3(b) $f = \left(\frac{x}{y}\right)^z$

$$\frac{\partial f}{\partial x} = \frac{z x^{z-1}}{y^z}$$

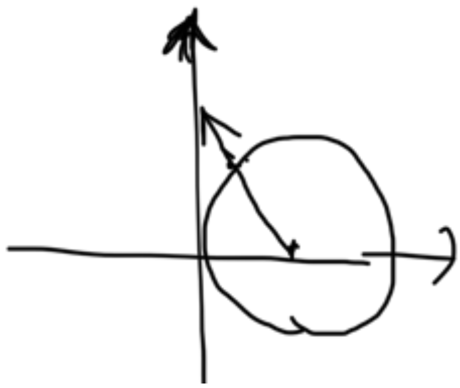
$$\frac{\partial f}{\partial y} = \frac{-z x^z}{y^{z+1}}$$

$$\frac{\partial f}{\partial z} = \ln\left(\frac{x}{y}\right)\left(\frac{x}{y}\right)^z$$

44.2) $f = \operatorname{arctg}(y/x)$, $M\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$x^2 + y^2 = 2x$

$(x-1)^2 + y^2 = 1$



$$\bar{u} = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

$$\left.\frac{\partial f}{\partial \bar{u}}\right|_M = (\bar{\nabla} f, \bar{u})|_M =$$

$$= -\frac{1}{2} \frac{-y_M/x_M^2}{1+\left(\frac{y_M}{x_M}\right)^2} + \frac{\sqrt{3}}{2} \frac{1/x_M}{1+\left(\frac{y_M}{x_M}\right)^2} =$$

$$= \frac{\frac{1}{2} \cdot \frac{\sqrt{3}/2}{(1/2)^2} + \frac{\sqrt{3}}{2} \cdot \frac{1}{1/2}}{1+\left(\frac{\sqrt{3}/2}{1/2}\right)^2} = \frac{2\sqrt{3}}{4} = \boxed{\frac{\sqrt{3}}{2}}$$

19. 2) $f = |y| \sin x$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0$$

$$\lim_{\rho\to 0} \frac{|y|\,\overset{\text{б.м}}{\sin x}}{\underset{\text{огр.}}{\rho}} = 0$$

4) $f = \operatorname{ch}(x^2 y)^{1/5}$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0$$

$$\lim_{\rho\to 0} \frac{\operatorname{ch}(x^2 y)^{1/5} - 1}{\rho} =$$

$$= \lim_{\rho\to 0} \frac{\frac{1}{2}(x^2 y)^{2/5} + o\left((x^2 y)^{2/5}\right)}{\rho} =$$

$$= \lim \; 1 \; {}^{1/5} + o(\rho^{6/5}) \; 0$$

$$= \lim_{\rho\to 0} \frac{1}{2}\rho^{3} + \frac{\ldots}{\rho} = 0$$

20. 1) $f = \sqrt{|xy|}$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0$$

$$\lim_{\rho\to 0} \frac{\sqrt{|xy|}}{\rho} = \begin{cases} \xrightarrow{x=y} \frac{1}{2} \\ \xrightarrow{y=0} 0 \end{cases}$$

3) $f = \sqrt[3]{x^3+y^3}$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 1$$

$$\lim_{\rho\to 0} \frac{\sqrt[3]{x^3+y^3} - x - y}{\rho} =$$

$$= \lim_{\rho \to 0} \sqrt[3]{\cos^3\varphi + \sin^3\varphi} - \cos\varphi - \sin\varphi$$

$\boxed{1.1}$ a) $f(x,y) = tg(\sqrt[3]{x^2y^2}) + e^{x+y}$

$$\frac{\partial f}{\partial x}\Big|_{(0,0)} = 1 = \frac{\partial f}{\partial y}\Big|_{(0,0)}$$

$$\lim_{\rho \to 0} \frac{tg(x^{2/3}y^{2/3}) + e^{x+y} - x - y - 1}{\rho} =$$

$$= \lim_{\rho \to 0} \frac{tg(\rho^{4/3}\cos^{2/3}\varphi\sin^{2/3}\varphi)}{\rho} \to 0 +$$

$$+ \lim_{\rho \to 0} \frac{o(x+y)}{\rho} = 0$$

δ) $f(x,y) = \sin(|x|^{\alpha} y^{1/3})$

$0 < \alpha < 2/3$

$$\lim_{\rho \to 0} \frac{\sin(|x|^{\alpha} y^{1/3})}{\rho} = \lim_{\rho \to 0} \frac{\sin(\rho^{\alpha + 1/3} |\cos^{\alpha} \varphi| \sin^{1/3} \varphi)}{\rho}$$

$\nexists$

$\alpha = 2/3 \quad y = 0 \to 0$

$x = y \to 1/\sqrt{2}$

$\alpha > 2/3 \quad \lim_{\rho \to 0} \frac{\sin(|x|^{\alpha} y^{1/3})}{\rho} = 0$

72. $f(x,y,z)$

$$f(x,y,z) = \cos x \cos y \cos z -$$

$$- \cos(x+y+z)$$

$$f(x,y,z) = \frac{\partial f}{\partial x_i} x_i +$$

$$+ \frac{1}{2} \frac{\partial^2 f}{\partial x_i^2} x_i^2 + \frac{\partial^2 f}{\partial x_i \partial x_j} x_i x_j + o(\rho^2)$$

$$= xy + yz + xz + o(\rho^2)$$

73. $$f(x,y,z) = \ln(xy+z^2) =$$

$$\partial_x f = \partial_y f = 0$$

$$\partial_z f = 2$$

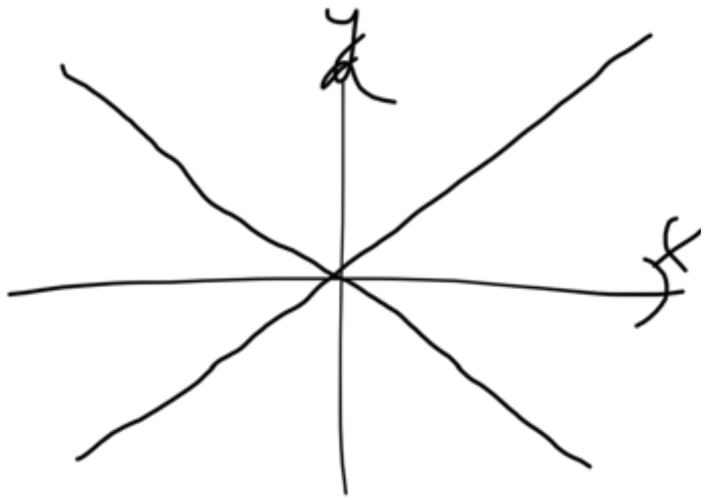
$$1 = \partial^2_{xy} f, \quad 0 = \partial^2_x f = \partial^2_y f$$

$\partial_{xz} f = \partial_{yz} f = 0$

$$\partial^2_{z} f = \partial_z \frac{2z}{xy+z^2} =$$

$$= \frac{2xy+2z^2-(2z)^2}{(xy+z^2)^2} = -2$$

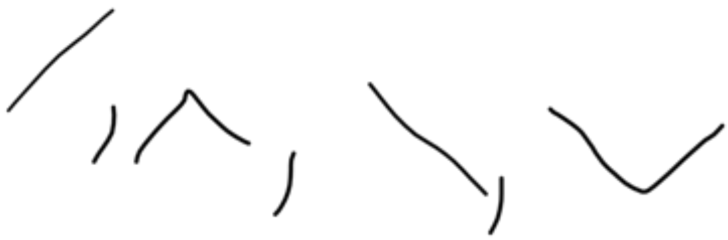
$$f(x,y,z) = 2(z-1) + xy - 2(z-1)^2 + o(\rho^2)$$



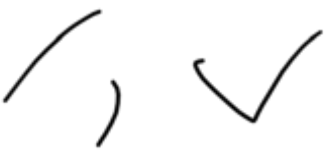
$\underline{\text{T.2}}$ $y^2 = x^2$

а) α_0

б) 4 :



в) 2 :



г) 1 :

ч 1. /

$\boxed{Т.3}$
$$\begin{cases} u = e^x \cos y \\ v = e^x \sin y \end{cases}$$

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial v}{\partial x} = e^x \sin y, \quad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = e^{2x} > 0 \quad \forall (x,y) \in \mathbb{R}^2$$

$$u^2 + v^2 = e^{2x}$$

Не инъективно, т.к. $(u,v) \Leftrightarrow (x, y+2\pi k)$

Если: $u = v = 0 \to$ беды

$E_f = \mathbb{R}^2 \setminus \{0,0\}$

66. $u^3 - 3(x+y)u^2 + z^3 = 0$

$u^3 - 3(x_0+y_0)u^2 + z_0^3 = 0 \to u_0 = \ldots$

$3u_0^2 du - 3(dx+dy)u_0^2 -$

$-6(x_0+y_0)u_0 du + 3z_0^2 dz = 0$

$(u_0^2 - 2(x_0+y_0)u_0)du =$

$= (dx+dy)u_0^2 - z_0^2 dz$

$du = \frac{(dx+dy)u_0^2 - z_0^2 dz}{}$

$$u_0^2 - 2(x_0+y_0)\,y_0$$

78. $z = u^3 + v^3$, $u \neq v$

$u = u(x,y)$, $v = v(x,y)$

$u + v = x$, $u^2 + v^2 = y$

$$dz = 3u^2\,du + 3v^2\,dv$$

$$\begin{cases} u + v = x \\ uv = \frac{1}{2}(x^2 - y) \end{cases}$$

$$\lambda^2 - x\lambda + \frac{1}{2}(x^2 - y) = 0$$

$$\lambda_{1,2} = \frac{1}{2}\left[x \pm \sqrt{x^2 - 2x^2 + 2y}\right] =$$

$$= \frac{1}{2}\left[\qquad \sqrt{\qquad} \right]$$

$$z(x \pm \sqrt{2y - x^4})$$

$$du + dv = dx$$

$$u\,du + v\,dv = dy/2$$

$$\begin{pmatrix} 1 & 1 \\ u & v \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} dx \\ dy/2 \end{pmatrix}$$

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \frac{1}{v-u} \begin{pmatrix} v & -1 \\ -u & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy/2 \end{pmatrix}$$

$$= \frac{1}{v-u} \begin{pmatrix} v\,dx - dy/2 \\ -u\,dx + dy/2 \end{pmatrix}$$

$$dz = \frac{3u^2}{v-u}\left(v\,dx - dy/2\right) + \frac{3v^2}{v-u}\left(-u\,dx + \frac{dy}{2}\right)$$

где u, v — корни

$$\lambda^2 - x\lambda + \frac{1}{2}(x^2 - y) = 0$$

$$dz = -3uv\,dx + \frac{3\,dy}{2}(v+u) =$$

$$= -\frac{3}{2}(x^2 - y)\,dx + \frac{3}{2}x\,dy$$

§2 (1,2) $f(x,y,z) = 0$ (x_0, y_0, z_0)

$$x = x(y,z)$$
$$y = y(x,z)$$
$$z = z(x,y)$$

$$\frac{\partial x(y_0,z_0)}{\partial y} \cdot \frac{\partial y(x_0,z_0)}{\partial z} \cdot \frac{\partial z(x_0,y_0)}{\partial x} = ?$$

$$\partial \qquad \partial z \qquad \partial x^{2} \;-\!-\; 1 .$$

$$x = x_0 + \frac{\partial x}{\partial y}dy + \frac{\partial x}{\partial z}dz + o(\rho)$$

$$y = y_0 + \frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial z}dz + o(\rho)$$

$$z = z_0 + \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy + o(\rho)$$

$$\underbrace{\begin{pmatrix} -1 & x_y & x_z \\ y_x & -1 & y_z \\ z_x & z_y & -1 \end{pmatrix}}_{A} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = 0$$

$$\det A = 0 = -1 + x_y y_z z_x +$$

$$+ x_z z_y y_x + x_z z_x + y_z z_y + x_y y_x$$

$$\frac{1}{x_y y_z z_x}$$

$$x_y y_z z_x + \frac{1}{x_y y_z z_x} = -2$$

$$\Rightarrow x_y y_z z_x = -1$$

$\boxed{103.1}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} =$$

$$= \begin{vmatrix} 3x^2-3y^2 & -6yx \\ 6xy & 3x^2-3y^2 \end{vmatrix} = 9(x^2-y^2)^2 + 36x^2y^2$$

$= \sqrt{1} x + y y$

$\forall \psi (2,3)$ 2) $x + y + u = e^u$

$$dx + dy + du = e^u du$$

$$d^2u = e^u du^2 + e^u d^2u$$

$$d^2u = \frac{e^u}{1-e^u} du^2$$

3) $u = \ln|yu - x|$

$$du = \frac{u dy + y du - dx}{yu - x}$$

$$d^2u (yu - x) + du (u dy + y du - dx)$$

$$= \underbrace{dudy + dydu}_{2dydu} + yd^2u$$

$$d^2u = \frac{2dydu - du(udy + ydu - dx)}{yu - x - y}$$

2\. 15) $\int \sin^2\frac{x}{2}\,dx = \boxed{\frac{x}{2} - \frac{\sin x}{2} + C}$

16) $\int ctg^2 x\,dx = \boxed{-ctg x - x + C}$

(2.2) $\int x\sqrt{x+1}\,dx =$

$$= \boxed{\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C}$$

17.4) $\int x \ln x \, dx = \boxed{\frac{x^2}{2}\ln x - \frac{x^2}{4} + C}$

24.3) $I = \int e^{ax} \sin bx \, dx =$

$$= \begin{bmatrix} & D & I \\ + & \sin bx & e^{ax} \\ - & b\cos bx & \frac{1}{a}e^{ax} \\ + & -b^2 \sin bx & \frac{1}{a^2}e^{ax} \end{bmatrix} =$$

$= \frac{1}{a}\sin bx \, e^{ax}$

$$a \sin bx\, e^{ax} - \frac{b}{a^2} e^{ax} \cos bx$$

$$- \frac{b^2}{a^2} I + C$$

$$I = \boxed{\frac{\frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx}{1 + \frac{b^2}{a^2}} + C}$$

$\boxed{1.4}$

$$\int \frac{x^2 - 5x + 9}{x^2 - 5x + 6}\, dx =$$

$$= x + \int \frac{3}{(x-2)(x-3)}\, dx =$$

$$= \boxed{x + 3 \ln \left| \frac{x-3}{x-2} \right| + C}$$

2.1

$$\int \frac{dx}{(x-1)(x+2)(x+3)} =$$

$$= \boxed{\frac{1}{12}\ln|x-1| - \frac{1}{3}\ln|x+2| + \frac{1}{4}\ln|x+3| + C}$$

3.2

$$P = \frac{x^2+2}{(x-1)(x+1)^2} =$$

$$= \frac{3/4}{x-1} + \frac{\frac{1}{4}x - \frac{5}{4}}{(x+1)^2}$$

$$\int P\,dx = \quad 3\ \ell$$

$$\int f\,dx = \boxed{\frac{-1}{4}\ln|x-1| + \frac{1}{4}\ln|x+1| + \frac{3}{2}\frac{1}{x+1} + C}$$

<u>4.2</u> $$p = \frac{1}{x^3+1} = \frac{1/3}{x+1} + \frac{-\frac{1}{3}x + 2/3}{x^2-x+1}$$

$$= \frac{1/3}{x+1} - \frac{1}{3}\frac{x-1/2}{(x-1/2)^2+\frac{3}{4}} + \frac{1/2}{(x-\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\int p\,dx = \boxed{\frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left(x^2-x+1\right) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C}$$

3.2 $\int \frac{x^3\,dx}{1-x^4} =$

$$= -x + \int \frac{1/2}{1-x^2} + \frac{1/2}{1+x^2}\,dx$$

$$= \boxed{-x - \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{2}\tan^{-1}x + C}$$

2.2 $\int \frac{dx}{3x + x^{2/3}} = \left[\begin{matrix} x = u^3 \\ dx = 3u^2\,du \end{matrix}\right]$

$$= \int \frac{3u^2\,du}{3u^3+u^2} = \int \frac{3\,du}{3u+1} =$$

$$= \boxed{\ln\left|\tfrac{1}{3}+u\right| + C}$$

<u>5.1</u> $$\int \frac{1-x+x^2}{\sqrt{1+x-x^2}}\,dx =$$

$$= -\int \sqrt{\frac{5}{4}-\left(x-\frac{1}{2}\right)^2}\,dx +$$

$$+2\int \frac{dx}{\sqrt{\frac{5}{4}-\left(x-\frac{1}{2}\right)^2}} =$$

$$= 2\sin^{-1}\left(\frac{x-1/2}{\sqrt{5}/2}\right) - I \quad \ominus$$

$$\left[\, x-\tfrac{1}{2} = \frac{\sqrt{5}}{2}\sin\theta \,\right]$$

$$dx = \frac{\sqrt{5}}{2}\cos\theta\, d\theta$$

$$I = \frac{5}{4}\int \cos^2\theta\, d\theta = \frac{5}{8}\theta + \frac{5}{16}\sin 2\theta + C$$

$$= \frac{11}{8}\sin^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{5}/2}\right) - \frac{\cancel{5}\,\sqrt{5}}{4 \cdot \cancel{16}}\,\frac{\cancel{2}\cdot 2}{\cancel{\sqrt{5}}}\left(x-\frac{1}{2}\right)\sqrt{1-\frac{4}{5}\left(x-\frac{1}{2}\right)^2} + C$$

$$= \boxed{\frac{11}{8}\sin^{-1}\left(\frac{x-1/2}{\sqrt{5}/2}\right) + \frac{\sqrt{5}}{4}\left(x-\frac{1}{2}\right)\sqrt{1-\frac{4}{5}\left(x-\frac{1}{2}\right)^2} + C}$$

18-31

$$\int \frac{\sqrt[3]{1+\sqrt{x}}}{\sqrt{x}}\,dx =$$

$$\left[\begin{array}{l} 1+\sqrt[4]{x}=u^3 \\ \frac{1}{2}\frac{dx}{\sqrt{x}} = 2(u^3-1)\cdot 3u^2\,du \end{array}\right]$$

$$= 12\int u^3\cdot(u^3-1)\,du = \frac{12}{7}u^7 - 3u^4 + C =$$

$$= \boxed{\frac{12}{7}\left(1+\sqrt[4]{x}\right)^{7/3} - 3\left(1+\sqrt[4]{x}\right)^{4/3} + C}$$

№2 $\displaystyle\int \frac{dx}{x^3\sqrt[3]{2-x^3}} = \left[\begin{array}{l} \frac{1}{x}=t \\ -\frac{1}{x^2}dx = dt \end{array}\right]$

$$= -\int \frac{t\,dt}{\sqrt[3]{2-\frac{1}{t^3}}} = -\int \frac{t^2\,dt}{\sqrt[3]{2t^3-1}} =$$

$$= -\frac{1}{3}\int \frac{dt^3}{(2t^3-1)^{1/3}} = -\frac{1}{4}(2t^3-1)^{4/3}+C$$

$$= \boxed{-\frac{1}{4}\left(\frac{2}{x^3}-1\right)^{2/3}+C}$$

4.1 $\int \cos^3 x\,dx = \frac{1}{4}\int \cos 3x + 3\cos x \, dx$

$$= \boxed{\frac{1}{12}\sin 3x + \frac{3}{4}\sin x + C}$$

15.6 $\int \frac{dx}{1-\operatorname{th}x} = \int \frac{\operatorname{ch}x\,dx}{\operatorname{ch}x-\operatorname{sh}x}$

$\int \left(\frac{1}{2}e^{2x}+1\right)dx = \boxed{\frac{1}{4}e^{2x}+x/2+C}$

16.1 $\int \frac{dx}{2\cos^2 x+\sin x\cos x+\sin^2 x} =$

$t=\operatorname{tg}x$

$= \int \frac{dt}{2+t+t^2} =$

$= \int \frac{dt}{(t+1/2)^2+\left(\frac{\sqrt{7}}{2}\right)^2} =$

$= \frac{2}{\sqrt{7}}\tan^{-1}\frac{t+1/2}{\sqrt{7}/2}+C =$

$$= \boxed{\frac{2}{\sqrt{7}} \tan^{-1} \frac{tg\,x + 1/2}{\sqrt{7}/2} + C}$$

<u>21.2</u>

$$\int \frac{dx}{4+\cos x} = \left[\begin{array}{l} t = tg\,\frac{x}{2} \\ dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx \end{array} \right]$$

$$= \int \frac{2dt/1+t^2}{4 + \frac{1-t^2}{1+t^2}} = 2\int \frac{dt}{3t^2+5} =$$

$$= \frac{2}{3} \int \frac{dt}{t^2 + \left(\sqrt{\frac{5}{3}}\right)^2} =$$

$$= \frac{2}{3}\sqrt{\frac{3}{5}}\, \tan^{-1}\left(t\sqrt{\frac{3}{5}}\right) + C =$$

$$= \boxed{\frac{2}{\sqrt{15}} \tan^{-1}\left(\sqrt{\tfrac{5}{3}}\,\tfrac{4}{9}\,2\right) + C}$$

$\underline{144}$

$$y = \int \left(\frac{\sin x}{e^x}\right)^2 dx = \int \frac{1}{2} e^{-2x} dx -$$

$$-\frac{1}{2}\int \cos 2x \, e^{-2x} dx =$$

$$= -\frac{1}{4} e^{-2x} - \frac{1}{2} I$$

	D	I
+	$\cos 2x$	e^{-2x}
−	$-2\sin 2x$	$-\frac{1}{2}e^{-2x}$
+	$-4\cos 2x$	$+\frac{1}{4}e^{-2x}$

$$I = -\frac{1}{2}e^{-2x}\cos 2x + \frac{1}{2}\sin 2x\, e^{-2x}$$

$$- I$$

$$I = \frac{1}{4}e^{-2x}\left[\sin 2x - \cos 2x\right] + C$$

$$y = \boxed{-\frac{1}{4}e^{-2x} + \frac{1}{8}e^{-2x}\left[\cos 2x - \sin 2x\right] + C}$$

<u>18D.</u>

$$\int \frac{\sin^{-1}x\, dx}{(1-x^2)\sqrt{1-x^2}} =$$

$$= \begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{bmatrix} =$$

$$= \int \frac{\theta \cos\theta \, d\theta}{\cos^3\theta} = \int \theta \sec^2\theta \, d\theta =$$

$$\left[\begin{array}{ccc} & D & I \\ + & \theta & \sec^2\theta \\ - & 1 & tg\theta \\ + & 0 & -\ln|\cos\theta| \end{array}\right]$$

$$= \theta \, tg\theta + \ln|\cos\theta| + C =$$

$$= \boxed{\sin^{-1}x \, \frac{x}{\sqrt{1-x^2}} + \frac{1}{2}\ln|1-x^2| + C}$$

180

$$\int \frac{x^2 \cos^{-1}(x\sqrt{x})}{(1-x^3)^2}\,dx =$$

$$= \left[\begin{array}{l} x^{3/2} = \cos\theta \\ \frac{3}{2}\sqrt{x}\,dx = -\sin\theta\,d\theta \end{array}\right] =$$

$$= \int \frac{-\frac{2}{3}\theta \cos\theta \sin\theta\,d\theta}{\sin^4\theta} =$$

$$= \left(\begin{array}{ccc} & D & I \\ + & -\frac{2}{3}\theta & \frac{\cos\theta}{\sin^3\theta} \\ - & -\frac{2}{3} & -\frac{1}{2}\frac{1}{\sin^2\theta} \end{array}\right)$$

$$+ \qquad \theta \qquad \frac{1}{2}\ln|\csc\theta+\cot\theta|$$

$$= \frac{1}{3}\,\theta\Big/_{\sin^2\theta} + \frac{1}{3}\ln|\csc\theta+\cot\theta| + C$$

$$= \frac{1}{3}\,\frac{\cos^{-1}x^{3/2}}{\quad} + \frac{1}{}\ln|1+x^{3/2}|$$

$$2.1)\; S_n = \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \dots + \frac{1}{(n+2)(n+3)} =$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \dots + \frac{1}{n+2} - \frac{1}{n+3}$$

$$= \frac{1}{3} - \frac{1}{n+3}$$

$$S = \frac{1}{3}$$

$$10.2)\; S = \sum_{n=1}^{\infty} a^n \sin n\alpha, \quad \alpha \in \mathbb{R},\; a \in \mathbb{R}$$

$$|a| < 1$$

$$S = \operatorname{Im} \sum_{n=1}^{\infty} \left(a e^{i\alpha}\right)^n =$$

$$= \operatorname{Im}\left(\frac{ae^{i\alpha}}{1-ae^{i\alpha}}\right) =$$

$$= \operatorname{Im} \frac{ae^{i\alpha} - a^2}{1 - a^2 - 2a\cos\alpha} =$$

$$= \boxed{\frac{a\sin\alpha}{1+a^2-2a\cos\alpha}}$$

<u>25.9</u>

$$a_n = \frac{1}{n^\alpha \ln^\beta n}, \quad n \geq 2$$

$$\alpha \leq 1$$

$$\sum_{n=2}^{\infty} a_n \geq \sum_{n=2}^{\infty} \frac{1}{n \ln^\beta n} \sim$$

$$\sim \sum_{n=2}^{\infty} \frac{\cancel{2^n}}{\cancel{2^n}\, n^{\beta}\, 2^n} = \sum_{n=2}^{\infty} \frac{1}{n^{\beta}}$$

$\beta \le 1 \Rightarrow$ расх

$$\sum_{n=2}^{\infty} a_n \sim \sum_{n=2}^{\infty} \frac{2^n}{2^{\alpha n}\, n^{\beta}} = \sum_{n=2}^{\infty} \frac{2^{(1-\alpha)n}}{n^{\beta}}$$

$\alpha < 1 \Rightarrow$ расх. при $\forall \beta$

$\alpha = 1 \Rightarrow$ расх. при $\beta \le 1$

и сх. при $\beta > 1$

$$\alpha > 1 \Rightarrow \sum_{n=2}^{\infty} \frac{n^a}{b^n} = \xi_n$$

$$\lim_{n\to\infty} \frac{\xi_{n+1}}{\xi_n} = \frac{1}{b} < 1 \Rightarrow \text{сх. при } \forall \beta$$

2 (6, 7)

6) $a_n = \dfrac{\ln n + \sin n}{n^2 + 2\ln n}$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\ln n}{n^2 + 2\ln n} + \sum_{n=1}^{\infty} \frac{\sin n}{n^2 + 2\ln n}$$

$$\wedge \qquad\qquad \wedge$$

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2} \qquad\qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

сх. $\qquad\qquad$ сх.

7) $a_n = \dfrac{n+2}{n^2\left(4 + 3\sin\left(\pi n/3\right)\right)}$

$$\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} \frac{n+2}{n^2} \to \infty$$

$\Rightarrow$ расх.

9(6,8) б) α - ?

$$a_n = \left(\left(\frac{sh(1/n)}{\sin(1/n)} \right)^{3n} - 1 \right)^{\alpha}$$

$$\frac{sh(\frac{1}{n})}{\sin(\frac{1}{n})} = \frac{\frac{1}{n} + \frac{1}{6}\frac{1}{n^3} + o(\frac{1}{n^4})}{\frac{1}{n} - \frac{1}{6}\frac{1}{n^3} + o\frac{1}{n^4})} =$$

$$= \frac{1 + \frac{1}{6}\frac{1}{n^2} + o(\frac{1}{n^3})}{1 - \frac{1}{6}\frac{1}{n^2} + o(\frac{1}{n^3})} =$$

$$= 1 + \frac{1}{3}\frac{1}{n^2} + O(\frac{1}{n^3})$$

$$\left(\frac{\operatorname{sh}(1/n)}{\sin(1/n)}\right)^{3n} - 1 =$$

$$= \frac{1}{n^2} + O\left(\frac{1}{n^3}\right)$$

$$a_n = \frac{1}{n^{2\alpha}}\left(1 + O\left(\frac{1}{n}\right)\right)^{\alpha}$$

$\sum\limits_{n=1}^{\infty} a_n$ сх. тогда $\alpha > \frac{1}{2}$

8) $a_n = \left(1 - \left(\cos\frac{1}{n}\right)^{\frac{1}{n}}\right)^{\alpha}$

$$\left(\cos\frac{1}{n}\right)^{1/n} = \left(1 - \frac{1}{2n^2} + o\left(\frac{1}{n^3}\right)\right)^{\frac{1}{n}} =$$

$$= \exp\left(\frac{\ln\left(1 - \frac{1}{2n^2} + o\left(\frac{1}{n^3}\right)\right)}{n}\right) =$$

$$= \exp \frac{-\frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)}{n} =$$

$$= \exp\left(-\frac{1}{2n^3} + o\left(\frac{1}{n^3}\right)\right) =$$

$$= 1 - \frac{1}{2n^3} + o\left(\frac{1}{n^3}\right)$$

$$a_n = \left(-\frac{1}{2n^3} + o\left(\frac{1}{n^3}\right)\right)^\alpha =$$

$$= \frac{1}{n^{3\alpha}}\left(-\frac{1}{2} + o(1)\right)^\alpha$$

Сх. при $\alpha > 1/3$

<u>19.8</u>

$$a_n = \frac{(2n)!!}{n!} \operatorname{arctg}\frac{1}{3^n}$$

$$\frac{\frac{(2n+2)!!}{(n+1)!}\operatorname{arctg}\frac{1}{3^{n+1}}}{\frac{(2n)!!}{n!}\operatorname{arctg}\frac{1}{3^n}} = \frac{2n+2}{n+1}\,\frac{\operatorname{arctg}\frac{1}{3^{n+1}}}{\operatorname{arctg}\frac{1}{3^n}}$$

$$\Big\downarrow_{n\to\infty}$$

$$2/3 < 1 \Rightarrow \text{Сх.}$$

20(1) $a_n = \frac{(3n)!}{(n!)^3 4^{3n}}$

$$\frac{\frac{(3n+3)!}{((n+1)!)^3 4^{3n+3}}}{\frac{(3n)!}{(n!)^3 4^{3n}}} = \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3 \cdot 4^3} \to$$

$$\xrightarrow[n\to\infty]{} \left(\frac{3}{4}\right)^3 < 1 \Rightarrow \text{K.}$$

21. (6, 12) 6) $a_n = 3^{n+1}\left(\frac{n+2}{n+3}\right)^{n^2}$

$$\sqrt[n]{a_n} = 3 \cdot 3^{1/n}\left(\frac{n+2}{n+3}\right)^n =$$

$$= 3 \cdot 3^{1/n} \left(1 - \frac{1}{n+3}\right)^n \xrightarrow[n\to\infty]{} \frac{3}{e} \Rightarrow \text{расх.}$$

12) $$a_n = \left(n \operatorname{sh} \frac{1}{n}\right)^{-n^3}$$

$$\sqrt[n]{a_n} = \left(n \operatorname{sh} \frac{1}{n}\right)^{-n^2} =$$

$$= \left(1 + \frac{1}{6}\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)\right)^{-n^2} \xrightarrow[n\to\infty]{} \frac{1}{e^{1/6}} < 1$$

$$\Rightarrow \text{сх.}$$

<u>3f*</u>

$$\sum_{n=1}^{\infty} a_n \text{ — } cx. \Rightarrow$$

$$\sum_{n=1}^{\infty} 2^n a_{2^n} \text{ — } cx.$$

$$\Rightarrow 2^n a_{2^n} \xrightarrow[n\to\infty]{} 0$$

$$\Rightarrow n a_n \xrightarrow[n\to\infty]{} 0$$

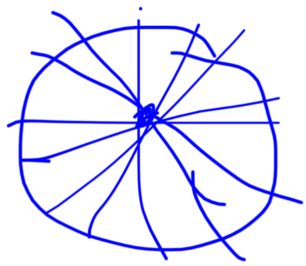
T. 5

a_n мон стр. к 0.

$$S = \sum_{n=1}^{\infty} a_n \sin n\alpha \quad \text{сх.} \ \forall \alpha \in \mathbb{R}?$$

0) $\alpha = \pi k \Rightarrow S = 0$

1) $\frac{\alpha}{\pi} = \frac{k}{m} \in \mathbb{Q}$



$$\sin n\alpha \in \{x_1, \dots, x_m\}$$

$$S = \sum_{n=0}^{\infty} (-1)^n x_1 a_{1+n\lfloor \frac{m}{2} \rfloor} +$$

$$+ \sum_{n=0}^{\infty} (-1)^n x_2 a_{2+n\lfloor \frac{m}{2} \rfloor}$$

$$\dots$$

$$+ \sum_{n=0}^{\infty} (-1)^n x_{\lfloor \frac{m}{2} \rfloor} a_{\lfloor \frac{m}{2} \rfloor + n\lfloor \frac{m}{2} \rfloor}$$

$$S_i = \sum_{n=0}^{\infty} (-1)^n x_i \, a_{i+n\lfloor \frac{m}{2} \rfloor}$$

$$|S| = \left| \sum_{n=N}^{\infty} (-1)^n x_i \, a_{i+n\lfloor \frac{m}{2} \rfloor} \right| =$$

$$= \left| x_i \sum_{n=N}^{\infty} (-1)^n a_{i+n\lfloor \frac{m}{2} \rfloor} \right|$$

$$\exists N : \forall n \geq N \quad a_{i+n\lfloor \frac{m}{2} \rfloor} < \frac{\varepsilon}{x_i}$$

$$\Rightarrow |S| < \varepsilon \Rightarrow \alpha.$$

$$S = \sum_{i=1}^{\lfloor m/2 \rfloor} S_i \longrightarrow \alpha.$$

$$c\alpha.$$

2) $\frac{\alpha}{\pi} \notin \mathbb{Q}$

$$\sin n\alpha = \frac{\cos\frac{2n+1}{2}\alpha - \cos\frac{2n-1}{2}\alpha}{2\sin\frac{\alpha}{2}}$$

$$2\sin\frac{\alpha}{2} S = \sum_{n=1}^{\infty} a_n \cos\frac{2n+1}{2}\alpha - a_n \cos\frac{2n-1}{2}\alpha$$

$$= -a_1\cos\frac{\alpha}{2} + \underbrace{(a_1 - a_2)\cos\frac{3\alpha}{2} + (a_2 - a_3)\cos\frac{5\alpha}{2} + \dots}_{r}$$

$$|r| \le |a_1 - a_2 + a_2 - a_3 + \dots a_N| \xrightarrow[N\to\infty]{} |a_1| \Rightarrow \text{сх.}$$

$\sum_{n=1}^{\infty} a_n \cos n\alpha$ аналог.

3.(4,5) 4) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n+2)\sqrt[4]{n+1}}$

$\displaystyle\sum_{n=1}^{N} (-1)^n$ огр.

$$f(x) = \frac{x}{(x+2)(x+1)^{1/4}}$$

$$f'(x) = f(x)\left(\frac{1}{x} - \frac{1}{x+2} - \frac{1/4}{x+1}\right)$$

$$\frac{f'}{f} = \frac{\cancel{x^2} + 3x + 2 - \cancel{x^2} - x - \frac{1}{4}x^2 - \frac{1}{2}x}{x(x+1)(x+2)} =$$

$$= \frac{-\frac{1}{4}x^2 + \frac{3}{2}x + 2}{x(x+1)(x+2)} = -\frac{x^2 - 6x - 8}{4x(x+1)(x+2)} =$$

$$x > 3 + \sqrt{17} \Rightarrow f' < 0$$

$\Rightarrow$ по пр. Лейбн. $\displaystyle\sum_{n=1}^{\infty} a_n$ сх.

5) $$\sum_{n=1}^{\infty} \cos\left(\frac{\pi}{4}+\pi n\right)\sin\frac{1}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2}}\sin\frac{1}{n}$$

$\sin\frac{1}{n}$ ↓ 0 мон.

$$= \frac{1}{\sqrt{2}}\underbrace{\sum_{n=1}^{\infty}(-1)^n\sin\frac{1}{n}}_{\text{сх. по пр. Лейбница}}$$

8. (3, 4)   3) $$\sum_{n=1}^{\infty} \frac{\sin n}{\sqrt{n}+\sin n}$$

$$a_n = \frac{\sin n}{\sqrt{n}}\left(1+\frac{\sin n}{\sqrt{n}}\right)^{-1} =$$

$$= \frac{\sin n}{\sqrt{n}} - \frac{\sin^2 n}{n} + \alpha_n$$

$$|\alpha_n| \leq \frac{C\sin^3 n}{n^{3/2}} \leq \frac{C}{n^{3/2}} \to \sum_{n=1}^{\infty}\alpha_n \text{ сх.}$$

$$\frac{\sin n}{\sqrt{n}} - \frac{1}{2n} + \frac{\cos 2n}{2n} + \alpha_n$$

сх. расх. сх. сх.

$$\Rightarrow \sum_{n=1}^{\infty} a_n \quad \text{расх.}$$

4) $$\sum_{n=1}^{\infty} \underbrace{\sin\left(\frac{\sin n}{\sqrt[3]{n}}\right)}_{a_n}$$

$$\sin t = t + O(t^3)$$

$$a_n = \frac{\sin n}{\sqrt[3]{n}} + \alpha_n$$

$$|\alpha_n| \le \frac{C \sin^3 n}{n} = \frac{C(3\sin n - \sin 3n)}{4n}$$

$$= \frac{3C}{4} \cdot \frac{\sin n}{n} - \frac{C \sin 3n}{4n}$$

$$\Rightarrow \sum_{n=1}^{\infty} \alpha_n \quad cx.$$

$$\sum_{n=1}^{\infty} \frac{\sin n}{2\sqrt{n}} \quad cx. \quad \text{by T.S.}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \quad cx.$$

<u>9.2</u> $$\sum_{n=1}^{\infty} \sin\left(\pi\sqrt{n^2+1}\right) = \sum_{n=1}^{\infty} a_n$$

$$a_n = (-1)^n \sin\left(\pi\left(\sqrt{n^2+1}-n\right)\right)$$

$$\sqrt{t^2+1} - t = t\left[\sqrt{1+\frac{1}{t^2}} - 1\right] =$$

$$= t\left[\frac{1}{2t^2} + O\left(\frac{1}{t^4}\right)\right] = \frac{1}{2t} + O\left(\frac{1}{t^3}\right)$$

$$a_n = (-1)^n \sin\left(\frac{\pi}{2t} + \alpha_n\right) =$$

$$= (-1)^n \left[\sin\frac{\pi}{2n}\cos\alpha_n + \sin\alpha_n \cos\frac{\pi}{2n}\right]$$

$$|\alpha_n| \le \frac{C}{n^3}$$

$$|\sin\alpha_n| \le |\alpha_n| \le \frac{C}{n^3}$$

$$\left|\sin\alpha_n \cos\frac{\pi}{2n}\right| \le \frac{C}{n^3} \to \text{сх.}$$

$$\sin\frac{\pi}{2n}\cos\alpha_n \xrightarrow[n\to\infty]{} 0$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} (-1)^n \sin\frac{\pi}{2n}\cos\alpha_n \quad \text{сх. по пр. Лейб.}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \quad \text{сх.}$$

4. $\sum\limits_{n=1}^{\infty} c_n$, $c_n = a_n - b_n$

1) $\sum\limits_{n=1}^{\infty} a_n$ расх., $\sum\limits_{n=1}^{\infty} b_n$ сх.

$a_n \xrightarrow[n\to\infty]{} 0$, иначе очев.

$$\left|\sum_{n=m}^{k} c_n\right| = \left|\sum_{n=m}^{k} a_n - \sum_{n=m}^{k} b_n\right| \geq$$

$$\geq \left|\,\underset{\substack{\vee \\ \varepsilon/2}}{\left|\sum_{n=m}^{k} a_n\right|} - \underbrace{\left|\sum_{n=m}^{k} b_n\right|}_{\substack{\wedge \\ \varepsilon/2}}\right| > \varepsilon$$

2) $a_n = \frac{1}{n}$, $b_n = \frac{1}{n} - \frac{1}{n^2}$

расх.

$$\sum_{n=1}^{\infty} |a_n - b_n| = \sum_{n=1}^{\infty} \frac{1}{n^2} \to \text{сх.}$$

I.6 $\sum_{n=1}^{\infty} a_n = A$ $\alpha.$, $\sum_{n=1}^{\infty} b_n = B$ $\alpha.$ $\cancel{ad}$.

$\overset{?}{\Longrightarrow} \sum_{n=1}^{\infty} a_n b_n$ $\alpha.$

$$B_n = \sum_{k=1}^{n} b_k$$

$$\left|\sum_{n=N}^{M} a_n b_n\right| = \left|\sum_{n=N}^{M} a_n (B_n - B_{n-1})\right| =$$

$$= \left|\sum_{n=N}^{M} a_n B_n - \sum_{n=N}^{M} a_n B_{n-1}\right| =$$

$$\left|\sum_{n=N}^{M} a_n B_n - \sum_{n=N-1}^{M-1} a_{n+1} B_n\right| =$$

$$= \left|a_M B_M - a_N B_{N-1} + \sum_{n=N}^{M-1} (a_n - a_{n+1}) B_n\right| \leq$$

$$\leq \left|a_M B_M - a_N B_{N-1}\right| + \underbrace{\left|\sum_{n=M}^{M-1} (a_{n+1} - a_n) B_n\right|}_{S}$$

$$\exists N' \ \forall m \geq N' \ |B_m - B| < \varepsilon$$

$$S = \left| B \sum_{n=N}^{M-1} (a_{n+1} - a_n) + \sum_{n=N}^{M-1} (a_{n+1} - a_n)(B_n - B) \right| \leq$$

$$\leq |B| |a_M - a_N| + \varepsilon |a_M - a_N| =$$

$$= |a_M - a_N| (|B| + \varepsilon)$$

$$\exists N'' \ \forall n \geq N'' \quad |a_n - a| < \frac{\varepsilon}{(|B| + \varepsilon) 2}$$

$$S < \varepsilon$$

$$|a_M B_M - a_N B_{N-1}| \leq$$

$$\leq |a_M| \|B_M - B_{N-1}\| + |B_{N-1}| |a_M - a_N| <$$

$$< \frac{\cancel{2} \varepsilon^2}{(|B| + \varepsilon) \cdot \cancel{2}} + \frac{\varepsilon^2}{|B| + \varepsilon} = \frac{2 \varepsilon^2}{|B| + \varepsilon}$$

$$\forall \varepsilon > 0 \quad \exists \tilde{N} = \max\{N', N''\} \quad \forall N, M > \tilde{N}$$

$$\left| \sum_{n=N}^{M} a_n b_n \right| < \varepsilon \left(1 + \frac{2\varepsilon}{|B| + \varepsilon} \right) \xrightarrow[\varepsilon \to 0]{} 0$$