Bruce C. Berndt

Ramanujan's Notebooks

Part IV

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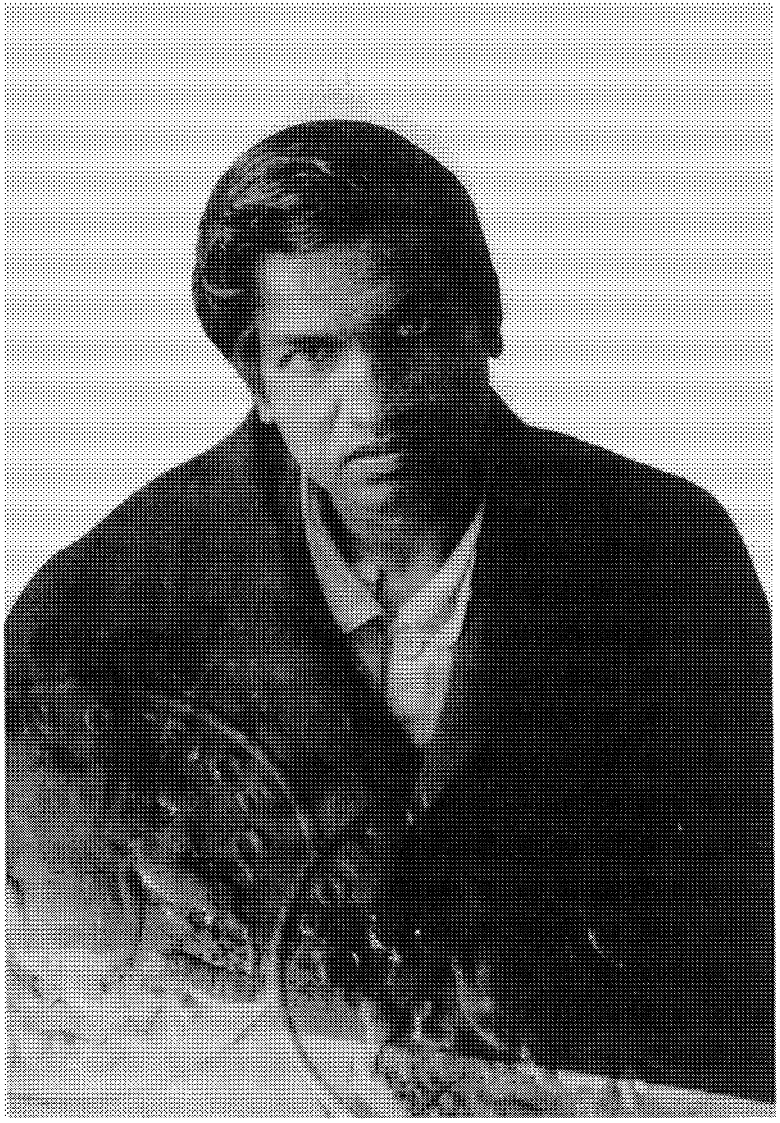

Passport photo of Srinivasa Ramanujan.
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Preface

During the years 1903–1914, Ramanujan recorded most of his mathematical discoveries without proofs in notebooks. Although many of his results were already found in the literature, most were not. Almost a decade after Ramanujan's death in 1920, G. N. Watson and B. M. Wilson began to edit Ramanujan's notebooks, but they never completed the task. A photostat edition, with no editing, was published by the Tata Institute of Fundamental Research in Bombay in 1957.

This book is the fourth of five volumes devoted to the editing of Ramanujan's notebooks. Part I, published in 1985, contains an account of Chapters 1–9 in the second notebook as well as a description of Ramanujan's quarterly reports. Part II, published in 1989, comprises accounts of Chapters 10–15 in the second notebook. Part III, published in 1991, provides an account of Chapters 16–21 in the second notebook. This is the first of two volumes devoted to proving the results found in the unorganized portions of the second notebook and in the third notebook. We also shall prove those results in the first notebook that are not found in the second or third notebooks. For those results that are known, we provide references in the literature where proofs may be found. Otherwise, we give complete proofs.

Urbana, Illinois *Bruce C. Berndt*
February, 1993

Contents

Introduction

> If you have built castles in the air, your work need not be lost; that is where they should be. Now put the foundations under them.
>
> H. D. Thoreau—*Walden*

Ramanujan built many castles. Although some may have been lost, most were preserved. Since his death in 1920, many mathematicians have been constructing the foundations for these magnificent structures. We continue this task in the present volume.

In the first three volumes of this series (Berndt [2], [4], [6]), we gave references or proofs for all of the results claimed by Ramanujan in the 21 chapters of his second notebook. At the end of this notebook, after the 21 chapters of organized material, there are exactly 100 pages of unorganized results. In the Tata Institute's publication (Ramanujan [22]) of the second notebook, three pages of further results precede the 21 chapters. Between 1920 and the publication of [22] in 1957, these three pages evidently were shifted from the unorganized part to the beginning, for in G. N. Watson's copy of the second notebook, the three pages appear among the unorganized pages at the end of the second notebook. Ramanujan's third notebook, published by the Tata Institute in the same volume as the second notebook, contains only 33 pages of unorganized material. In this volume and the next (Berndt [9]), we provide proofs or give references for all of the results found in the unorganized pages of the second and third notebooks.

The first notebook, published as volume 1 by the Tata Institute, is a preliminary version of the second. However, the first notebook contains several results not found in the second. In Parts IV and V, we also provide proofs for these theorems as well.

Since the results in the aforementioned 136 pages of miscellaneous material in the second and third notebooks were not organized by Ramanujan into chapters, we have taken the liberty of doing so. Generally, within each chapter, we have recorded the results in chronological order as they appear in the unorganized pages. This volume contains ten of the fifteen chapters into which we have organized Ramanujan's theorems. Part V will contain chapters on continued fractions, Ramanujan's theory of elliptic functions to alternative bases, class invariants and singular moduli, asymptotic analysis and approximations, and infinite series.

In this volume, we also provide an account of the 16 chapters of organized material in the first notebook. More precisely, for each result in the first notebook that can be found in the second, we indicate where in the second notebook the corresponding result is located. Furthermore, we provide proofs for those theorems in the 16 chapters of Notebook 1 that cannot be found in the second notebook. Most of the results in these 16 chapters that Ramanujan failed to record in Notebook 2 are either wrong or relatively easy to prove. However, some are more interesting and more challenging to prove. The miscellaneous material in the first notebook contains substantially more results not found in the second notebook; we prove these in Part V.

Brief descriptions of the contents of the ten chapters in this volume will now be given.

The first chapter, Chapter 22, is devoted to elementary results. Most require only high school algebra to prove. Despite the modest title, Chapter 22 contains many very interesting results. Several were submitted by Ramanujan as problems in the *Journal of the Indian Mathematical Society*. Many entries in the chapter pertain to polynomial equations or systems of equations. In particular, we mentioned Entries 4, 5, and 32 on certain systems of equations wherein the solutions are represented as infinite nested radicals.

Although Ramanujan is known to most mathematicians as a number theorist, Chapters 1–21 in the second notebook contain little number theory, although much of this material, for example, the chapters on theta-functions, is related to number theory. Ramanujan's interest in number theory appears to have commenced only one or two years before he wrote G. H. Hardy in January, 1913. Most of the discoveries in number theory that Ramanujan made before departing for England in 1914 are found in the 136 pages of miscellaneous material in the second and third notebooks. Chapter 23 contains results in number theory, except that discoveries in the theory of prime numbers are reserved for the following chapter. While in England under the influence of Hardy, Ramanujan worked primarily on number theory. As we shall see in Chapters 23 and 24, many of the principal ideas in Ramanujan's papers in the theory of numbers published in England had their geneses in India. Thus, most of Chapters 23 and 24 concerns previously published material. Fortunately, in the unorganized pages of the second and third notebooks, Ramanujan provides sketches of some of his methods. In particular, Ramanujan indicates his proof of the asymptotic formula for the number of integers less than or equal to x that can be represented as a sum of two squares. He also sketches a general method which he undoubtedly employed in determining a formula for the coefficients of the reciprocal of the classical theta-function and later used to obtain the first version of the famous Hardy–Ramanujan asymptotic formula for the partition function $p(n)$.

Nonetheless, Chapter 23 contains some fascinating new theorems. Perhaps one of the two most interesting results in the chapter is the new theorem in Entry 18 on representing primes by certain quadratic forms. This result was recently proved for us by K. S. Williams [2]. Another amazing result is Entry 45, which is a version of an astounding polynomial identity in two variables. Chapter 23 contains several results on equal sums of powers, with Entries 43 and 45 being the two most interesting results. We think that S. Bhargava's beautiful proof [2] of Entry 45 is the one Ramanujan must have found.

As indicated above, Chapter 24 is devoted to Ramanujan's theory of prime numbers. Except for a couple of theorems, most of the material is not new. The unorganized pages of the second and third notebooks in conjunction with the four letters that Ramanujan wrote to Hardy before departing for England (Berndt and Rankin [1]) provide considerable insight into Ramanujan's thinking about primes.

In contrast to Chapters 23 and 24, almost all of the results in the long chapter on theta-functions and modular equations (Chapter 25) are new. Entries 23–26 provide Lambert series identities of a type that we have never seen before. G. E. Andrews [5] has devised proofs of these remarkable theorems, which we reproduce here. Entry 32 gives theta-function identities of a sort unlike the myriad of other theta-function identities found in the literature. Entries 51–72 offer beautiful eta-function identities. Although several eta-function identities have appeared in the literature, none are as symmetric and elegant as those of Ramanujan in these entries.

All the results in the short Chapter 26 are new and unlike any other results found in the literature. Ramanujan offers ten theorems on the inversion of the lemniscate integral and similar integrals.

Chapter 27 is on q-series, and undoubtedly the highlight of this chapter is Entry 7, which provides the asymptotic behavior of a large class of q-series as q tends to $1-$. We reproduce R. McIntosh's [1] recent and elegant proof of this wonderful theorem.

Chapter 28 is devoted to integrals, and, somewhat surprisingly, many of the evaluations appear not to have been given previously. Our favorite result is Entry 41, which is related to the dilogarithm and which was greatly generalized by the author and R. J. Evans [1].

Chapter 29 is devoted to special functions, and little is new here. However, there is one remarkable new theorem; Entry 15 provides the evaluation of a large class of 0-balanced hypergeometric series. We are pleased to present D. Bradley's [1] exquisite proof of this theorem.

Partial fraction decompositions constitute the sole topic of Chapter 30. Although the results are not deep, some of the calculations are rather challenging, and the results evince beautiful symmetry.

We have placed in Chapter 31 several results of either an elementary or miscellaneous nature in analysis. Some are of interest because they indicate the lack of Ramanujan's theoretical training.

We conclude this book with a description of the 16 organized chapters comprising Ramanujan's first notebook. Most of the new results are found in Chapters 12–16. Many pertain to hypergeometric functions. The last entry of Chapter 16 provides a generalization of Ramanujan's famous formula for $\zeta(2n+1)$.

Different people will have different counts for the number of results in each of the chapters described above, for several entries and examples have more than one part. Generally, we regard a raft of related examples as a single result. With this in mind, the following table provides our count of the number of results in each chapter.

Chapter	Number of Results
22	47
23	108
24	24
25	86
26	10
27	9
28	63
29	39
30	15
31	36
16 chapters of notebook 1	54
Total	491

Many of the theorems that Ramanujan communicated in his letters of January 16, 1913 and February 27, 1913 to Hardy may be found in Chapters 22–31. We list these results in the following table.

Location in *Collected Papers*	Location in *Notebooks*
Most claims in prime number theory on pages xxiii, xxvii, 349, 351, 352	Chapter 24
p. xxiv, (1)	Chapter 23, Entry 15
p. xxiv, (3)	Chapter 23, Entry 9
p. xxiv, (4)	Chapter 23, Entry 13
p. xxvi, VI (6)	Chapter 28, Entry 4

Some of Ramanujan's published papers and questions posed to readers of the *Journal of the Indian Mathematical Society* have their origins in

Chapters 22–31. In some instances, only a small portion of the paper actually arises from material in the notebooks. The following table lists these papers and questions and the corresponding locations in the chapters of this volume.

Paper or Question	Location in Part IV
[1] Question 283	Chapter 22, Entry 19
[2] Some properties of Bernoulli's numbers	Chapter 23, tables between Sections 20 and 21; Entries 29–33
[3] Note on a set of simultaneous equations	Chapter 22, Entry 19
[4] Question 284	Chapter 22, Entry 17
[5] Question 463	Chapter 28, Entry 10
[6] Question 507	Chapter 22, Entries 4, 5
[7] Question 524	Chapter 22, Entry 29
[8] Question 441	Chapter 23, Entry 4
[9] Question 605	Chapter 29, Corollaries 8.1, 8.2
[10] Modular equations and approximations to π	Chapter 22, Entry 33; Chapter 23, Entries 35–37; Chapter 29, Entries 20, 21
[11] On the product $\prod_{n=0}^{n=\infty}\left[1+\left(\frac{x}{a+nd}\right)^3\right]$	Chapter 29, Entries 1–4
[12] Question 722	Chapter 22, Section 32
[13] Some definite integrals	Chapter 28, Entries 4, 30, 31
[14] Some definite integrals connected with Gauss's sums	Chapter 28, Entries 9–12, 27
[15] New expressions for Riemann's functions $\xi(s)$ and $\Xi(t)$	Chapter 28, Entry 4
[16] Highly composite numbers	Chapter 23, Tables at beginning; Entry 28 and following table; Section 34
[18] Question 427	Chapter 22, Entry 3
[19] Question 723	Chapter 23, Entries 22–24
[20] Question 783	Chapter 28, Entry 41

In the sequel, equation numbers refer to equations in the same chapter, unless another chapter is indicated. Unless otherwise stated, in Chapters 22–31, page numbers refer to those in the pagination of the Tata Institute's publication of Ramanujan's notebooks [22]; occasionally these page numbers will be unattended by the reference [22]. Parts I, II, III, and V refer to the author's accounts [2], [4], [6], and [9], respectively, of Ramanujan's notebooks.

Some of the results in this volume were previously published by the author, usually in collaboration. Entry 45 of Chapter 23 is the subject of a paper by the author and Bhargava [2]. A considerably shorter, preliminary version of Chapter 24 was published in [3]. Some of the results in Chapter

25 were published by the author [9] and the author and L.-C. Zhang [1], [2]. Chapter 26 forms the major portion of a paper by the author and Bhargava [1]. Entry 6 of Chapter 27 was proved by the author in [5]. Entry 41 and its generalization comprise a paper coauthored with Evans [1].

Many mathematicians have supplied proofs and comments for several entries of this volume. G. E. Andrews found formidable proofs for four of Ramanujan's most difficult formulas in Chapter 25. R. A. Askey offered helpful comments on most chapters. S. Bhargava both individually and in collaboration proved several of Ramanujan's bewildering formulas. D. Bradley deciphered and proved Ramanujan's enigmatic claim on hypergeometric series in Chapter 29. As in the three previous volumes, R. J. Evans supplied proofs of difficult formulas. R. McIntosh found a much better proof than the author's for Ramanujan's asymptotic formula for q-series. B. Reznick supplied both proofs and valuable remarks. K. Venkatachaliengar offered a nice proof and helpful comments. F. Wheeler, W. Galway, and J. Keiper provided a thorough investigation of Entry 2 in Chapter 24. K. S. Williams devised a beautiful proof of Entry 18 in Chapter 23. L.-C. Zhang collaborated with the author on some of Ramanujan's difficult theta-function identities. For proofs and comments, we also are indebted to P. T. Bateman, H. G. Diamond, K. Ford, R. W. Gosper, D. Grayson, H. Halberstam, A. Hildebrand, D. K. Mick, G. Myerson, M. Newman, A. M. Odlyzko, K. G. Ramanathan, R. A. Rankin, J. L. Selfridge, D. Shanks, J. Steinig, S. S. Wagstaff, Jr., and W. B. Zhang. All of these mathematicians made important contributions to this volume, and we are very grateful to each of them.

G. N. Watson found proofs for some of the formulas established in this volume. We thank the Master and Fellows of Trinity College, Cambridge for a copy of Watson's notes.

Nancy Anderson, Mathematics Librarian at the University of Illinois, helped unearth several references for us, and we thank her for her generous help.

The manuscript for this book was typed by Hilda Britt, Dee Wrather, and the author, and we thank them for their superb and efficient typing.

The author bears the responsibility for all the errors in this volume and wishes to be notified of such, whether they be minor or serious.

CHAPTER 22

Elementary Results

In this chapter we will examine Ramanujan's findings that require only elementary high-school algebra for an understanding. A few of the results, e.g., Entries 5 and 32, however, require some knowledge of calculus for the proofs. Chapter 23, which examines Ramanujan's discoveries in number theory, with the exception of his work on prime numbers, also contains some theorems that are accessible to those with only a background in elementary algebra. The latter results primarily involve equal sums of powers, and so it seems appropriate to place them in a chapter on number theory rather than in the present chapter. Possibly a few results in Chapter 22 fit better in Chapter 31 on elementary analysis, and possibly some would move a few results from that chapter to this chapter. Thus, the choice of material for this chapter is admittedly somewhat arbitrary.

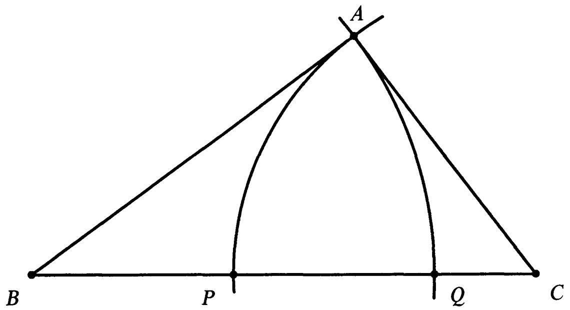
Several theorems in this chapter concern solutions of sytems of equations. In particular, we mention Entries 4, 17, 18, 19, and 32. Entries 4 and 32 are particularly interesting since they lead to infinite nested radicals. Several examples are given to illustrate the beauty of these results.

Entry 20 appears to give a new method for solving quartic equations. Although a certain classical algorithm for solving quartic equations depends upon solving a cubic equation, Ramanujan's method for reducing a quartic equation to a cubic equation seems to be different.

Some of the results in this chapter are related to problems that Ramanujan posed for readers of the *Journal of the Indian Mathematical Society*. Generally, the references to Ramanujan's problems at the end of this book refer to pages where the solutions were printed, if, indeed, a solution was published.

We have found it useful at times to use the computer algebra system *Mathematica*, which is listed in the references under its founder S. Wolfram.

At the top of page 263, the following geometrical figure is found:



The triangle ABC is a right triangle. The two curves are arcs of circles centered at B and C. Thus, $c := BA = BQ$ and $b := CA = CP$. Let $a = BC$. Underneath the figure, three equalities are given. (We have changed the notation in (1.2) below.)

Entry 1 (p. 263). *In the notation above,*

$$PQ^2 = 2BP \cdot QC \tag{1.1}$$

and

$$(b + c - \sqrt{b^2 + c^2})^2 = 2(\sqrt{b^2 + c^2} - b)(\sqrt{b^2 + c^2} - c). \tag{1.2}$$

Also, for any numbers a and b,

$$\{\sqrt[3]{(a + b)^2} - \sqrt[3]{a^2 - ab + b^2}\}^3 = 3(\sqrt[3]{a^3 + b^3} - a)(\sqrt[3]{a^3 + b^3} - b). \tag{1.3}$$

Proof. The following proof is due to T. Dharmarajan and P. K. Srinivasan [1, pp. 11–13].

First,

$$AB^2 = BQ^2 = (BP + PQ)^2 = BP^2 + PQ^2 + 2BP \cdot PQ. \tag{1.4}$$

Similarly,

$$AC^2 = CQ^2 + QP^2 + 2CQ \cdot QP. \tag{1.5}$$

By the Pythagorean theorem, (1.4), and (1.5),

$$\begin{aligned} BC^2 &= AB^2 + AC^2 \\ &= BP^2 + 2PQ^2 + CQ^2 + 2BP \cdot PQ + 2CQ \cdot PQ. \end{aligned} \tag{1.6}$$

On the other hand,

$$\begin{aligned} BC^2 &= (BP + PQ + QC)^2 \\ &= BP^2 + PQ^2 + CQ^2 + 2BP \cdot PQ + 2BP \cdot CQ + 2PQ \cdot CQ. \end{aligned} \tag{1.7}$$

Equating (1.6) and (1.7), we deduce (1.1).

Observe that $PQ = b + c - a$, $BP = a - b$, and $QC = a - c$. Substituting these expressions into (1.1), we deduce (1.2).

Of course, (1.2) is an algebraical identity which can also be proved by expanding both sides. Ramanujan's geometrical proof is very clever.

Identity (1.3) apparently has no connection with geometry. Ramanujan evidently sought a "cubic" analogue of (1.2).

Now,

$$\begin{aligned}\{\sqrt[3]{(a+b)^2} - \sqrt[3]{a^2 - ab + b^2}\}^3 &= (a+b)^2 - 3(a+b)^{4/3}(a^2 - ab + b^2)^{1/3}\\ &\quad + 3(a+b)^{2/3}(a^2 - ab + b^2)^{2/3}\\ &\quad - a^2 + ab - b^2\\ &= 3ab - 3(a+b)(a^3 + b^3)^{1/3}\\ &\quad + 3(a^3 + b^3)^{2/3}\\ &= 3\{(a^3 + b^3)^{1/3} - a\}\{(a^3 + b^3)^{1/3} - b\},\end{aligned}$$

which establishes (1.3).

Entry 2 (p. 263). *If*

$$Bk^4 - 4Ak^3 - 8Bkp - 4Ap = 0, \tag{2.1}$$

then

$$\sqrt{A + B\sqrt[3]{p}} = \sqrt{\frac{B}{p + k^3}}\,(\tfrac{1}{2}k^2 + k\sqrt[3]{p} - \sqrt[3]{p^2}). \tag{2.2}$$

Proof. Squaring both sides of (2.2), we find that (2.2) holds if and only if (2.1) is true.

Entries 1 and 2 were also verified by D. Somasundaram [5].

Entry 3 (p. 266). *If*

$$u = \tfrac{5}{2}(x+y)(p+q) - 2xp \quad \text{and} \quad v = 2qy - (x+y)(p+q),$$

then

$$(2x^2 + 3xy + 5y^2)(2p^2 + 3pq + 5q^2) = 2u^2 + 3uv + 5v^2.$$

Proof. We shall show that Entry 3 is a consequence of the following more general lemma.

Lemma 3.1. *Let*

$$w = \frac{(Dyq - zr) + D(yr + zq)}{A} \quad \text{and} \quad v = \frac{(Dyq - zr) - (yr + zq)}{A}. \tag{3.1}$$

Then

$$w^2 + Dv^2 = \frac{1 + D}{A^2}(z^2 + Dy^2)(r^2 + Dq^2).$$

PROOF. We have

$$\begin{aligned} A^2(w^2 + Dv^2) &= (1 + D)(Dyq - zr)^2 + (D + D^2)(yr + zq)^2 \\ &= (1 + D)(D^2y^2q^2 + z^2r^2 + Dy^2r^2 + Dz^2q^2), \end{aligned}$$

from which the desired result follows.

PROOF OF ENTRY 3. We shall apply Lemma 3.1 with $w = 4u + 3v$, $z = 4x + 3y$, $r = 4p + 3q$, $A = 16$, and $D = 31$. We first verify the lemma's hypotheses. By elementary calculations,

$$\frac{(Dyq - zr) + D(yr + zq)}{A} = 13yq - xp + 7xq + 7yp = 4u + 3v$$

and

$$\frac{(Dyq - zr) - (yr + zq)}{A} = yq - xp - xq - yp = v.$$

Hence, both equalities in (3.1) have been demonstrated. Applying Lemma 3.1, we find that

$$\begin{aligned} 8(2u^2 + 5v^2 + 3uv) &= (4u + 3v)^2 + 31v^2 \\ &= \tfrac{1}{8}((4x + 3y)^2 + 31y^2)((4p + 3q)^2 + 31q^2) \\ &= 8(2x^2 + 3xy + 5y^2)(2p^2 + 3pq + 5q^2), \end{aligned}$$

which completes the proof.

Entry 3 is also a question that Ramanujan [18], [23, p. 326] posed to readers of the *Journal of the Indian Mathematical Society*.

Entry 4 (p. 307). *Let*

$$x^2 = y + a, \qquad y^2 = z + a, \quad \text{and} \quad z^2 = x + a. \tag{4.1}$$

Then

$$x^3 + \tfrac{1}{2}x^2\{1 + \sqrt{4a - 7}\} - \tfrac{1}{2}x\{2a + 1 - \sqrt{4a - 7}\} + \tfrac{1}{2}\{a - 2 - a\sqrt{4a - 7}\} = 0 \tag{4.2}$$

and

$$x^3+\tfrac{1}{2}x^2\{1-\sqrt{4a-7}\}-\tfrac{1}{2}x\{2a+1+\sqrt{4a-7}\}+\tfrac{1}{2}\{a-2+a\sqrt{4a-7}\}=0. \tag{4.3}$$

It is impossible to read the blurred equations (4.2) and (4.3) in Ramanujan's Notebooks [22]. However, Entry 4 is a portion of a problem that Ramanujan [6], [23, pp. 327–329] submitted to the *Journal of the Indian Mathematical Society*. In fact, x is a root of a polynomial of degree 8. The quadratic factor x^2-x-a is easy to find. The polynomials in (4.2) and (4.3) are the remaining factors. Evidently, no solutions, other than Ramanujan's own solution, were received. Although the factors (4.2) and (4.3) are correctly given in Ramanujan's published solution [6], in his *Collected Papers* [23, p. 328], there are two incorrect signs. Since Ramanujan's solution is otherwise clear, we shall refrain from further commentary. However, in the sequel, we thoroughly discuss the roots of (4.2) and (4.3).

Theorem 4.1. *For brevity, set* $A=\sqrt{4a-7}$. *The roots of* (4.2) *are given by*

$$x_1:=-\frac{1+A}{6}-\tfrac{2}{3}\sqrt{4a-A}\,\sin\left(\tfrac{1}{3}\tan^{-1}\frac{2A-1}{3\sqrt{3}}\right),$$

$$x_2:=-\frac{1+A}{6}-\tfrac{2}{3}\sqrt{4a-A}\,\sin\left(\frac{\pi}{3}-\tfrac{1}{3}\tan^{-1}\frac{2A-1}{3\sqrt{3}}\right),$$

and

$$x_3:=-\frac{1+A}{6}+\tfrac{2}{3}\sqrt{4a-A}\,\sin\left(\frac{\pi}{3}+\tfrac{1}{3}\tan^{-1}\frac{2A-1}{3\sqrt{3}}\right),$$

and the solutions of (4.3) *are given by*

$$x_4:=\frac{A-1}{6}+\tfrac{2}{3}\sqrt{4a+A}\,\sin\left(\tfrac{1}{3}\tan^{-1}\frac{2A+1}{3\sqrt{3}}\right),$$

$$x_5:=\frac{A-1}{6}+\tfrac{2}{3}\sqrt{4a+A}\,\sin\left(\frac{\pi}{3}-\tfrac{1}{3}\tan^{-1}\frac{2A+1}{3\sqrt{3}}\right),$$

and

$$x_6:=\frac{A-1}{6}-\tfrac{2}{3}\sqrt{4a+A}\,\sin\left(\frac{\pi}{3}+\tfrac{1}{3}\tan^{-1}\frac{2A+1}{3\sqrt{3}}\right).$$

Proof. The representations of the roots in terms of trigonometric functions suggest that we use an algorithm, due to F. Viète (Dickson [1, pp. 36–37]), for solving cubic equations that depends upon the identity

$$\sin^3 \theta - \tfrac{3}{4} \sin \theta + \tfrac{1}{4} \sin 3\theta = 0. \tag{4.4}$$

We first consider (4.2). Making the substitution $x = s - (1 + A)/6$, and remembering that $A^2 = 4a - 7$, we find, after some elementary algebra and simplification, that

$$s^3 + \left(\frac{A - 4a}{3}\right)s + \frac{12a - 14 - A - 8Aa}{27} = 0. \tag{4.5}$$

Similarly, setting $x = s - (1 - A)/6$ in (4.3) and observing that the calculations are the same as above, but with A replaced by $-A$, we deduce that

$$s^3 - \left(\frac{A + 4a}{3}\right)s + \frac{12a - 14 + A + 8Aa}{27} = 0. \tag{4.6}$$

Next, letting $s = \frac{2}{3}t\sqrt{4a - A}$ in (4.5) and $s = \frac{2}{3}t\sqrt{4a + A}$ in (4.6), we deduce that, respectively,

$$t^3 - \tfrac{3}{4}t + \frac{12a - 14 - A - 8Aa}{8(4a - A)^{3/2}} = 0 \tag{4.7}$$

and

$$t^3 - \tfrac{3}{4}t + \frac{12a - 14 + A + 8Aa}{8(4a + A)^{3/2}} = 0. \tag{4.8}$$

Now set $t = \sin \theta$ in both (4.7) and (4.8). Then, using (4.4), we deduce that, respectively,

$$\sin 3\theta = \frac{12a - 14 - A - 8Aa}{2(4a - A)^{3/2}} =: u_1$$

and

$$\sin 3\theta = \frac{12a - 14 + A + 8Aa}{2(4a + A)^{3/2}} =: u_2.$$

Hence, the solutions of (4.7) and (4.8) are given by

$$t = \sin(\tfrac{1}{3} \sin^{-1} u_1), \quad \sin\left(\tfrac{1}{3} \sin^{-1} u_1 + \frac{2\pi}{3}\right), \quad \sin\left(\tfrac{1}{3} \sin^{-1} u_1 + \frac{4\pi}{3}\right) \tag{4.9}$$

and

$$t = \sin(\tfrac{1}{3} \sin^{-1} u_2), \quad \sin\left(\tfrac{1}{3} \sin^{-1} u_2 + \frac{2\pi}{3}\right), \quad \sin\left(\tfrac{1}{3} \sin^{-1} u_2 + \frac{4\pi}{3}\right), \tag{4.10}$$

respectively.

We next show that u_1 and u_2 can be greatly simplified. In particular, we shall prove that

$$\sin^{-1}(-u_1) = \tan^{-1}\frac{2A-1}{3\sqrt{3}} \tag{4.11}$$

and

$$\sin^{-1}(u_2) = \tan^{-1}\frac{2A+1}{3\sqrt{3}}. \tag{4.12}$$

Observe that the right triangle with sides of lengths $2A \pm 1$ and $3\sqrt{3}$ has an hypotenuse of length $2\sqrt{4a \pm A}$. Thus, to demonstrate the truth of (4.11) and (4.12), it suffices to prove that

$$\frac{2A-1}{2\sqrt{4a-A}} = \frac{-12a+14+A+8Aa}{2(4a-A)^{3/2}}$$

and

$$\frac{2A+1}{2\sqrt{4a+A}} = \frac{12a-14+A+8Aa}{2(4a+A)^{3/2}},$$

respectively. But these last two equalities are easily verified by cross-multiplication.

Retracing our substitutions and employing (4.9)–(4.12), we find that the solutions of (4.2) and (4.3) are described by, respectively,

$$-\tfrac{2}{3}\sqrt{4a-A}\sin\left(\tfrac{1}{3}\tan^{-1}\frac{2A-1}{3\sqrt{3}}\right) - \frac{1+A}{6},$$

$$-\tfrac{2}{3}\sqrt{4a-A}\sin\left(\tfrac{1}{3}\tan^{-1}\frac{2A-1}{3\sqrt{3}} - \frac{2\pi}{3}\right) - \frac{1+A}{6},$$

$$-\tfrac{2}{3}\sqrt{4a-A}\sin\left(\tfrac{1}{3}\tan^{-1}\frac{2A-1}{3\sqrt{3}} - \frac{4\pi}{3}\right) - \frac{1+A}{6},$$

and

$$\tfrac{2}{3}\sqrt{4a+A}\sin\left(\tfrac{1}{3}\tan^{-1}\frac{2A+1}{3\sqrt{3}}\right) - \frac{1-A}{6},$$

$$\tfrac{2}{3}\sqrt{4a+A}\sin\left(\tfrac{1}{3}\tan^{-1}\frac{2A+1}{3\sqrt{3}} + \frac{2\pi}{3}\right) - \frac{1-A}{6},$$

$$\tfrac{2}{3}\sqrt{4a+A}\sin\left(\tfrac{1}{3}\tan^{-1}\frac{2A+1}{3\sqrt{3}} + \frac{4\pi}{3}\right) - \frac{1-A}{6}.$$

Further simplification yields the roots claimed in Theorem 4.1.

For example, let $a = 8$ so that $A = 5$. Then

$$x_1 = -2\sqrt{3}\sin 20^\circ - 1, \qquad x_2 = -2\sqrt{3}\sin 40^\circ - 1,$$

$$x_3 = 2\sqrt{3}\cos 10^\circ - 1, \qquad x_4 = \tfrac{2}{3}\sqrt{37}\sin\left(\tfrac{1}{3}\tan^{-1}\frac{11}{3\sqrt{3}}\right) + \tfrac{2}{3},$$

$$x_5 = \tfrac{2}{3}\sqrt{37}\sin\left(\frac{\pi}{3} - \tfrac{1}{3}\tan^{-1}\frac{11}{3\sqrt{3}}\right) + \tfrac{2}{3},$$

$$x_6 = -\tfrac{2}{3}\sqrt{37}\sin\left(\frac{\pi}{3} + \tfrac{1}{3}\tan^{-1}\frac{11}{3\sqrt{3}}\right) + \tfrac{2}{3}.$$

From the equalities (4.1), we find that

$$x = \sqrt{a+y} = \sqrt{a+\sqrt{a+z}} = \sqrt{a+\sqrt{a+\sqrt{a+x}}}$$
$$= \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\cdots}}}}. \tag{4.13}$$

Thus, we are led to infinite sequences of nested radicals. The square roots should be considered two-valued. Because of the different choices of signs, there are eight distinct infinite radicals. If we choose all positive signs or all negative signs, then the infinite radicals satisfy a quadratic equation, namely $x^2 - x - a = 0$. (In Chapter 12 (Part II [4, pp. 108–112]), Ramanujan briefly considers infinite radicals. In particular, examples when all the signs are positive are examined.) In the next extry, Ramanujan considers the remaining six cases. However, before establishing Entry 5, we must derive some sufficient conditions to ensure the convergence of infinite nested sequences of radicals.

Theorem 5.1. *Let $a \geq 2$ and define a sequence $\{a_n\}$ by*

$$a_1 = \sqrt{a},\ a_2 = \sqrt{a-\sqrt{a}},\ a_3 = \sqrt{a-\sqrt{a+\sqrt{a}}},$$
$$a_4 = \sqrt{a-\sqrt{a+\sqrt{a+\sqrt{a}}}}, \ldots,$$

where the sequence of signs $-, +, +, \ldots$ appearing in the nested radicals has period 3. *Then $\{a_n\}$ converges.*

It is necessary and sufficient that

$$a \geq \sqrt{a+\sqrt{a+\sqrt{a}}}$$

to obtain real square roots for each a_n, $n \geq 1$. By successively squaring, we find that this condition is equivalent to the inequality

$$F(a) := (a^4 - 2a^3 + a^2 - a)^2 - a \geq 0. \tag{5.1}$$

Using *Mathematica*, we find that (5.1) is valid for $a \geq 1.9408$. Thus, the lower bound for a in Theorem 5.1 can be improved. For technical reasons required in the proof, we need a lower bound of 2, although this can be improved slightly.

Proof. For $0 \leq t \leq a$ define

$$f(t) = \sqrt{a - \sqrt{a + t}} \quad \text{and} \quad g(t) = \sqrt{a + t}.$$

Clearly, $f(t)$ is decreasing and $g(t)$ is increasing on $[0, a]$. Thus,

$$a_1 > f(0) = a_2 > f(\sqrt{a}) = a_3 > f(\sqrt{a + \sqrt{a}}) = f(g(a_1)) = a_4$$

and

$$a_4 = f(g(a_1)) < f(g(a_2)) = a_5 < f(g(a_3)) = a_6 < f(g(a_4)) = a_7.$$

In general, it easily follows by induction that, for each nonnegative integer n,

$$\left.\begin{array}{c} a_{6n+1} > a_{6n+2} > a_{6n+3} > a_{6n+4} \\ \text{and} \\ a_{6n+4} < a_{6n+5} < a_{6n+6} < a_{6n+7}. \end{array}\right\} \tag{5.2}$$

By successively squaring and either adding or subtracting a, and by using induction, we also find that

$$0 < a_4 < a_{10} < a_{16} < \cdots$$

and

$$a_1 > a_7 > a_{13} > \cdots.$$

Hence,

$$0 < a_4 < a_{10} < \cdots < a_{6n+4} < a_{6n+7} < a_{6n+1} < \cdots < a_7 < a_1 = \sqrt{a}.$$

Thus, $\{a_{6n+4}\}$ is a bounded, monotonically increasing sequence, and $\{a_{6n+1}\}$ is a bounded, monotonically decreasing sequence. Therefore, both $\{a_{6n+1}\}$ and $\{a_{6n+4}\}$ converge, say, to α and β, respectively.

Next, observe that $a_{6n+7} = f(g(a_{6n+4}))$ and $a_{6n+4} = f(g(a_{6n+1}))$. Letting n tend to ∞, we deduce that

$$\alpha = f(g(\beta)) \quad \text{and} \quad \beta = f(g(\alpha)),$$

since f and g are continuous. Thus, if $h = f \circ g$, we see that α and β are fixed points of $h \circ h =: H$.

For convenience, set $h_1(t) = \sqrt{a + t}$, $h_2(t) = \sqrt{a + h_1(t)}$, $h_3(t) = \sqrt{a - h_2(t)}$, $h_4(t) = \sqrt{a + h_3(t)}$, $h_5(t) = \sqrt{a + h_4(t)}$, and $H(t) = h_6(t) = \sqrt{a - h_5(t)}$. Then

$$H'(t) = \frac{1}{2^6 h_6(t) h_5(t) \cdots h_1(t)}.$$

On $0 \leq t \leq \sqrt{a}$, h_1, h_2, and h_6 are increasing and h_3, h_4, and h_5 are decreasing. Thus, for $0 \leq t \leq \sqrt{a}$,

$$H'(t) \leq \frac{1}{2^6 h_6(0) h_5(\sqrt{a}) h_4(\sqrt{a}) h_3(\sqrt{a}) h_2(0) h_1(0)}.$$

Set $h_j(a; t) = h_j(t)$, $1 \leq j \leq 6$. It is easy to see that $h_1(a; 0)$, $h_2(a; 0)$, $h_3(a; \sqrt{a})$, $h_4(a; \sqrt{a})$, $h_5(a; \sqrt{a})$, and $h_6(a; 0)$ are increasing functions of a for $a \geq 2$. Thus,

$$\begin{aligned} H'(t) &\leq \frac{1}{2^6 h_6(2; 0) h_5(2; \sqrt{2}) h_4(2; \sqrt{2}) h_3(2; \sqrt{2}) \sqrt{2 + \sqrt{2}}\sqrt{2}} \\ &\leq \frac{1}{2^6 \cdot (0.34)(1.86)(1.48)(0.19)\sqrt{2 + \sqrt{2}}\sqrt{2}} < 1. \end{aligned}$$

Thus, $0 < H'(t) < 1$ for $0 \leq t \leq \sqrt{a}$, and so the equation $H(t) = t$ can have *at most* one root. Hence, $\alpha = \beta$. Therefore $\{a_{3n+1}\}$ converges. Utilizing (5.2), we conclude that $\{a_n\}$ converges.

Define five additional sequences $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$, and $\{f_n\}$ by

$$b_1 = \sqrt{a}, \quad b_2 = \sqrt{a + \sqrt{a}}, \quad b_3 = \sqrt{a + \sqrt{a - \sqrt{a}}},$$
$$b_4 = \sqrt{a + \sqrt{a - \sqrt{a + \sqrt{a}}}}, \ldots,$$

$$c_1 = \sqrt{a}, \quad c_2 = \sqrt{a + \sqrt{a}}, \quad c_3 = \sqrt{a + \sqrt{a + \sqrt{a}}},$$
$$c_4 = \sqrt{a + \sqrt{a + \sqrt{a - \sqrt{a}}}}, \ldots,$$

$$d_1 = \sqrt{a}, \quad d_2 = \sqrt{a - \sqrt{a}}, \quad d_3 = \sqrt{a - \sqrt{a + \sqrt{a}}},$$
$$d_4 = \sqrt{a - \sqrt{a + \sqrt{a - \sqrt{a}}}}, \ldots,$$

$$e_1 = \sqrt{a}, \quad e_2 = \sqrt{a + \sqrt{a}}, \quad e_3 = \sqrt{a + \sqrt{a - \sqrt{a}}},$$
$$e_4 = \sqrt{a + \sqrt{a - \sqrt{a - \sqrt{a}}}}, \ldots,$$

and

$$f_1 = \sqrt{a}, \qquad f_2 = \sqrt{a - \sqrt{a}}, \qquad f_3 = \sqrt{a - \sqrt{a - \sqrt{a}}},$$

$$f_4 = \sqrt{a - \sqrt{a - \sqrt{a + \sqrt{a}}}}, \ldots,$$

where the sequences of signs are $+, -, +$; $+, +, -$; $-, +, -$; $+, -, -$; and $-, -, +$, respectively. By using arguments similar to that above, we can prove that each of these five sequences converges for $a \geq 2$. For $\{b_n\}$ and $\{c_n\}$, in order to obtain real square roots, we need the same lower bound established for $\{a_n\}$, i.e., 1.9408.

For the sequences $\{d_n\}$, $\{e_n\}$, and $\{f_n\}$, to obtain real square roots for each d_n, e_n, and f_n, $n \geq 1$, it is necessary and sufficient that

$$a \geq \sqrt{a + \sqrt{a}},$$

i.e.,

$$G(a) := (a^2 - a)^2 - a \geq 0. \tag{5.3}$$

Employing *Mathematica*, we find that (5.3) holds provided that $a \geq 1.7549$.

In the next entry, Ramanujan associates each of the six sequences of nested radicals defined above with the proper root of Theorem 4.1.

Entry 5 (pp. 305–306). *Let* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$, and $\{f_n\}$ *denote the sequences of nested radicals defined above. Let* $x_1, x_2, \ldots, x_6$ *be the roots defined in Theorem* 4.1. *Then*

$$a' := \lim_{n\to\infty} a_n = x_4, \tag{5.4}$$

$$b' := \lim_{n\to\infty} b_n = x_5, \tag{5.5}$$

$$c' := \lim_{n\to\infty} c_n = -x_6, \tag{5.6}$$

$$d' := \lim_{n\to\infty} d_n = -x_1, \tag{5.7}$$

$$e' := \lim_{n\to\infty} e_n = -x_2, \tag{5.8}$$

and

$$f' := \lim_{n\to\infty} f_n = x_3. \tag{5.9}$$

Proof. An examination of the sequences shows that $\{a_n\}$, $\{b_n\}$, and $\{f_n\}$ must converge to roots while $\{c_n\}$, $\{d_n\}$, and $\{e_n\}$ must converge to negatives of roots.

Using MACSYMA, we expand $x_1, x_2, \ldots, x_6$ "around ∞." Thus, for $a \geq 2$ (at least), we find that

$$x_1 = -\sqrt{a} + \tfrac{1}{2} + \frac{3}{8\sqrt{a}} + \frac{17}{128a^{3/2}} + \cdots,$$

$$x_2 = -\sqrt{a} - \tfrac{1}{2} + \frac{3}{8\sqrt{a}} - \frac{1}{4a} + \cdots,$$

$$x_3 = \sqrt{a} - \tfrac{1}{2} + \frac{1}{8\sqrt{a}} + \cdots,$$

$$x_4 = \sqrt{a} - \tfrac{1}{2} - \frac{3}{8\sqrt{a}} - \frac{1}{4a} + \cdots,$$

$$x_5 = \sqrt{a} + \tfrac{1}{2} - \frac{3}{8\sqrt{a}} - \frac{17}{128a^{3/2}} + \cdots,$$

and

$$x_6 = -\sqrt{a} - \tfrac{1}{2} - \frac{1}{8\sqrt{a}} + \cdots.$$

Next, we use *Mathematica* to expand the limits of the six infinite nested sequences of radicals. Thus, for $a \geq 2$,

$$a' = \sqrt{a} - \tfrac{1}{2} - \frac{3}{8\sqrt{a}} - \frac{1}{4a} + \cdots,$$

$$b' = \sqrt{a} + \tfrac{1}{2} - \frac{3}{8\sqrt{a}} - \frac{17}{128a^{3/2}} + \cdots,$$

$$c' = \sqrt{a} + \tfrac{1}{2} + \frac{1}{8\sqrt{a}} + \cdots,$$

$$d' = \sqrt{a} - \tfrac{1}{2} - \frac{3}{8\sqrt{a}} - \frac{17}{128a^{3/2}} + \cdots,$$

$$e' = \sqrt{a} + \tfrac{1}{2} - \frac{3}{8\sqrt{a}} + \frac{1}{4a} + \cdots,$$

and

$$f' = \sqrt{a} - \tfrac{1}{2} + \frac{1}{8\sqrt{a}} + \cdots.$$

Comparing these two sets of six expansions, we are able to complete the matchings in (5.4)–(5.9).

We are very grateful to R. W. Gosper, Jr. and D. Grayson for advice on employing computer algebra in the expansions given above.

For certain values of a, we can verify (5.4)–(5.9) in a more elementary manner. For example, if $a = 2$, (5.4)–(5.9) are easy to establish. An an illustration, we establish (5.4). Note that $A = 1$. Thus, we want to show that

$$(2 - (2 + (2 + (2 - \cdots)^{1/2})^{1/2})^{1/2})^{1/2} = 2\sin(\pi/18). \tag{5.10}$$

We have

$$\begin{aligned}
2\sin(\pi/18) &= \sqrt{4\sin^2(\pi/18)} \\
&= \sqrt{2(1 - \cos(\pi/9))} \\
&= (2 - (4\cos^2(\pi/9))^{1/2})^{1/2} \\
&= (2 - (2(1 + \cos(2\pi/9)))^{1/2})^{1/2} \\
&= (2 - (2 + (4\cos^2(2\pi/9))^{1/2})^{1/2})^{1/2} \\
&= (2 - (2 + (2(1 + \cos(4\pi/9)))^{1/2})^{1/2})^{1/2} \\
&= (2 - (2 + (2 + (4\cos^2(4\pi/9))^{1/2})^{1/2})^{1/2})^{1/2} \\
&= (2 - (2 + (2 + (4\sin^2(\pi/18))^{1/2})^{1/2})^{1/2})^{1/2}.
\end{aligned}$$

By continued iteration, we establish (5.10).

When $a = 8$, and so $A = 5$, (5.4)–(5.9) may likewise be established in an elementary fashion. We shall prove (5.9). Hence, we wish to show that

$$\sqrt{8 - \sqrt{8 - \sqrt{8 + \sqrt{8 - \cdots}}}} = 2\sqrt{3}\cos 10^\circ - 1. \tag{5.11}$$

We have

$$\begin{aligned}
2\sqrt{3}\cos 10^\circ - 1 &= (12\cos^2 10^\circ - 4\sqrt{3}\cos 10^\circ + 1)^{1/2} \\
&= (6(1 + \cos 20^\circ) - 4\sqrt{3}\cos 10^\circ + 1)^{1/2} \\
&= (7 + 4\sqrt{3}\cos 30^\circ \cos 20^\circ - 4\sqrt{3}\cos 10^\circ)^{1/2} \\
&= (7 + 2\sqrt{3}\cos 50^\circ + 2\sqrt{3}\cos 10^\circ - 4\sqrt{3}\cos 10^\circ)^{1/2} \\
&= (7 + 2\sqrt{3}\cos 50^\circ - 2\sqrt{3}\cos 10^\circ)^{1/2} \\
&= (7 - 4\sqrt{3}\sin 30^\circ \sin 20^\circ)^{1/2} \\
&= (7 - 2\sqrt{3}\sin 20^\circ)^{1/2} \\
&= (8 - (1 + 4\sqrt{3}\sin 20^\circ + 12\sin^2 20^\circ)^{1/2})^{1/2} \\
&= (8 - (1 + 4\sqrt{3}\sin 20^\circ + 6(1 - \cos 40^\circ))^{1/2})^{1/2} \\
&= (8 - (7 + 4\sqrt{3}\sin 20^\circ - 4\sqrt{3}\cos 30^\circ \cos 40^\circ)^{1/2})^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&= (8 - (7 + 4\sqrt{3}\sin 20^\circ - 2\sqrt{3}\cos 70^\circ \\
&\quad - 2\sqrt{3}\cos 10^\circ)^{1/2})^{1/2} \\
&= (8 - (7 + 2\sqrt{3}\cos 70^\circ - 2\sqrt{3}\cos 10^\circ)^{1/2})^{1/2} \\
&= (8 - (7 - 4\sqrt{3}\sin 40^\circ \sin 30^\circ)^{1/2})^{1/2} \\
&= (8 - (7 - 2\sqrt{3}\sin 40^\circ)^{1/2})^{1/2} \\
&= (8 - (8 - (1 + 4\sqrt{3}\sin 40^\circ + 12\sin^2 40^\circ)^{1/2})^{1/2})^{1/2} \\
&= (8 - (8 - (1 + 4\sqrt{3}\sin 40^\circ + 6(1 - \cos 80^\circ))^{1/2})^{1/2})^{1/2} \\
&= (8 - (8 - (7 + 4\sqrt{3}\sin 40^\circ \\
&\quad - 4\sqrt{3}\cos 30^\circ \cos 80^\circ)^{1/2})^{1/2})^{1/2} \\
&= (8 - (8 - (7 + 4\sqrt{3}\sin 40^\circ - 2\sqrt{3}\cos 110^\circ \\
&\quad - 2\sqrt{3}\cos 50^\circ)^{1/2})^{1/2})^{1/2} \\
&= (8 - (8 - (7 + 2\sqrt{3}\cos 50^\circ - 2\sqrt{3}\cos 110^\circ)^{1/2})^{1/2})^{1/2} \\
&= (8 - (8 - (7 + 4\sqrt{3}\sin 80^\circ \sin 30^\circ)^{1/2})^{1/2})^{1/2} \\
&= (8 - (8 - (8 + 2\sqrt{3}\cos 10^\circ - 1)^{1/2})^{1/2})^{1/2}.
\end{aligned}
$$

Hence, by iteration, the proof of (5.11) is complete.

Entry 6 (p. 309). *We have*

$$\sqrt{2\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{11^2}\right)\left(1 - \frac{1}{19^2}\right)} = (1 + \tfrac{1}{7})(1 + \tfrac{1}{11})(1 + \tfrac{1}{19}).$$

This curious identity is readily verified by straightforward arithmetic. Is this an isolated result, or are there other identities of this type? This entry is also recorded at the bottom of p. 363 [22].

Entry 7 (p. 320) *If* $x = \alpha + \beta + \gamma(n - 1)$, $y = \beta + \gamma + \alpha(n - 1)$, *and* $z = \gamma + \alpha + \beta(n - 1)$, *then*

$$
\begin{aligned}
x^2 + nyz &= (n^3 - 3n^2 + 4)\alpha\beta + (n^2 - n + 1)(\alpha^2 + \beta^2 + \gamma^2) \\
&\quad + (n^2 + 2n - 2)(\alpha\beta + \beta\gamma + \gamma\alpha). \qquad (7.1)
\end{aligned}
$$

Similar identities can be found for $y^2 + nzx$ *and* $z^2 + nxy$.

Proof. The equality (7.1) follows easily by direct verification.

Entry 8 (p. 320). *Let* $a = x^2 + 2yz$, $b = y^2 + 2zx$, $c = z^2 + 2xy$, $\theta = xy + yz + zx$, *and* $t = (x-y)(y-z)(z-x)$. *Then*

$$x - y = \frac{t}{\theta - c}, \tag{8.1}$$

$$y - z = \frac{t}{\theta - a}, \tag{8.2}$$

$$z - x = \frac{t}{\theta - b}, \tag{8.3}$$

$$t^2 = (\theta - a)(\theta - b)(\theta - c), \tag{8.4}$$

$$\frac{1}{\theta - a} + \frac{1}{\theta - b} + \frac{1}{\theta - c} = 0, \tag{8.5}$$

$$a + b + c = (x + y + z)^2, \tag{8.6}$$

and

$$a^3 + b^3 + c^3 - 3abc = (x^3 + y^3 + z^3 - 3xyz)^2. \tag{8.7}$$

Proof. Formula (8.1) is equivalent to

$$(x-y)(y-z)(z-x) = (x-y)(xy + yz + zx - z^2 - 2xy),$$

which is verified by a trivial calculation. Formulas (8.2) and (8.3) follow by symmetry. Formula (8.4) is a trivial consequence of (8.1)–(8.3). From (8.1)–(8.3),

$$\frac{1}{\theta - c} + \frac{1}{\theta - a} + \frac{1}{\theta - b} = \frac{1}{t}(x - y + y - z + z - x) = 0,$$

which proves (8.5). Equality (8.6) follows immediately from the definitions of a, b, and c. Lastly, using the definitions of a, b, and c, we can easily verify (8.7) by elementary algebra.

Entry 9 (p. 320). *Let* $x = Ap + Bq + Cr$, $y = Bp + Cq + Ar$, *and* $z = Cp + Aq + Br$. *Then*

$$\begin{aligned} x^2 + 2yz = (A^2 + 2BC)(p^2 + 2qr) + (B^2 + 2CA)(q^2 + 2rp) \\ + (C^2 + 2AB)(r^2 + 2pq). \end{aligned} \tag{9.1}$$

Furthermore, if $x = \frac{1}{3}(p - 2q - 2r)$, $y = \frac{1}{3}(q - 2r - 2p)$, *and* $z = \frac{1}{3}(r - 2p - 2q)$, *then*

$$x^2 + 2yz = p^2 + 2qr, \tag{9.2}$$

$$y^2 + 2zx = q^2 + 2rp, \tag{9.3}$$

and

$$z^2 + 2xy = r^2 + 2pq. \tag{9.4}$$

PROOF. The identity (9.1) follows by a direct verification. Set $A = \frac{1}{3}$ and $B = C = -\frac{2}{3}$ in (9.1) to deduce (9.2). Lastly, (9.3) and (9.4) follow from analogues of (9.1) for $y^2 + 2zx$ and $z^2 + 2xy$.

Entry 10 (p. 325). *If α, β, and γ are roots of the equation*

$$x^3 - ax^2 + bx - 1 = 0, \tag{10.1}$$

then, for a suitable determination of roots,

$$\alpha^{1/3} + \beta^{1/3} + \gamma^{1/3} = (a + 6 + 3t)^{1/3} \tag{10.2}$$

and

$$(\alpha\beta)^{1/3} + (\beta\gamma)^{1/3} + (\gamma\alpha)^{1/3} = (b + 6 + 3t)^{1/3}, \tag{10.3}$$

where

$$t^3 - 3(a + b + 3)t - (ab + 6(a + b) + 9) = 0. \tag{10.4}$$

PROOF. Choose cube roots so that $(\alpha\beta\gamma)^{1/3} = 1$, and then let

$$z^3 - \theta z^2 + \varphi z - 1 = 0 \tag{10.5}$$

denote the cubic polynomial with roots $\alpha^{1/3}$, $\beta^{1/3}$, and $\gamma^{1/3}$. Cubing both sides of the equality

$$z^3 - 1 = \theta z^2 - \varphi z,$$

we find that

$$(z^3 - 1)^3 - \theta^3 z^6 + \varphi^3 z^3 + 3\theta\varphi z^3(z^3 - 1) = 0. \tag{10.6}$$

Since $\alpha^{1/3}$, $\beta^{1/3}$, and $\gamma^{1/3}$ are roots of (10.5), they are also roots of (10.6). As a cubic polynomial in z^3, (10.6) thus has the roots α, β, and γ.

Comparing (10.1) with (10.6), we deduce that

$$a = \theta^3 + 3 - 3\theta\varphi \tag{10.7}$$

and

$$b = \varphi^3 + 3 - 3\theta\varphi. \tag{10.8}$$

If we define t by

$$\theta^3 = a + 6 + 3t, \tag{10.9}$$

then, by (10.5) and (10.9),

$$\alpha^{1/3} + \beta^{1/3} + \gamma^{1/3} = \theta = (a + 6 + 3t)^{1/3},$$

which proves (10.2). Also, by (10.7)–(10.9),

$$\varphi^3 = b - 3 + 3\theta\varphi = b + \theta^3 - a = b + 6 + 3t. \tag{10.10}$$

Hence, (10.3) is established. From (10.7) and (10.9),

$$3 + t = \theta\varphi.$$

Thus, by (10.9) and (10.10),

$$(3 + t)^3 = \theta^3\varphi^3 = (a + 6 + 3t)(b + 6 + 3t).$$

Expanding both sides, collecting terms, and simplifying, we deduce (10.4).

Entry 11 (p. 328). *If*

$$x = y + \left(\frac{z}{y} - y^3\right)^{1/3}, \tag{11.1}$$

then

$$3y = x + \left(\frac{9z}{x} - x^3\right)^{1/3}. \tag{11.2}$$

Proof. Subtracting y from both sides of (11.1), cubing each side, and simplifying, we find that

$$x^3y - 3x^2y^2 + 3xy^3 - z = 0. \tag{11.3}$$

Multiplying both sides of (11.3) by 9 and rearranging terms, we deduce that

$$(3y)^3x - 3(3y)^2x^2 + 3(3y)x^3 - 9z = 0. \tag{11.4}$$

Now observe that (11.3) and (11.4) have identical forms with x, y, and z in (11.3) being replaced by $3y$, x, and $9z$, respectively, in (11.4). Thus, by using this symmetry in (11.1), we immediately deduce (11.2).

Entry 12 (p. 328). *Let x and y be positive, and define positive numbers α and β by*

$$1 + \frac{x}{y} = \alpha^{1/5} - \beta^{1/5} \tag{12.1}$$

and $\alpha\beta = 1$. Furthermore, define z by

$$\alpha - \beta = 11 + \frac{z}{y^6}. \tag{12.2}$$

Then

$$1+\frac{5y}{x}=\gamma^{1/5}-\delta^{1/5}, \tag{12.3}$$

where γ and δ are positive numbers such that $\gamma\delta=1$ and

$$\gamma-\delta=11+\frac{125z}{x^6}.$$

Proof. First, by (12.1), since $\alpha\beta=1$,

$$\begin{aligned}\left(1+\frac{x}{y}\right)^5&=\alpha-5\alpha^{3/5}+10\alpha^{1/5}-10\beta^{1/5}+5\beta^{3/5}-\beta\\&=(\alpha-\beta)-5\left\{\left(1+\frac{x}{y}\right)^3+3\left(1+\frac{x}{y}\right)\right\}+10\left(1+\frac{x}{y}\right)\\&=(\alpha-\beta)-5\left(1+\frac{x}{y}\right)^3-5\left(1+\frac{x}{y}\right).\end{aligned}$$

Hence, by (12.2),

$$11+\frac{z}{y^6}=\left(1+\frac{x}{y}\right)^5+5\left(1+\frac{x}{y}\right)^3+5\left(1+\frac{x}{y}\right). \tag{12.4}$$

Expanding and simplifying, we deduce that

$$z=25xy^5+25x^2y^4+15x^3y^3+5x^4y^2+x^5y.$$

This may be rewritten in the form

$$\begin{aligned}11+\frac{125z}{x^6}&=11+\left(\frac{5y}{x}\right)^5+5\left(\frac{5y}{x}\right)^4+15\left(\frac{5y}{x}\right)^3+25\left(\frac{5y}{x}\right)^2+25\left(\frac{5y}{x}\right)\\&=\left(1+\frac{5y}{x}\right)^5+5\left(1+\frac{5y}{x}\right)^3+5\left(1+\frac{5y}{x}\right).\end{aligned} \tag{12.5}$$

Observe that (12.4) and (12.5) have the same shape with x, y, and z in (12.4) being replaced by $5y$, x, and $125z$, respectively, in (12.5). Thus, if γ and δ are defined as in the hypotheses, we may use the aforementioned symmetry to deduce (12.3) from (12.1).

Entry 13 (p. 328). *If*

$$x=y+(z-y^2)^{1/2}, \tag{13.1}$$

then

$$2y=x+(2z-x^2)^{1/2}. \tag{13.2}$$

PROOF. Subtracting y from both sides and then squaring (13.1), we find that

$$x^2 - 2xy + 2y^2 - z = 0. \tag{13.3}$$

Multiplying both sides of (13.3) by 2 and rearranging terms, we arrive at

$$(2y)^2 - 2(2y)x + 2x^2 - 2z = 0. \tag{13.4}$$

Note that (13.3) and (13.4) have the same form, with x, y, and z in (13.3) being replaced by $2y$, x, and $2z$, respectively, in (13.4). Hence, by symmetry, (13.2) follows from (13.1).

The previous three entries demonstrate Ramanujan's keen eye in spotting beautiful algebraic relationships.

Entry 14 (p. 337). *Let f, F, φ, and ψ be functions related by the equality*

$$f(x) = \varphi(x)F(x + p) + \psi(x)F(x + q), \tag{14.1}$$

where p and q are constants. Let χ be a function satisfying the equality

$$\frac{\chi(x + p)}{\chi(x + q)} = \frac{\psi(x)}{\varphi(x)}. \tag{14.2}$$

Let F_1 be defined by $F(x) = F_1(x)\chi(x)$. Then

$$F_1(x + p) + F_1(x + q) = \frac{f(x)}{\varphi(x)\chi(x + p)}.$$

PROOF. Using the definition of F_1, (14.2), and (14.1), we find that

$$\begin{aligned} F_1(x + p) + F_1(x + q) &= \frac{F(x + p)}{\chi(x + p)} + \frac{F(x + q)}{\chi(x + q)} \\ &= \frac{F(x + p)}{\chi(x + p)} + \frac{\psi(x)F(x + q)}{\varphi(x)\chi(x + p)} \\ &= \frac{\varphi(x)F(x + p) + \psi(x)F(x + q)}{\varphi(x)\chi(x + p)} \\ &= \frac{f(x)}{\varphi(x)\chi(x + p)}, \end{aligned}$$

which completes the proof.

For the next entry, we quote Ramanujan.

Entry 15 (p. 337). *If a, b, c are constants in A. P. and u, v, w are functions of x in G. P. solve*

$$uF(x + a) + vF(x + b) + wF(x + c) = \varphi(x). \tag{15.1}$$

Find $\chi(x)$ *so that*

$$\frac{\chi(x + 3a/2)}{\chi(x + 3c/2)} = \left(\frac{w^3}{u^3}\right)^{1/2} = \frac{v^3}{u^3} \tag{15.2}$$

and substitute

$$F(x) = \chi(x)\{\sqrt{u}F_1(x + a/2) - \sqrt{u}F_1(x + c/2)\}. \tag{15.3}$$

We have been unable to meaningfully interpret this result.

Since a, b, and c are in arithmetic progression, there exists a constant h so that

$$b = a + h \quad \text{and} \quad c = a + 2h.$$

Since u, v, and w are in geometric progression, there exists a function r such that

$$v = ur \quad \text{and} \quad w = ur^2,$$

from which the second equality in (15.2) is readily verified.

Presumably, Ramanujan intends for us to substitute (15.3) into (15.1). Supposedly, after considerable simplification, we should be able to deduce significant information about F. As we shall see, very little simplification is possible.

From (15.1) and (15.3),

$$\begin{aligned}
&uF(x + a) + vF(x + b) + wF(x + c)\\
&\quad = u(x)\chi(x + a)\{\sqrt{u(x + a)}F_1(x + 3a/2) - \sqrt{u(x + a)}F_1(x + a + c/2)\}\\
&\qquad + v(x)\chi(x + b)\{\sqrt{u(x + b)}F_1(x + b + a/2) - \sqrt{u(x + b)}F_1(x + b + c/2)\}\\
&\qquad + w(x)\chi(x + c)\{\sqrt{u(x + c)}F_1(x + c + a/2) - \sqrt{u(x + c)}F_1(x + 3c/2)\}\\
&\quad = u(x)\sqrt{u(x + a)}\chi(x + a)\{F_1(x + 3a/2) - F_1(x + 3a/2 + h)\}\\
&\qquad + v(x)\sqrt{u(x + b)}\chi(x + b)\{F_1(x + 3a/2 + h) - F_1(x + 3a/2 + 2h)\}\\
&\qquad + w(x)\sqrt{u(x + c)}\chi(x + c)\{F_1(x + 3a/2 + 2h) - F_1(x + 3a/2 + 3h)\}.
\end{aligned}$$

It now appears that we should take the two expressions involving $F_1(x + 3a/2 + h)$, as well as the two terms involving $F_1(x + 3a/2 + 2h)$, and apply (15.2). However, no simplification occurs.

Since Ramanujan did not give the arguments of u, v, and w, there is some ambiguity. For example, is $uF(x + a) = u(x)F(x + a)$ or $(uF)(x + a)$? In (15.2), is x the argument on the right side?

It also seems strange that $\sqrt{u}$ is a factor of both expressions in (15.3). Why did not Ramanujan factor out $\sqrt{u}$? Perhaps the latter $\sqrt{u}$ in (15.3) should be replaced by $\sqrt{w}$.

We have employed various interpretations of this entry, but none has been successful.

Entry 16 (p. 338). *Let F, F_1, f, φ, and ψ denote functions satisfying the relations*

$$F(x+p)\{\varphi(x)+\psi(x)F(x+q)\}=f(x) \tag{16.1}$$

and

$$F(x)=\frac{F_1(x+q)}{F_1(x+p)}, \tag{16.2}$$

where p and q are constants. Then

$$\varphi(x)F_1(x+p+q)+\psi(x)F_1(x+2q)=f(x)F_1(x+2p). \tag{16.3}$$

Proof. By (16.1) and (16.2),

$$f(x)=\frac{F_1(x+p+q)}{F_1(x+2p)}\left\{\varphi(x)+\psi(x)\,\frac{F_1(x+2q)}{F_1(x+p+q)}\right\},$$

from which (16.3) easily follows.

Entry 17 (p. 338). *Solve the equations*

$$\frac{x^5-a}{x^2-y}=\frac{y^5-b}{y^2-x}=5(xy-1) \tag{17.1}$$

for x and y, where a and b are arbitrary.

The special case $a=6$, $b=9$ was submitted by Ramanujan [4], [23, pp. 322–323] as Question 284 in the *Journal of the Indian Mathematical Society*. In this special instance, the solutions can be described in terms of cosines. Ramanujan's own solution was published.

Question 284 was the fourth problem that Ramanujan published in the *Journal of the Indian Mathematical Society*. The first five problems that Ramanujan posed for the journal readers were published under the name of S. Ramanujam, which is an alternative spelling for Ramanujan. Both are translations of the same Sanskrit word Ramanujaha. In Ramanujan's native language Tamil, there are also two different spellings of Ramanujan. In subsequent problems and in all of his published papers, the spelling Ramanujan was used.

The solution of the more general problem follows along the same lines. Since Ramanujan's solution is short and elegant, we offer it below.

Solution to Entry 17. Let

$$x=\alpha+\beta+\gamma \quad \text{and} \quad y=\alpha\beta+\beta\gamma+\gamma\alpha, \tag{17.2}$$

where α, β, and γ satisfy the condition $\alpha\beta\gamma = 1$ but are otherwise arbitrary. Substituting into (17.1) and employing the requirement $\alpha\beta\gamma = 1$, we find that

$$(\alpha + \beta + \gamma)^5 - a = 5\{(\alpha + \beta + \gamma)^2 - (\alpha\beta + \beta\gamma + \gamma\alpha)\} \times \{(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma\}$$

and

$$(\alpha\beta + \beta\gamma + \gamma\alpha)^5 - b = 5\{(\alpha\beta + \beta\gamma + \gamma\alpha)^2 - \alpha\beta\gamma(\alpha + \beta + \gamma)\} \times \{(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha)\alpha\beta\gamma - \alpha^2\beta^2\gamma^2\}.$$

Comparing coefficients of like terms on each side of each equation, we deduce that, respectively,

$$\alpha^5 + \beta^5 + \gamma^5 = a \tag{17.3}$$

and

$$(\alpha\beta)^5 + (\beta\gamma)^5 + (\gamma\alpha)^5 = b. \tag{17.4}$$

Recall also that

$$(\alpha\beta\gamma)^5 = 1. \tag{17.5}$$

Thus, α^5, β^5, and γ^5 are roots of the cubic equation

$$t^3 - at^2 + bt - 1 = 0.$$

Clearly, (17.3)–(17.5) remain unchanged if α, β, and γ are replaced by $\alpha\rho_1$, $\beta\rho_2$, and $\gamma\rho_3$, respectively, where ρ_1, ρ_2, and ρ_3 are fifth roots of unity. However, because of the requirement $\alpha\beta\gamma = 1$, we must further require that $\rho_1\rho_2\rho_3 = 1$. If ρ is a primitive fifth root of unity, the possible values for x are then, by (17.2),

$$\alpha + \beta + \gamma, \quad \alpha + \beta\rho + \gamma\rho^4, \quad \alpha\rho + \beta + \gamma\rho^4, \quad \alpha\rho + \beta\rho^4 + \gamma,$$
$$\alpha\rho + \beta\rho + \gamma\rho^3, \quad \alpha\rho + \beta\rho^3 + \gamma\rho, \quad \alpha\rho^3 + \beta\rho + \gamma\rho.$$

Hence, altogether there are 25 solutions for x. Of course, using (17.2), we can also write down the 25 corresponding values for y.

Our solutions differ from those given by Ramanujan [4], [23, p. 323] in the special case $a = 6$, $b = 9$. He lists the values of x as

$$\alpha + \beta + \gamma, \quad \alpha + \beta\rho + \gamma\rho^4, \quad \alpha + \beta\rho^2 + \gamma\rho^3, \quad \alpha\rho + \beta\rho + \gamma\rho^3,$$
$$\alpha\rho + \beta\rho^2 + \gamma\rho^2, \quad \alpha\rho^2 + \beta\rho^4 + \gamma\rho^4, \quad \alpha\rho^3 + \beta\rho^3 + \gamma\rho^4,$$

where ρ is a primitive fifth root of unity. However, observe that the third member of this list may be generated from the second member by replacing ρ by ρ^2. Also, the seventh can be obtained from the fourth by replacing ρ by ρ^3. Lastly, the sixth arises from the fifth when ρ is replaced by

ρ^2. Ramanujan missed the roots $\alpha\rho + \beta + \gamma\rho^4$, $\alpha\rho + \beta\rho^4 + \gamma$, and $\alpha\rho + \beta\rho^3 + \gamma\rho$.

Entry 18 (p. 338). *If a and b are arbitrary, solve*

$$\frac{x^7 - a}{(x^2 - y)^2 + x} = \frac{y^7 - b}{(y^2 - x)^2 + y} = 7(xy - 1) \tag{18.1}$$

for x and y.

SOLUTION. Our procedure is similar to that above. Let

$$x = \alpha + \beta + \gamma \quad \text{and} \quad y = \alpha\beta + \beta\gamma + \gamma\alpha, \tag{18.2}$$

where $\alpha\beta\gamma = 1$. Substituting into (18.1), we find that

$$\begin{aligned}(\alpha + \beta + \gamma)^7 - a = 7\{&(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma\}\\ &\times \{[(\alpha + \beta + \gamma)^2 - (\alpha\beta + \beta\gamma + \gamma\alpha)]^2 + (\alpha + \beta + \gamma)\alpha\beta\gamma\}\end{aligned}$$

and

$$\begin{aligned}(\alpha\beta + \beta\gamma + \gamma\alpha)^7 - b = 7\{&(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha)\alpha\beta\gamma - \alpha^2\beta^2\gamma^2\}\\ &\times \{[(\alpha\beta + \beta\gamma + \gamma\alpha)^2 - (\alpha + \beta + \gamma)\alpha\beta\gamma]^2\\ &+ (\alpha\beta + \beta\gamma + \gamma\alpha)\alpha^2\beta^2\gamma^2\}.\end{aligned}$$

Examining coefficients of like terms on each side of these two equalities, we find that, respectively,

$$\alpha^7 + \beta^7 + \gamma^7 = a \tag{18.3}$$

and

$$(\alpha\beta)^7 + (\beta\gamma)^7 + (\gamma\alpha)^7 = b. \tag{18.4}$$

We also note that

$$(\alpha\beta\gamma)^7 = 1. \tag{18.5}$$

Thus, α^7, β^7, and γ^7 are roots of the cubic equation

$$t^3 - at^2 + bt - 1 = 0.$$

Now (18.3)–(18.5) are invariant under the multiplication of α, β, and γ by seventh roots of unity. However, because of the need for $\alpha\beta\gamma$ to equal 1, the product of these three seventh roots of unity must equal 1. By (18.2), the values of x are therefore given by

$$\begin{gathered}\alpha + \beta + \gamma, \quad \alpha + \beta\rho + \gamma\rho^6, \quad \alpha\rho + \beta\rho^6 + \gamma,\\ \alpha\rho + \beta + \gamma\rho^6, \quad \alpha\rho + \beta\rho + \gamma\rho^5, \quad \alpha\rho + \beta\rho^5 + \gamma\rho,\\ \alpha\rho^5 + \beta\rho + \gamma\rho, \quad \alpha\rho + \beta\rho^2 + \gamma\rho^4, \quad \alpha\rho + \beta\rho^4 + \gamma\rho^2,\end{gathered}$$

where ρ is a primitive seventh root of unity. Thus, there exist a total of 49 different values for x. By using (18.2), we can also record the corresponding 49 values for y.

The next entry is ambiguously recorded by Ramanujan. It is followed by a very brief indication of his method for obtaining the solutions. We shall offer the formulation found in Ramanujan's paper [3], [23, pp. 18–19].

Entry 19 (p. 338). *Solve the system of equations*

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= a_1, \\ x_1y_1 + x_2y_2 + \cdots + x_ny_n &= a_2, \\ x_1y_1^2 + x_2y_2^2 + \cdots + x_ny_n^2 &= a_3, \\ &\vdots \\ x_1y_1^{2n-1} + x_2y_2^{2n-1} + \cdots + x_ny_n^{2n-1} &= a_{2n}, \end{aligned} \tag{19.1}$$

where $x_1, x_2, \ldots, x_n$ *and* $y_1, y_2, \ldots, y_n$ *are* $2n$ *unknowns.*

We refer the reader to Ramanujan's [3], [23, pp. 18–19] clever solution of this system. The author and S. Bhargava sketched Ramanujan's solution in [3]. Earlier, Ramanujan [1], [23, p. 322] had submitted the special case $n = 3$ of Entry 19 as a problem to the *Journal of the Indian Mathematical Society*.

It is easy to see that the system (19.1) is equivalent to the single equation

$$\sum_{i=1}^{n} x_i(y_is + t)^{2n-1} = \sum_{j=0}^{2n-1} \binom{2n-1}{j} a_{j+1}s^jt^{2n-1-j}.$$

Thus, Ramanujan's problem is equivalent to the question: When can a binary $(2n - 1) - ic$ form be represented as a sum of n $(2n - 1)$th powers? In 1851, J. J. Sylvester [1], [2], [3, pp. 203–216, 265–283] found the following necessary and sufficient conditions for a solution: The system of n equations,

$$\begin{aligned} a_1u_1 + a_2u_2 + \cdots + a_{n+1}u_{n+1} &= 0, \\ a_2u_1 + a_3u_2 + \cdots + a_{n+2}u_{n+1} &= 0, \\ &\vdots \\ a_nu_1 + a_{n+1}u_2 + \cdots + a_{2n}u_{n+1} &= 0, \end{aligned}$$

must have a solution $u_1, u_2, \ldots, u_{n+1}$ such that the $n - ic$ form

$$p(w, z) := \sum_{j=0}^{n} u_{j+1}w^jz^{n-j}$$

can be represented as a product of n distinct linear forms. This is true for a general $2n$-tuple $(a_1, a_2, \ldots, a_{2n})$ in the sense of algebraic geometry.

Thus, the numbers $y_1, y_2, \ldots, y_n$ are related to the factorization of $p(w, z)$. Sylvester's theorem belongs to the subject of invariant theory, which was developed in the late nineteenth and early twentieth centuries. For a contemporary account, but with classical language, see a paper by J. P. S. Kung and G.-C. Rota [1].

Entry 20 (p. 338). *Let a, b, c, and d be arbitrary. Solve the system*

$$x^2 + ay = b, \qquad y^2 + cx = d.$$

By eliminating y, we find that the given system is equivalent to the biquadratic equation

$$a^2(d - cx) = (b - x^2)^2. \tag{20.1}$$

Of course, methods for solving quartic equations are well known (Hall and Knight [1, pp. 483–488]). Ramanujan, however, offers a new method. He suggests the substitutions $x = \alpha + \beta + \gamma$, $y = -(2/a)(\alpha\beta + \beta\gamma + \gamma\alpha)$, and $\alpha\beta\gamma =$ "suitable value."

Solution. Assume that $ac \neq 0$, for otherwise the solutions of the system are trivial. Setting $x = \alpha + \beta + \gamma$ and $y = -(2/a)(\alpha\beta + \beta\gamma + \gamma\alpha)$, as prescribed by Ramanujan, we find that

$$x^2 + ay = \alpha^2 + \beta^2 + \gamma^2 = b \tag{20.2}$$

and

$$\begin{aligned} y^2 + cx &= \frac{4}{a^2}(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha^2\beta\gamma + 2\alpha\beta^2\gamma + 2\alpha\beta\gamma^2) \\ &\quad + c(\alpha + \beta + \gamma) = d. \end{aligned} \tag{20.3}$$

This suggests that we set $\alpha\beta\gamma = -ca^2/8$. Then (20.3) reduces to the simpler equation

$$\tfrac{1}{4}a^2(y^2 + cx) = (\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 = \tfrac{1}{4}a^2 d. \tag{20.4}$$

We also know that

$$(\alpha\beta\gamma)^2 = \frac{a^4c^2}{64}. \tag{20.5}$$

Hence, from (20.2), (20.4), and (20.5), we conclude that α^2, β^2, and γ^2 are roots of the cubic equation

$$t^3 - bt^2 + \tfrac{1}{4}a^2dt - \tfrac{1}{64}a^4c^2 = 0.$$

Now, (20.2), (20.4), and (20.5) are invariant under sign changes of α, β, and γ. However, the condition $\alpha\beta\gamma = -ca^2/8$ requires that if sign changes are made, exactly two sign changes must be effected. Thus, the four values of x are given by

$$\alpha + \beta + \gamma, \quad \alpha - \beta - \gamma, \quad -\alpha - \beta + \gamma, \quad -\alpha + \beta - \gamma.$$

Of course, the corresponding values of y are then easily determined from the equality $-ay = 2(\alpha\beta + \beta\gamma + \gamma\alpha)$.

Since any biquadratic equation can be put in the form (20.1) by a suitable change of variable, Ramanujan has shown that any quartic equation can be solved by solving a certain cubic equation. This method appears not to have been heretofore observed.

Entry 21 (p. 339). *Let* $\tan^{-1} z$ *have its principal value, and let* n *denote a positive integer. Then*

$$(x^2 + 1)^{n/2} \sin(n \tan^{-1} x) = nx \prod_{k=1}^{[(n-1)/2]} \left(1 - \frac{x^2}{\tan^2(k\pi/n)}\right).$$

PROOF. First, let $n = 2m$ be even. Then there exist integers a_j, $1 \le j \le m$, such that

$$\sin n\theta = \cos\theta \sum_{j=1}^{m} a_j \sin^{2j-1}\theta. \tag{21.1}$$

Let $\theta = \tan^{-1} x$. Thus,

$$\sin\theta = \frac{x}{\sqrt{x^2+1}} \quad \text{and} \quad \cos\theta = \frac{1}{\sqrt{x^2+1}}.$$

Then, from (21.1),

$$(x^2 + 1)^m \sin(n \tan^{-1} x) = \sum_{j=1}^{m} a_j x^{2j-1}(x^2 + 1)^{m-j}. \tag{21.2}$$

Each term of the polynomial on the right side has degree $2m - 1 = n - 1$. For each positive integer n, $\sin(n \tan^{-1} x) = 0$ when

$$x = \tan(k\pi/n), \qquad -n/2 < k < n/2.$$

Thus, these values of x are precisely the $n - 1$ zeros of the polynomial on the right side of (21.2). Thus, for some constant c,

$$(x^2 + 1)^m \sin(n \tan^{-1} x) = cx \prod_{k=1}^{(n-2)/2} \left(1 - \frac{x^2}{\tan^2(k\pi/n)}\right).$$

Letting x tend to 0 on both sides above, we readily see that $c = n$. This completes the proof of Entry 21 when n is even.

The proof when $n = 2m + 1$ is odd is similar. In this case, there exists integers a_j, $0 \le j \le m$, so that

$$\sin n\theta = \sum_{j=0}^{m} a_j \sin^{2j+1}\theta. \tag{21.3}$$

With θ defined as above,

$$(x^2+1)^{n/2}\sin(n\tan^{-1}x)=\sum_{j=0}^{m}a_jx^{2j+1}(x^2+1)^{m-j}.$$

The right side is a polynomial of degree $2m+1=n$. The remainder of the proof proceeds as above.

Entry 22 (p. 339). *Let* $\sin^{-1}z$ *take its principal value, and let* n *be a positive integer. Then*

$$\frac{\sin(n\sin^{-1}x)}{\sqrt{1-x^2}}=nx\prod_{k=1}^{(n-2)/2}\left(1-\frac{x^2}{\sin^2(k\pi/n)}\right),\qquad \text{if } n \text{ is even,} \tag{22.1}$$

and

$$\sin(n\sin^{-1}x)=nx\prod_{k=1}^{(n-1)/2}\left(1-\frac{x^2}{\sin^2(k\pi/n)}\right),\qquad \text{if } n \text{ is odd.} \tag{22.2}$$

Proof. First, let $n=2m$ be even. Put $\theta=\sin^{-1}x$. Then $\cos\theta=\sqrt{1-x^2}$. From (21.1),

$$\sin n\theta=\sqrt{1-x^2}\sum_{j=1}^{m}a_jx^{2j-1}, \tag{22.3}$$

for certain integers a_j, $1\le j\le m$. Now $\sin(n\sin^{-1}x)$ is equal to 0 when $x=\sin(k\pi/n)$, $-m\le k\le m$. Since the polynomial on the right side of (22.3) has degree $2m-1$, there exists a constant c so that

$$\frac{\sin(n\sin^{-1}x)}{\sqrt{1-x^2}}=cx\prod_{k=1}^{m-1}\left(1-\frac{x^2}{\sin^2(k\pi/n)}\right).$$

Letting x tend to 0, we deduce that $c=n$. Thus, (22.1) has been proved.

If $n=2m+1$ is odd, then from (21.3),

$$\sin(n\sin^{-1}x)=\sum_{j=0}^{m}a_jx^{2j+1},$$

where each a_j, $0\le j\le m$, is a certain integer. The polynomial of degree $n=2m+1$ on the right side has the zeros $x=\sin(k\pi/n)$, $-m\le k\le m$. Hence, for a certain constant c,

$$\sin(n\sin^{-1}x)=cx\prod_{k=1}^{m}\left(1-\frac{x^2}{\sin^2(k\pi/n)}\right).$$

As above, $c=n$, and the proof of (22.2) is complete.

In fact, Ramanujan offers only (22.2), i.e., he evidently thought that (22.2) is valid for both even and odd n. Of course, the appearance of the factor $\sqrt{1-x^2}$ in (22.1) arises from the fact that when n is even, $\sin(n\sin^{-1}x)$ has branch points at $x = \pm 1$.

Entry 23 (p. 341). *If m and n are arbitrary, then*

$$\sqrt{m\sqrt[3]{4m-8n}+n\sqrt[3]{4m+n}}$$
$$= \tfrac{1}{3}\{\sqrt[3]{(4m+n)^2}+\sqrt[3]{4(m-2n)(4m+n)}-\sqrt[3]{2(m-2n)^2}\}. \quad (23.1)$$

First Proof. Square both sides of (23.1) and collect the coefficients of $\sqrt[3]{4m-8n}$, $\sqrt[3]{4m+n}$, and $\sqrt[3]{2(4m+n)^2(m-2n)^2}$ to verify the proposed identity.

Our first proof is not very satisfactory, for it is merely a verification and provides no insight into how Ramanujan discovered this elementary, but beautiful, identity. The second (and much better) proof presented here was found by Bruce Reznick and is perhaps the one discovered by Ramanujan.

Second Proof. Observe that

$$\begin{aligned}(u^2-2uv-2v^2)^2 &= u^4-4u^3v+8uv^3+4v^4\\ &= u(u^3+8v^3)+v(4v^3-4u^3). \qquad (23.2)\end{aligned}$$

Let

$$m = u^3+8v^3 \quad \text{and} \quad n = 4v^3-4u^3.$$

Thus,

$$4m-8n = 36u^3 \quad \text{and} \quad 4m+n = 36v^3,$$

or

$$u = 6^{-2/3}(4m-8n)^{1/3} \quad \text{and} \quad v = 6^{-2/3}(4m+n)^{1/3}.$$

Hence, by (23.2),

$$m(4m-8n)^{1/3}+n(4m+n)^{1/3}$$
$$= \tfrac{1}{36}\{(4m-8n)^{2/3}-2(4m-8n)^{1/3}(4m+n)^{1/3}-2(4m+n)^{2/3}\}^2,$$

from which (23.1) follows, provided the appropriate square root is chosen.

Note that (23.1) provides a formula for representing certain linear combinations of two cube roots as a square of a linear combination of three cube roots.

Examples (p. 341). "$\sqrt[3]{4}-1$, $5-\sqrt[3]{4}$, $\sqrt[3]{5}-\sqrt[3]{4}$, $\sqrt[3]{2^7}-\sqrt[3]{7}$, $2^5-\sqrt[3]{7}$, $2^3-\sqrt[3]{5}$, $8+\sqrt[3]{17}$, $\sqrt[3]{2^7}-5$, $2^4-5\sqrt[3]{3}$, $\sqrt[3]{28}-\sqrt[3]{27}$, $3\sqrt[3]{7}-\sqrt[3]{20}$, $5+\sqrt[3]{44}$, $11+\sqrt[3]{2^8}$, etc., are perfect squares."

We have quoted Ramanujan here. Each of the 13 numbers given is an algebraic integer in a certain cubic, bicubic, or tricubic field. If a denotes one of these 13 algebraic integers, then either a, $3a$, or $9a$, in fact, is a square of an algebraic integer in the same field. Each example is a special instance of Entry 23.

In the following table, we list the parameters m and n from Entry 23; either a, $3a$, or $9a$; and the corresponding representation as a perfect square:

m	n	a, $3a$, or $9a$	Perfect Square
1	-1	$3(2^{2/3}-1)$	$(1-2^{1/3}+2^{2/3})^2$
1	5	$5-2^{2/3}$	$(1-2^{1/3}-2^{2/3})^2$
1	1	$9(5^{1/3}-2^{2/3})$	$(5^{2/3}-(20)^{1/3}-2^{1/3})^2$
2	-1	$9(2^{7/3}-7^{1/3})$	$(7^{2/3}+2(14)^{1/3}-2^{5/3})^2$
16	-1	$2^5-7^{1/3}$	$(7^{2/3}+2\cdot 7^{1/3}-2)^2$
1	8	$3(2^3-5^{1/3})$	$(2-2\cdot 5^{1/3}-5^{2/3})^2$
4	1	$9(2^3+(17)^{1/3})$	$((17)^{2/3}+2(17)^{1/3}-2)^2$
2	-5	$3(2^{7/3}-5)$	$(1+2^{4/3}-2^{5/3})^2$
8	-5	$2^4-5\cdot 3^{1/3}$	$(3^{2/3}+2\cdot 3^{1/3}-2)^2$
1	-3	$9((28)^{1/3}-3)$	$(1+2^{2/3}7^{1/3}-2^{1/3}7^{2/3})^2$
1	3	$9(3\cdot 7^{1/3}-(20)^{1/3})$	$(7^{2/3}-(4\cdot 5\cdot 7)^{1/3}-(2\cdot 5^2)^{1/3})^2$
1	-5	$9(5+(44)^{1/3})$	$(1-(4\cdot 11)^{1/3}-(2(11)^2)^{1/3})^2$
2	-11	$3(11+2^{8/3})$	$(1-2^{5/3}-2^{7/3})^2$

Entries 23 and 24 have been independently established by E. Thandapani, G. Balasubramanian, and K. Balachandran [1].

Entry 24 (p. 342). *If m and n are arbitrary, then*

$$\sqrt[3]{(m^2+mn+n^2)\sqrt[3]{(m-n)(m+2n)(2m+n)}+3mn^2+n^3-m^3}$$
$$=\sqrt[3]{\frac{(m-n)(m+2n)^2}{9}}-\sqrt[3]{\frac{(2m+n)(m-n)^2}{9}}+\sqrt[3]{\frac{(m+2n)(2m+n)^2}{9}}. \tag{24.1}$$

PROOF. Cubing both sides of (24.1) and simplifying, we find that

$$\begin{aligned}(m^2 + mn + n^2)&\sqrt[3]{(m-n)(m+2n)(2m+n)} + 3mn^2 + n^3 - m^3 \\ &= \frac{(m-n)(m+2n)^2}{9} - \frac{(2m+n)(m-n)^2}{9} + \frac{(m+2n)(2m+n)^2}{9} \\ &\quad + \tfrac{1}{3}\{-(m-n)(m+2n) + (2m+n)(m-n) + (m+2n)(2m+n)\} \\ &\quad \times \sqrt[3]{(m-n)(m+2n)(2m+n)} - \tfrac{2}{3}(m-n)(m+2n)(2m+n) \\ &= (m^2+mn+n^2)\sqrt[3]{(m-n)(m+2n)(2m+n)} + 3mn^2 + n^3 - m^3,\end{aligned}$$

and the proof is complete.

As in our first proof of Entry 23, our verification provides no insight into Ramanujan's thinking. Entry 24 can be regarded as an identity between two sums of two cube roots. Alternatively, Entry 24 provides a formula expressing certain linear combinations of a cube root and rational number as a cube of a linear combination of three cube roots.

Entry 25 (p. 348). *Let $a_0, a_1, \ldots, a_n$ be numbers such that $a_0 = 1$ and $a_k \neq 0$, $1 \leq k \leq n$. Then, for each positive integer n,*

$$\sum_{k=0}^{n-1} \frac{a_0 a_1 \cdots a_k}{(x+a_1)(x+a_2)\cdots(x+a_{k+1})} = \frac{1}{x} - \frac{1}{x(1+x/a_1)(1+x/a_2)\cdots(1+x/a_n)}. \tag{25.1}$$

PROOF. We induct on n. If $n = 1$, the proposed identity is

$$\frac{1}{x+a_1} = \frac{1}{x} - \frac{1}{x(1+x/a_1)},$$

which is readily verified. Now assume that (25.1) is valid. We wish to show that (25.1) remains valid with n replaced by $n+1$. Thus, suppose $a_{n+1} \neq 0$. By the foregoing induction hypothesis,

$$\begin{aligned}\sum_{k=0}^{n} &\frac{a_0 a_1 \cdots a_k}{(x+a_1)(x+a_2)\cdots(x+a_{k+1})} \\ &= \frac{1}{x} - \frac{1}{x(1+x/a_1)(1+x/a_2)\cdots(1+x/a_n)} \\ &\quad + \frac{a_1 a_2 \cdots a_n}{(x+a_1)(x+a_2)\cdots(x+a_{n+1})} \\ &= \frac{1}{x} - \frac{1}{x(1+x/a_1)(1+x/a_2)\cdots(1+x/a_{n+1})},\end{aligned}$$

after a little simplification.

Entry 26 (p. 348). *Let n be any real number, and let $a_0, a_1, \ldots, a_{r-1}$, $b_0, b_1, \ldots, b_r$ be numbers such that $a_k \neq 0$, $1 \leq k \leq r-1$. Then for each positive integer r,*

$$\sum_{j=0}^{r-1} (a_j^n - b_{j+1}^n)\left(\frac{b_1 \cdots b_j}{a_1 \cdots a_j}\right)^n = a_0^n - \left(\frac{b_1 \cdots b_r}{a_1 \cdots a_{r-1}}\right)^n. \tag{26.1}$$

Observe that the introduction of the parameter n is adventitious. We thus will assume without loss of generality that $n = 1$. There are two misprints in Ramanujan's formulation.

Proof. Induct on r. For $r = 1$, (26.1) is trivial. Assume that (26.1) holds. Then by induction, if $a_r \neq 0$,

$$\begin{aligned}\sum_{j=0}^{r} (a_j - b_{j+1}) \frac{b_1 \cdots b_j}{a_1 \cdots a_j} &= a_0 - \frac{b_1 \cdots b_r}{a_1 \cdots a_{r-1}} + (a_r - b_{r+1}) \frac{b_1 \cdots b_r}{a_1 \cdots a_r} \\ &= a_0 - \frac{b_1 \cdots b_{r+1}}{a_1 \cdots a_r},\end{aligned}$$

which completes the proof.

Entry 27 (p. 356). *If*

$$x^3 + ax + b = y \tag{27.1}$$

and

$$y^3 + ay + b = x, \tag{27.2}$$

then

$$(x^3 + (a-1)x + b)(x^2 + \alpha x + \alpha^2 + 1 + a)(x^2 + \beta x + \beta^2 + 1 + a) \times (x^2 + \gamma x + \gamma^2 + 1 + a) = 0, \tag{27.3}$$

where α, β, and γ are the roots of the equation

$$z^3 + (a+2)z + b = 0. \tag{27.4}$$

Proof. If $x = y$, then, by (27.1) and (27.2), each satisfies the equation

$$x^3 + (a-1)x + b = 0,$$

from which (27.3) is apparent.

If $x \neq y$, then by adding (27.1) and (27.2), subtracting (27.2) from (27.1), and dividing out a common factor $(x - y)$, we see that the given system is equivalent to the system

$$x^3 + y^3 + (a-1)(x+y) + 2b = 0 \tag{27.5}$$

and

$$x^2 + xy + y^2 + a + 1 = 0. \tag{27.6}$$

Set $x + y = -\alpha$. It then follows from (27.6) that

$$xy = -x(x + \alpha) = \alpha^2 + 1 + a. \tag{27.7}$$

Hence,

$$x^3 + y^3 = (x + y)^3 - 3xy(x + y) = -\alpha^3 + 3\alpha(\alpha^2 + 1 + a),$$

and so (27.5) takes the form

$$-\alpha^3 + 3\alpha(\alpha^2 + 1 + a) - (a - 1)\alpha + 2b = 0.$$

Thus, α is a root of (27.4), and, from (27.7), we see that x is a root of the polynomial of degree 9 in (27.3), which must be identical to the polynomial in

$$(x^3 + ax + b)^3 + a(x^3 + ax + b) - x + b = 0,$$

arising from (27.1) and (27.2). This completes the proof.

Entry 28 (p. 356). *Let p, q, r, and s be given numbers, and define x, a, b, c, and d by*

$$\begin{aligned} x &= p + q + r + s, \\ a &= qr + ps, \\ b &= pq^2 + qs^2 + rp^2 + sr^2, \\ a^2 + c &= pr^3 + qp^3 + rs^3 + sq^3 + 3pqrs, \end{aligned}$$

and

$$d = p^5 + q^5 + r^5 + s^5 + 5(ps - qr)(pq^2 - qs^2 - rp^2 + sr^2).$$

Then

$$x^5 - 5ax^3 - 5bx^2 - 5cx - d = 0.$$

Proof. Let ρ denote a fifth root of unity. Set

$$P = \prod_{\rho} (x - p\rho - q\rho^2 - r\rho^3 - s\rho^4), \tag{28.1}$$

where the product is taken over all five fifth roots of unity. Calculating P, we find that

$$P = x^5 - 5ax^3 - 5bx^2 - 5cx - d,$$

where a, b, c, and d have the given stated values. Since, furthermore, $x = p + q + r + s$, we see from (28.1) that $P = 0$. This completes the proof.

For a related investigation of the quintic equation, see a paper by Cayley [1].

Entry 29 (p. 356). *We have*

$$(\cos 40^\circ)^{1/3} + (\cos 80^\circ)^{1/3} - (\cos 20^\circ)^{1/3} = \{\tfrac{3}{2}(\sqrt[3]{9} - 2)\}^{1/3} \tag{29.1}$$

and

$$(\sec 40^\circ)^{1/3} + (\sec 80^\circ)^{1/3} - (\sec 20^\circ)^{1/3} = \{6(\sqrt[3]{9} - 1)\}^{1/3}. \tag{29.2}$$

PROOF. Setting $\theta = 20^\circ, 40^\circ$, and 80° in the elementary trigonometric identity

$$\cos^3 \theta = \tfrac{1}{4}(3 \cos \theta + \cos 3\theta),$$

we find, respectively, that $2 \cos 20^\circ$, $-2 \cos 40^\circ$, and $-2 \cos 80^\circ$ are the three roots of the cubic equation

$$x^3 - 3x - 1 = 0.$$

We now apply Entry 10 of this chapter with $a = 0$ and $b = -3$. Then (10.4) becomes

$$t^3 + 9 = 0.$$

Hence, $t = -\sqrt[3]{9}$. Moreover, from (10.2) and (10.3), respectively,

$$(2 \cos 20^\circ)^{1/3} + (-2 \cos 40^\circ)^{1/3} + (-2 \cos 80^\circ)^{1/3} = (6 - 3\sqrt[3]{9})^{1/3} \tag{29.3}$$

and

$$(-4 \cos 20^\circ \cos 40^\circ)^{1/3} + (4 \cos 40^\circ \cos 80^\circ)^{1/3} + (-4 \cos 20^\circ \cos 80^\circ)^{1/3}$$
$$= (3 - 3\sqrt[3]{9})^{1/3}. \tag{29.4}$$

Multiplying both sides of (29.3) by $(-2)^{-1/3}$, we deduce (29.1). Multiplying both sides of (29.4) by

$$\frac{(-2)^{1/3}}{(8 \cos 20^\circ \cos 40^\circ \cos 80^\circ)^{1/3}} = (-2)^{1/3},$$

we readily deduce (29.2).

The identity (29.1) was proposed as a problem by Ramanujan in the *Journal of the Indian Mathematical Society* [7], [23, p. 329]. Another identity involving multiples of $\pi/7$ (instead of multiples of $\pi/9$) was also proposed by Ramanujan in the same problem.

Entry 30 (p. 356). *If* α, β, *and* γ *are the roots of*

$$x^3 - ax^2 + bx - 1 = 0,$$

then

$$(\alpha/\beta)^{1/3} + (\beta/\gamma)^{1/3} + (\gamma/\alpha)^{1/3} \quad \textit{and} \quad (\beta/\alpha)^{1/3} + (\gamma/\beta)^{1/3} + (\alpha/\gamma)^{1/3} \tag{30.1}$$

are the roots of

$$z^2 - tz + a + b + 3 = 0, \tag{30.2}$$

where

$$t^3 - 3(a + b + 3)t - (ab + 6a + 6b + 9) = 0. \tag{30.3}$$

The quantities (30.1) are also roots of

$$y^6 - (ab + 6a + 6b + 9)y^3 + (a + b + 3)^3 = 0. \tag{30.4}$$

Lastly,

$$\alpha^{1/3} + \beta^{1/3} + \gamma^{1/3}$$
$$= \left(a + 6 + 3\left\{\frac{ab + 9}{2} + 3a + 3b + \left(\frac{(ab + 9)^2}{4} - a^3 - b^3 - 27\right)^{1/2}\right\}^{1/3}\right.$$
$$\left. + 3\left\{\frac{ab + 9}{2} + 3a + 3b - \left(\frac{(ab + 9)^2}{4} - a^3 - b^3 - 27\right)^{1/2}\right\}^{1/3}\right)^{1/3}.$$

PROOF. This entry is a sequel to an earlier entry of this chapter, Entry 10. Thus, using (10.1) and (10.2), we find that

$$\begin{aligned}
&(\alpha/\beta)^{1/3} + (\beta/\gamma)^{1/3} + (\gamma/\alpha)^{1/3} + (\beta/\alpha)^{1/3} + (\gamma/\beta)^{1/3} + (\alpha/\gamma)^{1/3}\\
&\quad = (\alpha^2\gamma)^{1/3} + (\beta^2\alpha)^{1/3} + (\gamma^2\beta)^{1/3} + (\beta^2\gamma)^{1/3} + (\gamma^2\alpha)^{1/3} + (\alpha^2\beta)^{1/3}\\
&\quad = \tfrac{1}{3}\{(\alpha^{1/3} + \beta^{1/3} + \gamma^{1/3})^3 - (\alpha + \beta + \gamma) - 6(\alpha\beta\gamma)^{1/3}\}\\
&\quad = \tfrac{1}{3}\{(a + 6 + 3t) - a - 6\}\\
&\quad = t.
\end{aligned}$$

Also,

$$\begin{aligned}
&\{(\alpha/\beta)^{1/3} + (\beta/\gamma)^{1/3} + (\gamma/\alpha)^{1/3}\}\{(\beta/\alpha)^{1/3} + (\gamma/\beta)^{1/3} + (\alpha/\gamma)^{1/3}\}\\
&\quad = 3 + (\alpha^2/(\beta\gamma))^{1/3} + (\beta^2/(\gamma\alpha))^{1/3} + (\gamma^2/(\alpha\beta))^{1/3} + (\beta\gamma/\alpha^2)^{1/3}\\
&\qquad + (\gamma\alpha/\beta^2)^{1/3} + (\alpha\beta/\gamma^2)^{1/3}\\
&\quad = 3 + \alpha + \beta + \gamma + \beta\gamma + \gamma\alpha + \alpha\beta\\
&\quad = 3 + a + b.
\end{aligned}$$

Thus, we have shown that the expressions (30.1) are the roots of (30.2).

For brevity, set $c = a + b + 3$ and $d = ab + 6a + 6b + 9$, so that (30.2) and (30.3) assume the respective forms

$$z^2 - tz + c = 0 \quad \text{and} \quad t^3 - 3ct - d = 0.$$

Thus, $t = (z^2 + c)/z$, and so

$$\left(\frac{z^2 + c}{z}\right)^3 - 3c\,\frac{z^2 + c}{z} - d = 0,$$

which upon simplification yields the equation

$$z^6 - dz^3 + c^3 = 0.$$

Thus, (30.4) has been shown.

Lastly, employing Cardan's solution of the cubic equation (Hall and Knight [1, p. 480]), we find that the solutions of (30.3) are given by

$$\begin{aligned} t &= \left\{\frac{d}{2} + \left(\frac{d^2}{4} - c^3\right)^{1/2}\right\}^{1/3} + \left\{\frac{d}{2} - \left(\frac{d^2}{4} - c^3\right)^{1/2}\right\}^{1/3} \\ &= \left\{\frac{ab + 9}{2} + 3a + 3b + \left(\frac{(ab + 9)^2}{4} - a^3 - b^3 - 27\right)^{1/2}\right\}^{1/3} \\ &\quad + \left\{\frac{ab + 9}{2} + 3a + 3b - \left(\frac{(ab + 9)^2}{4} - a^3 - b^3 - 27\right)^{1/2}\right\}^{1/3}, \end{aligned} \tag{30.5}$$

since

$$\frac{d^2}{4} - c^3 = \frac{(ab + 9)^2}{4} - a^3 - b^3 - 27.$$

Substituting (30.5) into (10.2), we complete the proof.

Page 356 is the last page in the Tata Institute's publication of Ramanujan's second notebook [22].

Entry 31 (p. 366). *We have the following numerical values*:

$$\log\left(\frac{2\pi}{\log 2}\right) = 2.20438999,$$

$$\frac{2\pi^2}{\log 2} = 28.4776587,$$

and

$$\frac{2\pi}{\log 2} = 9.0647203.$$

We have corrected Ramanujan's value of $\log(2\pi/\log 2)$; he gives the erroneous value 2.20437894. It is unclear why Ramanujan recorded these three numerical evaluations.

Entry 32 (p. 367). *The equalities*

$$\begin{aligned} x^2 &= a + y, \\ y^2 &= a + z, \\ z^2 &= a + u, \end{aligned} \tag{32.1}$$

and

$$u^2 = a + x$$

determine a polynomial in x (*or* y *or* z *or* u) *of degree* 16. *This polynomial can be factored into a product of four quartic polynomials, one of which is* $x^4 - 2ax^2 - x + a^2 - a$. *Each of the remaining quartic polynomials has the form*

$$(x^2 + px + \tfrac{1}{2}\{p^2 - 2a - 1/p\})(x^2 + qx + \tfrac{1}{2}\{q^2 - 2a - 1/q\}), \tag{32.2}$$

where $pq = -1$ *and* $p + q$ *is a root of the polynomial equation*

$$z^3 + 3z = 4(1 + az), \tag{32.3}$$

or, alternatively, p *is a root of*

$$z^6 - 4az^4 - 4z^3 + 4az^2 - 1 = 0. \tag{32.4}$$

Entry 32 is a more detailed version of a question proposed by Ramanujan [12], [23, p. 332] in the *Journal of the Indian Mathematical Society*. Several years later, G. N. Watson [1] published a solution that is different from the one that we now give. A similar problem involving just three of the equalities (32.1) is found in Entry 4.

PROOF. By using the equalities (32.1) successively and in reverse order, we readily find that

$$a + x = (((x^2 - a)^2 - a)^2 - a)^2, \tag{32.5}$$

which is a polynomial of degree 16, as claimed by Ramanujan. Let $F(x)$ denote the difference of the right and left sides in (32.5).

Now,

$$\begin{aligned} x &= (a + y)^{1/2} = (a + (a + z)^{1/2})^{1/2} = (a + (a + (a + u)^{1/2})^{1/2})^{1/2} \\ &= (a + (a + (a + (a + x)^{1/2})^{1/2})^{1/2})^{1/2} \\ &= (a + (a + (a + (a + \ldots)^{1/2})^{1/2})^{1/2})^{1/2} \\ &= (a + (a + x)^{1/2})^{1/2}. \end{aligned} \tag{32.6}$$

Simplifying the last equality, we find that

$$x^4 - 2ax^2 - x + a^2 - a = 0. \tag{32.7}$$

This polynomial is the simple quartic factor claimed by Ramanujan. Of course, by the same reasoning,

$$x = (a + x)^{1/2},$$

and so $x^2 - x - a$ is a quadratic factor of $F(x)$ (and of (32.7)).

The reasoning given above is not completely rigorous, because it depends upon the (unproven) convergence of the infinite sequence of nested radicals (32.6). However, using *Mathematica*, we find that

$$\begin{aligned}F(x) = (x^2 - x - a)(x^2 + x - a + 1)(x^{12} &- 6ax^{10} + x^9 \\ &+ (15a^2 - 3a)x^8 - 4ax^7 + (-20a^3 + 12a^2 + 1)x^6 \\ &+ (6a^2 - 2a)x^5 + (15a^4 - 18a^3 + 3a^2 - 4a)x^4 + (-4a^3 + 4a^2 + 1)x^3 \\ &+ (-6a^5 + 12a^4 - 6a^3 + 5a^2 - a)x^2 + (a^4 - 2a^3 + a^2 - 2a)x \\ &+ a^6 - 3a^5 + 3a^4 - 3a^3 + 2a^2 + 1).\end{aligned}$$

Let $f(x)$ denote the polynomial of degree 12 given above. Let α, β, γ, δ, α', β', γ', δ', α'', β'', γ'', and δ'' denote the 12 roots of $f(x)$. Now y, z, and u are also roots of $f(x)$. Thus, without loss of generality, we may suppose that

$$\begin{aligned}
\alpha^2 &= a + \beta, & a'^2 &= a + \beta', & \alpha''^2 &= a + \beta'', \\
\beta^2 &= a + \gamma, & \beta'^2 &= a + \gamma', & \beta''^2 &= a + \gamma'', \\
\gamma^2 &= a + \delta, & \gamma'^2 &= a + \delta', & \gamma''^2 &= a + \delta'', \\
\delta^2 &= a + \alpha, & \delta'^2 &= a + \alpha', & \delta''^2 &= a + \alpha''.
\end{aligned}$$

We shall consider only α, β, γ, and δ below. Of course, identical treatments can be given for the other two sets of four roots.

Let

$$\begin{aligned}
r = S_1 &= \alpha + \beta + \gamma + \delta, \\
S_2 &= \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta, \\
S_3 &= \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta,
\end{aligned} \tag{32.8}$$

and

$$S_4 = \alpha\beta\gamma\delta.$$

First, it is easy to see that

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 4a + r.$$

Squaring r and using the last equality, we find that

$$r^2 = 4a + r + 2S_2,$$

or

$$S_2 = \tfrac{1}{2}(r^2 - r - 4a). \tag{32.9}$$

It is very difficult to directly determine S_3 as a function of r and a. We will proceed indirectly and first determine S_4.

Let

$$\begin{aligned} A = {} & \alpha^2(\beta\gamma + \beta\delta + \gamma\delta) + \beta^2(\alpha\gamma + \alpha\delta + \gamma\delta) \\ & + \gamma^2(\alpha\beta + \alpha\delta + \beta\delta) + \delta^2(\alpha\beta + \alpha\gamma + \beta\gamma) \end{aligned}$$

and

$$B = \alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2.$$

Then

$$S_2^2 = 2A + B + 6S_4$$

and

$$S_1S_3 = A + 4S_4.$$

Hence,

$$S_2^2 = 2S_1S_3 + B - 2S_4. \tag{32.10}$$

Now,

$$\begin{aligned} B &= (a + \beta)(a + \gamma) + (a + \beta)(a + \delta) + \cdots + (a + \delta)(a + \alpha) \\ &= 6a^2 + S_2 + 3aS_1. \end{aligned} \tag{32.11}$$

Using (32.11) in (32.10) and solving for S_4, we deduce that

$$S_4 = 3a^2 + \tfrac{1}{2}S_2 + \tfrac{1}{2}3aS_1 + S_1S_3 - \tfrac{1}{2}S_2^2. \tag{32.12}$$

Thus, so far, by (32.8) and (32.9), the fourth degree polynomial with roots α, β, γ, and δ is given by

$$x^4 - rx^3 + \tfrac{1}{2}(r^2 - r - 4a)x^2 - S_3x + S_4,$$

where S_3 and S_4 are related by (32.12).

Let us now assume that this polynomial can be represented in the form (32.2). Then $p + q = -r$ and

$$pq + \tfrac{1}{2}(p^2 - 2a - 1/p) + \tfrac{1}{2}(q^2 - 2a - 1/q) = \tfrac{1}{2}(r^2 - r - 4a). \tag{32.13}$$

Using the facts $pq = -1$ and $p + q = -r$, we can easily verify that, indeed, (32.13) is valid. Furthermore,

$$\begin{aligned} -2S_3 &= p^2q - 2aq - q/p + pq^2 - 2ap - p/q \\ &= -(p+q) - 2a(p+q) + p^2 + q^2 \\ &= r + 2ar + r^2 + 2 \end{aligned} \tag{32.14}$$

and

$$\begin{aligned} 4S_4 &= (p^2 - 2a + q)(q^2 - 2a + p) \\ &= p^3 + q^3 + p^2q^2 - 2a(p^2 + q^2) - 2a(p+q) + pq + 4a^2 \\ &= -r^3 - (3p^2q + 3pq^2) - 2ar^2 - 4a + 2ar + 4a^2 \\ &= -r^3 - 2ar^2 - 3r + 2ar + 4a^2 - 4a, \end{aligned} \tag{32.15}$$

where we have repeatedly used the equalities $pq = -1$ and $p + q = -r$.

We must show that (32.14) and (32.15) are compatible with (32.12). Suppose that we substitute (32.15) into (32.12) and solve for S_3. Using also (32.8) and (32.9) and omitting the algebra, we find that

$$S_3 = \tfrac{1}{8}r^3 - \tfrac{1}{2}r^2 - \tfrac{1}{8}r - \tfrac{3}{2}ar - \tfrac{1}{2}. \tag{32.16}$$

Equating the two formulas for S_3 given in (32.14) and (32.16), we find that

$$r^3 + 3r - 4ar + 4 = 0.$$

But since $p + q = -r$, this last equality is equivalent to (32.3). Thus, (32.12), (32.14), and (32.15) are in agreement.

Lastly, since $q = -1/p$, it is a simple matter to show that if $p + q$ satisfies (32.3), then p is a root of (32.4).

Corollary. *We have*

$$\begin{aligned} &(5 + (5 + (5 - (5 + (5 + (5 + (5 - (5 + \cdots)^{1/2})^{1/2})^{1/2})^{1/2})^{1/2})^{1/2})^{1/2})^{1/2} \\ &\qquad = \tfrac{1}{2}(2 + \sqrt{5} + \sqrt{15 - 6\sqrt{5}}) \end{aligned} \tag{32.17}$$

and

$$\begin{aligned} &(5 + (5 - (5 - (5 + (5 + (5 - (5 - (5 + \cdots)^{1/2})^{1/2})^{1/2})^{1/2})^{1/2})^{1/2})^{1/2})^{1/2} \\ &\quad = \tfrac{1}{4}(\sqrt{5} - 2 + \sqrt{13 - 4\sqrt{5}} + \sqrt{50 + 12\sqrt{5} - 2\sqrt{65 - 20\sqrt{5}}}). \end{aligned} \tag{32.18}$$

These infinite radical expansions were given by Ramanujan as part of the problem [12], [23, p. 332] from which Entry 32 arises. D. Mick [1] has also established (32.17).

Proof. Let $a_1 = \sqrt{5}$, $a_2 = \sqrt{5+\sqrt{5}}$, and, in general, let a_n, $n \geq 1$, denote the nth nested radical in (32.17). Next, let

$$f(t) = \sqrt{5+\sqrt{5+\sqrt{5-t}}} \quad \text{and} \quad g(t) = \sqrt{5+t},$$

where $0 \leq t \leq 5$. Clearly, f is decreasing and g is increasing on this interval. In particular,

$$\begin{aligned} f(0) = a_3 > f(\sqrt{5}) = f(g(0)) = a_4 > f(g(\sqrt{5})) \\ = a_5 > f(g(\sqrt{5+\sqrt{5}})) = a_6 > f(g(a_3)) = a_7 \end{aligned}$$

and

$$a_7 < f(g(a_4)) = a_8 < f(g(a_5)) = a_9 < f(g(a_6)) = a_{10} < f(g(a_7)) = a_{11}.$$

In general, it follows easily by induction that

$$\left.\begin{array}{l} a_{8n+3} > a_{8n+4} > a_{8n+5} > a_{8n+6} > a_{8n+7} \\ \text{and} \\ a_{8n+7} < a_{8n+8} < a_{8n+9} < a_{8n+10} < a_{8n+11} \end{array}\right\}, \tag{32.19}$$

for each nonnegative integer n. By successively squaring and then adding or subtracting 5, and by using induction we also can easily show that

$$2 < a_7 < a_{15} < a_{23} < \cdots < a_{8n+7} < a_{8n+11} < a_{8n+3} < \cdots < a_{19} < a_{11} < a_3 < 5,$$

for every nonnegative integer n. Thus $\{a_{8n+7}\}$ is a bounded, increasing sequence and so converges, say to α. Also, $\{a_{8n+3}\}$ is a bounded, decreasing sequence and so converges, say to β.

Observe that $a_{8n+7} = f(g(a_{8n+3}))$. Letting n tend to ∞ and using the continuity of f and g, we deduce that

$$\alpha = f(g(\beta)).$$

Since $a_{8n+11} = f(g(a_{8n+7}))$, we also deduce that

$$\beta = f(g(\alpha)).$$

If $h = f \circ g$, α and β are fixed points of $h \circ h$. It is easy to see that $0 < (h \circ h)'(t) < 1$ for $a_7 \leq t \leq a_3$. Thus, the equation $h \circ h(t) = t$ can have *at most* one root on this interval. Hence, $\alpha = \beta$. Therefore, $\{a_{4n+3}\}$ is convergent.

But now, from (32.19), we may also conclude that $\{a_n\}$ converges. Clearly, this limit is one of the 16 roots of $F(x)$ when $a = 5$.

A similar argument shows that the infinite sequence of radicals in (32.18) also converges to a root of $F(x)$ when $a = 5$.

Let $a = 5$ in Entry 32. Then by (32.3), $p + q$ is a root of

$$z^3 - 17z - 4 = (z^2 - 4z - 1)(z + 4) = 0. \tag{32.20}$$

Let $p + q = -4$. Then since $pq = -1$, we readily find that $p, q = -2 \pm \sqrt{5}$. Hence, the polynomial (32.2) is given by

$$(x^2 + (-2 + \sqrt{5})x + (-3 - 5\sqrt{5})/2)(x^2 + (-2 - \sqrt{5})x + (-3 + 5\sqrt{5})/2)$$
$$= x^4 - 4x^3 - 4x^2 + 31x - 29.$$

The four roots of this polynomial are

$$\tfrac{1}{2}(2 \pm \sqrt{5} \pm \sqrt{15 - 6\sqrt{5}}).$$

Thus, indeed $\frac{1}{2}(2 + \sqrt{5} + \sqrt{15 - 6\sqrt{5}}) = 2.74723827$ is a root of $F(x)$. It remains to show that this root has the infinite radical expansion given in (32.17).

Using *Mathematica*, we numerically calculated the 16 (real) roots of $F(x)$. The root $\frac{1}{2}(2 + \sqrt{5} + \sqrt{15 - 6\sqrt{5}})$ is the only one of these 16 roots which numerically satisfies the requisite equality

$$x = (5 + (5 + (5 - (5 + x)^{1/2})^{1/2})^{1/2})^{1/2}.$$

This then completes the proof of (32.17).

Return to (32.20) and now let $p + q$ be the root $2 - \sqrt{5}$. Since $pq = -1$, we find that

$$p, q = \tfrac{1}{2}(2 - \sqrt{5} \pm \sqrt{13 - 4\sqrt{5}}).$$

After a little algebraic calculation, we find that the polynomial (32.2) is given by

$$(x^2 + \tfrac{1}{2}(2 - \sqrt{5} + \sqrt{13 - 4\sqrt{5}})x + \tfrac{1}{4}\{-7 - 5\sqrt{5} + (1 - \sqrt{5})\sqrt{13 - 4\sqrt{5}}\})$$
$$\times (x^2 + \tfrac{1}{2}(2 - \sqrt{5} - \sqrt{13 - 4\sqrt{5}})x + \tfrac{1}{4}\{-7 - 5\sqrt{5} - (1 - \sqrt{5})\sqrt{13 - 4\sqrt{5}}\}).$$

The four roots of this polynomial are

$$\tfrac{1}{4}(-2 + \sqrt{5} \pm \sqrt{13 - 4\sqrt{5}}) \pm \sqrt{50 + 12\sqrt{5} \pm 2\sqrt{5}\sqrt{13 - 4\sqrt{5}}}),$$

where, in each root, the signs of the radicals $\sqrt{13 - 4\sqrt{5}}$ are opposite. Hence, the right side of (32.18), i.e.,

$$\tfrac{1}{4}(\sqrt{5} - 2 + \sqrt{13 - 4\sqrt{5}} + \sqrt{50 + 12\sqrt{5} - 2\sqrt{65 - 20\sqrt{5}}}) = 2.62140838,$$

indeed, is a root of $F(x)$. It remains to show that (32.18) has the given infinite radical representation. Using the numerically calculated roots of $F(x)$, we find that the right side of (32.18) is the only root which numerically solves the requisite equality

$$x = (5 + (5 - (5 - (5 + x)^{1/2})^{1/2})^{1/2})^{1/2}.$$

Hence, the proof of (32.18) is complete.

Although the numerical aspects in our proof of the corollary may be aesthetically objectionable to some, undoubtedly Ramanujan used similar reasoning.

J. M. Borwein and G. deBarra [1] have examined a class of nested radicals that includes those of Sections 5 and 32 above. See also a paper by S. Landau [1] for recent work on nested radicals, or consult a short, less technical description of Landau's work by B. Cipra [1].

Entry 33 (pp. 375, 376, 383). *The following are approximations to* π:

$$\left(3^4 + 2^4 + \frac{1}{2 + (\frac{2}{3})^2}\right)^{1/4} = (97\tfrac{9}{22})^{1/4}$$

$$= 3.14159265262\ldots, \tag{33.1}$$

$$\frac{9}{5} + \sqrt{\frac{9}{5}} = 3.14164\ldots, \tag{33.2}$$

$$\frac{63}{25}\,\frac{17 + 15\sqrt{5}}{7 + 15\sqrt{5}} = 3.14159265380\ldots, \tag{33.3}$$

$$\frac{7}{3}\left(1 + \frac{\sqrt{3}}{5}\right) = 3.14162\ldots, \tag{33.4}$$

$$\frac{19}{16}\sqrt{7} = 3.14180\ldots, \tag{33.5}$$

and

$$\frac{355}{113}\left(1 - \frac{0.0003}{3533}\right) = 3.14159265358979432\ldots. \tag{33.6}$$

For comparative purposes, we record that

$$\pi = 3.14159265358979323846264\,3\ldots.$$

Thus, the first approximation (33.1) is valid for eight decimal places. Ramanujan also gives this approximation in Section 3 of Chapter 18 [22] (Part III [6, p. 151]) and in his famous paper [10], [23, p. 35] on approximations to π. N. D. Mermin [1], [2] has suggested that Ramanujan might have discovered (33.1) by the following reasoning. In the decimal expansion $\pi^4 = 97.409091034002\ldots$, observe that the pair of digits 09 appears twice in succession followed by the pair 10, which is "close" to 09. Thus

$$97.40909090909\ldots = \frac{2143}{22} = 97\,\frac{9}{22}$$

is a natural approximation to π^4.

The approximations (33.2)–(33.5) are also found in Ramanujan's paper [10], [23, pp. 34–35] and are valid for 3, 9, 3, and 3 decimal places, respectively. Ramanujan's value for the left side of (33.5) is not quite correct; the correct decimal expansion is given by 3.141829682.... For a detailed account of how algebraic approximations to π can be derived from the theory of elliptic functions, see the books by J. M. and P. B. Borwein [1, Chap. 5] and the author [9, Chap. 34].

The last approximation (33.6) holds for 14 decimal places. It is also given in Ramanujan's paper [10], [23, p. 35] and in Chapter 18, Section 3 [22] (Part III [6, p. 151]). Observe that the approximation 355/113 arises from approximating π by the third convergent in the simple continued fraction for π,

$$\pi = 3 + \frac{1}{7+} \frac{1}{15+} \frac{1}{1+} \cdots.$$

According to P. Beckmann [1, p. 101], the approximation 355/113 is due to A. Anthoniszoon (1527–1607).

Other approximations to π, arising from certain values of the exponential function, may be found on pages 374–375 [23]. See Entries 35–37 of Chapter 23.

Lastly, we remark that informative, historical accounts of π have been given by Beckmann [1] and D. Castellanos [1], [2].

Entry 34 (p. 384). *We have*

$$2^{1/3} = 1.259921049894873164767208....$$

The first 18 of the 24 displayed decimal places are correct. In fact,

$$2^{1/3} = 1.259921049894873164267210....$$

Entry 35 (p. 384). *We have*

$$\frac{5}{4}\left(1 + \frac{24}{1000}\right)^{1/3} = \frac{63}{50}\left(1 + \frac{188}{1000000}\right)^{-1/3}.$$

In fact, elementary arithmetic shows that each side above is equal to $\sqrt[3]{2}$. Undoubtedly, Ramanujan used one of these representations for $\sqrt[3]{2}$ in determining his approximation to $\sqrt[3]{2}$ given in the previous entry. Entry 35 was also verified by D. Somasundaram [3], [4].

In a faded (or incompletely erased?) entry on page 387, Ramanujan claims that

$$(x + \tfrac{1}{4})^4 - (x - \tfrac{1}{4})^4 = (x + 1/\sqrt{a})^3 + (x - 1/\sqrt{a})^3, \qquad (36.1)$$

where it appears that $a = 3$, or possibly 2. Since the left and right sides of (36.1) equal

$$2x^3 + \tfrac{1}{8}x \quad \text{and} \quad 2x^3 + \frac{6}{a}x,$$

respectively, equality in (36.1) holds if and only if $a = 48$.

CHAPTER 23

Number Theory

Ramanujan's strong interest in number theory probably commenced no more than one or two years before his first letter to G. H. Hardy. Most of the results on number theory found in the notebooks lie in the 100 unorganized pages at the end of the second notebook and in the short third notebook consisting of 33 pages. Most of the material on these 133 pages was probably recorded in approximately the years 1912–1914, before Ramanujan departed for Cambridge. However, there is some evidence, especially from the material on number theory, that several entries in the third notebook were recorded in England.

Ramanujan communicated many claims in the theory of numbers in his letters to Hardy. Many of these are found in the notebooks. Because Ramanujan had only recently begun to think about properties of numbers, some of his claims are tentative. Some have question marks after them, giving evidence that Ramanujan himself knew that he did not have complete proofs and that he had arrived at his conclusions by some type of heuristic reasoning. In contrast to Ramanujan's work in other areas where he made very few serious mistakes, Ramanujan's reasoning in the theory of numbers was often speculative and faulty. But it should be emphasized that Ramanujan's results are usually correct, although his approximations generally are not as accurate as he thought.

Quite surprisingly, Ramanujan often employed Riemann–Stieltjes integrals, which he probably discovered himself. However, his working knowledge was not set on a firm foundation, and he frequently made errors. For example, he would show that two integrals are asymptotically equal as a certain parameter tends to a limit. He would then conclude that the integrands are equal. One of the measures might be a step function, e.g., the summatory function of an arithmetical function. By equating integrands, he would therefore obtain a formula for the "derivative" of this summatory function. He then would claim that an integration gives an asymptotic formula for the summatory function itself.

A majority of the results on number theory recorded by Ramanujan at the end of the second notebook and in the third notebook are well known to us today. Some are quite elementary, while most of the deeper results were communicated by Ramanujan in his letters to Hardy. In this chapter, we examine all of these contributions to number theory, except those pertaining to the theory of prime numbers, the subject of Chapter 24.

One of the first results in Ramanujan's first letter [23, p. xxiv] to Hardy is an asymptotic formula for counting the number of integers that can be represented as a sum of two squares. Unknown to Ramanujan, E. Landau had discovered and proved this asymptotic formula only a few years earlier. An account of Landau's proof is beautifully presented in Hardy's book [7, pp. 60–63]. Ramanujan's argument was not known and was thought to be unrigorous, because complex analysis was apparently needed to provide a rigorous proof. Quite surprisingly, it has not been heretofore realized that one entire page in the third notebook (page 363 of Volume 2 [22]) is devoted to Ramanujan's heuristic derivation of this asymptotic formula. There are only a handful of sketches of proofs in the three notebooks, and Ramanujan's "proof" of this asymptotic formula appears to be the most lengthy "proof" in the notebooks.

Ramanujan's very famous paper [16], [23, pp. 78–128] on highly composite numbers has been the cornerstone of considerable research for three-quarters of a century. In the notebooks, one can find his initial thoughts and discoveries on this subject. In particular, Ramanujan first defined a highly composite number to be an integer having only the prime factors 2, 3, 5, and 7.

Determining the general solution of the diophantine equation

$$A^3 + B^3 = C^3 + D^3 \tag{0.1}$$

is a famous problem in number theory. The most general solution is due to Euler. In Chapter 18 of the second notebook, Ramanujan offers a less general solution, but on page 387 of the third notebook, Ramanujan gives a version of Euler's solution that is perhaps simpler than those found by Euler and several later mathematicians. Ramanujan had remarkable algebraic skills which enabled him to derive several beautiful identities involving powers, and so he examined several other diophantine equations akin to (0.1). Many of these algebraic identities are new, and it has been difficult for us to ascertain the methods Ramanujan might have used. The most remarkable of all these algebraic identities is certainly Entry 45.

In the unorganized pages of the second and third notebooks, Ramanujan considers lattice point problems in addition to the one on sums of squares described above. For example, Ramanujan estimates two special cases of the sum

$$\sum_{m^\alpha n^\beta \le x} 1,$$

a problem examined by Landau [2] at about the same time. One of these instances arises in a theorem (Entry 38) providing elegant and sharp bounds for a sum involving the greatest integer function.

Ramanujan's first published paper [2] is on Bernoulli numbers, and the unorganized material contains further work of Ramanujan on Bernoulli numbers.

Toward the end of the third notebook (p. 390) are some elementary, arithmetical calculations. These calculations suggest two conjectures, one of which, it seems to us, is very remarkable.

Ramanujan's second notebook contains 21 chapters of reasonably organized material followed by 100 pages of mostly unorganized results. However, prior to the 21 chapters are three additional pages. The first two pages (pages 2–3 in the pagination of [22]) comprise a table of 87 pairs $(d(n), n)$, where $d(n)$ denotes the number of positive divisors of n, and n is a highly composite number. A number n is said to be *highly composite* if for every positive integer $m < n$, $d(m) < d(n)$. This table was extended to include the first 102 pairs in Ramanujan's paper [16] on highly composite numbers and several related topics. However, Ramanujan missed 293, 318, 625, 600, a number with 5040 divisors. This pair was added to the table when Ramanujan's *Collected Papers* [23] were published. The largest table of which we are aware was computed by G. Robin [1] and contains the first 5000 highly composite numbers.

Let $Q(x)$ denote the number of highly composite numbers less than or equal to x. In his epic memoir [16], Ramanujan proved that $\lim_{x\to\infty} Q(x)/\log x = +\infty$. In 1944, P. Erdös [1] showed that there exists a constant $c_1 > 0$ such $Q(x) \gg (\log x)^{1+c_1}$, while, on the other hand, J.-L. Nicolas [1] proved in 1971 that $Q(x) \ll (\log x)^{c_2}$, for some positive constant c_2. The precise rate of growth of $Q(x)$ remains unknown. A very comprehensive and informative survey on what was known about highly composite numbers and related numbers at the time of Ramanujan's one hundredth birthday has been written by Nicolas [2].

In the notes on Ramanujan's paper [16] in [23, p. 339], it is mentioned that, even though his paper [16] is quite long, part of it was suppressed to help save expenses of the financially troubled London Mathematical Society in 1915. The remainder of [16] was finally published in the publication of Ramanujan's "lost notebook" in 1987 [24, pp. 280–308]. Short accounts of this fragment have been given by Nicolas [2], [3]. The complete text with fuller annotation has been prepared by J.-L. Nicolas and G. Robin [1].

Elementary introductions to Ramanujan's composite numbers have been written by R. Honsberger [1] and R. V. Andree [1].

Ramanujan continues his work on highly composite numbers in the third notebook; in particular, on page 372 of [22].

On page 3 of the second notebook, Ramanujan shows how to generate an infinite set of solutions to

$$A^3 + B^3 + C^3 = D^3, \tag{1.1}$$

provided one solution $\{p, q, r, s\}$ is known. In Entry 50 in this chapter, Ramanujan gives a general solution of (1.1) in integers.

Entry 1 (p. 3). *If*

$$p^3 + q^3 + r^3 = s^3, \tag{1.2}$$

$$m = \pm(s+q)\left(\frac{s-q}{r+p}\right)^{1/2} \quad \textit{and} \quad n = \pm(r-p)\left(\frac{r+p}{s-q}\right)^{1/2}, \tag{1.3}$$

and a and b are arbitrary, then

$$(pa^2 + mab - rb^2)^3 + (qa^2 - nab + sb^2)^3 + (ra^2 - mab - pb^2)^3$$
$$= (sa^2 - nab + qb^2)^3. \tag{1.4}$$

Proof. Setting $x = a/b$, by (1.4), we then have to show that, if (1.2) and (1.3) are satisfied, then $f(x) \equiv 0$, where

$$f(x) = (px^2 + mx - r)^3 + (qx^2 - nx + s)^3 + (rx^2 - mx - p)^3 - (sx^2 - nx + q)^3.$$

Now, $x^6 f(1/x) = -f(x)$, and so f has the form

$$f(x) = A(x^6 - 1) + Bx(x^4 - 1) + Cx^2(x^2 - 1) + Dx^3.$$

Note that $D = 0$, since $D = f(1)$ and $f(1) = -f(1)$. It is easily verified that

$$A = p^3 + q^3 + r^3 - s^3, \tag{1.5}$$

$$B = 3[(p^2 - r^2)m + (s^2 - q^2)n], \tag{1.6}$$

and

$$C = 3[(p+r)m^2 + (q-s)n^2 - pr(p+r) + qs(q-s)]. \tag{1.7}$$

By (1.2) and (1.5), $A = 0$. Setting $B = 0$ and $C = 0$, and solving (1.6) and (1.7) simultaneously, we find that $f(x) \equiv 0$ if and only if (1.3) holds.

Entries 2 and 3 lead to Entry 4, another result on Euler's diophantine equation.

Entry 2 (p. 266). *If*

$$x + na^2 = y + nab = z + nb^2 = (a + b)^2,$$

then

$$x^2 + (n - 2)xz + z^2 = ny^2. \tag{2.1}$$

Proof. The most simple-minded approach is to observe that (2.1) is equivalent to the identity

$$\begin{aligned}\{(a + b)^2 - na^2\}^2 &+ (n - 2)\{(a + b)^2 - na^2\}\{(a + b)^2 - nb^2\} \\ &+ \{(a + b)^2 - nb^2\}^2 \\ &= n\{(a + b)^2 - nab\}^2,\end{aligned}$$

which can be readily verified by slightly tedious elementary algebra.

Using a more elegant approach, we observe that

$$(x - z)^2 = n^2(a^2 - b^2)^2 \tag{2.2}$$

and that

$$\begin{vmatrix} x & y \\ y & z \end{vmatrix} = \begin{vmatrix} (a + b)^2 - na^2 & (a + b)^2 - nab \\ (a + b)^2 - nab & (a + b)^2 - nb^2 \end{vmatrix} = -n(a + b)^2(a^2 - 2ab + b^2). \tag{2.3}$$

Multiplying both sides of (2.3) by n and using (2.2), we complete the proof.

Entry 3 (p. 266). *Let p, q, and r satisfy the equalities*

$$p + 3a^2 = q + 3ab = r + 3b^2 = (a + b)^2.$$

Let m and n be arbitrary. Then

$$n(mp + nq)^3 + m(mq + nr)^3 = m(np + mq)^3 + n(nq + mr)^3.$$

Proof. By elementary algebra,

$$\begin{aligned}n(mp + nq)^3 + m(mq + nr)^3 &- m(np + mq)^3 - n(nq + mr)^3 \\ &= (m^3n - mn^3)(p^3 - r^3 - 3pq^2 + 3rq^2) \\ &= (m^3n - mn^3)(p - r)(p^2 + r^2 + pr - 3q^2).\end{aligned} \tag{3.1}$$

Apply Entry 2 with $n = 3$, $p = x$, $q = y$, and $r = z$. Thus,

$$p^2 + pr + r^2 = 3q^2. \tag{3.2}$$

Using (3.2) in (3.1), we complete the proof.

Entry 4 (p. 266). *If a and b are arbitrary, then*

$$(3a^2 + 5ab - 5b^2)^3 + (4a^2 - 4ab + 6b^2)^3 + (5a^2 - 5ab - 3b^2)^3 = (6a^2 - 4ab + 4b^2)^3.$$

FIRST PROOF. In the previous entry, let $m = 8$ and $n = 1$. Thus,

$$(8p + q)^3 + 8(8q + r)^3 = 8(p + 8q)^3 + (q + 8r)^3, \tag{4.1}$$

where $p = b^2 - 2a^2 + 2ab$, $q = a^2 + b^2 - ab$, and $r = a^2 - 2b^2 + 2ab$. Substituting these values of p, q, and r in (4.1) and simplifying, we complete the proof.

SECOND PROOF. Let $p = 3$, $q = 4$, $r = 5$, and $s = 6$ in Entry 1. Then, with the choice of plus signs, $m = 5$ and $n = 4$. The desired identity is now immediate.

Entry 4 was submitted by Ramanujan [8], [23, p. 326] as a problem in the *Journal of the Indian Mathematical Society* and is mentioned by Hardy and Wright [1, p. 201].

Entry 4 was employed by C. Hooley [1] to obtain a lower bound for the number of integers less than x that can be represented as a sum of two cubes.

On p. 267, Ramanujan defines three functions. First, $I(p)$ denotes the greatest integer $\leq p$. Second, $G(p)$ denotes the least integer $\geq p$. Third, $N(p)$ denotes the nearest integer to p. In current notation, $I(p) = [p]$ or $\lfloor p \rfloor$ and $G(p) = \lceil p \rceil$. Ramanujan's definition of $N(p)$ is ambiguous if $p + \frac{1}{2}$ is an integer. However, it is apparent from the sequel that in such an instance, $N(p) = p + \frac{1}{2}$, instead of $p - \frac{1}{2}$.

Entry 5 (Theorem (1), p. 267). *For each real number p,*

$$N(p) = I(p + \tfrac{1}{2}).$$

PROOF. Let $p = [p] + \{p\}$, where $\{p\}$ denotes the fractional part of p. If $\{p\} < \frac{1}{2}$, then $N(p) = [p]$ and $I(p + \frac{1}{2}) = [p]$. If $\{p\} \geq \frac{1}{2}$, then $N(p) = [p] + 1$ and $I(p + \frac{1}{2}) = [p + 1]$.

Entry 6 (Theorem (2), p. 267). *Let p and n denote positive integers. Then the coefficient of x^n in the Maclaurin series of*

$$\frac{x^p}{(1 - x)(1 - x^p)}$$

equals $I(n/p)$.

PROOF. For $|x| < 1$,

$$\frac{x^p}{(1-x)(1-x^p)} = x^p \sum_{k=0}^{\infty} x^k \sum_{j=0}^{\infty} x^{pj} = \sum_{n=0}^{\infty} \left(\sum_{k+(j+1)p=n} 1 \right) x^n.$$

It is clear that the number of summands in the inner sum on the far right side equals $[n/p]$.

Entry 7 (Theorem (3), p. 267). *Let φ denote a continuous increasing function defined on $[1, \infty)$ and tending to ∞. Consider the formal power series of*

$$F(x) := \sum_{j=1}^{\infty} \frac{x^{G(\varphi^{-1}(j))}}{1-x},$$

where G is defined prior to Entry 5. Then the coefficient of x^n, $n \geq 1$, in the formal power series of $F(x)$ equals $I(\varphi(n))$.

PROOF. Now,

$$F(x) = \sum_{j=1}^{\infty} x^{G(\varphi^{-1}(j))} \sum_{k=0}^{\infty} x^k = \sum_{n=1}^{\infty} \left(\sum_{G(\varphi^{-1}(j))+k=n} 1 \right) x^n.$$

Let $S(n)$ denote the inner sum on the far right side above. Thus, $S(n)$ denotes the number of integral pairs (j, k), $j \geq 1$, $k \geq 0$, such that $G(\varphi^{-1}(j)) + k = n$. Since φ^{-1} is increasing, $S(n)$ is equal to the largest integer $j = j^*$ such that $G(\varphi^{-1}(j)) = n$. Thus, $G(\varphi^{-1}(j^* + 1)) \geq n + 1$. Hence, we see that $j^* = [\varphi(n)]$, and this completes the proof.

Entry 8 (Theorem (4), p. 267). *Let $d(n)$ denote the number of positive divisors of the positive integer n. Then, if p is a natural number,*

$$D(p) := \sum_{n \leq p} d(n) = \sum_{k \leq p} I(p/k) = 2 \sum_{k \leq \sqrt{p}} I(p/k) - I^2(\sqrt{p}).$$

These two elementary properties of $d(n)$ are well known and easy to prove; see, e.g., Hardy and Wright's text [1, pp. 264–265]. For a generalization, see the proof of Entry 20 below.

We quote Ramanujan in the next entry.

Entry 9 (Theorem (5), p. 267). *The above sum is odd or even according as $I(\sqrt{p})$ is odd or even and is approximately equal to $p(2c - 1 + \log p) + \frac{1}{2}$ the number of factors of $p + \frac{1}{4}$.*

First, it is clear from the second equality in Entry 8 that $D(p)$ is odd or even according as $I(\sqrt{p})$ is odd or even. Second, with the use of Entry 8, it is easy to show that

$$D(p) = p \log p + (2\gamma - 1)p + \Delta(p),$$

where γ (c in Ramanujan's notation) denotes Euler's constant and $\Delta(p) = O(\sqrt{p})$, as p tends to ∞. For example, see Hardy and Wright's book [1, pp. 264–265]. Third, Ramanujan is evidently claiming that a good approximation to the "error term" $\Delta(p)$ can be obtained by replacing $\Delta(p)$ by $\frac{1}{2}d(p)$. This approximation for $I(p)$ is too optimistic for, on the one hand, $d(p) = O(p^{\varepsilon})$, for every $\varepsilon > 0$ (Hardy and Wright [1, p. 260]), while, on the other hand, Hardy [2], [5, pp. 268–292] proved in 1916 that $\Delta(p) = O(p^{1/4})$ is impossible.

Ramanujan also briefly considers the "divisor problem" in Section 2 of Chapter 15 (Part II, [4, p. 304]). For a more complete discussion of the divisor problem, consult Ivić's book [1, Chap. 13] or Krätzel's treatise [2, pp. 228–230].

Entry 10 (Theorem (6), p. 267). *Let n and p be positive integers with $n > \sqrt{p}$. Then*

$$\sum_{k=1}^{n}\left[\frac{p}{k}\right] = n\left[\frac{p}{n}\right] + \sum_{k=1}^{n-[p/n]}\left[\frac{p}{k+[p/n]}\right].$$

Proof. The sum on the left side counts the number of lattice points L in the interior of the first quadrant on or below the hyperbola $xy = p$ with $x \le n$. The first expression on the right side counts those members of L in a rectangle of base n and height p/n. The sum on the right side counts the remainder of the points L lying above this rectangle.

In the next entry, we again quote Ramanujan.

Entry 11 (Theorem (7), p. 267). *If p be the* nth *prime number then $dp/dn = \log p$ nearly and hence $n = p/(\log p - 1)$ nearly.*

Of course, Entry 11 is not a mathematically sound declaration. The following heuristic and unrigorous reasoning may be an approximation to Ramanujan's thinking. Let $p(n)$ denote the nth prime. From the prime number theorem,

$$n \sim \frac{p(n)}{\log p(n)} \sim \frac{p(n)}{\log n},$$

as $p(n)$ tends to ∞. Hence

$$\frac{dp}{dn} \sim \log n + 1 \sim \log n \sim \log p(n),$$

as $p(n)$ tends to ∞. Thus,

$$\int dn \sim \int \frac{dp}{\log p},$$

and, after two integrations by parts,

$$n \sim \frac{p}{\log p}\left\{1 + \frac{1}{\log p}\right\}. \tag{11.1}$$

Now, by an elementary calculation,

$$\frac{p}{\log p - 1} - \frac{p}{\log p}\left\{1 + \frac{1}{\log p}\right\} = \frac{p}{\log^2 p(\log p - 1)} \sim \frac{p}{\log^3 p}, \tag{11.2}$$

as p tends to ∞. Using (11.2) in (11.1), we deduce Ramanujan's approximation for n.

In addition to being very unrigorous, our reasoning is also somewhat circular. However, we do not see how the claim $dp/dn = \log p$ can be made without using a result akin to the prime number theorem.

Entry 12 (Theorem (8), p. 267). *Let φ be a positive monotonically decreasing function with a continuous first derivative on $[\frac{3}{2}, \infty)$. Then*

$$S_1 := \sum_p \varphi(p) \quad \text{and} \quad S_2 := \sum_{n=2}^{\infty} \frac{\varphi(n)}{\log n}$$

are either both convergent or both divergent, where the sum on the left is over all primes p.

Proof. As customary, let $\pi(x)$ denote the number of primes $\leq x$. By an integration by parts,

$$\sum_{p \leq x} \varphi(p) = \int_{3/2}^{x+} \varphi(t)\, d\pi(t) = \varphi(x)\pi(x) - \int_{3/2}^{x} \varphi'(t)\pi(t)\, dt. \tag{12.1}$$

Thus, by (12.1) and the prime number theorem, S_1 converges if and only if

$$\begin{aligned}
&\lim_{x\to\infty}\left(\varphi(x)\pi(x) - \int_{3/2}^{x} \varphi'(t)\frac{t}{\log t}\, dt\right) \\
&= \lim_{x\to\infty}\left(\varphi(x)\pi(x) - \frac{\varphi(x)x}{\log x} + \frac{\varphi(\frac{3}{2})\frac{3}{2}}{\log(\frac{3}{2})} + \int_{3/2}^{x} \varphi(t)\left\{\frac{1}{\log t} - \frac{1}{\log^2 t}\right\} dt\right)
\end{aligned}$$

exists, i.e., if and only if

$$\lim_{x\to\infty}\left(\frac{\varphi(x)x}{\log^2 x} + \int_{3/2}^{x} \frac{\varphi(t)}{\log t}\, dt\right) \tag{12.2}$$

exists. In the last step, we employed the prime number theorem in the form

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + O\left(\frac{x}{\log^3 x}\right),$$

as x tends to ∞. Since φ is monotonically decreasing,

$$\int_{3/2}^{x} \frac{\varphi(t)}{\log t}\,dt \geq (x - \tfrac{3}{2})\frac{\varphi(x)}{\log x}.$$

Thus, by (12.2), S_1 converges if and only if

$$\int_{3/2}^{\infty} \frac{\varphi(t)}{\log t}\,dt$$

converges. Hence, by the integral test, S_1 converges if and only if S_2 converges.

In Ramanujan's [23, p. xxiv] first letter to Hardy [7, p. 60], it is asserted that "The number of such numbers (that can be expressed as a sum of two squares) greater than A and less than B

$$= K \int_{A}^{B} \frac{dx}{\sqrt{\log x}} + \theta(x),$$

where $K = 0.764\ldots$ and $\theta(x)$ is very small when compared with the previous integral. K and $\theta(x)$ have been exactly found through complicated...." Answering an enquiry of Hardy, Ramanujan [23, p. xxviii], in his second letter to Hardy, claims that "The order of $\theta(x)$ which you asked in your letter is $\sqrt{x/\log x}$." In his book [7, p. 61], Hardy informs us that "Ramanujan later (i) gave the exact value of K, viz.,

$$K = \left\{\frac{1}{2}\prod \frac{1}{1 - p^{-2}}\right\}^{1/2}, \tag{13.1}$$

where p runs through the primes $4m + 3\ldots.$" (The complete texts of the letters that Ramanujan wrote to Hardy in 1913–14 can be found in Berndt and Rankin's book [1].) As Hardy [7, pp. 61–63] has shown, Ramanujan's claim about the order of $\theta(x)$ is wrong.

In his notebooks [22], Ramanujan offers a less precise version of the aforementioned result communicated to Hardy.

Entry 13 (p. 307). *The number of integers between A and B representable as a sum of two squares is approximately equal to*

$$K \int_{A}^{B} \frac{dx}{\sqrt{\log x}},$$

where $K = 0.764$.

In fact, Ramanujan gives an exact expression for K, but it is unreadable in the published edition [22] of the notebooks. However, from the very faint marks that can be seen, we can almost certainly guess that Ramanujan recorded the expression given by (13.1).

Ramanujan's claim was, in fact, first proved by E. Landau [1], [6, pp. 59–66], [5, pp. 644, 649–669] in 1908. His elegant analytic proof is beautifully sketched by Hardy in [7, pp. 61–63].

At the end of his discussion, Hardy [7, p. 63] remarks "... and it would be very interesting to know just how Ramanujan came to this conclusion." G. N. Watson [3] likewise expressed wonder about Ramanujan's discovery when he wrote, "The most amazing thing about this formula is that it was discovered, apparently independently, by Ramanujan in his early days in India, and it appears in its appropriate place in his manuscript note-books." In fact, we shall present Ramanujan's argument, but before doing so, we shall state explicitly what is currently known about the Landau–Ramanujan problem.

Let $B(x)$ denote the number of positive integers $\leq x$ that are expressible as a sum of two squares. Then Landau showed that, as x tends to ∞,

$$B(x) = \frac{Kx}{\sqrt{\log x}}\left(1 + \frac{c}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right), \tag{13.2}$$

where K is given by (13.1). (The O-term can be replaced by further ascending powers of $1/\log x$.) Ramanujan's approximation of $B(x)$ by

$$I(x) := K \int_1^x \frac{dt}{\sqrt{\log t}}$$

does not have any advantage over the simpler approximation $Kx/\sqrt{\log x}$. Here, Ramanujan obviously was misled by a false analogy with the prime number theorem. By integrating $I(x)$ by parts, we see that $I(x)$ has an asymptotic expansion of the same form as (13.2). However, the coefficients in the asymptotic expansions of $B(x)$ and $I(x)$ do not agree beyond the first term. Nonetheless, D. Shanks [2] has observed that *numerically* $I(x)$ is a better approximation than $Kx/\sqrt{\log x}$.

G. K. Stanley [1] attempted to determine c, but she made several errors, which were discovered by Shanks [2]. Shanks [2] corrected Stanley's work and calculated the values $K = 0.764223654$ and $c = 0.581948659$. He also carefully analyzed the effectiveness of the approximations $Kx/\sqrt{\log x}$, $I(x)$, and the first two terms of (13.2). For related work, see papers of Shanks [3], [4] and Shanks and L. P. Schmid [1]. P. Shiu [1] has developed an analogue of the Meissel–Lehmer method for calculating $\pi(x)$ and has calculated $B(x)$ for $x = 10^k$, $1 \leq k \leq 12$.

By using a Tauberian theorem of Hardy, Littlewood, and Karamata (Schwarz [1, pp. 128–129]) or a more general Tauberian theorem of H.

Delange [1], one can deduce the asymptotic formula

$$B(x) \sim \frac{Kx}{\sqrt{\log x}}.$$

A. Wintner [1] also claimed to have established this asymptotic formula by a Tauberian theorem, but his argument is incorrect.

G. J. Rieger [1] gave an elementary proof of the estimate

$$B(x) = \frac{Kx}{\sqrt{\log x}}\left(1 + O\left(\frac{1}{\log \log x}\right)\right).$$

Observe that the error term is greater than the second expression on the right side of (13.2).

In an important paper, H. Iwaniec [1] employed sieve methods to prove that, for a certain constant B_k,

$$\sum_{\substack{n \le x \\ n = a^2 + b^2 \\ n \equiv \ell (\mathrm{mod}\, k)}} 1 = B_k \frac{x}{\sqrt{\log x}}\left(1 + O\left(\left(\frac{\log k}{\log x}\right)^{1/5}\right)\right).$$

This is currently the best result that can be established without using functions of a complex variable.

W. Heupel [1], R. D. James [1], and K. S. Williams [1] have considered the more general problem when the quadratic form $a^2 + b^2$ in the problem of Landau and Ramanujan is replaced by a primitive, positive, integral, binary quadratic form.

Ramanujan returns to this problem on sums of squares in his third notebook (p. 363 of [22]). In fact, all of page 363 is devoted to heuristically proving Entry 13. Ramanujan's argument hinges upon Entry 8 on page 362. We shall quote this result verbatim et litteratim.

Entry 14 (Entry 8, p. 362). *If A, B, C are quantities so taken that*

$$\frac{1}{A^k} + \frac{1}{B^k} + \frac{1}{C^k} + \cdots = \frac{a}{(k-\alpha)^r} \tag{14.1}$$

when $k = \alpha$ (the only pole) then the number of such quantities less than z is

$$\int \frac{a(\log z)^{r-1}}{z^{1-\alpha}\underline{|r-1}}\, dz. \tag{14.2}$$

For

$$\int^{\infty} \frac{dn}{z^k} = \frac{a}{(k-\alpha)^r} \quad \textit{and} \quad \int^{\infty} \frac{dz}{z^{k-\alpha+1}} = \frac{1}{k-\alpha}.$$

Differentiating r times with respect to k we get the above result.

Generally, (14.1) should be interpreted asymptotically as k tends to α. In the integrand of (14.2), $\underline{|r-1} = \Gamma(r)$. Also, n is the "counting" function for the ordered set $S = \{A, B, C, \ldots\}$, where $A > 0$. Although r presumably denotes a positive integer, in the applications which follows, $r = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$. Ramanujan (Part I [2, pp. 329–330]) was familiar with fractional differentiation and routinely assumed that differentiation formulas for positive integral r could be generalized for nonintegral r.

Note that Ramanujan uses the term "pole," although, as just indicated, r is not necessarily a positive integer. Since Ramanujan presumably had no familiarity with the theory of functions of a complex variable before departing for England in 1914, the appearance of the word "pole" suggests that Entry 14 was recorded later in England.

It will be convenient to alter Ramanujan's notation. Let

$$N(x) = \sum_{\substack{n \in S \\ |n| \le x}} 1.$$

Then by (14.1), for $k > \alpha$,

$$\int_{A-}^{\infty} \frac{dN(x)}{x^k} = \sum_{n \in S} \frac{1}{n^k} \sim \frac{a}{(k-\alpha)^r}, \tag{14.3}$$

as k tends to α. Differentiating the equality

$$\int_1^{\infty} \frac{dx}{x^{k-\alpha+1}} = \frac{1}{k-\alpha}, \qquad k > \alpha,$$

$r - 1$ times (not r times, as instructed by Ramanujan) with respect to k, we find that

$$\int_1^{\infty} \frac{(\log x)^{r-1}}{x^{k-\alpha+1}}\, dx = \frac{\Gamma(r)}{(k-\alpha)^r}, \qquad k > \alpha,$$

or

$$\int_1^{\infty} \frac{a \log^{r-1} x}{\Gamma(r) x^{k-\alpha+1}}\, dx = \frac{a}{(k-\alpha)^r}, \qquad k > \alpha. \tag{14.4}$$

Putting (14.4) in (14.3), evidently, Ramanujan now lets k tend to α and equates integrands to deduce that

$$\frac{dN(x)}{dx} = \frac{a \log^{r-1} x}{\Gamma(r) x^{1-\alpha}}.$$

Finally, upon integration, we conclude that

$$N(x) \sim \int \frac{a \log^{r-1} x}{\Gamma(r) x^{1-\alpha}}\, dx,$$

as x tends to ∞.

We emphasize that this argument is formal and not rigorous.

H. G. Diamond has kindly supplied the following example to show that Ramanujan's argument is spurious without the assumption of additional hypotheses.

For each nonnegative integer k and positive integer n, define

$$f(n)=\begin{cases}1, & \text{if } 2^{2k}\le n<2^{2k+1},\\ 0, & \text{if } 2^{2k+1}\le n<2^{2k+2},\end{cases}$$

and, for Re $s>1$,

$$F(s)=\sum_{n=1}^{\infty} f(n)n^{-s}=\sum_{k=0}^{\infty}\sum_{2^{2k}\le n<2^{2k+1}} n^{-s}.$$

For Re $s>1$, by the Euler–Maclaurin summation formula,

$$\sum_{2^{2k}\le n<2^{2k+1}} n^{-s}=\frac{1}{2}\int_{2^{2k}}^{2^{2k+2}} x^{-s}\,dx+O(4^{-k}).$$

Thus, for Re $s>1$,

$$\begin{aligned}F(s)&=\frac{1}{2}\sum_{k=0}^{\infty}\left(\int_{4^k}^{4^{k+1}} x^{-s}\,dx+O(4^{-k})\right)\\ &=\frac{1}{2}\int_{1}^{\infty} x^{-s}\,dx+O(1)\\ &=\frac{\frac{1}{2}}{s-1}+O(1).\end{aligned}$$

Thus, $F(s)\sim \frac{1}{2}/(s-1)$ as s tends to $1+$. However, $\sum_{n\le x} f(n)$ clearly has no asymptotic behavior. This then provides a counterexample to Ramanujan's general principle.

We now present Ramanujan's heuristic proof of Entry 13.

For $k>1$, let

$$S_k=\sum_{n=0}^{\infty}(2n+1)^{-k}=\prod_{p\text{ odd}}(1-p^{-k})^{-1}=(1-2^{-k})\zeta(k)$$

and

$$S_k'=\sum_{n=0}^{\infty}(-1)^n(2n+1)^{-k}=\prod_{p\equiv 1(\text{mod } 4)}(1-p^{-k})^{-1}\prod_{p\equiv 3(\text{mod } 4)}(1+p^{-k})^{-1},$$

where the products run over primes p as specified. Let S denote the set of positive integers that can be represented as a sum of two squares. Then by Entry 17, for $k > 1$,

$$\begin{aligned}
F(k) &:= \sum_{n \in S} n^{-k} \\
&= \sum_{n_0=0}^{\infty} 2^{-n_0 k} \sum_{n_1=0}^{\infty} 5^{-n_1 k} \sum_{n_2=0}^{\infty} 13^{-n_2 k} \cdots \sum_{m_1=0}^{\infty} 3^{-2m_1 k} \sum_{m_2=0}^{\infty} 7^{-2m_2 k} \cdots \\
&= (1 - 2^{-k})^{-1} \prod_{p \equiv 1 (\mathrm{mod}\ 4)} (1 - p^{-k})^{-1} \prod_{p \equiv 3 (\mathrm{mod}\ 4)} (1 - p^{-2k})^{-1} \\
&= (1 - 2^{-k})^{-1} \prod_{p\ \mathrm{odd}} (1 - p^{-k})^{-1} \prod_{p \equiv 3 (\mathrm{mod}\ 4)} (1 + p^{-k})^{-1}. \qquad (13.3)
\end{aligned}$$

A brief calculation shows that

$$\sqrt{S_k S_k'} = \prod_{p\ \mathrm{odd}} (1 - p^{-k})^{-1} \prod_{p \equiv 3 (\mathrm{mod}\ 4)} (1 + p^{-k})^{-1} \prod_{p \equiv 3 (\mathrm{mod}\ 4)} (1 - p^{-2k})^{1/2}.$$

Thus, by (13.3),

$$F(k) = (1 - 2^{-k})^{-1} \sqrt{S_k S_k'} \prod_{p \equiv 3 (\mathrm{mod}\ 4)} (1 - p^{-2k})^{-1/2}. \qquad (13.4)$$

Since

$$\frac{S_k}{S_k'} = \prod_{p \equiv 3 (\mathrm{mod}\ 4)} \frac{1 + p^{-k}}{1 - p^{-k}},$$

we see from (13.4) that

$$\begin{aligned}
F(k) &= (1 - 2^{-k})^{-1} S_k' \left(\frac{S_k}{S_k'}\right)^{1/2} \left(\prod_{p \equiv 3 (\mathrm{mod}\ 4)} \frac{1 + p^{-2k}}{1 - p^{-2k}}\right)^{1/4} \left(\prod_{p \equiv 3 (\mathrm{mod}\ 4)} \frac{1}{1 - p^{-4k}}\right)^{1/4} \\
&= (1 - 2^{-k})^{-1} S_k' \left(\frac{S_k}{S_k'}\right)^{1/2} \left(\frac{S_{2k}}{S_{2k}'}\right)^{1/4} \left(\frac{S_{4k}}{S_{4k}'}\right)^{1/8} \left(\prod_{p \equiv 3 (\mathrm{mod}\ 4)} \frac{1}{1 - p^{-8k}}\right)^{1/8} \\
&= \cdots \\
&= (1 - 2^{-k})^{-1} S_k' \prod_{j=0}^{\infty} \left(\frac{S_{2^j k}}{S_{2^j k}'}\right)^{1/2^{j+1}}. \qquad (13.5)
\end{aligned}$$

Now as k approaches 1, S_k' tends to $\pi/4$ and $S_k \sim \frac{1}{2}(k-1)^{-1}$. Thus, from (13.4), as k tends to 1,

$$F(k) \sim \sqrt{\frac{\pi}{2(k-1) \prod_{p \equiv 3 (\mathrm{mod}\ 4)} (1 - p^{-2})}} = \sqrt{\frac{\pi}{k-1}}\, K, \qquad (13.6)$$

by (13.1). At this point, Ramanujan's reasoning goes awry. He observes in (13.5) that $(S_{2^jk})^{1/2^{j+1}}$ has an algebraic singularity at $k = 2^{-j}$, $j \geq 0$. Thus, using (13.5) and repeating the argument that produced (13.6), he deduces, in some imprecise asymptotic sense, that for certain constants A, B, $C, \ldots,$

$$F(k) \sim \frac{A}{(k-1)^{1/2}} + \frac{B}{(2k-1)^{1/4}} + \frac{C}{(4k-1)^{1/8}} + \cdots,$$

where $A = \sqrt{\pi}K$.

Ramanujan now applies Entry 14 to each term of the "asymptotic" expansion above. Thus, with $r = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ and $\alpha = 1, \frac{1}{2}\,\frac{1}{4}, \ldots$, respectively,

$$\begin{aligned} B(x) \sim A \int \frac{(\log x)^{-1/2}}{\sqrt{\pi}}\,dx + \frac{B}{2^{1/4}} \int \frac{(\log x)^{-3/4}}{\Gamma(1/4)x^{1/2}}\,dx \\ + \frac{C}{4^{1/8}} \int \frac{(\log x)^{-7/8}}{\Gamma(1/8)x^{3/4}}\,dx + \cdots, \end{aligned} \tag{13.7}$$

as x tends to ∞.

Ramanujan concludes his argument on page 363 by claiming, in effect, that

$$B(x) = K \int \frac{dx}{\sqrt{\log x}} + \theta(x),$$

where $\theta(x) = O(x^{1/2}/(\log x)^{3/4})$, as x tends to ∞. (Note that this assertion is even stronger than that made in his letter to Hardy.) Ramanujan deduced this estimate by integrating by parts the second integral on the right side of (13.7).

In his first letter to Hardy, Ramanujan [23, p. xxiv] makes the following claim:

"The numbers of the form 2^p3^q less than n equal

$$\frac{1}{2}\frac{\log(2n)\log(3n)}{\log 2 \log 3}$$

where p and q may have any positive integral value including 0."

On page 309 in his second notebook, Ramanujan asserts the following more general result:

Entry 15 (p. 309). "*All nos. of the form a^pb^q within n equal*

$$\frac{1}{2}\frac{\log(an)\log(bn)}{\log a \log b}$$

$(+\frac{1}{2}$ *if n is of the required form*)."

(It is tacitly assumed that a and b are fixed natural numbers.)

Hardy, in his book [7] on Ramanujan's work, devotes Chapter 5 to this more general problem. With Hardy [7, p. 69], put

$$\eta = \log n, \qquad \omega = \log a, \qquad \omega' = \log b.$$

Then Entry 15 is equivalent to the assertion that the number of integral solutions to the inequalities

$$u \geq 0, \qquad v \geq 0, \qquad \omega u + \omega' v \leq \eta \tag{15.1}$$

is approximately

$$\frac{\eta^2}{2\omega\omega'} + \frac{\eta}{2\omega} + \frac{\eta}{2\omega'} + \frac{1}{2}, \tag{15.2}$$

with an additional additive factor of $\frac{1}{2}$ if there are nonnegative integers u and v such that $\omega u + \omega' v = \eta$. (Each of the additive factors $\frac{1}{2}$ must be taken with "a grain of salt," since the remaining terms in (15.2) do not necessarily equal an integer or $\frac{1}{2}$ times an integer.) Thus, we see that Ramanujan's claim is about the number of lattice points within or on a right triangle in the first quadrant bounded by the straight line $\omega u + \omega' v = \eta$ and the coordinate axes.

Ramanujan's assertion in Entry 15 is asymptotically correct, if one ignores the previously mentioned additive factors of $\frac{1}{2}$. More precisely, let $N(\eta)$ denote the number of solutions of (15.1), set

$$\Omega(\eta) = \frac{\eta^2}{2\omega\omega'} + \frac{\eta}{2\omega} + \frac{\eta}{2\omega'},$$

and put

$$N(\eta) = \Omega(\eta) + R(\eta).$$

First, from elementary considerations, it is very easy to show that $N(\eta) \sim \eta^2/(2\omega\omega')$, as η tends to ∞. Further results are more delicate to obtain. In particular, if ω and ω' are any positive numbers such that ω/ω' is irrational, then (Hardy [7, p. 74])

$$R(\eta) = o\left(\frac{\eta}{\log \eta}\right),$$

as η tends to ∞. More precise results can be obtained if ω/ω' satisfies additional assumptions or if ω/ω' is rational. For proofs, see Hardy's book [7] and a paper of Hardy and Littlewood [6] (Hardy [4, pp. 159–196]). Several further references can be found in the aforementioned sources as well as a slightly earlier paper of Hardy and Littlewood [5] (Hardy [4, pp. 136–157]) on this same problem. Recently, M. Hausman and H. N. Shapiro

[1] have found explicit sequences of n tending to ∞ for which Ramanujan's claim is true in the sense that $R(\eta)$ remains bounded for such sequences.

Like Entry 13, Entry 15 is one of only a few results in the notebooks for which Ramanujan provides a "proof." As we shall see, although Ramanujan's argument is very interesting, it is not rigorous.

First, Ramanujan claims that

$$\sum_{n=0}^{\infty} a^{-pn} = \frac{1}{p \log a} + \tfrac{1}{2} + \cdots. \tag{15.3}$$

Indeed, for $0 < p \log a < 2\pi$,

$$\sum_{n=0}^{\infty} a^{-pn} = \frac{1}{1-a^{-p}} = \frac{1}{1-e^{-p\log a}} = \frac{1}{p \log a} + \tfrac{1}{2} + \cdots.$$

Second, Ramanujan offers (15.3) again, but with a replaced by b. Since in Entry 15, a and b are fixed and p and q may be arbitrarily large, these two statements are enigmatic because of the requirements, $0 < p \log a$, $p \log b < 2\pi$. In fact, Ramanujan's notation is unfortunate, since p is being used in two distinct contexts, as we shall see below.

Third, Ramanujan asserts that "If the required no. of such nos. $= x$, then

$$\int^{\infty} \frac{dx}{n^p} = \int^{\infty} \frac{dn}{n^{p+1}}\left(\frac{\log n}{\log a \log b} + \tfrac{1}{2}\log a + \tfrac{1}{2}\log b\right) \tag{15.4}$$

when $p = 0$." Let us discern how Ramanujan might have deduced (15.4). For convenience, we shall alter Ramanujan's notation. For fixed natural numbers a and b, let $N(x)$ denote the number of positive integers of the form $a^p b^q$ that are less than or equal to x. For $\operatorname{Re} s > 0$,

$$\begin{aligned}\int_{1-}^{\infty} \frac{dN(x)}{x^s} = \sum_{m,n=1}^{\infty} \frac{1}{(a^m b^n)^s} &\sim \left(\frac{1}{s \log a} + \tfrac{1}{2}\right)\left(\frac{1}{s \log b} + \tfrac{1}{2}\right)\\ &\sim \frac{1}{s^2 \log a \log b} + \frac{1}{2s \log a} + \frac{1}{2s \log b},\end{aligned} \tag{15.5}$$

by (15.3), as s tends to 0.

Ramanujan now evidently asks, "What elementary function $f(x)$ do we need in the integrand

$$\int_1^{\infty} \frac{f(x)\,dx}{x^{s+1}}$$

to obtain the asymptotic expansion (15.5) as s tends to 0?" Now, quite certainly, (15.4) contains two misprints, because the choice of $f(n)$ indicated in (15.4) does not yield the desired asymptotic expansion. To correct

Ramanujan's choice of f, replace $\frac{1}{2}\log a + \frac{1}{2}\log b$ by $1/(2\log a) + 1/(2\log b)$ in (15.4). Then, with the help of an integration by parts, we easily find that

$$\int_1^\infty \frac{dx}{x^{s+1}}\left(\frac{\log x}{\log a \log b} + \frac{1}{2\log a} + \frac{1}{2\log b}\right)$$
$$= \frac{1}{s^2 \log a \log b} + \frac{1}{2s\log a} + \frac{1}{2s\log b}, \tag{15.6}$$

which is in agreement with the right side of (15.5).

Next, Ramanujan evidently lets p tend to 0 and equates integrands in (15.4) to "deduce" that

$$\frac{dx}{dn} = \frac{\log n}{n\log a \log b} + \frac{\log a}{2n} + \frac{\log b}{2n}. \tag{15.7}$$

Lastly, he supposedly integrates (15.7) to conclude that

$$x = \frac{1}{2}\frac{\log(an)\log(bn)}{\log a \log b}. \tag{15.8}$$

However, as previously intimated, the integration of (15.7) does not yield (15.8). But if we let s tend to 0 and equate integrands in (15.5) and (15.6), we formally find that

$$\frac{dN(x)}{dx} = \frac{\log x}{x\log a \log b} + \frac{1}{2x\log a} + \frac{1}{2x\log b}.$$

Formally integrating, we deduce that, as x tends to ∞,

$$N(x) \sim \frac{\log^2 x}{2\log a \log b} + \frac{\log x}{2\log a} + \frac{\log x}{2\log b},$$

which agrees with Ramanujan's assertion.

We emphasize that the argument above is formal and unrigorous.

Let $a = 2$, $b = 3$, and $N(x)$ be as given in the argument above (with $a = 2$ and $b = 3$). Define $\Delta(x)$ by

$$\Delta(x) = N(x) - \frac{\log(2x)\log(3x)}{2\log 2 \log 3}.$$

By using recursive formulas for $N(x)$ and $\Delta(x)$, S. S. Pillai [1] has extensively calculated $N(2^p3^q)$ and $\Delta(2^p3^q)$. By using these and additional calculations, Pillai and A. George [1] have shown that for $x \le 2^{1000}$, $-1.9 < \Delta(x) < 2.4$.

Except for a change in notation, we quote Ramanujan in the next entry.

Entry 16 (p. 310). *If*

$$\sum_{n=1}^\infty a_n e^{-nx} = \int_0^\infty e^{-tx} f(t)\,dt + \sum_{n=1}^\infty a_n, \tag{16.1}$$

then the average value of a_n equals $f(n)$ exactly.

We have been unable to provide a correct or meaningful interpretation of Entry 16. If (16.1) were correct as it stands, then the average value of a_n would be 0, i.e.,

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} a_n = 0,$$

since $\sum_{n=1}^{\infty} a_n$ converges. Moreover, if we let x tend to ∞ in (16.1), we deduce, under mild restrictions on f, that $\sum_{n=1}^{\infty} a_n = 0$. If $a_n \geq 0$, which is the case in most applications, and if $x > 0$, then f would have to be negative "more often" than positive. It does not seem likely that a nonnegative arithmetical function would have an average value $f(n)$ that is frequently negative.

As a first application, Ramanujan claims that the average value of the divisor function $d(n)$ is equal to $\log n + 2\gamma$ "exactly." Of course, 2γ should be replaced by $2\gamma - 1$. (See Entry 9.)

As a second application, Ramanujan deduces that the average value of $\sigma(n)$, the sum of the positive divisors of n, equals $(\pi^2/6)\, n - \frac{1}{2}$ "exactly." This assertion is correct (Hardy and Wright [1, p. 266]), if one deletes the term $-\frac{1}{2}$.

Entry 17 (p. 311).

(i) *Each positive integer can be represented as a sum of four squares.*
(ii) *A positive integer n can be represented as a sum of three squares if and only if n is not of the form $4^m(8k + 7)$, where k and m are nonnegative integers.*
(iii) *A positive integer n is expressible as a sum of two squares if and only if in the canonical factorization of n, each prime of the form $4m + 3$ that occurs appears an even number of times.*

Each of these results is quite famous, and proofs can be found in most elementary texts on number theory, e.g., those of Hardy and Wright [1, Chap. 20] and Niven, Zuckerman, and Montgomery [1, Sections 3.6, 6.4]. Part (i) is a famous theorem of Lagrange. Part (iii) is apparently due to Fermat, although it was discovered earlier by A. Girard, who evidently did not give a proof. See Dickson's *History* [2, pp. 227–230] for further historical information.

On the top of page 310, Ramanujan records some further observations about sums of squares. These statements appear to be trivial, if we interpret them properly. In fact, Ramanujan has struck them out but has added the word "Correct" as a heading. We refrain from making further comments.

At the top of page 311, Ramanujan makes the following assertion:

Entry 18 (p. 311). "*If a prime number of the form $An + B$ can be expressed as $ax^2 - by^2$, then a prime number of the form $An - B$ can be expressed as $bx^2 - ay^2$.*"

This is followed on pages 311 and 313 by a table of representations of primes by certain quadratic forms:

Table 1

	Form of Prime	Representation
1	$4n + 1$	$x^2 + y^2$
2	$8n + 1, 8n + 3$	$x^2 + 2y^2$
3	$8n + 1, 8n - 1$	$x^2 - 2y^2$
4	$6n + 1$	$x^2 + 3y^2$
5	$12n + 1$	$x^2 - 3y^2$
6	$20n + 1, 20n + 9$	$x^2 + 5y^2$
7	$10n + 1, 10n + 9$	$x^2 - 5y^2$
8	$14n + 1, 14n + 9, 14n + 25$	$x^2 + 7y^2$
9	$28n + 1, 28n + 9, 28n + 25$	$x^2 - 7y^2$
10	$30n + 1, 30n + 49$	$x^2 + 15y^2$
11	$60n + 1, 60n + 49$	$x^2 - 15y^2$
12	$30n - 7, 30n + 17$	$5x^2 + 3y^2$
13	$60n - 7, 60n + 17$	$5x^2 - 3y^2$
14	$24n + 1, 24n + 7$	$x^2 + 6y^2$
15	$24n + 1, 24n + 19$	$x^2 - 6y^2$
16	$24n + 5, 24n + 11$	$2x^2 + 3y^2$
17	$24n + 5, 24n - 1$	$2x^2 - 3y^2$

All of the results in the table are classical and can be found, for example, in tables on unnumbered pages at the back of A. M. Legendre's book [1]. A more contemporary account on representing primes by quadratic forms has been given by D. A. Cox [1]. However, Ramanujan's claim in Entry 18 evidently cannot be found in the classical literature. K. S. Williams [2] has used the theory of quadratic forms to give a proof of Entry 18 with appropriate hypotheses. F. Halter-Koch [1] has subsequently given a shorter proof, but it is more advanced and uses class field theory. We shall not give Williams' instructive proof here but only state his theorem. Following Williams, we also provide some examples to illustrate the theorem.

Theorem 18.1. *Let a, b, A, and B denote positive integers satisfying the conditions*

$$(a, b) = 1 = (A, B), \qquad ab \neq \textit{square}.$$

Suppose also that every prime $p \equiv B \pmod{A}$ with $(p, 2ab) = 1$ is expressible in the form $ax^2 - by^2$ for some integers x and y. Then every prime q such that $q \equiv -B \pmod{A}$ and $(q, 2ab) = 1$ is expressible in the form $bX^2 - aY^2$ for some integers X and Y.

Example 1. As indicated in Table 1, every prime $p \equiv 1 \pmod 8$ can be expressed in the form $x^2 - 2y^2$ (Nagell [1, p. 210]). For example, $17 =$

$5^2 - 2 \cdot 2^2$. Theorem 18.1 implies that every prime $q \equiv -1 \pmod 8$ is expressible in the form $2X^2 - Y^2$. Since $2X^2 - Y^2 = (2X + Y)^2 - 2(X + Y)^2$, every prime $q \equiv 7 \pmod 8$ is also expressible in the form $x^2 - 2y^2$. For example, $7 = 3^2 - 2 \cdot 1^2$.

Example 2. As indicated by Ramanujan in Table 1, every prime $p \equiv 1 \pmod{12}$ is expressible in the form $x^2 - 3y^2$ (Nagell [1, p. 211]). Thus, $13 = 4^2 - 3 \cdot 1^2$. Therefore, by Theorem 18.1, every prime $q \equiv -1 \pmod{12}$ is expressible in the form $3X^2 - Y^2$. For example, $11 = 3 \cdot 2^2 - 1^2$.

Suppose that a, b, A, and B are positive integers satisfying the conditions of Theorem 18.1. The following assertions follow from Williams' work [2]. Either:

(a) $ab \equiv 0, 3, 4, 6, 7 \pmod 8$,

or

(b) $ab \equiv 1, 2, 5 \pmod 8$ and ab has a divisor $\equiv 3 \pmod 4$,

or

(c) $ab \equiv 1, 2, 5 \pmod 8$, every odd prime divisor of ab is $\equiv 1 \pmod 4$, and the equation $T^2 - abU^2 = -1$ is solvable in integers T and U.

If (a) or (b) holds, then $T^2 - abU^2 = -1$ is not solvable in integers T and U. If either (a) or (b) holds and q is a prime with $q \equiv -B \pmod A$ and $(q, 2ab) = 1$, then q is expressible in the form $bX^2 - aY^2$ but not by $aX^2 - bY^2$. If (c) holds and q is a prime such that $q \equiv -B \pmod A$ and $(q, 2ab) = 1$, then q is represented by both forms $aX^2 - bY^2$ and $bX^2 - aY^2$.

Example 3. Let $a = 1$, $b = 7$, $A = 28$, and $B = 9$. These integers satisfy Theorem 18.1 and condition (a), since if p is a prime with $p \equiv 9 \pmod{28}$, then $p = x^2 - 7y^2$ for some integers x and y (Legendre [1, Table III]). For example, $37 = 10^2 - 7 \cdot 3^2$. We conclude that each prime $q \equiv -9 \pmod{28}$ is expressible in the form $7X^2 - Y^2$ but not by the form $X^2 - 7Y^2$. For example, $19 = 7 \cdot 2^2 - 3^2$.

Example 4. The integers $a = 3$, $b = 7$, $A = 42$, and $B = 17$ satisfy Theorem 18.1 and condition (b). Thus, if $p \equiv 17 \pmod{42}$, then $p = 3x^2 - 7y^2$ for some integers x and y (Legendre [1, Table III]). For example, $17 = 3 \cdot 8^2 - 7 \cdot 5^2$. We conclude that every prime $q \equiv -17 \pmod{42}$ is representable by the form $7X^2 - 3Y^2$ but not by $3X^2 - 7Y^2$. For example, $67 = 7 \cdot 5^2 - 3 \cdot 6^2$.

Example 5. The integers $a = 5$, $b = 13$, $A = 65$, and $B = 7$ satisfy Theorem 18.1 and condition (c), since for every prime $p \equiv 7 \pmod{65}$, $p = 5x^2 - 13y^2$ for some integers x and y (Legendre [1, Table III]). For example,

$7 = 5 \cdot 2^2 - 13 \cdot 1^2$. Note that $T^2 - 65U^2 = -1$ has the solutions $T = 8$ and $U = 1$. We conclude that every prime $q \equiv -7 \pmod{65}$ can be represented by both forms $5X^2 - 13Y^2$ and $13X^2 - 5Y^2$. For example,

$$383 = 5 \cdot 10^2 - 13 \cdot 3^2 = 13 \cdot 26^2 - 5 \cdot 41^2.$$

Entry 19 (p. 313). *Let g be the greatest common divisor of any one of $\{2^p - 1, 2^p, 2^p + 1\}$ and any one of $\{3^p - 1, 3^p, 3^p + 1\}$, where p denotes a fixed positive integer. Then $g \mid (n^p \pm 1)$ for any positive integer n.*

Clearly, there are nine separate assertions that are being made. Since $(2^p, 3^p) = 1$, we see that one of these assertions is trivial. However, for each of the remaining cases, it is easy to construct counterexamples. Thus, without additional hypotheses, Entry 19 is not meaningful.

J. L. Selfridge (R. K. Guy [1, p. 57]) has proposed a similar problem. For what positive integers a, b does $2^a - 2^b$ divide $n^a - n^b$ for every positive integer $n \geq 2$? Selfridge remarks that the pairs $(a, b) = (2, 1)$, $(2^2, 2)$, and $(2^{2^2}, 2^2)$ provide examples.

A closely related problem was posed by H. Ruderman [1]. Let a and b denote integers such that $a > b \geq 0$ and $(2^a - 2^b) \mid (3^a - 3^b)$. Show that $(2^a - 2^b) \mid (n^a - n^b)$ for all natural numbers n. A partial solution was obtained.

Both of these problems were completely solved by Q. Sun and M. Z. Zhang [1].

On the top of page 324 of his second notebook, Ramanujan writes "No. of the form $p^2q^3 = 2.1732542\sqrt{z} - 1.458455\sqrt[3]{z} = \sqrt{4.723034z} - \sqrt[3]{3.10227z}$." We remark that the second equality is readily verified. It would seem that Ramanujan is claiming that the number of positive integers of the form p^2q^3 that are less than or equal to z equals $2.1732542\sqrt{z} - 1.458455\sqrt[3]{z}$. Unfortunately, if our interpretation is correct, Ramanujan's claim is incorrect. However, we have no alternative explanation and so state a corrected version of Ramanujan's assertion, as we interpret it.

Entry 20 (p. 324). *Let $A(x) = \sum_{m^2n^3 \leq x} 1$. Then*

$$\begin{aligned} A(x) &= \zeta(\tfrac{3}{2})x^{1/2} + \zeta(\tfrac{2}{3})x^{1/3} + O(x^{1/5}) \\ &= 2.6123753x^{1/2} - 3.6009377x^{1/3} + O(x^{1/5}), \end{aligned}$$

as x tends to ∞, where ζ denotes the Riemann zeta-function.

In fact, Entry 20 is a special case of a general theorem proved by E. Landau [2], [7, p. 24] in 1913. Since the general result is no more difficult to prove than any special case and since Entry 39 below is also a particular instance of Landau's theorem, we shall give his proof.

Theorem 20.1. *Let α and β be fixed positive numbers such that $\alpha \neq \beta$. Then*

$$A(\alpha, \beta; x) := \sum_{m^\alpha n^\beta \leq x} 1 =: \zeta\left(\frac{\beta}{\alpha}\right)x^{1/\alpha} + \zeta\left(\frac{\alpha}{\beta}\right)x^{1/\beta} + \Delta(\alpha, \beta; x), \qquad (20.1)$$

where as x tends to ∞, $\Delta(\alpha, \beta; x) = O(x^{1/(\alpha+\beta)})$.

PROOF. We shall employ a familiar argument, called the Dirichlet hyperbola method, that is used to estimate $\sum_{n \leq x} d(n)$, where $d(n)$ denotes the number of positive divisors of n (Andrews [1, pp. 207–210]).

Observe that $A(\alpha, \beta; x)$ denotes the number of lattice points in the interior of the first quadrant lying on or below the curve $m^\alpha n^\beta = x$. Thus,

$$\begin{aligned} A(\alpha, \beta; x) &= \sum_{m \leq x^{1/(\alpha+\beta)}} \left[\left(\frac{x}{m^\alpha}\right)^{1/\beta}\right] + \sum_{n \leq x^{1/(\alpha+\beta)}} \left[\left(\frac{x}{n^\beta}\right)^{1/\alpha}\right] \\ &\quad - [x^{1/(\alpha+\beta)}][x^{1/(\alpha+\beta)}] \\ &= x^{1/\beta} \sum_{m \leq x^{1/(\alpha+\beta)}} m^{-\alpha/\beta} + x^{1/\alpha} \sum_{n \leq x^{1/(\alpha+\beta)}} n^{-\beta/\alpha} \\ &\quad - x^{2/(\alpha+\beta)} + O(x^{1/(\alpha+\beta)}), \end{aligned}$$

as x tends to ∞. We now use a general asymptotic formula for $\sum_{k \leq x} k^r$ at the beginning of Chapter 7. Accordingly, as x tends to ∞ (Part I [2, p. 150, Entry 1]),

$$\begin{aligned} A(\alpha, \beta; x) &= x^{1/\beta}\left(\zeta\left(\frac{\alpha}{\beta}\right) + \frac{x^{(1-\alpha/\beta)/(\alpha+\beta)}}{1-\alpha/\beta} + O(x^{-(\alpha/\beta)/(\alpha+\beta)})\right) \\ &\quad + x^{1/\alpha}\left(\zeta\left(\frac{\beta}{\alpha}\right) + \frac{x^{(1-\beta/\alpha)/(\alpha+\beta)}}{1-\beta/\alpha} + O(x^{-(\beta/\alpha)/(\alpha+\beta)})\right) \\ &\quad - x^{2/(\alpha+\beta)} + O(x^{1/(\alpha+\beta)}) \\ &= \zeta\left(\frac{\alpha}{\beta}\right)x^{1/\beta} + \zeta\left(\frac{\beta}{\alpha}\right)x^{1/\alpha} + O(x^{1/(\alpha+\beta)}). \end{aligned}$$

The numerical values of $\zeta(\frac{3}{2})$ and $\zeta(\frac{2}{3})$ can be found in tables of Hansen and Patrick [1] or can be calculated by using *Mathematica.*

From a still much more general theorem Landau [3], [7, pp. 308–342], it follows that

$$\sum_{n_0^{d_0} n_1^{d_1} \cdots n_r^{d_r} \leq x} 1 = \sum_{j=0}^{r} \rho_j x^{1/d_j} + O(x^{r/(d_0(r+2))} \log^r(x+1)), \qquad (20.2)$$

where $d_0, d_1, \ldots, d_r$ are fixed positive numbers such that $d_0 < d_1 < \cdots < d_r$, and where

$$\rho_j = \prod_{\substack{k=0 \\ k \neq j}}^{r} \zeta(d_k/d_j).$$

The exact orders of the error terms in (20.1) and (20.2) are unknown. For (20.1), H.-E. Richert [1] proved that, if $0 < \beta < \alpha$,

$$\Delta(\alpha, \beta; x) = \begin{cases} O(x^{2/(3(\alpha+\beta))}), & \text{if } \alpha < 2\beta, \\ O(x^{2/(9\beta)} \log x), & \text{if } \alpha = 2\beta, \\ O(x^{2/(2\alpha+5\beta)}), & \text{if } \alpha > 2\beta. \end{cases}$$

Richert's estimates can be slightly improved, but many special cases must be considered. See E. Krätzel's book [2, pp. 221–227] for more details. As the title of Richert's paper [1] intimates, results of this type are important in estimating $\sum_{n \le x} a(n)$, where $a(n)$ denotes the number of essentially distinct abelian groups of order n. For another application, see a paper of P. T. Bateman and E. Grosswald [1].

On the other side, Landau [4], [8, pp. 145–158] and Krätzel [1] proved that

$$\Delta(\alpha, \beta; x) = \Omega(x^{1/(2(\alpha+\beta))}).$$

Improvements were made by A. Schierwagen [1], [2] and J. L. Hafner [1]. In particular, Hafner proved that

$$\Delta(\alpha, \beta; x) = \Omega_{+}(x^{1/(2(\alpha+\beta))}(\log x)^{\beta/(2(\alpha+\beta))} \log\log x),$$

where $1 \le \alpha \le \beta$.

From the von Staudt–Clausen theorem, the denominator of the Bernoulli number B_{2n} (in lowest terms), $n \ge 1$, is precisely equal to the product of those primes p for which $(p-1)|2n$ (Part I [2, p. 123]). At the top of page 353, Ramanujan provides a list of the first ten Bernoulli numbers with denominators $2 \cdot 3 = 6$, $2 \cdot 3 \cdot 5 = 30$, $2 \cdot 3 \cdot 7 = 42$, and $2 \cdot 3 \cdot 11 = 66$, respectively.

6	30	42	66
B_2	B_4	B_6	B_{10}
B_{14}	B_8	B_{114}	B_{50}
B_{26}	B_{68}	B_{186}	B_{170}
B_{34}	B_{76}	B_{258}	B_{370}
B_{38}	B_{124}	B_{294}	B_{470}
B_{62}	B_{152}	B_{354}	B_{590}
B_{74}	B_{188}	B_{402}	B_{610}
B_{86}	B_{236}	B_{426}	B_{670}
B_{94}	B_{244}	B_{474}	B_{710}
B_{98}	B_{248}	B_{582}	B_{730}

The table contains one error; the denominator of B_{294} contains the factor 43. Thus, B_{294} should be deleted, and B_{654} should be added to the end of the third column, because 654 is the next smallest index when the denominator of B_n equals 42. Observe that the denominators 6, 30, 42, and 66 are the only denominators that are less than 100. In 1840, M. Ohm [1] calculated the Bernoulli numbers up to index 62. D. E. Knuth and T. J. Buckholtz [1] have calculated all Bernoulli numbers with index up to 250. We are very grateful to S. S. Wagstaff for providing us a table of Bernoulli numbers B_n, $n < 1000$, for which the denominator of B_n is less than 100.

To the right of the table described above, Ramanujan records another table.

1	7	13	17	19	31
37	43	47	49	59	61
67	71	73	79	91	97
101	103	107	109	127	133
137	139	149	151	157	167
169	179	181	193	197	199
211	217				

This table provides an extended list of Bernoulli numbers that have a denominator equal to 6, but only half of the index is recorded. Except for the omission of 163, the table is complete up to the index 434.

Entry 21 (Entry 5(i), p. 361). *The coefficients of* x^{100} *and* x^{95} *in the power series expansions of*

$$\frac{x^7}{(1-x^2)(1-x^3)} \quad \textit{and} \quad \frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)},$$

respectively, are each equal to $[95/2] - [95/3] = 16$.

PROOF. Observe that

$$\begin{aligned}\frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)} &= \frac{x^2(1-x^3) - x^3(1-x^2)}{(1-x)(1-x^2)(1-x^3)}\\ &= \frac{x^2}{(1-x^2)(1-x^3)}.\end{aligned} \tag{21.1}$$

By Entry 6, the coefficient of x^n in the Maclaurin series expansion of $x^p/\{(1-x)(1-x^p)\}$ is equal to $[n/p]$. Hence, the desired result follows from (21.1).

Entry 22 (Entry 5(ii), p. 361). *For each positive integer n,*

$$\left[\frac{n+4}{6}\right] - \left[\frac{n+3}{6}\right] + \left[\frac{n+2}{6}\right] = \left[\frac{n}{2}\right] - \left[\frac{n}{3}\right].$$

PROOF. From (21.1) and some additional elementary algebra,

$$\frac{x^2}{(1-x)(1-x^2)} - \frac{x^3}{(1-x)(1-x^3)} = \frac{x^2}{(1-x^2)(1-x^3)} = \frac{x^2 - x^3 + x^4}{(1-x)(1-x^6)}.$$

By Entry 6, the coefficient of x^n on the far left side is $[n/2] - [n/3]$, while the coefficient of x^n on the far right side is $[(n+4)/6] - [(n+3)/6] + [(n+2)/6]$. The desired equality thus readily follows.

Entry 23 (Entry 5(iii), p. 361). *For each positive integer n,*

$$[\sqrt{n+1} + \sqrt{n}] = [\sqrt{4n+2}]. \tag{23.1}$$

PROOF. Set $n = N^2 + a$, where N is a nonnegative integer and $0 \leq a < 2N + 1$. Let

$$n = N^2 + a = (N + \xi)^2,$$

$$n + \tfrac{1}{2} = N^2 + a + \tfrac{1}{2} = (N + \eta)^2,$$

and

$$n + 1 = N^2 + a + 1 = (N + \zeta)^2,$$

where $0 \leq \xi < 1$, $0 < \eta < 1$, and $0 < \zeta \leq 1$. Thus, (23.1) is equivalent to the equality

$$[\zeta + \xi] = [2\eta]. \tag{23.2}$$

If $a < N$, then $0 \leq \xi < \eta < \zeta < \frac{1}{2}$, and both sides of (23.2) equal 0.

If $a > N$, then $\frac{1}{2} < \xi < \eta < \zeta \leq 1$, and both sides of (23.2) equal 1.

If $a = N$, then $\xi < \frac{1}{2}$, but $\frac{1}{2} < \zeta, \eta < 1$. Thus, $[2\eta] = 1$. Put $\xi = \frac{1}{2} - x$ and $\zeta = \frac{1}{2} + z$. To prove that $[\zeta + \xi] = 1$, we must show that $[z - x] = 0$, i.e., we must prove that $x < z$. Returning to the definitions of ξ and ζ, we find that

$$x^2 - (2N + 1)x + \tfrac{1}{4} = 0$$

and

$$z^2 + (2N + 1)z - \tfrac{3}{4} = 0.$$

Thus,

$$(z - x)(2N + 1) = \tfrac{1}{2} - (x^2 + z^2). \tag{23.3}$$

Since $x, z < \frac{1}{2}$, the right side of (23.3) is positive. Thus $z > x$, as was to be shown.

Hence, in all three cases, (23.2) has been established.

Entry 24 (Entry 5(iv), p. 361). *Let* $\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}$, *as in Entry* 22 *of Chapter* 16 (*Part* III [6, p. 36]). *Then the coefficient of* x^n, $n \geq 0$, *in the Maclaurin expansion of* $\psi(x^2)/(1-x)$ *equals*

$$[\tfrac{1}{2} + \sqrt{n + \tfrac{1}{2}}] = [\tfrac{1}{2} + \sqrt{n + \tfrac{1}{4}}]. \tag{24.1}$$

Proof. We first establish the equality (24.1). Let N be an arbitrary fixed nonnegative integer. Then $\frac{1}{2} + \sqrt{n + \frac{1}{2}} \geq N$ if and only if $n \geq N^2 - N - \frac{1}{4}$, or if and only if $n \geq N^2 - N$, since n and N are nonnegative integers. It is also clear that $\frac{1}{2} + \sqrt{n + \frac{1}{4}} \geq N$ if and only if $n \geq N^2 - N$. Thus, (24.1) is immediate.

Next, if $|x| < 1$,

$$\frac{1}{1-x}\psi(x^2) = \sum_{n=0}^{\infty} \left(\sum_{\substack{k,m \geq 0 \\ k + m(m+1) = n}} 1 \right) x^n.$$

The coefficient of x^n on the right side equals one plus the largest integer m^* such that $m^*(m^* + 1) \leq n$. Solving this inequality, we find that

$$m^* \leq \frac{-1 + \sqrt{1 + 4n}}{2}.$$

Thus, the coefficient of x^n equals

$$1 + \left[\frac{-1 + \sqrt{1 + 4n}}{2} \right] = \left[\frac{1 + \sqrt{1 + 4n}}{2} \right],$$

which agrees with Ramanujan's assertion.

Except for the assertion about $\psi(x)$, Entries 5(ii)–(iv) on page 361 were submitted as a problem to the *Journal of the Indian Mathematical Society* by Ramanujan [19], [23, p. 332]. Solutions by H. Br. were published, and we have presented his solution to (iii). Entry 23 was problem no. 3 on the afternoon session of the Putnam Exam in 1948 (Gleason et al. [1, pp. 257–258]).

Entry 25 (Entry 6, p. 361). *Let* $n = \prod_{k=1}^{r} p_k^{\alpha_k}$ *denote the canonical factorization of the positive integer* n *into distinct primes* $p_1, p_2, \ldots, p_r$. *Then, if* $d(n)$ *denotes the number of positive divisors of* n,

$$d(n) = \prod_{k=1}^{r} (\alpha_k + 1).$$

This elementary result is well known. (For example, see the text of Niven, Zuckerman, and Montgomery [1, p. 189].)

Entry 26 (Entry 7(i), p. 362). *Let* $n = p^k$, *where* p *is a prime and* k *is a positive integer. Then* $d(n) \leq \log(pn)/\log p$.

Of course, the inequality sign $\leq$ can be replaced by an equality sign in Entry 26.

Entry 27 (Entry 7(ii), p. 362). *Let* $n = p^j q^k$, *where* p *and* q *are distinct primes and* j *and* k *are positive integers. Then*

$$d(n) \leq \frac{\log^2(pqn)}{4 \log p \log q}.$$

Entry 28 (Entry 7(iii), p. 362). *With the notation of Entry* 25,

$$d(n) \leq \frac{((1/r)\log(p_1 p_2 \cdots p_r n))^r}{\log p_1 \log p_2 \cdots \log p_r}.$$

Of course, Entries 26 and 27 are, respectively, the cases $r = 1$ and $r = 2$ of Entry 28. D. Somasundaram [2] established Entries 26 and 27. In fact, Entry 28 was proved by Ramanujan in his paper [16], [23, p. 81]. Since the proof of Entry 28 is so short, we give it here.

Proof. By the arithmetic–geometric mean inequality,

$$\begin{aligned}
&\frac{((1/r)\log(p_1 p_2 \cdots p_r n))^r}{\log p_1 \log p_2 \cdots \log p_r} \\
&\quad = \frac{((1/r)(\alpha_1 + 1)\log p_1 + (1/r)(\alpha_2 + 1)\log p_2 + \cdots + (1/r)(\alpha_r + 1)\log p_r)^r}{\log p_1 \log p_2 \cdots \log p_r} \\
&\quad \geq \frac{(\alpha_1 + 1)\log p_1 (\alpha_2 + 1)\log p_2 \cdots (\alpha_r + 1)\log p_r}{\log p_1 \log p_2 \cdots \log p_r} \\
&\quad = d(n).
\end{aligned}$$

On page 368 and part of page 369, there is found a table of those integers from 1 to 12,005 that contain only the primes 2, 3, 5, and 7 in their canonical factorizations. These are highly composite number which were discussed at the beginning of this chapter.

Entry 29 (p. 369). *Let* $1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5$, *and* α^6 *be the seven seventh roots of* 1. *Then*

$$\alpha + \alpha^2 + \alpha^4, \alpha^3 + \alpha^5 + \alpha^6 = \tfrac{1}{2}(-1 \pm i\sqrt{7}).$$

PROOF. By elementary algebra, it is easily seen that $\frac{1}{2}(-1 \pm i\sqrt{7})$ are the roots of the polynomial $x^2 + x + 2$. We must show that $\alpha + \alpha^2 + \alpha^4$ and $\alpha^3 + \alpha^5 + \alpha^6$ are roots of this same polynomial. Clearly, they are complex conjugates of each other, and their sum equals -1. By a straightforward multiplication, their product is found to equal 2, and the proof is complete.

Entry 30 (p. 369). *Let* α *be as above. For each complex number* x,

$$\varphi(x) := \sum_{k=0}^{6} \sin(\alpha^k x) = 7 \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{14n+7}}{(7n+14)!}$$

and

$$\psi(x) := \sum_{k=0}^{6} \cos(\alpha^k x) = 7 \sum_{n=0}^{\infty} \frac{(-1)^n x^{14n}}{(14n)!}.$$

PROOF. We have

$$\varphi(x) = \sum_{k=0}^{6} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha^k x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \sum_{k=0}^{6} \alpha^{k(2n+1)}.$$

The inner sum on the far-right side equals 0 unless $n \equiv 3 \pmod 7$ in which case the sum equals 7. The desired result now follows.

The formula for $\psi(x)$ is proved in a similar fashion.

Entry 31 (p. 369). *Let* φ, ψ, *and* α *be as defined in the previous two entries. Then*

$$\begin{aligned} 64 \prod_{k=0}^{6} \sin(\alpha^k x) = & -\varphi(2x) - \varphi(2(\alpha + \alpha^2 + \alpha^4)x) - \varphi(2(\alpha^3 + \alpha^5 + \alpha^6)x) \\ & + \varphi(2(\alpha + \alpha^6)x) + \varphi(2(\alpha^2 + \alpha^5)x) + \varphi(2(\alpha^3 + \alpha^4)x) \\ & + \varphi\left(\frac{2x}{\alpha + \alpha^6}\right) + \varphi\left(\frac{2x}{\alpha^2 + \alpha^5}\right) + \varphi\left(\frac{2x}{\alpha^3 + \alpha^4}\right) \end{aligned} \quad (31.1)$$

and

$$64 \prod_{k=0}^{6} \cos(\alpha^k x) = \psi(2x) + \psi(2(\alpha + \alpha^2 + \alpha^4)x) - \psi(2(\alpha^3 + \alpha^5 + \alpha^6)x)$$
$$+ \psi(2(\alpha + \alpha^6)x) + \psi(2(\alpha^2 + \alpha^5)x) + \psi(2(\alpha^3 + \alpha^4)x)$$
$$+ \psi\left(\frac{2x}{\alpha + \alpha^6}\right) + \psi\left(\frac{2x}{\alpha^2 + \alpha^5}\right) + \psi\left(\frac{2x}{\alpha^3 + \alpha^4}\right). \tag{31.2}$$

Proof. Observe that

$$64 \prod_{k=0}^{6} \sin(\alpha^k x) = -\frac{1}{2i} \prod_{k=0}^{6} (e^{i\alpha^k x} - e^{-i\alpha^k x})$$
$$= -\sum_{j=0}^{6} \sin(2\alpha^j x) + \sum_{j \neq k} \sin(2(\alpha^j + \alpha^k)x)$$
$$- \sum_{j \neq k \neq \ell} \sin(2(\alpha^j + \alpha^k + \alpha^\ell)x). \tag{31.3}$$

The second sum on the right side is over all unordered pairs (j, k), $0 \le j, k \le 6$. Thus, the sum contains $\binom{7}{2} = 21$ terms. The third sum is over all unordered triples (j, k, ℓ), $0 \le j, k, \ell \le 6$ and so has $\binom{7}{3} = 35$ terms. Since the first sum has seven terms and one of the original terms, i.e., $-\sin((1 + \alpha + \cdots + \alpha^6)x)$, equals 0, we have accounted for all $2^7 = 64$ terms.

Now the first expression on the right side of (31.3) clearly equals $-\varphi(2x)$. The second sum on the right side of (31.3) equals

$$\varphi(2(\alpha + \alpha^6)x) + \varphi(2(\alpha^2 + \alpha^5)x) + \varphi(2(\alpha^3 + \alpha^4)x),$$

while the third sum yields the sum of the remaining five terms in (31.1). For example, $\varphi(2x/(\alpha^3 + \alpha^4))$ equals the sum of the seven terms with $(j, k, \ell) = (1, 3, 5), (2, 4, 6), (0, 3, 5), (1, 4, 6), (1, 3, 6), (0, 2, 4), (0, 2, 5)$.

The proof of (31.2) follows along exactly the same lines. Observe that in the proof of (31.2) no minus signs arise, and so the signs of all expressions on the right side of (31.2) are positive.

After the last three entries, Ramanujan remarks "from which 7 interval formula can be found." As will be explained in the next two sections, a "7 interval formula" is a recursion formula for Bernoulli numbers wherein differences of successive indices are 14. Ramanujan does not state this "7 interval formula." These remarks also explain why we have chosen to place these last three entries in this chapter.

D. Somasundaram [7] has also supplied proofs of the past three entries.

Entry 32 (p. 369). *If, for* $n \ge 0$,

$$a_n = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n, \tag{32.1}$$

then

$$a_m a_n = a_{m+n} + a_{m-n},$$

where m and n are nonnegative integers with $m \geq n$.

PROOF. If $\alpha = 2 + \sqrt{3}$, then $\alpha^{-1} = 2 - \sqrt{3}$. Since

$$(\alpha^m + \alpha^{-m})(\alpha^n + \alpha^{-n}) = \alpha^{m+n} + \alpha^{-(m+n)} + \alpha^{m-n} + \alpha^{n-m},$$

the result immediately follows.

We provide the following table of values for a_n:

Table 2

n	a_n
0	2
1	4
2	14
3	52
4	194
5	724
6	2,702
7	10,084
8	37,634
9	140,452
10	524,174
11	1,956,244

The preceding entry is headed by the title "6 interval formula" in the notebooks. Indeed, in the terminology of Ramanujan's paper [2], [23, pp. 1–14], the next four results provide "6 interval formulas." As did Ramanujan, we will express these four theorems in the language of power series. However, the focus should be directed to the associated recursive formulas given by (33.13), (33.15), (33.16), and (33.17) below.

Entry 33 (p. 370). *Let* a_n *be defined by* (32.1), *and let* B_n *denote the* nth *Bernoulli number, where* $n \geq 0$. *Then, for* $|x| < 2\pi$,

$$6 \sum_{j=0}^{\infty} B_{12j} \frac{x^{12j}}{(12j)!} \sum_{k=0}^{\infty} (-1)^k (a_{6k+3} + 2^{6k+3}) \frac{x^{12k+6}}{(12k+6)!}$$
$$= \sum_{n=0}^{\infty} (-1)^n (a_{6n+3} + 2^{6n+3}) \frac{x^{12n+6}}{(12n+5)!}, \quad (33.1)$$

$$6 \sum_{j=0}^{\infty} B_{12j+2} \frac{x^{12j+2}}{(12j+2)!} \sum_{k=0}^{\infty} (-1)^k (a_{6k+3} + 2^{6k+3}) \frac{x^{12x+6}}{(12k+6)!}$$
$$= \sum_{n=0}^{\infty} (-1)^n (a_{6n+4} + 2^{6n+4}) \frac{x^{12n+8}}{(12n+7)!}, \quad (33.2)$$

$$6\sum_{j=0}^{\infty} B_{12j+6}\frac{x^{12j+6}}{(12j+6)!}\sum_{k=0}^{\infty}(-1)^k(a_{6k+3}+2^{6k+3})\frac{x^{12k+6}}{(12k+6)!}$$
$$=\sum_{n=0}^{\infty}(-1)^n(a_{6n+5}-2^{6n+6})\frac{x^{12n+12}}{(12n+11)!}, \quad (33.3)$$

and

$$-6\sum_{j=0}^{\infty} B_{12j+8}\frac{x^{12j+8}}{(12j+8)!}\sum_{k=0}^{\infty}(-1)^k(a_{6k+3}+2^{6k+3})\frac{x^{12k+6}}{(12k+6)!}$$
$$=\sum_{n=0}^{\infty}(-1)^n(a_{6n+6}-2^{6n+7})\frac{x^{12n+14}}{(12n+13)!}. \quad (33.4)$$

PROOF. By Euler's formula, $B_{2n}/(2n)! = 2(-1)^{n-1}(2\pi)^{-2n}\zeta(2n)$, where ζ denotes the Riemann zeta-function. Since $\zeta(2n)$ approaches 1 as n tends to ∞, it is easily seen that the first series on the left side of each of (33.1)–(33.4) converges for $|x| < 2\pi$. By (32.1), the remaining series in (33.1)–(33.4) converge for all x.

From the elementary theory of linear recurring sequences, $(2+\sqrt{3})^n$ and $(2-\sqrt{3})^n$ form a basis of solutions to

$$A_n = 4A_{n-1} - A_{n-2}, \qquad n \geq 2. \quad (33.5)$$

Thus, a_n is a solution of (33.5). By successively employing (33.5), we find that, for $n \geq 12$,

$$a_{n-12} = 780a_{n-7} - 209a_{n-6}$$

and, for $n \geq 7$,

$$a_n = 2911a_{n-6} - 780a_{n-7}.$$

Hence, for $n \geq 12$,

$$a_n = 2702a_{n-6} - a_{n-12}. \quad (33.6)$$

We shall now show that (33.1)–(33.4) follow from a result of D. H. Lehmer [1, p. 646]. Define a sequence $\{\beta_n\}$, $0 \leq n < \infty$, as follows. For $0 \leq n \leq 11$, β_n is given in Table 3 on page 84. For $n \geq 12$, β_n is defined by the recursive formula

$$\beta_n = -2702\beta_{n-6} - \beta_{n-12}. \quad (33.7)$$

From Tables 2 and 3 and from (33.6) and (33.7), we observe that, for $n \geq 0$,

$$a_{6n} = 2(-1)^n\beta_{6n}, \quad (33.8)$$

$$a_{6n+3} = 2(-1)^n\beta_{6n+2}, \quad (33.9)$$

$$a_{6n+4} = 2(-1)^n\beta_{6n+3}, \quad (33.10)$$

and

$$a_{6n+5} = 2(-1)^n\beta_{6n+5}. \quad (33.11)$$

Table 3

n	β_n
0	1
1	5
2	26
3	97
4	265
5	362
6	$-$ 1351
7	$-$ 13,775
8	$-$ 70,226
9	$-$262,087
10	$-$716,035
11	$-$978,122

With β_n defined as above, Lehmer [1] proved that, if $m \not\equiv 2 \pmod 3$, then

$$\sum_{k=0}^{[m/6]} B_{2m-12k}\binom{2m+6}{12k+6}(\beta_{6k+2} + (-1)^k 2^{6k+2}) = \tfrac{1}{3}(m+3)(\beta_{m+2} + (-1)^{[m/2]}2^{m+2}). \quad (33.12)$$

We now prove (33.1). Equating coefficients of x^{12n+6} on both sides of (33.1), we see that we are required to show that, for $n \geq 0$,

$$6\sum_{k=0}^{n} \frac{(-1)^k(a_{6k+3} + 2^{6k+3})B_{12n-12k}}{(12k+6)!\,(12n-12k)!} = \frac{(-1)^n(a_{6n+3} + 2^{6n+3})}{(12n+5)!},$$

or equivalently that

$$\sum_{k=0}^{n} (-1)^k\binom{12n+6}{12k+6}(a_{6k+3} + 2^{6k+3})B_{12n-12k} = (-1)^n(2n+1)(a_{6n+3} + 2^{6n+3}). \quad (33.13)$$

On the other hand, letting $m = 6n$ in (33.12), we find that, for $n \geq 0$,

$$\sum_{k=0}^{n} B_{12n-12k}\binom{12n+6}{12k+6}(\beta_{6k+2} + (-1)^k 2^{6k+2}) = (2n+1)(\beta_{6n+2} + (-1)^n 2^{6n+2}). \quad (33.14)$$

However, using (33.9), we readily see that (33.13) and (33.14) are precisely the same. Hence, (33.1) has been established.

Since the proofs of (33.2)–(33.4) are similar to that above, we shall just sketch the details.

Equating coefficients of x^{12n+8} on both sides of (33.2), we find that it suffices to show that, for $n \geq 0$,

$$\sum_{k=0}^{n} (-1)^k \binom{12n+8}{12k+6} (a_{6k+3} + 2^{6k+3}) B_{12n-12k+2} = \tfrac{1}{3}(-1)^n(6n+4)(a_{6n+4} + 2^{6n+4}). \qquad (33.15)$$

Letting $m = 6n + 1$ in (33.12) and utilizing (33.9) and (33.10), we see that (33.15) follows from Lehmer's result (33.12).

Equating coefficients of x^{12n+12} on both sides of (33.3), we see that we must show that, for $n \geq 0$,

$$\sum_{k=0}^{n} (-1)^k \binom{12n+12}{12k+6} (a_{6k+3} + 2^{6k+3}) B_{12n-12k+6} = (-1)^n(2n+2)(a_{6n+5} - 2^{6n+6}). \qquad (33.16)$$

Letting $m = 6n + 3$ in (33.12) and employing (33.9) and (33.11), we deduce (33.16).

Equating coefficients of x^{12n+14} in (33.4), we find that we must show that, for $n \geq 0$,

$$\sum_{k=0}^{n} (-1)^k \binom{12n+14}{12k+6} (a_{6k+3} + 2^{6k+3}) B_{12n-12k+8} = \tfrac{1}{3}(-1)^{n+1}(6n+7)(a_{6n+6} - 2^{6n+7}). \qquad (33.17)$$

Putting $m = 6n + 4$ in (33.12) and using (33.8) and (33.9), we easily derive (33.17).

This completes the proof of Entry 33.

Ramanujan's first published paper [2], [23, pp. 1–14] is devoted to Bernoulli numbers. A small portion of this material is found in Chapter 5 of the second notebook, and a perspicacious description of the entire paper's contents has been given by S. S. Wagstaff [1]. Ramanujan's paper contains recursive formulas for Bernoulli numbers, where the indices have gaps of lengths 4, 6, 8, and 10, or, in Ramanujan's terminology, 2, 3, 4, and 5 interval formulas. Thus, the recursive formulas given above are not found in [2]. In Entry 31 above, Ramanujan gives the necessary background material for developing recursive formulas in which the indices have gaps of length 14. In discussing the practicality of recursion formulas with gaps, Lehmer [1, pp. 643, 646] opined, "But it is very desirable to have large gaps,... The most practical recurrences have gaps of 12... Recurrences with gaps of 14 or more are not so practical as those given above."

Earlier work on recursive formulas with gaps was accomplished by F. J. Van den Berg [1] and R. Haussner [1]. Further details may be found in Wagstaff's paper [1]. More recently, a general recursion formula with gaps of length $2k$, $k \geq 1$, has been established by M. Chellali [1].

On page 371, Ramanujan offers a list of numbers of the form $x^3 \pm y^3$:

1	2	7	8	9	16	19	26	27	28	35	37	54	56	61
63	64	65	72	91	98	117	124	125	126	127	128	133	152	169
189	208	215	216	217	218	224	243	250	271	279	280	296	316	331
335	341	342	343	344	351	370	386	387	397	407	432	448	469	485
488	504	511	512	513	520	539	547	559	576	602	604	631	637	657
665	686	702	721	728	729	730	737	756	784	798	817	819	854	855
866	875	919	936	945	973	989	992	999	1000					

The largest value of x or y arising in this table is 18, which appears in the representation $18^3 - 17^3 = 919$. One entry, 989, is not of the form $x^3 \pm y^3$. The table contains two omissions, namely $468 = 7^3 + 5^3$ and $988 = 11^3 - 7^3$.

Page 372 is devoted to five rough, preliminary assertions, one of which is expunged, about highly composite numbers. Since these statements obviously represent Ramanujan's earliest thoughts on the subject, and since Ramanujan magnificently and fully developed the subject in his path setting paper [16], [23, pp. 78–128], we shall confine ourselves to just a brief, informal discussion of these few results.

With Ramanujan [23, p. 88], denote a highly composite number by

$$2^{a_2}3^{a_3}5^{a_5}\cdots p_1^{a_{p_1}},$$

where $a_2 \geq a_3 \geq a_5 \geq \cdots \geq a_{p_1} \geq 1$. Not surprisingly, in light of the table at the beginning of the second notebook, on page 372, Ramanujan assumes that $p_1 = 7$ and claims that "the highest composite number can be found from

$$2^{[\log n/\log 2]-1}3^{[\log n/\log 3]-1}5^{[\log n/\log 5]-1}7^{[\log n/\log 7]-1}$$

where n is any positive quantity and [ ] meaning that the nearest integer is taken." This is followed by a similar formula for "the highest composite near the region N," except that the square brackets do not appear and n is replaced by K for which a vague definition punctured by two question marks is given. Much of Ramanujan's paper [16] is devoted to a careful study of the values of a_j, and the values for a_2, a_3, a_5, and a_7 given above are rough, initial approximations. In particular, Ramanujan [16], [23, p. 92] showed that

$$a_\nu\left[\frac{\log \nu}{\log \lambda}\right] \leq a_\lambda \leq a_\mu + (2 + a_\mu)\left[\frac{\log \mu}{\log \lambda}\right]$$

for all values of λ, μ, and ν. For example, if $\nu = p_1$ and $\mu = P_1$, the prime immediately following p_1, then, since $a_{p_1} = 1$ and $a_{P_1} = 0$,

$$\left[\frac{\log p_1}{\log \lambda}\right] \leq a_\lambda \leq 2\left[\frac{\log P_1}{\log \lambda}\right].$$

Ramanujan obtained still more precise results by dividing the primes from 2 to p_1 into five subintervals.

Third, Ramanujan asserts that if N is a highly composite number, then

$$d(N) = O(e^{\log N/\log\log N}),$$

where $d(n)$ denotes the number of positive divisors of n. This result can be improved. Wigert [1] showed in 1906 that if $\varepsilon > 0$ is given, then

$$d(N) < 2^{\frac{\log N}{\log\log N}(1+\varepsilon)},$$

for *all* sufficiently large positive integers N. In his paper [16], Ramanujan established several estimates for $d(N)$. In particular, for all N [23, p. 85],

$$d(N) < 2^{\frac{\log N}{\log\log N} + O\left(\frac{\log N}{(\log\log N)^2}\right)},$$

and for infinitely many N [23, p. 86],

$$d(N) = 2^{\frac{\log N}{\log\log N} + O\left(\frac{\log N}{(\log\log N)^2}\right)}.$$

Lastly, Ramanujan claims that if $N = a!$, for some positive integer a, then

$$d(N) = O\left(e^{\frac{\log N}{(\log\log N)^2}}\right). \tag{34.1}$$

However, Ramanujan [23, p. 127] proved that

$$d(N) = C^{\frac{\log N}{(\log\log N)^2} + \frac{2\log N\log\log\log N}{(\log\log N)^3} + O\left(\frac{\log N}{(\log\log N)^3}\right)},$$

where

$$C = \prod_{n=1}^{\infty}\left(1 + \frac{1}{n}\right)^{1/n}.$$

Since $C > e$, (34.1) is incorrect.

Entry 35 (p. 374). *We have*

$$e^{(\pi/4)\sqrt{30}} \approx 4\sqrt{3}(5 + 4\sqrt{2}),$$
$$e^{(\pi/4)\sqrt{34}} \approx 12(4 + \sqrt{17}),$$
$$e^{(\pi/2)\sqrt{46}} \approx 144(147 + 104\sqrt{2}),$$

$$e^{(\pi/4)\sqrt{42}} \approx 4(21 + 8\sqrt{6}),$$

$$e^{(\pi/4)\sqrt{70}} \approx 12\sqrt{7}(5\sqrt{5} + 8\sqrt{2}),$$

$$e^{(\pi/4)\sqrt{78}} \approx 4\sqrt{3}(75 + 52\sqrt{2}),$$

$$e^{(\pi/4)\sqrt{102}} \approx 4\sqrt{3}(200 + 49\sqrt{17}),$$

and

$$e^{(\pi/4)\sqrt{130}} \approx 12(323 + 40\sqrt{65}).$$

Ramanujan actually uses equality signs above, but it is clear that he regarded the right sides as approximations to the left sides in all cases.

Entry 36 (p. 375). *The expressions*

$$\frac{12}{\sqrt{130}}\log\left(\frac{(3 + \sqrt{13})(\sqrt{8} + \sqrt{10})}{2}\right),$$

$$\frac{24}{\sqrt{142}}\log\left(\frac{\sqrt{10 + 11\sqrt{2}} + \sqrt{10 + 7\sqrt{2}}}{2}\right),$$

and

$$\frac{12}{\sqrt{190}}\log((3 + \sqrt{10})(\sqrt{8} + \sqrt{10}))$$

are approximations to π *valid for* 14, 15, *and* 18 *decimal places, respectively.*

Ramanujan actually asserts that the approximations are good to 15, 16, and 18 decimal places, respectively.

Entry 37 (p. 375). *We have*

$$e^{\pi\sqrt{22}} = 2508951.9982,$$

$$e^{\pi\sqrt{37}} = 199148647.999978,$$

and

$$e^{\pi\sqrt{58}} = 24591257751.99999982.$$

All three numerical approximations are correct.

Entries 35–37 may be found in Ramanujan's famous paper [10], [23, pp. 23–39]. In particular, Entries 35 and 36 are found on page 31 and Entry 37 is on page 26 of [23].

To determine the accuracy of the approximations in Entry 35, we employed *Mathematica* to calculate the following numerical values:

$$e^{(\pi/4)\sqrt{30}} = 73.83278,$$

$$4\sqrt{3}(5 + 4\sqrt{2}) = 73.83285,$$

$$e^{(\pi/4)\sqrt{34}} = 94.477239,$$

$$12(4 + \sqrt{17}) = 97.477267,$$

$$e^{(\pi/2)\sqrt{46}} = 42347.2610,$$

$$144(147 + 104\sqrt{2}) = 42347.2623,$$

$$e^{(\pi/4)\sqrt{42}} = 162.383665,$$

$$4(21 + 8\sqrt{6}) = 162.383671,$$

$$e^{(\pi/4)\sqrt{70}} = 714.16389604,$$

$$12\sqrt{7}(5\sqrt{5} + 8\sqrt{2}) = 714.16389611,$$

$$e^{(\pi/4)\sqrt{78}} = 1029.109108745,$$

$$4\sqrt{3}(75 + 52\sqrt{2}) = 1029.109108769,$$

$$e^{(\pi/4)\sqrt{102}} = 2785.3606180482,$$

$$4\sqrt{3}(200 + 49\sqrt{17}) = 2785.3606180495,$$

$$e^{(\pi/4)\sqrt{130}} = 7745.88371918324,$$

and

$$12(323 + 40\sqrt{65}) = 7745.88371918330.$$

For reference,

$$\pi = 3.14159265358979323846264 3\ldots.$$

Using *Mathematica*, we found the three expressions of Entry 36 to numerically equal,

$$3.1415926535897926, \; 3.14159265358979313,$$

and

$$3.14159265358979323819,$$

respectively.

D. Shanks [5] has discovered several, even more remarkable approximations to π. The most accurate of these is the approximation

$$\pi = \frac{6}{\sqrt{3502}} \log(2u) + 7.37 \cdot 10^{-82},$$

where u is a product of four, comparatively simple, quartic units.

As remarked in the notes in Ramanujan's *Collected Papers* [23, p. 336], approximating π by means of an equation of the type

$$e^{\pi\sqrt{n}} = m$$

was investigated first by C. Hermite [1]. See also a paper of L. Kronecker [1], [2, pp. 123–131] and H. J. S. Smith's *Report* [1, p. 357].

Ramanujan's approximations to π in Entries 35–37 arose from his work on class invariants and singular moduli in [10]. Ramanujan calculated an enormous number of class invariants, and a long chapter in Part V [9] is devoted to this topic. Also see the book [1] of J. M. and P. B. Borwein, wherein many of Ramanujan's formulas and approximations to π are examined, in particular, some of Ramanujan's approximations from class invariants. If we translate each approximation in Entry 36 into an approximation like those in Entry 35 and compare the numerical values in these eleven approximations, in each instance, we find that the quadratic or quartic irrationality is larger than the associated value of the exponential function. This is easily explained from the approximations' origins in class invariants.

We now briefly indicate another, closely related approach to approximations of this sort, although Ramanujan probably did not employ such reasoning.

Let $J(q)$ denote Klein's absolute invariant, where $q = e^{2\pi i\tau}$ and $\operatorname{Im} \tau > 0$. Put $j(q) = 1728J(q)$. All of the Fourier coefficients in the expansion

$$j(q) = \frac{1}{q} + 744 + 196884q + \cdots \tag{37.1}$$

are positive integers (Rankin [1, p. 199]). Now let d be a positive squarefree integer, and let

$$\tau = \begin{cases} i\sqrt{d}, & \text{if } d \equiv 1 \text{ or } 2 \pmod 4, \\ (1 + i\sqrt{d})/2, & \text{if } d \equiv 3 \pmod 4). \end{cases}$$

Thus,

$$q = \begin{cases} e^{-2\pi\sqrt{d}}, & \text{if } d \equiv 1 \text{ or } 2 \pmod 4, \\ -e^{-\pi\sqrt{d}}, & \text{if } d \equiv 3 \pmod 4). \end{cases}$$

Then $j(q)$ is an algebraic integer of degree $h(d)$, where $h(d)$ is the class number of $Q(\sqrt{-d})$ (Silverman [1, p. 339]). In particular, if $h(d) = 1$, then from (37.1), $e^{2\pi\sqrt{d}}$ or $-e^{\pi\sqrt{d}}$ is very close to the rational integer $j(q) - 744$, with an error approximately equal to $196884q$. The largest value for which $h(d)$ equals 1 is $d = 163$. In this case,

$$e^{\pi\sqrt{163}} = 262\ 537\ 412\ 640\ 768\ 743.\ 999\ 999\ 999\ 999\ 2.$$

For each of the three examples in Entry 37, $h(d) = 2$.

For an excellent description of Ramanujan's work that is connected with transcendental number theory, read M. Waldschmidt's centennial lecture [1].

In closing, we remark that by a theorem of A. O. Gelfond [1] proved in 1929, $e^{\pi\sqrt{d}}$ is transcendental for each positive integer d.

Entry 38 (p. 378). *Let a and n denote positive integers with $a \geq 2$. Then*

$$\frac{n}{a-1} - \frac{\log(n+1)}{\log a} \leq \sum_{k=1}^{\infty} \left[\frac{n}{a^k}\right] \leq \frac{n-1}{a-1}. \tag{38.1}$$

Moreover, both inequalities are the best possible.

The proof below appears in a paper by the author and Bhargava [3].

PROOF. Let

$$n = \sum_{j=0}^{m} b_j a^j, \qquad 0 \leq b_j \leq a-1, \quad b_m \neq 0.$$

Then

$$\begin{aligned} S := \sum_{k=1}^{\infty} \left[\frac{n}{a^k}\right] &= \sum_{j=1}^{m} b_j a^{j-1} + \sum_{j=2}^{m} b_j a^{j-2} + \cdots + b_m \\ &= b_m \sum_{j=0}^{m-1} a^j + b_{m-1} \sum_{j=0}^{m-2} a^j + \cdots + b_1 \\ &= b_m \frac{a^m - 1}{a-1} + b_{m-1} \frac{a^{m-1}-1}{a-1} + \cdots + b_1 \frac{a-1}{a-1} + \frac{b_0}{a-1} - \frac{b_0}{a-1} \\ &= \frac{n}{a-1} - \frac{1}{a-1} \sum_{j=0}^{m} b_j. \end{aligned} \tag{38.2}$$

Since $b_m \neq 0$,

$$S \leq \frac{n}{a-1} - \frac{1}{a-1},$$

which proves the latter inequality of (38.1). Note that if $n = a^m$, we have equality above.

We are very grateful to Bruce Reznick for the following proof of the former inequality in (38.1).

Let

$$b = \sum_{j=0}^{m} b_j.$$

Then, by (38.2), it suffices to prove that

$$b \leq (a-1)\frac{\log(n+1)}{\log a}. \tag{38.3}$$

Furthermore, we shall show that we have equality in (38.3) if and only if $n = a^{m+1} - 1$, i.e., $b_j = a - 1$, $0 \leq j \leq m$.

Let

$$b = k(a-1) + r, \qquad 0 \leq r \leq a - 2. \tag{38.4}$$

Then

$$\begin{aligned} n &\geq (a-1)a^0 + (a-1)a + (a-1)a^2 + \cdots + (a-1)a^{k-1} + ra^k \\ &= (r+1)a^k - 1. \end{aligned}$$

So,

$$(a-1)\frac{\log(n+1)}{\log a} \geq (a-1)\frac{\log((r+1)a^k)}{\log a} = k(a-1) + (a-1)\frac{\log(r+1)}{\log a}. \tag{38.5}$$

By (38.3)–(38.5), we shall be finished with the proof if we can show that

$$(a-1)\frac{\log(r+1)}{\log a} \geq r. \tag{38.6}$$

Furthermore, we shall show that equality in (38.6) holds if and only if $r = 0$. Hence, there exists equality in (38.3) if and only if $b = (m+1)(a-1)$, i.e., if and only if $n = a^{m+1} - 1$.

First, if $r = 0$, then clearly (38.6) is valid with equality holding.

If $r \geq 1$, then (38.6) can be written in the form

$$\frac{a-1}{\log a} \geq \frac{r}{\log(r+1)},$$

or

$$f(a-1) \geq f(r), \tag{38.7}$$

where

$$f(x) := \frac{x}{\log(x+1)}.$$

However, by elementary calculus, $f(x)$ is strictly increasing for positive integers x. Since $1 \le r \le a-2$, (38.7) is therefore valid with a strict inequality. This completes the proof.

Another proof of Entry 38 has been given by Bhargava, Adiga, and Somashekara [1].

Entry 38 is of significant interest, because when a is a prime p, the sum in (38.1) equals the highest power of p dividing $n!$ (Andrews [1, p. 104], Niven, Zuckerman, and Montgomery [1, p. 182]).

Entry 39 (p. 382). *Let $x > 0$ and suppose that n is a positive integer exceeding one. Then*

$$\sum_{k=1}^{\infty} \left[\frac{x}{k^n}\right] = \sum_{k=1}^{\infty} \left[\sqrt[n]{\frac{x}{k}}\right] \tag{39.1}$$

$$= \zeta(n)x + \zeta\left(\frac{1}{n}\right)x^{1/n} + O(x^{1/(n+1)}), \tag{39.2}$$

as x tends to ∞.

Proof. Observe that the sum on the left side of (39.1) denotes the number of lattice points in the interior of the first quadrant under the curve $u^n v = x$, where we are counting the lattice points on vertical lines $u = k$. The sum on the right side of (39.1) denotes the same number of lattice points, but now we are counting the lattice points on horizontal lines $v = k$.

We shall indicate a second proof of (39.1) that is not as elementary as the first proof. However, the idea of this second proof can be vastly generalized to yield a great variety of identities involving the greatest integer function. See a paper by the author and U. Dieter [1].

Since $[t]$ has a discontinuity at each positive integer k, with a "jump" of 1, we find that

$$\int_0^{\infty} \left[\frac{x}{t^n}\right] d[t] = \sum_{k=1}^{\infty} \left[\frac{x}{k^n}\right]. \tag{39.3}$$

On the other hand, by an integration by parts,

$$\int_0^{\infty} \left[\frac{x}{t^n}\right] d[t] = -\int_0^{\infty} [t]\, d\left[\frac{x}{t^n}\right]$$

$$= \sum_{k=1}^{\infty} \left[\sqrt[n]{\frac{x}{k}}\right], \tag{39.4}$$

since $[x/t^n]$ has a discontinuity at $t = \sqrt[n]{x/k}$, $1 \le k < \infty$, with a "jump" of -1. Combining (39.3) and (39.4), we complete the second proof of (39.1).

To prove (39.2), we merely apply Theorem 1 of Section 20 with $\alpha = 1$ and $\beta = n$.

Ramanujan's O-term in (39.2) is not what we have given but is $O(x^\varepsilon)$, where presumably ε is an arbitrarily small positive number. This is very interesting, because this is the only instance in the second or third notebook where this notion or notation is used. Ramanujan's assertion about the error term is actually false. By theorems of Landau [4], [8, pp. 145–158] and Krätzel [1], the error term is *not* $o(x^{1/\{2(n+1)\}})$ as x tends to ∞.

Entry 40 (p. 383). *Let $\sigma(n)$ denote the sum of the positive divisors of the positive integer n. Let $n = \prod_{j=1}^r p_j^{\alpha_j}$ be the canonical factorization of n into distinct primes $p_1, p_2, \ldots, p_r$. Then*

$$\sigma(n) = \prod_{j=1}^{r} \frac{p_j^{\alpha_j+1} - 1}{p_j - 1}.$$

This elementary result is well known. For example, see Niven, Zuckerman, and Montgomery's text [1, p. 191].

We quote Ramanujan for the next result.

Entry 41 (p. 383). "*The sum of the divisors of $N = e^\gamma N \log\log N$?*"

In 1913, T. H. Gronwall [1] proved that

$$\overline{\lim_{n\to\infty}} \frac{\sigma(n)}{n \log\log n} = e^\gamma,$$

where γ denotes Euler's constant. Thus, Ramanujan's assertion is a less precise version of Gronwall's theorem. Robin [2] has shown that the truth of the inequality

$$\sigma(n) < e^\gamma n \log\log n,$$

for every $n \geq 5041$ is equivalent to the Riemann hypothesis. See Nicolas' paper [2, p. 237] for further references.

Entry 42 (p. 384). *If s, t, m, and n are arbitrary, then*

$$\begin{aligned}(8s^2 + 40st - 24t^2)^4 + (6s^2 - 44st - 18t^2)^4 + (14s^2 - 4st - 42t^2)^4 \\ + (9s^2 + 27t^2)^4 + (4s^2 + 12t^2)^4 = (15s^2 + 45t^2)^4 \end{aligned} \tag{42.1}$$

and

$$\begin{aligned}(4m^2 - 12n^2)^4 + (3m^2 + 9n^2)^4 + (2m^2 - 12mn - 6n^2)^4 \\ + (4m^2 + 12n^2)^4 + (2m^2 + 12mn - 6n^2)^4 = (5m^2 + 15n^2)^4. \end{aligned} \tag{42.2}$$

Examples (p. 384). *We have*

$$4^4 + 6^4 + 8^4 + 9^4 + 14^4 = 15^4, \tag{42.3}$$

$$1^4 + 2^4 + 12^4 + 24^4 + 44^4 = 45^4, \tag{42.4}$$

$$4^4 + 21^4 + 22^4 + 26^4 + 28^4 = 35^4, \tag{42.5}$$

$$4^4 + 8^4 + 13^4 + 28^4 + 54^4 = 55^4, \tag{42.6}$$

$$1^4 + 8^4 + 12^4 + 32^4 + 64^4 = 65^4, \tag{42.7}$$

$$22^4 + 28^4 + 63^4 + 72^4 + 94^4 = 105^4, \tag{42.8}$$

$$4^5 + 5^5 + 6^5 + 7^5 + 9^5 + 11^5 = 12^5, \tag{42.9}$$

$$5^5 + 10^5 + 11^5 + 16^5 + 19^5 + 29^5 = 30^5, \tag{42.10}$$

$$2^4 + 39^4 + 44^4 + 46^4 + 52^4 = 65^4, \tag{42.11}$$

and

$$22^4 + 52^4 + 57^4 + 74^4 + 76^4 = 95^4. \tag{42.12}$$

Formula (42.1) is due to C. B. Haldeman [1, pp. 289, 290]. It is quite remarkable that Ramanujan used the same notation as Haldeman and recorded the terms in precisely the same order as Haldeman! One might conclude that Ramanujan saw Haldeman's paper or a secondary source quoting it. However, this seems highly unlikely, for Ramanujan had access to very few journals in India, and, moreover, Haldeman's paper [1] was published in a very obscure journal. It is also quite possible that Ramanujan made his discovery before Haldeman did. Thus, in conclusion, the identical notation must be an amazing coincidence.

Likewise, (42.2) is also due to Haldeman [1, p. 289]. A. Martin [3, pp. 325, 326, 331] also found a proof of (42.2), but, inexplicably, did not acknowledge Haldeman's priority, However, at the beginning of his paper [3], Martin remarks that "Mr. Cyrus B. Haldeman of Ross, Butler Co., O., contributed a valuable paper 'On biquadrates' which was published in the present volume of the Magazine, pp. 285–296." (The title of Haldeman's paper is slightly misquoted.) Ramanujan does not use Haldeman's notation but does employ Martin's notation in (42.2). Again, this must be an astonishing coincidence.

Both (42.1) and (42.2) are mentioned by Dickson [2, p. 650], who also sketches proofs of these formulas.

Example (42.3) was evidently first found by D. S. Hart [1]. It is also mentioned by Martin [1, p. 174] and Dickson [2, p. 649]. E. Barbette [1] showed that the only sum of distinct biquadrates, each not exceeding 14^4, equal to a biquadrate is given by (42.3). Note that (42.3) is obtained from (42.1) by setting $s = 1$ and $t = 0$.

Example (42.4) was found by Martin [1, p. 174].

Example (42.5) is given by Haldeman [1, p. 289] and Martin [3, pp. 326, 328] and can be deduced from (42.2) by putting $m = 2$ and $n = 1$.

Example (42.6) can be found in Martin's paper [1, p. 174].

Example (42.7) is evidently due to Hart who, just before his death, communicated it to Martin [1, p. 174]. See also Dickson's treatise [2, p. 649].

Putting $s = 2$ and $t = -1$ in (42.1), Haldeman [1, p. 290] derived Example (42.8).

Examples (42.9) and (42.10) were found by Martin [2, p. 201] who devised another general method. Barbette [1] showed that (42.9) is the only sum of distinct fifth powers $\leq 11^5$ that is equal to a fifth power.

Examples (42.11) and (42.12) were communicated by both Haldeman [1, p. 289] and Martin [3, p. 326] and can be deduced from (42.2) by setting $m = 1$, $n = 2$ and $m = 4$, $n = 1$, respectively.

Entry 43 (p. 385). *If* $a + b + c = 0$, *then*

(i) $2(ab + ac + bc)^2 = a^4 + b^4 + c^4$,

(ii) $2(ab + ac + bc)^4 = a^4(b - c)^4 + b^4(c - a)^4 + c^4(a - b)^4$,

(iii) $2(ab + ac + bc)^6 = (a^2b + b^2c + c^2a)^4 + (ab^2 + bc^2 + ca^2)^4 + (3abc)^4$,

and

(iv) $$2(ab + ac + bc)^8 = (a^3 + 2abc)^4(b - c)^4 + (b^3 + 2abc)^4(c - a)^4 + (c^3 + 2abc)^4(a - b)^4.$$

After stating (i)–(iv), Ramanujan writes "and so on" to indicate that he could derive further formulas of this type.

Observe that Entries 43(i)–(iv) provide formulas for writing a sum of three fourth powers as twice a second, fourth, sixth, or eighth power.

Formula (ii) is a special case of the apparently more general identity

$$\begin{aligned}(a^2 + 2ac - 2bc - b^2)^4 + (b^2 - 2ab - 2ac - c^2)^4 \\ + (c^2 + 2ab + 2bc - a^2)^4 \\ = 2(a^2 + b^2 + c^2 - ab + ac + bc)^4,\end{aligned}$$

discovered by F. Ferrari [1]; let $c = 0$ and replace a by $-a$ to deduce (ii). Proofs of (ii) have also been found by Gérardin [1], [2] and Martin [3, p. 351]. Further references to (ii) and related problems can be found in Dickson's *History* [2, pp. 654–656].

Although Ferrari's identity appears to be more general than (ii), the two identities are, in fact, equivalent. To see this, first replace c by $-c$ in Ferrari's identity, so that this new version is symmetric in a, b, and c. Thus,

$$\begin{aligned}(a^2 - 2ac + 2bc - b^2)^4 &+ (b^2 - 2ab + 2ac - c^2)^4 \\ &+ (c^2 + 2ab - 2bc - a^2)^4 \\ &\qquad = 2(a^2 + b^2 + c^2 - ab - ac - bc)^4,\end{aligned}$$

which we rewrite in the form

$$\begin{aligned}((a-c)^2 - (b-c)^2)^4 + ((b-a)^2 - (c-a)^2)^4 + ((c-b)^2 - (a-b)^2)^4 \\ = 2((c-b)(a-c) + (c-b)(b-a) + (a-c)(b-a))^4.\end{aligned}$$

If we now replace a, b, and c in Ramanujan's formula (ii) by $c - b$, $a - c$, and $b - a$, respectively, we obtain Ferrari's formula, as given above. Since we previously had shown that (ii) could be deduced from Ferrari's formula, the equivalence of the two formulas has been demonstrated.

Formulas (iii) and (iv) are due originally to Ramanujan.

S. Bhargava [2] has devised a beautiful proof of Entry 43, which is very likely to be the same proof found by Ramanujan. Moreover, the meaning of Ramanujan's addendum, "and so on," becomes clear, because Bhargava's Theorem 43.1 below can be easily employed to produce further identities of the sort given in Entry 43.

We first introduce some notation. We denote the nth composition of a function $f(x)$ by $f^{(n)}(x)$, where $f^{(0)}(x) = x$ and $f^{(1)}(x) = f(x)$. Set $f \circ g(x) = f(g(x))$. The symbol $\sum F(a, b, c)$ indicates that the sum is over all cyclic permutations of a, b, and c, i.e.,

$$\sum F(a, b, c) = F(a, b, c) + F(b, c, a) + F(c, a, b).$$

Theorem 43.1. *Let*

$$u := (a, b, c), \qquad |u| := a + b + c, \qquad \|u\| := a^4 + b^4 + c^4,$$

$$f(u) := (f_1(u), f_2(u), f_3(u)) := (a(b - c), b(c - a), c(a - b)), \tag{43.1}$$

and

$$g(u) := (g_1(u), g_2(u), g_3(u)) := \left(\sum a^2 b, \sum ab^2, 3abc\right). \tag{43.2}$$

Then, if $|u| = 0$,

$$\begin{aligned}\|f^{(m)} \circ g^{(n)}(u)\| &= 2(ab + bc + ca)^{2^{m+1}3^n} \\ &= \|g^{(n)} \circ f^{(m)}(u)\|,\end{aligned} \tag{43.3}$$

where $m, n \in \{0, 1, 2, \ldots\}$ *and composition is in terms of components.*

To prove Theorem 43.1, it will be convenient to first establish the following lemma.

Lemma 43.2. *If*

$$|u| = 0, \tag{43.4}$$

then

$$\|u\| = 2(ab + bc + ca)^2. \tag{43.5}$$

If $m, n \in \{0, 1, 2, \ldots\}$, *then*

$$\sum f_1^{(m)}(g^{(n)}(u))f_2^{(m)}(g^{(n)}(u)) = \pm(\sum ab)^{2^m 3^n} = \sum g_1^{(n)}(f^{(m)}(u))g_2^{(n)}(f^{(m)}(u)), \tag{43.6}$$

where the plus sign $+$ *is present only in the case* $m = 0$.

PROOF OF LEMMA 43.2. Squaring (43.4), we find that

$$a^2 + b^2 + c^2 = -2(ab + bc + ca).$$

Squaring the foregoing equality, we deduce that

$$a^4 + b^4 + c^4 = 4(ab + bc + ca)^2 - 2\{(ab + bc + ca)^2 - 2abc(a + b + c)\}.$$

Upon using (43.4), we deduce (43.5).

Next, upon using (43.1) and (43.2), and the assumption (43.4), we find that, respectively $|f(u)| = 0$ and $|g(u)| = 0$. By iteration, it follows that

$$|f^{(m)}(u)| = 0, \qquad m \geq 0, \tag{43.7}$$

and

$$|g^{(n)}(u)| = 0, \qquad n \geq 0. \tag{43.8}$$

Using (43.8) in (43.7) and then conversely (43.7) in (43.8), we deduce that

$$|f^{(m)} \circ g^{(n)}(u)| = 0 = |g^{(n)} \circ f^{(m)}(u)|, \tag{43.9}$$

for $m, n \in \{0, 1, 2, \ldots\}$.

Next, using (43.4) and elementary calculations, we find from (43.1) and (43.2) that

$$\sum f_1(u)f_2(u) = -(\sum ab)^2 \tag{43.10}$$

and

$$\sum g_1(u)g_2(u) = (\sum ab)^3, \tag{43.11}$$

respectively. Invoking (43.7) and replacing u by $f^{(m-1)}(u)$, $m \geq 1$, in (43.10), we deduce that

$$\sum f_1^{(m)}(u)f_2^{(m)}(u) = -(\sum f_1^{(m-1)}(u)f_2^{(m-1)}(u))^2.$$

Upon iteration, this yields

$$\sum f_1^{(m)}(u)f_2^{(m)}(u) = -(\sum ab)^{2^m},$$

where $m \geq 1$. Thus, for $m \geq 0$,

$$\sum f_1^{(m)}(u) f_2^{(m)}(u) = \pm\left(\sum ab\right)^{2^m}, \tag{43.12}$$

where the plus sign is taken only in the trivial case $m = 0$. Similarly, reversing the roles of f and g and employing (43.8) and (43.11), we find that, for $n \geq 0$,

$$\sum g_1^{(n)}(u) g_2^{(n)}(u) = \left(\sum ab\right)^{3^n}. \tag{43.13}$$

Invoking (43.8), we can replace u by $g^{(m)}(u)$ in (43.12) to deduce that

$$\begin{aligned}\sum f_1^{(m)}(g^{(n)}(u)) f_2^{(m)}(g^{(n)}(u)) &= \left(\sum g_1^{(m)}(u) g_2^{(n)}(u)\right)^{2^m} \\ &= \pm\left(\sum ab\right)^{2^m 3^n},\end{aligned}$$

by (43.13), where the plus sign is taken only when $m = 0$. Thus, the first equality in (43.6) has been proved. Similarly, invoking (43.7) to replace u by $f^{(m)}(u)$ in (43.13), we arrive at

$$\begin{aligned}\sum g_1^{(n)}(f^{(m)}(u)) g_2^{(n)}(f^{(m)}(u)) &= \left(\sum f_1^{(m)}(u) f_2^{(m)}(u)\right)^{3^n} \\ &= \pm\left(\sum ab\right)^{2^m 3^n},\end{aligned}$$

by (43.12). Thus, the second equality in (43.6) has been proved to complete the proof of the lemma.

Proof of Theorem 43.1. By (43.9), we may replace u by $f^{(m)} \circ g^{(n)}(u)$ in (43.5) to deduce that

$$\|f^{(m)} \circ g^{(n)}(u)\| = 2\left(\sum f_1^{(m)}(g^{(n)}(u)) f_2^{(m)}(g^{(n)}(u))\right)^2,$$

which immediately yields the first equality in (43.3) upon the use of the first equality in (43.6). Similarly, (43.9) permits us to replace u by $g^{(n)} \circ f^{(m)}(u)$ in (43.5) and deduce that

$$\|g^{(n)} \circ f^{(m)}(u)\| = 2\left(\sum g_1^{(n)}(f^{(m)}(u)) g_2^{(n)}(f^{(m)}(u))\right)^2,$$

which, by the second equality in (43.6), yields the second equality in (43.3).

It is now a simple matter to deduce Entry 43 from Bhargava's beautiful Theorem 43.1. In fact, we already note that (43.5) is identical to Entry 43(i).

Proof of Entry 43. Letting $(m, n) = (0, 0), (1, 0), (0, 1)$, and $(2, 0)$, respectively, in (43.3), we find that

$$\begin{aligned}2(ab + bc + ca)^2 &= \|u\| = a^4 + b^4 + c^4, \\ 2(ab + bc + ca)^4 &= \|f(u)\| = \sum a^4(b - c)^4, \\ 2(ab + bc + ca)^6 &= \|g(u)\| = \left(\sum a^2 b\right)^4 + \left(\sum ab^2\right)^4 + (3abc)^4,\end{aligned}$$

and

$$2(ab + bc + ca)^8 = \|f^{(2)}(u)\| = \sum (a^3 + 2abc)^4(b - c)^4, \tag{43.14}$$

where, in (43.14), we used the elementary calculation

$$f_1^{(2)}(u) = a(b - c)\{b(c - a) - c(a - b)\} = a(b - c)(a^2 + 2bc),$$

since $|u| = 0$, and similar expressions for $f_2^{(2)}(u)$ and $f_2^{(3)}(u)$. This completes the proof of Entry 43.

Using a completely different idea, Kevin Ford has generalized Ramanujan and Bhargava's theorem by representing $2(a^2 + ab + b^2)^{2k}$ as a sum of three fourth powers for *any* positive integer k.

Theorem 43.3. *Let a, b, and k be integers with $k \geq 1$. For $j = 0, 1, 2$, let*

$$S_j = \sum_{\substack{i=0 \\ i \equiv j \pmod 3}}^{k} (-1)^j \binom{k}{i} a^{k-i} b^i.$$

Then

$$2(a^2 + ab + b^2)^{2k} = (S_0 - S_1)^4 + (S_1 - S_2)^4 + (S_2 - S_0)^4.$$

Proof. We begin with the relation

$$A^4 + B^4 + (-A - B)^4 = 2(A^2 + AB + B^2)^2. \tag{43.15}$$

We want to find $A, B \in \mathbb{Z}[a, b]$ such that

$$A^2 + AB + B^2 = (a^2 + ab + b^2)^k. \tag{43.16}$$

Factoring both sides in $\mathbb{Z}[\omega]$, where $\omega = e^{2\pi i/3}$, we find that

$$(A - \omega B)(A - \omega^2 B) = \{(a - \omega b)(a - \omega^2 b)\}^k. \tag{43.17}$$

Setting

$$\begin{aligned} A - \omega B = (a - \omega b)^k &= \sum_{i=0}^{k} \binom{k}{i} (-\omega b)^i a^{k-i} \\ &= S_0 + \omega S_1 + \omega^2 S_2 = S_0 - S_2 - \omega(S_2 - S_1), \end{aligned}$$

we see that $A = S_0 - S_2$ and $B = S_2 - S_1$. Moreover,

$$\begin{aligned} (a - \omega^2 b)^k &= \sum_{i=0}^{k} \binom{k}{i} (-\omega^2 b)^i a^{k-i} \\ &= S_0 + \omega^2 S_1 + \omega S_2 = S_0 - S_2 + \omega^2(S_1 - S_2) = A - B\omega^2. \end{aligned}$$

Hence, (43.17) holds. The theorem now follows immediately from (43.15) and (43.16).

Entry 44 (p. 385). *If* $ad = bc$, *then*

$$(a+b+c)^n + (b+c+d)^n + (a-d)^n = (c+d+a)^n + (d+a+b)^n + (b-c)^n,$$

where $n = 2$ *or* 4.

PROOF. In each case, expand each side to verify the proposed equality.

We have not been able to find Entry 44 in the literature. There exist other algebraic identities yielding two equal sums of three biquadrates. However, Ramanujan's equality of two equal sums of three biquadrates appears to be the simplest and most elegant of all. See also Entry 49 below.

Examples (p. 385). *We have*

$$2^4 + 4^4 + 7^4 = 3^4 + 6^4 + 6^4,$$

$$3^4 + 7^4 + 8^4 = 1^4 + 2^4 + 9^4,$$

$$6^4 + 9^4 + 12^4 = 2^4 + 2^4 + 13^4,$$

$$3^4 + 9^4 = 5^4 + 5^4 + 6^4 + 8^4,$$

$$2^4 + 2^4 + 7^4 = 4^4 + 4^4 + 5^4 + 6^4,$$

$$3^4 + 9^4 + 14^4 = 7^4 + 8^4 + 10^4 + 13^4,$$

$$7^4 + 10^4 + 13^4 = 5^4 + 5^4 + 6^4 + 14^4,$$

$$1^4 + 2^4 + 4 \cdot 2^4 = 3^4,$$

$$3^4 + 6^4 + 4 \cdot 4^4 = 7^4,$$

$$7^4 + 8^4 + 4 \cdot 2^4 = 9^4,$$

$$3^4 + 14^4 + 4 \cdot 2^4 = 10^4 + 13^4,$$

and

$$3^4 + 7^4 + 4 \cdot 2^4 = 5^4 + 5^4 + 6^4.$$

The truth of each of these examples is readily verified.

Haldeman [1, pp. 286, 287] has found general formulas for a sum of four biquadrates to equal a sum of two biquadrates and for a sum of four biquadrates to equal a sum of three biquadrates. The second example above can be found in a paper of Martin [1, p. 183], where many similar examples are also found.

Entry 45 (p. 386). *Let a, b, c, and d be any numbers such that ad = bc. Then*

$$\begin{aligned}64\{&(a+b+c)^6+(b+c+d)^6-(c+d+a)^6-(d+a+b)^6\\&+(a-d)^6-(b-c)^6\}\\&\times\{(a+b+c)^{10}+(b+c+d)^{10}-(c+d+a)^{10}-(d+a+b)^{10}\\&+(a-d)^{10}-(b-c)^{10}\}\\=45\{&(a+b+c)^8+(b+c+d)^8-(c+d+a)^8-(d+a+b)^8\\&+(a-d)^8-(b-c)^8\}^2.\end{aligned} \tag{45.1}$$

The hypothesis $ad = bc$ was omitted by Ramanujan, although it does appear on page 385 as a hypothesis for Entry 44. If this assumption is removed, a numerical counterexample can be found by simply letting $a = d = 1$ and $b = c = 0$. Then the left side equals 16,498,944, while the right side equals 11,704,500.

Entry 45 is one of the most fascinating finite identities we have ever seen. We first verified it by using *Mathematica*, but, of course, this gives no insight into its origin. The proof of Berndt and Bhargava [2] below provides more insight but is still undoubtedly far from Ramanujan's proof. A more recent proof by T. S. Nanjundiah [1] may be closer to Ramanujan's proof.

Set, for each positive integer m,

$$\begin{aligned}F_{2m}(a,b,c,d)=(a+b+c)^{2m}&+(b+c+d)^{2m}-(c+d+a)^{2m}\\&-(d+a+b)^{2m}+(a-d)^{2m}-(b-c)^{2m}.\end{aligned}$$

Put $b = ax$, $c = ay$, and $d = axy$, which does not contravene the hypothesis $ad = bc$. Then it is easily seen that

$$F_{2m}(a,b,c,d)=a^{2m}f_{2m}(x,y),$$

where

$$\begin{aligned}f_{2m}(x,y)=(1+x+y)^{2m}&+(x+y+xy)^{2m}-(y+xy+1)^{2m}\\&-(xy+1+x)^{2m}+(1-xy)^{2m}-(x-y)^{2m}.\end{aligned} \tag{45.2}$$

Hence, (45.1) takes the form

$$64f_6(x,y)f_{10}(x,y)=45f_8^2(x,y). \tag{45.3}$$

From (45.2), we can easily see by inspection that $x = 0, 1, -1, -2, -\frac{1}{2}$ are zeros of $f_{2m}(x, y)$. (By symmetry, $y = 0, 1, -1, -2, -\frac{1}{2}$ are also zeros.) Since f_{2m} has degree (at most) $2m$ in each of the variables x and y, it follows that $f_2(x, y) \equiv 0 \equiv f_4(x, y)$. We have therefore just provided another proof of Entry 44 above. In fact, it is not difficult to see that $f_{2m}(x, y)$ has degree $2m - 1$ in either x or y when $m \geq 3$. Thus, in particular, we have shown that, for some constant c,

$$f_6(x,y)=cx(x^2-1)(x+2)(2x+1)y(y^2-1)(y+2)(2y+1).$$

Furthermore, it also follows that $f_6(x, y) \mid f_{2m}(x, y)$ for every integer $m \geq 3$, which accounts for one feature of (45.3).

In preparation for the theory below, set

$$\gamma_{k,j}^{(2m)} = \begin{cases} \binom{2m}{k}\left\{\binom{2m-k}{j} - \binom{2m-k}{j-k}\right\}, & \text{if } j > k, \\ \binom{2m}{k}\left\{\binom{2m-j}{j} - 2 + (-1)^j\right\}, & \text{if } j = k, \\ \binom{2m}{k}\left\{\binom{2m-k}{j} - \binom{k}{j}\right\}, & \text{if } j < k, \end{cases} \tag{45.4}$$

where $1 \leq j, k \leq m-1$. For brevity, we shall often delete the superscript $(2m)$ in the sequel.

Lemma 45.1. *Let $\gamma_{j,k}^{(2m)}$ be defined by* (45.4). *Suppose that there exist numbers $\xi_j^{(2m)}$, $\xi_k^{(2m)}$, and A_{2m} such that*

$$\gamma_{j,k}^{(2m)} = A_{2m}\xi_j^{(2m)}\xi_k^{(2m)}, \qquad 1 \leq j, k \leq m-1. \tag{45.5}$$

Then

$$f_{2m}(x, y) = A_{2m}x(1 - x^2)H_{2m}(x)y(1 - y^2)H_{2m}(y), \tag{45.6}$$

where

$$H_{2m}(x) = \sum_{k=1}^{m-1} \xi_k x^{k-1}(1 + x^2 + \cdots + x^{2m-2k-2}) \tag{45.7}$$

and $\xi_k = \xi_k^{(2m)}$. Furthermore, if $m \geq 3$,

$$H_{2m}(x) = \xi_1 \prod_{j=1}^{m-2} (1 + x^2 + \rho_j x), \tag{45.8}$$

for certain constants $\rho_1, \rho_2, \ldots, \rho_{m-2}$, depending upon m.

Proof of Lemma 45.1. By (45.2) and the binomial theorem,

$$f_{2m}(x, y) = \sum_{k=0}^{2m} \binom{2m}{k}\alpha_k(y)x^k, \qquad m \geq 1, \tag{45.9}$$

where

$$\begin{aligned} \alpha_k(y) := \alpha_k^{(2m)}(y) := (1 + y)^{2m-k} + (1 + y)^k y^{2m-k} \\ - y^k(1 + y)^{2m-k} - (1 + y)^k + (-y)^k - (-1)^k y^{2m-k}. \end{aligned} \tag{45.10}$$

It follows immediately from (45.10) that

$$\alpha_0(y) = \alpha_m(y) = \alpha_{2m}(y) = 0, \tag{45.11}$$

$$\alpha_k(y) = -\alpha_{2m-k}(y), \qquad 1 \leq k \leq 2m-1, \tag{45.12}$$

and

$$\alpha_k(y) = -y^{2m}\alpha_k(1/y), \qquad 1 \le k \le 2m-1. \tag{45.13}$$

From (45.11), it follows immediately that $f_2(x, y) \equiv 0$, as observed above. Thus, in the sequel, assume that $m \ge 2$. From (45.11) and (45.12), we see that (45.9) can now be written in the form

$$f_{2m}(x, y) = \sum_{k=1}^{m-1} \binom{2m}{k} \alpha_k(y)(x^k - x^{2m-k}). \tag{45.14}$$

Returning to (45.10), we set

$$\alpha_k^{(2m)}(y) = \sum_{j=0}^{2m} \beta_{k,j}^{(2m)} y^j. \tag{45.15}$$

Suppressing the superscripts, from (45.10), we find that

$$\beta_{k,0} = 0, \qquad 1 \le k \le m-1, \tag{45.16}$$

and, from (45.13), we see that

$$\beta_{k,j} = -\beta_{k,2m-j}, \qquad 1 \le k \le m-1, \quad 0 \le j \le 2m. \tag{45.17}$$

The relations (45.16) and (45.17) further imply that

$$\beta_{k,m} = \beta_{k,2m} = 0, \qquad 1 \le k \le m-1. \tag{45.18}$$

Utilizing (45.16)–(45.18) in (45.15), we deduce that

$$\alpha_k(y) = \sum_{j=1}^{m-1} \beta_{k,j}(y^j - y^{2m-j}), \qquad m \ge 2. \tag{45.19}$$

Comparing coefficients of y^j in (45.10) and (45.19), we find that, for $1 \le j, k \le m-1$,

$$\beta_{k,j} = \begin{cases} \dbinom{2m-k}{j} - \dbinom{2m-k}{j-k}, & \text{if } j > k, \\ \dbinom{2m-j}{j} - 2 + (-1)^j, & \text{if } j = k, \\ \dbinom{2m-k}{j} - \dbinom{k}{j}, & \text{if } j < k. \end{cases} \tag{45.20}$$

Utilizing (45.20) in (45.19), putting (45.19) in (45.14), and recalling (45.4), we deduce that

$$f_{2m}(x, y) = \sum_{k=1}^{m-1} \sum_{j=1}^{m-1} \gamma_{k,j}(x^k - x^{2m-k})(y^j - y^{2m-j}). \tag{45.21}$$

Using the assumption (45.5) in (45.21), we establish the factorization (45.6), with $H_{2m}(x)$ given by (45.7). From (45.21), we also see that if $x_0 \neq 0$ is a zero of $f_{2m}(x, y)$, then $1/x_0$ is also a zero. Hence, the factorization (45.8) follows, and so the proof is complete.

By (45.2), $f_{2m}(x, y) = f_{2m}(y, x)$. Hence, from (45.21), it follows that $\gamma_{k,j} = \gamma_{j,k}$, $1 \leq j, k \leq m - 1$. (Of course, this fact may also be readily verified from (45.4).)

By (45.4), $\gamma_{11}^{(4)} = 0$. It then follows from (45.6) and (45.7) that $f_4(x, y) \equiv 0$, which we had observed earlier.

In fact, the factorization (45.8) is actually not needed for the proof of Entry 45.

We now prove Entry 45.

Proof of Entry 45. As noted prior to the proof of the lemma above, it suffices to prove (45.3). In turn, we set $m = 3$, 4, and 5 and calculate the coefficients $\gamma_{k,j}$ from (45.4). In each case, we find the condition (45.5) satisfied. More specifically,

$$A_6 = 3, \qquad \xi_1 = 2, \qquad \xi_2 = 5;$$

$$A_8 = 8, \qquad \xi_1 = 2, \qquad \xi_2 = \xi_3 = 7;$$

and

$$A_{10} = 15, \qquad \xi_1 = 2, \qquad \xi_2 = 9, \qquad \xi_3 = 16, \qquad \xi_4 = 14.$$

Calculating the functions $H_6(x)$, $H_8(x)$, and $H_{10}(x)$ by means of (45.7), we find that

$$H_6(x) = 2(1 + x^2) + 5x = (x + 2)(2x + 1),$$

$$\begin{aligned} H_8(x) &= 2(1 + x^2 + x^4) + 7x(1 + x^2) + 7x^2 \\ &= 2(1 + x^2)^2 + 7x(1 + x^2) + 5x^2 \\ &= (2(1 + x^2) + 5x)((1 + x^2) + x) \\ &= (x + 2)(2x + 1)(x^2 + x + 1), \end{aligned}$$

and

$$\begin{aligned} H_{10}(x) &= 2(1 + x^2 + x^4 + x^6) + 9x(1 + x^2 + x^4) + 16x^2(1 + x^2) + 14x^3 \\ &= 2\{(1 + x^2)^3 - 2x^2(1 + x^2)\} + 9x\{(1 + x^2)^2 - x^2\} \\ &\quad + 16x^2(1 + x^2) + 14x^3 \\ &= 2(1 + x^2)^3 + 9x(1 + x^2)^2 + 12x^2(1 + x^2) + 5x^3 \\ &= (2(1 + x^2) + 5x)((1 + x^2) + x)^2 \\ &= (x + 2)(2x + 1)(x^2 + x + 1)^2. \end{aligned}$$

Hence, from (45.6) we deduce that

$$f_6(x, y) = 3x(1 - x^2)(x + 2)(2x + 1)y(1 - y^2)(y + 2)(2y + 1),$$

$$\begin{aligned} f_8(x, y) = {} & 8x(1 - x^2)(x + 2)(2x + 1)(x^2 + x + 1) \\ & \times y(1 - y^2)(y + 2)(2y + 1)(y^2 + y + 1), \end{aligned}$$

and

$$\begin{aligned} f_{10}(x, y) = {} & 15x(1 - x^2)(x + 2)(2x + 1)(x^2 + x + 1)^2 \\ & \times y(1 - y^2)(y + 2)(2y + 1)(y^2 + y + 1)^2. \end{aligned}$$

The remarkable identity (45.3) now easily follows.

Entry 46 (p. 386). *Let a, b, k, and x be any numbers such that*

$$3k = a^2 + ab + b^2. \tag{46.1}$$

Then

$$(ax^3 + k)^3 - (bx^3 + k)^3 = k\{(x^4 + ax)^3 - (x^4 + bx)^3\}. \tag{46.2}$$

PROOF. By expanding each side, we see that (46.2) is equivalent to the equality

$$\begin{aligned} & a^3x^9 + 3k(a^2x^6 - b^2x^6) + 3k^2(ax^3 - bx^3) - b^3x^9 \\ & \quad = k(a^3x^3 - b^3x^3) + 3k(ax^9 + a^2x^6 - bx^9 - b^2x^6). \end{aligned} \tag{46.3}$$

Using (46.1), we find that (46.3) reduces to a tautology after all terms are cancelled.

The culmination of the next three entries is a beautiful formula equating two equal sums of three biquadrates.

Entry 47 (p. 386). *For any number x,*

$$(x^4 + 1)^4 + 4\left(\frac{x^5 - 5x}{2}\right)^4 + 5(x^4 - 2)^4 = 3^4 + 4\left(\frac{x^5 + x}{2}\right)^4.$$

PROOF. Expand each side and equate coefficients of like powers of x.

Entry 48 (p. 386). *For any number x,*

$$(4x^5 - 5x)^4 + (4x^4 + 1)^4 + 5(4x^4 - 2)^4 = 3^4 + (4x^5 + x)^4.$$

PROOF. Replace x by $\sqrt{2}x$ in Entry 47.

Entry 49 (p. 386). *For any number x,*

$$3^4 + (2x^4 - 1)^4 + (4x^5 + x)^4 = (4x^4 + 1)^4 + (6x^4 - 3)^4 + (4x^5 - 5x)^4.$$

PROOF. Comparing the proposed formula with Entry 48, we find that it suffices to show that

$$5(4x^4 - 2)^4 = (6x^4 - 3)^4 - (2x^4 - 1)^4,$$

which is readily verified by dividing both sides by $(2x^4 - 1)^4$.

Dickson [2, pp. 653–655] cites several formulas providing solutions to

$$u^4 + v^4 + w^4 = x^4 + y^4 + z^4.$$

In particular, Haldeman [1, p. 286] gives a general formula. Several specific solutions can be found in Martin's paper [1].

Several problems on sums of powers examined in this chapter are special instances of the more general problem of representing real homogeneous polynomials of even degree as sums of mth powers of linear forms. Readers should consult B. Reznick's fascinating monograph [1] for more information, and, in particular, for more examples of the types examined in this chapter. For classical work on solving systems of equations involving sums of powers, see A. Gloden's book [1].

Entry 50 (p. 387). *If* $\alpha^2 + \alpha\beta + \beta^2 = 3\lambda\gamma^2$, *then*

$$(\alpha + \lambda^2\gamma)^3 + (\lambda\beta + \gamma)^3 = (\lambda\alpha + \gamma)^3 + (\beta + \lambda^2\gamma)^3.$$

PROOF. Expanding each side above, we find that the proposed equality is equivalent to the equality

$$\alpha^3 + \lambda^3\beta^3 + 3\lambda\gamma^2(\beta + \alpha\lambda^3) = \beta^3 + \lambda^3\alpha^3 + 3\lambda\gamma^2(\alpha + \beta\lambda^3).$$

Using the side condition $3\lambda\gamma^2 = \alpha^2 + \alpha\beta + \beta^2$, we may readily verify that the last displayed equality is, indeed, correct.

In fact, Entry 50 gives Euler's [1], [2, pp. 428–458] general solution in rational numbers to

$$A^3 + B^3 + C^3 = D^3. \tag{50.1}$$

Earlier, in Entry 20(iii) of Chapter 18 (Part III [6, p. 197]), and in Entry 4 of this chapter, Ramanujan gave less general solutions to (50.1). Another family of solutions can be found on page 341 in Ramanujan's lost notebook [24]. Many formulations of the aforementioned general solution of (50.1) have been discovered. In particular, those by J. P. M. Binet [1] and K. Schwering [1] are similar to that of Ramanujan but are more complicated.

Many further references to general solutions of (50.1) can be found in Dickson's *History* [2, pp. 550–561]. However, Ramanujan's solution in Entry 50 seems to represent the simplest of all formulations. It should be emphasized, however, that the general solutions of Euler and Ramanujan do not characterize all *positive integral* solutions to (50.1). This is still an unsolved problem. According to Dickson [2, p. 550], the problem of finding solutions to (50.1) goes back to Diophantus.

Both Hardy [7, p. 11] and Watson [2, p. 145] did not realize that Ramanujan had independently found Euler's solution.

See a paper by L. J. Lander, T. R. Parkin, and J. L. Selfridge [1] for a survey on the more general problem of equal sums of like powers.

On the other hand, Euler conjectured that there were no positive integral solutions to

$$A^4 + B^4 + C^4 = D^4. \tag{50.2}$$

It was not until 1988 that Euler's conjecture was shown to be false by N. D. Elkies [1], who found an infinite family of solutions. The scarcity of solutions to (50.2) should be compared to the abundance of solutions to

$$A^4 + B^4 + C^4 = 2D^4$$

given by Ramanujan in Entry 43(ii).

Entry 51 (p. 388). *Let $p(n)$ denote the number of unrestricted partitions of the positive integer n. Then*

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-1}, \qquad |x| < 1. \tag{51.1}$$

Entry 51 gives Euler's well-known generating function for $p(n)$ (Hardy and Wright [1, p. 274]).

Entry 52 (p. 388). *Let $\sigma(n)$ denote the sum of the positive divisors of the natural number n, and let $p(n)$ be as above. Then for every positive integer n,*

$$np(n) = \sum_{k=1}^{n} \sigma(k)p(n-k).$$

Proof. Differentiating (51.1) with respect to x, we deduce that, for $|x| < 1$,

$$\sum_{n=1}^{\infty} np(n)x^{n-1} = \prod_{n=1}^{\infty} (1 - x^n)^{-1} \sum_{r=1}^{\infty} \frac{rx^{r-1}}{1 - x^r},$$

or

$$\sum_{n=1}^{\infty} np(n)x^n = \sum_{j=0}^{\infty} p(j)x^j \sum_{k=1}^{\infty} \sigma(k)x^k.$$

Equating coefficients of x^n, $n \geq 1$, on both sides, we complete the proof.

The next entry formally generalizes the principal idea of Entry 51.

Entry 53 (p. 388). *Formally,*

$$\prod_{n=1}^{\infty} \frac{1}{1-a_n} = 1 + \frac{a_1 + a_2 + \cdots}{1 - a_1} + \frac{a_2^2 + a_2 a_3 + a_3^2 + \cdots}{(1-a_1)(1-a_2)} + \frac{a_3^3 + a_3^2 a_4 + \cdots}{(1-a_1)(1-a_2)(1-a_3)} + \frac{a_4^4 + a_4^3 a_5 + \cdots}{(1-a_1)(1-a_2)(1-a_3)(1-a_4)} + \cdots. \tag{53.1}$$

The numerator of the second expression on the right side contains all singletons a_j, $j \geq 1$, the numerator of the third expression contains all products $a_j a_k$, $j, k \geq 2$, the numerator of the fourth term contains all products $a_j a_k a_\ell$, $j, k, \ell \geq 3$, etc. Ramanujan has a minor misprint in his formulation.

Proof. The left side of (53.1) generates 1 plus all expressions $a_{j_1}^{\alpha_1} a_{j_2}^{\alpha_2} \cdots a_{j_r}^{\alpha_r}$, where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are positive integers and $r \geq 1$. It remains to show that the right side of (53.1) generates exactly the same set of products. Now

$$(a_1 + a_2 + \cdots) \sum_{n=0}^{\infty} a_1^n$$

generates all singletons a_j, $j \geq 1$, and all products of the form $a_j a_1^n$, $j \geq 1$, $n \geq 1$. Next,

$$(a_2^2 + a_2 a_3 + a_3^2 + \cdots) \sum_{n_1=0}^{\infty} a_1^{n_1} \sum_{n_2=0}^{\infty} a_2^{n_2}$$

generates all doubletons $a_j a_k$, j, $k \geq 2$, and all terms of the form $a_j a_k a_1^{n_1} a_2^{n_2}$, where j, $k \geq 2$ and $n_1, n_2 \geq 0$, with at least one of the indices n_1, n_2 not equal to 0. It should now be clear that each expression $a_{j_1}^{\alpha_1} a_{j_2}^{\alpha_2} \cdots \alpha_{j_r}^{\alpha_r}$ is generated once and only once on the right side of (53.1).

Entry 54 (p. 390). *We have*

$$\tfrac{1}{4} + 2 = (1\tfrac{1}{2})^2,$$

$$\tfrac{1}{4} + 2 \cdot 3 = (2\tfrac{1}{2})^2,$$

$$\tfrac{1}{4} + 2 \cdot 3 \cdot 5 = (5\tfrac{1}{2})^2,$$

$$\tfrac{1}{4} + 2 \cdot 3 \cdot 5 \cdot 7 = (14\tfrac{1}{2})^2,$$

and

$$\tfrac{1}{4} + 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17 = (714\tfrac{1}{2})^2.$$

Each of these equalities is readily verified.

As might be expected from the two omissions between the last two equalities, the suspected pattern does not persist. More precisely,

$$2256\tfrac{1}{4} = (47\tfrac{1}{2})^2 < \tfrac{1}{4} + 2\cdot 3\cdot 5\cdot 7\cdot 11 = 2310\tfrac{1}{4} < (48\tfrac{1}{2})^2 = 2352\tfrac{1}{4}$$

and

$$29{,}756\tfrac{1}{4} = (172\tfrac{1}{2})^2 < \tfrac{1}{4} + 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13 = 30{,}030\tfrac{1}{4} < (173\tfrac{1}{2})^2 = 30{,}102\tfrac{1}{4}.$$

Do there exist infinitely many positive integers n such that

$$\frac{1}{4} + \prod_{j=1}^{n} p_j = \left(\frac{N}{2}\right)^2,$$

where N is a positive odd integer and p_j denotes the jth prime? Equivalently, do there exist infinitely many positive pairs of integers (n, N), where N is odd, such that

$$\prod_{j=1}^{n} p_j = \frac{N-1}{2}\,\frac{N+1}{2}?$$

Observe that $(N-1)(N+1)/8$ is a triangular number.

Entry 55 (p. 390) *We have*

$$1 + 2 = 3, \qquad 2 + 3 = 5, \qquad 1 + 5 = 2\cdot 3, \qquad 3 + 7 = 2\cdot 5,$$

$$1 + 2\cdot 7 = 3\cdot 5, \qquad 2\cdot 5 + 11 = 3\cdot 7, \qquad 3\cdot 5 + 7 = 2\cdot 11,$$

$$2 + 3\cdot 11 = 5\cdot 7, \qquad 2\cdot 3\cdot 7 + 13 = 5\cdot 11, \qquad 3\cdot 5\cdot 11 + 17 = 2\cdot 7\cdot 13,$$

and

$$1 + 2\cdot 3\cdot 7\cdot 17 = 5\cdot 11\cdot 13.$$

Each of these equalities is trivially verified.

Ramanujan has represented certain products of distinct primes as a sum of a prime (regarding 1 as a prime) and another product of distinct primes. There are several gaps, e.g., $3 + 11 = 2\cdot 7$. The intent of Entry 55 is unclear. Perhaps Ramanujan is asking if every product of distinct primes can be represented as a sum of a prime (including 1) and another product of distinct primes.

CHAPTER 24

Ramanujan's Theory of Prime Numbers

In his famous letters of 16 January 1913 and 29 February 1913 to G. H. Hardy, Ramanujan [23, pp. xxiii–xxx, 349–353] made several assertions about prime numbers, including formulas for $\pi(x)$, the number of prime numbers less than or equal to x. Some of those formulas were analyzed by Hardy [3], [5, pp. 234–238] in 1937. A few years later, Hardy [7, Chapter II], in a very penetrating and lucid presentation, thoroughly discussed most of the results on primes found in these letters. In particular, Hardy related Ramanujan's fascinating, but unsound, argument for deducing the prime number theorem. Generally, Ramanujan thought that his formulas for $\pi(x)$ gave better approximations than they really did. As Hardy [7, p. 19] (Ramanujan [23, p. xxiv]) pointed out, some of Ramanujan's faulty thinking arose from his assumption that all of the zeros of the Riemann zeta-function $\zeta(s)$ are real.

In this regard, it is interesting to note two formulas on page 314 of Ramanujan's second notebook [22]. Writing, as he always does, S_x for $\zeta(x)$, Ramanujan offers partial fraction decompositions for

$$\frac{\pi}{2}\frac{p^x\varphi(x)}{S_x\cos(\pi x/2)} \quad \text{and} \quad \frac{\pi}{2}\frac{p^x\varphi(x)}{S_x\Gamma(\{x+1\}/2)\cos(\pi x/2)},$$

where φ is an unspecified function. Ramanujan's formulas include the words "+ terms involving roots of S_x," written in a different color or different shade of ink from the other parts of the formulas. Evidently, these additional phrases were added after Ramanujan reached England, where he learned from Hardy that $\zeta(x)$ possesses complex zeros. Except for this impreciseness, Ramanujan's two partial fraction decompositions are correct, provided $\varphi(x)$ is suitably well behaved. See Entries 11 and 12 of Chapter 30 for proofs.

As mentioned in the previous chapter, "asymptotic expansions" about several singularities and an inadequate understanding of Riemann–Stieltjes integration are two additional factors that contributed to Ramanujan's erroneous reasoning. He also seems to have been misled by false analogies between series and integrals.

It also should be remarked that, through S. N. Aiyar [1], Ramanujan communicated some of his discoveries about primes in a brief note in the *Journal of the Indian Mathematical Society.* This paper apparently never received much attention.

The unorganized portion of Ramanujan's second notebook and the third notebook contain additional discoveries in the theory of prime numbers not found in Ramanujan's letters to Hardy. Although his reasoning might have been unsound or unrigorous in many instances, Ramanujan's discoveries are truly remarkable, especially since, in India, he had little contact with mathematicians interested in prime number theory or with literature on the subject.

Ramanujan does not use the customary notation $\pi(x)$. In fact, he employs no notation at all but prefers to write, e.g., "the number of primes between A and B." We translate all of Ramanujan's statements into more contemporary terminology and notation and add hypotheses to ensure validity.

Entry 1 (p. 307).

$$\pi(B) - \pi(A) = \int_A^B \frac{dx}{\log x} \quad \text{"nearly."}$$

Of course, this reflects the prime number theorem

$$\pi(x) \sim \operatorname{Li}(x), \tag{1.1}$$

as x tends to ∞, where

$$\operatorname{Li}(x) = PV \int_0^x \frac{dt}{\log t} := \lim_{\varepsilon \to 0} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^x \right) \frac{dt}{\log t}.$$

Ramanujan was fond of using the expression "nearly." From Hardy's account [3], we know that Ramanujan considered Entry 1 to be a much better approximation than is warranted.

The next inequality is quite interesting and does not appear to have been previously given in the literature.

Entry 2 (p. 310). *For x sufficiently large,*

$$\pi^2(x) < \frac{ex}{\log x}\, \pi\left(\frac{x}{e}\right). \tag{2.1}$$

PROOF. From (1.1), with a suitable remainder term, and Entry 13 below,

$$\pi(x) = x \sum_{k=0}^{4} \frac{k!}{\log^{k+1} x} + O\left(\frac{x}{\log^6 x}\right),$$

as x tends to ∞. Thus, as x tends to ∞,

$$\pi^2(x) = x^2\left\{\frac{1}{\log^2 x} + \frac{2}{\log^3 x} + \frac{5}{\log^4 x} + \frac{16}{\log^5 x} + \frac{64}{\log^6 x}\right\} + O\left(\frac{x^2}{\log^7 x}\right) \tag{2.2}$$

and

$$\begin{aligned}\frac{ex}{\log x}\pi\left(\frac{x}{e}\right) &= \frac{x^2}{\log x}\sum_{k=0}^{4}\frac{k!}{(\log x - 1)^{k+1}} + O\left(\frac{x^2}{\log^7 x}\right)\\ &= x^2\left\{\frac{1}{\log^2 x} + \frac{2}{\log^3 x} + \frac{5}{\log^4 x} + \frac{16}{\log^5 x} + \frac{65}{\log^6 x}\right\} + O\left(\frac{x^2}{\log^7 x}\right).\end{aligned} \tag{2.3}$$

By comparing (2.2) and (2.3), we see that, indeed, (2.1) holds for sufficiently large x.

Using *Mathematica*, F. S. Wheeler [1], J. Keiper, and W. Galway attempted to numerically determine a value x_0 such that (2.1) is valid for all $x \geq x_0$. We relate part of their interesting investigations.

Observe that the function $\pi^2(x)/\pi(x/e)$, $x \geq 2e$, is a step function continuous from the right with increasing jumps at $x = p_k$, the kth prime, for $k \geq 4$, and decreasing jumps at $x = ep_k$, for $k \geq 2$. Also, $ex/\log x$ is increasing for $x > e$. Thus, if $[a, b]$ is any maximal closed interval in $[2, \infty)$ for which (2.1) is false, then a is a prime.

Wheeler [1] inverted the Prime[ ] function in *Mathematica* (Wolfram [1]), which gives the kth prime number for $1 \leq k \leq 105{,}097{,}565$. Of the 78,498 primes less than 10^6, he found that (2.1) is false for 15,089 primes, i.e., (2.1) is invalid for 19.2% of these primes. The largest value of x less than 10^6, for which (2.1) is false is approximately 926,721.651, where $\pi(x) = 73{,}242$. The next value for which (2.1) fails to hold is $x = p_{79{,}455} = 1{,}012{,}751$. The largest prime in *Mathematica*'s prime tables is $p_{105{,}097{,}565} = 2{,}147{,}483{,}647$. The largest prime in *Mathematica* for which (2.1) is false is $p_{104{,}098{,}917} = 2{,}126{,}020{,}723$. The inequality (2.1) becomes valid again at $x = 782{,}119{,}319e = 2{,}126{,}020{,}732.52\ldots$.

Programming on *Mathematica* and using the function PrimePi[ ], J. Keiper and W. Galway (independently) searched for sign changes for $p(x) := \pi^2(x) - (ex/\log x)\pi(x/e)$ beyond the internal table of Prime[ ]. In particular, Galway completed a search up to 10^{11}. The largest prime for which $p(x) > 0$ in this interval is $x = 38{,}358{,}837{,}677$.

In view of (2.1), it is natural to examine the possible validity of

$$\operatorname{Li}^2(x) < \frac{ex}{\log x}\operatorname{Li}\left(\frac{x}{e}\right). \tag{2.4}$$

By a straightforward calculation,

$$\frac{d}{dx}\left(\operatorname{Li}\left(\frac{x}{e}\right) - \frac{\operatorname{Li}^2(x)\log x}{ex}\right) = \frac{(x - \operatorname{Li}(x)\log(x/e))^2}{ex^2\log(x/e)}, \qquad x > e.$$

Thus, $\operatorname{Li}(x/e) - \operatorname{Li}^2(x)(\log x)/(ex)$ is monotonically increasing for $x > e$. Since (2.4) is valid for $x = 2418$, we conclude that (2.4) holds for all $x \geq 2418$.

This concludes our discussion of the very curious inequality (2.1).

Entry 3 (p. 315). *For* $|x| < 1$,

$$\sum_{k=1}^{\infty} x^k \log k = \sum_p \log p \sum_{k=1}^{\infty} \frac{x^{p^k}}{1 - x^{p^k}},$$

where the outer sum on the right side is over all primes p.

Proof. For a fixed prime p, the expressions involving $\log p$ on the left side equal

$$\sum_{\substack{r,s=1\\ s\not\equiv 0 \pmod p}}^{\infty} x^{p^r s}\log p^r = \log p \sum_{\substack{r,s=1\\ s\not\equiv 0 \pmod p}}^{\infty} r x^{p^r s}.$$

On the other hand, for $|x| < 1$,

$$\sum_{k=1}^{\infty} \frac{x^{p^k}}{1 - x^{p^k}} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} x^{p^k n} = \sum_{r=1}^{\infty} \sum_{\substack{s=1\\ s\not\equiv 0 \pmod p}}^{\infty} r x^{p^r s},$$

because the pairs $(k, n) = (1, p^{r-1}s), (2, p^{r-2}s), \ldots, (r, s)$, $s \not\equiv 0 \pmod p$, each contribute to the coefficient of $x^{p^r s}$. This completes the proof.

We do not know Ramanujan's reasons for recording Entry 3. It is quite similar to Entry 4 which follows. Although Entry 4 is not used in the sequel, it is related to Entry 10 which is crucial in Ramanujan's attempted proof of the prime number theorem.

Entry 4 (p. 315). *For* $x > 0$,

$$\sum_{k=1}^{\infty} (-1)^{k-1} e^{-kx}\log k = -\log 2 \sum_{k=1}^{\infty} \frac{1}{e^{2^k x} - 1} + \sum_{p \text{ odd}} \log p \sum_{k=1}^{\infty} \frac{1}{e^{p^k x} + 1}, \tag{4.1}$$

where the sum on p is over all odd primes p.

Proof. For a fixed prime $p \neq 2$, the terms containing $\log p$ on the left side equal

$$\sum_{\substack{r,s=1 \\ s \not\equiv 0 (\mathrm{mod}\, p)}}^{\infty} (-1)^{p^r s-1} e^{-p^r sx} \log p^r = \log p \sum_{\substack{r,s=1 \\ s \not\equiv 0 (\mathrm{mod}\, p)}}^{\infty} (-1)^{s-1} r e^{-p^r sx}.$$

On the other hand, for $p \neq 2$,

$$\sum_{k=1}^{\infty} \frac{1}{e^{p^k x}+1} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{n-1} e^{-p^k nx}$$
$$= \sum_{r=1}^{\infty} \sum_{\substack{s=1 \\ s \not\equiv 0 (\mathrm{mod}\, p)}}^{\infty} (-1)^{s-1} r e^{-p^r sx},$$

by the same argument used in the previous proof.

For $p = 2$, that part of the left side of (4.1) containing $\log 2$ equals

$$-\log 2 \sum_{\substack{r,s=1 \\ s\,\mathrm{odd}}}^{\infty} r e^{-2^r sx},$$

while on the right side of (4.1),

$$\sum_{k=1}^{\infty} \frac{1}{e^{2^k x}-1} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} e^{-2^k nx} = \sum_{\substack{r,s=1 \\ s\,\mathrm{odd}}}^{\infty} r e^{-2^r sx}.$$

This completes the proof.

Entry 5 (p. 316). *As s tends to $1+$,*

$$\sum_p \frac{\log p}{p^s - 1} \sim \frac{1}{s-1}, \tag{5.1}$$

where the sum is over all primes p.

Proof. Employing the Euler product representation for $\zeta(s)$ and logarithmically differentiating it, we find that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s - 1}, \qquad \operatorname{Re} s > 1. \tag{5.2}$$

Since $\zeta(s)$ has a simple pole at $s = 1$, the left side of (5.2) is asymptotic to $1/(s-1)$ as s tends to 1. Thus, (5.1) follows.

Entry 6 (p. 316). *Let $\mu(n)$ denote the Möbius function. For* $\operatorname{Re} s > 1$,

$$\sum_p \frac{\log p}{p^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{ns-1} + f(s), \tag{6.1}$$

where $f(s)$ is analytic for $\sigma := \operatorname{Re} s > 1$ and is given by

$$f(s) = -\sum_{n=1}^{\infty} \mu(n)\left\{\frac{\zeta'(ns)}{\zeta(ns)} + \frac{1}{ns-1}\right\}. \tag{6.2}$$

PROOF. For $\operatorname{Re} s > 1$,

$$\begin{aligned}
\sum_p \frac{\log p}{p^s} &= \sum_{k=1}^{\infty} \sum_p \frac{\log p}{p^{sk}} \sum_{d|k} \mu(d) \\
&= \sum_{m,n=1}^{\infty} \sum_p \frac{\log p}{p^{smn}} \mu(n) \\
&= \sum_{n=1}^{\infty} \mu(n) \sum_p \frac{\log p}{p^{sn} - 1} \\
&= \sum_{n=1}^{\infty} \mu(n)\left\{\frac{1}{ns-1} + f_n(s)\right\},
\end{aligned}$$

where, by (5.2), $f_n(s)$ is an analytic function for $\operatorname{Re} s > 1$ defined by

$$f_n(s) = -\frac{\zeta'(ns)}{\zeta(ns)} - \frac{1}{ns-1}, \qquad n \geq 1.$$

Since $-\zeta'(ns)/\zeta(ns) \ll 2^{-n\sigma}$, for $\sigma > 1$, we find that

$$\sum_{n=1}^{\infty} \mu(n) \frac{\zeta'(ns)}{\zeta(ns)}$$

converges absolutely and uniformly for $\operatorname{Re} s \geq 1 + \varepsilon$, where ε is any fixed positive number, and thus represents an analytic function for $\operatorname{Re} s > 1$. Examining (6.1) and (6.2), we see that it remains to show that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{ns-1}$$

converges uniformly for $\operatorname{Re} s \geq 1 + \varepsilon$.

Now,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\mu(n)}{ns-1} &= \sum_{n=1}^{\infty} \mu(n)\left\{\frac{1}{ns(ns-1)} + \frac{1}{ns}\right\} \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{ns(ns-1)},
\end{aligned} \tag{6.3}$$

since $\sum_{n=1}^{\infty} \mu(n)/n = 0$, a fact equivalent to the prime number theorem. The series on the right side of (6.3) converges absolutely and uniformly for $\operatorname{Re} s \geq 1 + \varepsilon$ by a routine application of the Weierstrass M-test. Hence, the series on the left side of (6.3) represents an analytic function for $\operatorname{Re} s > 1$. This completes the proof.

Ramanujan's formulation of Entry 6, which does not include the function $f(s)$, appears to indicate, in some sense, that he thought the first series on the left side of (6.1) is "asymptotic" to the series on the right side.

Entry 7 (p. 316). *For* $\operatorname{Re} s > 1$,

$$\int_1^\infty \frac{\log x}{x^s} \, d\pi(x) = \sum_{n=1}^\infty \frac{\mu(n)}{n} \int_1^\infty \frac{dx}{x^{s+(n-1)/n}} + f(s),$$

where $f(s)$ *is analytic for* $\operatorname{Re} s > 1$.

PROOF. By Entry 6, for $\operatorname{Re} s > 1$,

$$\begin{aligned}\int_1^\infty \frac{\log x}{x^s} \, d\pi(x) &= \sum_p \frac{\log p}{p^s} \\ &= \sum_{n=1}^\infty \frac{\mu(n)}{ns-1} + f(s) \\ &= \sum_{n=1}^\infty \frac{\mu(n)}{n} \int_1^\infty \frac{dx}{x^{s+(n-1)/n}} + f(s).\end{aligned}$$

As in Entry 6, the function $f(s)$ does not appear in Ramanujan's formulation of Entry 7.

Entry 8 (p. 316). *As* x *tends to* ∞,

$$\frac{d\pi(x)}{dx} \approx \frac{1}{x \log x} \sum_{n=1}^\infty \frac{\mu(n)}{n} x^{1/n}. \tag{8.1}$$

Except for changes in notation, we have quoted Ramanujan above.

Of course, Entry 8 is only a formal statement, but let us speculate how Ramanujan might have argued. Suppose we let s tend to 1 and equate integrands in Entry 7. Ignoring the possible influence of $f(s)$, we formally deduce that

$$\frac{\log x}{x} \frac{d\pi(x)}{dx} \approx \sum_{n=1}^\infty \frac{\mu(n)}{nx^{2-1/n}},$$

which readily implies (8.1).

Alternatively, employing Riemann's series for $\pi(x)$, given in (11.2) below, we find upon formal differentiation that

$$\frac{d\pi(x)}{dx} \approx \sum_{n=1}^\infty \frac{\mu(n)}{n} \frac{(1/n)x^{1/n-1}}{\log x^{1/n}} = \frac{1}{x \log x} \sum_{n=1}^\infty \frac{\mu(n)}{n} x^{1/n}.$$

However, we know without a doubt that Ramanujan deduced (11.2) from (8.1), for in a letter to Hardy reproduced with Ramanujan's lost notebook [24, pp. 374–375], he provides a sketch of his "proof" of the prime number theorem. In the last step (at the bottom of page 375), Ramanujan derives (11.1) (or (11.2)) from (8.1). Lastly, from this approximation of $\pi(x)$, Ramanujan deduces the prime number theorem. For further discussion of this letter, see Berndt and Rankin's book [1].

The next entry is rather mysterious, and we quote Ramanujan.

Entry 9 (p. 317). *If x be a function of n such that*

$$\int_1^\infty \sum_{k=1}^\infty e^{-ax^k} \log x \, dn = \frac{1}{a}, \tag{9.1}$$

then there will be n prime numbers within 1 *and x.*

Evidently, dn is meant to denote a discrete measure with a contribution either at each positive integer or at each prime. It seems hopeless to actually deduce that $n = \pi(x)$ from an equality such as (9.1).

What Ramanujan is probably trying to say is as follows: Observe that

$$\begin{aligned}\int_0^\infty \sum_{k=1}^\infty e^{-ax^k} \log x \, d\pi(x) &= \sum_p \sum_{k=1}^\infty e^{-ap^k} \log p \\ &= \sum_{n=2}^\infty \Lambda(n) e^{-an},\end{aligned} \tag{9.2}$$

where, as usual,

$$\Lambda(n) = \begin{cases}\log p, & \text{if } n = p^k, \\ 0, & \text{otherwise.}\end{cases}$$

We now interject into our discussion a Tauberian theorem of Hardy and Littlewood [3], [4] (Hardy [5, pp. 9–13; 20–97]) (see also Hardy's book [7, p. 34]). Let $a_n \geq 0$ and suppose that

$$\sum_{n=1}^\infty a_n e^{-an} \sim \frac{1}{a},$$

as a tends to $0+$. Then

$$\sum_{n \leq x} a_n \sim x,$$

as x tends to ∞.

Returning to (9.2), suppose that we can prove that

$$\varphi_1(a) := \sum_{n=2}^\infty \Lambda(n) e^{-an} \sim \frac{1}{a}, \tag{9.3}$$

as a tends to $0+$. Then by the Hardy–Littlewood Tauberian theorem quoted above,

$$\psi(x) := \sum_{p^m \leq x} \log p \sim x,$$

which is, of course, equivalent to the prime number theorem.

In detail, Hardy [7, pp. 35–38] describes Ramanujan's "proof" of (9.3), which Ramanujan had related to him in a letter dated 17 April 1913 (see Berndt and Rankin's book [1]). We shall provide a sketch of this argument here and perhaps provide a few additional insights based upon our examination of the notebooks [22] and the lost notebook [24].

The first primary step is Entry 10 below. Hardy [7, pp. 36–37] gives a proof of Entry 10 that is substantially equivalent to Ramanujan's proof, which, in fact, can be found in the lost notebook [24, pp. 373–374]. We give another proof, which is more elementary and somewhat shorter.

Entry 10 (p. 316). *For $x > 0$, define $\varphi(x)$ by*

$$\varphi_1(x) = \sum_p \log p \sum_{k=1}^{\infty} e^{-p^k x} = \log 2 \sum_{k=1}^{\infty} 2^k e^{-2^k x} + \varphi(x), \tag{10.1}$$

where the sum on the left side of (10.1) *is over all primes p. Then*

$$\sum_{k=1}^{\infty} (-1)^{k-1} \varphi(kx) = \sum_{\mu=1}^{\infty} (-1)^{\mu-1} e^{-\mu x} \log \mu. \tag{10.2}$$

PROOF. From (10.1),

$$\sum_{n=1}^{\infty} (-1)^{n-1} \varphi(nx) = \log 2 \sum_{n=1}^{\infty} (-1)^n \sum_{k=1}^{\infty} (2^k - 1) e^{-2^k nx}$$
$$+ \sum_{n=1}^{\infty} (-1)^{n-1} \sum_{p>2} \log p \sum_{k=1}^{\infty} e^{-p^k nx}. \tag{10.3}$$

Let $\mu = 2^m$, $m \geq 1$. The coefficient of $(\log 2)e^{-\mu x}$ on the right side of (10.3) equals

$$-(2^m - 1) + (2^{m-1} - 1) + (2^{m-2} - 1) + \cdots + (2^2 - 1) + (2 - 1)$$
$$= -2^m + 1 + 2^m - 2 - (m - 1) = -m,$$

which is in agreement with the right side of (10.2).

Next, let $\mu = 2^m r$, where $m \geq 0$ and r is an odd integer exceeding 1. First, suppose that $m \geq 1$ so that μ is even. By the same argument as above, the coefficient of $(\log 2)e^{-\mu x}$ on the right side of (10.3) equals $-m$. Now let p

be an odd prime dividing μ, and suppose that $p^\alpha \| \mu$. The coefficient of $(\log p)e^{-\mu x}$ on the right side of (10.3) equals

$$\sum_{j=1}^{\alpha} (-1) = -\alpha. \tag{10.4}$$

Adding up the contributions from each odd prime p and from 2, we see that the coefficient of $e^{-\mu x}$ on the right side of (10.3) equals $-\log \mu$, as required by (10.2).

Lastly, let μ be odd. We proceed as above. The only change is that the sum (10.4) is replaced by

$$\sum_{j=1}^{\alpha} 1 = \alpha.$$

Thus, the coefficient of $e^{-\mu x}$ on the right side of (10.3) equals $+\log \mu$, in agreement with (10.2).

Let $\Phi(x)$ denote the right side of (10.2). Hardy [7, p. 37] now relates Ramanujan's claim

$$\Phi(a) \sim \ell \tag{10.5}$$

as a tends to $0+$, where ℓ is given below. Hardy then states that, "He gives no reason, but the conclusion is correct and easy to prove." In Chapter 15 of his second notebook [22], Ramanujan records many general asymptotic expansions of this type. In particular, from Example (i) in Section 2 (Berndt [4, p. 303]), as a tends to $0+$,

$$\sum_{k=1}^{\infty} e^{-ka} \log k \sim \frac{-\gamma - \log a}{a} + \tfrac{1}{2}\log(2\pi), \tag{10.6}$$

where γ denotes Euler's constant. To derive (10.5) from (10.6), write

$$\begin{aligned}
\Phi(a) &= \sum_{k=1}^{\infty} e^{-ka} \log k - 2 \sum_{k=1}^{\infty} e^{-2ka} \log(2k) \\
&= \sum_{k=1}^{\infty} e^{-ka} \log k - 2 \sum_{k=1}^{\infty} e^{-2ka} \log k - 2 \log 2 \frac{e^{-2a}}{1 - e^{-2a}} \\
&\sim \frac{-\gamma - \log a}{a} + \tfrac{1}{2}\log(2\pi) - 2\left(\frac{-\gamma - \log(2a)}{2a} + \tfrac{1}{2}\log(2\pi)\right) \\
&\quad - 2 \log 2\left(\frac{1}{2a} - \frac{1}{2}\right) \\
&= \log 2 - \tfrac{1}{2}\log(2\pi),
\end{aligned}$$

as a tends to $0+$, by (10.6). Thus, in (10.5), $\ell = \log 2 - \frac{1}{2}\log(2\pi)$.

Next, from Entry 10 and (10.5), Ramanujan infers that

$$\varphi(a) \sim \ell \tag{10.7}$$

as a tends to $0+$. In fact, to prove the prime number theorem, one needs only the weaker assertion

$$\varphi(a) = o\left(\frac{1}{a}\right) \tag{10.8}$$

as a approaches $0+$. However, as Hardy [7, p. 38] demonstrates, $a\varphi(a)$ oscillates as a tends to $0+$, and so both (10.7) and (10.8) are false.

Ramanujan next asserts that

$$\varphi_2(a) := \log 2 \sum_{k=1}^{\infty} 2^k e^{-2^k a} \sim \frac{1}{a}, \tag{10.9}$$

as a approaches $0+$. Then, if Ramanujan were correct, from (10.1), (10.7), and (10.9),

$$\varphi_1(a) \sim \frac{1}{a} + \ell \sim \frac{1}{a}, \tag{10.10}$$

as a tends to $0+$. Indeed, recalling (9.3), we see that (10.10) was our goal. However, as Hardy [7, pp. 38–39] demonstrates, (10.9) is also false. Hardy conjectures, "He seems to have been deceived by an 'integral analogy'." Indeed, on pages 374–375 of [24], Ramanujan deduces from Entry 10 and (10.7) that (with minor changes in notation)

$$\text{“}\int_0^{\infty} \log t \sum_{k=1}^{\infty} e^{-t^k a}\, d\pi(t) - \log 2 \int_0^{\infty} 2^t e^{-2^t a}\, dt \tag{10.11}$$

is finite when a becomes 0, that is,

$$\int_0^{\infty} \log t \sum_{k=1}^{\infty} e^{-t^k a}\, d\pi(t) - \frac{1}{a} \quad \text{is finite when } a \text{ becomes 0.”} \tag{10.12}$$

(It is easily shown that the asymptotic estimate

$$\log 2 \int_0^{\infty} 2^t e^{-2^t a}\, dt \sim \frac{1}{a},$$

implied by (10.11) and (10.12), is correct.)

In the publication of the lost notebook [24], Ramanujan's complete letter of 22 January 1914 to Hardy is reproduced, and in this letter, responding to some of Hardy's objections to his proof of the prime number theorem, Ramanujan [24, p. 376] confesses that "my assertion in the previous letter that

$$\log 2 \sum_{k=1}^{\infty} 2^k e^{-2^k a} - \int_0^{\infty} 2^t e^{-2^t a}\, dt$$

is finite when $a = 0$ is wrong." Thus, from the observations made in the last two paragraphs, we see that, indeed, Ramanujan replaced the sum $\varphi_2(a)$ in (10.9) by the corresponding integral.

Returning to Ramanujan's argument on page 375 of the lost notebook, we find that Ramanujan now supposes that there exists a function $\eta(t)$ such that

$$\frac{d\pi(t)}{dt} = \frac{\eta(t)}{\log t}. \tag{10.13}$$

He then deduces from his assertion (10.12) that

$$\text{“}\int_0^{\infty} \sum_{k=1}^{\infty} e^{-t^k a}\eta(t)\, dt - \frac{1}{a} \quad \text{is finite when} \quad a = 0.\text{”} \tag{10.14}$$

Let us make another brief digression from Ramanujan's argument. Of course, the assertion (10.13) is highly objectionable. Responding to Hardy's protest, Ramanujan [24, p. 376], in the same letter quoted above, argues (with minor notational alterations)

"I think I am correct in using dn/dx which is not the differential coefficient of a discontinuous function but the differential coefficient of an average continuous function passing fairly (though not exactly) through the isolated points. I have used dn/dx in finding the number of numbers of the form 2^p3^q, $2^p + 3^q$, etc., less than x and have got correct results."

In the present context, $n = \pi(x)$. In regard to the first-mentioned application, Ramanujan's asymptotic formula for the number of numbers of the form 2^p3^q can be found in Entry 15 of Chapter 23.

Needless to say, Ramanujan's argument is unconvincing. A thorough examination of this letter may be found in a book by the author and Rankin [1].

Continuing the argument on page 375 of his lost notebook, Ramanujan deduces at once, from (10.14), that

$$\eta(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n x^{(n-1)/n}}. \tag{10.15}$$

This conclusion is rather tenuous, for the solution η of (10.14) is clearly not unique. Perhaps Ramanujan's knowledge of the "correct" result influenced his reasoning. Nonetheless, we shall verify that Ramanujan's choice of η does, indeed, formally satisfy (10.14).

Inverting the order of integration and summation twice and making the change of variable $u = t^k a$, we formally find that, for $a > 0$,

$$\begin{aligned}\int_0^\infty \sum_{k=1}^\infty e^{-t^k a}\eta(t)\,dt &= \sum_{k=1}^\infty \frac{1}{ka^{1/k}} \int_0^\infty e^{-u} u^{1/k-1} \sum_{n=1}^\infty \frac{\mu(n)}{n(u/a)^{(n-1)/(nk)}}\,du \\ &= \sum_{k=1}^\infty \frac{1}{k} \sum_{n=1}^\infty \frac{\mu(n)}{na^{1/(nk)}} \int_0^\infty e^{-u} u^{1/(nk)-1}\,du \\ &= \sum_{r=1}^\infty \frac{1}{ra^{1/r}} \sum_{n|r} \mu(n)\Gamma\left(\frac{1}{r}\right) \\ &= \frac{1}{a},\end{aligned}$$

since the last sum on n equals 0 unless $r = 1$ in which case the sum equals 1. Thus, formally, the choice of $\eta(x)$ in (10.15) satisfies (10.14). In fact, the "finite" expression in (10.14) is identically equal to 0 for $a > 0$.

The remainder of Ramanujan's argument is straightforward. From (10.13) and (10.15),

$$\frac{d\pi(x)}{dx} = \frac{1}{x \log x} \sum_{n=1}^\infty \frac{\mu(n)x^{1/n}}{n}. \tag{10.16}$$

Thus, we have once again reached Entry 8. Formally integrating both sides above, we deduce the prime number theorem, or, more precisely, Entry 11 below.

Entry 11 (p. 317). *As x tends to ∞,*

$$\pi(x) \approx \sum_{n=1}^\infty \frac{\mu(n)}{n} \int^{x^{1/n}} \frac{dt}{\log t}. \tag{11.1}$$

Ramanujan does not specify the lower limit above. If 0 is taken to be the lower limit, we obtain Riemann's series [1],

$$\pi(x) \approx \sum_{n=1}^\infty \frac{\mu(n)}{n} \operatorname{Li}(x^{1/n}) =: R(x). \tag{11.2}$$

As indicated above, Ramanujan integrated (8.1), or (10.16), to deduce (11.1). A rigorous derivation of Riemann's series approximation can be found in Hardy's book [7, pp. 40–41].

Ramanujan [23, p. 351] communicated (11.1) in his second letter to Hardy [7, p. 23], with the lower limit given by $\mu = 1.45136380$. Although no explanation is given by Ramanujan, μ is evidently that unique positive number such that

$$PV \int_0^\mu \frac{dt}{\log t} = 0. \tag{11.3}$$

Soldner (see Nielsen's treatise [1, p. 88]) calculated the value $\mu = 1.4513692346$. The author and R. J. Evans [2] used MACSYMA to calculate μ and found the more accurate value $\mu = 1.4513692349$.

In summary, Ramanujan's path to the prime number theorem is strewn with potholes of unrigorous thinking. Nonetheless, Ramanujan's heuristic reasoning is fascinating.

Entry 12 (p. 317). *As x tends to ∞,*

$$\pi(x) \approx \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}k}{B_{2k}(2k-1)} \left(\frac{\log x}{2\pi}\right)^{2k-1} =: G(x), \tag{12.1}$$

where B_j, $j \geq 2$, denotes the jth *Bernoulli number.*

We use the most common contemporary convention for Bernoulli numbers (Abramowitz and Stegun [1, p. 804]), which is different from the definitions of both Hardy and Ramanujan.

Entry 12 can also be found in Ramanujan's [23, p. xxvii] second letter to Hardy [7, p. 23].

The series $G(x)$ is closely related to an approximate series for $\pi(x)$ found by Gram [1], as we shall prove in the sequel.

It is not clear from the notebooks how accurate Ramanujan thought his approximations $R(x)$ and $G(x)$ to $\pi(x)$ were. (Ramanujan always used equality signs in instances where we would use the signs $\approx$, $\sim$, or $\cong$.) According to Hardy [7, p. 42], Ramanujan, in fact, claimed that, as x tends to ∞,

$$\pi(x) - R(x) = O(1) = \pi(x) - G(x),$$

both of which are false.

Entry 13 (p. 318). *Let n be a positive integer, let μ be as above, and put $\delta = n - \log x$. Define $\theta = \theta(n)$ by*

$$\int_{\mu}^{x} \frac{dt}{\log t} = x\left\{\sum_{k=0}^{n-2} \frac{k!}{\log^{k+1} x} + \frac{(n-1)!}{\log^n x}\theta\right\}.$$

Then as x tends to ∞,

$$\theta = \left(\tfrac{2}{3} - \delta\right) + \frac{1}{\log x}\left\{\frac{4}{135} - \frac{\delta^2(1-\delta)}{3}\right\}$$
$$+ \frac{1}{\log^2 x}\left\{\frac{8}{2835} + \frac{2\delta(1-\delta)}{135} - \frac{\delta(1-\delta^2)(2-3\delta^2)}{45}\right\} + \cdots.$$

This result is also found in Ramanujan's [23, p. 351] second letter to Hardy, but Hardy did not discuss or mention the result in [7]. In his letter, Ramanujan [23, p. 351] remarks that Entry 13 is best "for practical calculations." An equivalent formulation of Entry 10 is due to Gram [1], who expanded θ in powers of $1/n$ instead of $1/\log x$.

We now show that Entry 13 can be derived from a more general asymptotic formula proved by F. W. J. Olver [1, pp. 527–530].

PROOF. From Olver's book [1, p. 529], for $x > 0$ and any positive integer n,

$$e^{-x}PV\int_{-\infty}^{x}\frac{e^t}{t}\,dt = \sum_{k=0}^{n-1}\frac{k!}{x^{k+1}} + C_n(x)\frac{n!}{x^{n+1}}, \tag{13.1}$$

where, as n tends to ∞,

$$C_n(n+\zeta) \sim \sum_{k=0}^{\infty}\frac{\gamma_k(\zeta)}{n^k}, \tag{13.2}$$

where the coefficients $\gamma_k(\zeta)$ are polynomials in ζ, which is bounded. In particular [1, p. 530],

$$\gamma_0(\zeta) = \zeta - \tfrac{1}{3}, \tag{13.3}$$

$$\gamma_1(\zeta) = -\tfrac{1}{3}\zeta^3 + \tfrac{2}{3}\zeta^2 - \tfrac{1}{3}\zeta + \tfrac{4}{135}, \tag{13.4}$$

and

$$\gamma_2(\zeta) = \tfrac{1}{15}\zeta^5 - \tfrac{1}{9}\zeta^3 - \tfrac{2}{135}\zeta^2 + \tfrac{4}{135}\zeta + \tfrac{8}{2835}. \tag{13.5}$$

In (13.1), replace n by $n-1$, let $t = \log u$, and replace x by $\log x$ to deduce that

$$\int_0^x \frac{du}{\log u} = x\sum_{k=0}^{n-2}\frac{k!}{\log^{k+1}x} + x\frac{(n-1)!}{\log^n x}C_{n-1}(\log x). \tag{13.6}$$

By hypothesis, $\log x = n - \delta$. Thus, from (13.2), as x tends to ∞,

$$\begin{aligned}
C_{n-1}(\log x) &= C_{n-1}(n-\delta) = C_{n-1}(n-1+1-\delta)\\
&\sim \sum_{k=0}^{\infty}\frac{\gamma_k(1-\delta)}{(n-1)^k}\\
&= \sum_{k=0}^{\infty}\frac{\gamma_k(1-\delta)}{n^k}\sum_{j=0}^{\infty}\frac{(k)_j}{j!\,n^j}\\
&= \sum_{r=0}^{\infty}\frac{1}{n^r}\sum_{k=0}^{r}\frac{\gamma_k(1-\delta)(k)_{r-k}}{(r-k)!}\\
&= \gamma_0(1-\delta) + \frac{1}{n}\gamma_1(1-\delta) + \frac{1}{n^2}\{\gamma_1(1-\delta)+\gamma_2(1-\delta)\} + \cdots\\
&= \gamma_0(1-\delta) + \frac{\gamma_1(1-\delta)}{\log x+\delta} + \frac{\gamma_1(1-\delta)+\gamma_2(1-\delta)}{(\log x+\delta)^2} + \cdots
\end{aligned}$$

$$\sim \gamma_0(1-\delta) + \frac{\gamma_1(1-\delta)}{\log x} - \frac{\delta\gamma_1(1-\delta)}{\log^2 x} + \frac{\gamma_1(1-\delta)+\gamma_2(1-\delta)}{\log^2 x} + \cdots$$

$$= \gamma_0(1-\delta) + \frac{\gamma_1(1-\delta)}{\log x} + \frac{\gamma_2(-\delta)}{\log^2 x} + \cdots, \tag{13.7}$$

since from Olver's text [1, p. 530, eq. (4.12)], $\gamma_2(\zeta) + \zeta\gamma_1(\zeta) = \gamma_2(\zeta - 1)$. From (13.3)–(13.5),

$$\gamma_0(1-\delta) = \tfrac{2}{3} - \delta,$$

$$\gamma_1(1-\delta) = -\tfrac{1}{3}(1-\delta)^3 + \tfrac{2}{3}(1-\delta)^2 - \tfrac{1}{3}(1-\delta) + \tfrac{4}{135}$$
$$= \tfrac{4}{135} - \tfrac{1}{3}\delta^2(1-\delta),$$

and

$$\gamma_2(-\delta) = -\tfrac{1}{15}\delta^5 + \tfrac{1}{9}\delta^3 - \tfrac{2}{135}\delta^2 - \tfrac{4}{135}\delta + \tfrac{8}{2835}$$
$$= \frac{8}{2835} + \frac{2\delta(1-\delta)}{135} - \frac{\delta(1-\delta^2)(2-3\delta^2)}{45}.$$

Putting the values above in (13.7) and then substituting (13.7) into (13.6), we complete the proof.

It is very interesting that the coefficients $\gamma_0(1-\delta)$, $\gamma_1(1-\delta)$, $\gamma_2(-\delta), \ldots$ also appear as coefficients in the asymptotic expansion of another integral of Ramanujan (Berndt [4, p. 193, Entry 6]).

Entry 14 (p. 318). *Let μ be defined by* (11.3), *and let γ denote Euler's constant. Then*

$$\int_\mu^x \frac{dt}{\log t} = \gamma + \log\log x + \sum_{k=1}^\infty \frac{\log^k x}{k!\,k}.$$

PROOF. Observe that

$$\int_\mu^x \frac{dt}{\log t} = \mathrm{Li}(x) - \mathrm{Li}(\mu) = \mathrm{Li}(x)$$
$$= \int_0^x \frac{dt}{\log t} = -\int_{-\log x}^\infty \frac{e^{-u}}{u}\,du$$
$$= \int_0^1 \frac{1-e^{-u}}{u}\,du - \int_1^\infty \frac{e^{-u}}{u}\,du + \int_1^{\log x} \frac{du}{u} + \int_{-\log x}^0 \frac{1-e^{-u}}{u}\,du$$
$$= \gamma + \log\log x + \sum_{k=1}^\infty \frac{\log^k x}{k!\,k}, \tag{14.1}$$

where we have used a well-known representation for γ (Berndt [2, p. 103]).

The formula for $\mathrm{Li}(x)$ given by (14.1) is well known; e.g., see Nielsen's book [1, pp. 3, 11].

Employing (14.1), we shall now demonstrate the relationships among Riemann's series $R(x)$, Gram's series, and Ramanujan's series $G(x)$ in Entry 12.

By (14.1), for each positive integer n,

$$\mathrm{Li}(x^{1/n}) = \gamma - \log n + \log\log x + \sum_{k=1}^{\infty} \frac{\log^k x}{k!\, kn^k}.$$

Thus, from (11.2),

$$\begin{aligned} R(x) &= \gamma \sum_{n=1}^{\infty} \frac{\mu(n)}{n} - \sum_{n=1}^{\infty} \frac{\mu(n)\log n}{n} + \log\log x \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \\ &\quad + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{k=1}^{\infty} \frac{\log^k x}{k!\, kn^k} \\ &= 1 + \sum_{k=1}^{\infty} \frac{\log^k x}{k!\, k} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{k+1}} \\ &= 1 + \sum_{k=1}^{\infty} \frac{\log^k x}{k!\, k\zeta(k+1)} =: h(\log x) = h(y), \end{aligned} \tag{14.2}$$

where ζ denotes the Riemann zeta-function and $y = \log x$. In the derivation above, we used the two well-known facts (Landau [5, pp. 582–587])

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\mu(n)\log n}{n} = -1. \tag{14.3}$$

The series $h(\log x)$ is Gram's series. Thus, Riemann's series and Gram's series are identical.

Using Euler's formula (Berndt [2, p. 105]),

$$\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k} B_{2k}}{2(2k)!}, \qquad k \geq 1, \tag{14.4}$$

where B_j, $j \geq 0$, denotes the jth Bernoulli number, we find that

$$\begin{aligned} h(y) - h(-y) &= 2 \sum_{k=1}^{\infty} \frac{\log^{2k-1} x}{(2k-1)!\,(2k-1)\zeta(2k)} \\ &= 8 \sum_{k=1}^{\infty} \frac{(-1)^{k-1} k \log^{2k-1} x}{B_{2k}(2k-1)(2\pi)^{2k}}. \end{aligned}$$

Hence, $h(y) - h(-y) = G(x)$, Ramanujan's series in Entry 12. To show that $G(x)$ is a genuine approximation to $\pi(x)$, we need to prove that $h(-y) = o(1)$ as x tends to ∞.

Returning to (14.2), we find that

$$\begin{aligned}
h(-y) &= 1 + \sum_{k=1}^{\infty} \frac{(-y)^k}{k!\,k\zeta(k+1)} \\
&= 1 + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,k}\left(\frac{y}{n}\right)^k \\
&= 1 - \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \int_0^{y/n} \frac{1-e^{-u}}{u}\,du \\
&= -\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left(\int_0^{y/n} \frac{1-e^{-u}}{u}\,du + \log n\right) \\
&=: \sum_{n=1}^{\infty} \frac{\mu(n)}{n} F(y, n), \qquad (14.5)
\end{aligned}$$

by (14.3).

Now,

$$\begin{aligned}
F(y, n) - F(y, n+1) &= -\int_{y/(n+1)}^{y/n} \frac{1-e^{-u}}{u}\,du - \log\frac{n}{n+1} \\
&= \int_{y/(n+1)}^{y/n} \frac{e^{-u}}{u}\,du =: f(y, n). \qquad (14.6)
\end{aligned}$$

Put

$$g(n) = \sum_{k=1}^{n} \frac{\mu(k)}{k}.$$

Then, by (14.5), (14.6), and partial summation,

$$\begin{aligned}
h(-y) &= \sum_{n=1}^{\infty} \{g(n) - g(n-1)\} F(y, n) \\
&= \sum_{n=1}^{\infty} g(n)\{F(y, n) - F(y, n+1)\} \\
&= \sum_{n=1}^{\infty} g(n) f(y, n). \qquad (14.7)
\end{aligned}$$

By a theorem in Landau's treatise [5, p. 597], as n tends to ∞,

$$g(n) = O\left(\frac{1}{\log^2 n}\right).$$

Also,

$$0 \le f(y, n) \le \left(\frac{y}{n} - \frac{y}{n+1}\right)\frac{n+1}{y} e^{-y/(n+1)} = \frac{1}{n} e^{-y/(n+1)} \le \frac{1}{n}.$$

Employing these last two estimates in (14.7), we find that, for $1 \le y < \infty$,

$$h(-y) \ll \sum_{n=1}^{\infty} \frac{1}{n \log^2 n} < \infty.$$

Hence, $h(-y)$ is uniformly convergent for $1 \le y < \infty$. Thus, by (14.7) and (14.6),

$$\lim_{y\to\infty} h(-y) = \sum_{n=1}^{\infty} g(n) \lim_{y\to\infty} f(y, n) = 0,$$

which is what we wanted to prove.

The argument above is taken from Hardy's book [7, p. 45].

Entry 15 (p. 318). *We have*

$$\pi(x) \approx \int_0^\infty \frac{(\log x)^t \, dt}{t\Gamma(t+1)\zeta(t+1)} =: J(x), \tag{15.1}$$

where ζ denotes the Riemann zeta-function.

Entry 15 is due to Ramanujan. Observe that Entry 15 is precisely the integral analogue of Gram's series given in (14.2).

Entry 15 is the last of four approximate formulas for $\pi(x)$ communicated in Ramanujan's [23, pp. xxvii] second letter to Hardy [7, p. 23], who [3], [5, pp. 234–238] thoroughly discussed $J(x)$ in conjunction with $R(x)$ and $G(x)$, defined in (11.2) and (12.1), respectively. In particular, Hardy [3] showed that

$$J(x) = G(x) + o(1) = R(x) + o(1),$$

as x tends to ∞.

Ramanujan gives another version of Entry 15 by utilizing his "extended" Bernoulli numbers. In Chapter 6 of the second notebook [22] (see Part I [2, p. 125]), Ramanujan "interpolates" Euler's formula (14.4) for $\zeta(2k)$ by defining Bernoulli numbers B_s^* for any real index s by

$$B_s^* = \frac{2\Gamma(s+1)\zeta(s)}{(2\pi)^s}. \tag{15.2}$$

Thus, if s is an even positive integer $2k$, by Euler's formula (14.4), the right side of (15.2) equals $|B_{2k}|$, where B_j denotes the jth Bernoulli number. Using (15.2), we may rewrite (15.1) in the form

$$\pi(e^{2\pi a}) \approx \int_0^\infty \frac{a^x(1+x)}{\pi x B_{x+1}^*} \, dx = J(e^{2\pi a}).$$

(Ramanujan inadvertently placed an extra factor 2 in the denominator of the integrand above.)

Entry 16 (p. 323). *Let* μ *be defined by* (11.3), *and let* γ *denote Euler's constant. Then*

$$\int_\mu^x \frac{dt}{\log t} = \gamma + \log\log x + \sqrt{x} \sum_{n=0}^{\infty} \frac{(-1)^{n-1}\log^n x}{n!\,2^{n-1}} \sum_{k=0}^{[(n-1)/2]} \frac{1}{2k+1}.$$

Entry 16 is in some sense "between" Entries 13 and 14. For calculational purposes, Entry 13 is best. In contrast to Entry 13, Entry 14 is exact, but the convergence is very slow for large x. Entry 16 is exact and more useful than Entry 14 for calculations when x is large, because of the factor $\sqrt{x}$ on the right side.

Proof. In view of Entry 14, it suffices to prove that

$$\sum_{k=1}^{\infty} \frac{\log^k x}{k!\,k} = \sqrt{x} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\log^n x}{n!\,2^{n-1}} \sum_{k=0}^{[(n-1)/2]} \frac{1}{2k+1}. \tag{16.1}$$

Putting $z = \log x$, we rewrite (16.1) in the form

$$e^{-z/2} \sum_{k=1}^{\infty} \frac{z^k}{k!\,k} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}z^n}{n!\,2^{n-1}} \sum_{k=0}^{[(n-1)/2]} \frac{1}{2k+1}. \tag{16.2}$$

The coefficient of z^n, $n \geq 1$, on the left side of (16.2) is equal to

$$\frac{(-1)^n}{n!\,2^n} \sum_{k=1}^{n} \frac{(-1)^k 2^k}{k} \binom{n}{k}.$$

Thus, it suffices to prove that

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}2^k}{k} \binom{n}{k} = 2 \sum_{k=0}^{[(n-1)/2]} \frac{1}{2k+1}, \qquad n \geq 1. \tag{16.3}$$

For $n = 1$, (16.3) is trivial. By Pascal's formula and induction,

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{(-1)^{k+1}2^k}{k} \binom{n+1}{k} &= \sum_{k=1}^{n+1} \frac{(-1)^{k+1}2^k}{k} \left\{\binom{n}{k} + \binom{n}{k-1}\right\} \\
&= 2 \sum_{k=0}^{[(n-1)/2]} \frac{1}{2k+1} + \sum_{k=1}^{n+1} \frac{(-1)^{k+1}2^k}{n+1} \binom{n+1}{k} \\
&= 2 \sum_{k=0}^{[(n-1)/2]} \frac{1}{2k+1} + \frac{1}{n+1} - \frac{1}{n+1}(-1)^{n+1} \\
&= 2 \sum_{k=0}^{[(n-1)/2]} \frac{1}{2k+1} + \frac{1+(-1)^n}{n+1} \\
&= 2 \sum_{k=0}^{[n/2]} \frac{1}{2k+1}.
\end{aligned}$$

Thus, (16.3) is established, and the proof is complete.

On page 319, Ramanujan writes, in longhand, the sum

$$\sum_{k=1}^{400} \mu(k)k,$$

where $\mu(k)$ denotes the Möbius function. Vertical bars are inserted after certain terms according to the following rules. The first bar is inserted after -5. Between subsequent bars an equal number of positive and negative terms appear with the bars placed to minimize the number of terms in each interval. These bars are inserted in corresponding places in Ramanujan's communication [23, p. 351] of Riemann's formula (Entry 11 with the lower limit equaling μ) to Hardy. In his letter, Ramanujan advises Hardy to terminate calculations at the first bar after the individual terms in Ramanujan's formula become numerically less than 1 in absolute value. It should be mentioned that this rule does not guarantee a precise determination of $\pi(x)$. Ramanujan evidently does not adhere to his general rule for the first interval, because of the dominance of the leading term, $\int_\mu^x dt/\log t$.

In the aforementioned letter, immediately after Entry 11, Ramanujan [23, p. 351] gives a short table for $\pi(x)$.

x	$\pi(x)$	"Formula"
50	15	14.9
300	62	61.9
1000	168	168.2

This table is also found at the top of page 323 in the second notebook. Because the table is placed directly before Entry 16 in the notebook, if it were not for the table's appearance in the letter, one might wrongly conclude that Entry 16 was used for the calculations. Using *Mathematica*, we checked Ramanujan's calculations and found that just four terms of Riemann's series give the values 14.90, 61.85, and 168.22 as approximations for $\pi(x)$ when $x = 50$, 300, and 1000, respectively. Thus, indeed, Ramanujan's calculations are correct.

Entry 17 (p. 366). *For* $x > 1$,

$$\log 2\left(\sum_{n=0}^{\infty} 2^n e^{-2^n x} + \sum_{n=0}^{\infty} \frac{(-x)^n}{(2^{n+1}-1)n!}\right)$$
$$= \frac{1}{x}\left(1 + 0.0000098844 \cos\left(\frac{2\pi \log x}{\log 2} + 0.872711\right)\right). \qquad (17.1)$$

On the right side of (17.1), Ramanujan misrecorded the last constant as 0.872811. Note that the first expression on the left side is essentially $\varphi_2(x)$, which was defined in (10.9). This result was communicated in Ramanujan's letter of 22 January 1914 to Hardy in response to criticism made by Hardy in his letter of 24 December 1913 about Ramanujan's proof of the prime number theorem. In particular, see the paragraph in Section 10 between (10.12) and (10.13). For copies of these two letters and further commentary, see Berndt and Rankin's book [1].

In his book [7, pp. 39, 47], Hardy also discusses Entry 17. He quotes the formula

$$\log 2\left(\sum_{n=1}^{\infty} 2^n e^{-2^n x} + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^n}{n!\,(2^{n+1}-1)}\right) = \frac{1}{x}\left(1 + \sum_{k=-\infty}^{\infty}{}' \,\Gamma\left(1 + \frac{2k\pi i}{\log 2}\right) x^{-2k\pi i/\log 2}\right), \tag{17.2}$$

where $x > 0$, and the prime (′) indicates that the term for $k = 0$ is omitted from the sum. (In Hardy's book [7, p. 39, eq. (2.11.2)] there are two misprints; the sign on the bilateral sum is incorrect, and $\Gamma((1 + 2k\pi i)/\log 2)$ should be replaced by $\Gamma(1 + 2k\pi i/\log 2)$.) This result follows from a theorem in Hardy's paper [1], [6, pp. 146–165]. In particular, in the last equality of Section 13, set $a = 2$, replaced $\log(1/x)$ by x, and differentiate both sides with respect to x.

Now,

$$e^{-x} + \sum_{n=0}^{\infty} \frac{(-x)^n}{(2^{n+1}-1)n!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\left(1 + \frac{1}{2^{n+1}-1}\right) = \sum_{n=0}^{\infty} \frac{(-x)^n 2^{n+1}}{n!\,(2^{n+1}-1)}. \tag{17.3}$$

Using (17.3) in (17.1) and comparing the resulting formula with (17.2), we see that it remains to show that, for $x > 1$,

$$\sum_{k=-\infty}^{\infty}{}' \,\Gamma\left(1 + \frac{2k\pi i}{\log 2}\right) x^{-2k\pi i/\log 2} = 0.0000098844 \cos\left(\frac{2\pi \log x}{\log 2} + 0.872711\right). \tag{17.4}$$

In fact, the equality sign in (17.4) should be replaced by an approximation sign $\approx$, for, as will be seen, the right side of (17.4) arises from the terms $k = \pm 1$ on the left side.

Set

$$\Gamma\left(1 + \frac{2\pi i}{\log 2}\right) = \alpha + i\beta = 3.1766226452 \times 10^{-6} - 3.7861079986 \times 10^{-6} i,$$

where this and subsequent calculations were performed via *Mathematica*.

Then the sum of the two terms for $k = \pm 1$ on the left side of (17.4) equals

$$2\alpha \cos\left(\frac{2\pi \log x}{\log 2}\right) + 2\beta \sin\left(\frac{2\pi \log x}{\log 2}\right)$$
$$= 2\sqrt{\alpha^2 + \beta^2}\left(\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \cos\left(\frac{2\pi \log x}{\log 2}\right) + \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \sin\left(\frac{2\pi \log x}{\log 2}\right)\right)$$
$$= 2\sqrt{\alpha^2 + \beta^2} \cos\left(\frac{2\pi \log x}{\log 2} - \varphi\right), \tag{17.5}$$

where

$$\varphi = \cos^{-1} \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}.$$

Now

$$2\sqrt{\alpha^2 + \beta^2} = 0.000009884441351$$

and

$$\varphi = -0.8727109989.$$

Using these values in (17.5), we deduce the approximation indicated by (17.4).

In the third notebook (p. 371 in volume 2 of [22]), Ramanujan gives the following table:

Long Intervals of Composite Numbers

$p_n^{(1)}$	$p_n^{(2)}$	Difference
2	3	1
3	5	2
7	11	4
23	29	6
89	97	8
113	127	14
523	541	18
887	907	20
1,129	1,151	22
1,327	1,361	34
9,551	9,587	36
15,683	15,727	44
19,609	19,661	52
31,397	31,469	72
265,621	265,703	82
360,653	360,749	96
370,261	370,373	112
492,113	492,227	114
1,357,201	1,357,333	132
1,561,919	1,562,051	132
2,010,733	2,010,881	148

The intent in this table is to record all intervals $(p_n^{(1)}, p_n^{(2)})$, $n \geq 1$, up to 2,010,881, where $p_n^{(1)}$ and $p_n^{(2)}$ are consecutive primes and $p_n^{(2)} - p_n^{(1)} > p_{n-1}^{(2)} - p_{n-1}^{(1)}$, $n \geq 2$. There are, in fact, some gaps in this table. Ramanujan missed the pair (155,921, 156,007) with a difference of 86 and the pair (1,349,533, 1,349,651) with a difference of 118. As a consequence of the first omission, Ramanujan recorded the pair (265,621, 265,703) with a difference of 82. Inexplicably, Ramanujan records a second appearance for the difference 132.

Over the past few decades, Ramanujan's table has been extended by several authors, including L. J. Lander and T. R. Parkin [1], R. P. Brent [1], [2], and J. Young and A. Potler [1]. In particular, Young and Potler determined first occurrences of prime gaps up to the prime 72,635,119,999,997. All first occurrences of gaps from 2 to 674 were found. The largest gap they found is 778 following the prime 42,842,283,925,351.

Also on page 371, Ramanujan records a table of values for $\pi(x)$.

x	$\pi(x)$
$2 \cdot 10^4$	2,262
10^5	9,592
$2 \cdot 10^5$	17,984
$3 \cdot 10^5$	25,997
$4 \cdot 10^5$	33,860
$5 \cdot 10^5$	41,538
$6 \cdot 10^5$	49,098
$7 \cdot 10^5$	56,543
$8 \cdot 10^5$	63,951
$9 \cdot 10^5$	71,274
10^6	78,498
$2 \cdot 10^6$	148,931
$3 \cdot 10^6$	216,816
10^7	664,579
10^8	5,761,460

All but two values are correct. The two corrected values are $\pi(2{,}000{,}000) = 148{,}933$ and $\pi(100{,}000{,}000) = 5{,}761{,}455$.

The primary method for calculating $\pi(x)$ was discovered in 1870 by Meissel [1], who calculated $\pi(x)$ for several values of x. In fact, all of the values of $\pi(x)$ in Ramanujan's table, except those for $x = 2 \cdot 10^6$, $3 \cdot 10^6$, and 10^8, are found in Meissel's paper [1]. One year later, Meissel [2] published a paper entirely devoted to the calculation of $\pi(100{,}000{,}000)$. Unfortunately, he made an error and so claimed that $\pi(100{,}000{,}000) = 5{,}761{,}460$, which is exactly the value given in Ramanujan's table. It was not until 1883 that Meissel [3] published a correction giving the correct value

$\pi(100{,}000{,}000) = 5{,}761{,}455$. Thus, it appears that Ramanujan took his values of $\pi(x)$ from a secondary source published between 1870 and 1883 or from a later source unaware of Meissel's correction.

It is curious that in the introduction to his table of prime numbers published in 1914, D. N. Lehmer [1], who regards 1 as the first prime, remarks "This number he (Meissel) finds to be 5,761,461. As all of his other computations have been checked by actual count, and no errors have been discovered as yet, this number is worthy of confidence." However, later, in a table of values for $\pi(x)$, Lehmer records the correct value $\pi(100{,}000{,}000) = 5{,}761{,}456$.

In 1893, Gram [2] published an extensive table giving values of $\pi(x)$ in the range from 10^6 to 10^7 initially in intervals of 25,000 and later in intervals of 100,000. In particular, Gram records the values of $\pi(2{,}000{,}000)$ and $\pi(3{,}000{,}000)$.

Meissel's elementary methods have been considerably improved by J. C. Lagarias, V. S. Miller, and A. M. Odlyzko [1]. Analytic methods for computing $\pi(x)$ have been devised by Lagarias and Odlyzko [1]. They determined currently the largest known value of $\pi(x)$, namely, $\pi(4 \cdot 10^{16}) = 1{,}075{,}292{,}778{,}753{,}150$.

Lastly, we remark that Carr [1] reproduced Burkhardt's *Tables des Diviseurs* up to 99,000 and mentioned Glaisher and Dase's tables in his preface, written in 1880; the publication of [1] evidently did not occur until 1886. No mention is made of Meissel's calculations. Thus, Ramanujan learned about the existence of factor tables quite early, but his awareness of tables of primes and $\pi(x)$ probably came much later.

After his table for $\pi(x)$, Ramanujan remarks, "If p be any prime number and there are k primes between p and $p + \varphi(p, k)$, to find the max., min., and average values of φ." Ramanujan offers no results on such problems. However, recall that Ramanujan [21], [23, pp. 208, 209] gave a short, clever proof of Bertrand's postulate: for every integer $n > 1$, there exists at least one prime between n and $2n$. Perhaps the most famous problem of this sort is to determine the least value of θ such that there exists at least one prime between n and $n + O(n^\theta)$, for each sufficiently large positive integer n. At present, the smallest known value of θ is $\frac{6}{11} + \varepsilon$, for each $\varepsilon > 0$, established by S. Lou and Q. Yao [1].

On page 319, Ramanujan gives a table indicating the number of primes between 4 and 1000 in certain arithmetic progressions. (The table is given at the top of the following page.) All of these values are correct. Hudson [1] and Hudson and Brauer [1] have derived analogues of Meissel's formula for arithmetic progressions, so that more extensive calculations can be made.

On page 308, Ramanujan remarks that because the square of an odd prime is of the form $4n + 1$, $\pi_{4,3}(x) > \pi_{4,1}(x)$, where $\pi_{4,j}(x)$, $j = 1, 3$, denotes the number of primes less than or equal to x that are congruent to j modulo 4. Thus, Ramanujan observed the "quadratic effect" of primes. This quadra-

Arithmetic Progression	Number of Primes
$4n + 1$	80
$4n + 3$	86
$8n + 1$	37
$8n + 3$	43
$8n + 5$	43
$8n + 7$	43
$6n + 1$	80
$6n + 5$	86
$12n + 1$	36
$12n + 5$	44
$12n + 7$	44
$12n + 11$	42

tic effect is reflected by the term $-\frac{1}{2}\,\mathrm{Li}(x^{1/2})$ in Riemann's formula (11.2), which arises from the Möbius inversion of (Hardy [7, p. 40])

$$\prod(x) := \pi(x) + \tfrac{1}{2}\pi(x^{1/2}) + \tfrac{1}{3}\pi(x^{1/3}) + \cdots,$$

which is related to $\psi(x) := \sum_{p^m \le x} \log p$ as $\theta(x) := \sum_{p \le x} \log p$ is related to $\pi(x)$. Accordingly, because all prime squares are of the form $4n + 1$, one might expect that

$$\Delta(x) := \pi_{4,3}(x) - \pi_{4,1}(x) \approx \tfrac{1}{2}\pi(x^{1/2}).$$

The quadratic effect was first enunciated by P. L. Chebyshev [1] in a letter written in 1853. In recent times, this effect has been thoroughly discussed by Shanks [1], Hudson [2], and Bays and Hudson [1]. If we consider integers that are products of two odd primes, there are "more" congruent to 1 (mod 4) than to 3 (mod 4). In the aforementioned papers, Shanks, Hudson, and Bays demonstrate that a large disparity between the cardinalities of these two sets reflects a disparity between $\pi_{4,3}(x)$ and $\pi_{4,1}(x)$.

In his second letter to Hardy [7, p. 23], Ramanujan [23, p. 352], asserted that $\Delta(x)$ tends to ∞ as x tends to ∞. This is false. In fact, Hardy and Littlewood [4] (Hardy [5, pp. 20–97]) showed that $\Delta(x)$ changes sign infinitely often. The least value of x for which $\Delta(x) < 0$ is $x = 26{,}861$, found by Leech [1] in 1957. Shanks [1] calculated $\Delta(x)$ up to $x = 3{,}000{,}000$ and found that $\Delta(x) > 0$, $= 0$, and < 0, respectively, 99.84%, 0.05%, and 0.11% of the time. Further numerical calculations have been performed by Bays and Hudson [1], [2], [3]. Over the years, several conjectures have been made about the positivity of $\Delta(x)$, but most have been shown to be false.

Ramanujan (p. 308) also points to the quadratic effects in arithmetic progressions with moduli 6, 8, 10, 12, and 24. Thus, for example, there are "more" primes of the form $6n + 5$ than of the form $6n + 1$, while there are fewer primes of the form $8n + 1$ than of the forms $8n + 3$, $8n + 5$, and $8n + 7$, which Ramanujan says are "equal" in number. These assertions were also made in Ramanujan's [23, pp. 351–352] second letter to Hardy.

This chapter should be read in conjunction with Hardy's penetrating analysis [7, Chap. II]. We have tried to describe everything about primes recorded by Ramanujan in his notebooks [22] with closer attention paid to those aspects not covered by Hardy. Those wishing to learn more about the wonderful world of prime numbers should read the delightful article by Zagier [1] and the engaging book by Ribenboim [1]. Ramanujan would have loved both.

The author [3] previously published an abridged version of this chapter.

CHAPTER 25

Theta-Functions and Modular Equations

Chapters 16–21 in his second notebook [22] contain much of Ramanujan's prodigious outpouring of discoveries about theta-functions and modular equations. However, the unorganized pages in the second and third notebooks also embrace a large amount of Ramanujan's findings on these topics. In this chapter, we shall discuss most of this material. In Chapter 33 (Part V [9]), we relate Ramanujan's fascinating theories of elliptic functions and modular equations with alternative bases. Chapter 26 contains Ramanujan's theorems on inversion formulas for the lemniscate and allied integrals. Some material that normally would be placed in the present chapter is connected with continued fractions and so has been put in Chapter 32 (Part V [9]) on continued fractions.

Before outlining the contents of this chapter, we review some notation and the primary definitions of theta-functions originally given in Chapter 16 (Part III [6]). First set

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k), \qquad |q| < 1.$$

For each integer n, let

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}, \qquad |q| < 1.$$

It is tacitly assumed in the sequel that $|q| < 1$ always. Ramanujan's general theta-function $f(a, b)$ is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1. \tag{0.1}$$

Furthermore, define

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_\infty^2 (q^2; q^2)_\infty, \tag{0.2}$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{0.3}$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty, \tag{0.4}$$

and

$$\chi(q) := (-q; q^2)_\infty. \tag{0.5}$$

The functions $\varphi(q)$, $\psi(q)$, and $f(-q)$ are at the heart of Ramanujan's theory; the function $\chi(q)$ is chiefly used for notational purposes. Note that $f(-q) = q^{-1/24}\eta(z)$, where $q = e^{2\pi iz}$ and η denotes the Dedekind eta-function. We emphasize that theorems from Chapters 16–21 of Ramanujan's second notebook are frequently used in our proofs here, and so our book [6] is a necessary concomitant in reading this chapter.

A brief description of the results found in this chapter will now be given. We have organized and divided Ramanujan's results into three categories. The first part contains 17 entries of "general" results. These theorems generally do not specifically involve the functions $\varphi(q)$, $\psi(q)$, and $f(-q)$. The second section comprises 33 theorems, which include results on the three aforementioned functions and on modular equations. The last part contains 23 results on "P–Q eta-function identities," or "P–Q modular equations." These are identities involving quotients of eta-functions, which are designated by P or Q by Ramanujan.

We now describe a few of the many highlights in this chapter.

The classical literature on theta-functions contains many well-known Lambert series representations for products of theta-functions. Entries 23–26 offer four new such representations. However, in contrast to other results in the literature, two Lambert series appear in each identity. Moreover, they are considerably deeper than the classical theorems. G. E. Andrews [5] has found difficult, fascinating proofs of all four results based on Bailey pairs, an idea evidently not known to Ramanujan.

Chapters 19 and 20 in Ramanujan's second notebook [22] contain several new types of theta-function identities. In a remarkable paper [1], R. J. Evans employed modular forms in an elegant manner to establish some general theorems containing several of Ramanujan's identities in Chapters 19 and 20 as special cases. Another pair of beautiful identities of this genre is found in Entry 31 below. Entry 32 contains two wonderful identities of a form that we have not previously encountered in the literature. The author and L.-C. Zhang [2], employing some of Evans' ideas, have proved a general theorem encompassing these two identities as special cases.

Entry 38 comprises three interesting results that might have been placed in the first group of results. We have set these difficult identities in the second group, because they generalize theorems connected with modular equations of degree 5 found in Chapter 19.

Of the 23 P–Q modular equations, we have been able to prove 18 by employing classical methods. In particular, many theorems from Chapters 16–21 have been utilized in our proofs. Five P–Q identities, however, have remained impervious to classical attacks, and so we have had to resort to the theory of modular forms. Proofs revolving around Ramanujan's ideas would be preferable.

We note one incorrect result, Entry 27, which also has a peculiar form. The first five nonzero terms of the power series expansions of each side agree, and so it seems that a corrected version may exist, but we have been unable to find it.

Some of the calculations in this chapter are rather formidable if performed by hand. We have therefore relied upon the symbolic computational system *Mathematica* designed by S. Wolfram [1].

Entry 1 (p. 309). *Let $f(a, b)$ be defined by* (0.1). *Let m, n, p, r, and k be positive numbers such that $m + n = p + r = k$. Then as q tends to $1-$,*

$$\frac{f(-q^m, -q^n)}{f(-q^p, -q^r)} \sim \frac{\sin(\pi m/k)}{\sin(\pi p/k)}.$$

PROOF. From (0.1),

$$\begin{aligned} f(-q^m, -q^n) &= \sum_{j=-\infty}^{\infty} (-1)^j q^{(m+n)j^2/2+(m-n)j/2} \\ &= \sum_{j=-\infty}^{\infty} (-1)^j q^{aj^2+bj}, \end{aligned} \tag{1.1}$$

where $a = (m + n)/2$ and $b = (m - n)/2$. We shall set

$$f(-q^m, -q^n) = \theta_3(z, \tau) := \sum_{j=-\infty}^{\infty} \hat{q}^{j^2} e^{2jiz}, \qquad \hat{q} = e^{\pi i \tau}, \qquad \operatorname{Im} \tau > 0, \tag{1.2}$$

where comparing (1.1) and (1.2), we see that $\hat{q} = q^a$ and

$$\tau = -\frac{ia \log q}{\pi}. \tag{1.3}$$

Furthermore, if $z = x + iy$, we observe that

$$x = \pi/2 \quad \text{and} \quad y = -\tfrac{1}{2} b \log q. \tag{1.4}$$

We apply the transformation formula (Whittaker and Watson [1, p. 475])

$$\theta_3(z, \tau) = (-i\tau)^{-1/2} \exp\left(\frac{z^2}{\pi i \tau}\right)\theta_3(z/\tau, -1/\tau). \tag{1.5}$$

From (1.3),

$$(-i\tau)^{-1/2} = \left(\frac{-a \log q}{\pi}\right)^{-1/2}. \tag{1.6}$$

From (1.3) and (1.4),

$$\begin{aligned}\exp\left(\frac{z^2}{\pi i \tau}\right) &= \exp\left(\frac{\pi^2}{4a \log q} - \frac{b^2 \log q}{4a} - \frac{\pi i b}{2a}\right)\\ &\sim \exp\left(\frac{\pi^2}{4a \log q} - \frac{\pi i b}{2a}\right),\end{aligned} \tag{1.7}$$

as q approaches $1-$. Lastly,

$$\theta_3(z/\tau, -1/\tau) = \sum_{j=-\infty}^{\infty} \exp\left(\frac{\pi^2 j^2}{a \log q} - \frac{\pi^2 j}{a \log q} + \frac{\pi i b j}{a}\right). \tag{1.8}$$

Hence, from (1.6)–(1.8), the modulus of each term on the right side of (1.5) is asymptotic to

$$\left(\frac{-a \log q}{\pi}\right)^{-1/2} \exp\left(\frac{\pi^2}{a \log q}\{\tfrac{1}{4} + j^2 - j\}\right), \tag{1.9}$$

which is maximal when $j = 0, 1$. Thus, from (1.2), (1.5), and (1.9), as q tends to $1-$,

$$f(-q^m, -q^n) \sim 2\left(\frac{-a \log q}{\pi}\right)^{-1/2} \exp\left(\frac{\pi^2}{4a \log q}\right)\cos\left(\frac{\pi b}{2a}\right). \tag{1.10}$$

Since $a = (m + n)/2$, $b = (m - n)/2$, and $k = m + n$, an easy calculation shows that

$$\cos\left(\frac{\pi b}{2a}\right) = \sin\left(\frac{\pi m}{k}\right).$$

Thus, from (1.10), as q tends to $1-$,

$$f(-q^m, -q^n) \sim 2\left(\frac{-a \log q}{\pi}\right)^{-1/2} \exp\left(\frac{\pi^2}{4a \log q}\right)\sin\left(\frac{\pi m}{k}\right). \tag{1.11}$$

Finally, dividing (1.11) by the analogous asymptotic formula for $f(-q^p, -q^r)$, we complete the proof of Entry 1.

Entry 2 (p. 308). *Let m and n be positive. Then as q tends to $1-$,*

$$\frac{f(-q^m, -q^n)}{\varphi(-q^{(m+n)/2})} \sim \sin\left(\frac{\pi m}{m+n}\right).$$

PROOF. Apply Entry 1 with $p = r = (m + n)/2$. Thus,

$$\sin\left(\frac{\pi p}{k}\right) = 1.$$

Hence, Entry 2 follows immediately from Entry 1 and the definition (0.2) of φ.

Entry 3 (p. 321). *Suppose that $|ab| < 1$, and let $\left(\frac{n}{3}\right)$ denote the Legendre symbol. Then*

$$\begin{aligned} &f^3(a^2b, ab^2) + af^3(b, a^3b^2) + bf^3(a, a^2b^3) \\ &\qquad = f(a, b)\left(1 + 6 \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{a^n b^n}{1 - a^n b^n}\right). \end{aligned} \tag{3.1}$$

PROOF. Apply Entry 31 of Chapter 16 (Part III [6, p. 48]) with $n = 3$ and a and b replaced by ωa and $\omega^2 b$, respectively, where ω is a cube root of unity.

Then, in the notation of Entry 31, for $n \geq 0$,

$$U_n = \omega^{n(3n-1)/2} a^{n(n+1)/2} b^{n(n-1)/2}$$

and

$$V_n = \omega^{n(3n+1)/2} a^{n(n-1)/2} b^{n(n+1)/2}.$$

Thus,

$$f(\omega a, \omega^2 b) = f(a^6b^3, a^3b^6) + \omega a f(a^9b^6, b^3) + \omega^2 b f(a^3, a^6b^9). \tag{3.2}$$

Multiplying together the three equalities given in (3.2), we find that

$$\begin{aligned} f(a, b)f(\omega a, \omega^2 b)f(\omega^2 a, \omega b) &= f^3(a^6b^3, a^3b^6) + a^3 f(a^9b^6, b^3) + b^3 f(a^3, a^6b^9) \\ &\quad - 3abf(a^6b^3, a^3b^6)f(a^9b^6, b^3)f(a^3, a^6b^9). \end{aligned} \tag{3.3}$$

Applying the Jacobi triple product identity (Part III [6, p. 35, Entry 19]), we deduce that

$$\begin{aligned} &f(a, b)f(\omega a, \omega^2 b)f(\omega^2 a, \omega b) \\ &\quad = (-a; ab)_\infty(-b; ab)_\infty(-\omega a; ab)_\infty(-\omega^2 b; ab)_\infty(-\omega^2 a; ab)_\infty \\ &\qquad \times (-\omega b; ab)_\infty(ab; ab)_\infty^3 \\ &\quad = (-a^3; a^3b^3)_\infty(-b^3; a^3b^3)_\infty(ab; ab)_\infty^3 \\ &\quad = \frac{f(a^3, b^3)f^3(-ab)}{f(-a^3b^3)}, \end{aligned} \tag{3.4}$$

by (0.4). Invoking again the Jacobi triple product identity, we arrive at

$$\begin{aligned}
&f(a^6b^3, a^3b^6)f(a^9b^6, b^3)f(a^3, a^6b^9)\\
&\quad = (-a^6b^3; a^9b^9)_\infty(-a^3b^6; a^9b^9)_\infty(-a^9b^6; a^9b^9)_\infty(-b^3; a^9b^9)_\infty\\
&\qquad \times(-a^3; a^9b^9)_\infty(-a^6b^9; a^9b^9)_\infty(a^9b^9; a^9b^9)_\infty^3\\
&\quad = \frac{f(a^3, b^3)f^3(-a^9b^9)}{f(-a^3b^3)},
\end{aligned} \tag{3.5}$$

by (0.4). Putting (3.4) and (3.5) in (3.3), we deduce the identity

$$\begin{aligned}
&f^3(a^6b^3, a^3b^6) + a^3f(a^9b^6, b^3) + b^3f(a^3, a^6b^9)\\
&\quad = \frac{f(a^3, b^3)f^3(-ab)}{f(-a^3b^3)} + 3ab\,\frac{f(a^3, b^3)f^3(-a^9b^9)}{f(-a^3b^3)}\\
&\quad = \frac{f(a^3, b^3)}{f(-a^3b^3)}(f^3(-ab) + 3abf^3(-a^9b^9))\\
&\quad = f(a^3, b^3)\left(1 + 6\sum_{n=1}^\infty \left(\frac{n}{3}\right)\frac{(ab)^{3n}}{1-(ab)^{3n}}\right)
\end{aligned} \tag{3.6}$$

by Entry 1(v) in Chapter 20 (Part III [6, p. 346]). Replacing a^3 and b^3 in (3.6) by a and b, respectively, we complete the proof.

A proof of Entry 3 was given by the author in [8].

In order to prove Entries 4–6, which are corollaries of Entry 3, we rewrite (3.1) in the form

$$\begin{aligned}
&\frac{f^3(a^2b, ab^2) + af^3(b, a^3b^2) + bf^3(a, a^2b^3)}{f(a,b)} - 1\\
&\quad = 6\sum_{n=1}^\infty \left(\frac{n}{3}\right)\sum_{m=1}^\infty (ab)^{mn}\\
&\quad = 6\sum_{m=1}^\infty \sum_{n=0}^\infty \{(ab)^{m(3n+1)} - (ab)^{m(3n+2)}\}\\
&\quad = 6\sum_{m=1}^\infty \left(\frac{(ab)^m}{1-(ab)^{3m}} - \frac{(ab)^{2m}}{1-(ab)^{3m}}\right)\\
&\quad = 6\sum_{m=1}^\infty \frac{(ab)^m}{1+(ab)^m+(ab)^{2m}}.
\end{aligned} \tag{3.7}$$

Entry 4 (p. 321). *We have*

$$\begin{aligned}
&f^3(-q^6, -q^9) - qf^3(-q^4, -q^{11}) - q^4f(-q, -q^{14})\\
&\quad = f(-q, -q^4)\left(1 + 6\sum_{n=1}^\infty \frac{q^{5n}}{1+q^{5n}+q^{10n}}\right).
\end{aligned}$$

PROOF. Let $a = -q$ and $b = -q^4$ in (3.7).

Entry 5 (p. 321). *We have*

$$f^3(-q^7, -q^8) - q^2 f^3(-q^3, -q^{12}) - q^3 f^3(-q^2, -q^{13})$$
$$= f(-q^2, -q^3)\left(1 + 6\sum_{n=1}^{\infty} \frac{q^{5n}}{1 + q^{5n} + q^{10n}}\right).$$

PROOF. Set $a = -q^2$ and $b = -q^3$ in (3.7).

Entry 6 (p. 321). *We have*

$$f^3(-q^4, -q^5) - q f^3(-q^2, -q^7) - q^2 f^3(-q, -q^8)$$
$$= f(-q, -q^2)\left(1 + 6\sum_{n=1}^{\infty} \frac{q^{3n}}{1 + q^{3n} + q^{6n}}\right).$$

PROOF. Put $a = -q$ and $b = -q^2$ in (3.7).

Entry 7 (p. 328). *Let ω be a primitive cube root of unity. Then, for $|ab| < 1$,*

$$f(\omega a, \omega b) = \omega f(a, b) + (1 - \omega) f(a^6 b^3, a^3 b^6). \tag{7.1}$$

PROOF. From the definition (0.1),

$$\begin{aligned}
f(\omega a, \omega b) &= \sum_{n=-\infty}^{\infty} \omega^{n^2} a^{n(n+1)/2} b^{n(n-1)/2} \\
&= \omega \sum_{\substack{n=-\infty \\ n \equiv 1,2 \pmod 3}}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} + \sum_{\substack{n=-\infty \\ n \equiv 0 \pmod 3}}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \\
&= \omega f(a, b) + (1 - \omega) \sum_{n=-\infty}^{\infty} a^{3n(3n+1)/2} b^{3n(3n-1)/2} \\
&= \omega f(a, b) + (1 - \omega) \sum_{n=-\infty}^{\infty} (a^6 b^3)^{n(n+1)/2} (a^3 b^6)^{n(n-1)/2} \\
&= \omega f(a, b) + (1 - \omega) f(a^6 b^3, a^3 b^6),
\end{aligned}$$

which completes the proof.

Entry 8 (p. 328). *If $|ab| < 1$,*

$$\{f(a, b) - f(a^6 b^3, a^3 b^6)\}^3 = \frac{f(a^3, b^3)}{f(a^6 b^3, a^3 b^6)} f^3(a^2 b, a b^2) - f^3(a^6 b^3, a^3 b^6).$$

PROOF. First observe that, trivially, (7.1) also holds for $\omega = 1$. Write (7.1) in the form

$$\omega\{f(a, b) - f(a^6b^3, a^3b^6)\} = f(\omega a, \omega b) - f(a^6b^3, a^3b^6)$$

and multiply all three such equalities together to obtain

$$\begin{aligned}\{f(a, b) &- f(a^6b^3, a^3b^6)\}^3\\ &= \{f(a, b) - f(a^6b^3, a^3b^6)\}\{f(\omega a, \omega b) - f(a^6b^3, a^3b^6)\}\\ &\quad \times \{f(\omega^2 a, \omega^2 b) - f(a^6b^3, a^3b^6)\}\\ &= f(a, b)f(\omega a, \omega b)f(\omega^2 a, \omega^2 b) - f^3(a^6b^3, a^3b^6)\\ &\quad - \{f(a, b)f(\omega a, \omega b) + f(a, b)f(\omega^2 a, \omega^2 b)\\ &\quad + f(\omega a, \omega b)f(\omega^2 a, \omega^2 b)\}f(a^6b^3, a^3b^6)\\ &\quad + \{f(a, b) + f(\omega a, \omega b) + f(\omega^2 a, \omega^2 b)\}f^2(a^6b^3, a^3b^6). \qquad (8.1)\end{aligned}$$

Adding the three equalities represented in (7.1), we find that

$$f(a, b) + f(\omega \alpha, \omega b) + f(\omega^2 a, \omega^2 b) = 3f(a^6b^3, a^3b^6). \qquad (8.2)$$

Now let ω_1 and ω_2 be any two cube roots of unity. From (7.1),

$$\begin{aligned}f(\omega_1 a, \omega_1 b)f(\omega_2 a, \omega_2 b) &= \omega_1\omega_2 f^2(a, b) + (1 - \omega_1)(1 - \omega_2)f^2(a^6b^3, a^3b^6)\\ &\quad + (\omega_1 + \omega_2 - 2\omega_1\omega_2)f(a, b)f(a^6b^3, a^3b^6). \qquad (8.3)\end{aligned}$$

Let ω_1 and ω_2 be distinct and add together all three possible equalities given in (8.3). This gives us

$$\begin{aligned}f(a, b)f(\omega a, \omega b) + f(a, b)f(\omega^2 a, \omega^2 b) + f(\omega a, \omega b)f(\omega^2 a, \omega^2 b)&\\ = 3f^2(a^6b^3, a^3b^6). \qquad (8.4)\end{aligned}$$

Substituting (8.2) and (8.4) in (8.1), we deduce that

$$\{f(a,b) - f(a^6b^3, a^3b^6)\}^3 = f(a,b)f(\omega a, \omega b)f(\omega^2 a, \omega^2 b) - f^3(a^6b^3, a^3b^6). \qquad (8.5)$$

A comparison of (8.5) with Entry 8 indicates that it remains to show that

$$f(a, b)f(\omega a, \omega b)f(\omega^2 a, \omega^2 b) = \frac{f(a^3, b^3)f^3(a^2b, ab^2)}{f(a^6b^3, a^3b^6)}. \qquad (8.6)$$

From the Jacobi triple product identity,

$$\begin{aligned}f(a, b)&f(\omega a, \omega b)f(\omega^2 a, \omega^2 b)\\ &= \{(-a; ab)_\infty(-\omega a; \omega^2 ab)_\infty(-\omega^2 a; \omega ab)_\infty\}\\ &\quad \times \{(-b; ab)_\infty(-\omega b; \omega^2 ab)_\infty(-\omega^2 b; \omega ab)_\infty\}\\ &\quad \times \{(ab; ab)_\infty(\omega^2 ab; \omega^2 ab)_\infty(\omega ab; \omega ab)_\infty\}\end{aligned}$$

$$= \frac{(-ab^2;a^3b^3)_\infty^3(-b^3;a^3b^3)_\infty}{(-a^3b^6;a^9b^9)_\infty} \frac{(-a^2b;a^3b^3)_\infty^3(-a^3;a^3b^3)_\infty}{(-a^6b^3;a^9b^9)_\infty}$$
$$\times \frac{(a^3b^3;a^3b^3)_\infty^3(a^3b^3;a^3b^3)_\infty}{(a^9b^9;a^9b^9)_\infty}$$
$$= \frac{f^3(ab^2,a^2b)f(a^3,b^3)}{f(a^3b^6,a^6b^3)}.$$

Thus, (8.6) is established, and the proof is complete.

By applying the elementary result, Entry 11 of Chapter 22, we may deduce from Entry 8 that

$$\{3f(a^6b^3, a^3b^6) - f(a, b)\}^3 = 9\frac{f(a^3,b^3)}{f(a,b)} f^3(a^2b, ab^2) - f^3(a, b).$$

Entry 9 (p. 328). *If* $|ab| < 1$,

$$f(ai, bi) = \tfrac{1}{2}(1 + i)f(a, b) + \tfrac{1}{2}(1 - i)f(-a, -b). \tag{9.1}$$

Proof. From (0.1), we first observe that

$$f(ai, bi) = \sum_{n=-\infty}^{\infty} i^{n^2} a^{n(n+1)/2} b^{n(n-1)/2}$$
$$= \sum_{n=-\infty}^{\infty} a^{n(2n+1)} b^{n(2n-1)} + i \sum_{n=-\infty}^{\infty} a^{(2n+1)(n+1)} b^{(2n+1)n}. \tag{9.2}$$

Next, by Entries 30(ii), (iii), respectively, in Chapter 16 (Part III [6, p. 46]),

$$\tfrac{1}{2}\{f(a,b) + f(-a,-b)\} = f(a^3b, ab^3)$$
$$= \sum_{n=-\infty}^{\infty} a^{2n^2+n} b^{2n^2-n} \tag{9.3}$$

and

$$\tfrac{1}{2}\{f(a,b) - f(-a,-b)\} = af(b/a, a^5b^3)$$
$$= a \sum_{n=-\infty}^{\infty} a^{2n^2-3n} b^{2n^2-n}$$
$$= \sum_{n=-\infty}^{\infty} a^{2n^2+3n+1} b^{2n^2+n}. \tag{9.4}$$

Multiplying (9.4) by i and adding the resulting equality to (9.3), we deduce (9.1) by using (9.2).

Entry 10 (p. 328). *If* $|ab| < 1$,

$$\{f(a,b) - f(a^3b, ab^3)\}^2 = f(a^2, b^2)\varphi(ab) - f^2(a^3b, ab^3). \tag{10.1}$$

Proof. By Entry 30(ii) in Chapter 16 (Part III [6, p. 46]),

$$f(a^3b, ab^3) = \tfrac{1}{2}\{f(a,b) + f(-a,-b)\}. \tag{10.2}$$

Subtracting (10.2) from (9.1), rearranging, and adding $f(a,b)$ to both sides, we find that

$$\begin{aligned} f(a,b) - f(a^3b, ab^3) &= f(a,b) + \tfrac{1}{2}i\{f(a,b) - f(-a,-b)\} - f(ai, bi) \\ &= f(a,b) + iaf(b/a, a^5b^3) - f(ai, bi), \end{aligned} \tag{10.3}$$

where we used the first equality in (9.4). By conjugation,

$$f(a,b) - f(a^3b, ab^3) = f(a,b) - iaf(b/a, a^5b^3) - f(-ai, -bi). \tag{10.4}$$

Multiplying (10.3) by (10.4), we find that

$$\begin{aligned} &\{f(a,b) - f(a^3b, ab^3)\}^2 \\ &\quad = f^2(a,b) + a^2f^2(b/a, a^5b^3) + f(ai, bi)f(-ai, -bi) \\ &\qquad - f(a,b)\{f(ai,bi) + f(-ai,-bi)\} \\ &\qquad + iaf(b/a, a^5b^3)\{f(ai,bi) - f(-ai,-bi)\} \\ &\quad = f^2(a,b) + a^2f^2(b/a, a^5b^3) + f(ai,bi)f(-ai,-bi) \\ &\qquad - 2f(a,b)f(a^3b, ab^3) - 2a^2f^2(b/a, a^5b^3) \\ &\quad = f^2(a,b) - a^2f^2(b/a, a^5b^3) + f(a^2, b^2)\varphi(ab) \\ &\qquad - 2f(a,b)f(a^3b, ab^3), \end{aligned} \tag{10.5}$$

where in the penultimate equality we employed Entries 30(ii), (iii), and in the last equality we used Entry 30(iv), all from Chapter 16 (Part III [6, p. 46]).

Comparing (10.1) with (10.5), we see that it remains to show that

$$f^2(a,b) - a^2f^2(b/a, a^5b^3) - 2f(a,b)f(a^3b, ab^3) = -f^2(a^3b, ab^3). \tag{10.6}$$

Using Entries 30(iii), (ii) in Chapter 16 once again, we find that the left side of (10.6) equals

$$\begin{aligned} &f^2(a,b) - \tfrac{1}{4}\{f(a,b) - f(-a,-b)\}^2 - 2f(a,b)f(a^3b, ab^3) \\ &\quad = \tfrac{3}{4}f^2(a,b) + \tfrac{1}{2}f(a,b)f(-a,-b) - \tfrac{1}{4}f^2(-a,-b) \\ &\qquad - 2f(a,b)f(a^3b, ab^3) \\ &\quad = \tfrac{3}{4}f^2(a,b) + \tfrac{1}{2}f(a,b)\{2f(a^3b, ab^3) - f(a,b)\} \\ &\qquad - \tfrac{1}{4}\{2f(a^3b, ab^3) - f(a,b)\}^2 - 2f(a,b)f(a^3b, ab^3) \\ &\quad = -f^2(a^3b, ab^3). \end{aligned}$$

Thus, (10.6) has been established, and the proof of Entry 10 has been completed.

Applying Entry 13 of Chapter 22 to (10.1), we obtain the auxiliary identity

$$\{2f(a^3b, ab^3) - f(a, b)\}^2 = 2f(a^2, b^2)\varphi(ab) - f^2(a, b).$$

Entry 11 (p. 328). *If* $|ab| < 1$ *and* $|cd| < 1$,

$$\begin{aligned} &f(ai, bi)f(ci, di) - f(-ai, -bi)f(-ci, -di) \\ &\qquad = i\{f(a, b)f(c, d) - f(-a, -b)f(-c, -d)\}. \end{aligned} \tag{11.1}$$

Entry 11 is rather curious, for it indicates that i can be "factored out" of the left side of (11.1).

Proof. By Entry 9,

$$\begin{aligned} &f(ai, bi)f(ci, di) - f(-ai, -bi)f(-ci, -di) \\ &\qquad = \tfrac{1}{4}\{(1+i)f(a, b) + (1-i)f(-a, -b)\} \\ &\qquad\quad \times \{(1+i)f(c, d) + (1-i)f(-c, -d)\} \\ &\qquad\quad -\tfrac{1}{4}\{(1+i)f(-a, -b) + (1-i)f(a, b)\} \\ &\qquad\quad \times \{(1+i)f(-c, -d) + (1-i)f(c, d)\}. \end{aligned}$$

Noting that the right side is imaginary and multiplying out the right side, we complete the proof.

Entry 12 (p. 328). *If* $|ab| < 1$ *and* $|cd| < 1$,

$$\begin{aligned} &f(ai, bi)f(ci, di) + f(-ai, -bi)f(-ci, -di) \\ &\qquad = f(a, b)f(-c, -d) + f(-a, -b)f(c, d). \end{aligned}$$

Proof. The proof proceeds in the same fashion as that for Entry 11.

Entry 13 (p. 329). *If* $|ab| < 1$,

$$\psi(a)\psi(b) = \sum_{n=0}^{\infty} (ab)^{n(n+1)/2} f\left(\frac{a^{n+1}}{b^n}, \frac{b^{n+1}}{a^n}\right). \tag{13.1}$$

Proof. As in the proof of Entry 29 in Chapter 16 (Part III [6, p. 45]),

$$f(a, c)f(b, d) + f(-a, -c)f(-b, -d)$$

$$= 2 \sum_{\substack{m,n=-\infty \\ m+n \text{ even}}}^{\infty} a^{m(m+1)/2} b^{n(n+1)/2} c^{m(m-1)/2} d^{n(n-1)/2}$$

$$= 2 \sum_{M,N=-\infty}^{\infty} a^{(M+N)(M+N+1)/2} b^{(M-N)(M-N+1)/2} c^{(M+N)(M+N-1)/2}$$

$$\times d^{(M-N)(M-N-1)/2}, \tag{13.2}$$

where we have set $M + N = m$ and $M - N = n$ so that $m + n$ is even. Now let $c = d = 1$. From Entry 18(ii) in Chapter 16 and (0.3), $f(1, a) = 2f(a, a^3) = 2\psi(a)$. Thus, (13.2) reduces to

$$2\psi(a)\psi(b) = \sum_{M,N=-\infty}^{\infty} a^{(M+N)(M+N+1)/2} b^{(M-N)(M-N+1)/2}$$

$$= \sum_{M=-\infty}^{\infty} (ab)^{M(M+1)/2} \sum_{N=-\infty}^{\infty} (ab)^{N^2/2} (a/b)^{N(2M+1)/2}$$

$$= \sum_{M=-\infty}^{\infty} (ab)^{M(M+1)/2} f\left(\frac{a^{M+1}}{b^M}, \frac{b^{M+1}}{a^M}\right)$$

$$= 2 \sum_{M=0}^{\infty} (ab)^{M(M+1)/2} f\left(\frac{a^{M+1}}{b^M}, \frac{b^{M+1}}{a^M}\right),$$

since the Mth term of the last series is invariant under the replacement of M by $-M - 1$. Thus, (13.1) follows.

Entry 14 (p. 329). *Let* $|ab| < 1$. *Then*

(i) $$f(a, b)f(a^3, b^3) - f(-a, -b)f(-a^3, -b^3) = 2af(b/a, a^2)\psi(a^3b^3)$$

and

(ii) $$f(a, b)f(a^2b, ab^2) - f(-a, -b)f(-a^2b, -ab^2) = 2af(b/a, a^4b^2)\psi(ab).$$

Proof. Setting $\mu = 2$ and $\nu = 1$ in (36.2) of Chapter 16 (Part III [6, p. 68]), we find that

$$\tfrac{1}{2}\{f(Aq^3, q^3/A)f(Bq, q/B) - f(-Aq^3, -q^3/A)f(-Bq, -q/B)\}$$

$$= Aq^3 f\left(AB^3q^{18}, \frac{q^6}{AB^3}\right) f\left(\frac{A}{B} q^{10}, \frac{B}{A} q^{-2}\right)$$

$$+ A^2Bq^{13} f\left(AB^3q^{30}, \frac{1}{AB^3q^6}\right) f\left(\frac{A}{B} q^{14}, \frac{B}{A} q^{-6}\right)$$

$$= Bqf\left(AB^3q^{18}, \frac{q^6}{AB^3}\right)f\left(\frac{A}{B}q^2, \frac{B}{A}q^6\right)$$

$$+\frac{q}{B}f\left(AB^3q^6, \frac{q^{18}}{AB^3}\right)f\left(\frac{A}{B}q^6, \frac{B}{A}q^2\right), \tag{14.1}$$

where we have made three applications of Entry 18(iv) in Chapter 16 (Part III [6, p. 34]) with $n = 1$ in each case.

First, set $q = (ab)^{1/2}$, $A = (a/b)^{3/2}$, and $B = (b/a)^{1/2}$ in (14.1). Then (14.1) yields

$$\begin{aligned}
&\tfrac{1}{2}\{f(a^3, b^3)f(b, a) - f(-a^3, -b^3)f(-b, -a)\}\\
&\quad = bf(a^9b^9, a^3b^3)f(a^3/b, ab^5) + af(a^3b^3, a^9b^9)f(b^3/a, a^5b)\\
&\quad = \psi(a^3b^3)\{bf(a^3/b, ab^5) + af(b^3/a, a^5b)\}\\
&\quad = \psi(a^3b^3)\{\tfrac{1}{2}a[f(b/a, a^2) - f(-b/a, -a^2)]\\
&\qquad + \tfrac{1}{2}a[f(b/a, a^2) + f(-b/a, -a^2)]\}\\
&\quad = a\psi(a^3b^3)f(b/a, a^2),
\end{aligned}$$

where in the penultimate equality we utilized Entries 30(iii), (ii) of Chapter 16 (Part III [6, p. 46]). Thus, (i) has been established.

Secondly, in (14.1) set $q = (ab)^{1/2}$ and $A = B = (a/b)^{1/2}$ to deduce that

$$\begin{aligned}
&\tfrac{1}{2}\{f(a^2b, ab^2)f(a, b) - f(-a^2b, -ab^2)f(-a, -b)\}\\
&\quad = af(a^{11}b^7, ab^5)f(ab, a^3b^3) + bf(a^5b, a^7b^{11})f(a^3b^3, ab)\\
&\quad = \psi(ab)\{af(ab^5, a^{11}b^7) + bf(a^5b, a^7b^{11})\}\\
&\quad = \psi(ab)\left\{\frac{a}{2}[f(b/a, a^4b^2) + f(-b/a, -a^4b^2)]\right.\\
&\qquad \left. + \frac{a}{2}[f(b/a, a^4b^2) - f(-b/a, -a^4b^2)]\right\}\\
&\quad = a\psi(ab)f(b/a, a^4b^2),
\end{aligned}$$

where in the penultimate line we applied Entries 30(ii), (iii) of Chapter 16. Thus, (ii) has been proved.

Entry 15 (p. 329). *For* $|ab| < 1$,

$$\sum_{n=1}^{\infty} n(ab)^{n(n-1)/2}(a^n - b^n) = f(a, b)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{a^n - b^n}{1 - a^nb^n}. \tag{15.1}$$

PROOF. Let $a = q\alpha$ and $b = q/\alpha$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} n(ab)^{n(n-1)/2}(a^n - b^n) &= \sum_{n=1}^{\infty} nq^{n^2}(\alpha^n - \alpha^{-n}) \\
&= \sum_{n=-\infty}^{\infty} nq^{n^2}\alpha^n \\
&= \alpha \frac{d}{d\alpha} \sum_{n=-\infty}^{\infty} q^{n^2}\alpha^n \\
&= f(q\alpha, q/\alpha)\alpha \frac{d}{d\alpha} \log f(q\alpha, q/\alpha) \\
&= f(q\alpha, q/\alpha)\alpha \frac{d}{d\alpha} \log\{(-q\alpha; q^2)_\infty(-q/\alpha; q^2)_\infty(q^2; q^2)_\infty\} \\
&= f(q\alpha, q/\alpha) \sum_{n=0}^{\infty} \left\{\frac{q^{2n+1}\alpha}{1 + q^{2n+1}\alpha} - \frac{q^{2n+1}/\alpha}{1 + q^{2n+1}/\alpha}\right\} \\
&= f(q\alpha, q/\alpha) \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (-1)^{m-1}\{(q^{2n+1}\alpha)^m - (q^{2n+1}/\alpha)^m\} \\
&= f(q\alpha, q/\alpha) \sum_{m=1}^{\infty} (-1)^{m-1} \frac{q^m\alpha^m - q^m\alpha^{-m}}{1 - q^{2m}} \\
&= f(a, b) \sum_{m=1}^{\infty} (-1)^{m-1} \frac{a^m - b^m}{1 - (ab)^m},
\end{aligned}$$

and the proof of (15.1) is accomplished.

We quote Ramanujan in the next entry.

Entry 16 (p. 329).

$$\sum_{n=-\infty}^{\infty} c^{n(n+1)/2}d^{n(n-1)/2}f(ap^{n(n+1)/2}q^{n(n-1)/2}, bp^{n(n-1)/2}q^{n(n+1)/2})$$

is unaltered by interchanging a and c and at the same time b and d. It is better to take p and q to be of the form x^M and x^N, where M and N are of opposite signs.

Using the definition (0.1) of f, we see that the series above is expressible as a double series

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a^{m(m+1)/2}b^{m(m-1)/2}c^{n(n+1)/2}d^{n(n-1)/2}p^{mn(mn+1)/2}q^{mn(mn-1)/2}.$$

It is evident from this representation that the professed symmetry holds.

It is not clear what applications Ramanujan had contemplated in his remarks on x^M and x^N.

Entry 17 (p. 354). *Let* $|ab| < 1$, *and let* n *be any real number such that* $n \neq -(ab)^k$, *for any integer* k. *Then*

$$\frac{f(a/n, bn)f^3(-ab)}{nf(-a, -b)f(nab, 1/n)} = \frac{1}{n+1} + \sum_{k=1}^{\infty} \left(\frac{a^k}{n + (ab)^k} + \frac{b^k}{1 + n(ab)^k} \right). \tag{17.1}$$

Proof. We shall employ Ramanujan's ${}_1\psi_1$ summation, Entry 17 of Chapter 16 (Part III [6, p. 32]). Putting $\alpha = 1/\beta = -n$, $q = (ab)^{1/2}$, and $z = (a/b)^{1/2}/n$, we find that the left side of (17.1) in Chapter 16 becomes

$$\begin{aligned}
1 + \sum_{k=1}^{\infty} &\frac{(-1/n; ab)_k (n\sqrt{ab})^k}{(-ab/n; ab)_k} \left(\frac{1}{n} \sqrt{\frac{a}{b}} \right)^k + \sum_{k=1}^{\infty} \frac{(-n; ab)_k (\sqrt{ab}/n)^k}{(-nab; ab)_k} \left(n \sqrt{\frac{b}{a}} \right)^k \\
&= 1 + \sum_{k=1}^{\infty} \frac{(1 + 1/n)}{1 + (ab)^k/n} a^k + \sum_{k=1}^{\infty} \frac{(1 + n)}{1 + n(ab)^k} b^k \\
&= 1 + (n+1) \sum_{k=1}^{\infty} \left(\frac{a^k}{n + (ab)^k} + \frac{b^k}{1 + n(ab)^k} \right).
\end{aligned} \tag{17.2}$$

On the other hand, the right side of (17.1) in Chapter 16 equals

$$\frac{(-a/n; ab)_\infty (-bn; ab)_\infty (ab; ab)_\infty^2}{(a; ab)_\infty (b; ab)_\infty (-nab; ab)_\infty (-ab/n; ab)_\infty} = \frac{(n+1) f(a/n, bn) f^3(-ab)}{nf(-a, -b) f(1/n, abn)}. \tag{17.3}$$

Combining (17.2) and (17.3), we achieve (17.1).

Entry 17 is a generalization of the corollary of Entry 33(iii) in Chapter 16 (Part III [6, p. 54]).

This concludes the first part of the chapter where general theorems are proved. In the second part, specific theta-function identities and modular equations are established.

Entry 18 (Formula (1), p. 264). *For* $|q| < 1$,

$$\varphi^2(-q) = 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 + q^n}.$$

Entry 19 (Formula (2), p. 264). *If* $|q| < 1$,

$$\psi(q)\varphi(q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \frac{1 + q^{2n+1}}{1 - q^{2n+1}}.$$

Entry 20 (Formula (3), p. 264). *If* $|q| < 1$,

$$\psi^2(q) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \frac{1+q^{2n+1}}{1-q^{2n+1}}.$$

Entry 21 (Formula (4), p. 264). *For* $|q| < 1$,

$$\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = \sum_{n=1}^{\infty} (-1)^{n+1} q^{n(n+1)/2} \frac{1+q^n}{(1-q^n)^2}.$$

Entries 18–21 are, respectively, Entries 8(v)–(viii) in Chapter 17, and proofs have been given (Part III [6, pp. 116–118]).

Entry 22 (Formula (5), p. 264). *If* $|q| < 1$,

$$q\psi(q)\psi(q^4) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)(n+2)/2}}{1-q^{2n+1}}. \tag{22.1}$$

PROOF. By (0.3),

$$\begin{aligned}\psi(q)\psi(q^4) &= \frac{(q^2;q^2)_\infty (q^8;q^8)_\infty}{(q;q^2)_\infty (q^4;q^8)_\infty} \\ &= \frac{(q^2;q^8)_\infty (q^6;q^8)_\infty (q^8;q^8)_\infty^2}{(q;q^8)_\infty (q^3;q^8)_\infty (q^5;q^8)_\infty (q^7;q^8)_\infty}.\end{aligned} \tag{22.2}$$

We apply Ramanujan's ${}_1\psi_1$ summation in the form (17.6) of Chapter 16 (Part III [6, p. 34]). In the notation of (17.6), replace q by q^8 and set $z = q$, $a = q^5$, and $b = q^{13}$. Then

$$\begin{aligned}\frac{(q^6;q^8)_\infty (q^2;q^8)_\infty (q^8;q^8)_\infty^2}{(q;q^8)_\infty (q^7;q^8)_\infty (q^{13};q^8)_\infty (q^3;q^8)_\infty} &= \sum_{n=-\infty}^{\infty} \frac{(q^5;q^8)_n}{(q^{13};q^8)_n} q^n \\ &= \sum_{n=-\infty}^{\infty} \frac{1-q^5}{1-q^{8n+5}} q^n.\end{aligned} \tag{22.3}$$

Thus, by (22.2) and (22.3),

$$\begin{aligned}\psi(q)\psi(q^4) &= \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{8n+5}} \\ &= \sum_{n=0}^{\infty} \frac{q^n}{1-q^{8n+5}} + \sum_{n=1}^{\infty} \frac{q^{-n}}{1-q^{-8n+5}} \\ &= \sum_{n=0}^{\infty} \frac{q^n}{1-q^{8n+5}} - \sum_{n=1}^{\infty} \frac{q^{7n-5}}{1-q^{8n-5}} \\ &= \sum_{n=0}^{\infty} \frac{q^n}{1-q^{8n+5}} - \sum_{n=0}^{\infty} \frac{q^{7n+2}}{1-q^{8n+3}}.\end{aligned} \tag{22.4}$$

If we expand $1/(1-q^{8n+5})$ in a geometric series, the array of powers for the first sum on the right side of (22.4) is

$$\begin{array}{cccccc} 1 & q^5 & q^{10} & q^{15} & q^{20} & \cdots \\ q & q^{14} & q^{27} & q^{40} & q^{53} & \cdots \\ q^2 & q^{23} & q^{44} & q^{65} & q^{86} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}.$$

Summing by the column–row method, we deduce that

$$\sum_{n=0}^{\infty} \frac{q^n}{1-q^{8n+5}} = \sum_{n=0}^{\infty} \frac{q^{n(2n+3)}}{1-q^{4n+1}}. \tag{22.5}$$

If we expand $1/(1-q^{8n+3})$ in a geometric series, the array of powers for the second sum on the right side of (22.4) is

$$\begin{array}{cccccc} q^2 & q^5 & q^8 & q^{11} & q^{14} & \cdots \\ q^9 & q^{20} & q^{31} & q^{42} & q^{53} & \cdots \\ q^{16} & q^{35} & q^{54} & q^{73} & q^{92} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}.$$

Summing by rows and columns, we deduce that

$$\sum_{n=0}^{\infty} \frac{q^{7n+2}}{1-q^{8n+3}} = \sum_{n=0}^{\infty} \frac{q^{(n+2)(2n+1)}}{1-q^{4n+3}}. \tag{22.6}$$

Putting (22.5) and (22.6) in (22.4), we arrive at

$$\begin{aligned} q\psi(q)\psi(q^4) &= q\sum_{n=0}^{\infty} \frac{q^{n(2n+3)}}{1-q^{4n+1}} - q\sum_{n=0}^{\infty} \frac{q^{(n+2)(2n+1)}}{1-q^{4n+3}} \\ &= \sum_{n=0}^{\infty} \frac{q^{(2n+1)(n+1)}}{1-q^{4n+1}} - \sum_{n=0}^{\infty} \frac{q^{(n+1)(2n+3)}}{1-q^{4n+3}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)(n+2)/2}}{1-q^{2n+1}}, \end{aligned}$$

which completes the proof.

Although the next four formulas also appear on page 264 and are numbered 6–9 to indicate evidently that they are a continuation of the same sort of results as those preceeding, formulas (6)–(9) are much deeper and considerably more difficult to prove. Ramanujan's statement of Entry 23 contains a slight misprint.

Entry 23 (Formula (6), p. 264). *For* $|q| < 1$,

$$\frac{1}{\varphi^2(-q)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2 q^{n(n+1)/2}}{1+q^n} = \sum_{n=1}^{\infty} \frac{q^{n(2n-1)}(1+q^{2n-1})}{(1-q^{2n-1})^2}. \tag{23.1}$$

Entry 24 (Formula (7), p. 264). *For* $|q| < 1$,

$$\frac{1}{\psi^2(q)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)^2 q^{n(n-1)}(1+q^{2n-1})}{1-q^{2n-1}} = 1 + 8 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(1+q^n)^2}. \quad (24.1)$$

Entry 25 (Formula (8), p. 264). *For* $|q| < 1$,

$$\frac{1}{\psi^2(q)} \sum_{n=1}^{\infty} \frac{(-1)^n(2n-1)q^{n(n+1)-1}}{(1-q^{2n-1})^2} = \sum_{n=1}^{\infty} \frac{(-1)^n n q^{n^2}(1+q^{2n})}{1-q^{2n}}. \quad (25.1)$$

Entry 26 (Formula (9), p. 264). *For* $|q| < 1$,

$$\frac{1}{\varphi^2(-q)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n q^{n(n+1)/2}(1-q^n)}{(1+q^n)^2} = \sum_{n=1}^{\infty} \frac{n q^{n(n+1)/2}}{1-q^n}. \quad (26.1)$$

Since the work of Jacobi [1], [2] in his *Fundamenta Nova*, many representations for theta-functions and products of theta-functions as Lambert series have been established in the literature. In particular, in addition to Entries 18–22 above, Ramanujan [22] derived many such identities, most of which are proved in our book [6]. Entries 23–26, however, are much different from all other Lambert series identities. Most notably, *two* Lambert series appear in each of the four formulas above. Unfortunately, we have no insight whatsoever into Ramanujan's arguments. The following formidable proofs are due to G. E. Andrews [5]. Although Andrews' proofs crucially employ basic hypergeometric series, an area in which Ramanujan was an expert and made many profound contributions, it is doubtful that these proofs closely resemble those of Ramanujan. In particular, Bailey pairs and Bailey's [1, p. 69, eq. (3)] nonterminating extension of the q-analogue of Whipple's theorem play key roles in Andrews' proofs, and these tools likely were not in Ramanujan's arsenal. Moreover, from their placement, it would seem that Ramanujan's proofs of these theorems arise from the theory of theta-functions, rather than from basic hypergeometric series.

Before commencing Andrews' proofs of Entries 23–26, we record the necessary background material on basic hypergeometric series. Let $|t|$, $|q| < 1$. Suppose that $a_1, a_2, \ldots, a_r$ and $b_1, \ldots, b_s$ are complex numbers such that $b_j \neq q^{-n}$, $1 \leq j \leq s$, where n is a nonnegative integer. The basic hypergeometric series ${}_r\varphi_s$ is defined, for $r = s + 1$, by

$${}_r\varphi_s\begin{pmatrix} a_1, a_2, \ldots, a_r; q, t \\ b_1, b_2, \ldots, b_s \end{pmatrix} = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n t^n}{(q, b_1, b_2, \ldots, b_s; q)_n},$$

where

$$(A_1, A_2, \ldots, A_k; q)_n = \prod_{i=1}^{k} \prod_{j=0}^{n-1} (1 - A_i q^j).$$

Also, set

$$(A_1, A_2, \ldots, A_k; q)_\infty = \prod_{i=1}^{k} \prod_{j=0}^{\infty} (1 - A_i q^j).$$

We always tacitly assume that $|q| < 1$. The variable t will assume many forms depending upon the parameters $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s$, but it is always understood that these parameters are chosen so that $|t| < 1$.

Bailey's [1, p. 69, eq. (3)] nonterminating extension of the q-analogue of Whipple's theorem is given by

$$\begin{aligned}
&{}_8\varphi_7\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, d, e, f, g, h; q, a^2q^2/(defgh) \\ \sqrt{a}, -\sqrt{a}, aq/d, aq/e, aq/f, aq/g, aq/h \end{matrix}\right) \\
&\quad = \frac{(aq, aq/(fg), aq/(fh), aq/(gh); q)_\infty}{(aq/f, aq/g, aq/h, aq/(fgh); q)_\infty}\, {}_4\varphi_3\left(\begin{matrix} aq/(de), f, g, h; q, q \\ aq/d, aq/e, fgh/a \end{matrix}\right) \\
&\quad + \frac{(aq, aq/(de), f, g, h, a^2q^2/(dfgh), a^2q^2/(efgh); q)_\infty}{(aq/d, aq/e, aq/f, aq/g, aq/h, a^2q^2/(defgh), fgh/(aq); q)_\infty} \\
&\quad \times {}_4\varphi_3\left(\begin{matrix} aq/(gh), aq/(fh), aq/(fg), a^2q^2/(defgh); q, q \\ aq^2/(fgh), a^2q^2/(dfgh), a^2q^2/(efgh) \end{matrix}\right).
\end{aligned} \tag{23.2}$$

If we replace h by q^{-N} in (23.2), where N is a nonnegative integer, the latter expression on the right side of (23.2) vanishes because of the appearance of the argument h in the infinite product multiplying ${}_4\varphi_3$. We thus obtain Watson's q-analogue of Whipple's theorem (Bailey [1, p. 69, eq. (2)]),

$$\begin{aligned}
&{}_8\varphi_7\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, d, e, f, g, q^{-N}; q, a^2q^{2+N}/(defg) \\ \sqrt{a}, -\sqrt{a}, aq/d, aq/e, aq/f, aq/g, aq^{N+1} \end{matrix}\right) \\
&\qquad = \frac{(aq, aq/(fg); q)_N}{(aq/f, aq/g; q)_N}\, {}_4\varphi_3\left(\begin{matrix} aq/(de), f, g, q^{-N}; q, q \\ aq/d, aq/e, fgq^{-N}/a \end{matrix}\right).
\end{aligned} \tag{23.3}$$

Next, we state the limiting form of Jackson's theorem, i.e.,

$$\begin{aligned}
&{}_6\varphi_5\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, d, e, f; q, aq/(def) \\ \sqrt{a}, -\sqrt{a}, aq/d, aq/e, aq/f \end{matrix}\right) \\
&\qquad = \frac{(aq, aq/(de), aq/(df), aq/(ef); q)_\infty}{(aq/d, aq/e, aq/f, aq/(def); q)_\infty},
\end{aligned} \tag{23.4}$$

which can be deduced from (23.3) (Slater [1, p. 96, eq. (3.3.1.3)]).

In our proof of (23.1), we shall need a three-term connection relation for ${}_3\varphi_2$'s, namely (Sears [1, p. 175, eq. (10.2)])

$$
{}_3\varphi_2\left(\begin{matrix} a, b, c; q, ef/(abc) \\ e, f \end{matrix}\right) = \frac{(e/a, e/b; q)_\infty}{(e, e/(ab); q)_\infty}\, {}_3\varphi_2\left(\begin{matrix} a, b, f/c; q, q \\ qab/e, f \end{matrix}\right)
$$
$$
+ \frac{(a, b, f/c, ef/(ab); q)_\infty}{(e, ab/e, f, ef/(abc); q)_\infty}\, {}_3\varphi_2\left(\begin{matrix} e/a, e/b, ef/(abc); q, q \\ eq/(ab), ef/(ab) \end{matrix}\right). \tag{23.5}
$$

The next identity follows from combining (23.2) and (23.4) and is essential to the proof of Entry 25.

Lemma 23.1. *We have*

$$
- \sum_{n=1}^{\infty} \frac{(1 - aq^{2n})(f, g; q)_n}{(1 - aq^n)(1 - q^n)(aq/f, aq/g; q)_n} \left(\frac{aq}{fg}\right)^n
$$
$$
+ \sum_{n=1}^{\infty} \frac{(1 - aq^{2n})(d, e, f, g; q)_n}{(1 - aq^n)(1 - q^n)(aq/d, aq/e, aq/f, aq/g; q)_n} \left(\frac{a^2q^2}{defg}\right)^n
$$
$$
= \sum_{n=1}^{\infty} \frac{(aq/(de), f, g; q)_n q^n}{(1 - q^n)(aq/d, aq/e, fg/a; q)_n}
$$
$$
+ \frac{(aq, aq/(de), f, g, q, a^2q^2/(dfg), a^2q^2/(efg); q)_\infty}{(aq/d, aq/e, aq/f, aq/g, aq, a^2q^2/(defg), fg/(aq); q)_\infty}
$$
$$
\times\, {}_4\varphi_3\left(\begin{matrix} aq/g, aq/f, aq/(fg), a^2q^2/(defg); q, q \\ aq^2/(fg), a^2q^2/(dfg), a^2q^2/(efg) \end{matrix}\right). \tag{23.6}
$$

Proof. Set ${}_r\varphi_s^*(-) := {}_r\varphi_s(-) - 1$. Subtracting

$$
\frac{(aq, aq/(fg), aq/(fh), aq/(gh); q)_\infty}{(aq/f, aq/g, aq/h, aq/(fgh); q)_\infty}
$$

from both sides of (23.2), we find that

$$
1 - \frac{(aq, aq/(fg), aq/(fh), aq/(gh); q)_\infty}{(aq/f, aq/g, aq/h, aq/(fgh); q)_\infty}
$$
$$
+ {}_8\varphi_7^*\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, d, e, f, g, h; q, a^2q^2/(defgh) \\ \sqrt{a}, -\sqrt{a}, aq/d, aq/e, aq/f, aq/g, aq/h \end{matrix}\right)
$$
$$
= \frac{(aq, aq/(fg), aq/(fh), aq/(gh); q)_\infty}{(aq/f, aq/g, aq/h, aq/(fgh); q)_\infty}\, {}_4\varphi_3^*\left(\begin{matrix} aq/(de), f, g, h; q, q \\ aq/d, aq/e, fgh/a \end{matrix}\right)
$$
$$
+ \frac{(aq, aq/(de), f, g, h, a^2q^2/(dfgh), a^2q^2/(efgh); q)_\infty}{(aq/d, aq/e, aq/f, aq/g, aq/h, a^2q^2/(defgh), fgh/(aq); q)_\infty}
$$
$$
\times\, {}_4\varphi_3\left(\begin{matrix} aq/(gh), aq/(fh), aq/(fg), a^2q^2/(defgh); q, q \\ aq^2/(fgh), a^2q^2/(dfgh), a^2q^2/(efgh) \end{matrix}\right). \tag{23.7}
$$

Next, rewrite (23.4) in the form

$$1-\frac{(aq, aq/(fg), aq/(fh), aq/(gh); q)_\infty}{(aq/f, aq/g, aq/h, aq/(fgh); q)_\infty}$$
$$= -{}_6\varphi_5^*\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, f, g, h; q, aq/(fgh) \\ \sqrt{a}, -\sqrt{a}, aq/f, aq/g, aq/h \end{matrix}\right)$$

and substitute this in the left side of (23.7). Each term in the resulting identity has a factor of $(1-h)$. Divide both sides by $(1-h)$ and then set $h=1$. We then obtain precisely (23.6) to complete the proof.

Lemma 23.2. *We have*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(1-aq^{2n})(f;q)_n a^n q^{n(n+1)/2}}{(1-aq^n)(1-q^n)(aq/f;q)_n}$$
$$+\sum_{n=1}^{\infty}\frac{(-1)^n(1-aq^{2n})(d,e,f;q)_n a^{2n} q^{n(n+3)/2}}{(1-aq^n)(1-q^n)(aq/d, aq/e, aq/f;q)_n(def)^n}$$
$$=\sum_{n=1}^{\infty}\frac{(aq/(de), f;q)_n(aq/f)^n}{(1-q^n)(aq/d, aq/e;q)_n}. \tag{23.8}$$

Proof. In Lemma 23.1, replace g by q^{-N}, where N is a positive integer. The latter expression on the right side of (23.6) vanishes because of the argument g in the numerator of the product. Letting N tend to ∞, we readily deduce (23.8).

Let $L_{j-2}(q)$ denote the left side of $(2j.1)$, $3 \le j \le 6$. Our objective now is to employ Bailey pairs to obtain new representations for $L_j(q)$, $1 \le j \le 4$, that do not involve theta-functions. We first state a weak form of Bailey's lemma (Andrews [4, pp. 25–26, eq. (3.27)]).

Lemma 23.3. *Let α_r, $0 \le r < \infty$, be a sequence of complex numbers. For each n, $n \ge 0$, define*

$$\beta_n=\sum_{r=0}^{n}\frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}. \tag{23.9}$$

Then, for $\rho_1\rho_2 \ne 0$,

$$\sum_{n=0}^{\infty}(\rho_1,\rho_2;q)_n\left(\frac{aq}{\rho_1\rho_2}\right)^n\beta_n$$
$$=\frac{(aq/\rho_1, aq/\rho_2;q)_\infty}{(aq, aq/(\rho_1\rho_2);q)_\infty}\sum_{n=0}^{\infty}\frac{(\rho_1,\rho_2;q)_n}{(aq/\rho_1, aq/\rho_2;q)_n}\left(\frac{aq}{\rho_1\rho_2}\right)^n\alpha_n, \tag{23.10}$$

provided that the series and products converge.

The sequences $\{a_n\}$, $\{b_n\}$ are said to form a Bailey pair if they satisfy (23.9).

Lemma 23.4. *If* $a = 1$ *in* (23.9), *then*

$$\alpha_n = \begin{cases} (-1)^n(z^n q^{n(n-1)/2} + z^{-n} q^{n(n+1)/2}), & n > 0, \\ 1, & n = 0, \end{cases} \tag{23.11}$$

and

$$\beta_n = \frac{(z, q/z; q)_n}{(q; q)_{2n}}, \qquad n \geq 0, \tag{23.12}$$

form a Bailey pair.

Proof. We must show that (23.11) and (23.12) satisfy (23.9) when $a = 1$. To that end,

$$\begin{aligned} \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(q;q)_{n+r}} &= \frac{1}{(q;q)_n^2} + \sum_{r=1}^{n} \frac{(-1)^r(z^r q^{r(r-1)/2} + z^{-r} q^{r(r+1)/2})}{(q;q)_{n-r}(q;q)_{n+r}} \\ &= \sum_{r=-n}^{n} \frac{(-1)^r z^r q^{r(r-1)/2}}{(q;q)_{n-r}(q;q)_{n+r}} \\ &= \frac{(z, q/z; q)_n}{(q;q)_{2n}}, \end{aligned}$$

where we have used a summation due to P. A. MacMahon [1, p. 75]. This completes the proof.

We now differentiate (23.11) and (23.12) with respect to z, multiply both sides of each equality by -1, and then set $z = 1$. It is easy to see that this operation produces another Bailey pair

$$\alpha_n = (-1)^{n-1} n q^{n(n-1)/2}(1 - q^n), \qquad n \geq 0, \tag{23.13}$$

and

$$\begin{aligned} \beta_n &= -\frac{d}{dz} \frac{(1-z)(zq;q)_{n-1}(q/z;q)_n}{(q;q)_{2n}}\bigg|_{z=1} \\ &= \begin{cases} 0, & n = 0, \\ \dfrac{(q;q)_{n-1}(q;q)_n}{(q;q)_{2n}}, & n > 0. \end{cases} \end{aligned} \tag{23.14}$$

Lemma 23.5. *Recall that $L_4(q)$ denotes the left side of* (26.1). *Then*

$$L_4(q) = \sum_{n=1}^{\infty} \frac{(-q, -q, q; q)_{n-1}(q; q)_n q^n}{(q; q)_{2n}}$$

$$= \sum_{n=1}^{\infty} \frac{(-q, -q, q; q)_{n-1}(q; q)_n q^n}{(q, -q, \sqrt{q}, -\sqrt{q}; q)_n}.$$

PROOF. Set $a = 1$ and $\rho_1 = \rho_2 = -1$ in (23.10) and insert the Bailey pair (23.13) and (23.14). Upon dividing both sides by 4, we deduce that

$$\sum_{n=1}^{\infty} (-q; q)_{n-1}^2 q^n \frac{(q; q)_{n-1}(q; q)_n}{(q; q)_{2n}} = \frac{1}{\varphi^2(-q)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n q^{n(n+1)/2}(1 - q^n)}{(1 + q^n)^2}$$

$$= L_4(q),$$

where we have used (22.4) in Chapter 16. This completes the proof.

Next, we apply the operator $-(d/dz)z(d/dz)$ to the Bailey pair (23.11), (23.12) and then set $z = 1$ to produce a new Bailey pair α_n', β_n'. More precisely,

$$\alpha_n' = (-1)^{n-1}(n^2 z^{n-1} q^{n(n-1)/2} + n^2 z^{-n-1} q^{n(n+1)/2})|_{z=1}$$

$$= (-1)^{n-1} n^2 q^{n(n-1)/2}(1 + q^n), \qquad n \geq 0. \tag{23.15}$$

To calculate β_n', we first observe that, for any twice differentiable function F,

$$\frac{d}{dz} z \frac{d}{dz}(1 - z)F(z)|_{z=1} = \frac{d}{dz}\{z(1 - z)F'(z) - zF(z)\}|_{z=1}$$

$$= -F(1) - 2F'(1). \tag{23.16}$$

Thus, differentiating (23.12) and employing (23.16), we find that $\beta_0' = 0$ and, for $n > 0$,

$$\beta_n' = -\frac{d}{dz} z \frac{d}{dz}(1 - z) \frac{(zq; q)_{n-1}(q/z; q)_n}{(q; q)_{2n}}\Bigg|_{z=1}$$

$$= \frac{(q; q)_{n-1}(q; q)_n}{(q; q)_{2n}} + 2\frac{(q; q)_{n-1}(q; q)_n}{(q; q)_{2n}}\left\{\sum_{j=1}^{n-1} \frac{-q^j}{1 - q^j} + \sum_{j=1}^{n} \frac{q^j}{1 - q^j}\right\}$$

$$= \frac{(q; q)_{n-1}(q; q)_n}{(q; q)_{2n}} + 2\frac{(q; q)_{n-1}(q; q)_n}{(q; q)_{2n}} \frac{q^n}{1 - q^n}$$

$$= \frac{(q; q)_{n-1}^2}{(1 - q^n)(q; q)_{2n-1}}. \tag{23.17}$$

Lemma 23.6. *We have*

$$L_1(q) = \sum_{n=1}^{\infty} \frac{(-q, -q, q, q; q)_{n-1} q^n}{(1 - q^n)(q; q)_{2n-1}}.$$

PROOF. Set $a = 1$ and $\rho_1 = \rho_2 = -1$ in (23.10) and insert the Bailey pair (23.15), (23.17). Upon dividing both sides by 4, we deduce that

$$\sum_{n=1}^{\infty} (-q;q)_{n-1}^2 q^n \frac{(q;q)_{n-1}^2}{(1-q^n)(q;q)_{2n-1}} = \frac{1}{\varphi^2(-q)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2 q^{n(n+1)/2}}{1+q^n}$$
$$= L_1(q),$$

where again we used (22.4) in Chapter 16. This finishes the proof.

To obtain analogous representations for $L_2(q)$ and $L_3(q)$, we need to derive a further Bailey pair.

Lemma 23.7. *For $n \geq 0$,*

$$\alpha_n = \frac{(-1)^{n-1}(z^{n+1}q^{n^2+n} - z^{-n}q^{n^2+n})}{1-q^2} \tag{23.18}$$

and

$$\beta_n = \frac{(z;q^2)_{n+1}(q^2/z;q^2)_n}{(q^2;q^2)_{2n+1}} \tag{23.19}$$

is a Bailey pair.

PROOF. Replace q by q^2 and set $a = q^2$ in (23.9). With α_n given by (23.18), we must show that β_n is given by (23.19). Hence,

$$\sum_{r=0}^{n} \frac{\alpha_r}{(q^2;q^2)_{n-r}(q^4;q^2)_{n+r}} = \sum_{r=0}^{n} \frac{(-1)^r(z^{-r}q^{r^2+r} - z^{r+1}q^{r^2+r})}{(q^2;q^2)_{n-r}(q^2;q^2)_{n+r+1}}$$
$$= \sum_{r=-n-1}^{n} \frac{(-1)^r z^{-r} q^{r^2+r}}{(q^2;q^2)_{n-r}(q^2;q^2)_{n+r+1}}$$
$$= \frac{(z;q^2)_{n+1}(q^2/z;q^2)_n}{(q^2;q^2)_{2n+1}},$$

where again we have used a result in MacMahon's book [1, p. 75]. Thus, (23.19) has been shown.

Differentiating (23.18) and (23.19) with respect to z and then putting $z = 1$ produces a new Bailey pair α_n'', β_n'' given by

$$\alpha_n'' = \frac{(-1)^{n-1}(2n+1)q^{n^2+n}}{1-q^2} \tag{23.20}$$

and

$$\beta_n'' = \frac{d}{dz}(1-z)\frac{(zq^2, q^2/z; q^2)_n}{(q^2;q^2)_{2n+1}}\bigg|_{z=1} = -\frac{(q^2,q^2;q^2)_n}{(q^2;q^2)_{2n+1}}. \tag{23.21}$$

Lemma 23.8. *We have*

$$L_3(q) = -\frac{q}{1-q^2}\,{}_4\varphi_3\left(\begin{matrix} q, q, q^2, q^2; q^2, q^2 \\ -q^2, q^3, -q^3 \end{matrix}\right).$$

Proof. In (23.10), replace q by q^2 and then set $a = q^2$ and $\rho_1 = \rho_2 = q$. Inserting the Bailey pair (23.20), (23.21) and multiplying both sides by q, we find that

$$\begin{aligned}
-\sum_{n=0}^{\infty} (q; q^2)_n^2 q^{2n+1} \frac{(q^2; q^2)_n^2}{(q^2; q^2)_{2n+1}}
&= \frac{(q^3, q^3; q^2)_\infty}{(q^4, q^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(1-q)^2 q^{2n+1}}{(1-q^{2n+1})^2} \frac{(-1)^{n-1}(2n+1)q^{n^2+n}}{1-q^2} \\
&= \frac{1}{\psi^2(q)} \sum_{n=1}^{\infty} \frac{(-1)^n(2n-1)q^{n^2+n-1}}{(1-q^{2n-1})^2} \\
&= L_3(q),
\end{aligned}$$

where we have employed Entry 22(ii) in Chapter 16. This completes the proof.

For the Bailey pair α_n''', β_n''' required for $L_2(q)$, we replace z by zq^2 in Lemma 23.7, multiply both (23.18) and (23.19) by $z^{-1/2}$, apply the operator $(d/dz)z(d/dz)$, and lastly set $z = 1$. This gives us

$$\begin{aligned}
\alpha_n''' &:= \frac{1}{1-q^2} \frac{d}{dz} z \frac{d}{dz} (-1)^{n-1} (z^{n+1/2} q^{n^2+3n+2} - z^{-n-1/2} q^{n^2-n})\big|_{z=1} \\
&= \frac{(-1)^n (n+\frac{1}{2})^2 q^{n^2-n}(1-q^{4n+2})}{1-q^2}.
\end{aligned} \tag{23.22}$$

A simple calculation shows that $\beta_0''' = \frac{1}{4}$. To calculate β_n''' for $n > 0$, we employ (23.16) to determine that

$$\begin{aligned}
\beta_n''' &= \frac{d}{dz} z \frac{d}{dz} (1-z) \left(\frac{-z^{-3/2}(zq^2; q^2)_{n+1}(q^2/z; q^2)_{n-1}}{(q^2; q^2)_{2n+1}} \right)\Bigg|_{z=1} \\
&= \frac{(q^2; q^2)_{n+1}(q^2; q^2)_{n-1}}{(q^2; q^2)_{2n+1}} \left(-2 + 2\sum_{j=1}^{n+1} \frac{-q^{2j}}{1-q^{2j}} + 2\sum_{j=1}^{n-1} \frac{q^{2j}}{1-q^{2j}} \right) \\
&= -\frac{2(q^2; q^2)_{n+1}(q^2; q^2)_{n-1}}{(q^2; q^2)_{2n+1}} \left(1 + \frac{q^{2n}}{1-q^{2n}} + \frac{q^{2n+2}}{1-q^{2n+2}} \right) \\
&= -\frac{2(q^2; q^2)_{n+1}(q^2; q^2)_{n-1}}{(q^2; q^2)_{2n+1}} \frac{1-q^{4n+2}}{(1-q^{2n})(1-q^{2n+2})} \\
&= -\frac{2(q^2; q^2)_{n-1}^2}{(q^2; q^2)_{2n}}.
\end{aligned} \tag{23.23}$$

Lemma 23.9. *We have*

$$L_2(q) = 1 - 8 \sum_{n=1}^{\infty} (q; q^2)_n^2 q^{2n} \frac{(q^2; q^2)_{n-1}^2}{(q^2; q^2)_{2n}}.$$

PROOF. In (23.10), replace q by q^2, set $a = q^2$ and $\rho_1 = \rho_2 = q$, and substitute the Bailey pair α_n''', β_n''' given by $\beta_0''' = \frac{1}{4}$, (23.22), and (23.23). After multiplying both sides by 4, we find that

$$\begin{aligned}
1 - 8 \sum_{n=1}^{\infty} & (q; q^2)_n^2 q^{2n} \frac{(q^2; q^2)_{n-1}^2}{(q^2; q^2)_{2n}} \\
&= \frac{(q^3, q^3; q^2)_\infty}{(q^4, q^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(1-q)^2 q^{2n}}{(1-q^{2n+1})^2} \frac{(-1)^n (2n+1)^2 q^{n^2-n}(1-q^{4n+2})}{1-q^2} \\
&= \frac{1}{\psi^2(q)} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n}(2n+1)^2(1+q^{2n+1})}{1-q^{2n+1}} \\
&= L_2(q),
\end{aligned}$$

and this completes the proof.

Using the new representations for $L_j(q)$, $1 \le j \le 4$, derived above, we will now give proofs of Entries 23–26. For brevity, the right sides of (2j.1) will be denoted by $R_{j-2}(q)$, $3 \le j \le 6$, respectively.

PROOF OF ENTRY 23. We first transform $R_1(q)$. Applying (23.3) with q replaced by q^2, $a = q^2$, $d = f = g = q$ and letting e and N tend to ∞, we deduce that

$$\begin{aligned}
R_1(q) &= \sum_{n=0}^{\infty} \frac{q^{(n+1)(2n+1)}(1+q^{2n+1})}{(1-q^{2n+1})^2} \\
&= \frac{(q^2; q^2)_\infty^2}{(q; q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{(q; q^2)_n q^{2n+1}}{(q^2; q^2)_n (1-q^{2n+1})}.
\end{aligned} \tag{23.24}$$

On the other hand, by Lemma 23.6,

$$\begin{aligned}
L_1(q) &= \sum_{n=0}^{\infty} \frac{(q^2; q^2)_n^2 q^{n+1}}{(1-q^{n+1})(q^2; q^2)_n (q; q^2)_{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(q^2; q^2)_n^2 q^{n+1}(1+q^{n+1})}{(1-q^{2n+2})(q^2; q^2)_n (q; q^2)_{n+1}} \\
&= \frac{q}{(1-q)(1-q^2)} \left\{ {}_3\varphi_2\left(\begin{matrix} q^2, q^2, q^2; q^2, q \\ q^3, q^4 \end{matrix}\right) + q\, {}_3\varphi_2\left(\begin{matrix} q^2, q^2, q^2; q^2, q^2 \\ q^3, q^4 \end{matrix}\right) \right\}.
\end{aligned}$$

We now apply (23.5) with q replaced by q^2, $a = b = c = q^2$, $e = q^3$, and $f = q^4$ to deduce that

$$\begin{aligned} L_1(q) &= \frac{q}{(1-q)(1-q^2)} \frac{(q^2, q^2, q^2, q^3; q^2)_\infty}{(q^3, q, q^4, q; q^2)_\infty} {}_3\varphi_2\left(\begin{matrix} q, q, q; q^2, q^2 \\ q, q^3 \end{matrix}\right) \\ &= \frac{(q^2; q^2)_\infty^2}{(q; q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{(q; q^2)_n q^{2n+1}}{(q^2; q^2)_n (1 - q^{2n+1})} \\ &= R_1(q), \end{aligned}$$

by (23.24), and so our proof is complete.

PROOF OF ENTRY 24. In Lemma 23.2, replace q by q^2, divide both sides of (23.8) by $(1-f)$, and lastly set $a = f = 1$, $d = -1$, and $e = -q$. This yields

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1+q^{2n})q^{n^2+n}}{(1-q^{2n})^2} + 2\sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(1-q^{2n})^2}$$

$$= \sum_{n=1}^{\infty} \frac{(q; q^2)_n (q^2; q^2)_{n-1} q^{2n}}{(1-q^{2n})(-q^2, -q; q^2)_n}. \tag{24.2}$$

Algebraically combining the two sums on the left side, we may rewrite (24.2) in the form

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n^2+n}}{(1+q^n)^2} &= \sum_{n=1}^{\infty} \frac{(q; q^2)_n^2 (q^2; q^2)_{n-1}^2 q^{2n}}{(q, q^2, -q^2, -q; q^2)_n} \\ &= \tfrac{1}{8}(1 - L_2(q)), \end{aligned} \tag{24.3}$$

by Lemma 23.9. Rearranging (24.3), we arrive at (24.1) to complete the proof.

PROOF OF ENTRY 25. In Lemma 23.1, replace q by q^2, divide both sides by $(1-f)$, and lastly set $a = f = 1$, $e = -1$, $d = -q$, and $g = q$. Accordingly, we find that

$$- \sum_{n=1}^{\infty} \frac{q^n(1+q^{2n})}{(1-q^{2n})^2} + 2\sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2}$$

$$\begin{aligned} &= \sum_{n=1}^{\infty} \frac{(q; q^2)_n (q^2; q^2)_{n-1} q^{2n}}{(1-q^{2n})(-q, -q^2; q^2)_n} - \frac{q}{1-q^2} {}_4\varphi_3\left(\begin{matrix} q, q, q^2, q^2; q^2, q^2 \\ q^3, -q^2, -q^3 \end{matrix}\right) \\ &= \sum_{n=1}^{\infty} \frac{(q; q^2)_n (q^2; q^2)_{n-1} q^{2n}}{(1-q^{2n})(-q, -q^2; q^2)_n} + L_3(q), \end{aligned} \tag{25.2}$$

by Lemma 23.8. Using the first equality of (24.3) in (25.2) and solving for $L_3(q)$, we find that

$$L_3(q) = -\sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+n}}{(1+q^n)^2}. \tag{25.3}$$

It remains to show that $R_3(q)$ is equal to the right side of (25.3).

Now, by (25.1),

$$\begin{aligned} R_3(q) &= \sum_{n=1}^{\infty} \frac{(-1)^n n q^{n^2}}{1-q^{2n}} + \sum_{n=1}^{\infty} \frac{(-1)^n n q^{n^2+2n}}{1-q^{2n}} \\ &= \sum_{n=1}^{\infty} \frac{2nq^{4n^2}}{1-q^{4n}} - \sum_{n=0}^{\infty} \frac{(2n+1)q^{4n^2+4n+1}}{1-q^{4n+2}} \\ &\quad + \sum_{n=1}^{\infty} \frac{2nq^{4n^2+4n}}{1-q^{4n}} - \sum_{n=1}^{\infty} \frac{(2n+1)q^{4n^2+8n+3}}{1-q^{4n+2}} \\ &=: S_1(q) - S_2(q) + S_3(q) - S_4(q). \end{aligned} \tag{25.4}$$

We shall examine each of these four sums in turn. It will be helpful to use the elementary identity

$$\sum_{n=1}^{m} nx^n = \frac{x - x^{m+1}}{(1-x)^2} - \frac{mx^{m+1}}{1-x}, \qquad m \geq 0, \tag{25.5}$$

which is obtained by differentiating the standard formula for summing a finite geometric series.

First, by (25.5),

$$\begin{aligned} S_1(q) &= 2\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} nq^{4n^2+4nm} \\ &= 2\sum_{n=1}^{\infty} \sum_{m=n}^{\infty} nq^{4mn} \\ &= 2\sum_{m=1}^{\infty} \sum_{n=1}^{m} nq^{4mn} \\ &= 2\sum_{m=1}^{\infty} \left(\frac{q^{4m} - q^{4m(m+1)}}{(1-q^{4m})^2} - \frac{mq^{4m(m+1)}}{1-q^{4m}} \right) \\ &= 2\sum_{m=1}^{\infty} \frac{q^{4m} - q^{4m(m+1)}}{(1-q^{4m})^2} - S_3(q). \end{aligned} \tag{25.6}$$

Second, by (25.5),

$$\begin{aligned}
S_2(q) - \sum_{n=0}^{\infty} \frac{q^{(2n+1)^2}}{1-q^{4n+2}} &= 2\sum_{n=0}^{\infty} \frac{nq^{(2n+1)^2}}{1-q^{4n+2}} \\
&= 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} nq^{(2n+1)(2n+1+2m)} \\
&= 2\sum_{n=0}^{\infty}\sum_{m=n}^{\infty} nq^{(2n+1)(2m+1)} \\
&= 2\sum_{m=0}^{\infty} q^{2m+1}\sum_{n=0}^{m} nq^{(4m+2)n} \\
&= 2\sum_{m=0}^{\infty} q^{2m+1}\left(\frac{q^{4m+2}-q^{(m+1)(4m+2)}}{(1-q^{4m+2})^2} - \frac{mq^{(m+1)(4m+2)}}{1-q^{4m+2}}\right) \\
&= 2\sum_{m=0}^{\infty} \frac{q^{6m+3}(1-q^{4m^2+2m})}{(1-q^{4m+2})^2} \\
&\quad -S_4(q) + \sum_{m=0}^{\infty} \frac{q^{(2m+1)(2m+3)}}{1-q^{4m+2}}. \qquad (25.7)
\end{aligned}$$

Obtaining $S_1(q)+S_3(q)$ from (25.6) and $S_2(q)+S_4(q)$ from (25.7) and substituting these formulas in (25.4), we find that

$$\begin{aligned}
R_3(q) &= 2\sum_{m=1}^{\infty} \frac{q^{4m}-q^{4m(m+1)}}{(1-q^{4m})^2} - \sum_{n=0}^{\infty} \frac{q^{(2n+1)^2}}{1-q^{4n+2}} \\
&\quad -2\sum_{m=0}^{\infty} \frac{q^{6m+3}(1-q^{4m^2+2m})}{(1-q^{4m+2})^2} - \sum_{m=0}^{\infty} \frac{q^{(2m+1)(2m+3)}}{1-q^{4m+2}} \\
&= 2\sum_{m=1}^{\infty} \frac{q^{4m}}{(1-q^{4m})^2} - 2\sum_{m=0}^{\infty} \frac{q^{6m+3}}{(1-q^{4m+2})^2} - 2\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+2)}}{(1-q^{2n})^2} \\
&\quad - \sum_{n=0}^{\infty} \frac{q^{(2n+1)^2}(1+q^{4n+2})}{1-q^{4n+2}} \\
&=: 2T_1(q) - 2T_2(q) - 2T_3(q) - T_4(q). \qquad (25.8)
\end{aligned}$$

We thus must examine each of these four sums.

First,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{2n+1}}{1-q^{4n+2}} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q^{(2n+1)(2m+1)} \\
&= \left(\sum_{n=0}^{\infty}\sum_{m=0}^{n} + \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\right) q^{(2n+1)(2m+1)}
\end{aligned}$$

$$= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} q^{(2n+1)(2m+1)} + \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} q^{(2n+1)(2m+1)}$$

$$= \sum_{m=0}^{\infty} \frac{q^{(2m+1)^2}}{1-q^{4m+2}} + \sum_{n=0}^{\infty} \frac{q^{(2n+1)^2+4n+2}}{1-q^{4n+2}}$$

$$= T_4(q). \tag{25.9}$$

Second, in Lemma 23.2, replace q by q^2, divide both sides of (23.8) by $(1-f)$, and lastly set $a = f = d = e = 1$. Using this result in the third equality below, we find that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+n}}{(1+q^n)^2} &= \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+n}(1-q^n)^2}{(1-q^{2n})^2} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+n}(1+q^{2n})}{(1-q^{2n})^2} - 2T_3(q) \\ &= -\sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2} - 2T_3(q). \end{aligned} \tag{25.10}$$

Third,

$$\begin{aligned}-\sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^2} &= -\sum_{n=1}^{\infty} \frac{q^n(1-2q^n+q^{2n})}{(1-q^{2n})^2} \\ &= -\sum_{n=1}^{\infty} \frac{q^n}{(1-q^{2n})^2} + 2\sum_{n=1}^{\infty} \frac{q^{4n}}{(1-q^{4n})^2} + 2\sum_{n=0}^{\infty} \frac{q^{4n+2}}{(1-q^{4n+2})^2} \\ &\quad - \sum_{n=1}^{\infty} \frac{q^{6n}}{(1-q^{4n})^2} - \sum_{n=0}^{\infty} \frac{q^{6n+3}}{(1-q^{4n+2})^2} \\ &= -\sum_{n=1}^{\infty} \frac{q^n}{(1-q^{2n})^2} + 2T_1(q) + 2\sum_{n=0}^{\infty} \frac{q^{4n+2}}{(1-q^{4n+2})^2} \\ &\quad - \sum_{n=1}^{\infty} \frac{q^{6n}}{(1-q^{4n})^2} - T_2(q). \end{aligned} \tag{25.11}$$

Utilizing (25.9)–(25.11) to eliminate $T_4(q)$, $T_3(q)$, and $T_1(q)$, respectively, from (25.8), we find that

$$\begin{aligned}R_3(q) = &-\sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^2} + \sum_{n=1}^{\infty} \frac{q^n}{(1-q^{2n})^2} - 2\sum_{n=0}^{\infty} \frac{q^{4n+2}}{(1-q^{4n+2})^2} \\ &+ \sum_{n=1}^{\infty} \frac{q^{6n}}{(1-q^{4n})^2} - T_2(q) + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+n}}{(1+q^n)^2} \\ &+ \sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2} - \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1-q^{4n+2}}. \end{aligned}$$

Employing (25.3), (25.5), and the definition of $T_2(q)$, we further find that

$$\begin{aligned}
R_3(q) &= L_3(q) + \sum_{n=1}^{\infty} \frac{q^n}{(1-q^{2n})^2} - 2\sum_{n=0}^{\infty} \frac{q^{4n+2}}{(1-q^{4n+2})^2} + \sum_{n=1}^{\infty} \frac{q^{6n}}{(1-q^{4n})^2} \\
&\quad - \sum_{n=0}^{\infty} \frac{q^{6n+3}}{(1-q^{4n+2})^2} + \sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2} - \sum_{n=0}^{\infty} \frac{q^{2n+1}(1-q^{4n+2})}{(1-q^{4n+2})^2} \\
&= L_3(q) + \sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{4n})^2} - 2\sum_{n=0}^{\infty} \frac{q^{4n+2}}{(1-q^{4n+2})^2} \\
&\quad + \sum_{n=1}^{\infty} \frac{q^{6n}}{(1-q^{4n})^2} + \sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2} \\
&= L_3(q) + \sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{4n})^2} - \sum_{n=0}^{\infty} \frac{q^{4n+2}}{(1-q^{4n+2})^2} \\
&\quad + \sum_{n=1}^{\infty} \frac{q^{6n}}{(1-q^{4n})^2} + \sum_{n=1}^{\infty} \frac{q^{4n}}{(1-q^{4n})^2} \\
&= L_3(q) + \sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{4n})^2} - \left(\sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2} - \sum_{n=1}^{\infty} \frac{q^{4n}}{(1-q^{4n})^2}\right) \\
&\quad + \sum_{n=1}^{\infty} \frac{q^{6n}}{(1-q^{4n})^2} + \sum_{n=1}^{\infty} \frac{q^{4n}}{(1-q^{4n})^2} \\
&= L_3(q) + \sum_{n=1}^{\infty} \frac{q^{2n}(1+2q^{2n}+q^{4n})}{(1-q^{4n})^2} - \sum_{n=1}^{\infty} \frac{q^{2n}}{(1-q^{2n})^2} \\
&= L_3(q).
\end{aligned}$$

Proof of Entry 26. First, from Lemma 23.5, we see that

$$L_4(q) = \frac{q}{(1-q^2)}\, {}_4\varphi_3\left(\begin{matrix} -q, -q, q, q; q, q \\ -q^2, q^{3/2}, -q^{3/2} \end{matrix}\right). \tag{26.2}$$

Next, in (23.2), put $a = q$, $f = d = h = \sqrt{q}$, $g = -\sqrt{q}$, and $e = -q$. Upon simplification, we arrive at

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(1-\sqrt{q})^2 q^n}{(1-q^{n+1/2})^2} &= \frac{(q^2;q^2)_\infty^2}{(1-q)(1+\sqrt{q})(q^3;q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{(1-\sqrt{q})(q;q^2)_n q^n}{(1-q^{n+1/2})(q^2;q^2)_n} \\
&\quad + \frac{(1-\sqrt{q})^2(1+\sqrt{q})}{(1-q^2)(1+1/\sqrt{q})}\, {}_4\varphi_3\left(\begin{matrix} -q, q, -q, q; q, q \\ -q^{3/2}, q^{3/2}, -q^2 \end{matrix}\right).
\end{aligned}$$

Multiplying both sides by $\sqrt{q}(1-\sqrt{q})^{-2}$ gives us

$$\sum_{n=0}^{\infty} \frac{q^{n+1/2}}{(1-q^{n+1/2})^2} = \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{n+1/2}}{(1-q^{n+1/2})(q^2;q^2)_n} + L_4(q), \tag{26.3}$$

by (26.2).

We now extract the rational parts of the two series in (26.3). First,

$$\sum_{n=0}^{\infty} \frac{q^{n+1/2}}{(1-q^{n+1/2})^2} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} mq^{m(n+1/2)}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} 2mq^{m(2n+1)} + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (2m+1)q^{m(2n+1)+n+1/2}$$

$$=: 2\sum_{n=0}^{\infty} \frac{q^{2n+1}}{(1-q^{2n+1})^2} + q^{1/2}H(q). \tag{26.4}$$

Second,

$$\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{n+1/2}}{(1-q^{n+1/2})(q^2;q^2)_n}$$

$$= \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{(q;q^2)_n}{(q^2;q^2)_n} q^{m(n+1/2)}$$

$$=: \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{(q;q^2)_n}{(q^2;q^2)_n} q^{m(2n+1)} + q^{1/2}K(q)$$

$$= \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n+1}}{(1-q^{2n+1})(q^2;q^2)_n} + q^{1/2}K(q). \tag{26.5}$$

Hence, using (26.4) and (26.5), we equate the rational parts on both sides of (26.3) to deduce that

$$2\sum_{n=0}^{\infty} \frac{q^{2n+1}}{(1-q^{2n+1})^2} = \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n+1}}{(1-q^{2n+1})(q^2;q^2)_n} + L_4(q). \tag{26.6}$$

Next, in (23.3), replace q by q^2, set $a = q^2$ and $d = f = g = q$, and let e and N tend to ∞. Upon multiplication of both sides by $q(1+q)(1-q)^{-2}$, we find that

$$\sum_{n=0}^{\infty} \frac{(1+q^{2n+1})q^{(2n+1)(n+1)}}{(1-q^{2n+1})^2} = \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n+1}}{(q^2;q^2)_n(1-q^{2n+1})}. \tag{26.7}$$

Combining (26.6) and (26.7), we infer that

$$L_4(q) = 2\sum_{n=0}^{\infty} \frac{q^{2n+1}}{(1-q^{2n+1})^2} - \sum_{n=0}^{\infty} \frac{(1+q^{2n+1})q^{(2n+1)(n+1)}}{(1-q^{2n+1})^2}. \tag{26.8}$$

It remains to demonstrate that the right side of (26.8) equals $R_4(q)$.

From the definition of $R_4(q)$ in (26.1),

$$\begin{aligned}R_4(q) &= \sum_{n=1}^{\infty} \frac{nq^{n(n+1)/2}}{1-q^n}\\ &= \sum_{n=1}^{\infty} \frac{2nq^{n(2n+1)}}{1-q^{2n}} + \sum_{n=0}^{\infty} \frac{(2n+1)q^{(n+1)(2n+1)}}{1-q^{2n+1}}\\ &=: U_1(q) + U_2(q).\end{aligned} \tag{26.9}$$

Using (25.5), we find that

$$\begin{aligned}U_1(q) &= \sum_{n=1}^{\infty} 2nq^{n(2n-1)} \sum_{m=1}^{\infty} q^{2nm}\\ &= \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} 2nq^{n(2m+1)}\\ &= 2\sum_{m=1}^{\infty} \sum_{n=1}^{m} nq^{n(2m+1)}\\ &= 2\sum_{m=0}^{\infty} \left(\frac{q^{2m+1} - q^{(2m+1)(m+1)}}{(1-q^{2m+1})^2} - \frac{mq^{(2m+1)(m+1)}}{1-q^{2m+1}}\right)\\ &= 2\cdot\sum_{m=0}^{\infty} \frac{q^{2m+1} - q^{(2m+1)(m+1)}}{(1-q^{2m+1})^2} - U_2(q) + \sum_{m=0}^{\infty} \frac{q^{(2m+1)(m+1)}}{1-q^{2m+1}}.\end{aligned} \tag{26.10}$$

Combining (26.9) and (26.10), we conclude that

$$\begin{aligned}R_4(q) &= 2\sum_{m=0}^{\infty} \frac{q^{2m+1}}{(1-q^{2m+1})^2} - 2\sum_{m=0}^{\infty} \frac{q^{(2m+1)(m+1)}}{(1-q^{2m+1})^2}\\ &\quad + \sum_{m=0}^{\infty} \frac{q^{(2m+1)(m+1)}(1-q^{2m+1})}{(1-q^{2m+1})^2}\\ &= 2\sum_{m=0}^{\infty} \frac{q^{2m+1}}{(1-q^{2m+1})^2} - \sum_{m=0}^{\infty} \frac{q^{(2m+1)(m+1)}(1+q^{2m+1})}{(1-q^{2m+1})^2}\\ &= L_4(q),\end{aligned}$$

by (26.8). This completes the proof.

For further discussion of Entries 23–26, see the concluding section of Andrews' paper [5].

Entry 27 (p. 300). *For* $|q| < 1$,

$$\begin{aligned}2\{\varphi(q)\varphi(q^7)\varphi(q^9)\varphi(q^{63}) &+ \varphi(-q)\varphi(-q^7)\varphi(-q^9)\varphi(-q^{63})\\ &\qquad + 4q^4 f^2(-q^6) f^2(-q^{42})\}\\ &= \{\varphi(q)\varphi(q^{63}) + \varphi(-q)\varphi(-q^{63}) + 4q^{16}\psi(q^2)\psi(q^{126})\}^2.\end{aligned} \tag{27.1}$$

This proposed identity is very unusual and unlike any other identity in the notebooks. In the language of modular equations, six moduli, with degrees 1, 3, 7, 9, 21, and 63, appear. Unfortunately, Entry 27 appears to be incorrect, and we are unable to find a corrected version of it. Expanding the left and right sides of (27.1), we obtain the series

$$4+16q^4+16q^8+32q^{16}+16q^{18}+32q^{20}+16q^{22}+\cdots$$

and

$$4+16q^4+16q^8+32q^{16}+16q^{18}+64q^{20}+48q^{22}+\cdots,$$

respectively. Thus, quite remarkably, both expansions agree through the eighteenth power.

We use Ramanujan's notation in (28.1) below. The function φ in (28.1) has no connection with the theta-function φ.

Entry 28 (p. 300). *Let*

$$\varphi(q):=1+6\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n}{1-q^n}, \tag{28.1}$$

where $\left(\frac{n}{3}\right)$ *denotes the Legendre symbol. Then*

$$\varphi(q)+\varphi(-q)=2\varphi(q^4).$$

PROOF. Observe that

$$\begin{aligned}\varphi(q)+\varphi(-q)&=2+6\left\{\sum_{n=0}^{\infty}\left(\frac{q^{6n+1}}{1-q^{6n+1}}-\frac{q^{6n+1}}{1+q^{6n+1}}\right)-2\sum_{n=0}^{\infty}\frac{q^{6n+2}}{1-q^{6n+2}}\right.\\&\qquad\left.+2\sum_{n=0}^{\infty}\frac{q^{6n+4}}{1-q^{6n+4}}+\sum_{n=0}^{\infty}\left(-\frac{q^{6n+5}}{1-q^{6n+5}}+\frac{q^{6n+5}}{1+q^{6n+5}}\right)\right\}\\&=2+12\left\{\sum_{n=0}^{\infty}\frac{q^{12n+2}}{1-q^{12n+2}}-\sum_{n=0}^{\infty}\frac{q^{6n+2}}{1-q^{6n+2}}\right.\\&\qquad\left.+\sum_{n=0}^{\infty}\frac{q^{6n+4}}{1-q^{6n+4}}-\sum_{n=0}^{\infty}\frac{q^{12n+10}}{1-q^{12n+10}}\right\}\\&=2+\left\{-\sum_{n=0}^{\infty}\frac{q^{12n+8}}{1-q^{12n+8}}+\sum_{n=0}^{\infty}\frac{q^{12n+4}}{1-q^{12n+4}}\right\}\\&=2\varphi(q^4),\end{aligned}$$

and the proof is complete.

Entry 29 (p. 300). *Let* $\varphi(q)$ *be defined by* (28.1). *Then*

$$\varphi^2(q) + \varphi(q)\varphi(-q) + \varphi^2(-q) = 3\varphi^2(q^2). \tag{29.1}$$

We are grateful to K. Venkatachaliengar for the following proof, which replaces the author's original, more complicated proof.

PROOF. We write (29.1) in the equivalent form

$$\begin{aligned} 6\varphi^2(q^2) &= \{\varphi(q) + \varphi(-q)\}^2 + \varphi^2(q) + \varphi^2(-q) \\ &= 4\varphi^2(q^4) + \varphi^2(q) + \varphi^2(-q), \end{aligned} \tag{29.2}$$

by Entry 28.

From Ramanujan's paper [17], [23, p. 139, eq. (19)],

$$\varphi^2(q) = 1 + 12\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 36\sum_{n=1}^{\infty} \frac{nq^{3n}}{1-q^{3n}}. \tag{29.3}$$

Thus, from (29.3),

$$\begin{aligned} &4\varphi^2(q^4) + \varphi^2(q) + \varphi^2(-q) \\ &= 4\left(1 + 12\sum_{n=1}^{\infty} \frac{nq^{4n}}{1-q^{4n}} - 36\sum_{n=1}^{\infty} \frac{nq^{12n}}{1-q^{12n}}\right) \\ &\quad +1 + 12\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 36\sum_{n=1}^{\infty} \frac{nq^{3n}}{1-q^{3n}} \\ &\quad +1 + 12\sum_{n=1}^{\infty} \frac{n(-q)^n}{1-(-q)^n} - 36\sum_{n=1}^{\infty} \frac{n(-q)^{3n}}{1-(-q)^{3n}} \\ &= 6 + 72\sum_{n=1}^{\infty} \frac{2nq^{4n}}{1-q^{4n}} + 48\sum_{n=0}^{\infty} \frac{(2n+1)q^{4n+2}}{1-q^{4n+2}} \\ &\quad +12\sum_{n=0}^{\infty}\left(\frac{(2n+1)q^{2n+1}}{1-q^{2n+1}} - \frac{(2n+1)q^{2n+1}}{1+q^{2n+1}}\right) - 72\sum_{n=1}^{\infty} \frac{2nq^{12n}}{1-q^{12n}} \\ &\quad -144\sum_{n=1}^{\infty} \frac{nq^{6n}}{1-q^{6n}} - 36\sum_{n=0}^{\infty}\left(\frac{(2n+1)q^{6n+3}}{1-q^{6n+3}} - \frac{(2n+1)q^{6n+3}}{1+q^{6n+3}}\right) \\ &= 6 + 72\sum_{n=1}^{\infty} \frac{2nq^{4n}}{1-q^{4n}} + 72\sum_{n=0}^{\infty} \frac{(2n+1)q^{4n+2}}{1-q^{4n+2}} \\ &\quad -216\sum_{n=1}^{\infty} \frac{2nq^{12n}}{1-q^{12n}} - 216\sum_{n=0}^{\infty} \frac{(2n+1)q^{12n+6}}{1-q^{12n+6}} \\ &= 6 + 72\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} - 216\sum_{n=1}^{\infty} \frac{nq^{6n}}{1-q^{6n}} \\ &= 6\varphi^2(q^2), \end{aligned}$$

by (29.3). Hence, (29.2) has been established, and the proof of Entry 29 has been completed.

The next three results are a sequel to the several beautiful theorems in Entries 18(i), (ii) in Chapter 19 (Part III [6, p. 305]).

Entry 30 (p. 300). *For* $|q| < 1$,

$$\frac{f(-q^3,-q^4)}{q^{3/7}f(-q,-q^6)} - \frac{q^{1/7}f(-q^2,-q^5)}{f(-q^3,-q^4)} + \frac{q^{2/7}f(-q,-q^6)}{f(-q^2,-q^5)} - 2$$
$$= \frac{1}{2}\left\{\frac{3f(q^{-1/7})}{q^{2/7}f(-q^7)} + \left(\frac{4f^3(-q^{1/7})}{q^{6/7}f^3(-q^7)} + \frac{21f^2(-q^{1/7})}{q^{4/7}f^2(-q^7)} + \frac{28f(-q^{1/7})}{q^{2/7}f(-q^7)}\right)^{1/2}\right\}. \tag{30.1}$$

Proof. We shall utilize the notation at the beginning of the proof of Entry 18(i) in Chapter 19 (Part III [6, p. 306]). Thus, put

$$\left.\begin{aligned} \alpha &= \frac{f(-q^2,-q^5)}{q^{2/7}f(-q,-q^6)}, &\qquad \beta &= -\frac{f(-q^3,-q^4)}{q^{1/7}f(-q^2,-q^5)}, \\ \gamma &= \frac{q^{3/7}f(-q,-q^6)}{f(-q^3,-q^4)}, \quad \text{and} &\qquad v &= \frac{f(-q^{1/7})}{q^{2/7}f(-q^7)}. \end{aligned}\right\} \tag{30.2}$$

Then, by (30.1), we are required to show that

$$\frac{1}{\gamma} + \frac{1}{\beta} + \frac{1}{\alpha} - 2 = \tfrac{1}{2}\{3v + (4v^3 + 21v^2 + 28v)^{1/2}\}. \tag{30.3}$$

By entry 18(i) in Chapter 19, we know that

$$\alpha + \beta + \gamma = v + 1. \tag{30.4}$$

As in the proof of Entry 18(i), let α, β, and γ be the roots of the polynomial

$$z^3 - pz^2 + sz - r,$$

where, by (30.4), $p = v + 1$. Then $1/\alpha$, $1/\beta$, and $1/\gamma$ are roots of the equation

$$1 - pz + sz^2 - rz^3 = 0.$$

Thus,

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{s}{r}. \tag{30.5}$$

Now, by (18.4) in Chapter 19, $r = -1$, and by (18.13) in the same chapter,

$$s = -\frac{3v + 4}{2} - \tfrac{1}{2}(4v^3 + 21v^2 + 28v)^{1/2}. \tag{30.6}$$

Hence, (30.3) follows immediately from (30.5), (30.6), and the fact $r = -1$.

Ramanujan has a slight misprint in his formulation of Entry 30; he has -1 instead of -2 on the left side of (30.1).

Although Ramanujan does not define u, v, and w in the next two results, it is natural to conclude that $u = \alpha^7$, $v = -\beta^7$, and $w = \gamma^7$, as in Section 18 of Chapter 19, where α, β, and γ are redefined above in (30.2). However, this interpretation is incorrect; Ramanujan's u, v, and w should be replaced by α, β, and γ. We therefore will record Entry 31 in this amended notation.

Entry 31 (p. 300). *Let* α, β, *and* γ *be defined as in* (30.2). *Then*

$$\frac{\alpha^2}{\gamma} + \frac{\beta^2}{\alpha} - \frac{\gamma^2}{\beta} = 8 + \frac{f^4(-q)}{qf^4(-q^7)} \tag{31.1}$$

and

$$\frac{\beta}{\gamma^2} - \frac{\alpha}{\beta^2} - \frac{\gamma}{\alpha^2} = 5 + \frac{f^4(-q)}{qf^4(-q^7)}. \tag{31.2}$$

Proof. To prove (31.1) and (31.2), we shall apply a general result of R. J. Evans [1, Theorem 4.1] (Part III [6, p. 339, Theorem 0.4]). For p odd and >1, let

$$G(m; pz) := (-1)^m q^{m(3m-p)/(2p)} \frac{f(-q^{2m}, -q^{p-2m})}{f(-q^m, -q^{p-m})}, \tag{31.3}$$

where $q = e^{2\pi iz}$ and m is an integer. Let $\{\Gamma, r, v\}$ denote the space of modular forms on the modular subgroup Γ of weight r and multiplier system v. Let ε_r and β_r, $1 \le r \le s$, be nonzero integers with

$$\varepsilon_1\beta_1^2 + \cdots + \varepsilon_s\beta_s^2 \equiv 0 \pmod{p}. \tag{31.4}$$

Then

$$g(z) := \sum_m \prod_{r=1}^{s} G(m\beta_r; z)^{\varepsilon_r} \in \{\Gamma^0(p), 0, 1\}, \tag{31.5}$$

where the sum is over all $m \pmod{p}$. Moreover, $g(z)$ has no poles on the upper half-plane $\mathscr{H}$ or at the cusp 0.

Translating (31.1) and (31.2) into the notation of (31.3), we find that, respectively

$$\frac{G_1^2(z)}{G_3(z)} + \frac{G_2^2(z)}{G_1(z)} + \frac{G_3^2(z)}{G_2(z)} = -8 - \frac{\eta^4(z/7)}{\eta^4(z)} \tag{31.6}$$

and

$$\frac{G_2(z)}{G_3^2(z)} + \frac{G_1(z)}{G_2^2(z)} + \frac{G_3(z)}{G_1^2(z)} = 5 + \frac{\eta^4(z/7)}{\eta^4(z)}, \tag{31.7}$$

where we have put $G_m(z) = G(m; z)$, and where $\eta(z)$ denotes the Dedekind eta-function. For $p \equiv 1 \pmod 6$, Evans [1, eq. (5.23)] (Part III [6, p. 343]) showed that $\eta^4(z/p)/\eta^4(z) \in \{\Gamma^0(p), 0, 1\}$.

Apply the theorem above with $p = 7$, $s = 2$, $\varepsilon_1 = 2$, $\varepsilon_2 = -1$, $\beta_1 = 2$, and $\beta_2 = 1$. Hence, (31.4) is satisfied. Using the facts (Evans [1, eqs. (2.28), (2.29)]; Part III [6, pp. 338–339, eqs. (0.51), (0.52)])

$$G(p - m; z) = G(m; z) \tag{31.8}$$

and

$$G(m; z) = 2, \qquad \text{if} \quad p \mid m, \tag{31.9}$$

we see that

$$g(z) = 2 + 2\left(\frac{G_1^2(z)}{G_3(z)} + \frac{G_2^2(z)}{G_1(z)} + \frac{G_3^2(z)}{G_2(z)}\right).$$

Thus, by (31.5), the left side of (31.6) is in $\{\Gamma^0(7), 0, 1\}$ and has no poles on $\mathscr{H}$ or at the cusp 0. To prove (31.6), all we need do is show that the difference of the two sides has a zero at ∞.

Now (Evans [1, eq. (5.6)]; Part III [6, p. 343, eq. (0.77)])

$$G_1(z) = -q^{-2/49}(1 + q^{1/7} + O(q^{5/7})), \tag{31.10}$$

$$G_2(z) = q^{-1/49}(1 + q^{2/7} + O(q^{3/7})), \tag{31.11}$$

and

$$G_3(z) = -q^{3/49}(1 - q^{1/7} + O(q^{3/7})). \tag{31.12}$$

Hence,

$$\begin{aligned} \frac{G_1^2(z)}{G_3(z)} + \frac{G_2^2(z)}{G_1(z)} + \frac{G_3^2(z)}{G_2(z)} &= -q^{-1/7}(1 + 3q^{1/7}) - 1 + O(q^{1/7}) \\ &= -q^{-1/7} - 4 + O(q^{1/7}). \end{aligned} \tag{31.13}$$

On the other hand, since, by (0.4),

$$\eta(z) = q^{1/24}(1 - q - q^2 + O(q^5)), \tag{31.14}$$

we easily find that

$$\frac{\eta^4(z/7)}{\eta^4(z)} = q^{-1/7}(1 - 4q^{1/7} + O(q^{2/7})). \tag{31.15}$$

Hence, by (31.13) and (31.15), the difference of the left and right sides of (31.6) has a zero at ∞, and so (31.6) has been established.

Secondly, we apply Evans' theorem with $p = 7$, $s = 2$, $\varepsilon_1 = 1$, $\varepsilon_2 = -2$, $\beta_1 = 1$, and $\beta_2 = 2$, so that (31.4) is satisfied. Using (31.8) and (31.9), we conclude, by the same reasoning as before, that the left side of (31.7) belongs to $\{\Gamma^0(7), 0, 1\}$ and has no poles on $\mathscr{H}$ or at the cusp 0. Thus, We shall be done if we can prove that the difference of the left and right sides of (31.7) has a zero at ∞. By (31.10)–(31.12),

$$\frac{G_2(z)}{G_3^2(z)} + \frac{G_1(z)}{G_2^2(z)} + \frac{G_3(z)}{G_1^2(z)} = q^{-1/7}(1 + 2q^{1/7}) - 1 + O(q^{1/7})$$

$$= q^{-1/7} + 1 + O(q^{1/7}). \tag{31.16}$$

By (31.15),

$$5 + \frac{\eta^4(z/7)}{\eta^4(z)} = q^{-1/7} + 1 + O(q^{1/7}).$$

Thus, the difference of the two sides of (31.7) has a zero at ∞, and the proof of (31.7) is complete.

H. H. Chan [1] has employed (31.1) in giving a beautiful new proof of Ramanujan's congruence $p(7n + 5) \equiv 0 \pmod 7$ for the partition function $p(n)$.

Entry 32 (p. 300). *Let*

$$u = q^{1/56} f(-q^3, -q^4), \qquad v = q^{9/56} f(-q^2, -q^5),$$

and

$$w = q^{25/26} f(-q, -q^6).$$

Then

(i) $$uvw = q^{5/8} f(-q) f^2(-q^7),$$

(ii) $$\frac{u^2}{v} - \frac{v^2}{w} + \frac{w^2}{u} = 0,$$

and

(iii) $$\frac{v}{u^2} - \frac{w}{v^2} + \frac{u}{w^2} = \frac{f(-q)}{q^{7/8} f^2(-q^7)} \left(\frac{f^4(-q)}{f^4(-q^7)} + 13q + 49q^2 \frac{f^4(-q^7)}{f^4(-q)} \right)^{1/3}.$$

The identities (ii) and (iii) have forms different from any other theta-function identities in the notebooks [22]. A general theorem containing (ii) and (iii) as special cases has been proved by Berndt and L.-C. Zhang [2].

Part (i) is easily seen to be equivalent to Entry 18(iv) in Chapter 19.

PROOFS OF (ii) AND (iii). As in our account of Chapter 20 [6, pp. 337–345], we follow R. J. Evans [1] and define, for Im $z > 0$,

$$F(u, v; z) = -i \sum_{n=-\infty}^{\infty} (-1)^n \exp(i\pi z(n + u + \tfrac{1}{2})^2 + i\pi v(2n + u + 1)). \tag{32.1}$$

In particular, for $v = 0$,

$$F(u; z) := F(u, 0; z) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+u+1/2)^2/2}, \tag{32.2}$$

where $q = \exp(2\pi iz)$. It is easily checked that

$$f(-q^u, -q^{1-u}) = -iq^{-(u-1/2)^2/2} F(u; z). \tag{32.3}$$

Employing (32.3), we readily verify that

$$\frac{f^2(-q^{3/7}, -q^{4/7})}{f(-q^{2/7}, -q^{5/7})} = -iq^{1/56} \frac{F^2(3/7; z)}{F(2/7; z)}, \tag{32.4}$$

$$\frac{f^2(-q^{2/7}, -q^{5/7})}{f(-q^{1/7}, -q^{6/7})} = -iq^{1/56} \frac{F^2(2/7; z)}{F(1/7; z)}, \tag{32.5}$$

and

$$\frac{f^2(-q^{1/7}, -q^{6/7})}{f(-q^{3/7}, -q^{4/7})} = -iq^{-1/8} \frac{F^2(1/7; z)}{F(3/7; z)}. \tag{32.6}$$

Thus, by (32.4)–(32.6), (ii) may be transcribed into the proposed identity

$$g(z) := \frac{F^2(3/7; z)}{F(2/7; z)} - \frac{F^2(2/7; z)}{F(1/7; z)} + \frac{F^2(1/7; z)}{F(3/7; z)} \equiv 0. \tag{32.7}$$

Cubing both sides of (iii), recalling that $\eta(z) = q^{1/24} f(-q)$, and using (32.4)–(32.6), we find that (iii) is equivalent to the proposed identity

$$-i \frac{\eta^6(z)}{\eta^3(z/7)} \left\{ \frac{F(2/7; z)}{F^2(3/7; z)} - \frac{F(1/7; z)}{F^2(2/7; z)} + \frac{F(3/7; z)}{F^2(1/7; z)} \right\}^3$$
$$= \frac{\eta^4(z/7)}{\eta^4(z)} + 13 + 49 \frac{\eta^4(z)}{\eta^4(z/7)}. \tag{32.8}$$

Next, we examine (32.7) and (32.8) under modular transformations in $\Gamma^0(7)$. To do this, we need the transformation formula (Stark [1, eq. (17)]; Part III [6, p. 339, eqs. (0.54), (0.55)])

$$F(u, v; Vz) = v_\eta^3(V)\sqrt{cz + d}\, F(u_V, v_V; z), \tag{32.9}$$

where $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, v_η is the multiplier system for the Dedekind eta-function $\eta(z)$, and

$$(u_V, v_V) = (u, v)V = (au + cv, bu + dv). \tag{32.10}$$

We also shall need the translation formula (Evans [1, eq. (2.12)]; Part III [6, p. 338, eq. (0.45)])

$$F(u + r, v + s; z) = (-e^{\pi iu})^s(-e^{-\pi iv})^r(-1)^{rs}F(u, v; z), \tag{32.11}$$

where r and s are arbitrary integers.

Recall that g is defined in (32.7). Let $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^0(7)$. Then by (32.9)–(32.11), since $7 \mid b$,

$$\begin{aligned}
g(Vz) &= v_\eta^3(V)\sqrt{cz+d}\left(\frac{F^2(3a/7, 3b/7; z)}{F(2a/7, 2b/7; z)} - \frac{F^2(2a/7, 2b/7; z)}{F(a/7, b/7; z)}\right. \\
&\qquad \left. + \frac{F^2(a/7, b/7; z)}{F(3a/7, 3b/7; z)}\right) \\
&= v_\eta^3(V)\sqrt{cz+d}\left(\frac{(-e^{\pi i3a/7})^{6b/7}F^2(3a/7; z)}{(-e^{\pi i2a/7})^{2b/7}F(2a/7; z)}\right. \\
&\qquad - \frac{(-e^{\pi i2a/7})^{4b/7}F^2(2a/7; z)}{(-e^{\pi ia/7})^{b/7}F(a/7; z)} \\
&\qquad \left. + \frac{(-e^{\pi ia/7})^{2b/7}F^2(a/7; z)}{(-e^{\pi i3a/7})^{3b/7}F(3a/7; z)}\right) \\
&= v_\eta^3(V)\sqrt{cz+d}\left((-1)^{4b/7}e^{2\pi iab/7}\frac{F^2(3a/7; z)}{F(2a/7; z)}\right. \\
&\qquad - (-1)^{b/7}e^{\pi iab/7}\frac{F^2(2a/7; z)}{F(a/7; z)} \\
&\qquad \left. + (-1)^{-b/7}e^{-\pi iab/7}\frac{F^2(a/7; z)}{F(3a/7; z)}\right) \\
&= v_\eta^3(V)\sqrt{cz+d}\left(\frac{F^2(3a/7; z)}{F(2a/7; z)} - (-1)^{B(a+1)}\frac{F^2(2a/7; z)}{F(a/7; z)}\right. \\
&\qquad \left. + (-1)^{B(a+1)}\frac{F^2(a/7; z)}{F(3a/7; z)}\right),
\end{aligned} \tag{32.12}$$

where $b = 7B$. Since $(a, b) = 1$, $7 \nmid a$. We now examine the right side of (32.12) for each of the 12 possible residue classes of a modulo 14. To do this, we need the two consequences of (32.11) (Evans [1, eqs. (2.14), (2.15)]; Part III [6, p. 338, eqs. (0.46), (0.47)])

$$F(u+1; z) = -F(u; z)$$

and

$$F(-u; z) = -F(u; z).$$

After a time consuming, but straightforward examination of all 12 cases, we conclude that

$$g(Vz) = v_\eta^3(V)\sqrt{cz+d}\left(\frac{a}{7}\right)g(z), \tag{32.13}$$

where $\left(\frac{a}{7}\right)$ denotes the Legendre symbol. By the Jacobi triple product identity, $F(u; z)$ is an analytic, nonvanishing function of z on $\mathscr{H}$. Thus, g is a modular form on $\Gamma^0(7)$ of weight $\frac{1}{2}$ and multiplier system $v_\eta^3(V)\left(\frac{a}{7}\right)$ that has no poles on $\mathscr{H}$.

Recall (Part III [6, p. 330, eq. (0.14)]; Knopp [1, p. 51]) that the multiplier system v_η is given by

$$v_\eta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\dfrac{d}{|c|}\right)\zeta_{24}^{\{bd(1-c^2)+c(a+d)-3c\}}, & \text{if } c \text{ is odd,} \\ \left(\dfrac{c}{|d|}\right)\zeta_{24}^{\{ac(1-d^2)+d(b-c)+3(d-1)\}}, & \text{if } d \text{ is odd and either } c \geq 0 \text{ or } d \geq 0, \\ -\left(\dfrac{c}{|d|}\right)\zeta_{24}^{\{ac(1-d^2)+d(b-c)+3(d-1)\}}, & \text{if } d \text{ is odd, } c < 0, d < 0, \end{cases} \tag{32.14}$$

where $\zeta_n = \exp(2\pi i/n)$. Letting $V(z) = -1/z$, we see from (32.9), (32.10), and (32.14) that

$$F(u; -1/z) = e^{-3\pi i/4}\sqrt{z}F(0, -u; z). \tag{32.15}$$

Recall also that (Evans [1, eq. (4.18)]; Part III [6, p. 341, eq. (0.70)])

$$q^{-1/8}F(0, -m/7; z) = -2\sum_{n=0}^{\infty}(-1)^n \sin\left(\frac{\pi m(2n+1)}{7}\right)q^{(n^2+n)/2}, \tag{32.16}$$

where m is any integer. It follows from (32.15) and (32.16) that $g(z)$ has no pole at the cusp 0.

If $h(z)$ denotes the expression within curly brackets in (32.8), then by identical reasoning as that used above,

$$h(Vz) = v_\eta^{-3}(V)(cz+d)^{-1/2}\left(\frac{a}{7}\right)h(z), \tag{32.17}$$

where $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^0(7)$.

We next examine the effect of transformations in $\Gamma^0(7)$ acting on $\eta^6(z)/\eta^3(z/7)$, which appears in (32.8). In fact, it will also be convenient to multiply (32.7) by $\eta^3(z/7)/\eta^6(z)$.

Now, for $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$,

$$\eta(Vz) = v_\eta(V)\sqrt{cz+d}\,\eta(z). \tag{32.18}$$

For $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^0(7)$,

$$\eta(Vz/7) = v_\eta\begin{pmatrix} a & b/7 \\ 7c & d \end{pmatrix}\sqrt{cz+d}\,\eta(z/7). \tag{32.19}$$

We shall prove that

$$v_\eta^3\begin{pmatrix} a & b/7 \\ 7c & d \end{pmatrix} = \left(\frac{a}{7}\right)v_\eta^{-3}(V). \tag{32.20}$$

Suppose first that $c = 2n+1$ is odd. Put $b = 7B$. By (32.14),

$$\begin{aligned}
v_\eta^3\begin{pmatrix} a & b/7 \\ 7c & d \end{pmatrix} &= \left(\frac{d}{7|c|}\right)\zeta_8^{(bd/7)(1-49c^2)+7c(a+d)-21c} \\
&= \left(\frac{a}{7}\right)\left(\frac{ad}{7}\right)\left(\frac{d}{|c|}\right)\zeta_8^{(bd/7)(1-c^2)-c(a+d)+3c} \\
&= \left(\frac{a}{7}\right)\left(\frac{1+bc}{7}\right)\left(\frac{d}{|c|}\right)\zeta_8^{-Bd4n(n+1)-c(a+d)+3c} \\
&= \left(\frac{a}{7}\right)\left(\frac{d}{|c|}\right)\zeta_8^{bd4n(n+1)-c(a+d)+3c} \\
&= \left(\frac{a}{7}\right)\left(\frac{d}{|c|}\right)\zeta_8^{-bd(1-c^2)-c(a+d)+3c} \\
&= \left(\frac{a}{7}\right)v_\eta^{-3}(V),
\end{aligned}$$

which proves (32.20) in the case that c is odd.

Second, suppose that $d = 2n + 1$ is odd. Suppose also that $c \geq 0$; the proof is identical if $c < 0$. By (32.14),

$$v_\eta^3\begin{pmatrix} a & b/7 \\ 7c & d \end{pmatrix} = \left(\frac{7c}{|d|}\right)\zeta_8^{7ac(1-d^2)+d(b/7-7c)+3(d-1)}$$

$$= \left(\frac{7c}{|d|}\right)\zeta_8^{-ac(1-d^2)+dc+dB+3(d-1)}. \tag{32.21}$$

Since $7 \nmid d$, because $(7c, d) = 1$, we may apply the law of quadratic reciprocity to deduce that

$$\left(\frac{a}{7}\right)\left(\frac{7c}{|d|}\right) = \left(\frac{c}{|d|}\right)\left(\frac{a}{7}\right)\left(\frac{7}{|d|}\right)$$

$$= \left(\frac{c}{|d|}\right)\left(\frac{a}{7}\right)\left(\frac{|d|}{7}\right)(-1)^{3(|d|-1)/2}$$

$$= \left(\frac{c}{|d|}\right)\left(\frac{d}{7}\right)\left(\frac{|d|}{7}\right)(-1)^{3(d \operatorname{sgn} d-1)/2}$$

$$= \left(\frac{c}{|d|}\right)(-1)^{(\operatorname{sgn} d-1)/2+3(d \operatorname{sgn} d-1)/2}$$

$$= \left(\frac{c}{|d|}\right)(-1)^{\operatorname{sgn} d(1+3d)/2}$$

$$= \left(\frac{c}{|d|}\right)(-1)^{3n \operatorname{sgn} d}$$

$$= \left(\frac{c}{|d|}\right)(-1)^n. \tag{32.22}$$

Now,

$$\zeta_8^{bd+Bd+6(d-1)} = \zeta_8^{7Bd+Bd+12n} = \zeta_8^{4n} = (-1)^n. \tag{32.23}$$

Putting (32.22) and (32.23) in (32.21), we conclude that

$$v_\eta^3\begin{pmatrix} a & b/7 \\ 7c & d \end{pmatrix} = \left(\frac{a}{7}\right)\left(\frac{c}{|d|}\right)\zeta_8^{-ac(1-d^2)+dc-bd-3(d-1)}$$

$$= \left(\frac{a}{7}\right)v_\eta^{-3}(V),$$

and again (32.20) is established.

Hence, from (32.19) and (32.20),

$$\eta^3(Vz/7) = \left(\frac{a}{7}\right)v_\eta^{-3}(V)(cz+d)^{3/2}\eta^3(z/7). \tag{32.24}$$

Combining (32.18) and (32.24), we conclude that

$$\frac{\eta^6(Vz)}{\eta^3(Vz/7)} = \left(\frac{a}{7}\right) v_\eta^9(V)(cz + d)^{3/2} \frac{\eta^6(z)}{\eta^3(z/7)}. \tag{32.25}$$

In order to prove (32.7), it suffices to prove that

$$G(z) := \frac{\eta^3(z/7)}{\eta^6(z)} g^3(z) \equiv 0. \tag{32.26}$$

By (32.25), (32.13), and the remarks made after (32.13), $G(z) \in \{\Gamma^0(7), 0, 1\}$ and has no poles on $\mathscr{H}$ or at the cusp 0. By the discussion from Part III [6, p. 342] or from Evans' paper [1, pp. 111–112], all we need do to prove (32.26) is to show that $G(z)$ has a zero at ∞.

From (32.2),

$$F(1/7; z) = iq^{25/392}(1 - q^{1/7} + O(q^{6/7})), \tag{32.27}$$

$$F(2/7; z) = iq^{9/392}(1 - q^{2/7} + O(q^{5/7})), \tag{32.28}$$

and

$$F(3/7; z) = iq^{1/392}(1 - q^{3/7} + O(q^{4/7})). \tag{32.29}$$

It follows that

$$\begin{aligned} g(z) &= i(q^{-1/56} + O(q^{23/56}) - q^{-1/56} + O(q^{15/56}) + O(q^{1/8})) \\ &= O(q^{1/8}). \end{aligned} \tag{32.30}$$

From (31.14), it follows easily that

$$\frac{\eta(z/7)}{\eta^2(z)} = q^{-13/168}(1 - q^{1/7} + O(q^{2/7})). \tag{32.31}$$

Hence, from (32.30) and (32.31),

$$G(z) = O(q^{1/7}).$$

Thus, $G(z)$ has a zero at the cusp $z = \infty$. This completes the proof of (ii).

Recall that $h(z)$ is the expression within curly brackets in (32.8). By (32.17) and (32.25),

$$H_1(z) := -i \frac{\eta^6(z)}{\eta^3(z/7)} h^3(z) \in \{\Gamma^0(7), 0, 1\}. \tag{32.32}$$

Furthermore, $H_1(z)$ is analytic on $\mathscr{H}$. Also set

$$H_2(z) := \frac{\eta^4(z/7)}{\eta^4(z)} + 13 + 49 \frac{\eta^4(z)}{\eta^4(z/7)}. \tag{32.33}$$

We shall examine $H_1(z)$ and $H_2(z)$ in a neighborhood of the cusp $z = 0$. To do this, we need the evaluation

$$\frac{\sin(2\pi/7)}{\sin^2(3\pi/7)} - \frac{\sin(\pi/7)}{\sin^2(2\pi/7)} + \frac{\sin(3\pi/7)}{\sin^2(\pi/7)} = 2\sqrt{7}, \tag{32.34}$$

which can be achieved by a laborious computation using the exponential representations for $\sin(k\pi/7)$, $1 \leq k \leq 3$. We also need the well-known transformation

$$\eta(-1/z) = e^{-\pi i/4}\sqrt{z}\eta(z) = e^{-\pi i/4}\sqrt{z}q^{1/24}(q;q)_\infty, \tag{32.35}$$

which is a consequence of (32.18) and (32.14).

Hence, from (32.32), (32.15), (32.16), (32.35), and (32.34),

$$\begin{aligned} H_1\left(-\frac{1}{z}\right) &= \left\{\frac{\sin(2\pi/7)}{\sin^2(3\pi/7)} - \frac{\sin(\pi/7)}{\sin^2(2\pi/7)} + \frac{\sin(3\pi/7)}{\sin^2(\pi/7)}\right\}^3 \frac{1}{56\sqrt{7}q} \frac{(q;q)_\infty^6}{(q^7;q^7)_\infty^3} + O(1) \\ &= \frac{1}{q} + O(1), \end{aligned} \tag{32.36}$$

as q tends to 0. From (32.35),

$$\frac{\eta^4(-1/z)}{\eta^4(-1/(7z))} = \frac{(q^4;q^4)_\infty^4}{49q(q^7;q^7)_\infty^4}. \tag{32.37}$$

So, from (32.33) and (32.37),

$$H_2\left(-\frac{1}{z}\right) = \frac{1}{q} + O(1) \tag{32.38}$$

as q tends to 0. Thus,

$$H(z) := H_1(z) - H_2(z) \tag{32.39}$$

has no pole at the cusp $z = 0$. It remains to show that $H(z)$ has a zero at the cusp ∞.

From (32.27)–(32.29),

$$\frac{F(2/7;z)}{F^2(3/7;z)} = -iq^{1/56}(1 + O(q^{2/7})),$$

$$\frac{F(1/7;z)}{F^2(2/7;z)} = -iq^{1/56}(1 - q^{1/7} + O(q^{2/7})),$$

and

$$\frac{F(3/7;z)}{F^2(1/7;z)} = -iq^{-1/8}(1 + 2q^{1/7} + O(q^{2/7})).$$

Thus, recalling the definition of $h(z)$ in (32.8), we find that

$$h(z) = -iq^{-1/8}(1 + 2q^{1/7} + O(q^{2/7}))$$

and so

$$h^3(z) = iq^{-3/8}(1 + 6q^{1/7} + O(q^{2/7})). \tag{32.40}$$

By (31.14), we readily see that

$$\frac{\eta^6(z)}{\eta^3(z/7)} = q^{13/56}(1 + 3q^{1/7} + O(q^{2/7})). \tag{32.41}$$

Hence, from (32.40) and (32.41),

$$\begin{aligned} H_1(z) &= q^{-1/7}(1 + 3q^{1/7} + O(q^{2/7}))(1 + 6q^{1/7} + O(q^{2/7})) \\ &= q^{-1/7}(1 + 9q^{1/7} + O(q^{2/7})). \end{aligned} \tag{32.42}$$

Next, from (31.14),

$$\frac{\eta^4(z/7)}{\eta^4(z)} = q^{-1/7}(1 - 4q^{1/7} + O(q^{2/7}))$$

and

$$\frac{\eta^4(z)}{\eta^4(z/7)} = O(q^{1/7}).$$

Hence,

$$\begin{aligned} H_2(z) &= q^{-1/7}(1 - 4q^{1/7} + O(q^{2/7})) + 13 + 49 \cdot O(q^{1/7}) \\ &= q^{-1/7}(1 + 9q^{1/7} + O(q^{2/7})). \end{aligned} \tag{32.43}$$

Therefore, by (32.42) and (32.43), $H(z) = O(q^{1/7})$. Hence, the proof of (iii) has been completed.

We conclude this section by noting a consequence of (ii), which is equivalent to (32.7). It follows trivially that $g(-1/z) \equiv 0$. In particular, the leading coefficient in the q-expansion of $g(-1/z)$ is equal to 0. Using (32.15) and (32.16), we may then easily deduce the following nontrivial trigonometric identity.

Corollary 32.1. *We have*

$$\frac{\sin^2(3\pi/7)}{\sin(2\pi/7)} - \frac{\sin^2(2\pi/7)}{\sin(\pi/7)} + \frac{\sin^2(\pi/7)}{\sin(3\pi/7)} = 0.$$

Entry 33 (p. 310). *For* $|q| < 1$,

(i) $$\frac{\varphi^3(q^{1/3})}{\varphi(q)} = \frac{\varphi^3(q)}{\varphi(q^3)} + 6q^{1/3}\frac{f^3(q^3)}{f(q)} + 12q^{2/3}\frac{f^3(-q^6)}{f(-q^2)}$$

and

(ii) $$\frac{\psi^3(q^{1/3})}{\psi(q)} = \frac{\psi^3(q)}{\psi(q^3)} + 3q^{1/3}\frac{f^3(-q^3)}{f(-q)} + 3q^{2/3}\frac{f^3(-q^6)}{f(-q^2)}.$$

Proof. For convenience, we replace q by q^3. Let α, β, and γ have degrees 1, 3, and 9, respectively. Let m and m' denote the multipliers associated with the pairs α, β and β, γ, respectively. Recall that $z_n = \varphi^2(q^n)$.

We first prove (i). By (3.10), (3.11), and (3.13) in Chapter 20 (Part III [6, p. 354]),

$$\begin{aligned}\frac{\varphi^3(q^3)}{\varphi^3(q^9)}\left(\frac{\varphi^3(q)}{\varphi^3(q^3)} - \frac{\varphi(q^3)}{\varphi(q^9)}\right) &= m'^{3/2}(m^{3/2} - m'^{1/2})\\ &= (1+2t)^3 - (1+8t^3)\\ &= 6t + 12t^2\\ &= 6\left(\frac{\gamma^3(1-\gamma)^3}{256\beta(1-\beta)}\right)^{1/24} + 12\left(\frac{\gamma^3(1-\gamma)^3}{256\beta(1-\beta)}\right)^{1/12}. \end{aligned} \tag{33.1}$$

By Entries 12(i) and (iii), respectively, in Chapter 17 (Part III [6, p. 124]),

$$q\frac{f^3(q^9)}{f(q^3)} = \frac{z_9^{3/2}}{z_3^{1/2}}\left(\frac{\gamma^3(1-\gamma)^3}{256\beta(1-\beta)}\right)^{1/24} \tag{33.2}$$

and

$$q^2\frac{f^3(-q^{18})}{f(-q^6)} = \frac{z_9^{3/2}}{z_3^{1/2}}\left(\frac{\gamma^3(1-\gamma)^3}{256\beta(1-\beta)}\right)^{1/12}. \tag{33.3}$$

From (33.1)–(33.3), it transpires that

$$\begin{aligned}6q\frac{f^3(q^9)}{f(q^3)} + 12q^2\frac{f^3(-q^{18})}{f(-q^6)} &= \frac{z_9^{3/2}}{z_3^{1/2}}\frac{\varphi^3(q^3)}{\varphi^3(q^9)}\left(\frac{\varphi^3(q)}{\varphi^3(q^3)} - \frac{\varphi(q^3)}{\varphi(q^9)}\right)\\ &= z_3\left(\frac{\varphi^3(q)}{\varphi^3(q^3)} - \frac{\varphi(q^3)}{\varphi(q^9)}\right)\\ &= \frac{\varphi^3(q)}{\varphi(q^3)} - \frac{\varphi^3(q^3)}{\varphi(q^9)},\end{aligned}$$

which completes the proof of (i).

Commencing the proof of (ii), we use Entry 11(ii) in Chapter 17 and then (3.7)–(3.11) and (3.13) in Chapter 20 (Part III [6, pp. 123, 354]) to deduce that

$$\begin{aligned}
&\frac{\varphi(q^3)}{\varphi^3(q^9)}\left(\frac{\psi^3(-q^3)}{\psi(-q^9)}-\frac{\psi^3(-q)}{\psi(-q^3)}\right)\\
&\quad=\frac{z_3^2}{2z_9^2}\left(\frac{\beta^3(1-\beta)^3}{\gamma(1-\gamma)}\right)^{1/8}-\frac{z_1^{3/2}}{2z_9^{3/2}}\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8}\\
&\quad=\frac{m'^2}{2}\left(\frac{\beta^3(1-\beta)^3}{\gamma(1-\gamma)}\right)^{1/8}-\frac{(mm')^{3/2}}{2}\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8}\\
&\quad=\frac{1+8t^3}{2}\left(16t^3\left(\frac{1-t^3}{1+8t^3}\right)^3\right)^{3/8}\left(\frac{1+8t^3}{16t^9(1-t^3)}\right)^{1/8}\\
&\quad\quad-\frac{(1+2t)^3}{2}\left(16t\left(\frac{1-t}{1+2t}\right)^8\frac{1-t^3}{1+8t^3}\right)^{3/8}\left(\frac{(1+8t^3)^3}{16t^3(1-t^3)^3}\right)^{1/8}\\
&\quad=(1-t^3)-(1-t)^3\\
&\quad=3t-3t^2\\
&\quad=3\left(\frac{\gamma^3(1-\gamma)^3}{256\beta(1-\beta)}\right)^{1/24}-3\left(\frac{\gamma^3(1-\gamma)^3}{256\beta(1-\beta)}\right)^{1/12}.
\end{aligned}\tag{33.4}$$

On the other hand, by Entries 12(i), (iii) in Chapter 17 (Part III [6, p. 124]),

$$\begin{aligned}
&3q\frac{f^3(q^9)}{f(q^3)}-3q^2\frac{f^3(-q^{18})}{f(-q^6)}\\
&\quad=\frac{3z_9^{3/2}}{z_3^{1/2}}\left(\frac{\gamma^3(1-\gamma)^3}{256\beta(1-\beta)}\right)^{1/24}-\frac{3z_9^{3/2}}{z_3^{1/2}}\left(\frac{\gamma^3(1-\gamma)^3}{256\beta(1-\beta)}\right)^{1/12}.
\end{aligned}\tag{33.5}$$

Combining (33.4) and (33.5), we conclude that

$$\frac{\psi^3(-q^3)}{\psi(-q^9)}-\frac{\psi^3(-q)}{\psi(-q^3)}=3q\frac{f^3(q^9)}{f(q^3)}-3q^2\frac{f^3(-q^{18})}{f(-q^6)}.$$

Replacing q by $-q$ above, we complete the proof of (ii).

Entry 34 (p. 314). *Let*

$$u=\frac{f(-q)f(-q^2)}{q^{1/4}f(-q^3)f(-q^6)}\quad\text{and}\quad v=\frac{f(-q^{1/3})f(-q^{2/3})}{q^{1/3}f(-q^3)f(-q^6)}.$$

Then

$$u^4=v^3+3v^2+9v.\tag{34.1}$$

PROOF. As in the previous proof, let α, β, and γ have degrees 1, 3, and 9, respectively, and let m and m' denote the multipliers associated with the pairs α, β and β, γ, respectively. Let

$$U^4 = \frac{f^4(q)f^4(-q^2)}{qf^4(q^3)f^4(-q^6)} \quad \text{and} \quad V = \frac{f(q^{1/3})f(-q^{2/3})}{q^{1/3}f(q^3)f(-q^6)}.$$

Thus, replacing q by $-q$ in (34.1), we observe that it suffices to prove that

$$U^4 = V^3 - 3V^2 + 9V. \tag{34.2}$$

By Entries 12(i), (iii) in Chapter 17 (Part III [6, p. 124]),

$$U^4 = m'^4\left(\frac{\beta(1-\beta)}{\gamma(1-\gamma)}\right)^{1/2} \tag{34.3}$$

and

$$V = mm'\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/8}. \tag{34.4}$$

Employing (3.8), (3.9), and (3.11) of Chapter 20 (Part III [6, p. 354]) in (34.3), we find that

$$U^4 = \frac{(1+8t^3)(1-t^3)}{t^3}, \tag{34.5}$$

and utilizing (3.7) and (3.9)–(3.11) from Chapter 20 in (34.4), we see that

$$\begin{aligned}
V^3 - 3V^2 + 9V &= (1+2t)^6\left(16t\left(\frac{1-t}{1+2t}\right)^8 \frac{1-t^3}{1+8t^3}\,\frac{1+8t^3}{16t^9(1-t^3)}\right)^{3/8} \\
&\quad -3(1+2t)^4\left(16t\left(\frac{1-t}{1+2t}\right)^8 \frac{1-t^3}{1+8t^3}\,\frac{1+8t^3}{16t^9(1-t^3)}\right)^{1/4} \\
&\quad +9(1+2t)^2\left(16t\left(\frac{1-t}{1+2t}\right)^8 \frac{1-t^3}{1+8t^3}\,\frac{1+8t^3}{16t^9(1-t^3)}\right)^{1/8} \\
&= \frac{(1+2t)^3(1-t)^3}{t^3} - 3\,\frac{(1+2t)^2(1-t)^2}{t^2} + 9\,\frac{(1+2t)(1-t)}{t} \\
&= \frac{(1+2t)(1-t)}{t^3}(4t^4 + 2t^3 + 3t^2 - t + 1) \\
&= \frac{(1+2t)(1-t)}{t^3}(4t^2 - 2t + 1)(t^2 + t + 1) \\
&= \frac{(1+8t^3)(1-t^3)}{t^3}.
\end{aligned} \tag{34.6}$$

Thus, (34.2) follows immediately from (34.5) and (34.6), and the proof is finished.

Entry 35 (p. 321). *We have*

$$\frac{f(-q^{3/5})f^3(-q^5)}{f(-q^{15})} = f^3(-q^7, -q^8) - q^{3/5}f^3(-q^5) - q^{6/5}f^3(-q^4, -q^{11})$$
$$+q^3f^3(-q^2, -q^{13}) + q^{21/5}f^3(-q, -q^{14}).$$

Entry 36 (p. 321). *We have*

(i) $$f^3(-q^7, -q^8) + q^3f^3(-q^2, -q^{13}) = f^3(-q^5)\frac{f(-q^6, -q^9)}{f(-q^3, -q^{12})}$$

and

(ii) $$f^3(-q^4, -q^{11}) - q^3f^3(-q, -q^{14}) = f^3(-q^5)\frac{f(-q^3, -q^{12})}{f(-q^6, -q^9)}.$$

We prove Entry 36 first, because Entry 35 follows from Entry 36.

Proof of Entry 36. Apply the quintuple product identity, (38.2) of Chapter 16 (Part III [6, p. 80]), with q replaced by $q^{5/2}$ and $B = -\omega q^{3/2}$ and $-\omega q^{1/2}$, in turn, where ω is a cube root of unity. Accordingly, we find that

$$f(-q^7, -q^8) - \omega^2q^3f(-q^{-2}, -q^{17}) = f(-q^5)\frac{f(-\omega^2q^3, -\omega q^2)}{f(-\omega q^4, -\omega^2q)} \tag{36.1}$$

and

$$f(-q^4, -q^{11}) - \omega^2qf(-q, -q^{14}) = f(-q^5)\frac{f(-\omega^2q, -\omega q^4)}{f(-\omega q^3, -\omega^2q^2)}, \tag{36.2}$$

respectively.

Apply Entry 18(iv) of Chapter 16 (Part III [6, p. 34]), with $n = 1$, to $f(-q^{-2}, -q^{17})$. Then take the product of each side of (36.1) over all three cube roots of unity. We therefore find that

$$f^3(-q^7, -q^8) + q^3f^3(-q^{13}, -q^2) = f^3(-q^5)\prod_{\omega}\frac{f(-\omega^2q^3, -\omega q^2)}{f(-\omega q^4, -\omega^2q)}. \tag{36.3}$$

Applying the Jacobi triple product identity (Entry 19 of Chapter 16) to each theta-function on the right side of (36.3), we find, after simplification, that

$$f^3(-q^7, -q^8) + q^3f^3(-q^2, -q^{13}) = f^3(-q^5)\frac{(q^9; q^{15})_\infty(q^6; q^{15})_\infty}{(q^{12}; q^{15})_\infty(q^3; q^{15})_\infty}$$
$$= f^3(-q^5)\frac{f(-q^6, -q^9)}{f(-q^3, -q^{12})},$$

by two more applications of the Jacobi triple product identity. Thus, (i) has been established.

The proof of (ii) emanating from (36.2) is almost identical, except that no application of Entry 18(iv) of Chapter 16 is necessary.

PROOF OF ENTRY 35. We recall Entry 12(v) of Chapter 19 (Part III [6, p. 270]), viz.,

$$\frac{f(-q^{1/5})}{f(-q^5)} = \frac{f(-q^2, -q^3)}{f(-q, -q^4)} - q^{1/5} - q^{2/5} \frac{f(-q, -q^4)}{f(-q^2, -q^3)}.$$

Replacing q by q^3 and then multiplying both sides by $f^3(-q^5)$, we derive that

$$\begin{aligned}\frac{f(-q^{3/5})f^3(-q^5)}{f(-q^{15})} &= \frac{f(-q^6, -q^9)}{f(-q^3, -q^{12})} f^3(-q^5) - q^{3/5} f^3(-q^5) \\ &\quad - q^{6/5} \frac{f(-q^3, -q^{12})}{f(-q^6, -q^9)} f^3(-q^5).\end{aligned}$$

If we now substitute the left sides of Entries 36(i), (ii) into the right side above, we complete the proof.

Entry 37 (p. 328). *For* $|q| < 1$,

$$\varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6) = 1 + 6 \sum_{n=0}^{\infty} \left(\frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right).$$

Entry 37 was proved in the course of proving Entry 3(i) of Chapter 21. In particular, see (3.6) there (Part III [6, p. 462]).

The next result is a generalization of Entry 11(i) in Chapter 19 (Part III [6, p. 265]).

Entry 38 (p. 329). *There exist functions* μ *and* ν *such that*

(i) $$f(a, b) = f(a^{15}b^{10}, a^{10}b^{15}) + \mu^{1/5} + \nu^{1/5},$$

where

(ii) $$(\mu\nu)^{1/5} = f(a^4b, ab^4)f(a^3b^2, a^2b^3) - f^2(a^{15}b^{10}, a^{10}b^{15})$$

and

(iii) $$\begin{aligned}\mu + \nu = {} & \frac{f(a^5, b^5)}{f(a^{15}b^{10}, a^{10}b^{15})} f^5(a^3b^2, a^2b^3) \\ & - 5f^2(a^4b, ab^4)f^2(a^3b^2, a^2b^3)f(a^{15}b^{10}, a^{10}b^{15}) \\ & + 15f(a^4b, ab^4)f(a^3b^2, a^2b^3)f^3(a^{15}b^{10}, a^{10}b^{15}) \\ & - 11f^5(a^{15}b^{10}, a^{10}b^{15}).\end{aligned}$$

PROOF. We apply Entry 31 in Chapter 16 with $U_k = a^{k(k+1)/2}b^{k(k-1)/2}$, $V_k = a^{k(k-1)/2}b^{k(k+1)/2}$, and $n = 5$, where $k \geq 0$. Using the formulation given in (31.2) of Chapter 16 (Part III [6, p. 48]), we find that

$$f(a, b) = f(a^{15}b^{10}, a^{10}b^{15}) + af(a^5b^{10}, a^{20}b^{15}) + bf(a^{10}b^5, a^{15}b^{20}) + a^3bf(b^5, a^{25}b^{20}) + ab^3f(a^5, a^{20}b^{25}). \tag{38.1}$$

Examining (i), we see that (38.1) suggests taking

$$\mu^{1/5} = af(a^5b^{10}, a^{20}b^{15}) + bf(a^{10}b^5, a^{15}b^{20}) \tag{38.2}$$

and

$$\nu^{1/5} = a^3bf(b^5, a^{25}b^{20}) + ab^3f(a^5, a^{20}b^{25}), \tag{38.3}$$

where we choose those branches which are positive when a and b are positive and small. It remains to show that these choices of μ and ν satisfy (ii) and (iii).

To prove (ii), we work from first principles. It is evident that

$$f(xA, x/A)f(xB, x/B) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x^{m^2+n^2}A^mB^n, \qquad |x| < 1.$$

Now suppose that

$$2m + n \equiv 0, \ \pm 1, \quad \text{or} \quad \pm 2 \ (\text{mod } 5).$$

Then since $2n - m = 2(2m + n) - 5m$, it is clear that

$$2n - m \equiv 0, \ \pm 2, \quad \text{or} \quad \mp 1 \ (\text{mod } 5),$$

respectively. Hence, for any given pair of integral values of m and n, integral values of p and q can be found to satisfy one pair of the five pairs of simultaneous equations

$$\left.\begin{aligned} 2m + n &= 5p \\ 2n - m &= 5q \end{aligned}\right\}, \qquad \left.\begin{aligned} 2m + n &= 5p \pm 1 \\ 2n - m &= 5q \pm 2 \end{aligned}\right\}, \qquad \left.\begin{aligned} 2m + n &= 5p \pm 2 \\ 2n - m &= 5q \mp 1 \end{aligned}\right\}.$$

Conversely, given any pair of integral values of p and q, each of the five sets of simultaneous equations has a unique integral solution pair m, n. Hence,

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x^{m^2+n^2}A^mB^n = &\sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} x^{5(p^2+q^2)}A^{2p-q}B^{p+2q} \\ &+ \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} x^{5(p^2+q^2)+2p+4q+1}A^{2p-q}B^{p+2q+1} \\ &+ \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} x^{5(p^2+q^2)-2p-4q+1}A^{2p-q}B^{p+2q-1} \end{aligned}$$

$$+\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} x^{5(p^2+q^2)+4p-2q+1}A^{2p-q+1}B^{p+2q}$$

$$+\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} x^{5(p^2+q^2)-4p+2q+1}A^{2p-q-1}B^{p+2q},$$

that is to say,

$$\begin{aligned} f(xA, x/A)f(xB, x/B) &= f(x^5A^2B, x^5A^{-2}B^{-1})f(x^5A^{-1}B^2, x^5AB^{-2}) \\ &\quad + xBf(x^7A^2B, x^3A^{-2}B^{-1})f(x^9A^{-1}B^2, xAB^{-2}) \\ &\quad + xB^{-1}f(x^3A^2B, x^7A^{-2}B^{-1})f(xA^{-1}B^2, x^9AB^{-2}) \\ &\quad + xAf(x^9A^2B, xA^{-2}B^{-1})f(x^3A^{-1}B^2, x^7AB^{-2}) \\ &\quad + xA^{-1}f(xA^2B, x^9A^{-2}B^{-1})f(x^7A^{-1}B^2, x^3AB^{-2}). \end{aligned} \tag{38.4}$$

Now let $x = (ab)^{5/2}$, $A = (a/b)^{1/2}$, and $B = (a/b)^{3/2}$. Then (38.4) yields

$$\begin{aligned} f(a^3b^2, a^2b^3)f(a^4b, ab^4) &= f^2(a^{15}b^{10}, a^{10}b^{15}) \\ &\quad + a^4bf(a^{20}b^{15}, a^5b^{10})f(a^{25}b^{20}, b^5) \\ &\quad + ab^4f(a^{10}b^5, a^{15}b^{20})f(a^5, a^{20}b^{25}) \\ &\quad + a^3b^2f(a^{25}b^{20}, b^5)f(a^{10}b^5, a^{15}b^{20}) \\ &\quad + a^2b^3f(a^5, a^{20}b^{25})f(a^{20}b^{15}, a^5b^{10}) \\ &= f^2(a^{15}b^{10}, a^{10}b^{15}) + (\mu\nu)^{1/5}, \end{aligned}$$

by (38.2) and (38.3). Thus, (ii) is established.

Lastly, we prove (iii). Let ε denote a fifth root of unity. Replacing a and b by εa and εb, respectively, in (i), we find that

$$f(\varepsilon a, \varepsilon b) = f(a^{15}b^{10}, a^{10}b^{15}) + \varepsilon\mu^{1/5} + \varepsilon^4\nu^{1/5}. \tag{38.5}$$

We take the product of both sides of (38.5) over all five fifth roots of unity and employ (ii) to derive that

$$\begin{aligned} \prod_{\varepsilon} f(\varepsilon a, \varepsilon b) &= \prod_{\varepsilon} \{f(a^{15}b^{10}, a^{10}b^{15}) + \varepsilon\mu^{1/5} + \varepsilon^4\nu^{1/5}\} \\ &= f^5(a^{15}b^{10}, a^{10}b^{15}) + \mu + \nu - 5f^3(a^{15}b^{10}, a^{10}b^{15})(\mu\nu)^{1/5} \\ &\quad + 5f(a^{15}b^{10}, a^{10}b^{15})(\mu\nu)^{2/5} \\ &= f^5(a^{15}b^{10}, a^{10}b^{15}) + \mu + \nu \\ &\quad - 5f^3(a^{15}b^{10}, a^{10}b^{15})\{f(a^4b, ab^4)f(a^3b^2, a^2b^3) \\ &\quad - f^2(a^{15}b^{10}, a^{10}b^{15})\} \\ &\quad + 5f(a^{15}b^{10}, a^{10}b^{15})\{f(a^4b, ab^4)f(a^3b^2, a^2b^3) \\ &\quad - f^2(a^{15}b^{10}, a^{10}b^{15})\}^2 \end{aligned}$$

$$\begin{aligned}&= \mu + \nu + 11f^5(a^{15}b^{10}, a^{10}b^{15})\\ &\quad -15f^3(a^{15}b^{10}, a^{10}b^{15})f(a^4b, ab^4)f(a^3b^2, a^2b^3)\\ &\quad + 5f(a^{15}b^{10}, a^{10}b^{15})f^2(a^4b, ab^4)f^2(a^3b^2, a^2b^3).\end{aligned} \tag{38.6}$$

Comparing (38.6) with (iii), we shall be done if we can show that

$$\prod_{\varepsilon} f(\varepsilon a, \varepsilon b) = \frac{f(a^5, b^5)f^5(a^3b^2, a^2b^3)}{f(a^{15}b^{10}, a^{10}b^{15})}. \tag{38.7}$$

Expand each of the five theta-functions on the left side by the Jacobi triple product identity, Entry 19 of Chapter 16. Direct calculations show that

$$\begin{aligned}&(-a; ab)_\infty(-\varepsilon a; \varepsilon^2 ab)_\infty(-\varepsilon^2 a; \varepsilon^4 ab)_\infty(-\varepsilon^3 a; \varepsilon ab)_\infty(-\varepsilon^4 a; \varepsilon^3 ab)_\infty\\ &\qquad = \frac{(-a^5; a^5b^5)_\infty(-a^3b^2; a^5b^5)_\infty^5}{(-a^{15}b^{10}; a^{25}b^{25})_\infty},\end{aligned}$$

$$\begin{aligned}&(-b; ab)_\infty(-\varepsilon b; \varepsilon^2 ab)_\infty(-\varepsilon^2 b; \varepsilon^4 ab)_\infty(-\varepsilon^3 b; \varepsilon ab)_\infty(-\varepsilon^4 b; \varepsilon^3 ab)_\infty\\ &\qquad = \frac{(-b^5; a^5b^5)_\infty(-a^2b^3; a^5b^5)_\infty^5}{(-a^{10}b^{15}; a^{25}b^{25})_\infty},\end{aligned}$$

and

$$\begin{aligned}&(ab; ab)_\infty(\varepsilon^2 ab; \varepsilon^2 ab)_\infty(\varepsilon^4 ab; \varepsilon^4 ab)_\infty(\varepsilon ab; \varepsilon ab)_\infty(\varepsilon^3 ab; \varepsilon^3 ab)_\infty\\ &\qquad = \frac{(a^5b^5; a^5b^5)_\infty^6}{(a^{25}b^{25}; a^{25}b^{25})_\infty}.\end{aligned}$$

Multiplying together the last three equalities, we deduce (38.7) upon applying the Jacobi triple product identity.

Entry 39 (p. 330). *Let*

$$u = \frac{f(-q)f(-q^5)}{q^{1/2}f(-q^3)f(-q^{15})} \quad \textit{and} \quad v = \frac{f(-q^{1/3})f(-q^{5/3})}{q^{2/3}f(-q^3)f(-q^{15})}.$$

Then

$$u^4 - 3u^2v = v^3 + 3v^2 + 9v. \tag{39.1}$$

We are unable to prove Entry 39 by using classical methods. We thus defer the proof of Entry 39 until the end of this chapter where modular forms will be utilized.

Entry 40 (p. 330). *If*

$$\alpha = \frac{p(2+p)^2}{(1+p)^3} \quad \textit{and} \quad \beta = p^2(2+p),$$

then

$$(\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/2} = 1. \tag{40.1}$$

The equation (40.1) is evidently not a modular equation. Entry 40 is probably an unsuccessful attempt to construct a modular equation with simple parametric representations of the moduli. Ramanujan's efforts are reminiscent of his early work on modular equations in Chapter 15 (Part II [4, pp. 333–335]).

PROOF. By simple algebra,

$$(\alpha\beta)^{1/3} = \frac{p(2+p)}{1+p}. \tag{40.2}$$

Also,

$$\begin{aligned}\{(1-\alpha)(1-\beta)\}^{1/2} &= \left(\frac{1-p-p^2}{(1+p)^3}(1-2p^2-p^3)\right)^{1/2}\\ &= \left(\frac{(1-p-p^2)^2}{(1+p)^2}\right)^{1/2} = \frac{1-p-p^2}{1+p}.\end{aligned} \tag{40.3}$$

Adding (40.2) and (40.3), we deduce (40.1).

Entry 41 (p. 330). *Let α and β have degrees* 1 *and* 9, *respectively. Let t be the parameter defined in the proofs of Entries* 3(i)–(iii) *in Chapter* 20 (*Part* III [6, *p.* 354]). *Then*

(i) $$\alpha(1-\alpha) = 16t\left(\frac{1-t}{1+2t}\right)^8 \frac{1-t^3}{1+8t^3},$$

(ii) $$\beta(1-\beta) = 16t^9 \frac{1-t^3}{1+8t^3},$$

and

(iii) $$m = (1+2t)^2.$$

Furthermore, for t sufficiently small and positive,

$$\begin{aligned}&{}_2F_1\left(\tfrac{1}{4}, \tfrac{1}{4}; 1; 64t\left(\frac{1-t}{1+2t}\right)^8 \frac{1-t^3}{1+8t^3}\right)\\ &\qquad = \frac{(1+2t)^2}{\sqrt{1+8t^3}}\, {}_2F_1\left(\tfrac{1}{4}, \tfrac{1}{4}; 1; 64t^3\left(\frac{1-t^3}{1+8t^3}\right)^3\right)\\ &\qquad = (1+2t)^2\, {}_2F_1\left(\tfrac{1}{4}, \tfrac{1}{4}; 1; 64t^9 \frac{1-t^3}{1+8t^3}\right).\end{aligned} \tag{41.1}$$

These parametric representations were previously derived in the course of proving Entries 3(i)–(iii) in Chapter 20 (Part III [6, p. 354]). Parts (i) and (iii) are identical to (3.7) and (3.9), respectively, in Chapter 20. Part (iii) is established by multiplying (3.10) and (3.11) in Chapter 20.

Proof of (41.1). By (i) and (ii), the equality of the first and third members of (41.1) may be written in the form

$$ {}_2F_1(\tfrac{1}{4},\tfrac{1}{4};1;4\alpha(1-\alpha)) = (1+2t)^2\,{}_2F_1(\tfrac{1}{4},\tfrac{1}{4};1;4\beta(1-\beta)). \tag{41.2}$$

Setting $\alpha = x/(x+1)$ and using Example (ii) in Section 33 and Entry 32(ii) of Chapter 11 in Part II [4, pp. 95, 92], we find that

$$\begin{aligned} {}_2F_1(\tfrac{1}{4},\tfrac{1}{4};1;4\alpha(1-\alpha)) &= {}_2F_1\left(\tfrac{1}{4},\tfrac{1}{4};1;\frac{-4x}{(1-x)^2}\right) \\ &= \sqrt{1-x}\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;x) \\ &= {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\alpha). \end{aligned} \tag{41.3}$$

Of course, an analogue of (41.3) with α replaced by β is also valid. Hence, (41.2) is equivalent to the equality

$$ {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\alpha) = (1+2t)^2\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\beta). \tag{41.4}$$

Now remember that α and β have degrees 1 and 9, respectively, and by (3.10) and (3.11) in Chapter 20 (Part III [6, p. 354]), the multiplier associated with α and β equals $(1+2t)^2$. Thus, (41.4) is simply the defining equation for the multiplier. Thus, the truth of (41.2) is evident.

We next show that the second and third members of (41.1) are equal. In contrast to the proof above which arises from modular equations of degree 9, the present proof has its origin in modular equations of degree 3.

With $q = 2t^3$, we now set

$$\alpha(1-\alpha) = 16t^3\left(\frac{1-t^3}{1+8t^3}\right)^3 = q\left(\frac{2-q}{1+4q}\right)^3. \tag{41.5}$$

If α and β have degrees 1 and 3, respectively, then by Entry 5(xv) in Chapter 19 (Part III [6, p. 231]),

$$\beta(1-\beta) = q^3\left(\frac{2-q}{1+4q}\right) = 16t^9\left(\frac{1-t^3}{1+8t^3}\right). \tag{41.6}$$

Hence, the equality to be proved can be put in the form

$$ {}_2F_1(\tfrac{1}{4},\tfrac{1}{4};1;4\alpha(1-\alpha)) = \sqrt{1+8t^3}\,{}_2F_1(\tfrac{1}{4},\tfrac{1}{4};1;4\beta(1-\beta)). \tag{41.7}$$

But, by (41.3), (41.7) is equivalent to the equality

$$ {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\alpha) = \sqrt{1+8t^3}\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\beta). \tag{41.8}$$

By (41.5) and (41.6), an elementary calculation shows that

$$\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8} = 2t^3. \tag{41.9}$$

On the other hand, from [6, p. 233, eq. (5.3)],

$$\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8} = \frac{m^2-1}{4}, \tag{41.10}$$

where m is the multiplier of degree 3. Combining (41.9) and (41.10), we find that $m = \sqrt{1+8t^3}$. Hence, by the definition of a multiplier, (41.8) is valid. This completes the proof of the last equality of (41.1).

Entry 42 (p. 332). *For* $0 < \alpha, \beta < 1$, *suppose that* α, β, *and* a *satisfy the equality*

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2a\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4} = 1. \tag{42.1}$$

Then we have the parametrizations

$$\{\alpha(1-\beta)\}^{1/2} = \frac{4ap}{(1-p^2)^2+(2ap)^2} \quad \textit{and} \quad \{\beta(1-\alpha)\}^{1/2} = \frac{4ap^3}{(1-p^2)^2+(2ap)^2}. \tag{42.2}$$

For $a = 1$, (42.1) is a modular equation of degree 3. This can be seen by squaring the modular equation of degree 3 given in Entry 5(ii) of Chapter 19 (Part III [6, p. 230]).

Proof. Define the parameter $p > 0$ by

$$p^4 := \frac{\beta(1-\alpha)}{\alpha(1-\beta)}. \tag{42.3}$$

Solving the equality above for β, we find that

$$\beta = \frac{p^4\alpha}{1-\alpha+\alpha p^4}. \tag{42.4}$$

It follows that

$$\alpha(1-\beta) = \frac{\alpha(1-\alpha)}{1-\alpha+\alpha p^4}. \tag{42.5}$$

Substituting (42.4) and (42.5) into (42.1) and multiplying both sides by $\sqrt{1-\alpha+\alpha p^4}$, we find that

$$p^2\alpha + 1 - \alpha + 2ap\{\alpha(1-\alpha)\}^{1/2} = (1-\alpha+\alpha p^4)^{1/2}. \tag{42.6}$$

If we set

$$x := \left(\frac{\alpha(1-\alpha)}{1-\alpha+\alpha p^4}\right)^{1/2}, \tag{42.7}$$

then, upon squaring, (42.6) can be put in the form

$$\begin{aligned}(1-\alpha+\alpha p^4)(1-4apx+4a^2p^2x^2) &= (p^2\alpha+1-\alpha)^2 \\ &= (1-\alpha+\alpha p^4)-(1-p^2)^2\alpha(1-\alpha).\end{aligned}$$

Dividing both sides by $1-\alpha+\alpha p^4$ and using (42.7), we obtain the quadratic equation

$$1-4apx+4a^2p^2x^2 = 1-(1-p^2)^2x^2.$$

Solving for x, we deduce that

$$x = \frac{4ap}{(1-p^2)^2+(2ap)^2}. \tag{42.8}$$

The first equality of (42.2) now follows immediately from (42.5), (42.7), and (42.8). The second equality of (42.2) then follows from (42.3) and the first equality of (42.2).

Entry 43 (p. 332). *Suppose that the hypotheses of Entry* 42 *hold, but with the additional assumption that* $|a| \geq 1$. *Let* μ *and* ν *denote positive acute angles such that*

$$\alpha = \sin^2(\mu+\nu) \quad \textit{and} \quad \beta = \sin^2(\mu-\nu). \tag{43.1}$$

Then for some φ, $0 \leq \varphi \leq \pi/2$, *we have the parametrizations*

$$\sin\nu = \frac{\sin\varphi}{\sqrt{1-a^{-2}}} \quad \textit{and} \quad \sin(2\mu) = \frac{\sin(2\varphi)}{\sqrt{1-a^{-2}}}. \tag{43.2}$$

Ramanujan's parametrizations are slightly in error, because he has a^{-1} instead of a^{-2} in each formula above.

PROOF. From (43.1) and (42.1),

$$\begin{aligned}1-\cos 2\nu &= 1-\sin(\mu+\nu)\sin(\mu-\nu)-\cos(\mu+\nu)\cos(\mu-\nu) \\ &= 1-(\alpha\beta)^{1/2}-\{(1-\alpha)(1-\beta)\}^{1/2} \\ &= 2a\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4} \\ &= 2a\{\sin^2(\mu+\nu)\sin^2(\mu-\nu)\cos^2(\mu+\nu)\cos^2(\mu-\nu)\}^{1/4} \\ &= 2a\{(\sin^2\mu-\sin^2\nu)(\cos^2\mu-\sin^2\nu)\}^{1/2} \\ &= 2a\{\sin^2\mu\cos^2\mu-\sin^2\nu+\sin^4\nu\}^{1/2} \\ &= 2a\{\tfrac{1}{4}\sin^2(2\mu)-\sin^2\nu\cos^2\nu\}^{1/2}.\end{aligned}$$

Thus,

$$4 \sin^4 \nu = a^2(\sin^2(2\mu) - 4 \sin^2 \nu \cos^2 \nu). \tag{43.3}$$

Now define φ by the equation

$$\sin(2\mu) = 2 \sin \nu \cos \varphi. \tag{43.4}$$

Then, from (43.3),

$$\begin{aligned}\sin^2 \nu &= a^2(\cos^2 \varphi - \cos^2 \nu)\\ &= a^2(\sin^2 \nu - \sin^2 \varphi),\end{aligned}$$

or

$$\sin^2 \varphi = (1 - a^{-2}) \sin^2 \nu.$$

Thus, since ν is an acute angle and $1 - a^{-2} \geq 1$, the first equality of (43.2) follows. It is also now clear that we can assume that $0 \leq \varphi \leq \pi/2$. The second equality in (43.2) now follows from (43.4) and the first equality in (43.2).

Entry 44 (p. 355). *For* $|q| < 1$,

$$\frac{2\psi(q)\psi(q^2)\psi(q^3)}{\psi(q^6)} - \varphi^2(q^3) = 1 + 2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{q^{6n+1}}{1 - q^{6n+1}} + \frac{q^{6n+5}}{1 - q^{6n+5}}\right). \tag{44.1}$$

Proof. Expanding the summands in geometric series, reversing the order of summation, applying the corollary of Entry 33(iii) in Chapter 16 with $a = q$ and $b = q^5$, and lastly utilizing Example (v) in Section 31 of Chapter 16 (Part III [6, pp. 54, 51]), we arrive at

$$\begin{aligned}F(q) &:= 1 + 2 \sum_{n=0}^{\infty} (-1)^n \left(\frac{q^{6n+1}}{1 - q^{6n+1}} + \frac{q^{6n+5}}{1 - q^{6n+5}}\right)\\ &= 1 + 2 \sum_{n=0}^{\infty} (-1)^n \sum_{m=1}^{\infty} \{(q^{6n+1})^m + (q^{6n+5})^m\}\\ &= \varphi^2(-q^6) \frac{f(q, q^5)}{f(-q, -q^5)}\\ &= \varphi^2(-q^6) \frac{\psi(-q^3)\chi(q)}{\psi(q^3)\chi(-q)},\end{aligned}$$

where χ is defined by (0.5). Let α and β have degrees 1 and 3, respectively. Invoking Entries 10(iii), 11(i), (ii), and 12(v), (vi) from Chapter 17 (Part III [6, pp. 122–124]), after much simplification, we find that

$$F(q) = z_3 \frac{(1 - \beta)^{3/8}}{(1 - \alpha)^{1/8}}.$$

Next, we turn to Chapter 19 and apply the second part of Entry 5(i), both parts of Entry 5(iii), and then the first part of Entry 5(i) (Part III [6, p. 230]). Thus, if m denotes the multiplier, we find that

$$\begin{aligned}
F(q) &= z_3\left(1 + \left(\frac{\beta^3}{\alpha}\right)^{1/8}\right)\\
&= z_3\left(\frac{m+1}{2}\right)\\
&= z_3\left(\frac{3-m}{2} - 1 + m\right)\\
&= z_3\left(m\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} - 1 + m\right)\\
&= z_3\left(m\left(\frac{\alpha^3}{\beta}\right)^{1/8} - m - 1 + m\right)\\
&= z_1\left(\frac{\alpha^3}{\beta}\right)^{1/8} - z_3\\
&= 2\frac{\psi(q)\psi(q^2)\psi(q^3)}{\psi(q^6)} - \varphi^2(q^3),
\end{aligned}$$

by Entries 10(i) and 11(i), (iii) in Chapter 17 (Part III [6, pp. 122–123]). This completes the proof of (44.1).

Entry 45 (p. 355). *We have*

$$\psi(q)\psi(q^3) + \psi(-q)\psi(-q^3) = 2\psi(q^4)\varphi(q^6). \tag{45.1}$$

Proof. We apply Schröter's formula (36.4) in Chapter 16 with $\mu = 2$ and $\nu = 1$ and then Entries 18(ii), (iv) in Chapter 16 (Part III [6, pp. 68, 34]) to find that

$$2\psi(q^6)\psi(q^2) = 2\varphi(q^{12})\psi(q^8) + 2q^2\psi(q^{24})\varphi(q^4).$$

Replacing q^2 by q yields

$$\psi(q^3)\psi(q) = \varphi(q^6)\psi(q^4) + q\psi(q^{12})\varphi(q^2). \tag{45.2}$$

Now replace q by $-q$ in (45.2) and add this new equality and (45.2) to obtain (45.1).

Entry 46 (p. 355). *For* $|q| < 1$,

$$\varphi(q)\varphi(q^3) + \varphi(-q)\varphi(-q^3)$$
$$= 2\left\{1 + 6\sum_{n=0}^{\infty}\left(\frac{q^{24n+4}}{1-q^{24n+4}} - \frac{q^{24n+8}}{1-q^{24n+8}} + \frac{q^{24n+16}}{1-q^{24n+16}} - \frac{q^{24n+20}}{1-q^{24n+20}}\right)\right\}. \tag{46.1}$$

PROOF. By Entries 3(i), (ii) in Chapter 19 (Part III [6, p. 223]),

$$\begin{aligned}F(q)&:=\varphi(q^4)\varphi(q^{12})+4q^4\psi(q^8)\psi(q^{24})\\&=1+2\sum_{n=0}^{\infty}\left(\frac{q^{24n+4}}{1-q^{24n+4}}-\frac{q^{24n+8}}{1+q^{24n+8}}+\frac{q^{24n+16}}{1+q^{24n+16}}-\frac{q^{24n+20}}{1-q^{24n+20}}\right)\\&\quad+4\sum_{n=0}^{\infty}\left(\frac{q^{24n+4}}{1-q^{48n+8}}-\frac{q^{24n+20}}{1-q^{48n+40}}\right).\end{aligned}$$

In the last sum, use the simple identity

$$\frac{x}{1-x^2}=\frac{1}{2}\left(\frac{x}{1+x}+\frac{x}{1-x}\right),$$

and then, wherever possible, employ the elementary fact

$$\frac{x}{1+x}=\frac{x}{1-x}-\frac{2x^2}{1-x^2}.$$

We then arrive at

$$\begin{aligned}F(q)&=1+2\sum_{n=0}^{\infty}\left(\frac{q^{24n+4}}{1-q^{24n+4}}-\frac{q^{24n+8}}{1+q^{24n+8}}+\frac{q^{24n+16}}{1+q^{24n+16}}-\frac{q^{24n+20}}{1-q^{24n+20}}\right)\\&\quad+2\sum_{n=0}^{\infty}\left(\frac{q^{24n+4}}{1-q^{24n+4}}+\frac{q^{24n+4}}{1+q^{24n+4}}-\frac{q^{24n+20}}{1-q^{24n+20}}-\frac{q^{24n+20}}{1+q^{24n+20}}\right)\\&=1+4\sum_{n=0}^{\infty}\left(\frac{q^{24n+4}}{1-q^{24n+4}}-\frac{q^{24n+20}}{1-q^{24n+20}}\right)\\&\quad+2\sum_{n=0}^{\infty}\left(-\frac{q^{24n+8}}{1-q^{24n+8}}+\frac{2q^{48n+16}}{1-q^{48n+16}}+\frac{q^{24n+16}}{1-q^{24n+16}}-\frac{2q^{48n+32}}{1-q^{48n+32}}\right.\\&\quad\left.+\frac{q^{24n+4}}{1-q^{24n+4}}-\frac{2q^{48n+8}}{1-q^{48n+8}}-\frac{q^{24n+20}}{1-q^{24n+20}}+\frac{2q^{48n+40}}{1-q^{48n+40}}\right)\\&=1+6\sum_{n=0}^{\infty}\left(\frac{q^{24n+4}}{1-q^{24n+4}}-\frac{q^{24n+20}}{1-q^{24n+20}}-\frac{q^{48n+8}}{1-q^{48n+8}}-\frac{q^{48n+32}}{1-q^{48n+32}}\right.\\&\quad\left.+\frac{q^{48n+16}}{1-q^{48n+16}}+\frac{q^{48n+40}}{1-q^{48n+40}}\right)\\&=1+6\sum_{n=0}^{\infty}\left(\frac{q^{24n+4}}{1-q^{24n+4}}-\frac{q^{24n+8}}{1-q^{24n+8}}+\frac{q^{24n+16}}{1-q^{24n+16}}-\frac{q^{24n+20}}{1-q^{24n+20}}\right).\end{aligned}\tag{46.2}$$

On the other hand, by Entries 25(i), (ii) in Chapter 16 (Part III [6, p. 40]),

$$\begin{aligned}\varphi(q^4)\varphi(q^{12}) &+ 4q^4\psi(q^8)\psi(q^{24})\\ &= \tfrac{1}{4}\{\varphi(q)+\varphi(-q)\}\{\varphi(q^3)+\varphi(-q^3)\} + \tfrac{1}{4}\{\varphi(q)-\varphi(-q)\}\{\varphi(q^3)-\varphi(-q^3)\}\\ &= \tfrac{1}{2}\{\varphi(q)\varphi(q^3)+\varphi(-q)\varphi(-q^3)\}. \end{aligned}\tag{46.3}$$

Combining (46.2) and (46.3), we deduce (46.1) to complete the proof.

Entry 47 (p. 370). *We have*

$$\lim_{q\to 1-} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^3q^n}{1-q^n} = \frac{1}{16}. \tag{47.1}$$

PROOF. From Entry 14(v) in Chapter 17 (Part III [6, p. 130]),

$$1 - 16\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^3q^n}{1-q^n} = z^4(1-x)^2. \tag{47.2}$$

Recall that $x = k^2$, where k is the modulus, $z = 2K/\pi$, and $q = \exp(-\pi K'/K)$. Thus, as q tends to 1, K tends to ∞, i.e., k approaches 1. Now, from Whittaker and Watson's text [1, p. 522],

$$K \sim \log\frac{4}{\sqrt{1-k^2}}$$

as k tends to $1-$. Hence,

$$\lim_{k\to 1-} z^4(1-x)^2 = 0. \tag{47.3}$$

Thus, letting q tend to $1-$ on both sides of (47.2) and using (47.3), we immediately deduce (47.1).

The next two results are notationally interesting because Ramanujan employs the classical notations q, K, L, k', and ℓ. On no other page of the notebooks does Ramanujan use this notation.

Entry 48 (p. 382). *If ℓ has degree* 2, *then*

$$2\left(1 - 24\sum_{n=1}^{\infty}\frac{nq^{4n}}{1-q^{4n}}\right) - \left(1 - 24\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\right) = \frac{4KL}{\pi^2}(k'+\ell). \tag{48.1}$$

PROOF. Since we shall utilize previous results of Ramanujan, we first convert (48.1) into his customary notation. Set, as in (9.2) of Chapter 17,

$$L(q^2) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}. \tag{48.2}$$

(The Eisenstein series $L(q^2)$ in (48.2) should not be confused with the complete elliptic integral L of the first kind in (48.1).) Thus, (48.1) may be written in the more familiar form

$$2L(q^4) - L(q^2) = z_1 z_2(\sqrt{1-\alpha} + \sqrt{\beta}), \tag{48.3}$$

where β has degree 2.

Applying Entry 13(ix) in Chapter 17 (Part III [6, p. 127]), we find that

$$2L(q^4) - L(q^2) = z_1^2(1 - \tfrac{1}{2}\alpha). \tag{48.4}$$

Comparing (48.3) and (48.4), we see that we are required to prove that

$$m = 2\frac{\sqrt{1-\alpha} + \sqrt{\beta}}{2-\alpha}, \tag{48.5}$$

where m is the multiplier for degree 2. By (24.12) in Chapter 18 (Part III [6, p. 213]),

$$\begin{aligned} 2\frac{\sqrt{1-\alpha} + \sqrt{\beta}}{2-\alpha} &= 2\frac{\sqrt{1-\alpha} + (2 - 2\sqrt{1-\alpha} - \alpha)/\alpha}{2-\alpha} \\ &= 2\frac{(\alpha-2)\sqrt{1-\alpha} - (\alpha-2)}{(2-\alpha)\alpha} \\ &= \frac{-2\sqrt{1-\alpha} + 2}{\alpha} = m, \end{aligned}$$

by (24.11) in Chapter 18. Thus, (48.5) has been established, and the proof has been accomplished.

Entry 49 (p. 382). *If ℓ has degree* 4, *then*

$$4\left(1 - 24\sum_{n=1}^{\infty} \frac{nq^{8n}}{1-q^{8n}}\right) - \left(1 - 24\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}}\right) = \frac{12KL}{\pi^2}(\sqrt{k'} + \sqrt{\ell})^2. \tag{49.1}$$

Proof. Converting (49.1) into Ramanujan's more familiar notation, we see that

$$4L(q^8) - L(q^2) = 3z_1 z_4\{(1-\alpha)^{1/4} + \beta^{1/4}\}^2, \tag{49.2}$$

where we have used (48.2). Applying the process of duplication (Part III [6, p. 125]) to (48.4), we find that

$$\begin{aligned} 4L(q^8) - L(q^2) &= 2\{2L(q^8) - L(q^4)\} + 2L(q^4) - L(q^2) \\ &= 2\left\{\tfrac{1}{4}z_1^2(1 + \sqrt{1-\alpha})^2\left(1 - \frac{1}{2}\left(\frac{1-\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\right)^2\right)\right\} + z_1^2(1 - \tfrac{1}{2}\alpha) \\ &= \tfrac{1}{2}z_1^2(1 - \tfrac{1}{2}\alpha + 3\sqrt{1-\alpha}) + z_1^2(1 - \tfrac{1}{2}\alpha) \\ &= \tfrac{3}{2}z_1^2(1 - \tfrac{1}{2}\alpha + \sqrt{1-\alpha}). \end{aligned} \tag{49.3}$$

Comparing (49.2) and (49.3), we realize that it suffices to prove that

$$m = \frac{2\{(1-\alpha)^{1/4} + \beta^{1/4}\}^2}{1 - \frac{1}{2}\alpha + \sqrt{1-\alpha}}, \tag{49.4}$$

where m is the multiplier for degree 4.

By (24.22) in Chapter 18 (Part III [6, p. 215]),

$$\begin{aligned}
\frac{2\{(1-\alpha)^{1/4} + \beta^{1/4}\}^2}{1 - \frac{1}{2}\alpha + \sqrt{1-\alpha}} &= \frac{2\left((1-\alpha)^{1/4} + \dfrac{1-(1-\alpha)^{1/4}}{1+(1-\alpha)^{1/4}}\right)^2}{1 - \frac{1}{2}\alpha + \sqrt{1-\alpha}} \\
&= \frac{2(1+\sqrt{1-\alpha})^2}{(1+(1-\alpha)^{1/4})^2(1 - \frac{1}{2}\alpha + \sqrt{1-\alpha})} \\
&= \frac{4}{(1+(1-\alpha)^{1/4})^2} = m,
\end{aligned}$$

by (24.23) in Chapter 18. This proves (49.4) and completes the proof.

We conclude the second portion of this chapter by proving two formulas on page 4 of the second notebook; Chapter 1 begins with page 5.

Entry 50 (p. 4). *We have*

(i)
$$\frac{\chi^3(q)}{\chi(q^3)} = 1 + 3q\frac{\psi(-q^9)}{\psi(-q)}$$

and

(ii)
$$\frac{\chi^5(q)}{\chi(q^5)} = 1 + 5q\frac{\psi^2(-q^5)}{\psi^2(-q)}.$$

PROOF. Let α, β, and γ have degrees 1, 3, and 9, respectively, and let m and m' denote the multipliers for the pairs α, β and β, γ, respectively. By Entry 11(ii) in Chapter 17 and (3.7) and (3.9)–(3.11) in Chapter 20 (Part III [6, pp. 123, 354]),

$$\begin{aligned}
3q\frac{\psi(-q^9)}{\psi(-q)} &= \frac{3}{\sqrt{mm'}}\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8} \\
&= \frac{3}{1+2t}\,t\,\frac{1+2t}{1-t} \\
&= \frac{1+2t}{1-t} - 1.
\end{aligned} \tag{50.1}$$

By Entry 12(v) in Chapter 17 and (3.7)–(3.8) in Chapter 20 (Part III [6, pp. 124, 354]),

$$\frac{\chi^3(q)}{\chi(q^3)} = \frac{2^{1/3}(\beta(1-\beta))^{1/24}}{(\alpha(1-\alpha))^{1/8}} = \frac{1+2t}{1-t}. \tag{50.2}$$

Combining (50.1) and (50.2), we deduce (i).

Now let α and β have degrees 1 and 5, respectively, and let m denote the associated multiplier. As in (13.3) of Chapter 19 (Part III [6, p. 284]), put

$$\rho = (m^3 - 2m^2 + 5m)^{1/2}. \tag{50.3}$$

By Entry 11(ii) in Chapter 17, (13.13) in Chapter 19 (Part III [6, pp. 123, 286]), and (50.3) above,

$$\begin{aligned}
5q\frac{\psi^2(-q^5)}{\psi^2(-q)} &= 5\frac{z_5}{z_1}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} \\
&= \frac{5}{m}\frac{4m^2-\rho^2}{(5-m)^2} \\
&= \frac{5}{m}\frac{-m^3+6m^2-5m}{(5-m)^2} \\
&= \frac{5(m-1)}{5-m} \\
&= \frac{4m}{5-m} - 1.
\end{aligned} \tag{50.4}$$

By Entry 12(v) in Chapter 17, (13.4)–(13.5) in Chapter 19 (Part III [6, pp. 124, 284]), and (50.3) above,

$$\begin{aligned}
\frac{\chi^5(q)}{\chi(q^5)} &= 2^{2/3}\left(\frac{\beta(1-\beta)}{\alpha^5(1-\alpha)^5}\right)^{1/24} \\
&= 2^{2/3}\left(\frac{16m^4}{25\rho^2-(m^2+5m)^2}\right)^{1/3} \\
&= \frac{4m}{(125-75m+15m^2-m^3)^{1/3}} \\
&= \frac{4m}{5-m}.
\end{aligned} \tag{50.5}$$

It is now evident that (ii) follows from (50.4) and (50.5).

This concludes the second part of the chapter.

The third part is devoted to proving Ramanujan's P–Q modular equations. Altogether, there are 23 of these beautiful theorems. We are able to prove 18 of them by employing the theory of theta-functions in the spirit of Ramanujan; for the remaining five, we are forced to use the theory of modular forms. In contrast to the first two parts wherein little rearrangement of the results was undertaken, we have reorganized the P–Q modular equations. In particular, several more difficult theorems are found prior to many easier results in the notebooks. We first prove those modular equations involving two moduli and lastly establish those with four moduli. We have also arranged the equations by increasing degree.

Entry 51 (p. 327). *Let*

$$P = \frac{f^2(-q)}{q^{1/6} f^2(-q^3)} \quad \textit{and} \quad Q = \frac{f^2(-q^2)}{q^{1/3} f^2(-q^6)}.$$

Then

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3. \tag{51.1}$$

PROOF. Rewrite (51.1) in the form

$$P^4Q^4 + 9P^2Q^2 = Q^6 + P^6. \tag{51.2}$$

Set

$$R = \frac{f^2(q)}{q^{1/6} f^2(q^3)}.$$

If we replace q by $-q$ in (51.2), then P^2 is replaced by $-R^2$ and Q is replaced by $-Q$. Thus, (51.2) is equivalent to

$$R^4Q^4 - 9R^2Q^2 = Q^6 - R^6. \tag{51.3}$$

By Entries 12(i), (iii), respectively, in Chapter 17 (Part III [6, p. 124]),

$$R = \frac{z_1}{z_3}\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12} \quad \text{and} \quad Q = \frac{z_1}{z_3}\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/6}, \tag{51.4}$$

where α and β have degrees 1 and 3, respectively. It follows that

$$m = \frac{R^2}{Q}, \tag{51.5}$$

where $m = z_1/z_3$, the multiplier. Thus, by (5.1) in Chapter 19 (Part III

[6, p. 232]) and (51.5) above,

$$\begin{aligned}\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2} &= \left(\frac{(1-\alpha)^3}{1-\beta}\frac{\alpha^3}{\beta}\frac{\alpha}{\beta^3}\frac{1-\alpha}{(1-\beta)^3}\right)^{1/8}\\ &= \frac{3-m}{2m}\frac{3+m}{2m}\frac{2}{m-1}\frac{2}{m+1}\\ &= \frac{9-m^2}{m^2(m^2-1)}\\ &= \frac{9-R^4/Q^2}{(R^4/Q^2)(R^4/Q^2-1)}. \end{aligned} \tag{51.6}$$

We also may deduce from (51.4) and (51.5) that

$$R^2Q^2 = m^4\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2} = \frac{R^8}{Q^4}\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2},$$

or

$$\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2} = \frac{Q^6}{R^6}. \tag{51.7}$$

Equating the right sides of (51.6) and (51.7), we deduce (51.3), after a modest amount of algebraic manipulation.

Entry 52 (p. 327). *Let*

$$P = \frac{f(-q^2)}{q^{1/24}f(-q^3)} \quad \textit{and} \quad Q = \frac{f(-q)}{q^{5/24}f(-q^6)}.$$

Then

$$(PQ)^2 - \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - 8\left(\frac{P}{Q}\right)^3. \tag{52.1}$$

Proof. We first record (52.1) in the equivalent form

$$P^5Q^5 - 9PQ = Q^6 - 8P^6. \tag{52.2}$$

Set

$$R = \frac{f(-q^2)}{q^{1/24}f(q^3)} \quad \text{and} \quad S = \frac{f(q)}{q^{5/24}f(-q^6)}.$$

By Entries 12(i), (iii) in Chapter 17,

$$R = \sqrt{m}\left(\frac{\alpha^2(1-\alpha)^2}{16\beta(1-\beta)}\right)^{1/24} \quad \text{and} \quad S = \sqrt{m}\left(\frac{16\alpha(1-\alpha)}{\beta^2(1-\beta)^2}\right)^{1/24}, \tag{52.3}$$

where α and β have degrees 1 and 3, respectively, and m denotes the multiplier of degree 3. Therefore, by (5.1) in Chapter 19 and (52.3) above,

$$\frac{R^5}{S}=\frac{m^2}{2}\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8}=\frac{m^2}{2}\frac{9-m^2}{4m^2}=\frac{9-m^2}{8} \tag{52.4}$$

and

$$\frac{R}{S^5}=\frac{1}{2m^2}\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}=\frac{1}{2m^2}\frac{m^2-1}{4}=\frac{m^2-1}{8m^2}. \tag{52.5}$$

Solving each of (52.4) and (52.5) for m^2 and equating the resulting two formulas, we find that

$$9-\frac{8R^5}{S}=\frac{1}{1-8R/S^5}. \tag{52.6}$$

Now replacing q by $-q$, we see that R^5/S and R/S^5 are transformed into P^5/Q and $-P/Q^5$, respectively. The equality (52.6) then is converted into

$$9-\frac{8P^5}{Q}=\frac{1}{1+8P/Q^5},$$

or

$$(9Q-8P^5)(Q^5+8P)=Q^6,$$

which, upon simplification, reduces to (52.2).

Entry 53 (p. 325). *If*

$$P=\frac{f(-q)}{q^{1/6}f(-q^5)} \quad \textit{and} \quad Q=\frac{f(-q^2)}{q^{1/3}f(-q^{10})},$$

then

$$PQ+\frac{5}{PQ}=\left(\frac{Q}{P}\right)^3+\left(\frac{P}{Q}\right)^3. \tag{53.1}$$

Proof. We first rewrite (53.1) in the equivalent form

$$P^4Q^4+5P^2Q^2=Q^6+P^6. \tag{53.2}$$

Set

$$R=\frac{f(q)}{q^{1/6}f(q^5)}.$$

Replacing q by $-q$, we see that P^2 is changed into $-R^2$ and Q^2 is invariant. Thus, (53.2) is transformed into

$$R^4Q^4 - 5R^2Q^2 = Q^6 - R^6. \tag{53.3}$$

By Entries 12(i), (iii), respectively, in Chapter 17,

$$R^2 = \frac{z_1}{z_5}\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12} \quad \text{and} \quad Q^2 = \frac{z_1}{z_5}\left(\frac{\alpha(1-q)}{\beta(1-\beta)}\right)^{1/6}, \tag{53.4}$$

where α and β have degrees 1 and 5, respectively. If m denotes the multiplier of degree 5, we deduce from (53.4) that

$$m = \frac{R^4}{Q^2}. \tag{53.5}$$

On the other hand, from Entry 13(iv) in Chapter 19 (Part III [6, p. 281]) as well as (53.4) and (53.5) above,

$$\begin{aligned}\frac{R^4/Q^2 - 1}{5Q^2/R^4 - 1} = \frac{m-1}{5/m-1} &= \left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\frac{\beta(1-\beta)}{\alpha^5(1-\alpha)^5}\right)^{1/24} \\ &= \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} = \frac{R^6}{Q^6}.\end{aligned} \tag{53.6}$$

By elementary algebra, (53.3) follows from (53.6), and the proof is complete.

Entry 54 (p. 327). *If*

$$P = \frac{f(-q^2)}{q^{1/8}f(-q^5)} \quad \textit{and} \quad Q = \frac{f(-q)}{q^{3/8}f(-q^{10})},$$

then

$$PQ - \frac{5}{PQ} = \left(\frac{Q}{P}\right)^2 - 4\left(\frac{P}{Q}\right)^2. \tag{54.1}$$

PROOF. Let

$$R = \frac{f(-q^2)}{q^{1/8}f(q^5)} \quad \text{and} \quad S = \frac{f(q)}{q^{3/8}f(-q^{10})}.$$

By Entries 12(i), (iii) in Chapter 17,

$$R = \sqrt{m}\left(\frac{\alpha^2(1-\alpha)^2}{16\beta(1-\beta)}\right)^{1/24} \quad \text{and} \quad S = \sqrt{m}\left(\frac{16\alpha(1-\alpha)}{\beta^2(1-\beta)^2}\right)^{1/24}, \tag{54.2}$$

where α and β have degrees 1 and 5, respectively, and m is the multiplier of degree 5.

Recall, from (13.3) in Chapter 19 (Part III [6, p. 284]), the notation

$$\rho = (m^3 - 2m^2 + 5m)^{1/2}. \tag{54.3}$$

By (13.4) and (13.5) in Chapter 19 and (54.3), we find that

$$\begin{aligned}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/8} &= \frac{5\rho + m^2 + 5m}{4m^2}\,\frac{5\rho - m^2 - 5m}{4m^2}\\ &= \frac{25(m^3 - 2m^2 + 5m) - (m^2 + 5m)^2}{16m^4}\\ &= \frac{(5-m)^3}{16m^3}.\end{aligned} \tag{54.4}$$

Thus, by (54.2) and (54.4),

$$\frac{R^3}{S} = \frac{m}{2^{2/3}}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24} = \frac{m}{2^{2/3}}\,\frac{5-m}{2^{4/3}m} = \frac{5-m}{4}, \tag{54.5}$$

since, by Entry 13(iv) in Chapter 19 (Part III [6, p. 281]), $m < 5$.

We next derive a similar representation for R/S^3. By (13.4) and (13.5) in Chapter 19 and (54.3),

$$\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/8} = \frac{\rho - m - 1}{4}\,\frac{\rho + m + 1}{4} = \frac{\rho^2 - (m+1)^2}{16} = \frac{(m-1)^3}{16}. \tag{54.6}$$

Hence, by (54.2) and (54.6),

$$\frac{R}{S^3} = \frac{1}{m2^{2/3}}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24} = \frac{1}{m2^{2/3}}\,\frac{m-1}{2^{2/3}} = \frac{m-1}{4m}, \tag{54.7}$$

since, by Entry 13(iv) in Chapter 19, $m > 1$.

Solving each of (54.5) and (54.7) for m and equating the two representations, we find that

$$5 - \frac{4R^3}{S} = \frac{1}{1 - 4R/S^3}. \tag{54.8}$$

The replacement of q by $-q$ transforms R/S^3 into $-P/Q^3$ and R^3/S into P^3/Q. Thus, (54.8) is transformed into the equality

$$5 - \frac{4P^3}{Q} = \frac{1}{1 + 4P/Q^3},$$

or

$$(5Q - 4P^3)(Q^3 + 4P) = Q^4,$$

which reduces to

$$Q^4 - 4P^4 = P^3Q^3 - 5PQ,$$

which is equivalent to (54.1).

Entry 55 (p. 327). *Let*

$$P = \frac{f^2(-q)}{q^{1/2} f^2(-q^7)} \quad \textit{and} \quad Q = \frac{f^2(-q^2)}{q f^2(-q^{14})}.$$

Then

$$PQ + \frac{49}{PQ} = \left(\frac{Q}{P}\right)^3 - 8\frac{Q}{P} - 8\frac{P}{Q} + \left(\frac{P}{Q}\right)^3. \tag{55.1}$$

Ramanujan incorrectly wrote $q^{3/2}$ instead of q in the denominator of the definition of Q.

Proof. Let

$$R = \frac{f^2(q)}{q^{1/2} f^2(q^7)}.$$

By Entry 12(i), (iii) in Chapter 17,

$$R = m\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12} \quad \text{and} \quad Q = m\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/6}, \tag{55.2}$$

where α and β have degrees 1 and 7, respectively, and m is the multiplier of degree 7. It follows that

$$m = \frac{R^2}{Q} \tag{55.3}$$

and

$$\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2} = \frac{Q^6}{R^6}. \tag{55.4}$$

Next, recall Entry 19(v) in Chapter 19 (Part III [6, p. 314]), viz.,

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} - 8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3} \tag{55.5}$$

and

$$\frac{49}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2} - 8\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/3}. \tag{55.6}$$

Multiply both sides of (55.5) by $\alpha^{1/2}(1-\alpha)^{1/2}$ and both sides of (55.6) by $\beta^{1/2}(1-\beta)^{1/2}$. Combining the resulting two equalities, we deduce that

$$\begin{aligned}
&\alpha^{1/2}(1-\alpha)^{1/2}\left\{m^2+\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}+8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3}\right\}\\
&\quad=\beta^{1/2}(1-\alpha)^{1/2}+\alpha^{1/2}(1-\beta)^{1/2}\\
&\quad=\beta^{1/2}(1-\beta)^{1/2}\left\{\frac{49}{m^2}+\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}+8\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/3}\right\}.
\end{aligned}\tag{55.7}$$

Dividing the extremal sides of (55.7) by $\beta^{1/2}(1-\beta)^{1/2}$ and employing (55.3) and (55.4), we deduce that

$$\frac{Q^6}{R^6}\left(\frac{R^4}{Q^2}+\frac{R^6}{Q^6}+8\frac{R^4}{Q^4}\right)=49\frac{Q^2}{R^4}+\frac{Q^6}{R^6}+8\frac{Q^4}{R^4},$$

or

$$R^4Q^4+R^6+8R^4Q^2=49R^2Q^2+Q^6+8R^2Q^4. \tag{55.8}$$

Replacing q by $-q$, we observe that R^2 is converted to $-P^2$ and Q^2 is invariant. Thus, (55.8) is transformed into

$$P^4Q^4-P^6+8P^4Q^2=-49P^2Q^2+Q^6-8P^2Q^4,$$

which is easily seen to be equivalent to (55.1). This finishes the proof.

Entry 56 (p. 327). *If*

$$P=\frac{f(-q)}{q^{1/3}f(-q^9)} \quad \textit{and} \quad Q=\frac{f(-q^2)}{q^{2/3}f(-q^{18})},$$

then

$$P^3+Q^3=P^2Q^2+3PQ. \tag{56.1}$$

PROOF. Let

$$R=\frac{f(q)}{q^{1/3}f(q^9)}.$$

By Entries 12(i), (iii), respectively, in Chapter 17,

$$R=\sqrt{mm'}\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/24} \quad \text{and} \quad Q=\sqrt{mm'}\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/12}, \tag{56.2}$$

where α and γ have degrees 1 and 9, respectively, $m=z_1/z_3$, and $m'=z_3/z_9$. It follows that

$$\sqrt{mm'}=\frac{R^2}{Q}. \tag{56.3}$$

Now, from (3.10) and (3.11) in Chapter 20 (Part III [6, p. 354]), $\sqrt{mm'} = 1 + 2t$. Thus, from (56.3),

$$t = \frac{\sqrt{mm'} - 1}{2} = \frac{R^2 - Q}{2Q}. \tag{56.4}$$

Also, from (56.2),

$$\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)} = \frac{Q^{24}}{R^{24}}. \tag{56.5}$$

On the other hand, from (3.7) and (3.9) in Chapter 20,

$$\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)} = \left(\frac{1-t}{t(1+2t)}\right)^8. \tag{56.6}$$

Hence, from (56.3)–(56.6),

$$\frac{Q^3}{R^3} = \frac{1-t}{t(1+2t)} = \frac{Q}{R^2}\frac{3Q - R^2}{R^2 - Q},$$

or

$$Q^2R^2 - Q^3 = 3QR - R^3. \tag{56.7}$$

Replacing q by $-q$ takes R into $-P$ and fixes Q. Thus, (56.7) is transformed into (56.1), which completes the proof.

Entry 57 (p. 327). *If*

$$P = \frac{f(-q)}{q^{1/2}f(-q^{13})} \quad \textit{and} \quad Q = \frac{f(-q^2)}{qf(-q^{26})},$$

then

$$PQ + \frac{13}{PQ} = \left(\frac{Q}{P}\right)^3 - 4\frac{Q}{P} - 4\frac{P}{Q} + \left(\frac{P}{Q}\right)^3. \tag{57.1}$$

PROOF. Let

$$R = \frac{f(q)}{q^{1/2}f(q^{13})}.$$

Then, by Entries 12(i), (iii), respectively, in Chapter 17,

$$R = \sqrt{m}\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/24} \quad \text{and} \quad Q = \sqrt{m}\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12}, \tag{57.2}$$

where α and β have degrees 1 and 13, respectively, and m is the multiplier of degree 13. It follows that

$$m = \frac{R^4}{Q^2} \tag{57.3}$$

and

$$\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4} = \frac{Q^6}{R^6}. \tag{57.4}$$

By Entries 8(iii), (iv) in Chapter 20 (Part III [6, p. 376]),

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} - 4\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}$$

and

$$\frac{13}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4} - 4\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/6}.$$

By an argument analogous to that for (55.7),

$$\{\alpha(1-\alpha)\}^{1/4}\left\{m + \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} + 4\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}\right\}$$
$$= \{\beta(1-\beta)\}^{1/4}\left\{\frac{13}{m} + \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4} + 4\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/6}\right\}. \tag{57.5}$$

Dividing both sides of (57.5) by $\{\beta(1-\beta)\}^{1/4}$ and employing (57.3) and (57.4), we find that

$$\frac{Q^6}{R^6}\left(\frac{R^4}{Q^2} + \frac{R^6}{Q^6} + 4\frac{R^4}{Q^4}\right) = 13\frac{Q^2}{R^4} + \frac{Q^6}{R^6} + 4\frac{Q^4}{R^4},$$

or

$$R^4Q^4 + R^6 + 4R^4Q^2 = 13R^2Q^2 + Q^6 + 4R^2Q^4. \tag{57.6}$$

The replacement of q by $-q$ takes R^2 into $-P^2$ and leaves Q^2 invariant. Thus, (57.6) is transformed into

$$P^4Q^4 - P^6 + 4P^4Q^2 = -13P^2Q^2 + Q^6 - 4P^2Q^4,$$

which is readily seen to be equivalent to (57.1).

Entry 58 (p. 325). *Let*

$$P = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \quad \textit{and} \quad Q = \frac{f(-q^{2/5})}{q^{2/5}f(-q^{10})}.$$

Then

$$PQ + \frac{25}{PQ} = \left(\frac{Q}{P}\right)^3 - 4\left(\frac{Q}{P}\right)^2 - 4\left(\frac{P}{Q}\right)^2 + \left(\frac{P}{Q}\right)^3 \tag{58.1}$$

and

$$P^2Q^2 + 5PQ = P^3 - 2P^2Q - 2PQ^2 + Q^3. \tag{58.2}$$

PROOF. Let

$$R = \frac{f(q^{1/5})}{q^{1/5}f(q^5)},$$

and let α and β be the squares of the moduli corresponding to $q^{1/5}$ and q^5, so that β has degree 25. From Entries 12(i), (iii), respectively, in Chapter 17, it follows that

$$R = \sqrt{m}\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/24} \quad \text{and} \quad Q = \sqrt{m}\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12}, \tag{58.3}$$

where m is the multiplier for degree 25. From (58.3), we can deduce that

$$m = \frac{R^4}{Q^2} \tag{58.4}$$

and

$$\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/24} = \frac{Q}{R}. \tag{58.5}$$

From Entries 15(i), (ii) of Chapter 19 (Part III [6, p. 291]), we deduce that

$$\{\alpha(1-\alpha)\}^{1/8}\left\{\sqrt{m} + \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} + 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12}\right\}$$
$$= \{\beta(1-\beta)\}^{1/8}\left\{\frac{5}{\sqrt{m}} + \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} + 2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12}\right\}, \tag{58.6}$$

in the same way that we previously derived (55.7) and (57.5). Dividing both sides of (58.6) by $\{\beta(1-\beta)\}^{1/8}$ and employing (58.4) and (58.5), we arrive at

$$\frac{Q^3}{R^3}\left(\frac{R^2}{Q} + \frac{R^3}{Q^3} + 2\frac{R^2}{Q^2}\right) = 5\frac{Q}{R^2} + \frac{Q^3}{R^3} + 2\frac{Q^2}{R^2},$$

or

$$R^2Q^2 + R^3 + 2R^2Q = 5RQ + Q^3 + 2RQ^2. \tag{58.7}$$

If q is replaced by $-q$, R is replaced by $-P$ and Q is invariant. Thus, (58.7) is transformed into (58.2).

Squaring both sides of (58.2) and collecting together like terms, we find that

$$P^4Q^4 + 25P^2Q^2 = P^6 - 4P^5Q - 4PQ^5 + Q^6,$$

which is readily seen to be equivalent to (58.1).

On pages 313, 323, 324, 325, 327, and 330, Ramanujan records a total of nine P–Q modular equations with moduli of orders 1, 3, 5, and 15. We prove these nine results in the next nine entries.

Entry 59 (p. 330). *If*

$$P = \frac{f(-q^3)f(-q^5)}{q^{1/3}f(-q)f(-q^{15})} \quad \textit{and} \quad Q = \frac{f(-q^6)f(-q^{10})}{q^{2/3}f(-q^2)f(-q^{30})},$$

then

$$PQ + \frac{1}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3 + 4. \tag{59.1}$$

Proof. Let

$$R = \frac{f(q^3)f(q^5)}{q^{1/3}f(q)f(q^{15})}.$$

By Entries 12(i), (iii), respectively, in Chapter 17,

$$R = \left(\frac{z_3z_5}{z_1z_{15}}\right)^{1/2}\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/24} \tag{59.2}$$

and

$$Q = \left(\frac{z_3z_5}{z_1z_{15}}\right)^{1/2}\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/12}, \tag{59.3}$$

where α, β, γ, and δ, have degrees 1, 3, 5, and 15, respectively. It readily follows from (59.2) and (59.3) that

$$\frac{Q}{R} = \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/24} \tag{59.4}$$

and

$$\frac{R^2}{Q} = \sqrt{\frac{m'}{m}}, \tag{59.5}$$

where $m = z_1/z_3$ and $m' = z_5/z_{15}$.

By Entries 11(viii), (ix), respectively, in Chapter 20 (Part III [6, p. 384]),

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} = \sqrt{\frac{m'}{m}} \tag{59.6}$$

and

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} = -\sqrt{\frac{m}{m'}}. \tag{59.7}$$

Multiply both sides of (59.6) by $\{\beta\gamma(1-\beta)(1-\gamma)\}^{1/8}$ and multiply both sides of (59.7) by $\{\alpha\delta(1-\alpha)(1-\delta)\}^{1/8}$. Combining the resulting two equalities, we find that

$$\begin{aligned}\{\beta\gamma(1-\beta)(1-\gamma)\}^{1/8}&\left\{\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} + \sqrt{\frac{m'}{m}}\right\}\\ &= \{\alpha\delta(1-\alpha)(1-\delta)\}^{1/8}\left\{\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} - \sqrt{\frac{m}{m'}}\right\}.\end{aligned} \tag{59.8}$$

Dividing both sides of (59.8) by $\{\alpha\delta(1-\alpha)(1-\delta)\}^{1/8}$ and employing (59.4) and (59.5), we find that

$$\left(\frac{Q}{R}\right)^3\left\{\left(\frac{R}{Q}\right)^3 + \frac{R^2}{Q}\right\} = \left(\frac{Q}{R}\right)^3 - \frac{Q}{R^2},$$

or

$$R^3 + R^2Q^2 = Q^3 - RQ. \tag{59.9}$$

If we replace q by $-q$, R is converted to $-P$ and Q is unaffected. Thus, (59.9) is transformed into

$$P^2Q^2 - PQ = P^3 + Q^3. \tag{59.10}$$

Squaring both sides of (59.10), we deduce that

$$P^4Q^4 + P^2Q^2 = P^6 + Q^6 + 4P^3Q^3,$$

which immediately implies (59.1).

Entry 60 (p. 330). *If*

$$P = \frac{f(-q)f(-q^5)}{q^{1/2}f(-q^3)f(-q^{15})} \quad \textit{and} \quad Q = \frac{f(-q^2)f(-q^{10})}{qf(-q^6)f(-q^{30})},$$

then

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 - 4\frac{Q}{P} - 4\frac{P}{Q} + \left(\frac{P}{Q}\right)^3. \tag{60.1}$$

PROOF. The proof is very similar to the previous proof, and so we shall provide fewer details. Let

$$R = \frac{f(q)f(q^5)}{q^{1/2}f(q^3)f(q^{15})}.$$

By Entries 12(i), (iii), respectively, in Chapter 17,

$$R = \sqrt{mm'}\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/24}$$

and

$$Q = \sqrt{mn'}\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/12}.$$

It follows that

$$\frac{Q}{R} = \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/24} \tag{60.2}$$

and

$$\frac{R^2}{Q} = \sqrt{mm'}. \tag{60.3}$$

Take Entries 11(x), (xi) in Chapter 20 (Párt III [6, p. 384]) and multiply both sides of the former equation by $\{\alpha\gamma(1-\alpha)(1-\gamma)\}^{1/4}$ and both sides of the latter by $\{\beta\delta(1-\beta)(1-\delta)\}^{1/4}$. Combining the two equations, we find that

$$\begin{aligned}\{\alpha\gamma(1-\alpha)(1-\gamma)\}^{1/4}&\left\{mm' + \left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} + 4\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/6}\right\}\\ = \{\beta\delta(1-\beta)(1-\delta)\}^{1/4}&\left\{\frac{9}{mm'} + \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/4} + 4\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/6}\right\}.\end{aligned} \tag{60.4}$$

Using (60.2) and (60.3) in (60.4), we eventually deduce that

$$R^4Q^4 + 4R^4Q^2 + R^6 = 9R^2Q^2 + 4R^2Q^4 + Q^6. \tag{60.5}$$

Replacing q by $-q$ takes R^2 into $-P^2$ and fixes Q^2. Thus, (60.5) yields

$$P^4Q^4 + 4P^4Q^2 - P^6 = -9P^2Q^2 - 4P^2Q^4 + Q^6,$$

which is readily seen to be equivalent to (60.1).

The proofs of the remaining seven P–Q modular equations for α, β, γ, and δ of degrees 1, 3, 5, and 15, respectively, are more difficult. Because many facts from Section 11 of Chapter 20 are needed, it seems appropriate to record all of the necessary material here (Part III [6, pp. 388–393]).

Recall that $m = z_1/z_3$ and $m' = z_3/z_{15}$. Set

$$t = \left(\frac{z_3 z_5}{z_1 z_{15}}\right)^{1/2},$$

so that

$$m' = mt^2. \tag{61.1}$$

Furthermore, we set $\mu = z_1/z_5$ and $\mu' = z_3/z_{15}$. Thus, from (61.1),

$$\mu' = \mu t^2. \tag{61.2}$$

All references of the form (11.––) refer to Section 11 of Chapter 20. By (11.10),

$$\begin{aligned} \alpha &= \frac{(m-1)(3+m)^3}{16m^3}, & \beta &= \frac{(m-1)^3(3+m)}{16m}, \\ \gamma &= \frac{(m'-1)(3+m')^3}{16m'^3}, & \delta &= \frac{(m'-1)^3(3+m')}{16m'}, \\ 1-\alpha &= \frac{(m+1)(3-m)^3}{16m^3}, & 1-\beta &= \frac{(m+1)^3(3-m)}{16m}, \\ 1-\gamma &= \frac{(m'+1)(3-m')^3}{16m'^3}, & 1-\delta &= \frac{(m'+1)^3(3-m')}{16m'}. \end{aligned} \tag{61.3}$$

From (11.13), (11.14), (11.17), and (11.18), respectively,

$$\left(1+\frac{1}{t}\right)^5(1-t) = (m^2-1)(9m'^{-2}-1), \tag{61.4}$$

$$\left(1+\frac{1}{t}\right)(1-t)^5 = (m'^2-1)(9m^{-2}-1), \tag{61.5}$$

$$m^2 + \frac{9}{m^2t^4} = \frac{t^6+5t^5+5t^4-5t^2+5t-1}{t^5}, \tag{61.6}$$

and

$$\mu + \frac{5}{\mu t^2} = \frac{t^4+3t^3+3t-1}{t^3}. \tag{61.7}$$

From (11.19), (11.20), (11.21), and (11.22), respectively,

$$2t^5m^2 = t^6+5t^5+5t^4-5t^2+5t-1-4t^2(t^2+2t-1)RS, \tag{61.8}$$

$$2t^3\mu = t^4+3t^3+3t-1-4t^2RS, \tag{61.9}$$

where

$$4t^2R^2 = t^4 + t^3 + 2t^2 - t + 1 \tag{61.10}$$

and

$$4t^2S^2 = t^4 + 5t^3 + 2t^2 - 5t + 1 = (t^2 + 4t - 1)(t^2 + t - 1). \tag{61.11}$$

From (11.24), (11.25), (11.31), (11.33), and (11.34), respectively,

$$\{(m^2 - 1)(m'^2 - 1)\}^{1/3} = \frac{t^4 + 3t^3 + 2t^2 - 3t + 1 - 4t^2RS}{2t^2} = (R - S)^2, \tag{61.12}$$

$$\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} = \frac{(R-S)^6}{(t^{-1}-t)^3}, \tag{61.13}$$

$$\left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/4} = \frac{(t^{-1}-t)^6}{(m^2-1)^2(9m^{-2}-1)^2}, \tag{61.14}$$

$$(m^2 - 1)(9m^{-2} - 1) = (t^{-1} - t)\{(2 - t^{-1} + t)R + (t^{-1} + t)S\}^2, \tag{61.15}$$

and

$$(2 - t^{-1} + t)R + (t^{-1} + t)S = \frac{(t^{-1} - t)^2}{(2 - t^{-1} + t)R - (t^{-1} + t)S}. \tag{61.16}$$

Entry 61 (p. 330). *If*

$$P = \frac{f(-q^6)f(-q^5)}{q^{1/4}f(-q^2)f(-q^{15})} \quad \textit{and} \quad Q = \frac{f(-q^3)f(-q^{10})}{q^{3/4}f(-q)f(-q^{30})},$$

then

$$PQ + 1 + \frac{1}{PQ} = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2. \tag{61.17}$$

Proof. Set

$$M = \frac{f(-q^6)f(q^5)}{q^{1/4}f(-q^2)f(q^{15})} \quad \text{and} \quad N = \frac{f(q^3)f(-q^{10})}{q^{3/4}f(q)f(-q^{30})}.$$

Observe that if we replace q by $-q$ in (61.17), then PQ is transformed into $-MN$ and $(P/Q)^2$ is converted into $-(M/N)^2$. Thus, it suffices to prove that

$$MN - 1 + \frac{1}{MN} = \left(\frac{M}{N}\right)^2 + \left(\frac{N}{M}\right)^2. \tag{61.18}$$

By Entries 12(i), (iii) in Chapter 17,

$$M = \sqrt{\frac{z_3 z_5}{z_1 z_{15}}} \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12} \left(\frac{\gamma(1-\gamma)}{\delta(1-\delta)}\right)^{1/24} \tag{61.19}$$

and

$$N = \sqrt{\frac{z_3 z_5}{z_1 z_{15}}} \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/24} \left(\frac{\gamma(1-\gamma)}{\delta(1-\delta)}\right)^{1/12}. \tag{61.20}$$

It follows that

$$MN = \frac{z_3 z_5}{z_1 z_{15}} \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} = \frac{m'}{m} \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8},$$

where $m = z_1/z_3$ and $m' = z_5/z_{15}$.

By (61.3) and considerable simplification,

$$\begin{aligned} MN &= \frac{m'}{m} \left(\frac{m^4(m-1)^2(m+1)^2(3+m')^2(3-m')^2}{m'^4(3+m)^2(3-m)^2(m'-1)^2(m'+1)^2}\right)^{1/8} \\ &= \sqrt{\frac{m'}{m}} \left(\frac{(m^2-1)(9-m'^2)}{(9-m^2)(m'^2-1)}\right)^{1/4}. \end{aligned}$$

Hence, by (61.1), (61.4), and (61.5),

$$\begin{aligned} MN &+ \frac{1}{MN} - 1 \\ &= \frac{m'}{m} \left(\frac{(m^2-1)(9m'^{-2}-1)}{(9m^{-2}-1)(m'^2-1)}\right)^{1/4} + \frac{m}{m'} \left(\frac{(9m^{-2}-1)(m'^2-1)}{(m^2-1)(9m'^{-2}-1)}\right)^{1/4} - 1 \\ &= t^2 \frac{1+1/t}{1-t} + \frac{1-t}{t^2(1+1/t)} - 1 \\ &= \frac{t(t+1)}{1-t} + \frac{1-t}{t(t+1)} - 1 \\ &= \frac{t^4+3t^3+2t^2-3t+1}{t(1-t^2)}. \end{aligned} \tag{61.21}$$

Next, from (61.19) and (61.20),

$$\frac{M}{N} = \left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/24}.$$

So from (61.13),

$$\left(\frac{M}{N}\right)^2 + \left(\frac{N}{M}\right)^2 = \left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/12} + \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/12}$$
$$= \frac{(R-S)^2}{t^{-1}-t} + \frac{t^{-1}-t}{(R-S)^2}, \tag{61.22}$$

where R and S are defined by (61.10) and (61.11), respectively. For brevity, set

$$A := t^4 + t^3 + 2t^2 - t + 1 \tag{61.23}$$

and

$$B := t^4 + 5t^3 + 2t^2 - 5t + 1. \tag{61.24}$$

Thus, from (61.10), (61.11), and (61.12), respectively,

$$4t^2R^2 = A, \qquad 4t^2S^2 = B, \tag{61.25}$$

and

$$(R-S)^2 = \frac{\frac{1}{2}(A+B) - 4t^2RS}{2t^2} = \frac{(\sqrt{A}-\sqrt{B})^2}{4t^2}.$$

Returning to (61.22) and employing the last equality above, we deduce that

$$\left(\frac{M}{N}\right)^2 + \left(\frac{N}{M}\right)^2$$
$$= \frac{(\sqrt{A}-\sqrt{B})^2}{4t(1-t^2)} + \frac{4t(1-t^2)}{(\sqrt{A}-\sqrt{B})^2}$$
$$= \frac{(\sqrt{A}-\sqrt{B})^4 + 16t^2(1-t^2)^2}{4t(1-t^2)(\sqrt{A}-\sqrt{B})^2}$$
$$= \frac{A^2 + B^2 + 6AB - 4(A+B)\sqrt{AB} + 16t^2(1-t^2)^2}{4t(1-t^2)(A+B-2\sqrt{AB})}$$
$$= \frac{(A^2 + B^2 + 6AB + 16t^2(1-t^2)^2 - 4(A+B)\sqrt{AB})(A+B+2\sqrt{AB})}{4t(1-t^2)((A+B)^2 - 4AB)}$$
$$= \frac{(A+B)\{(A-B)^2 + 16t^2(1-t^2)^2\} + \{-2(A-B)^2 + 32t^2(1-t^2)^2\}\sqrt{AB}}{4t(1-t^2)(A-B)^2}.$$

Note that $A - B = 4t(1-t^2)$. Hence,

$$\left(\frac{M}{N}\right)^2 + \left(\frac{N}{M}\right)^2 = \frac{(A+B)32t^2(1-t^2)^2}{64t^3(1-t^2)^3}$$
$$= \frac{A+B}{2t(1-t^2)}$$
$$= \frac{t^4 + 3t^3 + 2t^2 - 3t + 1}{t(1-t^2)}. \tag{61.26}$$

Comparing (61.26) and (61.21), we see that we have established (61.18) to complete the proof.

Entry 62 (p. 324). *Let*

$$P = \frac{f(-q)}{q^{1/12}f(-q^3)} \quad \textit{and} \quad Q = \frac{f(-q^5)}{q^{5/12}f(-q^{15})}.$$

Then

$$(PQ)^2 + 5 + \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3. \tag{62.1}$$

PROOF. Let

$$T = \frac{f(q)}{q^{1/12}f(q^3)} \quad \text{and} \quad U = \frac{f(q^5)}{q^{5/12}f(q^{15})}.$$

If we replace q by $-q$ in (62.1), then $(PQ)^2$ and $(P/Q)^3$ are transformed, respectively, into $-(TU)^2$ and $-(T/U)^3$. Equivalently, we shall then prove that

$$(TU)^2 + \frac{9}{(TU)^2} = \left(\frac{U}{T}\right)^3 - \left(\frac{T}{U}\right)^3 + 5. \tag{62.2}$$

By Entry 12(i) in Chapter 17,

$$T = \sqrt{\frac{z_1}{z_3}}\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/24} \quad \text{and} \quad U = \sqrt{\frac{z_5}{z_{15}}}\left(\frac{\gamma(1-\gamma)}{\delta(1-\delta)}\right)^{1/24}.$$

Hence, by (61.3), (61.1), (61.4), and (61.5),

$$\begin{aligned}
\left(\frac{T}{U}\right)^3 &= \left(\frac{m}{m'}\right)^{3/2}\left(\frac{m'^4(3+m)^2(3-m)^2(m'-1)^2(m'+1)^2}{m^4(m-1)^2(m+1)^2(3+m')^2(3-m')^2}\right)^{1/8} \\
&= \left(\frac{m}{m'}\right)^{3/2}\left(\frac{(9m^{-2}-1)(m'^2-1)}{(m^2-1)(9m'^{-2}-1)}\right)^{1/4} \\
&= \frac{1-t}{t^3(1+1/t)}.
\end{aligned} \tag{62.3}$$

It follows from (62.3) that

$$\begin{aligned}
\left(\frac{U}{T}\right)^3 - \left(\frac{T}{U}\right)^3 + 5 &= \frac{t^3(1+1/t)}{1-t} - \frac{1-t}{t^3(1+1/t)} + 5 \\
&= \frac{t^6 + 2t^5 - 4t^4 + 4t^2 + 2t - 1}{t^2(1-t^2)}.
\end{aligned} \tag{62.4}$$

Next,

$$TU = \sqrt{\frac{z_1 z_5}{z_3 z_{15}}} \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/24}.$$

So, by (61.1) and (61.13),

$$(TU)^2 = mm'\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/12} = m^2t^2\frac{t^{-1}-t}{(R-S)^2}, \tag{62.5}$$

where R and S are given in (61.23)–(61.25). Observe that

$$4t^2(R^2 - S^2) = -4t(t^2-1). \tag{62.6}$$

Thus, from (62.6),

$$\frac{1}{(R-S)^2} = \frac{t^2(R+S)^2}{(1-t^2)^2}. \tag{62.7}$$

Hence, by (62.5), (62.7), and (61.25),

$$\begin{aligned}(TU)^2 + \frac{9}{(TU)^2} &= m^2t^2\frac{t^{-1}-t}{(R-S)^2} + \frac{9}{m^2t^2}\frac{(R-S)^2}{t^{-1}-t}\\ &= m^2t^2\frac{t}{1-t^2}(R+S)^2 + \frac{9}{m^2t^2}\frac{t}{1-t^2}(R-S)^2\\ &= \frac{t}{4(1-t^2)}\left\{m^2(A+B+2\sqrt{AB}) + \frac{9}{m^2t^4}(A+B-2\sqrt{AB})\right\}\\ &= \left(m^2 + \frac{9}{m^2t^4}\right)\frac{t(A+B)}{4(1-t^2)} + \left(m^2 - \frac{9}{m^2t^4}\right)\frac{t\sqrt{AB}}{2(1-t^2)}.\end{aligned} \tag{62.8}$$

By (61.6) and (61.8), respectively,

$$m^2 + \frac{9}{m^2t^4} = \frac{t^6 + 5t^5 + 5t^4 - 5t^2 + 5t - 1}{t^5} =: \frac{C}{t^5} \tag{62.9}$$

and

$$2t^5m^2 = C - 4t^2(t^2 + 2t - 1)RS,$$

where R and S are given in (61.23)–(61.25). Thus,

$$\begin{aligned}m^2 - \frac{9}{m^2t^4} = 2m^2 - \left(m^2 + \frac{9}{m^2t^4}\right) &= -\frac{4(t^2+2t-1)RS}{t^3}\\ &= -\frac{(t^2+2t-1)\sqrt{AB}}{t^5}.\end{aligned} \tag{62.10}$$

Employing (62.9) and (62.10) in (62.8), we deduce that

$$(TU)^2 + \frac{9}{(TU)^2} = \frac{(A+B)C}{4t^4(1-t^2)} - \frac{(t^2+2t-1)AB}{2t^4(1-t^2)}$$

$$= \frac{1}{4t^4(1-t^2)}\{(A+B)C - 2(t^2+2t-1)AB\}$$

$$= \frac{t^6+2t^5-4t^4+4t^2+2t-1}{t^2(1-t^2)}. \tag{62.11}$$

Comparing (62.4) and (62.11), we see that we have established (62.2) and hence (62.1) as well.

Entry 63 (p. 325). *If*

$$P = \frac{f^3(-q)}{q^{1/2}f^3(-q^5)} \quad \textit{and} \quad Q = \frac{f^3(-q^3)}{q^{3/2}f^3(-q^{15})},$$

then

$$PQ + \frac{125}{PQ} = \left(\frac{Q}{P}\right)^2 - 9\frac{Q}{P} - 9\frac{P}{Q} - \left(\frac{P}{Q}\right)^2. \tag{63.1}$$

Proof. Let

$$T = \frac{f^3(q)}{q^{1/2}f^3(q^5)} \quad \text{and} \quad U = \frac{f^3(q^3)}{q^{3/2}f^3(q^{15})}.$$

By replacing q by $-q$, we see that PQ and P/Q in (63.1) are transformed into TU and $-T/U$, respectively. Thus, it suffices to prove that

$$TU + \frac{125}{TU} = \left(\frac{U}{T}\right)^2 + 9\frac{U}{T} + 9\frac{T}{U} - \left(\frac{T}{U}\right)^2. \tag{63.2}$$

By Entry 12(i) in Chapter 17,

$$T = \left(\frac{z_1}{z_5}\right)^{3/2}\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/8} \quad \text{and} \quad U = \left(\frac{z_3}{z_{15}}\right)^{3/2}\left(\frac{\beta(1-\beta)}{\delta(1-\delta)}\right)^{1/8}.$$

By the same calculation as in the previous proof, i.e., by (62.3),

$$\frac{T}{U} = \left(\frac{z_1 z_{15}}{z_3 z_5}\right)^{3/2}\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} = \frac{1-t}{t^3(1+1/t)}.$$

Hence,

$$\begin{aligned}
&\left(\frac{U}{T}\right)^2 + 9\frac{U}{T} + 9\frac{T}{U} - \left(\frac{T}{U}\right)^2 \\
&\quad = \frac{t^4(1+t)^2}{(1-t)^2} + \frac{9t^2(1+t)}{1-t} + \frac{9(1-t)}{t^2(1+t)} - \frac{(1-t)^2}{t^4(1+t)^2} \\
&\quad = \frac{t^{12}+4t^{11}-3t^{10}-14t^9+t^8+18t^7+18t^5-t^4-14t^3+3t^2+4t-1}{t^4(1-t^2)^2}.
\end{aligned} \tag{63.3}$$

Next, from (61.2), (61.14), and (61.16),

$$\begin{aligned}
TU &= \left(\frac{z_1 z_3}{z_5 z_{15}}\right)^{3/2}\left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{1/8} \\
&= \mu^3 t^3 \frac{(m^2-1)(9m^{-2}-1)}{(t^{-1}-t)^3} \\
&= \frac{\mu^3 t^3}{(t^{-1}-t)^2}\{(2-t^{-1}+t)R + (t^{-1}+t)S\}^2.
\end{aligned}$$

Utilizing (61.16), we thus find that

$$\begin{aligned}
TU + \frac{125}{TU} &= \frac{\mu^3 t^3}{(t^{-1}-t)^2}\{(2-t^{-1}+t)R + (t^{-1}+t)S\}^2 \\
&\quad + \frac{125}{\mu^3 t^3 (t^{-1}-t)^2}\{(2-t^{-1}+t)R - (t^{-1}+t)S\}^2 \\
&= \frac{1}{(t^{-1}-t)^2}\left(\mu^3 t^3 + \frac{125}{\mu^3 t^3}\right)\{(2-t^{-1}+t)^2R^2 + (t^{-1}+t)^2S^2\} \\
&\quad + \frac{2}{(t^{-1}-t)^2}\left(\mu^3 t^3 - \frac{125}{\mu^3 t^3}\right)(2-t^{-1}+t)(t^{-1}+t)RS.
\end{aligned} \tag{63.4}$$

From (61.7) and (61.9), respectively,

$$\mu + \frac{5}{\mu t^2} = \frac{t^4+3t^3+3t-1}{t^3} =: \frac{D}{t^3}$$

and

$$2t^3\mu = D - 4t^2 RS.$$

Hence, from

$$\left(\mu t + \frac{5}{\mu t}\right)^3 = \mu^3 t^3 + \frac{125}{\mu^3 t^3} + 15\left(\mu t + \frac{5}{\mu t}\right),$$

we deduce that

$$\mu^3 t^3 + \frac{125}{\mu^3 t^3} = \frac{D^3}{t^6} - 15\frac{D}{t^2}. \tag{63.5}$$

Also, we may deduce that

$$\begin{aligned}
\mu^3 t^3 - \frac{125}{\mu^3 t^3} &= 2\mu^3 t^3 - \left(\mu^3 t^3 + \frac{125}{\mu^3 t^3}\right) \\
&= 2\left(\frac{D - 4t^2 RS}{2t^2}\right)^3 - \frac{D^3}{t^6} + 15\frac{D}{t^2} \\
&= \frac{1}{4t^6}(D^3 - 12D^2 t^2 RS + 48Dt^4 R^2 S^2 - 64t^6 R^3 S^3) - \frac{D^3}{t^6} + \frac{15}{t^2}D \\
&= \frac{1}{4t^6}(-3D^3 - 3D^2\sqrt{AB} + 3DAB - (AB)^{3/2}) + \frac{15}{t^2}D, \qquad (63.6)
\end{aligned}$$

by (61.25).

Using (63.5) and (63.6) in (63.4) and employing (61.25) once again, we find that

$$\begin{aligned}
TU &+ \frac{125}{TU} \\
&= \frac{(D^3 - 15Dt^4)\{(2t - 1 + t^2)^2 A + (1 + t^2)^2 B\}}{4t^8(1 - t^2)^2} \\
&\quad + \frac{(-3D^3 - 3D^2\sqrt{AB} + 3DAB - (AB)^{3/2} + 60Dt^4)(2t - 1 + t^2)(1 + t^2)\sqrt{AB}}{8t^8(1 - t^2)^2} \\
&= \frac{\begin{aligned}2(D^3 - 15Dt^4)\{(2t - 1 + t^2)^2 A \\ + (1 + t^2)^2 B\} + (-3D^2 - AB)(2t - 1 + t^2)(1 + t^2)AB\end{aligned}}{8t^8(1 - t^2)^2} \\
&\quad + \frac{(-3D^3 + 3DAB + 60Dt^4)(2t - 1 + t^2)(1 + t^2)\sqrt{AB}}{8t^8(1 - t^2)^2}. \qquad (63.7)
\end{aligned}$$

An easy calculation shows that

$$-D^2 + AB + 20t^4 = 0, \tag{63.8}$$

and so the expression involving $\sqrt{AB}$ in (63.7) is identically equal to 0. Employing *Mathematica* in (63.7), we find that

$$\begin{aligned}
TU &+ \frac{125}{TU} \\
&= \frac{\begin{aligned}-8t^4 + 32t^5 + 24t^6 - 112t^7 - 8t^8 + 144t^9 \\ + 144t^{11} + 8t^{12} - 112t^{13} - 24t^{14} + 32t^{15} + 8t^{16}\end{aligned}}{8t^8(1 - t^2)^2}.
\end{aligned}$$

Comparing the equality above with (63.3), we see that we have established (63.2) to complete the proof.

Entry 64 (p. 323). *If*

$$P=\frac{f(-q^3)}{q^{1/12}f(-q^5)} \quad \textit{and} \quad Q=\frac{f(-q)}{q^{7/12}f(-q^{15})},$$

then

$$(PQ)^3-\frac{125}{(PQ)^3}=\left(\frac{Q}{P}\right)^4+\left(\frac{Q}{P}\right)^2-9\left(\frac{P}{Q}\right)^2-81\left(\frac{P}{Q}\right)^4. \tag{64.1}$$

Proof. Let

$$M=\frac{f(q^3)}{q^{1/12}f(q^5)} \quad \text{and} \quad N=\frac{f(q)}{q^{7/12}f(q^{15})}.$$

Replacing q by $-q$ in (64.1), we find that PQ and $(P/Q)^2$ are transformed into MN and $-(M/N)^2$, respectively. Thus, it suffices to prove that

$$(MN)^3-\frac{125}{(MN)^3}=\left(\frac{N}{M}\right)^4-\left(\frac{N}{M}\right)^2+9\left(\frac{M}{N}\right)^2-81\left(\frac{M}{N}\right)^4. \tag{64.2}$$

By Entry 12(i) in Chapter 17,

$$M=\sqrt{\frac{z_3}{z_5}}\left(\frac{\beta(1-\beta)}{\gamma(1-\gamma)}\right)^{1/24} \quad \text{and} \quad N=\sqrt{\frac{z_1}{z_{15}}}\left(\frac{\alpha(1-\alpha)}{\delta(1-\delta)}\right)^{1/24}.$$

Thus, employing also (61.2), (61.14), and (61.15), we find that

$$\begin{aligned}(MN)^3=\left(\frac{z_1z_3}{z_5z_{15}}\right)^{3/2}\left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{1/8}&=(\mu t)^3\frac{(m^2-1)(9m^{-2}-1)}{(t^{-1}-t)^3}\\ &=\frac{(\mu t)^3}{(t^{-1}-t)^2}\{(2-t^{-1}+t)R\\ &\quad+(t^{-1}+t)S\}^2,\end{aligned} \tag{64.3}$$

where R and S are defined in (61.10) and (61.11). By (61.14) and (61.16),

$$\frac{1}{(MN)^3}=\frac{\{(2-t^{-1}+t)R-(t^{-1}+t)S\}^2}{(\mu t)^3(t^{-1}-t)^2}. \tag{64.4}$$

Utilizing (64.3) and (64.4), we deduce that

$$\begin{aligned}(MN)^3-\frac{125}{(MN)^3}&=\left(\mu^3t^3-\frac{1}{\mu^3t^3}\right)\frac{\{(2t-1+t^2)^2R^2+(1+t^2)^2S^2\}}{(1-t^2)^2}\\ &\quad+\left(\mu^3t^3+\frac{1}{\mu^3t^3}\right)\frac{2(2t-1+t^2)(1+t^2)RS}{(1-t^2)^2}.\end{aligned} \tag{64.5}$$

From (61.7) and (61.9), respectively,

$$\mu + \frac{5}{\mu t^2} = \frac{t^4 + 3t^3 + 3t - 1}{t^3} = \frac{D}{t^3} \tag{64.6}$$

and

$$2t^3\mu = D - 4t^2RS. \tag{64.7}$$

Now,

$$\left(\mu t + \frac{5}{\mu t}\right)^3 = \mu^3 t^3 + \frac{125}{\mu^3 t^3} + 15\left(\mu t + \frac{5}{\mu t}\right).$$

Hence, by (64.6),

$$\mu^3 t^3 + \frac{125}{\mu^3 t^3} = \frac{D^3}{t^6} - 15\frac{D}{t^2}. \tag{64.8}$$

Also, by (63.6),

$$\mu^3 t^3 - \frac{125}{\mu^3 t^3} = \frac{1}{4t^6}(-3D^3 - 3D^2\sqrt{AB} + 3DAB - (AB)^{3/2}) + \frac{15}{t^2}D. \tag{64.9}$$

Employing (61.25), (64.8), and (64.9) in (64.5), we deduce that

$$\begin{aligned}
(MN)^3 - \frac{125}{(MN)^3} &= \frac{1}{16t^8(1-t^2)^2}\{(-3D^2 - 3D^2\sqrt{AB} + 3DAB - (AB)^{3/2} \\
&\quad + 60Dt^4)\{(2t - 1 + t^2)^2 A + (1 + t^2)^2 B\} \\
&\quad + 8(D^3 - 15Dt^4)(2t - 1 + t^2)(1 + t^2)\sqrt{AB}\} \\
&= \frac{1}{16t^8(1-t^2)^2}\{(-3D^2 + 3DAB + 60Dt^4) \\
&\quad \times \{(2t - 1 + t^2)^2 A + (1 + t^2)^2 B\} \\
&\quad + ((-3D^2 - AB)\{(2t - 1 + t^2)^2 A + (1 + t^2)^2 B\} \\
&\quad + 8(D^3 - 15Dt^4)(2t - 1 + t^2)(1 + t^2))\sqrt{AB}\}.
\end{aligned}$$

Recalling (63.8) and employing *Mathematica* to calculate the expression multiplying $\sqrt{AB}$, we find that

$$(MN)^3 - \frac{125}{(MN)^3} = \frac{-t^8 - t^7 + 6t^6 - t^5 - 6t^4 + t^3 + 6t^2 + t - 1}{t^4(1-t^2)^2}\sqrt{AB}. \tag{64.10}$$

We now turn to the right side of (64.2). By (61.13),

$$\left(\frac{M}{N}\right)^2 = \frac{1}{mm'}\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/12} = \frac{1}{mm'}\frac{(R-S)^2}{t^{-1}-t}. \tag{64.11}$$

But by (62.7),

$$\frac{(R-S)^2}{t^{-1}-t}=\frac{t^{-1}-t}{(R+S)^2}. \tag{64.12}$$

Hence, by (64.11) and (64.12),

$$\left(\frac{N}{M}\right)^2=mm'\frac{(R+S)^2}{t^{-1}-t}. \tag{64.13}$$

Thus, by (64.11), (64.13), and (61.1),

$$\begin{aligned}9\left(\frac{M}{N}\right)^2-\left(\frac{N}{M}\right)^2&=\frac{9}{m^2t^2}\frac{R^2+S^2-2RS}{t^{-1}-t}-m^2t^2\frac{R^2+S^2+2RS}{t^{-1}-t}\\&=\frac{t^2}{t^{-1}-t}\left\{\left(\frac{9}{m^2t^4}-m^2\right)(R^2+S^2)-\left(m^2+\frac{9}{m^2t^4}\right)2RS\right\}.\end{aligned} \tag{64.14}$$

By (61.6),

$$m^2+\frac{9}{m^2t^4}=\frac{t^6+5t^5+5t^4-5t^2+5t-1}{t^5}=:\frac{C}{t^5}. \tag{64.15}$$

Thus,

$$m^2-\frac{9}{m^2t^4}=2m^2-\left(m^2+\frac{9}{m^2t^4}\right)=2m^2-\frac{C}{t^5}. \tag{64.16}$$

But by (61.8),

$$2t^5m^2=C-4t^2(t^2+2t-1)RS. \tag{64.17}$$

Using (64.17) in (64.16), we find that

$$m^2-\frac{9}{m^2t^4}=-\frac{4(t^2+2t-1)RS}{t^3}. \tag{64.18}$$

Employing (64.15) and (64.18) in (64.14), we deduce that

$$\begin{aligned}9\left(\frac{M}{N}\right)^2-\left(\frac{N}{M}\right)^2&=\frac{t^2}{t^{-1}-t}\frac{4(t^2+2t-1)RS}{t^3}(R^2+S^2)-\frac{2CRS}{t^5}\\&=\frac{1}{4t^4(1-t^2)}((t^2+2t-1)(A+B)-2C)\sqrt{AB},\end{aligned} \tag{64.19}$$

by (61.25).

Returning to (64.11) and (64.13), we find that

$$\left(\frac{N}{M}\right)^4 - 81\left(\frac{M}{N}\right)^4 = m^4t^4\frac{(R+S)^4}{(t^{-1}-t)^2} - \frac{81}{m^4t^4}\frac{(R-S)^4}{(t^{-1}-t)^2}$$
$$= \frac{1}{(t^{-1}-t)^2}\left\{\left(m^4t^4 + \frac{81}{m^4t^4}\right)(4R^3S + 4RS^3) + \left(m^4t^4 - \frac{81}{m^4t^4}\right)(R^4 + 6R^2S^2 + S^4)\right\}. \tag{64.20}$$

Squaring (64.15), we readily find that

$$m^4 + \frac{81}{m^4t^8} = \frac{C^2}{t^{10}} - \frac{18}{t^4}. \tag{64.21}$$

By (61.17) and (64.21),

$$m^4 - \frac{81}{m^4t^8} = 2m^4 - \left(m^4 + \frac{81}{m^4t^8}\right)$$
$$= \frac{1}{2}\left(\frac{C}{t^5} - \frac{4(t^2+2t-1)RS}{t^3}\right)^2 - \frac{C^2}{t^{10}} + \frac{18}{t^4}$$
$$= -\frac{C^2}{2t^{10}} - \frac{4C(t^2+2t-1)RS}{t^8} + \frac{8(t^2+2t-1)^2R^2S^2}{t^6} + \frac{18}{t^4}$$
$$= \frac{1}{2t^{10}}(-C^2 - 2C(t^2+2t-1)\sqrt{AB} + (t^2+2t-1)^2AB + 36t^6), \tag{64.22}$$

by (61.25).

Putting (64.21) and (64.22) in (64.20) and employing (61.25) again, we deduce that

$$\left(\frac{N}{M}\right)^4 - 81\left(\frac{M}{N}\right)^4$$
$$= \frac{1}{4(1-t^2)^2t^8}(\{(C^2-18t^6)(A+B) - \tfrac{1}{4}C(t^2+2t-1)(A^2+6AB+B^2)\}\sqrt{AB}$$
$$+ \tfrac{1}{8}(-C^2 + (t^2+2t-1)^2AB + 36t^6)(A^2+6AB+B^2)).$$

Using *Mathematica*, we find that

$$-C^2 + (t^2+2t-1)^2AB + 36t^6 \equiv 0.$$

Thus,

$$\left(\frac{N}{M}\right)^4 - 81\left(\frac{M}{N}\right)^4$$
$$= \frac{\{(C^2 - 18t^6)(A + B) - \frac{1}{4}C(t^2 + 2t - 1)(A^2 + 6AB + B^2)\}\sqrt{AB}}{4(1 - t^2)^2 t^8}. \tag{64.23}$$

In conclusion, from (64.19) and (64.23), we have shown that

$$\left(\frac{N}{M}\right)^4 - 81\left(\frac{M}{N}\right)^4 + 9\left(\frac{M}{N}\right)^2 - \left(\frac{N}{M}\right)^2$$
$$= \frac{\begin{array}{r}\{(C^2 - 18t^6)(A + B) - \frac{1}{4}C(t^2 + 2t - 1)(A^2 + 6AB + B^2)\\ + (1 - t^2)t^4((t^2 + 2t - 1)(A + B) - 2C)\}\sqrt{AB}\end{array}}{4(1 - t^2)^2 t^8}$$
$$= \frac{-t^8 - t^7 + 6t^6 - t^5 - 6t^4 + t^3 + 6t^2 + t - 1}{(1 - t^2)^2 t^4}\sqrt{AB}, \tag{64.24}$$

where once again we utilized *Mathematica*. Comparing (64.10) and (64.24), we see that we have completed the proof of (64.2).

Entry 65 (p. 313). *If*

$$P = \frac{f(-q)f(-q^2)}{q^{1/2}f(-q^5)f(-q^{10})} \quad \textit{and} \quad Q = \frac{f(-q^3)f(-q^6)}{q^{3/2}f(-q^{15})f(-q^{30})},$$

then

$$PQ + \frac{25}{PQ} = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2 - 3\left(\frac{Q}{P} + \frac{P}{Q} + 2\right). \tag{65.1}$$

Proof. Set

$$M = \frac{f(q)f(-q^2)}{q^{1/2}f(q^5)f(-q^{10})} \quad \text{and} \quad N = \frac{f(q^3)f(-q^6)}{q^{3/2}f(q^{15})f(-q^{30})}.$$

Changing the sign of q transforms PQ into MN and P/Q into $-M/N$. Thus, to prove (65.1), it suffices to show that

$$MN + \frac{25}{MN} = \left(\frac{N}{M}\right)^2 + \left(\frac{M}{N}\right)^2 + 3\left(\frac{N}{M} + \frac{M}{N} - 2\right). \tag{65.2}$$

By Entries 12(i), (iii) in Chapter 17,

$$M = \frac{z_1}{z_5}\left(\frac{\alpha(1 - \alpha)}{\gamma(1 - \gamma)}\right)^{1/8} \quad \text{and} \quad N = \frac{z_3}{z_{15}}\left(\frac{\beta(1 - \beta)}{\delta(1 - \delta)}\right)^{1/8}. \tag{65.3}$$

Thus, by (65.3) and (61.3),

$$\frac{M}{N}=\frac{m}{m'}\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8}=\frac{m}{m'}\left(\frac{(9m^{-2}-1)(m'^2-1)}{(m^2-1)(9m'^{-2}-1)}\right)^{1/4}.$$

Thus, by (61.1), (61.4), and (61.5),

$$\begin{aligned}
&\left(\frac{N}{M}\right)^2+\left(\frac{M}{N}\right)^2+3\left(\frac{N}{M}+\frac{M}{N}-2\right)\\
&=\left(\frac{m'}{m}\right)^2\left(\frac{(m^2-1)(9m'^{-2}-1)}{(9m^{-2}-1)(m'^2-1)}\right)^{1/2}+\left(\frac{m}{m'}\right)^2\left(\frac{(9m^{-2}-1)(m'^2-1)}{(m^2-1)(9m'^{-2}-1)}\right)^{1/2}\\
&\quad+3\frac{m'}{m}\left(\frac{(m^2-1)(9m'^{-2}-1)}{(9m^{-2}-1)(m'^2-1)}\right)^{1/4}+3\frac{m}{m'}\left(\frac{(9m^{-2}-1)(m'^2-1)}{(m^2-1)(9m'^{-2}-1)}\right)^{1/4}-6\\
&=\frac{t^2(1+t)^2}{(1-t)^2}+\frac{(1-t)^2}{t^2(1+t)^2}+\frac{3t(1+t)}{1-t}+\frac{3(1-t)}{t(1+t)}-6\\
&=\frac{t^8+t^7-6t^6+t^5+26t^4-t^3-6t^2-t+1}{t^2(1-t)^2(1+t)^2}.
\end{aligned}\tag{65.4}$$

Next, by (65.3), (61.2), (61.14), and (61.15),

$$\begin{aligned}
MN&=\mu^2t^2\left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{1/8}\\
&=\frac{\mu^2t^2}{(t^{-1}-t)^2}\{(2-t^{-1}+t)R+(t^{-1}+t)S\}^2.
\end{aligned}\tag{65.5}$$

Thus, by (65.5), (61.16), and (61.25),

$$\begin{aligned}
MN+\frac{25}{MN}&=\frac{\mu^2t^2}{(1-t^2)^2}\{(2t-1+t^2)R+(1+t^2)S\}^2\\
&\quad+\frac{25}{\mu^2t^2(1-t^2)^2}\{(2t-1+t^2)R-(1+t^2)S\}^2\\
&=\frac{\mu^2}{4(1-t^2)^2}\{(2t-1+t^2)^2A+(1+t^2)^2B\\
&\quad+2(2t-1+t^2)(1+t^2)\sqrt{AB}\}\\
&\quad+\frac{25}{4\mu^2t^4(1-t^2)^2}\{(2t-1+t^2)^2A+(1+t^2)^2B\\
&\quad-2(2t-1+t^2)(1+t^2)\sqrt{AB}\}\\
&=\frac{1}{4(1-t^2)^2}\left\{\left(\mu^2+\frac{25}{\mu^2t^4}\right)\{(2t-1+t^2)^2A+(1+t^2)^2B\}\right.\\
&\quad\left.+2\left(\mu^2-\frac{25}{\mu^2t^4}\right)(2t-1+t^2)(1+t^2)\sqrt{AB}\right\}.
\end{aligned}\tag{65.6}$$

Proceeding as in the proof of Entry 63, set, by (61.7),

$$\mu + \frac{5}{\mu t^2} = \frac{t^4 + 3t^3 + 3t - 1}{t^3} = \frac{D}{t^3}. \tag{65.7}$$

Thus,

$$\mu^2 + \frac{25}{\mu^2 t^4} = \frac{D^2}{t^6} - \frac{10}{t^2}. \tag{65.8}$$

By (61.9) and (61.25), $2t^3\mu = D - \sqrt{AB}$. Using also (65.8), we deduce that

$$\begin{aligned}
\mu^2 - \frac{25}{\mu^2 t^4} &= 2\mu^2 - \left(\mu^2 + \frac{25}{\mu^2 t^4}\right) \\
&= \frac{1}{2}\left(\frac{D - \sqrt{AB}}{t^3}\right)^2 - \frac{D^2}{t^6} + \frac{10}{t^2} \\
&= -\frac{D^2}{2t^6} + \frac{AB}{2t^6} + \frac{10}{t^2} - \frac{D\sqrt{AB}}{t^6} \\
&= -\frac{D\sqrt{AB}}{t^6},
\end{aligned} \tag{65.9}$$

by (63.8). Substituting (65.8) and (65.9) into (65.6), we deduce that

$$\begin{aligned}
MN + \frac{25}{MN} = \frac{1}{4t^6(1 - t^2)^2} &\{(D^2 - 10t^4)\{(2t - 1 + t^2)^2 A + (1 + t^2)^2 B\} \\
&-2ABD(2t - 1 + t^2)(1 + t^2)\}.
\end{aligned} \tag{65.10}$$

The calculation of the right side of (65.10) can be simplified by using (63.8) and the easily established identity

$$\begin{aligned}
F &:= (2t - 1 + t^2)^2 A + (1 + t^2)^2 B \\
&= 2D(2t - 1 + t^2)(1 + t^2) - 16t^4.
\end{aligned} \tag{65.11}$$

Hence, (65.10) reduces to

$$\begin{aligned}
MN + \frac{25}{MN} &= \frac{1}{4t^6(1 - t^2)^2}\{(AB + 10t^4)F - AB(F + 16t^4)\} \\
&= \frac{10F - 16AB}{4t^2(1 - t^2)^2} \\
&= \frac{t^8 + t^7 - 6t^6 + t^5 + 26t^4 - t^3 - 6t^2 - t + 1}{t^2(1 - t^2)^2}.
\end{aligned} \tag{65.12}$$

A comparison of (65.12) with (65.4) indicates that (65.2) has been established.

Entry 66 (p. 327). *Let*

$$P = \frac{\psi(q)}{q^{1/2}\psi(q^5)} \quad \textit{and} \quad Q = \frac{\psi(q^3)}{q^{3/2}\psi(q^{15})}.$$

Then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^2 + 3\frac{Q}{P} + 3\frac{P}{Q} - \left(\frac{P}{Q}\right)^2. \tag{66.1}$$

PROOF. Let

$$M = \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad \text{and} \quad N = \frac{\psi(-q^3)}{q^{3/2}\psi(-q^{15})}.$$

If q is replaced by $-q$, then PQ is transformed into MN while P/Q is converted to $-M/N$. Thus, to prove (66.1), it suffices to demonstrate that

$$MN + \frac{5}{MN} = \left(\frac{N}{M}\right)^2 - 3\frac{N}{M} - 3\frac{M}{N} - \left(\frac{M}{N}\right)^2. \tag{66.2}$$

By Entry 11(ii) in Chapter 17 (Part III [6, p. 123]),

$$M = \left(\frac{z_1}{z_5}\right)^{1/2}\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/8} \quad \text{and} \quad N = \left(\frac{z_3}{z_{15}}\right)^{1/2}\left(\frac{\beta(1-\beta)}{\delta(1-\delta)}\right)^{1/8}. \tag{66.3}$$

Thus, by (66.3), (61.2), (61.14), and (61.15),

$$\begin{aligned} MN &= \left(\frac{z_1 z_3}{z_5 z_{15}}\right)^{1/2}\left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{1/8} \\ &= \mu t\,\frac{(m^2-1)(9m^{-2}-1)}{(t^{-1}-t)^3} \\ &= \mu t\,\frac{\{(2-t^{-1}+1)R+(t^{-1}+t)S\}^2}{(t^{-1}-t)^2}. \end{aligned} \tag{66.4}$$

By (66.4) and (61.16),

$$\frac{1}{MN} = \frac{\{(2-t^{-1}+t)R-(t^{-1}+t)S\}^2}{\mu t(t^{-1}-t)^2}. \tag{66.5}$$

Thus, from (66.4) and (66.5),

$$\begin{aligned} MN + \frac{5}{MN} = \frac{1}{(1-t^2)^2}\bigg\{&\left(\mu t + \frac{5}{\mu t}\right)\{(2t-1+t^2)R^2 + (1+t^2)S^2\} \\ &+2\left(\mu t - \frac{5}{\mu t}\right)\{(2t-1+t^2)(1+t^2)RS\}\bigg\}. \end{aligned} \tag{66.6}$$

Recalling the definition of D in (65.7), we have

$$\mu - \frac{5}{\mu t^2} = 2\mu - \left(\mu + \frac{5}{\mu t^2}\right) = \frac{D - 4t^2RS}{t^3} - \frac{D}{t^3} = -\frac{\sqrt{AB}}{t^3}, \tag{66.7}$$

by (61.9) and (61.25). Substituting (65.7) and (66.7) into (66.6) and using again (61.25), we deduce that

$$MN + \frac{5}{MN} = \frac{1}{4t^4(1-t^2)^2}\{D\{(2t - 1 + t^2)^2A + (1 + t^2)^2B\} \\ -2(2t - 1 + t^2)(1 + t^2)AB\}.$$

Simplifying by the use of (65.11) and (63.8), we find that

$$\begin{aligned} MN + \frac{5}{MN} &= \frac{1}{4t^4(1-t^2)^2}\{2(D^2 - AB)(2t - 1 + t^2)(1 + t^2) - 16Dt^4\} \\ &= \frac{10(2t - 1 + t^2)(1 + t^2) - 4D}{(1-t^2)^2} \\ &= \frac{6t^4 + 8t^3 + 8t - 6}{(1-t^2)^2}. \end{aligned} \tag{66.8}$$

Next, by (66.3), (61.1), (61.3), (61.4), and (61.5),

$$\begin{aligned} \frac{M}{N} &= \left(\frac{z_1z_{15}}{z_3z_5}\right)^{1/2}\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} \\ &= \left(\frac{m}{m'}\right)^{1/2}\left(\frac{(9m^{-2}-1)(m'^2-1)}{(m^2-1)(9m'^{-2}-1)}\right)^{1/4} \\ &= \frac{1}{t}\frac{1-t}{1+1/t} = \frac{1-t}{1+t}. \end{aligned}$$

Hence,

$$\frac{M}{N} + \frac{N}{M} = \frac{2 + 2t^2}{1 - t^2}$$

and

$$\left(\frac{N}{M}\right)^2 - \left(\frac{M}{N}\right)^2 = \left(\frac{1+t}{1-t}\right)^2 - \left(\frac{1-t}{1+t}\right)^2 = \frac{8t^3 + 8t}{(1-t^2)^2}.$$

Therefore,

$$\begin{aligned} \left(\frac{N}{M}\right)^2 - \left(\frac{M}{N}\right)^2 - 3\frac{N}{M} - 3\frac{M}{N} &= \frac{8t^3 + 8t}{(1-t^2)^2} - \frac{3(2 + 2t^2)}{1 - t^2} \\ &= \frac{6t^4 + 8t^3 + 8t - 6}{(1-t^2)^2}. \end{aligned} \tag{66.9}$$

The modular equation (66.2) now follows immediately from (66.8) and (66.9).

Entry 67 (p. 327). *If*

$$P = \frac{\varphi(q)}{\varphi(q^5)} \quad and \quad Q = \frac{\varphi(q^3)}{\varphi(q^{15})},$$

then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^2 + 3\frac{Q}{P} + 3\frac{P}{Q} - \left(\frac{P}{Q}\right)^2. \tag{67.1}$$

Ramanujan incorrectly recorded the entry above in the form

$$5PQ + \frac{1}{PQ} = \left(\frac{P}{Q}\right)^2 + 3\frac{P}{Q} + 3\frac{Q}{P} - \left(\frac{Q}{P}\right)^2.$$

Proof. By (61.2) and (61.9),

$$\begin{aligned} PQ = \left(\frac{z_1 z_3}{z_5 z_{15}}\right)^{1/2} = \mu t &= \frac{t^4 + 3t^3 + 3t - 1 - 4t^2 RS}{2t^2} \\ &= \frac{D - \sqrt{AB}}{2t^2}, \end{aligned} \tag{67.2}$$

by (65.7) and (61.25). Thus, by (67.2) and (63.8),

$$\begin{aligned} PQ + \frac{5}{PQ} &= \frac{D - \sqrt{AB}}{2t^2} + \frac{10t^2}{D - \sqrt{AB}} \\ &= \frac{D - \sqrt{AB}}{2t^2} + \frac{10t^2(D + \sqrt{AB})}{20t^4} = \frac{D}{t^2}. \end{aligned} \tag{67.3}$$

On the other hand, by (61.1),

$$\frac{P}{Q} = \left(\frac{z_1 z_{15}}{z_5 z_3}\right)^{1/2} = \left(\frac{m}{m'}\right)^{1/2} = \frac{1}{t}.$$

Thus,

$$\begin{aligned} \left(\frac{Q}{P}\right)^2 + 3\frac{Q}{P} + 3\frac{P}{Q} - \left(\frac{P}{Q}\right)^2 &= t^2 + 3t + \frac{3}{t} - \frac{1}{t^2} \\ &= \frac{t^4 + 3t^3 + 3t - 1}{t^2}. \end{aligned} \tag{67.4}$$

The desired conclusion (67.1) now follows from (67.3) and (67.4).

This completes the proofs of Ramanujan's nine beautiful P–Q modular equations for the quadruple of degrees 1, 3, 5, and 15.

We have been unable to prove the remaining five P–Q eta-function identities by employing the classical theory of theta-functions and modular equations in the spirit of Ramanujan. We therefore rely on the theory of modular forms to prove Entries 68–72 as well as Entry 39, which we have also been unsuccessful in proving by classical means.

Entry 68 (p. 323). *Let*

$$P=\frac{f(-q)}{q^{1/4}f(-q^7)} \quad and \quad Q=\frac{f(-q^3)}{q^{3/4}f(-q^{21})}.$$

Then

$$PQ+\frac{7}{PQ}=\left(\frac{Q}{P}\right)^2-3+\left(\frac{P}{Q}\right)^2. \tag{68.1}$$

Entry 69 (p. 323). *Let*

$$P=\frac{f(-q)}{q^{1/12}f(-q^3)} \quad and \quad Q=\frac{f(-q^7)}{q^{7/12}f(-q^{21})}.$$

Then

$$(PQ)^3+\frac{27}{(PQ)^3}=\left(\frac{Q}{P}\right)^4-7\left(\frac{Q}{P}\right)^2+7\left(\frac{P}{Q}\right)^2-\left(\frac{P}{Q}\right)^4. \tag{69.1}$$

Entry 70 (p. 323). *Let*

$$P=\frac{f(-q^3)}{q^{1/6}f(-q^7)} \quad and \quad Q=\frac{f(-q)}{q^{5/6}f(-q^{21})}.$$

Then

$$\left(\frac{Q}{P}\right)^3-27\left(\frac{P}{Q}\right)^3=(PQ)^2-PQ+\frac{7}{PQ}-\frac{49}{(PQ)^2}. \tag{70.1}$$

Entry 71 (p. 303). *Let*

$$P=\frac{f(-q)}{q^{1/4}f(-q^7)} \quad and \quad Q=\frac{f(-q^5)}{q^{5/4}f(-q^{35})}.$$

Then

$$(PQ)^2-5+\frac{49}{(PQ)^2}=\left(\frac{Q}{P}\right)^3-5\left(\frac{Q}{P}\right)^2-5\left(\frac{P}{Q}\right)^2-\left(\frac{P}{Q}\right)^3. \tag{71.1}$$

Entry 72 (p. 322). *Let*

$$P = \frac{f(-q)}{q^{1/2}f(-q^{13})} \quad \text{and} \quad Q = \frac{f(-q^3)}{q^{3/2}f(-q^{39})}.$$

Then

$$PQ + \frac{13}{PQ} = \left(\frac{Q}{P}\right)^2 - 3\frac{Q}{P} - 3 - 3\frac{P}{Q} + \left(\frac{P}{Q}\right)^2. \tag{72.1}$$

Recall that $\Gamma(1)$ denotes the full modular group, and that the space of modular forms of weight r and multiplier system v on Γ is denoted by $\{\Gamma, r, v\}$, where Γ is a subgroup of $\Gamma(1)$ of finite index. As customary, let

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \colon c \equiv 0 \pmod{N}\right\},$$

where N is a positive integer. For positive integers m and n, set

$$R_{m,n}(z) = \frac{\eta(z)\eta(nz)}{\eta(mz)\eta(mnz)} \tag{68.2}$$

and

$$S_{m,n}(z) = \frac{\eta(mz)\eta(nz)}{\eta(z)\eta(mnz)}, \tag{68.3}$$

where $\eta(z)$ denotes the Dedekind eta-function. The following two lemmas show that, under appropriate conditions, $R^r_{m,n}(z)$ and $S^r_{m,n}(z)$ are modular functions on $\Gamma_0(mn)$, i.e., modular forms of weight 0 and multiplier system identically equal to 1.

Lemma 68.1. *Let m and n designate positive, odd integers, and suppose that r is an integer such that*

$$r(m-1)(n+1) \equiv 0 \pmod{24}. \tag{68.4}$$

Then

$$R^r_{m,n}(z) \in \{\Gamma_0(mn), 0, 1\}.$$

Moreover, $R^r_{m,n}(z)$ is analytic on $\mathscr{H} := \{z \colon \operatorname{Im} z > 0\}$.

PROOF. The last assertion in Lemma 68.1 is obvious from the definition of $R_{m,n}(z)$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(mn)$. Then, for $z \in \mathscr{H}$,

$$\eta(Az) = v_\eta(A)(cz+d)^{1/2}\eta(z) \tag{68.5}$$

and, for $s|c$,

$$\eta(sAz) = \eta\left(\frac{a(sz) + sb}{\frac{c}{s}(sz) + d}\right) = v_\eta\begin{pmatrix} a & sb \\ c/s & d \end{pmatrix}(cz + d)^{1/2}\eta(sz), \tag{68.6}$$

where the multiplier system v_η is given by (32.14). Thus, by (68.2), (68.5), and (68.6),

$$R_{m,n}(Az) = v(A)R_{m,n}(z),$$

where

$$v(A) = \frac{v_\eta\begin{pmatrix} a & b \\ c & d \end{pmatrix} v_\eta\begin{pmatrix} a & nb \\ c/n & d \end{pmatrix}}{v_\eta\begin{pmatrix} a & mb \\ c/m & d \end{pmatrix} v_\eta\begin{pmatrix} a & mnb \\ c/(mn) & d \end{pmatrix}}.$$

Suppose first that c is odd. Then, from (32.14), by a straightforward calculation,

$$v(A) = \zeta_{24}^{(m-1)(n+1)\{-bdmn - bdc^2 + c(a+d) - 3c\}/(mn)}.$$

Hence, by (68.4), $v^r(A) \equiv 1$, as desired.

Second, suppose that c is even. Then, by (32.14),

$$v(A) = \zeta_{24}^{(m-1)(n+1)\{ac(1-d^2) - bdmn - cd\}/(mn)}.$$

Again, by (68.4), $v^r(A) \equiv 1$.

Thus, in both instances,

$$R^r_{m,n}(Az) = R^r_{m,n}(z),$$

and the proof is complete.

Lemma 68.2. *Let m and n denote positive, odd integers, and suppose r is an integer such that*

$$r(m-1)(n-1) \equiv 0 \pmod{24}.$$

Then

$$S^r_{m,n}(z) \in \{\Gamma_0(mn), 0, 1\}.$$

Moreover, $S^r_{m,n}(z)$ is analytic on $\mathscr{H}$.

PROOF. The proof is analogous to that of Lemma 68.1.

Let $\operatorname{ord}(f; z)$ denote the invariant order of a modular from f at z. The order of f with respect to Γ is denoted and defined by

$$\operatorname{Ord}_\Gamma(f; z) := \frac{1}{\ell}\operatorname{ord}(f; z),$$

where ℓ ($\ell = 1, 2,$ or 3) is the order of f at z as a fixed point of Γ (Rankin [1, pp. 45, 91]). We now state the valence formula (Rankin [1, p. 98, Theorem 4.1.4]). If $f \in \{\Gamma, r, v\}$ and $\mathscr{F}$ is any fundamental set for Γ, then, provided f is not constant,

$$\sum_{z \in \mathscr{F}} \mathrm{Ord}_\Gamma(f; z) = r\rho_\Gamma, \tag{68.7}$$

where

$$\rho_\Gamma := \frac{1}{12}(\Gamma(1) : \Gamma).$$

Next, from Schoeneberg's book [1, p. 102], if σ_∞ denotes the number of inequivalent cusps of $\Gamma_0(N)$, then

$$\sigma_\infty = \sum_{d \mid N} \varphi((d, N/d)), \tag{68.8}$$

where φ denotes Euler's φ-function and (a, b) denotes the greatest common divisor of a and b. In particular, if $N = mn$, where m and n are distinct odd primes, $\sigma_\infty = 4$. Moreover (Newman [1, p. 337]), a complete set of inequivalent cusps are 0, ∞, $1/m$, and $1/n$.

Let r/s, $(r, s) = 1$, denote a cusp. Then for any pair of positive integers m, n (e.g., see Biagioli's paper [1, p. 282]),

$$\mathrm{ord}\left(\eta(mnz); \frac{r}{s}\right) = \frac{(mn, s)^2}{24mn}. \tag{68.9}$$

We now indicate our plan of attack. We shall apply (68.7) to the difference of the left and right sides of (68.1), (69.1), (70.1), (71.1), and (72.1). In each case, the weight $r = 0$. Except for possibly the four cusps, every order is nonnegative. Calculating the orders of the finite cusps by (68.9), we can use (68.7) to obtain an upper bound for the order of the cusp ∞. We then actually calculate the expansion around ∞ and find that the order at the cusp ∞ exceeds the upper bound just determined. Thus, we obtain a contradiction unless the modular function is a constant, which is readily seen to equal 0. This therefore establishes the proffered identity.

Proofs of Entries 68–72. We first prove that the functions appearing in Entries 68–72 are, indeed, modular functions on $\Gamma_0(mn)$. For Entry 68, by Lemma 68.1, $PQ \in \{\Gamma_0(21), 0, 1\}$, and by Lemma 68.2, $(P/Q)^2 \in \{\Gamma_0(21), 0, 1\}$. Similarly, for Entry 69, $(PQ)^3$, $(P/Q)^2 \in \{\Gamma_0(21), 0, 1\}$. For Entry 70, by Lemma 68.1, PQ, $(P/Q)^3 \in \{\Gamma_0(21), 0, 1\}$. For Entry 71, by Lemma 68.1, $(PQ)^2 \in \{\Gamma_0(35), 0, 1\}$, and by Lemma 68.2, $P/Q \in \{\Gamma_0(35), 0, 1\}$. Lastly, for Entry 72, by Lemma 68.1, $PQ \in \{\Gamma_0(39), 0, 1\}$, while by Lemma 68.2, $P/Q \in \{\Gamma_0(39), 0, 1\}$.

Employing (68.9), we prepare tables of orders for certain modular forms appearing in Entries 68–72. For brevity, L(n.1) and R(n.1) denote, respectively, the left and right sides of equality (n.1), $68 \leq n \leq 72$.

Let $f_j(q)$, $q = \exp(2\pi i z)$, denote the difference of the left and right sides of $(67 + j.1)$, $1 \leq j \leq 5$. Then from (68.7) and the tables on the following two pages,

$$\begin{aligned} 0 = \sum_{z \in \mathscr{F}} \operatorname{Ord}_{\Gamma_0(21)}(f_1; z) &\geq \operatorname{ord}(f_1; \infty) + \operatorname{ord}(f_1; \tfrac{1}{3}) + \operatorname{ord}(f_1; \tfrac{1}{7}) + \operatorname{ord}(f; 0) \\ &\geq \operatorname{ord}(f_1; \infty) - \tfrac{1}{7} - \tfrac{1}{3} - \tfrac{1}{21} \\ &= \operatorname{ord}(f_1; \infty) - \tfrac{11}{21}, \end{aligned} \tag{68.10}$$

$$0 \geq \operatorname{ord}(f_2; \infty) - \tfrac{2}{7} - \tfrac{2}{3} - \tfrac{2}{21} = \operatorname{ord}(f_2; \infty) - \tfrac{22}{21}, \tag{68.11}$$

$$0 \geq \operatorname{ord}(f_3; \infty) - \tfrac{2}{7} - \tfrac{2}{3} - \tfrac{2}{21} = \operatorname{ord}(f_3; \infty) - \tfrac{22}{21}, \tag{68.12}$$

$$0 \geq \operatorname{ord}(f_4; \infty) - \tfrac{3}{7} - \tfrac{3}{5} - \tfrac{3}{35} = \operatorname{ord}(f_4; \infty) - \tfrac{39}{35}, \tag{68.13}$$

and

$$0 \geq \operatorname{ord}(f_5; \infty) - \tfrac{2}{13} - \tfrac{2}{3} - \tfrac{2}{39} = \operatorname{ord}(f_5; \infty) - \tfrac{34}{39}. \tag{68.14}$$

Suppose that we can show that

$$f_j(q) = 0(q), \qquad j = 1, 5, \tag{68.15}$$

and

$$f_j(q) = 0(q^2), \qquad j = 2, 3, 4, \tag{68.16}$$

as q tends to 0 (or as z tends to ∞). Then, in the first instance, we have contradictions to (68.10) and (68.14), and, in the second instance, (68.11)–(68.13) are contradicted, unless $f_j(q)$ is a constant in each case. Letting q tend to 0, we see from (68.15) and (68.16) that this constant equals 0.

Using *Mathematica*, we calculated the expansions of the left and right sides of (n.1), $68 \leq n \leq 72$, about $z = \infty (q = 0)$ and found that

$$L(68.1) = \frac{1}{q} - 1 + O(q) = R(68.1),$$

$$L(69.1) = \frac{1}{q^2} - \frac{3}{q} + 8q + O(q^2) = R(69.1),$$

$$L(70.1) = \frac{1}{q^2} - \frac{3}{q} + 8q + O(q^2) = R(70.1),$$

$$L(71.1) = \frac{1}{q^3} - \frac{2}{q^2} - \frac{1}{q} - 3 + q + O(q^2) = R(71.1),$$

and

$$L(72.1) = \frac{1}{q^2} - \frac{1}{q} - 1 + O(q) = R(72.1).$$

Entries 68–70		
Function f	cusp ζ	ord$(f;\zeta)$
$\eta(z)$	$\frac{1}{3}, \frac{1}{7}, 0$	$\frac{1}{24}$
$\eta(3z)$	$\frac{1}{3}$	$\frac{1}{8}$
	$\frac{1}{7}, 0$	$\frac{1}{72}$
$\eta(7z)$	$\frac{1}{3}, 0$	$\frac{1}{168}$
	$\frac{1}{7}$	$\frac{7}{24}$
$\eta(21z)$	$\frac{1}{3}$	$\frac{1}{56}$
	$\frac{1}{7}$	$\frac{7}{72}$
	0	$\frac{1}{504}$

Entry 68		
Function f	cusp ζ	ord$(f;\zeta)$
P	$\frac{1}{3}, 0$	$\frac{1}{28}$
	$\frac{1}{7}$	$-\frac{1}{4}$
Q	$\frac{1}{3}$	$\frac{3}{28}$
	$\frac{1}{7}$	$-\frac{1}{12}$
	0	$\frac{1}{84}$
PQ	$\frac{1}{3}$	$\frac{1}{7}$
	$\frac{1}{7}$	$-\frac{1}{3}$
	0	$\frac{1}{21}$
P/Q	$\frac{1}{3}$	$-\frac{1}{14}$
	$\frac{1}{7}$	$-\frac{1}{6}$
	0	$\frac{1}{42}$
L(68.1)	$\frac{1}{3}$	$-\frac{1}{7}$
	$\frac{1}{7}$	$-\frac{1}{3}$
	0	$-\frac{1}{21}$
R(68.1)	$\frac{1}{3}$	$-\frac{1}{7}$
	$\frac{1}{7}$	$-\frac{1}{3}$
	0	$-\frac{1}{21}$

Entry 69		
Function f	cusp ζ	ord$(f;\zeta)$
P	$\frac{1}{3}$	$-\frac{1}{12}$
	$\frac{1}{7}, 0$	$\frac{1}{36}$
Q	$\frac{1}{3}$	$-\frac{1}{84}$
	$\frac{1}{7}$	$\frac{7}{36}$
	0	$\frac{1}{252}$
PQ	$\frac{1}{3}$	$-\frac{2}{21}$
	$\frac{1}{7}$	$\frac{2}{9}$
	0	$\frac{2}{63}$
P/Q	$\frac{1}{3}$	$-\frac{1}{14}$
	$\frac{1}{7}$	$-\frac{1}{6}$
	0	$\frac{1}{42}$
L(69.1)	$\frac{1}{3}$	$-\frac{2}{7}$
	$\frac{1}{7}$	$-\frac{2}{3}$
	0	$-\frac{2}{21}$
R(69.1)	$\frac{1}{3}$	$-\frac{2}{7}$
	$\frac{1}{7}$	$-\frac{2}{3}$
	0	$-\frac{2}{21}$

Entry 70		
Function f	cusp ζ	ord$(f;\zeta)$
P	$\frac{1}{3}$	$\frac{5}{42}$
	$\frac{1}{7}$	$-\frac{5}{18}$
	0	$\frac{1}{126}$
Q	$\frac{1}{3}$	$\frac{1}{42}$
	$\frac{1}{7}$	$-\frac{1}{18}$
	0	$\frac{5}{126}$
PQ	$\frac{1}{3}$	$\frac{1}{7}$
	$\frac{1}{7}$	$-\frac{1}{3}$
	0	$\frac{1}{21}$
P/Q	$\frac{1}{3}$	$\frac{2}{21}$
	$\frac{1}{7}$	$-\frac{2}{9}$
	0	$-\frac{2}{63}$
L(70.1)	$\frac{1}{3}$	$-\frac{2}{7}$
	$\frac{1}{7}$	$-\frac{2}{3}$
	0	$-\frac{2}{21}$
R(70.1)	$\frac{1}{3}$	$-\frac{2}{7}$
	$\frac{1}{7}$	$-\frac{2}{3}$
	0	$-\frac{2}{21}$

Entry 71			Entry 72		
Function f	cusp ζ	ord$(f;\zeta)$	Function f	cusp ζ	ord$(f;\zeta)$
$\eta(z)$	$\frac{1}{5}, \frac{1}{7}, 0$	$\frac{1}{24}$	$\eta(z)$	$\frac{1}{5}, \frac{1}{13}, 0$	$\frac{1}{24}$
$\eta(5z)$	$\frac{1}{5}$	$\frac{5}{24}$	$\eta(3z)$	$\frac{1}{3}$	$\frac{1}{8}$
	$\frac{1}{7}, 0$	$\frac{1}{120}$		$\frac{1}{13}, 0$	$\frac{1}{72}$
$\eta(7z)$	$\frac{1}{5}, 0$	$\frac{1}{168}$	$\eta(13z)$	$\frac{1}{3}, 0$	$\frac{1}{312}$
	$\frac{1}{7}$	$\frac{7}{24}$		$\frac{1}{13}$	$\frac{13}{24}$
$\eta(35z)$	$\frac{1}{5}$	$\frac{5}{168}$	$\eta(39z)$	$\frac{1}{3}$	$\frac{1}{104}$
	$\frac{1}{7}$	$\frac{7}{120}$		$\frac{1}{13}$	$\frac{13}{72}$
	0	$\frac{1}{840}$		0	$\frac{1}{936}$
P	$\frac{1}{5}, 0$	$\frac{1}{28}$	P	$\frac{1}{3}, 0$	$\frac{1}{26}$
	$\frac{1}{7}$	$-\frac{1}{4}$		$\frac{1}{13}$	$-\frac{1}{2}$
Q	$\frac{1}{5}$	$\frac{5}{28}$	Q	$\frac{1}{3}$	$\frac{3}{26}$
	$\frac{1}{7}$	$-\frac{1}{20}$		$\frac{1}{13}$	$-\frac{1}{6}$
	0	$\frac{1}{140}$		0	$\frac{1}{78}$
PQ	$\frac{1}{5}$	$\frac{3}{14}$	PQ	$\frac{1}{3}$	$\frac{2}{13}$
	$\frac{1}{7}$	$-\frac{3}{10}$		$\frac{1}{13}$	$-\frac{2}{3}$
	0	$\frac{3}{70}$		0	$\frac{2}{39}$
P/Q	$\frac{1}{5}$	$-\frac{1}{7}$	P/Q	$\frac{1}{3}$	$-\frac{1}{13}$
	$\frac{1}{7}$	$-\frac{1}{5}$		$\frac{1}{13}$	$-\frac{1}{3}$
	0	$\frac{1}{35}$		0	$\frac{1}{39}$
L(71.1)	$\frac{1}{5}$	$-\frac{3}{7}$	L(72.1)	$\frac{1}{3}$	$-\frac{2}{13}$
	$\frac{1}{7}$	$-\frac{3}{5}$		$\frac{1}{13}$	$-\frac{2}{3}$
	0	$-\frac{3}{35}$		0	$-\frac{2}{39}$
R(71.1)	$\frac{1}{5}$	$-\frac{3}{7}$	R(72.1)	$\frac{1}{3}$	$-\frac{2}{13}$
	$\frac{1}{7}$	$-\frac{3}{5}$		$\frac{1}{13}$	$-\frac{2}{3}$
	0	$-\frac{3}{35}$		0	$-\frac{2}{39}$

Hence, (68.15) and (68.16) are valid, and the proofs of Entries 68–72 are complete.

We conclude this chapter by proving Entry 39.

PROOF OF ENTRY 39. By Lemma 68.1, $v \in \{\Gamma_0(45), 0, 1\}$. Observe that

$$u = R_{9,5}(z)R_{3,\frac{5}{3}}^{-1}(z).$$

Thus, by Lemma 68.1, $u^2 \in \{\Gamma_0(45), 0, 1\}$. Hence, both sides of (39.1) belong to $\{\Gamma_0(45), 0, 1\}$.

We noted above that a fundamental region for $\Gamma_0(45)$ has eight inequivalent cusps. To identify eight such cusps, we proceed as in Schoeneberg's book [1, pp. 86–87]. Let r/s and r_1/s_1 be two rational numbers in reduced form. Let

$$A := \begin{pmatrix} r & b \\ s & d \end{pmatrix} \quad \text{and} \quad A_1 := \begin{pmatrix} r_1 & b_1 \\ s_1 & d_1 \end{pmatrix}$$

be in $\Gamma(1)$. Thus, $A(i\infty) = r/s$ and $A_1(i\infty) = r_1/s_1$. Let $U := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then the transformations

$$V := A_1 U^k A^{-1}, \qquad k \in \mathbb{Z},$$

are the modular transformations for which $V(r/s) = r_1/s_1$. An elementary calculation shows that $V \in \Gamma_0(45)$ if an only if

$$s_1 d - s s_1 k - s d_1 \equiv 0 \pmod{45}. \tag{39.2}$$

Using (39.2), we see that $0, \frac{1}{3}, \frac{1}{5}, \frac{1}{9}, \frac{1}{15}, \frac{2}{3}, \frac{2}{15}$, and ∞ constitute a complete set of eight inequivalent cusps.

Next, we employ (68.9) to calculate the orders of u and v at each finite cusp. These orders are then used to determine lower bounds for the orders of the left and right sides of (39.1), denoted by L(39.1) and R(39.1), respectively, at each finite cusp. We emphasize that, because of the cancellation of terms, we do not always obtain the exact order for $v^3 + 3v^2 + 9v$. The following table summarizes our calculations; l.b. is an abbreviation for lower bound.

cusp ζ	ord(u; ζ)	ord(v; ζ)	ord(L(39.1); ζ)	l.b. ord(R(39.1); ζ)
0	$\frac{1}{90}$	$\frac{2}{45}$	$\frac{2}{45}$	$\frac{2}{45}$
$\frac{1}{3}$	$\frac{1}{10}$	0	$\frac{1}{5}$	0
$\frac{1}{5}$	$\frac{1}{18}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$
$\frac{1}{9}$	$-\frac{3}{10}$	$-\frac{2}{5}$	$-\frac{6}{5}$	$-\frac{6}{5}$
$\frac{1}{15}$	$\frac{1}{2}$	0	1	0
$\frac{2}{3}$	$\frac{1}{10}$	0	$\frac{1}{5}$	0
$\frac{2}{15}$	$\frac{1}{2}$	0	1	0

Let $f_6(q)$, $q = \exp(2\pi i z)$, denote the difference of the left and right sides of (39.1). Then from (68.7) and the foregoing table,

$$\begin{aligned} 0 \geq \sum_{\zeta} \operatorname{ord}(f_6; \zeta) &\geq \operatorname{ord}(f_6; \infty) + \tfrac{2}{45} + 0 + \tfrac{2}{9} - \tfrac{6}{5} + 0 + 0 + 0 \\ &= \operatorname{ord}(f_6; \infty) - \tfrac{14}{15}, \end{aligned}$$

where the sum is taken over the eight inequivalent cusps. Arguing as we did in the proofs of the last five entries, we shall be finished with the proof if we can show that

$$f_6(q) = O(q), \tag{39.3}$$

as q tends to 0. Using *Mathematica*, we find that

$$L(39.1) = \frac{1}{q^6} - \frac{3}{q^5} + \frac{3}{q^4} - \frac{1}{q^3} + \frac{6}{q^2} - \frac{6}{q} - 4 + O(q^2) = R(39.1).$$

Thus, (39.3) is true, and the proof of Entry 39 is complete.

Many other authors have considered products of eta-functions, but generally with less emphasis on finding identities. In particular, see papers by A. J. F. Biagioli [1], [2], H. Cohn and M. I. Knopp [1], D. Ford and J. McKay [1], B. Gordon and D. Sinor [1], E. Hecke [1], [2, pp. 919–940], G. Köhler [1], M. Newman [1], [2], and H. M. Stark [2].

Some of the P–Q eta-function identities were proved in a paper of the author and L.-C. Zhang [1].

CHAPTER 26

Inversion Formulas for the Lemniscate and Allied Functions

On pages 283, 285, and 286 in his second notebook [22], Ramanujan states ten inversion formulas for series. As an example, we quote (in a somewhat more compact notation) the entry at the top of page 285. If

$$\frac{\theta\mu}{\sqrt{2}} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n v^{4n+1}}{n!\,(4n+1)}, \tag{0.1}$$

where μ is the constant obtained by putting $v = 1$ and $\theta = \pi/2$, then

$$\frac{\mu^2}{2v^2} = \frac{1}{\sin^2\theta} - \frac{1}{\pi} - 8\sum_{n=1}^{\infty} \frac{n\cos(2n\theta)}{e^{2\pi n} - 1}. \tag{0.2}$$

In (0.1), we have used the rising factorial notation

$$(a)_n = a(a+1)(a+2)\dots(a+n-1).$$

By expanding $(1 - t^4)^{-1/2}$ in a binomial series and integrating termwise, we find that, for $0 \le v \le 1$,

$$F(v) := \int_0^v \frac{dt}{\sqrt{1-t^4}} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n v^{4n+1}}{n!\,(4n+1)}. \tag{0.3}$$

This integral is the famous lemniscate integral, originally studied by James Bernoulli and Count Giulio Fagnano. For a very informative history of the lemniscate integral and its importance in the early theory of elliptic integrals and functions, see R. Ayoub's historical survey article [1]. Moreover, C. L. Siegel [1] considered the lemniscate integral to be so important that he began his development of the theory of elliptic functions with a thorough discussion of it. Thus, (0.2) gives an interesting inversion formula for the lemniscate function.

On page 285, Ramanujan [22] gives five additional inversion formulas involving the lemniscate integral. These are followed on page 286 by three inversion formulas associated with the integral

$$G(v) := \int_0^v \frac{dt}{\sqrt{1+t^4}}. \tag{0.4}$$

As above, Ramanujan records these three theorems in terms of series instead of integrals. Now, in fact, the integrals in (0.3) and (0.4) are closely related. If we set

$$v = \frac{\sqrt{2}x}{\sqrt{1+x^4}},$$

then an easy calculation shows that

$$\int_0^v \frac{dt}{\sqrt{1-t^4}} = \sqrt{2}\int_0^x \frac{dt}{\sqrt{1+t^4}}. \tag{0.5}$$

The relation (0.5) is very important, for it represents the key intermediary step in the famous problem of doubling the arc length of the lemniscate. For discussions of this historically important problem, see the aforementioned paper of Ayoub [1] and text of Siegel [1].

Lastly, on page 283, Ramanujan [22] offers an inversion formula for the integral

$$H(v) := \int_0^v \frac{dt}{\sqrt{1-t^6}}.$$

Again, Ramanujan expresses his theorem in series notation.

In this chapter, we provide proofs of each of these ten inversion formulas. (Ramanujan gave no hints of his proofs.) We emphasize that we know no other theorems in the literature like these ten results which were originally proved in a paper coauthored with S. Bhargava [1]. This paper and another by Bhargava [1] contain further inversion formulas for F and H.

Results from the theory of elliptic functions play key roles in our proofs. We shall use results from both the Weierstrass theory and Jacobi's theory of the Jacobian elliptic functions. In particular, we shall employ facts about $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, $\operatorname{sd} u$, and $\operatorname{am} u$. Recall that (Whittaker and Watson [1, p. 493])

$$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1 \tag{0.6}$$

and

$$k^2 \operatorname{sn}^2 u + \operatorname{dn}^2 u = 1, \tag{0.7}$$

where k, $0 < |k| \le 1$, $k \ne \pm 1$, is the modulus of $\operatorname{sn} u$ and $\operatorname{dn} u$.

The Fourier series of several elliptic functions will be utilized. Following Ramanujan, set $z = 2K/\pi$, where K is the complete elliptic integral of the first kind. As usual, let $q = e^{-\pi K'/K}$, where K' is the complete elliptic integral of the first kind associated with the complementary modulus $k' = \sqrt{1-k^2}$. Also, E denotes the complete elliptic integral of the second kind. Each of the following Fourier series expansions is valid for a wider range of θ than what is indicated, but the greater generality is not needed here. From Whittaker and Watson's text [1, pp. 511–512, 535],

$$\operatorname{am}(z\theta) = \theta + \sum_{n=1}^{\infty} \frac{2q^n \sin(2n\theta)}{n(1+q^{2n})}, \qquad 0 \le \theta \le \pi/2, \tag{0.8}$$

$$\frac{z^2}{\operatorname{sn}^2(z\theta)} = \csc^2\theta + z^2 - \frac{4KE}{\pi^2} - 8\sum_{n=1}^{\infty} \frac{nq^{2n}\cos(2n\theta)}{1-q^{2n}}, \qquad 0 < \theta \le \pi/2, \tag{0.9}$$

$$\operatorname{cn}(z\theta) = \frac{2\pi}{Kk}\sum_{n=0}^{\infty} \frac{q^{n+1/2}\cos\{(2n+1)\theta\}}{1+q^{2n+1}}, \qquad 0 \le \theta \le \pi/2, \tag{0.10}$$

$$\operatorname{sd}(z\theta) = \frac{2\pi}{Kkk'}\sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1/2}\sin\{(2n+1)\theta\}}{1+q^{2n+1}}, \qquad 0 \le \theta \le \pi/2, \tag{0.11}$$

$$z\,\operatorname{dn}(z\theta) = 1 + 4\sum_{n=1}^{\infty} \frac{q^n\cos(2n\theta)}{1+q^{2n}}, \qquad 0 \le \theta \le \pi/2, \tag{0.12}$$

and

$$z\frac{\operatorname{dn}(z\theta)}{\operatorname{cn}(z\theta)} = \sec\theta + 4\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}\cos\{(2n+1)\theta\}}{1-q^{2n+1}}, \qquad 0 \le \theta \le \pi/2. \tag{0.13}$$

Lastly, we shall need Euler's transformation (Bailey [1, p. 2]),

$${}_2F_1(a, b; c; z) = (1-z)^{c-a-b}\,{}_2F_1(c-a, c-b; c; z) \tag{0.14}$$

for the ordinary hypergeometric function ${}_2F_1$.

In the sequel, it will always be tacitly assumed that $0 \le \theta \le \pi/2$ and $0 \le v \le 1$. Occasionally, because of singularities, end points must be omitted from the domains.

Inversion Formulas for the Lemniscate Function F

Entry 1 (p. 285). *Let θ, v, and μ be defined by*

$$\frac{\theta\mu}{\sqrt{2}} = \int_0^v \frac{dt}{\sqrt{1-t^4}} = F(v), \tag{1.1}$$

where $0 \le \theta \le \pi/2$, $0 \le v \le 1$, and μ is a constant defined by setting $\theta = \pi/2$ and $v = 1$. Then for $0 < \theta \le \pi/2$,

$$\frac{\mu^2}{2v^2} = \csc^2\theta - \frac{1}{\pi} - 8\sum_{n=1}^{\infty} \frac{n\cos(2n\theta)}{e^{2\pi n}-1}. \tag{1.2}$$

First Proof. We first calculate μ. Letting $\theta = \pi/2$, $v = 1$, and $u = t^4$ in (1.1), we find that

$$\frac{\pi\mu}{2\sqrt{2}} = \frac{1}{4}\int_0^1 u^{-3/4}(1-u)^{-1/2}\,du = \tfrac{1}{4}B(\tfrac{1}{4}, \tfrac{1}{2}) = \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{2\pi}},$$

when $B(x, y)$ denotes the beta function. Thus,

$$\mu = \frac{\Gamma^2(\frac{1}{4})}{2\pi^{3/2}}. \tag{1.3}$$

The elliptic integral $F(v)$ can be inverted by the classical theory of elliptic integrals. More precisely, from Whittaker and Watson's text [1, p. 494],

$$v = 2^{-1/2}\,\mathrm{sd}(\mu\theta), \tag{1.4}$$

where $\mathrm{sd}\,u = \mathrm{sn}\,u/\mathrm{dn}\,u$ and $k = 1/\sqrt{2}$. Since $k = 1/\sqrt{2}$ (Whittaker and Watson [1, p. 499]),

$$z = \frac{2K}{\pi} = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{2}) = \frac{\Gamma^2(\frac{1}{4})}{2\pi^{3/2}},$$

where ${}_2F_1$ denotes the ordinary hypergeometric function, and where we have employed Kummer's theorem (Bailey [1, p. 11]). Hence, from (1.3), $z = \mu$, and, from (1.4), $v = 2^{-1/2}\,\mathrm{sd}(z\theta)$.

Using the identity (0.7), we deduce that

$$\frac{1}{v^2} = \frac{2}{\mathrm{sn}^2(z\theta)} - 1. \tag{1.5}$$

We now utilize the Fourier series for $\mathrm{sn}^{-2}(z\theta)$ given in (0.9). The integral E appearing in (1.9) is easily determined by Legendre's relation (Whittaker and Watson [1, p. 520]), since $k = 1/\sqrt{2}$ and so $K = K'$. Accordingly, we find that

$$2EK - K^2 = \frac{\pi}{2}.$$

Thus, by (0.9), for $0 < \theta \le \pi/2$,

$$\frac{z^2}{\mathrm{sn}^2(z\theta)} = \csc^2\theta + \frac{z^2}{2} - \frac{1}{\pi} - 8\sum_{n=1}^{\infty}\frac{n\cos(2n\theta)}{q^{-2n}-1}. \tag{1.6}$$

Now since $K = K'$, $q = e^{-\pi}$. Remembering that $\mu = z$, we complete the proof of (1.2) by putting (1.6) in (1.5).

Our first proof used the Jacobian theory of elliptic functions. Our second proof of Entry 1 utilizes the Weierstrass theory.

SECOND PROOF. In (1.1), let $u = \mu^2/(2t^2)$. It follows that

$$\theta = \int_{\mu^2/(2v^2)}^{\infty} \frac{du}{\sqrt{4u^3 - \mu^4 u}}. \tag{1.7}$$

From (1.7), in the standard notation of the Weierstrass normal form (Chandrasekharan [1, pp. 29, 94], there exists a pair of linearly independent periods (ω_1, ω_2) with $\operatorname{Im}(\omega_2/\omega_1) > 0$ such that

$$g_2(\omega_1, \omega_2) := 60 \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m\omega_1 + n\omega_2)^4} = \mu^4 = \frac{\Gamma^8(\frac{1}{4})}{16\pi^6}, \tag{1.8}$$

by (1.3), where the prime (′) on the summation sign indicates that the sum is over all integral pairs (m, n) except $(0, 0)$. Also, from (1.7),

$$g_3(\omega_1, \omega_2) := 140 \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m\omega_1 + n\omega_2)^6} \equiv 0. \tag{1.9}$$

Now $g_3(1, i) = 0$, and the pair of arguments $(1, i)$ yielding the value 0 is unique except for multiplication by a nonzero number (Chandrasekharan [1, p. 88]). Hence, from (1.8) and (1.9),

$$\begin{aligned}\frac{\Gamma^8(\frac{1}{4})}{16\pi^6} = g_2(\omega_1, \omega_2) &= 60\omega_1^{-4} \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m + n\tau)^4} \\ &= 60\omega_1^{-4} E_4^*(\tau) \\ &= 120\omega_1^{-4} \zeta(4) M(q^2),\end{aligned} \tag{1.10}$$

where $E_4^*(\tau)$ is an Eisenstein series, $\tau = \omega_2/\omega_1 = i$, ζ denotes the Riemann zeta-function, $q = e^{\pi i \tau} = e^{-\pi}$, and

$$M(q^2) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1 - q^{2n}}.$$

See R. A. Rankin's book [1, p. 194] for the material on Eisenstein series quoted above. The function M is one of a triple of basic functions studied by Ramanujan in his notebooks [22] (Part II [4, Chap. 15, Section 9]; Part III [6, Chap. 17, Section 13]) and in his famous paper [17], [23, pp. 136–162], where M is replaced by Q.

By Entry 13(i) in Chapter 17 (Part III [6, p. 126]) (see also Ramanujan's paper [17], [23, p. 140]),

$$M(q^2) = z^4(1 - k^2 + k^4). \tag{1.11}$$

Here, $q = e^{-\pi}$, and so $k = 1/\sqrt{2}$. Thus, $z = \mu$ is given by (1.3). It follows from (1.11) that

$$M(e^{-2\pi}) = \frac{3\Gamma^8(\frac{1}{4})}{64\pi^6}. \tag{1.12}$$

Using (1.12) and the value $\zeta(4) = \pi^4/90$ in (1.10), we conclude that

$$g_2(\omega_1, \omega_2) = \frac{\Gamma^8(\frac{1}{4})}{16\omega_1^4\pi^2}. \tag{1.13}$$

Comparing (1.8) and (1.13), we deduce that $\omega_1 = \pi$, and so $\omega_2 = \pi i$.

We now apply the inversion theorem for elliptic integrals in the Weierstrass normal form (Chandrasekharan [1, p. 94, Theorem 7]). Accordingly,

$$\frac{\mu^2}{2v^2} = \mathscr{P}(\theta; \pi, i\pi), \tag{1.14}$$

where $\mathscr{P}(\theta) := \mathscr{P}(\theta; \omega_1, \omega_2)$ denotes the Weierstrass $\mathscr{P}$-function.

Recall from the theory of the Weierstrass $\mathscr{P}$-function that e_1, e_2, and e_3 are the roots of

$$4t^3 - g_2 t - g_3 = 0.$$

If the roots are real, then $e_1 > e_2 > e_3$ (Chandrasekharan [1, p. 33], Whittaker and Watson [1, p. 443]). In our situation, $e_1 = \mu^2/2$, $e_2 = 0$, and $e_3 = -\mu^2/2$. In general, $\mathscr{P}$ and sn are connected by the formula (Whittaker and Watson [1, p. 505])

$$\mathscr{P}(\theta; \omega_1, \omega_2) = e_3 + (e_1 - e_3)\,\mathrm{sn}^{-2}\{\theta(e_1 - e_3)^{1/2}\}.$$

Thus, in the present case,

$$\mathscr{P}(\theta; \pi, i\pi) = -\frac{\mu^2}{2} + \frac{\mu^2}{\mathrm{sn}^2(\mu\theta)}.$$

Using the equality above in (1.14), we may now proceed as we did in the previous proof when we derived (1.5) and (1.6).

Alternatively, we can employ the representation (Whittaker and Watson [1, p. 460])

$$\mathscr{P}(\theta; \omega_1, \omega_2) = -\frac{2\eta_1}{\omega_1} + \left(\frac{\pi}{\omega_1}\right)^2 \csc^2\left(\frac{\pi\theta}{\omega_1}\right) - \frac{8\pi^2}{\omega_1^2} \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} \cos\left(\frac{2n\pi\theta}{\omega_1}\right), \tag{1.15}$$

where $-2\,\mathrm{Re}(-i\omega_2/\omega_1) < \mathrm{Re}(-2i\theta/\omega_1) < 2\,\mathrm{Re}(-i\omega_2/\omega_1)$, $\eta_1 = \zeta(\omega_1)$, and now ζ denotes the Weierstrass ζ-function. In our case, $\omega_1 = \pi$, $\omega_2 = i\pi$, and $q = \varepsilon^{-\pi}$. Using (1.15) in (1.14) and restricting ourselves to the interval $0 < \theta \leq \pi/2$, we deduce that

$$\frac{\mu^2}{2v^2} = -\frac{2\eta_1}{\pi} + \csc^2\theta - 8\sum_{n=1}^{\infty} \frac{n\cos(2n\theta)}{e^{2\pi n} - 1}.$$

In view of (1.2), it remains to show that $\eta_1 = \frac{1}{2}$.

To calculate η_1, it will be convenient to use the formula

$$\eta_1 = \frac{\pi^2}{\omega_1}\left(\frac{1}{6} + \sum_{n=1}^{\infty} \csc^2(n\pi\omega_2/\omega_1)\right), \tag{1.16}$$

found in D. F. Lawden's book [1, p. 184]. Thus, with $\omega_1 = \pi$ and $\omega_2 = i\pi$,

$$\eta_1 = \pi\left(\frac{1}{6} - \sum_{n=1}^{\infty} \operatorname{csch}^2(n\pi)\right). \tag{1.17}$$

However, Ramanujan [22] and the author [4, p. 245, eq. (1.16)], [1, Prop. 2.26] have shown that

$$\sum_{n=1}^{\infty} \operatorname{csch}^2(n\pi) = \frac{1}{6} - \frac{1}{2\pi}.$$

Using this in (1.17), we deduce that $\eta_1 = \frac{1}{2}$, as desired.

Entry 2 (p. 285). *Let θ, v, and μ be as in Entry* 1. *Then, if* $0 < \theta < \pi/2$,

$$-\frac{\mu}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n v^{4n-1}}{n!\,(4n-1)} = \cot\theta + \frac{\theta}{\pi} + 4\sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{e^{2\pi n} - 1}. \tag{2.1}$$

Proof. We first indicate that (2.1) is valid in the limit as θ approaches 0. Letting θ tend to 0, so that also v tends to 0, we find that we must show that

$$\lim_{\theta\to 0}\left(\frac{\mu}{\sqrt{2}v} - \cot\theta\right) = 0.$$

By a routine application of l'Hôpital's rule, the limit above is indeed true.

It thus suffices to prove that the derivatives of both sides of (2.1) with respect to θ are equal. Differentiating (2.1), we see that it is sufficient to prove that

$$-\frac{\mu}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n v^{4n-2}}{n!} \frac{dv}{d\theta} = -\csc^2\theta + \frac{1}{\pi} + 8\sum_{n=1}^{\infty} \frac{n\cos(2n\theta)}{e^{2\pi n} - 1}.$$

By Entry 1, it therefore suffices to prove that

$$\frac{\mu}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n v^{4n-2}}{n!} \frac{dv}{d\theta} = \frac{\mu^2}{2v^2}. \tag{2.2}$$

But by (1.1),

$$\frac{\mu}{\sqrt{2}} \frac{d\theta}{dv} = \frac{1}{\sqrt{1 - v^4}},$$

and since θ is a monotonically increasing function of v for $\theta \le v < 1$,

$$\frac{dv}{d\theta} = \frac{\mu}{\sqrt{2}}\sqrt{1 - v^4}. \tag{2.3}$$

Using (2.3) in (2.2), we see that it suffices to prove that

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n v^{4n-2}}{n!} \sqrt{1 - v^4} = \frac{1}{v^2}. \tag{2.4}$$

But the series on the left side of (2.4) equals $v^{-2}(1 - v^4)^{-1/2}$, and so we are done.

Following Ramanujan [22] (Berndt [6, p. 26, Entry 22(iii)]), we define

$$f(-q) := \prod_{n=1}^{\infty} (1 - q^n), \qquad |q| < 1. \tag{3.1}$$

Observe that if $q = e^{\pi i \tau}$, $\operatorname{Im} \tau > 0$, then $f(-q^2) = e^{-\pi i \tau/12}\eta(\tau)$, where $\eta(\tau)$ denotes the classical Dedekind eta-function.

Ramanujan interjected two small question marks between the two lines of his version of (3.2) below. Indeed, his equality is slightly in error, because he evidently neglected the term $2 \log(f(-e^{-2\pi}))$ which doubtless arose in his proof.

Entry 3 (p. 285). *Let θ, v, and μ be defined by Entry* 1. *If* $0 < \theta \le \pi/2$, *then*

$$\log v + \frac{\pi}{6} - \tfrac{1}{2}\log 2 + \sum_{n=1}^{\infty} \frac{(\frac{1}{4})_n v^{4n}}{(\frac{3}{4})_n 4n}$$
$$= \log(\sin\theta) + \frac{\theta^2}{2\pi} - 2\sum_{n=1}^{\infty} \frac{\cos(2n\theta)}{n(e^{2\pi n} - 1)}. \tag{3.2}$$

Proof. Let θ tend to 0 on both sides of (3.2). Since v also approaches 0, we find that

$$\lim_{\theta \to 0} \log\left(\frac{v}{\sqrt{2}\sin\theta}\right) = -\frac{\pi}{6} - 2\sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n} - 1)}. \tag{3.3}$$

By (2.3),

$$\lim_{\theta \to 0} \frac{v}{\sqrt{2}\sin\theta} = \lim_{\theta \to 0} \frac{dv/d\theta}{\sqrt{2}\cos\theta} = \lim_{\theta \to 0} \frac{\mu\sqrt{1 - v^4}}{2\cos\theta} = \frac{\mu}{2}.$$

Also, by Entry 23(iii) in Chapter 16 of Ramanujan's second notebook [22], (Berndt [6, p. 38]),

$$\log(f(-e^{-2\pi})) = -\sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n} - 1)}.$$

Thus, (3.3) is valid if and only if

$$\log\frac{\mu}{2} = -\frac{\pi}{6} + 2\log(f(-e^{-2\pi})). \tag{3.4}$$

By Entry 12(iii) in Chapter 17 of Ramanujan's second notebook [22] (Berndt [6, p. 124]),

$$f(-q^2) = \sqrt{z}2^{-1/3}\{k^2(1-k^2)/q\}^{1/12}.$$

Here $q = e^{-\pi}$, $k = 1/\sqrt{2}$, and $z = \mu$. Thus,

$$\log\left(\frac{2}{\mu}f^2(-e^{-2\pi})\right) = \log q^{-1/6} = \frac{\pi}{6}.$$

Thus, (3.4) holds, and (3.2) is valid for $\theta = 0$.

It therefore suffices to show that the derivatives of each side of (3.2) with respect to θ are equal. Differentiating both sides of (3.2), we find that

$$\left(\frac{1}{v} + \sum_{n=1}^{\infty}\frac{(\frac{1}{4})_n v^{4n-1}}{(\frac{3}{4})_n}\right)\frac{dv}{d\theta} = \cot\theta + \frac{\theta}{\pi} + 4\sum_{n=1}^{\infty}\frac{\sin(2n\theta)}{e^{2\pi n}-1}.$$

By (2.1) and (2.3), it suffices to prove that

$$\sum_{n=0}^{\infty}\frac{(\frac{1}{4})_n v^{4n-1}}{(\frac{3}{4})_n}\sqrt{1-v^4} = -\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n v^{4n-1}}{n!\,(4n-1)}.$$

This follows from Euler's transformation (0.14) by taking $a = \frac{1}{4}$, $b = 1$, $c = \frac{3}{4}$, and $z = v^4$. This completes the proof.

Entry 4 (p. 285). *Let θ, v, and μ be as defined in Entry* 1. *For* $0 \le \theta \le \pi/2$,

$$\tfrac{1}{2}\tan^{-1} v = \sum_{n=0}^{\infty}\frac{\sin\{(2n+1)\theta\}}{(2n+1)\cosh\{(2n+1)\pi/2\}}. \tag{4.1}$$

PROOF. First, it is clear that (4.1) holds for $\theta = 0 = v$. Thus, it suffices to show that the derivatives of both sides of (4.1) with respect to θ are equal.

Differentiating (4.1), we find that

$$\sum_{n=0}^{\infty}\frac{\cos\{(2n+1)\theta\}}{\cosh\{(2n+1)\pi/2\}} = \frac{1}{2}\frac{1}{1+v^2}\frac{dv}{d\theta} = \frac{\mu}{2\sqrt{2}}\sqrt{\frac{1-v^2}{1+v^2}}, \tag{4.2}$$

by (2.3). Recalling the Fourier series (0.10), we see that (4.2) is equivalent to the equality

$$\mathrm{cn}(z\theta) = \sqrt{\frac{1-v^2}{1+v^2}},$$

since $k = 1/\sqrt{2}$ and $z = \mu$. Since $v = 2^{-1/2}\,\mathrm{sd}(z\theta)$, by (1.4), we are thus required to prove that

$$\frac{2 - \mathrm{sc}^2(z\theta)}{2 + \mathrm{sd}^2(z\theta)} = \mathrm{cn}^2(z\theta),$$

or, since $\mathrm{sd}\, u = \mathrm{sn}\, u/\mathrm{dn}\, u$,

$$2\,\mathrm{dn}^2(z\theta) - \mathrm{sn}^2(z\theta) = \mathrm{cn}^2(z\theta)(2\,\mathrm{dn}^2(z\theta) + \mathrm{sn}^2(z\theta)). \tag{4.3}$$

However, using (0.6) and (0.7), we readily see that (4.3) is true, and this completes the proof.

Note that if we set $\theta = \pi/2$ and $v = 1$ in (4.1), we obtain the evaluation

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\cosh\{(2n+1)\pi/2\}} = \frac{\pi}{8}, \tag{4.4}$$

a result established by Ramanujan in Entry 15 of Chapter 14 in his second notebook [22] (Berndt [4, p. 262]).

Entry 5 (p. 285). *Let θ, v, and μ be as in Entry* 1. *If* $0 \le \theta \le \pi/2$, *then*

$$\tfrac{1}{4}\cos^{-1} v^2 = \sum_{n=0}^{\infty} \frac{(-1)^n \cos\{(2n+1)\theta\}}{(2n+1)\cosh\{(2n+1)\pi/2\}}. \tag{5.1}$$

Proof. Both sides of (5.1) are equal to 0 when $\theta = \pi/2$ ($v = 1$). Hence, it suffices to show that the derivatives with respect to θ of both sides of (5.1) are equal.

Differentiating (5.1) and using (2.3) and (1.4), we find that

$$\sum_{n=0}^{\infty} \frac{(-1)^n \sin\{(2n+1)\theta\}}{\cosh\{(2n+1)\pi/2\}} = \frac{v}{2\sqrt{1-v^4}}\frac{dv}{d\theta} = \frac{vz}{2\sqrt{2}} = \tfrac{1}{4}z\,\mathrm{sd}(z\theta), \tag{5.2}$$

since $\mu = z$. However, (5.2) follows immediately from (0.11), since $k = k' = 1/\sqrt{2}$. This completes the proof.

Ramanujan's formulation of Entry 5 contains a slight misprint; he has written $\frac{1}{2}$ instead of $\frac{1}{4}$ on the left side of (5.1).

Setting $\theta = 0$ and $v = 0$ in (5.1) yields once again (4.4).

Entry 6 (p. 285). *Let θ, v, and μ be defined as in Entry* 1. *If* $0 \le \theta \le \pi/2$, *then*

$$\frac{\sqrt{2}}{4\mu}\sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!\,(4n+3)} v^{4n+3} = \frac{\pi\theta}{8} - \sum_{n=0}^{\infty} \frac{(-1)^n \sin\{(2n+1)\theta\}}{(2n+1)^2\cosh\{(2n+1)\pi/2\}}. \tag{6.1}$$

PROOF. It is clear that (6.1) holds for $\theta = 0$ $(v = 0)$. It thus suffices to demonstrate that the derivatives with respect to θ of both sides of (6.1) are equal.

Differentiating (6.1), we arrive at

$$\frac{\sqrt{2}}{4\mu}\sum_{n=0}^{\infty}\frac{2^{2n}(n!)^2}{(2n+1)!}v^{4n+2}\frac{dv}{d\theta} = \frac{\pi}{8} - \sum_{n=0}^{\infty}\frac{(-1)^n\cos\{(2n+1)\theta\}}{(2n+1)\cosh\{(2n+1)\pi/2\}},$$

or, by (2.3) and (5.1),

$$\sum_{n=0}^{\infty}\frac{2^{2n}(n!)^2}{(2n+1)!}v^{4n+2}\sqrt{1-v^4} = \frac{\pi}{2} - \cos^{-1}v^2. \tag{6.2}$$

Now,

$$\frac{\pi}{2} - \cos^{-1}v^2 = \int_0^{v^2}\frac{dt}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{n!}\int_0^{v^2}t^{2n}\,dt = \sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n v^{4n+2}}{n!\,(2n+1)}.$$

Thus, from (6.2), it suffices to prove that

$$\sum_{j=0}^{\infty}\frac{2^{2j}(j!)^2}{(2j+1)!}v^{4j+2}\sqrt{1-v^4} = \sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{n!\,(2n+1)}v^{4n+2}.$$

This last equality follows from Euler's transformation (0.14) with $a = b = \frac{1}{2}$, $c = \frac{3}{2}$, and $z = v^4$.

Inversion Formulas for G

Entry 7 (p. 286). *Let* $0 \le \theta \le \pi/2$ *and* $0 \le v \le 1$. *Put*

$$\frac{\theta\mu}{2} = \int_0^v \frac{dt}{\sqrt{1+t^4}} = G(v), \tag{7.1}$$

where the constant μ *is defined by setting* $\theta = \pi/2$ *and* $v = 1$. *Then*

$$2\tan^{-1}v = \theta + \sum_{n=1}^{\infty}\frac{\sin(2n\theta)}{n\cosh(n\pi)}. \tag{7.2}$$

PROOF. Set

$$u = 2\tan^{-1}t.$$

A short calculation shows that

$$\frac{du}{dt} = \frac{2}{1+t^2}, \tag{7.3}$$

and another straightforward calculation yields

$$\sqrt{1-\tfrac{1}{2}\sin^2 u}=\frac{\sqrt{1+t^4}}{1+t^2}. \tag{7.4}$$

Utilizing (7.3) and (7.4) in (7.1), we find that

$$\theta\mu=\int_0^{2\tan^{-1}v}\frac{du}{\sqrt{1-\tfrac{1}{2}\sin^2 u}}. \tag{7.5}$$

This is an incomplete elliptic integral of the first kind with $k=1/\sqrt{2}$.

Setting $v=1$ and $\theta=\pi/2$ in (7.5), we find that

$$\frac{\pi}{2}\mu=\int_0^{\pi/2}\frac{du}{\sqrt{1-\tfrac{1}{2}\sin^2 u}}=K(1/\sqrt{2}).$$

Thus, $\mu=2K(1/\sqrt{2})/\pi$, as before, and moreover μ is again given by (1.3).

We now apply the inversion theorem for elliptic integrals in the Jacobi form (Whittaker and Watson [1, p. 494]). Thus, in the usual Jacobian notation, we deduce from (7.5) that

$$\operatorname{am}(z\theta)=2\tan^{-1}v. \tag{7.6}$$

Since $k=1/\sqrt{2}$, it follows as before that $q=e^{-\pi}$. Hence, using the Fourier expansion (0.8) in (7.6), we complete the proof.

Entry 8 (p. 286). *Let θ, v, and μ be as given in Entry 7. Then, for $0\le\theta\le\pi/2$,*

$$\frac{\pi}{8}-\tfrac{1}{2}\tan^{-1}v^2=\sum_{n=0}^{\infty}\frac{(-1)^n\cos\{(2n+1)\theta\}}{(2n+1)\cosh\{(2n+1)\pi/2\}}. \tag{8.1}$$

PROOF. Trivially, (8.1) holds for $\theta=\pi/2$ and $v=1$. (By (4.4), we also see that (8.1) is valid for $\theta=0$ and $v=0$.) Thus, it suffices to prove that the derivatives of both sides of (8.1) with respect to θ are equal.

By differentiation,

$$\frac{v}{1+v^4}\frac{dv}{d\theta}=\sum_{n=0}^{\infty}\frac{(-1)^n\sin\{(2n+1)\theta\}}{\cosh\{(2n+1)\pi/2\}}. \tag{8.2}$$

By (7.1),

$$\frac{\mu}{2}\frac{d\theta}{dv}=\frac{1}{\sqrt{1+v^4}}.$$

Since θ is a monotonically increasing function of v for $0\le v\le 1$,

$$\frac{dv}{d\theta}=\frac{z}{2}\sqrt{1+v^4}, \tag{8.3}$$

since $\mu = z$. Using (8.3) and (0.11), we see that (8.2) may be rewritten in the form

$$\frac{2v}{\sqrt{1+v^4}} = \operatorname{sd}(z\theta). \tag{8.4}$$

Differentiating (7.2) with respect to θ and employing (8.2), we find that, since $\mu = z$,

$$z\frac{\sqrt{1+v^4}}{1+v^2} = 1 + 2\sum_{n=1}^{\infty} \frac{\cos(2n\theta)}{\cosh(n\pi)} = z\operatorname{dn}(z\theta), \tag{8.5}$$

by (0.12).

Next, since $k = 1/\sqrt{2}$, it follows from (0.7) and (8.5) that

$$\operatorname{sn}^2(z\theta) = 2 - 2\operatorname{dn}^2(z\theta) = \frac{4v^2}{(1+v^2)^2}. \tag{8.6}$$

Thus, by (8.5) and (8.6),

$$\frac{2v}{\sqrt{1+v^4}} = \frac{2v}{1+v^2}\frac{1+v^2}{\sqrt{1+v^4}} = \frac{\operatorname{sn}(z\theta)}{\operatorname{dn}(z\theta)} = \operatorname{sd}(z\theta).$$

Hence, (8.4) has been proved, and the proof is complete.

Entry 9 (p. 286). *Let θ, v, and μ be defined as in Entry 7. Then, for $0 \le \theta < \pi/2$,*

$$\log\left(\frac{1+v}{1-v}\right) = \log\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right) + 4\sum_{n=0}^{\infty} \frac{(-1)^n \sin\{(2n+1)\theta\}}{(2n+1)(e^{(2n+1)\pi}-1)}. \tag{9.1}$$

PROOF. Clearly, (9.1) is valid for $\theta = 0 = v$. Thus, it suffices to prove that the derivatives with respect to θ of both sides of (9.1) are equal. Hence, upon differentiation,

$$z\frac{\sqrt{1+v^4}}{1-v^2} = \sec\theta + 4\sum_{n=0}^{\infty} \frac{(-1)^n\cos\{(2n+1)\theta\}}{e^{(2n+1)\pi}-1} = z\frac{\operatorname{dn}(z\theta)}{\operatorname{cn}(z\theta)}, \tag{9.2}$$

where we have used (8.3) and the Fourier series (0.13).

Now, by (0.6) and (8.6),

$$\operatorname{cn}^2(z\theta) = 1 - \operatorname{sn}^2(z\theta) = \left(\frac{1-v^2}{1+v^2}\right)^2. \tag{9.3}$$

Hence, by (8.5) and (9.3),

$$\frac{\sqrt{1+v^4}}{1-v^2} = \frac{\sqrt{1+v^4}}{1+v^2}\frac{1+v^2}{1-v^2} = \frac{\operatorname{dn}(z\theta)}{\operatorname{cn}(z\theta)}.$$

Thus, (9.2) has been demonstrated, and the proof is complete.

Ramanujan recorded Entry 9 with an extraneous factor of $\frac{1}{2}$ on the left side of (9.1).

Further results can be obtained by exploiting the integral identity (0.5). More precisely, replacing θ and v by θ_1 and x, respectively, in (7.1), recalling that the values of μ in (1.1) and (7.1) are identical, and employing (7.1), (0.5), and (1.1), we find that

$$\frac{\theta_1\mu}{\sqrt{2}} = \sqrt{2}\int_0^x \frac{dt}{\sqrt{1+t^4}} = \int_0^v \frac{dt}{\sqrt{1-t^4}} = \frac{\theta\mu}{\sqrt{2}},$$

where $v = \sqrt{2}x/\sqrt{1+x^4}$. In other words,

$$\theta_1 = \theta_1(x) = \theta(v) = \theta\left(\frac{\sqrt{2}x}{\sqrt{1+x^4}}\right) = \theta.$$

Thus, inversion formulas involving v can be converted into inversion formulas involving x, and conversely.

Inversion Formula for H

Entry 10 (p. 283). *Let θ, v, and μ be defined by the equation*

$$\tfrac{2}{3}\theta\mu = \int_0^v \frac{dt}{\sqrt{1-t^6}} = H(v), \tag{10.1}$$

where $0 \le \theta \le \pi/2$ and $0 \le v \le 1$. Then

$$\frac{4\mu^2}{9v^2} = \frac{1}{\sin^2\theta} - \frac{2}{\pi\sqrt{3}} + 8\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n\cos(2n\theta)}{e^{\pi n\sqrt{3}} - (-1)^n}. \tag{10.2}$$

Proof. Putting $\theta = \pi/2$, $v = 1$, and $u = t^6$ in (10.1), we deduce that

$$\mu = \frac{1}{2\pi}\int_0^1 u^{-5/6}(1-u)^{-1/2}\,du = \frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{2})}{2\pi\Gamma(\frac{2}{3})} = \frac{\sqrt{\pi}}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}, \tag{10.3}$$

where we applied the reflection formula for the gamma function.

Next, let $u = 4\mu^2/(9t^2)$ in (10.1) to deduce that

$$\theta = \int_{4\mu^2/(9v^2)}^{\infty} \frac{du}{\sqrt{4u^3 - 4(2\mu/3)^6}}. \tag{10.4}$$

Appealing to the classical theory of elliptic functions, in particular, the

Weierstrass normal form, we know that there exists a pair of linearly independent periods ω_1, ω_2 with $\mathrm{Im}(\omega_2/\omega_1) > 0$, such that

$$g_2(\omega_1, \omega_2) \equiv 0 \tag{10.5}$$

and

$$g_3(\omega_1, \omega_2) = 4\left(\frac{2\mu}{3}\right)^6, \tag{10.6}$$

where g_2 and g_3 are defined by (1.8) and (1.9), respectively. Now $g_2(1, \rho) = 0$, where $\rho = \exp(2\pi i/3)$ (Chandrasekharan [1, p. 88]). Moreover, the pair of arguments $(1, \rho)$ yielding the value 0 is unique up to the multiplication by a nonzero constant. Thus, from (10.3), (10.5), and (10.6),

$$\begin{aligned} 4\left(\frac{2\mu}{3}\right)^6 = g_3(\omega_1, \omega_2) &= 140\omega_1^{-6} \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m + n\rho)^6} \\ &= 140\omega_1^{-6} E_6^*(\tau) \\ &= 280\omega_1^{-6}\zeta(6)N(q^2), \end{aligned} \tag{10.7}$$

where $E_6^*(\tau)$ is an Eisenstein series, $\tau = \omega_2/\omega_1 = \rho$, ζ denotes the Riemann zeta-function, $q = \exp(\pi i\tau)$, and

$$N(q^2) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^{2n}}{1 - q^{2n}}.$$

For the facts quoted in (10.7), see Rankin's book [1, p. 194], and for further information about N, see Ramanujan's work [22], [17], [23, pp. 136–162] (Berndt [4, Chap. 15, Section 9], [6, Chap. 17, Section 13]). In his paper [17], Ramanujan replaced N by R.

By Entry 13(ii) in Chapter 17 of Ramanujan's second notebook [22] (Berndt [6, p. 126]),

$$N(q^2) = z^6(1 + k^2)(1 - \tfrac{1}{2}k^2)(1 - 2k^2), \tag{10.8}$$

where

$$q = e^{\pi i\tau} = e^{\pi i\omega_2/\omega_1} = e^{\pi i\rho} = e^{-\pi \exp(\pi i/6)}. \tag{10.9}$$

(See also Ramanujan's paper [17], [23, p. 140].) Thus, $K'/K = \exp(\pi i/6)$. We must determine k and z.

From (10.5), (1.10), and (1.11), we find that $1 - k^2 + k^4 = 0$. Thus, $k^2 = \exp(\pm\pi i/3)$. To prove that $k^2 = \exp(-\pi i/3)$, we use the result (Erdélyi [1, p. 105, eqs. (55), (56)]),

$$\frac{{}_2F_1(a + \frac{1}{3}, 3a; 2a + \frac{2}{3}; \exp(\pm\pi i/3))}{{}_2F_1(a + \frac{1}{3}, 3a; 2a + \frac{2}{3}; \exp(\mp\pi i/3))} = \exp(\pm\pi ia), \tag{10.10}$$

provided that $2a + \frac{2}{3} \neq 0, -1, -2, \ldots$. Suppose that $k^2 = \exp(\pi i/3)$, so that $1 - k^2 = \exp(-\pi i/3)$. Then, putting $a = \frac{1}{6}$ in (10.10), we find that $K'/K = \exp(-\pi i/6)$, which does not agree with the value for K'/K found above. However, if we assume that $k^2 = \exp(-\pi i/3)$, so that $1 - k^2 = \exp(\pi i/3)$, we find from (10.10) that $K'/K = \exp(\pi i/6)$, in agreement with what we calculated above. Hence $k^2 = \exp(-\pi i/3)$. Furthermore (Erdélyi [1, p. 105, eq. (56)]),

$$z = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; e^{-\pi i/3}) = \frac{2\sqrt{\pi}3^{-\pi i/12}}{3^{3/4}\Gamma(\frac{5}{6})\Gamma(\frac{2}{3})}. \tag{10.11}$$

Returning to (10.8) and using the values $k = e^{-\pi i/6}$ and z in (10.11), we find that

$$N(q^2) = -\frac{2^6\pi^3 i}{3^{9/2}\Gamma^6(\frac{2}{3})\Gamma^6(\frac{5}{6})}\frac{3\sqrt{3}i}{2} = \frac{2^5\pi^3}{3^3\Gamma^6(\frac{2}{3})\Gamma^6(\frac{5}{6})}. \tag{10.12}$$

Finally, from (10.7) and (10.12),

$$4\left(\frac{2\mu}{3}\right)^6 = \frac{280\pi^6}{945\omega_1^6}\frac{2^5\pi^3}{3^3\Gamma^6(\frac{2}{3})\Gamma^6(\frac{5}{6})} = \frac{2^8\pi^9}{3^6\omega_1^6\Gamma^6(\frac{2}{3})\Gamma^6(\frac{5}{6})} = 4\left(\frac{2\mu}{3}\right)^6\frac{\pi^6}{\omega_1^6},$$

by (10.3). Hence, $\omega_1 = \pi$ and $\omega_2 = \rho\pi$, where $\rho = e^{2\pi i/3}$.

We now invoke the inversion formula for elliptic integrals in the Weierstrass normal form (Chandrasekharan [1, p. 94]). Hence, from (10.4),

$$\frac{4\mu^2}{9v^2} = \wp(\theta; \pi, \rho\pi), \tag{10.13}$$

where $\wp$ denotes the Weierstrass $\wp$-function.

From (1.15),

$$\wp(\theta; \pi, \rho\pi) = -\frac{2\eta_1}{\pi} + \csc^2\theta - 8\sum_{n=1}^{\infty}\frac{nq^{2n}\cos(2n\theta)}{1 - q^{2n}}, \tag{10.14}$$

where, by (1.16),

$$\eta_1 = \pi\left(\frac{1}{6} + \sum_{n=1}^{\infty}\csc^2(n\pi\rho)\right) = \pi\left(\frac{1}{6} + \frac{1}{\pi\sqrt{3}} - \frac{1}{6}\right) = \frac{1}{\sqrt{3}}, \tag{10.15}$$

where we have used an evaluation found in Berndt's paper [1, p. 164, Prop. 2.27]. Recalling that q is given by (10.9), we easily find that

$$\frac{q^{2n}}{1 - q^{2n}} = \frac{(-1)^n}{e^{\pi n\sqrt{3}} - (-1)^n}, \qquad n \geq 1. \tag{10.16}$$

Putting (10.15) and (10.16) in (10.14) and then combining (10.13) and (10.14), we complete the proof of (10.2).

CHAPTER 27

q-Series

In view of Chapter 16 (Ramanujan [22], Berndt [6]), wherein several theorems on q-series are offered, and of the "lost notebook" (Ramanujan [24]), which is almost completely devoted to q-series, it is surprising that the unorganized portion of the second notebook and the third notebook contain only a small amount of material on q-series. In fact, only five identities are to be found here (Entries 1–5 below). More interesting are the asymptotic formulas given in Entries 6–8. Thus, in this chapter, only eight results are proved. Additional q-analysis can be found in Chapter 32, entitled Continued Fractions (Berndt [9]), or in the memoir of Andrews, Berndt, Jacobsen, and Lamphere [1]. Further related material can be found in Chapter 25 on theta-functions.

In the sequel, we employ the customary notation, $(a)_0 := 1$,

$$(a)_n := (a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \qquad n = 1, 2, \ldots,$$

and

$$(a)_\infty := (a; q)_\infty = \lim_{n \to \infty} (a; q)_n, \qquad |q| < 1.$$

Before Ramanujan found a rigorous proof of the Rogers–Ramanujan identities, he derived considerable evidence for their validity. In a fragment found in the publication of the "lost notebook" (Ramanujan [24, p. 358]), Ramanujan offers four reasons why the first Rogers–Ramanujan identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q; q^5)_\infty (q^4; q^5)_\infty}, \qquad |q| < 1, \tag{0.1}$$

must be true. In particular, as q tends to 1, the logarithms of the left and right sides of (0.1) are both asymptotic to $\pi^2/\{15(1 - q)\}$.

On page 366 in the third notebook and on page 359 of the "lost notebook" [24], Ramanujan records an imprecise asymptotic formula for a certain general q-series generalizing the series on the left side of (0.1). On page 365 of the third notebook, Ramanujan offers another general asymptotic expansion, which does not fall under the purview of the aforementioned result. We offer two proofs, one due to R. McIntosh [1], of the first asymptotic formula. The latter-mentioned asymptotic expansion is easier to prove, and we first published that proof in [5].

In the sequel, B_n, $n \geq 0$, denotes the nth Bernoulli number, defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n, \qquad |x| < 2\pi. \tag{0.2}$$

The dilogarithm $\mathrm{Li}_2(z)$ may be defined for all complex z by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-u)}{u}\, du \tag{0.3}$$

$$= \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \qquad |z| \leq 1, \tag{0.4}$$

where the principal value of the logarithm is taken here and throughout the sequel.

We begin by proving the aforementioned five identities, of which Entry 3 is perhaps the most interesting.

Entry 1 (p. 354). *Let* $|q| < 1$. *Suppose that* $a \neq 0$ *and* $1 - bq^n \neq 0$ *for* $n \geq 1$. *Then*

$$\frac{(-aq; q)_\infty}{(bq; q)_\infty} = \sum_{n=0}^{\infty} \frac{(-b/a; q)_n a^n q^{n(n+1)/2}}{(q)_n (bq)_n}. \tag{1.1}$$

PROOF. Recall Heine's theorem given in Entry 4 of Chapter 16 (Berndt [6, p. 14]); if $|abc| < 1$, then

$$\sum_{n=0}^{\infty} \frac{(1/b)_n (1/c)_n (abc)^n}{(q)_n (a)_n} = \frac{(ab)_\infty (ac)_\infty}{(a)_\infty (abc)_\infty}.$$

If b tends to 0, then $(1/b)_n b^n$ approaches $(-1)^n q^{n(n-1)/2}$, and so

$$\sum_{n=0}^{\infty} \frac{(1/c)_n (-ac)^n q^{n(n-1)/2}}{(q)_n (a)_n} = \frac{(ac)_\infty}{(a)_\infty}. \tag{1.2}$$

Replacing a by bq and c by $-a/b$ in (1.2), we readily deduce (1.1).

Entry 1 is equivalent to a result found in Andrews' paper [2, eq. (5.5)] and, according to Andrews, is due to Cauchy.

Entry 2 (p. 354). *Let $|q| < 1$, and suppose that $(aq)_n \neq 0$ for $n \geq 1$. Then*

$$(aq)_\infty \sum_{n=1}^{\infty} \frac{na^n q^{n^2}}{(q)_n (aq)_n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} a^n q^{n(n+1)/2}}{1-q^n}. \tag{2.1}$$

Proof. Recall Entry 9 of Chapter 16 (Berndt [6, p. 18]), i.e.,

$$(aq)_\infty \sum_{n=0}^{\infty} \frac{b^n q^{n^2}}{(q)_n (aq)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n (b/a)_n a^n q^{n(n+1)/2}}{(q)_n}. \tag{2.2}$$

Differentiating (2.2) with respect to b, we find that

$$(aq)_\infty \sum_{n=1}^{\infty} \frac{nb^{n-1} q^{n^2}}{(q)_n (aq)_n} = \sum_{n=1}^{\infty} \frac{(-1)^n \dfrac{d}{db}(b/a)_n a^n q^{n(n+1)/2}}{(q)_n}. \tag{2.3}$$

Now set $b = a$ and observe that

$$\frac{d}{db}\left(\frac{b}{a}\right)_n \bigg|_{b=a} = -\frac{1}{a}(q)_{n-1}, \qquad n \geq 1. \tag{2.4}$$

Substituting (2.4) in (2.3) and multiplying both sides by a, we complete the proof.

Entry 3 (p. 354). *Let $|q| < 1$, $a \neq 0$, and $(b)_n \neq 0$, $n \geq 1$. Then*

$$\sum_{n=1}^{\infty} \frac{a^n - b^n}{1-q^n} = \sum_{n=1}^{\infty} \frac{a^n (b/a)_n}{(1-q^n)(b)_n}. \tag{3.1}$$

Proof. Let $F(a, b)$ denote the right side of (3.1). Then

$$\begin{aligned} F(a,b) - F(aq,bq) &= \sum_{n=1}^{\infty} \frac{a^n (b/a)_n}{(1-q^n)(b)_n} - \sum_{n=1}^{\infty} \frac{(aq)^n (b/a)_n}{(1-q^n)(bq)_n} \\ &= \sum_{n=1}^{\infty} \frac{a^n (b/a)_n}{(1-q^n)(b)_{n+1}} \{(1-bq^n) - q^n(1-b)\} \\ &= \sum_{n=1}^{\infty} \frac{a^n (b/a)_n}{(b)_{n+1}}. \end{aligned} \tag{3.2}$$

Recall Heine's transformation

$$\sum_{n=0}^{\infty} \frac{(b/a)_n (c)_n}{(d)_n (q)_n} a^n = \frac{(b)_\infty (c)_\infty}{(a)_\infty (d)_\infty} \sum_{n=0}^{\infty} \frac{(d/c)_n (a)_n}{(b)_n (q)_n} c^n, \tag{3.3}$$

from Entry 6 of Chapter 16 (Berndt [6, p. 15]), where $|a|$, $|c|$, $|q| < 1$.

Replacing b by bq, c by q, and d by bq^2 in (3.3), we see that

$$\sum_{n=0}^{\infty} \frac{(bq/a)_n a^n}{(bq^2)_n} = \frac{(bq)_\infty (q)_\infty}{(a)_\infty (bq^2)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n q^n}{(q)_n}$$

$$= \frac{(1-bq)(q)_\infty}{(a)_\infty} \frac{(aq)_\infty}{(q)_\infty} = \frac{1-bq}{1-a},$$

where we employed the q-binomial theorem, (2.1) of Chapter 16 (Berndt [6, p. 14]). Thus,

$$\sum_{n=1}^{\infty} \frac{(bq/a)_{n-1} a^{n-1}}{(bq)_n} = \frac{1}{1-a},$$

or

$$\sum_{n=1}^{\infty} \frac{(b/a)_n a^n}{(b)_{n+1}} = \frac{(1-b/a)a}{(1-b)(1-a)} = \frac{a}{1-a} - \frac{b}{1-b}. \tag{3.4}$$

Combining (3.2) and (3.4), we deduce that

$$F(a, b) - F(aq, bq) = \frac{a}{1-a} - \frac{b}{1-b}.$$

Hence,

$$F(aq^n, bq^n) - F(aq^{n+1}, bq^{n+1}) = \frac{aq^n}{1-aq^n} - \frac{bq^n}{1-bq^n}. \tag{3.5}$$

Now sum both sides of (3.5) on n, $0 \le n < \infty$, and observe that $F(aq^n, bq^n)$ tends to 0 as n tends to ∞. We therefore conclude that

$$F(a, b) = \sum_{n=0}^{\infty} \left(\frac{aq^n}{1-aq^n} - \frac{bq^n}{1-bq^n} \right)$$

$$= \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \{(aq^n)^m - (bq^n)^m\}$$

$$= \sum_{m=1}^{\infty} \left(\frac{a^m}{1-q^m} - \frac{b^m}{1-q^m} \right),$$

and the proof is complete.

Entry 4 (p. 354). *For* $|q| < 1$ *and* $(aq)_n \neq 0$, $n \geq 1$,

$$\sum_{n=1}^{\infty} \frac{(aq)^n}{1-q^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} a^n q^{n(n+1)/2}}{(1-q^n)(aq)_n}. \tag{4.1}$$

PROOF. Letting a tend to 0 in Entry 3 and observing that

$$\lim_{a \to 0} (b/a)_n a^n = (-b)^n q^{n(n-1)/2},$$

we find that

$$-\sum_{n=1}^{\infty} \frac{b^n}{1-q^n} = \sum_{n=1}^{\infty} \frac{(-b)^n q^{n(n-1)/2}}{(1-q^n)(b)_n}. \tag{4.2}$$

Replacing b by aq in (4.2), we deduce (4.1).

An equivalent version of Entry 4 has been established by K. Uchimura [1, eq. (3)].

Entry 5 (p. 355). *For* $|q| < 1$ *and* $(a)_n \neq 0$, $n \geq 1$,

$$\sum_{n=1}^{\infty} \frac{na^n}{1-q^n} = \sum_{n=1}^{\infty} \frac{(q)_{n-1} a^n}{(1-q^n)(a)_n}. \tag{5.1}$$

Proof. Return to (3.1) and divide both sides by $1 - b/a$, $a \neq b$, to find that

$$a \sum_{n=1}^{\infty} \frac{a^n - b^n}{(a-b)(1-q^n)} = \sum_{n=1}^{\infty} \frac{a^n (bq/a)_{n-1}}{(1-q^n)(b)_n}. \tag{5.2}$$

Letting b tend to a in (5.2), we readily achieve (5.1).

Ramanujan's primary asymptotic formulas are given in Entries 6 and 7 below. Entries 8 and 9 are corollaries of Entry 7.

We have reformulated Ramanujan's version of Entry 6 by replacing x, in Ramanujan's notation, by $-\log q$ and taking the logarithms of both sides of Ramanujan's formula.

Entry 6 (p. 365). *Let* $|a|$, $|q| < 1$. *Then as* q *tends to* $1-$,

$$\begin{aligned} &\log\left(\sqrt{1+a} \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q)_n}\right) \\ &\quad \sim \frac{1}{\log q} \operatorname{Li}_2(-a) + \frac{B_2}{2!}\left(-\frac{\log q}{1+a}\right) a + \frac{B_4}{4!}\left(-\frac{\log q}{1+a}\right)^3 (a - a^2) \\ &\quad + \frac{B_6}{6!}\left(-\frac{\log q}{1+a}\right)^5 (a - 11a^2 + 11a^3 - a^4) + \cdots, \end{aligned}$$

where B_n, $n \geq 0$, *and* Li_2 *are defined by* (0.2) *and* (0.3), *respectively.*

Proof. By Euler's second identity (Andrews [3, p. 19]),

$$f(a; q) := \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q)_n} = (-aq)_\infty.$$

Thus, for $|a|, |q| < 1$,

$$\begin{aligned}
\log(\sqrt{1+a}f(a;q)) &= \sum_{n=0}^{\infty} \log(1+aq^{n+1}) + \tfrac{1}{2}\log(1+a) \\
&= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{m-1}(aq^{n+1})^m}{m} + \frac{1}{2}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}a^m}{m} \\
&= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}a^m}{m}\left(\sum_{n=0}^{\infty} q^{(n+1)m} + \frac{1}{2}\right) \\
&= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}a^m}{m}\left(\frac{q^m}{1-q^m} + \frac{1}{2}\right) \\
&= \left(\sum_{m \le -1/\log q} + \sum_{m > -1/\log q}\right)\frac{(-1)^{m-1}a^m}{m}\left(\frac{1}{q^{-m}-1} + \frac{1}{2}\right).
\end{aligned} \tag{6.1}$$

For $m > -1/\log q$,

$$q^{-m} - 1 \ge q^{1/\log q} - 1 = e - 1.$$

Thus,

$$\begin{aligned}
&\left|\sum_{m > -1/\log q} \frac{(-a)^m}{m}\left(\frac{1}{q^{-m}-1} + \frac{1}{2}\right)\right| \\
&\qquad \le e^{-\log|a|/\log q} \sum_{m > -1/\log q} \frac{|a|^{m+1/\log q}}{m}\left(\frac{1}{e-1} + \frac{1}{2}\right) \\
&\qquad \ll e^{-\log|a|/\log q}.
\end{aligned}$$

Hence, from (6.1),

$$\log(\sqrt{1+a}f(a;q)) = \sum_{m \le -1/\log q} \frac{(-1)^{m-1}a^m}{m}\left(\frac{1}{q^{-m}-1} + \frac{1}{2}\right) + O(e^{-\log|a|/\log q}). \tag{6.2}$$

Now, for $|x| < \pi$,

$$\frac{1}{e^{-x}-1} + \tfrac{1}{2} = -\tfrac{1}{2}\coth\tfrac{1}{2}x = -\frac{1}{2}\left(\frac{2}{x} + 2\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}x^{2k-1}\right).$$

(In fact, see Entry 13 of Chapter 5 in Ramanujan's second notebook [22,

p. 51], or see Part I [2, p. 121].) Hence, since $|m \log q| \le 1 < \pi$,

$$\sum_{m \le -1/\log q} \frac{(-1)^{m-1}a^m}{m}\left(\frac{1}{q^{-m}-1}+\frac{1}{2}\right)$$
$$= -\frac{1}{2}\sum_{m \le -1/\log q} \frac{(-1)^{m-1}a^m}{m}\coth\left(\frac{m\log q}{2}\right)$$
$$= \sum_{m \le -1/\log q} \frac{(-1)^{m-1}a^m}{m}\left\{-\frac{1}{m\log q} - \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}(m\log q)^{2k-1}\right\}. \tag{6.3}$$

We now calculate the contribution of each of the first four expressions in curly brackets on the right side of (6.3).

The first expression in curly brackets, by (0.4), contributes

$$\frac{1}{\log q}\sum_{m \le -1/\log q}\frac{(-a)^m}{m^2} = \frac{1}{\log q}\operatorname{Li}_2(-a) + O(e^{-\log|a|/\log q}), \tag{6.4}$$

as q tends to $1-$.

The contribution of the term with $k = 1$ in (6.3) is equal to

$$\sum_{m \le -1/\log q}\frac{(-a)^m}{m}\frac{B_2}{2!}(m\log q) = \frac{B_2\log q}{2!}\left(\sum_{m=1}^{\infty}(-a)^m + O(e^{-\log|a|/\log q})\right)$$
$$= -\frac{B_2 a\log q}{2!(1+a)} + O(e^{-\log|a|/\log q}\log q). \tag{6.5}$$

For $k = 2$, the contribution is

$$\sum_{m \le -1/\log q}\frac{(-a)^m}{m}\frac{B_4}{4!}(m\log q)^3 = \frac{B_4}{4!}\log^3 q\sum_{m=1}^{\infty}(-a)^m m^2 + O(e^{-\log|a|/\log q}\log^3 q)$$
$$= \frac{B_4(a^2-a)\log^3 q}{4!(1+a)^3} + O(e^{-\log|a|/\log q}\log^3 q). \tag{6.6}$$

Lastly, we calculate the contribution of the term with $k = 3$ in (6.3). This term equals

$$\frac{B_6}{6!}\log^5 q\sum_{m=1}^{\infty}(-a)^m m^4 + O(e^{-\log|a|/\log q}\log^5 q)$$
$$= \frac{B_6(a^4 - 11a^3 + 11a^2 - a)\log^5 q}{6!(1+a)^5} + O(e^{-\log|a|/\log q}\log^5 q). \tag{6.7}$$

Substituting (6.4)–(6.7) in (6.3) and then (6.3) into (6.2), we complete the proof of Entry 6.

McIntosh [1] has both generalized Entry 6 and established a more explicit version in terms of Eulerian numbers.

On page 284, Ramanujan records a preliminary, less explicit version of Entry 6. The asymptotic expansion is supplied for the infinite product representation, via Euler's second identity, instead of the infinite q-series of Entry 6. We quote Ramanujan's formulation (in contemporary notation).

Entry 6′ (Formula (12), p. 284).

$$e^{-x/24}(1-e^{-a})^{1/2}(1-e^{-a-x})(1-e^{-a-2x})(1-e^{-a-3x})\dots$$
$$= \frac{(a/x)^{a/x}}{e^{a/x}\Gamma\left(\frac{a}{x}+1\right)}\sqrt{\frac{2\pi a}{x}}\, e^{-(1/x)L_2(e^{-a})-\theta}, \tag{6.8}$$

where

$$\theta = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} x^{2n+1}\left\{\frac{B_{2n}}{2n}\frac{a}{1!} + \frac{B_{2n+2}}{2n+2}\frac{a^3}{3!} + \cdots\right\}.$$

Although not immediately apparent, Entry 6′ is to be regarded as an asymptotic expansion for the left side of (6.8) as x tends to $0+$. As a first step in presenting (6.8) in a more recognizable form, we take logarithms of both sides of (6.8) and apply Stirling's formula in the form (Gradshteyn and Ryzhik [1, p. 940])

$$\log\Gamma\left(\frac{a}{x}+1\right) \sim \frac{a}{x}\log\frac{a}{x} - \frac{a}{x} + \tfrac{1}{2}\log\frac{2\pi a}{x} + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)}\left(\frac{x}{a}\right)^{2n-1},$$

as x tends to $0+$. Then, for $a > 0$, the logarithmic form of (6.8) can be written in the equivalent form

$$\tfrac{1}{2}\log(1-e^{-a}) + \sum_{n=1}^{\infty}\log(1-e^{-a-nx})$$
$$\sim -\frac{1}{x}L_2(e^{-a}) + \frac{x}{24} - \theta - \sum_{n=1}^{\infty}\frac{B_{2n}}{2n(2n-1)}\left(\frac{x}{a}\right)^{2n-1}, \tag{6.9}$$

as x approaches $0+$. To identify (6.9) with Entry 6, replac e^{-a} by $-a$, where $a<0$, and set $q = e^{-x}$ in (6.9). Thus, (6.9) gives an asymptotic expansion in powers of $\log q$, as q tends to $1-$, with the leading term in agreement with that in Entry 6. However, the coefficients of $(\log q)^{2n-1}$, $n \geq 1$, are considerably less explicit in (6.9) than in Entry 6. First, note that these coefficients are given in terms of $\log(-a)$ instead of a. Second, and much more importantly, from the definition of θ, these coefficients are represented by infinite series! Therefore, since the form of the asymptotic expansion in Entry 6′, or (6.9), is much inferior to that in Entry 6, it does not seem worthwhile to pursue it further.

Our first proof of Entry 7 below generalizes a method of G. Meinardus [1], while the second proof is due to McIntosh [1].

Entry 7 ([22, p. 366], [24, p. 359]). *Let* $a > 0$, $|q| < 1$, *and* b *and* c *be integers with* $b > 0$. *Let* z *denote the positive root of* $az^{2b} + z = 1$. *Then as* q *tends to* $1-$,

$$\log(f(a;q)) := \log\left(\sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n}\right) \sim -\frac{1}{\log q}(\mathrm{Li}_2(az^{2b}) + b\log^2 z) + c\log z - \tfrac{1}{2}\log(z + 2b(1-z)), \tag{7.1}$$

where Li_2 *is defined by* (0.3).

In both the third and lost notebooks, Ramanujan's formulation of Entry 7 is not as precise as (7.1). In particular, the first expression on the right side of (7.1) is replaced by

$$-\frac{1}{\log q}\int_0 \frac{\log(1/z)}{a}\,da$$

in both notebooks. Indeed, the upper limit on this integral is absent. Assuming that the upper limit is intended to be a, we find that

$$\begin{aligned}\int_0^a \frac{\log(1/z)}{a}\,da &= -\int_0^a \frac{\log(1-az^{2b})}{a}\,da \\ &= -\int_0^{az^{2b}} \frac{\log(1-u)}{u}\,du = \mathrm{Li}_2(az^{2b}),\end{aligned}$$

by (0.3). Thus, if our assumption is correct, the term $-b\log^2 z/\log q$ was omitted by Ramanujan in his formulation. (In fact, Ramanujan uses n for b.)

For positive integers ℓ, m, and n with $(\ell, m) = 1$, Meinardus [1] derived the asymptotic behavior of

$$\sum_{k=0}^{\infty} \frac{q^{\ell nk(k-1)/2+mk}}{(q^\ell;q^\ell)_k}$$

as q tends to $1-$. We shall use the same method to establish (7.1). Because the proof is very long and the details are similar to those in Meinardus's proof, we shall not provide all of the details.

First Proof. For $|wa| < 1$, set

$$F(a, w; q) := \sum_{k=0}^{\infty} \frac{(wa)^k}{(q)_k}$$

and

$$G(w; q) = \sum_{r=-\infty}^{\infty} (wq^{-c})^r q^{br^2}.$$

If $R(g; z_0)$ denotes the residue of a function g at a pole z_0, it follows that

$$R\left(\frac{1}{w} F(a, w; q)G(w; q); 0\right) = \sum_{k=0}^{\infty} \frac{a^k q^{bk^2+ck}}{(q)_k}.$$

Hence, by the residue theorem,

$$f(a; q) = \frac{1}{2\pi i} \int_C \frac{1}{w} F(a, w; q)G(w; q)\, dw, \tag{7.2}$$

where C is a positively oriented circle of radius $\rho < 1$ centered at the origin. The radius ρ will be prescribed later.

We next obtain asymptotic estimates for $F(a, w; q)$ and $G(w; q)$ as q tends to 1. First, we record the well-known theta transformation formula (Part II [4, Chap. 14, Entry 7, pp. 252–253]),

$$\sum_{k=-\infty}^{\infty} e^{-\alpha k^2 - \beta k} = \sqrt{\frac{\pi}{\alpha}}\, e^{\beta^2/(4\alpha)} \sum_{k=-\infty}^{\infty} e^{-\pi^2 k^2/\alpha - \pi i \beta k/\alpha},$$

where $\alpha > 0$. Let $\alpha = -b \log q$ and $\beta = c \log q - \log w$. Then, as q tends to 1,

$$\begin{aligned} G(w; q) &\sim \sqrt{\frac{\pi}{-b \log q}} \exp\left(-\frac{(c \log q - \log w)^2}{4b \log q}\right) \\ &\sim \sqrt{\frac{\pi}{-b \log q}} \exp\left(\frac{c}{2b} \log \omega - \frac{\log^2 w}{4b \log q}\right). \end{aligned} \tag{7.3}$$

Second, by trivial modifications in Meinardus's proof [1, p. 294],

$$F(a, w; q) \sim \exp\left(-\frac{1}{\log q} \operatorname{Li}_2(wa) - \tfrac{1}{2} \log(1 - wa)\right) \tag{7.4}$$

as q tends to 1, where $|wa| < 1$.

As in Meinardus's proof, letting $w = \rho e^{i\varphi}$, we find that the primary contribution in (7.2) arises from a small neighborhood of $\varphi = 0$. Thus, for

φ_0 sufficiently small,

$$f(a;q) \sim \frac{1}{2\pi i}\int_{\rho e^{-i\varphi_0}}^{\rho e^{i\varphi_0}} \frac{1}{w} F(a, w; q)G(w; q)\, dw$$
$$= \frac{1}{2\pi}\int_{-\varphi_0}^{\varphi_0} F(a, \rho e^{i\varphi}; q)G(\rho e^{i\varphi}; q)\, d\varphi$$
$$\sim \frac{1}{2\sqrt{-\pi b \log q}}\int_{-\varphi_0}^{\varphi_0} e^{\psi(\varphi)}\, d\varphi, \tag{7.5}$$

by (7.3) and (7.4), where

$$\psi(\varphi) = -\frac{\log^2(\rho e^{i\varphi})}{4b\log q} - \frac{1}{\log q}\mathrm{Li}_2(\rho a e^{i\varphi}) + \frac{c\log(\rho e^{i\varphi})}{2b} - \tfrac{1}{2}\log(1-\rho a e^{i\varphi}). \tag{7.6}$$

In a neighborhood of $\varphi = 0$,

$$\mathrm{Li}_2(\rho a e^{i\varphi}) = \mathrm{Li}_2(\rho a) + i\varphi\sum_{k=1}^{\infty}\frac{(\rho a)^k}{k} - \tfrac{1}{2}\varphi^2\sum_{k=1}^{\infty}(\rho a)^k + O(\varphi^3)$$

and

$$\log^2(\rho e^{i\varphi}) = \{\log\rho + \log(1 + i\varphi - \tfrac{1}{2}\varphi^2 + O(\varphi^3))\}^2$$
$$= \{\log\rho + (i\varphi - \tfrac{1}{2}\varphi^2) - \tfrac{1}{2}(i\varphi - \tfrac{1}{2}\varphi^2)^2 + O(\varphi^3)\}^2$$
$$= \{\log\rho + i\varphi + O(\varphi^3)\}^2$$
$$= \log^2\rho + 2i\varphi\log\rho - \varphi^2 + O(\varphi^3).$$

Thus, as φ tends to 0,

$$\mathrm{Li}_2(\rho a e^{i\varphi}) + \frac{1}{4b}\log^2(\rho e^{i\varphi}) = \mathrm{Li}_2(\rho a) + \frac{1}{4b}\log^2\rho$$
$$+ i\varphi\left(-\log(1-\rho a) + \frac{1}{2b}\log\rho\right) - \varphi^2\left(\frac{1}{4b} + \frac{\rho a}{2(1-\rho a)}\right) + O(\varphi^3). \tag{7.7}$$

We now choose ρ to be that unique positive number such that

$$\rho = (1-\rho a)^{2b}. \tag{7.8}$$

Define a positive number z by

$$z = 1 - \rho a. \tag{7.9}$$

Then by (7.8) and (7.9), $az^{2b} + z = 1$. Using (7.8) and (7.9) in (7.7), we find that, in a neighborhood of $\varphi = 0$,

$$\mathrm{Li}_2(\rho a e^{i\varphi}) + \frac{1}{4b}\log^2(\rho e^{i\varphi})$$
$$= \mathrm{Li}_2(\rho a) + \frac{1}{4b}\log^2\rho - \varphi^2\left(\frac{1}{4b} + \frac{\rho a}{2(1-\rho a)}\right) + O(\varphi^3)$$
$$= \mathrm{Li}_2(az^{2b}) + b\log^2 z - \frac{\varphi^2}{4b}(1 + 2abz^{2b-1}) + O(\varphi^3). \tag{7.10}$$

Next, in a neighborhood of $\varphi = 0$,

$$\frac{c}{2b}\log(\rho e^{i\varphi}) - \tfrac{1}{2}\log(1-\rho a e^{i\varphi}) = \frac{c}{2b}\log\rho - \tfrac{1}{2}\log(1-\rho a) + O(\varphi)$$
$$= (c - \tfrac{1}{2})\log z + O(\varphi), \tag{7.11}$$

by (7.8) and (7.9).

Employing (7.10) and (7.11) in (7.6), we conclude that

$$\psi(\varphi) = -\frac{1}{\log q}(\mathrm{Li}_2(az^{2b}) + b\log^2 z) + \frac{\varphi^2}{4b\log q}(1 + 2abz^{2b-1})$$
$$+ (c - \tfrac{1}{2})\log z + O\left(\frac{\varphi^3}{\log q}\right) + O(\varphi),$$

in a neighborhood of $\varphi = 0$. Then using the equality above in (7.5), we deduce, as in Meinardus's paper [1], that, as q tends to 1,

$$f(a;q) \sim \frac{1}{2\sqrt{-\pi b\log q}}\exp\left(-\frac{1}{\log q}(\mathrm{Li}_2(az^{2b}) + b\log^2 z)\right)z^{c-1/2}$$
$$\times \int_{-\varphi_0}^{\varphi_0}\exp\left(\frac{\varphi^2}{4b\log q}(1 + 2abz^{2b-1})\right)d\varphi$$
$$\sim \frac{\exp(-(1/\log q)\mathrm{Li}_2(az^{2b}) + b\log^2 z))z^{c-1/2}}{2\sqrt{-\pi b\log q}}\sqrt{\frac{-\pi 4b\log q}{1 + 2abz^{2b-1}}},$$

or

$$\log f(a;q) \sim -\frac{1}{\log q}(\mathrm{Li}_2(az^{2b}) + b\log^2 z) + (c - \tfrac{1}{2})\log z$$
$$-\tfrac{1}{2}\log(1 + 2b(1-z)/z),$$

from which (7.1) readily follows.

We emphasize that several details in the proof above have been omitted. In particular, no error analysis for our approximations has been given. To do this, φ_0 must be chosen to depend upon q in a suitable fashion. Meinardus's method can likely be extended to give further terms in the asymptotic expansion of $f(a;q)$. However, the additional technical difficulties that arise appear to be formidable. Meinardus's ideas have also been generalized by M.-P. Chen [1].

A second approach to proving Entry 7 has been devised by R. McIntosh [1]. His method is applicable in a wide variety of problems. It also has the advantage of being able to provide additional terms in the asymptotic expansion, although computer algebra is advisable to effect the necessary calculations. We remark that in the third notebook, but not in the lost notebook, Ramanujan indicates two further terms in the asymptotic expansion of $f(a; q)$, but he does not explicitly state them.

Second Proof. The sequence of summands in $f(a; q)$ is unimodal, and we first determine where these summands are largest. Setting the ratio of the nth and $(n-1)$st terms of $f(a; q)$ equal to 1, we readily see that

$$aq^{2bn-b+c} + q^n - 1 = 0.$$

This equation implicitly defines n as a function of q. In particular, as q tends to $1-$, n tends to ∞. Thus, as q tends to 1, q^n approaches z, where z is the positive root of the equation

$$az^{2b} + z - 1 = 0. \tag{7.12}$$

For each $q < 1$, define $N = N(q)$ by $q^N = z$ and set $n = Nt$. Thus,

$$\frac{a^n q^{bn^2+cn}}{(q)_n} = \frac{a^{Nt} z^{bNt^2+ct}}{(z^{1/N})_{Nt}}.$$

Note that t tends to 1 as n tends to ∞. We will utilize the Euler–Maclaurin summation formula (Olver [1, p. 285]) to determine the asymptotic behavior of $\log(z^{1/N})_n$ as N tends to ∞. If $f(x) = \log(1 - z^{x/N})$, then, for each nonnegative integer m,

$$\begin{aligned}\log(z^{1/N})_{Nt} = \sum_{k=1}^{Nt} f(k) &= \int_1^{Nt} f(x)\,dx + \tfrac{1}{2}f(1) + \tfrac{1}{2}f(Nt)\\ &+ \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!}\{f^{(2j-1)}(Nt) - f^{(2j-1)}(1)\}\\ &+ \frac{1}{(2m+1)!}\int_1^{Nt} B_{2m+1}(x - [x]) f^{(2m+1)}(x)\,dx, \end{aligned} \tag{7.13}$$

where, for each $k \geq 0$, B_k denotes the kth Bernoulli number and $B_k(x)$ denotes the kth Bernoulli polynomial.

First,

$$\begin{aligned}\int_1^{Nt} f(x)\,dx &= N\int_{1/N}^{t} \log(1 - z^s)\,ds\\ &= N\int_0^t \log(1 - z^s)\,ds - N\int_0^{1/N} \log(1 - z^s)\,ds\\ &= \frac{N}{\log z}\int_1^{z^t} \frac{\log(1-u)}{u}\,du - N\int_0^{1/N} \log(1 - z^s)\,ds\\ &= \frac{N}{\log z}\{\mathrm{Li}_2(1) - \mathrm{Li}_2(z^t)\} - N\int_0^{1/N} \log(1 - z^s)\,ds\\ &= \frac{N}{\log z}\left\{\frac{\pi^2}{6} - \mathrm{Li}_2(z^t)\right\} - N\int_0^{1/N} \log(1 - z^s)\,ds, \end{aligned} \tag{7.14}$$

by (0.3). Now for $|s \log z| < 2\pi$,

$$\begin{aligned}\log(1 - z^s) &= \log(1 - e^{s \log z}) \\ &= -\log\left(\frac{-s \log z}{1 - e^{s \log z}}\right) + \log(-s \log z) \\ &= \sum_{k=1}^{\infty} \frac{(-s \log z)^k B_k}{k \cdot k!} + \log s + \log(-\log z), \qquad (7.15)\end{aligned}$$

by Entry 11 of Chapter 5 (Part I [2, p. 119]). Using (7.15) in (7.14), we conclude that, for $|(\log z)/N| < 2\pi$,

$$\begin{aligned}\int_1^{Nt} f(x)\,dx = \frac{N}{\log z}\left\{\frac{\pi^2}{6} - \mathrm{Li}_2(z^t)\right\} + \log N + 1 - \log(-\log z) \\ - \sum_{k=1}^{\infty} \frac{(-\log z)^k B_k}{k(k+1)!\, N^k}. \qquad (7.16)\end{aligned}$$

Second, by (7.15),

$$f(1) = \log(1 - z^{1/N}) = \sum_{k=1}^{\infty} \frac{(-\log z)^k B_k}{k \cdot k!\, N^k} - \log N + \log(-\log z). \qquad (7.17)$$

Third, by (7.15) again, for $j \geq 1$,

$$f^{(2j-1)}(x) = \frac{1}{N^{2j-1}}\left\{\frac{(2j-2)!}{(x/N)^{2j-1}} + \sum_{k=2j-1}^{\infty} \frac{(-\log z)^k x^{k-2j+1} B_k}{k(k-2j+1)!\, N^{k-2j+1}}\right\},$$

provided that $|x(\log z)/N| < 2\pi$. So, for $|(\log z)/N| < 2\pi$,

$$f^{(2j-1)}(1) = (2j-2)! + \sum_{k=2j-1}^{\infty} \frac{(-\log z)^k B_k}{k(k-2j+1)!\, N^k}.$$

Thus, since $B_1 = -\frac{1}{2}$ and $B_{2j-1} = 0$ when $j > 1$,

$$\begin{aligned}\sum_{j=1}^{m} \frac{B_{2j}}{(2j)!} f^{(2j-1)}(1) &= \sum_{j=1}^{m} \frac{B_{2j}}{(2j)(2j-1)} + \frac{\log z}{24N} \\ &\quad + \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!} \sum_{k=j}^{\infty} \frac{(-\log z)^k B_{2k}}{(2k)(2k-2j+1)!\, N^{2k}} \\ &= \sum_{j=1}^{m} \frac{B_{2j}}{(2j)(2j-1)} + \frac{\log z}{24N} \\ &\quad + \sum_{k=1}^{m} \frac{(-\log z)^{2k} B_{2k}}{(2k)N^{2k}} \sum_{j=1}^{k} \frac{B_{2j}}{(2j)!\,(2k-2j+1)!} + O\left(\frac{1}{N^{2m+2}}\right)\end{aligned}$$

$$= \sum_{j=1}^{m} \frac{B_{2j}}{(2j)(2j-1)} + \frac{\log z}{24N}$$
$$+ \sum_{k=1}^{m} \frac{(-\log z)^{2k} B_{2k}}{(2k)(2k+1)!\, N^{2k}} \sum_{j=1}^{k} \binom{2k+1}{2j} B_{2j} + O\left(\frac{1}{N^{2m+2}}\right)$$
$$= \sum_{k=1}^{m} \frac{B_{2k}}{(2k)(2k-1)} + \frac{\log z}{24N}$$
$$+ \sum_{k=1}^{m} \frac{(2k-1)(-\log z)^{2k} B_{2k}}{(4k)(2k+1)!\, N^{2k}} + O\left(\frac{1}{N^{2m+2}}\right), \tag{7.18}$$

where we have employed the equality

$$\sum_{j=1}^{m} \binom{2k+1}{2j} B_{2j} = k - \tfrac{1}{2},$$

which is easily deducible from the well-known recursion formula (Gradshteyn and Ryzhik [1, p. 1077])

$$\sum_{j=0}^{n} \binom{n}{j} B_j = B_n, \qquad n \geq 2.$$

Since $\log(1 - z^s) = \log s + h(s)$, where $h(s) = \log\{(1 - z^s)/s\}$ is analytic for all real s, it follows that

$$f^{(2m+1)}(x) = \frac{(2m)!}{x^{2m+1}} + h_m(x), \tag{7.19}$$

where $h_m(x) = h^{(2m+1)}(x/N)N^{-2m-1}$. Thus, by (7.19),

$$\frac{1}{(2m+1)!} \int_1^{Nt} B_{2m+1}(x - [x]) f^{(2m+1)}(x)\, dx$$
$$= \frac{1}{2m+1} \int_1^{Nt} \frac{B_{2m+1}(x - [x])}{x^{2m+1}}\, dx + \frac{1}{(2m+1)!} \int_1^{Nt} B_{2m+1}(x - [x]) h_m(x)\, dx. \tag{7.20}$$

Recall that (Abramowitz and Stegun [1, p. 804]) $B_n'(x) = nB_{n-1}(x)$, $n \geq 1$. Thus, by an integration by parts,

$$\int_1^{Nt} B_{2m+1}(x - [x]) h_m(x)\, dx$$
$$= \frac{1}{2m+2} B_{2m+2}(x - [x]) h_m(x) \Big|_1^{Nt} - \frac{1}{2m+2} \int_1^{Nt} B_{2m+2}(x - [x]) h_m'(x)\, dx$$
$$= O\left(\frac{1}{N^{2m+1}}\right) + O\left(\frac{1}{N^{2m+1}}\right)$$
$$= O\left(\frac{1}{N^{2m+1}}\right), \tag{7.21}$$

uniformly for $t \geq \delta > 0$. Putting (7.21) in (7.20), we deduce that

$$\frac{1}{(2m+1)!}\int_1^{Nt} B_{2m+1}(x-[x])f^{(2m+1)}(x)\,dx$$
$$= \frac{1}{2m+1}\int_1^{Nt}\frac{B_{2m+1}(x-[x])}{x^{2m+1}}\,dx + O\left(\frac{1}{N^{2m+1}}\right), \quad (7.22)$$

uniformly for $t \geq \delta$. From a standard proof of Stirling's formula for $\log n!$ (for example, see N. G. de Bruijn's book [1, pp. 47–48]), it follows that

$$\frac{1}{2m+1}\int_1^{Nt}\frac{B_{2m+1}(x-[x])}{x^{2m+1}}\,dx = \tfrac{1}{2}\log(2\pi) - 1 + \sum_{k=1}^{m}\frac{B_{2k}}{(2k)(2k-1)} + O\left(\frac{1}{N^{2m+1}}\right),$$

uniformly for $t \geq \delta$. Using this in (7.22), we have

$$\frac{1}{(2m+1)!}\int_1^{Nt} B_{2m+1}(x-[x])f^{(2m+1)}(x)\,dx$$
$$= \tfrac{1}{2}\log(2\pi) - 1 + \sum_{k=1}^{m}\frac{B_{2k}}{(2k)(2k-1)} + O\left(\frac{1}{N^{2m+1}}\right), \quad (7.23)$$

uniformly for $t \geq \delta$.

We now utilize (7.16)–(7.18) and (7.23) in (7.13) to deduce that

$$\log(z^{1/N})_{Nt} = \frac{N}{\log z}\left\{\frac{\pi^2}{6} - \mathrm{Li}_2(z^t)\right\} + \tfrac{1}{2}\log N + \tfrac{1}{2}F(t) - \tfrac{1}{2}\log(-\log z)$$
$$+ \tfrac{1}{2}\log(2\pi) - \frac{\log z}{24N} + \sum_{k=1}^{m}\frac{B_{2k}F^{(2k-1)}(t)}{(2k)!\,N^{2k-1}} + O\left(\frac{1}{N^{2m+1}}\right),$$

uniformly for $t \geq \delta$, where $F(x) = \log(1 - z^x)$.

Hence,

$$\log\left(\frac{a^{Nt}z^{bNt^2+ct}}{(z^{1/N})_{Nt}}\right) = Nt\log a + Nbt^2\log z - \frac{N}{\log z}\left\{\frac{\pi^2}{6} - \mathrm{Li}_2(z^t)\right\}$$
$$- \tfrac{1}{2}\log N + ct\log z - \tfrac{1}{2}F(t) + \tfrac{1}{2}\log(-\log z)$$
$$- \tfrac{1}{2}\log(2\pi) + \frac{(1+z^t)\log z}{24(1-z^t)N} - \sum_{k=2}^{m}\frac{B_{2k}F^{(2k-1)}(t)}{(2k)!\,N^{2k-1}}$$
$$+ O\left(\frac{1}{N^{2m+1}}\right), \quad (7.24)$$

uniformly for $t \geq \delta$. In particular, (7.24) implies that

$$\lim_{N\to\infty}\left(\frac{a^{Nt}z^{bNt^2+ct}}{(z^{1/N})_{Nt}}\right)^{1/N} = a^t z^{bt^2}\exp\left(-\frac{1}{\log z}\left\{\frac{\pi^2}{6}-\operatorname{Li}_2(z^t)\right\}\right), \tag{7.25}$$

and the convergence is uniform for $t \geq \delta$.

We now show that the convergence in (7.25) is uniform for $t \geq 0$. Let $F_N(t)$ and $F(t)$ denote the functions appearing on the left and right sides of (7.25), respectively, so that (7.25) can be written as $\lim_{N\to\infty} F_N(t) = F(t)$. Observe that on $[0, \frac{1}{2}]$, $F(t)$ and $F_N(t)$ are monotonically increasing for $N \geq N'$, say. Also, note that $F(0) = 1 = F_N(0)$, $N \geq 1$. Let $\varepsilon > 0$ be fixed. Then, since F is continuous and monotonically increasing, there exists a number δ such that $0 < \delta \leq \frac{1}{2}$ and

$$1 = F(0) \leq F(t) \leq F(\delta) \leq 1 + \varepsilon,$$

for $0 \leq t \leq \delta$. Also, since $F_N(t)$ tends to $F(t)$ uniformly for $t \geq \delta$, there exists a positive integer $N_0 \geq N'$ such that

$$|F_N(t) - F(t)| \leq \varepsilon$$

for $N \geq N_0$ and $t \geq \delta$. Thus, for $N \geq N_0$ and $0 \leq t \leq \delta$,

$$F_N(t) - F(t) \leq F_N(\delta) - F(0) \leq \varepsilon + F(\delta) - F(0) \leq 2\varepsilon$$

and

$$F(t) - F_N(t) \leq F(\delta) - F_N(0) = F(\delta) - 1 \leq \varepsilon.$$

Thus, for $N \geq N_0$ and $0 \leq t \leq \delta$, $|F(t) - F_N(t)| \leq 2\varepsilon$, and therefore $F_N(t)$ tends to $F(t)$ uniformly for $t \geq 0$.

Let

$$g(t) := a^t z^{bt^2}\exp\left(-\frac{1}{\log z}\left\{\frac{\pi^2}{6}-\operatorname{Li}_2(z^t)\right\}\right).$$

Then an elementary calculation shows that

$$\begin{aligned} g'(t) &= g(t)\left\{\log a + 2bt\log z + \sum_{n=1}^{\infty}\frac{z^{tn}}{n}\right\} \\ &= g(t)\{\log a + 2bt\log z - \log(1 - z^t)\}, \end{aligned} \tag{7.26}$$

which, by (7.12), equals 0 only when $t = 1$. Thus, the right side of (7.25) achieves its maximum at $t = 1$.

Set $g(0, N) = 1$ and, for $t \neq 0$,

$$g(t, N) := \frac{a^{Nt}z^{bNt^2+ct}}{(z^{1/N})_{Nt}}.$$

Let $\varepsilon > 0$ be given. We shall prove that

$$\sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n} = \sum_{n \in S} g\left(\frac{n}{N}, N\right)\{1 + o(e^{-\delta N})\} \tag{7.27}$$

as q tends to $1-$, where $S = \{n: |n/N - 1| \le \varepsilon\}$ and δ is a positive constant depending upon ε. Thus, as q tends to $1-$, or N tends to ∞, the primary contribution to $f(a; q)$ arises from those n which are close to N.

We now prove (7.27). As we saw in the argument leading to (7.25), $g(t, N)^{1/N}$ tends to $g(t)$, uniformly for $t \ge 0$, as N tends to ∞. First, we shall prove that there exists a number t_0 such that

$$\frac{\displaystyle\sum_{n=Nt_0}^{\infty} g\left(\frac{n}{N}, N\right)}{\displaystyle\sum_{n=0}^{\infty} g\left(\frac{n}{N}, N\right)} < e^{-\delta_1 N}, \tag{7.28}$$

where δ_1 is a positive constant depending upon t_0, and where N is sufficiently large. As we demonstrated above, $g(t)$ achieves its maximum at $t = 1$. Thus, for each N sufficiently large, there exists an n such that

$$g\left(\frac{n}{N}, N\right)^{1/N} > \tfrac{1}{2} g(1).$$

Hence,

$$\sum_{n=0}^{\infty} g\left(\frac{n}{N}, N\right) > (\tfrac{1}{2} g(1))^N. \tag{7.29}$$

Since $0 < z < 1$, we can choose t_0 large enough so that

$$\frac{az^{2bt_0}}{1 - z^{t_0}} < \tfrac{1}{2}. \tag{7.30}$$

Now,

$$\frac{g\left(\frac{n}{N}, N\right)}{g\left(\frac{n-1}{N}, N\right)} = \frac{az^{2bn/N + (c-b)/N}}{1 - z^{n/N}}.$$

Thus, from (7.30),

$$\frac{g\left(\frac{n}{N}, N\right)}{g\left(\frac{n-1}{N}, N\right)} < \tfrac{1}{2}$$

for $n > Nt_0$ and N sufficiently large. Thus,

$$\sum_{n=Nt_0}^{\infty} g\left(\frac{n}{N}, N\right) < 2g(t_0, N). \tag{7.31}$$

Choose ε_0 so that $0 < \varepsilon_0 < \frac{1}{2}g(1)$. Since $g(t, N)^{1/N}$ tends to $g(t)$ as N tends to ∞, and since $g(t)$ tends to 0 as t tends to ∞, it follows that we can choose t_0 sufficiently large such that

$$g(t_0, N)^{1/N} < \varepsilon_0$$

for all N sufficiently large. Thus, from (7.31),

$$\sum_{n=Nt_0}^{\infty} g\left(\frac{n}{N}, N\right) < 2\varepsilon_0^N, \tag{7.32}$$

and so, from (7.29), (7.32), and our choice of ε_0,

$$\frac{\displaystyle\sum_{n=Nt_0}^{\infty} g\left(\frac{n}{N}, N\right)}{\displaystyle\sum_{n=0}^{\infty} g\left(\frac{n}{N}, N\right)} < \frac{2\varepsilon_0^N}{(\frac{1}{2}g(1))^N} < e^{-\delta_1 N},$$

for some positive constant $\delta_1 > 0$ and N sufficiently large, i.e., (7.28) has been established.

Next, we shall show that there exists a positive constant $\delta_2 > 0$ such that

$$\frac{\displaystyle\sum_{\substack{n=0\\ n\notin S}}^{Nt_0} g\left(\frac{n}{N}, N\right)}{\displaystyle\sum_{n=0}^{\infty} g\left(\frac{n}{N}, N\right)} < e^{-\delta_2 N}, \tag{7.33}$$

for N sufficiently large. Let $u = \sup_{|t-1|\geq\varepsilon} g(t)$, and so $u < g(1)$. Choose v and w such that $u < v < w < g(1)$. Since $g(n/N, N)^{1/N}$ tends to $g(t)$ as N tends to ∞, uniformly for $n \geq 0$, it follows that for N sufficiently large,

$$g\left(\frac{n}{N}, N\right) < v^N$$

for all $n \notin S$. Also, for each N sufficiently large, there exists an n such that

$$g\left(\frac{n}{N}, N\right) > w^N.$$

Therefore,

$$\frac{\displaystyle\sum_{\substack{n=0\\ n\notin S}}^{Nt_0} g\left(\frac{n}{N}, N\right)}{\displaystyle\sum_{n=0}^{\infty} g\left(\frac{n}{N}, N\right)} < \frac{\displaystyle\sum_{\substack{n=0\\ n\notin S}}^{Nt_0} v^N}{w^N} \leq \frac{(Nt_0+1)v^N}{w^N} < e^{-\delta_2 N},$$

for some positive constant, δ_2, since $v < w$. Thus, (7.33) has been established. Combining (7.28) and (7.33), we complete the proof of our claim (7.27).

Returning to (7.24), put

$$\begin{aligned} H(t) &:= Nt \log a + Nbt^2 \log z - \frac{N}{\log z}\left\{\frac{\pi^2}{6} - \operatorname{Li}_2(z^t)\right\} + ct \log z \\ &\quad - \tfrac{1}{2}F(t) + \tfrac{1}{2}\log(-\log z) - \tfrac{1}{2}\log(2\pi) + \frac{(1+z^t)\log z}{24(1-z^t)N} \\ &\quad - \sum_{k=2}^{m} \frac{B_{2k}F^{(2k-1)}(t)}{(2k)!\,N^{2k-1}} \\ &:= h_0(t)N + h_1(t) + \sum_{k=1}^{m} \frac{h_{2k}(t)}{N^{2k-1}}, \end{aligned} \tag{7.34}$$

where $F(t) = \log(1 - z^t)$. Thus, by (7.24), we see that

$$\log(g(t, N)) = H(t) - \tfrac{1}{2}\log N + O\left(\frac{1}{N^{2m+1}}\right).$$

Hence, from (7.27),

$$\sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n} = \frac{1}{\sqrt{N}} \sum_{n\in S} e^{H(n/N)}\left\{1 + O\left(\frac{1}{N^{2m+1}}\right)\right\}, \tag{7.35}$$

where $S = \{n : |n/N - 1| \leq \varepsilon\}$.

In anticipation of another application of the Euler–Maclaurin summation formula, we shall examine

$$\frac{1}{\sqrt{N}} \int_{N(1-\varepsilon)}^{N(1+\varepsilon)} e^{H(s/N)}\, ds = \int_{-\varepsilon\sqrt{N}}^{\varepsilon\sqrt{N}} e^{H(1+x/\sqrt{N})}\, dx.$$

Since $H(t)$ is analytic in a neighborhood of $t = 1$, by Taylor's theorem and (7.34), for $|t - 1| < \varepsilon$,

$$H(t) = \sum_{j=0}^{\infty} \left\{h_0^{(j)}(1)N + h_1^{(j)}(1) + \sum_{k=1}^{m} \frac{h_{2k}^{(j)}(1)}{N^{2k-1}}\right\} \frac{(t-1)^j}{j!},$$

or

$$H\left(1+\frac{x}{\sqrt{N}}\right)=\sum_{j=0}^{\infty}\left\{h_0^{(j)}(1)N+h_1^{(j)}(1)+\sum_{k=1}^{m}\frac{h_{2k}^{(j)}(1)}{N^{2k-1}}\right\}\frac{x^j}{j!\,N^{j/2}}. \tag{7.36}$$

Now,

$$h_0'(1)=\frac{d}{dt}\left(t\log a+bt^2\log z-\frac{1}{\log z}\left\{\frac{\pi^2}{6}-\mathrm{Li}_2(z^t)\right\}\right)\Bigg|_{t=1}=0$$

by the same argument as used in (7.26). Hence, from (7.36),

$$H\left(1+\frac{x}{\sqrt{N}}\right)=h_0(1)N+h_1(1)+\sum_{k=1}^{m}\frac{h_{2k}(1)}{N^{2k-1}}+\frac{h_0''(1)}{2}x^2+\sum_{k=1}^{\infty}\frac{u_k(x)}{N^{k/2}},$$

where $u_k(x)$, $k\geq 1$, is a polynomial in x. The polynomial $u_k(x)$, $k\geq 1$, is an even or odd polynomial according as k is even or odd, respectively. Exponentiating, we find that

$$e^{H(1+x/\sqrt{N})}=\exp\left(h_0(1)N+h_1(1)+\sum_{k=1}^{m}\frac{h_{2k}(1)}{N^{2k-1}}\right)e^{h_0''(1)x^2/2}\left\{1+\sum_{k=1}^{\infty}\frac{p_k(x)}{N^{k/2}}\right\},$$

where $p_k(x)$, $k\geq 1$, is a polynomial in x that is even or odd, according as k is even or odd, respectively. Hence,

$$\int_{-\varepsilon\sqrt{N}}^{\varepsilon\sqrt{N}}e^{H(1+x/\sqrt{N})}\,dx$$
$$=\exp\left(h_0(1)N+h_1(1)+\sum_{k=1}^{m}\frac{h_{2k}(1)}{N^{2k-1}}\right)\int_{-\varepsilon\sqrt{N}}^{\varepsilon\sqrt{N}}e^{h_0''(1)x^2/2}\left\{1+\sum_{k=1}^{\infty}\frac{p_k(x)}{N^{k/2}}\right\}dx. \tag{7.37}$$

By symmetry, the contributions of the terms involving $p_{2k+1}(x)$, $k\geq 0$, equal 0.

Now observe that

$$h_0''(1)=\left(2b+\frac{z}{1-z}\right)\log z<0, \tag{7.38}$$

since $0<z<1$. Thus, for each nonnegative integer j,

$$\int_{-\infty}^{\infty}e^{h_0''(1)x^2/2}x^{2j}\,dx=\left(-\frac{2}{h_0''(1)}\right)^{j+1/2}\Gamma(j+\tfrac{1}{2}), \tag{7.39}$$

and it follows that

$$\int_{-\varepsilon\sqrt{N}}^{\varepsilon\sqrt{N}}e^{h_0''(1)x^2/2}\frac{p_{2j}(x)}{N^j}\,dx=O\left(\frac{1}{N^j}\right).$$

Thus, from (7.37), so far we have shown that

$$\int_{-\varepsilon\sqrt{N}}^{\varepsilon\sqrt{N}} e^{H(1+x/\sqrt{N})}\,dx$$

$$= \exp\left(h_0(1)N + h_1(1) + \sum_{k=1}^{m} \frac{h_{2k}(1)}{N^{2k-1}}\right)$$

$$\times \left\{\int_{-\varepsilon\sqrt{N}}^{\varepsilon\sqrt{N}} e^{h_0''(1)x^2/2}\left\{1 + \sum_{k=1}^{2m} \frac{p_{2k}(x)}{N^k}\right\} dx + O\left(\frac{1}{N^{2m+1}}\right)\right\}$$

$$= \exp\left(h_0(1)N + h_1(1) + \sum_{k=1}^{m} \frac{h_{2k}(1)}{N^{2k-1}}\right)$$

$$\times \left\{\int_{-\infty}^{\infty} e^{h_0''(1)x^2/2}\left\{1 + \sum_{k=1}^{2m} \frac{p_{2k}(x)}{N^k}\right\} dx + O\left(\frac{1}{N^{2m+1}}\right)\right\}, \tag{7.40}$$

since clearly the integrals over $(-\infty, -\varepsilon\sqrt{N})$ and $(\varepsilon\sqrt{N}, \infty)$ are $O(e^{-\delta N})$, for some positive $\delta > 0$, as N tends to ∞. By using (7.39) in (7.40), we can develop an asymptotic series for the integral on the left side of (7.40).

We now apply the Euler–Maclaurin summation formula to the sum on the right side of (7.35). For brevity, set $G(t) = \exp(H(t))$. Then for every nonnegative integer m,

$$\sum_{n\in S} e^{H(n/H)} = \int_{N(1-\varepsilon)}^{N(1+\varepsilon)} e^{H(x/N)}\,dx + \frac{G(1-\varepsilon)}{2} + \frac{G(1+\varepsilon)}{2}$$

$$+ \sum_{k=1}^{m} \frac{B_{2k}\{G^{(2k-1)}(1+\varepsilon) - G^{(2k-1)}(1-\varepsilon)\}}{(2k)!\,N^{2k-1}}$$

$$+ \frac{1}{(2m+1)!}\int_{N(1-\varepsilon)}^{N(1+\varepsilon)} B_{2m+1}(x-[x])\frac{G^{(2m+1)}(x/N)}{N^{2m+1}}\,dx, \tag{7.41}$$

where, for simplicity, we have made the assumption that $N\varepsilon$ is a positive integer. Integrating by parts and using the same type of argument that we utilized in (7.21), we readily find that

$$\int_{N(1-\varepsilon)}^{N(1+\varepsilon)} B_{2m+1}(x-[x])G^{(2m+1)}(x/N)\,dx = O(1),$$

as N tends to ∞. From the definition of $H(t)$ given in (7.34), we observe that

$$H(t) = h_0(t)N + O(1),$$

as N tends to ∞, or

$$G(t) = e^{H(t)} = O(e^{h_0(t)N}),$$

where $h_0(t) = \log g(t)$. As demonstrated in (7.26), $g(t)$ attains its maximum at

$t = 1$. Thus, the terms in (7.41) involving $G(1 \pm \varepsilon)$ and $G^{(2k-1)}(1 \pm \varepsilon)$, $1 \le k \le m$, are exponentially small in comparison with the leading term in (7.40). Therefore, from (7.41),

$$\sum_{n \in S} e^{H(t/n)} = \int_{N(1-\varepsilon)}^{N(1+\varepsilon)} e^{H(x/N)}\, dx\{1 + o(e^{-\delta N})\}, \tag{7.42}$$

for some positive number δ, as N tends to ∞.

Hence, using (7.40) and (7.42) in (7.35), we complete our proof of the asymptotic formula

$$\sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n} = \exp\left(h_0(1)N + h_1(1) + \sum_{k=1}^{m} \frac{h_{2k}(1)}{N^{2k-1}}\right) \int_{-\infty}^{\infty} e^{h_0''(1)x^2/2} \times \left\{1 + \sum_{k=1}^{2m} \frac{p_{2k}(x)}{N^k}\right\} dx \left\{1 + O\left(\frac{1}{N^{2m+1}}\right)\right\}. \tag{7.43}$$

We now calculate the first few terms of this expansion. From the definition of $h_0(t)$ in (7.34), from (7.12), and from the functional equation

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \log x \log(1-x),$$

found in Entry 6(iii) in Chapter 9 (Part I [2, p. 247]),

$$\begin{aligned} h_0(1) &= \log a + b \log z - \frac{1}{\log z}\left\{\frac{\pi^2}{6} - \mathrm{Li}_2(z)\right\} \\ &= \log\left(\frac{az^b}{1-z}\right) - \frac{\mathrm{Li}_2(1-z)}{\log z} \\ &= -\frac{1}{\log z}(b \log^2 z + \mathrm{Li}_2(az^{2b})). \end{aligned}$$

Hence, from (7.43), since $q^N = z$, the first term in the asymptotic expansion of $\log(f(a; q))$, as q tends to $1-$, equals

$$-\frac{1}{\log q}(\mathrm{Li}_2(az^{2b}) + b \log^2 z), \tag{7.44}$$

which is in agreement with (7.1).

From (7.43), (7.34), (7.39), and (7.38), the second term in the asymptotic expansion of $\log(f(a; q))$ equals

$$\begin{aligned} h_1(1) &+ \log\left(\int_{-\infty}^{\infty} e^{h_0''(1)x^2/2}\, dx\right) \\ &= c \log z - \tfrac{1}{2}\log(1-z) + \tfrac{1}{2}\log(-\log z) - \tfrac{1}{2}\log(2\pi) \\ &\quad + \log\left\{\frac{-2(1-z)\pi}{(2b(1-z)+z)\log z}\right\}^{1/2} \\ &= c \log z - \tfrac{1}{2}\log\{z + 2b(1-z)\}. \end{aligned} \tag{7.45}$$

Adding (7.44) and (7.45), we deduce the asymptotic expansion (7.1). This finally completes the second proof of Entry 7.

A complete asymptotic expansion for $\log(f(a; q))$ (generally) involves an infinite series of positive powers of $\log q$ in addition to the terms displayed in (7.1). R. McIntosh [1] used the computer algebra system MAPLE to calculate further terms of this asymptotic expansion. For example, for $b \neq \frac{1}{2}$, the coefficient of $\log q$ is

$$\frac{(y-1)\{10b^2y^2 + (12c - 4b - 7)\,by + 6c^2 - 6c + b + 1\}}{12(2b-1)},$$

where $y = \{z + 2b(1-z)\}^{-1}$. McIntosh's calculations suggest conjectures about the general coefficient of $(\log q)^n$, $n \geq 1$.

McIntosh's techniques are applicable to a large class of q-series. In particular, he has derived asymptotic expansions for many mock theta-functions as q approaches $1-$.

Entry 8 (p. 365). *Let* Li_2 *denote the dilogarithm defined by* (0.3). *Suppose that* $a > 0$ *and* $|q| < 1$. *Let* x *denote the positive root of* $x^2 + x = a$. *Then as* q *tends to* $1-$,

$$\log\left(\sqrt{\frac{1+2x}{1+x}}\sum_{n=0}^{\infty}\frac{a^n q^{n^2}}{(q)_n}\right) \sim -\frac{1}{\log q}\{\tfrac{1}{2}\log^2(1+x) - \mathrm{Li}_2(-x)\}.$$

Proof. This result is a special case of Entry 7. Let $c = 0$ and $b = 1$ there. If $x = az$, the condition $az^2 + z = 1$ translates into the equality $x^2 + x = a$. Now,

$$z + 2b(1-z) = 2 - z = 2 - \frac{x}{a} = 2 - \frac{x}{x^2+x} = \frac{1+2x}{1+x}.$$

Also $\log^2 z = \log^2(1+x)$. Lastly, by Entry 6(i) of Chapter 9 (Part I [2, p. 247]),

$$\mathrm{Li}_2(az^2) = \mathrm{Li}_2(1-z) = \mathrm{Li}_2\left(\frac{x}{x+1}\right) = -\mathrm{Li}_2(-x) - \tfrac{1}{2}\log^2(x+1).$$

Using these calculations in Entry 7, we complete the proof of Entry 8.

In a fragment published on page 358 of the "lost notebook" [24], Ramanujan offers several facts pointing to the truth of the Rogers–Ramanujan identity (0.1), which, at the time of writing, Ramanujan had not yet proved. In particular, as q tends to 1, the logarithms of the two sides of (0.1) are asymptotically equal. More precisely,

$$\log\left(\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n}\right) \sim \frac{\pi^2}{15(1-q)}, \tag{8.1}$$

as q tends to $1-$. We shall now demonstrate that (8.1) follows from Entry 8.

In Entry 8, let $a = 1$. Then $x = \frac{1}{2}(\sqrt{5} - 1)$. From Lewin's book [1, p. 7], or from Part I [2, p. 247, Entry 6(iv); p. 248, Examples (ii), (iii)],

$$\operatorname{Li}_2(-x) = \operatorname{Li}_2\left(\frac{1-\sqrt{5}}{2}\right) = -\frac{\pi^2}{15} + \frac{1}{2}\log^2\left(\frac{\sqrt{5}-1}{2}\right).$$

Thus,

$$\tfrac{1}{2}\log^2(1+x) - \operatorname{Li}_2(-x) = \frac{\pi^2}{15}. \tag{8.2}$$

Hence, from Entry 8 and (8.2), as q tends to $1-$,

$$\log\left(\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}\right) \sim -\frac{1}{\log q}\frac{\pi^2}{15} - \tfrac{1}{2}\log\left(\frac{5-\sqrt{5}}{2}\right), \tag{8.3}$$

from which (8.1) immediately follows.

McIntosh [1] has improved (8.3) by proving that, as q tends to $1-$,

$$\log\left(\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n}\right) = -\frac{1}{\log q}\frac{\pi^2}{15} - \tfrac{1}{2}\log\left(\frac{5-\sqrt{5}}{2}\right) + \frac{\log q}{60} + E_1(q),$$

where $E_1(q)$ is exponentially small. The small error term arises from (0.1) and the transformation formula for the infinite product.

Recall the second Rogers–Ramanujan identity

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

By setting $b = c = 1$ in Entry 7 and using McIntosh's further work, we find that, as q tends to $1-$,

$$\log\left(\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n}\right) = -\frac{1}{\log q}\frac{\pi^2}{15} - \tfrac{1}{2}\log\left(\frac{5+\sqrt{5}}{2}\right) - \frac{11\log q}{60} + E_2(q),$$

where $E_2(q)$ is exponentially small.

We close this chapter with an application of Entry 6. R. Garwin [1] has devised a probabilistic model for predicting nuclear war that is briefly discussed by McGeorge Bundy in his book [1, pp. 615–616]. The analysis has been further developed by R. Avenhaus, S. J. Brams, J. Fichtner, and D. M. Kilgour [1] and by W. Beyer and B. K. Swartz [1]. Let r denote the risk (probability) of nuclear war in a fixed time interval, and let n be the fraction of reduction of the risk in each successive time interval. Then the probability of nuclear war sometime, now or in the future, is, according to this probabilistic model,

$$p := 1 - \prod_{k=0}^{\infty} (1 - n^k r). \tag{6.8}$$

If we set $q = n$ and $aq = -r$, then by Euler's identity, given at the beginning of the proof of Entry 6, and (6.8), we see that Entry 6 provides the asymptotic estimate of p,

$$p \sim 1 - \frac{\exp\left(\dfrac{\operatorname{Li}_2(r/n)}{n-1}\right)}{\sqrt{1 - \dfrac{r}{n}}},$$

as n tends to $1-$. In particular, as n tends to 1, the probability of nuclear war tends to 1.

For another asymptotic expansion as q tends to 1, that also involves the dilogarithm, of a q-series arising in Ramanujan's work, see Zagier's beautiful paper [2].

We are very grateful to R. J. Evans for a helpful suggestion in the proof of Entry 6 and to G. E. Andrews for many informative remarks.

CHAPTER 28

Integrals

Integrals are scattered throughout Ramanujan's notebooks. In particular, an abundance of integrals appears in Chapter 13 (Berndt [4]), which is highlighted by several elegant asymptotic expansions of integrals. In this chapter, we examine the nearly 50 results on integrals appearing in the 100 pages at the end of the second notebook, and in the 33-page third notebook.

A reader of this chapter who has Ramanujan's notebooks close at hand will find other integrals in the aforementioned 133 pages. Some of these integrals arise from summation formulas, for example, variants of the Abel-Plana summation formula due to Ramanujan. We have placed such entries in Chapter 35, which is on infinite series (Part V [8]). Other integrals arise from special functions and so are found in Chapter 29, which is devoted to this topic. Further integrals appear in Chapter 31 on miscellaneous analysis. Still other integrals are associated with Ramanujan's theory of prime numbers and so are found in Chapter 24.

Thus, the classification of material related to integrals is difficult and at times somewhat arbitrary. The dilogarithm, in particular, presents difficulties. On the one hand, the dilogarithm is frequently defined by an integral; some of Ramanujan's results on integrals are associated with the dilogarithm. On the other hand, it could be argued that results related to the dilogarithm properly belong to Chapter 29 on special functions. At this time of writing, however, the dilogarithm has not been universally elevated to the class of special functions, although in the past decade or two, its occurrences in mathematics, as well as physics, have been increasingly frequent. Thus, we have included in this chapter not only integral formulas related to the dilogarithm, but some related results involving the dilogarithm as well. But not all results involving the dilogarithm are found here. The dilogarithm appears in Chapter 32 (Part V [8]) on continued fractions as well as in certain asymptotic expansions of q-series found in Chapter 27.

A few of the many interesting integral formulas found amongst the unorganized notebook pages will now be briefly described. One of the most interesting results in this chapter is a beautiful integral functional equation related to the dilogarithm. This functional equation was originally proposed as a problem by Ramanujan in the *Journal of the Indian Mathematical Society* [20], [23, p. 334]. The result was greatly generalized by the author and R. J. Evans [1], and we present this generalization here.

Entry 4 below presents a beautiful reciprocity theorem for a certain integral. Ramanujan actually proved this formula in two papers [13], [15]. Entry 5 gives a marvelous approximation for this integral and makes manifest Ramanujan's extraordinary ability to discover simplicity and elegance in mathematical relations. Entries 4 and 6 provide asymptotic expansions of integrals with the remarkable property that the expansions as a parameter tends to zero have the same forms as those when the parameter tends to infinity.

Analytic number theorists should take a look at Entry 21 to see if they recognize a well-known theorem in disguise. This entry demonstrates Ramanujan's uncanny expertise in discovering unusual formulations.

One of the last pages of Ramanujan's third notebook ([22, p. 391]) is unusual for at least two reasons. First, it is upside down. Second (and more importantly), some of the integrals are from complex analysis, a subject which Ramanujan evidently never learned. At the top (bottom?) of the page appear the words "contour integration" in juxtaposition to a beautiful integral formula, which, indeed, can be established by contour integration. Questions immediately arise. Did someone inform Ramanujan that this formula could be established by contour integration? Or, contrary to commonly held belief, did Ramanujan actually learn about functions of a complex variable, in particular, about the residue theorem, and prove this formula himself?

Page 391, however, is devoted primarily to integral transforms, some of which, e.g., Mellin's transformation formulas, involve integrals over vertical lines in the complex plane. We conjecture that this page arises from Ramanujan's conversations with Hardy in England. We relate in the sequel further evidence that Ramanujan learned a small amount of complex function theory.

Entry 1 (p. 265). *If* $n > 0$, *then*

$$\int_0^\infty e^{-nx} \sin(nx) \cot x \, dx = 2n^2 \sum_{k=1}^\infty \frac{k}{4k^4 + n^4}.$$

Proof. If N is a positive integer, upon using the Riemann–Lebesgue lemma, we find that

$$\begin{aligned}
&\int_0^\infty e^{-nx}\sin(nx)\cot x\,dx\\
&\quad=\int_0^\infty e^{-nx}\frac{\sin(nx)}{\sin x}\cos x\,dx\\
&\quad=\lim_{N\to\infty}\int_0^\infty e^{-nx}\frac{\sin(nx)}{\sin x}\{\cos x-\cos((2N+1)x)\}\,dx\\
&\quad=2\lim_{N\to\infty}\int_0^\infty e^{-nx}\frac{\sin(nx)}{\sin x}\sin((N+1)x)\sin(Nx)\,dx\\
&\quad=2\lim_{N\to\infty}\int_0^\infty e^{-nx}\sin(nx)\sum_{k=1}^{N}\sin(2kx)\,dx\\
&\quad=2\lim_{N\to\infty}\sum_{k=1}^{N}\int_0^\infty e^{-nx}\sin(nx)\sin(2kx)\,dx\\
&\quad=\lim_{N\to\infty}\sum_{k=1}^{N}\int_0^\infty e^{-nx}\{\cos((n-2k)x)-\cos((n+2k)x)\}\,dx\\
&\quad=\sum_{k=1}^{\infty}\left(\frac{n}{n^2+(n-2k)^2}-\frac{n}{n^2+(n+2k)^2}\right)\\
&\quad=2n^2\sum_{k=1}^{\infty}\frac{k}{4k^4+n^4}.
\end{aligned}$$

This completes the proof.

Entry 1 is closely related to Entry 27 below.

Entry 1 might be compared with the corollary of Entry 20(i) in Chapter 14 (Part II [4, p. 274]).

Entry 2 (p. 265). *For $a > 0$,*

$$\int_0^\infty e^{-ax}\sin(ax)\coth x\,dx=\frac{1}{2a}+2a\sum_{k=1}^{\infty}\frac{1}{a^2+(a+2k)^2}.$$

PROOF. We have

$$\begin{aligned}
\int_0^\infty e^{-ax}\sin(ax)\coth x\,dx&=\int_0^\infty e^{-ax}\sin(ax)\left(1+\frac{2}{e^{2x}-1}\right)dx\\
&=\frac{1}{2a}+2\int_0^\infty\frac{e^{-(a+2)x}\sin(ax)}{1-e^{-2x}}\,dx\\
&=\frac{1}{2a}+2\sum_{k=0}^{\infty}\int_0^\infty e^{-(a+2+2k)x}\sin(ax)\,dx\\
&=\frac{1}{2a}+2\sum_{k=1}^{\infty}\frac{a}{a^2+(a+2k)^2},
\end{aligned}$$

where the inversion in order of summation and integration is justified by absolute convergence.

Entry 3 (p. 265). *If* $a > 0$, *then*

$$\int_0^\infty e^{-ax} \sin(ax)\{\cot x + \coth x\}\, dx = \frac{\pi}{2} \frac{\sinh(\pi a)}{\cosh(\pi a) - \cos(\pi a)}.$$

Proof. From Entries 1 and 2,

$$\int_0^\infty e^{-ax} \sin(ax)\{\cot x + \coth x\}\, dx$$

$$= 2a^2 \sum_{k=1}^\infty \frac{k}{4k^4 + a^4} + \frac{1}{2a} + 2a \sum_{k=1}^\infty \frac{1}{a^2 + (a + 2k)^2}. \tag{3.1}$$

On the other hand, from Entry 24 of Chapter 14 (Part II [4, p. 291]), for any complex number a,

$$\frac{\pi e^{-\pi a}}{a\{\cosh(\pi a) - \cos(\pi a)\}}$$

$$= \frac{1}{\pi a^3} - \frac{1}{a^2} + \frac{\pi}{2a}$$

$$-4 \sum_{k=1}^\infty \frac{1}{a^2 + (a + 2k)^2} + 8a \sum_{k=1}^\infty \frac{k}{(e^{2\pi k} - 1)(a^4 + 4k^4)}$$

and

$$\frac{-\pi a^{\pi a}}{a\{\cosh(\pi a) - \cos(\pi a)\}}$$

$$= -\frac{1}{\pi a^3} - \frac{1}{a^2} - \frac{\pi}{2a}$$

$$-4 \sum_{k=1}^\infty \frac{1}{a^2 + (-a + 2k)^2} - 8a \sum_{k=1}^\infty \frac{k}{(e^{2\pi k} - 1)(a^4 + 4k^4)}.$$

Adding the last two equalities, we find that

$$\frac{\pi}{2} \frac{\sinh(\pi a)}{\{\cosh(\pi a) - \cos(\pi a)\}}$$

$$= \frac{1}{2a} + a \sum_{k=1}^\infty \frac{1}{a^2 + (a + 2k)^2} + a \sum_{k=1}^\infty \frac{1}{a^2 + (-a + 2k)^2}$$

$$= \frac{1}{2a} + 2a \sum_{k=1}^\infty \frac{1}{a^2 + (a + 2k)^2} + a \sum_{k=1}^\infty \left(\frac{1}{a^2 + (a - 2k)^2} - \frac{1}{a^2 + (a + 2k)^2} \right)$$

$$= \frac{1}{2a} + 2a \sum_{k=1}^\infty \frac{1}{a^2 + (a + 2k)^2} + 2a^2 \sum_{k=1}^\infty \frac{k}{4k^4 + a^4}. \tag{3.2}$$

Upon comparing (3.1) and (3.2), we see that we have completed the proof.

For Entries 4 and 5, define, for $\alpha > 0$,

$$I(\alpha) = \alpha^{-1/4}\left(1 + 4\alpha \int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} dx\right). \tag{4.1}$$

Entry 4 (Formula (1), p. 268). *If α, $\beta > 0$ and $\alpha\beta = \pi^2$, then*

$$I(\alpha) = I(\beta).$$

Entry 4 appears in Ramanujan's [23, p. xxvi] first letter to Hardy and, in fact, was proved by Ramanujan in two papers [13], [23, pp. 53–58] and [15], [23, pp. 72–77]. This result was also established by Preece [1]. Entry 5 below was proved in a paper by Berndt and Evans [2].

Entry 5 (Formula (1), p. 268). *If $I(\alpha)$ is defined by* (4.1) *and β is as in Entry* 4, *then*

$$I(\alpha) = \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{2}{3}\right)^{1/4} \tag{5.1}$$

"*nearly.*"

We have partially quoted Ramanujan in Entry 5. More precisely, we shall show that as β tends to ∞ (or α tends to 0),

$$I(\alpha) \sim \frac{1}{\alpha^{1/4}} + \frac{\alpha^{3/4}}{6} - \frac{\alpha^{7/4}}{60} + \cdots. \tag{5.2}$$

Furthermore, we shall show that, for α sufficiently small and positive (or β sufficiently large),

$$\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{2}{3}\right)^{1/4} = \frac{1}{\alpha^{1/4}} + \frac{\alpha^{3/4}}{6} - \left(\frac{1}{24} - \frac{1}{4\pi^2}\right)\alpha^{7/4} + \cdots. \tag{5.3}$$

Now,

$$\frac{1}{60} = 0.01666\ldots$$

and

$$\frac{1}{24} - \frac{1}{4\pi^2} = 0.01633\ldots.$$

Thus, the agreement between (5.2) and (5.3) is truly remarkable! How did Ramanujan ever discern such a simple, elegant approximation to $I(\alpha)$?

PROOF. Employing the standard generating function for the Bernoulli numbers B_n, $0 \le n < \infty$ (Abramowitz and Stegun [1, p. 804], and Watson's lemma (Copson [2, p. 49]), we find that, as β tends to ∞,

$$\begin{aligned} I(\beta) &\sim \beta^{-1/4} + 4\beta^{3/4} \sum_{n=0}^{\infty} \frac{B_n(2\pi)^{n-1}}{n!} \int_0^\infty e^{-\beta x^2} x^n \, dx \\ &= \beta^{-1/4} + \frac{2\beta^{3/4}}{\pi} \int_0^\infty e^{-\beta x^2} \, dx - 2\beta^{3/4} \int_0^\infty e^{-\beta x^2} x \, dx \\ &\quad + \frac{2\pi\beta^{3/4}}{3} \int_0^\infty e^{-\beta x^2} x^2 \, dx - \frac{2\pi^3\beta^{3/4}}{45} \int_0^\infty e^{-\beta x^2} x^4 \, dx + \cdots \\ &= \beta^{-1/4} + I_1 + I_2 + I_3 + I_4 + \cdots, \end{aligned} \tag{5.4}$$

say. The integral terms I_k, $1 \le k \le 4$, are easily calculated, and we find that

$$I_1 = \alpha^{-1/4}, \qquad I_2 = -\beta^{-1/4}, \qquad I_3 = \tfrac{1}{6}\alpha^{3/4}, \quad \text{and} \quad I_4 = -\tfrac{1}{60}\alpha^{7/4}.$$

Using these values in (5.4), we deduce the expansion (5.2).

On the other hand, for α sufficiently small,

$$\begin{aligned} \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{2}{3}\right)^{1/4} &= \alpha^{-1/4}\left(1 + \frac{2\alpha}{3} + \frac{\alpha^2}{\pi^2}\right)^{1/4} \\ &= \alpha^{-1/4}\left(1 + \frac{1}{4}\left(\frac{2\alpha}{3} + \frac{\alpha^2}{\pi^2}\right) - \frac{3}{32}\left\{\frac{2\alpha}{3} + \frac{\alpha^2}{\pi^2}\right\}^2 + \cdots\right) \\ &= \alpha^{-1/4}\left(1 + \frac{\alpha}{6} + \left\{\frac{1}{4\pi^2} - \frac{1}{24}\right\}\alpha^2 + \cdots\right). \end{aligned}$$

Hence, (5.3) is established, and the proof is finished.

Entry 6 (Formula (3), p. 268). *Let* $\alpha, \beta > 0$, $\alpha\beta = 4\pi^2$, *and* $m > 1$. *Put*

$$I(\alpha) = \alpha^{(m+1)/2} \int_0^\infty \frac{x^m}{e^{2\pi x} - 1} \frac{dx}{e^{\alpha x} - 1}.$$

Then as α *tends to* 0,

$$I(\alpha) \sim \alpha^{(m-1)/2} \sum_{n=0}^{\infty} \frac{B_n \Gamma(m+n)\zeta(m+n)\alpha^n}{(2\pi)^{m+n} n!}, \tag{6.1}$$

and as β *tends to* 0,

$$I(\alpha) \sim \beta^{(m-1)/2} \sum_{n=0}^{\infty} \frac{B_n \Gamma(m+n)\zeta(m+n)\beta^n}{(2\pi)^{m+n} n!}, \tag{6.2}$$

where B_n, $0 \le n < \infty$, *denotes the* nth *Bernoulli number, and* ζ *denotes the Riemann zeta-function.*

Ramanujan did not express his expansions in terms of ζ but alternatively employed his generalized Bernoulli numbers, obtained by "interpolating" Euler's formula for $\zeta(2n)$, $1 \leq n < \infty$. (See Chapter 5 of Part I [2, pp. 125–127] for a discussion of Ramanujan's generalized Bernoulli numbers.)

It is fascinating that the asymptotic expansions (6.1) and (6.2) have identical forms. Once again, we are witness to Ramanujan's incredibly uncanny ability in discovering elegant symmetry and beauty in mathematics.

PROOF. As in the proof of Entry 5, we shall employ the Maclaurin series generating function for Bernoulli numbers, but now we need a variant of Watson's lemma, instead of Watson's lemma itself. From its definition, we see that $I(\alpha)$ does not have the form required by Watson's lemma, for $e^{-\alpha x}$ has been replaced by $1/(e^{\alpha x} - 1)$. However, by slightly modifying the usual proof of Watson's lemma, e.g., in Copson [2, pp. 49–50] or Olver's [1, pp. 71–72] books, we arrive at the same conclusion with Γ replaced by $\Gamma\zeta$ at suitable arguments.

Hence, by the aforementioned easily proved variant of Watson's lemma, as α tends to ∞,

$$\begin{aligned} I(\alpha) &\sim \alpha^{(m+1)/2} \sum_{n=0}^{\infty} \frac{B_n(2\pi)^{n-1}}{n!} \int_0^{\infty} \frac{x^{m+n-1}}{e^{\alpha x} - 1}\, dx \\ &= \alpha^{(m+1)/2} \sum_{n=0}^{\infty} \frac{B_n(2\pi)^{n-1}}{n!} \frac{\Gamma(m+n)\zeta(m+n)}{\alpha^{m+n}} \\ &= \beta^{(m-1)/2} \sum_{n=0}^{\infty} \frac{B_n \Gamma(m+n)\zeta(m+n)\beta^n}{(2\pi)^{m+n} n!}, \end{aligned}$$

where in the penultimate line we used a well-known integral evaluation (Gradshteyn and Ryzhik [1, p. 325, formula 1])

$$\int_0^{\infty} \frac{x^{k-1}}{e^{\alpha x} - 1}\, dx = \frac{\Gamma(k)\zeta(k)}{\alpha^k}, \qquad \operatorname{Re}\alpha > 0, \quad \operatorname{Re} k > 1, \tag{6.3}$$

$$= (-1)^{n-1}\left(\frac{2\pi}{\alpha}\right)^{2n} \frac{B_{2n}}{4n}, \qquad \operatorname{Re}\alpha > 0, \quad k = 2n, \quad n = 1, 2, \ldots, \tag{6.4}$$

and in the last line employed the fact $\alpha\beta = 4\pi^2$. Thus, (6.2) has been established.

Next, let $u = 2\pi x/\beta$. Then

$$\begin{aligned} I(\alpha) &= \alpha^{(m+1)/2}\left(\frac{\beta}{2\pi}\right)^{m+1} \int_0^{\infty} \frac{u^m}{e^{\beta u} - 1} \frac{du}{e^{2\pi u} - 1} \\ &= \beta^{(m+1)/2} \int_0^{\infty} \frac{u^m}{e^{2\pi u} - 1} \frac{du}{e^{\beta u} - 1} \\ &= I(\beta). \end{aligned}$$

Hence, by symmetry and (6.2), we conclude that $I(\alpha)$ has the asymptotic expansion (6.1).

Entry 7 (Formula (3), p. 269). *If $n > 0$, then*

$$\int_0^\infty \frac{x\sin(2nx)}{e^{x^2}-1}\,dx = \frac{\sqrt{\pi}n}{2}\sum_{k=1}^\infty \frac{e^{-n^2/k}}{k^{3/2}} \tag{7.1}$$

$$= \pi \sum_{k=0}^{\infty}{}' e^{-2n\sqrt{k\pi}}\cos(2n\sqrt{k\pi}), \tag{7.2}$$

where the prime (′) on the summation sign in (7.2) *indicates that the term with $k = 0$ is multiplied by $\frac{1}{2}$.*

Proof. Expanding $1/(1 - e^{-x^2})$ in a geometric series and inverting the order of summation and integration by absolute convergence, we find that

$$\int_0^\infty \frac{x\sin(2nx)}{e^{x^2}-1}\,dx = \sum_{k=1}^\infty \int_0^\infty xe^{-kx^2}\sin(2nx)\,dx$$
$$= \frac{\sqrt{\pi}n}{2}\sum_{k=1}^\infty \frac{e^{-n^2/k}}{k^{3/2}},$$

where we have used a classical integral evaluation (Gradshteyn and Ryzhik [1, p. 495, formula 1]).

Having proved (7.1), we now show that the right sides of (7.1) and (7.2) are equal by an application of the Poisson summation formula

$$\sum_{a\le k\le b}{}' f(k) = \int_a^b f(x)\,dx + 2\sum_{k=1}^\infty \int_a^b f(x)\cos(2\pi kx)\,dx, \tag{7.3}$$

where f is a continuous function of bounded variation on $[a, b]$, and where the prime on the summation sign on the left side indicates that if a or b is an integer, then only $\frac{1}{2}f(a)$ or $\frac{1}{2}f(b)$, respectively, is counted.

In (7.3), let $f(x) = \exp(-2n\sqrt{x\pi})\cos(2n\sqrt{x\pi})$ and $a = 0$, and let b tend to ∞, which is easily justified. Thus,

$$\sum_{k=0}^{\infty}{}' e^{-2n\sqrt{k\pi}}\cos(2n\sqrt{k\pi}) = \int_0^\infty e^{-2n\sqrt{x\pi}}\cos(2n\sqrt{x\pi})\,dx$$
$$+ 2\sum_{k=1}^\infty \int_0^\infty e^{-2n\sqrt{x\pi}}\cos(2n\sqrt{x\pi})\cos(2\pi kx)\,dx. \tag{7.4}$$

Letting $u = \sqrt{x}$, we find that

$$\int_0^\infty e^{-2n\sqrt{x\pi}}\cos(2n\sqrt{x\pi})\,dx = 2\int_0^\infty ue^{-2nu\sqrt{\pi}}\cos(2nu\sqrt{\pi})\,du = 0, \tag{7.5}$$

where we have employed a formula from the tables of Gradshteyn and Ryzhik [1, p. 490, formula 6]. By the same change of variable,

$$\int_0^\infty e^{-2n\sqrt{x\pi}}\cos(2n\sqrt{x\pi})\cos(2\pi kx)\,dx$$
$$= 2\int_0^\infty ue^{-2nu\sqrt{\pi}}\cos(2nu\sqrt{\pi})\cos(2\pi ku^2)\,du$$
$$= \frac{ne^{-n^2/k}}{4\sqrt{\pi}k^{3/2}}, \tag{7.6}$$

where we have utilized another formula from Gradshteyn and Ryzhik's tables [1, p. 499, formula 3.965, no. 2]. Putting (7.5) and (7.6) in (7.4), we complete the proof.

Entry 8 (Formula (4), p. 269). *Let χ denote the primitive character of modulus 4. If $n > 0$, then*

$$\int_0^\infty \frac{x\sin(2nx)}{e^{x^2}+e^{-x^2}}\,dx = \frac{\sqrt{\pi}n}{2}\sum_{k=1}^\infty \frac{\chi(k)}{k^{3/2}}e^{-n^2/k} \tag{8.1}$$
$$= \frac{\pi}{2}\sum_{k=1}^\infty \chi(k)e^{-n\sqrt{k\pi}}\sin(n\sqrt{k\pi}). \tag{8.2}$$

Proof. The proof is similar to that of Entry 7. Expanding $1/(1+e^{-2x^2})$ in a geometric series and inverting the order of summation and integration by absolute convergence, we find that

$$\int_0^\infty \frac{x\sin(2nx)}{e^{x^2}+e^{-x^2}}\,dx = \sum_{k=0}^\infty (-1)^k\int_0^\infty x\sin(2nx)e^{-(2k+1)x^2}\,dx$$
$$= \frac{\sqrt{\pi}n}{2}\sum_{k=1}^\infty \frac{\chi(k)}{k^{3/2}}e^{-n^2/k},$$

where we have used the same integral formula that we employed in proving (7.1).

To prove that the right sides of (8.1) and (8.2) are equal, we utilize the Poisson summation formula for Fourier sine transforms (Titchmarsh [1, p. 66])

$$\sum_{a\le k\le b}{}' \chi(k)f(k) = \sum_{k=1}^\infty \chi(k)\int_a^b f(x)\sin(\pi kx/2)\,dx, \tag{8.3}$$

where f is a continuous function of bounded variation on $[a, b]$, and where the prime on the summation sign on the left side has the same meaning as in (7.3).

Let $f(x) = \exp(-n\sqrt{x\pi})\sin(n\sqrt{x\pi})$ and $a = 0$, and let b tend to ∞ in (8.3). Then

$$\sum_{k=1}^{\infty} \chi(k)e^{-n\sqrt{k\pi}}\sin(n\sqrt{k\pi}) = \sum_{k=1}^{\infty} \chi(k)\int_0^{\infty} e^{-n\sqrt{x\pi}}\sin(n\sqrt{x\pi})\sin(\pi kx/2)\,dx$$
$$= 2\sum_{k=1}^{\infty} \chi(k)\int_0^{\infty} ue^{-nu\sqrt{\pi}}\sin(nu\sqrt{\pi})\sin(\pi ku^2/2)\,du.$$

Upon evaluating the last integrals by using a formula from Gradshteyn and Ryzhik's tables [1, p. 499, formula 3.965, no. 1], we complete the proof.

The next result is recorded in the form given by Ramanujan.

Entry 9 (Formula (1), p. 274). *Let* B_n, $0 \le n < \infty$, *denote the* nth *Bernoulli number. Then*

$$\sum_{k=0}^{\infty} \frac{(-1)^k B_{4k+2}\theta^{2k+1}}{(2k+1)!} = \sqrt{\frac{\theta}{2\pi}}\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{4k} B_{4k}}{(2k)!\,\theta^{2k}}$$
$$- \sqrt{\frac{\theta}{2\pi}}\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{4k+2} B_{4k+2}}{(2k+1)!\,\theta^{2k+1}}. \tag{9.1}$$

Since $|B_{2n}| \sim 2(2\pi)^{-2n}(2n)!$ as n tends to ∞, we note that each series above diverges for every finite, nonzero value of θ. Now, in fact, (9.1) arises from the equality (9.5) below between integrals. The series on the left side of (9.1) is an asymptotic series as θ tends to 0 for the integral on the left side of (9.5), while the two series on the right side of (9.1) are asymptotic series as θ tends to ∞ for the two expressions, respectively, on the right side of (9.5). Thus, our proof below will verify this last interpretation.

Proof. Following Ramanujan [14], [23, p. 60], we define, for each nonzero real number n,

$$\varphi^*(n) = \int_0^{\infty} \frac{\cos(\pi nx)}{e^{2\pi\sqrt{x}} - 1}\,dx \tag{9.2}$$

and

$$\psi^*(n) = \frac{1}{2\pi n} + \int_0^{\infty} \frac{\sin(\pi nx)}{e^{2\pi\sqrt{x}} - 1}\,dx. \tag{9.3}$$

Then [14], [23, p. 60]

$$\varphi^*(n) = \frac{1}{n}\sqrt{\frac{2}{n}}\,\psi^*\!\left(\frac{1}{n}\right) - \psi^*(n)$$

and

$$\varphi^*(n) = \psi^*(n) - \frac{1}{n}\sqrt{\frac{2}{n}}\,\varphi^*\left(\frac{1}{n}\right). \tag{9.4}$$

Adding the last two equalities, we find that

$$2\varphi^*(n) = \frac{1}{n}\sqrt{\frac{2}{n}}\,\psi^*\left(\frac{1}{n}\right) - \frac{1}{n}\sqrt{\frac{2}{n}}\,\varphi^*\left(\frac{1}{n}\right),$$

and then replacing n by θ/π, we conclude that

$$2\theta\varphi^*\left(\frac{\theta}{\pi}\right) = \pi\sqrt{\frac{2\pi}{\theta}}\,\psi^*\left(\frac{\pi}{\theta}\right) - \pi\sqrt{\frac{2\pi}{\theta}}\,\varphi^*\left(\frac{\pi}{\theta}\right). \tag{9.5}$$

Replacing x by x^2 and employing (6.4), we find that, for each positive integer N, as θ tends to 0,

$$\begin{aligned}
\varphi^*\left(\frac{\theta}{\pi}\right) &= 2\int_0^\infty \frac{x\cos(\theta x^2)}{e^{2\pi x}-1}\,dx \\
&= 2\int_0^\infty \frac{x}{e^{2\pi x}-1}\left(\sum_{k=0}^{N}\frac{(-1)^k(\theta x^2)^{2k}}{(2k)!} + O((\theta x^2)^{2N+2})\right)dx \\
&= 2\sum_{k=0}^{N}\frac{(-1)^k\theta^{2k}}{(2k)!}\int_0^\infty \frac{x^{4k+1}}{e^{2\pi x}-1}\,dx + O(\theta^{2N+2}) \\
&= \frac{1}{2}\sum_{k=0}^{N}\frac{(-1)^k\theta^{2k}B_{4k+2}}{(2k+1)!} + O(\theta^{2N+2}).
\end{aligned}$$

Hence, as θ tends to 0,

$$2\theta\varphi^*\left(\frac{\theta}{\pi}\right) \sim \sum_{k=0}^{\infty}\frac{(-1)^kB_{4k+2}\theta^{2k+1}}{(2k+1)!}. \tag{9.6}$$

It follows immediately from (9.6) that, as θ tends to ∞,

$$\pi\sqrt{\frac{2\pi}{\theta}}\,\varphi^*\left(\frac{\pi}{\theta}\right) \sim \sqrt{\frac{\theta}{2\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k\pi^{4k+2}B_{4k+2}}{(2k+1)!\,\theta^{2k+1}}. \tag{9.7}$$

A similar argument shows that, as θ tends to ∞,

$$\begin{aligned}
\pi\sqrt{\frac{2\pi}{\theta}}\,\psi^*\left(\frac{\pi}{\theta}\right) &= \pi\sqrt{\frac{2\pi}{\theta}}\left\{\frac{\theta}{2\pi^2} + \int_0^\infty \frac{\sin(\pi^2x/\theta)}{e^{2\pi\sqrt{x}}-1}\,dx\right\} \\
&= \pi\sqrt{\frac{2\pi}{\theta}}\left\{\frac{\theta}{2\pi^2} + 2\int_0^\infty \frac{x\sin(\pi^2x^2/\theta)}{e^{2\pi x}-1}\,dx\right\}
\end{aligned}$$

$$\sim \sqrt{\frac{\theta}{2\pi}}\left\{1+\frac{4\pi^2}{\theta}\sum_{k=0}^{\infty}\frac{(-1)^k\pi^{4k+2}}{(2k+1)!\,\theta^{2k+1}}\int_0^\infty \frac{x^{4k+3}}{e^{2\pi x}-1}\,dx\right\}$$

$$=\sqrt{\frac{\theta}{2\pi}}\left\{1+\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\pi^{4k+4}B_{4k+4}}{(2k+2)!\,\theta^{2k+2}}\right\}. \tag{9.8}$$

Substituting (9.6)–(9.8) into (9.5), we complete our formal derivation of (9.1).

Entry 10 (Formula (2), p. 274). *Let $\varphi(n) = \varphi^*(n/\pi)$, where φ^* is defined by* (9.2). *Then, if n denotes any nonzero, real number,*

$$\int_0^\infty \frac{\sin(nx)}{e^{2\pi\sqrt{x}}-1}\,dx = \varphi(n) - \frac{1}{2n} + \frac{\pi}{n}\sqrt{\frac{2\pi}{n}}\,\varphi\left(\frac{\pi^2}{n}\right).$$

Entry 10 is just a rewritten version of (9.4). Prior to proving Entry 10 in [14], [23, pp. 59–67], Ramanujan [5] submitted Entry 10 as a problem to the *Journal of the Indian Mathematical Society.*

Entry 11 (Formula (3), p. 274). *Let $\varphi(n)$ be as defined in Entry* 10, *and define $\psi(n)$ by*

$$\frac{1}{4\pi} + 2\sum_{k=1}^{\infty}\frac{k\cos(nk^2)}{e^{2\pi k}-1} = \varphi(n) + \psi(n). \tag{11.1}$$

Then, if $a > 0$,

$$\int_0^\infty e^{-2a^2 n}\psi(n)\,dn = \frac{\pi}{e^{4\pi a} - 2e^{2\pi a}\cos(2\pi a) + 1}. \tag{11.2}$$

Proof. Apply the Poisson summation formula (7.3) with $a = 0$, $b = \infty$, and $f(x) = 2x\cos(nx^2)/(e^{2\pi x}-1)$. Since $f(0) = 1/\pi$, we find that

$$\begin{aligned}\frac{1}{2\pi} + 2\sum_{k=1}^{\infty}\frac{k\cos(nk^2)}{e^{2\pi k}-1} &= \int_0^\infty \frac{2x\cos(nx^2)}{e^{2\pi x}-1}\,dx \\ &\quad + 4\sum_{k=1}^{\infty}\int_0^\infty \frac{x\cos(nx^2)\cos(2\pi kx)}{e^{2\pi x}-1}\,dx \\ &= \varphi(n) + F(n),\end{aligned}$$

where

$$F(n) = 4\sum_{k=1}^{\infty}\int_0^\infty \frac{x\cos(nx^2)\cos(2\pi kx)}{e^{2\pi x}-1}\,dx.$$

Hence, referring to (11.1), we observe that

$$\psi(n) = F(n) - \frac{1}{4\pi}. \tag{11.3}$$

For the moment, set $f_n(x) = x\cos(nx^2)/(e^{2\pi x} - 1)$. Now

$$\int_0^\infty e^{-2a^2n} F(n)\,dn = 4\sum_{k=1}^\infty \int_0^\infty e^{-2a^2n}\left(\int_0^\infty f_n(x)\cos(2\pi kx)\,dx\right)dn, \tag{11.4}$$

provided we can show that

$$\sum_{k=1}^\infty \int_0^\infty e^{-2a^2n}\left|\int_0^\infty f_n(x)\cos(2\pi kx)\,dx\right|dn \tag{11.5}$$

converges. Integrating by parts twice and noting that $f_n'(0) = -\frac{1}{2}$, we find that

$$\begin{aligned}\int_0^\infty f_n(x)\cos(2\pi kx)\,dx &= \frac{1}{2(2\pi k)^2} - \frac{1}{(2\pi k)^2}\int_0^\infty f_n''(x)\cos(2\pi kx)\,dx \\ &= O\left(\frac{1}{k^2}\right).\end{aligned}$$

Hence, (11.5) converges, and the inversion in order of summation and integration in (11.4) is justified.

Clearly, the double integral in (11.4) converges absolutely. Thus, inverting the order of integration in (11.4) is justified. Hence,

$$\begin{aligned}\int_0^\infty e^{-2a^2n}F(n)\,dn &= 4\sum_{k=1}^\infty \int_0^\infty \frac{x\cos(2\pi kx)}{e^{2\pi x}-1}\,dx \int_0^\infty e^{-2a^2n}\cos(nx^2)\,dn \\ &= 8a^2\sum_{k=1}^\infty \int_0^\infty \frac{x\cos(2\pi kx)}{(e^{2\pi x}-1)(4a^4+x^4)}\,dx. \qquad (11.6)\end{aligned}$$

Apply the Poisson summation formula (7.3) once again but now with, in the notation (7.3), $a = 0$, $b = \infty$, and

$$f(x) = \frac{4a^2x}{(e^{2\pi x}-1)(4a^4+x^4)}.$$

Since $f(0) = 1/(2\pi a^2)$, we find that

$$\begin{aligned}&\frac{1}{4\pi a^2} + 4a^2\sum_{k=1}^\infty \frac{k}{(e^{2\pi k}-1)(4a^4+k^4)} \\ &= 4a^2\int_0^\infty \frac{x\,dx}{(e^{2\pi x}-1)(4a^4+x^4)} + 8a^2\sum_{k=1}^\infty \int_0^\infty \frac{x\cos(2\pi kx)}{(e^{2\pi x}-1)(4a^4+x^4)}\,dx. \qquad (11.7)\end{aligned}$$

By (11.3), (11.6), and (11.7), so far, we have shown that, for $a > 0$,

$$\int_0^\infty e^{-2a^2 n}\psi(n)\,dn = \frac{1}{8\pi a^2} + 4a^2 \sum_{k=1}^\infty \frac{k}{(e^{2\pi k}-1)(4a^4+k^4)} - 4a^2 \int_0^\infty \frac{x\,dx}{(e^{2\pi x}-1)(4a^4+x^4)}. \tag{11.8}$$

Let $G(a)$ denote the right side of (11.2). Observe that

$$G(a) = \frac{\pi e^{-2\pi a}}{2\cosh(2\pi a) - 2\cos(2\pi a)}. \tag{11.9}$$

From Entry 24 of Chapter 14 (Part II [4, p. 291]), for each complex number a,

$$G(a) = \frac{1}{8\pi a^2} - \frac{1}{4a} + \frac{\pi}{4} - a\sum_{k=1}^\infty \frac{1}{a^2+(a+k)^2} + 4a^2 \sum_{k=1}^\infty \frac{k}{(e^{2\pi k}-1)(4a^4+k^4)}. \tag{11.10}$$

Comparing (11.8) and (11.10), we see that we must show that, for $a > 0$,

$$4a^2 \int_0^\infty \frac{x\,dx}{(e^{2\pi x}-1)(4a^4+x^4)} = \frac{1}{4a} - \frac{\pi}{4} + a\sum_{k=1}^\infty \frac{1}{a^2+(a+k)^2}. \tag{11.11}$$

We now invoke formula (2), p. 269, of the *Notebooks* [22] (Part V [9, Chap. 35]), with $n = 0$. Accordingly, for $a > 0$,

$$\sum_{k=1}^\infty \frac{k}{k^4+4a^4} = \frac{\pi}{8a^2} - 2\int_0^\infty \frac{x\,dx}{(e^{2\pi x}-1)(4a^4+x^4)} + \frac{\pi}{4a^2}\,\frac{\cos(2\pi a) - e^{-2\pi a}}{\cosh(2\pi a) - \cos(2\pi a)}. \tag{11.12}$$

Recalling the representation for $G(a)$ given in (11.9), substituting (11.10) into (11.12), and solving for the integral, we find that, for $a > 0$,

$$\int_0^\infty \frac{x\,dx}{(e^{2\pi x}-1)(4a^4+x^4)} = \frac{\pi}{8a^2}\,\frac{\cos(2\pi a)}{\cosh(2\pi a)-\cos(2\pi a)} - \frac{1}{32\pi a^4} + \frac{1}{16a^3} - \frac{1}{2}\sum_{k=1}^\infty \frac{k}{k^4+4a^4} + \frac{1}{4a}\sum_{k=1}^\infty \frac{1}{a^2+(a+k)^2} - \sum_{k=1}^\infty \frac{k}{(e^{2\pi k}-1)(4a^4+k^4)}. \tag{11.13}$$

Comparing (11.11) and (11.13), we see that it suffices to prove that, for $a > 0$,

$$\frac{\pi}{2a^2} \frac{\cos(2\pi a)}{\cosh(2\pi a) - \cos(2\pi a)} = \frac{1}{8\pi a^4} - \frac{\pi}{4a^2} + 2 \sum_{k=1}^{\infty} \frac{k}{k^4 + 4a^4} + 4 \sum_{k=1}^{\infty} \frac{k}{(e^{2\pi k} - 1)(4a^4 + k^4)}. \tag{11.14}$$

Let $g(a)$ denote the left side of (11.14). Now g has a double pole at $a = 0$ and simple poles at $a = \pm k(1 \pm i)/2$ for each positive integer k. A brief calculation shows that the principal part of g about $a = 0$ equals

$$\frac{1}{8\pi a^4} - \frac{\pi}{4a^2}. \tag{11.15}$$

Let $R(g, z_0) = R(z_0)$ denote the residue of g at a pole z_0. Elementary calculations yield

$$R(k(1 \pm i)/2) = \pm \frac{\coth(\pi k)}{2k^2 i(1 \pm i)} = \pm \frac{1}{2k^2 i(1 \pm i)} \left(\frac{2}{e^{2\pi k} - 1} + 1 \right)$$

and

$$R(-k(1 \pm i)/2) = \mp \frac{\coth(\pi k)}{2k^2 i(1 \pm i)} = \mp \frac{1}{2k^2 i(1 \pm i)} \left(\frac{2}{e^{2\pi k} - 1} + 1 \right),$$

where $k > 0$. Thus, the contributions of the four poles $\pm k(1 \pm i)/2$ to the partial fraction decomposition of g equal

$$\begin{aligned} &\frac{1}{k^2 i(e^{2\pi k} - 1)} \left(\frac{1}{(1 + i)(a - k(1 + i)/2)} - \frac{1}{(1 - i)(a - k(1 - i)/2)} \right. \\ &\qquad \left. - \frac{1}{(1 + i)(a + k(1 + i)/2)} + \frac{1}{(1 - i)(a + k(1 - i)/2)} \right) \\ &\qquad + \frac{1}{2k^2 i} \left(\frac{1}{(1 + i)(a - k(1 + i)/2)} - \frac{1}{(1 - i)(a - k(1 - i)/2)} \right. \\ &\qquad \left. - \frac{1}{(1 + i)(a + k(1 + i)/2)} + \frac{1}{(1 - i)(a + k(1 - i)/2)} \right) \\ &= \frac{k}{(e^{2\pi k} - 1)(a^4 + k^4/4)} + \frac{k}{2(a^4 + k^4/4)}, \end{aligned}$$

after a straightforward calculation. Summing the foregoing principal parts on k, $1 \le k < \infty$, and adding to this sum the principal part (11.15), we establish the partial fraction decomposition (11.14) to complete the proof of this entry.

We quote Ramanujan in the next entry.

Entry 12 (Formula (4), p. 274). *The part without the transcendental part of $\varphi(\pi n)$ can be found from the series*

$$\frac{1}{n\sqrt{n}}\left\{\sin\left(\frac{\pi}{4}+\frac{\pi}{n}\right)+2\sin\left(\frac{\pi}{4}+\frac{4\pi}{n}\right)+3\sin\left(\frac{\pi}{4}+\frac{9\pi}{n}\right)+\&c.\right\}$$
$$-(\cos(\pi n)+2\cos(4\pi n)+3\cos(9\pi n)+\&c.).$$

We do not know what Ramanujan intends to convey by "the transcendental part." Also, the upper indices of the two sums are not indicated. But nonetheless, these series do arise in Ramanujan's [14], [23, pp. 66–67] formulas for evaluating $\varphi(\pi n)$. More precisely, if $n = a/b$ is a positive rational number with $(a, b) = 1$, then

$$\varphi\left(\frac{\pi a}{b}\right)=\frac{1}{4}\sum_{k=1}^{b}(b-2k)\cos\left(\frac{k^2\pi a}{b}\right)-\frac{b}{4a}\sqrt{\frac{b}{a}}\sum_{k=1}^{a}(a-2k)\sin\left(\frac{\pi}{4}+\frac{k^2 b\pi}{a}\right), \tag{12.1}$$

if a and b are both odd, while

$$\varphi\left(\frac{\pi a}{b}\right)=\frac{1}{4\pi a}\sum_{k=1}^{b}\sin\left(\frac{k^2\pi a}{b}\right)-\frac{1}{2}\sum_{k=1}^{b}k\left(1-\frac{k}{b}\right)\cos\left(\frac{k^2\pi a}{b}\right)$$
$$+\frac{b}{2a}\sqrt{\frac{b}{a}}\sum_{k=1}^{a}k\left(1-\frac{k}{a}\right)\sin\left(\frac{\pi}{4}+\frac{k^2\pi b}{a}\right), \tag{12.2}$$

if one of the pair a, b is odd. Recall that $\sum_{k=1}^{b}\cos(k^2\pi a/b)$ and $\sum_{k=1}^{b}\sin(k^2\pi a/b)$ are the real and imaginary parts, respectively, of the Gauss sum $\sum_{k=1}^{b}\exp(\pi i k^2 a/b)$.

Examples. *Let φ be defined as in Entry* 10. *Then*

$$\varphi(0)=\frac{1}{12},\qquad \varphi\left(\frac{\pi}{2}\right)=\frac{1}{4\pi},\qquad \varphi(\pi)=\frac{2-\sqrt{2}}{8},$$
$$\varphi(2\pi)=\frac{1}{16},\qquad \varphi\left(\frac{2\pi}{5}\right)=\frac{8-3\sqrt{5}}{16},\qquad \varphi\left(\frac{\pi}{5}\right)=\frac{6+\sqrt{5}}{4}-\frac{5\sqrt{10}}{8},$$
$$\varphi(\infty)=0,\quad\text{and}\quad \varphi\left(\frac{2\pi}{3}\right)=\frac{1}{3}-\sqrt{3}\left(\frac{3}{16}-\frac{1}{8\pi}\right).$$

Proof. It follows from the Riemann–Lebesgue lemma that $\varphi(\infty) = 0$.
From (6.4),

$$\varphi(0)=\int_0^\infty \frac{dx}{e^{2\pi\sqrt{x}}-1}=2\int_0^\infty \frac{u\,du}{e^{2\pi u}-1}=\frac{1}{12}.$$

The values for $\varphi(\pi)$ and $\varphi(\pi/5)$ may be calculated from (12.1), and the values of $\varphi(\pi/2)$, $\varphi(2\pi)$, $\varphi(2\pi/5)$, and $\varphi(2\pi/3)$ can be determined from (12.2). We note that in calculating $\varphi(\pi/5)$ and $\varphi(2\pi/5)$, we need the value $\cos(\pi/5) = (\sqrt{5}+1)/4$. Since the calculations are routine, we omit them.

Some of the eight values listed above were recorded by Ramanujan in [14], or [23, p. 67].

Entry 13 (Formula (5), p. 275). *If*

$$\int_0^\infty e^{-2a^2 n} f(n)\, dn = \pi e^{-4ap}, \qquad a > 0,$$

then

$$f(n) = \frac{p}{n}\sqrt{\frac{2\pi}{n}}\, e^{-2p^2/n}, \qquad |\arg p| < \pi/4.$$

Entry 13 records the inverse Laplace transform of πe^{-4ap} and is well known (Erdélyi [2, p. 245, formula (1)]).

Entry 14 (Formula (6), p. 275). *For $n > 0$, let $\psi(n)$ be defined by* (11.1). *Then*

$$n\sqrt{\frac{n}{2}}\,\psi(n\pi) = \sum_{r=1}^\infty \Bigg(re^{-2\pi r^2/n} + \sum_{j=1}^\infty e^{-2\pi r(r+j)/n} \times \{(2r+j)\cos(\pi j(2r+j)/n) + j\sin(\pi j(2r+j)/n)\}\Bigg). \tag{14.1}$$

Proof. We shall use Entries 11 and 13.

Recalling the definition of $G(a)$ given in (11.9), we find that, for $\operatorname{Re} a > 0$,

$$\begin{aligned} G(a) &= \frac{\pi e^{-2\pi a}}{4\sin\{\pi a(1+i)\}\sin\{\pi a(1-i)\}} = \frac{\pi e^{-4\pi a}}{(1-e^{2\pi i a(1+i)})(1-e^{-2\pi i a(1-i)})} \\ &= \pi e^{-4\pi a}\sum_{m=0}^\infty e^{2\pi i a m(1+i)} \sum_{n=0}^\infty e^{-2\pi i a n(1-i)} \\ &= \pi e^{-4\pi a}\sum_{k=0}^\infty \sum_{m=0}^k e^{-2\pi a\{k - i(2m-k)\}}, \end{aligned}$$

where we have set $m + n = k$, $0 \le k < \infty$, $0 \le m \le k$.

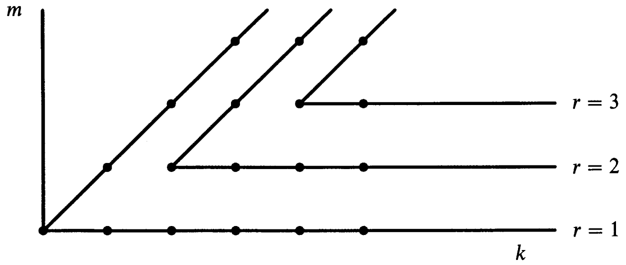
To each term above, we apply Entry 13 with $p = \pi(2 + k - i(2m - k))/2$. Then, in the notation of Entry 13,

$$f_{k,m}(n) := f(n) = \frac{\pi}{n}\sqrt{\frac{\pi}{2n}}(2 + k - i(2m-k))e^{-\pi^2(2+k-i(2m-k))^2/(2n)},$$

or

$$n\sqrt{\frac{n}{2}}\, f_{k,m}(\pi n) = \tfrac{1}{2}(2 + k - i(2m - k))e^{-\pi(2+k-i(2m-k))^2/(2n)}. \tag{14.2}$$

We now set $k = 2r + j - 2$ and consider the two values $m = r - 1, r + j - 1$, where $r \geq 1$ and $j \geq 0$. (The two values of m coalesce if $j = 0$.) Observe that if $r = 1$ and j varies from 0 to ∞, we obtain all lattice points on the real axis and on the line $m = k$; if $r = 2$ and j varies from 0 to ∞, we obtain the lattice points on $m = 1$ with $m \geq 2$ and on the diagonal immediately below the main diagonal; etc.



Hence, as r ranges from 1 to ∞ and j varies from 0 to ∞, we realize a bijective map of the lattice points $\{(k, m)\colon 0 \leq m \leq k, 0 \leq k < \infty\}$.

The term (14.2) corresponding to $j = 0$ is given by

$$n\sqrt{\frac{n}{2}}\, f_{2r-2,r-1}(\pi n) = re^{-2\pi r^2/n}. \tag{14.3}$$

We now add the contributions of $k = 2r + j - 2$ and $m = r - 1, r + j - 1$ for each fixed pair (r, j), where $r, j \geq 1$, in (14.2). Hence, we obtain

$$\begin{aligned}
n\sqrt{\frac{n}{2}}&(f_{2r+j-2,r-1}(\pi n) + f_{2r+j-2,r+j-1}(\pi n)) \\
&= \tfrac{1}{2}(2r + j + ij)e^{-\pi(2r+j+ij)^2/(2n)} + \tfrac{1}{2}(2r + j - ij)e^{-\pi(2r+j-ij)^2/(2n)} \\
&= \tfrac{1}{2}(2r + j + ij)e^{-\pi(2r^2+2rj+ij(2r+j))/n} \\
&\quad + \tfrac{1}{2}(2r + j - ij)e^{-\pi(2r^2+2rj-ij(2r+j))/n} \\
&= e^{-2\pi r(r+j)/n}\{(2r + j)\cos(\pi j(2r + j)/n) + j\sin(\pi j(2r + j)/n)\}.
\end{aligned} \tag{14.4}$$

Summing (14.3) over r, $1 \leq r < \infty$, and summing (14.4) over r and j, $1 \leq r < \infty$, $1 \leq j < \infty$, and employing Entry 11, we complete the proof of (14.1).

Entry 15 (Formula (8), p. 277). *Let p, n, and a be real with $0 < p < a$ and $n < 1$. Then*

$$I(p, n, a) := \frac{1}{p} \int_0^1 x^{a-1}(1 - x^{p/(1-nx)})\,dx = \sum_{k=0}^{\infty} \frac{n^k(a+k)^{k-1}}{(p+a+k)^{k+1}}.$$

Proof. In the sequel, we shall apply the generalized binomial theorem in two forms. The integral evaluation used below is easily established by induction on k. All inversions of limiting operations are easily justified by absolute convergence. Thus,

$$\begin{aligned}
I(p, n, a) &= \frac{1}{p} \int_0^1 x^{a-1}(1 - e^{p(\log x)/(1-nx)})\,dx \\
&= -\frac{1}{p} \sum_{k=1}^{\infty} \frac{p^k}{k!} \int_0^1 x^{a-1} \log^k x \left(\frac{1}{1-nx}\right)^k dx \\
&= -\sum_{k=1}^{\infty} \frac{p^{k-1}}{k!} \int_0^1 x^{a-1} \log^k x \sum_{j=0}^{\infty} \frac{(k)_j}{j!} (nx)^j\,dx \\
&= -\sum_{k=1}^{\infty} \frac{p^{k-1}}{k!} \sum_{j=0}^{\infty} \frac{(k)_j n^j}{j!} \int_0^1 x^{a+j-1} \log^k x\,dx \\
&= -\sum_{k=1}^{\infty} \frac{p^{k-1}}{k!} \sum_{j=0}^{\infty} \frac{(k)_j n^j}{j!} \frac{(-1)^k k!}{(a+j)^{k+1}} \\
&= \sum_{j=0}^{\infty} \frac{n^j}{(a+j)^2} \sum_{k=0}^{\infty} \frac{(k+1)_j}{j!} \left(\frac{-p}{a+j}\right)^k \\
&= \sum_{j=0}^{\infty} \frac{n^j}{(a+j)^2} \left(\frac{1}{1 + p/(a+j)}\right)^{j+1} \\
&= \sum_{j=0}^{\infty} \frac{n^j(a+j)^{j-1}}{(p+a+j)^{j+1}},
\end{aligned}$$

and the proof is complete.

Entry 16 (Formula (1), p. 279). *We have*

$$\int_0^{\infty} x^{n-1} e^{-x} \sin x\,dx = 2^{-n/2}\Gamma(n) \sin(n\pi/4), \qquad \operatorname{Re} n > -1,$$

and

$$\int_0^{\infty} x^{n-1} e^{-x} \cos x\,dx = 2^{-n/2}\Gamma(n) \cos(n\pi/4), \qquad \operatorname{Re} n > 0.$$

These two formulas are special cases of the known results

$$\int_0^\infty x^{n-1}e^{-ax}\sin(bx)\,dx = \frac{\Gamma(n)}{(a^2+b^2)^{n/2}}\sin\left(n\tan^{-1}\frac{b}{a}\right), \qquad \operatorname{Re} n > -1, \quad (16.1)$$

and

$$\int_0^\infty x^{n-1}e^{-ax}\cos(bx)\,dx = \frac{\Gamma(n)}{(a^2+b^2)^{n/2}}\cos\left(n\tan^{-1}\frac{b}{a}\right), \qquad \operatorname{Re} n > 0, \quad (16.2)$$

respectively, where Re $a > |\operatorname{Im} b|$ (Gradshteyn and Ryzhik [1, p. 490, formulas 5,6]). Elegant proofs of (16.1) and (16.2), depending on differentiation under the integral sign, may be found in Fichtenholz's text [1, p. 817].

Entry 17 (Formula (2), p. 279). *If a is real, $|a| \le \pi$, and $n > 0$, then*

$$\int_0^\infty \frac{\sinh(ax)}{\sinh(\pi x)}\frac{dx}{n^2+x^2} = \frac{\sin a}{n}\int_0^1 \frac{x^n}{1+2x\cos a + x^2}\,dx.$$

Proof. From formula (30) in Ramanujan's paper [14], [23, p. 64], for $|a| \le \pi$ and $n > 0$,

$$\int_0^\infty \frac{\sinh(ax)}{\sinh(\pi x)}\frac{dx}{n^2+x^2} = -\frac{i}{2n^2} + \frac{i\pi e^{-ina}}{2n\sin(\pi n)} - i\sum_{k=1}^\infty \frac{(-1)^k e^{-ika}}{n^2-k^2}.$$

Equating real parts, we find that, under the given hypotheses,

$$\begin{aligned}\int_0^\infty \frac{\sinh(ax)}{\sinh(\pi x)}\frac{dx}{n^2+x^2} &= \frac{\pi\sin(na)}{2n\sin(n\pi)} - \sum_{k=1}^\infty \frac{(-1)^k\sin(ka)}{n^2-k^2}\\ &= \frac{1}{n}\sum_{k=1}^\infty \frac{(-1)^{k-1}k\sin(ka)}{k^2-n^2} - \sum_{k=1}^\infty \frac{(-1)^k\sin(ka)}{n^2-k^2}\\ &= \frac{1}{n}\sum_{k=1}^\infty \frac{(-1)^{k-1}\sin(ka)}{n+k}, \end{aligned} \qquad (17.1)$$

where we have used a standard partial fraction decomposition for $\sin(na)/\sin(n\pi)$ (Gradshteyn and Ryzhik [1, p. 40, formula 7]). Formula (17.1) may also be found in the tables of Gradshteyn and Ryzhik [1, p. 352, formula 7].

On the other hand,

$$\begin{aligned} I := \frac{\sin a}{n}\int_0^1 \frac{x^n\,dx}{1+2x\cos a+x^2} &= \frac{\sin a}{n}\int_0^1 \frac{x^n\,dx}{(1+xe^{-ia})(1+xe^{ia})}\\ &= \frac{\sin a}{n}\int_0^1 x^n \sum_{j,k=0}^\infty (-x)^{j+k}e^{ia(k-j)}\,dx. \end{aligned}$$

Now let $j = r - k - 1$, $1 \le r < \infty$. Then

$$\begin{aligned}
I &= \frac{\sin a}{n} \int_0^1 x^n \sum_{r=1}^{\infty} (-x)^{r-1} \sum_{k=0}^{r-1} e^{ia(2k-r+1)}\, dx \\
&= \frac{e^{2ia}-1}{2in} \int_0^1 x^n \sum_{r=1}^{\infty} (-x)^{r-1} e^{-ira} \frac{e^{2ira}-1}{e^{2ia}-1} \\
&= \frac{1}{n} \int_0^1 x^n \sum_{r=1}^{\infty} (-x)^{r-1} \sin(ra)\, dx \\
&= \frac{1}{n} \sum_{r=1}^{\infty} (-1)^{r-1} \sin(ra) \int_0^1 x^{n+r-1}\, dx \\
&= \frac{1}{n} \sum_{r=1}^{\infty} \frac{(-1)^{r-1} \sin(ra)}{n+r}. \qquad (17.2)
\end{aligned}$$

Comparing (17.1) and (17.2), we see that we have finished the proof.

Ramanujan next claims that

$$\int_0^{\infty} \cos(nx) \log(1+x^2)\, dx = -\frac{\pi}{n} e^{-n}. \qquad (18.1)$$

Now, by an integration by parts,

$$\lim_{N\to\infty} \int_0^N \frac{x \sin(nx)}{1+x^2}\, dx = \lim_{N\to\infty} \left(\tfrac{1}{2} \sin(nx) \log(1+x^2) \Big|_0^N - \frac{n}{2} \int_0^N \cos(nx) \log(1+x^2)\, dx \right). \qquad (18.2)$$

Since the limit on the left side of (18.2) exists, i.e., the infinite integral converges, and since the limit of the first expression on the right side clearly does not exist, we conclude that the integral in (18.1) does not converge. However, (18.2) points to a corrected version of (18.1), which we now state.

Entry 18 (Corrected Version of Formula (9), p. 281). *If a, b, and n are positive, then*

$$\int_0^{\infty} \cos(nx) \log\left(\frac{a^2+x^2}{b^2+x^2}\right) dx = \frac{\pi}{n}(e^{-bn} - e^{-an}). \qquad (18.3)$$

PROOF. By a routine application of the calculus of residues and an integration by parts, we find that, for n, a, and b positive,

$$\begin{aligned}\frac{\pi}{2}e^{-bn}-\frac{\pi}{2}e^{-an} &= \int_0^\infty \frac{x\sin(nx)}{b^2+x^2}\,dx - \int_0^\infty \frac{x\sin(nx)}{a^2+x^2}\,dx \\ &= \lim_{N\to\infty}\left(\tfrac{1}{2}\sin(nx)\log\left(\frac{b^2+x^2}{a^2+x^2}\right)\Bigg|_0^N\right. \\ &\quad \left.+\frac{n}{2}\int_0^N \cos(nx)\log\left(\frac{a^2+x^2}{b^2+x^2}\right)dx\right) \\ &= \frac{n}{2}\int_0^\infty \cos(nx)\log\left(\frac{a^2+x^2}{b^2+x^2}\right)dx. \qquad (18.4)\end{aligned}$$

Multiplying both sides of (18.4) by $2/n$, we complete the proof.

In fact, the integrals in the first line of (18.4) and in (18.3) may be found in Gradshteyn and Ryzhik's tables [1, p. 406, formula 3.723, no. 3; p. 583, formula 4.382, no. 3, respectively]. We have given this short proof of (18.3) to provide some insight into the origin of (18.1).

Entry 19 (Formula (10), p. 283). *If $n > 0$, then*

$$I_n := \int_0^n \left(\frac{n}{x}\right)^x dx = \sum_{k=1}^\infty \frac{n^k}{k^k}.$$

Proof. This result is very easy to prove. Setting $x = nu$, we find that

$$\begin{aligned} I_n &= n\int_0^1 e^{-nu\log u}\,du \\ &= \sum_{k=0}^\infty \frac{(-1)^k n^{k+1}}{k!}\int_0^1 u^k \log^k u\,du \\ &= \sum_{k=0}^\infty \frac{(-1)^k n^{k+1}}{k!}\frac{(-1)^k k!}{(k+1)^{k+1}} \\ &= \sum_{k=0}^\infty \frac{n^{k+1}}{(k+1)^{k+1}}.\end{aligned}$$

The proof is complete.

Entry 19 provides a beautiful example of an identity between an integral and series with the integrand and summands having the same form, but with the interval of integration finite and the series infinite! Note the similarity between Entries 15 and 19; in fact, we used the same integral evaluation in each proof. Entry 19 can also be found in the tables of Gradshteyn and Ryzhik [1, p. 310, formula 3.342].

The following result is complementary to Entry 22(ii) of Chapter 14 (Part II [4, pp. 278–279]).

Entry 20 (Formula (2), p. 288). *Let $n \geq 0$. Then*

$$\int_0^\infty \frac{\sin(2nx)\,dx}{x(\cosh(\pi x) + \cos(\pi x))} = \frac{\pi}{4} - 2\sum_{k=0}^\infty \frac{(-1)^k e^{-(2k+1)n}\cos\{(2k+1)n\}}{(2k+1)\cosh\{(2k+1)\pi/2\}}.$$

Proof. From our book [4, p. 279, eq. (22.7)],

$$\frac{1}{\cosh(\pi x) + \cos(\pi x)} = \frac{1}{\pi}\sum_{k=0}^\infty \frac{(-1)^k(2k+1)^3}{\cosh\{(2k+1)\pi/2\}(x^4 + (2k+1)^4/4)}, \tag{20.1}$$

where x is any complex number, and where a slight misprint has been corrected. Multiplying both sides of (20.1) by $\sin(2nx)/x$, integrating over $(0, \infty)$, inverting the order of summation and integration by absolute convergence, and employing an integral evaluation (Gradshteyn and Ryzhik [1, p. 412, formula 3.734, no. 1]) that is easily established by contour integration, we find that, for $n \geq 0$,

$$\begin{aligned}
&\int_0^\infty \frac{\sin(2nx)\,dx}{x(\cosh(\pi x) + \cos(\pi x))} \\
&\qquad = \frac{1}{\pi}\sum_{k=0}^\infty \frac{(-1)^k(2k+1)^3}{\cosh\{(2k+1)\pi/2\}} \int_0^\infty \frac{\sin(2nx)\,dx}{x(x^4 + (2k+1)^4/4)} \\
&\qquad = 2\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)\cosh\{(2k+1)\pi/2\}}(1 - e^{-(2k+1)n}\cos\{(2k+1)n\}) \\
&\qquad = \frac{\pi}{4} - 2\sum_{k=0}^\infty \frac{(-1)^k e^{-(2k+1)n}\cos\{(2k+1)n\}}{(2k+1)\cosh\{(2k+1)\pi/2\}},
\end{aligned}$$

where, in the last step, we employed Entry 25(vii) in Chapter 14 (Part II [4, p. 295]). This completes the proof.

Entry 21 (Formula (5), p. 289). *If $n > 1$, then*

$$\int_0^\infty \frac{dx}{e^{x^n} + e^{-x^n}} = \left(\frac{\pi}{2}\right)^{(1/n)-1} \Gamma\left(\frac{1}{n}\right) \cos\left(\frac{\pi}{2n}\right) \int_0^\infty \frac{x^{n-2}\,dx}{e^{x^n} + e^{-x^n}}. \tag{21.1}$$

Let $\chi(k)$ denote the primitive character of modulus 4, and let

$$L(s) = \sum_{k=1}^\infty \chi(k)k^{-s}, \qquad \operatorname{Re} s > 0, \tag{21.2}$$

be the associated L-function. As we shall see in the proof below, (21.1) is equivalent to the functional equation of $L(s)$, to wit,

$$L(s) = \cos(\pi s/2)(\pi/2)^{s-1}\Gamma(1 - s)L(1 - s), \tag{21.3}$$

where, since $L(s)$ can be analytically continued to an entire function, s is any complex number. For example, see Davenport's book [1, p. 71, eq. (11)]. Thus, (21.1) is a truly fascinating equality! One can only wonder in awe how Ramanujan could have ever discovered this camouflage.

PROOF. Expanding $1/(1+e^{-2x^n})$ in a geometric series, inverting the order of integration and summation by absolute convergence, and setting $u = (2k+1)x^n$, we find that, for $n > 0$,

$$\begin{aligned} \int_0^\infty \frac{dx}{e^{x^n}+e^{-x^n}} &= \sum_{k=0}^{\infty}(-1)^k \int_0^\infty e^{-(2k+1)x^n}\,dx \\ &= \frac{1}{n}\sum_{k=1}^{\infty}\chi(k)k^{-1/n}\int_0^\infty u^{-1+1/n}e^{-u}\,du \\ &= \frac{1}{n}\Gamma\left(\frac{1}{n}\right)L\left(\frac{1}{n}\right), \end{aligned} \tag{21.4}$$

by (21.2).

On the other hand, using the same steps as above, we deduce that, for $n > 1$,

$$\begin{aligned} \int_0^\infty \frac{x^{n-2}\,dx}{e^{x^n}+e^{-x^n}} &= \sum_{k=0}^{\infty}(-1)^k \int_\sigma^\infty x^{n-2}e^{-(2k+1)x^n}\,dx \\ &= \frac{1}{n}\sum_{k=1}^{\infty}\chi(k)k^{-1+1/n}\int_0^\infty u^{-1/n}e^{-u}\,du \\ &= \frac{1}{n}\Gamma\left(1-\frac{1}{n}\right)L\left(1-\frac{1}{n}\right). \end{aligned} \tag{21.5}$$

Thus, by (21.4) and (21.5), (21.1) is equivalent to the equation

$$L\left(\frac{1}{n}\right) = \left(\frac{\pi}{2}\right)^{(1/n)-1}\cos\left(\frac{\pi}{2n}\right)\Gamma\left(1-\frac{1}{n}\right)L\left(1-\frac{1}{n}\right). \tag{21.6}$$

With $s = 1/n$, (21.3) and (21.6) are identical, and so the proof is complete.

Entry 22 (p. 318). *Formally,*

$$\int_0^\infty \frac{\varphi(x)}{x^x}\,dx = \sum_{k=-\infty}^{\infty}\frac{\varphi(k)}{k^k}. \tag{22.1}$$

It is doubtful that Ramanujan intended Entry 22 to be anything more than a proposed equality for which he probably tried to find examples. In Chapter 13 (Part II [4, pp. 226–227]), Ramanujan briefly considered a similar problem. Of course, instances where equality holds in (22.1) are extremely rare. The next entry provides an example where Ramanujan formally used (22.1) to obtain a correct result, when properly interpreted.

Entry 23 (p. 318). *As a tends to ∞,*

$$\int_a^\infty \left(\frac{a}{x}\right)^x dx \sim \sum_{k=0}^\infty \left(\frac{-k}{a}\right)^k. \tag{23.1}$$

Before giving a rigorous proof, we shall indicate how Ramanujan undoubtedly argued. Applying Entry 19 (rigorously) and Entry 22 (nonrigorously), we find that

$$\begin{aligned}\int_a^\infty \left(\frac{a}{x}\right)^x dx &= \int_0^\infty \left(\frac{a}{x}\right)^x dx - \int_0^a \left(\frac{a}{x}\right)^x dx \\ &= \sum_{k=-\infty}^\infty \frac{a^k}{k^k} - \sum_{k=1}^\infty \frac{a^k}{k^k} \\ &= \sum_{k=-\infty}^0 \frac{a^k}{k^k} \\ &= \sum_{k=0}^\infty \left(\frac{-k}{a}\right)^k.\end{aligned}$$

This completes Ramanujan's "proof." Of course, the series in (23.1) diverges for every finite value a.

PROOF. First, letting $x = au$, we find that

$$\int_a^\infty \left(\frac{a}{x}\right)^x dx = a\int_1^\infty e^{-au\log u}\, du. \tag{23.2}$$

We shall use Example 2 in Section 15 of Chapter 3 (Part I [2, p. 75]) to obtain an asymptotic expansion for the right side of (23.2). In the notation of Example 2, let $m = 1, n = -1, p = 1, a = t$, and $x = u$. Thus, for $t = u \log u$ and $|t| \le 1/e$,

$$u = \sum_{k=0}^\infty \frac{(1-k)^{k-1}t^k}{k!},$$

or

$$u - 1 = \sum_{k=1}^\infty \frac{(-1)^{k-1}(k-1)^{k-1}t^k}{k!}. \tag{23.3}$$

Using (23.3), we apply Laplace's method (Olver [1, pp. 85–86]) to obtain an asymptotic expansion as a tends to ∞ for the integral on the right side of (23.2). To help the reader, we remark that (23.3) corresponds to (8.04) in Olver's book. Also, note the parenthetical remark prior to the statement of Theorem 8.1. on page 86 of Olver's text. Hence, by a direct application of Laplace's theorem, as a tends to ∞,

$$\int_1^\infty e^{-au\log u}\,du \sim \sum_{k=0}^\infty \Gamma(k+1)(k+1)\frac{(-1)^k k^k}{(k+1)!\,a^{k+1}}$$
$$= \sum_{k=0}^\infty \frac{(-1)^k k^k}{a^{k+1}}.$$

Using this in (23.2), we complete the proof.

Another approach to Entry 23 is via Watson's lemma. For $-1/e < t < \infty$, let u be the unique number in $(1/e, \infty)$ such that $t = u\log u$. Let $\varphi(t) = 1/(\log u + 1)$. Then, by Watson's lemma (Olver [1, p. 71]), as a tends to ∞,

$$\int_a^\infty \left(\frac{a}{x}\right)^x dx = a\int_1^\infty e^{-au\log u}\,du = a\int_0^\infty e^{-at}\varphi(t)\,dx$$
$$\sim \sum_{k=0}^\infty \frac{\varphi^{(k)}(0)}{a^k}. \tag{23.4}$$

Note that $\varphi(0) = 1$. From (23.1) and (23.4), we conclude that

$$\varphi^{(k)}(0) = (-k)^k, \qquad k \geq 1. \tag{23.5}$$

It is an interesting exercise to give a direct proof of (23.5), and, in fact, the author [7] submitted this problem to the *American Mathematical Monthly*. Several novel and interesting solutions were received. We present two of them.

The first was found independently by R. J. Evans, T. S. Norfolk, and J. H. Steelman. It uses the following well-known result, easily proved by induction on n: for integers m and n such that $m \geq n \geq 0$,

$$\sum_{k=0}^m (-1)^k k^n \binom{m}{k} = \begin{cases} 0, & \text{if } m > n, \\ (-1)^n n!, & \text{if } m = n. \end{cases} \tag{23.6}$$

Set $x = \log u$ so that $t = xe^x$. Thus,

$$t^k = x^k e^{xk} = \sum_{j=0}^\infty \frac{k^j x^{j+k}}{j!} = \sum_{n=k}^\infty \frac{k^{n-k}x^n}{(n-k)!}. \tag{23.7}$$

By setting $0^0 = 1$, employing (23.7), inverting the order of summation, and using (23.6), we find that, for $|t| < 1/e$,

$$\sum_{k=0}^\infty \frac{(-k)^k}{k!}t^k = \sum_{k=0}^\infty \sum_{n=k}^\infty \frac{(-k)^k k^{n-k}x^n}{k!\,(n-k)!} = \sum_{n=0}^\infty \sum_{k=0}^n (-1)^k k^n \binom{n}{k}\frac{x^n}{n!}$$
$$= \sum_{n=0}^\infty (-1)^n x^n = \frac{1}{1+x} = \varphi(t). \tag{23.8}$$

By Taylor's theorem, we conclude that $\varphi^{(k)}(0) = (-k)^k$, $k \geq 0$, as desired.

The second solution is by N. J. Fine. Let $f(t)$ denote the far left side of (23.8). Then, for t sufficiently small,

$$\begin{aligned} f(t) &= \sum_{k,n=0}^{\infty} \frac{(-n)^k}{k!} t^k \frac{1}{2\pi i} \int_{|z|=2} \frac{dz}{z^{n-k+1}} \\ &= \frac{1}{2\pi i} \int_{|z|=2} \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \sum_{k=0}^{\infty} \frac{(-nzt)^k}{k!} \, dz \\ &= \frac{1}{2\pi i} \int_{|z|=2} \sum_{n=0}^{\infty} \frac{e^{-nzt}}{z^{n+1}} = \frac{1}{2\pi i} \int_{|z|=2} \frac{dz}{z - e^{-tz}}. \end{aligned} \tag{23.9}$$

For $|t|$ sufficiently small, the integrand has a simple pole at $z = z_0$, where $z_0 = e^{-tz_0}$, $|z_0| < 2$, and z_0 is unique. The residue at z_0 equals $1/(1 + te^{-tz_0})$. Setting $u' = 1/z_0$, we find that $u' = e^{t/u'}$. Hence, $u' = u$. Thus, from (23.9) and the residue theorem,

$$f(t) = \frac{1}{1 + te^{-t/u}} = \varphi(t).$$

Thus, by the definition of f and Taylor's theorem, $\varphi^{(k)}(0) = (-k)^k$, as was sought.

C. C. Grosjean [1] has carefully studied (22.1) with $\varphi(x) = a^x f(x)$ and with the equality sign replaced by an asymptotic symbol $\sim$ as a tends to ∞. In the context of Entry 23, $f(x) \equiv 1$. Grosjean showed that asymptotic expansions could be established when $f(x)$ is any polynomial in x and when $f(x)$ belongs to a wide class of functions representable by Maclaurin series on $(-\infty, \infty)$.

The next three entries are found in Ramanujan's *Quarterly Reports* and were discussed by us in Part I [2, pp. 334–335]. There we showed how each result arose from Ramanujan's "Master Theorem." Hardy [7, p. 206] proved Entry 24 by invoking the "Master Theorem." We have nothing further to add about Entry 25 and regard it as a formal identity. However, we shall give new proofs of Entries 24 and 26 that are not dependent upon the "Master Theorem."

Ramanujan's statement of Entry 24 is somewhat incomplete.

Entry 24 (p. 322). *Let* $0 < p < 1$. *Suppose that* $x^{p-1}F(x)$ *and* $x^{-p}f(x)$ *are absolutely integrable on* $(0, \infty)$. *If*

$$\int_0^{\infty} F(ax) f(bx) \, dx = \frac{1}{a+b} \tag{24.1}$$

for all nonnegative numbers a *and* b *with* $a + b \neq 0$, *then*

$$\int_0^{\infty} x^{p-1} F(x) \, dx \int_0^{\infty} y^{-p} f(y) \, dy = \frac{\pi}{\sin(\pi p)}.$$

PROOF. Replacing x by x^2 and y by y^2 and then putting $x = r\cos\theta$ and $y = r\sin\theta$, $0 \le r < \infty$, $0 \le \theta \le \pi/2$, we find that

$$\int_0^\infty x^{p-1}F(x)\,dx \int_0^\infty y^{-p}f(y)\,dy = 4\int_0^\infty x^{2p-1}F(x^2)\,dx \int_0^\infty y^{-2p+1}f(y^2)\,dy$$

$$= 4\int_0^{\pi/2}\int_0^\infty \cos^{2p-1}\theta\,\sin^{1-2p}\theta F(r^2\cos^2\theta)f(r^2\sin^2\theta)r\,dr\,d\theta. \quad (24.2)$$

From (24.1),

$$\int_0^\infty F(r^2\cos^2\theta)f(r^2\sin^2\theta)r\,dr = \frac{1}{2}\int_0^\infty F(u\cos^2\theta)f(u\sin^2\theta)\,du$$

$$= \frac{1}{2(\cos^2\theta + \sin^2\theta)} = \frac{1}{2}.$$

Using this calculation in (24.2), we conclude that

$$\int_0^\infty x^{p-1}F(x)\,dx \int_0^\infty y^{-p}f(y)\,dy = 2\int_0^{\pi/2} \cos^{2p-1}\theta\,\sin^{1-2p}\theta\,d\theta$$

$$= \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(\pi p)},$$

since $0 < p < 1$. This completes the proof.

Observe that we did not use the full strength of our hypotheses; we only needed (24.1) to be valid for nonnegative a, b such that $a + b = 1$.

Entry 24 was clearly motivated by the example $F(x) = f(x) = e^{-x}$. It would be interesting to find further examples and to characterize those functions F and f for which Entry 24 is valid.

Entry 25 (p. 322). *Let F and f satisfy the hypotheses of Entry 24. If*

$$\psi(n) = \frac{1}{2}\int_0^\infty \varphi(x)\{F(nxi) + F(-nxi)\}\,dx,$$

then

$$\frac{\pi}{2}\varphi(n) = \frac{1}{2}\int_0^\infty \psi(x)\{f(nxi) + f(-nxi)\}\,dx.$$

Entry 26 (p. 322). *Let a, b, m, n, and p be positive numbers such that $a/m = (n-b)/n = p$.* Suppose that *$x^{a-1}F(x^m)$ and $x^{b-1}f(x^n)$ are absolutely integrable on $(0, \infty)$. If*

$$\int_0^\infty F(\alpha x)f(\beta x)\,dx = \frac{1}{\alpha + \beta}$$

for all nonnegative numbers α *and* β *with* $\alpha + \beta \neq 0$, *then*

$$I(a, b, m, n) := \int_0^\infty x^{a-1} F(x^m)\,dx \int_0^\infty y^{b-1} f(y^n)\,dy = \frac{\pi}{mn\sin(\pi p)}.$$

Observe that if $m = n = 1$, Entry 26 reduces to Entry 24.

PROOF. The proof is like that for Entry 24. Let $x^m = u^2$ and $y^n = v^2$. Then

$$I(a, b, m, n) = \frac{4}{mn} \int_0^\infty u^{(2a/m)-1} F(u^2)\,du \int_0^\infty v^{(2b/n)-1} f(v^2)\,dv.$$

Setting $u = r\cos\theta$ and $v = r\sin\theta$, we find that

$$\begin{aligned} I(a, b, m, n) &= \frac{4}{mn} \int_0^{\pi/2} \cos^{(2a/m)-1}\theta \sin^{(2b/n)-1}\theta\,d\theta \\ &\quad \times \int_0^\infty F(r^2\cos^2\theta) f(r^2\sin^2\theta) r^{(2a/m)+(2b/n)-1}\,dr \\ &= \frac{4}{mn} \int_0^{\pi/2} \cos^{2p-1}\theta \sin^{1-2p}\theta\,d\theta \int_0^\infty F(r^2\cos^2\theta) f(r^2\sin^2\theta) r\,dr, \end{aligned}$$

where we used the hypotheses on a, b, m, n, and p. The remainder of the proof is exactly the same as for Entry 24, and so we omit it.

Entry 27 (p. 324). *If* $a > 0$ *and* $0 \le b < 2$, *then*

$$2\int_0^\infty \frac{\sinh(bx)\sin(ax)}{e^{2x}-1}\,dx = \int_0^\infty e^{-ax}\cot x \sin(bx)\,dx.$$

PROOF. Inverting the order of summation and integration by absolute convergence, we find that

$$\begin{aligned} 2\int_0^\infty \frac{\sinh(bx)\sin(ax)}{e^{2x}-1}\,dx &= 2\sum_{k=1}^\infty \int_0^\infty e^{-2kx}\sinh(bx)\sin(ax)\,dx \\ &= \sum_{k=1}^\infty \int_0^\infty (e^{-(2k-b)x} - e^{-(2k+b)x})\sin(ax)\,dx \\ &= \sum_{k=1}^\infty \left(\frac{a}{(2k-b)^2+a^2} - \frac{a}{(2k+b)^2+a^2}\right) \\ &= \sum_{k=1}^\infty \int_0^\infty e^{-ax}(\cos\{(2k-b)x\} - \cos\{(2k+b)x\})\,dx \\ &= 2\lim_{N\to\infty} \sum_{k=1}^N \int_0^\infty e^{-ax}\sin(2kx)\sin(bx)\,dx \end{aligned}$$

$$= 2 \lim_{N \to \infty} \int_0^\infty e^{-ax} \sin(bx) \sum_{k=1}^{N} \sin(2kx)\, dx$$

$$= 2 \lim_{N \to \infty} \int_0^\infty e^{-ax} \sin(bx) \sin\{(N+1)x\}$$

$$\times \sin(Nx) \csc x\, dx$$

$$= \lim_{N \to \infty} \int_0^\infty e^{-ax} \frac{\sin(bx)}{\sin x} (\cos x - \cos\{(2N+1)\})\, dx$$

$$= \int_0^\infty e^{-ax} \sin(bx) \cot x,$$

by the Riemann–Lebesgue lemma. Thus, the proof is complete.

Entry 28 (p. 332). *Let*

$$\varphi(x) = \sum_{k=0}^{\infty} u(k) x^k \quad \textit{and} \quad \psi(x) = \sum_{k=0}^{\infty} v(k) x^k$$

in some neighborhood of $x = 0$. *Suppose that* $\lim_{x \to \infty} \varphi(x) = \lim_{x \to \infty} \psi(x) = 0$ *and that* $\varphi(0) = \psi(0)$. *Then, if* $a, b > 0$,

$$\int_0^\infty \frac{\varphi(ax) - \psi(bx)}{x}\, dx = \varphi(0) \left\{ \frac{d}{ds} \log\left(\frac{v(s)}{u(s)}\right) \bigg|_{s=0} + \log\left(\frac{b}{a}\right) \right\}.$$

Entry 28 appears in a more general form in Ramanujan's *Quarterly Reports* and yields a vast generalization of Frullani's theorem. Additional hypotheses are necessary to ensure the validity of Entry 28. Since these are considerably lengthy and since a rigorous proof of Entry 28 is given by us in [2, pp. 313–314], we refer the reader to [2].

Entry 29 (p. 332). *Let* $F(z)$ *be analytic in a region* R^* *containing the closed half-plane* $\operatorname{Re} z \geq 0$, *except possibly for* $z = 0$. *As* z *tends to* ∞ *in* R^*, *assume that* $F(z) = O(z^\alpha)$ *for some real number* $\alpha < 1$, *and as* z *tends to* 0 *in* R^*, *assume that* $F(z) = O(z^{\delta - 1})$ *for some number* $\delta > 0$. For $\operatorname{Re} a \geq 0$, *define*

$$\varphi(a) = \frac{2}{\pi} \int_0^\infty e^{-z^2} F(z/a)\, dz. \tag{29.1}$$

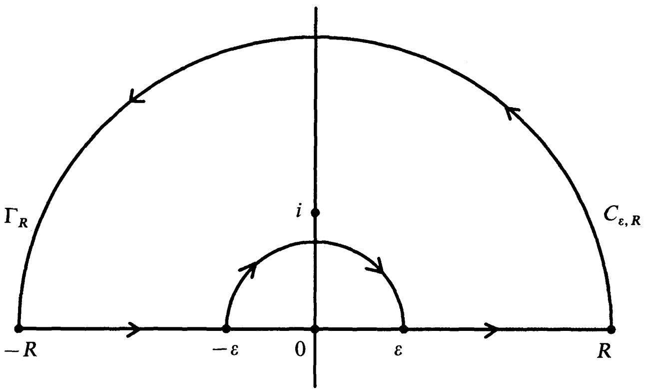
Then, for $z > 0$,

$$F(z) = \int_0^\infty e^{-u^2} \left\{ \varphi\left(\frac{ui}{z}\right) + \varphi\left(-\frac{ui}{z}\right) \right\} du. \tag{29.2}$$

Proof. Substituting (29.2) into (29.1) and replacing u by uz, we find that, for $w > 0$,

$$\begin{aligned}
&\int_0^\infty e^{-u^2}\left\{\varphi\left(\frac{ui}{w}\right) + \varphi\left(-\frac{ui}{w}\right)\right\} du \\
&\qquad = \frac{2}{\pi}\int_0^\infty \int_0^\infty e^{-z^2(u^2+1)}\left\{F\left(\frac{w}{ui}\right) + F\left(-\frac{w}{ui}\right)\right\} z\, dz\, du \\
&\qquad = \frac{1}{\pi}\int_0^\infty \left\{F\left(-\frac{iw}{u}\right) + F\left(\frac{iw}{u}\right)\right\}\left(\int_0^\infty e^{-z(u^2+1)}\, dz\right) du \\
&\qquad = \frac{1}{\pi}\int_0^\infty \frac{F(-iw/u) + F(iw/u)}{u^2+1}\, du \\
&\qquad = \frac{1}{\pi}\int_{-\infty}^\infty \frac{F(iw/u)}{u^2+1}\, du. \qquad (29.3)
\end{aligned}$$

We now integrate $F(iw/u)/(u^2 + 1)$ over the positively oriented contour $C_{\varepsilon,R}$ consisting of the points $\Gamma_R \cup [-R, -\varepsilon] \cup \Gamma_\varepsilon \cup [\varepsilon, R]$, where $0 < \varepsilon < 1$, $R > 1$, $\Gamma_R = \{u\colon |u| = R, 0 \le \arg u \le \pi\}$, and $\Gamma_\varepsilon = \{u\colon |u| = \varepsilon, 0 \le \arg u \le \pi\}$. From our hypotheses on F and the residue theorem,



$$\int_{C_{\varepsilon,R}} \frac{F(iw/u)}{u^2+1}\, du = \pi F(w). \qquad (29.4)$$

Now, by our assumptions on the growth of F,

$$\int_{\Gamma_R} \frac{F(iw/u)}{u^2+1}\, du \ll \frac{R^{1-\delta}\pi R}{R^2-1} = o(1), \qquad (29.5)$$

as R tends to ∞, since $\delta > 0$, and

$$\int_{\Gamma_\varepsilon} \frac{F(iw/u)}{u^2+1}\, du \ll \frac{\varepsilon^{-\alpha}\pi\varepsilon}{1-\varepsilon^2} = o(1), \qquad (29.6)$$

as ε tends to 0, since $\alpha < 1$. Hence, letting R tend to ∞ and ε tend to 0 and employing (29.4)–(29.6), we conclude that

$$\int_{-\infty}^{\infty} \frac{F(iw/u)}{u^2+1}\, du = \pi F(w).$$

Using this in (29.3), we complete the proof.

As an example, let $F(z) = z^{-m}$, $z > 0$, $-1 < m < 1$. Then

$$\varphi(a) = \frac{2a^m}{\pi} \int_0^{\infty} e^{-z^2} z^{-m}\, dz = \frac{a^m}{\pi} \int_0^{\infty} e^{-u} u^{-(m+1)/2}\, du = \frac{a^m}{\pi} \Gamma\left(\frac{1-m}{2}\right).$$

For $z > 0$, the right side of (29.2) is then equal to

$$\begin{aligned} &\frac{\Gamma(\frac{1}{2}(1-m))}{\pi} \int_0^{\infty} e^{-u^2}\{(ui/z)^m + (-ui/z)^m\}\, du \\ &\qquad = \frac{\Gamma(\frac{1}{2}(1-m))\cos(\frac{1}{2}\pi m)}{\pi z^m} \int_0^{\infty} e^{-v} v^{(m-1)/2}\, dv \\ &\qquad = z^{-m}\pi^{-1}\cos(\tfrac{1}{2}\pi m)\Gamma(\tfrac{1}{2}(1-m))\Gamma(\tfrac{1}{2}(1+m)) \\ &\qquad = z^{-m}, \end{aligned}$$

as predicted by Entry 29.

We are grateful to R. J. Evans for suggesting a simplification in our original proof of Entry 29 and for providing the example above.

Entry 30 (p. 333). *If $n > 0$ and a is a positive integer, then*

$$\int_0^{\infty} \frac{\cos(nx)\, dx}{\prod_{k=0}^{a-1}\left\{1 + \frac{x^2}{(2k+1)^2}\right\}} = \frac{2\Gamma^2(a+\frac{1}{2})}{\Gamma(a+1)\Gamma(a)} \sum_{k=0}^{\infty} e^{-(2k+1)n} \frac{(1-a)_k}{(1+a)_k}.$$

PROOF. By using a partial fraction decomposition for a certain product of gamma functions, multiplying both sides by $\cos(2nx)$, and integrating termwise, Ramanujan [13], [23, p. 53] proved that

$$\int_0^{\infty} \frac{\cos(2nx)\, dx}{\prod_{k=0}^{a-1}\left\{1 + \frac{x^2}{(k+b)^2}\right\}} = \frac{\pi\Gamma(2b)\Gamma^2(b+a)}{\Gamma^2(b)\Gamma(a)\Gamma(2b+a)} \sum_{k=0}^{\infty} e^{-2(b+k)n} \frac{(2b)_k(1-a)_k}{k!\,(a+2b)_k}, \tag{30.1}$$

where $n > 0$, $b > 0$, and a is a positive integer. Let $b = \frac{1}{2}$ and replace $2x$ by x in (30.1). We therefore see that (30.1) then reduces to the required result.

Entry 31 (p. 334). *Let $m > 0$ and $0 < a < b + 1$. Then*

$$\int_0^\infty \prod_{k=0}^\infty \left\{\frac{1+\dfrac{x^2}{(b+k+1)^2}}{1+\dfrac{x^2}{(a+k)^2}}\right\} \cos(nx)\,dx$$
$$= \frac{\pi\Gamma(2a)\Gamma^2(b+1)}{\Gamma^2(a)\Gamma(b+a+1)\Gamma(b-a+1)} \sum_{k=0}^\infty e^{-(a+k)n} \frac{(a-b)_k(2a)_k}{(a+b+1)_k k!}.$$

With two trivial changes of parameters, Entry 31 is precisely the formula at the beginning of the second section of Ramanujan's paper [13], [23, p. 54]. The proof is similar to that of (30.1).

Entry 32 (p. 334). *Let $\varphi(x)$ be continuous on $[0, \infty)$ and suppose that $\lim_{x\to\infty} \varphi(x) =: \varphi(\infty)$ exists. Then*

$$\lim_{n\to 0} n \int_0^\infty x^{n-1}\varphi(x)\,dx = \varphi(0) - \varphi(\infty).$$

Proof. By Frullani's theorem, or Ramanujan's generalization thereof (Part I [2, p. 313]),

$$\begin{aligned}\lim_{n\to\infty} \int_0^\infty x^{n-1}\{\varphi(x) - \varphi(ex)\}\,dx &= \{\varphi(0) - \varphi(\infty)\}\log e \\ &= \varphi(0) - \varphi(\infty). \end{aligned} \tag{32.1}$$

Putting $ex = u$ in the second integral below, we find that

$$\begin{aligned}\int_0^\infty x^{n-1}\varphi(x)\,dx - \int_0^\infty x^{n-1}\varphi(ex)\,dx &= \int_0^\infty x^{n-1}\varphi(x)\,dx - e^{-n}\int_0^\infty u^{n-1}\varphi(u)\,du \\ &= (1 - e^{-n}) \int_0^\infty x^{n-1}\varphi(x)\,dx \\ &= \{n + O(n^2)\} \int_0^\infty x^{n-1}\varphi(x)\,dx,\end{aligned}$$

as n tends to 0. Hence,

$$\lim_{n\to 0} \int_0^\infty x^{n-1}\{\varphi(x) - \varphi(ex)\}\,dx = \lim_{n\to 0} n \int_0^\infty x^{n-1}\varphi(x)\,dx. \tag{32.2}$$

Using (32.2) in (32.1), we complete the proof.

As examples, we record that

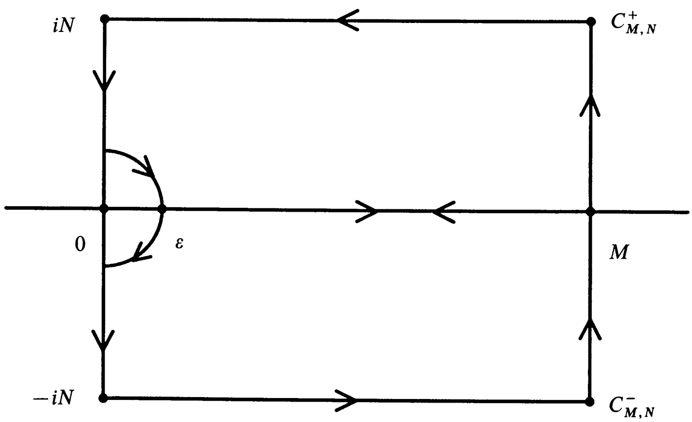
$$\lim_{n\to 0} n \int_0^\infty x^{n-1} \tanh x\,dx = -1$$

and

$$\lim_{n\to 0} n\int_0^\infty x^{n-1}R(x)\,dx = R(0) - R(\infty),$$

where $R(x)$ is any rational function such that the degree of the numerator does not exceed the degree of the denominator and such that $R(x)$ has no poles on $[0, \infty)$.

In the statements and proofs of Entries 33 and 34, we shall employ the following contours. For $M, N > 0$, let $C^+_{M,N}$ and $C^-_{M,N}$ denote the positively oriented "rectangles" with vertices at $(0, 0)$, $(M, 0)$, $(M, \pm N)$, and $(0, \pm N)$, respectively, with quarter circular indentations of radius $\varepsilon > 0$ about the origin.



Entry 33 (p. 334). *Let $\varphi(z)$ be analytic in the half-plane* Re $z \geq 0$. *Let $n > 0$. Assume that the integral of $(-iz)^{n-1}\varphi(z)$ over $[M, M + iN] \cup [M + iN, iN]$ and the integral of $(iz)^{n-1}\varphi(z)$ over $[M, M - iN] \cup [M - iN, -iN]$ tend to 0 as M and N tend to ∞. Then*

$$\int_0^\infty x^{n-1}\{\varphi(ix) + \varphi(-ix)\}\,dx = 2\cos(\pi n/2)\int_0^\infty x^{n-1}\varphi(x)\,dx. \tag{33.1}$$

PROOF. By Cauchy's theorem,

$$\int_{C^+_{M,N}} (-iz)^{n-1}\varphi(z)\,dz + \int_{C^-_{M,N}} (iz)^{n-1}\varphi(z)\,dz = 0. \tag{33.2}$$

Since $\varphi(z) = O(1)$ as z tends to 0 and since $n > 0$, it is easy to see that, as ε tends to 0, the integrals over the quarter circles of radius ε tend to 0 as ε tends to 0. Thus, letting ε tend to 0 and M and N tend to ∞ in (33.2), we deduce that

$$
\begin{aligned}
0 &= -\int_0^{i\infty} (-iz)^{n-1}\varphi(z)\,dz + \int_0^{\infty} (-iz)^{n-1}\varphi(z)\,dz \\
&\quad + \int_0^{-i\infty} (iz)^{n-1}\varphi(z)\,dz - \int_0^{\infty} (iz)^{n-1}\varphi(z)\,dz \\
&= -i\int_0^{\infty} x^{n-1}\{\varphi(ix) + \varphi(-ix)\}\,dx - 2i\sin\{\pi(n-1)/2\}\int_0^{\infty} x^{n-1}\varphi(x)\,dx.
\end{aligned}
\tag{33.3}
$$

Simplifying (33.3), we arrive at (33.1) to complete the proof.

Entry 34 (p. 334). *Let $\varphi(z)$ be subject to the same hypotheses as in Entry* 33. *Let $n > 0$. Then*

$$
\int_0^{\infty} x^{n-1}\{\varphi(ix) - \varphi(-ix)\}\,dx = -2i\sin(\pi n/2)\int_0^{\infty} x^{n-1}\varphi(x)\,dx.
$$

Proof. By Cauchy's theorem,

$$
\int_{C_{M,N}^{+}} (-iz)^{n-1}\varphi(z)\,dz - \int_{C_{M,N}^{-}} (iz)^{n-1}\varphi(z)\,dz = 0.
$$

The remainder of the proof proceeds in exactly the same fashion as for Entry 33, and so we forego the details.

As an example, let $\varphi(z) = e^{-z}$ and $0 < n < 1$. Then Entries 33 and 34 yield, respectively,

$$
\int_0^{\infty} x^{n-1}\cos x\,dx = \cos(\pi n/2)\int_0^{\infty} x^{n-1}e^{-x}\,dx = \cos(\pi n/2)\Gamma(n)
$$

and

$$
\int_0^{\infty} x^{n-1}\sin x\,dx = \sin(\pi n/2)\int_0^{\infty} x^{n-1}e^{-x}\,dx = \sin(\pi n/2)\Gamma(n),
$$

both of which are well known (Gradshteyn and Ryzhik [1, p. 421, formula 9; p. 420, formula 4]).

Ramanujan made a sign error in his statement of Entry 34.

The hypotheses on $\varphi(z)$ in Entries 33 and 34 could have been made more explicit. In view of the limited applications, however, we have offered more general, less explicit hypotheses.

Entry 35 (p. 361). *If, for* Re $a > 0$ *and* $r > -1$,

$$
\int_0^{\infty} \varphi(z)e^{-az}\,dz = a^{-r-1},
$$

then, for $z > 0$,

$$\varphi(z) = \frac{z^r}{\Gamma(r+1)}.$$

Proof. Entry 35 gives the inverse Laplace transform of a^{-r-1}, i.e., for $c > 0$ and $r > -1$,

$$\varphi(z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} s^{-r-1}e^{sz}\,ds = \frac{z^r}{\Gamma(r+1)},$$

which is a well-known integral representation for $1/\Gamma(r+1)$ due to Laplace (Copson [1, p. 231]).

On page 365, Ramanujan offers several results on the dilogarithm $\mathrm{Li}_2(z)$ defined by

$$\begin{aligned}\mathrm{Li}_2(z) &= -\int_0^z \frac{\mathrm{Log}(1-w)}{w}\,dw, \qquad z \in \mathbb{C}, \\ &= \sum_{k=1}^{\infty} \frac{z^k}{k^2}, \qquad |z| \le 1,\end{aligned} \tag{36.1}$$

where the principal branch of $\log(1-w)$ is chosen. Thus, $\mathrm{Li}_2(z)$ is analytic on $\mathbb{C} - [1, \infty)$. Although only three of the results involving $\mathrm{Li}_2(z)$ on page 365 pertain directly to integrals, as indicated in the introduction, we shall prove all of these results in this chapter.

Entry 36 (p. 365). *Let* $n \in \mathbb{C} - [-1, 0]$. *Then*

$$\mathrm{Li}_2\left(-\frac{1}{n}\right) + \mathrm{Li}_2\left(\frac{1}{n+1}\right) = -\tfrac{1}{2}\,\mathrm{Log}^2\left(1 + \frac{1}{n}\right),$$

where the principal value of $\log z$ *is taken.*

Proof. Entry 36 is simply a version of a well-known transformation formula for $\mathrm{Li}_2(z)$, which is also Entry 6(i) in Chapter 9 (Part I [2, p. 247]), viz., for $z \in \mathbb{C} - (-\infty, 0]$,

$$\mathrm{Li}_2(1-z) + \mathrm{Li}_2\left(1 - \frac{1}{z}\right) = -\tfrac{1}{2}\,\mathrm{Log}^2 z. \tag{36.2}$$

Set $z = 1 + 1/n$ to complete the proof.

Entry 37 (p. 365). *If* p, q, r, *and* s *are real with* $p + qx \not\equiv 0$ *and* $r + sx \not\equiv 0$, *then*

$$\int \frac{\log(p+qx)}{r+sx}\,dx$$

can be calculated in terms of Li_2 *and elementary functions.*

Proof. Let $u = r + sx$, and put $a = p - qr/s$ and $b = q/s$. Then

$$\int \frac{\log(p+qx)}{r+sx}\,dx = \frac{1}{s}\int \frac{\log(a+bu)}{u}\,du$$
$$= \frac{1}{s}\log a \log u + \frac{1}{s}\int \frac{\log(1+bu/a)}{u}\,du$$
$$= \frac{1}{s}\log a \log u + \frac{1}{s}\int \frac{\log(1-t)}{t}\,dt.$$

Recalling the integral representation for $\mathrm{Li}_2(z)$ given in (36.1), we complete the verification of Ramanujan's claim.

Entry 38 (p. 365). *If $p, q, r,$ and s are real with $p + qx \not\equiv 0$, and $r + sx \not\equiv 0$, then*

$$\int \log(p+qx)\log(r+sx)\,dx$$

can be evaluated in terms of Li_2 *and elementary functions.*

Proof. Letting $u = r + sx$, $a = p - qr/s$, and $b = q/s$, and integrating by parts, we find that

$$\int \log(p+qx)\log(r+sx)\,dx$$
$$= \frac{1}{s}\int \log(a+bu)\log u\,du$$
$$= \frac{1}{s}\left(\log(a+bu)\{u\log u - u\} - b\int \frac{u\log u - u}{a+bu}\,du\right).$$

Now,

$$b\int \frac{u\log u - u}{a+bu}\,du = \int \frac{(a+bu)\log u - a\log u - (a+bu) + a}{a+bu}\,du$$
$$= \int \log u\,du - a\int \frac{\log u}{a+bu}\,du - \int du + a\int \frac{du}{a+bu}.$$

The first, third, and fourth integrals on the far right side are elementary, while, by Entry 37, the second integral can be evaluated in terms of Li_2 and elementary functions. This completes the proof.

Ramanujan next asserts that five linear combinations of dilogarithms can be evaluated, but he does not record their values.

Entry 39 (p. 365). *We have*

$$\mathrm{Li}_2(\tfrac{1}{3}) - \tfrac{1}{6}\,\mathrm{Li}_2(\tfrac{1}{9}) = \frac{\pi^2}{18} - \tfrac{1}{6}\,\mathrm{Log}^2 3, \tag{39.1}$$

$$\mathrm{Li}_2(-\tfrac{1}{2}) + \tfrac{1}{6}\,\mathrm{Li}_2(\tfrac{1}{9}) = -\frac{\pi^2}{18} + \mathrm{Log}\,2\,\mathrm{Log}\,3 - \tfrac{1}{2}\,\mathrm{Log}^2 2 - \tfrac{1}{3}\,\mathrm{Log}^2 3, \tag{39.2}$$

$$\mathrm{Li}_2(\tfrac{1}{4}) + \tfrac{1}{3}\,\mathrm{Li}_2(\tfrac{1}{9}) = \frac{\pi^2}{18} + 2\,\mathrm{Log}\,2\,\mathrm{Log}\,3 - 2\,\mathrm{Log}^2 2 - \tfrac{2}{3}\,\mathrm{Log}^2 3, \tag{39.3}$$

$$\mathrm{Li}_2(-\tfrac{1}{3}) - \tfrac{1}{3}\,\mathrm{Li}_2(\tfrac{1}{9}) = -\frac{\pi^2}{18} + \tfrac{1}{6}\,\mathrm{Log}^2 3, \tag{39.4}$$

and

$$\mathrm{Li}_2(-\tfrac{1}{8}) + \mathrm{Li}_2(\tfrac{1}{9}) = -\tfrac{1}{2}\,\mathrm{Log}^2 \tfrac{9}{8}. \tag{39.5}$$

Proof. We shall repeatedly use (36.2) as well as another classical functional equation

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(-z) = \tfrac{1}{2}\,\mathrm{Li}_2(z^2), \qquad |z| \le 1, \tag{39.6}$$

which is Entry 6(iv) in Chapter 9 (Part I [2, p. 247]).

First, using (39.6), (36.2), and the value

$$\mathrm{Li}_2(\tfrac{1}{2}) = \frac{\pi^2}{12} - \tfrac{1}{2}\,\mathrm{Log}^2 2,$$

found in Section 6 of Chapter 9 (Part I [2, p. 248]), we find that

$$\begin{aligned}
\mathrm{Li}_2(\tfrac{1}{3}) - \tfrac{1}{6}\,\mathrm{Li}_2(\tfrac{1}{9}) &= \mathrm{Li}_2(\tfrac{1}{3}) - \tfrac{1}{3}\{\mathrm{Li}_2(\tfrac{1}{3}) + \mathrm{Li}_2(-\tfrac{1}{3})\} \\
&= \tfrac{2}{3}\,\mathrm{Li}_2(\tfrac{1}{3}) - \tfrac{1}{3}\,\mathrm{Li}_2(-\tfrac{1}{3}) \\
&= \tfrac{2}{3}\,\mathrm{Li}_2(\tfrac{1}{3}) + \tfrac{1}{3}\{\mathrm{Li}_2(\tfrac{1}{4}) + \tfrac{1}{2}\,\mathrm{Log}^2 \tfrac{4}{3}\} \\
&= \tfrac{2}{3}\,\mathrm{Li}_2(\tfrac{1}{3}) + \tfrac{2}{3}\{\mathrm{Li}_2(\tfrac{1}{2}) + \mathrm{Li}_2(-\tfrac{1}{2})\} + \tfrac{1}{6}\,\mathrm{Log}^2 \tfrac{4}{3} \\
&= \tfrac{2}{3}\{\mathrm{Li}_2(\tfrac{1}{3}) + \mathrm{Li}_2(\tfrac{1}{2})\} - \tfrac{2}{3}\{\mathrm{Li}_2(\tfrac{1}{3}) + \tfrac{1}{2}\,\mathrm{Log}^2 \tfrac{3}{2}\} + \tfrac{1}{6}\,\mathrm{Log}^2 \tfrac{4}{3} \\
&= \tfrac{2}{3}\left(\frac{\pi^2}{12} - \tfrac{1}{2}\,\mathrm{Log}^2 2\right) - \tfrac{1}{3}\,\mathrm{Log}^2 \tfrac{3}{2} + \tfrac{1}{6}\,\mathrm{Log}^2 \tfrac{4}{3} \\
&= \frac{\pi^2}{18} - \tfrac{1}{6}\,\mathrm{Log}^2 3,
\end{aligned} \tag{39.7}$$

which completes the proof of (39.1).

Using the same three formulas utilized above, we deduce that

$$\begin{aligned}
\mathrm{Li}_2(-\tfrac{1}{2}) + \tfrac{1}{6}\mathrm{Li}_2(\tfrac{1}{9}) &= \mathrm{Li}_2(-\tfrac{1}{2}) + \tfrac{1}{3}\{\mathrm{Li}_2(\tfrac{1}{3}) + \mathrm{Li}_2(-\tfrac{1}{3})\}\\
&= \mathrm{Li}_2(-\tfrac{1}{2}) + \tfrac{1}{3}\mathrm{Li}_2(\tfrac{1}{3}) - \tfrac{1}{3}\{\mathrm{Li}_2(\tfrac{1}{4}) + \tfrac{1}{2}\mathrm{Log}^2\tfrac{4}{3}\}\\
&= \mathrm{Li}_2(-\tfrac{1}{2}) + \tfrac{1}{3}\mathrm{Li}_2(\tfrac{1}{3}) - \tfrac{2}{3}\{\mathrm{Li}_2(\tfrac{1}{2}) + \mathrm{Li}_2(-\tfrac{1}{2})\} - \tfrac{1}{6}\mathrm{Log}^2\tfrac{4}{3}\\
&= -\tfrac{2}{3}\mathrm{Li}_2(\tfrac{1}{2}) + \tfrac{1}{3}\{\mathrm{Li}_2(-\tfrac{1}{2}) + \mathrm{Li}_2(\tfrac{1}{3})\} - \tfrac{1}{6}\mathrm{Log}^2\tfrac{4}{3}\\
&= -\tfrac{2}{3}\left(\frac{\pi^2}{12} - \tfrac{1}{2}\mathrm{Log}^2 2\right) - \tfrac{1}{6}\mathrm{Log}^2\tfrac{3}{2} - \tfrac{1}{6}\mathrm{Log}^2\tfrac{4}{3}\\
&= -\frac{\pi^2}{18} + \mathrm{Log}\,2\,\mathrm{Log}\,3 - \tfrac{1}{2}\mathrm{Log}^2 2 - \tfrac{1}{3}\mathrm{Log}^2 3,
\end{aligned}$$

and the proof of (39.2) is accomplished.

Employing (39.6), (36.2), and the third and last lines of (39.7), we find that

$$\begin{aligned}
\mathrm{Li}_2(\tfrac{1}{4}) + \tfrac{1}{3}\mathrm{Li}_2(\tfrac{1}{9}) &= \mathrm{Li}_2(\tfrac{1}{4}) + \tfrac{2}{3}\{\mathrm{Li}_2(\tfrac{1}{3}) + \mathrm{Li}_2(-\tfrac{1}{3})\}\\
&= \mathrm{Li}_2(\tfrac{1}{4}) + \tfrac{2}{3}\mathrm{Li}_2(\tfrac{1}{3}) - \tfrac{2}{3}\{\mathrm{Li}_2(\tfrac{1}{4}) + \tfrac{1}{2}\mathrm{Log}^2\tfrac{4}{3}\}\\
&= \tfrac{1}{3}\mathrm{Li}_2(\tfrac{1}{4}) + \tfrac{2}{3}\mathrm{Li}_2(\tfrac{1}{3}) + \tfrac{1}{6}\mathrm{Log}^2\tfrac{4}{3} - \tfrac{1}{2}\mathrm{Log}^2\tfrac{4}{3}\\
&= \frac{\pi^2}{18} - \tfrac{1}{6}\mathrm{Log}^2 3 - \tfrac{1}{2}\mathrm{Log}^2\tfrac{4}{3},
\end{aligned}$$

and (39.3) readily follows.

Employing exactly the same equalities as in the proof of (39.3), we find that

$$\begin{aligned}
\mathrm{Li}_2(-\tfrac{1}{3}) - \tfrac{1}{3}\mathrm{Li}_2(\tfrac{1}{9}) &= \mathrm{Li}_2(-\tfrac{1}{3}) - \tfrac{2}{3}\{\mathrm{Li}_2(\tfrac{1}{3}) + \mathrm{Li}_2(-\tfrac{1}{3})\}\\
&= -\tfrac{2}{3}\mathrm{Li}_2(\tfrac{1}{3}) - \tfrac{1}{3}\{\mathrm{Li}_2(\tfrac{1}{4}) + \tfrac{1}{2}\mathrm{Log}^2\tfrac{4}{3}\}\\
&= -\frac{\pi^2}{18} + \tfrac{1}{6}\mathrm{Log}^2 3,
\end{aligned}$$

which completes the proof of (39.4).

Lastly, (39.5) follows from (36.2) by setting $z = \frac{9}{8}$.

Entry 40 (p. 365). *We have*

$$I := \int_0^1 \frac{\log\left(\dfrac{1+\sqrt{1+4x}}{2}\right)}{x}\,dx = \frac{\pi^2}{15}.$$

PROOF. Let $u = (1 + \sqrt{1 + 4x})/2$, so that $x = u^2 - u$. Then integrating by parts, setting $u = 1/v$, using (36.1), and employing the value (Part I [2, p. 248])

$$\mathrm{Li}_2\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{10} - \mathrm{Log}^2\left(\frac{\sqrt{5}-1}{2}\right),$$

we find that

$$
\begin{aligned}
I &= \int_1^{(\sqrt{5}+1)/2} \frac{\log u}{u^2 - u}(2u-1)\,du \\
&= -\int_1^{(\sqrt{5}+1)/2} \frac{\log(u^2-u)}{u}\,du \\
&= -\int_1^{(\sqrt{5}+1)/2} \left(\frac{\log u}{u} + \frac{\log(u-1)}{u}\right) du \\
&= -\tfrac{1}{2}\log^2\left(\frac{\sqrt{5}+1}{2}\right) + \int_1^{(\sqrt{5}-1)/2} \frac{\log(1-v) - \log v}{v}\,dv \\
&= -\tfrac{1}{2}\log^2\left(\frac{\sqrt{5}+1}{2}\right) - \mathrm{Li}_2\left(\frac{\sqrt{5}-1}{2}\right) + \mathrm{Li}_2(1) - \tfrac{1}{2}\log^2\left(\frac{\sqrt{5}-1}{2}\right) \\
&= -\frac{\pi^2}{10} + \frac{\pi^2}{6} = \frac{\pi^2}{15}.
\end{aligned}
$$

Hence, the proof is complete.

In fact, the integral I of Entry 40 is precisely the integral $\varphi(2)$ of Entry 41 below.

Entry 41 (p. 373). *Let $n \geq 0$, put $v = u^n - u^{n-1}$, and define*

$$\varphi(n) = \int_0^1 \frac{\log u}{v}\,dv.$$

Then

$$\varphi(0) = \frac{\pi^2}{6}, \tag{41.1}$$

$$\varphi(1) = \frac{\pi^2}{12}, \tag{41.2}$$

and

$$\varphi(2) = \frac{\pi^2}{15}. \tag{41.3}$$

Furthermore, for $n > 0$,

$$\varphi(n) + \varphi\left(\frac{1}{n}\right) = \frac{\pi^2}{6}. \tag{41.4}$$

The functional equation (41.4) is a truly elegant and beautiful result. We wonder how Ramanujan discovered it. Ramanujan submitted (41.4) as a problem to the *Journal of the Indian Mathematical Society*, and one solution was received [20], [23, p. 334]. In fact, (41.4) is a special case of the following much more general theorem established by the author and R. J. Evans [1].

Theorem 1. *Let g be a strictly increasing, differentiable function on $[0, \infty)$ with $g(0) = 1$ and $g(\infty) = \infty$. For $n > 0$ and $t \geq 0$, define*

$$v(t) = \frac{g(t)^n}{g(t^{-1})}. \tag{41.5}$$

Suppose that

$$\varphi(n) := \int_0^1 \log g(t) \frac{dv}{v} \tag{41.6}$$

converges. Then

$$\varphi(n) + \varphi\left(\frac{1}{n}\right) = 2\varphi(1). \tag{41.7}$$

Let $g(t) = 1 + t$. Observing that $v(t) = t(1 + t)^{n-1}$, setting $u = 1 + t$, and using (41.2), which is proved below, we find that Theorem 1 reduces to (41.4).

Further examples for which Theorem 1 is valid can be obtained from setting $g(t) = \log(t + e)$ and $g(t) = 1 + a_1 t + a_2 t^2 + \cdots + a_k t^k$, where $k \geq 1$ and $a_j > 0$, $1 \leq j \leq k$.

Proof of Theorem 1. Since $g(\infty) = \infty$, it follows from (41.5) that $v(0) = 0$. Also, $v(w) = 1$, where w is defined by

$$n = \frac{\log g(w^{-1})}{\log g(w)}. \tag{41.8}$$

Since $g(0) = 1$ and g is increasing, we see from (41.8) that each $n > 0$ determines a unique $w > 0$, and conversely each $w > 0$ determines a unique $n > 0$. We also see from (41.8) that interchanging n and n^{-1} corresponds to interchanging w and w^{-1}. Define a real-valued function F on $[0, \infty)$ by

$$F(w) = \varphi(n). \tag{41.9}$$

Then (41.7) is equivalent to

$$F(w) + F(w^{-1}) = 2F(1). \tag{41.10}$$

Differentiating (41.10), we find that

$$F'(w) = w^{-2} F'(w^{-1}). \tag{41.11}$$

Since (41.10) is clearly valid for $w = 1$, it then suffices to prove (41.11).

By (41.5),

$$\frac{dv}{v} = nd(\log g(t)) - d(\log g(t^{-1})). \tag{41.12}$$

Therefore by (41.9), (41.6), (41.8), and (41.12),

$$\begin{aligned} F(w) &= \frac{\log g(w^{-1})}{\log g(w)} \int_{t=0}^{w} \log g(t)\, d(\log g(t)) - \int_{t=0}^{w} \log g(t)\, d(\log g(t^{-1})) \\ &= \tfrac{1}{2} \log g(w^{-1}) \log g(w) - \int_{t=0}^{w} \log g(t)\, d(\log g(t^{-1})). \end{aligned} \tag{41.13}$$

Thus,

$$\begin{aligned} F'(w) &= \frac{1}{2}\left\{\log g(w^{-1}) \frac{d}{dw} \log g(w) + \log g(w) \frac{d}{dw} \log g(w^{-1})\right\} \\ &\quad - \log g(w) \frac{d}{dw} \log g(w^{-1}) \\ &= \frac{1}{2}\left\{\log g(w^{-1}) \frac{d}{dw} \log g(w) - \log g(w) \frac{d}{dw} \log g(w^{-1})\right\}. \end{aligned}$$

The equality (41.11) now easily follows, and so the proof of Theorem 1 is complete.

Setting $g(t) = 1 + t$ in (41.13), we find that

$$\begin{aligned} F(w) &= \tfrac{1}{2}\{\log(1 + w) - \log w\} \log(1 + w) - \int_0^w \log(1 + t)\left\{\frac{1}{1 + t} - \frac{1}{t}\right\} dt \\ &= -\tfrac{1}{2} \log w \log(1 + w) + \int_0^w \frac{\log(1 + t)}{t}\, dt. \end{aligned} \tag{41.14}$$

Expanding $\log(1 + t)$ in its Maclaurin series and integrating termwise, we deduce that

$$F(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

This proves (41.2).

By (36.1), we may rewrite (41.14) in the form

$$F(w) = -\tfrac{1}{2} \log w \log(1 + w) - \mathrm{Li}_2(-w). \tag{41.15}$$

Hence, we easily deduce from Theorem 1 (or (41.4)) that

$$\frac{\pi^2}{6} = F(w) + F(w^{-1}) = -\tfrac{1}{2} \log^2 w - \mathrm{Li}_2(-w) - \mathrm{Li}_2(-w^{-1}),$$

which is a well-known functional equation for the dilogarithm (Part I [2, p. 247, Entry 6(ii)]; Lewin [1, p. 4]). Thus, Theorem 1 can be regarded as a vast generalization of this classical functional equation. One might then ask how much of the general theory of the dilogarithm can be generalized for the more general function $\varphi(n)$ defined by (41.6). Ramanujan, in fact, considered some different generalizations of $\mathrm{Li}_2(z)$ (Ramanujan [22, Chap. 9], Part I [2, pp. 249–260]).

Observe that if we let n tend to ∞ in (41.4), we deduce (41.1), since $\varphi(\infty) = 0$. Of course, it is not difficult to prove (41.1) directly from the definition of $\varphi(0)$.

Finally, we prove (41.3). By (41.15) and (41.9) with $n = 2$ and $w = (\sqrt{5} - 1)/2$, we have

$$\varphi(2) = \tfrac{1}{2}\log^2 w - \mathrm{Li}_2(-w) = \frac{\pi^2}{15},$$

where we have used a well-known value for $\mathrm{Li}_2(-w)$ (Lewin [1, p. 7]).

As mentioned in the introduction, the words "contour integration" appear beside the next entry in the notebooks.

Entry 42 (p. 391). *For $n > 0$,*

$$\int_0^\infty \frac{\cos(nx)}{x^2+1}\log x\,dx + \frac{\pi}{2}\int_0^\infty \frac{\sin(nx)}{x^2+1}\,dx = 0. \tag{42.1}$$

Proof. Define a branch of $\log z$ by $-\pi/2 < \arg z \leq 3\pi/2$. Let

$$f(z) = \frac{e^{inz}\log z}{z^2+1}.$$

We shall integrate $f(z)$ over the same indented contour $C_{\varepsilon,R}$ that we used in Section 29, where again $0 < \varepsilon < 1$ and $R > 1$. By the residue theorem,

$$\frac{1}{2\pi i}\int_{C_{\varepsilon,R}} f(z)\,dz = \frac{e^{-n}(i\pi/2)}{2i} = \frac{e^{-n}\pi}{4}. \tag{42.2}$$

It is easy to see that

$$\int_{\Gamma_\varepsilon} f(z)\,dz = o(1), \tag{42.3}$$

as ε tends to 0, and that

$$\int_{\Gamma_R} f(z)\,dz = o(1), \tag{42.4}$$

as R tends to ∞. Thus, letting ε tend to 0 and R tend to ∞ and using (42.2)–(42.4), we find that

$$\frac{e^{-n}\pi^2 i}{2} = \int_{-\infty}^{0} \frac{e^{inx}(\log|x| + i\pi)}{x^2+1}\,dx + \int_0^\infty \frac{e^{inx}\log x}{x^2+1}\,dx$$
$$= \int_0^\infty \frac{\{\cos(nx) - i\sin(nx)\}(\log x + i\pi)}{x^2+1}\,dx$$
$$+ \int_0^\infty \frac{\{\cos(nx) + i\sin(nx)\}\log x}{x^2+1}\,dx. \tag{42.5}$$

Equating real parts on both sides of (42.5), we deduce (42.1).

Note that by equating imaginary parts on both sides of (42.5), we deduce the well-known evaluation

$$\int_0^\infty \frac{\cos(nx)}{x^2+1}\,dx = \frac{\pi e^{-n}}{2}.$$

Entry 43 (p. 391). *If* $n > 0$, *then*

$$\int_0^\infty e^{-x}x^{n-1}\,dx = \Gamma(n), \tag{43.1}$$

$$\int_{-\infty}^\infty \frac{n^{x-1}}{\Gamma(x)}\,dx = e^n, \tag{43.2}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(z)n^{-z}\,dz = e^{-n}, \qquad c > 0, \tag{43.3}$$

and

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^x x^{-n}\,dx = \frac{1}{\Gamma(n)}, \qquad c > 0. \tag{43.4}$$

Of course, (43.1) is Euler's integral representation of $\Gamma(n)$. The integral in (43.2) diverges; Ramanujan also stated (43.2) in Chapter 13 (Part II [4, p. 227]). Formula (43.3) gives the well-known inverse Mellin transform of $\Gamma(z)$ (Copson [1, p. 231]). Formula (43.4), offers the inverse Laplace transform of x^{-n}, which we previously examined in Entry 35.

Entry 44 (p. 391). *If*

$$\psi(s) = \int_0^\infty t^{s-1}\varphi(t)\,dt$$

converges absolutely on the line Re $s = c$, *and if* $\varphi(t)$ *is a continuous function of bounded variation in a neighborhood of* $t = x$, *with* $x > 0$, *then*

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \psi(s)x^{-s}\,ds = \varphi(x).$$

Entry 44 is the classical inversion theorem for Mellin transforms; the conditions for its validity are taken from Widder's book [1, pp. 246–247].

It seems unlikely that any pair φ, ψ of functions can satisfy the following inversion formula claimed by Ramanujan.

Entry 45 (p. 391). *Under certain conditions, the equality*

$$\int_{-\infty}^{\infty} n^{x-1}\varphi(x)\,dx = \psi(n) \tag{45.1}$$

implies that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-n}\psi(x)\,dx = \varphi(n). \tag{45.2}$$

It seems plausible that Ramanujan argued in the following way. Assume that (45.1) holds. Letting $x = \log u$ and putting $\varphi(\log u) = \varphi^*(u)$, we find that

$$\begin{aligned}\psi(n) &= \int_0^\infty n^{\log u - 1}\,\frac{\varphi(\log u)}{u}\,du\\ &= \frac{1}{n}\int_0^\infty u^{\log n - 1}\varphi^*(u)\,du.\end{aligned}$$

Letting $m = \log n$ and setting $\psi^*(m) = e^m\psi(e^m)$, we see that

$$\int_0^\infty u^{m-1}\varphi^*(u)\,du = e^m\psi(e^m) = \psi^*(m).$$

Let $N = \log\log n$. Invoke Mellin's inversion formula in Entry 44 to deduce that

$$\begin{aligned}\varphi(N) = \varphi(\log\log n) = \varphi(\log m) = \varphi^*(m) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} m^{-z}\psi^*(z)\,dz\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (\log n)^{-z}e^z\psi(e^z)\,dz\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{(1-N)z}\psi(e^z)\,dz. \qquad (45.3)\end{aligned}$$

Up to this point, we could certainly prescribe hypotheses on φ and ψ to legitimize the foregoing manipulations. Ramanujan now evidently makes a change of variable $e^z = s = \sigma + it$, where σ and t are real. It is here that Ramanujan's lack of knowledge of the theory of functions of a complex variable handicaps him. The change of variable must be justified. Ramanujan evidently assumes that the path of integration $\operatorname{Re} z = c$ is transformed into another vertical line in the complex plane. In fact, the transformation $s = e^z$ maps the line $\operatorname{Re} z = c$ onto infinitely many copies of the circle C defined by $\sigma^2 + t^2 = e^{2c}$. Hence, from (45.3), formally and nonrigorously,

$$\varphi(N) = \frac{1}{2\pi i}\int_C s^{-N}\psi(s)\,ds.$$

Thus, we conclude our argument showing how Ramanujan probably deduced (45.2).

It seems likely that Ramanujan deduced the erroneous formula (43.2) from (43.4) by applying the converse of Entry 45.

Entry 46 (p. 391). *Let φ be absolutely integrable on $[0, R]$ for every positive number R. Suppose that*

$$\int_0^\infty e^{-sx}\varphi(x)\,dx = \psi(s)$$

converges absolutely for $\operatorname{Re} s = c$. *If $\varphi(t)$ is a continuous function of bounded variation in a neighborhood of $t = x \geq 0$, then*

$$\varphi(x) = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} e^{xs}\psi(s)\,ds.$$

Entry 46 gives the complex inversion formula for Laplace transforms (Widder [1, p. 66]). The next entry offers the inversion formula for the bilateral Laplace transform (Widder [1, p. 241]).

Entry 47 (p. 391). *Let φ be absolutely integrable on every finite interval. Suppose that*

$$\psi(s) = \int_{-\infty}^\infty e^{-sx}\varphi(x)\,dx$$

converges absolutely on the line $\operatorname{Re} s = c$ *and that $\varphi(t)$ is a continuous function of bounded variation on some neighborhood of $t = x$. Then*

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} e^{xs}\psi(s)\,ds = \varphi(x).$$

The next result is unreadable in the notebooks [22], probably because the original ink faded. From the part that is visible, we are confident that Ramanujan stated his "Master Theorem." We refer to our book [2, pp. 298–299] for conditions guaranteeing its validity.

Entry 48 (p. 391). *We have*

$$\int_0^\infty x^{n-1} \sum_{k=0}^\infty \frac{\varphi(k)(-x)^k}{k!}\, dx = \Gamma(n)\varphi(-n).$$

Entry 49 (p. 391). *Formally,*

$$\int_{-\infty}^\infty \frac{\varphi(x)}{\Gamma(x+1)}\, dx = \sum_{n=0}^\infty \frac{\varphi(n)}{n!}.$$

This formula is discussed in Part II [4, p. 226].

CHAPTER 29

Special Functions

In this chapter, we collect together those results in the unorganized portions of the second and third notebooks that pertain to special functions. The first ten entries concern the gamma function. All ten entries are either known results or can easily be derived from standard theorems on the gamma function. The next four theorems arise from the theory of Bessel functions. These four theorems also are either classical or can be simply deduced from standard results on Bessel functions. The last section of the chapter is devoted to hypergeometric functions. By far, the most interesting result is contained in Section 15. Here Ramanujan offers a tantalizingly incomplete statement about a class of Saalschützian hypergeometric series. D. Bradley [1] has provided what is probably the best theorem that can be deduced from Ramanujan's enigmatic statement. In particular, a large, new class of Saalschützian series is summed in closed form. We complete this chapter by listing Ramanujan's series for $1/\pi$, arising from certain hypergeometric functions.

Throughout the chapter, we use the familiar notation

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

for each integer n.

The Gamma Function

Entries 1, 2 and 6 below have also been established by D. Somasundaram [6].

Entry 1 (p. 279). *For* $\operatorname{Re}(n) > 0$ *and* x *real,*

$$\tfrac{1}{2}\log\left(2\pi\sqrt{n^2+x^2}\prod_{k=1}^{\infty}\left(1+\frac{x^2}{(n+k)^2}\right)\right)$$
$$= -\frac{n}{2}\log(n^2+x^2) + \log\Gamma(n+1) + x\arctan\frac{x}{n}$$
$$+ n - \int_0^{\infty}\frac{\arctan\left(\dfrac{2nz}{n^2+x^2-z^2}\right)}{e^{2\pi z}-1}\,dz,$$

where the principal branches of $\log z$ *and* $\arctan z$ *are chosen.*

Proof. Recall Binet's formula for $\log\Gamma(a)$ (Whittaker and Watson [1, p. 251]). If $\operatorname{Re}(a) > 0$,

$$\log\Gamma(a) = (a-\tfrac{1}{2})\log a - a + \tfrac{1}{2}\log(2\pi) + 2\int_0^{\infty}\frac{\arctan(z/a)}{e^{2\pi z}-1}\,dz. \tag{1.1}$$

We set $a = n + xi$ and $a = n - xi$ in turn in (1.1) and then add the two formulas. In doing this, we observe that

$$\arctan\left(\frac{z}{n+xi}\right) + \arctan\left(\frac{z}{n-xi}\right) = \arctan\left(\frac{2nz}{n^2+x^2-z^2}\right)$$

and

$$(n+xi-\tfrac{1}{2})\log(n+xi) + (n-xi-\tfrac{1}{2})\log(n-xi)$$
$$= \tfrac{1}{2}(2n-1)\log(n^2+x^2) - 2x\arctan\frac{x}{n}.$$

Thus,

$$-\tfrac{1}{2}\log\{\Gamma(n+xi)\Gamma(n-xi)\} = \left(\frac{1}{4}-\frac{n}{2}\right)\log(n^2+x^2) + x\arctan\frac{x}{n}$$
$$+ n + \log(2\pi) - \int_0^{\infty}\frac{\arctan\left(\dfrac{2nz}{n^2+x^2-z^2}\right)}{e^{2\pi z}-1}\,dz. \tag{1.2}$$

Next, from Euler's product formula for the gamma function (Whittaker and Watson [1, p. 237]),

$$-\tfrac{1}{2}\log\{\Gamma(n+xi)\Gamma(n-xi)\}$$

$$= -\tfrac{1}{2}\lim_{N\to\infty}\log$$

$$\times\left(\frac{(N!)^2 N^{n+xi}N^{n-xi}}{(n+xi)(n+xi+1)\cdots(n+xi+N)(n-xi)(n-xi+1)\cdots(n-xi+N)}\right)$$

$$= \tfrac{1}{2}\lim_{N\to\infty}\log$$

$$\times\left((n^2+x^2)\left(1+\frac{x^2}{(n+1)^2}\right)\cdots\left(1+\frac{x^2}{(n+N)^2}\right)\frac{(n+1)^2(n+2)^2\cdots(n+N)^2}{(N!)^2N^{2n}}\right)$$

$$= \tfrac{1}{2}\log\left((n^2+x^2)\prod_{k=1}^{\infty}\left(1+\frac{x^2}{(n+k)^2}\right)\right) - \log\Gamma(n+1). \tag{1.3}$$

Putting (1.3) in (1.2) and rearranging, we complete the proof.

Binet's formula (1.1), when a is a positive integer, is given by Ramanujan in Section 17 of Chapter 13 (Part II [4, p. 221]). The general formula (1.1) was employed by Ramanujan in his paper [11], [23, p. 51], wherein formulas similar to Entry 1 are derived.

Ramanujan's formulation of Entry 1 omits the factor x in front of $\arctan(x/n)$.

Entry 2 (p. 279). *We have*

$$\frac{\Gamma^2(n+1)}{\Gamma(n+xi+1)\Gamma(n-xi+1)} = \prod_{k=1}^{\infty}\left(1+\frac{x^2}{(n+k)^2}\right).$$

Entry 2 is equivalent to (1.3) and so was proved above.

Entry 3 (Formula (1), p. 287). *Let*

$$\varphi(m,n) = \prod_{k=1}^{\infty}\left\{1+\left(\frac{m+n}{k+m}\right)^3\right\}.$$

Then

$$\varphi(m,n)\varphi(n,m)$$

$$= \frac{\Gamma^3(m+1)\Gamma^3(n+1)}{\Gamma(2m+n+1)\Gamma(2n+m+1)}\,\frac{\cosh\{\pi(m+n)\sqrt{3}\} - \cos\{\pi(m-n)\}}{2\pi^2(m^2+mn+n^2)}.$$

Entry 4 (Formula (2), p. 287). *We have*

$$\prod_{k=1}^{\infty}\left(1+\left(\frac{n}{k}\right)^3\right)\prod_{k=1}^{\infty}\left(1+3\left(\frac{n}{n+2k}\right)^2\right) = \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}(n+1))}\,\frac{\cosh(\pi n\sqrt{3}) - \cos(\pi n)}{2^{n+2}\pi^{3/2}n}.$$

Entries 3 and 4 were proved by Ramanujan in his paper [11], [23, pp. 50, 51].

Entry 5 (Formula (3), p. 287). *Let $n > 0$. As n tends to ∞,*

$$\log\left(\prod_{k=1}^{\infty}\left(1+\frac{n^3}{k^3}\right)\right)+\tfrac{3}{2}\log(2\pi n)-\log(2\cosh(\pi n\sqrt{3})-2\cos(\pi n))$$

$$\sim -\frac{\pi n}{\sqrt{3}}-\sum_{m=0}^{\infty}\frac{B_{6m+4}}{(6m+4)(2m+1)n^{6m+3}},$$

where B_j, $0 \le j < \infty$, denotes the jth *Bernoulli number.*

Proof. We apply Entry 2 with $x = \sqrt{3}n/2$ and n replaced by $n/2$. Substituting the resulting equality in Entry 4, we deduce that

$$\prod_{k=1}^{\infty}\left(1+\frac{n^3}{k^3}\right)\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+\frac{\sqrt{3}ni}{2}+1\right)\Gamma\left(\frac{n}{2}-\frac{\sqrt{3}ni}{2}+1\right)}$$

$$=\frac{1}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}\frac{\cosh(\pi n\sqrt{3})-\cos(\pi n)}{2^{n+1}\pi^{3/2}n^2}.$$

Take logarithms of both sides and use Stirling's formula (Part I [2, p. 175]). For $\Gamma((n+1)/2)$, it is convenient to apply Stirling's formula in the form given by Ramanujan in Corollary 4 of Section 25 in Chapter 7 (Part I [2, p. 176]). Thus, as n tends to ∞,

$$\log\left(\prod_{k=1}^{\infty}\left(1+\frac{n^3}{k^3}\right)\right)-\log(2\cosh(\pi n\sqrt{3})-2\cos(\pi n))$$

$$=-(n+2)\log 2-\tfrac{3}{2}\log\pi-2\log n+\log\Gamma\left(\frac{n}{2}+\frac{\sqrt{3}ni}{2}+1\right)$$

$$+\log\Gamma\left(\frac{n}{2}-\frac{\sqrt{3}ni}{2}+1\right)-\log\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)-\log\Gamma\left(\frac{n}{2}+1\right)$$

$$\sim -(n+2)\log 2-\tfrac{3}{2}\log\pi-2\log n$$

$$+\left(\frac{n}{2}+\frac{\sqrt{3}ni}{2}+\frac{1}{2}\right)\log\left(\frac{n}{2}+\frac{\sqrt{3}ni}{2}\right)$$

$$-\left(\frac{n}{2}+\frac{\sqrt{3}ni}{2}\right)+\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)\left(\frac{n}{2}+\frac{\sqrt{3}ni}{2}\right)^{2k-1}}$$

$$+\left(\frac{n}{2}-\frac{\sqrt{3}ni}{2}+\frac{1}{2}\right)\log\left(\frac{n}{2}-\frac{\sqrt{3}ni}{2}\right)-\left(\frac{n}{2}-\frac{\sqrt{3}ni}{2}\right)$$
$$+\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)\left(\frac{n}{2}-\frac{\sqrt{3}ni}{2}\right)^{2k-1}}$$
$$-\frac{n}{2}\log\frac{n}{2}+\frac{n}{2}-\sum_{k=1}^{\infty}\frac{B_{2k}(2^{1-2k}-1)}{2k(2k-1)(n/2)^{2k-1}}-\left(\frac{n}{2}+\frac{1}{2}\right)\log\frac{n}{2}+\frac{n}{2}$$
$$-\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)(n/2)^{2k-1}}$$
$$=-\tfrac{3}{2}\log(2\pi n)-\frac{\pi n}{\sqrt{3}}+\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)n^{2k-1}}$$
$$\times\,(e^{-\pi i(2k-1)/3}+e^{\pi i(2k-1)/3}-(1-2^{2k-1})-2^{2k-1})$$
$$=-\tfrac{3}{2}\log(2\pi n)-\frac{\pi n}{\sqrt{3}}-\sum_{m=0}^{\infty}\frac{3B_{6m+4}}{(6m+4)(6m+3)n^{6m+3}}.$$

The desired result now follows.

Entry 6 (Formula (4), p. 287). *Let $n > 0$. As n tends to ∞,*

$$\log\left(\frac{e^n\Gamma(n+1)}{n^n\sqrt{2\pi n}}\right)-\tfrac{1}{2}\log\left(\prod_{k=1}^{\infty}\left\{1+\left(\frac{n}{n+k}\right)^2\right\}\right)$$
$$+\frac{n}{2}\left(\frac{\pi}{2}-\log 2\right)-\tfrac{1}{4}\log 2\sim\sum_{k=1}^{\infty}\frac{(-1)^{k(k-1)/2}B_{2k}}{2k(2k-1)2^k n^{2k-1}},\qquad(6.1)$$

where B_j, $0\le j<\infty$, denotes the jth *Bernoulli number.*

Proof. Apply Entry 2 with $x = n$ and substitute in the left side of (6.1). Applying Stirling's formula, we find that, as n tends to ∞,

$$\log\left(\frac{e^n\Gamma(n+1)}{n^n\sqrt{2\pi n}}\right)-\tfrac{1}{2}\log\left(\prod_{k=1}^{\infty}\left\{1+\left(\frac{n}{n+k}\right)^2\right\}\right)+\frac{n}{2}\left(\frac{\pi}{2}-\log 2\right)-\tfrac{1}{4}\log 2$$
$$\sim\frac{n}{2}\left(\frac{\pi}{2}-\log 2\right)-\tfrac{1}{4}\log 2+n-n\log n-\tfrac{1}{2}\log(2\pi n)$$
$$+\tfrac{1}{2}\Bigg((n+ni+\tfrac{1}{2})\log(n+ni)-(n+ni)+\log(2\pi)$$
$$+\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)(n+ni)^{2k-1}}\Bigg)$$

$$+\tfrac{1}{2}\Bigg((n - ni + \tfrac{1}{2})\log(n - ni) - (n - ni) + \log(2\pi)$$

$$+ \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)(n-ni)^{2k-1}}\Bigg)$$

$$= \frac{n}{2}\left(\frac{\pi}{2} - \log 2\right) - \tfrac{1}{4}\log 2 - \frac{\pi n}{4} + \tfrac{1}{2}(n + \tfrac{1}{2})\log 2$$

$$+ \sum_{k=1}^{\infty} \frac{B_{2k}}{4k(2k-1)n^{2k-1}}\left(\frac{1}{(1+i)^{2k-1}} + \frac{1}{(1-i)^{2k-1}}\right)$$

$$= \sum_{k=1}^{\infty} \frac{B_{2k}}{k(2k-1)2^k\sqrt{2}n^{2k-1}} \cos\left\{\frac{(2k-1)\pi}{4}\right\}$$

$$= \sum_{k=1}^{\infty} \frac{B_{2k}(-1)^{k(k-1)/2}}{2k(2k-1)2^k n^{2k-1}},$$

which completes the proof.

Entry 7 (p. 345). *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ denote complex numbers such that $\sum_{k=1}^n a_k = \sum_{k=1}^n b_k$. Then, as x tends to ∞,*

$$\prod_{k=1}^{\infty} \frac{\Gamma(x + a_k + 1)}{\Gamma(x + b_k + 1)} \sim 1 + \frac{\sum_{k=1}^n a_k^2 - \sum_{k=1}^n b_k^2}{2x}. \tag{7.1}$$

Proof. We use Stirling's formula in the form (Part I [2, p. 175]),

$$\Gamma(x+1) \sim \sqrt{2\pi}x^{x+1/2}e^{-x}\left(1 + \frac{1}{12x} - \cdots\right), \tag{7.2}$$

as x tends to ∞. Observe that, as x tends to ∞,

$$(x+a)^{x+a+1/2} \sim x^{x+a+1/2}\exp\left(a + \frac{1}{2x}(a^2 + a) + \cdots\right)$$

$$\sim x^{x+a+1/2}e^a\left(1 + \frac{1}{2x}(a^2 + a) + \cdots\right),$$

and so, by (7.2),

$$\Gamma(x + a + 1) \sim \sqrt{2\pi}x^{x+a+1/2}e^{-x+a}\left(1 + \frac{1}{2x}(a^2 + a + \tfrac{1}{6}) + \cdots\right).$$

Using this expansion for each of the gamma functions on the left side of (7.1) and invoking the hypothesis $\sum_{k=1}^n a_k = \sum_{k=1}^n b_k$, we easily complete the proof of (7.1).

Ramanujan's formulation of Entry 7 contains an extraneous factor of $1/(12x)$ on the right side of (7.1).

Entry 8 (p. 346). *Let $A_k > 0$, $1 \le k \le m$, $B_k > 0$, $1 \le k \le n$, and $a_1, \ldots, a_m$, $b_1, \ldots, b_n$ be arbitrary complex numbers. If*

$$\sum_{k=1}^{m} A_k = \sum_{k=1}^{n} B_k, \tag{8.1}$$

$$\prod_{k=1}^{m} A_k^{A_k} = \prod_{k=1}^{n} B_k^{B_k}, \tag{8.2}$$

and

$$\tfrac{1}{2}m + \sum_{k=1}^{m} a_k = \tfrac{1}{2}n + \sum_{k=1}^{n} b_k, \tag{8.3}$$

then

$$\lim_{x\to\infty} \frac{\prod_{k=1}^{m} \Gamma(A_k x + a_k + 1)}{\prod_{k=1}^{n} \Gamma(B_k x + b_k + 1)} = (2\pi e)^{(m-n)/2} \frac{\prod_{k=1}^{m} A_k^{a_k+1/2}}{\prod_{k=1}^{n} B_k^{b_k+1/2}}. \tag{8.4}$$

PROOF. The proof of (8.4) is a straightforward exercise with the use of (7.2). Each of the conditions (8.1)–(8.3) is necessary.

Ramanujan inadvertently omitted the factor $e^{(m-n)/2}$ on the right side of (8.4).

To find numbers such that (8.1)–(8.3) are satisfied, Ramanujan first remarks that it is easy to find $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ so that (8.3) is satisfied. However, finding $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_n$ to simultaneously satisfy (8.1) and (8.2) is slightly more difficult. He says to first choose $A_1, \ldots, A_m, B_1, \ldots, B_n$ so that (8.2) is satisfied. With these choices, suppose that (8.1) does not hold, but

$$\sum_{k=1}^{m} A_k - \sum_{k=1}^{n} B_k = r,$$

where r is a nonzero *integer*. If $r < 0$, then multiply the left side of (8.2) by $-r$ factors of 1^1. If $r > 0$, then multiply the right side of (8.2) by r factors of 1^1. Thus, (8.2) is invariant, but a proper number of 1's has been added to either $\sum_{k=1}^{m} A_k$ or $\sum_{k=1}^{n} B_k$ to satisfy (8.1). In terms of gamma functions, either the numerator on the left side of (8.4) is multiplied by $\Gamma^{-r}(x+1)$ or the denominator is multiplied by $\Gamma^{r}(x+1)$, respectively.

At the bottom of page 346 and the top of page 347, Ramanujan gives several examples. First, Ramanujan observes that

$$2^2 \cdot 6^2 = 3^3 \cdot 3^3 \cdot 4^4.$$

For (8.1) to be satisfied, we must multiply the left side by $1^1 \cdot 1^1$. Further examples are

$$1^1 \cdot 1^1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \cdot 5^5 \cdot 6^6 = 3^3 \cdot 3^3 \cdot 3^3 \cdot 4^4 \cdot 4^4 \cdot 5^5,$$
$$1^1 \cdot 8^8 \cdot 9^9 = 3^3 \cdot 3^3 \cdot 12^{12}, \tag{8.5}$$
$$1^1 \cdot 3^3 \cdot 12^{12} \cdot 20^{20} = 5^5 \cdot 15^{15} \cdot 16^{16},$$

and

$$1^1 \cdot 4^4 \cdot 20^{20} \cdot 30^{30} = 6^6 \cdot 24^{24} \cdot 25^{25}.$$

Corollary 8.1 (p. 347). *If a, b, and c are arbitrary complex numbers, then*

$$\lim_{x\to\infty} \frac{\Gamma(x+a-b+1)\Gamma(8x+2b+1)\Gamma(9x+a+b)}{\Gamma(3x+a-c+1)\Gamma(3x+a-b+c+1)\Gamma(12x+3b+1)} = \sqrt{\tfrac{2}{3}}.$$

Corollary 8.1 is a special case of (8.1), with (8.5) corresponding to (8.2).

Corollary 8.1 was submitted as a problem by Ramanujan [9], [23, p. 330] to the *Journal of the Indian Mathematical Society.*

Corollary 8.2 (p. 347). *If a, b, and c are arbitrary complex numbers, then*

$$\lim_{x\to\infty} \frac{\Gamma(3x+a+1)\Gamma(3x+b+1)\Gamma(12x+3c+1)}{\Gamma(x+\frac{1}{2}(a+b-c)+1)\Gamma(8x+2c+1)\Gamma(9x+\frac{1}{2}(a+b+3c)+1)} = \sqrt{\tfrac{3}{2}}.$$

Corollary 8.2 is also a special case of Entry 8, with (8.5) corresponding to (8.2).

In fact, for some unknown reason, Ramanujan crossed out Corollary 8.2.

Entry 9 (p. 347). *With $z = x + iy$, where x and y are real,*

$$\Gamma(z+1) = \Gamma(x+1)\frac{\exp\left(i \lim_{n\to\infty}\left(y\log n - \sum_{k=1}^{n} \arctan\frac{y}{x+k}\right)\right)}{\sqrt{\prod_{k=1}^{\infty}\left(1+\left(\frac{y}{x+k}\right)^2\right)}}.$$

PROOF. From Euler's product representation for $\Gamma(z)$,

$$\begin{aligned}\Gamma(z) &= \lim_{n\to\infty} \frac{n^z}{z}\prod_{k=1}^{n}\frac{k}{z+k}\\ &= \lim_{n\to\infty}\frac{n^x}{x}\left(\prod_{k=1}^{n}\frac{k}{x+k}\right)\frac{n^{iy}}{1+iy/x}\prod_{k=1}^{n}\frac{1}{1+\dfrac{iy}{x+k}}\end{aligned}$$

$$= \Gamma(x) \lim_{n\to\infty} \exp(iy \log n) \prod_{k=0}^{n} \frac{\exp\left(-i \arctan \dfrac{y}{x+k}\right)}{\sqrt{1 + \left(\dfrac{y}{x+k}\right)^2}}$$

$$= \Gamma(x) \lim_{n\to\infty} \frac{\exp(iy \log n) \sum_{k=0}^{n} \exp\left(-i \arctan \dfrac{y}{x+k}\right)}{\sqrt{\prod_{k=0}^{n} \left(1 + \left(\dfrac{y}{x+k}\right)^2\right)}}.$$

Replacing z by $z + 1$, we complete the proof.

Entry 10 (p. 365). *For each complex number z,*

$$\Gamma(z + 1)e^{\gamma z} = \prod_{n=1}^{\infty} (1 + z/n)^{-1} e^{z/n},$$

where γ denotes Euler's constant.

Entry 10 is the well-known Weierstrass product representation for the gamma function and can be easily derived from Euler's product representation (Whittaker and Watson [1, p. 237]).

Bessel Functions

Entry 11 (Formula (1), p. 282). *Let $I_\nu(x)$ denote the Bessel function of imaginary argument of order ν. Then, if n is not an integer,*

$$I_n(2x) - I_{-n}(2x)$$
$$\sim -\frac{\sin(n\pi)e^{-2x}}{\sqrt{\pi x}} \left(1 + \frac{n^2 - 1^1/2^2}{4x \cdot 1!} + \frac{(n^2 - 1^2/2^2)(n^2 - 3^2/2^2)}{(4x)^2 2!} + \cdots\right),$$

as x tends to ∞.

Proof. Recall that the modified Bessel function $K_n(x)$ is defined by (Watson [4, p. 78])

$$K_n(x) = \tfrac{1}{2}\pi \frac{I_{-n}(x) - I_n(x)}{\sin(n\pi)}, \tag{11.1}$$

if n is not an integer. Recall also the asymptotic expansion (Watson [4, p. 202]),

$$K_n(x) \sim \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}\left(1 + \frac{4n^2 - 1^2}{8x \cdot 1!} + \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{(8x)^2 2!} + \cdots\right), \quad (11.2)$$

as x tends to ∞. Combining (11.1) and (11.2), we deduce Entry 11.

Entry 12 (Formula (1), p. 282). *If n is not an integer,*

$$I_n(2x) - I_{-n}(2x) = -\frac{\sin(\pi n)}{\pi} \int_0^\infty u^{n-1} e^{-x(u+1/u)}\, du.$$

Proof. This formula is an easy consequence of (11.1) and the well-known formula (Watson [4, p. 182]),

$$K_n(2x) = \frac{1}{2} \int_0^\infty u^{n-1} e^{-x(u+1/u)}\, du, \quad (12.1)$$

valid for any complex number n.

Entry 13 (Formula (2), p. 282). *If n is arbitrary,*

$$\int_0^\infty t^{2n} e^{-t^2 - x^2/t^2}\, dt \sim \frac{\sqrt{\pi} e^{-2x}}{2} x^n\left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k(-n)_k(n+1)_k}{(4x)^k k!}\right),$$

as x tends to ∞.

Proof. Letting $u = t^2/x$ in (12.1), we find that

$$K_n(2x) = \left(\frac{1}{x}\right)^n \int_0^\infty t^{2n-1} e^{-t^2 - x^2/t^2}\, dt.$$

Replacing n by $n + \frac{1}{2}$, we see that

$$x^{n+1/2} K_{n+1/2}(2x) = \int_0^\infty t^{2n} e^{-t^2 - x^2/t^2}\, dt. \quad (13.1)$$

Employing the asymptotic expansion (11.2) in (13.1), we complete the proof.

Entry 14 (Formula (7), p. 283). *Let $\varphi(x)$ be continuous on $[0, \infty)$. Suppose that p and a are positive and that n is any complex number with* $\mathrm{Re}(n) > 0$. *If*

$$\int_0^\infty e^{-p^2 x} \varphi(x)\, dx = \frac{e^{-2ap}}{p^{n+1}}, \quad (14.1)$$

then

$$\varphi(x) = \frac{x^n}{\sqrt{\pi x}} \frac{e^{-a^2/x}}{\Gamma(n)} \int_0^\infty e^{-at - xt^2/4} t^{n-1}\, dt, \quad (14.2)$$

and, as a tends to ∞,

$$\varphi(x) \sim \frac{x^n}{a^n\sqrt{\pi x}} e^{-a^2/x} \sum_{k=0}^{\infty} \frac{(-1)^k (n)_{2k}}{4^k k!\, a^{2k}} x^k. \tag{14.3}$$

PROOF. Taking the inverse Laplace transform of (14.1), we find that (Erdélyi [2, p. 246, eq. (9)])

$$\varphi(x) = \sqrt{\frac{2}{\pi}}(2x)^{(n-1)/2} e^{-a^2/(2x)} D_{-n}\left(\sqrt{\frac{2}{x}}\, a\right), \tag{14.4}$$

where D_ν denotes the parabolic cylinder function of order ν. From Gradshteyn and Ryzhik's tables [1, p. 1064, eq. 9.241, no. 2],

$$D_{-n}(x) = \frac{e^{-x^2/4}}{\Gamma(n)} \int_0^\infty e^{-xt - t^2/2} t^{n-1}\, dt, \qquad \operatorname{Re}(n) > 0. \tag{14.5}$$

Thus, by (14.4) and (14.5), for $\operatorname{Re}(n) > 0$,

$$\varphi(x) = \sqrt{\frac{2}{\pi}}(2x)^{(n-1)/2} \frac{e^{-a^2/x}}{\Gamma(n)} \int_0^\infty e^{-at\sqrt{2/x} - t^2/2} t^{n-1}\, dt.$$

Replacing t by $t\sqrt{x/2}$ above, we readily deduce (14.2).

The asymptotic expansion (14.3) follows from using the asymptotic series for $D_{-n}(\sqrt{2/x}a)$ (Gradshteyn and Ryzhik [1, p. 1065, eq. 9.246, no. 1]),

$$D_{-n}\left(\sqrt{\frac{2}{x}}\, a\right) \sim e^{-a^2/(2x)} \left(\sqrt{\frac{2}{x}}\, a\right)^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k (n)_{2k} x^k}{4^k k!\, a^{2k}},$$

as a tends to ∞, in (14.4).

Hypergeometric Series

At the top of page 284 in his second notebook, Ramanujan writes, "The difference between

$$\frac{\Gamma(\beta - m + 1)}{\Gamma(\alpha + \beta - m + 1)}$$

and

$$\frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} + \frac{\alpha m}{1!} \frac{\Gamma(\beta + n + 1)}{\Gamma(\alpha + \beta + n + 2)}$$
$$+ \frac{\alpha(\alpha + 1)}{2!} m(m + 2n + 1) \frac{\Gamma(\beta + 2n + 1)}{\Gamma(\alpha + \beta + 2n + 3)}$$

$$+\frac{\alpha(\alpha+1)(\alpha+2)}{3!}m(m+3n+1)(m+3n+2)\frac{\Gamma(\beta+3n+1)}{\Gamma(\alpha+\beta+3n+4)}+\cdots\text{''} \tag{15.1}$$

The sentence is not completed. We might surmise that this difference is zero or expressible in closed form for certain sets of values for the parameters α, β, m, and n. It seems certain that Ramanujan felt that he could further extend whatever result he might have had. It seems likely that n is meant to be a nonnegative integer. With this assumption, we write (15.1) in the more familiar notation

$$\begin{aligned}
&\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\sum_{j=0}^{\infty}\frac{(\alpha)_j(m)_{(n+1)j}(\beta+1)_{nj}}{j!\,(m+1)_{nj}(\alpha+\beta+1)_{(n+1)j}}\\
&=\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\,{}_{2n+2}F_{2n+1}\\
&\times\left(\begin{matrix}\alpha,\dfrac{m}{n+1},\dfrac{m+1}{n+1},\ldots,\dfrac{m+n}{n+1},\dfrac{\beta+1}{n},\dfrac{\beta+2}{n},\ldots,\dfrac{\beta+n}{n}\\ \dfrac{m+1}{n},\dfrac{m+2}{n},\ldots,\dfrac{m+n}{n},\dfrac{\alpha+\beta+1}{n+1},\dfrac{\alpha+\beta+2}{n+1},\ldots,\dfrac{\alpha+\beta+n+1}{n+1}\end{matrix};1\right).
\end{aligned} \tag{15.2}$$

Note that this hypergeometric function is Saalschützian, or balanced, if $n > 0$.

We examine two special cases. Suppose first that $n = 0$. Then the series in (15.2) is

$$\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\sum_{j=0}^{\infty}\frac{(\alpha)_j(m)_j}{(\alpha+\beta+1)_j j!}=\frac{\Gamma(\beta+1-m)}{\Gamma(\alpha+\beta+1-m)},$$

upon the use of Gauss's theorem (Bailey [1, p. 2]). Thus, we conclude that, in the case $n = 0$, the difference equals 0 in Ramanujan's incomplete statement.

Secondly, let $n = 1$ and $\alpha = -k$, where k is a nonnegative integer. We shall prove that, also in this case, Ramanujan's difference equals zero. More precisely, we prove that

$$\begin{aligned}
&\frac{\Gamma(\beta+1)}{\Gamma(-k+\beta+1)}\,{}_4F_3\left(\begin{matrix}-k,\frac{1}{2}m,\frac{1}{2}(m+1),\beta+1\\ m+1,\frac{1}{2}(-k+\beta+1),\frac{1}{2}(-k+\beta+2)\end{matrix};1\right)\\
&\qquad=\frac{\Gamma(\beta+1-m)}{\Gamma(-k+\beta+1-m)}.
\end{aligned} \tag{15.3}$$

We are grateful to R. A. Askey for the following proof.

From an unpublished manuscript of Askey and M. E. H. Ismail [1],

$$
{}_4F_3\left(\begin{matrix}\tfrac12 a, \tfrac12(a+1), c, -k\\ \tfrac12 d, \tfrac12(d+1), a+c-d+1-k\end{matrix}; 1\right) = \frac{(d-a)_k(d-c)_k}{(d-a-c)_k(d)_k}\, {}_3F_2\left(\begin{matrix}a, c, -k\\ d-c, d+k\end{matrix}; 1\right). \tag{15.4}
$$

(In fact, (15.4) can be deduced from a formula in Bailey's book [1, p. 32, eq. (1)]; replace k by d, a by $-a+d$, c by a, b by c, and m by k.) Let $a = m$, $c = \beta + 1$, and $d = -k + \beta + 1 + \varepsilon$ in (15.4), where $\varepsilon > 0$. Taking the limit on both sides as ε tends to 0 and using Vandermonde's theorem (Bailey [1, p. 3]), we find that

$$
\begin{aligned}
&\frac{\Gamma(\beta+1)}{\Gamma(-k+\beta+1)}\, {}_4F_3\left(\begin{matrix}\tfrac12 m, \tfrac12(m+1), \beta+1, -k\\ \tfrac12(\alpha+\beta+1), \tfrac12(\alpha+\beta+2), m+1\end{matrix}; 1\right)\\
&\qquad = \frac{\Gamma(\beta+1)}{\Gamma(-k+\beta+1)}\frac{(-m-k+\beta+1)_k(-k)_k}{(-m-k)_k(-k+\beta+1)_k}\\
&\qquad\quad \times \lim_{\varepsilon\to 0} {}_3F_2\left(\begin{matrix}m, \beta+1, -k\\ -k+\varepsilon, \beta+1+\varepsilon\end{matrix}; 1\right)\\
&\qquad = \frac{\Gamma(-m+\beta+1)(-k)_k}{(-m-k)_k\Gamma(-m-k+\beta+1)}\frac{(-k-m)_k}{(-k)_k}\\
&\qquad = \frac{\Gamma(-m+\beta+1)}{\Gamma(-m-k+\beta+1)}.
\end{aligned}
$$

Thus, (15.3) has been established, and Ramanujan's difference in (15.1) is equal to 0 when $n = 1$ and $\alpha = -k$, where k is a nonnegative integer.

D. Bradley [1] probably has provided the best possible completion of Ramanujan's unfinished sentence. Let

$$
S(z) := S(\alpha, \beta, m, z) := m \sum_{j=0}^{\infty} \frac{\Gamma(m + j(z+1))\Gamma(\beta+1+jz)}{\Gamma(m+jz+1)\Gamma(\alpha+\beta+1+j(z+1))} \frac{(\alpha)_j}{j!}, \tag{15.5}
$$

where α, β, m, and z are complex numbers. Thus, when $z = n$ is a nonnegative integer, (15.5) reduces to

$$
S(n) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \sum_{j=0}^{\infty} \frac{(m)_{(n+1)j}(\beta+1)_{nj}(\alpha)_j}{(m+1)_{nj}(\alpha+\beta+1)_{(n+1)j}\, j!}, \tag{15.6}
$$

i.e., Ramanujan's series (15.2). When α is a nonpositive integer, (15.5) terminates, and, in this case, Bradley has proved the following beautiful theorem.

Theorem 15. *Let $\alpha = -k$, where k is a nonnegative integer. Then, for every complex number z,*

$$S(z) = \frac{\Gamma(\beta + 1 - m)}{\Gamma(\alpha + \beta + 1 - m)}. \tag{15.7}$$

PROOF. Since

$$\begin{aligned} S(z) &= m \sum_{j=0}^{k} \frac{\Gamma(m + j(z+1))\Gamma(\beta + 1 + jz)(-k)_j}{\Gamma(m + jz + 1)\Gamma(-k + \beta + 1 + j(z+1))j!} \\ &= \sum_{j=0}^{k} (m + jz + 1)_{j-1}(-k + \beta + 1 + j(z+1))_{k-j}(-k)_j/j!, \end{aligned}$$

is a polynomial in z of degree $k - 1$, it suffices to prove (15.7) for k distinct values of z. In fact, we shall prove (15.7) for all positive integers z, since the argument is no more difficult than the argument for only k values of z. Thus, let $z = n$ denote a positive integer. By (15.6) and the reflection formula for the gamma function,

$$\frac{\Gamma(\alpha+\beta+1-m)}{\Gamma(\beta+1-m)} S(n) = \frac{\Gamma(m-\beta)\Gamma(-\alpha-\beta)}{\Gamma(-\beta)\Gamma(m-\alpha-\beta)} \sum_{j=0}^{k} \frac{(m)_{(n+1)j}(\beta+1)_{nj}(\alpha)_j}{(m+1)_{nj}(\alpha+\beta+1)_{(n+1)j}j!}. \tag{15.8}$$

Next, By Gauss's theorem, for $\operatorname{Re}(-\alpha - \beta - j(n+1)) > 0$,

$$\begin{aligned} {}_2F_1\left(\begin{matrix} m + j(n+1), \alpha + j \\ m - \beta + j \end{matrix}; 1\right) &= \frac{\Gamma(m - \beta + j)\Gamma(-\alpha - \beta - j(n+1))}{\Gamma(m - \alpha - \beta)\Gamma(-\beta - jn)} \\ &= \frac{\Gamma(m - \beta)(m - \beta)_j\Gamma(-\alpha - \beta)(-\beta - jn)_{nj}}{\Gamma(m - \alpha - \beta)(-\alpha - \beta - j(n+1))_{(n+1)j}\Gamma(-\beta)} \\ &= \frac{\Gamma(m - \beta)\Gamma(-\alpha - \beta)}{\Gamma(-\beta)\Gamma(m - \alpha - \beta)} \frac{(m - \beta)_j(\beta + 1)_{nj}(-1)^j}{(\alpha + \beta + 1)_{(n+1)j}}. \end{aligned} \tag{15.9}$$

Observe that the condition $\operatorname{Re}(-\alpha - \beta - j(n+1)) > 0$ is certainly satisfied if $\operatorname{Re}(\beta) < -nk$, which we now assume. Thus, by (15.8) and (15.9),

$$\begin{aligned} S(n) &= \frac{\Gamma(\beta + 1 - m)}{\Gamma(\alpha + \beta + 1 - m)} \sum_{j=0}^{k} \frac{(\alpha)_j(m)_{(n+1)j}(-1)^j}{(m - \beta)_j(m+1)_{nj}j!} \, {}_2F_1\left(\begin{matrix} m + j(n+1), \alpha + j \\ m - \beta + j \end{matrix}; 1\right) \\ &= \frac{\Gamma(\beta + 1 - m)}{\Gamma(\alpha + \beta + 1 - m)} \sum_{j=0}^{k} \frac{(\alpha)_j(m)_{(n+1)j}(-1)^j}{(m - \beta)_j(m+1)_{nj}j!} \sum_{s=0}^{\infty} \frac{(m + j(n+1))_s(\alpha + j)_s}{(m - \beta + j)_s s!} \\ &= \frac{\Gamma(\beta + 1 - m)}{\Gamma(\alpha + \beta + 1 - m)} \sum_{j=0}^{k} \frac{(-1)^j}{(m+1)_{nj}j!} \sum_{s=0}^{\infty} \frac{(m)_{s+j(n+1)}(\alpha)_{s+j}}{(m - \beta)_{s+j}s!} \end{aligned}$$

$$= \frac{\Gamma(\beta + 1 - m)}{\Gamma(\alpha + \beta + 1 - m)} \sum_{j=0}^{k} \frac{(-1)^j}{(m+1)_{nj} j!} \sum_{r=j}^{\infty} \frac{(m)_{r+nj} (\alpha)_r}{(m-\beta)_r (r-j)!}$$

$$= \frac{\Gamma(\beta + 1 - m)}{\Gamma(\alpha + \beta + 1 - m)} \sum_{j=0}^{k} \frac{(\alpha)_r (m)_r}{(m-\beta)_r r!} \sum_{j=0}^{r} \frac{(m+r)_{nj}}{(m+1)_{nj}} (-1)^j \binom{r}{j}. \tag{15.10}$$

Note that inverting the order of summation is trivially justified, since both sums are finite. In view of (15.7), we must show that the last double sum in (15.10) equals 1.

Let E denote the inner sum on the far right side of (15.10). For $r = 0$, clearly, $E = 1$. For $r \geq 1$, we shall prove that $E = 0$. Thus, the double sum equals 1, and the proof will have been completed when $\operatorname{Re}(\beta) < -nk$.

Now

$$\frac{(m+r)_{nj}}{(m+1)_{nj}} = \frac{\Gamma(m + r + nj)\Gamma(m + 1)}{\Gamma(m + 1 + nj)\Gamma(m + r)} = \frac{(m + nj + 1)_{r-1}}{(m+1)_{r-1}}.$$

Thus,

$$E = \sum_{j=0}^{r} \frac{(m + nj + 1)_{r-1}}{(m+1)_{r-1}} (-1)^j \binom{r}{j}.$$

Since

$$(k)_j = D^j x^{k+j-1}\big|_{x=1}, \qquad j \geq 0,$$

we can write E in the form

$$\begin{aligned} E &= \frac{1}{(m+1)_{r-1}} \sum_{j=0}^{r} D^{r-1} x^{(m+nj+1)+(r-1)-1} (-1)^j \binom{r}{j}\Bigg|_{x=1} \\ &= \frac{1}{(m+1)_{r-1}} D^{r-1} x^{m+r-1} \sum_{j=0}^{r} x^{nj} (-1)^j \binom{r}{j}\Bigg|_{x=1} \\ &= \frac{1}{(m+1)_{r-1}} D^{r-1} x^{m+r-1} (1 - x^n)^r \Bigg|_{x=1} \\ &= 0, \end{aligned}$$

as required.

Hence, we have shown that (15.7) holds for all positive integers n such that $\operatorname{Re}(\beta) < -nk$. Since both sides of (15.7) are meromorphic functions of β for fixed n, m, and $\alpha = -k$, the restriction $\operatorname{Re}(\beta) < -nk$ may be removed by analytic continuation, and so the proof is complete.

Except for the previously discussed case of $z = 0$, using computer algebra, we have found no examples of nonterminating series when (15.7) holds.

Entry 16 (p. 332). *If a is a positive integer, then*

$$\frac{\pi\Gamma(a+1)\Gamma(a)}{4\Gamma^2(a+\frac{1}{2})\prod_{k=0}^{a-1}\left(1+\frac{z^2}{(2k+1)^2}\right)} = \sum_{n=0}^{a-1}\frac{(2n+1)(-a+1)_n}{(a+1)_n(z^2+(2n+1)^2)}. \tag{16.1}$$

Proof. If $f(z)$ denotes the left side of (16.1), we see that $f(z)$ has simple poles at $z = \pm i(2n+1)$, where $n = 0, 1, \ldots, a-1$. Now,

$$\begin{aligned}
R_{(2n+1)i} &= \frac{\pi\Gamma(a+1)\Gamma(a)}{4\Gamma^2(a+\frac{1}{2})}\frac{(2n+1)}{2i}\prod_{\substack{k=0\\k\neq n}}^{a-1}\frac{(2k+1)^2}{(2k+1)^2-(2n+1)^2}\\
&= \frac{\pi\Gamma(a+1)\Gamma(a)(2n+1)}{8i\Gamma^2(a+\frac{1}{2})2^{2a-2}}\prod_{\substack{k=0\\k\neq n}}^{a-1}\frac{(2k+1)^2}{(k-n)(k+n+1)}\\
&= \frac{\Gamma(a+1)\Gamma(a)}{2i(2n+1)}\prod_{\substack{k=0\\k\neq n}}^{a-1}\frac{1}{(k-n)(k+n+1)}\\
&= \frac{(-1)^n\Gamma(a+1)\Gamma(a)}{2i\Gamma(n+1)\Gamma(a-n)}\frac{\Gamma(n+1)}{\Gamma(a+n+1)}\\
&= \frac{(-a+1)_n\Gamma(a-n)\Gamma(a+1)}{2i\Gamma(a-n)\Gamma(a+n+1)}\\
&= \frac{(-a+1)_n}{2i(a+1)_n}.
\end{aligned}$$

Since $R_{-(2n+1)i} = -R_{(2n+1)i}$, we find that

$$\begin{aligned}
f(z) &= \frac{1}{2i}\sum_{n=0}^{a-1}\frac{(-a+1)_n}{(a+1)_n}\left(\frac{1}{z-(2n+1)i}-\frac{1}{z+(2n+1)i}\right)\\
&= \sum_{n=0}^{a-1}\frac{(-a+1)_n(2n+1)}{(a+1)_n(z^2+(2n+1)^2)},
\end{aligned}$$

which is the desired result.

Although Ramanujan clearly intended a to be a positive integer in Entry 16, R. A. Askey has informed us that this restriction may be eased.

Entry 16′ (p. 332). *If* $\mathrm{Re}(a) > 0$, *then*

$$\sum_{n=0}^{\infty}\frac{(2n+1)(-a+1)_n}{(a+1)_n(z^2+(2n+1)^2)} = \frac{\Gamma(a+1)\Gamma(a)\Gamma(\frac{1}{2}(1+iz))\Gamma(\frac{1}{2}(1-iz))}{4\Gamma(a+\frac{1}{2}(1+iz))\Gamma(a+\frac{1}{2}(1-iz))}.$$

Before proving Entry 16′, we demonstrate that Entry 16′ reduces to Entry 16 when a is a natural number. To that end,

$$\frac{\Gamma(\frac{1}{2}(1+iz))\Gamma(\frac{1}{2}(1-iz))}{\Gamma(a+\frac{1}{2}(1+iz))\Gamma(a+\frac{1}{2}(1-iz))}$$

$$=\frac{4^a}{(1^2+z^2)(3^2+z^2)\cdots((2a-1)^2+z^2)}$$

$$=\frac{4^a}{1^2\cdot 3^2\cdots(2a-1)^2(1+z^2)(1+z^2/3^2)\cdots(1+z^2/(2a-1)^2)}$$

$$=\frac{\pi}{\left(\Gamma\left(\frac{1}{2}\right)\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2a-1}{2}\right)^2\prod_{k=0}^{a-1}\left(1+\frac{z^2}{(2k+1)^2}\right)}$$

$$=\frac{\pi}{\Gamma^2(a+\frac{1}{2})\prod_{k=0}^{a-1}\left(1+\frac{z^2}{(2k+1)^2}\right)},$$

and so Entry 16 follows from Entry 16′.

Proof of Entry 16′. Apply Dougall's theorem, which is Entry 5 in Chapter 10 of Ramanujan's second notebook (Part II [4, p. 11]). In that entry, set $n=1$, $x=a-1$, and $y=-\frac{1}{2}(1-iz)$, and replace z by $-\frac{1}{2}(1+iz)$. We then find that, for $\operatorname{Re}(a)>0$,

$$\sum_{k=0}^{\infty}\frac{(\frac{3}{2})_k(-a+1)_k(\frac{1}{2}(1+iz))_k(\frac{1}{2}(1-iz))_k}{(\frac{1}{2})_k(a+1)_k(\frac{1}{2}(3+iz))_k(\frac{1}{2}(3-iz))_k}$$
$$=\frac{\Gamma(a+1)\Gamma(a)\Gamma(\frac{1}{2}(3+iz))\Gamma(\frac{1}{2}(3-iz))}{\Gamma(2)\Gamma(a+\frac{1}{2}(1+iz))\Gamma(a+\frac{1}{2}(1-iz))}.$$

Multiplying both sides by $\{(1+iz)(1-iz)\}^{-1}$ and simplifying, we deduce Entry 16′.

Entry 17 (pp. 353–354). *Let x and y be complex numbers such that $\operatorname{Re}(2x+2y+1)>0$. Let m and n denote arbitrary complex numbers. Then*

$$\frac{\Gamma^2(x+1)\Gamma^2(y+1)\Gamma(2x+2y+1)}{\Gamma^4(x+y+1)}$$
$$\times\prod_{k=0}^{\infty}\frac{\left(1+\frac{(m+n)^2}{(x+y+1+k)^2}\right)\left(1+\frac{(m-n)^2}{(x+y+1+k)^2}\right)}{\left(1+\frac{m^2}{(x+1+k)^2}\right)\left(1+\frac{n^2}{(y+1+k)^2}\right)}$$
$$=1+\sum_{k=1}^{\infty}\frac{(-x-mi)_k(-x+mi)_k}{(y+1+ni)_k(y+1-ni)_k}+\sum_{k=1}^{\infty}\frac{(-y-ni)_k(-y+ni)_k}{(x+1+mi)_k(x+1-mi)_k}. \quad (17.1)$$

Proof. By using Entry 2 and elementary manipulation, we transform (17.1) into the equivalent identity

$$\frac{\Gamma(2x+2y+1)\Gamma(x+1+mi)\Gamma(x+1-mi)\Gamma(y+1+ni)\Gamma(y+1-ni)}{\Gamma(x+y+1+(m+n)i)\Gamma(x+y+1-(m+n)i) \times \Gamma(x+y+1+(m-n)i)\Gamma(x+y+1-(m-n)i)}$$
$$= \sum_{k=-\infty}^{\infty} \frac{(-x-mi)_k(-x+mi)_k}{(y+1+ni)_k(y+1-ni)_k}. \quad (17.2)$$

We now recall Dougall's formula in the bilateral form (Henrici [1, p. 52]). If $1 + \mathrm{Re}(a+b) < \mathrm{Re}(c+d)$, then

$$\sum_{k=-\infty}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(d+k)} = \frac{\pi^2}{\sin(\pi a)\sin(\pi b)} \frac{\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(d-a)\Gamma(c-b)\Gamma(d-b)}. \quad (17.3)$$

Let $a = -x - mi$, $b = -x + mi$, $c = y + 1 + ni$, and $d = y + 1 - ni$ in (17.3). Thus, we require that $\mathrm{Re}(2x + 2y + 1) > 0$. With the use of the reflection formula, $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, it is now a straightforward exercise to show that (17.3) yields (17.2).

Entry 18 (p. 355). *For* $0 \le x \le \frac{1}{2}$,

$${}_3F_2(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; 1, 1; 4x(1-x)) = {}_2F_1^2(\tfrac{1}{2}, \tfrac{1}{2}; 1; x).$$

Proof. From Entries 33(iii) and (ii), respectively, in Chapter 11 (Part II [4, pp. 95, 94]), for $0 \le x < 1$,

$${}_3F_2(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; 1, 1; 4x(1-x)) = {}_2F_1^2(\tfrac{1}{4}, \tfrac{1}{4}; 1; 4x(1-x)), \quad (18.1)$$

and

$${}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{2}(1-\sqrt{1-x})) = {}_2F_1(\tfrac{1}{4}, \tfrac{1}{4}; 1; x). \quad (18.2)$$

Replacing x by $4x(1-x)$ in (18.2), we find that, for $0 \le x \le \frac{1}{2}$,

$${}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x) = {}_2F_1(\tfrac{1}{4}, \tfrac{1}{4}; 1; 4x(1-x)). \quad (18.3)$$

Combining (18.1) and (18.3), we complete the proof.

Recall that

$$L(q^2) := 1 - 24 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} \quad (19.1)$$

and

$$L(q^2) = (1-2x)z^2 + 6x(1-x)z\frac{dz}{dx}, \tag{19.2}$$

where

$$q = \exp\left(-\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}\right) \tag{19.3}$$

and

$$z = z(x) = {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;x) \tag{19.4}$$

(Part III [6, pp. 121, 120, 102, 101]).

Entry 19 (p. 355). *If L is defined by* (19.1) *and q is given by* (19.3), *then*

$$\frac{L(q^2)}{1-2x} = \sum_{n=0}^{\infty}\frac{(3n+1)(\frac{1}{2})_n^3\{4x(1-x)\}^n}{(n!)^3}, \qquad 0 \le x \le \tfrac{1}{2}.$$

PROOF. By (19.2), we want to show that

$$z^2 + \frac{6x(1-x)}{1-2x}z\frac{dz}{dx} = \sum_{n=0}^{\infty}\frac{(3n+1)(\frac{1}{2})_n^3\{4x(1-x)\}^n}{(n!)^3}. \tag{19.5}$$

By (19.4) and Entry 18, (19.5) is equivalent to the identity

$$\frac{z}{2(1-2x)}\frac{dz}{dx} = \sum_{n=1}^{\infty}\frac{n(\frac{1}{2})_n^3\{4x(1-x)\}^{n-1}}{(n!)^3}. \tag{19.6}$$

Using (19.4) and differentiating Entry 18, we find that

$$2z\frac{dz}{dx} = \sum_{n=1}^{\infty}\frac{n(\frac{1}{2})_n^3}{(n!)^3}\{4x(1-x)\}^{n-1}(4-8x),$$

which immediately yields (19.6), and so the proof of Entry 19 is complete.

For the last two entries of this chapter, we list all of Ramanujan's series for $1/\pi$ found in the 133 unorganized pages of the second and third notebooks.

Entry 20 (p. 355). *We have*

$$\frac{4}{\pi} = \sum_{n=0}^{\infty}\frac{(6n+1)(\frac{1}{2})_n^3}{4^n(n!)^3}, \tag{20.1}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty}\frac{(42n+5)(\frac{1}{2})_n^3}{(64)^n(n!)^3}, \tag{20.2}$$

and

$$\frac{32}{\pi}=\sum_{n=0}^{\infty}\frac{(42\sqrt{5}n+5\sqrt{5}+30n-1)(\frac{1}{2})_n^3}{(64)^n(n!)^3}\left(\frac{\sqrt{5}-1}{2}\right)^{8n}. \tag{20.3}$$

These three representations for $1/\pi$ are found in Ramanujan's paper [10], [23, pp. 36, 37]. Both Entries 18 and 19 are necessary for Ramanujan's proofs [23, p. 36], which are very sketchily indicated. (Our proof of Entry 19 is different from that briefly sketched by Ramanujan in [10], [23, p. 36].)

Entry 21 (p. 378). *We have*

$$\frac{27}{4\pi}=\sum_{n=0}^{\infty}\frac{(15n+2)(\frac{1}{2})_n(\frac{1}{3})_n(\frac{2}{3})_n}{(n!)^3}\left(\frac{2}{27}\right)^n, \tag{21.1}$$

$$\frac{15\sqrt{3}}{2\pi}=\sum_{n=0}^{\infty}\frac{(33n+4)(\frac{1}{2})_n(\frac{1}{3})_n(\frac{2}{3})_n}{(n!)^3}\left(\frac{4}{125}\right)^n, \tag{21.2}$$

$$\frac{5\sqrt{5}}{2\pi\sqrt{3}}=\sum_{n=0}^{\infty}\frac{(11n+1)(\frac{1}{2})_n(\frac{1}{6})_n(\frac{5}{6})_n}{(n!)^3}\left(\frac{4}{125}\right)^n, \tag{21.3}$$

$$\frac{85\sqrt{85}}{18\pi\sqrt{3}}=\sum_{n=0}^{\infty}\frac{(133n+8)(\frac{1}{2})_n(\frac{1}{6})_n(\frac{5}{6})_n}{(n!)^3}\left(\frac{4}{85}\right)^n, \tag{21.4}$$

$$\frac{4}{\pi}=\sum_{n=0}^{\infty}\frac{(-1)^n(20n+3)(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3 2^{2n+1}}, \tag{21.5}$$

$$\frac{4}{\pi\sqrt{3}}=\sum_{n=0}^{\infty}\frac{(-1)^n(28n+3)(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3 3^n 4^{n+1}}, \tag{21.6}$$

$$\frac{4}{\pi}=\sum_{n=0}^{\infty}\frac{(-1)^n(260n+23)(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3(18)^{2n+1}}, \tag{21.7}$$

$$\frac{4}{\pi\sqrt{5}}=\sum_{n=0}^{\infty}\frac{(-1)^n(644n+41)(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3 5^n(72)^{2n+1}}, \tag{21.8}$$

$$\frac{4}{\pi}=\sum_{n=0}^{\infty}\frac{(-1)^n(21460n+1123)(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3(882)^{2n+1}}, \tag{21.9}$$

$$\frac{2\sqrt{3}}{\pi}=\sum_{n=0}^{\infty}\frac{(8n+1)(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3 9^n}, \tag{21.10}$$

$$\frac{1}{2\pi\sqrt{2}}=\sum_{n=0}^{\infty}\frac{(10n+1)(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3 9^{2n+1}}, \tag{21.11}$$

$$\frac{1}{3\pi\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(40n + 3)(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3(49)^{2n+1}}, \tag{21.12}$$

$$\frac{2}{\pi\sqrt{11}} = \sum_{n=0}^{\infty} \frac{(280n + 19)(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3(99)^{2n+1}}, \tag{21.13}$$

and

$$\frac{1}{2\pi\sqrt{2}} = \sum_{n=0}^{\infty} \frac{(26390n + 1103)(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(n!)^3(99)^{4n+2}}. \tag{21.14}$$

Each of these fourteen formulas is stated without proof in Ramanujan's paper [10], [23, pp. 37, 38]. In contrast to Entry 20, the formulas of Entry 21 arise from "corresponding theories in which q is replaced by one or other of the functions ... " [10], [23, p. 37]. In reference to (19.3), ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$ is replaced by either ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; x)$, ${}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; x)$, or ${}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; x)$. Indeed, almost six pages at the beginning of the unorganized section of the second notebook are devoted to developing these alternative theories of elliptic functions. See a paper of the author, S. Bhargava, and F. G. Garvan [1] for proofs of Ramanujan's claims in these alternative theories.

The formulas in Entries 20 and 21 were not proved until 1987, when J. M. and P. B. Borwein proved them in their book [1, pp. 177–187]. In three further papers [2], [3], and [4], they established several additional formulas of this type. D. V. and G. V. Chudnovsky [1] not only also proved formulas of this sort, but they, moreover, found representations for other transcendental constants, some involving gamma functions, by hypergeometric series of the same kin.

By taking a few terms of the series in Entries 20 and 21, one can obtain excellent rational or algebraic approximations to π. By truncating the series, one can also compute decimal approximations to π. For example, in (21.14), each series term gives approximately eight digits of π. In 1987, R. W. Gosper employed (21.14) in calculating 17,000,000 digits of π. One of the series for $1/\pi$ found by the Borweins in [3] adds roughly 25 digits of π per term, while another series in their paper [4] yields about 50 digits of π per term.

CHAPTER 30

Partial Fraction Expansions

Ramanujan evidently had an affinity for partial fraction expansions, which can be found in several places in his notebooks. The heaviest concentrations lie in Chapters 14 and 18 and in the unorganized pages at the end of the second notebook. See our books [4] and [6] for accounts of the material in Chapters 14 and 18, respectively. In this chapter, we prove the 15 partial fraction decompositions found in the unorganized pages of the second notebook.

Perhaps the most interesting partial fraction expansion is that in Entry 8, which yields a new series representation for $\psi(z) := \Gamma'(z)/\Gamma(z)$.

In employing the Mittag–Leffler theorem throughout this chapter, we shall find that a certain function f is equal to the sum of its principal parts plus an entire function, which can be shown to be identically equal to 0 by letting z tend to ∞. This argument is the same in each application, and so we shall normally omit it.

In the sequel, $R_a = R_a(f)$ denotes the residue of a function f at a pole a.

Ramanujan offers Entries 1 and 2 below in terms of his extended Bernoulli numbers, which are featured in Chapter 5 (Ramanujan [22], Part I [2, p. 125]). These numbers B_s^* are defined for an arbitrary complex number s by

$$B_s = \frac{2\Gamma(s+1)}{(2\pi)^s}\zeta(s), \tag{1.1}$$

where $\zeta(s)$ denotes the Riemann zeta-function. Note that (1.1) interpolates Euler's famous formula

$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!}B_{2n}, \tag{1.2}$$

where n is any positive integer and B_k, $0 \leq k < \infty$, denotes the kth Bernoulli number. We restate Ramanujan's first two results in more familiar notation.

Entry 1 (Formula (4), p. 268). *Suppose that $\varphi(z)$ is an entire function, and let $\alpha\beta = \pi^2$, with $\alpha, \beta > 0$. Define*

$$f(z) := -\frac{\pi\alpha^{z/2}\Gamma(z+1)\zeta(z)\varphi(z)}{\sin(\frac{1}{2}\pi z)(2\pi)^z\Gamma(\frac{1}{2}z+1)}.$$

Assume that $f(z)$ tends to 0 as $|z|$ tends to ∞. Then

$$f(z) = \sum_{n=0}^{\infty} \frac{\alpha^n\varphi(2n)B_{2n}}{(z-2n)n!} + \sqrt{\frac{\alpha}{\pi}}\left\{\frac{\varphi(1)}{1-z} - \sum_{n=1}^{\infty} \frac{\beta^n\varphi(-2n+1)B_{2n}}{(2n+1+z)n!}\right\}. \tag{1.3}$$

Proof. First, recall that

$$\zeta(-2m) = 0 \tag{1.4}$$

for each positive integer m (Titchmarsh [2, p. 19]). Thus $f(z)$ has a simple pole at $z = 2n$ for each *nonnegative* integer n. If n is *positive*, the residue of f at $z = 2n$ is equal to

$$R_{2n} = -\frac{2\alpha^n(2n)!\,\zeta(2n)\varphi(2n)}{(-1)^n(2\pi)^{2n}n!} = \frac{\alpha^n B_{2n}\varphi(2n)}{n!}, \tag{1.5}$$

where we employed (1.2). The residue at $z = 0$ equals

$$R_0 = -2\zeta(0)\varphi(0) = \varphi(0), \tag{1.6}$$

since

$$\zeta(0) = -\tfrac{1}{2} \tag{1.7}$$

(Titchmarsh [2, p. 19]). Second, $\Gamma(z+1)/\Gamma(\frac{1}{2}z+1)$ has a simple pole at each negative odd integer $z = -2n-1$, $n \geq 0$, and the residue of f at $-2n-1$ equals

$$R_{-2n-1} = \frac{(-1)^n\pi\alpha^{-(2n+1)/2}\zeta(-2n-1)\varphi(-2n-1)}{(2n)!\,(2\pi)^{-2n-1}\Gamma(-n+\frac{1}{2})}. \tag{1.8}$$

Recall that (Titchmarsh [2, p. 19])

$$\zeta(1-2m) = -\frac{B_{2m}}{2m}, \tag{1.9}$$

for each positive integer m. Also, from the functional equation of $\Gamma(z)$ and the value $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we easily find that

$$\Gamma(-n+\tfrac{1}{2}) = \frac{(-1)^n\sqrt{\pi}2^{2n}n!}{(2n)!}. \tag{1.10}$$

Utilizing (1.9) and (1.10) in (1.8), we deduce that, for $n \geq 0$,

$$R_{-2n-1} = -\sqrt{\pi}\left(\frac{2\pi}{\sqrt{\alpha}}\right)^{2n+1} \frac{B_{2n+2}\varphi(-2n-1)}{(2n+2)2^{2n}n!}$$
$$= -\sqrt{\frac{\alpha}{\pi}}\,\beta^{n+1}\,\frac{B_{2n+2}\varphi(-2n-1)}{(n+1)!}. \tag{1.11}$$

Thirdly, $f(z)$ has a simple pole at $z = 1$ with residue

$$R_1 = -\frac{\pi\sqrt{\alpha}\varphi(1)}{2\pi\Gamma(\frac{3}{2})} = -\frac{\sqrt{\alpha}\varphi(1)}{\sqrt{\pi}}. \tag{1.12}$$

Thus, summing all the principal parts arising from (1.5), (1.6), (1.11), and (1.12), we find that

$$f(z) = \sum_{n=1}^{\infty} \frac{\alpha^n B_{2n}\varphi(2n)}{n!\,(z-2n)} + \frac{\varphi(0)}{z}$$
$$-\sqrt{\frac{\alpha}{\pi}}\sum_{n=0}^{\infty} \frac{\beta^{n+1}B_{2n+2}\varphi(-2n-1)}{(n+1)!\,(z+2n+1)} - \sqrt{\frac{\alpha}{\pi}}\frac{\varphi(1)}{z-1} + g(z),$$

for some entire function $g(z)$. Since $f(z)$ tends to 0 as $|z|$ approaches ∞, we see that $g(z)$ also tends to 0 as $|z|$ tends to ∞. Thus, $g(z)$ is a bounded entire function which, by Liouville's theorem, must be a constant. This constant is obviously equal to 0, and so the proof of (1.3) is complete.

Entry 2 (Formula (4), p. 268). *Suppose that $\varphi(z)$ is an entire function. Define, for any complex number a,*

$$f(z) := \frac{\pi a^z \Gamma(z+1)\zeta(z)\varphi(z)}{\sin(\frac{1}{2}\pi z)(2\pi)^z},$$

where ζ denotes the Riemann zeta-function. Assume that $f(z)$ approaches 0 as $|z|$ tends to ∞. Then

$$f(z) = -\frac{\varphi(0)}{z} + \frac{a\varphi(1)}{2(z-1)} - \sum_{n=1}^{\infty} \frac{a^{2n}B_{2n}\varphi(2n)}{z-2n}$$
$$+ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n\zeta(n+1)\varphi(-n)}{a^n(z+n)}. \tag{2.1}$$

Proof. We see that $f(z)$ has a simple pole at each even nonnegative integer $2n$. First, if n is positive,

$$R_{2n} = \frac{2a^{2n}\Gamma(2n+1)\zeta(2n)\varphi(2n)}{(-1)^n(2\pi)^{2n}}$$
$$= -a^{2n}B_{2n}\varphi(2n), \tag{2.2}$$

by (1.2). Second, the residue at $z = 0$ is equal to

$$R_0 = -\varphi(0), \tag{2.3}$$

by (1.7).

Third, $f(z)$ has a simple pole at each negative integer. However, we need to separate the calculation of the residues into two cases according as n is even or odd. First, for $n \geq 0$, by (1.9) and (1.2),

$$\begin{aligned} R_{-2n-1} &= \frac{\pi a^{-2n-1}\zeta(-2n-1)\varphi(-2n-1)}{(-1)^{n+1}(2n)!\,(2\pi)^{-2n-1}} \\ &= \frac{(-1)^n\pi(2\pi)^{2n+1}B_{2n+2}\varphi(-2n-1)}{a^{2n+1}(2n+2)(2n)!} \\ &= \frac{(2n+1)\zeta(2n+2)\varphi(-2n-1)}{a^{2n+1}}. \end{aligned} \tag{2.4}$$

To calculate the residue at $z = -2n$, $n > 0$, we first recall the functional equation of $\zeta(s)$ in its asymmetric form (Titchmarsh [2, p. 16]),

$$\Gamma(z)\zeta(z) = 2^{z-1}\pi^z \sec(\tfrac{1}{2}\pi z)\zeta(1-z).$$

Thus,

$$\lim_{z\to -2n} \Gamma(z+1)\zeta(z) = \frac{(-1)^{n+1}n\zeta(2n+1)}{(2\pi)^{2n}}. \tag{2.5}$$

Therefore, with the aid of (2.5),

$$\begin{aligned} R_{-2n} &= \frac{2a^{-2n}\varphi(-2n)}{(-1)^n(2\pi)^{-2n}}\,\frac{(-1)^{n+1}n\zeta(2n+1)}{(2\pi)^{2n}} \\ &= -\frac{(2n)\zeta(2n+1)\varphi(-2n)}{a^{2n}}. \end{aligned} \tag{2.6}$$

Combining (2.4) and (2.6), we conclude that, for $n \geq 1$,

$$R_{-n} = \frac{(-1)^{n-1}n\zeta(n+1)\varphi(-n)}{a^n}. \tag{2.7}$$

Lastly, $f(z)$ has a simple pole at $z = 1$ with residue

$$R_1 = \frac{a\varphi(1)}{2}. \tag{2.8}$$

To conclude the proof of (2.1), we form the partial fraction expansion of $f(z)$ by using (2.2), (2.3), (2.7), and (2.8).

Of course, Entries 1 and 2 can be modified if φ is not an entire function. It would be interesting to find examples of φ for which either Entry 1 or Entry 2 is valid.

Entry 3 (Formula (1), p. 280). *We have*

$$f(z):=\frac{\pi^2}{\sin^2(\pi z)(e^{2\pi z}-1)}=\frac{1}{2\pi z^3}-\frac{1}{2z^2}+\frac{\pi}{3z}-\sum_{n=1}^{\infty}\frac{1}{(z+n)^2}$$
$$+4z\sum_{n=1}^{\infty}\frac{n}{(e^{2\pi n}-1)(z^2-n^2)^2}-2\pi z^3\sum_{n=1}^{\infty}\frac{1}{\sinh^2(\pi n)(z^4-n^4)}. \tag{3.1}$$

Proof. The function $f(z)$ has simple poles at $z = ni$ and double poles at $z = n$ for each nonzero integer n. At $z = 0$, $f(z)$ has a triple pole.

By a straightforward calculation, we find that $f(z)$ has the Laurent expansion

$$f(z)=\frac{1}{2\pi z^3}-\frac{1}{2z^2}+\frac{\pi}{3z}+\cdots, \qquad 0<|z|<1. \tag{3.2}$$

Next, we calculate the principal parts of f at the double poles n. First,

$$\lim_{z\to n}(z-n)^2 f(z)=\frac{1}{e^{2\pi n}-1}. \tag{3.3}$$

Suppose that $F(z) = p(z)/q(z)$, where p and q are analytic at $z = a$ and $p(a) \neq 0$. If F has a double pole at $z = a$, then (Churchhill [1, p. 160])

$$R_a=2\frac{p'(a)}{q''(a)}-\frac{2p(a)q'''(a)}{3\{q''(a)\}^2}. \tag{3.4}$$

Let $p(z) = \pi^2/(e^{2\pi z} - 1)$ and $q(z) = \csc^2(\pi z)$. From (3.4), we then readily find that

$$R_n=-\frac{\pi}{2\sinh^2(\pi n)}. \tag{3.5}$$

If, for $n < 0$, we write

$$\frac{1}{e^{2\pi n}-1}=-\frac{1}{e^{-2\pi n}-1}-1, \tag{3.6}$$

then, from (3.3), (3.5), and (3.6), the sum of the principal parts corresponding to the double poles at $z = \pm n$, $n > 0$, equals

$$\begin{aligned} &\frac{1}{(e^{2\pi n}-1)(z-n)^2} - \frac{1}{(e^{2\pi n}-1)(z+n)^2} \\ &\qquad - \frac{1}{(z+n)^2} - \frac{\pi}{2\sinh^2(\pi n)}\left(\frac{1}{z-n} + \frac{1}{z+n}\right) \\ &\qquad = \frac{4zn}{(e^{2\pi n}-1)(z^2-n^2)^2} - \frac{1}{(z+n)^2} - \frac{\pi z}{\sinh^2(\pi n)(z^2-n^2)}. \end{aligned} \tag{3.7}$$

Lastly, an elementary calculation yields, for $n \neq 0$,

$$R_{ni} = -\frac{\pi}{2\sinh^2(\pi n)}.$$

The sum of the principal parts around $z = \pm ni$, $n \neq 0$, then equals

$$-\frac{\pi}{2\sinh^2(\pi n)}\left(\frac{1}{z-ni} + \frac{1}{z+ni}\right) = -\frac{\pi z}{\sinh^2(\pi n)(z^2+n^2)}. \tag{3.8}$$

Summing the principal parts (3.7) and (3.8) over n, $1 \leq n < \infty$, and adding to them the principal part (3.2) around 0, we readily deduce (3.1).

Entry 4 (Formula (3), p. 280). *If*

$$f(z) := \frac{\pi}{z^2\sqrt{3}} \frac{\sinh(\pi z\sqrt{3})\sinh(\pi z) + \sin(\pi z\sqrt{3})\sin(\pi z)}{(\cosh(\pi z\sqrt{3}) - \cos(\pi z))(\cosh(\pi z) - \cos(\pi z\sqrt{3}))},$$

then

$$f(z) = \frac{1}{2\pi z^4} + \sum_{n=1}^{\infty} \frac{n\coth(\pi n)}{z^4 + z^2n^2 + n^4} + \sum_{n=1}^{\infty} \frac{n\coth(\pi n)}{z^4 - z^2n^2 + n^4}. \tag{4.1}$$

Proof. For brevity, set $\pi z\sqrt{3} = 2a$, $\pi z = 2b$,

$$N(a, b) = \sinh(2a)\sinh(2b) + \sin(2a)\sin(2b),$$

and

$$D(a, b) = \{\cosh(2a) - \cos(2b)\}\{\cosh(2b) - \cos(2a)\},$$

so that

$$f(z) = \frac{\pi}{z^2\sqrt{3}} \frac{N(a, b)}{D(a, b)}. \tag{4.2}$$

Furthermore, define

$$F(a, b) = \cosh(a - ib)\sinh(a + ib)\cosh(b + ia)\sinh(b - ia) \tag{4.3}$$

and

$$G(a, b) = \sinh(a - ib)\sinh(a + ib)\sinh(b + ia)\sinh(b - ia). \tag{4.4}$$

By separating each of the four hyperbolic functions into its real and imaginary parts, multiplying, and simplifying, we readily find that

$$F(a, b) = \tfrac{1}{4}\{\sinh(2a) + i\sin(2b)\}\{\sinh(2b) - i\sin(2a)\}.$$

It follows that

$$F(a, b) + F(b, a) = \tfrac{1}{2}N(a, b). \tag{4.5}$$

Similarly, after separating each of the four hyperbolic functions into its real and imaginary parts, we eventually find that

$$\begin{aligned} G(a, b) &= \tfrac{1}{4}\{\cosh(2a) - \cos(2b)\}\{\cosh(2b) - \cos(2a)\} \\ &= \tfrac{1}{4}D(a, b). \end{aligned} \tag{4.6}$$

Thus, from (4.2)–(4.6),

$$\begin{aligned} f(z) &= \frac{\pi}{2z^2\sqrt{3}} \frac{F(a, b) + F(b, a)}{G(a, b)} \\ &= \frac{\pi}{2z^2\sqrt{3}}(\coth(a - ib)\coth(b + ia) + \coth(b - ia)\coth(a + ib)) \\ &= \frac{\pi}{2z^2\sqrt{3}}(\coth\{\tfrac{1}{2}\pi z(\sqrt{3} - i)\}\coth\{\tfrac{1}{2}\pi z(1 + i\sqrt{3})\} \\ &\quad + \coth\{\tfrac{1}{2}\pi z(1 - i\sqrt{3})\}\coth\{\tfrac{1}{2}\pi z(\sqrt{3} + i)\}). \end{aligned} \tag{4.7}$$

Examining the first product of hyperbolic functions on the right side of (4.7), we find, after elementary calculations, that it has simple poles at

$$z = \tfrac{1}{2}n(\sqrt{3} + i)i \quad \text{and} \quad z = \tfrac{1}{2}n(1 - i\sqrt{3})i, \tag{4.8}$$

for each nonzero integer n. Straightforward calculations show that

$$R_{n(\sqrt{3}+i)i/2} = \frac{\coth(\pi n)}{n^2(\sqrt{3} + i)\sqrt{3}} \tag{4.9}$$

and

$$R_{n(1-i\sqrt{3})i/2} = -\frac{\coth(\pi n)}{n^2(1 - i\sqrt{3})\sqrt{3}}. \tag{4.10}$$

Summing the principal parts for the two poles $z = \pm\frac{1}{2}n(\sqrt{3} + i)i$, $n > 0$, we arrive at

$$\frac{i\coth(\pi n)}{n\sqrt{3}(z^2 + \frac{1}{2}n^2(1 + i\sqrt{3}))}. \tag{4.11}$$

Next, adding the principal parts for the two poles $z = \pm\frac{1}{2}n(1 - i\sqrt{3})i$, $n > 0$, we obtain

$$-\frac{i\coth(\pi n)}{n\sqrt{3}(z^2 - \frac{1}{2}n^2(1 + i\sqrt{3}))}. \tag{4.12}$$

Examining the second product of hyperbolic functions on the far right side of (4.7), we see that it has simple poles at the conjugates of the poles given by (4.8). The residues are the conjugates of those given in (4.9) and (4.10), respectively. The sums of principal parts, analogous to those in (4.11) and (4.12), are

$$-\frac{i\coth(\pi n)}{n\sqrt{3}(z^2 + \frac{1}{2}n^2(1 - i\sqrt{3}))} \tag{4.13}$$

and

$$\frac{i\coth(\pi n)}{n\sqrt{3}(z^2 - \frac{1}{2}n^2(1 - i\sqrt{3}))}, \tag{4.14}$$

respectively.

Summing (4.11)–(4.14), we find that the sum of the principal parts arising from the poles $\pm\frac{1}{2}n(\sqrt{3} + i)i$, $\pm\frac{1}{2}n(1 - i\sqrt{3})i$, $\pm\frac{1}{2}n(\sqrt{3} - i)i$, and $\pm\frac{1}{2}n(1 + i\sqrt{3})i$, $n > 0$, equals

$$\frac{n\coth(\pi n)}{z^4 + z^2n^2 + n^4} + \frac{n\coth(\pi n)}{z^4 - z^2n^2 + n^4}. \tag{4.15}$$

Lastly, by (4.7), the Laurent expansion of $f(z)$ in a neighborhood of its quadruple pole, $z = 0$, equals

$$\begin{aligned}
&\frac{\pi}{2z^2\sqrt{3}}\left\{\left(\frac{2}{\pi(\sqrt{3} - i)z} + \frac{\pi(\sqrt{3} - i)z}{6} + \cdots\right)\left(\frac{2}{\pi(1 + i\sqrt{3})z} + \frac{\pi(1 + i\sqrt{3})z}{6} + \cdots\right)\right.\\
&\qquad\left. + \left(\frac{2}{\pi(1 - i\sqrt{3})z} + \frac{\pi(1 - i\sqrt{3})z}{6} + \cdots\right)\left(\frac{2}{\pi(\sqrt{3} + i)z} + \frac{\pi(\sqrt{3} + i)z}{6} + \cdots\right)\right\}\\
&= \frac{\pi}{2z^2\sqrt{3}}\left\{\frac{4}{\pi^2(2\sqrt{3} + 2i)z^2} + \frac{4}{\pi^2(2\sqrt{3} - 2i)z^2}\right\} + O(1)\\
&= \frac{1}{2\pi z^4} + O(1).
\end{aligned} \tag{4.16}$$

Summing (4.15) on n, $1 \le n < \infty$, and adding to it the principal part from (4.16), we arrive at the partial fraction decomposition (4.1).

Entry 4 is truly a beautiful partial fraction expansion!

Entry 5 (Formula (2), p. 289). *Let m be real and n be a complex number with neither equal to* 0. *Then*

$$\frac{\pi}{2n}\frac{\sec(\pi m/(2n))}{e^{\pi/n}-1}=\frac{1}{2}+\sum_{k=1}^{\infty}\frac{\operatorname{sech}(\pi mk)}{1+4k^2n^2}$$
$$-2m\sum_{k=0}^{\infty}\frac{(-1)^k}{(e^{(2k+1)\pi/m}-1)(m^2-n^2(2k+1)^2)}$$
$$-\sum_{k=0}^{\infty}\frac{(-1)^k}{m+n(2k+1)}. \tag{5.1}$$

It is not clear why Ramanujan stated the expansion (5.1) in this retrorse fashion. With $z=1/n$, it seems preferable to offer the partial fraction expansion of

$$f(z):=\frac{\pi}{2z}\frac{\sec(\frac{1}{2}\pi mz)}{e^{\pi z}-1}.$$

Proof. The function $f(z)$ has a double pole at $z=0$, simple poles at $z=2ki$, where k is any nonzero integer, and simple poles at $z=(2k+1)/m$, where k is any integer.

First, a straightforward calculation yields

$$f(z)=\frac{1}{2z^2}-\frac{\pi}{4z}+\cdots,\qquad 0<|z|<\min(2,1/m). \tag{5.2}$$

Second, another elementary calculation gives

$$R_{2ki}=\frac{\operatorname{sech}(\pi mk)}{4ki},$$

for each nonzero integer k. Thus, the sum of the principal parts for the two poles $z=\pm 2ki$, $k>0$, equals

$$\frac{\operatorname{sech}(\pi mk)}{z^2+4k^2}. \tag{5.3}$$

Third, we find that, for each nonnegative integer k,

$$R_{(2k+1)/m}=-\frac{(-1)^k}{(2k+1)(e^{(2k+1)\pi/m}-1)}$$

and

$$R_{-(2k+1)/m}=-\frac{(-1)^k}{(2k+1)(e^{-(2k+1)\pi/m}-1)}$$
$$=\frac{(-1)^k}{2k+1}\left(\frac{1}{e^{(2k+1)\pi/m}-1}+1\right).$$

Hence, the principal parts arising from the two simple poles $z = \pm(2k+1)/m$, $k = 0, 1, 2, \ldots$, sum to

$$-\frac{(-1)^k}{(2k+1)(e^{(2k+1)\pi/m}-1)}\left(\frac{1}{z-(2k+1)/m}-\frac{1}{z+(2k+1)/m}\right)$$
$$+\frac{(-1)^k}{(2k+1)(z+(2k+1)/m)}$$
$$=-\frac{2m(-1)^k}{(e^{(2k+1)\pi/m}-1)(m^2z^2-(2k+1)^2)}$$
$$+\frac{(-1)^k}{z}\left(\frac{1}{2k+1}-\frac{1}{(mz+2k+1)}\right). \tag{5.4}$$

Summing (5.3) on k, $1 \le k < \infty$, summing (5.4) over k, $0 \le k < \infty$, and adding the principal part from (5.2) to these two sums, we find that

$$f(z) = \frac{1}{2z^2} - \frac{\pi}{4z} + \sum_{k=1}^{\infty} \frac{\operatorname{sech}(\pi m k)}{z^2+4k^2}$$
$$-2m\sum_{k=0}^{\infty}\frac{(-1)^k}{(e^{(2k+1)\pi/m}-1)(m^2z^2-(2k+1)^2)}$$
$$+\frac{\pi}{4z}-\frac{1}{z}\sum_{k=0}^{\infty}\frac{(-1)^k}{mz+2k+1}. \tag{5.5}$$

After cancelling the terms $\pm\pi/(4z)$, set $z = 1/n$ and multiply both sides of (5.5) by $1/n^2$. We immediately obtain (5.1) to complete the proof.

Entry 6 (Formula (1), p. 291). *Let*

$$\varphi(\alpha, \beta) = \frac{\pi}{e^{4\pi\alpha} - 2e^{2\pi\alpha}\cos(2\pi\beta) + 1} + \alpha\left\{\frac{1}{2(\alpha^2+\beta^2)} + \sum_{n=1}^{\infty}\frac{1}{\alpha^2+(n+\beta)^2}\right\}$$
$$-4\alpha\beta\sum_{n=1}^{\infty}\frac{n}{(e^{2\pi n}-1)(\alpha^2+(n+\beta)^2)(\alpha^2+(n-\beta)^2)},$$

where α and β are arbitrary complex numbers. Then

$$\varphi(\alpha, \beta) + \varphi(\beta, \alpha) = \frac{\pi}{2} + \frac{\alpha\beta}{\pi(\alpha^2+\beta^2)^2}$$
$$+\frac{\pi}{2}\frac{\cosh\{2\pi(\alpha-\beta)\} - \cos\{2\pi(\alpha-\beta)\}}{\{\cosh(2\pi\alpha) - \cos(2\pi\beta)\}\{\cosh(2\pi\beta) - \cos(2\pi\alpha)\}}. \tag{6.1}$$

PROOF. For brevity, set

$$D(\alpha, \beta) = \{\cosh(2\pi\alpha) - \cos(2\pi\beta)\}\{\cosh(2\pi\beta) - \cos(2\pi\alpha)\}.$$

First, observe that

$$\begin{aligned}F(\alpha,\beta):=&\frac{\pi}{e^{4\pi\alpha}-2e^{2\pi\alpha}\cos(2\pi\beta)+1}+\frac{\pi}{e^{4\pi\beta}-2e^{2\pi\beta}\cos(2\pi\alpha)+1}\\&-\frac{\pi}{2}\frac{\cosh\{2\pi(\alpha-\beta)\}-\cos\{2\pi(\alpha-\beta)\}}{D(\alpha,\beta)}\\=&\frac{\pi}{2}\left(\frac{e^{-2\pi\alpha}}{\cosh(2\pi\alpha)-\cos(2\pi\beta)}+\frac{e^{-2\pi\beta}}{\cosh(2\pi\beta)-\cos(2\pi\alpha)}\right.\\&\left.-\frac{\cosh\{2\pi(\alpha-\beta)\}-\cos\{2\pi(\alpha-\beta)\}}{D(\alpha,\beta)}\right)\\=&\frac{\pi}{2}\frac{e^{-2\pi(\alpha+\beta)}-e^{-2\pi\alpha}\cos(2\pi\alpha)-e^{-2\pi\beta}\cos(2\pi\beta)+\cos\{2\pi(\alpha-\beta)\}}{D(\alpha,\beta)}.\end{aligned}\tag{6.2}$$

Now

$$\begin{aligned}&e^{-2\pi(\alpha+\beta)}-e^{-2\pi\alpha}\cos(2\pi\alpha)-e^{-2\pi\beta}\cos(2\pi\beta)+\cos\{2\pi(\alpha-\beta)\}\\&=\cosh\{2\pi(\alpha+\beta)\}-\sinh\{2\pi(\alpha+\beta)\}-(\cosh(2\pi\alpha)-\sinh(2\pi\alpha))\cos(2\pi\alpha)\\&\quad-(\cosh(2\pi\beta)-\sinh(2\pi\beta))\cos(2\pi\beta)+\cos\{2\pi(\alpha-\beta)\}\\&=D(\alpha,\beta)+\sinh(2\pi\alpha)\sinh(2\pi\beta)-\sinh\{2\pi(\alpha+\beta)\}\\&\quad+\sinh(2\pi\alpha)\cos(2\pi\alpha)+\sinh(2\pi\beta)\cos(2\pi\beta)+\sin(2\pi\alpha)\sin(2\pi\beta)\\&=D(\alpha,\beta)+\sinh(2\pi\alpha)\sinh(2\pi\beta)+\sin(2\pi\alpha)\sin(2\pi\beta)\\&\quad-\sinh(2\pi\alpha)(\cosh(2\pi\beta)-\cos(2\pi\alpha))-\sinh(2\pi\beta)(\cosh(2\pi\alpha)-\cos(2\pi\beta)).\end{aligned}\tag{6.3}$$

Substituting (6.3) into (6.2) and then employing (4.5) and (4.7), we find that

$$\begin{aligned}F(\alpha,\beta)=&\frac{\pi}{2}\left(1+\frac{\sinh(2\pi\alpha)\sinh(2\pi\beta)+\sin(2\pi\alpha)\sin(2\pi\beta)}{D(\alpha,\beta)}\right.\\&\left.-\frac{\sinh(2\pi\alpha)}{\cosh(2\pi\alpha)-\cos(2\pi\beta)}-\frac{\sinh(2\pi\beta)}{\cosh(2\pi\beta)-\cos(2\pi\alpha)}\right)\\=&\frac{\pi}{2}+\frac{\pi}{4}(\coth\{\pi(\alpha-i\beta)\}\coth\{\pi(\beta+i\alpha)\}\\&+\coth\{\pi(\beta-i\alpha)\}\coth\{\pi(\alpha+i\beta)\})\\&-\frac{\pi}{2}\frac{\sinh(2\pi\alpha)}{\cosh(2\pi\alpha)-\cos(2\pi\beta)}-\frac{\pi}{2}\frac{\sinh(2\pi\beta)}{\cosh(2\pi\beta)-\cos(2\pi\alpha)}.\end{aligned}\tag{6.4}$$

Comparing (6.4) with (6.1), we see that we must prove that

$$\frac{\pi}{4}(\coth\{\pi(\alpha - i\beta)\}\coth\{\pi(\beta + i\alpha)\} + \coth\{\pi(\beta - i\alpha)\}\coth\{\pi(\alpha + i\beta)\})$$

$$-\frac{\pi}{2}\frac{\sinh(2\pi\alpha)}{\cosh(2\pi\alpha) - \cos(2\pi\beta)} - \frac{\pi}{2}\frac{\sinh(2\pi\beta)}{\cosh(2\pi\beta) - \cos(2\pi\alpha)}$$

$$= \frac{\alpha\beta}{\pi(\alpha^2 + \beta^2)^2} - \alpha\left\{\frac{1}{2(\alpha^2 + \beta^2)} + \sum_{n=1}^{\infty}\frac{1}{\alpha^2 + (n + \beta)^2}\right\}$$

$$-\beta\left\{\frac{1}{2(\alpha^2 + \beta^2)} + \sum_{n=1}^{\infty}\frac{1}{\beta^2 + (n + \alpha)^2}\right\}$$

$$+4\alpha\beta\sum_{n=1}^{\infty}\frac{n}{(e^{2\pi n} - 1)(\alpha^2 + (n + \beta)^2)(\alpha^2 + (n - \beta)^2)}$$

$$+4\alpha\beta\sum_{n=1}^{\infty}\frac{n}{(e^{2\pi n} - 1)(\beta^2 + (n + \alpha)^2)(\beta^2 + (n - \alpha)^2)}. \tag{6.5}$$

Put

$$g(\alpha) = -\frac{\pi}{2\alpha}\frac{\sinh(2\pi\alpha)}{\cosh(2\pi\alpha) - \cos(2\pi\beta)},$$

which has simple poles at $\alpha = (n \pm \beta)i$, for each integer n, provided that no two poles coalesce. The residues at these poles are easily calculated and are given by

$$R_{(n-\beta)i} = \frac{i}{4(n - \beta)} \quad \text{and} \quad R_{(n+\beta)i} = \frac{i}{4(n + \beta)}.$$

The sum of the principal parts arising from the four poles $(\pm n \pm \beta)i$, $n > 0$, equals

$$\frac{i}{4}\left(\frac{1}{(n - \beta)(\alpha - (n - \beta)i)} - \frac{1}{(n + \beta)(\alpha + (n + \beta)i)}\right.$$

$$\left.+\frac{1}{(n + \beta)(\alpha - (n + \beta)i)} - \frac{1}{(n - \beta)(\alpha + (n - \beta)i)}\right)$$

$$= -\frac{1}{2}\left(\frac{1}{(\alpha^2 + (n - \beta)^2)} + \frac{1}{(\alpha^2 + (n + \beta)^2)}\right)$$

$$= -\frac{\alpha^2 + \beta^2 + n^2}{(\alpha^2 + (n - \beta)^2)(\alpha^2 + (n + \beta)^2)}. \tag{6.6}$$

The sum of the two principal parts corresponding to the simple poles $\alpha = \pm\beta i$ equals

$$\frac{i}{4}\left(-\frac{1}{\beta(\alpha + \beta i)} + \frac{1}{\beta(\alpha - \beta i)}\right) = -\frac{1}{2(\alpha^2 + \beta^2)}. \tag{6.7}$$

Summing (6.6) over n, $1 \le n < \infty$, adding (6.7) to this sum, applying the Mittag–Leffler theorem, and lastly multiplying both sides by $-\alpha$, we conclude that

$$\frac{\pi}{2}\frac{\sinh(2\pi\alpha)}{\cosh(2\pi\alpha)-\cos(2\pi\beta)} = \alpha\sum_{n=1}^{\infty}\frac{\alpha^2+\beta^2+n^2}{(\alpha^2+(n-\beta)^2)(\alpha^2+(n+\beta)^2)} + \frac{\alpha}{2(\alpha^2+\beta^2)}. \tag{6.8}$$

Next, examine

$$h(\alpha) := \frac{\pi}{4\alpha}\coth\{\pi(\alpha - i\beta)\}\coth\{\pi(\beta+i\alpha)\},$$

which has simple poles at $\alpha = 0$, $(n+\beta)i$, and $n + i\beta$, for each nonzero integer n, and a double pole at $\alpha = i\beta$. First, the principal part of $h(\alpha)$ about $\alpha = 0$ is easily seen to be

$$\frac{\pi i}{4\alpha}\cot(\pi\beta)\coth(\pi\beta). \tag{6.9}$$

Next, simple calculations yield

$$R_{(n+\beta)i} = \frac{i\coth(\pi n)}{4(n+\beta)} \quad \text{and} \quad R_{n+i\beta} = -\frac{i\coth(\pi n)}{4(n+i\beta)},$$

for each nonzero integer n. Thus, the sum of the principal parts around the four poles $(\pm n + \beta)i$ and $\pm n + i\beta$, $n > 0$, is equal to

$$\begin{aligned}
&\frac{i\coth(\pi n)}{4}\left(\frac{1}{(n+\beta)(\alpha - ni - \beta i)} + \frac{1}{(n-\beta)(\alpha + ni - \beta i)}\right.\\
&\qquad\left. - \frac{1}{(n+\beta i)(\alpha - n - \beta i)} - \frac{1}{(n-\beta i)(\alpha + n - \beta i)}\right)\\
&= \frac{in\coth(\pi n)}{2}\left(\frac{\alpha - 2\beta i}{(n^2-\beta^2)((\alpha-\beta i)^2+n^2)} + \frac{-\alpha+2\beta i}{(n^2+\beta^2)((\alpha-\beta i)^2-n^2)}\right).
\end{aligned} \tag{6.10}$$

Lastly, we must calculate the principal part of $h(\alpha)$ arising from the double pole at $\alpha = \beta i$. It will be convenient to set $\alpha = z + \beta i$. We then calculate the principal part around $z = 0$ of

$$\begin{aligned}
H(z) := h(\alpha) &= -\frac{\pi i}{4(z+\beta i)}\coth(\pi z)\cot(\pi z)\\
&= -\frac{\pi}{4\beta}\left(1 - \frac{z}{\beta i} + \cdots\right)\left(\frac{1}{\pi z} + \frac{\pi z}{3} + \cdots\right)\left(\frac{1}{\pi z} - \frac{\pi z}{3} + \cdots\right)\\
&= -\frac{1}{4\beta\pi}\left(\frac{1}{(\alpha-\beta i)^2} - \frac{1}{\beta i(\alpha - \beta i)} + \cdots\right).
\end{aligned} \tag{6.11}$$

Summing (6.10) on n, $1 \le n < \infty$, adding (6.9) and the principal part from (6.11) to this sum, invoking the Mittag–Leffler theorem, and then multiplying both sides by α, we deduce that

$$\begin{aligned}
&\frac{\pi}{4}\coth\{\pi(\alpha - i\beta)\}\coth\{\pi(\beta + i\alpha)\} \\
&\quad = \frac{\pi i}{4}\cot(\pi\beta)\coth(\pi\beta) - \frac{\alpha}{4\beta\pi}\left(\frac{1}{(\alpha-\beta i)^2} - \frac{1}{\beta i(\alpha - \beta i)}\right) \\
&\qquad + \frac{\alpha i}{2}\sum_{n=1}^{\infty} n\coth(\pi n) \\
&\qquad \times \left(\frac{\alpha - 2\beta i}{(n^2-\beta^2)(\alpha^2-\beta^2+n^2-2\alpha\beta i)} + \frac{-\alpha + 2\beta i}{(n^2+\beta^2)(\alpha^2-\beta^2-n^2-2\alpha\beta i)}\right) \\
&\quad = \frac{\pi i}{4}\cot(\pi\beta)\coth(\pi\beta) - \frac{\alpha}{4\beta\pi}\left(\frac{(\alpha+\beta i)^2}{(\alpha^2+\beta^2)^2} - \frac{\alpha + \beta i}{\beta i(\alpha^2+\beta^2)}\right) \\
&\qquad + \frac{\alpha i}{2}\sum_{n=1}^{\infty} n\coth(\pi n)\left(\frac{(\alpha - 2\beta i)(\alpha^2-\beta^2+n^2+2\alpha\beta i)}{(n^2-\beta^2)((\alpha^2-\beta^2+n^2)^2+4\alpha^2\beta^2)}\right. \\
&\qquad\qquad \left. - \frac{(\alpha-2\beta i)(\alpha^2-\beta^2-n^2+2\alpha\beta i)}{(n^2+\beta^2)((\alpha^2-\beta^2-n^2)^2+4\alpha^2\beta^2)}\right). \qquad (6.12)
\end{aligned}$$

In order to simplify our calculation above, we shall assume that α and β are real. Note then that both sides of (6.5) are real. Thus, we only need to determine the real part of the right side of (6.12). Hence,

$$\begin{aligned}
&\operatorname{Re}\left(\frac{\pi}{4}\coth\{\pi(\alpha - i\beta)\}\coth\{\pi(\beta + i\alpha)\}\right) \\
&= -\frac{\alpha}{4\beta\pi}\left(\frac{\alpha^2-\beta^2}{(\alpha^2+\beta^2)^2} - \frac{1}{\alpha^2+\beta^2}\right) \\
&\quad + \frac{\alpha}{2}\sum_{n=1}^{\infty} n\coth(\pi n)\left(\frac{-2\alpha^2\beta + 2\alpha^2\beta - 2\beta^3 + 2\beta n^2}{(n^2-\beta^2)((\alpha^2-\beta^2+n^2)^2+4\alpha^2\beta^2)}\right. \\
&\qquad\qquad \left. + \frac{2\alpha^2\beta - 2\alpha^2\beta + 2\beta^3 + 2\beta n^2}{(n^2+\beta^2)((\alpha^2-\beta^2-n^2)^2+4\alpha^2\beta^2)}\right) \\
&= \frac{\alpha\beta}{2\pi(\alpha^2+\beta^2)^2} \\
&\quad + \alpha\beta\sum_{n=1}^{\infty} n\coth(\pi n)\left(\frac{1}{(\alpha^2-\beta^2+n^2)^2+4\alpha^2\beta^2} + \frac{1}{(\alpha^2-\beta^2-n^2)^2+4\alpha^2\beta^2}\right).
\end{aligned}$$

(6.13)

So far, from (6.8) and (6.13), we have shown that

$$\operatorname{Re}\left(\frac{\pi}{4}\coth\{\pi(\alpha - i\beta)\}\coth\{\pi(\beta + i\alpha)\}\right) - \frac{\pi}{2}\frac{\sinh(2\pi\alpha)}{\cosh(2\pi\alpha) - \cos(2\pi\beta)}$$
$$= \alpha\beta\sum_{n=1}^{\infty} n\left(\frac{2}{e^{2\pi n} - 1} + 1\right)\left(\frac{1}{(\alpha^2 + (n-\beta)^2)(\alpha^2 + (n+\beta)^2)}\right.$$
$$\left. + \frac{1}{(\beta^2 + (n-\alpha)^2)(\beta^2 + (n+\alpha)^2)}\right)$$
$$-\alpha\sum_{n=1}^{\infty}\frac{\alpha^2 + \beta^2 + n^2}{(\alpha^2 + (n-\beta)^2)(\alpha^2 + (n+\beta)^2)} - \frac{\alpha}{2(\alpha^2 + \beta^2)} + \frac{\alpha\beta}{2\pi(\alpha^2 + \beta^2)^2}. \tag{6.14}$$

Hence, by (6.14),

$$\frac{\pi}{4}\left(\coth\{\pi(\alpha - i\beta)\}\coth\{\pi(\beta + i\alpha)\} + \coth\{\pi(\beta - i\alpha)\}\coth\{\pi(\alpha + i\beta)\}\right)$$
$$-\frac{\pi}{2}\frac{\sinh(2\pi\alpha)}{\cosh(2\pi\alpha) - \cos(2\pi\beta)} - \frac{\pi}{2}\frac{\sinh(2\pi\beta)}{\cosh(2\pi\beta) - \cos(2\pi\alpha)}$$
$$= 4\alpha\beta\sum_{n=1}^{\infty}\frac{n}{(e^{2\pi n} - 1)(\alpha^2 + (n-\beta)^2)(\alpha^2 + (n+\beta)^2)}$$
$$+ 4\alpha\beta\sum_{n=1}^{\infty}\frac{n}{(e^{2\pi n} - 1)(\beta^2 + (n-\alpha)^2)(\beta^2 + (n+\alpha)^2)}$$
$$+ 2\alpha\beta\sum_{n=1}^{\infty}\frac{n}{(\alpha^2 + (n-\beta)^2)(\alpha^2 + (n+\beta)^2)}$$
$$+ 2\alpha\beta\sum_{n=1}^{\infty}\frac{n}{(\beta^2 + (n-\alpha)^2)(\beta^2 + (n+\alpha)^2)}$$
$$-\alpha\sum_{n=1}^{\infty}\frac{\alpha^2 + \beta^2 + n^2}{(\alpha^2 + (n-\beta)^2)(\alpha^2 + (n+\beta)^2)}$$
$$-\beta\sum_{n=1}^{\infty}\frac{\alpha^2 + \beta^2 + n^2}{(\beta^2 + (n-\alpha)^2)(\beta^2 + (n+\alpha)^2)}$$
$$-\frac{\alpha}{2(\alpha^2 + \beta^2)} - \frac{\beta}{2(\alpha^2 + \beta^2)} + \frac{\alpha\beta}{\pi(\alpha^2 + \beta^2)^2}. \tag{6.15}$$

Lastly, we observe that

$$2\alpha\beta\sum_{n=1}^{\infty}\frac{n}{(\alpha^2 + (n-\beta)^2)(\alpha^2 + (n+\beta)^2)} - \alpha\sum_{n=1}^{\infty}\frac{\alpha^2 + \beta^2 + n^2}{(\alpha^2 + (n-\beta)^2)(\alpha^2 + (n+\beta)^2)}$$
$$= -\alpha\sum_{n=1}^{\infty}\frac{1}{\alpha^2 + (n+\beta)^2}. \tag{6.16}$$

Using (6.16) and its analogue, with the roles of α and β reversed, in (6.15), we complete the proof of (6.5). During the proof we assumed that no poles coalesce and that α and β are real. These restrictions may now be removed by analytic continuation.

Entry 7 (Formula (2), p. 291). *Let*

$$\varphi(\alpha, \beta) = \frac{\pi/2}{e^{2\pi\alpha} + 2e^{\pi\alpha}\cos(\pi\beta) + 1} + \alpha \sum_{n=0}^{\infty} \frac{1}{\alpha^2 + (2n+1+\beta)^2}$$
$$+ 4\alpha\beta \sum_{n=0}^{\infty} \frac{2n+1}{(e^{(2n+1)\pi} + 1)(\alpha^2 + (2n+1+\beta)^2)(\alpha^2 + (2n+1-\beta)^2)},$$

where α and β are arbitrary complex numbers. Then

$$\varphi(\alpha, \beta) + \varphi(\beta, \alpha) = \frac{\pi}{4} + \frac{\pi}{4} \frac{\cosh\{\pi(\alpha-\beta)\} - \cos\{\pi(\alpha-\beta)\}}{\{\cosh(\pi\alpha) + \cos(\pi\beta)\}\{\cosh(\pi\beta) + \cos(\pi\alpha)\}}. \tag{7.1}$$

Proof. The details are similar to those in the previous proof. For brevity, set

$$D(\alpha, \beta) = \{\cosh(\pi\alpha) + \cos(\pi\beta)\}\{\cosh(\pi\beta) + \cos(\pi\alpha)\}$$

and define

$$\begin{aligned}\frac{4}{\pi} F(\alpha, \beta) :&= \frac{e^{-\pi\alpha}}{\cosh(\pi\alpha) + \cos(\pi\beta)} + \frac{e^{-\pi\beta}}{\cosh(\pi\beta) + \cos(\pi\alpha)} \\ &\quad - \frac{\cosh\{\pi(\alpha-\beta)\} - \cos\{\pi(\alpha-\beta)\}}{D(\alpha, \beta)} \\ &= \frac{e^{-\pi(\alpha+\beta)} + e^{-\pi\alpha}\cos(\pi\alpha) + e^{-\pi\beta}\cos(\pi\beta) + \cos\{\pi(\alpha-\beta)\}}{D(\alpha, \beta)} \\ &= 1 - \frac{\sinh(\pi\alpha)}{\cosh(\pi\alpha) + \cos(\pi\beta)} - \frac{\sinh(\pi\beta)}{\cosh(\pi\beta) + \cos(\pi\alpha)} \\ &\quad + \frac{\sinh(\pi\alpha)\sinh(\pi\beta) + \sin(\pi\alpha)\sin(\pi\beta)}{D(\alpha, \beta)}.\end{aligned} \tag{7.2}$$

By an elementary calculation,

$$\begin{aligned}&\{\cosh(2a) + \cos(2b)\}\{\cosh(2b) + \cos(2a)\} \\ &\qquad = \cosh(a - ib)\cosh(a + ib)\cosh(b - ia)\cosh(b + ia).\end{aligned} \tag{7.3}$$

Employing (7.3), (4.5), and (4.3) in (7.2), we find that

$$\frac{4}{\pi}F(\alpha,\beta)=1-\frac{\sinh(\pi\alpha)}{\cosh(\pi\alpha)+\cos(\pi\beta)}-\frac{\sinh(\pi\beta)}{\cosh(\pi\beta)+\cos(\pi\alpha)}$$
$$+\tfrac{1}{2}\tanh\{\tfrac{1}{2}\pi(\alpha+\beta i)\}\tanh\{\tfrac{1}{2}\pi(\beta-\alpha i)\}$$
$$+\tfrac{1}{2}\tanh\{\tfrac{1}{2}\pi(\beta+\alpha i)\}\tanh\{\tfrac{1}{2}\pi(\alpha-\beta i)\}. \tag{7.4}$$

In view of the definition of $F(\alpha, \beta)$ in (7.2), the sought conclusion (7.1), and the representation (7.4), it suffices to prove that

$$\frac{\pi}{4}\left(\frac{\sinh(\pi\alpha)}{\cosh(\pi\alpha)+\cos(\pi\beta)}+\frac{\sinh(\pi\beta)}{\cosh(\pi\beta)+\cos(\pi\alpha)}\right.$$
$$-\tfrac{1}{2}\tanh\{\tfrac{1}{2}\pi(\alpha+\beta i)\}\tanh\{\tfrac{1}{2}\pi(\beta-\alpha i)\}$$
$$\left.-\tfrac{1}{2}\tanh\{\tfrac{1}{2}\pi(\beta+\alpha i)\}\tanh\{\tfrac{1}{2}\pi(\alpha-\beta i)\}\right)$$
$$=\alpha\sum_{n=0}^{\infty}\frac{1}{\alpha^2+(2n+1+\beta)^2}+\beta\sum_{n=0}^{\infty}\frac{1}{\beta^2+(2n+1+\alpha)^2}$$
$$+4\alpha\beta\sum_{n=0}^{\infty}\frac{2n+1}{(e^{(2n+1)\pi}+1)(\alpha^2+(2n+1+\beta)^2)(\alpha^2+(2n+1-\beta)^2)}$$
$$+4\alpha\beta\sum_{n=0}^{\infty}\frac{2n+1}{(e^{(2n+1)\pi}+1)(\beta^2+(2n+1+\alpha)^2)(\beta^2+(2n+1-\alpha)^2)}. \tag{7.5}$$

We first calculate the partial fraction expansion of

$$g(\alpha):=\frac{\pi}{4\alpha}\frac{\sinh(\pi\alpha)}{\cosh(\pi\alpha)+\cos(\pi\beta)}.$$

Now, $g(\alpha)$ has simple poles at $\alpha=(2n+1\pm\beta)i$, for each integer n. If we temporarily assume that $2m+1+\beta\neq 2n+1-\beta$, for every pair of integers m, n, then these poles are simple. The residues equal

$$R_{(2n+1\pm\beta)i}=-\frac{i}{4(2n+1\pm\beta)}.$$

Thus, the sum of the principal parts of $g(\alpha)$ arising from the four simple poles $(\pm r\pm\beta)i$, where $r=2n+1$ and $n\geq 0$, equals

$$-\frac{i}{4}\left(\frac{1}{(r-\beta)(\alpha-(r-\beta)i)}-\frac{1}{(r+\beta)(\alpha+(r+\beta)i)}\right.$$
$$\left.+\frac{1}{(r+\beta)(\alpha-(r+\beta)i)}-\frac{1}{(r-\beta)(\alpha+(r-\beta)i)}\right)$$
$$=\frac{1}{2}\left(\frac{1}{\alpha^2+(r-\beta)^2}+\frac{1}{\alpha^2+(r+\beta)^2}\right)$$
$$=\frac{\alpha^2+\beta^2+r^2}{(\alpha^2+(r-\beta)^2)(\alpha^2+(r+\beta)^2)}.$$

It follows that

$$g(\alpha) = \sum_{n=0}^{\infty} \frac{\alpha^2 + \beta^2 + (2n+1)^2}{(\alpha^2 + (2n+1-\beta)^2)(\alpha^2 + (2n+1+\beta)^2)}. \tag{7.6}$$

Next, consider

$$h(\alpha) := \frac{\pi}{8\alpha} \tanh\{\tfrac{1}{2}\pi(\alpha + \beta i)\} \tanh\{\tfrac{1}{2}\pi(\beta - \alpha i)\}.$$

Observe that $h(\alpha)$ has simple poles at $\alpha = 0$, $\alpha = (2n + 1 - \beta)i$, and $\alpha = -(2n+1) - \beta i$, for each integer n, provided that β is not an odd integer. The principal part of h about $\alpha = 0$ is readily seen to equal

$$\frac{i\pi}{8\alpha} \tan(\tfrac{1}{2}\pi\beta) \tanh(\tfrac{1}{2}\pi\beta). \tag{7.7}$$

Calculating the residues of $h(\alpha)$ at the remaining poles, we find that

$$R_{(2n+1-\beta)i} = -\frac{i}{4(2n+1-\beta)} \tanh\{\tfrac{1}{2}\pi(2n+1)\}$$

and

$$R_{-(2n+1)-\beta i} = \frac{i}{4(2n+1+\beta i)} \tanh\{\tfrac{1}{2}\pi(2n+1)\}.$$

Therefore, the sum of the principal parts arising from the four poles $(\pm r - \beta)i$ and $\pm r - \beta i$, where $r = 2n + 1$ and $n \geq 0$, equals

$$\begin{aligned}
&-\frac{i}{4}\tanh(\tfrac{1}{2}\pi r)\left(\frac{1}{(r-\beta)(\alpha - ri + \beta i)} + \frac{1}{(r+\beta)(\alpha + ri + \beta i)}\right.\\
&\qquad\qquad \left. - \frac{1}{(r+\beta i)(\alpha + r + \beta i)} - \frac{1}{(r - \beta i)(\alpha - r + \beta i)}\right)\\
&= -\frac{ir}{2}\tanh(\tfrac{1}{2}\pi r)\left(\frac{\alpha + 2\beta i}{(r^2 - \beta^2)((\alpha+\beta i)^2 + r^2)} - \frac{\alpha + 2\beta i}{(r^2+\beta^2)((\alpha+\beta i)^2 - r^2)}\right)\\
&= -\frac{ir}{2}\tanh(\tfrac{1}{2}\pi r)\left(\frac{(\alpha + 2\beta i)(\alpha^2 + r^2 - \beta^2 - 2\alpha\beta i)}{(r^2 - \beta^2)((\alpha^2 + r^2 - \beta^2)^2 + 4\alpha^2\beta^2)}\right.\\
&\qquad \left. - \frac{(\alpha + 2\beta i)(\alpha^2 - r^2 - \beta^2 - 2\alpha\beta i)}{(r^2 + \beta^2)((\alpha^2 - r^2 - \beta^2)^2 + 4\alpha^2\beta^2)}\right).
\end{aligned} \tag{7.8}$$

Proceeding as in the proof of Entry 6, so that we might simplify our calculations, we shall temporarily assume that α and β are real. Thus, both sides of (7.5) are real. We therefore only need to determine the real part of the far right side of (7.8). After a straightforward calculation, we find that the real part of the far right side of (7.8) equals

$$\beta r \tanh(\tfrac{1}{2}\pi r)\left(\frac{1}{(\alpha^2 + r^2 - \beta^2)^2 + 4\alpha^2\beta^2} + \frac{1}{(\alpha^2 - r^2 - \beta^2)^2 + 4\alpha^2\beta^2}\right)$$
$$= \beta r \tanh(\tfrac{1}{2}\pi r)\left(\frac{1}{(\alpha^2 + (r-\beta)^2)(\alpha^2 + (r+\beta)^2)} + \frac{1}{(\beta^2 + (r-\alpha)^2)(\beta^2 + (r+\alpha)^2)}\right). \tag{7.9}$$

Hence, summing (7.9) on n, $0 \le n < \infty$, and using (7.7), we find that

$$\begin{aligned}\operatorname{Re}(h(\alpha)) = \beta \sum_{n=0}^{\infty} (2n+1)\tanh\{\tfrac{1}{2}\pi(2n+1)\} \\ \times \left(\frac{1}{(\alpha^2 + (2n+1-\beta)^2)(\alpha^2 + (2n+1+\beta)^2)}\right. \\ \left. + \frac{1}{(\beta^2 + (2n+1-\alpha)^2)(\beta^2 + (2n+1+\alpha)^2)}\right).\end{aligned} \tag{7.10}$$

Amalgamating (7.6) and (7.10) and their analogues with α and β interchanged, we deduce that

$$\begin{aligned}\alpha g(\alpha) + \beta g(\beta) - \alpha h(\alpha) - \beta h(\beta) = \alpha \sum_{n=0}^{\infty} \frac{\alpha^2 + \beta^2 + (2n+1)^2}{(\alpha^2 + (2n+1-\beta)^2)(\alpha^2 + (2n+1+\beta)^2)} \\ + \beta \sum_{n=0}^{\infty} \frac{\alpha^2 + \beta^2 + (2n+1)^2}{(\beta^2 + (2n+1-\alpha)^2)(\beta^2 + (2n+1+\alpha)^2)} \\ - 2\alpha\beta \sum_{n=0}^{\infty} (2n+1)\tanh\{\tfrac{1}{2}\pi(2n+1)\} \\ \times \left(\frac{1}{(\alpha^2 + (2n+1-\beta)^2)(\alpha^2 + (2n+1+\beta)^2)}\right. \\ \left. + \frac{1}{(\beta^2 + (2n+1-\alpha)^2)(\beta^2 + (2n+1+\alpha)^2)}\right).\end{aligned} \tag{7.11}$$

Now, with $r = 2n + 1$,

$$\tanh(\tfrac{1}{2}\pi r) = 1 - \frac{2}{e^{\pi r} + 1} \tag{7.12}$$

and

$$\alpha(\alpha^2 + \beta^2 + r^2) - 2\alpha\beta r = \alpha(\alpha^2 + (r-\beta)^2). \tag{7.13}$$

Using (7.12), (7.13), and the analogue of (7.13) with α and β interchanged, we find that (7.11) reduces to (7.5). By analytic continuation, the restrictions imposed on α and β during our proof of (7.5) may now be removed. Thus, the proof of Entry 7 is complete.

Ramanujan's formulation of Entry 7 is somewhat imprecise.

Entry 8 (Formula (7), p. 293). *As customary, let* $\psi(z) = \Gamma'(z)/\Gamma(z)$, *and let* γ *denote Euler's constant. Then*

$$\psi(z+1)+\gamma = \frac{1}{2z} - \frac{1}{2\pi z^2} + \frac{\pi\cot(\pi z)}{e^{2\pi z}-1} + \sum_{k=1}^{\infty} \frac{z^2}{k(k^2+z^2)} + 4\sum_{k=1}^{\infty} \frac{kz^2}{(e^{2\pi k}-1)(k^4-z^4)}. \tag{8.1}$$

Harkening back to his convention in Chapter 6 on divergent series (Part I [2, p. 138]), Ramanujan writes Entry 8 in an unorthodox fashion; for each positive integer n,

$$\sum_{k=1}^{n} \frac{1}{k} = \frac{1}{2n} - \frac{1}{2\pi n^2} + \frac{\pi\cot(\pi n)}{e^{2\pi n}-1} + \sum_{k=1}^{\infty} \frac{n^2}{k(k^2+n^2)} + 4\sum_{k=1}^{\infty} \frac{kn^2}{(e^{2\pi k}-1)(k^4-n^4)}. \tag{8.2}$$

If we formally set $z = n$ in (8.1) and use the classical result (Part I [2, p. 138])

$$\psi(z+1) = -\gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{z+k}\right), \tag{8.3}$$

we see that (8.1) reduces to (8.2).

The series representation for $\psi(z)$ in Entry 8 appears to be new. The expansion is more rapidly convergent than other classical representations, such as (8.3).

Proof. Let

$$f(z) := \frac{\pi\cot(\pi z)}{z(e^{2\pi z}-1)},$$

which has simple poles at each nonzero integer and at each nonzero integral multiple of i. At $z = 0$, $f(z)$ has a triple pole.

First, it is easy to show that the principal part of f about $z = 0$ equals

$$\frac{1}{2\pi z^3} - \frac{1}{2z^2}. \tag{8.4}$$

Next, elementary calculations yield, for each positive integer k,

$$R_k = \frac{1}{k(e^{2\pi k} - 1)}, \tag{8.5}$$

$$R_{-k} = -\frac{1}{k(e^{-2\pi k} - 1)} = \frac{1}{k}\left(\frac{1}{e^{2\pi k} - 1} + 1\right), \tag{8.6}$$

$$R_{ki} = \frac{\cot(\pi k i)}{2ki} = -\frac{1}{k}\left(\frac{1}{e^{2\pi k} - 1} + \frac{1}{2}\right), \tag{8.7}$$

and

$$R_{-ki} = -\frac{1}{k}\left(\frac{1}{e^{2\pi k} - 1} + \frac{1}{2}\right). \tag{8.8}$$

From (8.4)–(8.8), we deduce the partial fraction decomposition

$$\begin{aligned}
f(z) &= \frac{1}{2\pi z^3} - \frac{1}{2z^2} \\
&\quad + \sum_{k=1}^{\infty} \frac{1}{k}\left\{\frac{1}{(e^{2\pi k} - 1)(z - k)} + \frac{1}{(e^{2\pi k} - 1)(z + k)} + \frac{1}{z + k}\right\} \\
&\quad - \sum_{k=1}^{\infty} \frac{1}{k}\left\{\left(\frac{1}{e^{2\pi k} - 1} + \frac{1}{2}\right)\frac{1}{z - ki} + \left(\frac{1}{e^{2\pi k} - 1} + \frac{1}{2}\right)\frac{1}{z + ki}\right\} \\
&= \frac{1}{2\pi z^3} - \frac{1}{2z^2} + \sum_{k=1}^{\infty} \frac{1}{k}\left\{\frac{2z}{(e^{2\pi k} - 1)(z^2 - k^2)} + \frac{1}{z + k}\right\} \\
&\quad - \sum_{k=1}^{\infty} \frac{1}{k}\left\{\frac{2z}{(e^{2\pi k} - 1)(z^2 + k^2)} + \frac{z}{(z^2 + k^2)}\right\} \\
&= \frac{1}{2\pi z^3} - \frac{1}{2z^2} + \sum_{k=1}^{\infty}\left\{\frac{4kz}{(e^{2\pi k} - 1)(z^4 - k^4)} + \frac{1}{k(z + k)} - \frac{z}{k(z^2 + k^2)}\right\} \\
&= \frac{1}{2\pi z^3} - \frac{1}{2z^2} - \sum_{k=1}^{\infty} \frac{4kz}{(e^{2\pi k} - 1)(k^4 - z^4)} - \sum_{k=1}^{\infty} \frac{z}{k(z^2 + k^2)} \\
&\quad + \frac{1}{z}(\Gamma(z + 1) + \gamma),
\end{aligned}$$

by (8.3). The desired expansion (8.1) is now apparent.

Entry 9 (Formula (8), p. 293). *If n is a positive integer, then*

$$\sum_{k=1}^{n} \frac{1}{2k-1} = \frac{\pi}{2}\frac{\tan(\pi n)}{e^{2\pi n}+1} + \sum_{k=0}^{\infty} \frac{4n^2}{(2k+1)\{4n^2+(2k+1)^2\}} \tag{9.1}$$
$$-16\sum_{k=0}^{\infty} \frac{n^2(2k+1)}{(e^{(2k+1)\pi}+1)\{(2k+1)^4-16n^4\}}.$$

We have replaced n by $2n$ in Ramanujan's formulation.

Observe that (9.1) is an analogue of (8.2), except that, in contrast to (9.1), (8.2) is devoid of meaning. By (8.3), the left side of (9.1) can be written as

$$\psi(2n+1) - \tfrac{1}{2}\psi(n+1) + \tfrac{1}{2}\gamma.$$

We shall establish a partial fraction decomposition from which (9.1) follows by setting $z = n$.

Proof. Let

$$f(z) := \frac{\pi \tan(\pi z)}{2z(e^{2\pi z}-1)},$$

which has simple poles at $z = \frac{1}{2}(2k+1)$ and $z = \frac{1}{2}(2k+1)i$, for each integer k. For $k \geq 0$, routine calculations show that

$$R_{(2k+1)/2} = -\frac{1}{(2k+1)(e^{(2k+1)\pi}+1)}, \tag{9.2}$$

$$R_{-(2k+1)/2} = \frac{1}{(2k+1)(e^{-(2k+1)\pi}+1)} = -\frac{1}{2k+1}\left(\frac{1}{e^{(2k+1)\pi}+1} - 1\right), \tag{9.3}$$

$$R_{(2k+1)i/2} = -\frac{\tanh\{\frac{1}{2}(2k+1)\pi\}}{2(2k+1)} = -\frac{1}{2(2k+1)}\left(1 - \frac{2}{e^{(2k+1)\pi}+1}\right), \tag{9.4}$$

and

$$R_{-(2k+1)i/2} = \frac{1}{2(2k+1)}\left(\frac{2}{e^{(2k+1)\pi}+1} - 1\right). \tag{9.5}$$

From (9.2)–(9.5), we calculate the partial fraction decomposition

$$f(z) = -\sum_{k=0}^{\infty} \frac{1}{2k+1}\left\{\frac{1}{(e^{(2k+1)\pi}+1)(z-(2k+1)/2)} + \frac{1}{(e^{(2k+1)\pi}+1)(z+(2k+1)/2)} - \frac{1}{z+(2k+1)/2}\right\}$$
$$+\sum_{k=0}^{\infty} \frac{1}{2(2k+1)}\left\{\left(\frac{2}{e^{(2k+1)\pi}+1} - 1\right)\frac{1}{z-(2k+1)i/2} + \left(\frac{2}{e^{(2k+1)\pi}+1} - 1\right)\frac{1}{z+(2k+1)i/2}\right\}$$

$$= -\sum_{k=0}^{\infty} \frac{1}{2k+1} \left\{ \frac{2z}{(e^{(2k+1)\pi}+1)(z^2-(2k+1)^2/4)} - \frac{1}{z+(2k+1)/2} \right\}$$
$$+ \sum_{k=0}^{\infty} \frac{1}{2k+1} \left\{ \frac{2z}{(e^{(2k+1)\pi}+1)(z^2+(2k+1)^2/4)} - \frac{z}{z^2+(2k+1)^2/4} \right\}$$
$$= -\sum_{k=0}^{\infty} \frac{(2k+1)z}{(e^{(2k+1)\pi}+1)(z^4-(2k+1)^4/16)}$$
$$- \sum_{k=0}^{\infty} \left(\frac{z}{(2k+1)(z^2+(2k+1)^2/4)} - \frac{1}{(2k+1)(z+(2k+1)/2)} \right)$$
$$= -\sum_{k=0}^{\infty} \frac{16(2k+1)z}{(e^{(2k+1)\pi}+1)(16z^4-(2k+1)^4)} - \sum_{k=0}^{\infty} \frac{4z}{(2k+1)(4z^2+(2k+1)^2)}$$
$$+ \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{1}{2k+1} - \frac{1}{2z+2k+1} \right). \tag{9.6}$$

Setting $z = n$, where n is a positive integer, in (9.6), we deduce (9.1).

Entry 10 (Formula (9), p. 293). *We have*

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{z+2k+1} = \frac{1}{2z} - \frac{\pi}{2} \frac{\sec(\frac{1}{2}\pi z)}{e^{\pi z}-1} + 2z \sum_{k=0}^{\infty} \frac{(-1)^k}{(e^{(2k+1)\pi}-1)\{(2k+1)^2-z^2\}}$$
$$+ z \sum_{k=1}^{\infty} \frac{\operatorname{sech}(k\pi)}{4k^2+z^2}. \tag{10.1}$$

Entry 10 follows immediately from Entry 5.

We shall consider the next two expansions only in a formal sense.

Entry 11 (p. 314). *If $\varphi(z)$ is entire and p is any complex number, then*

$$\frac{\pi}{2} \frac{p^z \varphi(z)}{\zeta(z)\cos(\frac{1}{2}\pi z)} = \pi \sum_{n=1}^{\infty} \left(\frac{-2\pi}{p} \right)^n \frac{\varphi(-n)}{n!\,\zeta(n+1)(z+n)}$$
$$- \sum_{n=1}^{\infty} \frac{(-1)^n p^{2n+1} \varphi(2n+1)}{\zeta(2n+1)(z-2n-1)}$$
$$+ \textit{terms involving roots of } \zeta(z). \tag{11.1}$$

Entry 12 (p. 314). *If $\varphi(z)$ is entire and p is any complex number, then*

$$\frac{\pi}{2} \frac{p^z \varphi(z)}{\Gamma(\frac{1}{2}(z+1))\zeta(z)\cos(\frac{1}{2}\pi z)} = \sqrt{\pi} \sum_{n=1}^{\infty} \left(\frac{\pi}{p} \right)^{2n} \frac{(-1)^n \varphi(-2n)}{n!\,\zeta(2n+1)(z+2n)}$$
$$- \sum_{n=1}^{\infty} \frac{(-1)^n p^{2n+1} \varphi(2n+1)}{n!\,\zeta(2n+1)(z-2n-1)}$$
$$+ \textit{terms involving roots of } \zeta(z). \tag{12.1}$$

Ramanujan did not employ the notation $\zeta(x)$, but used S_x instead, a common notation in his time. The phrases "+ terms involving roots of S_x" are recorded in a different color or shade of ink. We conjecture that Entries 11 and 12 were discovered in India, and that after arriving in England, Ramanujan learned from Hardy that $\zeta(z)$ has complex zeros in addition to the "trivial zeros" $-2, -4, \ldots$, which Ramanujan would have discovered from the functional equation of $\zeta(z)$. Ramanujan recorded this functional equation as Entry 4 of Chapter 7, and we have reconstructed his nonrigorous "proof" (Part I [2, pp. 153–154]). Rigorous formulations of Entries 11 and 12 would require explicit determinations of the "terms involving roots of $\zeta(z)$" and hypotheses on the function $\varphi(z)$, including its rate of growth as $|z|$ tends to ∞.

In fact, Ramanujan recorded two additional terms in each of (11.1) and (12.1), namely

$$\frac{p\varphi(1)}{S_1(1-z)} \quad \text{and} \quad \frac{\pi\varphi(0)}{S_1 z}$$

and

$$-\frac{p\varphi(1)}{S_1(1-z)} \quad \text{and} \quad \sqrt{\pi}\,\frac{\varphi(0)}{S_1 z},$$

respectively. Now, of course, S_1 "$=\infty$," and so in reality these four terms are equal to 0, unless $\varphi(z)$ has simple poles at $z = 0$ and $z = 1$. As the calculations below evince, if this assumption is made, Ramanujan's "extra" terms are correct.

Proof of Entry 11. Let

$$f(z) := \frac{\pi}{2}\,\frac{p^z\varphi(z)}{\zeta(z)\cos(\frac{1}{2}\pi z)}.$$

Observe that f has simple poles at $z = -2n$, where n is a positive integer, because $\zeta(z)$ has zeros at these values (Titchmarsh [2, p. 19]). There are simple poles at each integer $2n + 1$, $n \neq 0$, for $\zeta(z)$ has a simple pole at $z = 1$. To calculate the residues, it seems best to use the functional equation of $\zeta(z)$ in the form (Titchmarsh [2, p. 25])

$$\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin(\tfrac{1}{2}\pi z). \tag{11.2}$$

Straightforward calculations yield

$$R_{-2n} = \pi\left(\frac{2\pi}{p}\right)^{2n}\frac{\varphi(-2n)}{(2n)!\,\zeta(2n+1)}, \qquad n \geq 1, \tag{11.3}$$

$$R_{2n+1} = -\frac{(-1)^n p^{2n+1}\varphi(2n+1)}{\zeta(2n+1)}, \qquad n \geq 1, \tag{11.4}$$

and

$$R_{-2n-1}=\frac{(-1)^n p^{-2n-1}\varphi(-2n-1)}{\zeta(-2n-1)}$$
$$=-\pi\left(\frac{2\pi}{p}\right)^{2n+1}\frac{\varphi(-2n-1)}{(2n+1)!\,\zeta(2n+2)},\qquad n\geq 0, \tag{11.5}$$

where in the last equality we employed (11.2). Expanding $f(z)$ into partial fractions with the use of (11.3)–(11.5), we deduce that

$$f(z)=\pi\sum_{n=1}^{\infty}\left(\frac{2\pi}{p}\right)^{2n}\frac{\varphi(-2n)}{(2n)!\,\zeta(2n+1)(z+2n)}-\sum_{n=1}^{\infty}\frac{(-1)^n p^{2n+1}\varphi(2n+1)}{\zeta(2n+1)(z-2n-1)}$$
$$-\pi\sum_{n=1}^{\infty}\left(\frac{2\pi}{p}\right)^{2n+1}\frac{\varphi(-2n-1)}{(2n+1)!\,\zeta(2n+2)(z+2n+1)}.$$

Combining the first and third sums on the right side above, we formally complete the derivation of (11.1), except that we have not considered the nonreal zeros of $\zeta(z)$.

Proof of Entry 12. Let

$$f(z):=\frac{\pi}{2}\frac{p^z\varphi(z)}{\Gamma(\frac{1}{2}(z+1))\zeta(z)\cos(\frac{1}{2}\pi z)},$$

which has simple poles at $z=-2n$ and $z=2n+1$, for each positive integer n. As in the previous proof, the first set of poles arises from the "trivial" zeros of $\zeta(z)$. Note that $z=2n+1$ for $n\leq 0$ is not a pole of $f(z)$, since $\Gamma(\frac{1}{2}(z+1))\zeta(z)$ has simple poles at these points.

By using the reflection formula for $\Gamma(z)$ in the form

$$\Gamma(\tfrac{1}{2}+z)\Gamma(\tfrac{1}{2}-z)=\frac{\pi}{\cos(\pi z)},$$

and then the duplication formula for $\Gamma(z)$, we find that

$$\Gamma(2n+1)\Gamma(\tfrac{1}{2}-n)=(-1)^n\sqrt{\pi}\,2^{2n}n!. \tag{12.2}$$

Thus, by (11.3) and (12.2),

$$R_{-2n}=\left(\frac{\pi}{p}\right)^{2n}\frac{(-1)^n\sqrt{\pi}\varphi(-2n)}{n!\,\zeta(2n+1)},\qquad n\geq 1. \tag{12.3}$$

By the same calculation as in (11.4),

$$R_{2n+1}=-\frac{(-1)^n p^{2n+1}\varphi(2n+1)}{n!\,\zeta(2n+1)},\qquad n\geq 1. \tag{12.4}$$

Hence, from (12.3) and (12.4), we deduce at once the partial fraction decomposition (12.1) for $f(z)$, except for the contribution of the nonreal zeros of $\zeta(z)$.

Entry 13 (p. 323). *We have*

$$\frac{\pi}{8z^3}\frac{\sinh(2\pi z)+\sin(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}=\frac{1}{8z^4}+\sum_{n=1}^{\infty}\frac{1}{4z^4+n^4}.$$

PROOF. Let

$$f(z):=\frac{\sinh(2\pi z)+\sin(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}.$$

Observe that $f(z)$ has a simple pole at the origin with residue $1/\pi$ and simple poles at $z=\pm n(1\pm i)/2$, for each positive integer n, with residue $1/(2\pi)$ in each case. A simple calculation shows that the contributions of the four poles $\pm n(1\pm i)/2$, $n\geq 1$, to the partial fraction expansion of $f(z)$ equals

$$\frac{8z^3}{\pi(4z^4+n^4)}.$$

Thus,

$$\frac{\pi f(z)}{8z^3}=\frac{1}{8z^4}+\sum_{n=1}^{\infty}\frac{1}{4z^4+n^4}+\frac{g(z)}{z^3},$$

where $g(z)$ is entire. Letting $|z|$ tend to ∞ and noting that $g(z)/z^3$ must be an even function of z^2, we conclude that $g(z)\equiv 0$, as desired.

Entry 14 (p. 333). *We have*

$$\frac{\pi}{4z}\frac{\sinh(2\pi z)-\sin(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}=\sum_{n=1}^{\infty}\frac{n^2}{4z^4+n^4}.$$

Entry 15 was recorded by Ramanujan [22] in Section 8 of Chapter 15 in a slightly different form. See Part II [4, pp. 314–315] for a proof.

Entry 15 (p. 333). *We have*

$$\frac{\pi}{4z^2}\frac{\sinh(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}$$
$$=\frac{1}{8z^3}+\sum_{n=1}^{\infty}\frac{n}{4z^4+n^4}+\frac{1}{2z}\sum_{n=1}^{\infty}\frac{1}{z^2+(z+n)^2}.$$

PROOF. Using Entries 13 and 14, we find that

$$\begin{aligned}
&\frac{\pi}{4z^2}\frac{\sinh(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}\\
&\quad=\frac{1}{2z}\frac{\pi}{4z}\frac{\sinh(2\pi z)-\sin(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}+z\frac{\pi}{8z^3}\frac{\sinh(2\pi z)+\sin(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}\\
&\quad=\frac{1}{2z}\sum_{n=1}^{\infty}\frac{n^2}{4z^4+n^4}+\frac{1}{8z^3}+z\sum_{n=1}^{\infty}\frac{1}{4z^4+n^4}\\
&\quad=\frac{1}{8z^3}+\sum_{n=1}^{\infty}\frac{n}{4z^4+n^4}+\frac{1}{2z}\sum_{n=1}^{\infty}\frac{2z^2+n^2-2nz}{4z^4+n^4}\\
&\quad=\frac{1}{8z^3}+\sum_{n=1}^{\infty}\frac{n}{4z^4+n^4}+\frac{1}{2z}\sum_{n=1}^{\infty}\frac{1}{2z^2+2nz+n^2},
\end{aligned}$$

from which the desired result follows.

CHAPTER 31

Elementary and Miscellaneous Analysis

We have attempted to make the title of this chapter broad enough to encompass all the results contained therein. More precisely, we have placed in this chapter those results connected with analysis that are either very elementary or do not seem to fit in any other chapter. It must be admitted that some entries in this chapter are not particularly interesting. Some entries are trivial or are well known to almost all mathematicians, and one may ask why Ramanujan recorded them. There are at least two rejoinders. First, the third notebook appears to contain findings from Ramanujan's youth, possibly as early as 1903, in addition to theorems perhaps recorded in England. Second, the recording of such items as the definition of a limit and the connection between a function's singularities and the radius of convergence of its power series evince the absence of *theoretical* analysis in Ramanujan's training.

This chapter, however, does contain some items of interest. For example, Entries 6–9 pertain to the composition of functions, and although we have contented ourselves to only formal proofs in some instances, the formulas claimed by Ramanujan are engaging. Although quite elementary, Entries 24–28 are also appealing.

Entry 1 (p. 306). *For* $x > 0$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n^2}{n^3 + x^3} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n + x} + \frac{4}{3}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n - x)}{(2n - x)^2 + 3x^2}.$$

PROOF. For each positive integer n, an elementary calculation shows that

$$\begin{aligned}\frac{n^2}{n^3+x^3} &= \frac{1}{3(n+x)} + \frac{2n-x}{3(x^2-nx+n^2)} \\ &= \frac{1}{3(n+x)} + \frac{4(2n-x)}{3\{(2n-x)^2+3x^2\}},\end{aligned}$$

from which the proposed equality follows.

Entry 2 (p. 313). *Suppose that*

$$\sum_{n=1}^{\infty} u_n \quad \textit{and} \quad \sum_{n=1}^{\infty} v_n$$

are divergent, while

$$\sum_{n=1}^{\infty} (u_n - v_n)$$

is convergent. "Then u_n and v_n are nearly equal and also $1/u_n$ and $1/v_n$ are nearly equal when n is great."

We have quoted Ramanujan above in this apparently meaningless entry. Since $u_n - v_n$ tends to 0 as n tends to ∞, in some sense, u_n and v_n, as well as $1/u_n$ and $1/v_n$ are "nearly equal."

Entry 3 (p. 317). *If*

$$\psi(p) = \sum_{n=1}^{\infty} \varphi(np),$$

then

$$\varphi(p) = \sum_{n=1}^{\infty} \mu(n)\psi(np),$$

where $\mu(n)$ denotes the Möbius function.

This inversion theorem is well known (Hardy and Wright [1, p. 237, Theorem 270]).

Entry 4 (p. 317). *If*

$$\sum_{n=1}^{\infty} (-1)^{n-1}\varphi(np) = \sum_{n=1}^{\infty} \psi(np), \tag{4.1}$$

then

$$\varphi(p) = \sum_{n=0}^{\infty} 2^n\psi(2^n p) \tag{4.2}$$

and

$$\psi(p) = \varphi(p) - 2\varphi(2p). \tag{4.3}$$

We have again quoted Ramanujan. The equality (4.1) does not uniquely determine φ in terms of ψ, nor is ψ uniquely determined by φ. To see this, merely replace $\varphi(np)$ in (4.1) by $\varphi(np) + f(n)$, where

$$\sum_{n=1}^{\infty} (-1)^{n-1} f(n) = 0.$$

Thus, we shall only show that (4.2) and (4.3) satisfy (4.1).

Proof. Substituting (4.2) in the left side of (4.1), we obtain

$$\sum_{k=1}^{\infty} (-1)^{k-1} \varphi(kp) = \sum_{k=1}^{\infty} (-1)^{k-1} \sum_{j=0}^{\infty} 2^j \psi(2^j kp). \tag{4.4}$$

We find the coefficient of $\psi(np)$, for each positive integer n, on the right side of (4.4). Write $n = 2^m s$, where s is odd. The coefficient of $\psi(np)$ equals

$$\begin{aligned}(-1)^{s-1} 2^m + (-1)^{2s-1} 2^{m-1} + (-1)^{4s-1} 2^{m-2} + \cdots + (-1)^{2^m s-1} 2^0 \\ = 2^m - 2^{m-1} - 2^{m-2} - \cdots - 1 \\ = 1,\end{aligned}$$

i.e., (4.1) holds.

Using (4.2), we easily deduce (4.3).

Page 331 is devoted to elementary results on the composition of functions and is related to material described in Ramanujan's third quarterly report (Part I [2, pp. 323–330]).

Entry 5 (p. 331). *Let $f(x)$ be a strictly increasing function with a continuous first derivative on $(-\infty, \infty)$. Let $\varphi(x)$ be determined by the relation*

$$\varphi(x) = f'(x)\varphi\{f(x)\}. \tag{5.1}$$

Then

$$\int_a^{f(a)} \varphi(x)\, dx \tag{5.2}$$

is constant for $a \in (-\infty, \infty)$.

Entry 5 is easy to prove; see Part I [2, pp. 328–329].

Define $f^2(x) := f(f(x))$ and, in general, $f^n(x) = f(f^{n-1}(x))$, for each positive integer $n \geq 2$. For each positive integer n, define f^{-n} by $f^{-n}(f^n(x)) = x$.

Entry 6 (p. 331). *Let* φ *be given by* (5.1), *and let* C *denote the integral in* (5.2). *Then, if* m *and* n *are any integers,*

$$f^m(f^n(x)) = f^{m+n}(x) \tag{6.1}$$

and

$$\int_{f^n(x)}^{f^m(x)} \varphi(t)\, dt = (m-n)C. \tag{6.2}$$

PROOF. The equality (6.1) is obvious.

In Part I [2, p. 329], we proved that (6.2) is valid for any two *nonnegative* integers m and n. A similar argument holds in the more general case. First, by (5.1),

$$\int_{f^{-1}(a)}^{a} \varphi(x)\, dx = \int_{f^{-1}(a)}^{a} \varphi\{f(x)\} f'(x)\, dx = \int_{a}^{f(a)} \varphi(t)\, dt = C, \tag{6.3}$$

by (5.2). By (6.3) and induction, if $n \geq 0$,

$$\int_{f^{-(n+1)}(a)}^{f^{-n}(a)} \varphi(x)\, dx = \int_{f^{-(n+1)}(a)}^{f^{-n}(a)} \varphi\{f(x)\} f'(x)\, dx = \int_{f^{-n}(a)}^{f^{-(n-1)}(a)} \varphi(t)\, dt = C.$$

Hence, for $m, n \geq 0$,

$$\int_{f^{-n}(a)}^{f^{-m}(a)} \varphi(x)\, dx = (n-m)C = (-m-(-n))C.$$

It follows that (6.2) is valid for any two *integers* m and n.

Entry 7 (p. 331). *Assume the hypotheses of Entries* 5 *and* 6. *Suppose also that* $f^{\pm n}(x)$ *tends to* ∞ *as either* n *or* x *tends to* ∞. *Lastly, assume that* $\lim_{x\to\infty} \varphi(x) = 0$. *Then the order of*

$$F(x) := \int_a^x \varphi(t)\, dt \quad \textit{is less than that of} \quad f^{\pm n}(x).$$

PROOF. By the definition of *order*, we must prove that

$$\lim_{x\to\infty} \frac{F(x)}{f^{\pm n}(x)} = 0. \tag{7.1}$$

From (6.2) and the fact that $f^n(a)$ tends to ∞ as n tends to ∞, we conclude that $F(x)$ tends to ∞ as x tends to ∞. Thus, by (7.1) and l'Hôpital's rule, it suffices to prove that

$$\lim_{x\to\infty} \frac{\varphi(x)}{(d/dx) f^{\pm n}(x)} = 0. \tag{7.2}$$

By the chain rule,

$$\begin{aligned}\frac{d}{dx} f^n(x) &= f'(f^{n-1}(x)) \frac{d}{dx} f^{n-1}(x)\\ &= f'(f^{n-1}(x)) f'(f^{n-2}(x)) \frac{d}{dx} f^{n-2}(x)\\ &= \cdots\\ &= f'(f^{n-1}(x)) f'(f^{n-2}(x)) \ldots f'(f(x)) f'(x). \end{aligned} \tag{7.3}$$

Thus, from (7.3) and (5.1),

$$\begin{aligned}\frac{\varphi(x)}{(d/dx)f^n(x)} &= \frac{\varphi(x)}{f'(f^{n-1}(x)) f'(f^{n-2}(x)) \ldots f'(f(x)) f'(x)}\\ &= \frac{\varphi(f(x))}{f'(f^{n-1}(x)) f'(f^{n-2}(x)) \ldots f'(f(x))}\\ &= \frac{\varphi(f^2(x))}{f'(f^{n-1}(x)) f'(f^{n-2}(x)) \ldots f'(f^2(x))}\\ &= \ldots\\ &= \varphi(f^n(x)). \end{aligned} \tag{7.4}$$

Letting x tend to ∞ in (7.4) and using our hypotheses, we have completed the proof of (7.2) when the plus sign is taken.

The proof when the minus sign is assumed in (7.2) is similar. First, by the chain rule,

$$\frac{d}{dx} f^{-n}(x) = (f^{-1})'(f^{-n+1}(x))(f^{-1})'(f^{-n+2}(x)) \ldots (f^{-1})'(x). \tag{7.5}$$

Applying the chain rule to the equality $f(f^{-1}(x)) = x$, we find that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Hence, by (5.1),

$$\varphi(f^{-1}(x)) = \frac{\varphi(x)}{(f^{-1})'(x)}. \tag{7.6}$$

Therefore, by (7.5) and (7.6),

$$\begin{aligned}\frac{\varphi(x)}{(d/dx)f^{-n}(x)} &= \frac{\varphi(x)}{(f^{-1})'(f^{-n+1}(x))(f^{-1})'(f^{-n+2}(x)) \ldots (f^{-1})'(x)}\\ &= \frac{\varphi(f^{-1}(x))}{(f^{-1})'(f^{-n+1}(x))(f^{-1})'(f^{-n+2}(x)) \ldots (f^{-1})'(f^{-1}(x))}\end{aligned}$$

$$= \frac{\varphi(f^{-2}(x))}{(f^{-1})'(f^{-n+1}(x))(f^{-1})'(f^{-n+2}(x))\dots(f^{-1})'(f^{-2}(x))}$$

$$= \dots$$

$$= \varphi(f^{-n}(x)). \tag{7.7}$$

Letting x tend to ∞ in (7.7) and invoking our hypotheses, we deduce (7.2) in the presence of the minus sign.

For the next sequence of four results, Ramanujan *assumes* that

$$f^r(x) = \sum_{n=0}^{\infty} \frac{\psi_n(x)r^n}{n!}, \tag{8.1}$$

where r is a *continuous* variable and $\psi_n(x)$ is differentiable. We shall proceed formally.

Entry 8 (p. 331). *Under the assumptions of* (8.1) *and Entries* 5 *and* 6,

(1) $$\psi_0(x) = f^0(x) = x,$$

(2) $$\frac{df^r(x)}{dr} = \psi_1(x)\frac{df^r(x)}{dx},$$

(3) $$\psi_n(x) = \psi_1(x)\psi'_{n-1}(x), \qquad n \geq 1,$$

and

(4) $$\psi_1(x) = \frac{C}{\varphi(x)}.$$

Proof. First, (1) is immediate from (8.1).

To prove (2), we consider

$$f^{r+k}(x) = f^r(f^k(x)), \tag{8.2}$$

where r and k will be formally regarded as continuous variables. To the left side of (8.2), we apply Taylor's theorem; on the right side, we use (8.1). Thus,

$$\sum_{n=0}^{\infty} \frac{d^n f^{r+k}(x)/dk^n\,|_{r=0}\, r^n}{n!} = \sum_{n=0}^{\infty} \frac{\psi_n(f^k(x))r^n}{n!}.$$

Equating coefficients of r, we find that

$$\frac{df^k(x)}{dk} = \psi_1(f^k(x)). \tag{8.3}$$

Set $y = f^k(x)$ and $z = f^k(y)$. Thus, (8.3) becomes

$$\frac{dy}{dk} = \psi_1(y). \tag{8.4}$$

Thus, by (8.4),

$$\frac{dz}{dk} = \frac{dz}{dy}\frac{dy}{dk} = \frac{dz}{dy}\psi_1(y). \tag{8.5}$$

Recalling the definition of z, we see that (8.5) is the same as (2) with r replaced by k.

Next, by (8.1) and (2),

$$\frac{d}{dr}\sum_{n=0}^{\infty}\frac{\psi_n(x)r^n}{n!} = \psi_1(x)\frac{d}{dx}\sum_{n=0}^{\infty}\frac{\psi_n(x)r^n}{n!},$$

or

$$\sum_{n=1}^{\infty}\frac{\psi_n(x)r^{n-1}}{(n-1)!} = \psi_1(x)\sum_{n=0}^{\infty}\frac{\psi_n'(x)r^n}{n!}.$$

Equating coefficients of r^{n-1}, $n \geq 1$, on both sides, we deduce (3).

Lastly, from (6.2),

$$\int_x^{f^r(x)} \varphi(t)\,dt = rC.$$

Formally differentiating with respect to r, we find that

$$\varphi(f^r(x))\frac{d}{dr}f^r(x) = C,$$

or, by (2),

$$\varphi(f^r(x))\frac{d}{dx}f^r(x)\psi_1(x) = C. \tag{8.6}$$

Now, by repeated use of (5.1),

$$\begin{aligned}
\varphi(x) &= f'(x)\varphi(f(x)) \\
&= f'(x)f'(f(x))\varphi(f^2(x)) \\
&= \frac{d}{dx}f^2(x)f'(f^2(x))\varphi(f^3(x)) \\
&= \dots \\
&= \frac{d}{dx}f^r(x)\varphi(f^r(x)).
\end{aligned} \tag{8.7}$$

Substituting (8.7) into (8.6), we complete the proof of (4).

Formula (2) is found in Ramanujan's third quarterly report.

Entry 9 (p. 331). *Let φ denote a continuous, strictly increasing function. Recursively define a sequence of functions $\psi_n(x)$, $n \geq 0$,* by

$$\psi_0(x) = x, \qquad \psi_n(x)\varphi(x) = \psi'_{n-1}(x). \tag{9.1}$$

Let

$$v = \int_\alpha^\beta \varphi(x)\,dx. \tag{9.2}$$

Then

$$\beta(v) = \sum_{n=0}^\infty \psi_n(\alpha)\frac{v^n}{n!}. \tag{9.3}$$

PROOF. Since φ is strictly increasing, (9.2) is invertible, and so $\beta(v)$ is well defined. Clearly, $\beta(0) = \alpha = \psi_0(\alpha)$, by (9.1). Next, by (9.2) and (9.1),

$$\frac{dv}{d\beta} = \varphi(\beta) = \frac{\psi'_0(\beta)}{\psi_1(\beta)} = \frac{1}{\psi_1(\beta)}. \tag{9.4}$$

Since v is monotonically increasing and since $\beta = \alpha$ when $v = 0$,

$$\beta'(v) = \psi_1(\beta) \quad \text{and} \quad \beta'(0) = \psi_1(\alpha),$$

respectively.

Proceeding by induction, assume that

$$\beta^{(n)}(v) = \psi_n(\beta), \qquad n \geq 1. \tag{9.5}$$

By (9.5), (9.1), and (9.4),

$$\beta^{(n+1)}(v) = \psi'_n(\beta)\frac{d\beta}{dv} = \psi_{n+1}(\beta)\varphi(\beta)\frac{1}{\varphi(\beta)} = \psi_{n+1}(\beta).$$

Since $\beta = \alpha$ when $v = 0$,

$$\beta^{(n+1)}(0) = \psi_{n+1}(\alpha),$$

which completes the proof of (9.3).

Page 336 is devoted to somewhat imprecisely stated theorems about general expansions. We quote Ramanujan below.

Entry 10 (p. 336). *If $\varphi(x)$ vanishes for a, b, c, d, etc., of x, then the expansion of the function*

$$\frac{1}{\varphi(x)}+\frac{1}{(a-x)\varphi'(a)}+\frac{1}{(b-x)\varphi'(b)}+\frac{1}{(c-x)\varphi'(c)}+\cdots \tag{10.1}$$

is convergent for all values of x.

Ramanujan has assumed that all of the zeros of $\varphi(x)$ are simple so that the poles of $1/\varphi(x)$ are simple. He has then subtracted from $1/\varphi(x)$ the sum of the principal parts of $1/\varphi(x)$. Contrary to Ramanujan's declaration, the series (10.1) does not necessarily converge. Thus, Entry 10 is a restricted version of the Mittag–Leffler theorem (Ahlfors [1, pp. 185–186]).

Again, we quote Ramanujan below.

Entry 11 (p. 336). *If $\varphi(x)$ vanishes for a, b, c, d, etc., of x, then the coefficient of x^{n-1} in the expansion of $1/\varphi(x)$*

$$=-\frac{1}{a^n\varphi'(a)}-\frac{1}{b^n\varphi'(b)}-\frac{1}{c^n\varphi'(c)}-\cdots-\theta(n),$$

where $\lim_{n\to\infty} K^n\theta(n)=0$ for any value of K and $\theta(n)$ is 0 in many cases.

Proof. Assuming again that all zeros of $\varphi(x)$ are simple, we write Entry 10 in the form

$$\frac{1}{\varphi(x)}=\frac{1}{\varphi'(a)(x-a)}+\frac{1}{\varphi'(b)(x-b)}+\cdots-g(x), \tag{11.1}$$

where $g(x)$ is entire. Ramanujan sets

$$g(x)=\sum_{n=1}^{\infty}\theta(n)x^{n-1}. \tag{11.2}$$

Assuming that $|x|<\min(|a|,|b|,\ldots)$, we deduce from (11.1) and (11.2) that

$$\frac{1}{\varphi(x)}=-\frac{1}{a\varphi'(a)}\sum_{n=0}^{\infty}\left(\frac{x}{a}\right)^n-\frac{1}{b\varphi'(b)}\sum_{n=0}^{\infty}\left(\frac{x}{b}\right)^n-\cdots$$
$$-\sum_{n=1}^{\infty}\theta(n)x^{n-1}.$$

Ramanujan's first assertion in Entry 11 now follows.

Because $g(x)$ is entire, $\lim_{n\to\infty} K^n\theta(n)=0$ for every K.

Lastly, in many cases, $g(x)\equiv 0$, e.g., when $\varphi(x)$ is a rational function with the degree of the numerator exceeding that of the denominator. Thus, Ramanujan's last claim has validity.

Entry 12 (p. 336). *Let*

$$\varphi(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where $a_n > 0$ *for* n *sufficiently large,* $\lim_{n\to\infty} a_n/a_{n+1}$ *exists, and* R *is the radius of convergence. Then*

$$\lim_{n\to\infty} \varphi\left(\frac{a_n}{a_{n+1}}\right) = \infty.$$

PROOF. Since $a_n > 0$ for n sufficiently large, $R = \lim_{n\to\infty} a_n/a_{n+1}$ is a singularity of $\varphi(x)$ and $\lim_{x\to R-} \varphi(x) = \infty$. The desired conclusion now follows.

We quote Ramanujan in the next entry.

Entry 13 (p. 336). *If* $\varphi(x) = \infty$ *for the values of* a, b, c, d, *etc. of* x *and if* $|a|$ *be the nearest to* 0, then

(1) *the expansion of* $\varphi(x)$ *is convergent if* $|x| < |a|$ *and divergent if* $|x| > |a|$,
(2) $\lim_{n\to\infty} a^n \cdot$ *coefficient of* x^{n-1} *in the expansion*

$$= \frac{\varphi'(a)}{\{\varphi(a)\}^2} = -\left[\frac{d}{dx}\frac{1}{\varphi(x)}\right]_{x=a}.$$

The first assertion is a version of the basic theorem stating that if $\varphi(x)$ is analytic for $|x| < |a|$, where a is the nearest singularity to 0, then the Taylor series for $\varphi(x)$ about $x = 0$ converges for $|x| < |a|$ and diverges for $|x| > |a|$. (In fact, Ramanujan wrote x for $|x|$ in (1).)

The displayed expressions under (2) are best written in the form

$$\lim_{\substack{x\to a \\ |x|<|a|}} \frac{\varphi'(x)}{\{\varphi(x)\}^2} =: L.$$

Write

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad |z| < |a|, \tag{13.1}$$

and

$$\varphi(z) = \sum_{n=-m}^{\infty} b_n (z-a)^n,$$

for $0 < |z - a| < r$, say. We assume that $\varphi(z)$ has an analytic continuation to $|z| < R$, where $R > |a|$ and a is the only singularity of $\varphi(z)$ on $|z| < R$. We shall prove that

$$\lim_{n\to\infty} a^n a_{n-1} = \begin{cases} -b_1, & \text{if } m = 1, \\ \infty, & \text{if } m \geq 2. \end{cases} \tag{13.2}$$

Thus, in Ramanujan's statement of (2), L should be replaced by $1/L$.

PROOF OF (13.2). We prove (13.2) for $m = 1$; the proof for $m \geq 2$ is similar.

The function

$$f(z) := \varphi(z) - \frac{b_{-1}}{z - a}$$

is analytic on $|z| < R$. With the use of (13.1), we see that

$$f(z) = \sum_{n=0}^{\infty} \left(a_n + \frac{b_{-1}}{a^{n+1}}\right) z^n, \qquad |z| < R.$$

Since a lies on the interior of the circle of convergence for $f(z)$,

$$\lim_{n\to\infty} \left(a_n + \frac{b_{-1}}{a^{n+1}}\right) a^n = 0,$$

from which (13.2) in the case $m = 1$ readily follows.

Entry 14 (p. 337). *Let f and g be given functions, and let a_n and b_n, $1 \leq n < \infty$, be constants. Suppose that F is a function satisfying the equality*

$$\sum_{n=1}^{\infty} a_n F(x + b_n) = \int_\alpha^\beta f(t)^x g(t)\, dt. \tag{14.1}$$

Then

$$F^*(x) := \int_\alpha^\beta \frac{f(t)^x g(t)}{\sum_{n=1}^{\infty} a_n f(t)^{b_n}}\, dt \tag{14.2}$$

is a solution of (14.1).

PROOF. Inverting the order of summation and integration and proceeding formally, we find that

$$\begin{aligned} \sum_{n=1}^{\infty} a_n F^*(x + b_n) &= \int_\alpha^\beta \frac{\sum_{n=1}^{\infty} a_n f(t)^{x+b_n} g(t)}{\sum_{n=1}^{\infty} a_n f(t)^{b_n}}\, dt \\ &= \int_\alpha^\beta f(t)^x g(t)\, dt, \end{aligned}$$

which completes the proof.

Ramanujan's formulation implicitly implies that the solution (14.2) is unique. It seems to be extremely difficult to supply nontrivial hypotheses to ensure a unique solution. For this reason, we have regarded Entry 14 as only a formal statement.

Entry 15 (p. 339). *Let $\{a_n\}$, $1 \le n < \infty$, be a convergent sequence. Then*

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - \lim_{n\to\infty} a_n.$$

Entry 15 enunciates the well-known principle of telescoping series. Ramanujan next records the same elementary theorem in functional notation, which we refrain from explicitly stating. This is yet followed by another (somewhat vague) articulation of the telescoping principle.

Entry 16 (p. 340). *We have*

$$\sum_{k=1}^{\infty} (a_k + b_{k-1} - b_k) = \lim_{n\to\infty} \left(\sum_{k=1}^{n} a_k + b_0 - b_n \right),$$

provided that the limit on the right side exists.

Of course, Entry 16 is merely a variant of Entry 15.

On page 341, Ramanujan considers the effect of rearranging the terms of an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n, \qquad a_n > 0, \quad 1 \le n < \infty.$$

He takes the first $\varphi(1)$ positive terms $-a_2$, i.e.,

$$a_1 + a_3 + \cdots + a_{2\varphi(1)-1} - a_2.$$

Then he takes the next $\varphi(2)$ positive terms $-a_4$, i.e.,

$$a_{2\varphi(1)+1} + a_{2\varphi(1)+3} + \cdots + a_{2\varphi(1)+2\varphi(2)-1} - a_4.$$

Taking a total of n such groups of terms, he claims that "consequently the sum is made greater by

$$\frac{1}{2} \int_n^{\int \varphi(n/2)\,dn} a_x\,dx \tag{17.1}$$

when n becomes ∞". As is to be expected, no hypotheses are offered for a_n and φ. There are two ways to interpret this derangement. First, the sum

$$\sum_{k=1}^{2n} (-1)^{k-1} a_k$$

is increased by

$$\sum_{k=n+1}^{S_n} a_{2k-1},$$

where $S_n = \sum_{j=1}^{n} \varphi(j)$. Second, the sum

$$\sum_{k=1}^{2S_n} (-1)^{k-1} a_k \tag{17.2}$$

is increased by

$$\sum_{k=n+1}^{S_n} a_{2k} := T_n. \tag{17.3}$$

The form (17.1) favors the latter interpretation. Of course, either interpretation yields roughly the same increase.

So that we might establish a version of Ramanujan's claim, we assume (as Ramanujan implicitly did) that a_n can be extended to a continuous, monotonically decreasing function a_x of the positive continuous variable x tending to ∞.

Entry 17 (p. 341). *Consider the second interpretation of the derangement described above, and assume the notation and hypotheses that we have set. Then the sum* (17.2) *is increased by*

$$\frac{1}{2}\int_{n/2}^{S_n/2} a_x \, dx + o(1), \tag{17.4}$$

as n tends to ∞.

Note that (17.4) is different from (17.1). Ramanujan has replaced the sum S_n by an integral, and both ranges of integration are different from ours.

PROOF. The proof is simple. Since a_{2x} is monotonically decreasing, elementary geometric considerations show that

$$\int_{n+1}^{S_n+1} a_{2x} \, dx \le T_n \le \int_{n}^{S_n} a_{2x} \, dx.$$

Replacing $2x$ by x and using the hypothesis that a_n tends to 0, we conclude that

$$T_n = \frac{1}{2}\int_{n/2}^{S_n/2} a_x \, dx + o(1),$$

as n tends to ∞.

The following quoted entry records Ramanujan's definition of a limit of a sequence.

Entry 18 (p. 348). $\lim_{n\to\infty} u_n$ *is said to be c when* $u_n - c$ *cannot be made greater than any arbitrary small quantity h by making n sufficiently great.*

The next entry is a rephrased version of Ramanujan's definition of a *legitimate convergent series.*

Entry 19 (p. 348). *The function* $\varphi(x)$ *is a legitimate convergent series if*

$$\varphi(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x), \tag{19.1}$$

for some sequence $\{f_k(x)\}$, $1 \le k < \infty$.

Ramanujan then writes: "The remaining cases are *illegitimate convergent*, *legitimate divergent*, and *illegitimate divergent* series." He does not define these terms, and so we offer some "guesses" for Ramanujan's definitions. An *illegitimate convergent* series is one where the limit in (19.1) exists but does not equal $\varphi(x)$, at least for some values of x. A *legitimate divergent* series is a series where the partial sums in (19.1) tend to ∞ as n tends to ∞. Lastly, an *illegitimate divergent* series is one where the partial sums in (19.1) do not tend to any finite or infinite limit as n approaches ∞.

At the top of page 361, near the beginning of his third notebook [22], Ramanujan records two queries seeking Tauberian theorems of certain types, and then states a specific Tauberian theorem with no hypotheses. We quote all three statements.

Entry 20 (p. 361).

$$\sum_{n=1}^{\infty} a_n e^{-b_n x} - \int^{\infty} a_z e^{-b_z x}\, dz$$

is finite when x is 0?

Entry 21 (p. 361).

$$\sum_{k=1}^{\infty} \frac{1}{a_k^{n+1}} - \int \frac{dz}{a_z^{n+1}}$$

is finite when x is 0?

Entry 22 (p. 361). *If*

$$\sum_{n=1}^{\infty} a_n e^{-nx} - \frac{c}{x^r}$$

is finite when x is 0, then the average value of a_n is

$$\frac{cn^{r-1}}{\Gamma(r)}.$$

We rephrase Entry 22 and correct it by replacing $\Gamma(r)$ by $\Gamma(r+1)$. If

$$\sum_{n=1}^{\infty} a_n e^{-nx} \sim \frac{c}{x^r}$$

as x tends to $0+$, then

$$\sum_{n \le z} a_n \sim \frac{cz^r}{\Gamma(r+1)},$$

as z tends to ∞.

Entries 20 and 21 indicate that Ramanujan wanted to generalize Theorem 22, but evidently he possessed no genuine theorems or conjectures. Many Tauberian theorems have been established in the literature, and, indeed, a proper formulation of Entry 22 can be generalized. For instance, see three papers of G. H. Hardy and J. E. Littlewood [1], [2], [4] (Hardy [6, pp. 510–527; 542–554], [5, pp. 20–97]) for several types of Tauberian theorems. In particular, see Theorem D in [2, pp. 141–142]. Consult N. Wiener's monograph [1] for proofs of many Tauberian theorems established prior to 1932. Important and useful extensions were established by S. Ikehara [1] in 1931 and H. Delange [1] in 1954. Tauberian theorems are frequently needed in analytic number theory, and undoubtedly problems in number theory were the genesis of Ramanujan's questions (cf. Chapter 23, Entries 13–15). For applications to analytic number theory, see, for example, the books of W. Narkiewicz [1] and W. Schwarz [1].

On page 364, Ramanujan records once again a version of the telescoping series principle given in Entry 15, and we do not restate it here.

Entry 23 (p. 364). *Let $\{a_n\}$, $1 \le n < \infty$, be a convergent sequence with $\lim_{n\to\infty} a_n = a \ne 0$ and $a_n \ne 0$, $n \ge 1$. Then*

$$\prod_{n=1}^{\infty} \frac{a_n}{a_{n+1}} = \frac{a_1}{a}.$$

Entry 23 is the well-known principle about telescoping products. In the entries that follow, Ramanujan relates some clever applications of telescoping series and products.

Entry 24 (p. 364). *If x is real and if $x/2^k$, $1 \le k < \infty$, is not an odd multiple of $\pi/2$, then*

$$\sum_{k=1}^{\infty} \frac{\tan(x/2^k)}{2^k} = \frac{1}{x} - \cot x.$$

Proof. Recall the elementary trigonometric identity

$$\tfrac{1}{2}\tan\tfrac{1}{2}x = \tfrac{1}{2}\cot\tfrac{1}{2}x - \cot x.$$

Replace x by $x/2^{k-1}$, multiply both sides by $1/2^{k-1}$, and sum on k, $1 \le k \le n$, to find that

$$\begin{aligned}\sum_{k=1}^{n}\frac{\tan(x/2^k)}{2^k} &= \sum_{k=1}^{n}\frac{\cot(x/2^k)}{2^k} - \sum_{k=1}^{n}\frac{\cot(x/2^{k-1})}{2^{k-1}}\\ &= \frac{\cot(x/2^n)}{2^n} - \cot x. \end{aligned}\tag{24.1}$$

Letting n tend to ∞ in (24.1), we complete the proof.

In fact, (24.1) is given in the tables of Gradshteyn and Ryzhik [1, p. 32, formula 1.371, no. 1].

Entry 25 (p. 364). *For every real number x,*

$$\sum_{k=1}^{\infty}\frac{\sin(x/3^k)}{3^k(1+2\cos(x/3^k))} = \frac{1}{2x} - \tfrac{1}{4}\cot\tfrac{1}{2}x.$$

Proof. By a straightforward exercise in trigonometry,

$$\frac{\sin(2x)}{1+2\cos(2x)} = \tfrac{1}{4}(\cot x - 3\cot(3x)).$$

Replace $2x$ by $x/3^k$, multiply both sides by $1/3^k$, and sum on k, $1 \le k \le n$. Thus,

$$\begin{aligned}\sum_{k=1}^{n}\frac{\sin(x/3^k)}{3^k(1+2\cos(x/3^k))} &= \frac{1}{4}\sum_{k=1}^{n}\frac{1}{3^k}\left\{\cot\left(\frac{x}{2\cdot 3^k}\right) - 3\cot\left(\frac{x}{2\cdot 3^{k-1}}\right)\right\}\\ &= \frac{1}{4}\left\{\frac{1}{3^n}\cot\left(\frac{x}{2\cdot 3^n}\right) - \cot\left(\frac{x}{2}\right)\right\}.\end{aligned}$$

Letting n tend to ∞, we complete the proof.

Entry 26 (p. 364). *For every real number x,*

$$\sum_{k=0}^{\infty}\frac{\sin^3(3^k x)}{3^k} = \tfrac{3}{4}\sin x.$$

Proof. We employ the elementary trigonometric identity

$$\sin^3 x = \tfrac{3}{4}\sin x - \tfrac{1}{4}\sin(3x).$$

Replacing x by $3^k x$, multiplying both sides by $1/3^k$, and summing on k, $0 \le k \le n$, we deduce that

$$\begin{aligned}\sum_{k=0}^{n} \frac{\sin^3(3^k x)}{3^k} &= \frac{1}{4} \sum_{k=0}^{n} \frac{1}{3^k} \{3 \sin(3^k x) - \sin(3^{k+1} x)\} \\ &= \frac{3}{4} \sin x - \frac{1}{4} \frac{\sin(3^{n+1} x)}{3^n}.\end{aligned}$$

Letting n tend to ∞, we finish the proof.

Entry 27 (p. 364). *If x is real and if $x/2^k$, $k \ge 0$, is not an integral multiple of π, then*

$$\sum_{k=0}^{\infty} \left\{\csc\left(\frac{x}{2^k}\right) - \frac{2^k}{x}\right\} = \frac{1}{x} - \cot x.$$

Proof. Since $\csc(x/2^k) - 2^k/x = O(x/2^k)$ as k tends to ∞, the series above indeed converges.

We begin with the elementary trigonometric identity

$$\csc x = \cot \tfrac{1}{2}x - \cot x.$$

Subtract $1/x$ from each side, replace x by $x/2^k$, and sum on k, $0 \le k \le n$, to deduce that

$$\begin{aligned}\sum_{k=0}^{n} \left\{\csc\left(\frac{x}{2^k}\right) - \frac{2^k}{x}\right\} &= \sum_{k=0}^{n} \left\{\cot\left(\frac{x}{2^{k+1}}\right) - \cot\left(\frac{x}{2^k}\right) - \frac{2^k}{x}\right\} \\ &= -\cot x + \cot\left(\frac{x}{2^{n+1}}\right) - \frac{2^{n+1} - 1}{x}.\end{aligned}$$

Letting n tend to ∞ and using the fact

$$\cot\left(\frac{x}{2^{n+1}}\right) - \frac{2^{n+1}}{x} = O\left(\frac{x}{2^{n+1}}\right),$$

as n tends to ∞, we complete the proof.

Entry 28 (p. 364). *Let x be real and suppose that $x/2^k$, $k \ge 0$, is not a multiple of $\pi/2$. Then*

$$\prod_{k=0}^{\infty} \left\{\frac{2^k}{x} \tan\left(\frac{x}{2^k}\right)\right\}^{2^k} = \frac{2x}{\sin(2x)}.$$

Proof. We begin with the trivial identity

$$\frac{1}{x} \tan x = \frac{2 \sin^2 x}{x \sin(2x)}.$$

Replace x by $x/2^k$, raise each side to the 2^kth power, and take the product of each side on k, $0 \le k \le n$. This gives us

$$\begin{aligned}
\prod_{k=0}^{n} \left\{\frac{2^k}{x}\tan\left(\frac{x}{2^k}\right)\right\}^{2^k} &= \prod_{k=0}^{n} \left(\frac{2^{k+1}\sin^2(x/2^k)}{x\sin(x/2^{k-1})}\right)^{2^k} \\
&= \frac{1}{\sin(2x)} \frac{2^{\sum_{k=0}^{n}(k+1)2^k}}{x^{\sum_{k=0}^{n}2^k}} \sin^{2^{n+1}}\left(\frac{x}{2^n}\right) \\
&= \frac{1}{\sin(2x)} \frac{2^{1+n2^{n+1}}}{x^{2^{n+1}-1}} \sin^{2^{n+1}}\left(\frac{x}{2^n}\right) \\
&= \frac{2x}{\sin(2x)} \left(\frac{2^n}{x}\right)^{2^{n+1}} \sin^{2^{n+1}}\left(\frac{x}{2^n}\right).
\end{aligned}$$

Letting n tend to ∞ on each side above, we complete the proof.

Entries 25–28 are apparently new. D. Somasundaram [1] has also given proofs of Entries 24–28. His proofs are similar to ours, except that our proof of Entry 28 is somewhat simpler.

Entry 29 (p. 364). *For* $x > 0$,

$$\frac{1}{\log x} + \frac{1}{1-x} = \sum_{k=1}^{\infty} \frac{1}{2^k(1+x^{1/2^k})}.$$

Proof. For $x > 0$,

$$\begin{aligned}
\frac{1}{1-x} = \frac{1}{(1+\sqrt{x})(1-\sqrt{x})} &= \frac{1}{2}\left(\frac{1}{1+\sqrt{x}} + \frac{1}{1-\sqrt{x}}\right) \\
&= \frac{1}{2(1+\sqrt{x})} + \frac{1}{2^2}\left(\frac{1}{1+x^{1/4}} + \frac{1}{1-x^{1/4}}\right) \\
&= \dots \\
&= \sum_{k=1}^{n} \frac{1}{2^k(1+x^{1/2^k})} + \frac{1}{2^n(1-x^{1/2^n})}.
\end{aligned}$$

Letting n tend to ∞ above, we complete the proof.

Entry 30 (p. 364). *For* $x > 0$,

$$\frac{1}{\log x} + \frac{1}{1-x} = \sum_{k=1}^{\infty} \frac{2+x^{1/3^k}}{3^k(1+x^{1/3^k}+x^{2/3^k})}.$$

PROOF. For $x > 0$,

$$\begin{aligned}
\frac{1}{1-x} &= \frac{1}{(1+x^{1/3}+x^{2/3})(1-x^{1/3})} \\
&= \frac{1}{3}\left(\frac{2+x^{1/3}}{1+x^{1/3}+x^{2/3}} + \frac{1}{1-x^{1/3}}\right) \\
&= \frac{2+x^{1/3}}{3(1+x^{1/3}+x^{2/3})} + \frac{1}{3^2}\left(\frac{2+x^{1/9}}{1+x^{1/9}+x^{2/9}} + \frac{1}{1-x^{1/9}}\right) \\
&= \ldots \\
&= \sum_{k=1}^{n} \frac{2+x^{1/3^k}}{3^k(1+x^{1/3^k}+x^{2/3^k})} + \frac{1}{3^n(1-x^{1/3^n})}.
\end{aligned}$$

Letting n tend to ∞ above, we complete the proof.

Location in Notebook 2 of the Material in the 16 Chapters of Notebook 1

Ramanujan's second notebook is a revised, considerably enlarged edition of the first notebook, which contains 16 chapters of organized material and over 100 pages of unorganized material. The 16 chapters are recorded on odd-numbered pages from page 1 to page 263. Shortly thereafter Ramanujan began to record results on both odd and even numbered pages until the end of the notebook. He then evidently began to use the previously blank even-numbered pages, beginning at the back and proceeding toward the front. We record below the location in the second notebook of the material in the 16 organized chapters of the first notebook. Almost all of the material in the 16 organized chapters is recorded in the second notebook. Most of the entries in the 16 chapters of the second notebook that are not found in the second notebook are either easy to prove or are incorrect. The unorganized portions of the first notebook contain considerably more substantial results not recorded in the second notebook. These will be examined in Part V [9].

The page numbers listed for each chapter below are odd-numbered pages only. Thus, for example, Chapter 1 is recorded on pages 1, 3, and 5.

Chapter 1 (pages 1–5)
Magic Squares

All material in Chapter 1 can be found in Chapter 1 of the second notebook, except for some numerical examples of certain types of magic squares.

Chapter 2 (pages 7–19) Sums Related to the Harmonic Series or Inverse Tangent Function

The results in the first seven sections of Chapter 2 correspond to those in the first seven sections of Chapter 2 in the second notebook, while Section 8 in Chapter 2 of the first notebook is equivalent to Section 12 of Chapter 2 in the second.

Chapter 3 (pages 21–33) Bell Polynomials and Numbers

The material in Chapter 3 of the first notebook is located in the first nine sections of Chapter 3 in the second notebook.

Chapter 4 (pages 35–41) Series Arising from Lagrange Inversion

Entry 1 is identical to Entry 12 in Chapter 3 in Notebook 2. Section 2 is contained in Sections 13 and 14 of Chapter 3 in the second notebook. Ramanujan has two sections numbered 3. Most of the first Section 3 is contained in Section 15 of Chapter 3 in the second notebook. Those results not found in Section 15 are simple variations on results found there, and, in most cases, Ramanujan clearly indicates how to prove these variations, and so it does not seem worthwhile to state or prove these simple variations here. The second Section 3 is contained in Section 16 of Chapter 3 in Notebook 2. The content of Section 4 is found in Section 17 of Chapter 3 in the second notebook.

Chapter 5 (pages 43–51) Iterates of the Exponential Function

Chapter 5 is entirely contained in Chapter 4 of the second notebook. More specifically, Entry 1 is the same as Entry 1 of Chapter 4, Section 2 is contained in Sections 2 and 3 of Chapter 4, Section 3 is almost identical with Section 4 of Chapter 4, Entry 4 is identical to Entry 5 of Chapter 4, Section 5 is contained in Section 6 of Chapter 4, Section 6 is contained in Sections 6 and 7 of Chapter 4, and Entry 7 is the same as Entry 8 of Chapter 4.

Chapter 6 (pages 53–65)
Bernoulli Numbers and Eulerian Polynomials and Numbers

Entries 1 and 2 are found in Section 1 of Chapter 5 of Notebook 2. Entry 3 is the same as Entry 2 of Chapter 5. The long Section 4 is contained in Sections 3–8 of Chapter 5. Section 5 is found in Sections 11 and 12 of Chapter 5. The material from Section 6 is contained in Sections 13–18 of Chapter 5. Section 7 is found in Section 22 of Chapter 5. Section 8 is located in Section 9 of Chapter 2 in the second notebook. Section 9 can be found in either Section 23 of Chapter 5 or Section 10 of Chapter 2 in the second notebook. Section 10 can be located in Section 11 of the second chapter of Notebook 2. At the bottom of page 65, Ramanujan gives the well-known Wallis product for $\pi/2$ (Gradshteyn and Ryzhik [1, p. 12]) and a familiar infinite product representation for $\sqrt{2}$ (Gradshteyn and Ryzhik [1, p. 12]).

Chapter 7 (pages 67–77)
The Riemann Zeta-Function and Allied Functions

Entry 1 corresponds to Entries 23 and 24 of Chapter 5 in the second notebook, and Entry 2 is the same as Entry 25 in Chapter 5. Except for Examples 3 and 4 in Section 6, the content of Sections 3–6 is found in Section 25 of Chapter 5. Example 4 is in Section 28, while Example 3 asserts that

$$\frac{1+E_2}{1-E_2}=\frac{\sum_{k=0}^{\infty}(4k+1)^{-2}}{\sum_{k=0}^{\infty}(4k+3)^{-2}},$$

where E_2 is the second Euler number. This is easy to prove, and Ramanujan supplies a sketch of the proof. Section 7 is contained in Sections 27 and 28 of Chapter 5. Section 8 is found in Sections 28 and 30 of Chapter 5, except for Examples 3(ii) and 5. Although Example 3(ii) is not recorded in the second notebook, it is an illustration of Corollary 2 in Section 28 of Chapter 5. Example 5 simply gives the series representation for the logarithmic derivative of the Riemann zeta-function.

Chapter 8 (pages 79–99)
Ramanujan's Theory of Divergent Series

Chapter 8 corresponds almost exactly to Chapter 6 in the second notebook. The numbering of entries and the discourse are different, with slightly more explanation given at the beginning of Chapter 8 than at the start of Chapter 6. However, the orders of presentation are exactly the same.

Section 17 of Chapter 6 is not found in Chapter 8, while Chapter 8 ends with the identity

$$\sum_{n=1}^{\infty}\left(\frac{1}{x^n-1}-\frac{1}{x^{n^2}}\right)=2\sum_{n=1}^{\infty}\frac{1}{x^{n(n+1)}-x^{n^2}},$$

which is not found in the second notebook. However, it is easily deducible from Entry 15(i) of Chapter 5 of the second notebook or from Entry 7(i) in Chapter 8 of the first notebook.

Chapter 9 (pages 101–117) Sums of Powers

The first four sections correspond to the first four sections of Chapter 7 in the second notebook, although the contents of Sections 2 and 3 have been inverted. However, the content of Section 5 closely matches that in Section 10 of Chapter 4 of Notebook 2, but one example is not found in the latter source. Ramanujan claims that

$$\frac{\pi}{2\sqrt{x}}-\frac{1}{2x^{3/2}}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6x^{7/2}}-\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10x^{11/2}}+\cdots=\sqrt{2} \tag{1}$$

when $x=0$. This is one of four examples in this section that purportedly illustrates a method for finding the "value" of an asymptotic series without actually finding the generating function of this series. As we pointed out in Part I [2, p. 101], this method is not rigorous, and for each of the examples that appear in the second notebook, we found the claimed value by determining the generating function. Ramanujan evidently did not include (1) in the second notebook, because he noticed that (1), in fact, is false, although it can be reformulated. First observe that the left side of (1) can be written in the form

$$\frac{1}{\sqrt{x}}\left(\frac{\pi}{2}-\frac{1}{2x}\,{}_2F_1\left(\frac{3}{4},\frac{5}{4};\frac{3}{2};-\frac{1}{x^2}\right)\right). \tag{2}$$

Now recall that (Gradshteyn and Ryzhik [1, p. 1043])

$$\begin{aligned}{}_2F_1(a,b;c;z)&=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}\left(-\frac{1}{z}\right)^a {}_2F_1(a,\,a+1-c;\,a+1-b;\,1/z)\\&\quad+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\left(-\frac{1}{z}\right)^b {}_2F_1(b,\,b+1-c;\,b+1-a;\,1/z).\end{aligned} \tag{3}$$

Using (3), we find, after some simplification, that (2) equals

$$\frac{1}{\sqrt{x}}\left(\frac{\pi}{2}-\frac{1}{2x}\left(\sqrt{2}x^{3/2}\,{}_2F_1\left(\frac{3}{4},\frac{1}{4};\frac{1}{2};-x^2\right)-\frac{x^{5/2}}{\sqrt{2}}\,{}_2F_1\left(\frac{5}{4},\frac{3}{4};\frac{3}{2};-x^2\right)\right)\right)$$
$$=\frac{1}{\sqrt{x}}\left(\frac{\pi}{2}-\sqrt{\frac{x}{2}}+\left(\frac{x}{2}\right)^{3/2}+\cdots\right). \quad (4)$$

It is now clear that, as x tends to 0, (4) does not have a limit.

Entry 6 in Chapter 9 is the same as Entry 5 in Chapter 7 of Notebook 2. Entry 6 is followed by eleven examples which are found in Section 6 of Chapter 7. These, in turn, are followed by nine further examples which are not found in the second notebook, but, instead, in Section 6 of Chapter 7 in the second notebook, Ramanujan states two general theorems (Entries 6(i), (ii)), which give rise to these nine examples. Sections 7–11 in Chapter 9 are almost identical to Sections 7–11 in Chapter 7 of Notebook 2, except that the last four examples of Section 11 of Chapter 9 are in Section 12 of Chapter 7. Otherwise, Sections 12–13 in both chapters coincide. Entries 14–17 of Chapter 9 are found in Sections 15 and 16 of Chapter 7. Following Entry 17 in Chapter 9 are nine examples. Except for Example 5, these results are located in Sections 17–20 of Chapter 7. In Example 5, Ramanujan claims that

$$\frac{\pi}{2}\left(\frac{1}{2}+\sum_{n=0}^{\infty}\frac{(-1)^{n(n+1)/2}}{\sqrt{n}+\sqrt{n+2}}\right)=\sum_{n=0}^{\infty}\frac{1}{(4n+1)^{3/2}}, \quad (5)$$

which we now prove.

Using Example 4 in Section 17 of Chapter 9, or the corollary in Section 18 of Chapter 7 (Part I [2, p. 172]), and Example 4 in Section 4 of Chapter 9, i.e., Corollary 4 in Section 4 of Chapter 7 (Part I [2, p. 154]), we find that the left side of (5) equals

$$\frac{\pi}{2}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{2n-1}+\sqrt{2n+1}}+\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{2n}+\sqrt{2n+2}}\right)$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{3/2}}+\frac{\pi}{2\sqrt{2}}\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$$
$$=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{3/2}}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{3/2}},$$

from which (5) immediately follows.

Chapter 10 (pages 119–141)
The Gamma Function and Analogues

The first 13 sections of Chapter 10 correspond to the first 13 sections of Chapter 8 in the second notebook. Except for (e)–v., the results (a)–(e) in

Section 14 are in Section 13 of Chapter 8 in Notebook 2. Entry (e)–v is merely a statement of the Euler–Maclaurin summation formula. Examples 1–8 in Section 14 are in Section 14 of Chapter 8, but in the first notebook, the results are stated in terms of power series, while in the second notebook, they are given in terms of integrals. In the first notebook, there now follow eleven examples. The first eight are simply the cases $x = 1$ of the eight previous identities and do not appear in the second notebook. The next three examples are found in Section 14 of Chapter 8 of the second notebook. Ten further examples appear in Section 14 of the first notebook, and these are also found in Section 14 of the second. Entries 15 and 16 of the two chapters correspond.

The material in Entries 17–20, however, is found in Chapter 7 of the second notebook. More specifically, Sections 17, 18, and 19 are found in Sections 23, 26, and 24, respectively, of Chapter 7. Except for two examples, the content of Section 20 is in Section 25 of Chapter 7. One of the examples is simply the case $z = i$ of the reflection formula for the gamma function,

$$\Gamma(z)\Gamma(1 - z) = \pi \csc(\pi z).$$

The other example is the case $z = i$ of Legendre's duplication formula for the gamma function,

$$\Gamma(z + 1) = 2^z \pi^{-1/2} \Gamma(\tfrac{1}{2}\{z + 1\})\Gamma(\tfrac{1}{2}z + 1).$$

Entries 21–25 of Chapter 10 are contained in Section 17 of Chapter 8 in Notebook 2. Entries 26–28 are found in Section 18 of Chapter 8. Sections 29–34 correspond to Sections 19–24 of Chapter 8.

Chapter 11 (pages 143–169) Infinite Series Identities and Transformations

The first three entries of Chapter 11 correspond to the first three entries of Chapter 9 of the second notebook. However, Entry 4 is not found in the second notebook. Entry 4(ii) is meaningless, since neither series converges. In Entry 4(i), Ramanujan claims that

$$\sum_{n=2}^{\infty} \frac{(-1)^n H_n \sin(nx)}{n} = \frac{x}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos(nx)}{n}, \tag{6}$$

where, for each integer $n \geq 1$, $H_n = \sum_{k=1}^{n} 1/k$. To prove (6), first square the Maclaurin series for $\log(1 + z)$ and then set $z = e^{ix}$, $|x| < \pi$, to deduce that

$$\log^2(1 + e^{ix}) = 2 \sum_{n=2}^{\infty} \frac{(-1)^n H_{n-1} e^{inx}}{n}.$$

Taking the imaginary parts of both sides above, we find that, for $|x| < \pi$,

$$\sum_{n=2}^{\infty} \frac{(-1)^n H_{n-1} \sin(nx)}{n} = \tfrac{1}{2}x \log(2 \cos(x/2)).$$

On the other hand, it is well known that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos(nx)}{n} = \log(2 \cos(x/2)), \qquad |x| < \pi.$$

The last two equalities imply that (6) is valid for $|x| < \pi$.

Entry 5 in Chapter 11 is identical to Entry 5 of Chapter 9 of the second notebook, while Entries 6 and 7 are in Section 4 of Chapter 9. The contents of Sections 8–17 are roughly equivalent to those in Sections 6–15 of Chapter 9. However, in Section 17, Ramanujan records three well-known results that he obviously felt he need not record in the second notebook, namely, the Maclaurin series for $\arcsin x$ and $\arctan x$ and the derivatives of $(\arcsin x)^n$ and $(\arctan x)^n$. Sections 18–25 of Chapter 11 correspond to Sections 16–23 of Chapter 9 in Notebook 2. Section 26 is the same as Section 33 of Chapter 9. Sections 27–30 of each chapter correspond. Sections 31 of the two chapters are the same, except that Examples 6 and 7 of the first notebook are not in the second. In these examples, Ramanujan says "Find" $\psi(\frac{2}{3})$, $\psi(\frac{3}{4})$, $\psi(\frac{5}{6})$, $\psi(\frac{3}{8}) - \psi(\frac{1}{8})$, $\psi(\frac{1}{8})$, and $\psi(\frac{1}{12})$, but he does not give the values. However, these values are easily found in the same manner as those in the previous examples.

The content of Section 32 of Chapter 11 is found in Section 32 of Chapter 9 of the second notebook, except for Examples 6 and 10. Example 10 is the same as Entry 34 of Chapter 9. Example 6 does not appear in the second notebook. We will not record it here, because it simply arises from Corollary (ii) in Section 32 of Chapter 9 (Part I [2, p. 288]) by setting $u = \tan x$ in this corollary.

Chapter 12 (pages 171–187) Hypergeometric Series, I

Chapter 12 corresponds to Chapter 10 of the second notebook. However, ten results in Chapter 12 are apparently not to be found in the second notebook.

Entries 1–5 of Chapter 12 are the same as Entries 1, 4, 2, 3, and 5, respectively, of Chapter 10 in the second notebook, with the Examples in Section 4 of Notebook 1 also in Section 4 of Notebook 2. Entries 6–12 are, respectively, Corollaries 1, 3, and 2, Entry 7, and Corollaries 4–6 in Section 7 of Chapter 10. Entries 13–17 are not given in the second notebook.

Entry 13. *If* $\operatorname{Re}(x + n) > 1$, *then*

$$n\,{}_4F_3\left[\begin{matrix} \frac{1}{2}n + 1, & 1, & 1, & -x \\ \frac{1}{2}n, & n, & x + n + 1 \end{matrix}; 1\right] = \frac{(n-1)(x+n)}{x+n-1}.$$

PROOF. Set $y = -1$ in Entry 7, or Corollary 3 in Section 7 of Chapter 10 of the second notebook (Part II [4, p. 16]), and the result easily follows.

Entry 14. *If* Re $x > 0$, *then*

$$n\,{}_3F_2\left[\begin{matrix}\frac{1}{2}n+1, & n, & -x \\ \frac{1}{2}n, & x+n+1\end{matrix};1\right]=0.$$

PROOF. Set $y = -n$ in the same result employed in the proof above.

Entry 15. *If* Re$(2x + n) > 0$, *then*

$$\sum_{k=0}^{\infty}\frac{(-x)_k}{(x+n+1)_k}=\frac{x+n}{2x+n}.$$

PROOF. In Gauss's theorem, Entry 8 of Chapter 10 of the second notebook (Part II [4, p. 25]), set $y = -1$ and replace n by $n + x$. The desired result easily follows.

Entry 16. *If* Re$(2x + n) > 1$, *then*

$$n\,{}_3F_2\left[\begin{matrix}\frac{1}{2}n+1, & 1, & -x \\ \frac{1}{2}n, & x+n+1\end{matrix};1\right]=\frac{(n-1)(x+n)}{2x+n-1}.$$

PROOF. Set $y = -1$ in Entry 10, i.e., Corollary 4 in Section 7 of Chapter 10 of the second notebook (Part II [4, p. 16]), and the sought result easily follows.

Entries 13 and 16 should be compared.

Entry 17. *If* Re$(2x + n) > 0$, *then*

$$n\,{}_3F_2\left[\begin{matrix}\frac{1}{2}n+1, & 1, & -x \\ \frac{1}{2}n, & x+n+1\end{matrix};-1\right]=x+n.$$

PROOF. In Entry 11, i.e., Corollary 5 in Section 7 of Chapter 10 in the second notebook (Part II [4, p. 16]), let $y = -1$, and the result readily follows.

Entries 18–25 are identical to Corollaries 7–14 of Entry 7 in Chapter 10. Entries 26–29 and Entry 31 do not appear in the second notebook.

Entry 26. *If* Re $n > 2$, *then*

$$n\,{}_4F_3\left[\begin{matrix}\frac{1}{2}n+1, & 1, & 1, & 1 \\ \frac{1}{2}n, & n, & n\end{matrix};1\right]=\frac{(n-1)^2}{n-2}.$$

PROOF. In Entry 5, or Entry 5 of Chapter 10 in Notebook 2 (Part II [4, p. 11]), set $x = y = z = -1$.

Entry 27. *If* Re $n > 2$, *then*

$$\sum_{k=0}^{\infty} \frac{k!}{(n)_k} = \frac{n-1}{n-2}.$$

PROOF. In Gauss's theorem, Entry 8 in Chapter 10 of the second notebook (Part II [4, p. 25]), put $x = y = -1$ and replace n by $n - 1$.

Entry 28. *If* Re $n > 3$, *then*

$$n\,{}_3F_2\left[\begin{matrix} \frac{1}{2}n+1, & 1, & 1 \\ \frac{1}{2}n, & n, & \end{matrix}; 1\right] = \frac{(n-1)^2}{n-3}.$$

PROOF. Again we turn to Entry 5, but now with $y = z = -1$ and $x = -(n+1)/2$. The result now readily follows.

Entry 29. *If* Re $n > 2$, *then*

$$n\,{}_3F_2\left[\begin{matrix} \frac{1}{2}n+1, & 1, & 1 \\ \frac{1}{2}n, & n & \end{matrix}; -1\right] = n - 1.$$

PROOF. In Entry 11, or Corollary 5 in Section 7 of Chapter 10 in the second notebook (Part II [4, p. 16]), set $x = y = -1$.

Entry 31. *If* Re $n < \frac{2}{3}$, *then*

$${}_3F_2\left[\begin{matrix} n, & n, & n \\ 1, & 1 & \end{matrix}; 1\right] = \frac{6\sin(\pi n)\sin(\frac{1}{2}\pi n)\Gamma^3(\frac{1}{2}n+1)}{\pi^2 n^2(1+2\cos(\pi n))\Gamma(\frac{3}{2}n+1)}.$$

PROOF. In Entry 19, or Corollary 8 in Section 7 of the second notebook (Part II [4, p. 17]), set $x = -n$. The result follows after elementary simplification.

Entry 30 is the same as Corollary 15 of Entry 7 of Chapter 10 in Notebook 2. Entries 32–40 are identical to Corollaries 16–24 of Entry 7. Eighteen examples follow Entry 40 in Chapter 12. The first fourteen are identical to the first fourteen examples found in Section 7 of Chapter 10 of Notebook 2. Example 15 is the same in each chapter, except that the hypergeometric function ${}_3F_2(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; 1)$ is evaluated in Notebook 2, but not in Notebook 1. Similarly, Example 16 in Notebook 1 is identical to Example 19 of Notebook 2, except that the same hypergeometric function is evaluated in the latter instance, but not the former. Example 17 is the same as Example 16 in Notebook 2, if one uses the evaluation in Example 19 of Notebook 2. Lastly, Example 18 is the same as Example 17 in the second notebook, if Example 15 of Notebook 2 is employed.

Entries 41–43 of Chapter 12 are the same as Entries 8–10 in Chapter 10 of the second notebook. After Entry 43, there are eight examples, and these roughly correspond to the seven examples after Entry 10 in Chapter 10. Examples 1 and 2 in both chapters are identical. Example 3 in Chapter 12 is the same as Example 3 in Chapter 10, except that, in the second notebook, x was replaced by $n + 1$, and the result was simplified via the duplication formula for the gamma function. Example 4 in Chapter 12 is an incorrect version of Example 4 in Chapter 10. To obtain a correct statement, $-n$ in the second notebook should be replaced by x in the first notebook, and then the result should be transformed by employing the reflection formula for the gamma function. Example 5 is the same in each chapter. Example 6 of Chapter 12 corresponds to Example 6 in Section 10 of Chapter 10, but the formulations are quite different. In the first notebook, Example 6 arises from Example 3 by differentiation with respect to x. Example 7 does not appear in the second notebook and is incorrect. More specifically, Ramanujan incorrectly expresses

$$\sum_{k=0}^{\infty} \frac{(x)_k}{(2k+1)^2 k!}$$

in terms of gamma functions. A correct formula for the sum above can be derived by replacing x by $-x$ and setting $n = \frac{1}{2}$ in Example 5 of Section 10 of Chapter 10 of Notebook 2 (Part II [4, p. 26]). We then find that

$$\sum_{k=0}^{\infty} \frac{(x)_k}{(2k+1)^2 k!} = \frac{\sqrt{\pi}\,\Gamma(1-x)}{2\Gamma(\frac{3}{2}-x)} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k+1-2x} \right).$$

Example 8 is the same as Example 7 in Chapter 10 of the second notebook.

Entries 44–47 in Chapter 12 are equivalent to Entries 11–14 in Chapter 10 of the first notebook.

Chapter 13 (pages 189–207) Hypergeometric Series, II

The first seven entries of Chapter 13 correspond to Entries 16–23 of Chapter 10 in the second notebook. However, Example 3 in Section 7 of Chapter 13 is not found in the second notebook. Here Ramanujan claims that

$$ {}_3F_2(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; 1, 1; 1) = 2\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; -1) = {}_2F_1^2(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{2}). \tag{7}$$

By Example 18 in Section 7 of Chapter 10 of Notebook 2 (Part II [4, p. 24]), the first member of (7) equals $\pi/\Gamma^4(\frac{3}{4})$, and by Entry 34 of Chapter 10 of the second notebook (Part II [4, p. 42]), the third member of (7) also equals $\pi/\Gamma^4(\frac{3}{4})$. However, by Example 19 in Section 7 of Chapter 10 of the second notebook (Part II [4, p. 24]), the second member of (7) equals $\sqrt{2\pi}/\Gamma^2(\frac{3}{4})$. Thus, only the extremal members of (7) are equal.

Entry 8 in Chapter 13 is identical to Entry 24 of Chapter 10 in the second notebook. In Entry 9, Ramanujan claims that

$$\sum_{k=0}^{\infty} \frac{(r)_k}{x^{r+k}(n)_{k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k (r)_k}{(n+k)(x-1)^{n+k} k!},$$

where $|x|$, $|x-1| > 1$ and $\operatorname{Re} n > 0$. Letting $m = n + 1 - r$ in Entry 24 of Chapter 10 of Notebook 2 (Part II [4, p. 38]), we find that we must show that, for $|x| > 1$,

$$\sum_{k=0}^{\infty} \frac{(r)_k}{x^{r+k}(n)_{k+1}} = \sum_{k=0}^{\infty} \frac{(n+1-r)_k}{(n+k)k!\, x^{n+k}}. \tag{8}$$

But (8) cannot be correct, unless $r = n$, for otherwise the two sides of (8) have different behaviors as $x \to \infty$. If $r = n$, (8) is a tautology. Ramanujan's confusion about the roles of r and n becomes even clearer when one notes that in his recording of the right side of (8), the first power of $x - 1$ is $-r$, while in the next two terms, the powers are $-n-1$ and $-n-2$. Entries 10–15 of Chapter 13 correspond to Entries 25–30 of Chapter 10 in the second notebook.

Entries 16–18 of Chapter 13 are the same as Entries 1–3 of Chapter 11 in the second notebook. Section 19 in Chapter 13 is the same as Section 7 of Chapter 11 of Notebook 2, except that three examples in Section 19 do not appear in the second notebook. In the first example, Ramanujan claims that a certain hypergeometric series evaluated at π equals 0. However, the series has radius of convergence equal to 1. The next two examples are purported hypergeometric series identities at the value π. But in each case, one of the two series does not converge at π. Thus, clearly, Ramanujan saw that his previous claims were in error.

Entries 20(i), (ii) are the special cases $r = \frac{1}{2}$ of Entry 5 of Chapter 11 and $r = m = \frac{1}{2}$ of Entry 2 of Chapter 11 in the second notebook (Part II [4, pp. 50, 49]), respectively. Examples 1 and 2 in Section 20 are not recorded by Ramanujan in the second notebook. In Example 1, Ramanujan asserts that

$${}_2F_1\left(r, m; 2m; \frac{4x}{(1+x)^2}\right) = \frac{(1-x^2)^{2m}}{(1-x)^{2r}}\, {}_2F_1(2m-r, m-r+\tfrac{1}{2}; m+\tfrac{1}{2}; x^2). \tag{9}$$

Identity (9) is easily proved by combining Entry 3 of Chapter 11 and

Entry 26 of Chapter 10, both in the second notebook (Part II [4, pp. 50, 39]). In Example 2, Ramanujan records that

$$ {}_2F_1\left(r, m; 2m; \frac{2x}{1+x}\right) = \frac{(1-x^2)^m}{(1-x)^r} {}_2F_1(\tfrac{1}{2}(2m-r), \tfrac{1}{2}(2m-r+1); m+\tfrac{1}{2}; x^2). \tag{10} $$

Example 2 is easily proved by combining Entry 2 of Chapter 11 with Entry 26 of Chapter 10, both in the second notebook (Part II [4, pp. 49, 39]). Examples 3 and 4 in Section 20 are identical to Entries 4 and 5, respectively, of Chapter 11 of Notebook 2.

Entry 21 is the same as Entry 15 of Chapter 11 of the second notebook. Inexplicably, Ramanujan again records Entry 19 of the present chapter as Corollary 1. Corollary 2 is the instance $\gamma = \delta = n + \frac{1}{2}$ of Entry 15 of Chapter 11, with x replaced by $x/8$. Examples 1 and 2 in Section 21 are easily derived from Corollaries 1 and 2 by setting $n = \frac{1}{2}$ and replacing x by $x/2$ and $2x$, respectively.

Sections 22–26 correspond to Sections 16–20 of Chapter 11 of the second notebook. Sections 27 and 28 are the same as Sections 34 and 33, respectively, of Chapter 10 of Notebook 2. Sections 29 and 30 are the same as Sections 12 and 13, respectively, of Chapter 11 in Notebook 2.

Chapter 14 (pages 209–225) Continued Fractions

Entries 1–4 are identical to the first four entries of Chapter 12 of the second notebook. Entries 5–14 are a reordering of Entries 7–16 in Chapter 12 of Notebook 2. Entry 15 is an equivalent version of Entry 17 in Chapter 12 of the second notebook. Entries 17–19 correspond to Entries 19–21 in Chapter 12 of Notebook 2. Sections 20–24 correspond to Sections 33, 25, 26, 29, and 30, respectively, in Chapter 12. Entries 25–27 are identical to Entries 32(ii), (i), (iii), respectively, in Chapter 12 of the second notebook. The corollary in Section 27, however, is not found in the second notebook.

Corollary. *Let $Z(x) = \zeta(3, x) - 1/x^3$, where $\zeta(s, x)$ denotes the Hurwitz zeta-function. Then, for some function $\varphi(x)$ and $x > 0$,*

$$ Z(x) = \frac{1}{2(x^2+x)+1} + \frac{1}{24(x^2+x)^3 + 60(x^2+x)^2 + \frac{72}{5}(x^2+x) + \frac{1}{40}\varphi(x)}. \tag{11} $$

Moreover,

$$ \varphi(0) = 198, \qquad \varphi(\tfrac{1}{2}) = 571, \qquad \varphi(1) = 1015, \qquad \varphi(\tfrac{3}{2}) = 1384, $$
$$ \varphi(2) = 1679, \qquad \varphi(\tfrac{5}{2}) = 1916, \quad \textit{and} \quad \varphi(3) = 2093 $$

"*nearly*," *and* $\varphi(\infty) = 2880$. *Moreover*, "*if h is a positive proper fraction*,"

$$\frac{\varphi(2+h)-\varphi(2)}{\varphi(3)-\varphi(2)} = \frac{3h\varphi(2)}{2\varphi(3)+\frac{1}{2}h\{3\varphi(2)-2\varphi(3)\}} \tag{12}$$

"*nearly.*"

Proof. The form of (11) suggests that Ramanujan did not employ $Z(x)$ directly but instead truncated the first continued fraction of Entry 27, or Entry 32(iii) of Chapter 12 in the second notebook (Part II [4, p. 153]). More precisely, if $y = 2(x^2 + x)$,

$$\frac{1}{y+}\,\frac{1^3}{1+}\,\frac{1^3}{3y+}\,\frac{2^3}{1+}\,\frac{2^3}{5y+}\,\frac{3^3}{1+}\,\frac{3^3}{7y} = \frac{105y^3+1455y^2+3620y+216}{105y^4+1560y^3+5040y^2+3456y}. \tag{13}$$

Then

$$\begin{aligned}
&\frac{105y^3+1455y^2+3620y+216}{105y^4+1560y^3+5040y^2+3456y} - \frac{1}{y+1}\\
&\qquad = \frac{35y^2+380y+216}{105y^5+1665y^4+6600y^3+8496y^2+3456y}\\
&\qquad = \frac{1}{\dfrac{105y^5+1665y^4+6600y^3+8496y^2+3456y}{35y^2+380y+216}}\\
&\qquad = \frac{1}{3y^3+15y^2+\frac{36}{5}y+\dfrac{2520y^2+1900.8y}{35y^2+380y+216}}.
\end{aligned}$$

Thus, in the notation of (11), if $Z(x)$ is replaced by the continued fraction in (13), we find that

$$\tfrac{1}{40}\varphi(x) = \frac{2520y^2+1900.8y}{35y^2+380y+216}. \tag{14}$$

Hence,

$$\lim_{x\to\infty} \varphi(x) = 2880,$$

as claimed by Ramanujan.

If we employ (14) to calculate $\varphi(x)$ for the remaining seven values of x, we do not obtain Ramanujan's values. For example, by (14), $\varphi(\frac{1}{2})$ is about 394, in contrast to that claimed. Thus, Ramanujan returned to (11) to calculate $\varphi(x)$ for $x = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$. With the help of *Mathematica*, we used (11) to obtain the following values, most given to the nearest integer:

$$\varphi(0) = 198, \quad \varphi(\tfrac{1}{2}) = 591, \quad \varphi(1) = 1015, \quad \varphi(\tfrac{3}{2}) = 1383.46,$$
$$\varphi(2) = 1679, \quad \varphi(\tfrac{5}{2}) = 1909, \quad \text{and} \quad \varphi(3) = 2086.$$

Thus, the values for $x = 0$, 1, and 2 are precisely those given by Ramanujan, while for $x = \frac{3}{2}$ our calculated value is very close to what Ramanujan claimed. We think that Ramanujan's value 571 for $\varphi(\frac{1}{2})$ is probably a misprint of 591. His values for $\varphi(\frac{5}{2})$ and $\varphi(3)$ are slightly more in error.

The approximation (12) was probably empirically discovered by Ramanujan. If $\varphi(x) = ax$, for some constant a, (12) is a trivial identity, but, otherwise, (12) appears to lack validity. Using $h = \frac{1}{2}$ and the values for $\varphi(2)$, $\varphi(\frac{5}{2})$, and $\varphi(3)$ claimed by Ramanujan, we find that the left and right sides of (12) equal, respectively, 0.572464 and 0.572549, which are remarkably close. We calculated these quotients for $h = \frac{1}{8}$, $\frac{1}{4}$, and $\frac{3}{4}$ and obtained the following values for the left and right sides of (12):

h	Left Side	Right Side
$\frac{1}{8}$	0.1548	0.1490
$\frac{1}{4}$	0.2998	0.2942
$\frac{3}{4}$	0.7961	0.8402.

Although for $h = \frac{1}{4}$ the values closely agree, this is not the case for $h = \frac{1}{8}$ and $h = \frac{3}{4}$. In conclusion, we think that Ramanujan empirically devised (12) to reflect his calculations of $\varphi(2)$, $\varphi(\frac{5}{2})$, and $\varphi(3)$.

Most of the material in Sections 28 and 29 has no relevance to continued fractions and can be found in Section 24 of Chapter 13 and Entry 10(ii) of Chapter 15, both in the second notebook. In an example concluding Section 29, Ramanujan claims that

$$\sum_{k=0}^{\infty} \frac{(-m)_k}{(n+1)_k}(-x)^k = \frac{\Gamma(m+1)\Gamma(n+1)}{x^n\Gamma(m+n+1)}(1+x)^{m+n} - \theta,$$

where $\theta < n/(x(m+1))$, when m and x are "very great." It is also tacitly assumed that $n > 1$. With these conditions on m, n, and x, Ramanujan is, indeed, correct, and his claim follows easily from eq. (41.2) of Part II [4, p. 164]. Ramanujan next gives a continued fraction representation for θ, and this is equivalent to Entry 41 in Chapter 12 the second notebook.

The first three identities of Section 30 are found in Section 42 of Chapter 12 of Notebook 2, while the two examples are alternative versions of the two corollaries in Section 43 of Chapter 12 of Notebook 2. Entry 31 corresponds to Entry 43 of Chapter 12; the corollary of Entry 31 is not recorded in the second notebook but is merely the instance $x = 1$ of Entry 31.

Most of Section 32 is found in Section 44 of Chapter 12 of the second notebook, but several parts are not recorded in the second notebook. We designate these parts by (i)–(iv) below.

Entry 32(i). *Let*

$$\varphi(x) = e^x\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^k}{k!\,k} - \gamma - \log 2\right), \tag{15}$$

where γ denotes Euler's constant. Then

$$\varphi(2) - \varphi^2(1) = \frac{1}{170}$$

"*nearly.*"

PROOF. Using *Mathematica*, we find that $\varphi(2) = 0.3613286169$ and $\varphi(1) = 0.5963473623$, and so

$$\varphi(2) - \varphi^2(1) = 0.005698440378.$$

On the other hand,

$$\frac{1}{170} = 0.005882352941\ldots,$$

which justifies Ramanujan's approximation.

Entry 32(ii). *Let $\varphi(x)$ be given by* (15). *Then there exists a function $\theta = \theta(x)$ such that*

$$\varphi(x) = \frac{1}{x+1} + \frac{1}{x^3 + 5x^2 + 2x + \theta(x)},$$

$\theta(0) = 0$, *and* $\theta(\infty) = 14$.

PROOF. We are quite certain that Ramanujan obtained this formula for $\varphi(x)$ by truncating one of the continued fractions for $\varphi(x)$ in Section 32, i.e., Entry 44(iii) in Chapter 12 of the second notebook (Part II [4, p. 169]). More precisely,

$$\frac{1}{x+1}\,\frac{}{+}\,\frac{1}{x+1}\,\frac{}{+}\,\frac{1}{x+1}\,\frac{}{+}\,\frac{2}{x+1}\,\frac{}{+}\,\frac{2}{x+1}\,\frac{}{+}\,\frac{3}{x+1}\,\frac{}{+}\,\frac{3}{x+1}\,\frac{}{+}\,\frac{4}{x+1}\,\frac{}{+}\,\frac{4}{x} = \frac{x^4 + 19x^3 + 102x^2 + 154x + 24}{x^5 + 20x^4 + 120x^3 + 240x^2 + 120x}.$$

Then

$$\frac{x^4+19x^3+102x^2+154x+24}{x^5+20x^4+120x^3+240x^2+120x}-\frac{1}{x+1}$$
$$=\frac{x^3+16x^2+58x+24}{x^6+21x^5+140x^4+360x^3+360x^2+120x}$$
$$=\frac{1}{\dfrac{x^6+21x^5+140x^4+360x^3+360x^2+120x}{x^3+16x^2+58x+24}}$$
$$=\frac{1}{x^3+5x^2+2x+\dfrac{14x^3+124x^2+72x}{x^3+16x^2+58x+24}}. \tag{16}$$

Thus, with

$$\theta(x)=\frac{14x^3+124x^2+72x}{x^3+16x^2+58x+24}, \tag{17}$$

we easily see that $\theta(0)=0$ and $\theta(\infty)=14$, as claimed.

Entry 32(iii). *Let $n=1/(14-\theta(x))$. Then*

$$x=100n-8.6+\frac{0.81}{9n+0.14} \tag{18}$$

"*very nearly.*"

Proof. It is clear that Ramanujan obtained (18) empirically. Using (17), we used *Mathematica* to expand the right side of (18) in descending powers of x and found that

$$100n-8.6+\frac{0.81}{9n+0.14}=x+0.00642144-8.95418/x+61.6369/x^2+\cdots,$$

which is amazingly close to x, if x is large.

Entry 32(iv). *If $x>8$, then*

$$x=\frac{0.08\theta^2+6.34\theta-4.45}{14-\theta} \tag{19}$$

"*to 2 places of decimals.*" *If $x<7$, then*

$$14-\theta=\frac{1000}{11x+75} \tag{20}$$

"*to a place of decimal.*" *Furthermore*, "$x = 16.74$ *when* $\theta = 10$ *and* $\theta = 5.6$ *when* $x = 4$."

PROOF. All calculations and expansions given below were performed by *Mathematica.*

The approximations (19) and (20) were doubtless found experimentally. If we take x to be given by (18), we find that

$$x(14 - \theta) = -5.9146 + 6.71570\theta + 0.049835\theta^2 + \dots, \tag{21}$$

which does not compare well with (19). Note from (20) that

$$x(14 - \theta) = \frac{75\theta - 50}{11} = 6.81818\theta = 4.545,$$

which does not agree very well with (21). Furthermore, if we invert (17), we find that

$$x = \frac{1}{3}\theta + \frac{25}{324}\theta^2 - \frac{313}{34992}\theta^3 + \cdots,$$

and so

$$x(14 - \theta) = \frac{14}{3}\theta + 0.74690\theta^2 + \cdots.$$

Despite the lack of agreement amongst these formulas, Ramanujan's approximations are very accurate. Using (19), we find that $x(10) = 16.74$, as claimed, and using (20), we find that $\theta(4) = 5.6$, as claimed. If we employ (17), we find that $\theta(16.74) = 9.99331$ and $\theta(4) = 5.5$, which corroborates Ramanujan's approximations, although the latter is not as accurate as he claimed. We calculated a few additional values to check the accuracy of Ramanujan's assertions (19) and (20). By (17), $\theta(2) = 3.54717$, while by (20), $\theta(2) = 3.69072$. By (17), $\theta(10) = 8.46442$, while by (19), $x(8.46442) = 9.92599$. By (17), $\theta(30) = 11.3928$, while by (19), $x(11.3928) = 29.9801$.

Section 33 is contained in Section 46 of Chapter 12 of the second notebook.

Chapter 15 (pages 227–249)
Hypergeometric Series and Theta-Functions

Entries 1–3 correspond to Entries 32(i)–(iii), and the corollary in Section 3 corresponds to Entry 32(iv) in Chapter 11 of Ramanujan's second notebook. Entries 4–6 are the same as Entries 33(i)–(iii) in Chapter 11 in the second notebook. Section 7 contains Entry 34 and Entry 34(i) of Chapter 11, while

Entry 8, which contains some misprints, corresponds to Entry 34(ii) of Chapter 11 of the second notebook. Following Entry 8 are 21 examples.

Example 1 is identical to Example (i) in Section 33 of Chapter 11. Example 2 cannot be found in the second notebook, but is given in Part II [4, p. 95, line 5b] in the course of proving Example (ii) of Section 33, which is the same as Example 3 here. Example 4 is a combination of Entry 33(ii) and Entry 33(v), while Example 5 is identical to Example (iii) in Section 33 of Chapter 11. Examples 6 and 7 are Examples (i) and (ii), respectively, in Section 34 of Chapter 11. The top of page 231 in the first notebook is badly frayed, and so it is impossible to decipher Example 8. Several of the remaining examples are not given in the second notebook and so will be proved here. All references in the proofs below are to the second notebook.

Example 9.

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{80}{81}\right) = \frac{18}{\sqrt{62}}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{31^2}\right).$$

Proof. In Entry 2 of Chapter 11 (Part II [4, p. 49]), set $r = m = \frac{1}{2}$ and $x = \frac{1}{31}$ to obtain the equality

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{16}\right) = \frac{18}{\sqrt{62}}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{31^2}\right). \tag{22}$$

Next apply Entry 5 of Chapter 11 (Part II [4, p. 50]) with $r = \frac{1}{2}$ and $x = \frac{1}{4}$ to find that

$${}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{16}) = \tfrac{4}{5}\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; (\tfrac{4}{5})^2). \tag{23}$$

Applying Entry 5 again, but with $r = \frac{1}{2}$ and $x = \frac{4}{5}$, we find that

$${}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; (\tfrac{4}{5})^2) = \tfrac{5}{9}\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{80}{81}). \tag{24}$$

Combining (22)–(24), we complete the proof.

Example 10.

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{15}{16}\right) = \frac{32}{\sqrt{322}}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{161^2}\right).$$

Proof. Applying Entry 2 of Chapter 11 (Part II [4, p. 49]) with $r = m = \frac{1}{2}$ and $x = \frac{1}{161}$, we find that

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{81}\right) = \frac{18}{\sqrt{322}}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{161^2}\right). \tag{25}$$

Invoking Entry 5 of Chapter 11 (Part II [4, p. 50]) with $r = \frac{1}{2}$ and $x = \frac{1}{9}$ and $x = \frac{3}{5}$, respectively, we find that

$$ {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{81}) = \tfrac{9}{10}\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{9}{25}) \tag{26} $$

and

$$ {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{9}{25}) = \tfrac{5}{8}\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{15}{16}). \tag{27} $$

Combining (25)–(27), we complete the proof.

Example 11 is Corollary 1 in Section 29 of Chapter 11 of the second notebook. Example 12 is the same as Example 16 in Section 7 of Chapter 10 in the second notebook, but is incorrectly given in the first notebook.

Example 13.

$$ {}_3F_2(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; 1, 1; -1) = \frac{\Gamma^2(\frac{9}{8})}{\Gamma^2(\frac{5}{4})\Gamma^2(\frac{7}{8})}. $$

Proof. Set $x = -1$ in Entry 33(iii) of Chapter 11 (Part II [4, p. 95]) and then apply Kummer's theorem, Corollary 13 in Section 7 of Chapter 10 (Part II [4, p. 17]). The desired result easily follows.

Example 14 is identical to Example 14 in Section 7 of Chapter 10.

Example 15.

$$ {}_2F_1(\tfrac{1}{4}, \tfrac{1}{4}; 1; 1) = \frac{\sqrt{\pi}}{\Gamma^2(\frac{3}{4})}. $$

Proof. This result is a direct consequence of Gauss's theorem, Entry 8 of Chapter 10 of the second notebook (Part II [4, p. 25]).

Example 16.

$$ {}_2F_1(\tfrac{1}{4}, \tfrac{1}{4}; 1; -1) = \frac{\Gamma(\frac{9}{8})}{\Gamma(\frac{5}{4})\Gamma(\frac{7}{8})}. $$

Proof. The result follows from Kummer's theorem, Corollary 13, Section 7, Chapter 10, and, in fact, is the same application of Kummer's theorem that was made in the proof of Example 13.

Example 17.

$$ {}_3F_2(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}; 1, 1; 1) = \frac{\sqrt{\pi}}{2^{1/4}\Gamma(\frac{3}{4})\Gamma^2(\frac{7}{8})}. $$

PROOF. Applying Corollary 8 in Section 7 of Chapter 10 (Part II [4, p. 17]) with $x = -\frac{1}{4}$ and $n = \frac{1}{4}$, we find that

$$ {}_3F_2(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}; 1, 1; 1) = \frac{\Gamma(\frac{9}{8})\Gamma(\frac{5}{8})}{\Gamma(\frac{5}{4})\Gamma(\frac{3}{4})\Gamma^2(\frac{7}{8})}. $$

Simplifying with the use of the duplication formula for the gamma function, we complete the proof.

Example 18.

$$ 1 - 9\left(\frac{1}{4}\right)^3 + 17\left(\frac{1\cdot 5}{4\cdot 8}\right)^3 - 25\left(\frac{1\cdot 5\cdot 9}{4\cdot 8\cdot 12}\right)^3 + \cdots = \frac{2\sqrt{2}}{\pi}. $$

PROOF. The result follows from Corollary 17 in Section 7 of Chapter 10 (Part II [4, p. 18]) with $n = \frac{1}{4}$.

Example 19 is identical to Example 15 of Section 7 in Chapter 10.

Example 20.

$$ \int_0^{\pi/2}\int_0^{\pi/2} \frac{d\theta\, d\varphi}{\sqrt{1 - \sin^2\theta \sin^2\varphi \sin^2\psi}} = \left\{\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \sin^2\theta \sin^2(\psi/2)}}\right\}^2. \tag{28} $$

PROOF. Expanding the first integrand in a binomial series and the second integrand below in a hypergeometric series, we find that (e.g., see Part II [4, p. 79, eq. (26.3)], we find that

$$ \begin{aligned} \int_0^{\pi/2}\int_0^{\pi/2} \frac{d\theta\, d\varphi}{\sqrt{1 - \sin^2\theta \sin^2\varphi \sin^2\psi}} &= \frac{\pi}{2}\int_0^{\pi/2} {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \sin^2\theta \sin^2\psi)\, d\theta \\ &= \frac{\pi^2}{4}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{n!^3} \sin^{2n}\psi. \end{aligned} \tag{29} $$

Comparing (29) with (28), we see that it suffices to prove that

$$ {}_3F_2(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; 1, 1; \sin^2\psi) = {}_2F_1^2(\tfrac{1}{2}, \tfrac{1}{2}; 1; \sin^2(\psi/2)). \tag{30} $$

By Entries 33(iii) and 33(ii) in Chapter 11 (Part II [4, p. 95]), we deduce that

$$ \begin{aligned} {}_3F_2(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; 1, 1; \sin^2\psi) &= {}_2F_1^2(\tfrac{1}{4}, \tfrac{1}{4}; 1; \sin^2\psi) \\ &= {}_2F_1^2(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{2}(1 - \sqrt{1 - \sin^2\psi})) \\ &= {}_2F_1^2(\tfrac{1}{2}, \tfrac{1}{2}; 1; \sin^2(\psi/2)), \end{aligned} $$

which is what we wanted to prove.

Example 21.

$$\int_0^{\pi/4}\left\{\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-\sin^2\varphi\sin^2\theta}}\right\}^2 d\varphi=\frac{\pi^3}{16}\,{}_4F_3(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2};1,1,1;1).$$

Proof. Using (30) above and inverting the order of summation and integration, we find that

$$\int_0^{\pi/4}\left\{\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-\sin^2\varphi\sin^2\theta}}\right\}^2 d\varphi=\frac{\pi^2}{4}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^3}{n!^3}\int_0^{\pi/4}\sin^{2n}(2\varphi)\,d\varphi$$

$$=\frac{\pi^3}{16}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^4}{n!^4},$$

which completes the proof.

Entry 9 corresponds to Entry 23 of Chapter 11. Sections 10 and 11 are the same as Sections 24 and 25, respectively, in Chapter 11. Entry 12, the corollary, and Example 1 in Section 12 are found in Section 26 of Chapter 11, while Example 2 corresponds to Entry 15 in Chapter 10, and a proper interpretation of Example 3 can be found in the corollary of Section 15 of Chapter 10 of the second notebook.

Section 13 has seven parts. The first part of (i) is identical to Entry 35(i) of Chapter 10.

Entry 13(i) (Second Part). *Let*

$$\varphi(n)=\frac{\Gamma^2(n+\frac{1}{2})}{\Gamma(n)\Gamma(n+1)}\,{}_3F_2(\tfrac{1}{2},\tfrac{1}{2},n;1,n+1;1)$$

and $\psi(z)=\Gamma'(z)/\Gamma(z)$. *Then*

$$\pi\varphi\left(\frac{n+1}{4}\right)-2\psi\left(\frac{n+2}{4}\right)+\psi\left(\frac{n+1}{2}\right)=5\log 2+\gamma-\frac{2}{8n^2+13}\tag{31}$$

"very nearly," where γ *denotes Euler's constant.*

Proof. By the first part of this entry, or Entry 35(i) of Chapter 10 in the second notebook (Part II [4, p. 43]),

$$\pi\varphi\left(\frac{n+1}{4}\right)-\psi\left(\frac{n+1}{2}\right)\sim 3\log 2+\gamma+\frac{3}{4n^2}-\frac{99}{32n^4}+\frac{999}{32n^6}+\cdots,\tag{32}$$

as n tends to ∞. Comparing (31) and (32), we find that it suffices to show that

$$2\psi\left(\frac{n+2}{4}\right)-2\psi\left(\frac{n+1}{2}\right)$$

has an asymptotic expansion "close" to that for

$$-2\log 2 + \frac{2}{8n^2+13} + \frac{3}{4n^2} - \frac{99}{32n^4} + \frac{999}{32n^6} + \cdots,$$

as n tends to ∞.

Now by Stirling's formula (Abramowitz and Stegun [1, p. 259]),

$$\psi(z) \sim \log z - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kz^{2k}},$$

as z tends to ∞, where B_j, $2 \le j < \infty$, denotes the jth Bernoulli number. Thus, as n tends to ∞,

$$\begin{aligned}
2\left(\psi\left(\frac{n+2}{4}\right) - \psi\left(\frac{n+1}{2}\right)\right) \sim\ & 2\left(\log\frac{n}{4}\left(1+\frac{2}{n}\right) - \log\frac{n}{2}\left(1+\frac{1}{n}\right)\right. \\
& - \frac{2}{n(1+2/n)} + \frac{1}{n(1+1/n)} \\
& \left. + \sum_{k=1}^{\infty} \frac{B_{2k}}{2kn^{2k}}\left(\frac{2^{2k}}{(1+1/n)^{2k}} - \frac{4^{2k}}{(1+2/n)^{2k}}\right)\right) \\
=\ & -2\log 2 + 2\sum_{k=2}^{\infty} \frac{(-1)^{k-1}(2^k-1)}{kn^k} \\
& -2\sum_{k=1}^{\infty} \frac{(-1)^k(2^{k+1}-1)}{n^{k+1}} \\
& +2\sum_{k=1}^{\infty} \frac{B_{2k}}{2kn^{2k}}\left(\frac{2^{2k}}{(1+1/n)^{2k}} - \frac{4^{2k}}{(1+2/n)^{2k}}\right) \\
=\ & -2\log 2 + \frac{1}{n^2} - \frac{7}{2n^4} + \frac{31}{n^6} + \cdots. \qquad (33)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
-2\log 2 + \frac{2}{8n^2+13} + \frac{3}{4n^2} - \frac{99}{32n^4} + \frac{999}{32n^6} + \cdots \\
= -2\log 2 + \frac{1}{n^2} - \frac{7}{2n^4} + \frac{8161}{256n^6} + \cdots. \qquad (34)
\end{aligned}$$

Since $8161/256 = 31.8789\ldots$, we see that the expansions in (33) and (34) closely agree, and thus Ramanujan's approximation described by the words, "very nearly," is justified.

Although (ii) is not explicitly found in the second notebook, it is discussed in Part II [4, p. 42]; in particular, see (35.1) on page 42. Parts (iii) and (iv) form Entry 35(iv) in Chapter 10. Part (v) is established in Part II [4, p. 46]. Lastly, (vi) and (vii) comprise Entries 35(ii) and (iii) in Chapter 10.

Entry 14 is the same as Entry 17 in Chapter 6, while Corollary 1 and the first part of Corollary 2 are also in Section 17 of Chapter 6 of the second notebook. The second part of Corollary 2 is Entry 15(i) of Chapter 6. Corollary 3 is a version of Entry 15(i) with x replaced by $\exp(-2x)$.

Entry 15 begins several sections on theta-functions. Entry 15 is identical to Entry 2 of Chapter 16 in the second notebook. Section 16 has 15 parts followed by four examples. First, (i)–(iv) are found in Entry 18 of Chapter 16. Part (v) is the same as Entry 19 of Chapter 16. Parts (vi) and (vii) are Entries 28 and 31, respectively, in Chapter 16. Parts (viii) and (ix) correspond to Entries 29(i),(ii) in Chapter 16. Parts (x)–(xv) are identical to Entries 30(i)–(vi) in Chapter 16. Examples 1–4 are not explicitly stated in the second notebook. We use the notation in Chapter 16; in particular, see (18.1) and Entry 22 in Part III [6, pp. 34, 36].

Example 1.

$$f(-q, q) = \varphi(-q^4).$$

Proof. Expanding $f(-q, q)$ by the Jacobi triple product identity (Part III [6, p. 35, Entry 19]), we find, after some elementary manipulation, that

$$f(-q, q) = (q^4; q^8)_\infty (q^4; q^4)_\infty = \varphi(-q^4),$$

by eq. (22.4) in [6, p. 37].

Example 2.

$$f(q, q^7) f(q^3, q^5) = \psi(q)\psi(q^4).$$

Proof. Example 2 follows from Entry 30(i) of Chapter 16 (Part III [6, p. 46]) by setting $a = q$ and $b = q^3$ there.

Example 3.

$$2f(q^3, q^5) = \psi(\sqrt{q}) + \psi(-\sqrt{q}).$$

Proof. Set $a = \sqrt{q}$ and $b = q^{3/2}$ in Entry 30(ii) of Chapter 16 (Part III [6, p. 46]).

Example 4.

$$2f(q, q^7) = \frac{\psi(\sqrt{q}) - \psi(-\sqrt{q})}{\sqrt{q}}.$$

PROOF. Set $a = \sqrt{q}$ and $b = q^{3/2}$ in Entry 30(iii) of Chapter 16 (Part III [6, p. 46]).

Section 17 has 14 parts followed by three examples. Parts (i) and (ii) are contained in Entry 22 of Chapter 16. Parts (iii) and (iv) are found in Entry 23 of Chapter 16.

Entry 17(v).

$$\frac{\psi(q)}{\varphi(q)} = \frac{(-q^2; q^2)_\infty}{(-q; q^2)_\infty}.$$

PROOF. This product representation follows from the product representations for $\psi(q)$ and $\varphi(q)$ found in Entry 22(ii) and (22.4), respectively, in Part III [6, pp. 36–37].

Parts (vi)–(xii) are identical to Entries 25(i)–(vii) of Chapter 16.

Entry 17(xiii).

$$\psi^2(q) + \psi^2(-q) = 2\psi(q^2)\varphi(q^4).$$

PROOF. By Entries 25(iv) and 25(i) in Chapter 16 (Part III [6, p. 40]),

$$\psi^2(q) + \psi^2(-q) = \varphi(q)\psi(q^2) + \varphi(-q)\psi(q^2) = 2\psi(q^2)\varphi(q^4).$$

Part (xiv) is the corollary in Section 25 of Chapter 16.

Examples 1 and 2 are in Entries 24(i) and 25(iii), respectively, in Chapter 16.

Example 3.

$$\frac{\psi(q)\psi(-q)}{\psi(q^2)\psi(-q^2)} = \frac{\psi(-q^2)}{\psi(q^4)}. \tag{35}$$

PROOF. Using Example 2 above, or Entry 25(iii) of Chapter 16 (Part III [6, p. 40]), we find that

$$\frac{\psi(q)\psi(-q)}{\psi(q^2)\psi(-q^2)} = \frac{\psi(q^2)\varphi(-q^2)}{\psi(q^4)\varphi(-q^4)}. \tag{36}$$

Comparing (35) and (36), we find that we must show that

$$\psi(q^2)\varphi(-q^2) = \psi(-q^2)\varphi(-q^4).$$

The verification of this last identity can be readily achieved by using Entry 22(ii), (22.3), and (22.4) in Part III [6, pp. 36–37].

Entry 18 has 14 parts. Parts (i)–(vii) correspond to Entries 2(i)–(vii) in Chapter 17 of the second notebook; after (v), Ramanujan records a note, which can also be found in Section 2 of Chapter 17. Part (viii) and the example which follows it are Examples 2 and 1, respectively, in Section 2 of Chapter 17. Part (ix) is identical to Entry 3 of Chapter 17. Part (x) is not explicitly given in the second notebook, but it can be easily deduced, by iteration, from the lemma on page 99 of Part III [6]. Parts (xi) and (xii) are Entries 4(i), (ii), while (xiii) and (xiv) are Entries 5 and 6, respectively, in Chapter 17.

Entry 19 is merely another version of (xiv) above, while the corollary which follows is identical to the corollary in Section 6 of Chapter 17 in the second notebook.

Entry 20 is identical to Entry 35 in Chapter 13 of the second notebook, while the corollary which follows is the same corollary as that given in Section 19 above. Examples 1–3 are identical to Examples (i), (ii), and (iv), respectively, in Section 6 of Chapter 17.

Chapter 16 (pages 251–263)
Infinite Series Identities

Section 1 is roughly equivalent to Section 1 in Chapter 15 of the second notebook. Entries 2–5 correspond to Entries 2, 4, 5, and 6 of Chapter 15, with the corollary of Section 5 identical to the corollary in Section 6 of Chapter 15.

Entry 6 gives a version of the product representation for $f(qe^{i\theta}, qe^{-i\theta})$ (Whittaker and Watson [1, p. 469, last line]).

Corollary 1. *We have*

$$(q^3; q^6)_\infty = \frac{\varphi(-q^9) + qf(-q^3, -q^{15})}{\psi(q)} \tag{37}$$

$$= \frac{\varphi(-q^9) + \omega qf(-q^3, -q^{15})}{\psi(\omega q)} \tag{38}$$

$$= \frac{f(-q^3, -q^{15})}{\psi(q^9)} \tag{39}$$

$$= \frac{\varphi(-q^9)}{f(q^3, q^6)}, \tag{40}$$

where ω is a cube root of unity.

Proof. We first prove (37). By Entry 18(iv) of Chapter 16 (Part III [6, p. 34]),

$$\varphi(-q^9) + qf(-q^3, -q^{15}) = \varphi(-q^9) - q^4 f(-q^{21}, -q^{-3}). \tag{41}$$

Next, we apply the quintuple product identity (Part III [6, p. 80]) with $B = q^2$ and q replaced by $-q^3$. Hence, also using the Jacobi triple product identity and the fact $f(-q) = (q;q)_\infty$ (Part III [6, p. 36, Entry 22(iii)]), we find that

$$\begin{aligned}
\varphi(-q^9) - q^4 f(-q^{21}, -q^{-3}) &= \frac{f(-q^6)f(-q^4, -q^2)}{f(-q^5, -q)} \\
&= \frac{(q^6;q^6)_\infty(q^2;q^6)_\infty(q^4;q^6)_\infty}{(q;q^6)_\infty(q^5;q^6)_\infty} \\
&= \frac{(q^2;q^2)_\infty(q^3;q^6)_\infty}{(q;q^2)_\infty} \\
&= \psi(q)(q^3;q^6)_\infty, \qquad (42)
\end{aligned}$$

by Entry 22(ii) of Chapter 16 (Part III [6, p. 36]). Putting (42) in (41), we complete the proof of (37).

Since the left side of (37) is invariant under the replacement of q by ωq, (38) follows immediately from (37).

Upon applying the Jacobi triple product identity to $f(-q^3, -q^{15})$ and using the product representation of $\psi(q^9)$ in Entry 22(ii) of Chapter 16 (Part III [6, p. 36]), we immediately complete the proof of (39).

To prove (40), first use the product representation for $\varphi(-q^9)$ given by (22.4) in Chapter 16 of Part III [6, p. 37] and the Jacobi triple product identity for $f(q^3, q^6)$. Then use (22.3) of Chapter 16 in Part III [6, p. 37] and elementary manipulation. We thus find that

$$\begin{aligned}
\frac{\varphi(-q^9)}{f(q^3, q^6)} &= \frac{1}{(-q^9;q^9)_\infty(-q^3;q^9)_\infty(-q^6;q^9)_\infty} \\
&= \frac{(q^9;q^{18})_\infty(q^3;q^9)_\infty(q^6;q^9)_\infty}{(q^6;q^{18})_\infty(q^{12};q^{18})_\infty} \\
&= (q^9;q^{18})_\infty(q^3;q^{18})_\infty(q^{15};q^{18})_\infty \\
&= (q^3;q^6)_\infty.
\end{aligned}$$

In Corollary 2, Ramanujan defines $\varphi(n)$ to be the infinite product of Entry 6 with $2\cos\theta$ replaced by $\sqrt{n}$, i.e.,

$$\varphi(n) = (q^2;q^2)_\infty(1 + q\sqrt{n} + q^2)(1 + q^3\sqrt{n} + q^6)(1 + q^5\sqrt{n} + q^{10})\cdots.$$

He then asks to find $\varphi(1)$, $\varphi(2)$, $\varphi(3)$, and $\varphi(4)$ in ascending powers of q. These are simple exercises with $\theta = \pi/3$, $\pi/4$, $\pi/6$, 0, respectively, in the product representation of $f(qe^{i\theta}, qe^{-i\theta})$ of Entry 6.

Entry 7 is identical to Entry 27(iv) in Chapter 16.

Entry 8 gives the transformation formula for $\log \eta(z)$. In Entry 8(ii) of Chapter 14 in the second notebook, Ramanujan states a generalization which includes Entry 8 as a special case. Entry 9 is identical to Corollary (i) in Section 8 of Chapter 14, while the following corollary is the example in Section 8 of Chapter 14. Entry 10 corresponds to Entry 7 in Chapter 14. Entry 11 is an equivalent formulation of Entry 8(ii) in Chapter 14, while Entry 12 is precisely Entry 8(ii) of Chapter 14. Entry 13 corresponds to Entry 8(i) in Chapter 14. Entry 14 and the three corollaries following it are the same as Entry 13 and the corollaries following it in Chapter 14. Entry 15 is identical to Entry 21(i) of Chapter 14. Corollaries 1 and 2 are equivalent versions of Entries 25(i), (ii), respectively, in Chapter 14.

Entry 16 is crossed out by Ramanujan.

Entries 17 and 18 do not appear in the second notebook.

Entry 17. *Let n and x be positive numbers with $n < 2\pi/x$. Then*

$$1 + 2\sum_{k=1}^{\infty} \frac{\cos(knx)}{1 + k^2x^2} = \frac{\pi}{x}\coth\left(\frac{\pi}{x}\right)\cosh n - \frac{\pi}{x}\sinh n.$$

PROOF. We shall apply the Poisson summation formula (Titchmarsh [1, p. 60], Part II [4, p. 252, eq. (6.1)]) to the function $2\cos(nxt)/(1 + t^2x^2)$. Accordingly, we find that

$$1 + 2\sum_{k=1}^{\infty} \frac{\cos(knx)}{1 + k^2x^2} = 2\int_0^{\infty} \frac{\cos(nxt)}{1 + t^2x^2}\,dt + 4\sum_{k=1}^{\infty}\int_0^{\infty} \frac{\cos(nxt)\cos(2\pi kt)}{1 + t^2x^2}\,dt. \tag{43}$$

First, for $n > 0$ (Gradshteyn and Ryzhik [1, p. 406]),

$$\int_0^{\infty} \frac{\cos(nxt)}{1 + t^2x^2}\,dt = \frac{\pi}{2x}e^{-n}. \tag{44}$$

Next, by (44) again,

$$\begin{aligned}\int_0^{\infty} \frac{\cos(nxt)\cos(2\pi kt)}{1 + t^2x^2}\,dt &= \frac{1}{2}\int_0^{\infty} \frac{\cos(2\pi k + nx)t + \cos(2\pi k - nx)t}{1 + t^2x^2}\,dt \\ &= \frac{\pi}{4x}e^{-(2\pi k + nx)/x} + \frac{\pi}{4x}e^{-(2\pi k - nx)/x},\end{aligned} \tag{45}$$

provided that $2\pi k - nx > 0$. Putting (44) and (45) into (43), we find that

$$\begin{aligned}1 + 2\sum_{k=1}^{\infty} \frac{\cos(knx)}{1 + k^2x^2} &= \frac{\pi}{x}e^{-n} + \frac{\pi}{x}\sum_{k=1}^{\infty}\left(e^{-(2\pi k + nx)/x} + e^{-(2\pi k - nx)/x}\right) \\ &= \frac{\pi}{x}e^{-n} + \frac{\pi}{x}(e^{-n} + e^{n})\frac{e^{-2\pi/x}}{1 - e^{-2\pi/x}} \\ &= \frac{\pi}{x}\left(\cosh n \coth\left(\frac{\pi}{x}\right) - \sinh n\right),\end{aligned}$$

which completes the proof.

In Section 18, Ramanujan states a general theorem, which includes Entry 17 as a special case. He then writes "N.B. Within certain limits of x only this theorem is true; so we must be very careful in applying this theorem." He then gives another example, which is very similar to Entry 17. We will first establish the example, which contains two misprints in the first notebook.

Example. *Let m, n, and x be positive with $0 < m + n, m - n < 2\pi/x$. Then*

$$1 + 2\sum_{k=1}^{\infty} \frac{\cos(kmx)\cos(knx)}{1 + k^2x^2} = \frac{\pi}{x}\coth\left(\frac{\pi}{x}\right)\cosh m \cosh n - \frac{\pi}{x}\sinh m \cosh n.$$

Proof. Applying Entry 17, with n replaced by $m + n$ and $m - n$, respectively, we find that

$$\begin{aligned} 1 + 2\sum_{k=1}^{\infty} \frac{\cos(kmx)\cos(knx)}{1 + k^2x^2} &= 1 + \sum_{k=1}^{\infty} \frac{\cos(m+n)kx + \cos(m-n)kx}{1 + k^2x^2} \\ &= \frac{\pi}{2x}\coth\left(\frac{\pi}{x}\right)(\cosh(m+n) + \cosh(m-n)) \\ &\quad - \frac{\pi}{2x}(\sinh(m+n) + \sinh(m-n)) \\ &= \frac{\pi}{x}\coth\left(\frac{\pi}{x}\right)\cosh m \cosh n - \frac{\pi}{x}\sinh m \cosh n. \end{aligned}$$

We first state, as did Ramanujan, his general formula in Section 18 with no hypotheses.

Entry 18. *We have*

$$\begin{aligned} \varphi(0) + \sum_{k=1}^{\infty} \frac{\varphi(kxi) + \varphi(-kxi)}{1 + k^2x^2} &= \frac{\pi}{2x}\coth\left(\frac{\pi}{x}\right)\{\varphi(1) + \varphi(-1)\} \\ &\quad + \frac{\pi}{2x}\{\varphi(1) - \varphi(-1)\}. \end{aligned} \tag{46}$$

Apparently, Ramanujan rightly considered (46) to be valid for only an extremely small class of functions and so eliminated it from the second edition of his notebooks. By Entry 17, one obvious class of functions φ for which (46) is valid is the set of all trigonometric polynomials in $\{\cos(nx)\}$. Provided the requisite order of summation can be reversed, (46) is valid for certain Fourier cosine series in $\{\cos(nx)\}$. We now state and prove a general theorem with different hypotheses, but for which the two examples in Sections 17 and 18 are *not* special cases.

Theorem. *Suppose that φ is an entire function. Define, for $x > 0$,*

$$f(z) := \frac{\{\coth(\pi z/x) + 1\}\varphi(z)}{1 - z^2}.$$

Suppose that C_N, $1 \le N < \infty$, is a sequence of positively oriented squares tending to infinity, centered at the origin, and having their horizontal sides at some bounded distance from the points kix, for each integer k. Assume that

$$\lim_{N \to \infty} \int_{C_N} f(z)\, dz = 0.$$

Then (46) *holds.*

PROOF. We apply the residue theorem. The function $f(z)$ has simple poles at $z = kix$, where k is any integer, and at $z = \pm 1$. Let R_a denote the residue of $f(z)$ at a pole a. Straightforward calculations yield, for $k > 0$,

$$R_{kxi} = \frac{(x/\pi)\varphi(kxi)}{1 + k^2x^2}, \qquad R_{-kxi} = \frac{(x/\pi)\varphi(-kxi)}{1 + k^2x^2}, \qquad R_0 = (x/\pi)\varphi(0),$$

$$R_1 = -\frac{1}{2}\left(\coth\left(\frac{\pi}{x}\right) + 1\right)\varphi(1), \quad \text{and} \quad R_{-1} = \frac{1}{2}\left(-\coth\left(\frac{\pi}{x}\right) + 1\right)\varphi(-1).$$

Applying the residue theorem, letting N tend to ∞, and using our hypotheses, we find that

$$\frac{x}{\pi}\varphi(0) + \frac{x}{\pi}\sum_{k=1}^{\infty} \frac{\varphi(kxi) + \varphi(-kxi)}{1 + k^2x^2} - \tfrac{1}{2}\coth\left(\frac{\pi}{x}\right)\{\varphi(1) + \varphi(-1)\}$$
$$- \tfrac{1}{2}\{\varphi(1) - \varphi(-1)\} = 0.$$

After a modest amount of rearrangement, we obtain (46).

Entry 19 is identical to Entry 8(iii) in Chapter 14 of the second notebook.

In Ramanujan's formulation of Entry 20, which is not in the second notebook, he has an additional parameter n. However, this parameter is superfluous, since the function $\varphi(nx)$ can be replaced by $\varphi(x)$, and so we omit it. Our formulation of Entry 20 is similar to that we gave for Entry 8(iii) (Entry 19 above) in our book [4, pp. 253–254].

Entry 20. *Let α, β, $t > 0$ with $\alpha\beta = \pi$ and $t = \alpha/\beta$. Let C denote the positively oriented parallelogram with vertices $\pm i$ and $\pm t$. Let $\varphi(z)$ be an entire function. Let m be a positive integer and put $M = m + \frac{1}{2}$. Define, for each positive integer n,*

$$f_m(z) := \frac{\varphi(2\beta Mz)}{z^{2n+1}(e^{-2\pi Mz} - 1)(e^{2\pi iMz/t} - 1)}$$

and assume that $f_m(z)/M^{2n}$ *tends to* 0 *boundedly on* $C - \{\pm i, \pm t\}$ *as m tends to* ∞. *Let* B_j, $0 \le j < \infty$, *denote the* jth *Bernoulli number. Then*

$$\frac{(-1)^n}{(2\beta)^{2n}} \sum_{k=1}^{\infty} \frac{\varphi(2\beta ki) + \varphi(-2\beta ki)}{k^{2n+1}(e^{2k\beta^2} - 1)} + \frac{(-1)^n}{(2\beta)^{2n}} \sum_{k=1}^{\infty} \frac{\varphi(2\beta ki)}{k^{2n+1}}$$

$$- \frac{1}{(2\alpha)^{2n}} \sum_{k=1}^{\infty} \frac{\varphi(2\alpha k) + \varphi(-2\alpha k)}{k^{2n+1}(e^{2k\alpha^2} - 1)} - \frac{1}{(2\alpha)^{2n}} \sum_{k=1}^{\infty} \frac{\varphi(2\alpha k)}{k^{2n+1}}$$

$$= -\frac{\pi i}{2} \frac{\varphi^{(2n)}(0)}{(2n)!} + \alpha \sum_{k=0}^{n} (-1)^{n+k} \frac{\varphi^{(2k+1)}(0)}{(2k+1)!} \frac{B_{2n-2k}}{(2n-2k)!} \beta^{2n-2k}$$

$$- i\beta \sum_{k=0}^{n} \frac{\varphi^{(2k+1)}(0)}{(2k+1)!} \frac{B_{2n-2k}}{(2n-2k)!} \alpha^{2n-2k}$$

$$+ 2 \sum_{k=0}^{n+1} \frac{\varphi^{(2k)}(0)}{(2k)!} \sum_{j=0}^{n+1-k} (-1)^j \frac{B_{2j}}{(2j)!} \frac{B_{2n+2-2k-2j}}{(2n+2-2k-2j)!} \alpha^{2n+2-2k-2j} \beta^{2j}.$$

Proof. We integrate $f_m(z)$ over C and apply the residue theorem. On the interior of C, f_m has simple poles at $z = \pm ik/M$ and at $z = \pm kt/M$, $1 \le k \le m$. Furthermore, f_m has a pole of order $2n + 3$ at the origin. Let R_a denote the residue of $f_m(z)$ at a pole a. Straightforward calculations give, for $1 \le k \le m$,

$$R_{ik/M} = \frac{(-1)^n M^{2n} \varphi(2\beta ki)}{2\pi i k^{2n+1}} \left(\frac{1}{e^{2\pi k/t} - 1} + 1 \right),$$

$$R_{-ik/M} = \frac{(-1)^n M^{2n} \varphi(-2\beta ki)}{2\pi i k^{2n+1}(e^{2\pi k/t} - 1)},$$

$$R_{kt/M} = -\frac{M^{2n} \varphi(2\beta kt)}{2\pi i k^{2n+1} t^{2n}} \left(\frac{1}{e^{2\pi kt} - 1} + 1 \right),$$

and

$$R_{-kt/M} = -\frac{M^{2n} \varphi(-2\beta kt)}{2\pi i k^{2n+1} t^{2n}(e^{2\pi kt} - 1)}.$$

Using the generating function

$$\frac{z}{e^z - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} z^j, \qquad |z| < 2\pi,$$

we find that

$$f_m(z) = \frac{it}{(2\pi M)^2 z^{2n+3}} \sum_{k=0}^{\infty} \frac{\phi^{(k)}(0)}{k!} (2\beta M z)^k \sum_{r=0}^{\infty} \frac{B_r}{r!} (-2\pi M z)^r \sum_{j=0}^{\infty} \frac{B_j}{j!} (2\pi i M z/t)^j$$

$$= \frac{it}{(2\pi M)^2 z^{2n+3}} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (2\beta M z)^k$$

$$\times \left(-\frac{\pi^2 i M^2 z^2}{t} + \pi M z \sum_{j=0}^{\infty} \frac{B_{2j}}{(2j)!} (2\pi i M z/t)^{2j} \right.$$

$$\left. - \frac{\pi i M z}{t} \sum_{r=0}^{\infty} \frac{B_{2r}}{(2r)!} (2\pi M z)^{2r} + \sum_{s=0}^{\infty} \sum_{j=0}^{s} \frac{B_{2j}}{(2j)!} \frac{B_{2s-2j}}{(2s-2j)!} \left(\frac{i}{t}\right)^{2j} (2\pi M z)^{2s} \right).$$

Hence,

$$R_0 = \tfrac{1}{4}(2\beta M)^{2n} \frac{\varphi^{(2n)}(0)}{(2n)!}$$

$$+ \tfrac{1}{2} i (2M)^{2n} \sum_{k=0}^{n} (-1)^{n+k} \frac{\varphi^{(2k+1)}(0)}{(2k+1)!} \frac{B_{2n-2k}}{(2n-2k)!} \beta^{4n-2k-1}$$

$$+ \frac{(2M)^{2n}}{2\pi} \beta^{2n+1} \sum_{k=0}^{n} \frac{\varphi^{(2k+1)}(0)}{(2k+1)!} \frac{B_{2n-2k}}{(2n-2k)!} \alpha^{2n-2k}$$

$$+ \frac{i(2M)^{2n}}{\pi} \sum_{k=0}^{n+1} \frac{\varphi^{(2k)}(0)}{(2k)!}$$

$$\times \sum_{j=0}^{n+1-k} (-1)^j \frac{B_{2j}}{(2j)!} \frac{B_{2n+2-2k-2j}}{(2n+2-2k-2j)!} \alpha^{2n+2-2k-2j} \beta^{2n+2j}.$$

Thus, by the residue theorem,

$$\frac{1}{2\pi i (2M)^{2n}} \int_C f_m(z)\, dz$$

$$= \frac{(-1)^n}{2^{2n} 2\pi i} \sum_{k=1}^{m} \frac{\varphi(2\beta k i) + \varphi(-2\beta k i)}{k^{2n+1}(e^{2k\beta^2} - 1)} + \frac{(-1)^n}{2^{2n} 2\pi i} \sum_{k=0}^{m} \frac{\varphi(2\beta k i)}{k^{2n+1}}$$

$$- \frac{1}{2\pi i} \left(\frac{\beta}{2\alpha}\right)^{2n} \sum_{k=1}^{m} \frac{\varphi(2\alpha k) + \varphi(-2\alpha k)}{k^{2n+1}(e^{2k\alpha^2} - 1)} - \frac{1}{2\pi i} \left(\frac{\beta}{2\alpha}\right)^{2n} \sum_{k=1}^{m} \frac{\varphi(2\alpha k)}{k^{2n+1}}$$

$$+ \tfrac{1}{4} \beta^{2n} \frac{\varphi^{(2n)}(0)}{(2n)!} + \tfrac{1}{2} i \sum_{k=0}^{n} (-1)^{n+k} \frac{\varphi^{(2k+1)}(0)}{(2k+1)!} \frac{B_{2n-2k}}{(2n-2k)!} \beta^{4n-2k-1}$$

$$+ \frac{1}{2\pi} \beta^{2n+1} \sum_{k=0}^{n} \frac{\varphi^{(2k+1)}(0)}{(2k+1)!} \frac{B_{2n-2k}}{(2n-2k)!} \alpha^{2n-2k}$$

$$+ \frac{i}{\pi} \sum_{k=0}^{n+1} \frac{\varphi^{(2k)}(0)}{(2k)!} \sum_{j=0}^{n+1-k} (-1)^j \frac{B_{2j}}{(2j)!} \frac{B_{2n+2-2k-2j}}{(2n+2-2k-2j)!} \alpha^{2n+2-2k-2j} \beta^{2n+2j}.$$

Letting M tend to ∞ and using the bounded convergence theorem, we conclude that the limit on the left side above equals 0. Multiplying both sides above by $2\pi i/\beta^{2n}$ and rearranging, we complete the proof.

To illustrate Entry 20, let $\varphi(z) \equiv 1$. Replace α and β by $\sqrt{\alpha}$ and $\sqrt{\beta}$, respectively. Let $\zeta(z)$ denote the Riemann zeta-function. It is easy to check that the hypotheses of Entry 20 are satisfied. Thus, for $\alpha\beta = \pi^2$ and any positive integer n,

$$\alpha^{-n}\left(\sum_{k=1}^{\infty} \frac{1}{k^{2n+1}(e^{2k\alpha}-1)} + \tfrac{1}{2}\zeta(2n+1)\right)$$
$$= (-\beta)^{-n}\left(\sum_{k=1}^{\infty} \frac{1}{k^{2n+1}(e^{2k\beta}-1)} + \tfrac{1}{2}\zeta(2n+1)\right)$$
$$- 2^{2n}\sum_{j=0}^{n+1} (-1)^j \frac{B_{2j}}{(2j)!}\frac{B_{2n+2-2j}}{(2n+2-2j)!}\alpha^{n+1-j}\beta^j.$$

This is Ramanujan's famous formula for $\zeta(2n+1)$, given in Entry 15 of Chapter 16 of the first notebook and Entry 21(i) of Chapter 14 of the second notebook (Part II [4, pp. 275–276]).

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