AOSS 555 Final Project:
Modeling Seismic Wave Propagation with Radial Basis Functions

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Introduction

Imaging the interior of the Earth from seismic waves requires an accurate means of predicting how seismic waves would propagate through the Earth for any given realization of its elastic properties. In the past decade there has been a great deal of interest in using numerical methods to calculate how seismic waves propagate through the earth. Among the numerous methods are finite difference methods, finite element methods, spectral element methods, and method of circles. In this paper I exlplore the potential for radial basis function.

Radial basis functions have two appealing properties for a seismologist. They allow for an arbitrary topology with any distribution of collocation points and any geometry. Although Global seismologists assume a disk or spherical domain, which could be handled with trigonemetric pseudospectral methods, seismology also can be on regional scales where complex domains resulting from topography or sedimentary basins could complicate things. it is also a remarkably easy method to understand.

Seismic Solve the problem

$$u_t = F(u), \tag{1}$$

where

$$F(u) = \nu u_{xx} - u u_x, \tag{2}$$

u is 2π periodic, $\nu = 1/10$, and the initial condition are

$$u(x, 0) = 1 - \sin(x). \tag{3}$$

Use the Fast Fourier Transform and an explicit time-marching method to integrate from t=0 to t=2. Present graphs illustrating

1. the evolution of the Fourier coefficients with time and
2. the evolution of u(x,t) with time.

Solution

I solve eq. (1) by first discretizing the time domain into M time steps as

$$t_j = \frac{2j}{M}, \quad j = \{0, 1, ..., M-1\} \tag{4}$$

and then I find $u(x, t_{j+1})$ with an explicit Runge-Kutta scheme. For each iteration, eq. (2) is evaluated at $u(x, t_j)$ as descibed in the following paragraph.

I approximate $u(x, t_j)$ with a complex exponential series containing N terms:

$$u(x, t_j) \approx \sum_{k=-N/2}^{N/2-1} \alpha_{jk} e^{ikx}. \tag{5}$$

The choice of exponential basis functions ensures that the 2π periodic condition is satisfied. I define my N collocation points as

$$x_n = \frac{2\pi n}{N}, \quad n = \{0, 1, ..., N-1\}, \tag{6}$$

and then find α_{jk} for the current time step by making use of the discrete Fourier transform:

$$\alpha_{jk} = \mathrm{DFT}[u(x_n, t_j)]_k = \frac{1}{N} \sum_{n=0}^{N-1} u(x_n, t_j) e^{-ikx_n}, \quad k = \{-N/2, ..., N/2-1\}. \tag{7}$$

I then evaluate eq. (2) substituting u with the series in eq. (5) and using the coefficients found from eq. (7). For computational efficiency, the derivatives inside eq. (2) are evaluated in the Fourier domain. Namely, I use the properties

$$\mathrm{DFT}[u_x(x_n, t_j)]_k = (ik)\mathrm{DFT}[u(x_n, t_j)]_k = (ik)\alpha_{jk} \tag{8}$$

and

$$\mathrm{DFT}[u_{xx}(x_n, t_j)]_k = (ik)^2\mathrm{DFT}[u(x_n, t_j)]_k = (ik)^2\alpha_{jk} \tag{9}$$

to evaluate eq. (2) as

$$F(u(x_n, t_j)) = \mathrm{IDFT}[\nu(ik)^2\alpha_{jk}]_n - u(x_n, t_j)\mathrm{IDFT}[(ik)\alpha_{jk}]_n, \tag{10}$$

where IDFT is the inverse discrete Laplace transform, which I define as

$$u(x_n, t_j) = \mathrm{IDFT}[\alpha_{jk}]_n = \sum_{k=-N/2}^{N/2-1} \alpha_{jk} e^{ikx_n}, \quad n = \{0, 1, ..., N-1\}. \tag{11}$$

In total, evaluating eq. (2) requires three Fourier transforms and the computational cost for each time step is $O(N \log N)$ when using the Fast Fourier Transform algorithm.

The procedure described above is demonstrated in the below Python script.

Results

The solution for $u(x,t)$ using $M = 1000$ and $N = 200$ is shown in figure 1. As time progresses, the initial sine wave moves in the positive x direction while also becoming steeper on the leeward side. The amplitude of the wave also decreases over time as $u(x,t)$ approaches its steady state value of 1.

Figure 2 shows the magnitude of the Fourier coefficients, α_{jk}, over time. The coefficients are spectrally accurate throughout the time interval from 0 to 2. However, the amplitude of the high frequency coefficients increases over time and it is likely that the solution for $u(x,t)$ would become unstable if I continued time stepping much further past $t = 2$.