AOSS 555 Final Project:
Modeling Seismic Wave Propagation with Radial Basis Functions

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Assignment

Solve the problem

$$u_t = F(u), \tag{1}$$

where

$$F(u) = \nu u_{xx} - u u_x, \tag{2}$$

u is 2π periodic, $\nu = 1/10$, and the initial condition are

$$u(x, 0) = 1 - \sin(x). \tag{3}$$

Use the Fast Fourier Transform and an explicit time-marching method to integrate from t=0 to t=2. Present graphs illustrating

1. the evolution of the Fourier coefficients with time and
2. the evolution of u(x,t) with time.

Solution

I solve eq. (1) by first discretizing the time domain into M time steps as

$$t_j = \frac{2j}{M}, \quad j = \{0, 1, ..., M-1\} \tag{4}$$

and then I find $u(x, t_{j+1})$ with an explicit Runge-Kutta scheme. For each iteration, eq. (2) is evaluated at $u(x, t_j)$ as descibed in the following paragraph.

I approximate $u(x, t_j)$ with a complex exponential series containing N terms:

$$u(x, t_j) \approx \sum_{k=-N/2}^{N/2-1} \alpha_{jk} e^{ikx}. \tag{5}$$

The choice of exponential basis functions ensures that the 2π periodic condition is satisfied. I define my N collocation points as

$$x_n = \frac{2\pi n}{N}, \quad n = \{0, 1, ..., N-1\}, \tag{6}$$

and then find α_{jk} for the current time step by making use of the discrete Fourier transform:

$$\alpha_{jk} = \mathrm{DFT}[u(x_n, t_j)]_k = \frac{1}{N}\sum_{n=0}^{N-1} u(x_n, t_j)e^{-ikx_n}, \quad k = \{-N/2, ..., N/2-1\}. \tag{7}$$

I then evaluate eq. (2) substituting u with the series in eq. (5) and using the coefficients found from eq. (7). For computational efficiency, the derivatives inside eq. (2) are evaluated in the Fourier domain. Namely, I use the properties

$$\mathrm{DFT}[u_x(x_n, t_j)]_k = (ik)\mathrm{DFT}[u(x_n, t_j)]_k = (ik)\alpha_{jk} \tag{8}$$

and

$$\mathrm{DFT}[u_{xx}(x_n, t_j)]_k = (ik)^2\mathrm{DFT}[u(x_n, t_j)]_k = (ik)^2\alpha_{jk} \tag{9}$$

to evaluate eq. (2) as

$$F(u(x_n, t_j)) = \mathrm{IDFT}[\nu(ik)^2\alpha_{jk}]_n - u(x_n, t_j)\mathrm{IDFT}[(ik)\alpha_{jk}]_n, \tag{10}$$

where IDFT is the inverse discrete Laplace transform, which I define as

$$u(x_n, t_j) = \mathrm{IDFT}[\alpha_{jk}]_n = \sum_{k=-N/2}^{N/2-1} \alpha_{jk}e^{ikx_n}, \quad n = \{0, 1, ..., N-1\}. \tag{11}$$

In total, evaluating eq. (2) requires three Fourier transforms and the computational cost for each time step is $O(N \log N)$ when using the Fast Fourier Transform algorithm.

The procedure described above is demonstrated in the below Python script.

Results

The solution for $u(x, t)$ using $M = 1000$ and $N = 200$ is shown in figure 1. As time progresses, the initial sine wave moves in the positive x direction while also becoming steeper on the leeward side. The amplitude of the wave also decreases over time as $u(x, t)$ approaches its steady state value of 1.

Figure 2 shows the magnitude of the Fourier coefficients, α_{jk}, over time. The coefficients are spectrally accurate throughout the time interval from 0 to 2. However, the amplitude of the high frequency coefficients increases over time and it is likely that the solution for $u(x, t)$ would become unstable if I continued time stepping much further past $t = 2$.