In the most general sense, we seek to estimate a smoothed signal, u_{smooth}, from observed data, u_{obs}, by incorporating prior knowledge of the signals statistical properties. In this paper, we describe the smoothed and observed signal as

$$u_{\text{smooth}} = u_{\text{obs}} + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \mathbf{C}_{\text{obs}}) \tag{1}$$

$$u_{\text{smooth}} = u_{\text{prior}}, \quad u_{\text{prior}} \sim \mathcal{N}(0, \mathbf{C}_{\text{prior}}). \tag{2}$$

We find u_{smooth} that simulataneously satisfies eq. (1) and eq. (2) in a least-squares sense by minimization the objection function

$$F(u_{\text{smooth}}) = ||u_{\text{smooth}} - u_{\text{obs}}||^2_{\mathbf{C}_{\text{obs}}} + ||u_{\text{smooth}}||^2_{\mathbf{C}_{\text{prior}}}. \tag{3}$$

The solution that minimizes eq. 3 is simply

$$u_{\text{smooth}} = (\mathbf{C}_{\text{obs}}^{-1} + \mathbf{C}_{\text{prior}}^{-1})^{-1}\mathbf{C}_{\text{obs}}^{-1}u_{\text{obs}}. \tag{4}$$

The challenge is in choosing an appropriate $\mathbf{C}_{\text{prior}}$. Below we discuss selection of $\mathbf{C}_{\text{prior}}$ is one-dimensional problems, which then naturally leads to an extension to selection of $\mathbf{C}_{\text{prior}}$ for higher dimensions.

One-dimensional smoothing

When the signal is only assumed to covary in time, u_{prior} is commonly treated as Brownian motion (eg. Me Segall, McQuire, Murray, etc.). We treat a u_{prior} as Brownian motion by assuming that its velocity is white noise with constant variance λ^2. That is to say

$$\mathbf{D}_1 u_{\text{prior}} = q, \quad q \sim \mathcal{N}(0, \lambda^2) \tag{5}$$

where $\mathbf{D}_N$ denotes an N'th order differentiation matrix. We then model u_{prior} as Brownian motion by setting $\mathbf{C}_{\text{prior}}$ to be

$$\mathbf{C}_{\text{prior}} = \lambda^2(\mathbf{D}_1^T\mathbf{D}_1)^{-1}. \tag{6}$$

It is also common to treate u_{prior} as integrated Brownian motion. In such case, the appropriate choice of $\mathbf{C}_{\text{prior}}$ would just use $\mathbf{D}_2$ rather than $\mathbf{D}_1$. There is still a need to select λ, which described how rapidly we expect the smoothed signal to vary. There are numerous methods for selecting λ. For example, one could use maximum likelihood methods, a trade-off curve, or simply vary σ until the smoothed signal looks appropriate when compared to the observations. We note that smoothing is fundamentally a low-pass filter and so it is perhaps most useful to choose σ based on its cut-off attenuation frequencies. We then consider the solution for u_{smooth} in the frequency domain.

For the purpose analytical tractability, we assume that u_{smooth} and ϵ are stationary stochastic processes and furthermore

$$\mathbf{C}_{\text{obs}} = \sigma^2\mathbf{I} \tag{7}$$

and

$$\mathbf{C}_{\text{prior}} = \lambda^2(\mathbf{D}_N^T\mathbf{D}_N)^{-1}. \tag{8}$$

The solution for u_{smooth} in the frequency domain is

$$\hat{u}_{\text{smooth}}(\omega) = \frac{\left(\frac{1}{\sigma}\right)^2}{\left(\frac{1}{\sigma}\right)^2 + \left(\frac{(2\pi\omega)^N}{\lambda}\right)^2}\hat{u}_{\text{obs}}(\omega). \tag{9}$$

We make the change of variables

$$\lambda = (2\pi\omega_c)^N\sigma \tag{10}$$

where ω_c is now our free parameter. This simplifies eq. 9 to

$$\hat{u}_{\text{smooth}}(\omega) = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2N}}\hat{u}_{\text{obs}}(\omega). \tag{11}$$

We can recognize eq. 11 as a N'th order Butterworth filter, and ω_c is the cut-off frequency. Thus selecting the hyperparameter, λ, can be translated through eq. 10 into a more tangible question of identifying an upper bound on the frequency content of the underlying signal. In the limit as $N \to \infty$ eq. 11 becomes an ideal low-pass filter which removes all frequencies above ω_c and leaves lower frequencies unaltered. Of course, an ideal low-pass filter is often undesirable in practice because it will tend to produce ringing artifacts in the smoothed solutions. When modeling u_{prior} as a Brownian motion or integrated Brownian motion, where $N = 1$ and $N = 2$ respectively, the transfer function is tapered across ω_c, which ameliorates ringing in the smoothed solution.