$$I=\int\limits_0^{\frac{\pi}{2}}\frac{\tan^{-1}(\sin t)}{\sin t}dt=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int\limits_0^{\frac{\pi}{2}}(\sin t)^{2n}\,dt=$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cdot\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}$$

$$\frac{1}{2n+1}=\frac{1}{2}\cdot\frac{1}{n+\frac{1}{2}}=\frac{1}{2}\cdot\frac{(1/2)_n}{\frac{1}{2}(3/2)_n}=\frac{(1/2)_n}{(3/2)_n}$$

$$\frac{(2n-1)!!}{(2n)!!}=\frac{(n-\frac{1}{2})\cdot\ldots\cdot\frac{1}{2}}{n\cdot\ldots\cdot 1}=\frac{(1/2)_n}{(1)_n}$$

$$I=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(1/2)_n(1/2)_n}{(3/2)_n(1)_n}(-1)^n=\frac{\pi}{2}{}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ \frac{3}{2}\end{matrix};-1\right)$$

$${}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};z\right)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int\limits_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

$$a=b=\frac{1}{2};\quad c=\frac{3}{2};\quad z=-1$$

$${}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ \frac{3}{2}\end{matrix};-1\right)=\frac{\Gamma(3/2)}{\Gamma(1/2)\Gamma(1)}\int\limits_0^1 t^{-1/2}(1-t)^0(1+t)^{-1/2}\,dt=$$

$$=\frac{1}{2}\int\limits_0^1\frac{dt}{\sqrt{t^2+t}}=\frac{1}{2}\int\limits_0^1\frac{dt}{\sqrt{(t+\frac{1}{2})^2-(\frac{1}{2})^2}}=$$

$$=\left[\begin{matrix}t+\frac{1}{2}=\frac{1}{2}\cosh\theta\\ dt=\frac{1}{2}\sinh\theta\,d\theta\end{matrix}\right]=\frac{1}{2}\int\limits_0^{\cosh^{-1}3}\frac{\frac{1}{2}\sinh\theta\,d\theta}{\frac{1}{2}\sinh\theta}=\frac{1}{2}\cosh^{-1}3=\ln(1+\sqrt{2})$$

$$I=\frac{\pi}{2}\ln(1+\sqrt{2})$$