n, n-m) = \sum_{k=0}^{2m} C_{m,k} \binom{n}{2m-k}, \tag{6.6}$$

where $C_{1,0} = -1$, $C_{1,k} = 0$ if $k \ne 0$, and

$$C_{m+1,k} = -(2m-k+1)(C_{m,k} + C_{m,k-1}). \tag{6.7}$$

The coefficients $C_{m,k}$ for $1 \le m \le 6$, $0 \le k \le 5$, are found in a table on page 152 of Jordan's book [1].

For $n \ge 1$, the coefficient of a^{-2} equals

$$-s(n+1, n) \frac{(n-1)!}{(n+1)!} = \frac{n(n+1)}{2} \frac{1}{(n+1)n} = \frac{1}{2}.$$

If $n = 0$, the series in (6.5) also yields $\frac{1}{2}$ for the coefficient of a^{-2}.

For $n \ge 2$, the coefficient of a^{-3} equals

$$\begin{aligned} s(n+1, n-1) \frac{2(n-2)!}{(n+1)!} &= \left\{ 3\binom{n+1}{4} + 2\binom{n+1}{3} \right\} \frac{2}{(n+1)n(n-1)} \\ &= \frac{n-2}{4} + \frac{2}{3} = \frac{n}{4} + \frac{1}{6}. \end{aligned}$$

For $n = 0, 1$, we obtain the same value.

For the remainder of the calculations, we assume that $n \ge N - 1$. In each case, the same formula for the coefficient of a^{-N} also holds for $0 \le n \le N - 2$, from an examination of the double sum in (6.5).

The coefficient of a^{-4}, if $n \geq 3$, is equal to

$$
\begin{aligned}
&-s(n+1, n-2)\frac{3!\,(n-3)!}{(n+1)!} \\
&= \left\{15\binom{n+1}{6} + 20\binom{n+1}{5} + 6\binom{n+1}{4}\right\} \frac{3!\,(n-3)!}{(n+1)!} \\
&= \frac{(n-3)(n-4)}{8} + (n-3) + \tfrac{3}{2} = \frac{n}{4} + \frac{n(n-1)}{8}.
\end{aligned}
$$

We employed MACSYMA in (6.5) to calculate the remaining coefficients as polynomials in n. We then collected terms to write Ramanujan's coefficients as polynomials in n to verify that the polynomials agree for each coefficient. The coefficients of a^{-5}, a^{-6}, and a^{-7} are, respectively,

$$
\begin{aligned}
&s(n+1, n-3)\frac{4!\,(n-4)!}{(n+1)!} \\
&= \left\{105\binom{n+1}{8} + 210\binom{n+1}{7} + 130\binom{n+1}{6} + 24\binom{n+1}{5}\right\} \frac{4!\,(n-4)!}{(n+1)!} \\
&= \frac{15n^3 + 15n^2 - 10n - 8}{240}, \\
&-s(n+1, n-4)\frac{5!\,(n-5)!}{(n+1)!} \\
&= \left\{945\binom{n+1}{10} + 25220\binom{n+1}{9} + 2380\binom{n+1}{8}\right. \\
&\qquad \left. + 924\binom{n+1}{7} + 120\binom{n+1}{6}\right\} \frac{5!\,(n-5)!}{(n+1)!} = \frac{3n^4 + 2n^3 - 7n^2 - 6n}{96},
\end{aligned}
$$

and

$$
\begin{aligned}
&s(n+1, n-5)\frac{6!\,(n-6)!}{(n+1)!} \\
&= \left\{10395\binom{n+1}{12} + 34650\binom{n+1}{11} + 44100\binom{n+1}{10} + 26432\binom{n+1}{9}\right. \\
&\qquad \left. + 7308\binom{n+1}{8} + 720\binom{n+1}{7}\right\} \frac{6!\,(n-6)!}{(n+1)!} \\
&= \frac{63n^5 - 315n^3 - 224n^2 + 140n + 96}{4032}.
\end{aligned}
$$

The coefficients of $s(n+1, n-6)$ are not found in Jordan's book [1]. Thus, we used (6.6) and (6.7) to calculate the needed coefficients. Therefore, the coefficient of a^{-8} equals

$$
\begin{aligned}
& -s(n+1, n-6)\frac{7!\,(n-7)!}{(n+1)!} \\
&= \left\{135135\binom{n+1}{14} + 540540\binom{n+1}{13} + 866250\binom{n+1}{12} + 705320\binom{n+1}{11}\right. \\
&\quad \left. + 303660\binom{n+1}{10} + 64224\binom{n+1}{9} + 5040\binom{n+1}{8}\right\}\frac{7!\,(n-7)!}{(n+1)!} \\
&= \frac{9n^6 - 9n^5 - 75n^4 - 23n^3 + 114n^2 + 80n}{1152}.
\end{aligned}
$$

This completes the proof.

Although our first proof is a natural one, Ramanujan's formulas for the coefficients of a^{-N} indicate that another approach utilizing less calculation was employed by him. In our second proof, calculations lead to coefficients in the form given by Ramanujan, but the calculations are even more difficult to perform by hand than in our first proof.

Second Proof. For $n \geq 0$ and $a + k > 0$,

$$(a+k)^{-n-2} = \frac{1}{(n+1)!}\int_0^\infty e^{-(a+k)t} t^{n+1}\,dt.$$

Using also (5.2) and inverting the order of summation and integration by absolute convergence, we find that

$$
\begin{aligned}
u_n(a) &= \int_0^\infty \sum_{k=0}^\infty \frac{(k+1)_n (e^{-t})^k}{(n+1)!} e^{-at} t^{n+1}\,dt \\
&= \int_0^\infty \frac{e^{-at} t^{n+1}}{(n+1)(1-e^{-t})^{n+1}}\,dt \\
&= \int_0^\infty e^{-at}\varphi_n(t)\,dt, \qquad (6.8)
\end{aligned}
$$

where we have used the generalized binomial theorem and set

$$\varphi_n(t) = \frac{1}{n+1}\left(\frac{t}{1-e^{-t}}\right)^{n+1}.$$

Applying Watson's Lemma (Olver [1, p. 71]), we deduce that

$$u_n(a) \sim \sum_{k=0}^\infty \varphi_n^{(k)}(0) a^{-k-1},$$

as a tends to ∞. Thus, it remains to calculate $\varphi_n^{(k)}(0)$, $0 \leq k \leq 7$. Since

$$\varphi_n(t) = \frac{1}{n+1}\left(\sum_{r=0}^\infty \frac{B_r}{r!}(-t)^r\right)^{n+1},$$

where B_r denotes the rth Bernoulli number, we may readily calculate the coefficients. With the help of MACSYMA, we easily verified Ramanujan's coefficients of a^{-N}, $1 \le N \le 8$.

Entry 7 (Formula (5), pp. 272–273). *Let a, $p > 0$. As p tends to ∞,*

$$S(a,p) := \sum_{n=0}^{\infty} \frac{(a+n)^{n-1}}{(2p+a+n)^{n+1}} \sim \frac{1}{2ap} - e^{-2p} \sum_{n=0}^{\infty} \frac{(-1)^n P_{2n}}{(a+p)^{2n+1}}, \tag{7.1}$$

where $P_{2n} := P_{2n}(p)$, $n \ge 1$, is a polynomial in p of degree $n-1$. In particular,

$$P_0(p) = \frac{1}{2p},$$

$$P_2(p) = \frac{1}{6},$$

$$P_4(p) = \frac{1}{30} + \frac{p}{6},$$

$$P_6(p) = \frac{1}{42} + \frac{p}{6} + \frac{5p^2}{18},$$

$$P_8(p) = \frac{1}{30} + \frac{3p}{10} + \frac{7p^2}{9} + \frac{35p^3}{54},$$

$$P_{10}(p) = \frac{5}{66} + \frac{5p}{6} + \frac{17p^2}{6} + \frac{35p^3}{9} + \frac{35p^4}{18},$$

$$P_{12}(p) = \frac{691}{2730} + \frac{691p}{210} + \frac{616p^2}{45} + \frac{451p^3}{18} + \frac{385p^4}{18} + \frac{385p^5}{54},$$

and

$$P_{14}(p) = \frac{7}{6} + \frac{35p}{2} + \frac{7709p^2}{90} + \frac{26026p^3}{135} + \frac{2002p^4}{9} + \frac{7007p^5}{54} + \frac{5005p^6}{162}.$$

Moreover, for $n \ge 1$,

$$\begin{aligned} P_{2n}(p) &= \frac{(2n)!}{2\cdot 6^n n!}\Bigg(p^{n-1} + \frac{n(n-1)}{10}p^{n-2} + \frac{n(n-1)(n-2)}{200}\left\{(n-3)+\frac{20}{7}\right\}p^{n-3} \\ &\quad + \frac{n(n-1)(n-2)(n-3)}{6000}\left\{(n-4)(n-5)+\frac{60}{7}(n-4)+\frac{90}{7}\right\}p^{n-4} \\ &\quad + \frac{n(n-1)(n-2)(n-3)(n-4)}{240000}\Big\{(n-5)(n-6)(n-7) \\ &\quad + \frac{120}{7}(n-5)(n-6) + \frac{3720}{49}(n-5) + \frac{6000}{77}\Big\}p^{n-5} + \cdots\Bigg). \end{aligned} \tag{7.2}$$

Lastly, for $n \geq 2$ and n even,

$$
\begin{aligned}
&(-1)^{n/2-1}P_n(p) \\
&= B_n + (n+1)B_n p + \left\{\frac{(n+1)(n+2)}{3}B_n - \frac{n(n-1)}{6}B_{n-2}\right\}p^2 \\
&\quad + \left\{\frac{(n+1)(n+2)(n+3)}{18}B_n - \frac{n^2(n-1)}{9}B_{n-2}\right\}p^3 \\
&\quad + \left\{\frac{(n+1)(n+2)(n+3)(n+4)}{180}B_n - \frac{n^2(n^2-1)}{36}B_{n-2}\right. \\
&\quad \left. + \frac{n(n-1)(n-2)(n-3)}{120}B_{n-4}\right\}p^4 \\
&\quad + \left\{\frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{2700}B_n\right. \\
&\quad - \frac{n^2(n^2-1)(n+2)}{270}B_{n-2} \\
&\quad \left. + \frac{n(n-1)(n-2)(n-3)(23n-25)}{5400}B_{n-4}\right\}p^5 + \cdots, \qquad (7.3)
\end{aligned}
$$

where B_j, $j \geq 0$, denotes the jth Bernoulli number and where $P_{2n}(p)$ has degree $n-1$, for $n \geq 0$.

Before proving Entry 7, we shall show that the case $\theta_\infty = \frac{3}{5}$ of Entry 4 follows from Entry 7.

Completion of the Proof of Entry 4. Let

$$A(p) = e^p\left(S(1, p/2) - \frac{1-e^{-p}}{p}\right).$$

In Entry 4, Ramanujan is claiming that, if the rational function within the large parentheses of (4.1) is expanded in powers of $1/p$ when $\theta = \frac{3}{5}$, then the first five terms coincide with the asymptotic expansion of $A(p)$ in powers of $1/p$ as p tends to ∞. Expanding this rational function in powers of $1/p$, we therefore must prove that

$$A(p) \sim \frac{1}{p} - \frac{2}{p^2} + \frac{16}{3p^3} - \frac{56}{3p^4} + \frac{3712}{45p^5} + \cdots, \qquad (4.4)$$

as p tends to ∞.

Now let $a = 1$ and replace p by $p/2$ in (7.1). Then, as p tends to ∞,

$$
\begin{aligned}
e^p\left(S(1, p/2) - \frac{1}{p}\right) &= -\sum_{n=0}^{3}\frac{(-1)^n P_{2n}(p)}{(1+p/2)^{2n+1}} + O\left(\frac{1}{p^6}\right) \\
&= -\frac{2}{p^2(1+2/p)} + \frac{4/3}{p^3(1+2/p)^3} - \frac{32\left(\frac{1}{30}+\frac{p}{12}\right)}{p^5(1+2/p)^5}
\end{aligned}
$$

$$+\frac{128\left(\frac{1}{42}+\frac{p}{12}+\frac{5p^2}{72}\right)}{p^7(1+2/p)^7}+O\left(\frac{1}{p^6}\right)$$
$$=-\frac{2}{p^2}+\frac{16}{3p^3}-\frac{56}{3p^4}+\frac{3712}{45p^5}+O\left(\frac{1}{p^6}\right). \tag{4.5}$$

Comparing (4.4) and (4.5), we complete the proof.

Proof of Entry 7. From Entry 5 and (6.8), for $0 < p < a$,

$$\begin{aligned} S(a,p) &= \sum_{n=0}^{\infty}\frac{(-2p)^n}{n!}u_n(a) \\ &= -\frac{1}{2p}\int_0^{\infty} e^{-at}\sum_{n=0}^{\infty}\frac{(-2pt)^{n+1}}{(n+1)!\,(1-e^{-t})^{n+1}}dt \\ &= -\frac{1}{2p}\int_0^{\infty} e^{-at}\left\{\exp\left(\frac{2pt}{e^{-t}-1}\right)-1\right\}dt \\ &= \frac{1}{2ap}-\frac{1}{2p}\int_0^{\infty} e^{-at}\exp\left(\frac{2pt}{e^{-t}-1}\right)dt. \end{aligned} \tag{7.4}$$

Referring to the definition of $S(a, p)$ in (7.1), we see that $S(a, p)$ represents an analytic function of a and p for $\operatorname{Re} a > 0$ and $\operatorname{Re} p > 0$. Likewise, the right side of (7.4) is analytic for $\operatorname{Re} a > 0$ and $\operatorname{Re} p > 0$. Thus, by analytic continuation, (7.4) is valid for all a and p with $\operatorname{Re} a > 0$ and $\operatorname{Re} p > 0$.

Multiplying both sides of (7.4) by $-e^{2p}$, we see that (7.1) is equivalent to the asymptotic expansion

$$\frac{1}{2p}\int_0^{\infty} e^{-(a+p)t}\exp\left(2p+pt+\frac{2pt}{e^{-t}-1}\right)dt \sim \sum_{n=0}^{\infty}\frac{(-1)^n P_{2n}(p)}{(a+p)^{2n+1}}, \tag{7.5}$$

as p tends to ∞. Since the first term on the right side of (7.5) is equal to

$$\frac{1}{2p}\int_0^{\infty} e^{-(a+p)t}dt,$$

the asymptotic expansion (7.5) is equivalent to

$$\frac{1}{2p}\int_0^{\infty} e^{-(a+p)t}\{\exp(p\,w(t))-1\}\,dt \sim \sum_{n=1}^{\infty}\frac{(-1)^n P_{2n}(p)}{(a+p)^{2n+1}}, \tag{7.6}$$

where

$$w(t) := 2+t+\frac{2t}{e^{-t}-1} = 2-t\coth(t/2) = -2\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}t^{2n}, \qquad |t| < 2\pi, \tag{7.7}$$

where B_j, $j \geq 2$, denotes the jth Bernoulli number.

Let us now *define* the polynomials $P_{2n}(p)$, $n \geq 1$, by the expansion

$$\frac{e^{pw(t)}-1}{2p} = \sum_{n=1}^{\infty} \frac{(-1)^n P_{2n}(p)}{(2n)!} t^{2n}, \qquad |t| < 2\pi. \tag{7.8}$$

We therefore shall prove that the polynomials $P_{2n}(p)$ have the properties enunciated in Entry 7.

By employing MACSYMA, we verified that $P_2(p)$, $P_4(p), \ldots, P_{14}(p)$ are indeed given by the formulas displayed in Entry 7. Second, we remark that it is easy to see from (7.8) that the polynomial $P_{2n}(p)$ has degree $n-1$, $n \geq 1$.

Next, we prove (7.3). From (7.8),

$$\sum_{n=1}^{\infty} \frac{(-1)^n P_{2n}(p)}{(2n)!} t^{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{w^n p^{n-1}}{n!}, \qquad |t| < 2\pi. \tag{7.9}$$

In particular, by (7.7),

$$\sum_{n=1}^{\infty} \frac{(-1)^n P_{2n}(0)}{(2n)!} t^{2n} = \frac{w}{2} = -\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} t^{2n}, \qquad |t| < 2\pi. \tag{7.10}$$

Equating coefficients of t^{2n}, $n \geq 1$, we find that $P_{2n}(0) = (-1)^{n-1} B_{2n}$, as claimed by Ramanujan in (7.3).

Next, differentiating (7.9) with respect to p and setting $p = 0$, we find that

$$\sum_{n=1}^{\infty} \frac{(-1)^n P'_{2n}(0)}{(2n)!} t^{2n} = \frac{w^2}{4}, \tag{7.11}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n P''_{2n}(0)}{(2n)!\,2!} t^{2n} = \frac{w^3}{12}, \tag{7.12}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n P'''_{2n}(0)}{(2n)!\,3!} t^{2n} = \frac{w^4}{48}, \tag{7.13}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n P^{(4)}_{2n}(0)}{(2n)!\,4!} t^{2n} = \frac{w^5}{240}, \tag{7.14}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n P^{(5)}_{2n}(0)}{(2n)!\,5!} t^{2n} = \frac{w^6}{1440}. \tag{7.15}$$

Since the coefficient of p^m, $m \geq 0$, in the Taylor series of $P_n(p)$ about $p = 0$ equals $P_n^{(m)}(0)/m!$, we can calculate the coefficients of p^m on the right side of (7.3), for $1 \leq m \leq 5$, by equating coefficients of t^{2n} on both sides in each of the foregoing five equalities. These calculations are facilitated by observing from (7.7) that

$$w^2 = 2w + 2tw' + t^2.$$

Hence,

$$\begin{aligned}
w^3 &= 2w^2 + t(w^2)' + wt^2, \\
w^4 &= 2w^3 + 2t(w^3/3)' + w^2t^2, \\
w^5 &= 2w^4 + 2t(w^4/4)' + w^3t^2,
\end{aligned}$$

and

$$w^6 = 2w^5 + 2t(w^5/5)' + w^4t^2.$$

Using these five equalities in (7.11)–(7.15), utilizing (7.10), and employing MACSYMA, we can readily verify that each of the coefficients given by Ramanujan in (7.3) is correct.

Lastly, we prove (7.2). Replacing p by $1/p$ and t by $t\sqrt{p}$ in (7.8), we find that

$$\tfrac{1}{2}p\left(\exp\left(w\left(t\sqrt{p}\right)/p\right) - 1\right) = \sum_{n=1}^{\infty} \frac{(-1)^n P_{2n}(1/p)}{(2n)!}(t\sqrt{p})^{2n},$$

or

$$\exp\left(w(t\sqrt{p})/p\right) - 1 = 2\sum_{n=1}^{\infty} \frac{(-1)^n Q_{2n}(p)}{(2n)!}t^{2n}, \tag{7.16}$$

where $Q_{2n}(p) := p^{n-1}P_{2n}(1/p)$ is a polynomial in p of degree $n-1$. Thus, the coefficient of p^{n-m} in $P_{2n}(p)$ equals the coefficient of p^{m-1} in $Q_{2n}(p)$.

Now, by (7.7),

$$u(p,t) := w(t\sqrt{p})/p = -\frac{2}{p}\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}(t\sqrt{p})^{2n} = -2\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}p^{n-1}t^{2n}. \tag{7.17}$$

In particular,

$$u(0,t) = -\frac{2B_2t^2}{2} = -\frac{t^2}{6}. \tag{7.18}$$

Thus, from (7.16),

$$2\sum_{n=1}^{\infty} \frac{(-1)^n Q_{2n}(0)}{(2n)!}t^{2n} = \sum_{k=1}^{\infty} \frac{u^k(0,t)}{k!} = \sum_{k=1}^{\infty} \frac{(-1)^k t^{2k}}{6^k k!}.$$

Equating coefficients of t^{2n}, $n \geq 1$, on both sides, we find that

$$Q_{2n}(0) = \frac{(2n)!}{2\cdot 6^n n!},$$

as claimed by Ramanujan.

Next, by (7.17),

$$u_p(0,t) = -\frac{B_4t^4}{12} = \frac{t^4}{360}. \tag{7.19}$$

Thus, from (7.16), (7.18), and (7.19),

$$2\sum_{n=1}^{\infty}\frac{(-1)^n Q'_{2n}(0)}{(2n)!}t^{2n}=\sum_{k=1}^{\infty}\frac{u^{k-1}(0,t)u_p(0,t)}{(k-1)!}=\sum_{k=1}^{\infty}\frac{(-t^2/6)^{k-1}(t^4/360)}{(k-1)!}.$$

Equating coefficients of t^{2n}, $n \geq 1$, on both sides, we find that

$$Q'_{2n}(0)=\frac{(2n)!}{2\cdot 6^{n-2}360(n-2)!}=\frac{(2n)!}{2\cdot 6^n n!}\frac{n(n-1)}{10},$$

which again agrees with Ramanujan.

Next, by (7.17),

$$u_{pp}(0,t)=-\frac{4B_6t^6}{6!}=-\frac{t^6}{6^3\cdot 35}. \tag{7.20}$$

Thus, from (7.16) and (7.18)–(7.20),

$$\begin{aligned}2\sum_{n=1}^{\infty}\frac{(-1)^n Q''_{2n}(0)}{(2n)!}t^{2n}&=\sum_{k=1}^{\infty}\left(\frac{u^{k-2}(0,t)u_p^2(0,t)}{(k-2)!}+\frac{u^{k-1}(0,t)u_{pp}(0,t)}{(k-1)!}\right)\\&=\sum_{k=1}^{\infty}\left(\frac{(-t^2/6)^{k-2}(t^4/360)^2}{(k-2)!}+\frac{(-t^2/6)^{k-1}(-t^6/(6^3\cdot 35))}{(k-1)!}\right).\end{aligned}$$

Thus, equating coefficients of t^{2n}, we find that

$$\begin{aligned}\frac{Q''_{2n}(0)}{2!}&=\frac{(2n)!}{4}\left(\frac{1}{6^{n-4}(360)^2(n-4)!}+\frac{1}{6^n35(n-3)!}\right)\\&=\frac{(2n)!}{2\cdot 6^n n!}\left(\frac{n(n-1)(n-2)(n-3)}{200}+\frac{n(n-1)(n-2)}{70}\right),\end{aligned}$$

as claimed by Ramanujan.

The remaining two coefficients recorded by Ramanujan are similarly calculated, and we omit the details. (We used MACSYMA to effect the calculations.)

We now return to the task of establishing (7.6). We would like to employ Watson's Lemma, but we cannot do so because $\exp(pw(t))-1$ depends upon p. Thus, a completely new procedure seems necessary. We prove a very general theorem (Theorem 7.1 below) from a paper by the author and R. J. Evans [2] that includes (7.1) as a special case.

Define

$$T(x):=\sum_{n=0}^{\infty}\frac{\Gamma(n+s)}{n!}\frac{(a+n)^{n-r}}{(x+a+n)^{n+s}}, \tag{7.21}$$

where r is a fixed positive integer, and a and s are fixed complex numbers with positive real parts. Note that when $r=s=1$, $T(2p)=S(a,p)$, where $S(a,p)$ is defined in (7.1).

Theorem 7.1. *Let N denote a positive integer. Then as $x \to \infty$ in the sector $|\operatorname{Arg} x| \le \frac{1}{2}\pi - \delta$, where $\delta > 0$,*

$$T(x) = \sum_{k=0}^{r-1} A_k x^{-k-s} - e^{-x}\left(\sum_{m=0}^{N-1} \frac{C_m(x)}{(a+x/2)^{m+1}} + O(x^{-1-r-N/2})\right), \tag{7.22}$$

where

$$A_k = \sum_{j=0}^{k} (-1)^{k-j}\Gamma(s+k)\binom{k}{j}(a+j)^{k-r}, \tag{7.23}$$

and where the functions $C_m(x)$ (defined in (7.32)) have the estimate

$$C_m(x) = O(x^{[m/2]-r}), \tag{7.24}$$

as x tends to ∞.

The arbitrarily small positive number δ is fixed throughout the sequel. Observe that (7.22) is a genuine asymptotic expansion, in view of (7.24). Note also that x can be replaced by $x + b$ in (7.22), for any constant b. Thus, for example, if the sign of a is reversed in the denominator of (7.21), then $C_m(x)/(a + x/2)^{m+1}$ is replaced by $2^{m+1}C_m(x - 2a)/x^{m+1}$.

If $r = s = 1$, the leading sum on the right side of (7.22) equals $1/x$. Furthermore, assuming the validity of (7.22), we conclude that

$$C_m(2p) = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ (-1)^n P_{2n}(p), & \text{if } m = 2n \text{ is even.} \end{cases}$$

Before beginning the proof of Theorem 7.1, we need to define several functions and prove an auxiliary lemma.

Consider the confluent hypergeometric function

$${}_1F_1(s, s+r; z) = \sum_{m=0}^{\infty} \frac{(s)_m z^m}{(s+r)_m m!}, \qquad |z| < \infty. \tag{7.25}$$

This function is related to $U(s, s + r; z)$, the confluent hypergeometric function of the second kind, by

$${}_1F_1(s, s+r; z) = \frac{\Gamma(s+r)}{\Gamma(r)} e^{i\pi s} U(s, s+r; z) + \frac{\Gamma(s+r)}{\Gamma(s)} (-1)^r e^z U(r, s+r; -z), \tag{7.26}$$

where $\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$ (see, e.g., Olver's book [1, p. 257, eq. (10.09)] or N. N. Lebedev's book [1, p. 270, eq. (9.12.4)]). In many texts (e.g., Lebedev [1, p. 263]), U is designated by Ψ. As $z \to \infty$ with $|\arg z| \le \frac{3}{2}\pi - \delta$, we have the asymptotic expansion (Olver [1, p. 256])

$$U(r, s+r; z) \sim \sum_{m=0}^{\infty} \frac{(-1)^m (r)_m (1-s)_m}{m!\, z^{m+r}}. \tag{7.27}$$

Since r is a positive integer, $U(s, s+r; z)$ can be expressed as a Laguerre polynomial (Erdélyi [2, pp. 188–189, eqs. (7), (14)]). Thus,

$$U(s, s+r; z) = \sum_{k=0}^{r-1} \frac{(-1)^k (s)_k (1-r)_k}{z^{k+s}}. \tag{7.28}$$

For brevity, write, for $t \geq 0$,

$$w := w(t) := \frac{t}{1-e^{-t}}, \tag{7.29}$$

so that by (0.1) of Chapter 37,

$$w = \sum_{m=0}^{\infty} \frac{B_m}{m!}(-t)^m, \qquad |t| < 2\pi. \tag{7.30}$$

For $t \geq 0$ and $\operatorname{Re} x > 0$, define

$$f(t, x) := e^{x(1-w+t/2)}(-t)^{r-1} w^s U(r, s+r; wx). \tag{7.31}$$

Finally, the functions $C_m(x)$ in Theorem 7.1 are defined by

$$C_m(x) = f^{(m)}(0, x), \qquad \operatorname{Re} x > 0, \tag{7.32}$$

where the superscript m denotes the mth derivative with respect to t.

We remark that in the case $r = 1$,

$$f(t, x) = x^{-s} e^{x+xt/2} \Gamma(s, wx),$$

where $\Gamma(s, z)$ denotes the incomplete gamma function

$$\Gamma(s, z) := \int_z^{\infty} e^{-t} t^{s-1} dt, \qquad \operatorname{Re} s > 0.$$

This follows from (7.31) and the formula (Erdélyi [2, p. 136, eq. (15)])

$$\Gamma(s, z) = e^{-z} z^s U(1, s+1; z).$$

Define, for each integer $m \geq 0$,

$$U_m := U_m(a) := \sum_{n=0}^{\infty} \frac{\Gamma(m+s+n)}{n!\,(a+n)^{m+s+r}}. \tag{7.33}$$

Note that $U_m(a)$ generalizes the function $u_n(a)$, defined in (5.2). From Euler's integral representation of the gamma function,

$$\frac{1}{(a+n)^{m+s+r}} = \frac{1}{\Gamma(m+s+r)} \int_0^{\infty} e^{-nt-at} t^{m+s+r-1} dt.$$

Thus,

$$U_m = \frac{\Gamma(m+s)}{\Gamma(m+s+r)} \int_0^{\infty} e^{-at} t^{m+s+r-1} \sum_{n=0}^{\infty} \frac{\Gamma(m+s+n)}{\Gamma(m+s)n!} e^{-tn} dt, \tag{7.34}$$

where absolute convergence justifies the interchange of integration and summation. The sum on n in (7.34) equals $(1-e^{-t})^{-m-s}$, and so

$$U_m = \frac{\Gamma(m+s)}{\Gamma(m+s+r)} \int_0^\infty e^{-at} t^{r-1} w^{m+s} dt, \tag{7.35}$$

where w is defined in (7.29).

Recall that $T(x)$ is defined in (7.21) for $\operatorname{Re} x > 0$. Assuming for the moment that $|x| < |a|$, we find that

$$\begin{aligned} T(x) &= \sum_{n=0}^\infty \frac{\Gamma(n+s)}{n!\,(a+n)^{s+r}} \left(1 + \frac{x}{a+n}\right)^{-n-s} \\ &= \sum_{n=0}^\infty \frac{\Gamma(n+s)}{n!\,(a+n)^{s+r}} \sum_{m=0}^\infty \frac{\Gamma(m+s+n)}{m!\,\Gamma(n+s)} \left(\frac{-x}{a+n}\right)^m . \end{aligned} \tag{7.36}$$

By (7.33) and (7.36),

$$T(x) = \sum_{m=0}^\infty \frac{(-x)^m}{m!} U_m, \qquad |x| < |a|, \tag{7.37}$$

where absolute convergence justifies the interchange of summation. Note that (5.1) in Entry 5 is the case $r = s = 1$ of (7.37).

Put (7.35) in (7.37) to deduce that, for $|x| < |a|$,

$$\begin{aligned} T(x) &= \sum_{m=0}^\infty \frac{(-x)^m}{m!} \frac{\Gamma(m+s)}{\Gamma(m+s+r)} \int_0^\infty e^{-at} t^{r-1} w^{m+s} dt \\ &= \int_0^\infty e^{-at} t^{r-1} w^s \sum_{m=0}^\infty \frac{(-xw)^m}{m!} \frac{\Gamma(m+s)}{\Gamma(m+s+r)} dt, \end{aligned} \tag{7.38}$$

where the interchange of integration and summation can be justified by absolute convergence. By (7.25) and (7.38), for $|x| < |a|$,

$$T(x) = \frac{\Gamma(s)}{\Gamma(s+r)} \int_0^\infty e^{-at} t^{r-1} w^s \, {}_1F_1(s, s+r; -wx)\, dt. \tag{7.39}$$

As $|x| \to \infty$ with $|\operatorname{Arg} x| \le \frac{1}{2}\pi - \delta$, $-wx \to \infty$ with $\frac{1}{2}\pi + \delta \le \arg(-wx) \le \frac{3}{2}\pi - \delta$. Thus, by (7.26)–(7.28), the integral in (7.39) is convergent and analytic in each variable a, x in the right half-plane. From (7.21), $T(x)$ is also seen to be analytic in each of a, x in the right half-plane. Thus, (7.39) holds for all x with $\operatorname{Re} x > 0$.

The proof of Lemma 7.2 below makes heavy use of Faa di Bruno's formula (J. Riordan [1, p. 36], S. Roman [1])

$$\frac{d^n}{dt^n} h(g(t)) = \sum \frac{n!\, h_k(g(t))}{k_1!\, k_2! \cdots k_n!} \left(\frac{g_1}{1!}\right)^{k_1} \left(\frac{g_2}{2!}\right)^{k_2} \cdots \left(\frac{g_n}{n!}\right)^{k_n}, \tag{7.40}$$

where the sum is over all integers $k_1, k_2, \ldots, k_n$ for which

$$n = k_1 + 2k_2 + \cdots + nk_n, \qquad k_i \ge 0, 1 \le i \le n, \tag{7.41}$$

and where $k = k_1 + k_2 + \cdots + k_n$,

$$h_k(z) = \frac{d^k}{dz^k}h(z), \qquad \text{and} \qquad g_i := g_i(t) := \frac{d^i}{dt^i}g(t). \tag{7.42}$$

Lemma 7.2. *Fix an integer $N \geq 1$. As $x \to \infty$ with $|\operatorname{Arg} x| \leq \frac{1}{2}\pi - \delta$,*

$$f^{(N)}(t, x) = O\left(x^{-r+[N/2]}\sum_{j=0}^{N}|xt|^j\right), \tag{7.43}$$

uniformly for t in $[0, 1]$.

Proof. Let $0 \leq t \leq 1$ and $n \geq 0$. We shall obtain uniform estimates for the nth derivatives of each factor $(-t)^{r-1}$, w^s, $e^{x(1-w+t/2)}$, and $U(r, s+r; wx)$ of $f(t, x)$ in (7.31) and then combine them to deduce (7.43) from Leibniz's rule.

First, for each $n \geq 0$,

$$\frac{d^n}{dt^n}(-t)^{r-1} = O(1), \tag{7.44}$$

since r is a positive integer. Next, by (7.30), we find that, for each $k \geq 0$,

$$\frac{d^k}{dt^k}w = O(1). \tag{7.45}$$

Consequently, by (7.40) with $h(z) = z^s$ and $g(t) = w$,

$$\frac{d^n}{dt^n}w^s = O(1). \tag{7.46}$$

For $|\operatorname{Arg} z| \leq \frac{1}{2}\pi - \delta$, $U(r, s+r; z)$ is analytic (Olver [1, p. 257, eq. (10.04)]), and so we can differentiate (7.27) (Olver [1, pp. 9–10, Theorem 4.2]) to obtain, for $k \geq 0$ and $|z|$ sufficiently large,

$$\frac{d^k}{dz^k}U(r, s+r; z) \sim \sum_{m=0}^{\infty}\frac{(r)_{m+k}(-1)^{m+k}(1-s)_m}{m!\, z^{m+r+k}} = O(z^{-k-r}). \tag{7.47}$$

Now apply (7.40) with $h(z) = U(r, s+r; z)$ and $g(t) = wx$ to deduce from (7.45) and (7.47) that, as x tends to ∞ with $|\operatorname{Arg} x| \leq \frac{1}{2}\pi - \delta$,

$$\frac{d^n}{dt^n}U(r, r+s; wx) = O(x^{-r}), \tag{7.48}$$

uniformly for $0 \leq t \leq 1$.

A final application of (7.40) with $h(z) = e^{zx}$ and $g(t) = 1 + t/2 - w$ yields

$$\frac{d^n}{dt^n}e^{x(1+t/2-w)} = e^{xg(t)}\sum B(k_1, k_2, \ldots, k_n)g_1^{k_1}g_2^{k_2}\cdots g_n^{k_n}x^{k_1+k_2+\cdots+k_n}, \tag{7.49}$$

where the sum is over all integers k_i satisfying (7.41), where the coefficients $B(k_1, k_2, \ldots, k_n)$ are independent of x, t, and where g_i is defined by (7.42). By

(7.30),

$$g(t) = -\sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} t^{2m},$$

and so

$$g_i = O(t), \qquad \text{for all } odd\ i \geq 1 \tag{7.50}$$

and

$$g_j = O(1), \qquad \text{for all } j \geq 1. \tag{7.51}$$

Since $g(t) \leq 0$ for $0 \leq t \leq 1$,

$$e^{xg(t)} = O(1). \tag{7.52}$$

By (7.41),

$$k_2 + k_4 + k_6 + \cdots \leq \tfrac{1}{2}(k_1 + 2k_2 + \cdots + nk_n) = \tfrac{1}{2}n. \tag{7.53}$$

Combining (7.49)–(7.53), we see that

$$\begin{aligned} \frac{d^n}{dt^n} e^{x(1+t/2-w)} &\ll \sum |x|^{k_1+k_2+\cdots+k_n} t^{k_1+k_3+\cdots} \\ &\ll \sum |xt|^{k_1+k_3+\cdots} |x|^{k_2+k_4+\cdots} \ll |x|^{[n/2]} \sum_{i=0}^{n} |xt|^i. \end{aligned} \tag{7.54}$$

The result now follows from (7.44), (7.46), (7.48), (7.54), and Leibniz's rule.

Proof of Theorem 7.1. By (7.26) and (7.39),

$$T(x) = A(x) - B(x), \tag{7.55}$$

where

$$A(x) = \frac{\Gamma(s)}{\Gamma(r)} \int_0^{\infty} e^{-at} t^{r-1} (-w)^s U(s, s+r; -wx)\, dt \tag{7.56}$$

and

$$B(x) = \int_0^{\infty} e^{-at} (-t)^{r-1} w^s e^{-wx} U(r, s+r; wx)\, dt, \tag{7.57}$$

where $\frac{1}{2}\pi < \arg(-wx) < \frac{3}{2}\pi$.

We first examine $A(x)$, which yields the dominant part of the asymptotic expansion of $T(x)$. Using (7.28) in (7.56), we find that

$$\begin{aligned} A(x) &= \frac{\Gamma(s)}{\Gamma(r)} \sum_{k=0}^{r-1} (-1)^k (s)_k (1-r)_k \int_0^{\infty} e^{-at} t^{r-1} (-w)^s (-wx)^{-s-k} dt \\ &= \frac{\Gamma(s)}{\Gamma(r)} \sum_{k=0}^{r-1} \frac{(-1)^k (s)_k (1-r)_k}{x^{s+k}} \int_0^{\infty} e^{-at} t^{r-k-1} (e^{-t} - 1)^k dt \end{aligned}$$

$$= \frac{\Gamma(s)}{\Gamma(r)} \sum_{k=0}^{r-1} \sum_{j=0}^{k} \frac{(-1)^j (s)_k (1-r)_k}{x^{s+k}} \binom{k}{j} \int_0^\infty e^{-t(a+j)} t^{r-k-1} dt, \tag{7.58}$$

where we have expanded $(e^{-t} - 1)^k$ by the binomial theorem. It follows easily from (7.58) that

$$A(x) = \sum_{k=0}^{r-1} A_k x^{-k-s},$$

in agreement with (7.22) and (7.23).

Now (7.24) follows by putting $t = 0$ in (7.43). Thus, by (7.22), (7.31), (7.55), and (7.57), it remains to show that

$$\int_0^\infty e^{-t(a+x/2)} f(t,x)\,dt = \sum_{m=0}^{N-1} \frac{C_m(x)}{(a+x/2)^{m+1}} + O(x^{-1-r-N/2}). \tag{7.59}$$

By (7.27) and (7.31),

$$f(t,x) \ll e^{x(1-w+t/2)} t^{r-1} w^s (wx)^{-r},$$

and so

$$e^{-t(a+x/2)} f(t,x) \ll e^{-at} e^{x(1-w)} x^{-r} t^{s-1}, \tag{7.60}$$

uniformly for $t \geq 1$. Since, for $t \geq 1$, we know that $1 - w < -\frac{1}{2}$, it follows from (7.60) that

$$\int_1^\infty e^{-t(a+x/2)} f(t,x)\,dt \ll e^{-x/2} x^{-r} \int_1^\infty e^{-t\,\mathrm{Re}\,a} t^{\mathrm{Re}\,s-1} dt \ll e^{-x/2}. \tag{7.61}$$

In view of (7.59) and (7.61), it remains to show that

$$\int_0^1 e^{-t(a+x/2)} f(t,x)\,dt = \sum_{m=0}^{N-1} \frac{C_m(x)}{(a+x/2)^{m+1}} + O(x^{-1-r-N/2}). \tag{7.62}$$

Integrating by parts N times, we find that

$$\int_0^1 e^{-t(a+x/2)} f(t,x)\,dt = \sum_{m=0}^{N-1} \frac{f^{(m)}(0,x) - f^{(m)}(1,x) e^{-(a+x/2)}}{(a+x/2)^{m+1}} + (a+x/2)^{-N} \int_0^1 e^{-t(a+x/2)} f^{(N)}(t,x)\,dt.$$

By Lemma 7.2,

$$e^{-(a+x/2)} f^{(m)}(1,x) \ll e^{-(a+x/2)} x^{3m/2} \ll e^{-x/3}.$$

Thus, to prove (7.62), it remains to prove that

$$\int_0^1 e^{-t(a+x/2)} f^{(N)}(t,x)\,dt = O(x^{N/2-r-1}).$$

Again, by Lemma 7.2,

$$\int_0^1 e^{-t(a+x/2)} f^{(N)}(t,x)\,dt \ll x^{N/2-r} \int_0^1 e^{-t\,\mathrm{Re}(a+x/2)} \sum_{j=0}^{N} |xt|^j dt$$

$$\ll x^{N/2-r} \sum_{j=0}^{N} |x|^j \int_0^\infty e^{-t\,\mathrm{Re}(a+x/2)} t^j dt$$

$$= x^{N/2-r} \sum_{j=0}^{N} \frac{|x|^j j!}{(\mathrm{Re}(a+x/2))^{j+1}}$$

$$\ll x^{N/2-r} \sum_{j=0}^{N} \frac{|x|^j j!}{|x|^{j+1}} \ll x^{N/2-r-1}.$$

This completes the proof of Theorem 7.1.

We show now that $C_m(x)$ possesses an asymptotic expansion in descending powers of x.

As in the proof of Lemma 7.2, we shall estimate $C_m(x) = f^{(m)}(0,x)$ by combining Leibniz's rule with formulas for the nth derivatives of $(-t)^{r-1}$, w^s, $e^{x(1-w+t/2)}$, and $U(r, s+r; wx)$. The nth derivatives of $(-t)^{r-1}$ and w^s at $t=0$ are constants. Since, for the function $g(t)$ in (7.49), we have $g(0) = 0$, the nth derivative of $e^{x(1+t/2-w)}$ at $t=0$ is, by (7.49), a polynomial in x. It remains to show that the nth derivative of $U(r, s+r; wx)$ at $t=0$ has an asymptotic expansion in descending powers of x. By (7.40) with $h(z) = U(r, s+r; z)$ and $g(t) = wx$, we find that

$$\frac{d^n}{dt^n} U(r, s+r; wx)\bigg|_{t=0} = \sum_{k=0}^{n} E_k x^k \frac{d^k}{dz^k} U(r, s+r; z)\bigg|_{z=x} \tag{7.63}$$

for some constants E_k. Using the asymptotic formula (7.47) in (7.63), we obtain the desired result.

If s is a positive integer, we can deduce the stronger result that $C_m(x)$ is a Laurent polynomial. To see this, note that when s is an integer,

$$U(r, s+r; z) = \sum_{k=0}^{s-1} \frac{(-1)^k (r)_k (1-s)_k}{z^{k+r}}$$

by (7.28) with r and s interchanged. Thus, $U(r, s+r; z)$ and its derivatives with respect to z are Laurent polynomials in z, and the result follows from (7.63) as before.

After stating (7.3), Ramanujan provides what is evidently a hint for proving (7.3). However, we have been unable to use Ramanujan's advice in establishing (7.3). Correcting three misprints, we state Ramanujan's "hint" as a separate entry.

Entry 8 (p. 273). *Let n be a nonnegative integer and suppose that* $0 < p < a$. *Then*

$$e^{2p}\frac{(a-p+n)^{n-1}}{(a+p+n)^{n+1}} = \frac{1}{(a+n)^2}\exp\left(\frac{2ap}{a+n}+\frac{p^2}{(a+n)^2}-\frac{2np^3}{3(a+n)^3}\right.$$
$$\left.+\frac{p^4}{2(a+n)^4}-\frac{2np^5}{5(a+n)^5}+\frac{p^6}{3(a+n)^6}-\cdots\right)$$
$$=\frac{1}{(a+n)^2}\left(1+2p\frac{a}{a+n}+2p^2\frac{a^2+\frac{1}{2}}{(a+n)^2}+\frac{4p^3}{3}\left\{\frac{a^3+2a}{(a+n)^3}\right.\right.$$
$$\left.-\frac{1}{2(a+n)^2}\right\}+\frac{2p^4}{3}\left\{\frac{a^4+5a^2+\frac{3}{2}}{(a+n)^4}-\frac{2a}{(a+n)^3}\right\}$$
$$\left.+\frac{4p^5}{15}\left\{\frac{a^5+10a^3+23a/2}{(a+n)^5}-\frac{5a^4+4}{(a+n)^4}\right\}+\cdots\right). \tag{8.1}$$

Proof. Write

$$e^{2p}\frac{(a-p+n)^{n-1}}{(a+p+n)^{n+1}} = \frac{e^{2p}}{(a+n)^2}\frac{\left(1-\dfrac{p}{a+n}\right)^{n-1}}{\left(1+\dfrac{p}{a+n}\right)^{n+1}}$$
$$=\frac{e^{2p}}{(a+n)^2}\exp\left((n-1)\log\left(1-\frac{p}{a+n}\right)-(n+1)\log\left(1+\frac{p}{a+n}\right)\right)$$
$$=\frac{e^{2p}}{(a+n)^2}\exp\left((n-1)\left\{-\frac{p}{a+n}-\frac{p^2}{2(a+n)^2}-\frac{p^3}{3(a+n)^3}-\frac{p^4}{4(a+n)^4}\right.\right.$$
$$\left.\left.-\cdots\right\}-(n+1)\left\{\frac{p}{a+n}-\frac{p^2}{2(a+n)^2}+\frac{p^3}{3(a+n)^3}-\frac{p^4}{4(a+n)^4}+\cdots\right\}\right)$$
$$=\frac{1}{(a+n)^2}\exp\left(\frac{2ap}{a+n}+\frac{p^2}{(a+n)^2}-\frac{2np^3}{3(a+n)^3}\right.$$
$$\left.+\frac{p^4}{2(a+n)^4}-\frac{2np^5}{5(a+n)^5}+\frac{p^6}{3(a+n)^6}-\cdots\right).$$

This proves the first equality in (8.1).

If $A(a,p,n)$ denotes the expression in large parentheses on the far right side above, then

$$\exp(A(a,p,n))$$
$$=1+\left(\frac{2ap}{a+n}+\frac{p^2}{(a+n)^2}-\frac{2np^3}{3(a+n)^3}+\frac{p^4}{2(a+n)^4}-\frac{2np^5}{5(a+n)^5}\right.$$
$$\left.+\cdots\right)+\frac{1}{2}\left(\frac{2ap}{a+n}+\frac{p^2}{(a+n)^2}-\frac{2np^3}{3(a+n)^3}+\frac{p^4}{2(a+n)^4}-\cdots\right)^2$$

$$
\begin{aligned}
&+\frac{1}{6}\left(\frac{2ap}{a+n}+\frac{p^2}{(a+n)^2}-\frac{2np^3}{3(a+n)^3}+\cdots\right)^3\\
&+\frac{1}{24}\left(\frac{2ap}{a+n}+\frac{p^2}{(a+n)^2}-\cdots\right)^4+\frac{1}{120}\left(\frac{2ap}{a+n}+\cdots\right)^5+\cdots\\
=\ &1+\frac{2ap}{a+n}+\frac{p^2}{(a+n)^2}\left(1+2a^2\right)+\frac{p^3}{(a+n)^3}\left(-\frac{2n}{3}+2a+\frac{4a^3}{3}\right)\\
&+\frac{p^4}{(a+n)^4}\left(\frac{1}{2}-\frac{4an}{3}+\frac{1}{2}+2a^2+\frac{2a^4}{3}\right)\\
&+\frac{p^5}{(a+n)^5}\left(-\frac{2n}{5}+a-\frac{2n}{3}-\frac{4a^2n}{3}+a+\frac{4a^3}{3}+\frac{4a^5}{15}\right)+\cdots\\
=\ &1+2p\frac{a}{a+n}+2p^2\frac{a^2+\frac{1}{2}}{(a+n)^2}+\frac{4p^3}{3}\left\{\frac{a^3+2a}{(a+n)^3}-\frac{1}{2(a+n)^2}\right\}\\
&+\frac{2p^4}{3}\left\{\frac{a^4+5a^2+\frac{3}{2}}{(a+n)^4}-\frac{2a}{(a+n)^3}\right\}\\
&+\frac{4p^5}{15}\left\{\frac{a^5+10a^3+23a/2}{(a+n)^5}-\frac{5a^2+4}{(a+n)^4}\right\}+\cdots.
\end{aligned}
$$

This concludes the proof.

The next result is somewhat enigmatic. Ramanujan offers an asymptotic series for $\sum_{k=1}^{n} 1/k$ as n tends to ∞, but he expresses the asymptotic expansion in powers of $1/m$ instead of $1/n$, where $m=\frac{1}{2}n(n+1)$. We cannot find a "natural" method to produce such an asymptotic series. Therefore, we take Ramanujan's expansion, convert it into powers of $1/n$, and show that it agrees with Euler's well-known asymptotic series for a partial sum of the harmonic series.

Entry 9 (Formula (5), p. 276). *Let $m=\frac{1}{2}n(n+1)$, where n is a positive integer. Then as m approaches ∞,*

$$
\begin{aligned}
\sum_{k=1}^{n}\frac{1}{k} \sim \tfrac{1}{2}\log(2m)+\gamma+\frac{1}{12m}-\frac{1}{120m^2}+\frac{1}{630m^3}-\frac{1}{1680m^4}+\frac{1}{2310m^5}\\
-\frac{191}{360360m^6}+\frac{29}{30030m^7}-\frac{2833}{1166880m^8}+\frac{140051}{17459442m^9}-\cdots,
\end{aligned}
\tag{9.1}
$$

where γ denotes Euler's constant.

Proof. We rewrite (9.1) in the form

$$
\sum_{k=1}^{n}\frac{1}{k}\sim\log n+\gamma+\tfrac{1}{2}\log\left(1+\frac{1}{n}\right)+\frac{1}{6n(n+1)}-\frac{1}{30n^2(n+1)^2}
$$

$$+\frac{4}{315n^3(n+1)^3}-\frac{1}{105n^4(n+1)^4}+\frac{16}{1155n^5(n+1)^5}$$
$$-\frac{8\cdot 191}{45045n^6(n+1)^6}+\frac{64\cdot 29}{15015n^7(n+1)^7}-\frac{8\cdot 2833}{36465n^8(n+1)^8}$$
$$+\frac{256\cdot 140051}{8729721n^9(n+1)^9}+\cdots. \tag{9.2}$$

Using *Mathematica*, we expand $\log(1+1/n)$ and $(n+1)^{-k}$, $1\le k\le 9$, in powers of $1/n$ and collect coefficients of like powers of $1/n$. We then find that (9.2) can be put in the shape

$$\sum_{k=1}^{n}\frac{1}{k}\sim\log n+\gamma+\frac{1}{2n}-\frac{1}{12n^2}+\frac{1}{120n^4}-\frac{1}{252n^6}$$
$$+\frac{1}{240n^8}-\frac{1}{132n^{10}}+\frac{691}{32760n^{12}}-\frac{1}{12n^{14}}$$
$$+\frac{3617}{8160n^{16}}-\frac{43867}{14364n^{18}}+\cdots. \tag{9.3}$$

On the other hand, from a result of Euler found in Entry 2 of Chapter 8 (Part I [1, p. 182]),

$$\sum_{k=1}^{n}\frac{1}{k}\sim\log n+\gamma-\sum_{k=1}^{\infty}\frac{B_k}{kn^k}, \tag{9.4}$$

as n tends to ∞, where B_k, $k\ge 1$, denotes the kth Bernoulli number. Calculating the first 18 terms in the sum on the right side of (9.4), we find that they are in agreement with those in (9.3).

D. W. DeTemple and S.–H. Wang [1] found an analogue of (9.4) with n replaced by $n+\frac{1}{2}$.

We quote Ramanujan in the next entry.

Entry 10 (Formula (13), p. 284). *The property of the function*

$$\sum_{n=1}^{\infty}\frac{\log n}{n^2+x^2}$$

and the integral

$$\int_0^{\infty}\frac{t\,dt}{(e^{2\pi t}-1)(t+x)}.$$

Ramanujan did not inform us what *property* he had in mind. Since these two functions are not equal, it would seem that he is claiming that they are asymptotically equal as x tends to ∞. However, as we shall demonstrate, this is not the case.

Theorem 10.1. *As x tends to ∞,*

$$F(x) := \sum_{n=1}^{\infty} \frac{\log n}{n^2+x^2} \sim \frac{\pi \log x}{2x}. \tag{10.1}$$

Proof. By partial summation,

$$\sum_{n=1}^{N} \frac{\log n}{n^2+x^2} = \frac{\log N!}{N^2+x^2} + \int_1^N \frac{2t \log \Gamma(t+1)}{(t^2+x^2)^2} dt.$$

Letting N tend to ∞, we deduce that

$$F(x) = 2\int_1^{\infty} \frac{t \log \Gamma(t+1)}{(t^2+x^2)^2} dt, \tag{10.2}$$

since, by Stirling's formula,

$$\log \Gamma(u+1) \sim (u+\tfrac{1}{2}) \log u - u + O(1), \tag{10.3}$$

as u tends to ∞. Using (10.3) in (10.2), we find that

$$\begin{aligned}
F(x) &= 2\int_1^{\infty} \frac{t\left\{(t+\frac{1}{2})\log t - t + O(1)\right\}}{(t^2+x^2)^2} dt \\
&= 2\int_0^{\infty} \frac{t(t+\frac{1}{2})\log t - t^2}{(t^2+x^2)^2} dt + O(x^{-2}) \\
&= 2\int_0^{\infty} \frac{\log t}{t^2+x^2} dt - 2x^2 \int_0^{\infty} \frac{\log t}{(t^2+x^2)^2} dt + \int_0^{\infty} \frac{t \log t}{(t^2+x^2)^2} dt \\
&\quad - \int_0^{\infty} \frac{dt}{t^2+x^2} + x^2 \int_0^{\infty} \frac{dt}{(t^2+x^2)^2} \\
&= I_1 + I_2 + \cdots + I_5,
\end{aligned} \tag{10.4}$$

say.

First, from elementary considerations,

$$I_4 + I_5 = -\frac{\pi}{2x} + \frac{\pi}{4x} = -\frac{\pi}{4x}. \tag{10.5}$$

Next, from Gradshteyn and Ryzhik's *Tables* [1, p. 564, formula 4.231, no. 8],

$$I_1 = \frac{2}{x}\int_0^{\infty} \frac{\log(xu)}{u^2+1} du = \frac{2}{x}\left(\frac{\pi \log x}{2} + \int_0^{\infty} \frac{\log u}{u^2+1} du\right) = \frac{\pi \log x}{x}, \tag{10.6}$$

$$\begin{aligned}
I_2 &= -\frac{2}{x}\int_0^{\infty} \frac{\log(xu)}{(u^2+1)^2} du = -\frac{2}{x}\left(\int_0^{\infty} \frac{\log u}{(u^2+1)^2} du + \log x \int_0^{\infty} \frac{du}{(u^2+1)^2}\right) \\
&= -\frac{\pi \log x}{2x} + O\left(\frac{1}{x}\right),
\end{aligned} \tag{10.7}$$

and

$$I_3 = \frac{1}{x^2}\int_0^{\infty} \frac{u \log(xu)}{(u^2+1)^2} du = O\left(\frac{\log x}{x^2}\right). \tag{10.8}$$

Putting (10.5)–(10.8) in (10.4), we complete the proof of (10.1).

So that we may find an asymptotic expansion of the integral in Entry 10, we first establish an analogue of Watson's Lemma (Olver [1, p. 71]).

Lemma 10.2. *Suppose that*

$$f(t) \sim \sum_{n=1}^{\infty} a_n t^n, \tag{10.9}$$

as t approaches 0. Then, as x tends to ∞,

$$\int_0^{\infty} \frac{f(t)}{e^{xt}-1} dt \sim \sum_{n=1}^{\infty} \frac{a_n n!\, \zeta(n+1)}{x^{n+1}},$$

provided that the integral converges for x sufficiently large, where $\zeta(z)$ denotes the Riemann zeta–function.

Proof. Let

$$f_m(t) := f(t) - \sum_{n=1}^{m-1} a_n t^n.$$

Then

$$\begin{aligned} \int_0^{\infty} \frac{f(t)}{e^{xt}-1} dt &= \int_0^{\infty} \frac{f_m(t)}{e^{xt}-1} dt + \sum_{n=1}^{m-1} a_n \int_0^{\infty} \frac{t^n}{e^{xt}-1} dt \\ &= \int_0^{\infty} \frac{f_m(t)}{e^{xt}-1} dt + \sum_{n=1}^{m-1} \frac{a_n n!\, \zeta(n+1)}{x^{n+1}}. \end{aligned} \tag{10.10}$$

The integral on the right side converges for x sufficiently large, because the corresponding integral with $f_m(t)$ replaced by $f(t)$ converges for all x sufficiently large.

As t tends to 0, $f_m(t) = O(t^m)$. Thus, for some positive constants k_m and K_m,

$$|f_m(t)| \leq K_m t^m, \qquad 0 < t \leq k_m.$$

Hence,

$$\left| \int_0^{k_m} \frac{f_m(t)}{e^{xt}-1} dt \right| \leq K_m \int_0^{k_m} \frac{t^m}{e^{xt}-1} dt < K_m \frac{m!\, \zeta(m+1)}{x^{m+1}}. \tag{10.11}$$

Let X be a value of x for which the integral on the right side of (10.10) converges. Now

$$F_m(t) := \int_{k_m}^{t} \frac{f_m(u)}{e^{Xu}-1} du$$

is continuous and bounded on $[k_m, \infty)$. Let $L_m = \sup_{k_m \le t < \infty} |F_m(t)|$. Then, for $x > X$,

$$\begin{aligned}\int_{k_m}^{\infty} \frac{f_m(t)}{e^{xt}-1}dt &= \int_{k_m}^{\infty} \frac{f_m(t)}{e^{Xt}-1}\frac{e^{Xt}-1}{e^{xt}-1}dt \\ &= F_m(t)\frac{e^{Xt}-1}{e^{xt}-1}\bigg|_{k_m}^{\infty} - \int_{k_m}^{\infty} F_m(t)\frac{d}{dt}\left(\frac{e^{Xt}-1}{e^{xt}-1}\right)dt \\ &= -\int_{k_m}^{\infty} F_m(t)\frac{d}{dt}\left(\frac{e^{Xt}-1}{e^{xt}-1}\right)dt.\end{aligned}$$

An elementary calculation shows that, for x sufficiently large,

$$\frac{d}{dt}\left(\frac{e^{Xt}-1}{e^{xt}-1}\right) < 0, \qquad t \ge k_m.$$

Thus,

$$\left|\int_{k_m}^{\infty} \frac{f_m(t)}{e^{xt}-1}dt\right| \le -L_m \int_{k_m}^{\infty} \frac{d}{dt}\left(\frac{e^{Xt}-1}{e^{xt}-1}\right)dt = L_m\frac{e^{Xk_m}-1}{e^{xk_m}-1}. \tag{10.12}$$

Taking (10.10)–(10.12) together, we deduce that

$$\int_0^{\infty} \frac{f(t)}{e^{xt}-1}dt = \sum_{n=1}^{m-1} \frac{a_n n!\, \zeta(n+1)}{x^{n+1}} + O\left(\frac{1}{x^{m+1}}\right),$$

as x tends to ∞, which completes the proof of the lemma.

Theorem 10.3. *As x tends to ∞,*

$$G(x) := \int_0^{\infty} \frac{t\,dt}{(e^{2\pi t}-1)(t+x)} \sim \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!\, \zeta(n+2)}{(2\pi)^{n+2} x^{n+1}}. \tag{10.13}$$

Proof. Write

$$G(x) = \frac{x}{(2\pi)^2}\int_0^{\infty} \frac{t\,dt}{(e^{xt}-1)(t/(2\pi)+1)}.$$

Since

$$\frac{t}{t/(2\pi)+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2\pi)^n} t^{n+1}, \qquad 0 \le t < 2\pi,$$

we may apply Lemma 10.2 with $f(t) = t/(t/(2\pi)+1)$ to deduce that, as x tends to ∞,

$$G(x) \sim \frac{x}{(2\pi)^2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n!\, \zeta(n+1)}{(2\pi)^{n-1} x^{n+1}}.$$

The proposed result is now immediate.

In particular, the leading term in (10.13) is $1/(24x)$. Since, in Theorem 10.1, $F(x) \sim \pi \log x/(2x)$, $F(x)$ and $G(x)$ are asymptotically quite different. At $x = 0$, $F(x)$ is analytic, while $G(x) \sim -\log x/(2\pi)$, as x tends to 0. In conclusion, we have not been able to discern a property of $F(x)$ and $G(x)$ held in common.

We quote Ramanujan in the next entry.

Entry 11 (p. 307).

$$\sum_{n=0}^{\infty} e^{-a^n x} = -\frac{\gamma + \log(x/\sqrt{a})}{\log a} \quad \textit{nearly.} \tag{11.1}$$

Here, γ denotes Euler's constant. Ramanujan evidently intends (11.1) to be an asymptotic formula as x tends to $0+$. Clearly, $a > 1$.

In 1907, Hardy [1, p. 283], [4, p. 160] proved that

$$\begin{aligned}\sum_{n=0}^{\infty} e^{-a^n x} = &-\frac{\log x}{\log a} + \frac{1}{2} - \frac{\gamma}{\log a} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n!\,(a^n - 1)} \\ &+ \frac{1}{\log a} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \Gamma\left(-\frac{2n\pi i}{\log a}\right) x^{2n\pi i/\log a}.\end{aligned} \tag{11.2}$$

(We have corrected a sign error in Hardy's formulation.) The first three terms on the right side of (11.2) are identical to the right side of (11.1). By Stirling's formula, the latter series on the right side of (11.2) converges absolutely for $0 \le x < \infty$.

Entry 12 (p. 307). *Let $a, b > 1$. Suppose further that*

$$\frac{2n\pi i}{\log a} \neq \frac{2m\pi i}{\log b},$$

for every pair of nonzero integers m, n. Then, for $x > 0$,

$$\begin{aligned}\sum_{m,n=0}^{\infty} e^{-a^n b^m x} = &\frac{\log^2 x}{2\log a \log b} + \log x \left(\frac{\gamma}{\log a \log b} - \frac{1}{2\log a} - \frac{1}{2\log b}\right) \\ &+ \frac{1}{12}\left(\frac{\log b}{\log a} + \frac{\log a}{\log b} + \frac{\pi^2 + 6\gamma^2}{\log a \log b}\right) - \frac{\gamma}{2}\left(\frac{1}{\log a} + \frac{1}{\log b}\right) + \frac{1}{4} \\ &+ \sum_{n=1}^{\infty} \frac{(-x)^n}{n!(a^n - 1)(b^n - 1)} + \frac{1}{\log a} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\Gamma\left(-\frac{2n\pi i}{\log a}\right) x^{2n\pi i/\log a}}{1 - b^{2n\pi i/\log a}} \\ &+ \frac{1}{\log b} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\Gamma\left(-\frac{2n\pi i}{\log b}\right) x^{2n\pi i/\log b}}{1 - a^{2n\pi i/\log b}},\end{aligned} \tag{12.1}$$

where γ denotes Euler's constant.

Equality (12.1) is a double series analogue of Hardy's theorem (11.2). The three series on the right side of (12.1) do not appear in Ramanujan's formulation. There is one further discrepancy in that Ramanujan omitted the expression

$$\frac{\pi^2}{12 \log a \log b}$$

on the right side of (12.1).

Proof. Set

$$f(x) := \sum_{m,n=0}^{\infty} e^{-a^n b^m x}.$$

Then, for $\sigma = \operatorname{Re} s > 0$,

$$\begin{aligned}\int_0^{\infty} f(x)x^{s-1}dx &= \sum_{m,n=0}^{\infty} \int_0^{\infty} e^{-a^n b^m x} x^{s-1} dx \\ &= \sum_{m,n=0}^{\infty} b^{-ms}a^{-ns}\Gamma(s) = \frac{\Gamma(s)}{(1-a^{-s})(1-b^{-s})}.\end{aligned}$$

By Mellin's inversion formula,

$$f(x) = \frac{1}{2\pi i}\int_{\alpha-\infty}^{\alpha+\infty} \frac{\Gamma(s)x^{-s}}{(1-a^{-s})(1-b^{-s})}ds, \qquad \alpha > 0. \tag{12.2}$$

Let

$$I_{M,T} := \int_{C_{M,T}} \frac{\Gamma(s)x^{-s}}{(1-a^{-s})(1-b^{-s})}ds, \tag{12.3}$$

where $C_{M,T}$ is a positively oriented rectangle with vertices at $\alpha \pm iT$ and $-M \pm iT$, where $T > 0$ and $M = N + \frac{1}{2}$, where N is a positive integer. We choose $T = T_n$, $n \geq 1$, tending to ∞ so that

$$|T_n \log a - k\pi| \geq \pi/3 \tag{12.4}$$

and

$$|T_n \log b - k\pi| \geq \pi/3, \tag{12.5}$$

for every positive integer k.

We evaluate $I_{M,T}$ by the residue theorem. The integrand in (12.3) has a triple pole at the origin and simple poles at $s = -n$, for each positive integer n. Furthermore, there are simple poles at $s = -2n\pi i/\log a$ and $s = -2m\pi i/\log b$, where m and n are nonzero integers. The latter two sets of poles are simple poles by hypothesis.

To calculate the residue at 0, we use the expansions

$$\Gamma(s)=\frac{1}{s}-\gamma+\left(\frac{\pi^2}{12}+\frac{\gamma^2}{2}\right)s-\left(\frac{\zeta(3)}{3}+\frac{\gamma\pi^2}{12}+\frac{\gamma^3}{6}\right)s^2+\cdots, \tag{12.6}$$

$$\frac{1}{1-a^{-s}}=\frac{1}{s\log a}\frac{(-s\log a)}{e^{-s\log a}-1}=\frac{1}{s\log a}+\frac{1}{2}+\frac{\log a}{12}s+\cdots, \tag{12.7}$$

$$\frac{1}{1-b^{-s}}=\frac{1}{s\log b}+\frac{1}{2}+\frac{\log b}{12}s+\cdots, \tag{12.8}$$

and

$$x^{-s}=1-s\log x+\tfrac{1}{2}s^2\log^2 x-\tfrac{1}{6}s^3\log^3 x+\cdots. \tag{12.9}$$

The expansion (12.6) can be deduced from a well-known formula found in the *Tables* of Gradshteyn and Ryzhik [1, p. 944, formula 8.321, no. 1]. After a lengthy calculation, we find that

$$R_0=\frac{\log^2 x}{2\log a\log b}+\log x\left(\frac{\gamma}{\log a\log b}-\frac{1}{2\log a}-\frac{1}{2\log b}\right)$$
$$+\frac{1}{12}\left(\frac{\log b}{\log a}+\frac{\log a}{\log b}+\frac{\pi^2+6\gamma^2}{\log a\log b}\right)-\frac{\gamma}{2}\left(\frac{1}{\log a}+\frac{1}{\log b}\right)+\frac{1}{4}. \tag{12.10}$$

The remaining residues are much easier to calculate. For each positive integer n,

$$R_{-n}=\frac{(-x)^n}{n!\,(a^n-1)(b^n-1)}. \tag{12.11}$$

For each nonzero integer n,

$$R_{-2n\pi i/\log a}=\frac{\Gamma\left(-\dfrac{2n\pi i}{\log a}\right)x^{2n\pi i/\log a}}{\log a(1-b^{2n\pi i/\log a})} \tag{12.12}$$

and

$$R_{-2n\pi i/\log b}=\frac{\Gamma\left(-\dfrac{2n\pi i}{\log b}\right)x^{2n\pi i/\log b}}{\log b(1-a^{2n\pi i/\log b})}. \tag{12.13}$$

Hence, using (12.10)–(12.13) in the residue theorem, we find that

$$I_{M,T}=R_0+\sum_{n=1}^{N}\frac{(-x)^n}{n!\,(a^n-1)(b^n-1)}$$
$$+\frac{1}{\log a}\sum_{|2n\pi i/\log a|<T}\frac{\Gamma\left(-\dfrac{2n\pi i}{\log a}\right)x^{2n\pi i/\log a}}{1-b^{2n\pi i/\log a}}$$

$$+\frac{1}{\log b}\sum_{|2n\pi i/\log b|<T}\frac{\Gamma\left(-\dfrac{2n\pi i}{\log b}\right)x^{2n\pi i/\log b}}{1-a^{2n\pi i/\log b}}. \tag{12.14}$$

By (12.2), (12.3), (12.10), and (12.14), if we can show that the integrals over the two horizontal sides tend to 0 as T tends to ∞ and that the integral over the left vertical side approaches 0 as M tends to ∞, then (12.1) follows.

By (12.4) and (12.5), we see that, for $s=\sigma\pm iT$,

$$|1-a^{-s}|\geq\sqrt{3}/2 \qquad \text{and} \qquad |1-b^{-s}|\geq\sqrt{3}/2,$$

respectively. Recall Stirling's formula

$$\Gamma(\sigma+it)=\sqrt{2\pi}e^{-\pi|t|/2}|t|^{\sigma-1/2}\{1+O(1/|t|)\},$$

uniformly for $-M\leq\sigma\leq\alpha$, as $|t|$ tends to ∞. Hence,

$$\int_{-M}^{\alpha}\frac{\Gamma(\sigma\pm iT)x^{-(\sigma\pm iT)}}{(1-a^{-(\sigma\pm iT)})(1-b^{-(\sigma\pm iT)})}d\sigma=o(1),$$

as T tends to ∞.

By the reflection formula for the gamma function,

$$\Gamma(-M+it)=\frac{(-1)^{N+1}\pi}{\Gamma(M+1-it)\cosh(\pi t)}.$$

Also, for $s=-M+it$, $|1-a^{-s}|\geq a^M/2$ and $|1-b^{-s}|\geq b^M/2$. Hence,

$$\int_{-\infty}^{\infty}\frac{\Gamma(-M+it)x^{M-it}}{(1-a^{M-it})(1-b^{M-it})}dt=o(1),$$

as M tends to ∞. This completes the proof.

Entry 13 (p. 307). *Let $a, b, c>1$, and assume that no two of the numbers*

$$\frac{2n\pi i}{\log a},\qquad \frac{2m\pi i}{\log b},\qquad \textit{and}\qquad \frac{2k\pi i}{\log c}$$

are equal, where n, m, and k are nonzero integers. Then, for $x>0$,

$$\begin{aligned}\sum_{k,m,n=0}^{\infty}e^{-a^nb^mc^kx} &= -\frac{\log^3 x}{6\log a\log b\log c}\\ &+\frac{\log^2 x}{2}\left\{\frac{1}{2}\left(\frac{1}{\log a\log b}+\frac{1}{\log b\log c}+\frac{1}{\log c\log a}\right)\right.\\ &\left.-\frac{\gamma}{\log a\log b\log c}\right\}-\log x\left\{\frac{1}{4}\left(\frac{1}{\log a}+\frac{1}{\log b}+\frac{1}{\log c}\right)\right.\\ &-\frac{\gamma}{2}\left(\frac{1}{\log a\log b}+\frac{1}{\log b\log c}+\frac{1}{\log c\log a}\right)\\ &+\frac{1}{12}\left(\frac{\log c}{\log a\log b}+\frac{\log a}{\log b\log c}+\frac{\log b}{\log c\log a}\right)\end{aligned}$$

$$+\left(\frac{\pi^2}{12}+\frac{\gamma^2}{2}\right)\frac{1}{\log a\log b\log c}\Bigg\}$$

$$+\frac{1}{24}\left(\frac{\log b+\log c}{\log a}+\frac{\log c+\log a}{\log b}+\frac{\log a+\log b}{\log c}\right)$$

$$+\left(\frac{\pi^2}{24}+\frac{\gamma^2}{4}\right)\left(\frac{1}{\log a\log b}+\frac{1}{\log b\log c}+\frac{1}{\log c\log a}\right)$$

$$-\frac{\gamma}{12}\left(\frac{\log c}{\log a\log b}+\frac{\log a}{\log b\log c}+\frac{\log b}{\log c\log a}\right)$$

$$-\frac{\gamma}{4}\left(\frac{1}{\log a}+\frac{1}{\log b}+\frac{1}{\log c}\right)-\frac{4\zeta(3)+\gamma\pi^2+2\gamma^3}{12\log a\log b\log c}+\frac{1}{8}$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n!\,(a^n-1)(b^n-1)(c^n-1)}$$

$$+\frac{1}{\log a}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{\Gamma\left(-\dfrac{2n\pi i}{\log a}\right)x^{2n\pi i/\log a}}{(1-b^{2n\pi i/\log a})(1-c^{2n\pi i/\log a})}$$

$$+\frac{1}{\log b}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{\Gamma\left(-\dfrac{2n\pi i}{\log b}\right)x^{2n\pi i/\log b}}{(1-c^{2n\pi i/\log b})(1-a^{2n\pi i/\log b})}$$

$$+\frac{1}{\log c}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{\Gamma\left(-\dfrac{2n\pi i}{\log c}\right)x^{2n\pi i/\log c}}{(1-a^{2n\pi i/\log c})(1-b^{2n\pi i/\log c})},\tag{13.1}$$

where γ denotes Euler's constant, and $\zeta(s)$ denotes the Riemann zeta–function.

The four infinite series on the right side of (13.1) do not appear in Ramanujan's formulation. Furthermore, the terms

$$-\frac{\pi^2\log x}{12\log a\log b\log c}$$

and

$$\frac{\pi^2}{24}\left(\frac{1}{\log a\log b}+\frac{1}{\log b\log c}+\frac{1}{\log c\log a}\right)-\frac{4\zeta(3)+\gamma\pi^2}{12\log a\log b\log c}$$

are not found in Ramanujan's version.

The proof of Entry 13 follows along the same lines as the proof of Entry 12. In particular, (12.6)–(12.9) are needed to calculate the residue of the quadruple pole at the origin. Therefore, we forego the proof.

In Entry 14 we quote Ramanujan.

Entry 14 (p. 314). *Let perimeter of ellipse* $= \pi(a+b)(1+h)$, *then*

$$\left(\frac{a-b}{a+b}\right)^2 = 4h - \frac{3h^2}{2+\sqrt{1-3h}} \tag{14.1}$$

very nearly. According to the above approximation, the perimeter of a parabola $= 3.99944(a+b)$ *for* $4(a+b)$.

Proof. Set $\lambda = (a-b)/(a+b)$. If L denotes the perimeter of the ellipse given by $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$, then (Part III [3, p. 146])

$$L = \pi(a+b)\,{}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; \lambda^2) =: \pi(a+b)(1+h). \tag{14.2}$$

Hence,

$$h = \frac{1}{4}\lambda^2 + \frac{1}{4^3}\lambda^4 + \frac{1}{4^4}\lambda^6 + \frac{25}{4^7}\lambda^8 + \frac{49}{4^8}\lambda^{10} + \frac{441}{4^{10}}\lambda^{12} + \cdots. \tag{14.3}$$

From (14.1), we find that, according to Ramanujan,

$$3h^4 + (16-4\lambda^2)h^2 + (\lambda^4 - 8\lambda^2)h + \lambda^4 \approx 0. \tag{14.4}$$

Beginning with the approximate solution $h = \lambda^4/4$, we solved (14.4) by the method of successive approximations to deduce that

$$h \approx \frac{1}{4}\lambda^2 + \frac{1}{4^3}\lambda^4 + \frac{1}{4^4}\lambda^6 + \frac{25}{4^7}\lambda^8 + \frac{49}{4^8}\lambda^{10} + \frac{439}{4^{10}}\lambda^{12} + \cdots. \tag{14.5}$$

Comparing (14.3) and (14.5), we find that the two power series agree up to the coefficient of λ^{12}, where the difference is remarkably only $2\lambda^{12}/4^{10}$. Thus, Ramanujan's claim is certainly justified.

The second claim in Entry 14 remained an enigma to us for a few years before R. J. Evans deciphered the proper interpretation.

Since the eccentricity for a parabola equals 1, set $b = 0$ in (14.1). Solving (14.1) via *Mathematica* when the left side equals 1, we find that $h = 0.2730576913$. Using this value of h in (14.2), we find that

$$L \approx \pi(a+b)(1+h) = 3.99943(a+b).$$

Taking $h = 0.27306$, we would obtain the approximation $3.99944(a+b)$ claimed by Ramanujan.

Lastly, we know that

$$L = 4\int_0^{\pi/2} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt.$$

If $b = 0$, then

$$L = 4\int_0^{\pi/2} a\sin t\,dt = 4a = 4(a+b).$$

This concludes the explanation of Ramanujan's strange claim.

The work in the following section first appeared in a paper by the author and Evans [1].

Before stating and proving Entry 15, we introduce some notation and offer a preliminary lemma. Ramanujan defines the logarithmic integral $\mathrm{Li}(x)$ by

$$\mathrm{Li}(x) = \mathrm{PV}\int_0^x \frac{dt}{\log t}, \qquad x > 0.$$

Define the unique positive number μ by

$$\mathrm{Li}(\mu) = 0. \tag{15.1}$$

Ramanujan [9] and Soldner (N. Nielsen [1, p. 88]) numerically calculated μ. We used MACSYMA to also calculate μ. The table below summarizes the three calculations:

Ramanujan	1.45136380
Soldner	1.4513692346
MACSYMA	1.4513692349

Lemma 15.1. *Let μ be defined by (15.1), and let γ denote Euler's constant. Then, for $x > 1$,*

$$\mathrm{Li}(x) = \gamma + \log\log x + \sum_{k=1}^{\infty} \frac{\log^k x}{k!\,k}.$$

Lemma 15.1 can be found in Ramanujan's notebooks on the same page as Entry 15, and a proof is given in Part IV [4, p. 126]. For another proof, see Nielsen's book [1, pp. 3, 11].

We first give Ramanujan's version of Entry 15: If

$$S := \sum_{k=1}^{n} \frac{1}{k}\left(1 + \frac{1}{p}\right)^k = \log p, \tag{15.2}$$

then $n = (p + \frac{1}{2})\log\mu - \frac{1}{2}$.

We might interpret Ramanujan's statement as giving an estimate for S when $n = [(p + \frac{1}{2})\log\mu - \frac{1}{2}]$. With such an interpretation, the error made in the approximation by $\log p$ is $O(1/p)$, as p tends to ∞. However, if p is chosen so that $n = (p + \frac{1}{2})\log\mu - \frac{1}{2}$ is a positive integer, then, as stated in our version of Entry 15 below, the error term is $O(1/p^2)$. Amazingly, Ramanujan found the precise linear function of p that yields an error term of $O(1/p^2)$. Thus, if the constant $\frac{1}{2}\log\mu - \frac{1}{2}$ in the definition of n is replaced by any other constant, the error term is $O(1/p)$.

Entry 15 (p. 318). *Let S be defined by (15.2), and let $n = (p + \frac{1}{2})\log\mu - \frac{1}{2}$ be a positive integer. Then, as p tends to ∞,*

$$S = \log p + O(p^{-2}). \tag{15.3}$$

Proof. Setting $y = \log \mu$, applying Lemma 15.1, and using (15.1), we find that

$$0 = \gamma + \log y + \sum_{k=1}^{\infty} \frac{y^k}{k!\,k} = \gamma + \log y + \int_0^y \frac{e^t - 1}{t} dt. \tag{15.4}$$

Setting $x = 1 + 1/p$ and using the definition of S, a familiar estimate for a partial sum of the harmonic series (Olver [1, p. 292]), and (15.4), we deduce that, as n tends to ∞,

$$\begin{aligned} S &= \sum_{k=1}^{n} \frac{x^k - 1}{k} + \sum_{k=1}^{n} \frac{1}{k} \\ &= \sum_{k=1}^{n} \frac{x^k - 1}{k} + \log n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right) \\ &= \sum_{k=1}^{n} \frac{x^k - 1}{k} + \log\left(\frac{n}{y}\right) + \frac{1}{2n} - \int_0^y \frac{e^t - 1}{t} dt + O\left(\frac{1}{p^2}\right). \end{aligned} \tag{15.5}$$

Next, applying the Euler–Maclaurin summation formula from (0.5) of Chapter 37, we deduce that, as p tends to ∞,

$$\sum_{k=1}^{n} \frac{x^k - 1}{k} = -\frac{1}{2}\log x + \frac{x^n - 1}{2n} + \int_0^n \frac{x^t - 1}{t} dt + O\left(\frac{1}{p^2}\right). \tag{15.6}$$

Employing (15.6) in (15.5), we find that

$$\begin{aligned} S &= \int_0^n \frac{x^t - 1}{t} dt - \int_0^y \frac{e^t - 1}{t} dt - \frac{1}{2}\log x + \log\left(\frac{n}{y}\right) + \frac{x^n}{2n} + O\left(\frac{1}{p^2}\right) \\ &= \int_y^{n \log x} f(t) dt - \frac{1}{2}\log x + \log\left(p + \frac{1}{2} - \frac{1}{2y}\right) + \frac{x^n}{2n} + O\left(\frac{1}{p^2}\right), \end{aligned} \tag{15.7}$$

where $f(t) = (e^t - 1)/t$. Now,

$$-\tfrac{1}{2}\log x = -\frac{1}{2p} + O\left(\frac{1}{p^2}\right),$$

$$\log\left(p + \frac{1}{2} - \frac{1}{2y}\right) = \log p + \frac{1}{2p} - \frac{1}{2py} + O\left(\frac{1}{p^2}\right),$$

and

$$\frac{x^n}{2n} = \frac{(1 + 1/p)^{py + (y-1)/2}}{2py + (y - 1)} = \frac{e^y}{2py} + O\left(\frac{1}{p^2}\right),$$

as p tends to ∞. Substituting these estimates in (15.7), we find that

$$S = \int_y^{n \log x} f(t)\, dt + \frac{f(y)}{2p} + \log p + O\left(\frac{1}{p^2}\right),$$

as p tends to ∞. Thus, it remains to prove that

$$\int_{n \log x}^{y} f(t)\, dt = \frac{f(y)}{2p} + O\left(\frac{1}{p^2}\right). \tag{15.8}$$

Now,

$$n \log x = y - \frac{1}{2p} + O\left(\frac{1}{p^2}\right). \tag{15.9}$$

By the first mean value theorem for integrals and (15.9),

$$\int_{n \log x}^{y} f(t)\, dt = f(u)(y - n \log x) = \frac{f(u)}{2p} + O\left(\frac{1}{p^2}\right), \tag{15.10}$$

for some value u such that $n \log x < u < y$. By the mean value theorem, there exists a value v such that $u < v < y$ and

$$f(u) = f(y) + (u - y) f'(v) = f(y) + O\left(\frac{1}{p}\right), \tag{15.11}$$

where the last equality follows from (15.9), since f' is bounded on (u, y). Using (15.11) in (15.10), we complete the proof of (15.8).

So as not to interrupt the proof of Entry 16 below with two calculations, we now set them aside in two lemmas.

Lemma 16.1. *If a and θ are positive and n is a nonnegative integer, then*

$$I_n := \frac{1}{\sqrt{\pi\theta}} \int_{-\infty+ai}^{\infty+ai} z^{2n} e^{-z^2/\theta} dz = \frac{\theta^n (2n)!}{2^{2n} n!}.$$

Proof. Setting $u = z/\sqrt{\theta}$ and applying Cauchy's theorem, we find that

$$\begin{aligned} I_n &= \frac{\theta^n}{\sqrt{\pi}} \int_{-\infty+ai/\sqrt{\theta}}^{\infty+ai/\sqrt{\theta}} u^{2n} e^{-u^2} du \\ &= \frac{\theta^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} u^{2n} e^{-u^2} du \\ &= \frac{\theta^n}{\sqrt{\pi}} \int_{0}^{\infty} t^{n-1/2} e^{-t} dt \\ &= \frac{\theta^n}{\sqrt{\pi}} \Gamma(n + \tfrac{1}{2}) = \frac{\theta^n (2n)!}{2^{2n} n!}. \end{aligned}$$

Lemma 16.2. *If a and θ are positive, and n is a nonnegative integer, then*

$$J_n := \frac{1}{\sqrt{\pi\theta}} \int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta} e^{(2n+1)iz} dz = e^{-(n+1/2)^2\theta}.$$

Proof. Setting $z = u\sqrt{\theta}$ and applying Cauchy's theorem, we find that

$$\begin{aligned} J_n &= \frac{1}{\sqrt{\pi}} \int_{-\infty+ai/\sqrt{\theta}}^{\infty+ai/\sqrt{\theta}} \exp\left(-\left(u - (n+\tfrac{1}{2})i\sqrt{\theta}\right)^2 - (n+\tfrac{1}{2})^2\theta \right) du \\ &= \frac{1}{\sqrt{\pi}} e^{-(n+1/2)^2\theta} \int_{-\infty}^{\infty} e^{-u^2} du \\ &= e^{-(n+1/2)^2\theta}. \end{aligned}$$

Entry 16 (p. 324). *As t tends to* $0+$,

$$2\sum_{n=0}^{\infty}(-1)^n \left(\frac{1-t}{1+t}\right)^{n(n+1)} \sim 1 + t + t^2 + 2t^3 + 5t^4 + 17t^5 + \cdots. \tag{16.1}$$

The function on the left side of (16.1) is not a theta–function but is a false theta–function in the sense of Rogers [3]. In fact, we shall obtain a more explicit asymptotic expansion, which enables the calculation of further terms in the asymptotic series. It is interesting that Ramanujan appended an abbreviation for "asymptotically" after the series on the right side. We are unaware of any other instance in the notebooks where Ramanujan used this word. Usually, he wrote "nearly" or "very nearly."

Proof. Let

$$e^{-\theta} := \frac{1-t}{1+t},$$

so that θ is small and positive. If $a > 0$ and n is a nonnegative integer, by Lemma 16.2,

$$e^{-(n+1/2)^2\theta} = \frac{1}{\sqrt{\pi\theta}} \int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta} e^{(2n+1)iz} dz.$$

Multiply both sides by $2(-1)^n$ and sum on $n, 0 \le n < \infty$. Upon inverting the order of integration and summation, we see that the resulting series on the right side converges absolutely and uniformly on $(-\infty + ai, \infty + ai)$ and that the resulting integral also converges absolutely, and so the inversion is justified. Hence,

$$\begin{aligned} 2\sum_{n=0}^{\infty}(-1)^n e^{-(n+1/2)^2\theta} &= \frac{2}{\sqrt{\pi\theta}} \int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta} \sum_{n=0}^{\infty}(-1)^n e^{(2n+1)iz} dz \\ &= \frac{1}{\sqrt{\pi\theta}} \int_{-\infty+ai}^{\infty+ai} \frac{e^{-z^2/\theta}}{\cos z} dz. \end{aligned} \tag{16.2}$$

Now recall that (Abramowitz and Stegun [1, p. 804])

$$\sec z = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} z^{2n}, \qquad |z| < \pi/2,$$

where E_j, $j \geq 0$, denotes the jth Euler number. Thus, for $|z| \leq \pi/4$,

$$\left|\sec z - \sum_{n=0}^{N} \frac{(-1)^n E_{2n}}{(2n)!} z^{2n}\right| \leq C_1 |z|^{2N+2},$$

where the positive constant C_1 depends on N but not on z. If $0 < a \leq 1$ and $|z| \geq \pi/4$ on the contour $(-\infty + ai, \infty + ai)$, then

$$\frac{a}{\cos z}, \qquad \frac{1}{z^{2N+2}}, \qquad \text{and} \qquad \frac{1}{z^{2N+2}} \sum_{n=0}^{N} \frac{(-1)^n E_{2n}}{(2n)!} z^{2n}$$

are bounded functions of both a and z. In particular, observe that

$$\cos((2n-1)\pi/2 + ia) = i(-1)^n \sinh a,$$

where n is an integer, and so $a/\cos z$ remains bounded as a tends to 0. Hence, for all points on the contour,

$$\left|\frac{a}{z^{2N+2}} \left(\sec z - \sum_{n=0}^{N} \frac{(-1)^n E_{2n}}{(2n)!} z^{2n}\right)\right| \leq C_2, \tag{16.3}$$

for some positive constant C_2, which is independent of both z and a.

Therefore, by (16.2) and Lemma 16.1,

$$\begin{aligned} 2\sum_{n=0}^{\infty}(-1)^n e^{-(n+1/2)^2\theta} &= \frac{1}{\sqrt{\pi\theta}} \sum_{n=0}^{N} \frac{(-1)^n E_{2n}}{(2n)!} \int_{-\infty+ai}^{\infty+ai} z^{2n} e^{-z^2/\theta} dz + R_N \\ &= \sum_{n=0}^{N} \frac{(-1)^n E_{2n}\theta^n}{2^{2n} n!} + R_N, \end{aligned} \tag{16.4}$$

where, by (16.3),

$$\begin{aligned} |R_N| &= \left|\frac{1}{\sqrt{\pi\theta}} \int_{-\infty+ai}^{\infty+ai} e^{-z^2/\theta} \left(\sec z - \sum_{n=0}^{N} \frac{(-1)^n E_{2n}}{(2n)!} z^{2n}\right) dz\right| \\ &\leq \frac{C_2}{a\sqrt{\pi\theta}} \int_{-\infty}^{\infty} \left|e^{-(x+ai)^2/\theta}(x+ai)^{2N+2}\right| dx \\ &= \frac{C_2}{a\sqrt{\pi\theta}} \int_{-\infty}^{\infty} e^{(a^2-x^2)/\theta}(x^2+a^2)^{N+1} dx \\ &\leq \frac{C_2}{a\sqrt{\pi\theta}} \int_{-\infty}^{\infty} e^{(a^2-x^2)/\theta} \left\{(2x^2)^{N+1} + (2a^2)^{N+1}\right\} dx, \end{aligned}$$

since $x^2 + a^2$ does not exceed the larger of $2x^2$ and $2a^2$. Setting $u = x^2/\theta$ and evaluating the resulting integrals in terms of gamma functions, we find that

$$|R_N| < \frac{2^{N+1} C_2 e^{a^2/\theta}}{a\sqrt{\pi\theta}} \left\{\Gamma(N + \tfrac{3}{2})\theta^{N+3/2} + \Gamma(\tfrac{1}{2}) a^{2N+2} \theta^{1/2}\right\}.$$

We now choose $a = \sqrt{\theta}$. Hence,

$$|R_N| < \frac{2^{N+1}C_2 e}{\sqrt{\pi}} \left\{\Gamma(N+\tfrac{3}{2}) + \sqrt{\pi}\right\} \theta^{N+1/2}. \tag{16.5}$$

In conclusion, by (16.4) and (16.5), we have obtained the asymptotic expansion

$$2\sum_{n=0}^{\infty}(-1)^n \left(\frac{1-t}{1+t}\right)^{n(n+1)} \sim \left(\frac{1+t}{1-t}\right)^{1/4} \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{2^{2n} n!} \log^n\left(\frac{1+t}{1-t}\right). \tag{16.6}$$

Using the values $E_{2n} = 1, -1, 5, -61, 1385$, and -50521, where $n = 0, 1, 2, 3$, 4, and 5, respectively, and employing *Mathematica*, we calculated the coefficients displayed on the right side of (16.1).

Using operator methods, D. Bradley found a shorter route to (16.6).

It is curious that each of the six coefficients on the right side of (16.1) is a positive integer. Using *Mathematica*, we calculated the first 50 coefficients and found all of them to be positive integers. Using his own multiple–precision integer arithmetic package, Brent calculated the first 1000 coefficients to further substantiate the conjecture that all coefficients are positive integers. Brent easily showed that the coefficients are positive, while Galway [1] later established the more difficult assertion that they are integral.

Entry 17 (p. 333). *For $n \geq 0$ and $x > 0$, define*

$$u_n := u_n(x) := \Gamma(n+1)\left(e^{-n^2x/4} + \sum_{k=1}^{\infty} \frac{(-1)^k (n+2k)(n+1)_{k-1} e^{-(n+2k)^2x/4}}{k!}\right).$$

Then

$$u_{n+2} = \frac{n^2}{4}u_n + \frac{du_n}{dx} \tag{17.1}$$

and

$$u_n = \frac{1}{2^{n-1}}\left(\frac{\pi}{x}\right)^{n+1/2} e^{-\pi^2/(4x)}\left(1 - \frac{n(n-1)x}{\pi^2} + \cdots\right). \tag{17.2}$$

Proof. We first prove (17.1). After a simple calculation,

$$\begin{aligned}
&\frac{n^2}{4}u_n + \frac{du_n}{dx} \\
&= \Gamma(n+1)\sum_{k=1}^{\infty} \frac{(-1)^k (n+2k)(n+1)_{k-1} e^{-(n+2k)^2x/4}}{k!}\left(\frac{n^2}{4} - \frac{(n+2k)^2}{4}\right) \\
&= \Gamma(n+1)\Bigg((n+2)(n+1)e^{-(n+2)^2x/4} \\
&\quad + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}(n+2k)(n+1)_k e^{-(n+2k)^2x/4}}{(k-1)!}\Bigg)
\end{aligned}$$

$$= \Gamma(n+3)\left(e^{-(n+2)^2x/4}\sum_{k=1}^{\infty}\frac{(-1)^k(n+2+2k)(n+1)_{k+1}e^{-(n+2+2k)^2x/4}}{k!(n+1)(n+2)}\right)$$

$$= u_{n+2}.$$

Thus, (17.1) has been shown.

We are pleased to present M. L. Glasser's proof of (17.2). We induct on n. First observe that

$$u_0 = 1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-k^2x} = \varphi(-e^{-x}) = \vartheta_4(0, ix/\pi),$$

in the notation of Ramanujan (Part III [3, p. 36]) and, e.g., H. Rademacher [1, p. 166], respectively. Now (Rademacher [1, p. 177]),

$$\begin{aligned} u_0 = \vartheta_4(0, ix/\pi) &= \sqrt{\frac{\pi}{x}}\vartheta_2(0, -\pi/(ix)) \\ &= \sqrt{\frac{\pi}{x}}\sum_{n=-\infty}^{\infty} e^{-\pi^2(2n+1)^2/(4x)} \\ &= 2\sqrt{\frac{\pi}{x}}e^{-\pi^2/(4x)}\left(1 + e^{-2\pi^2/x} + \cdots\right), \end{aligned}$$

which agrees with (17.2) for $n = 0$.

Next recall that (Rademacher [1, p. 166])

$$\vartheta_1(v, ix/\pi) = -i\sum_{n=-\infty}^{\infty}(-1)^n e^{-(n+1/2)^2x}e^{(2n+1)\pi i v} \tag{17.3}$$

and (Rademacher [1, p. 177])

$$\vartheta_1(v, ix/\pi) = i\sqrt{\frac{\pi}{x}}e^{-\pi^2v^2/x}\vartheta_1(v\pi/(ix), -\pi/(ix)). \tag{17.4}$$

Hence, by (17.3) and (17.4),

$$\begin{aligned} u_1 &= \sum_{n=0}^{\infty}(-1)^n(2n+1)e^{-(2n+1)^2x/4} \\ &= \frac{1}{2\pi}\frac{d}{dv}\vartheta_1(v, ix/\pi)\bigg|_{v=0} \\ &= \frac{1}{2\pi}\frac{d}{dv}\left(i\sqrt{\frac{\pi}{x}}e^{-\pi^2v^2/x}\vartheta_1(v\pi/(ix), -\pi/(ix))\right)\bigg|_{v=0} \\ &= \frac{1}{2\sqrt{\pi x}}\frac{d}{dv}\left(\sum_{n=-\infty}^{\infty}(-1)^n e^{-\pi^2(n+1/2)^2/x-\pi^2v^2/x+(2n+1)\pi^2v/x}\right)\bigg|_{v=0} \\ &= \frac{1}{2\sqrt{\pi x}}\sum_{n=-\infty}^{\infty}(-1)^n\frac{(2n+1)\pi^2}{x}e^{-\pi^2(n+1/2)^2/x} \end{aligned}$$

$$= \left(\frac{\pi}{x}\right)^{3/2} e^{-\pi^2/(4x)} \left(1 - 3e^{-2\pi^2/x} + \cdots\right),$$

which agrees with (17.2) in the case $n = 1$.

Having proved (17.2) for $n = 0, 1$, we assume that (17.2) is valid and that it is also valid with n replaced by $n+1$. We shall then use these two equalities and (17.1) to establish (17.2) with n replaced by $n + 2$. By straightforward differentiation,

$$\begin{aligned} u_{n+2} &= \frac{n^2}{4}u_n + \frac{du_n}{dx} \\ &= \frac{1}{2^{n-1}}\left(\frac{\pi}{x}\right)^{n+1/2} e^{-\pi^2/(4x)} \left(\frac{n^2}{4} - \frac{n^3(n-1)x}{4\pi^2} + \cdots - \frac{2n+1}{2x}\right. \\ &\quad \left. + \frac{(2n+1)n(n-1)}{2\pi^2} - \cdots + \frac{\pi^2}{4x^2} - \frac{n(n-1)}{4x} + \cdots - \frac{n(n-1)}{\pi^2} + \cdots\right) \\ &= \frac{1}{2^{n-1}}\left(\frac{\pi}{x}\right)^{n+1/2} e^{-\pi^2/(4x)} \left(\frac{\pi^2}{4x^2} - \frac{1}{x}\left(\frac{2n+1}{2} + \frac{n(n-1)}{4}\right) + \cdots\right) \\ &= \frac{1}{2^{n+1}}\left(\frac{\pi}{x}\right)^{n+5/2} e^{-\pi^2/(4x)} \left(1 - \frac{x}{\pi^2}(n+2)(n+1) + \cdots\right), \end{aligned}$$

which is precisely (17.2) with n replaced by $n + 2$.

The function u_n can be regarded as a modified theta–function or a modified hypergeometric function. Recalling the proofs of (17.2) for $n = 0$ and $n = 1$, in general, we can regard (17.2) as an inversion formula for a modified theta–function. It would be interesting to find further terms and even more interesting to find an exact inversion formula. Entry 17 can also be regarded as an analogue of the many asymptotic expansions for exponential series found in the first seven sections of Chapter 15 (Part II [2, pp. 303–314]). V. Kowalenko, N. E. Frankel, M. L. Glasser, and T. Taucher [1] have considerably generalized and extended these results. In particular, Ramanujan's results do not cover certain "exceptional cases," and these are encompassed in their theorems.

The origins of the two families of approximations in the next two entries were a mystery to us until Askey [1] explained their roots in Gaussian quadrature with respect to a discrete measure. Thus, the following proofs are due to Askey, although we have amplified the details.

Entry 18 (p. 349). *Let*

$$S(x, n) := \sum_{k=0}^{n-1} \varphi(x - n + 1 + 2k).$$

Then $S(x, n)/n$ has the following successive approximations:

$$\varphi(x), \tag{18.1}$$

$$\frac{1}{2}\left\{\varphi\left(x + \sqrt{\frac{n^2-1}{3}}\right) + \varphi\left(x - \sqrt{\frac{n^2-1}{3}}\right)\right\}, \tag{18.2}$$

$$\frac{5(n^2-1)\left\{\varphi\left(x+\sqrt{\frac{3n^2-7}{5}}\right)+\varphi\left(x-\sqrt{\frac{3n^2-7}{5}}\right)\right\}+8(n^2-4)\varphi(x)}{6(3n^2-7)}, \tag{18.3}$$

and

$$\left(\frac{1}{4}-\frac{n^2-16}{6\beta}\right)\left\{\varphi\left(x+\sqrt{\frac{\alpha+\beta}{7}}\right)+\varphi\left(x-\sqrt{\frac{\alpha+\beta}{7}}\right)\right\}$$
$$+\left(\frac{1}{4}+\frac{n^2-16}{6\beta}\right)\left\{\varphi\left(x+\sqrt{\frac{\alpha-\beta}{7}}\right)+\varphi\left(x-\sqrt{\frac{\alpha-\beta}{7}}\right)\right\}, \tag{18.4}$$

where

$$\alpha = 3n^2-13 \qquad \textit{and} \qquad \beta=\sqrt{\tfrac{4}{5}(6n^4-45n^2+164)}. \tag{18.5}$$

Proof. Let $f(t)$ be a continuous function on $[a,b]$, and let $d\alpha(t)$ be a nonnegative measure on $[a,b]$. The problem of Gaussian quadrature is to approximate

$$\int_a^b f(t)\,d\alpha(t) \tag{18.6}$$

by a finite sum which is exact for all polynomials of as high a degree as possible. Let $a<t_1<t_2<\cdots<t_k<b$ and set

$$w_k(t)=\prod_{i=1}^{k}(t-t_i)$$

and

$$w_{j,k}(t)=\frac{w_k(t)}{w_k'(t_j)(t-t_j)}, \qquad 1\le j\le k. \tag{18.7}$$

Then

$$L_k^f(t):=\sum_{j=1}^{k} f(t_j)w_{j,k}(t) \tag{18.8}$$

is a polynomial of degree at most $k-1$ such that

$$L_k^f(t_j)=f(t_j), \qquad 1\le j\le k. \tag{18.9}$$

Now let $f(t)$ be a polynomial of degree not exceeding $k-1$. Then $f(t)=L_k^f(t)$, since, by (18.9), these two polynomials of degree $k-1$, or less, agree at k points. Hence, by (18.8),

$$\int_a^b f(t)\,d\alpha(t)=\sum_{j=1}^{k} f(t_j)\int_a^b w_{j,k}(t)\,d\alpha(t). \tag{18.10}$$

Thus, we have an exact quadrature formula for polynomials of degree $k-1$.

We now want to keep the number of terms, k, constant and choose the points t_j, $1 \leq j \leq k$, so that (18.10) is exact for polynomials of degree as high as possible. We will show that, if the points t_j, $1 \leq j \leq k$, are appropriately chosen, (18.10) is exact for polynomials of degree $2k-1$. Then

$$f(t) - L_k^f(t) = w_k(t) r_{k-1}(t), \tag{18.11}$$

where $r_{k-1}(t)$ is a polynomial of degree $k-1$.

Now let $\{w_k(t)\}$, $0 \leq k < \infty$, be a set of polynomials with $w_k(t)$ of degree k, which are orthogonal on $[a, b]$ with respect to the measure $d\alpha(t)$. Then, of course, $w_k(t)$ is orthogonal to all polynomials of degree $< k$. Then, by (18.11),

$$\int_a^b f(t)\, d\alpha(t) - \int_a^b L_k^f(t)\, d\alpha(t) = \int_a^b w_k(t) r_{k-1}(t)\, d\alpha(t) = 0. \tag{18.12}$$

If

$$\lambda_j := \lambda_{j,k} := \int_a^b w_{j,k}(t)\, d\alpha(t), \tag{18.13}$$

then, by (18.10) and (18.12),

$$\int_a^b f(t)\, d\alpha(t) = \sum_{j=1}^{k} \lambda_j f(t_j),$$

and we have an exact quadrature formula of k terms for a polynomial of degree $2k-1$. In general, for an arbitrary continuous function $f(t)$ on $[a, b]$, the Gaussian quadrature approximation of (18.6) equals

$$\sum_{j=1}^{k} \lambda_j f(t_j). \tag{18.14}$$

Other representations for λ_j, given by (18.13), exist; e.g., see G. Szegö's book [1, p. 48]. Two of these representations show immediately that $\lambda_j > 0$.

We now apply this theory to $f(t) = \varphi(x - n + 1 + 2t)$. Since $S(x, n)$ is a sum of n terms, we want $d\alpha(t)$ to be a discrete measure weighted at the integral points $0, 1, 2, \ldots, n-1$. This leads us to the Hahn polynomials, which were introduced by P. L. Tchebychef [1] in 1875 and which are constant multiples of

$$\begin{aligned} Q_k(t; \alpha, \beta, N) &:= {}_3F_2\left[\begin{matrix} -k, k+\alpha+\beta+1, -t \\ \alpha+1, -N \end{matrix}; 1\right] \\ &= \sum_{j=0}^{k} \frac{(-k)_j (k+\alpha+\beta+1)_j (-t)_j}{(\alpha+1)_j (-N)_j\, j!}, \qquad 0 \leq k < \infty. \end{aligned} \tag{18.15}$$

For $0 \leq m, n \leq N$, they satisfy an orthogonality relation of the form (A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov [1, p. 33])

$$\sum_{t=0}^{N} Q_m(t; \alpha, \beta, N) Q_n(t; \alpha, \beta, N) \frac{(\alpha+1)_t}{t!} \frac{(\beta+1)_{N-t}}{(N-t)!} = c_{m,n} \delta_{m,n},$$

for certain constants $c_{m,n}$, $0 \le m, n < \infty$.

In our application, we want $\alpha = \beta = 0$, so that the weights equal 1 at each nonnegative integer k, $0 \le k \le N := n - 1$. Thus, $w_k(t) = c_k Q_k(t; 0, 0, N)$ for some constant c_k, $0 \le k \le N$. Although the value of c_k is not needed in applications to Gaussian quadrature, we see that, by (18.15),

$$c_k = \frac{(-1)^k (2k)!\,(N-k)!}{(k!)^2 N!}.$$

We are now ready to calculate the four approximations (18.1)–(18.4) claimed by Ramanujan.

Let $k = 1$. By (18.7), $w_{1,1}(t) = 1$, and by (18.13),

$$\lambda_1 = \int_0^{n-1} d\alpha(t) = n.$$

By (18.15),

$$Q_1(t; 0, 0, N) = 1 - \frac{2t}{N}.$$

Hence, $t_1 = N/2$. Hence, by (18.14), the first approximation to $S(x, n)$ is

$$\lambda_1 f(t_1) = n\varphi(x - n + 1 + (n-1)) = n\varphi(x),$$

as claimed.

Second, let $k = 2$. Then, by (18.15),

$$Q_2(t; 0, 0, N) = 1 - \frac{6t}{N} + \frac{6t(t-1)}{N(N-1)}.$$

We therefore want the roots of

$$6t^2 - 6tN + N(N-1) = 0,$$

and they are

$$t_1, t_2 = \frac{1}{2}\left(n - 1 \pm \sqrt{\frac{n^2-1}{3}}\right).$$

Hence,

$$f(t_1), f(t_2) = \varphi\left(x \pm \sqrt{\frac{n^2-1}{3}}\right).$$

By (18.7), we find that

$$w_{1,2}(t) = \frac{t - t_2}{\sqrt{\dfrac{n^2-1}{3}}} \qquad \text{and} \qquad w_{2,2}(t) = -\frac{t - t_1}{\sqrt{\dfrac{n^2-1}{3}}}.$$

Hence, by (18.13),

$$\lambda_1 = \sqrt{\frac{3}{n^2-1}} \int_0^{n-1} (t - t_2)\, d\alpha(t)$$
$$= \sqrt{\frac{3}{n^2-1}} \left(\sum_{j=0}^{n-1} j - \frac{n}{2}\left(n - 1 - \sqrt{\frac{n^2-1}{3}}\right)\right) = \frac{n}{2},$$

and, by a similar calculation,

$$\lambda_2 = -\sqrt{\frac{3}{n^2-1}} \int_0^{n-1} (t - t_1)\, d\alpha(t) = \frac{n}{2}.$$

Thus, by (18.14), the second approximation of $S(x, n)$ equals

$$\frac{n}{2}\varphi\left(x + \sqrt{\frac{n^2-1}{3}}\right) + \frac{n}{2}\varphi\left(x - \sqrt{\frac{n^2-1}{3}}\right),$$

as claimed in (18.2).

Third, let $k = 3$. Then, by (18.15),

$$Q(t; 0, 0, N) = 1 - \frac{12t}{N} + \frac{30t(t-1)}{N(N-1)} - \frac{20t(t-1)(t-2)}{N(N-1)(N-2)}.$$

We therefore must solve

$$20t^3 - 30Nt^2 + (4 - 6N + 12N^2)t - N(N-1)(N-2) = 0.$$

These roots are found to be

$$2t_1 = n - 1 - \sqrt{\frac{3n^2-7}{5}}, \quad 2t_2 = n - 1, \quad \text{and} \quad 2t_3 = n - 1 + \sqrt{\frac{3n^2-7}{5}}.$$

Thus,

$$f(t_1) = \varphi\left(x - \sqrt{\frac{3n^2+7}{5}}\right), \; f(t_2) = \varphi(x), \; \text{and}\; f(t_3) = \varphi\left(x + \sqrt{\frac{3n^2+7}{5}}\right). \tag{18.16}$$

Also, by (18.7),

$$w_{1,3}(t) = \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)}$$
$$= \frac{10}{3n^2-7}\left(t^2 - \left\{(n-1) + \frac{1}{2}\sqrt{\frac{3n^2-7}{5}}\right\} t \right.$$
$$\left. + \frac{1}{4}\left\{(n-1)^2 + (n-1)\sqrt{\frac{3n^2-7}{5}}\right\}\right),$$

$$w_{2,3}(t) = \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)}$$
$$= -\frac{20}{3n^2-7}\left(t^2-(n-1)t+\tfrac{1}{10}n^2-\tfrac{1}{2}n+\tfrac{3}{5}\right),$$

and

$$w_{3,3}(t) = \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}$$
$$= \frac{10}{3n^2-7}\left(t^2-\left\{(n-1)-\frac{1}{2}\sqrt{\frac{3n^2-7}{5}}\right\}t\right.$$
$$\left.+\frac{1}{4}\left\{(n-1)^2-(n-1)\sqrt{\frac{3n^2-7}{5}}\right\}\right).$$

Thus, by (18.13),

$$\lambda_1 = \frac{10}{3n^2-7}\int_0^{n-1}\left(t^2-\left\{(n-1)+\frac{1}{2}\sqrt{\frac{3n^2-7}{5}}\right\}t\right.$$
$$\left.+\frac{1}{4}\left\{(n-1)^2+(n-1)\sqrt{\frac{3n^2-7}{5}}\right\}\right)d\alpha(t)$$
$$= \frac{10}{3n^2-7}\left(\frac{(n-1)n(2n-1)}{6}-\left\{n-1+\frac{1}{2}\sqrt{\frac{3n^2-7}{5}}\right\}\frac{n(n-1)}{2}\right.$$
$$\left.+\frac{1}{4}\left\{(n-1)^2+(n-1)\sqrt{\frac{3n^2-7}{5}}\right\}n\right)$$
$$= \frac{5n(n^2-1)}{6(3n^2-7)}. \tag{18.17}$$

By almost an identical calculation,

$$\lambda_3 = \frac{5n(n^2-1)}{6(3n^2-7)}. \tag{18.18}$$

Lastly, by (18.13),

$$\lambda_2 = -\frac{20}{3n^2-7}\int_0^{n-1}\left(t^2-(n-1)t+\tfrac{1}{10}n^2-\tfrac{1}{2}n+\tfrac{3}{5}\right)d\alpha(t)$$
$$= -\frac{20}{3n^2-7}\left(\tfrac{1}{6}(n-1)n(2n-1)-\tfrac{1}{2}(n-1)^2n+\left(\tfrac{1}{10}n^2-\tfrac{1}{2}n+\tfrac{3}{5}\right)n\right)$$
$$= \frac{4n(n^2-4)}{3(3n^2-7)}. \tag{18.19}$$

Using (18.16)–(18.19) in (18.14), we obtain Ramanujan's approximation (18.3). Fourth, let $k = 4$. Then, by (18.15),

$$Q_4(t;0,0,N) = 1 - \frac{20t}{N} + \frac{90t(t-1)}{N(N-1)} - \frac{140t(t-1)(t-2)}{N(N-1)(N-2)} + \frac{70t(t-1)(t-2)(t-3)}{N(N-1)(N-2)(N-3)}.$$

We used *Mathematica* to determine the roots of $Q_4(t;0,0,N) = 0$ and found that

$$t_1 = \frac{1}{2}\left(N + \sqrt{\frac{\alpha+\beta}{7}}\right), \qquad t_2 = \frac{1}{2}\left(N - \sqrt{\frac{\alpha+\beta}{7}}\right),$$
$$t_3 = \frac{1}{2}\left(N + \sqrt{\frac{\alpha-\beta}{7}}\right), \qquad t_4 = \frac{1}{2}\left(N - \sqrt{\frac{\alpha-\beta}{7}}\right), \tag{18.20}$$

where α and β are given in (18.5). For brevity, set $A = (\alpha+\beta)/7$ and $B = (\alpha-\beta)/7$. By (18.13),

$$\begin{aligned}
\lambda_1 &= \int_0^{n-1} \frac{(t-t_2)(t-t_3)(t-t_4)}{(t_1-t_2)(t_1-t_3)(t_1-t_4)}\,dt \\
&= \frac{14}{\beta\sqrt{A}} \int_0^{n-1} \left(t^3 - (t_2+t_3+t_4)t^2 + (t_2t_3+t_3t_4+t_4t_2)t - t_2t_3t_4\right) d\alpha(t) \\
&= \frac{14}{\beta\sqrt{A}} \left(\frac{n^2(n-1)^2}{4} - \left(\frac{3(n-1)}{2} - \frac{1}{2}\sqrt{A}\right)\frac{(n-1)n(2n-1)}{6}\right. \\
&\quad + \left(\frac{3(n-1)^2}{4} - \frac{1}{2}(n-1)\sqrt{A} - \frac{1}{4}B\right)\frac{n(n-1)}{2} \\
&\quad \left. - \frac{1}{8}(n-1-\sqrt{A})((n-1)^2 - B)n\right) \\
&= \frac{14}{\beta}\left(\frac{n(n^2-1)}{24} - \frac{n(\alpha-\beta)}{56}\right) \\
&= n\left(\frac{16-n^2}{6\beta} + \frac{1}{4}\right).
\end{aligned} \tag{18.21}$$

The calculations of λ_2, λ_3, and λ_4 are similar, and we find that

$$\begin{aligned}
\lambda_2 &= -\frac{14}{\beta\sqrt{A}} \left(\frac{n^2(n-1)^2}{4} - \left(\frac{3(n-1)}{2} + \frac{1}{2}\sqrt{A}\right)\frac{(n-1)n(2n-1)}{6}\right. \\
&\quad + \left(\frac{3(n-1)^2}{4} + \frac{1}{2}(n-1)\sqrt{A} - \frac{1}{4}B\right)\frac{n(n-1)}{2} \\
&\quad \left. - \frac{1}{8}(n-1+\sqrt{A})((n-1)^2 - B)n\right) \\
&= n\left(\frac{16-n^2}{6\beta} + \frac{1}{4}\right),
\end{aligned} \tag{18.22}$$

$$\lambda_3 = -\frac{14}{\beta\sqrt{B}}\left(\frac{n^2(n-1)^2}{4} - \left(\frac{3(n-1)}{2} - \frac{1}{2}\sqrt{B}\right)\frac{(n-1)n(2n-1)}{6}\right.$$
$$+\left(\frac{3(n-1)^2}{4} - \frac{1}{2}(n-1)\sqrt{B} - \frac{1}{4}A\right)\frac{n(n-1)}{2}$$
$$\left. - \frac{1}{8}(n-1-\sqrt{B})((n-1)^2 - A)n\right)$$
$$= n\left(\frac{-16+n^2}{6\beta} + \frac{1}{4}\right), \tag{18.23}$$

and

$$\lambda_4 = \frac{14}{\beta\sqrt{B}}\left(\frac{n^2(n-1)^2}{4} - \left(\frac{3(n-1)}{2} + \frac{1}{2}\sqrt{B}\right)\frac{(n-1)n(2n-1)}{6}\right.$$
$$+\left(\frac{3(n-1)^2}{4} + \frac{1}{2}(n-1)\sqrt{B} - \frac{1}{4}A\right)\frac{n(n-1)}{2}$$
$$\left. - \frac{1}{8}(n-1+\sqrt{B})((n-1)^2 - A)n\right)$$
$$= n\left(\frac{-16+n^2}{6\beta} + \frac{1}{4}\right). \tag{18.24}$$

Using (18.20)–(18.24) in (18.14), we obtain Ramanujan's last approximation (18.4).

After his four approximations, Ramanujan illustrates his theorem with five examples. In all examples he uses a different notation from that of Entry 18.

Corollary 18.1 (p. 349). *We have the following approximations:*

$$\sum_{k=1}^{7} u_k \approx \tfrac{7}{2}(u_2 + u_6), \tag{18.25}$$

$$\sum_{k=1}^{26} u_k \approx 13(u_6 + u_{21}), \tag{18.26}$$

$$\sum_{k=1}^{13} u_k \approx \tfrac{13}{25}(7u_2 + 11u_7 + 7u_{12}), \tag{18.27}$$

$$\sum_{k=1}^{22} u_k \approx \tfrac{11}{289}(161u_3 + 256u_{23/2} + 161u_{20}), \tag{18.28}$$

and

$$\sum_{k=1}^{21}\varphi(k) \approx \tfrac{7}{958}\Big(506\,(\varphi(2)+\varphi(20))+931 \times \Big(\varphi(11+2\sqrt{22/7})+\varphi(11-2\sqrt{22/7})\Big)\Big). \tag{18.29}$$

Proof. First, in the notation of Entry 18, set $n=7$ and $u_k=\varphi(x-8+2k)$, $1\le k\le 7$. Then $\sqrt{(n^2-1)/3}=4$, $\varphi(x-4)=u_2$, and $\varphi(x+4)=u_6$. Hence, (18.25) follows from (18.2).

Second, let $n=26$ and $u_k=\varphi(x-27+2k)$, $1\le k\le 26$. Then $\sqrt{(n^2-1)/3}=15$, $\varphi(x-15)=u_6$, and $\varphi(x+15)=u_{21}$. Hence, (18.26) follows from (18.2).

Third, set $n=13$ and $u_k=\varphi(x-14+2k)$, $1\le k\le 13$. Then $\sqrt{(3n^2-7)/5}=10$,

$$\frac{5(n^2-1)}{6(3n^2-7)}=\frac{7}{25}, \qquad \frac{8(n^2-4)}{6(3n^2-7)}=\frac{11}{25},$$

$\varphi(x-10)=u_2$, $\varphi(x+10)=u_{12}$, and $\varphi(x)=u_7$. Hence, (18.27) follows from (18.3).

Fourth, let $n=22$ and $u_k=\varphi(x-23+2k)$, $1\le k\le 22$. Then $\sqrt{(3n^2-7)/5}=17$,

$$\frac{5(n^2-1)}{6(3n^2-7)}=\frac{161}{2\cdot 289}, \qquad \frac{8(n^2-4)}{6(3n^2-7)}=\frac{128}{289},$$

$\varphi(x-17)=u_3$, $\varphi(x+17)=u_{20}$, and $\varphi(x)=u_{23/2}$. Thus, (18.28) follows from (18.3).

Fifth, let $n=21$ and $u_k=\varphi(x-22+2k)$, $1\le k\le 21$. Then $\alpha=1310$, $\beta=958$,

$$\sqrt{\frac{\alpha+\beta}{7}}=18, \qquad \sqrt{\frac{\alpha-\beta}{7}}=4\sqrt{\frac{22}{7}},$$

$$\frac{1}{4}-\frac{n^2-16}{6\beta}=\frac{506}{3\cdot 958}, \qquad \frac{1}{4}+\frac{n^2-16}{6\beta}=\frac{931}{3\cdot 958},$$

$\varphi(x-18)=u_2$, $\varphi(x+18)=u_{20}$, $\varphi(x-4\sqrt{22/7})=u_{11-2\sqrt{22/7}}$, and $\varphi(x+4\sqrt{22/7})=u_{11+2\sqrt{22/7}}$. Lastly, replace u_a by $\varphi(a)$. Then (18.29) follows form (18.4).

To the best of our knowledge, with the exception of this and the following entry, Ramanujan's notebooks, published papers, and unpublished papers give no indication that Ramanujan had any knowledge of Gaussian quadrature or orthogonal polynomials. Thus, Entry 18 is very remarkable, for it shows that Ramanujan must have derived some of the principal underlying ideas in these theories.

Entry 19 (p. 352). *Let $\varphi(t)$ be continuous for $t \geq 0$. Then the function*

$$e^{-x}\sum_{n=0}^{\infty}\varphi(n)\frac{x^n}{n!}$$

has the successive approximations

$$\varphi(x), \tag{19.1}$$

$$\frac{\sqrt{1+4x}-1}{2\sqrt{1+4x}}\varphi\left(x+\frac{1+\sqrt{1+4x}}{2}\right)+\frac{\sqrt{1+4x}+1}{2\sqrt{1+4x}}\varphi\left(x+\frac{1-\sqrt{1+4x}}{2}\right), \tag{19.2}$$

and

$$\begin{aligned}\tfrac{2}{3}\varphi(x)&+\frac{\sqrt{1+12x}-1}{6\sqrt{1+12x}}\varphi\left(x+\frac{1+\sqrt{1+12x}}{2}\right)\\&+\frac{\sqrt{1+12x}+1}{6\sqrt{1+12x}}\varphi\left(x+\frac{1-\sqrt{1+12x}}{2}\right).\end{aligned} \tag{19.3}$$

As with the previous entry, Askey [1] first observed that the origin of these approximations is found in Gaussian quadrature related to the discrete Charlier polynomials. It is again remarkable that, with no apparent knowledge of orthogonal polynomials, Ramanujan found these approximations.

Proof. We apply the general theory of Gaussian quadrature outlined in the proof of Entry 18. Here $f(t)=\varphi(t)$, and the interval $[a,b]$ is replaced by $[0,\infty)$. The discrete measure is realized at the nonnegative integers and is weighted by the Poisson distribution

$$\frac{e^{-x}x^n}{n!}, \qquad 0 \leq n < \infty. \tag{19.4}$$

The corresponding polynomials are the Charlier [1] polynomials defined by (Nikiforov, Suslov, and Uvarov [1, p. 35])

$$C_n(t;x) := {}_2F_0(-n,-t;-1/x), \qquad 0 \leq n < \infty.$$

They satisfy an orthogonality relation

$$e^{-x}\sum_{t=0}^{\infty}C_m(-t;x)C_n(-t;x)\frac{x^t}{t!}=c'_{mn}\delta_{mn},$$

for certain constants c'_{mn}, $0 \leq m, n < \infty$.

We now calculate the interpolation points t_j and the coefficients λ_j in the approximation (18.14).

First,

$$C_1(t;x) = {}_2F_0(-1,-t;-1/x) = 1-t/x,$$

which has $t = x$ as its zero. Since $w_{1,1}(t) \equiv 1$, by (18.13),

$$\lambda_1 = \int_0^\infty d\alpha(t) = e^{-x} \sum_{n=0}^\infty \frac{x^n}{n!} = 1.$$

Hence, our first approximation equals $1 \cdot \varphi(x)$, as claimed in (19.1). This approximation is exact for constant and linear polynomials.

Second,

$$C_2(t; x) = 1 - \frac{2t}{x} + \frac{t(t-1)}{x^2},$$

which has the zeros

$$t_1, t_2 = x + \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4x}. \tag{19.5}$$

So, by (18.7),

$$w_{1,2}(t) = \frac{t - x - \frac{1}{2} + \frac{1}{2}\sqrt{1+4x}}{\sqrt{1+4x}}$$

and

$$w_{2,2}(t) = -\frac{t - x - \frac{1}{2} - \frac{1}{2}\sqrt{1+4x}}{\sqrt{1+4x}}.$$

Hence,

$$\begin{aligned}
\lambda_1 &= \frac{1}{\sqrt{1+4x}} \int_0^\infty \left(t - x - \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4x}\right) d\alpha(t) \\
&= \frac{1}{\sqrt{1+4x}} e^{-x} \left(\sum_{n=0}^\infty n \frac{x^n}{n!} + \left(-x - \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4x}\right) \sum_{n=0}^\infty \frac{x^n}{n!} \right) \\
&= \frac{1}{\sqrt{1+4x}} e^{-x} \left(xe^x + \left(-x - \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1+4x}\right) e^x \right) \\
&= \frac{\sqrt{1+4x} - 1}{2\sqrt{1+4x}}.
\end{aligned} \tag{19.6}$$

Similarly,

$$\lambda_2 = \frac{\sqrt{1+4x} + 1}{2\sqrt{1+4x}}. \tag{19.7}$$

The approximation (19.2) now follows from (19.5)–(19.7). This approximation is exact for all polynomials of degree 3 or less.

Next,

$$C_3(t; x) = 1 - \frac{3t}{x} + \frac{3t(t-1)}{x^2} - \frac{t(t-1)(t-2)}{x^3}.$$

The three roots are very complicated, and upon examining (19.3), we see that Ramanujan did not choose these roots as interpolation points. If he had done so, his approximations would have been exact for all polynomials of degree 5 or less. To simplify the approximation, it is natural to choose $t_1 = x$ as one of the interpolation points, because it is the expected value of the Poisson distribution. Amazingly, Ramanujan found the proper interpolation points,

$$t_2, t_3 = x + \frac{1 \pm \sqrt{1+12x}}{2},$$

so that his approximation is exact for all polynomials of degree 4 or less. It is a tedious calculation to verify that (19.3) is exact for $\varphi(x) = 1, x, x^2, x^3, x^4$, and we resorted to *Mathematica* to check the right sides of (19.3) for x^3 and x^4. Thus, (19.3) is not a Gaussian quadrature formula.

In his first approximation, Ramanujan actually writes

$$\text{“}\sum_{n=0}^{\infty} \varphi(n)\frac{x^n}{n!} = e^x\varphi(x) \cdot \exp\left(\sum_{n=1}^{\infty} \frac{D^n}{n!}x^{n-1}\right) = e^x\varphi(x)$$

as the first approximation.” Undoubtedly, D denotes a differential operator, and so the latter equality is trivially true. However, we have no idea why Ramanujan introduced this series of differential operators. Perhaps this provides a hint to Ramanujan's derivations.

In Entry 10 of Chapter 3 (Part I [1, pp. 57–65]), Ramanujan provides an asymptotic expansion for $e^{-x}\sum_{n=0}^{\infty}\varphi(n)x^n/n!$ as x tends to ∞. As to be expected, the form of this expansion is quite different from the approximations given in Entry 19.

On page 350, Ramanujan claims that

$$a^{n/2}\prod_{k=1}^{\infty}\left(1+\frac{a^n}{\varphi^n(k)}\right) = c\exp\left(n\int_{\varphi(0)/a}^{\infty}\frac{\varphi^{-1}(ax)}{x(1+x^n)}dx\right) \tag{20.1}$$

“when a is very great. The above theorem is very useful to know.” Prior to writing (20.1), Ramanujan gives the special case when $n = 2$. As might be expected, Ramanujan does not give the value of c or any hypotheses about φ. Although the form (20.1) was perhaps convenient for applications that Ramanujan may have had in mind, we shall make a simplification. Suppose we let $f(x) = (\varphi(x)/a)^n$. Next, reintroduce a by replacing $f(x)$ by $\varphi(x)/a$. After a change of variable in the integral, we find that

$$\sqrt{a}\prod_{k=1}^{\infty}\left(1+\frac{a}{\varphi(k)}\right) = c\exp\left(\int_{\varphi(0)/a}^{\infty}\frac{\varphi^{-1}(ax)}{x(1+x)}dx\right), \tag{20.2}$$

which is simply the case $n = 1$ of (20.1). Thus, it is no loss of generality to assume at the outset that $n = 1$.

The following theorem is not as explicit as we would prefer, but its formulation, due to J. L. Hafner, is better than the author's original version.

Entry 20 (p. 350). *Assume that the product on the left side of (20.2) converges, $\varphi(x)$ is monotonically increasing, and $\varphi(0) > 0$. Then (20.2) is valid, when c is given (approximately) by (20.9) below.*

Proof. Taking the logarithm of each side of (20.2), we find it suffices to show that

$$\frac{1}{2}\log a + \sum_{k=1}^{\infty} \log\left(1 + \frac{a}{\varphi(k)}\right) = \log c + \int_{\varphi(0)/a}^{\infty} \frac{\varphi^{-1}(ax)}{x(1+x)} dx. \tag{20.3}$$

Since $\varphi(0) > 0$ and $\varphi(x)$ is monotonically increasing,

$$\int_1^{\infty} \log\left(1 + \frac{a}{\varphi(x)}\right) dx \leq \sum_{k=1}^{\infty} \log\left(1 + \frac{a}{\varphi(k)}\right) \leq \int_0^{\infty} \log\left(1 + \frac{a}{\varphi(x)}\right) dx.$$

By the intermediate value theorem, there exists a number x_a, $0 \leq x_a \leq 1$, such that

$$\begin{aligned} S_a := \sum_{k=1}^{\infty} \log\left(1 + \frac{a}{\varphi(k)}\right) &= \int_{x_a}^{\infty} \log\left(1 + \frac{a}{\varphi(x)}\right) dx \\ &= \int_0^{\infty} \log\left(1 + \frac{a}{\varphi(x)}\right) dx - \int_0^{x_a} \log\left(1 + \frac{a}{\varphi(x)}\right) dx \\ &= I_1 - I_2, \end{aligned} \tag{20.4}$$

say. By examining the inverse function of $\log(1 + a/\varphi(x))$, we see that

$$I_1 = \int_0^{\log(1+a/\varphi(0))} \varphi^{-1}\left(\frac{a}{e^y - 1}\right) dy.$$

Setting $x = 1/(e^y - 1)$, we find that

$$I_1 = \int_{\varphi(0)/a}^{\infty} \frac{\varphi^{-1}(ax)}{x(1+x)} dx.$$

Thus, from (20.4),

$$S_a = \int_{\varphi(0)/a}^{\infty} \frac{\varphi^{-1}(ax)}{x(1+x)} dx - I_2. \tag{20.5}$$

Comparing (20.5) with (20.3), we see that (20.3) has been proved with

$$\log c = -I_2 + \tfrac{1}{2}\log a. \tag{20.6}$$

We shall make the determination of c slightly more explicit.

For $0 \leq x \leq x_a \leq 1$,

$$\log(1+a/\varphi(x)) = \log a - \log\varphi(x) + \log(1+\varphi(x)/a) = \log a - \log\varphi(x) + O(1/a),$$

as a tends to ∞. Thus,

$$I_2 = x_a \log a - \int_0^{x_a} \log\varphi(x)\, dx + O(1/a) = x_a \log a - I_3 + O(1/a), \tag{20.7}$$

say. Since φ is increasing,

$$x_a \log \varphi(0) \leq I_3 \leq x_a \log \varphi(x_a),$$

so that by the second mean value theorem for integrals,

$$0 \leq I_3 - x_a \log \varphi(0) = \int_0^{x_a} \log \frac{\varphi(x)}{\varphi(0)} dx = (x_a - \xi_a) \log \frac{\varphi(x_a)}{\varphi(0)}, \tag{20.8}$$

for some number ξ_a, $0 \leq \xi_a \leq x_a$. Combining (20.6)–(20.8), we deduce that

$$c = \sqrt{a} \left(\frac{\varphi(0)}{a} \right)^{x_a} \left(\frac{\varphi(x_a)}{\varphi(0)} \right)^{x_a - \xi_a} (1 + O(1/a)). \tag{20.9}$$

This completes the proof.

It seems likely that, in many cases,

$$x_a = \frac{1}{2} + o\left(\frac{1}{\log a} \right) \qquad \text{and} \qquad x_a - \xi_a = o(1),$$

as a tends to ∞. In such cases, (20.9) yields

$$c = \sqrt{\varphi(0)}\, \{1 + o(1)\}.$$

The next entry is recorded in Ramanujan's Quarterly Reports and is discussed in detail in Part I [1, pp. 311–312].

Entry 21 (p. 351). *Consider the equation*

$$F(-x) := \sum_{k=0}^{\infty} \frac{(-x^3)^k}{(3k)!} = \frac{e^{-x} + e^{-\omega x} + e^{-\omega^2 x}}{3} = 0, \tag{21.1}$$

where $\omega := \exp(2\pi i/3)$. *Then there exist an infinite number of positive roots. They are close to the zeros* $\pi(2n+1)/\sqrt{3}$ *of* $\cos(x\sqrt{3}/2)$, *where* n *is a nonnegative integer. More precisely, if*

$$h = e^{-\pi(2n+1)\sqrt{3}/2},$$

then these roots are given by

$$\begin{aligned} x = \frac{\pi(2n+1)}{\sqrt{3}} &- \frac{1}{2} \left(h^2 + \frac{13}{3!} h^4 + \frac{28 \cdot 31}{5!} h^6 + \frac{49 \cdot 52 \cdot 57}{7!} h^8 \right. \\ &\left. + \frac{76 \cdot 79 \cdot 84 \cdot 91}{9!} h^{10} + \cdots \right) \\ &+ \frac{(-1)^n}{\sqrt{3}} \left(h + \frac{7}{3!} h^3 + \frac{19 \cdot 21}{5!} h^5 + \frac{37 \cdot 39 \cdot 43}{7!} h^7 + \frac{61 \cdot 63 \cdot 67 \cdot 73}{9!} h^9 \right. \\ &\left. + \frac{91 \cdot 93 \cdot 97 \cdot 103 \cdot 111}{11!} h^{11} + \cdots \right). \end{aligned} \tag{21.2}$$

Lastly, all roots of (21.1) are given by x, ωx, *and* $\omega^2 x$, *where* x *is given by (21.2) and* $\omega = \exp(2\pi i/3)$.

Entry 22 (p. 351). *Let $x_0, x_1, x_2, \ldots$ be the real roots of (21.1). Then*

$$F(x) = \prod_{n=0}^{\infty} \left(1 + \frac{x^3}{x_n^3}\right). \tag{22.1}$$

Proof. The real roots of $F(x) = 0$ are $-x_0, -x_1, -x_2, \ldots$. Then $-x_n, -x_n\omega$, $-x_n\omega^2$, $0 \leq n < \infty$, constitute all the roots of $F(x) = 0$, where $\omega = \exp(2\pi i/3)$. Now

$$(x + x_n)(x + x_n\omega)(x + x_n\omega^2) = x^3 + x_n^3.$$

We now apply the Weierstrass product formula. From the definition of F in (21.1) and the fact that the infinite product in (22.1) converges, since $x_n \sim \pi(2n+1)/\sqrt{3}$ as n tends to ∞, we easily deduce that $F(x)$ has the representation (22.1).

Entry 23 (p. 370). *As x tends to* $1-$,

$$\sum_{n=1}^{\infty} \left(\frac{x^{3n+1}}{1 + x^{4n+1}} - \frac{x^{3n+2}}{1 + x^{4n+3}}\right) \sim \frac{1}{4} + \frac{1}{4\sqrt{2}} \log(1 + \sqrt{2}) - \frac{\pi}{8\sqrt{2}}.$$

Proof. First,

$$\sum_{n=1}^{\infty} \left(\frac{x^{3n+1}}{1 + x^{4n+1}} - \frac{x^{3n+2}}{1 + x^{4n+3}}\right) = \sum_{n=1}^{\infty} \frac{(1 - x)(1 - x^{4n+2})x^{3n+1}}{(1 + x^{4n+1})(1 + x^{4n+3})}. \tag{23.1}$$

We apply the Euler–Maclaurin summation formula, (0.5) of Chapter 37, to

$$f(t) := \frac{(1 - x)(1 - x^{4t+2})x^{3t+1}}{(1 + x^{4t+1})(1 + x^{4t+3})}.$$

Thus,

$$\sum_{n=1}^{\infty} f(n) = \int_0^{\infty} f(t)\,dt - \frac{1}{2}\left(\frac{x}{1+x} - \frac{x^2}{1+x^3}\right) + \int_0^{\infty} \left(t - [t] - \tfrac{1}{2}\right) f'(t)\,dt \tag{23.2}$$

Let $x^t = u$. Since

$$f'(t) = f(t) \log x \left(3 - \frac{4x^2}{1 - u^4x^2} - \frac{4x}{1 + u^4x} - \frac{4x^3}{1 + u^4x^3}\right),$$

we find that

$$\begin{aligned}
&\int_0^{\infty} \left(t - [t] - \tfrac{1}{2}\right) f'(t)\,dt \\
&= x(x-1) \int_0^1 \left(\frac{\log u}{\log x} - \left[\frac{\log u}{\log x}\right] - \frac{1}{2}\right) \frac{u^2(1 - u^4x^2)}{(1 + u^4x)(1 + u^4x^3)} \\
&\qquad \times \left(3 - \frac{4x^2}{1 - u^4x^2} - \frac{4x}{1 + u^4x} - \frac{4x^3}{1 + u^4x^3}\right) du
\end{aligned}$$

$$\ll x(1-x)\int_0^1 \frac{u^2(1-u^4x^2)}{(1+u^4x)(1+u^4x^3)}$$
$$\times\left(3+\frac{4x^2}{1-u^4x^2}+\frac{4x}{1+u^4x}+\frac{4x^3}{1+u^4x^3}\right)du$$
$$= o(1), \tag{23.3}$$

as x tends to $1-$. Setting $x^t = u$ also in the first integral on the right side of (23.2) and using (23.3), we find that, as x tends to $1-$,

$$\sum_{n=1}^{\infty} f(n) = -\frac{x(1-x)}{\log x}\int_0^1 \frac{u^2(1-u^4x^2)}{(1+u^4x)(1+u^4x^3)}du + o(1)$$
$$= \int_0^1 \frac{u^2(1-u^4)}{(1+u^4)^2}du + o(1). \tag{23.4}$$

Using *Mathematica*, we find that

$$\int_0^1 \frac{u^2(1-u^4)}{(1+u^4)^2}du = \frac{1}{4} - \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{2-\sqrt{2}}{\sqrt{2}}\right) - \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{2+\sqrt{2}}{\sqrt{2}}\right)$$
$$- \frac{\sqrt{2}}{16}\log(2-\sqrt{2}) + \frac{\sqrt{2}}{16}\log(2+\sqrt{2})$$
$$= \frac{1}{4} - \frac{\sqrt{2}}{8}\frac{\pi}{8} - \frac{\sqrt{2}}{8}\frac{3\pi}{8} + \frac{\sqrt{2}}{8}\log(1+\sqrt{2})$$
$$= \frac{1}{4} - \frac{\pi}{8\sqrt{2}} + \frac{1}{4\sqrt{2}}\log(1+\sqrt{2}). \tag{23.5}$$

Using (23.5) in (23.4) and combining the result with (23.1), we complete the proof of Ramanujan's asymptotic formula.

39

Miscellaneous Results in the First Notebook

In this last chapter we collect together some miscellaneous results from the unorganized portions of the first notebook. Most are from analysis, with some pertaining to hypergeometric functions.

We use the familiar notation associated with hypergeometric functions; e.g., see Part II [2, p. 8]. In particular, for each nonnegative integer n,

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Page numbers after entries refer to the pagination of the Tata Institute's publication of the first notebook [9].

Entry 1 (p. 72). *If*

$$f(x) := \sum_{n=1}^{\infty} \varphi(nx), \tag{1.1}$$

then

$$\sum_{n=1}^{\infty} \varphi(n^2 x) = \sum_{n=1}^{\infty} (-1)^{\Omega(n)} f(nx), \tag{1.2}$$

where $\Omega(1) = 0$ *and, for* $n > 1$, $\Omega(n)$ *denotes the total number of prime factors of* n *counting multiplicities.*

Proof. By (1.1),

$$\sum_{n=1}^{\infty} (-1)^{\Omega(n)} f(nx) = \sum_{n=1}^{\infty} (-1)^{\Omega(n)} \sum_{m=1}^{\infty} \varphi(mnx) = \sum_{r=1}^{\infty} \left(\sum_{d|r} (-1)^{\Omega(d)} \right) \varphi(rx). \tag{1.3}$$

It is easy to see that $(-1)^{\Omega(n)}$ is completely multiplicative. Hence, $\sum_{d|r}(-1)^{\Omega(d)}$ is multiplicative. For each prime p and positive integer n,

$$\sum_{d|p^n}(-1)^{\Omega(d)} = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Hence, by multiplicativity,

$$\sum_{d|r}(-1)^{\Omega(d)} = \begin{cases} 1, & \text{if } r \text{ is a perfect square,} \\ 0, & \text{otherwise.} \end{cases}$$

Using the equality above in (1.3), we deduce (1.2).

Entry 2 (p. 94). *If n is a positive integer,*

$$\int_0^\infty \frac{\sin^n x}{x^n}dx = \pm\frac{(-1)^n\pi}{2^n\Gamma(n)}\sum_{k=0}^{[(n-1)/2]}(-n)_k(n-2k)^{n-1},$$

where the plus sign is taken if n is even, and the minus sign is chosen if n is odd.

In fact, Ramanujan claimed different values for the integral and crossed out the entry. The evaluation above appears to have been given first by O. Schlömilch [1] in 1860, as pointed out by M. L. Glasser. In the book by D. S. Mitrinović and J. D. Kečkić [1], the integral of Entry 2 is evaluated by contour integration. In 1980, E. T. H. Wang [1] submitted Entry 2 as a problem; solutions by R. L. Young and T. M. Apostol and several references to the problem's appearances in the literature were given.

Entry 3 (p. 94). *For $0 < x < 1$,*

$${}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; -x\right) = \frac{\left(\sqrt{1+x}+\sqrt{x}\right)^{1/3} + \left(\sqrt{1+x}-\sqrt{x}\right)^{1/3}}{2\sqrt{1+x}} \tag{3.1}$$

and

$${}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{3}{2}; -x\right) = 3\frac{\left(\sqrt{1+x}+\sqrt{x}\right)^{1/3} - \left(\sqrt{1+x}-\sqrt{x}\right)^{1/3}}{2\sqrt{x}}. \tag{3.2}$$

Proof. From Entry 35(iii) of Chapter 11 (Part II [2, p. 99]) with x^2 replaced by $-x$,

$${}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; -x\right) = \frac{\cos\left(\frac{1}{3}\sin^{-1}(i\sqrt{x})\right)}{\sqrt{1+x}}. \tag{3.3}$$

Now,

$$\begin{aligned}
\cos\left(\tfrac{1}{3}\sin^{-1}(i\sqrt{x})\right) &= \cos\left(\frac{i}{3}\log\left\{\sqrt{1+x}-\sqrt{x}\right\}\right) \\
&= \tfrac{1}{2}\left(\sqrt{1+x}-\sqrt{x}\right)^{-1/3} + \tfrac{1}{2}\left(\sqrt{1+x}-\sqrt{x}\right)^{1/3} \\
&= \tfrac{1}{2}\left(\sqrt{1+x}+\sqrt{x}\right)^{1/3} + \tfrac{1}{2}\left(\sqrt{1+x}-\sqrt{x}\right)^{1/3}.
\end{aligned} \tag{3.4}$$

Using (3.4) in (3.3), we deduce (3.1).

Next, Entry 35(ii) of Chapter 11 (Part II [2, p. 99]), with x^2 replaced by $-x$, states that

$${}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{3}{2}; -x\right) = \frac{3}{i\sqrt{x}}\sin\left(\tfrac{1}{3}\sin^{-1}\left(i\sqrt{x}\right)\right).$$

The remainder of the proof of (3.2) is similar to that for (3.1).

Entry 4 (p. 104). *Let x and y be complex numbers such that x/y is not purely imaginary. Let $\varphi(z)$ be an entire function such that*

$$f(z) := \tfrac{1}{4}\pi\sec\left(\tfrac{1}{2}\pi xz\right)\operatorname{sech}\left(\tfrac{1}{2}\pi yz\right)\{\varphi(xyz)-\varphi(-xyz)\}$$

tends to 0 as z tends to ∞. Then

$$\begin{aligned}
&\tfrac{1}{4}\pi\sec\left(\tfrac{1}{2}\pi x\right)\operatorname{sech}\left(\tfrac{1}{2}\pi y\right)\{\varphi(xy)-\varphi(-xy)\} \\
&= x\sum_{n=0}^{\infty}\frac{(-1)^n\operatorname{sech}\left(\dfrac{\pi(2n+1)y}{2x}\right)}{(2n+1)^2-x^2}\{\varphi((2n+1)y)-\varphi(-(2n+1)y)\} \\
&\quad - iy\sum_{n=0}^{\infty}\frac{(-1)^n\operatorname{sech}\left(\dfrac{\pi(2n+1)x}{2y}\right)}{(2n+1)^2+y^2}\{\varphi((2n+1)ix)-\varphi(-(2n+1)ix)\}.
\end{aligned} \tag{4.1}$$

Proof. Observe that $f(z)$ has simple poles at $z=(2n+1)/x$ and $z=(2n+1)i/y$ for each integer n. The poles do not coalesce because x/y is not purely imaginary. Letting R_a denote the residue of $f(z)$ at a pole a, we easily find that

$$R_{(2n+1)/x} = -\frac{(-1)^n}{2x}\operatorname{sech}\left(\frac{\pi(2n+1)y}{2x}\right)\{\varphi((2n+1)y)-\varphi(-(2n+1)y)\}$$

and

$$R_{(2n+1)i/y} = \frac{(-1)^n}{2yi}\operatorname{sech}\left(\frac{\pi(2n+1)x}{2y}\right)\{\varphi((2n+1)ix)-\varphi(-(2n+1)ix)\}.$$

We thus find that the sum of the principal parts arising from the poles $\pm(2n+1)/x$, $n \geq 0$, is

$$\frac{(-1)^n xz \operatorname{sech}\left(\dfrac{\pi(2n+1)y}{2x}\right)}{(2n+1)^2 - x^2z^2}\{\varphi((2n+1)y) - \varphi(-(2n+1)y)\},$$

while the sum of the principal parts arising from the poles $\pm(2n+1)i/y$, $n \geq 0$, equals

$$-\frac{(-1)^n yzi \operatorname{sech}\left(\dfrac{\pi(2n+1)x}{2y}\right)}{(2n+1)^2 + y^2z^2}\{\varphi((2n+1)ix) - \varphi(-(2n+1)ix)\}.$$

Since $f(z)$ tends to 0 as z tends to ∞, we conclude from the Mittag–Leffler theorem that

$$\begin{aligned} f(z) = xz\sum_{n=0}^{\infty} &\frac{(-1)^n \operatorname{sech}\left(\dfrac{\pi(2n+1)y}{2x}\right)}{(2n+1)^2 - x^2z^2}\{\varphi((2n+1)y) - \varphi(-(2n+1)y)\} \\ - iyz\sum_{n=0}^{\infty} &\frac{(-1)^n \operatorname{sech}\left(\dfrac{\pi(2n+1)x}{2y}\right)}{(2n+1)^2 + y^2z^2}\{\varphi((2n+1)ix) - \varphi(-(2n+1)ix)\}. \end{aligned}$$

Letting $z = 1$, we deduce (4.1) to complete the proof.

Entry 5 (p. 108). *If x is not an integer and $0 \leq \theta \leq 2\pi$, then*

$$\frac{\pi}{x}\{\cot(\pi x)\cos(\theta x) + \sin(\theta x)\} = \frac{1}{x^2} - 2\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n^2 - x^2}. \tag{5.1}$$

Proof. By Entry 34(i) in Chapter 13 (Part II [2, p. 237]), for $|\theta| \leq \pi$,

$$\frac{\pi}{x}\frac{\cos(\theta x)}{\sin(\pi x)} = \frac{1}{x^2} + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(n\theta)}{n^2 - x^2}. \tag{5.2}$$

Replacing θ by $\theta - \pi$ in (5.2), we readily deduce (5.1) to complete the proof.

Entry 6 (p. 120). *The maximum value of $a^x/\Gamma(x+1)$ is equal to*

$$\frac{a^{a-1/2}}{\Gamma(a+\frac{1}{2})}\exp\left(\frac{1}{1152a^3 + 323.2a}\right) \tag{6.1}$$

"very nearly."

This entry is the same as Example (i) in Section 25 of Chapter 13 in the second notebook. However, after proving a slightly weaker version of the result in Part II [2, p. 228], we unfortunately claimed that the appearance of "323.2a" in

Ramanujan's expression (6.1) is erroneous. We are very grateful to Richard Brent for pointing out to us that Ramanujan, indeed, is correct. Moreover, he kindly provided the proof below.

Proof. We shall prove more precisely that the maximum value equals

$$\frac{a^{a-1/2}}{\Gamma(a+\frac{1}{2})}\exp\left(\frac{1}{1152a^3}-\frac{101}{414720a^5}+O\left(\frac{1}{a^7}\right)\right), \tag{6.2}$$

as a tends to ∞. Since

$$\begin{aligned}\frac{1}{1152a^3}-\frac{101}{414720a^5}+O\left(\frac{1}{a^7}\right)&=\frac{1}{1152a^3}\left(1-\frac{101}{360a^2}+O\left(\frac{1}{a^4}\right)\right)\\&=\frac{1}{1152a^3\left(1+\frac{101}{360a^2}+O\left(\frac{1}{a^4}\right)\right)}\\&=\frac{1}{1152a^3+323.2a+O(1/a)},\end{aligned}$$

this would confirm Ramanujan's claim.

Let $f(a) := \log\Gamma(a+\frac{1}{2})$. By Corollary 1 in Section 6 of Chapter 8 in the second notebook (Part I [1, p. 184]),

$$f'(a)=\log a+\frac{1}{24a^2}-\frac{7}{960a^4}+O\left(\frac{1}{a^6}\right). \tag{6.3}$$

Further differentiations of the asymptotic expansion of $f'(a)$ are valid by a general theorem on the differentiation of asymptotic expansions (F. W. J. Olver [1, p. 21]), and we have

$$f''(a)=\frac{1}{a}-\frac{1}{12a^3}+O\left(\frac{1}{a^5}\right) \tag{6.4}$$

and

$$f'''(a)=-\frac{1}{a^2}+O\left(\frac{1}{a^4}\right). \tag{6.5}$$

Now,

$$\sup_{x\geq 0}\frac{a^x}{\Gamma(x+1)}=\frac{a^{a-1/2}}{\Gamma(a+\frac{1}{2})}R(a),$$

where

$$R(a)=\frac{a^\epsilon\Gamma(a+\frac{1}{2})}{\Gamma(a+\frac{1}{2}+\epsilon)}, \tag{6.6}$$

and where (see Part II [2, p. 228]), if the maximum is obtained at $x=x(a)$,

$$\epsilon:=x(a)+\tfrac{1}{2}-a=-\frac{1}{24a}+\frac{3}{640a^3}+O\left(\frac{1}{a^5}\right), \tag{6.7}$$

as a tends to ∞.

Using (6.3)–(6.5), we find from (6.6) that, as a tends to ∞,

$$\begin{aligned}\log R(a) &= \epsilon \log a + f(a) - f(a+\epsilon) \\ &= \epsilon \log a - \epsilon f'(a) - \tfrac{1}{2}\epsilon^2 f''(a) - \tfrac{1}{6}\epsilon^3 f'''(a) - \cdots \\ &= -\left(\frac{\epsilon}{24a^2} + \frac{\epsilon^2}{2a}\right) + \left(\frac{7\epsilon}{960a^4} + \frac{\epsilon^2}{24a^3} + \frac{\epsilon^3}{6a^2}\right) + O\left(\frac{1}{a^7}\right).\end{aligned} \tag{6.8}$$

From (6.7),

$$\epsilon^2 = \frac{1}{576a^2} - \frac{1}{2560a^4} + O\left(\frac{1}{a^6}\right)$$

and

$$\epsilon^3 = -\frac{1}{(24a)^3} + O\left(\frac{1}{a^5}\right),$$

as a tends to ∞. Using these expansions and (6.7) in (6.8), we conclude that

$$\log R(a) = \frac{1}{1152a^3} - \frac{101}{414720a^5} + O\left(\frac{1}{a^7}\right),$$

which is the required result in (6.2).

Entry 7 (p. 138). *Let $n > m > 0$ and let p be real. Put $f_j(x) = \cos x, \sin x$, for $j = 1, 2$, respectively. Then, for $j = 1, 2$,*

$$\begin{aligned}I_j &:= \int_{-\infty}^{\infty} \frac{e^{mx} f_j(px)}{(1+e^x)^n} dx \\ &= \frac{\dfrac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)} f_j\left(\displaystyle\sum_{k=0}^{\infty}\left\{\tan^{-1}\frac{p}{n-m+k} - \tan^{-1}\frac{p}{m+k}\right\}\right)}{\sqrt{\displaystyle\prod_{k=0}^{\infty}\left\{1+\frac{p^2}{(n-m+k)^2}\right\}\left\{1+\frac{p^2}{(m+k)^2}\right\}}}.\end{aligned} \tag{7.1}$$

Proof. It is not difficult to identify I_j in terms of beta functions. More precisely (A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev [1, p. 450, eq. 11]),

$$I_1 = \frac{1}{\Gamma(n)} \operatorname{Re}\{\Gamma(m+ip)\Gamma(n-m-ip)\} \tag{7.2}$$

and

$$I_2 = \frac{1}{\Gamma(n)} \operatorname{Im}\{\Gamma(m+ip)\Gamma(n-m-ip)\}. \tag{7.3}$$

Now, by Euler's product formula for $\Gamma(z)$,

$$\begin{aligned}
&\Gamma(m+ip)\Gamma(n-m-ip)\\
&= \lim_{N\to\infty} \frac{N^{m+ip}N!}{(m+ip)(m+ip+1)\cdots(m+ip+N)}\,\frac{N^{n-m+ip}N!}{(n-m-ip)\cdots(n-m-ip+N)}\\
&= \lim_{N\to\infty} \frac{N^m N!\; N^{n-m}N! \exp\left(-i\sum_{k=0}^{N}\tan^{-1}\frac{p}{m+k}+i\sum_{k=0}^{N}\tan^{-1}\frac{p}{n-m+k}\right)}{\sqrt{m^2+p^2}\cdots\sqrt{(m+N)^2+p^2}\sqrt{(n-m)^2+p^2}\cdots\sqrt{(n-m+N)^2+p^2}}\\
&= \frac{\Gamma(m)\Gamma(n-m)\exp\left(i\sum_{k=0}^{\infty}\left\{\tan^{-1}\frac{p}{n-m+k}-\tan^{-1}\frac{p}{m+k}\right\}\right)}{\prod_{k=0}^{\infty}\sqrt{1+\frac{p^2}{(m+k)^2}}\sqrt{1+\frac{p^2}{(n-m+k)^2}}}. \qquad (7.4)
\end{aligned}$$

Using (7.4) in (7.2) and (7.3), we readily deduce (7.1).

We consider the next result as a formal identity.

Entry 8 (p. 158). *If*

$$\int_0^h \varphi(x)\cos(2nx)\,dx = \frac{\sqrt{\pi}}{2}\psi(n),$$

then

$$\int_0^h e^{-x^2}\varphi(x)\,dx = \int_0^\infty e^{-x^2}\psi(x)\,dx.$$

Proof. We have

$$\begin{aligned}
\int_0^\infty e^{-x^2}\psi(x)\,dx &= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}dx\int_0^h \varphi(u)\cos(2xu)\,du\\
&= \frac{2}{\sqrt{\pi}}\int_0^h \varphi(u)\,du\int_0^\infty e^{-x^2}\cos(2xu)\,dx\\
&= \frac{2}{\sqrt{\pi}}\int_0^h \varphi(u)\frac{\sqrt{\pi}}{2}e^{-u^2}du,
\end{aligned}$$

upon using a familiar integral evaluation (Gradshteyn and Ryzhik [1, p. 515, formula 3.896, no. 4]). This completes the proof.

The next entry is somewhat obliquely stated by Ramanujan, who used a notation peculiar to him (see, e.g., Part I [1, p. 138]).

Entry 9 (p. 184). *If* $\operatorname{Re}\beta > \operatorname{Re}\alpha$, *then*

$$\int_0^\infty \frac{\Gamma(x+\alpha+1)}{\Gamma(x+\beta+1)}\{\psi(x+\beta+1)-\psi(x+\alpha+1)\}\,dx = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)},$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$.

Proof. We have

$$\int_0^\infty \frac{\Gamma(x+\alpha+1)}{\Gamma(x+\beta+1)}\{\psi(x+\beta+1)-\psi(x+\alpha+1)\}\,dx$$
$$= -\int_0^\infty \frac{d}{dx}\left\{\frac{\Gamma(x+\alpha+1)}{\Gamma(x+\beta+1)}\right\}dx = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)}.$$

Entry 10 (p. 194). *For each nonnegative integer* n,

$${}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, -n\\ \frac{3}{2}, 1\end{matrix}\right] = \frac{\sqrt{\pi}\,\Gamma(n+1)}{2\Gamma(n+\frac{3}{2})}\sum_{k=0}^{n}\frac{(\frac{1}{2})_k^2}{(k!)^2}.$$

Proof. This result follows in a straightforward manner from Entries 29(b), (c), (d) of Chapter 10 of Ramanujan's second notebook (Part II [2, pp. 39–40]).

We are grateful to R. A. Askey for providing the proof of Entry 11 given below.

Entry 11 (p. 206). *If* $0 < \operatorname{Re} n < \inf(\operatorname{Re}\alpha + 1, \operatorname{Re}\beta + 1)$, *then*

$$\int_0^\infty \frac{x^{n-1}}{1+x}\,{}_2F_1(\alpha,\beta;\alpha+\beta;-x)\,dx$$
$$= \frac{\Gamma(\alpha-n+1)\Gamma(\beta-n+1)}{\Gamma(\alpha+\beta-n)}\sum_{k=0}^\infty \frac{(\alpha)_k(\beta)_k}{(\alpha+\beta-n+k)(\alpha+\beta)_k k!}$$
$$= \frac{\Gamma(\alpha-n+1)\Gamma(n)}{\Gamma(\alpha)}\sum_{k=0}^\infty \frac{(\alpha)_k(n)_k}{(\alpha+k)(\alpha+\beta)_k k!}. \tag{11.1}$$

Proof. Let I denote the integral on the left side of (11.1). Using Pfaff's transformation (Part II [2, p. 36, Entry 19]), we find that

$$I = \int_0^\infty x^{n-1}(1+x)^{-1-\alpha}\,{}_2F_1\left(\alpha,\alpha;\alpha+\beta;\frac{x}{x+1}\right)dx.$$

Making the change of variable $t = x/(x+1)$, we deduce that

$$I = \int_0^1 t^{n-1}(1-t)^{-n+\alpha}\,{}_2F_1(\alpha,\alpha;\alpha+\beta;t)\,dt$$
$$= \sum_{k=0}^\infty \frac{(\alpha)_k^2}{(\alpha+\beta)_k k!}\int_0^1 t^{n+k-1}(1-t)^{-n+\alpha}dt$$
$$= \sum_{k=0}^\infty \frac{(\alpha)_k^2}{(\alpha+\beta)_k k!}\frac{\Gamma(n+k)\Gamma(-n+\alpha+1)}{\Gamma(\alpha+k+1)}$$
$$= \frac{\Gamma(n)\Gamma(-n+\alpha+1)}{\Gamma(\alpha)}\sum_{k=0}^\infty \frac{(\alpha)_k(n)_k}{(\alpha+k)(\alpha+\beta)_k k!},$$

which proves the second equality in (11.1).

To prove the first equality in (11.1), we first establish a general transformation for ${}_3F_2(a, b, c; d, e; 1)$. Suppose that $\operatorname{Re}(d+e-a-b-c) > 0$ and $0 < \operatorname{Re} c < \operatorname{Re} e$. Then, by inverting the order of integration and summation, we find that

$$\begin{aligned} J &:= \int_0^1 x^{c-1}(1-x)^{e-c-1}\, {}_2F_1(a, b; d; x)\, dx \\ &= \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(d)_k k!} \frac{\Gamma(c+k)\Gamma(e-c)}{\Gamma(e+k)} \\ &= \frac{\Gamma(c)\Gamma(e-c)}{\Gamma(e)}\, {}_3F_2(a, b, c; d, e; 1). \end{aligned} \tag{11.2}$$

On the other hand, by a fundamental transformation for hypergeometric functions (W. N. Bailey [1, p. 2]) and by the same argument as given above,

$$\begin{aligned} J &= \int_0^1 x^{c-1}(1-x)^{d+e-a-b-c-1}\, {}_2F_1(d-a, d-b; d; x)\, dx \\ &= \sum_{k=0}^{\infty} \frac{(d-a)_k(d-b)_k}{(d)_k k!} \frac{\Gamma(c+k)\Gamma(d+e-a-b-c)}{\Gamma(d+e-a-b+k)} \\ &= \frac{\Gamma(c)\Gamma(d+e-a-b-c)}{\Gamma(d+e-a-b)}\, {}_3F_2\left[\begin{matrix} d-a, d-b, c \\ d, d+e-a-b \end{matrix}\right]. \end{aligned} \tag{11.3}$$

Combining (11.2) and (11.3), we deduce that

$$ {}_3F_2\left[\begin{matrix} a, b, c \\ d, e \end{matrix}\right] = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-c)\Gamma(d+e-a-b)}\, {}_3F_2\left[\begin{matrix} d-a, d-b, c \\ d, d+e-a-b \end{matrix}\right]. \tag{11.4}$$

Now set $a = \alpha, b = n, c = \alpha, d = \alpha + \beta$, and $e = \alpha + 1$ in (11.4). Then

$$\begin{aligned} I &= \frac{\Gamma(n)\Gamma(\alpha-n+1)}{\Gamma(\alpha+1)}\, {}_3F_2\left[\begin{matrix} \alpha, \alpha, n \\ \alpha+\beta, \alpha+1 \end{matrix}\right] \\ &= \frac{\Gamma(n)\Gamma(\alpha-n+1)}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+1)\Gamma(\beta-n+1)}{\Gamma(1)\Gamma(\alpha+\beta-n+1)}\, {}_3F_2\left[\begin{matrix} \beta, \alpha+\beta-n, \alpha \\ \alpha+\beta, \alpha+\beta-n+1 \end{matrix}\right] \\ &= \frac{\Gamma(n)\Gamma(\alpha-n+1)\Gamma(\beta-n+1)}{\Gamma(\alpha+\beta-n+1)}\, {}_3F_2\left[\begin{matrix} \alpha, \beta, \alpha+\beta-n \\ \alpha+\beta, \alpha+\beta-n+1 \end{matrix}\right], \end{aligned}$$

which establishes the first equality in (11.1).

One might surmise that Entry 11 can be found in integral tables. However, we are unable to find these evaluations in the tables of Gradshteyn and Ryzhik [1] or Prudnikov, Brychkov, and Marichev [2], although on page 315 of the latter tables, some similar evaluations are given.

Entry 12 (p. 208). *Define $F^{(n)}(x)$, $n = 2^m$, where m is a nonnegative integer, by*

$$F^{(1)}(x) = x, \qquad F^{(2)}(x) = \frac{x^2}{(2-x)^2},$$

and $F^{(2n)}(x) = F^{(n)}(F^{(2)}(x))$. *Then*

$$F^{(n)}(x) = \frac{4x^n}{\left\{(1+\sqrt{1-x})^n + (1-\sqrt{1-x})^n\right\}^2}. \tag{12.1}$$

Proof. It is easily checked that (12.1) is valid for $m = 0, 1$. Proceeding by induction, we find that

$$\begin{aligned}
F^{(2n)}(x) &= F^{(n)}\left(\frac{x^2}{(2-x)^2}\right) \\
&= \frac{4\dfrac{x^{2n}}{(2-x)^{2n}}}{\left\{\left(1+\sqrt{1-\dfrac{x^2}{(2-x)^2}}\right)^n + \left(1-\sqrt{1-\dfrac{x^2}{(2-x)^2}}\right)^n\right\}^2} \\
&= \frac{4x^{2n}}{\left\{\left(2-x+2\sqrt{1-x}\right)^n + \left(2-x-2\sqrt{1-x}\right)^n\right\}^2} \\
&= \frac{4x^{2n}}{\left\{\left(1+\sqrt{1-x}\right)^{2n} + \left(1-\sqrt{1-x}\right)^{2n}\right\}^2},
\end{aligned}$$

which completes the induction.

Entry 13 (p. 265). *If* $\varphi(x)$ *is any polynomial, then formally*

$$\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{2})_n}{n!}\varphi(n) = \sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{2})_n}{n!}\varphi\left(-\frac{2n+1}{2}\right). \tag{13.1}$$

Proof. Let

$$\varphi(x) = \varphi_m(x) := x(x-1)(x-2)\cdots(x-m+1),$$

where m is any nonnegative integer. Since $\{\varphi_m(x)\}$, $0 \le m \le j$, form a basis for the vector space of all polynomials of degree $\le j$, it suffices to prove (13.1) for $\varphi(x) = \varphi_m(x)$. With this choice of $\varphi(x)$, the proposed identity becomes

$$\sum_{n=m}^{\infty}(-1)^{n+m}\frac{(\frac{1}{2})_n}{n!}(-n)_m = \sum_{n=0}^{\infty}(-1)^{n+m}\frac{(\frac{1}{2})_n}{n!}\left(\frac{2n+1}{2}\right)_m. \tag{13.2}$$

First,

$$\begin{aligned}
\sum_{n=m}^{\infty}(-1)^{n+m}\frac{(\frac{1}{2})_n}{n!}(-n)_m &= \sum_{j=0}^{\infty}(-1)^j\frac{(\frac{1}{2})_{m+j}}{(m+j)!}(-j-m)_m \\
&= (\tfrac{1}{2})_m\sum_{j=0}^{\infty}(-1)^{m+j}\frac{(m+\frac{1}{2})_j}{j!}.
\end{aligned} \tag{13.3}$$

Second,

$$\sum_{n=0}^{\infty}(-1)^{n+m}\frac{(\frac{1}{2})_n}{n!}\left(\frac{2n+1}{2}\right)_m = \sum_{n=0}^{\infty}(-1)^{n+m}\frac{(\frac{1}{2})_n}{n!}\frac{(\frac{1}{2})_m(m+\frac{1}{2})_n}{(\frac{1}{2})_n}. \tag{13.4}$$

Comparing (13.3) and (13.4), we see that we have established (13.2), as desired.

Observe that the series in Entry 13 do not converge.

Entry 14 (p. 265). *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Then*

$$\sqrt{\alpha}\left\{\frac{1}{4}+\sum_{n=0}^{\infty}\frac{(-1)^n}{e^{(2n+1)\alpha}-1}\right\} = \sqrt{\beta}\left\{\frac{1}{4}+\sum_{n=0}^{\infty}\frac{(-1)^n}{e^{(2n+1)\beta}-1}\right\}. \tag{14.1}$$

Proof. Let

$$L(s) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^{-s}, \qquad \operatorname{Re} s > 0.$$

It is well known that this Dirichlet L-function can be analytically continued to an entire function.

We apply Entry 21(iii) of Chapter 14 (Part II [2, p. 277]) in the case $n = 0$ to find that

$$\sqrt{\alpha}\left\{\frac{1}{2}L(0)+\sum_{n=0}^{\infty}\frac{(-1)^n}{e^{(2n+1)\alpha}-1}\right\} = \frac{1}{2}\sqrt{\beta}\sum_{n=1}^{\infty}\frac{1}{\cosh(\beta n)}+\frac{1}{4}\sqrt{\beta}, \tag{14.2}$$

where $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$. From the functional equation (H. Davenport [1, p. 71])

$$L(s) = \cos\left(\tfrac{1}{2}\pi s\right)\left(\tfrac{1}{2}\pi\right)^{s-1}\Gamma(1-s)L(1-s)$$

and the value $L(1) = \pi/4$, found as (32.7) in Chapter 37, we easily deduce that $L(0) = \frac{1}{2}$. Thus, (14.2) may be rewritten in the form

$$\sqrt{\alpha}\left\{\frac{1}{4}+\sum_{n=0}^{\infty}\frac{(-1)^n}{e^{(2n+1)\alpha}-1}\right\} = \sqrt{\beta}\left\{\frac{1}{4}+\sum_{n=1}^{\infty}\frac{1}{e^{\beta n}+e^{-\beta n}}\right\}. \tag{14.3}$$

But

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{e^{\beta n}+e^{-\beta n}} &= \sum_{n=1}^{\infty}e^{-\beta n}\sum_{j=0}^{\infty}(-1)^j e^{-2\beta nj}\\ &= \sum_{j=0}^{\infty}(-1)^j\sum_{n=1}^{\infty}e^{-(2j+1)\beta n} = \sum_{j=0}^{\infty}\frac{(-1)^j}{e^{(2j+1)\beta}-1}.\end{aligned} \tag{14.4}$$

Using (14.4) in (14.3), we deduce (14.1).

We consider the next result in a formal sense only.

Entry 15 (p. 265). *If*

$$\psi(n) = \int_0^\infty \varphi(x)\cos(nx)\,dx, \tag{15.1}$$

then, if r is an even nonnegative integer,

$$\psi^{(r)}(n) = \cos\left(\tfrac{1}{2}\pi r\right)\int_0^\infty x^r\varphi(x)\cos(nx)\,dx \tag{i}$$

and

$$\frac{\pi}{2}\varphi^{(r)}(n) = \cos\left(\tfrac{1}{2}\pi r\right)\int_0^\infty x^r\psi(x)\cos(nx)\,dx. \tag{ii}$$

Ramanujan put no restrictions on r. He also wrote $\psi^r(n)$ and $\varphi^r(n)$ for $\psi^{(r)}(n)$ and $\varphi^{(r)}(n)$, respectively.

Proof. Formally differentiating (15.1) r times, we deduce (i).

By Fourier's inversion theorem (Titchmarsh [2, pp. 16–17], Part I [1, p. 333]),

$$\frac{\pi}{2}\varphi(n) = \int_0^\infty \psi(x)\cos(nx)\,dx.$$

Formally differentiating r times, we deduce (ii).

Entry 16 (p. 267). *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi/4$. Then*

$$\alpha^{3/2}\sum_{n=0}^\infty (-1)^n(2n+1)e^{-(2n+1)^2\alpha^2} = \beta^{3/2}\sum_{n=0}^\infty (-1)^n(2n+1)e^{-(2n+1)^2\beta^2}. \tag{16.1}$$

Proof. We apply Poisson's summation formula for Fourier sine transforms (Titchmarsh [2, p. 66], Part II [2, p. 236]) to the function $x\exp\left(-x^2\right)$. Thus, if $\alpha, \beta > 0$ and $\alpha\beta = \pi/2$, then

$$\alpha^2\sum_{n=0}^\infty (-1)^n(2n+1)e^{-(2n+1)^2\alpha^2} = \sum_{n=0}^\infty (-1)^n\int_0^\infty xe^{-x^2}\sin\{(2n+1)\beta x\}\,dx. \tag{16.2}$$

But (Gradshteyn and Ryzhik [1, p. 529, formula 3.952, no. 1])

$$\int_0^\infty xe^{-x^2}\sin\{(2n+1)\beta x\}\,dx = \tfrac{1}{4}\sqrt{\pi}(2n+1)\beta e^{-(2n+1)^2\beta^2/4}.$$

Replace β by 2β and use the evaluation above in (16.2). Thus, for $\alpha, \beta > 0$ with $\alpha\beta = \pi/4$,

$$\alpha^2\sum_{n=0}^\infty (-1)^n(2n+1)e^{-(2n+1)^2\alpha^2} = \frac{\sqrt{\pi}\beta}{2}\sum_{n=0}^\infty (-1)^n(2n+1)e^{-(2n+1)^2\beta^2}.$$

Since $\sqrt{\alpha} = \sqrt{\pi}/(2\sqrt{\beta})$, we find that the proof of (16.1) is complete.

Entry 17 (p. 339). *If I_ν denotes the Bessel function of imaginary argument of order ν, then, if n is an integer,*

$$I_n(2x) - I_{-n}(2x) = 0. \tag{17.1}$$

Ramanujan's recording of Entry 17 is incomplete. He writes the difference of two series on the left side but does not indicate what this difference equals. The series are Bessel functions of imaginary argument, as we have indicated (Watson [15, p. 77]), and (17.1) is a well-known, easily proved equality (Watson [15, p. 79]).

Entry 18 (p. 350).

$$e^{-2\pi} \approx \frac{1}{540}\left(1+\frac{1}{120}\left\{1+\frac{1}{100}\left(1+\frac{1}{35}\left\{1+\frac{1}{90}+\frac{1}{90\cdot 25}\right\}\right)\right\}\right).$$

Most of Ramanujan's approximations for $\exp(-\alpha\pi)$ arise from modular equations or class invariants. However, it appears that this approximation to $\exp(-2\pi)$ was empirically derived by some method of successive approximations.

Proof. Using *Mathematica*, we find that

$$\begin{aligned}&\frac{1}{540}\left(1+\frac{1}{120}\left\{1+\frac{1}{100}\left(1+\frac{1}{35}\left\{1+\frac{1}{90}+\frac{1}{90\cdot 25}\right\}\right)\right\}\right)\\ &\quad = 0.0018674427317 26\ldots.\end{aligned}$$

On the other hand,

$$e^{-2\pi} = 0.0018674427317 07\ldots.$$

Thus, Ramanujan's approximation agrees to 13 decimal places.

We conclude by briefly mentioning two claims made by Ramanujan on page 112 that apparently have no precise meanings.

Entry 19.

$$\sum_{n=1}^{\infty}\frac{\varphi(\log n)}{n} = \int_0^{\infty}\varphi(x)\,dx + \sum_{n=0}^{\infty}\frac{c_n}{n!}\varphi^{(n)}(0).$$

Since Ramanujan does not specify the function φ or the constants c_n, we are unable to offer a definite interpretation of this formula. Possibly, Ramanujan applied the Euler–Maclaurin summation formula to the function $\varphi(\log x)/x$ on the interval $(1, \infty)$. A simple change of variable then gives the integral on the right side above. The constants c_n are therefore those that appear in the Euler–Maclaurin summation formula, and the series on the right side should probably be interpreted as an asymptotic series.

Entry 20.

$$2n + \sum_{k=2}^{\infty} \frac{k^n + k^{-n}}{(k-1)k} = \frac{1}{n} - \pi \cot(\pi n) + \cdots .$$

Since the series on the left side diverges and since the meaning of the "dots" on the right side is not divulged by Ramanujan, we are unable to offer a meaningful interpretation of this formula. We think that it pertains to material in the first part of Chapter 7 of the second notebook. In particular, see Part I [1, p. 161].

Location of Entries in the Unorganized Portions of Ramanujan's First Notebook

In Part IV [4] we provided the locations in the second and third notebooks of all entries in the 16 organized chapters of the first notebook. A small minority of these entries cannot be found in the second or third notebooks, and so we provided proofs for these results in [4]. Like the second notebook, the first notebook contains much unorganized material, in fact, considerably more than in the second notebook. The unorganized pages also contain a higher proportion of entries not found in the second notebook than the organized part of the first notebook. In the sequel, we indicate where proofs can be found for each correct result in the unorganized portions of the first notebook. Page numbers of the first notebook are given in boldface at the left margin. We assign numbers, in order, to each formula on each page. If the result appears in the second or third notebooks, we indicate where in these notebooks, and where in Parts I–V, it can be located. If an entry cannot be found in the second or third notebooks, we inform readers where a proof can be found in the present volume.

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1. This is a version of Entry 1 of Chapter 3 [1, p. 45].

2.,3. These are Corollaries 1 and 2, respectively, in Section 2 of Chapter 3 [1, p. 46].

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1. This is contained in Entry 10 of Chapter 3 [1, pp. 57–59].

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1. Entry 11(i) of Chapter 18 [3, p. 162].

2.,3. The two radical expressions are equivalent to the formulas for G_{1225} and G_{441}, respectively, given in the table of class invariants in Section 2 of Chapter 34.

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1.,2. Entries 24 and 24(i), respectively, of Chapter 14 [2, pp. 291–292].

3. Entry 36, Chapter 13 [2, p. 238].

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1.,3. These are equivalent to Entries 36(i) and (ii), respectively, of Chapter 11 [2, p. 100].

2.,4. Entries 36(iii) and (iv), respectively, of Chapter 11 [2, pp. 100–101].

52

1. Entry 49, Chapter 12 [2, p. 184].

2. Corollary 2, Section 44, Chapter 12 [2, p. 170].

54

1.,2. Entries 2(i), (ii), Chapter 13 [2, p. 188].

3. Entry 3, Chapter 13 [2, p. 188].

4.,5. Entry 4, Chapter 13 [2, p. 189].

6. Entry 5, Chapter 13 [2, p. 190].

56

1. Part (i) is the same as Entry 20, Chapter 5 [1, p. 123].

2. Part (ii) follows from Entries 19(i), (ii), Chapter 5 [1, pp. 122–123]. The word "multiple" on page 56 should be replaced by "factor."

58

1. Although this formula is not in the second notebook, it is formula 14 of Table 1 in Ramanujan's paper [7], [10, p. 141].

2. The value of the Bernoulli number B_{38} can be found in Ramanujan's paper [1], [10, p. 5] and on page 53 of his second notebook [9].

3. This deleted entry is a partial version of formula 15 in Table 1 mentioned above.

60

1.–3., 5.–9. The values of the Bernoulli numbers B_n, $n = 22, 24, 26, 28, 30, 32, 34, 36$ can be found in Ramanujan's paper [1], [10, p. 5] and second notebook [9, p. 53].

4. This is formula 12 in Table 1, mentioned in our commentary on page 58.

62

1. This is a trivial statement about the factorization of polynomials.

2.–4. These three formulas are renditions of the same result. The first two contain unexplained asterisks and are deleted by Ramanujan. The third version is imprecise and contains an error term that is not completely specified. See Entry 27(ii) of Chapter 7 [1, p. 178] for a correct version of these three formulas.

64

1.,2. Entry 11, Chapter 11 [2, p. 54].

3. This geometrical figure can be found in Section 19 of Chapter 18 [3, p. 190].

4.,5. Entries 19(iv), (iii), respectively, of Chapter 18 [3, pp. 184, 181].

66

1. See Section 24 of Chapter 18 [3, p. 211].

2. See Section 19 of Chapter 18 [3, p. 194].

3. Part of the corollary in Section 3 of Chapter 18 [3, p. 151].
4. Part of Entry 3, Chapter 18 [3, p. 146].

68

1.,2. Corollaries (i), (ii), Section 19 of Chapter 18 [3, pp. 185, 190].

70

1. Entry 14, Chapter 4 [1, p. 107]. This is also a special case of Entry 27 of Chapter 13 [2, p. 230].

2.,3. Both drawings appear to be versions of a figure of Chapter 18 [3, p. 194].

72

1. Entry 29, Chapter 5 [1, p. 130].
2. Entry 1 of Chapter 39.
3. Entry 6, Chapter 12 [2, p. 111].

74

1.–3. Entries 21(i), (iii), (iv), Chapter 18 [3, pp. 200–201].

78

1. Entry 4(i), Chapter 6 [1, p. 136].

2.,3. Examples 1, 2 in Section 4 of Chapter 6 [1, p. 137].

4. See Section 4 of Chapter 6 [1, p. 137].

80

1.–3. These three singular moduli are given in Theorem 9.9 of Chapter 34.

82

1.–4. Examples 9.4 of Chapter 34.

5.–7. Examples 9.7 of Chapter 34.

8.–10. Theorems 9.6, 9.3, and 9.5, respectively, in Chapter 34.

84

1. Entry 14, Chapter 15 [2, p. 332].

86

1. Entry 11(xv), Chapter 20 [3, p. 385].

2.–10. Entries 45, 42, 43, 44, 46, 48, 49, 47, 50, respectively, of Chapter 36.

88

1. Special case of Entry 11, Chapter 6 [1, p. 143].

2.,3. Two elementary abelian theorems for power series with no hypotheses. See Titchmarsh's treatise [1, pp. 229–231] for general abelian theorems.

4.,5. Entries 52 and 51, respectively, of Chapter 36.

90

1. Entry 5(xii), Chapter 19 [3, p. 231].
2. Entry 13(xiv), Chapter 19 [3, p. 282].
3. Entry 19(ix), Chapter 19 [3, p. 315].

4.–7. Entry 41, Chapter 36.

92

1. Entry 5(xiii), Chapter 19 [3, p. 231].
2. Entry 13(xv), Chapter 19 [3, p. 282].
3. Entry 19(x), Chapter 19 [3, p. 315].

94

1.,2. See Entry 2 of Chapter 39 for comments on these deleted formulas.
3. This is a special case of Entry 35(iii), Chapter 11 [2, p. 99].
4.,6. See Entry 3 of Chapter 39.
5. This is a special instance of Entry 35(ii), Chapter 11 [2, p. 99].

At the bottom of the page, Ramanujan apparently indicates that he has found the generalizations given in Entries 35(i)–(iii) of Chapter 11 of his second notebook (Part II [2, p. 99]).

96

1. Theorem 2.12 of Chapter 33.
2.–5. These are contained in Theorem 8.7 of Chapter 33.
6. All of Section 12 of Chapter 33 is devoted to an examination of this claim.
7.–9. See Theorem 8.1 of Chapter 33.

98

1.,2. Entries 9(iv), (iii), respectively, of Chapter 19 [3, p. 258].

100

1. Entry 35, Chapter 9 [1, p. 294].
2. Entry 11(iii), Chapter 13 [2, p. 217].
3. Entry 22(ii), Chapter 14 [2, p. 278].
4. Corollary, Section 22, Chapter 14 [2, p. 279].

102

1. Entry 18, Chapter 14 [2, pp. 267–268].
2. Ramanujan, in essence, states a general formula for the multiplication of two Laurent series.
3. Special case of Corollary 1, Section 18, Chapter 14 [2, p. 268].

104

1. Entry 19(iv), Chapter 14 [2, p. 273].
2. See Entry 4 of Chapter 39.

106

1. Entry 20(i), Chapter 14 [2, p. 274].
2. Entry 19(i), Chapter 14 [2, p. 271].
3. Corollary, Section 20, Chapter 14 [2, p. 274].
4. Entry 20(iv), Chapter 14 [2, p. 275].
5. Corollary of Entry 20(iv), Chapter 14 [2, p. 275].
6. Special case of Entries 29(i), (ii), Chapter 13 [2, p. 231].

108

1. Entry 35, Chapter 12 [2, pp. 156–157].

2. See Entry 5 of Chapter 39.
3. Entry 34(ii), Chapter 13 [2, p. 237].

110

1. Entry 40, Chapter 12 [2, p. 163].

112

1.,2. Entries 19 and 20, respectively, in Chapter 39.
3.–5. Entry 14, Corollaries 1, 2, Chapter 7 [1, pp. 166–168].

114

1. Entry 6, Chapter 13 [2, p. 193].
2. Entry 10, Chapter 13 [2, p. 207].
3. A more precise version of this entry can be found in Entry 9 of Chapter 11 [2, p. 51].

116

1.,2. Entries 11(i), (ii), Chapter 13 [2, pp. 215–216].

118

1. The entry is deleted and is a forerunner of Entry 7 of Chapter 13 [2, p. 195].

120

1. The first statement

$$\frac{a^x}{\Gamma(x+1)} = \frac{1}{\sqrt{2\pi}} \exp\left(\int \frac{x}{a} da\right)$$

is incorrect.

2. The second claim is a version of Example (i), Section 25, Chapter 13 [2, p. 228]. See also Entry 6 of Chapter 39 for a correction to our claim made in Part II [2].

122

1. Entry 26(ii), Chapter 13 [2, p. 229].
2.,3. Entries 17(iii), (iv), respectively, in Chapter 18 [3, p. 176].

124

1.,2. Entries 11 and 12, respectively, in Chapter 16 [3, pp. 21, 24].

126

1. Entry 31, Chapter 10 [2, p. 41].
2. Entry 7, Chapter 16 [3, p. 16].

128

1. Entry 8, Chapter 16 [3, p. 17].
2. Entry 4, Chapter 16 [3, p. 14].

130

1. Version of the corollary in Section 36 of Chapter 13 [2, p. 239].
2. Entry 5, Chapter 16 [3, p. 14].

132

1. Entry 6(vi), Chapter 9 [1, p. 247].
2. Entry 6, Chapter 16 [3, p. 15].
3. Entry 8, Chapter 13 [2, p. 202].
4. Corollary of Entry 48, Chapter 12 [2, p. 181].

134

1. Entry 9, Chapter 16 [3, p. 18].
2. Entry 15, Chapter 16 [3, p. 30].
3. This entry is very vague. It is possibly a less specific version of Entry 16, Chapter 16 [3, p. 31].

136

1. Corollary (i) in Section 9, Chapter 16 [3, p. 18].
2. Entry 17, Chapter 16 [3, p. 32].

138

1. Entry 21, Chapter 13 [2, p. 224].
2. See Entry 7 of Chapter 39 for a proof.

140

1.,2. Both entries are versions of Entry 21 of Chapter 13 [2, p. 224].

142

1. Version of Entry 17, Chapter 13 [2, pp. 220–221].
2. Deleted by Ramanujan.
3. Entry 20, Chapter 13 [2, p. 224].

4.,5. Entry 16 (Second Part) (iii), (i), respectively, of Chapter 18 [3, p. 174].

144

1. Deleted by Ramanujan.

2.,3. Entries 17(ii), (i), Chapter 18 [3, p. 176].

4.,5. Entry 16 (Second Part) (iv), (ii), respectively, of Chapter 18 [3, p. 174].

146

1. Entry 24, Chapter 12 [2, p. 139].

2.,3. Entries 45(i), (ii), Chapter 12 [2, p. 171].

4. Entry 16, Chapter 16 [3, p. 31].
5. Entry 7(vii), Chapter 17 [3, p. 106].

148

1. Entry 6(viii), Chapter 9 [1, p. 247].

2.–11. Entry 14, Chapter 18 [3, pp. 168–169].

12. Entry 18(iv), Chapter 18 [3, p. 179].

150

1. Corollary, Section 12, Chapter 18 [3, p. 164].
2. Special case of Entry 14, Chapter 12 [2, p. 121].
3. Corollary, Section 34, Chapter 12 [2, p. 156].

152

1.,2. Entry 47, Chapter 12 [2, p. 179].
3. Entry 13(i), Chapter 18 [3, p. 165].
4. Entry 22(iii), Chapter 18 [3, p. 207].
5. Deleted by Ramanujan.

154

1. Example (iv), Section 2, Chapter 15 [2, p. 305].
2. Entry 9, Chapter 13 [2, p. 205].
3. Entry 48, Chapter 12 [2, p. 181].
4. Entry 7, Chapter 13 [2, pp. 195–196].

156

1. Entry 12(ii), Chapter 18 [3, p. 163].
2. Entry 22(ii), Chapter 18 [3, p. 207].
3.,4. Entries 13(ii), (iii), Chapter 18 [3, p. 165].

158

1. Entry 39(i), Chapter 16 [3, p. 83].
2. Entry 38(iv), Chapter 16 [3, p. 80].
3. Entry 11(iii), Chapter 19 [3, pp. 265–266].
4. See Entry 8 of Chapter 39.

160

1. See Part II [2, p. 147] for comments.
2. Corollary, Section 15, Chapter 16 [3, p. 30].
3. Entry 3, Chapter 16 [3, p. 14].
4.,5. Entries 38(i), (ii), Chapter 16 [3, p. 77].

162

1. Entry 7(iii), Chapter 17 [3, p. 105].
2. Corollary 3.4 of Chapter 33.
3. Theorem 9.9 of Chapter 33.
4. Theorem 11.6 of Chapter 33.
5. Theorem 9.10 of Chapter 33.
6. Corollary 3.5 of Chapter 33.
7. Theorem 4.4 of Chapter 33.
8. Theorem 4.5 of Chapter 33.

164

1.–3. See Section 21, Chapter 13 [2, p. 225].
4. Entry 13, Chapter 16 [3, p. 27].
5. See Section 13, Chapter 16 [3, p. 28].

166

1.,2. Entries 31(i), (ii), Chapter 13 [2, pp. 233–234].

168

1. Entry 23, Chapter 12 [2, pp. 137–138].

2. Entry 7(viii), parts (a), (c), and (d), Chapter 17 [3, p. 107].

170

1. Entry 7(ii), Chapter 17 [3, p. 105].
2. Entry 7(ix), Chapter 17 [3, p. 108].
3. Entry 7(x), Chapter 17 [3, p. 110].
4. Entry 12(i), Chapter 18 [3, p. 163].
5. Entry 22(i), Chapter 18 [3, p. 206].
6. Entry 7(i), Chapter 17 [3, p. 104].

172

1. Entry 7(xii), Chapter 17 [3, p. 112].
2.,3. See Entries 79 and 80, respectively, of Chapter 36.
4. Entry 7(xiii), Chapter 17 [3, p. 113].
5. Entry 7(iv), Chapter 17 [3, p. 105].

174

1. Deleted by Ramanujan.
2. Entry 6, Chapter 10 [2, p. 12].

176

1.,2. Deleted by Ramanujan.
3. Entry 7(vi), Chapter 17 [3, p. 106].
4. See Entry 1 of Chapter 36.

178

1. Entry 14, Chapter 13 [2, p. 219].
2. Quarterly Reports [1, p. 331].
3. Special case of Entry 32(i), Chapter 13 [2, p. 235].
4. Entry 16, Chapter 15 [2, p. 338].

180

1. Entry 15, Chapter 13 [2, p.220].
2. Entry 22(i), Chapter 13 [2, p. 225].
3. Entry 13, Chapter 13 [2, p. 219].
4. Corollary of Entry 13, Chapter 13 [2, p. 219].
5. Corollary of Entry 21, Chapter 13 [2, p. 224].

182

1. Quarterly Reports [1, p. 298].
2. Entry 23, Chapter 13 [2, p. 226].
3. Entry 14, Chapter 16 [3, p. 29].
4. Entry 22(ii), Chapter 13 [2, p. 225].

184

1. See Entry 9 of Chapter 39.
2. See Part II [2, eq. (17.3), p. 265].
3.–5. Examples (i), (ii), (iii), Section 30, Chapter 13 [2, p. 233].

186

1.,2. Entries 36(i), (ii), Chapter 16 [3, pp. 65–66].

188

1. Entry 34, Chapter 12 [2, p. 156].
2. Entry 39, Chapter 12 [2, p. 159].
3.,4. Entry 38, Chapter 12 [2, p. 158].

190

1. Entry 20, Chapter 10 [2, pp. 36–37].
2. Part of second part of Entry 35, Chapter 12 [2, pp. 156–157].
3. Entry 27, Chapter 12 [2, p. 146].

192

1. First part of Entry 35, Chapter 12 [2, pp. 156–157].
2.,3. Entries 36 and 37, Chapter 12 [2, p. 158].

194

1. See Entry 10 of Chapter 39.
2. Entry 29(c), Chapter 10 [2, p. 40].
3. Entry 32, Chapter 10 [2, p. 41].

196

1. Entry 8, Chapter 11 [2, p. 51].
2. Entry 22, Chapter 11 [2, pp. 64–65].
3. Corollary, Section 22, Chapter 11 [2, p. 68].
4. Example, Section 17, Chapter 12 [2, p. 131].

198

For the meaning of the geometrical figure, see Section 7 of Chapter 19 [3, pp. 244–245].

1. Entry 22, Chapter 12 [2, p. 136].

200

For the significance of the geometrical figure, see Section 7 of Chapter 19 [3, p. 243].

1. Entry 17, Chapter 12 [2, pp. 124–125].
2.,3. Corollaries (i), (ii), Section 17, Chapter 12 [2, pp. 130–131].

202

1. In essence, Entry 28, Chapter 13 [2, p. 231].
2.,4.–6. See Section 24 of Chapter 13 [2, pp. 226–227].
3. Entry 20(iii), Chapter 18 [3, p. 197].

204

1. Entry 21, Chapter 11 [2, p. 64].
2. See Theorem 7.3 of Chapter 33.

206

1.,2. See Entry 11 of Chapter 39.

3. Example 2, Section 12, Chapter 11 [2, p. 58].

208

Except for one example, which we establish in Entry 12 of Chapter 39, all results on this page are found in Section 15 of Chapter 15 [2, pp. 335–337].

210

1. The first five lines on the page continue material from page 208 and can be found in Section 15 of Chapter 15.
2. Theorem 11.5 of Chapter 33.
3.,4. Theorems 4.2 and 4.3 of Chapter 33.
5.,6. Theorems 9.5 and 9.6 of Chapter 33.

212

1.,2. Entries 23(i), (ii), Chapter 18 [3, p. 208].
3. See Example (iii) in Section 17 of Chapter 9 [2, p. 266].
4. See Theorem 6.4 of Chapter 33.

214

1. This is a definition.
2.,3.,5. These are, respectively, Theorems 10.1, 10.3, and 10.2 of Chapter 33.
4. Theorem 9.11 of Chapter 33
6.,7. Part of Theorem 9.2 of Chapter 34.
8. Corollary 2.4 of Chapter 33.
9. Theorem 10.4 of Chapter 33.

216

1. Theorem 11.4 of Chapter 33.
2. Theorem 11.1 of Chapter 33.
3. Theorem 5.6 of Chapter 33.

218

1. Theorem 6.1 of Chapter 33.
2.–6. Theorem 7.1 of Chapter 33.
7.,8. Theorems 7.6 and 7.8 of Chapter 33.
9. Deleted by Ramanujan.

220

1. This follows from (2.6) and Theorem 2.10 in Chapter 33.
2. This is an incorrect version of part of Entry 3(i) of Chapter 21 [3, p. 460].
3. This is a misstatement of the first part of Entry 5(i) of Chapter 21 [3, p. 467].
4.,5. Theorem 2.13 of Chapter 33.

222

1.–4. Entries 8(i)–(iv), Chapter 19 [3, p. 249].
5. See Entry 14 of Chapter 36.

224

1.–3. Corollary, Section 37, Chapter 16 [3, p. 74].

4.,5. Entries 4(i), (ii) Chapter 19 [3, p. 226].

226

1.,3. Deleted by Ramanujan.

2. Part of Entry 34(iii), Chapter 11 [2, p. 97].

4. Entry 7(xi), Chapter 17 [3, p. 111].

228

1. This result is essentially equivalent to the example in Section 37 of Chapter 16 and can be found explicitly in (37.7) on page 76 of Part III [3].

2. Entry 37(iii), Chapter 16 [3, p. 73].

3. This result follows from Entries 37(i), (ii), Chapter 16 [3, p. 73].

4. Entry 32(v), Chapter 11 [2, p. 93].

230

1.–3. Entries 3(iii), (iv), (i), Chapter 19 [3, p. 223].

4.,8. Entries 17(i), (ii), Chapter 19 [3, p. 302].

5. Entry 9(i), Chapter 20 [3, p. 377].

6. See Entry 2 of Chapter 36.

7. Entry 29 of Chapter 36.

232

1. An incomplete version of Entry 28, Chapter 11 [2, p. 83].

2. Deleted by Ramanujan.

234

1. A deleted version of Entry 30, Chapter 11 [2, p. 87].

2. See the table in Section 2 of Chapter 34.

236

1. Corollary, Section 28, Chapter 11 [2, p. 85].

2. Special case of Entry 30, Chapter 11 [2, p. 87].

3. Entry 31(ii), Chapter 11 [2, p. 88].

238

1. Entry 30, Chapter 11 [2, p. 87].

2. Entry 14, Chapter 11 [2, p. 59].

240

1. Part of Corollary (ii), Section 31, Chapter 16 [3, p. 49].

2. Trivial algebraic identity.

3. Example (i), Section 31, Chapter 16 [3, p. 50].

4.,5. Corollary, Section 28, Chapter 16 [3, p. 44].

6. With the use of Entry 22(ii) and (22.4) in Chapter 16, it can easily be shown that this entry is equivalent to Example (v), Section 31, Chapter 16 [3, p. 51].

242

1. See Theorem 9.3 of Chapter 33.

2. Entry 9(v), Chapter 19 [3, p. 258].

3. Entry 31, Chapter 16 [3, p. 48].
4. Corollary, Section 30, Chapter 16 [3, p. 47].

244

1. This result is easily seen to be equivalent to the first part of Example (iv) in Section 31 of Chapter 16, although Ramanujan, in the numerator, neglected to write the factors $(x^3; x^8)_\infty (x^5; x^8)_\infty (x^8; x^8)_\infty$ [3, p. 51].
2. An incomplete version of the second part of Example (iv), Section 31, Chapter 16 [3, p. 51].
3. See Entry 81 of Chapter 36.
4. This is equivalent to the example in Section 9 of Chapter 17 [3, p. 122].
5. See Entry 82 of Chapter 36.
6.,7. Example (ii), Section 31, Chapter 16 [3, p. 50].

246

1.,2. See Entry 22 of Chapter 36.
3.–11. Entry 10, Chapter 17 [3, p. 122].
12.–17. Entries 11(i), (iii), (iv), (v), (vi), (viii), Chapter 17 [3, p. 123].

248

1. Example (i), Section 6, Chapter 17 [3, p. 103].
3. Example (iii), Section 6, Chapter 17 [3, p. 104].
1.–16. See Entry 1 of Chapter 35.

250

1.–9. See Entry 2 of Chapter 35.
10. Entry 27, Chapter 11 [2, p. 80].
11. Deleted by Ramanujan.

252

1. Entry 7, Chapter 15 [2, p. 313].
2. Special case of Theorem 6.1 of Chapter 15 [2, p. 310].

254

1.,2. See Entries 3 and 4 of Chapter 36.
3.–5. Entries 13(ii)–(iv), Chapter 15 [2, p. 330].

256

1.–3. Entries 12(ii)–(iv), Chapter 15 [2, p. 326].
4. Entry 13(i), Chapter 15 [2, p. 330].

258

1.–4. Entries 12(v)–(viii), Chapter 15 [2, p. 326].

260

1.,2. Entries 35(ii), (i), Chapter 16 [3, p. 63, 61].
3. Entry 33(i), Chapter 16 [3, p. 52].

262

1. Entry 34(i), Chapter 16 [3, p. 54].

2. Entry 33(ii), Chapter 16 [3, p. 53].
3. Corollary (i), Section 34, Chapter 16 [3, pp. 57–58].
4. Corollary (ii), Section 34, Chapter 16 [3, pp. 59–60].

264

1.–3. Entries 32(i)–(iii), Chapter 16 [3, pp. 51–52].
4. Entry 12(ix), Chapter 15 [2, p. 326].
5.,6. See Entries 38 and 39 of Chapter 36.

265

We use the numbering given by Ramanujan.
1. Entry 31, Chapter 12 [2, p. 150].
2. See Entry 13 of Chapter 39.
3.,4. Entries 8(i), (ii), Chapter 17 [3, p. 114].
5. See Entry 14 of Chapter 39.
6. Entry 12(i), Chapter 4; Quarterly Reports [1, pp. 107, 321].
7(i). Special case of the preceding entry.
7(ii). Entry 12(ii), Chapter 4 [1, p. 107]; special case of the corollary of Entry 21 of Chapter 13 [2, p. 224].
8. See Entry 15 of Chapter 39.

266

1.–4. Entries 8(ix), (xi), (xii), (x), Chapter 17 [3, pp. 114–115].
5. See Entry 31 of Chapter 36.
6. Special case of Entry 18(i), Chapter 13 [2, p. 221].
7. We have found no meaningful interpretation of this equality.
8. See Entry 5 of Chapter 36.
9. Entry 9, Chapter 19 [3, p. 257].

267

The numbering is continued from page 265.
9.,10. Entries 8(iii), (iv), Chapter 17 [3, p. 114].
11. See Entry 32 of Chapter 36.
12. Entry 24(ii), Chapter 16 [3, p. 39].
13. See Entry 16 of Chapter 39.
14. Special case of Entry 31(i), Chapter 13 [2, p. 233].
15.–17. Entries 16(i)–(iii), Chapter 13 [2, p. 220].
18. Special case of Theorem II in Ramanujan's Quarterly Reports [1, p. 313].

268

1. Entry 9(i), Chapter 17 [3, p. 120].
2. This is an incorrect version of Entry 9(iv) in Chapter 17 [3, p. 120].
3.,4. Entries 13(iii), (iv), Chapter 17 [3, p. 127].
5. Principle of duplication, Chapter 17 [3, p. 125].

269

1.,2. Entries 13(i), (ii), Chapter 17 [3, p. 126].
3.–6. Entries 14(i)–(iv), Chapter 17 [3, pp. 129–130].

7. Entry 13(viii), Chapter 17 [3, p. 127].

270

1.–5. Entries 17(i)–(v), Chapter 17 [3, p. 138].
6. Entry 11, Chapter 14 [2, p. 258].

271

1.–4. Entries 17(vi)–(ix), Chapter 17 [3, p. 138].
5. See Entry 40 of Chapter 36.
6. See Entry 83 of Chapter 36.

272

1.–4. Entries 14(v)–(viii), Chapter 17 [3, p. 130].
5.–8. Entries 15(i)–(iv), Chapter 17 [3, p. 132].

273

1.–4.,8.,9. Entries 13(viii), (x), (xi), (ix), (v), (vi), Chapter 17 [3, p. 127].
5.–7. Entries 14(ix)–(xi), Chapter 17 [3, p. 130].

274

1. Entry 21(iii), Chapter 14 [2, p. 277].
2. Entry 34(ii), Chapter 16 [3, p. 56].
3. See Entry 33 of Chapter 36.

275

1. Entry 33(iii), Chapter 16 [3, p. 53].
2.–6. Entries 16(i)–(v), Chapter 17 [3, p. 134].

276

1. Entry 14, Chapter 14 [2, p. 262].
2. Corollary, Section 12, Chapter 14 [2, p. 260].
3. Entry 12, Chapter 14 [2, p. 260].

277

1.,2. Entry 15, Chapter 14 [2, p. 262].
3. Entry 16(vii), Chapter 17 [3, p. 134].
4. Entry 12(iii), Chapter 17 [3, p. 124].
5. Entry 9(iv), Chapter 17 [3, p. 120].

278

1.–4. Entries 15(ix)–(xii), Chapter 17 [3, pp. 132–133].
5.–7. Entries 8(viii), (vi), (vii), Chapter 17 [3, p. 114].

279

1. Entry 11, Chapter 15 [2, p. 323].
2. Entry 21(ii), Chapter 14 [2, p. 276].

280

1. Entry 6, Chapter 18 [3, p. 153].
2.,3. See Entries 77 and 78 of Chapter 36.

4. Entry 7, Chapter 18 [3, pp. 154–155].

281

1. Example, Section 7, Chapter 18 [3, p. 156].

2. See Entry 76 of Chapter 36.

3. This is a formal application of the "change of sign" process.

4. The principal of "change of sign," as described in Section 13 of Chapter 17 [3, p. 126].

282

1. Entry 8(i), Chapter 17 [3, p. 114].

2.,3. Entries 4(iii), (iv), Chapter 19 [3, pp. 226–227].

4. This is equivalent to the second part of Entry 5(iii) of Chapter 19 [3, p. 230]. In particular, see the first equality in the proof of (iii) [3, p. 232].

5. This is equivalent to the first part of Entry 5(i) of Chapter 19 [3, p. 230]. In particular, see the equality at the top of page 232 [3], where $\varphi^3(-q^6)$ should be replaced by $\varphi(-q^6)$.

6. This is equivalent to the formula for G_3 in the table in Section 2 of Chapter 34.

7.–9. Entries 5(i), (iv), Chapter 19 [3, p. 230].

10. See Entry 6 of Chapter 36.

283

1.,2. See Entry 7 of Chapter 36.

3. This is equivalent to Entry 5(vi) of Chapter 19 [3, p. 230], with α replaced by $1-\beta$ and β replaced by $1-\alpha$.

4. In essence, this is contained in Entry 5(vi) of Chapter 19 [3, p. 230]; in particular, see (5.4) on page 233 [3].

5.,6. See Entries 8, 9 of Chapter 36.

284

1. Equivalent to the formula for G_9 in the table in Section 2 of Chapter 34.

2. See Entry 4 of Chapter 35.

3. Deleted by Ramanujan.

4.–6. See Entries 34–36 of Chapter 36.

7. See Entry 15 of Chapter 36.

8. Second part of Entry 5(i), Chapter 19 [3, p. 230].

285

1. Corollary, Section 23, Chapter 18 [3, p. 209].

2.,4. Examples, Section 23, Chapter 18 [3, pp. 209–210].

3. Ramanujan gives the value $1/\varphi^4(e^{-\pi}) = 0.71777$, which is correct and can be verified using the evaluation of $\varphi(e^{-\pi})$ given in Example (i) of Section 6 in Chapter 17 [3, p. 103].

5. See Entry 37 of Chapter 36.

6. See Entry 16 of Chapter 36.

7. Entry 13(iii), Chapter 19 [3, p. 280].

286

1. Entry 13(ii), Chapter 19 [3, p. 280].
2. See Entry 27 of Chapter 36.
3. See Entry 17 of Chapter 36.

287

1. Multiply the two equalities of Entry 5(iii), Chapter 19 [3, p. 230].
2. Multiply the two equalities of Entry 13(iv), Chapter 19 [3, p. 281].

3.–5. Equivalent to formulas for G_9, G_{25}, and G_5 given in the table in Section 2 of Chapter 34.

6. The value of α_7 is found in Theorem 9.9 of Chapter 34.
7. Entry 14(v), Chapter 19 [3, p. 289].
8. Entry 5 of Chapter 35.
9. Entry 14(iv), Chapter 19 [3, p. 289].

288

1.–4. Deleted by Ramanujan.

5. See Theorem 9.2 in Chapter 34.

289

1. We have not been able to discern the meaning of this entry.
2. Deleted by Ramanujan.
3. Equivalent to the formula for G_{15} in the table in Section 2 of Chapter 34.

4.,5. See Theorem 9.2 of Chapter 34.

6. Definition of a modular equation of degree 7.
7. Entry 19(i), Chapter 19 [3, p. 314].
8. Entry 5(ii), Chapter 19 [3, p. 230].
9. This modular equation of degree 1 is trivial since $\alpha = \beta$.
10. Ramanujan evidently intended to write a modular equation here, but there is no equality sign. Furthermore, the degree is not given. The proposed modular equation has the unusual feature that, in the first term, α and β appear with no radical signs about them.

290

1. Part of Entry 5(viii), Chapter 19 [3, p. 231].
2. Part of Entry 5(v), Chapter 19 [3, p. 230].
3. Entry 19(ii), Chapter 19 [3, p. 314].
4. Entry 13(v), Chapter 19 [3, p. 281].

5.,7. See Entry 10 of Chapter 36.

6. This modular equation of degree 7 follows from Entry 19(iii) of Chapter 19 [3, p. 314] by dividing the first part of Entry 19(iii) by the second part of the same theorem.

8. This modular equation of degree 7 follows from Entry 19(ii) of Chapter 19 [3, p. 314] by dividing the first part of Entry 19(ii) by the second part of the same theorem.

291

1. This modular equation of degree 3 follows from Entry 5(v) of Chapter 19 [3, p. 230] by dividing the first part of Entry 5(v) by the second part of the same theorem.

2. Entry 7(i), Chapter 20 [3, p. 363].
3. Entry 15(i), Chapter 20 [3, p. 411].
4. Entry 13(i), Chapter 19 [3, p. 280].
5. Part of Entry 5(vii), Chapter 19 [3, p. 230].
6. Part of Entry 13(xii), Chapter 19 [3, p. 281].
7. Entry 3(x), Chapter 20 [3, p. 352].
8. Entry 15(i), Chapter 19 [3, p. 291].
9. Entry 8(iii), Chapter 20 [3, p. 376].
10. Part of Entry 19(v), Chapter 19 [3, p. 314].
11. Entry 11(vi), Chapter 20 [3, p. 384].

292

1. Entry 5(viii), Chapter 19 [3, p. 231].
2. Part of Entry 13(vii), Chapter 19 [3, p. 281].
3. Entry 22(i), Chapter 20 [3, p. 439].
4. Equivalent to a formula for G_{13} given in the table in Section 2 of Chapter 34.
5. Theorem 9.14 of Chapter 34.
6. Entry 3(i), Chapter 20 [3, p. 352].
7. Part of Entry 5(ix), Chapter 19 [3, p. 231].
8. Part of Entry 13(x), Chapter 19 [3, p. 281].
9. Entry 5(xiv), Chapter 19 [3, p. 231].
10. Entry 14(i), Chapter 19 [3, p. 288].
11. This entry,

$$\sin(2u) = 4\sin(\tfrac{1}{2}v)\sqrt{\cos(2v) + 3\sin^2(\tfrac{1}{2}v)},$$

is difficult to read. Although not apparent, by using elementary trigonometry, it can be shown that this modular equation of degree 7 is equivalent to Entry 19(xi) of Chapter 19 [3, p. 315].

293

1.–3. Equivalent, respectively, to formulas for G_{21}, G_{49}, and G_{225} given in the table in Section 2 of Chapter 34.

4. Part of Entry 11(ii), Chapter 20 [3, p. 383].
5. Entry 2(i), Chapter 20 [3, p. 349].

6.–10. Entries 3(iv), (vi), (v), (ii), (iii), Chapter 20 [3, p. 353].

294

1.,3. Entries 2(ii), (iii), Chapter 20 [3, p. 349].

2.,4. Entries 9(v), (vi), Chapter 20 [3, p. 377].

5.,6. Equivalent to Entry 9(vii), Chapter 20 [3, p. 377].

7. Equivalent to the formula for G_{169} in the table in Section 2 of Chapter 34; for more details, see Watson's paper [7, p. 195].

8. Equivalent to the formula for G_{121} in the table of Chapter 34; for more details, see Watson's paper [7, p. 190].

9. This entry is difficult to read and is evidently incomplete. We offer some comments on it at the end of Section 8 of Chapter 34.

295

1.–3. See Entries 18, 19, and 11, respectively, of Chapter 36.

4. Deleted by Ramanujan.

5.–8. Equivalent, respectively, to formulas for G_{45}, G_{11}, G_{23}, and G_{19} in the table in Section 2 of Chapter 34.

296

1. Equivalent to the formula for G_{31} in the table of Chapter 34.

2. Entry 12(iii), Chapter 20 [3, p. 397].

3. Equivalent to the formula for G_{17} in the table of Chapter 34.

4. Deleted by Ramanujan.

5.,6. Entries 28 and 26, respectively, of Chapter 36.

7. Deleted by Ramanujan.

297

1.–4. See Entries 12, 20, 13, and 25, respectively, of Chapter 36.

5. See Entry 7 of Chapter 35.

6. Entry 9(iii), Chapter 20 [3, p. 377].

7.–11. Entries 11(i)–(v), Chapter 20 [3, pp. 383–384].

12. Part of Entry 19(iv), Chapter 19 [3, p. 314].

298

1. Entry 18(i), Chapter 20 [3, p. 423].

2. This is equivalent to Entry 12(iii) of Chapter 20 [3, p. 397]; the right-hand side, K, in the formulation given on page 298 is $\sqrt{m}$, in the customary notation for the multiplier m.

3. See Entry 75 of Chapter 36.

4. Entry 13(viii) of Chapter 19 [3, p. 281].

5. See Entry 30 of Chapter 36.

6. Part of Entry 19(vii), Chapter 19 [3, p. 314].

299

1. Entry 18(ii), Chapter 20 [3, p. 423].

2. Entry 4(iv), Chapter 20 [3, p. 359].

3.,4. Entries 17(i), (ii), Chapter 20 [3, p. 417].

5.–7. Entries 19(i), (iii), (ii), Chapter 20 [3, p. 426].

8.,9. Entries 17(iii), (iv), Chapter 20 [3, p. 417].

300

1. Part of Entry 5(ix), Chapter 19 [3, p. 231].

2. Part of Entry 13(x), Chapter 19 [3, p. 281].

3.,4. Part of Entry 5(x), Chapter 19 [3, p. 231].

5.,6.,10. Entry 24(ii), Chapter 18 [3, p. 214].

7.,8.,11. Entry 24(iii), Chapter 18 [3, pp. 214–215].
9.,12. See Entries 23 and 24 of Chapter 36.
13.–15. See, respectively, Theorems 7.2, 7.7, and 7.5 of Chapter 33.

301
1.–6. See Theorems 10.5–10.10, respectively, of Chapter 33.
7. Theorem 9.4 of Chapter 33.
8. Theorem 9.15 of Chapter 33.
9. This is an incorrect version of Entry 3(iii), Chapter 21 [3, p. 460].
10. This is an incorrect version of Entry 5(iii), Chapter 21 [3, p. 468].

302
1. Entry 19(i), Chapter 19 [3, p. 314].
2. Entry 15(i), Chapter 20 [3, p. 411].
3.,4. Deleted by Ramanujan.
5.–9.,12–14. See Entries 56, 53, 54, 55, 57, 58, 60, and 59, respectively, in Chapter 36.
10. Entry 5(ii) of Chapter 19 [3, p. 230].
11. Another version of Entry 7(i) of Chapter 20 [3, p. 363].

303
A discussion of the material on this page is given in the last part of Section 7 of Chapter 36.

304
1. The top left corner of the page in the original notebook has been torn away, and so the degree of the first modular equation is unknown. However, the form of the equation is exactly that of the modular equation of degree 23 in Entry 15(i) of Chapter 20 [3, p. 411].
2.–6. Entries 62, 63, 64, 65, and 61, respectively, of Chapter 36.

305
1. Incomplete version of Entry 4(ii) of Chapter 21 [3, p. 464], and is deleted by Ramanujan.
2.–5.,7. Equivalent to formulas for G_{27}, G_{37}, G_{39}, G_{97}, and G_{63}, respectively, in the table in Section 2 of Chapter 34.
6. Deleted by Ramanujan.

306
1.–3. Deleted by Ramanujan.
4. Part of Entry 19(vi), Chapter 19 [3, p. 314].
5. Entry 7(iv), Chapter 20 [3, p. 363].
6. Part of Entry 5(xi), Chapter 19 [3, p. 231].
7. Part of Entry 13(xiii), Chapter 19 [3, p. 282].
8. Entry 19(viii), Chapter 19 [3, p. 315].
9. Entry 7(ii), Chapter 20 [3, p. 363].
10. Entry 19(xi), Chapter 19 [3, p. 315].
11. Part of Entry 19(i), Chapter 19 [3, p. 314].

307

1.–3. Entries 11(ix), (viii), (xiv), Chapter 20 [3, pp. 384–385].
4. Entry 15(v), Chapter 19 [3, p. 291].
5.,6. Entry 19(iv), Chapter 20 [3, p. 426].
7.,8. Entries 3(xii), (xiii), Chapter 20 [3, pp. 352–353].

308

1.,2. Entries 13(i), (ii), Chapter 20 [3, p. 401].
3. Entry 5(ii), Chapter 20 [3, p. 360].
4. Deleted by Ramanujan.
5. Entry 15(iv), Chapter 19 [3, p. 291].
6. Deleted by Ramanujan.

309

1. Entry 15(iii), Chapter 19 [3, p. 291].
2. Entry 5(i), Chapter 20 [3, p. 360].
3.,4. Entries 69 and 68, respectively, of Chapter 36.
5. Entry 73 of Chapter 36.
6. Entry 13(iii) of Chapter 20 [3, p. 401].
7. Entry 14(i) of Chapter 20 [3, p. 408].
8. Entry 11(x) of Chapter 20 [3, p. 384].
9. Entry 13(i) of Chapter 20 [3, p. 401]. (There is a sign error; replace -4 by 4.)
10.–13. Entries 72, 74, 71, and 70, respectively, of Chapter 36.

310

1. See Entry 66 of Chapter 36.
2.,3. See Theorems 7.10 and 7.9, respectively, of Chapter 33.
4.–11. These singular moduli are given in Theorem 9.2 of Chapter 34.
12.,13. Theorem 9.16 in Chapter 34.

311

1. Equivalent to the formula for G_{33} given in the table in Section 2 of Chapter 34.
2.,3. See Entry 8.1 of Chapter 34.
4.,5. Entry 9.14 of Chapter 34.
6.–9. Entries 7–10, respectively, of Chapter 32.

312

1. See Entry 6 of Chapter 35.
2.–12. See Section 10 of Chapter 34.
13.,14. Deleted by Ramanujan.
15.,16. Part of Theorem 9.2 of Chapter 34.

313

1. Equivalent to the formula for G_{73} in the table in Section 2 of Chapter 34.
2. Deleted by Ramanujan.
3. Equivalent to the formula for G_{43} in the table of Chapter 34.

4. Entry 7(iii), Chapter 21 [3, p. 475].

5.–7. Part of Theorem 9.2 of Chapter 34.

314

1.–9. Equivalent to formulas for G_{69}, G_{117}, G_{333}, G_{81}, G_{147}, G_{363}, G_{217}, G_{205}, and G_{265}, respectively, in the table of Chapter 34.

315

1.–15. A table of values for G_n, $n = 57, 93, 177, 85, 133, 55, 65, 253, 145, 117, 333, 153, 77, 69, 213$. See the table of Chapter 34.

316

1. This is the definition of g_n.

2.–13. These twelve values for g_n, $n = 2, 6, 10, 16, 18, 22, 30, 58, 70, 46, 142, 82$, are found in the table of Chapter 34.

14.,15. See Entry 8.2 of Chapter 34.

16. Deleted by Ramanujan.

17.–22. These six values for g_n, $n = 42, 78, 102, 130, 190, 34$, are found in the table of Chapter 34.

317

1.–9. The nine values of G_n, $n = 289, 121, 169, 105, 165, 345, 385, 273, 357$, are given in the table of Chapter 34.

318

1.–3.,5.,6.,8.–10. These eight values for g_n, $n = 98, 90, 198, 522, 630, 50, 126, 26$, can be found in the table of Chapter 34.

4. See Entry 3.2 of Chapter 34.

7. This formula for g_{1170} has been deleted by Ramanujan.

319

1.,2.,5.,8.,9. Five values for g_n, $n = 66, 138, 238, 154, 310$, are given. These can be found in the table of Chapter 34.

3. This formula for g_{154} has been deleted by Ramanujan, but it is correct, except for one misprint; see the table of Chapter 34.

4. This formula for g_{114} is incorrect; see the table of Chapter 34 for a correct version.

6.,7.,10. These three values for g_n, $n = 62, 94, 158$, are also given in the table of Chapter 34, but the formulations are somewhat different.

11.,12.,13. Three values for G_n, $n = 465, 777, 1353$, are given. Some calculations are needed to show that the formula for G_{1353} given here is equivalent to that in the table of Chapter 34.

320

1.–6. The values for G_n, $n = 1645, 897, 1677, 141, 445, 553$, are given in the table of Chapter 34.

7.,8. The values for g_n, $n = 210, 330$, are given in the table of Chapter 34.

9. See Theorem 9.1 of Chapter 34.

10. See the introduction to Section 9 of Chapter 34.

321

1.,2. Entry 11(iii), Chapter 19 [3, pp. 265–266].
3.–6. Parts of Entries 1(iv), (v), (ii), (iii), Chapter 20 [3, pp. 345–346].
7. Entry 1, Chapter 18 [3, p. 144].
8. Entry 2, Chapter 18 [3, p. 145].
9. Entry 8(i), Chapter 18 [3, p. 157].
10. Entry 9, Chapter 18 [3, p. 159].

322

1. Entry 18(vi), Chapter 19 [3, p. 306].
2.,3. Part of Entry 3(i), Chapter 21 [3, p. 460].
4.,5. Entry 3(ii), Chapter 21 [3, p. 460].
6.,7. Entry 4(i), Chapter 21 [3, p. 463].
8. Entry 4(ii), Chapter 21 [3, p. 464].
9. Part of Entry 5(i), Chapter 21 [3, p. 467].

323

1. Entry 5(ii), Chapter 21 [3, p. 468].
2. Part of Entry 7(i), Chapter 21 [3, p. 475].
3. Entry 7(ii), Chapter 21 [3, p. 475].
4.–6. Entries 2(vii), (v), (viii), Chapter 20 [3, p. 349].
7.–9. Entries 8(i), (ii), (iii), Chapter 21 [3, p. 480].
10. Entry 9(i), Chapter 21 [3, p. 481].

324

1.,2. Entries 9(ii), (iii), Chapter 21 [3, p. 481].
3.,4. Entry 4(iii), Chapter 21 [3, p. 464].
5.–7. Entry 1(i), Chapter 20 [3, p. 345].
8.–10. Entry 1(ii), Chapter 19 [3, p. 221].

325

1. The formula for G_{301} can be found in the table of Chapter 34.
2.,3.,6.,7. Entries 10(i)–(iv) of Chapter 20 [3, p. 379].
4.,5. Entry 21 of Chapter 36.
8. Entry 1, Chapter 19 [3, p. 221].
9. Entry 18(i), Chapter 19 [3, p. 305].
10. Entry 6(iii), Chapter 20 [3, p. 363].

326

1.–4. Entry 18(i), Chapter 19 [3, p. 305]. Note that the definitions of u, v, and w are different in the first notebook.
5.–10. Entry 8(ii), Chapter 20 [3, pp. 372–373].

327

1.–4. Entry 12(i), Chapter 20 [3, p. 397].
5. Entry 5(iv), Chapter 20 [3, p. 360].

6.,7. Entry 18(v), Chapter 20 [3, p. 423].

328

1. Theorem 7.11 of Chapter 33.
2.–4. Entries 8–10, Chapter 30 [4, pp. 374–377].

329

1.–3. Entries 62, 69, and 63, respectively, of Chapter 25 [4, pp. 221, 236, 223].
4.,5. Entries 25 and 26 of Chapter 32.

330

1.–4. Entries 51, 49, 52, and 50, respectively, of Chapter 32.

331

1. Entry 21, Chapter 28 [4, p. 309].
2.–6. Entries 48, 14, 19, 20, and 21, respectively, of Chapter 32.

332

1. Entry 23 of Chapter 32.
2. Deleted by Ramanujan.

333

1.,2. Entries 6 and 7 of Chapter 30 [4, pp. 364, 370].

334

1. Entry 5 of Chapter 30 [4, p. 363].
2. An incomplete version of Entry 34 of Chapter 37.
3. Deleted by Ramanujan but proved in Entry 31 of Chapter 37.
4. Entry 20, Chapter 28 [4, p. 309].
5. See Entry 32 of Chapter 37.

335

1. See Entry 33 of Chapter 37.
2.,3. Entries 6 and 4, respectively, of Chapter 29 [4, pp. 338, 336].

336

1.,2. Entries 5 and 3, respectively, of Chapter 29 [4, pp. 337, 336].
3. Entry 18, Chapter 28 [4, p. 307].
4. See Entry 27 of Chapter 37.

337

1.,2. Entry 29 of Chapter 37.
3. Entry 30 of Chapter 37.
4. Entry 17, Chapter 28 [4, p. 306].
5. Deleted by Ramanujan; for a correct version see the tables of Gradshteyn and Ryzhik [1, p. 546, formula 4.113, no. 4].

338

The results on this page comprise the contents of Section 3 of Chapter 35.

339

1.–3. These results are also contained in Section 3 of Chapter 35.

4. A definition.

5. A triviality.

6.–8. These results are essentially Entries 4, 3, and 7, respectively, of Chapter 35.

9.–11. Entries 23, 24, and 22, respectively, of Chapter 25 [4, pp. 154, 155, 153].

12. See Entry 17 of Chapter 39.

340

1. Entry 12 of Chapter 35.

2. Entry 3, Chapter 30 [4, p. 359].

3.–8. Entries 1–6, Chapter 26 [4, pp. 245–255].

341

1.–3. Entries 7–9, Chapter 26 [4, pp. 255–257].

4. Entry 4, Chapter 30 [4, p. 360].

5. Entry 24, Chapter 32.

6. Entry 20 of Chapter 37.

342

1.–3. Deleted by Ramanujan.

4. Entry 23 of Chapter 37.

5. Entry 10, Chapter 26 [4, p. 258].

6. Entry 1, Chapter 29 [4, p. 335].

343

1.–3. Entries 24–26, respectively, of Chapter 37.

4. Entry 22 of Chapter 37.

5. An incomplete entry. We offer some comments on it at the end of Section 8 of Chapter 34.

6. The value for G_{765} is given in the table in Section 2 of Chapter 34.

344

1. Entry 6 of Chapter 32.

2. The value of G_{505} is given in the table of Chapter 34.

3.–5. Monic irreducible cubic polynomials satisfied by g_n, $n = 38, 26$, and 50. See the table in Section 2 of Chapter 34.

6. Entry 13 of Chapter 32.

345

1.–7. Monic irreducible cubic polynomials satisfied by G_n, $n = 23, 31, 11, 19,$ 27, 43, and 67. See the table in Section 2 of Chapter 34.

8.–16. Factors of singular moduli $\sqrt{\alpha_n}$, $n = 3, 5, 7, 9, 13, 15, 17, 21$, and 25. However, for $n = 21$, Ramanujan fails to record any factors. See Theorem 9.9 of Chapter 34.

346

1.–4. Factors of singular moduli $\sqrt{\alpha_n}$, $n = 7, 15, 39$, and 55, but for $n = 39$, Ramanujan left a blank space. See Theorem 9.9 of Chapter 34.

347–349

All the results on these three pages are found in Ramanujan's paper on Bernoulli numbers [1], [10, pp. 1–14].

350

For comments on Ramanujan's notes at the top of the page, see Berndt and Rankin's book [1, p. 10].

1. See Entry 18 of Chapter 39.

351

1. Ramanujan gives the first 22 digits of $\sqrt{2}$.

2.–10. These are irreducible polynomials for the reciprocals of the invariants G_n, $n = 3, 7, 11, 19, 23, 27, 31, 43$, and 67. See the table of Chapter 34. Ramanujan had evidently intended to calculate several further polynomials, as indicated by vacant spaces beside certain other values of n.

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$$1+\frac{(1-q^{\alpha})(1-q^{\beta})}{(1-q)(1-q^{\gamma})}x+\frac{(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})}{(1-q)(1-q^{2})(1-q^{\gamma})(1-q^{\gamma+1})}x^2+\cdots,$$

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