1 Задача 5.5

Вычислите неопределенный интеграл:

$$\int \frac{x^4+1}{x^6+1}\,dx$$

Ответ: $\int \frac{x^4+1}{x^6+1}\,dx = \frac{1}{3}(2\arctan x + \arctan\frac{x}{1-x^2}) + C$

$Q(x) = \frac{x^4+1}{x^6+1} \overset{(1)}{=} \frac{x^2}{x^3} \cdot \frac{x^2+\frac{1}{x^2}}{x^3+\frac{1}{x^3}} \overset{(2)}{=} \frac{1}{x} \cdot \frac{x^2+\frac{1}{x^2}}{(x+\frac{1}{x})(x^2+\frac{1}{x^2}-1)} \overset{(3)}{=} \frac{1}{x} \cdot \frac{(x^2+\frac{1}{x^2}-1)+1}{(x+\frac{1}{x})(x^2+\frac{1}{x^2}-1)} \overset{(4)}{=} \frac{1}{x} \cdot (\frac{1}{x+\frac{1}{x}} + \frac{1}{(x+\frac{1}{x})(x^2+\frac{1}{x^2}-1)}) \overset{(5)}{=} \frac{1}{x} \cdot (\frac{1}{x+\frac{1}{x}} + \frac{1}{(x+\frac{1}{x})((x+\frac{1}{x})^2-3)}) \overset{(6)}{=} \frac{1}{x} \cdot (\frac{1}{x+\frac{1}{x}} + \frac{1}{(x+\frac{1}{x})(x+\frac{1}{x}-\sqrt{3})(x+\frac{1}{x}+\sqrt{3})})$

$u = x + \frac{1}{x}$

$\frac{1}{u(u-\sqrt{3})(u+\sqrt{3})} \overset{(7)}{=} A \cdot \frac{1}{u} + B \cdot \frac{1}{u-\sqrt{3}} + C \cdot \frac{1}{u+\sqrt{3}} \overset{(8)}{=}$

$\overset{(8)}{=} \frac{1}{-\sqrt{3}\cdot\sqrt{3}} \cdot \frac{1}{u} + \frac{1}{2\sqrt{3}\cdot\sqrt{3}} \cdot \frac{1}{u-\sqrt{3}} + \frac{1}{(-2\sqrt{3})\cdot(-\sqrt{3})} \cdot \frac{1}{u+\sqrt{3}} \overset{(9)}{=} \frac{-1}{3} \cdot \frac{1}{u} + \frac{1}{6} \cdot \frac{1}{u-\sqrt{3}} + \frac{1}{6} \cdot \frac{1}{u+\sqrt{3}}$

$Q(x) \overset{(10)}{=} \frac{1}{x}(\frac{1}{x+\frac{1}{x}} + \frac{-1}{3} \cdot \frac{1}{x+\frac{1}{x}} + \frac{1}{6} \cdot \frac{1}{x+\frac{1}{x}-\sqrt{3}} + \frac{1}{6} \cdot \frac{1}{x+\frac{1}{x}+\sqrt{3}}) \overset{(11)}{=} \frac{1}{x}(\frac{2}{3} \cdot \frac{1}{x+\frac{1}{x}} + \frac{1}{6} \cdot \frac{1}{x+\frac{1}{x}-\sqrt{3}} + \frac{1}{6} \cdot \frac{1}{x+\frac{1}{x}+\sqrt{3}}) \overset{(12)}{=} \frac{2}{3} \cdot \frac{1}{x^2+1} + \frac{1}{6} \cdot \frac{1}{x^2-\sqrt{3}x+1} + \frac{1}{6} \cdot \frac{1}{x^2+\sqrt{3}x+1} \overset{(13)}{=} \frac{2}{3} \cdot \frac{1}{x^2+1} + \frac{1}{6} \cdot \frac{1}{(x-\frac{\sqrt{3}}{2})^2+(\frac{1}{2})^2} + \frac{1}{6} \cdot \frac{1}{(x+\frac{\sqrt{3}}{2})^2+(\frac{1}{2})^2}$

$\int \frac{x^4+1}{x^6+1}\,dx \overset{(14)}{=} \frac{2}{3}\int \frac{dx}{x^2+1} + \frac{1}{6}\int \frac{dx}{(x-\frac{\sqrt{3}}{2})^2+(\frac{1}{2})^2} + \frac{1}{6}\int \frac{dx}{(x+\frac{\sqrt{3}}{2})^2+(\frac{1}{2})^2} \overset{(15)}{=} \frac{2}{3}\int \frac{dx}{x^2+1} + \frac{1}{6}\int \frac{d(x-\frac{\sqrt{3}}{2})}{(x-\frac{\sqrt{3}}{2})^2+(\frac{1}{2})^2} + \frac{1}{6}\int \frac{d(x+\frac{\sqrt{3}}{2})}{(x+\frac{\sqrt{3}}{2})^2+(\frac{1}{2})^2} \overset{(16)}{=} \frac{2}{3}\arctan x + \frac{1}{6} \cdot \frac{1}{\frac{1}{2}}\arctan\frac{x-\frac{\sqrt{3}}{2}}{\frac{1}{2}} + \frac{1}{6} \cdot \frac{1}{\frac{1}{2}}\arctan\frac{x+\frac{\sqrt{3}}{2}}{\frac{1}{2}} + C \overset{(17)}{=} \frac{2}{3}\arctan x + \frac{1}{3}\arctan(2x+\sqrt{3}) + \frac{1}{3}\arctan(2x-\sqrt{3}) + C \overset{(18)}{=} \frac{1}{3}(2\arctan x + \arctan(2x+\sqrt{3}) + \arctan(2x-\sqrt{3})) + C$

$\tan(\arctan(2x+\sqrt{3}) + \arctan(2x-\sqrt{3})) \overset{(19)}{=} \frac{(2x+\sqrt{3})+(2x-\sqrt{3})}{1-(2x+\sqrt{3})(2x-\sqrt{3})} \overset{(20)}{=} \frac{4x}{1-(4x^2-3)} \overset{(21)}{=} \frac{4x}{4-4x^2} \overset{(22)}{=} \frac{x}{1-x^2}$

$\int \frac{x^4+1}{x^6+1}\,dx \overset{(23)}{=} \frac{1}{3}(2\arctan x + \arctan\frac{x}{1-x^2}) + C$