$$\prod_{n=1}^{\infty}\left(\frac{n!e^n}{\sqrt{2\pi n}n^n}\right)^{(-1)^{n-1}} = P$$

$$\ln P = \sum_{n=1}^{\infty}(-1)^{n-1}\ln n! - \sum_{n=1}^{\infty}(-1)^{n-1}\ln\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\ln n! = \sum_{n=1}^{\infty}(-1)^{n-1}\sum_{k=1}^{n}\ln k = \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{n-1}\ln k =$$

$$= \sum_{k=1}^{\infty}\sum_{n=k}^{\infty}(-1)^{n-1}\ln k = \sum_{k=1}^{\infty}\ln k\sum_{n=k}^{\infty}(-1)^{n-1} = \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\ln k$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\ln\sqrt{2\pi n}\left(\frac{n}{e}\right)^n = \sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{1}{2}\ln 2\pi + \frac{1}{2}\ln n + n\ln n - n\right] =$$

$$= \frac{1}{2}\eta(0)\ln 2\pi + \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\ln k + \sum_{n=1}^{\infty}(-1)^{n-1}n\ln n - \eta(-1) =$$

$$= \frac{1}{4}\ln 2\pi - \frac{1}{4} + \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\ln k + \sum_{n=1}^{\infty}(-1)^{n-1}n\ln n$$

$$\ln P = \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\ln k - \frac{1}{4}\ln 2\pi + \frac{1}{4} - \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\ln k - \sum_{n=1}^{\infty}(-1)^{n-1}n\ln n =$$

$$= -\frac{1}{4}\ln 2\pi + \frac{1}{4} - \sum_{n=1}^{\infty}(-1)^{n-1}n\ln n$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}n\ln n = \frac{d}{dx}\bigg|_{x=1}\sum_{n=1}^{\infty}(-1)^{n-1}n^x = \frac{d}{dx}\bigg|_{x=1}\eta(-x) =$$

$$= \frac{d}{dx}\bigg|_{x=1}(1-2^{1+x})\zeta(-x) = (1-2^{1+x})\zeta(-x)\left[\frac{-2^{1+x}\ln 2}{1-2^{1+x}} - \frac{\zeta'(-x)}{\zeta(-x)}\right]\bigg|_{x=1} =$$

$$= -4\zeta(-1)\ln 2 - (1-4)\zeta'(-1)$$