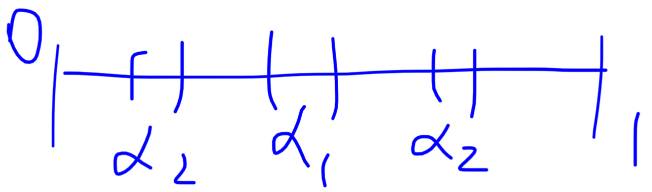
T.1



Kant. mn-bo C

$$\mu(C) = (1-\alpha_1)(1-\alpha_2)\cdot\ldots = \prod_{i=1}^{\infty}(1-\alpha_i) = P$$

$$\alpha_k = \frac{1}{(k+99)^2}$$

$$1-\alpha_k = \frac{(k+98)(k+100)}{(k+99)^2}$$

$$1 \geq \mu(C) = \prod_{k\in\mathbb{N}}(1-\alpha_k) =$$

$$= \frac{99\cdot\cancel{101}}{100^2}\cdot\frac{\cancel{100}\cdot\cancel{102}}{\cancel{101}^2}\cdot\frac{\cancel{101}\cdot\cancel{103}}{\cancel{102}^2}\cdot\ldots =$$

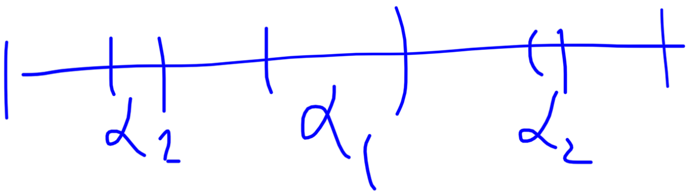
$$= 0{,}99$$

$$\mu(\mathbb{Q}\cap C)=0$$

$$X = C \setminus (\mathbb{Q}\cap C)$$

$$\mu(X) = \mu(C) - 0 = \mu(C) = 0{,}99$$

~~Т.2^4~~ $X = \{0, a_1 a_2 \ldots \mid a_i \neq 5 \ \forall i\}$



$$\alpha_i = \frac{1}{10} \quad \forall i$$

X — конт. мн-во

$$\mu(X) = \lim_{n\to\infty}\left(1-\frac{1}{10}\right)^n = 0$$

Т.3 $\partial X = \overline{X} \setminus \operatorname{int} X$

$$\underbrace{\overline{X}}_{\text{изм.}} \setminus \underbrace{\operatorname{int} X}_{\text{изм.}}$$

изм.

Пример: $X = \mathbb{Q} \cap [0,1]$

$\mu(X) = 1$

Т.4* $X \subset \mathbb{R}$ изм., $\mu(X) > 0$

Д-ть: $\exists x, y \in X$: $x - y \in \mathbb{Z}$

Д-во: $x \sim y \Longleftrightarrow x - y \in \mathbb{Z}$

$[\alpha] = \{x \mid x \sim \alpha \text{ и } x \in X\}$

$$X = \bigsqcup_{\alpha \in [0,1)} [\alpha]$$

Если $\forall x, y \in X$ $x - y \notin \mathbb{Z}$,

то в каждом классе ≤ 1 представителя

$\Rightarrow \mu(X) \le \mu([0,1]) = 1$ (?!)

<u>Т. 6*</u> Кант. лестница

<u>Т. 7</u> $\{X_k\}_{k=1}^{\infty}$ — изм.,

$$X_{k+1} \subset X_k \subset \mathbb{R}^n \quad \forall k \in \mathbb{N}$$

$$X = \bigcap_{k=1}^{\infty} X_k$$

Тогда $\mu(X) = \lim\limits_{k\to\infty} \mu(X_k)$, если

$$\mu(X_1) < +\infty.$$

<u>Д-во:</u> $$\mathbb{R}^n \setminus X = \underbrace{\bigcup_{k=1}^{\infty} \underbrace{\mathbb{R}^n \setminus X_k}_{\text{изм.}}}_{\text{изм.}}$$

$\Rightarrow X$ изм.

$$\mu(X) = \inf \mu(X_k) =$$

$$= \lim_{k\to\infty} \mu(X_k)$$

<u>Т.5</u> $\mathbb{R} \supset A$ - изм., $\mu(A)=0$

$f: \mathbb{R} \to \mathbb{R}$ - непр. дифф.

$\Rightarrow \mu(f(A)) = 0$

<u>Д-во</u>: $f'(x) \geq 0$. $\forall\, Y_k = A \cap U_\delta(k\delta)$

$$A = \bigsqcup_{k\in\mathbb{Z}} Y_k$$

$$f(Y_k) \subset U_\varepsilon(f(x)), \quad x = k\delta$$

$x_1, x_2 \in Y_k$

$$|f(x_1) - f(x_2)| \leq \max_{x \in U_\delta(\delta k)} \{f'(x)\} \cdot 2\delta$$

$$f(Y_k) \subset 2\delta \max_{x \in U_\delta(\delta k)} \{f'(x)\} U_\varepsilon(f(\delta k))$$

$$\mu(f(Y_k)) \leq 4\delta\varepsilon \max_{x \in U_\delta(\delta k)} \{f'(x)\} \quad \forall k.$$

$f'(x)$ непр. на $[\delta(k-1), \delta(k+1)] \to$ огр.

$$\mu(f(Y_k)) = 0 \quad \forall k$$

$$\mu(f(A)) \leq \sum_{k \in \mathbb{Z}} \mu(f(Y_k)) = 0$$

7.8 $X \subset \mathbb{R}$ кон. изм., то

$$\mu\left(X \setminus (X+t)\right) \xrightarrow[t\to 0]{} 0$$

Д-в

$\forall \varepsilon > 0 \ \exists A$-кл.:

$$\mu(X \Delta A) < \varepsilon/3$$

$$\mu\left((X+t)\Delta(A+t)\right) = \mu(X\Delta A + t) =$$

$$= \mu(X \Delta A) < \varepsilon/3$$

$$A = \bigsqcup_{k=1}^{n} [a_k, b_k]$$

$$A \Delta (A+t) \subset \bigcup_{k=1}^{n} \left([a_k, a_k+t] \cup [b_k, b_k+t]\right)$$

$$|t| < \delta$$

$$\mu(A \Delta (A+t)) \le 2nt < 2n\delta$$

$$\delta = \frac{\varepsilon}{6n}$$

$$\mu(X \setminus (X+t)) \le \mu(X \Delta (X+t)) \le$$

$$\le \mu(X \Delta A) + \mu(A \Delta (A+t)) + \mu((A+t) \Delta (X+t))$$

$$\le \varepsilon$$

Итого $\forall \varepsilon > 0 \ \exists \delta = \frac{\varepsilon}{2n} \ \forall t \in \mathring{U}_\delta(0) \hookrightarrow$

$$\mu(X \setminus (X+t)) < \varepsilon$$

$$\Rightarrow \mu(X \setminus (X+t)) \xrightarrow[t \to 0]{} 0$$

<u>Т.9^7</u>

$$X = \bigcup_{\alpha} X_\alpha$$

$$a \sim b \iff X \supseteq [a,b]$$

$$X = \bigsqcup_{\beta} X_\beta$$

классы экв.

т.к. $\forall \alpha$ X_α невырожд. отр. $\Rightarrow$

$\forall X_\beta$ $\exists q_\beta \in \mathbb{Q} \cap X_\beta$

$$X_\beta \longmapsto q_\beta$$

$\Rightarrow$ $\bigsqcup_\beta X_\beta$ не более чем счётно.

T.10* Витали

T.11* Витали

T.12 $f: A \to \mathbb{R}$ изм. на A.

$\forall a, b,\ -\infty < a < b < +\infty$ $f^{-1}(a,b)$ изм.

Д-во

$f \le c \Leftrightarrow \forall n \in \mathbb{N}\ f < c + \frac{1}{n}$

$$\{f \le c\} = \underbrace{\bigcap_{n\in\mathbb{N}} \underbrace{\{f < c + \tfrac{1}{n}\}}_{\text{изм.}}}_{\text{изм.}} \text{ - изм.}$$

$$\{a < f < b\} = \underbrace{\underbrace{\{f < b\}}_{\text{изм.}} \setminus \underbrace{\{f \le a\}}_{\text{изм.}}}_{\text{изм.}} \text{ - изм.}$$

Т.13 $f: \mathbb{R} \to \mathbb{R}$ изм. $\Rightarrow G_f$ изм. и

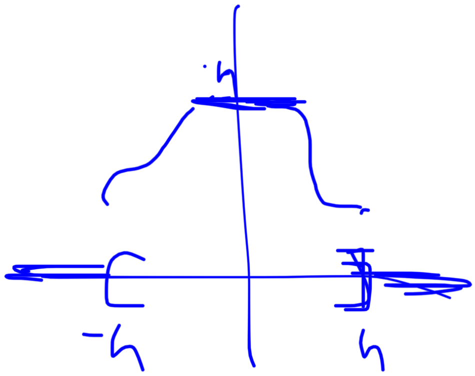
$$\mu(G_f) = 0$$

Д-во

$$G_f = \{ (x, f(x)) \mid x \in \mathbb{R} \}$$

$$f_n = \min(f(x), n)\, \chi_{[-n,n]}$$

$$G_f \subset \bigcup_{n \in \mathbb{N}} G_{f_n}$$



fix N.

$$I_j^m = \left[-N + \frac{j-1}{m}, -N + \frac{j}{m} \right]$$

$$A_j^m = f_n^{-1}(\overline{I_j^m}) \text{ — изм.}$$

$$G_{f_N} = \bigcap_{m=1}^{\infty} \bigcup_{j=1}^{2Nm} A_j^m \times I_j^m \quad \text{– ugyu.}$$

$$\mu(G_{f_n}) = \lim_{m\to\infty} \mu\left(\bigcup_{j=1}^{2Nm} A_j^m \times I_j^m\right) \le$$

$$\le \lim_{m\to\infty} \underbrace{\sum_{j=1}^{2Nm} \mu(A_j^m)}_{} \cdot 1/m \quad \le$$

$$\mu\left(f_n^{-1}([-N,N])\right)$$

$$\parallel$$

$$2N$$

$$2N \lim_{m\to\infty} \frac{1}{m} = 0$$

$$\mu(G_{f_n}) = 0 \quad \forall n$$

$$\mu(G_f) \le \sum_{n=1}^{\infty} \mu(G_{f_n}) = 0.$$

$\underline{T.14}$ $f, g : \mathbb{R} \to \mathbb{R}$ изм. по Леб.

a) $af + bg$

$$\{af < c\} = \{f < c\} \Rightarrow af\text{-изм.}$$

$\tilde f$ // af-изм.

bg-изм. // $\tilde g$

$$\tilde f + \tilde g < c \Leftrightarrow \exists q \in \mathbb{Q} \;\; \tilde f + \tilde g < q < c$$

$$\Leftrightarrow \exists q \in \mathbb{Q} \quad \begin{cases} \tilde f < c_1 \\ \tilde g < q - c_1 \end{cases}$$

$$\{\tilde f + \tilde g < c\} = \bigcup_{q \in \mathbb{Q}} \left(\{\tilde f < c_1\} \cap \{\tilde g < q - c_1\} \right)$$

б) fg : $\xi(u, v) = uv$ - непр.

$\xi(f, g)$ - изм.

б) $\min\{f,g\} < c \iff \left[\begin{array}{l} f<c \\ g<c \end{array}\right.$

$\{\min\{f,g\}<c\} = \{f<c\} \cup \{g<c\}$

T.15* $f, g : \mathbb{R} \to \mathbb{R}$ изм., $f>0$

$h(x) = f(x)^{g(x)}$ изм.?

$\xi(u,v) = u^{v}$ – непр. на $\mathbb{R}^{+} \times \mathbb{R}$

$h = \xi(\underbrace{f, g}_{\text{изм.}})$ – изм.

T.16 $f_k : X \to \mathbb{R}$ изм. $\forall k \in \mathbb{N}$

Д-ть: $\varphi(x) = \sup\limits_{k\in\mathbb{N}} f_k(x)$ изм.

$$\varphi \le c \iff \forall n\in\mathbb{N}\ \exists k: \begin{cases} f_k \ge c - \frac{1}{n} \\ f_n \le c \end{cases}$$

$$\{\varphi \le c\} = \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \left(\{f_n \le c\} \cap \{f_k \ge c - \tfrac{1}{n}\}\right)$$

T.17* неизм. $f: [0,1] \to \mathbb{R}$, f^2 изм.

$$f = \begin{cases} 1, & x \in B \text{ — неизм.} \\ -1, & x \in [0,1]\setminus B \end{cases}$$

тогда f — неизм., $f^2 \equiv 1$ — изм.

T. 18*. Канторова лестница

T. 19 $f: \mathbb{R} \to \mathbb{R}$ дифф. на $\mathbb{R}$

Д-ть: f' изм.

$$f' < c \iff \exists N \in \mathbb{N}\ \forall n \geq N \quad \frac{f(x+\frac{1}{n}) - f(x)}{1/n} < c$$

$$\{f' < c\} = \underbrace{\bigcup_{N \in \mathbb{N}} \underbrace{\bigcap_{n \geq N} \underbrace{\left\{ f(x+\tfrac{1}{n}) - f(x) < \tfrac{c}{n} \right\}}_{\text{изм.}}}_{\text{изм.}}}_{\text{изм.}}$$

f'_- анал.

$$\{f' < c\} = \{f'_+ - f'_- = 0\} \cap \{f'_+ < c\}$$

изм. изм.

T.20 $f, g : A \to \mathbb{R}$ - кон. - ст. ф-ции на A

$\alpha, \beta \in \mathbb{R}$

$$A = \bigsqcup_{k=1}^{n_1} X_k \qquad f\big|_{X_k} \equiv c_k$$

$$A = \bigsqcup_{l=1}^{n_2} Y_l \qquad g\big|_{Y_l} = \alpha_l$$

$$X_k = \bigsqcup_{l=1}^{n_2} \underbrace{(X_k \cap Y_l)}_{Z_{k,l}}$$

$$A = \bigsqcup_{k=1}^{n_1} \bigsqcup_{l=1}^{n_2} Z_{k,l}.$$

$$h_1 = af + bg$$

$$h_1\big|_{z_{k,l}} = ac_k + bd_l$$

$$h_2 = fg$$

$$h_2\big|_{z_{k,l}} = c_k d_l$$

$$h_3 = \min\{f, g\}$$

$$h_3\big|_{z_{k,l}} = \min\{c_k, g_l\}$$

T.21 $f(x) = e^x$, $\varepsilon > 0$

$g_\varepsilon, h_\varepsilon : [0,1] \to \mathbb{R}$

$$X_k = \left[\frac{k-1}{n}, \frac{k}{n}\right), \quad X_n = \left[\frac{n-1}{n}, 1\right]$$

$$g\big|_{X_k} = e^{\frac{k-1}{n}}, \quad h\big|_{X_k} = e^{\frac{k}{n}}$$

$$[0,1] = \bigsqcup_{k=1}^{n} X_k$$

$$\int_{[0,1]} (h-g)\,dx = \sum_{k=1}^{n} \int_{X_k} (h-g)\,dx =$$

$$= \sum_{k=1}^{n} \left(e^{\frac{k}{n}} - e^{\frac{k-1}{n}}\right) \cdot \frac{1}{n} =$$

$$= (e-1)/n < \varepsilon$$

$$n = \left\lceil \frac{e-1}{\varepsilon} \right\rceil$$

$$\int_{[0,1]} (h_\varepsilon(x) - g_\varepsilon(x))\,dx < \varepsilon$$

$$\int_{[0,1]} |f(x) - g_\varepsilon(x)|\,dx < \varepsilon$$

$$\Rightarrow \int_{[0,1]} f(x)\,dx = \lim_{\varepsilon \to 0} \int_{[0,1]} g_\varepsilon(x)\,dx =$$

$$= \int_0^1 e^x\,dx = e - 1$$

T.22 $\quad h = \lceil \frac{1}{\varepsilon} \rceil$

$$\int_{[1,2]} x\,dx = \frac{x^2}{2}\Big|_1^2 = 2 - \frac{1}{2} = \frac{3}{2}$$

T.23

$$D(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

$D(x)$ – изм., как $\chi_{\mathbb{Q}}$

изм.

$D(x) \geq 0 \quad \forall x$

$$\exists \int_{[0,1]} D(x)\,dx = \int_{[0,1]\cap\mathbb{Q}} 1\cdot dx + \int_{[0,1]\cap(\mathbb{R}\setminus\mathbb{Q})} 0\,dx =$$

$$= \mu([0,1]\cap\mathbb{Q}) + 0 = 0$$

и $\exists \int_{\mathbb{R}} D(x)\,dx = \mu(\mathbb{Q}) + 0 = 0$

T.24 $\quad f: X \to \mathbb{R}$ изм.

a) $I_*(f,X),\ I^*(f,X) < \infty$

и $\quad \exists \neq \forall \quad ?$

ог.-изм.

$$I_*(f,X) = \sup_g \int_X g\,dx, \quad g \le f \text{ всюду на } X$$

$$I^*(f,X) = \inf_h \int_X h\,dx \quad h \ge f \text{ всюду на } X$$

$$I^*(f) = I^*(f_+) - I_*(f_-)$$

$$I_*(f) = I_*(f_+) - I^*(f_-)$$

$$I^*(f) - I_*(f) = \overbrace{I^*(f_+) - I_*(f_+)}^{0} - \underbrace{\left(I^*(f_-) - I_*(f_-)\right)}_{0}$$

$\Rightarrow$ а) нет

б) $I^*(f) = +\infty \Rightarrow I^*(f_+) = +\infty$

$I_*(f_-) < +\infty$

$I_*(f) = -\infty \Rightarrow I^*(f_-) = +\infty$

$I_*(f_+) < +\infty$

f_+ и f_- – инт.

Значит. $I^*(f_+) = +\infty \Rightarrow I_*(f_+) = +\infty$

Противоречие

Ответ: нет.

T.25 f изм. и неотр. на X,

$\mu(X)>0$

$f(x)>0 \quad \forall x\in X$

Д-ть: $\int\limits_X f(x)dx>0$

Д-во:

$$X_k=\{f>\tfrac{1}{k}\}$$

$$X=\bigcup_{k\in\mathbb{N}} X_k$$

$$\mu(X)\leq\sum_{k\in\mathbb{N}}\mu(X_k)$$

$$\mu(X)>0 \Rightarrow \exists k:\ \mu(X_k)>0$$

$$\int\limits_X f dx=\underbrace{\int\limits_{X_k} f dx}_{\geq\frac{\mu(X_k)}{k}}+\underbrace{\int\limits_{X\setminus X_k} f dx}_{\geq 0}\geq\frac{\mu(X_k)}{k}>0$$

Т.2$\cancel{6}$ $f : \cancel{X} \to \mathbb{R}$ инт.

Д-ть: $\forall \varepsilon > 0$ $\exists$ кон.-ст. инт. ф-ция

$$f_\varepsilon : X \to \mathbb{R} : \int_X |f_\varepsilon(x) - f(x)| dx \le \varepsilon$$

Д-во

$$\int_X f dx = \inf_h \int_X h dx = \sup_g \int_X g dx$$

h, g — ст.-ступ.

$\forall \varepsilon > 0$ $\exists h_\varepsilon$ (ст.-ступ.) $h_\varepsilon \le f + \frac{\varepsilon}{2\mu(X)}$ п.в.

и $h_\varepsilon \ge f$ п.в.

$$h_\varepsilon - f < \frac{\varepsilon}{2\mu(X)}$$

$$X = \bigsqcup_{k=1}^{\infty} X_k, \quad h_\varepsilon \big|_{X_k} = c_k$$

$$\int_X h_\varepsilon \, dx = \sum_{k=1}^{\infty} c_k \mu X_k - \text{сх.} \Rightarrow$$

$$\Rightarrow \exists N: \sum_{k=N+1}^{\infty} c_k \mu X_k < \frac{\varepsilon}{2\mu(X)}$$

$$\Rightarrow h = h_\varepsilon \Big|_{\bigsqcup_{k=1}^{N} X_k} - \text{кон. ст. функ.}$$

$$: \ |h_\varepsilon - h| < \frac{\varepsilon}{2\mu(X)}$$

$$\Rightarrow |f - h| < \frac{\varepsilon}{\mu(X)}$$

$$\Rightarrow \int_X |f - h| \, dx \leq \varepsilon$$

т. 27 $f: \mathbb{R} \to \mathbb{R}$ инт. по Лебегу

то
$$\int\limits_{\mathbb{R}} f(x) e^{i\omega x}\, dx \xrightarrow[\omega\to\infty]{} 0$$

$$\int\limits_X |f(x) - f_\varepsilon(x)| \cos \omega x \, dx \leqslant$$

$$\leqslant \int\limits_X |f - f_\varepsilon|\, dx \leqslant \varepsilon/2$$

$$\int\limits_X f_\varepsilon(x) \cos \omega x \, dx =$$

$$X = \bigsqcup_{k=1}^{\infty} \underset{\text{кл.}}{X_k}, \quad f_\varepsilon = \begin{cases} c_k, & x \in X_k \\ & \text{и } k \leqslant N \\ 0, & \text{иначе} \end{cases}$$

$$= \sum_{k=1}^{N} C_k \int_{X_k} \cos\omega x\,dx = I$$

Хочу: $\exists\,\Omega \quad \forall \omega \geq \Omega \quad < \frac{\varepsilon}{2}$

$$I_j = \int_{X_j} \cos\omega x\,dx, \quad \mu X_j > 0$$

$$X_j = [x_{j-1}, x_j)$$

$$|I_j| = \left| \frac{\sin\omega x_j - \sin\omega x_{j-1}}{\omega} \right| \leq \frac{2}{\omega}$$

$$\exists\,\omega_j: \quad I_j < \frac{1}{2C_k N}$$

$$\Omega = \max\{\omega_j\}_{j=1}^{N}$$

Тогда $I < \varepsilon/2 \Rightarrow \int_X f(x)\cos\omega x\,dx < \varepsilon$

Т. 28* $f: \mathbb{R} \to \mathbb{R}$ — инт. по Леб.,

То $\lim\limits_{t \to 0} \int\limits_{\mathbb{R}} |f(x+t) - f(x)|\,dx = 0$

$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall t \in \mathring{U}_\delta(0) \hookrightarrow$

$$\int\limits_{\mathbb{R}} |f(x+t) - f(x)|\,dx \le \varepsilon$$

$$f_\varepsilon(x) \xrightarrow[\text{ср.}]{} f(x)$$

$$f_\varepsilon(x+t) \xrightarrow[\text{ср.}]{} f(x+t)$$

$$\int\limits_{\mathbb{R}} |f_\varepsilon(x+t) - f_\varepsilon(x)|\,dx =$$

$\bigsqcup\limits_{n=1}^{N} (X_n + t) = A \qquad \bigsqcup\limits_{l=1}^{N} X_l = B$

$$= \underbrace{\int_{A\Delta B}}_{\substack{t\to 0 \\ \downarrow \\ 0}} + \int_{A\cap B} + \underbrace{\int_{\mathbb{R}\setminus(A\cup B)}}_{\substack{t\to 0 \\ \downarrow \\ 0}}$$

$$A\cap B = \bigcup_{n,l=1}^{N} \underbrace{(X_n+t)\cap X_l}_{X_{n,l}}$$

$$x \in X_{n,l} \Rightarrow f_\varepsilon(x+t) - f_\varepsilon(x) = 0$$

$$\int_{A\cap B} |\cdots| dx \le \sum_{n,l=1}^{N} \underbrace{\int_{X_{n,l}}}_{0} |\cdots| dx = 0$$

$$\Rightarrow f_\varepsilon(x+t) \xrightarrow[\text{f.ü.}]{} f_\varepsilon(x), \ t\to 0$$

$$\Rightarrow \lim_{t\to 0} \int_{\mathbb{R}} |f(x+t) - f(x)| dx = 0$$

Т.29 $f: X \to \mathbb{R}$ изм.; X - изм. мн. $\subset \mathbb{R}^n$

$$R_k = \{k \le |f| < k+1\}$$

Д-ть: f инт. $\Longleftrightarrow \sum\limits_{k=1}^{\infty} k\mu(R_k) < +\infty$

Д-во $\Longleftarrow$ f инт. $\Longleftrightarrow |f|$ инт.

$$X_k := R_k$$

$$X = \bigsqcup_{k=1}^{\infty} X_k$$

$$\sum_{k=1}^{\infty} k\mu(R_k) \le$$

$$\le \sum_{k=1}^{\infty} \inf_{X_k} |f| \mu(X_k) \le \int_X f\,dx < +\infty$$

$\Leftarrow$

$$\int_X f\,dx = \sum_{k=1}^{\infty}\int_{X_k} f\,dx \le$$

$$\le \sum_{k=1}^{\infty}\sup_{X_k}\{|f|\}\,\mu X_k \le$$

$$\le \sum_{k=1}^{\infty}(k+1)\,\mu X_k = \mu X + \sum_{k=1}^{\infty} k\,\mu X_k$$

$$\le \underset{\substack{\wedge\\ +\infty}}{\mu X} + \sum_{k=1}^{\infty} \underset{\substack{\wedge\\ +\infty}}{k\,\mu R_k} < +\infty$$

T. 30* $\quad f: X \to \mathbb{R}$ изм. огр.

$\mathbb{R}^n \supseteq X$ изм. огр.

$$f(X) \subset [a, b]$$

$T: \quad a = y_0 < y_1 < \dots < y_I = b$

$$R_i = \{y_{i-1} \le f < y_i\}$$

$$\lambda(T) = \max_{i \in \overline{1,I}} (y_i - y_{i-1})$$

$$\Lambda(T) = \sum_{i=1}^{I} y_i \,\mu(R_i)$$

<u>Ø-ть</u> $\displaystyle\lim_{\lambda(T)\to 0} \Lambda(T) = \int_X f(x)\,dx$

то есть $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall T: \lambda(T) < \delta \Rightarrow$

$$\left| \Lambda(T) - \int_X f\,dx \right| < \varepsilon$$

<u>Ø-во</u> $\exists$ эл.-сум. f_ε: $f_\varepsilon \xrightarrow[\text{ср.}]{} f$,

т.е. $\displaystyle\int_X |f_\varepsilon - f|\,dx < \varepsilon/2$

$$f_\varepsilon = \sum_{k=1}^{N} c_k \chi_{\Pi_k}$$

Xory: $\left| \sum_{k=1}^{N} c_k \mu(\Pi_k) - \sum_{i=1}^{I} y_i \mu(f^{-1}[y_{i-1}, y_i)) \right|$

$$< \varepsilon/2$$

$f^{-1}[y_{i-1}, y_i)) - y\mu. \Rightarrow$

$\exists \mu.\ A_i:\ \mu(f^{-1}[y_{i-1}, y_i) \Delta A_i) < \varepsilon_i$

$$\left| \sum_{i=1}^{I} y_i (\mu(f^{-1}[y_{i-1}, y_i)) - \mu A_i) \right| < \varepsilon/4$$

$$\left| \sum_{k=1}^{N} c_k \mu \Pi_k - \sum_{i=1}^{I} y_i \mu A_i \right| \overset{?}{<} \varepsilon/4$$

$$\left| \mu \Pi_k - \sum_{i=1}^{I} \mu(\Pi_k \cap A_i) \right| < \varepsilon_k$$

$$\sum_{k=1}^{N}\sum_{i=1}^{I} |\underbrace{c_k - y_i}_{=0}|\, \mu(\Pi_k \cap A_i) < \varepsilon/4$$

$$\Longrightarrow \left| \Lambda(T) - \int_X f\,dx \right| < \varepsilon$$

Zyg!

54(5) $$\frac{d}{dx}\int_{x^2}^{x^3} \frac{dt}{\sqrt{1+t^4}} =$$

$$= \frac{3x^2}{\sqrt{1+x^{12}}} - \frac{2x}{\sqrt{1+x^8}}$$

108(1) $$\frac{1}{10\sqrt{2}} < \int_0^1 \frac{x^9}{\sqrt{1+x}}\,dx < \frac{1}{10}$$

$$\int_0^1 \frac{x^9}{\sqrt{1+x}}\,dx < \int_0^1 x^9\,dx = \frac{1}{10}$$

$$\vee$$

$$\int_0^1 \frac{1}{\sqrt{2}}\,x^9\,dx$$

$$\|$$

$$\frac{1}{10\sqrt{2}}$$

112 (1, 2)

1) $$\int_{-1}^1 \frac{d}{dx}\left(\operatorname{arctg}\frac{1}{x}\right)dx =$$

$$= \operatorname{arctg}\frac{1}{x}\Big|_{-1}^{1} = \frac{\pi}{2}$$

$\operatorname{arctg}\frac{1}{x}$ не опр. в 0.

2) $$\int_{-1}^1 \frac{dx}{x} = \ln|x|\Big|_{-1}^{1} = 0$$

$f(x) = \frac{1}{x}$ не огр. $\Rightarrow$ не инт. по Рим.

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$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}\right)=$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}\cdot\frac{1}{n} = \int_1^2 \frac{1}{x}\,dx = \ln 2$$

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f интегр. на $[-\ell, \ell]$.

Д-ть: 1) f – чет. $\Rightarrow \int_{-\ell}^{\ell} f\,dx = 2\int_0^{\ell} f\,dx$

2) f – нечет. $\Rightarrow \int_{-\ell}^{\ell} f\,dx = 0$

Д-во: 1) $I = \int_{-\ell}^{\ell} f\,dx =$

$$= \int_{-l}^{0} f\,dx + \int_{0}^{l} f\,dx \ominus$$

$$\Vert$$

$[x = -t]$

$$\int_{0}^{l} f\,dt$$

$$\ominus\ 2\int_{0}^{l} f\,dx$$

2) $$I = \int_{-l}^{0} f\,dx + \int_{0}^{l} f\,dx \ominus$$

$$\Vert$$

$[x = -t]$

$$-\int_{0}^{l} f(t)\,dt$$

$$\ominus\ 0.$$

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$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dx}{a\cos^2x+2b\cos x\sin x+c\sin^2x}\qquad ac-b^2>0$$

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sec^2x\,dx}{a+2b\,\mathrm{tg}\,x+c\,\mathrm{tg}^2x}=$$

$$=\int_{-\infty}^{\infty}\frac{dy}{cy^2+2by+a}=$$

$$=\frac{1}{c}\int_{-\infty}^{\infty}\frac{dy}{\left(y+\frac{b}{c}\right)^2+\left[\sqrt{\frac{ac-b^2}{c^2}}\right]^2}=$$

$$=\frac{1}{\cancel{c}}\frac{\cancel{c}}{\sqrt{ac-b^2}}\pi=\frac{\pi}{\sqrt{ac-b^2}}$$

44. Д-ть: $\forall a>0 \quad \lim\limits_{b\to\infty} \int\limits_b^{a+b} \frac{\sin x}{x} dx = 0$

$$\lim_{b\to\infty} \int\limits_b^{a+b} \frac{\sin x}{x} dx = I$$

$x = bt$

$$\lim_{b\to\infty} \int\limits_1^{1+\frac{a}{b}} \frac{\sin bt}{bt} b\,dt = \lim_{b\to\infty} \int\limits_1^{1+\frac{a}{b}} \frac{\sin bt}{t} dt$$

$$f(t) = \begin{cases} \frac{1}{t}, & t \in [1, 1+\frac{a}{b}] \\ 0, & \text{иначе} \end{cases}$$

$$I = \lim_{b\to\infty} \int\limits_{\mathbb{R}} f(t) \sin bt\, dt$$

$f(t)$ — инт. по Лебегу.

$\Rightarrow$ по Т.27 $\quad I = 0$

50. 3) $\frac{1}{3} < \int_0^1 3^{-x}\arccos x\,dx < 1$

$$\frac{1}{3}\int_0^1 \arccos x\,dx < \int_0^1 3^{-x}\arccos x\,dx < \int_0^1 \arccos x\,dx$$

$$\frac{1}{3}\int_0^1 \arccos x\,dx = -\frac{1}{3}\int_{\frac{\pi}{2}}^{0} t\sin t\,dt = \cancelto{0}{\frac{1}{3}t\cos t\Big|_{\frac{\pi}{2}}^{0}} - \frac{1}{3}\sin t\Big|_{\frac{\pi}{2}}^{0} = \frac{1}{3}$$

$$\int_0^1 \arccos x\,dx = 1$$

Т.33 $\exists\ \xi \in [-1,1]$ т.ч.

$$\int_{-1}^{1} f(x)\, g(x)\, dx = f(\xi) \int_{-1}^{1} g(x)\, dx,$$

$f : [-1,1] \to \mathbb{R}$ — непр. и

а) $g(x) = x$ б) $g(x) = \cos x^2$?

Д-во: б)

$$h(y) = \int_{-1}^{1} g(x)\big(f(x) - f(y)\big)\, dx =$$

$h(y)$ непр. за $[-1,1]$

$g(x) > 0$

$g(x)\big(f(x) - f(y_{min})\big) \geq 0 \quad \forall x$

$$g(x)\,(f(x)-f(y_{max})) \le 0 \quad \forall x$$

$$\Rightarrow h(y_{min})\cdot h(y_{max}) \le 0$$

„=" 0 $\Rightarrow \xi = y_{min}\ (y_{max})$

иначе $\exists\, \xi$ между y_{min} и y_{max}:

$$h(\xi)=0.$$

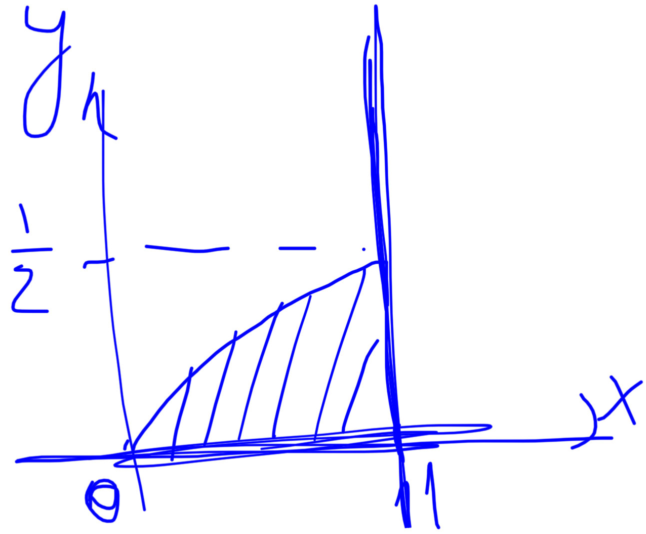
чтд!

а) $g(x)=x \rightarrow$ нет

Контрпример: $f(x)=x$

$$1 = \int_{-1}^{1} f g\,dx = f(\xi)\int_{-1}^{1} g(x)\,dx = 0 \ (?!)$$

4.3) $y = \frac{\sqrt{x}}{1+x^3}$, $y=0$, $x=1$



$$S = \int_0^1 \frac{\sqrt{x}}{1+x^3}\,dx = \left(\begin{matrix} x = t^2 \\ dx = 2t\,dt \end{matrix}\right) =$$

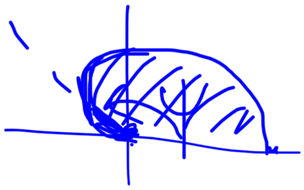
$$= 2\int_0^1 \frac{t^2\,dt}{1+t^6} = \frac{2}{3}\int_0^1 \frac{dy}{1+y^2} =$$

$$= \frac{2}{3}\,\frac{\pi}{4} = \frac{\pi}{6}$$

33 (4, 5)

4) $r = b + a\cos\varphi, \; a > b > 0$ (улитка)

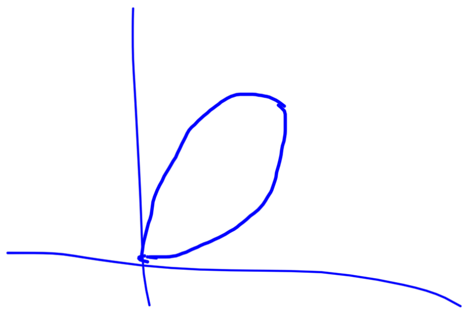
$\psi = \cos^{-1}(b/a)$



$$S = \frac{1}{2}\int_0^{\psi} r^2 \, d\varphi =$$

$$= \frac{b^2\psi}{2} + ab\sin\psi + \frac{a^2\psi}{4} + \frac{a^2\sin 2\psi}{8}$$

5) $r = a\sin 2\varphi$



$$S = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \, d\varphi = \frac{1}{4}a^2\frac{\pi}{2} - \frac{1}{8}a^2 =$$

$$= \frac{a^2}{8}(\pi - 1)$$

7213,5)

3) $x = a(\cos\varphi + \varphi\sin\varphi)$,

$y = a(\sin\varphi - \varphi\cos\varphi)$

$0 \leq \varphi \leq \varphi_0$

$$L = \int_0^{\varphi_0} \sqrt{x_\varphi'^2 + y_\varphi'^2}\, d\varphi =$$

$$= a\int_0^{\varphi_0} \sqrt{(\varphi\cos\varphi)^2 + (\varphi\sin\varphi)^2}\, d\varphi =$$

$$= a\int_0^{\varphi_0} \varphi\, d\varphi = \frac{a\varphi_0^2}{2}$$

5) $x = ch^3 t$, $y = sh^3 t$, $0 \leq t \leq t_0$

$$l = \int_0^{t_0} \sqrt{x_t'^2 + y_t'^2}\, dt =$$

$$= 3\int_0^{t_0} \sqrt{(ch^2 t \; sh\, t)^2 + (sh^2 t \; ch\, t)^2}\,dt$$

$$= 3\int_0^{t_0} \underbrace{sh\, t \; ch\, t}_{\frac{du}{4}} \sqrt{\underbrace{ch^2 t + sh^2 t}_{u}}\,dt =$$

$$= \frac{3}{4}\int_1^{u_0} \sqrt{u}\,du = \frac{\cancel{3}}{\cancel{4}_{2}}\,\frac{\cancel{2}}{\cancel{3}}\left(u_0^{3/2} - 1\right) =$$

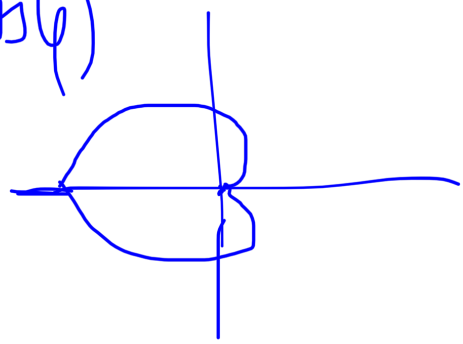
$$= \frac{1}{2}\left(\left(ch^2 t_0 + sh^2 t_0\right)^{3/2} - 1\right) =$$

$$= \frac{1}{2}\left(\left(ch\, 2t_0\right)^{3/2} - 1\right)$$

<u>§2 (3)*</u> $\quad r = a(1-\cos\varphi)$

$$\shortparallel$$

$$2a\sin^2\frac{\varphi}{2}$$



$$l = \int \sqrt{dr^2 + r^2 d\varphi^2} =$$

$$= \int_0^{\pi} \sqrt{r_\varphi'^2 + r^2}\, d\varphi =$$

$$= a \int_0^{\pi} \sqrt{\sin^2\varphi + 1 - 2\cos\varphi + \cos^2\varphi}\, d\varphi =$$

$$= 2a \int_0^{\pi} \sin\frac{\varphi}{2}\, d\varphi =$$

$$= -4a \cos\frac{\varphi}{2}\Big|_0^{\pi} = 4a$$

Т.34 $\forall k \in \mathbb{N}$ $f_k : [0,1] \to \mathbb{R}$ непр.

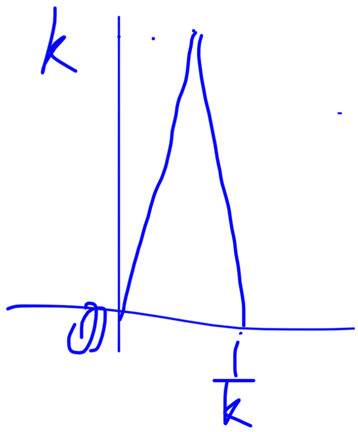
$\forall x \in [0,1]$ $\exists \lim\limits_{k\to\infty} f_k(x) = f(x) \in \mathbb{R}$

Верно ли, что

$$\lim_{k\to\infty} \int_0^1 f_k(x)\,dx = \int_0^1 f(x)\,dx$$

а) в общем случае → нет.

Контрпример:



$$\lim_{k\to\infty} f_k = 0$$

$$\underbrace{\lim_{k\to\infty} \int_0^1 f_k\,dx}_{= \frac{1}{2}} = \int_0^1 f\,dx = 0$$

(?!)

б) $\forall x \in [0,1]$ $\{f_k(x)\}$ мон. по k.

$\forall x \quad f_k \geq 0 \quad \{f_k\} \downarrow$

$\forall x \quad f(x) = \inf\limits_{k \in \mathbb{N}} \{f_k\}$

$\left\{ \int_0^1 f_k(x)\,dx \right\} \downarrow$

$\lim\limits_{k \to \infty} \int_0^1 f_k(x)\,dx = \inf\limits_{k \in \mathbb{N}} \left\{ \int_0^1 f_k(x)\,dx \right\} \overset{?}{=}$

$\overset{?}{=} \int_0^1 \inf\limits_{k \in \mathbb{N}} \{f_k(x)\}\,dx$

$f(x) \leq f_k(x) \quad \forall x \quad \forall k$ "

$\forall \varepsilon > 0 \quad \exists N, \quad \forall n \geq N \quad f(x) > f_k - \varepsilon$

$\Rightarrow \int_0^1 f\,dx\,dx \geq \int_0^1 f_k\,dx - \varepsilon$

$\Rightarrow \int f\,dx \le \int f_k\,dx \quad \forall k$

$\Rightarrow \quad \dots \quad$ ч.т.д.

(Ответ: да)

б) $\sup\limits_{k\in\mathbb{N}} \sup\limits_{x\in[0,1]} |f_k(x)| < +\infty$

Ответ: да

Д-во:

$\forall x \quad \lim\limits_{k\to\infty} f_k(x) = f(x) \in \mathbb{R} \iff$

$\forall \varepsilon > 0 \quad \exists N \quad \forall n \ge N \quad |f_n(x) - f(x)| < \varepsilon \quad \forall x$

$\Rightarrow \left| \int\limits_0^1 (f_n - f)\,dx \right| \le \int\limits_0^1 |f_n - f|\,dx \le \varepsilon$

2)* $\forall k \in \mathbb{N}$ $f_k(x)$ мон. возр

НУО $f_k(x) \uparrow$

Ответ: да

<u>Д-во:</u>

$$g(x) = \sup_{k \in \mathbb{N}} |f_k(x)|$$ — непр. на $[0,1]$ ф-ция

$\Rightarrow \forall k$ $|f_k| \leq g$ $\Rightarrow$ по т. Лебега

$$\lim_{k \to \infty} \int_0^1 f_k \, dx = \int_0^1 f \, dx$$

Т. 35* $f: [a,b] \to \mathbb{R}$ непр.,

$\exists M:$

дифф. на (a,b), и $|f'| < M$,

то $f(b) - f(a) = \int\limits_a^b f'(t)\,dt$

Д-во: Докажем сначала, что если

f — инт. на $[a,b]$ и непр. в $x_0 \in [a,b]$

$F(x) = \int\limits_a^x f(t)\,dt$

$\Rightarrow \exists F'(x_0) = f(x_0)$

fix $\varepsilon > 0$.

$\exists \delta > 0 \quad \forall x \in \overset{\circ}{U}_\delta(x_0) \quad |f(x) - f(x_0)| < \varepsilon$

$F(x) - F(x_0) = \int\limits_{x_0}^{x} f(t)\,dt$

$$F(x)-F(x_0)-(x-x_0)f(x_0)=$$

$$=\int_{x_0}^{x}(f(t)-f(x_0))dt$$

$$\Rightarrow \forall x\in \mathring{U}_\delta(x_0)$$

$$\left|\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)\right|=$$

$$=\frac{1}{|x-x_0|}\left|\int_{x_0}^{x}(f(t)-f(x_0))dt\right|\le$$

$$\le\frac{1}{|x-x_0|}\int_{x_0}^{x}|f(t)-f(x_0)|dt\le\varepsilon$$

✓

Следствие 1: любая первообр. f им. вид

$$F(x)=\int_a^x f(t)\,dt + C, \quad C\in\mathbb{R}$$

Следствие 2:

$$\int_a^b f(t)\,dt = F(b)-F(a)$$

Возвращаясь к изначальной задаче стоит отметить, что условия для ф-лы Ньютона-Лейбница почти выполнены ($\exists f'(x_0)\in\mathbb{R}$ всм. почти всюду)

$\Rightarrow \checkmark$.