$$S = 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{(4n+1)^2}$$

$$4n+1 = 4\left(n + \frac{1}{4}\right) = \frac{(5/4)_n}{(1/4)_n}$$

$$S = 1 + \sum_{n=1}^{\infty} \frac{(1/2)_n (1/4)_n^2}{(1)_n (5/4)_n} = {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{4}, \frac{1}{4} \\ \frac{5}{4}, \frac{5}{4} \end{matrix} ; 1 \right)$$

$${}_3F_2\left(\begin{matrix} a, b, c \\ 1+a-b, 1+a-c \end{matrix} ; 1 \right) = \frac{\Gamma(1+\frac{a}{2})\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)\Gamma(1+a-b-c)}$$

$$a = \frac{1}{2}, \quad b = \frac{1}{4}, \quad c = \frac{1}{4}$$

$$S = \frac{\Gamma(\frac{5}{4})\Gamma(\frac{5}{4})\Gamma(\frac{5}{4})\Gamma(\frac{3}{4})}{\Gamma(\frac{3}{2})} = \frac{2}{4^3} \frac{\Gamma^2(\frac{1}{4})\sqrt{2}\pi}{\sqrt{\pi}} = \frac{\sqrt{2\pi}}{32} \Gamma^2\left(\frac{1}{4}\right)$$