

# Revealing Transient Strain in Geodetic Data with the Radial Basis Function Finite Difference Method

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## 1 Introduction

Crustal strain rates are fundamentally important quantities for assessing seismic hazard, since knowing where and how quickly strain is accumulating gives insight into where we can expect stored elastic energy to be released seismically. It is then important to develop and improve upon methods for mapping strain in tectonically active regions because such maps could conceivably feed into seismic hazard models such as UCERF3 (Field et al., 2014).

Methods for estimating strain from geodetic data fall in one of two categories. There are model-based approaches which assume that strain is the result of loading on faults which have a known geometry, and there are data-based approaches which make no assumptions about the source of deformation. We will exclusively consider data-based approaches in this paper. The classic and simplest method for estimating strain is to assume that the strain rate is constant in time and spatially uniform within subnetworks of the geodetic data. Linear least squares is then used to find the components of the strain rate tensor for each subnetwork (e.g Frank, 1966; Prescott, 1976; Savage et al., 1986; Feigl et al., 1990; Murray and Lisowski, 2000). Several algorithms have been developed to improve upon this procedure. Shen et al. (1996) and Shen et al. (2015) discuss an algorithm where, instead of using the immediately adjacent stations to calculate strain at a position, the strain is computed with a weighted average over the entire network where the weighting is smaller for more distant stations. Another strategy is to fit a set of interpolating basis functions to the deformation field and then compute the strain from the analytical derivative of the interpolant (e.g. Beavan and Haines, 2001; Tape et al., 2009; Sandwell and Wessel, 2016).

The aforementioned studies have all been concerned with estimating long term strain rates. Time dependent strain would be useful for studying geophysical processes which occur over timescales of days to years such as slow slip events, postseismic relaxation, or volcanic deformation. Ohtani et al. (2010) identified transient strain events by fitting a set of spatial wavelet basis functions to the deformation field at discrete time epochs, and a Kalman filtering strategy was used to ensure that the coefficients for each basis function varied smoothly in time. Holt and Shcherbenko (2013) calculated time dependent strain by differentiating a bicubic interpolant that was fit to each epoch of a temporally smoothed deformation field.

Each of the methods described above are designed to overcome two complications that arise in estimating deformation gradients: (1) geodetic data are noisy and differentiation will only amplify the noise and (2) geodetic data are not observed on a regular grid, which prevents the use of standard finite difference methods for computing derivatives. In this paper we demonstrate that both of these complications can be elegantly handled with the Radial Basis Function-Finite Difference (RBF-FD) method.

The RBF-FD method was developed simultaneously and independently by Tolstykh and Shirobokov (2003), Shu et al. (2003), Cecil et al. (2004), and Wright and Fornberg (2006) as a computationally efficient way to solve large scale partial differential equations over irregular, multi-dimensional domains. The RBF-FD method can be thought of as a generalization of the traditional finite difference method, where the node layout is no longer restricted to regular grids. Indeed, the RBF-FD method can be used to estimate derivatives of discrete data located at arbitrary scattered positions in multi-dimensional space. The RBF-FD method is particularly appealing because it is algorithmically simple, regardless of the domain shape or node layout, and also because the method has performed well in numerous benchmark tests (?, and references therein).

In this paper, we do not use the RBF-FD method to solve a partial differential equation, but rather we use it to spatially smooth geodetic data and to compute deformation gradients. Our smoothing strategy can be viewed as a non-parametric, low-pass filter for scattered data where the degree of smoothness is controlled by a user specifies cutoff frequency. This can be contrasted with interpolation based methods where the resulting interpolant can be largely and perhaps unpredictably controlled by the choice of basis function. This process of smoothing and differentiating can be extended to estimate time dependent strain rates. In that case, we first temporally smooth and differentiate GPS displacement time series to get time dependent velocities. We

then spatially smooth and differentiate the resulting velocities for each time epoch to get time dependent strain rates.

The method proposed in this paper has numerous advantages which set it apart from other methods for computing strain rates. The method is computationally efficient and stable (there is no inversion of an ill-conditioned matrix). There are no hyper parameters or penalty parameters that need to be tuned for each application. As opposed to interpolation strategies such as Beavan and Haines (2001), Tape et al. (2009), or Ohtani et al. (2010), our method assumes that velocities are locally rather than globally continuous, which allows us to easily handle discontinuities resulting from, for example, a creeping fault.

We begin this paper by summarizing the RBF-FD scheme and explaining how we construct differentiation matrices for scattered data. We then introduce the RBF-FD filter, which is used to smooth the observed geodetic data prior to differentiation. We provide two real world demonstrations of our method for calculating strain rates. First we calculate the long term strain rates in Southern California from the CMM3 velocity data set (Shen et al., 2011), and we verify that our results are consistent with other studies. We then calculate time dependent strain rates in Cascadia from the GPS data provided by UNAVCO. In Cascadia, we analyze strain resulting from slow slip events and compare it to the long term tectonic strain accumulation. Slow slip events are found to produce compression in the Olympic Peninsula, which is in addition to the compression resulting from tectonic loading. Further south in Oregon, the slow slip events tend to release the compressional strain that is accumulated tectonically. While similar conclusions have been drawn from fault slip inversions for slow slip events, it is important to recognize that slip inversion are the product of inverting an ill-conditioned matrix making it difficult to determine whether slip inferences are real or just an artifact of the inversion. The strain rates presented in this paper are more direct observations and can be interpreted with a higher degree of confidence.

## 2 Method

In this section, we describe how GNSS displacement observations are used to identify transient crustal strain rates, which we denote as  $\dot{\epsilon}(z)$ , where  $z$  is the ordered pair  $(x, t)$ ,  $x$  are spatial coordinates in  $\mathbb{R}^2$ , and  $t$  is time. We consider  $\dot{\epsilon}$  to be spatially and temporally coherent deviations from the steady rate of strain accumulation from plate tectonics. We determine  $\dot{\epsilon}$  by first identifying transient displacements,  $\mathbf{u}(z)$ , which we then spatially and temporally differentiate. As we will show in Section X, estimates of  $\dot{\epsilon}$  turn out to be more effective at illuminating geophysical signal than estimates of  $\mathbf{u}$  or  $\dot{\mathbf{u}}$ . We make a prior assumption that each component of  $\mathbf{u}$  is a Gaussian process,

$$u_i(z) \sim \mathcal{N}(0, C_{u_i}), \quad (1)$$

where  $C_{u_i}(z, z')$  is a covariance function indicating how we expect  $u_i(z)$  to covary with  $u_i(z')$ . For simplicity, we treat each component of displacement independently and ignore any potential covariance. We then drop the component subscripts with the understanding that the same analysis is being repeated to estimate the east, north, and vertical components of  $\mathbf{u}$ . We further assume that  $C_u$  can be separated into positive definite spatial and temporal functions as

$$C_u((x, t), (x', t')) = X(x, x')T(t, t'), \quad (2)$$

The appropriate choice for  $X$  and  $T$  may vary depending on the geophysical signal we are trying to describe (e.g. postseismic deformation or deformation from slow slip events), and we discuss this matter in the next section.

We constrain  $u$  with GNSS data, which records  $u$  as well as other physical and non-physical processes which we are not interested in. We describe GNSS observations at position  $x_i$  and time  $t_j$  as a realization of the random variable

$$d_{ij} = u(x_i, t_j) + \epsilon(x_i, t_j) + w_{ij} + a_i^{(1)} + a_i^{(2)}t_j + a_i^{(3)}\sin(2\pi t_j) + a_i^{(4)}\cos(2\pi t_j) + a_i^{(5)}\sin(4\pi t_j) + a_i^{(6)}\cos(4\pi t_j), \quad (3)$$

where  $a_i^{(1)}$  is an offset that is unique for each GNSS monument,  $a_i^{(2)}$  is the steady rate of tectonic deformation at  $x_i$ , and the sinusoids describe seasonal deformation (using units of years for  $t$ ). We use  $w_{ij}$  to denote normally distributed, uncorrelated noise. Correlated noise which does not have a parametric representation is denoted by  $\epsilon$ . For example,  $\epsilon$  can consist of temporally correlated noise describing benchmark wobble (e.g., Wyatt, 1982, 1989), and/or spatially correlated noise describing common mode error (e.g., Wdowinski et al., 1997). For now, we will only assume that  $\epsilon \sim \mathcal{N}(0, C_\epsilon)$ . We consider the six coefficients in eq. (3) to be uncorrelated random variables distributed as  $\mathcal{N}(0, \kappa^2)$  in the limit as  $\kappa \rightarrow \infty$  (i.e., the coefficients have diffuse priors). Of course, the tectonic deformation,  $a_i^{(2)}$ , is spatially correlated and we could invoke a tectonic model to form a prior on  $a_i^{(2)}$ . However, in our application to Cascadia, we will be using displacement time series which are long enough to sufficiently constrain  $a_i^{(2)}$  for each station, and there is no need to incorporate a prior. Likewise, the seasonal

coefficients may be spatially correlated (Langbein, 2008), and it may be worth exploring and exploiting such a correlation in a future study.

We now consider the column vector of  $n$  GNSS observations,  $\mathbf{d}_*$ . Let  $\mathbf{y}$  be the set of  $(x_i, t_j)$  pairs describing where and when each of the GNSS observations have been made. Let  $\mathbf{a}$  be the vector of coefficients from eq. (3) for each of the  $m$  GNSS stations. We use  $\mathbf{P}$  to represent the  $n \times 6m$  matrix of corresponding basis functions evaluated at each point in  $\mathbf{y}$ . We also denote the vector of uncorrelated noise for each observation as  $\mathbf{w}$ , whose standard deviations are given by the formal data uncertainty  $\sigma$ . The observations  $\mathbf{d}_*$  can then be viewed as a realization of the random vector

$$\mathbf{d} = u(\mathbf{y}) + \epsilon(\mathbf{y}) + \mathbf{w} + \mathbf{P}\mathbf{a}, \quad (4)$$

which is distributed as  $\mathcal{N}(\mathbf{0}, \Sigma + \kappa^2 \mathbf{P}\mathbf{P}^T)$ , where

$$\Sigma = C_u(\mathbf{y}, \mathbf{y}) + C_\epsilon(\mathbf{y}, \mathbf{y}) + \text{diag}(\sigma^2). \quad (5)$$

It should be understood that the notation  $f(\mathbf{x})$  and  $f(\mathbf{x}, \mathbf{y})$  is shorthand for  $[f(x_i)]_{x_i \in \mathbf{x}}$  and  $[f(x_i, y_j)]_{(x_i, y_j) \in \mathbf{x} \times \mathbf{y}}$ , respectively.

We now describe how to condition the prior for each component of transient displacements,  $u$ , with  $\mathbf{d}_*$  to form a posterior estimate of transient displacements,  $\hat{u} = u|\mathbf{d}_*$ . We will assume that an appropriate covariance functions and corresponding hyperparameters for  $X$ ,  $T$ , and  $C_\epsilon$  have already been chosen. If  $\kappa$  is kept finite, then we can follow Rasmussen and Williams (2006) to find that  $\hat{u}$  is distributed as  $\mathcal{N}(\mu_{\hat{u}}, C_{\hat{u}})$ , where

$$\mu_{\hat{u}}(z) = C_u(z, \mathbf{y}) (\Sigma + \kappa^2 \mathbf{P}\mathbf{P}^T)^{-1} \mathbf{d}_* \quad (6)$$

and

$$C_{\hat{u}}(z, z') = C_u(z, z') - C_u(z, \mathbf{y}) (\Sigma + \kappa^2 \mathbf{P}\mathbf{P}^T)^{-1} C_u(\mathbf{y}, z'). \quad (7)$$

However, we are interested in the limit as  $\kappa \rightarrow \infty$ , and the form for eq. (6) and eq. (7) is not suitable for evaluating this limit. We can use the partitioned matrix inversion identity (e.g., Press et al., 2007) to rewrite eq. (6) and eq. (7) as

$$\mu_{\hat{u}}(z) = \begin{bmatrix} C_u(z, \mathbf{y}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Sigma & \mathbf{P} \\ \mathbf{P}^T & -\kappa^{-2} \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{d}_* \\ \mathbf{0} \end{bmatrix} \quad (8)$$

and

$$C_{\hat{u}}(z, z') = C_u(z, z') - \begin{bmatrix} C_u(z, \mathbf{y}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Sigma & \mathbf{P} \\ \mathbf{P}^T & -\kappa^{-2} \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} C_u(\mathbf{y}, z') \\ \mathbf{0} \end{bmatrix}. \quad (9)$$

Taking the limit as  $\kappa \rightarrow \infty$ , we get the solution for the mean and covariance of  $\hat{u}$ ,

$$\mu_{\hat{u}}(z) = \begin{bmatrix} C_u(z, \mathbf{y}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Sigma & \mathbf{P} \\ \mathbf{P}^T & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{d}_* \\ \mathbf{0} \end{bmatrix} \quad (10)$$

and

$$C_{\hat{u}}(z, z') = C_u(z, z') - \begin{bmatrix} C_u(z, \mathbf{y}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Sigma & \mathbf{P} \\ \mathbf{P}^T & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} C_u(\mathbf{y}, z') \\ \mathbf{0} \end{bmatrix}. \quad (11)$$

The posterior transients displacements,  $\hat{\mathbf{u}}$ , are found by evaluating eq. (10) and (11) for each displacement component. We differentiate the Gaussian process  $\hat{\mathbf{u}}$  (see Abrahamsen (1997)) to form an estimate of transient strain rate,  $\dot{\epsilon}$ , which is itself a spatially and temporally continuous Gaussian process. The components of the transient strain rate tensor,  $\dot{\epsilon}_{ij}$ , are given by

$$\dot{\epsilon}_{ij}(z) = \frac{1}{2} \frac{\partial}{\partial t} \left( \frac{\partial \hat{u}_i(z)}{\partial x_j} + \frac{\partial \hat{u}_j(z)}{\partial x_i} \right). \quad (12)$$

which are distributed as  $\mathcal{N}(\mu_{\dot{\epsilon}_{ij}}, C_{\dot{\epsilon}_{ij}})$ , where

$$\mu_{\dot{\epsilon}_{ij}}(z) = \frac{1}{2} \frac{\partial}{\partial t} \left( \frac{\partial \mu_{\hat{u}_i}(z)}{\partial x_j} + \frac{\partial \mu_{\hat{u}_j}(z)}{\partial x_i} \right) \quad (13)$$

and

$$C_{\dot{\epsilon}_{ij}}(z, z') = \frac{1}{4} \frac{\partial^2}{\partial t \partial t'} \left( \frac{\partial^2 C_{\hat{u}_i}(z, z')}{\partial x_j \partial x'_j} + \frac{\partial^2 C_{\hat{u}_j}(z, z')}{\partial x_i \partial x'_i} \right). \quad (14)$$

## 2.1 Outlier detection

In our formulation for estimating transient strain rates, we have assumed that noise in the data vector is normally distributed. This is not an appropriate assumption for GNSS displacement timeseries which are prone to more outliers than would be predicted for normally distributed noise. It follows that proposed methods for analyzing GNSS timeseries should be robust against outliers (e.g., Blewitt et al., 2016). In order to make our estimates of transient strain more robust, we automatically identify and remove outliers in the GNSS data as a pre-processing step.

Our method for detecting outliers is based on the data editing algorithm described in Gibbs (2011). We calculate the residuals between the observations and a best fitting model, and data with residuals that are anomalously large are identified as outliers. We treat  $\mathbf{d}_*$  as a sample of  $\mathbf{d}$  and assume that there is no temporally correlated noise (i.e.,  $\epsilon = 0$ ). The best fitting model for  $\mathbf{d}_*$  is considered to be the expected value of the random vector  $u(\mathbf{y}) + \mathbf{P}\mathbf{a}$  after conditioning it with non-outlier observations. We still consider  $u$  to have a separable covariance function as in eq. (2), and the choice for  $X$  and  $T$  does not need to be the same as that used in Section X. Since outliers are determined based on how well a spatially and temporally dependent model fits the data, we are able to identify anomalous observations which may not be immediately apparent based on inspection of individual timeseries.

To begin the algorithm, we let  $\Omega$  be the index set of non-outliers in  $\mathbf{d}_*$  and initiate it with all  $n$  indices. This algorithm is iterative, and for each iteration we calculate the residual vector

$$\mathbf{r} = \frac{\mathbf{d}_* - \mathbb{E}[(u(\mathbf{y}) + \mathbf{P}\mathbf{a})|\tilde{\mathbf{d}}_*]}{\boldsymbol{\sigma}} \quad (15)$$

$$= \frac{1}{\boldsymbol{\sigma}} \left( \mathbf{d}_* - \begin{bmatrix} C_u(\mathbf{y}, \tilde{\mathbf{y}}) & \mathbf{P} \end{bmatrix} \begin{bmatrix} C_u(\tilde{\mathbf{y}}, \tilde{\mathbf{y}}) + \text{diag}(\tilde{\boldsymbol{\sigma}}^2) & \tilde{\mathbf{P}} \\ \tilde{\mathbf{P}}^T & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\mathbf{d}}_* \\ \mathbf{0} \end{bmatrix} \right), \quad (16)$$

where the tilde indicates that only elements corresponding to indices in  $\Omega$  are retained (e.g.,  $\tilde{\mathbf{y}} = \{y_i\}_{i \in \Omega}$ ). We then update  $\Omega$  to be

$$\Omega = \{i : |r_i| < \eta \tilde{r}_{rms}\}, \quad r_i \in \mathbf{r} \quad (17)$$

where  $\tilde{r}_{rms}$  is the root-mean-square of  $\tilde{\mathbf{r}}$  and  $\eta$  is an outlier tolerance. We use  $\eta = 4$  in this study, which seems to accurately identify outliers without unnecessarily decimating the data. Iterations continue until the new  $\Omega$  is equal to the previous  $\Omega$ .

It should be noted that this algorithm does not identify jumps in GNSS time series, which are another common defect. Some, but not all, jumps can be automatically removed by parsing station logs and identifying dates of antenna changes. However, it is still necessary to manually inspect and remove jumps of unknown origin. That being said, this outlier detection algorithm significantly reduces the effort needed to manually clean GNSS data.

## 3 Application Cascadia Slow Slip Events

In this section we estimate transient strain from GNSS data in Cascadia. We are mainly interested in identifying transient strain resulting from slow slip events (SSEs) (e.g., Dragert et al., 2001). We show that Gaussian processes are an effective tool for (1) detecting SSEs and (2) understanding how the strain from SSEs are contributing to the strain accumulated over the seismic cycle. Our motivation for detecting SSEs is, in part, because there have been several instances where SSEs immediately preceded megathrust earthquakes (Roeloffs, 2006). In Cascadia, SSEs can be detected by monitoring for associated seismic tremor, which is actively being done by the Pacific Northwest Seismic Network (PNSN) (Wech, 2010). Here we demonstrate that strain rates derived from GNSS data can also be an effective tool for detecting SSEs. Furthermore, estimated strain rates provide a clear picture of how SSEs transfer elastic strain energy through the crust, which is also important information for assessing seismic hazard. We envision that GNSS derived transient strain rates could be useful for monitoring subduction zones which have SSEs but no associated tremor (Schwartz and Rokosky, 2007).

In this study, we use GNSS data which has been made publicly available by the University Navstar Consortium (UNAVCO) (Herring et al., 2016) and can be found at [www.unavco.org](http://www.unavco.org). We limit the dataset to the stations and time ranges which are pertinent to seven of the most recent SSEs in the Puget Sound region; the earliest SSE began in the August 2010, and the most recent SSE began in the February of 2017. We consider these most recent SSEs because the station coverage is sufficiently dense for us to make well constrained inferences of optimal prior models. The distribution of stations considered in this station is shown in Figure X.

### 3.1 Noise model

In this section we discuss our choice for the noise covariance function  $C_\epsilon$  which will be used when estimating transient strain. There have been numerous studies on temporally correlated noise in GNSS data (e.g., Zhang et al., 1997; Mao et al., 1999; Williams et al., 2004; Langbein, 2008). In these studies, temporally correlated noise was described with some combination of Brownian motion, first-order Gauss-Markov (FOGM) processes, and/or flicker noise. There is some physical justification for using Brownian motion as a noise model because it accurately describes the power spectrum of motion resulting from instability in geodetic monuments (e.g., Wyatt, 1982, 1989; Langbein and Johnson, 1997). Here we describe the time dependence of  $\epsilon$  as a FOGM process and consider  $\epsilon$  to be spatially uncorrelated. A FOGM process is a solution to the stochastic differential equation

$$\dot{\epsilon}(t) + \alpha\epsilon(t) = \beta w(t), \quad (18)$$

where  $w(t)$  is white noise with unit variance. The noise model is then a Gaussian process with zero mean and covariance function

$$C_\epsilon((x, t), (x', t')) = \frac{\beta^2}{2\alpha} \exp(-\alpha|t - t'|) \delta(\|x - x'\|_2). \quad (19)$$

Under the condition that  $\alpha = 0$  and  $v(0) = 0$ ,  $\epsilon$  degenerates to the more commonly used Brownian motion noise model.

We constrain the hyperparameters for  $\epsilon$ ,  $\alpha$  and  $\beta$ , with a set of 38 inland stations, which are shown in Figure X. These stations are sufficiently far from the subduction zone that they are unlikely to contain transient signal. We clean the data for these stations by removing jumps at times of equipment changes, and we remove outliers that have been detected with the algorithm described in Section 2.1. We then find  $\alpha$  and  $\beta$  for each station timeseries with the Restricted Maximum Likelihood (REML) method (e.g., Harville, 1974; Cressie, 1993) and Hines 2017. The REML method finds the hyperparameters which maximize the expression

$$\left( \frac{|P^T P|}{(2\pi)^{n-6m} |\Sigma| |P^T \Sigma P|} \right)^{\frac{1}{2}} e^{-\frac{1}{2} d_*^T K d_*}, \quad (20)$$

where

$$K = \Sigma^{-1} - \Sigma^{-1} P (P^T \Sigma^{-1} P)^{-1} P^T \Sigma^{-1}. \quad (21)$$

For this section,  $d_*$  is used to denote the displacement time series for just one station, and thus  $m = 1$ . We are also assuming  $C_u$  is zero when estimating the noise hyperparameters for this section. Harville (1974) showed that choosing the hyperparameters which maximize eq. (20) is equivalent to choosing the hyperparameters which maximize the probability of drawing  $d_*$  from  $d$ . We use the REML method over the maximum likelihood (ML) method (e.g., Langbein and Johnson, 1997) because the REML method properly accounts for the improper prior that we assigned to  $\mathbf{a}$ . The distribution of inferred  $\alpha$  and  $\beta$  are shown in Figure 1. The amplitude of FOGM noise for the east and north components is notable low and are clustered around  $0.5 \text{ mm/yr}^{0.5}$ . The corresponding estimates of  $\alpha$  tend to cluster around  $0 \text{ yr}^{-1}$ , suggesting that Brownian motion would have also been an appropriate noise model. The amplitude of FOGM noise for the vertical component is significantly larger with a median value of  $13.8 \text{ mm/yr}^{0.5}$ . The inferred values for alpha are higher for the vertical component with a median value of  $1.87 \text{ yr}^{-1}$ . In Figure X, we use the median values of  $\alpha$  and  $\beta$  to generate two random samples of FOGM noise. The samples span seven years and over these seven years the east and north samples drift by about 1 mm. In the context of detecting SSEs, which produce several mm's of surface displacement on the timescale of weeks, the estimated FOGM noise for the east and north component is negligible. In contrast, the estimated FOGM noise for the vertical component is significantly larger than the signal we would expect from SSEs. We suspect that the higher amplitude for the FOGM model in the vertical component is accommodating for deficiencies in our rather simple seasonal model. Based on this analysis, we henceforth ignore temporally correlated noise in the east and north component because of its low amplitude, and we do not estimate  $u$  for vertical deformation because of the low signal to noise ratio.

Another significant source of noise in GNSS data is common mode error (e.g., Wdowinski et al., 1997; Dong et al., 2006), which is noise that is highly spatially correlated. When not accounted for, common mode error manifests itself as spatially uniform undulations in  $\hat{u}$ . However, we are primarily interested in estimating strain which is insensitive to common mode error. We therefore do not include common mode error in our noise model.

For these reasons, we henceforth make the simplifying assumption that temporally and spatially correlated noise is negligible and  $\epsilon = 0$  for the easting and northing component of GNSS data.

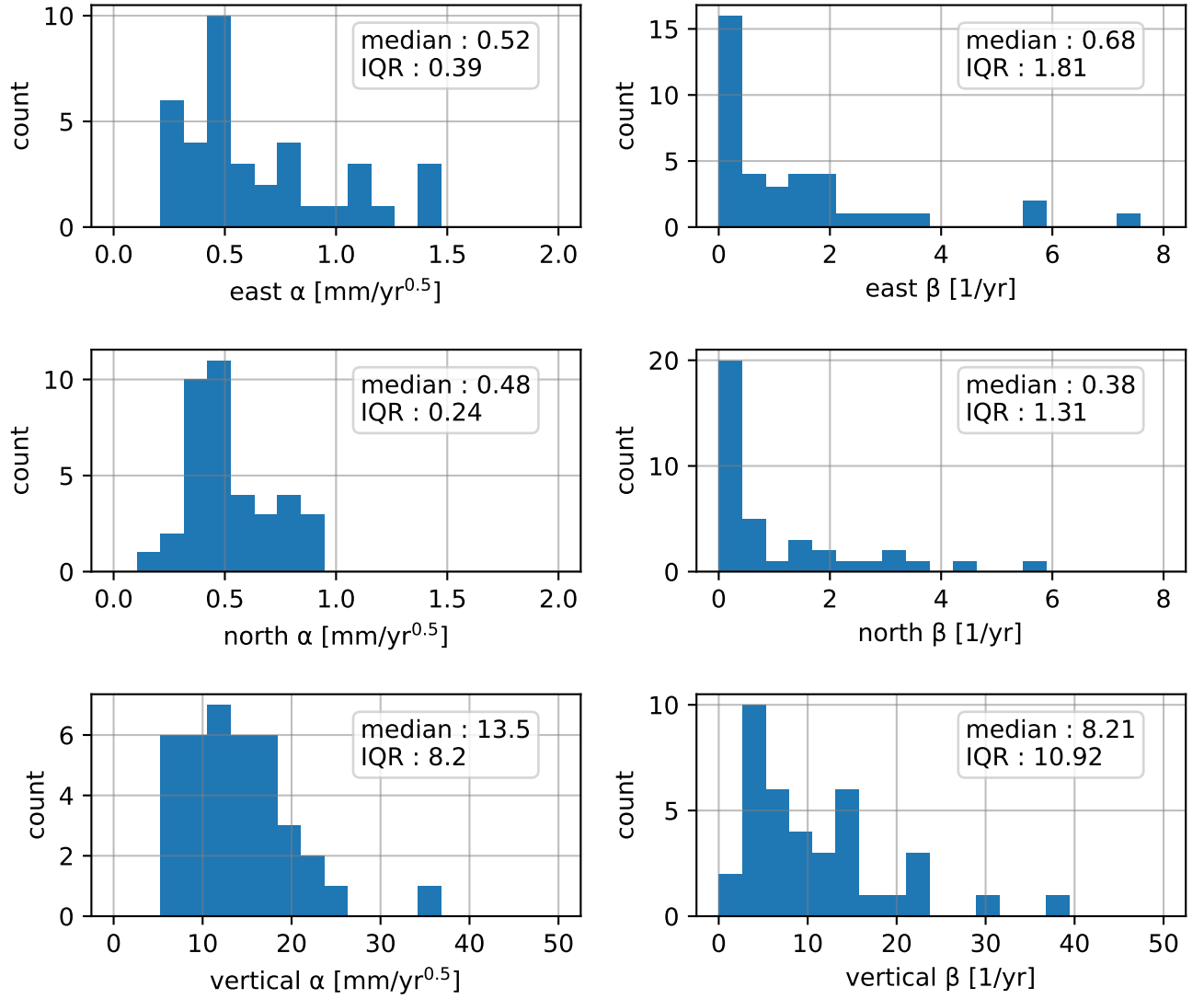


Figure 1: Distribution of estimated FOGM hyperparameters from eq. (19). Hyperparameters are independently estimated for each of the stations shown in Figure X and for each displacement component. “IQR” is the inter-quartile range.

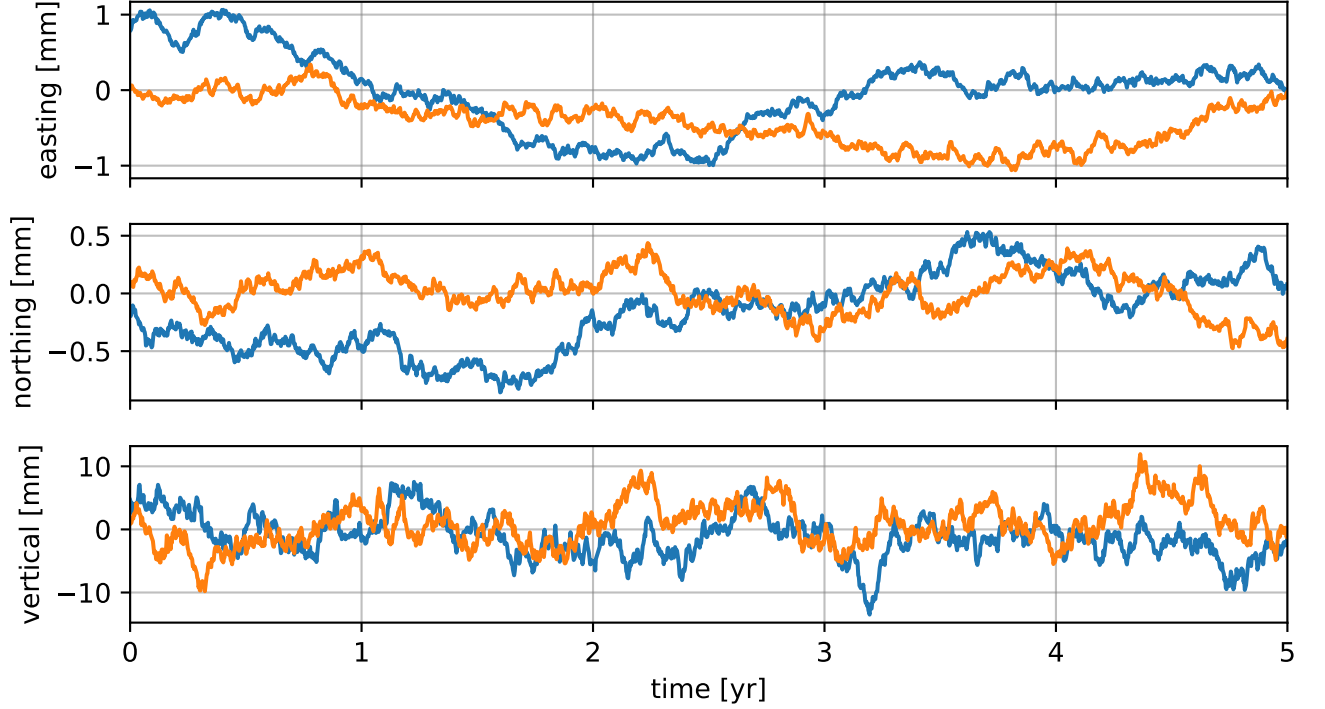


Figure 2: Two FOGM noise samples, where the hyperparameters have been set to the median values from Figure 1.

### 3.2 Transient displacement model

We next discuss our prior model for transient displacements, and, specifically, our choice for the covariance functions  $X(x, x')$  and  $T(t, t')$ . For  $X$ , we use the squared exponential (SE) covariance function,

$$X(x, x') = \exp\left(\frac{-\|x - x'\|_2^2}{2\lambda^2}\right). \quad (22)$$

The SE covariance function is commonly used in Kriging (e.g., Cressie, 1993) and Gaussian process regression (e.g., Rasmussen and Williams, 2006). The SE is a valid (i.e. positive definite) covariance function for any number of spatial dimensions, and it describes an isotropic Gaussian processes with realizations that are infinitely differentiable. In terms of geodetic applications, Kato et al. (1998) and El-Fiky and Kato (1999) demonstrated that the SE is an appropriate covariance model for describing long-term tectonic strain rates in Japan.

We consider three potential covariance functions to describe the temporal covariance of  $u$ . First, we consider the one-dimensional SE covariance function,

$$T(t, t') = \phi^2 \exp\left(\frac{-|t - t'|^2}{\theta^2}\right), \quad (23)$$

simply because the SE covariance function is versatile and commonly used for non-parametric modeling. Note that  $T$  includes the hyperparameter  $\phi$ , which serves to scale the covariance function  $C_u$ . Second, we consider integrated Brownian motion (IBM). IBM has zero mean and its covariance function can be found by integrating the covariance function for Brownian motion as

$$T(t, t') = \int_0^t \int_0^{t'} \phi^2 \min(\tau, \tau') d\tau' d\tau \quad (24)$$

$$= \frac{\phi^2}{2} \min(t, t')^2 \left( \max(t, t') - \frac{1}{3} \min(t, t') \right), \quad t, t' \geq 0. \quad (25)$$

IBM has been used, in the context of Kalman filtering, as a non-parametric model for the time dependence of geophysical signals (e.g., Segall and Mathews, 1997; McGuire and Segall, 2003; Ohtani et al., 2010; Hines and Hetland, 2016). It should be emphasized  $t = 0$  is a reference time at which the Gaussian process is exactly zero. For some geophysical signals, it is appropriate to have this reference time. For example, if we are trying to identify postseismic deformation then  $t$  should be zero at the time of the earthquake. However, if

$T$	direction	$\lambda$	$\phi$	$\theta$	diff. log(REML)
SE	east	$92 \pm 25$ km	$0.62 \pm 0.11$ mm	$0.026 \pm 0.011$ yr	-
SE	north	$91 \pm 53$ km	$0.43 \pm 0.05$ mm	$0.030 \pm 0.017$ yr	-
Wendland	east	$95 \pm 30$ km	$0.66 \pm 0.15$ mm	$0.093 \pm 0.044$ yr	$0.78 \pm 0.87$
Wendland	north	$92 \pm 57$ km	$0.46 \pm 0.10$ mm	$0.116 \pm 0.057$ yr	$0.08 \pm 0.58$
IBM	east	$110 \pm 130$ km	$290 \pm 420$ mm/yr <sup>1.5</sup>	-	$-16.4 \pm 7.8$
IBM	north	$150 \pm 560$ km	$110 \pm 250$ mm/yr <sup>1.5</sup>	-	$-10.1 \pm 2.3$

Table 1: Hyperparameters. The shown values are the medians and interquartile ranges.

we are interesting in detecting transient events, where there is no known start time, then IBM may not be an appropriate prior and an isotropic Gaussian process should be preferred. In the following analysis, we make the quite arbitrary choice that  $t$  is zero on the first epoch of  $\mathbf{d}_*$ . Our third option for  $T$  is the Wendland class of covariance functions (Wendland, 2005). Wendland covariance functions have compact support and hence their corresponding covariance matrices will be sparse. In our analysis, we exploit this sparsity with the CHOLMOD software package (Chen et al., 2008). Wendland functions are positive definite in  $\mathbb{R}^d$ , and they describes an isotropic Gaussian process with realizations that can be differentiated  $k$  times. The form of the covariance function depends on the choice of  $d$  and  $k$ . To describe the temporal covariance of  $u$ , we only require that  $d = 1$  and  $k = 1$ . The corresponding Wendland covariance function is

$$T(t, t') = \phi^2 \left( 1 - \frac{|t - t'|}{\theta} \right)_+^3 \left( \frac{3|t - t'|}{\theta} + 1 \right), \quad (26)$$

where

$$(t)_+ = \begin{cases} t, & t > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (27)$$

We next want to determine appropriate hyperparameters for  $X$  and each of the three candidate functions for  $T$ . First, we again clean the GNSS datasets by removing offsets at times of equipment changes and removing outliers with the method describe in Section 2.1. For the outlier detection algorithm, our prior model,  $u$ , is chosen to have a length-scale and time-scale which is able to approximately describe SSE displacements. We use the SE covariance function for  $X$  with  $\lambda = 100$  km, and we use the Wendland covariance function for  $T$ , due to its computational efficiency, with  $\theta = 0.1$  yr and  $\phi = 1$  mm. The outlier detection algorithm is particularly effective at removing outliers for stations at high elevation, which can be adversely affected by ice or snow during the winter (Lisowski et al., 2008). As an example, Figure 3 shows the outliers which were automatically detected for station SC03, which is located on Mount Olympus in Washington. After cleaning the data, we divide it into subsets which are four months long and centered on the time of an SSE. The times of seven SSEs are determined using tremor records from Wech (2010). We find the optimal hyperparameters for  $T$  and  $X$  for each subset of data. The REML method is once again used to determine the hyperparameters for  $T$  and  $X$  which maximize the probability of drawing  $\mathbf{d}_*$  from  $\mathbf{d}$ , where  $\mathbf{d}_*$  now refers to a data subset. We chose to make each data subsets four months long because it is long enough to encompass the SSE, and it is short enough to still be computationally tractable. However, four months is too short to resolve the sinusoids in  $\mathbf{d}$ , and they are omitted from  $\mathbf{d}$  for the REML analysis. The estimated hyperparameters for  $u$  are summarized in Table 1. When using SE or the Wendland function for  $T$ , the optimal hyperparameters appear have a unimodal distribution. When using IBM for  $T$  there were several anomalously large inferences, hence the larger IQR.

Next we identify which covariance function for  $T$  best describes the SSEs. One approach is to compare the REML likelihood for each covariance function, similar to the analysis in Langbein (2004). In Table 1, we summarize how the log REML likelihoods for the Wendland and IBM covariance functions compare to the SE covariance function. The tabulated values show the median and IQR differences in log REML likelihoods over each of the seven SSEs considered in this study. Negative values indicate that the observed data is less likely to have been sampled from the covariance function. Based on the differences in log REML likelihoods, the data is substantially more likely to come form the SE or Wendland covariance functions than from the IBM covariance function. The REML likelihoods do not definitively indicate whether the SE or Wendland covariance function is preferable.

As another test to determine the most appropriate covariance function for  $T$ , we compare the observations to the predicted displacements for each covariance function. let  $\hat{\mathbf{d}} = (u(\mathbf{y}) + \mathbf{P}\mathbf{a})|\mathbf{d}_*$  be the data prediction vector. It can be shown that  $\hat{\mathbf{d}}$  is normally distributed with mean

$$\boldsymbol{\mu}_{\hat{\mathbf{d}}} = \begin{bmatrix} C_u(\mathbf{y}, \mathbf{y}) & \mathbf{P} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma} & \mathbf{P} \\ \mathbf{P}^T & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{d}_* \\ \mathbf{0} \end{bmatrix} \quad (28)$$



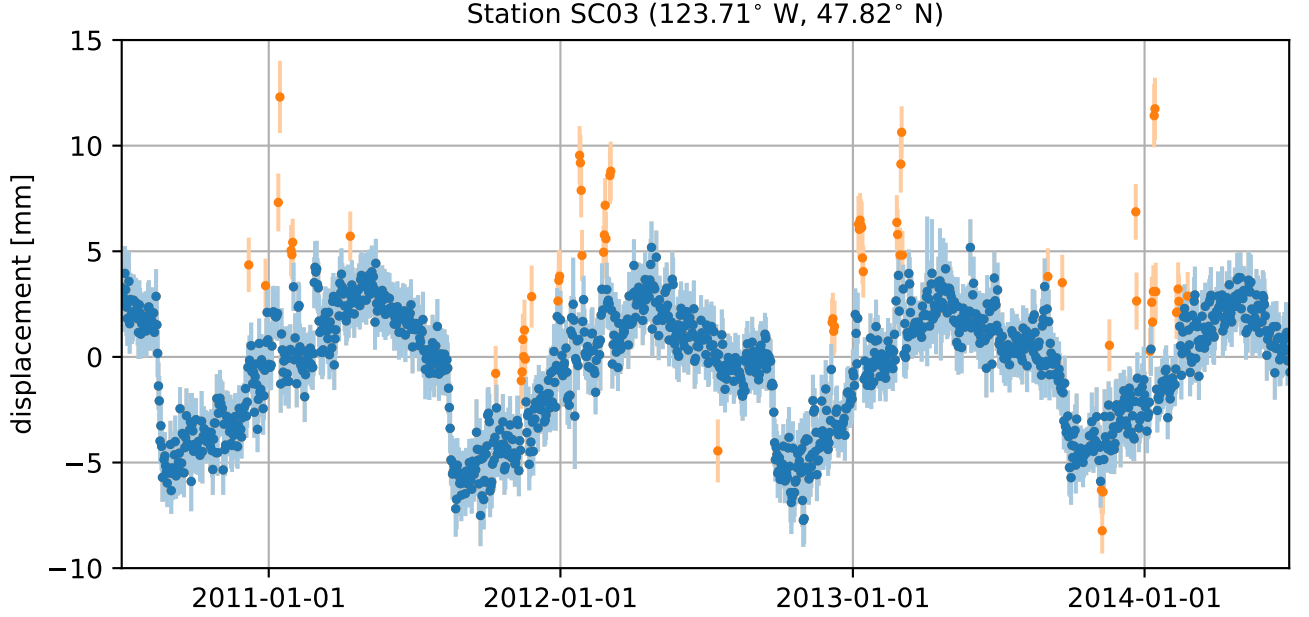


Figure 3: Detrended easting component of GNSS displacement data for station SC03, which is located on Mt. Olympus in Washington. The orange markers indicate outliers which were automatically detected using the algorithm from Section 2. The error bars show one standard deviation uncertainty.

and covariance

$$C_{\hat{d}} = C_u(\mathbf{y}, \mathbf{y}) - \begin{bmatrix} C_u(\mathbf{y}, \mathbf{y}) & \mathbf{P} \end{bmatrix} \begin{bmatrix} \Sigma & \mathbf{P} \\ \mathbf{P}^T & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} C_u(\mathbf{y}, \mathbf{y}) \\ \mathbf{P}^T \end{bmatrix}. \quad (29)$$

We compute  $\hat{\mathbf{d}}$  using SE, Wendland, and IBM covariance functions for  $T$  and the median hyperparameters from Table 1. Figure 1 compares the easting component of  $\mathbf{d}_*$  to  $\hat{\mathbf{d}}$  for the Winter 2016 SSE. For most stations, the data prediction vector appears to appropriately describe displacements throughout the SSE, regardless of the choice of  $T$ . The prediction for the IBM model contains slightly more high frequency, and perhaps spurious, features. The predictions for the Wendland and SE covariance functions are nearly indistinguishable. We ultimately settle on the Wendland covariance function for  $T$ . We choose the Wendland covariance function over the SE covariance function because of its computational advantages noted above.

### 3.3 Transient Strain Rates

Having established noise model and a prior for transient displacements, we use cleaned GNSS dataset described in Section 3.2 to calculate transient strain rates in the Puget Sound region from eq. (12). Our solution is shown in Figures X through X. Figures X through X show the estimates  $\dot{\epsilon}$  complete with uncertainties at select times over the course of the Winter 2016 SSE. Figure X, which shows strain rate at the height of the SSE, makes the implications of SSEs for seismic hazard clear. The SSE causes compression in the Olympic Peninsula and extension in East of Puget Sound. The SSE are thus concentrating strain energy trench-ward, and pushing the subduction zone closer to failure. This is not novel insight because the same conclusions has been drawn from fault slip models which show that SSEs tend to be just at the transition between locked and decoupled. Slow slip event depths from Dragert et al. (2001), Wech et al. (2009), Schmidt and Gao (2010), Bartlow et al. (2011) show slip concentrated at depths from 30 to 50 km detph. What is novel is that our estimates of strain rate from SSEs are nonparametric and less likely to be biased by systematic errors such as errors in the fault model. Are uncertainties are also likely to be more conservative since we do not make any dubious assumptions. Thus we can be more confident in our assessment of how much strain is accumulating from SSEs.

To demonstrate the effectiveness of our method for detecting SSEs, we show a timeseries of estimated strain components at the position indicated in figure X. We also compare the strain rate magnitude and the frequency of tremors in this region. The strain rate magnitudes have been normalized by its estimated uncertainty and so strain magnitudes less than  $\sim 2$  can be considered to be part of the background noise. The only time that the normalized strain rate magnitude exceeds 2 is when there is a corresponding peak in tremor. We are able to accurately detect the seven SSEs in over this period in addition to the inter-SSE events such as the one at X.

A video can be found as supp material

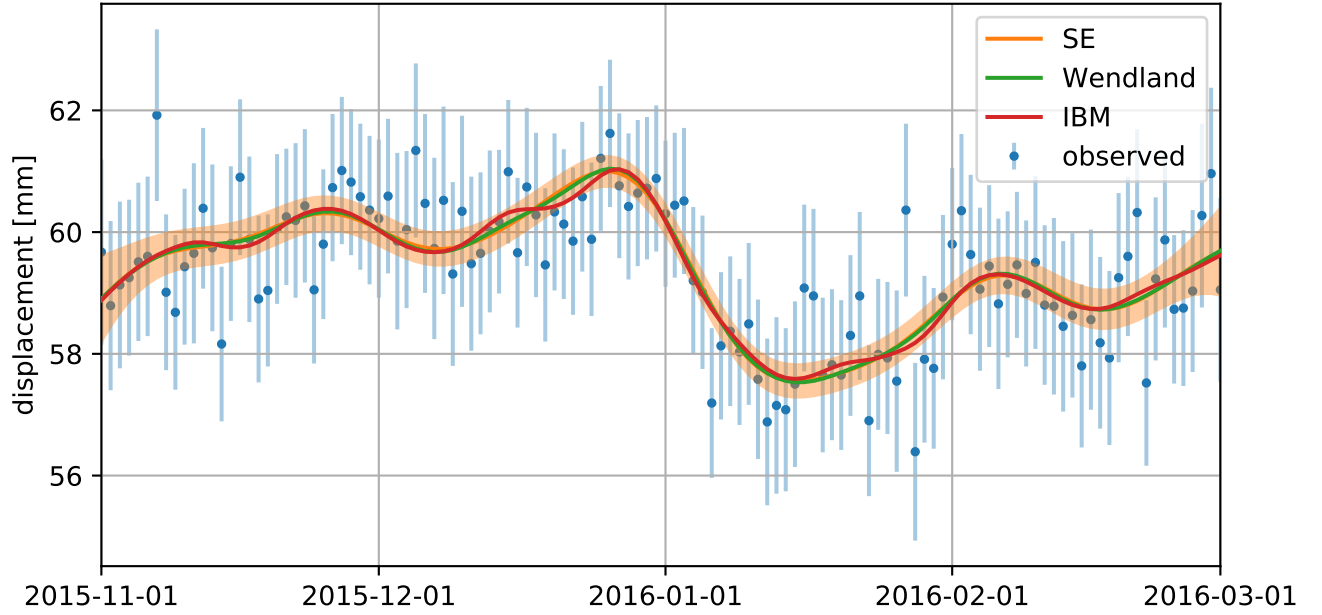


Figure 4: Fit.

made in , is accumulating to the trench perpendicular compression releases At the height of the SSE, the implications of

model exhibits the same relationship which can be described

Both of which have clear implications for assessing seismic hazard.

, which can have applications for seismic hazard assessment, Identifying strain resulting from ETS events serves two purposes: (1) ETS events have events which were first discovered by Dragert et al. (2001).

We demonstrate that Gaussian processes are effective tools for detecting transient strain by using them to detect Cascadia slow slip events.

#### BACKGROUND

Dragert et al. (2001) first discovered slow slip events

Explain strain concentration with this:

Interseismic locking depths from Flück et al. (1997), Murray and Lisowski (2000), McCaffrey et al. (2007) and McCaffrey et al. (2013), Burgette et al. (2009), Schmalzle et al. (2014) are consistent with a full locking down to about 20 km.

Studies which simultaneously modeled interseismic and ets are Holtkamp and Brudzinski (2010) and Schmalzle et al. (2014)

## 4 Discussion and Conclusion

The material presented in this paper is primarily focused on GPS data; however, we speculate that the RBF-FD scheme can be of particular use in denoising borehole strain meter (BSM) data. The Plate Boundary Observatory has deployed X BSMs along the Western United States. BSM data contains low frequency drift resulting from relaxation of the borehole (Gladwin et al., 1987), which can obscure the geophysical signal of interest. We suggest that the RBF-FD scheme may be useful in denoising BSM data. Since the RBF-FD scheme provides a straight-forward mapping from GPS displacements to strain at any target locations, it is possible to use GPS derived strains as *a priori* information for strain at BSM sites. GPS derived strains could then aid in discerning tectonic signal from noise in BSM data.

Additional uses for strain: transient detection, prior for strain meters,

Additional potential applications of the RBF-FD method: regularizing inverse problems,

## References

Abrahamsen, P. (1997). A review of Gaussian random fields and correlation functions. Technical report.

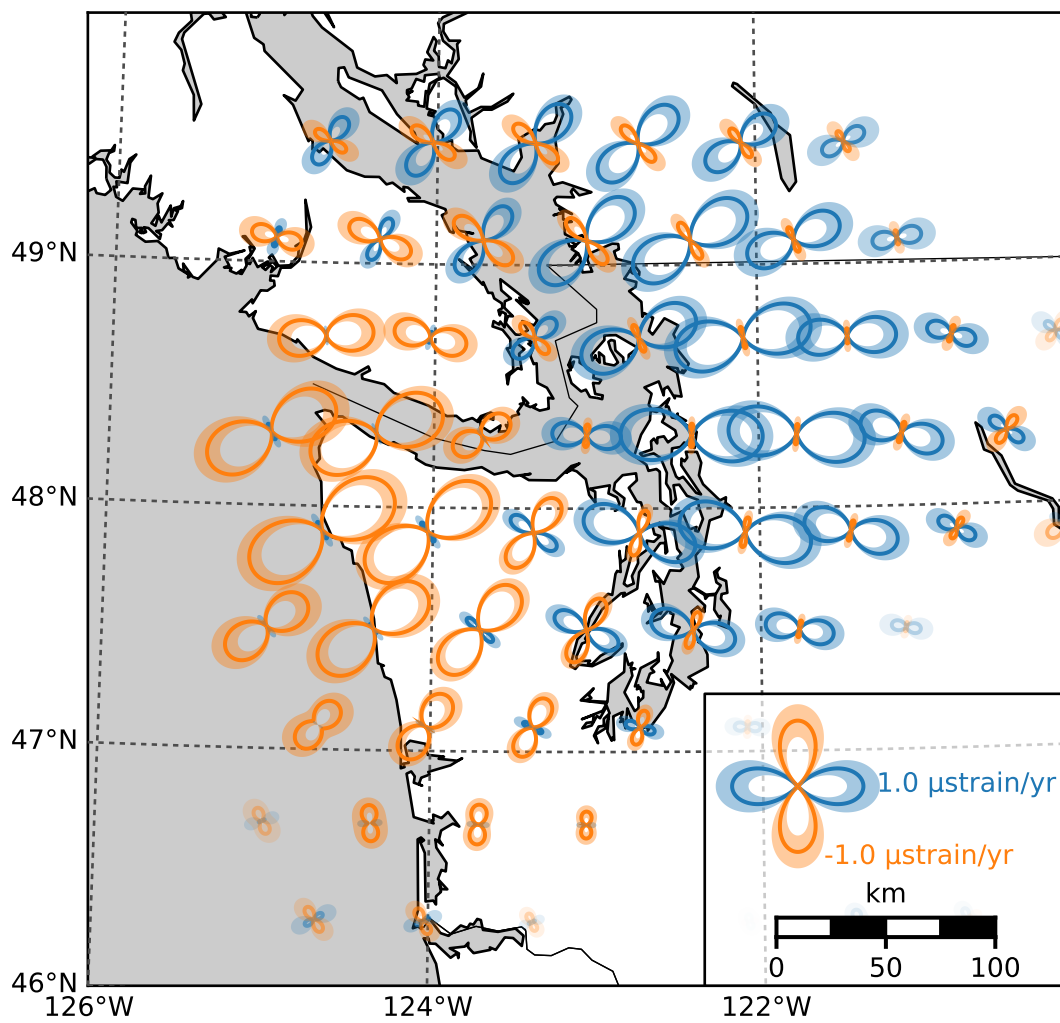


Figure 5: Strain map.

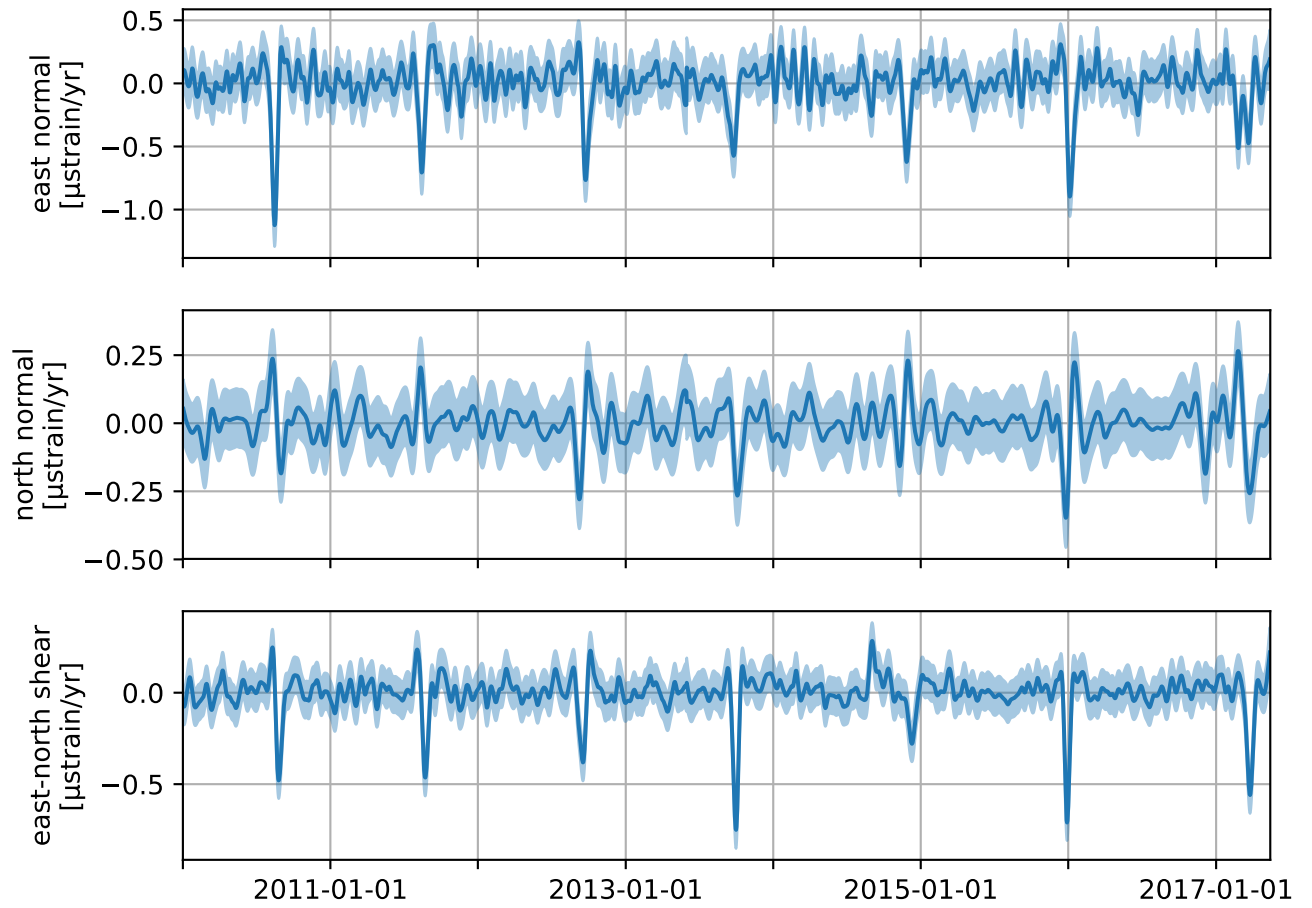


Figure 6: Strain time series.

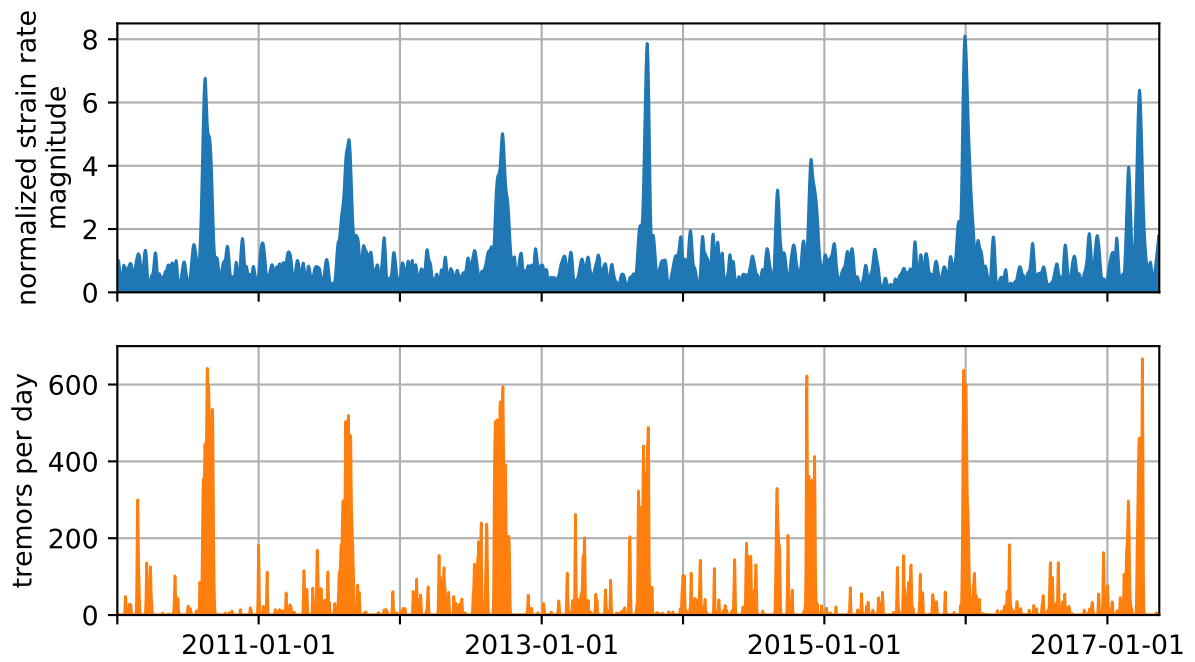


Figure 7: Strain magnitude time series.

- Bartlow, N. M., Miyazaki, S., Bradley, A. M., and Segall, P. (2011). Spacetime correlation of slip and tremor during the 2009 Cascadia slow slip event. *38(L18309)*:1–6.
- Beavan, J. and Haines, J. (2001). Contemporary horizontal velocity and strain rate fields of the Pacific-Australian plate boundary zone through New Zealand. *Journal of Geophysical Research*, 106(B1):741–770.
- Blewitt, G., Kreemer, C., Hammond, W. C., and Gazeaux, J. (2016). MIDAS robust trend estimator for accurate GPS station velocities without step detection. *Journal of Geophysical Research : Solid Earth*, 121:2054–2068.
- Burgette, R. J., Weldon, R. J., and Schmidt, D. A. (2009). Interseismic uplift rates for western Oregon and along-strike variation in locking on the Cascadia subduction zone. *Journal of Geophysical Research: Solid Earth*, 114(1):1–24.
- Cecil, T., Qian, J., and Osher, S. (2004). Numerical methods for high dimensional Hamilton-Jacobi equations using radial basis functions. *Journal of Computational Physics*, 196(1):327–347.
- Chen, Y., Davis, T. a., and Hager, W. W. (2008). Algorithm 887 : CHOLMOD , Supernodal Sparse Cholesky Factorization and Update/downdate. *ACM Transactions on Mathematical Software*, 35(3):1–12.
- Cressie, N. (1993). *Statistics for Spatial Data*. John Wiley & Sons, New York, rev. edition.
- Dong, D., Fang, P., Bock, Y., Webb, F., Prawirodirdjo, L., Kedar, S., and Jamason, P. (2006). Spatiotemporal filtering using principal component analysis and Karhunen-Loeve expansion approaches for regional GPS network analysis. *Journal of Geophysical Research: Solid Earth*, 111(3):1–16.
- Dragert, G., Wang, K., and James, T. S. (2001). A silent slip event on the deeper Cascadia subduction interface. *Science*, 292:1525–1528.
- El-Fiky, G. S. and Kato, T. (1999). Continuous distribution of the horizontal strain in the Tohoku district, Japan, predicted by least-squares collocation. *Journal of Geodynamics*, 27(2):213–236.
- Feigl, K. L., King, R. W., and Jordan, T. H. (1990). Geodetic measurement of tectonic deformation in the Santa Maria Fold and Thrust Belt, California. *Journal of Geophysical Research: Solid Earth*, 95(B3):2679–2699.
- Field, E. H., Arrowsmith, R. J., Biasi, G. P., Bird, P., Dawson, T. E., Felzer, K. R., Jackson, D. D., Johnson, K. M., Jordan, T. H., Madden, C., Michael, A. J., Milner, K. R., Page, M. T., Parsons, T., Powers, P. M., Shaw, B. E., Thatcher, W. R., Weldon, R. J., and Zeng, Y. (2014). Uniform California Earthquake Rupture Forecast, version 3 (UCERF3) -The time-independent model. *Bulletin of the Seismological Society of America*, 104(3):1122–1180.
- Flück, P., Hyndman, R. D., and Wang, K. (1997). 3-D dislocation model for great earthquakes of the Cascadia subduction zone. *Journal of Geophysical Research*, 102(B9):20539–20550.
- Frank, C. F. (1966). Deduction of earth strains from survey data. *Bulletin of the Seismological Society of America*, 56(1):35–42.
- Gibbs, B. P. (2011). *Advanced Kalman Filtering, Least-Squares and Modeling: A Practical Handbook*. John Wiley & Sons, Hoboken, NJ.
- Gladwin, M. T., Gwyther, R. L., Hart, R., Francis, M., and Johnston, M. J. S. (1987). Borehole tensor strain measurements in California. *Journal of Geophysical Research: Solid Earth*, 92(B8):7981–7988.
- Harville, D. A. (1974). Bayesian Inference for Variance Components Using Only Error Contrasts. *Biometrika*, 61(2):383–385.
- Herring, T. A., Melbourne, T. I., Murray, M. H., Floyd, M. A., Szeliga, W. M., King, R. W., Phillips, D. A., Puskas, C. M., Santillan, M., and Wang, L. (2016). Plate Boundary Observatory and related networks: GPS data analysis methods and geodetic product. *Reviews of Geophysics*, pages 1–50.
- Hines, T. T. and Hetland, E. A. (2016). Rapid and simultaneous estimation of fault slip and heterogeneous lithospheric viscosity from post-seismic deformation. *Geophysical Journal International*, 204(1):569–582.
- Holt, W. E. and Shcherbenko, G. (2013). Toward a Continuous Monitoring of the Horizontal Displacement Gradient Tensor Field in Southern California Using cGPS Observations from Plate Boundary Observatory (PBO). *Seismological Research Letters*, 84(3):455–467.
- Holtkamp, S. and Brudzinski, M. R. (2010). Determination of slow slip episodes and strain accumulation along the Cascadia margin. *Journal of Geophysical Research: Solid Earth*, 115(4):1–21.

- Kato, T., El-Fiky, G. S., Oware, E. N., and Miyazaki, S. (1998). Crustal strains in the Japanese islands as deduced from dense GPS array. *Geophysical Research Letters*, 25(18):3445–3448.
- Langbein, J. (2004). Noise in two-color electronic distance meter measurements revisited. *Journal of Geophysical Research: Solid Earth*, 109(4):1–16.
- Langbein, J. (2008). Noise in GPS displacement measurements from Southern California and Southern Nevada. *Journal of Geophysical Research: Solid Earth*, 113(5):1–12.
- Langbein, J. and Johnson, H. (1997). Correlated errors in geodetic time series: Implications for time-dependent deformation. *Journal of Geophysical Research*, 102(B1):591–603.
- Lisowski, M., Dzurisin, D., Denlinger, R. P., and Iwatsubo, E. Y. (2008). Analysis of GPS-Measured Deformation Associated with the 2004–2006 Dome-Building Eruption of Mount St. Helens, Washington. Technical Report September 1984.
- Mao, A., Harrison, G. A., and Dixon, H. (1999). Noise in GPS coordinate time series. *Journal of Geophysical Research*, 104(B2):2797–2816.
- McCaffrey, R., King, R. W., Payne, S. J., and Lancaster, M. (2013). Active tectonics of northwestern U.S. inferred from GPS-derived surface velocities. *Journal of Geophysical Research: Solid Earth*, 118:709–723.
- McCaffrey, R., Qamar, A. I., King, R. W., Wells, R., Khazaradze, G., Williams, C. A., Stevens, C. W., Vollick, J. J., and Zwick, P. C. (2007). Fault locking, block rotation and crustal deformation in the Pacific Northwest. *Geophysical Journal International*, 169(3):1315–1340.
- McGuire, J. J. and Segall, P. (2003). Imaging of aseismic fault slip transients recorded by dense geodetic networks. *Geophysical Journal International*, 155:778–788.
- Murray, M. H. and Lisowski, M. (2000). Strain accumulation along the Cascadia subduction zone in western Washington. *Geophysical Research Letters*, 27(22):3631–3634.
- Ohtani, R., McGuire, J. J., and Segall, P. (2010). Network strain filter: A new tool for monitoring and detecting transient deformation signals in GPS arrays. *Journal of Geophysical Research: Solid Earth*, 115(12):1–17.
- Prescott, W. H. (1976). An extension of Frank’s method for obtaining crustal shear strains from survey data. *Bulletin of the Seismological Society of America*, 66(6):1847–1853.
- Press, W. H., Flannery, B. P., Teukolsky, S. A., and Vetterling, W. T. (2007). *Numerical Recipes: The Art of Scientific Computing*. Cambridge University Press, Cambridge, 3 edition.
- Rasmussen, C. E. and Williams, C. K. I. (2006). *Gaussian processes for machine learning*. The MIT Press.
- Roeloffs, E. A. (2006). Evidence for Aseismic Deformation Rate Changes Prior To Earthquakes. *Annual Review of Earth and Planetary Sciences*, 34(1):591–627.
- Sandwell, D. T. and Wessel, P. (2016). Interpolation of 2-D vector data using constraints from elasticity. *Geophysical Research Letters*, pages 1–7.
- Savage, J. C., Prescott, W. H., and Gu, G. (1986). Strain accumulation in southern California, 1973–1984. *Journal of Geophysical Research*, 91(B7):7455–7473.
- Schmalzle, G. M., McCaffrey, R., and Creager, K. C. (2014). Central Cascadia subduction zone creep. *Geochemistry, Geophysics, Geosystems*, pages 1515–1532.
- Schmidt, D. A. and Gao, H. (2010). Source parameters and time-dependent slip distributions of slow slip events on the Cascadia subduction zone from 1998 to 2008. *Journal of Geophysical Research: Solid Earth*, 115(4):1–13.
- Schwartz, S. Y. and Rokosky, J. M. (2007). Slow slip events and seismic tremor at circum-Pacific subduction zones. *Reviews of Geophysics*, 45:1–32.
- Segall, P. and Mathews, M. (1997). Time dependent inversion of geodetic data. *Journal of Geophysical Research*, 102(B10):22391–22409.
- Shen, Z., Wang, M., Zeng, Y., and Wang, F. (2015). Optimal Interpolation of Spatially Discretized Geodetic Data. *Bulletin of the Seismological Society of America*, 105(4):2117–2127.

- Shen, Z. K., Jackson, D. D., Ge, B. X., and Bob, X. G. (1996). Crustal deformation across and beyond the Los Angeles basin from geodetic measurements. *Journal of Geophysical Research*, 101(B12):27927–27957.
- Shen, Z. K., King, R. W., Agnew, D. C., Wang, M., Herring, T. A., Dong, D., and Fang, P. (2011). A unified analysis of crustal motion in Southern California, 1970-2004: The SCEC crustal motion map. *Journal of Geophysical Research: Solid Earth*, 116(11):1–19.
- Shu, C., Ding, H., and Yeo, K. (2003). Local radial basis function-based differential quadrature method and its application to solve two-dimensional incompressible NavierStokes equations. *Computer Methods in Applied Mechanics and Engineering*, 192(7-8):941–954.
- Tape, C., Musé, P., Simons, M., Dong, D., and Webb, F. (2009). Multiscale estimation of GPS velocity fields. *Geophysical Journal International*, 179(2):945–971.
- Tolstykh, A. I. and Shirobokov, D. A. (2003). On using radial basis functions in a "finite difference mode" with applications to elasticity problems. *Computational Mechanics*, 33(1):68–79.
- Wdowinski, S., Zhang, J., Fang, P., and Genrich, J. (1997). Southern California Permanent GPS Geodetic Array: Spatial filtering of daily positions for estimating coseismic and postseismic displacements induced by the 1992 Landers earthquake. 102(97):57–70.
- Wech, A. G. (2010). Interactive Tremor Monitoring. *Seismological Research Letters*, 81(4):664 – 669.
- Wech, A. G., Creager, K. C., and Melbourne, T. I. (2009). Seismic and geodetic constraints on Cascadia slow slip. *Journal of Geophysical Research: Solid Earth*, 114(10):1–9.
- Wendland, H. (2005). *Scattered data approximation*.
- Williams, S. D. P., Bock, Y., Fang, P., Jamason, P., Nikolaidis, R. M., Prawirodirdjo, L., Miller, M., and Johnson, D. J. (2004). Error analysis of continuous GPS position time series. *Journal of Geophysical Research: Solid Earth*, 109(B3).
- Wright, G. B. and Fornberg, B. (2006). Scattered node compact finite difference-type formulas generated from radial basis functions. *Journal of Computational Physics*, 212(1):99–123.
- Wyatt, F. (1982). Displacement of Surface Monuments: Horizontal Motion. *Journal of Geophysical Research*, 87(B2):979–989.
- Wyatt, F. K. (1989). Displacement of surface monuments: Vertical motion. *Journal of Geophysical Research*, 94(B2):1655–1664.
- Zhang, J., Bock, Y., Johnson, H., Fang, P., Williams, S., Genrich, J., Wdowinski, S., and Behr, J. (1997). Southern California Permanent GPS Geodetic Array: Error analysis of daily position estimates and site velocities. *Journal of Geophysical Research*, 102(B8):18035–18055.