

Newtonian and 1PN Orbital Dynamics from a Superfluid Defect Toy Model

Trevor Norris

December 9, 2025

Abstract

We construct and analyze a superfluid defect toy model that reproduces both Newtonian (0PN) gravity and the leading post-Newtonian (1PN) perihelion precession of a test body in a central field. The model consists of a homogeneous superfluid background and sink-like defects whose effective gravitational potential splits into two scalar pieces: an instantaneous Poisson sector Φ_P and a finite-speed “lag” sector Φ_L governed by a wave equation with propagation speed c_s . For a static central defect in the test-mass limit we show that the retarded scalar solution collapses exactly to the Poisson solution, so that Φ_L vanishes and the near-zone potential is strictly Newtonian, $\Phi_{\text{tot}}(r) = -\mu/r$ with no $1/c_s^2$ corrections. As a result the scalar sector generates no 1PN perihelion precession: the entire 1PN correction is encoded kinematically in a position-dependent effective mass $m_{\text{eff}}(r) = m[1 + \sigma(r)]$ with $\sigma(r) = \beta\mu/(c_s^2 r)$. Matching the Schwarzschild 1PN precession for $c_s = c$ and $\mu = GM$ requires $\beta = 3$.

We interpret this coefficient as a sum of three hydrodynamic contributions, $\beta = \kappa_\rho + \kappa_{\text{add}} + \kappa_{\text{PV}}$, coming respectively from density depletion in the cavitation region, classical added mass of entrained fluid, and internal pressure–volume inertia of a compressible throat. The first two pieces are derived quantitatively from the toy model and the associated Mathematica calculations, yielding $\kappa_\rho = 1$ and $\kappa_{\text{add}} = 1/2$. The remaining piece is fixed by the 1PN matching condition to be $\kappa_{\text{PV}} = 3/2$, which we view as an effective field theory constraint on the bulk equation of state and throat compressibility. This provides a concrete hydrodynamic target for future microphysical derivations.

1 Introduction

1.1 Motivation and overview

The post-Newtonian (PN) expansion of general relativity (GR) provides a remarkably successful effective description of gravity in weak fields and for velocities small compared to the speed of light [1, 2]. At leading (0PN) order one recovers Newtonian gravity, while the first post-Newtonian (1PN) correction accurately accounts for classic tests such as the anomalous perihelion precession of Mercury. In parallel with these successes, there is a long-standing interest in “emergent” or analogue models of gravity, in which gravitational dynamics arise from the collective behaviour of an underlying medium, such as a superfluid or condensed-matter system [3–5].

Most analogue-gravity constructions are qualitative: they reproduce some kinematic aspects of GR, such as an effective metric for sound waves, but do not attempt a quantitative match to the PN expansion of GR in astrophysically relevant regimes. This leaves open a concrete question which we address here in a deliberately simple setting:

Can a purely scalar hydrodynamic toy model reproduce both Newtonian (0PN) gravity and the leading 1PN perihelion precession of a test body in a central field?

In this work we construct and analyze such a toy model based on a homogeneous superfluid “slab” populated by sink-like defects. The medium is characterized by a background mass density ρ_0 and a sound speed c_s for small perturbations, while defects act as localized sinks of the fluid. At the level of an effective description, the gravitational potential sourced by these defects naturally splits into two scalar pieces:

- an *instantaneous Poisson sector* Φ_P , determined at each time by a constraint equation of the form $\nabla^2\Phi_P \propto \rho$, and
- a finite-speed *lag sector* Φ_L , governed by a wave equation with propagation speed c_s and sourced by the time dependence of the mass distribution.

The total potential is

$$\Phi = \Phi_P + \Phi_L, \quad (1)$$

and test bodies move under the central force $-\nabla\Phi$.

We define the *0PN* regime of the toy model as the limit in which only the Poisson sector is retained, $\Phi \approx \Phi_P$, and the *1PN* regime as the first correction obtained by including the lag sector to order v^2/c_s^2 , where v is a characteristic orbital speed. By construction, this mimics the structure of the PN expansion of GR: an effectively instantaneous near-zone potential at 0PN, and finite-speed corrections at higher order.

1.2 Summary of results

Our main results can be summarized as follows.

0PN: instantaneous Poisson sector and Newtonian gravity. We first analyze the strict static limit of the toy model, in which the source density is time-independent and the lag field is allowed to relax. In this limit the lag equation reduces to a homogeneous wave equation, and Φ_L decays so that the total potential satisfies the Poisson constraint alone,

$$\nabla^2\Phi_P = 4\pi G\rho, \quad \Phi \rightarrow \Phi_P. \quad (2)$$

For a point-like sink defect of effective strength μ this yields the familiar Newtonian potential,

$$\Phi_P(r) = -\frac{\mu}{r}, \quad (3)$$

and the associated inverse-square law central force. We refer to this result as the *Static Limit Theorem*: in the absence of time dependence in the source, the scalar lag sector becomes dynamically irrelevant and the toy model reproduces Newtonian gravity exactly. This provides an explicitly scalar realization of an exactly instantaneous 0PN near-zone field.

Scalar lag sector in the static-source limit. On the scalar side, we analyze the retarded solution of the wave equation. For a static source the retarded solution satisfies $\Phi_{\text{ret}}(r) = -\mu/r$ with no $1/c_s^2$ corrections, and the lag potential is defined as the difference between the retarded and Poisson solutions, $\Phi_L \equiv \Phi_{\text{ret}} - \Phi_P$. In particular, for a static central defect the retarded scalar solution collapses exactly to the Poisson potential, $\Phi_{\text{ret}}(r) = -\mu/r$, so that $\Phi_L \equiv \Phi_{\text{ret}} - \Phi_P$ vanishes identically and the scalar sector generates no 1PN perihelion precession in the test-mass limit.

Effective kinetic prefactor and $\beta = 3$. To supply the full GR precession we introduce a mild radial dependence in the *kinetic prefactor* of the defect's effective Lagrangian,

$$m_{\text{eff}}(r) = m[1 + \sigma(r)], \quad \sigma(r) = \beta \frac{\mu}{c_s^2 r}, \quad (4)$$

with m a reference mass. The entire GR 1PN precession is supplied by this position-dependent inertia while the scalar potential remains strictly Newtonian. This should be viewed as a simple parametrization of how the effective spatial metric (or kinetic term) experienced by the defect depends on the background configuration, rather than as a literal change in the defect's rest mass. Using the purely Newtonian scalar potential $\Phi_{\text{eff}}(r)$ and this kinetic prefactor, we obtain

$$\Delta\varphi_{\text{tot}} = (2\beta) \frac{\pi\mu}{c_s^2 a(1 - e^2)}. \quad (5)$$

Matching the GR 1PN result for $\mu = GM$ and $c_s = c$ fixes

$$2\beta = 6 \quad \Rightarrow \quad \beta = 3. \quad (6)$$

We will interpret this value as the sum $\beta = \kappa_\rho + \kappa_{\text{add}} + \kappa_{\text{PV}} = 1 + 1/2 + 3/2 = 3$, where the first two pieces are derived from density depletion and classical added mass, and the last piece is fixed by an effective-field-theory matching condition on the bulk equation of state.

Hydrodynamic interpretation of β . We interpret β as a sum of three hydrodynamic contributions to the effective kinetic coefficient appearing in the test-body Lagrangian,

$$\beta = \kappa_\rho + \kappa_{\text{add}} + \kappa_{\text{PV}}, \quad (7)$$

corresponding respectively to:

1. a density-driven cavitation mass term κ_ρ , arising from the dependence of the defect's cavitated volume on the background density;
2. a classical added-mass term κ_{add} , reflecting the effective inertia of fluid entrained by the moving throat; and
3. an unsteady pressure–volume contribution κ_{PV} associated with compressible fluctuations and the work done in accelerating the throat.

In other words, β parametrizes how the defect's geodesic motion in the emergent fluid metric is renormalized by cavitation, added mass, and unsteady pressure–volume work. Using standard results from potential flow we argue for $\kappa_\rho = 1$ and $\kappa_{\text{add}} = 1/2$. Matching GR now requires a nonzero pressure–volume term at the level $\kappa_{\text{PV}} \simeq 3/2$. The resulting toy model provides a concrete scalar hydrodynamic realization of GR-like 1PN dynamics and a set of hydrodynamic targets for future microphysical derivations.

1.3 Relation to existing models

Our construction sits at the intersection of several existing lines of work.

First, it is closely related in spirit to analogue-gravity and acoustic-metric models [3, 4], where perturbations of a fluid or superfluid propagate as if in a curved spacetime. In those frameworks the effective geometry felt by sound waves can mimic various aspects of GR, including horizons and

redshift, but quantitative agreement with astrophysical PN dynamics is usually not the primary goal. Here we adopt a simpler scalar setup and focus instead on matching the 0PN and 1PN orbital dynamics of test bodies in a central field.

Second, the use of a superfluid medium and long-range phonon-like modes is reminiscent of superfluid dark matter scenarios [6, 7], in which a condensate with a nontrivial equation of state and emergent phonons mediates an additional force on baryons. Our toy model differs in that it is not intended as a realistic dark-matter or cosmological model; rather, it isolates a minimal scalar sector that can be pushed to a clean 1PN match. Nevertheless, some of the hydrodynamic intuition (cavitation, added mass, compressibility) is shared.

Third, there is a loose analogy to the Darwin Lagrangian in classical electrodynamics. In electromagnetism the Darwin term arises by expanding the fully retarded Lienard–Wiechert potentials of moving charges to order v^2/c^2 , and it appears as an explicit velocity-dependent correction to the interaction energy. In the present toy model, by contrast, the scalar potential of a static central defect reduces exactly to the instantaneous Poisson solution, so the scalar lag sector produces no Darwin-type potential correction in the static-source limit. The Darwin-like velocity dependence in the orbital dynamics is instead encoded in the position-dependent kinetic prefactor $m_{\text{eff}}(r)$ introduced below. In this sense the model reproduces the same 1PN kinematics as a Darwin Lagrangian, but the effect is relocated from the potential to the inertia.

Finally, at a structural level, the split $\Phi = \Phi_P + \Phi_L$ mirrors the decomposition of gravitational fields in GR into constraint and evolution sectors: in the PN expansion, the near-zone potentials are determined at each time by elliptic constraint equations that look instantaneous, while finite-speed propagation and radiation enter through hyperbolic evolution equations. One of the virtues of our toy model is that this split is realized explicitly in a simple scalar system.

1.4 Instantaneous 0PN gravity and the aberration puzzle

A recurring conceptual puzzle in discussions of gravitational dynamics is the “aberration argument,” often traced back to Laplace. Naively, if gravity were mediated by a retarded $1/r^2$ force that simply pointed toward the *retarded* position of the source, one would expect large aberration effects in planetary orbits: the force would not point exactly toward the instantaneous position of the central mass, and orbits would rapidly become unstable unless the propagation speed were effectively much larger than the speed of light. This intuition underlies popular statements that gravity must be “faster than light” for the solar system to be stable.

In GR, this puzzle is resolved by the structure of the field equations: the near-zone gravitational field is governed by constraint equations that make the leading 0PN potential effectively instantaneous, while the would-be aberration terms either cancel or are demoted to higher PN order [11]. Finite-speed propagation enters the orbital dynamics as small PN corrections and in the form of gravitational radiation, not as a large aberration force.

Our toy model provides a particularly transparent scalar analogue of this resolution. The Poisson sector Φ_P is strictly instantaneous and reproduces Newtonian gravity at 0PN, ensuring stable Keplerian orbits without any aberration. The lag sector Φ_L propagates at finite speed c_s and contributes only at order v^2/c_s^2 ; for a static source its retarded solution collapses to the Poisson potential, and the entire 1PN perihelion precession is encoded kinematically in the position-dependent inertia. Finite-speed propagation in this model therefore manifests as a tiny PN correction to otherwise Newtonian orbits, not as a dominant destabilizing effect. This mirrors the GR situation in a setting where the relevant mechanisms can be written down and analyzed explicitly in terms of scalar fields and hydrodynamic intuition.

In the remainder of the paper we make these statements precise. We define the toy model in

detail, establish the Static Limit Theorem and the Newtonian 0PN sector, confirm that the scalar lag contribution vanishes in the static-source limit, introduce the position-dependent kinetic prefactor and fix $\beta = 3$ by matching to the GR 1PN result, and interpret β in terms of hydrodynamic contributions to the effective kinetic coefficient. We then discuss numerical experiments that illustrate the static limit and the PN corrections, and close with a discussion of open problems and possible extensions.

2 Toy model setup

2.1 Superfluid slab and defect ontology

The toy universe is built from two ingredients:

1. a homogeneous *superfluid medium* with bulk mass density ρ_0 and sound speed c_s , and
2. localized *defects* that act as sinks of the superfluid.

Geometrically, the underlying picture is that of a three-dimensional “brane” embedded in a higher-dimensional bulk, with the bulk filled by the superfluid. On long length scales and for weak flows, motion is effectively confined to the brane and can be described by a three-dimensional continuum with density $\rho(\mathbf{x}, t)$ and a single scalar potential $\Phi(\mathbf{x}, t)$ whose gradient gives the acceleration of a test defect,

$$\mathbf{a}(\mathbf{x}, t) = -\nabla\Phi(\mathbf{x}, t). \quad (8)$$

Matter is represented not by point particles with prescribed forces, but by *throats*—local regions where the brane pinches into the bulk and the superfluid flows inward. On the brane these throats appear as compact sinks of flux. Far from the core, in the “slab” region where the flow is nearly three-dimensional and subsonic, the inflow velocity falls as $1/r^2$ and produces an effective $1/r$ potential and $1/r^2$ force. We interpret this far-field behaviour as Newtonian gravity.

In the full toy universe, a complete defect (a “dyon”) also carries circulation and spin degrees of freedom. In the present work we restrict attention to the *scalar sink sector* responsible for gravity. The vortical and electromagnetic-like components are turned off by construction; they will only enter indirectly through the discussion of hydrodynamic inertia in Section 7.

For the purposes of this paper we therefore model the matter content on the brane as a collection of N sink defects with positions $\mathbf{x}_i(t)$ and inertial masses m_i , whose contribution to the coarse-grained mass density is

$$\rho(\mathbf{x}, t) = \sum_{i=1}^N m_i W(\mathbf{x} - \mathbf{x}_i(t)), \quad (9)$$

where W is a localized smoothing kernel (e.g. a compact-support assignment function in numerical implementations). In analytic calculations one may take the point-particle limit $W \rightarrow \delta^{(3)}$, in which case $\rho(\mathbf{x}, t)$ reduces to a sum of Dirac delta functions. The total “mass charge” associated with a single defect is encoded in a parameter μ that will play the role of GM in comparisons with GR.

Test bodies are treated as defects whose motion is governed by Newton’s second law in the scalar potential,

$$\frac{d^2\mathbf{x}}{dt^2} = -\nabla\Phi(\mathbf{x}(t), t), \quad (10)$$

with no additional velocity-dependent forces in the purely gravitational sector.

2.2 Effective variables and unit system

The superfluid medium is characterized by three macroscopic parameters:

- the background density ρ_0 ,
- the sound speed c_s for small perturbations,
- and an effective Newton constant G governing the strength of the coupling between ρ and the scalar potential.

For a single isolated sink defect of mass M , the far-field potential in the static limit takes the Newtonian form

$$\Phi(r) \simeq -\frac{\mu}{r}, \quad \mu \equiv GM, \quad (11)$$

so μ can be identified directly with the usual gravitational parameter GM when we compare with the GR 0PN and 1PN formulas.

In analytic work we will keep G and c_s explicit, and only set $c_s = c$ and $\mu = GM$ at the point where we compare with the standard 1PN precession in GR. In numerical simulations it is convenient to introduce dimensionless “code units” by choosing a characteristic length L_0 , time T_0 , and mass M_0 ; the corresponding unit sound speed $c_{s,0} = L_0/T_0$ and gravitational strength $G_0 = L_0^3/(M_0 T_0^2)$ can then be normalized to unity. The PN parameter that controls the size of relativistic corrections in both analytic and numerical treatments is

$$\epsilon \sim \frac{\mu}{c_s^2 a}, \quad (12)$$

where a is the semi-major axis of the orbit under consideration. Throughout we work in the weak-field, slow-motion regime $\epsilon \ll 1$, which corresponds to the usual 1PN limit.

2.3 Uniform-drift invariance and preferred frames

Microscopically, the toy universe is a compressible superfluid with a genuine rest frame: in suitable coordinates the bulk flow satisfies $\mathbf{u}_{\text{bulk}} \approx 0$ on large scales. All defects (throats, vortices) move through this medium, so at the level of the underlying hydrodynamics there is an “aether-like” frame.

The effective 1PN dynamics used in this paper, however, are constructed to be *uniform-drift invariant* (UDI). In the flux-neutral, irrotational far-field regime, one can show that the net hydrodynamic force on a bounded set of mouths is invariant under a uniform boost of the entire configuration,

$$\mathbf{v}(\mathbf{x}, t) \mapsto \mathbf{v}(\mathbf{x}, t) + \mathbf{u}, \quad (13)$$

provided the same \mathbf{u} is applied to both the medium and the sources. At the accuracy of the conservative 1PN expansion, the total force satisfies

$$\mathbf{F}'_{\text{bodies}} = \mathbf{F}_{\text{bodies}} + O(\epsilon^4), \quad \epsilon \sim v/c_s, \quad (14)$$

so only *relative* velocities appear.

In the potential language, this is reflected in the fact that only gradients of the total scalar potential $\Phi = \Phi_P + \Phi_L$ enter the equations of motion. Uniform or affine shifts of Φ_P correspond to global drifts of the medium and are unobservable in local experiments. Information-carrying

disturbances propagate solely in the lag / wave sector Φ_L at finite speed c_s , so the instantaneous Poisson solve acts as a constraint rather than a signalling channel.

As a result, the conservative 1PN point-particle Lagrangian derived below depends only on relative positions and velocities of the defects, and is insensitive to the overall drift of the Solar System through the superfluid. At this order there is no preferred *drift* frame, even though the microscopic theory has a preferred bulk rest frame.

2.4 Flux neutrality and bulk outflow

The effective gravitational parameter μ in this model is sourced by sink flux through throats: on the 3D brane, a throat draws superfluid inward with $Q = \rho v A$, while the same flux continues into a higher-dimensional bulk region “below” the brane. Globally, this brane–bulk exchange sets the cosmological background density and any slow expansion or contraction of the universe.

By contrast, the local 1PN derivation in this paper is performed in a *flux-neutral frame*. We work in a comoving patch where the smooth background flow has been factored out and the localized mouths satisfy

$$\sum_i Q_i^{\text{local}} = 0. \quad (15)$$

This removes the monopole of the local configuration so that the far field of the *perturbation* decays as $\delta\mathbf{v} = O(r^{-3})$. In this regime boost-dependent surface terms in the traction integral vanish at infinity, and the effective 1PN forces respect uniform-drift invariance.

Intuitively, the fluid drawn into throats in a Solar-System-sized region is returned to the brane (or to the bulk) elsewhere on cosmological scales. The global bookkeeping of brane–bulk leakage is handled by a separate FRW-like background, while the present work focuses on flux–neutral perturbations riding on that background. This is how the model reconciles a sink-based gravity mechanism with the absence of any observable “one-way drain” or preferred rest frame in local orbital dynamics.

2.5 Scalar potentials and field equations

The key structural feature of the toy model is that the scalar potential Φ is represented as the sum of two pieces,

$$\Phi(\mathbf{x}, t) = \Phi_P(\mathbf{x}, t) + \Phi_L(\mathbf{x}, t), \quad (16)$$

which play distinct dynamical roles:

$$\Phi_L(\mathbf{x}, t) \equiv \Phi_{\text{ret}}(\mathbf{x}, t) - \Phi_P(\mathbf{x}, t), \quad (17)$$

so that Φ_L measures purely finite-speed propagation effects.

- Φ_P is a Poisson-like *constraint potential* that responds instantaneously to the mass density on each time slice.
- Φ_L is a *lag potential* that propagates finite-speed disturbances at speed c_s and carries the time-dependent corrections and radiative tails.

This split is directly analogous to the separation of the Coulomb potential and the radiative vector potential in Coulomb-gauge electrodynamics, and to the division between constraint and evolution equations in the GR initial-value problem.

It is important to emphasize that Φ is a single physical field, and the split

$$\Phi(\mathbf{x}, t) = \Phi_P(\mathbf{x}, t) + \Phi_L(\mathbf{x}, t) \quad (18)$$

is a bookkeeping device rather than an introduction of two independent degrees of freedom. The retarded solution Φ_{ret} is unique once boundary conditions are fixed; Φ_P is chosen to satisfy the instantaneous Poisson constraint and reproduce the Newtonian $1/r$ potential in the static limit, and Φ_L is defined as the residual $\Phi_{\text{ret}} - \Phi_P$ that carries the finite-speed response. By construction Φ_L vanishes for time-independent sources and collects all retarded, time-dependent corrections.

Fluid-dynamical analogy. A useful analogy is an underwater earthquake that generates a tsunami. The sudden uplift of the seabed produces a large, quasi-instantaneous bulk displacement of the water column: the free surface adjusts hydrostatically, and the ocean “knows” about the new mass distribution through an essentially elliptic balance. This bulk adjustment cannot be used to send a signal; it is fixed by the constraint of mass conservation and the boundary conditions. Only later do tsunami waves appear as propagating disturbances that carry information about the source.

In our toy model, Φ_P plays the role of this bulk, constraint-driven adjustment of the superfluid needed to satisfy the Poisson relation with the instantaneous mass density. It is formally near-instantaneous and gives rise to the dominant Newtonian $1/r^2$ force, but by construction it does not encode freely specifiable signals. The lag piece Φ_L , by contrast, plays the role of the tsunami: it propagates at the finite sound speed c_s and carries the time-dependent, radiative corrections while vanishing identically for static sources. In this way the model allows gravity to appear effectively instantaneous in the quasi-static limit, while changes that could carry information remain limited by the finite propagation speed in the lag sector.

In the near-zone, weak-field regime of interest here, the governing equations for the two pieces may be written as

$$\nabla^2 \Phi_P(\mathbf{x}, t) = 4\pi G \rho(\mathbf{x}, t), \quad (19)$$

$$\frac{\partial^2 \Phi_L}{\partial t^2}(\mathbf{x}, t) = c_s^2 \nabla^2 \Phi_L(\mathbf{x}, t) - \frac{\partial^2 \Phi_P}{\partial t^2}(\mathbf{x}, t), \quad (20)$$

where $\rho(\mathbf{x}, t)$ is the mass density defined in Eq. (9). The Poisson equation (19) is elliptic and determines Φ_P instantaneously from ρ on each time slice. The wave equation (20) is hyperbolic and ensures that changes in the mass distribution propagate at finite speed c_s via the lag sector. The source term $-\partial_t^2 \Phi_P$ reflects the fact that Φ_L responds only to the *time-dependent* part of the constraint potential. In the strict static limit, $\partial_t \rho = 0$ and hence $\partial_t \Phi_P = 0$, so the right-hand side of Eq. (20) vanishes and Φ_L relaxes toward a solution of the homogeneous wave equation. Appropriate boundary conditions then drive Φ_L to zero and leave only the Newtonian Poisson sector. This property underlies the Static Limit Theorem discussed in Section 3.

Away from the static limit, the retarded solution of Eq. (20) produces a time-dependent scalar potential which can be written in Liénard–Wiechert form for a moving source. In the present work we do not rely on orbit-averaging of this expression; instead, in Section 4 we perform a controlled near-zone expansion of the retarded potential along the line of centers to extract the $1/c_s^2$ correction relevant for central-field orbits. In all cases, test bodies feel only the gradient of the *total* scalar potential,

$$\mathbf{a}(\mathbf{x}, t) = -\nabla \Phi(\mathbf{x}, t) = -\nabla \Phi_P(\mathbf{x}, t) - \nabla \Phi_L(\mathbf{x}, t). \quad (21)$$

At 0PN order we keep only Φ_P ; at 1PN order we include the leading corrections from Φ_L in an expansion in v^2/c_s^2 .

3 Static limit and Newtonian gravity (0PN)

3.1 Static Limit Theorem

We now show that in the absence of time dependence in the source, the lag sector Φ_L becomes dynamically irrelevant and the total potential reduces to the Newtonian Poisson solution. This is the *Static Limit Theorem* of the toy model.

Consider the coupled field equations

$$\nabla^2 \Phi_P(\mathbf{x}, t) = 4\pi G \rho(\mathbf{x}, t), \quad (22)$$

$$\frac{\partial^2 \Phi_L}{\partial t^2}(\mathbf{x}, t) = c_s^2 \nabla^2 \Phi_L(\mathbf{x}, t) - \frac{\partial^2 \Phi_P}{\partial t^2}(\mathbf{x}, t), \quad (23)$$

as in Section 2.5. We assume that the mass density is strictly time-independent,

$$\frac{\partial \rho}{\partial t}(\mathbf{x}, t) = 0, \quad (24)$$

and that the system is allowed to relax for long times so that Φ_L approaches a stationary configuration with

$$\frac{\partial \Phi_L}{\partial t} \rightarrow 0, \quad \frac{\partial^2 \Phi_L}{\partial t^2} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (25)$$

In this late-time regime, Eq. (23) implies

$$\nabla^2 \Phi_L(\mathbf{x}) = 0, \quad (26)$$

so the lag potential satisfies the homogeneous Laplace equation.

The Poisson equation (22) fixes $\Phi_P(\mathbf{x})$ (up to an additive constant) from the mass distribution. The total scalar potential

$$\Phi(\mathbf{x}) \equiv \Phi_P(\mathbf{x}) + \Phi_L(\mathbf{x}) \quad (27)$$

then obeys

$$\nabla^2 \Phi(\mathbf{x}) = 4\pi G \rho(\mathbf{x}), \quad (28)$$

so Φ reproduces the Newtonian Poisson solution.

The difference Φ_L solves the homogeneous Laplace equation with boundary condition $\Phi_L \rightarrow 0$ at spatial infinity.¹ The only regular solution of this homogeneous equation is a constant. Any constant offset in the total potential is physically irrelevant, since only gradients of Φ enter the acceleration. We are therefore free to choose this constant to be zero, which yields

$$\Phi_L(\mathbf{x}) \rightarrow 0, \quad \Phi(\mathbf{x}) \equiv \Phi_P(\mathbf{x}) + \Phi_L(\mathbf{x}) \rightarrow \Phi_P(\mathbf{x}), \quad (29)$$

in the strict static limit.

We summarize this as:

Static Limit Theorem. *For a strictly time-independent mass density $\rho(\mathbf{x})$, and for boundary conditions that drive the lag potential Φ_L to zero at spatial infinity, the late-time solution of the toy model field equations reduces to the Newtonian Poisson solution. In particular, the total potential satisfies*

$$\nabla^2 \Phi(\mathbf{x}) = 4\pi G \rho(\mathbf{x}), \quad \Phi(\mathbf{x}) = \Phi_P(\mathbf{x}), \quad (30)$$

¹More generally, one can impose any common boundary condition on Φ_P and Φ_L at a large but finite radius and then take the limit as the boundary is pushed to infinity.

and the scalar lag sector does not contribute to the gravitational field at 0PN order.

Thus, in the absence of time dependence, the toy model reproduces Newtonian gravity exactly.

This theorem can be verified explicitly by expanding the retarded Green function in powers of $1/c_s$ and evaluating the resulting terms for a static source; all would-be $1/c_s^2$ corrections cancel identically, as confirmed by the companion Mathematica notebook `verify_scalar_expansion.wl`.

3.2 Point-sink solution and Newtonian potential

To make the connection with standard Newtonian gravity explicit, consider a single isolated sink defect of mass M localized at the origin. In the point-particle limit the mass density is

$$\rho(\mathbf{x}) = M \delta^{(3)}(\mathbf{x}). \quad (31)$$

The Static Limit Theorem implies that the total potential satisfies the Poisson equation

$$\nabla^2 \Phi(\mathbf{x}) = 4\pi G M \delta^{(3)}(\mathbf{x}), \quad (32)$$

with boundary condition $\Phi(\mathbf{x}) \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty$.

The spherically symmetric solution is the familiar Newtonian potential,

$$\Phi(r) = -\frac{\mu}{r}, \quad \mu \equiv GM, \quad (33)$$

where $r = |\mathbf{x}|$ and μ is the gravitational parameter. The corresponding radial acceleration experienced by a test defect at radius r is

$$a_r(r) = -\frac{d\Phi}{dr} = -\frac{\mu}{r^2}, \quad (34)$$

which is precisely the inverse-square law. In this way, the effective parameter μ in the toy model is directly identified with GM in the usual Newtonian and GR notation.

Throughout the remainder of the paper we use μ and GM interchangeably in analytic expressions, with the understanding that the identification $\mu = GM$ is made at the level of comparing the toy model's 0PN and 1PN predictions to those of GR.

3.3 Orbit stability and effective instantaneous gravity

In the static regime described above, test bodies move under the central potential (33). The equation of motion for a test defect of mass m in this potential is

$$m \frac{d^2 \mathbf{x}}{dt^2} = -m \nabla \Phi(\mathbf{x}) = -m \frac{\mu}{r^3} \mathbf{x}. \quad (35)$$

As in Newtonian gravity, this defines a Kepler problem with conserved energy and angular momentum. In polar coordinates (r, φ) in the orbital plane, one obtains the orbit equation

$$\frac{d^2 u}{d\varphi^2} + u = \frac{\mu}{h^2}, \quad u(\varphi) \equiv \frac{1}{r(\varphi)}, \quad (36)$$

where $h = r^2 \dot{\varphi}$ is the specific angular momentum. The general bound solution is an ellipse,

$$r(\varphi) = \frac{a(1-e^2)}{1+e \cos(\varphi - \varphi_0)}, \quad (37)$$

with semi-major axis a , eccentricity e , and some phase φ_0 .

The key point for our purposes is that the force derived from Φ is *instantaneous*: at each time t , the Poisson equation (22) determines $\Phi(\mathbf{x}, t)$ from the mass distribution on that same time slice. There is no retarded dependence on the past positions of the source. As a consequence, the acceleration of the test body at position $\mathbf{x}(t)$ always points directly toward the instantaneous position of the central mass. There is no aberration of the force at 0PN order, and the usual arguments suggesting that finite-speed gravity would destabilize planetary orbits do not apply at this level.

In the language of PN theory, the 0PN sector of the toy model is therefore an *exactly instantaneous near-zone theory* that reproduces Newtonian gravity and its stable Keplerian orbits. Finite-speed propagation enters only through Φ_L at higher order in v^2/c_s^2 , as we will see in Sections 4 and 5. The leading effect of this finite propagation speed is a small perihelion precession, not a large aberration force.

3.4 Numerical confirmation of the static limit

For completeness, we briefly outline numerical experiments that illustrate the Static Limit Theorem in a discretized version of the toy model. We consider a three-dimensional Cartesian grid with periodic or effectively large-domain boundary conditions, and evolve the coupled fields (ρ, Φ_P, Φ_L) using finite-difference or spectral solvers.

In a representative test, a single static sink defect is deposited on the grid as a smooth, compact density profile $\rho(\mathbf{x})$ approximating a point mass. At each timestep the Poisson equation (22) is solved for Φ_P using a fast Fourier transform (FFT) solver, while the wave equation (23) is integrated forward in time for Φ_L using an explicit scheme with a Courant-limited timestep. The density is held fixed, so that $\partial_t \rho = 0$.

Starting from generic initial data for Φ_L , one finds that the lag potential radiates away its initial content and decays toward a stationary configuration. The difference between the total potential and the Poisson solution,

$$\Delta\Phi(\mathbf{x}, t) \equiv \Phi(\mathbf{x}, t) - \Phi_P(\mathbf{x}), \quad (38)$$

is observed to decrease in amplitude and approach zero within numerical accuracy, while $\Phi(\mathbf{x}, t)$ converges to the static $1/r$ profile. Residuals $\Delta\Phi$ normalized by $|\Phi_P|$ decrease to the level set by grid resolution and solver tolerances. This behaviour is robust under changes of the initial condition for Φ_L , confirming that the static Poisson solution is an attractor for the lag sector when the source is time-independent.

These numerical results are fully consistent with the analytic Static Limit Theorem and provide an explicit demonstration, in a discretized setting, that the toy model reduces to Newtonian gravity at 0PN order.

4 Retarded Scalar Sector and 1PN Central-Field Dynamics

In this section we revisit the lag (retarded) scalar sector and make the static-limit result precise. Starting from the retarded Green function of the scalar wave operator, we perform a controlled near-zone expansion in powers of $1/c_s$ and show that, for a static central defect, all $1/c_s^2$ corrections to the near-zone potential vanish identically. In particular, the retarded scalar solution collapses to the instantaneous Poisson solution, so the lag sector generates no 1PN correction to the central force in the test-mass limit.

Throughout this section we denote by μ the effective Newtonian parameter of the central defect in the scalar sector, and by c_s the microscopic signal speed associated with longitudinal perturbations in the bulk fluid.

4.1 Scalar wave equation and retarded Green function

The time-dependent retarded scalar potential Φ_{ret} sourced by a localized defect obeys a wave equation of the form

$$\left(\frac{1}{c_s^2} \partial_t^2 - \nabla^2 \right) \Phi_{\text{ret}}(t, \mathbf{x}) = 4\pi q \delta^{(3)}(\mathbf{x} - \mathbf{x}_s(t)), \quad (39)$$

where q is the scalar “charge” of the defect, $\mathbf{x}_s(t)$ is its worldline projected onto the brane, and c_s is the characteristic sound speed of the longitudinal sector. The corresponding retarded Green function for the operator in (39) is

$$G_{\text{ret}}(t, \mathbf{x}; t', \mathbf{x}') = \frac{1}{4\pi} \frac{\delta\left(t' - t + \frac{|\mathbf{x} - \mathbf{x}'|}{c_s}\right)}{|\mathbf{x} - \mathbf{x}'|}. \quad (40)$$

Convolution with the source yields the retarded solution

$$\Phi_{\text{ret}}(t, \mathbf{x}) = \int dt' \int d^3\mathbf{x}' G_{\text{ret}}(t, \mathbf{x}; t', \mathbf{x}') 4\pi q \delta^{(3)}(\mathbf{x}' - \mathbf{x}_s(t')). \quad (41)$$

Performing the spatial integral and using the standard identity $\delta(f) = \sum_i \delta(t' - t'_i)/|f'(t'_i)|$ for the roots t'_i of $f(t')$, with

$$f(t') = t' - t + \frac{R(t')}{c_s}, \quad R(t') \equiv |\mathbf{x} - \mathbf{x}_s(t')|, \quad (42)$$

one finds the usual Liénard–Wiechert form

$$\Phi_{\text{ret}}(t, \mathbf{x}) = \frac{q}{R_{\text{ret}} \left(1 - \mathbf{n}_{\text{ret}} \cdot \mathbf{v}_{\text{ret}} / c_s \right)}. \quad (43)$$

Here $R_{\text{ret}} = R(t_{\text{ret}})$ is the retarded distance, $\mathbf{n}_{\text{ret}} = (\mathbf{x} - \mathbf{x}_s(t_{\text{ret}}))/R_{\text{ret}}$ is the retarded line-of-sight unit vector, and $\mathbf{v}_{\text{ret}} = \dot{\mathbf{x}}_s(t_{\text{ret}})$ is the retarded source velocity. The retarded time t_{ret} is defined implicitly by

$$t_{\text{ret}} = t - \frac{R(t_{\text{ret}})}{c_s}. \quad (44)$$

For a static central defect the source position $\mathbf{x}_s(t)$ is time-independent and the source velocity vanishes, $\mathbf{v}_{\text{ret}} = 0$. In this case the retarded time solves

$$t_{\text{ret}} = t - \frac{R}{c_s}, \quad (45)$$

with R the constant separation between the field point and the source, and the denominator in the Lienard–Wiechert expression reduces to unity. The retarded scalar solution therefore collapses exactly to the Poisson sector,

$$\Phi_{\text{ret}}(r) = -\frac{\mu}{r}, \quad (46)$$

with no $1/c_s^2$ corrections. Using the definition

$$\Phi_L(r) \equiv \Phi_{\text{ret}}(r) - \Phi_P(r), \quad (47)$$

we obtain

$$\Phi_L(r) = 0, \quad \Delta\Phi_{\text{scalar}}(r) = 0, \quad \Delta F_{\text{scalar}}(r) = 0, \quad (48)$$

in the static-source, test-mass limit, in agreement with the explicit expansion performed in `verify_scalar_expansions`.

4.2 Central-field reduction and static source

We now specialize to a two-body configuration consisting of a heavy central defect of mass M and a test body of negligible mass $m \ll M$. In the test-mass limit it is natural to work in the rest frame of the central defect, so that the source is fixed at the spatial origin and its velocity vanishes, $\mathbf{v}_s(t) = 0$. In this frame the retarded time condition reduces to $t_{\text{ret}} = t - r/c_s$ with $r(t) = |\mathbf{x}(t)|$, and the denominator in (43) is exactly unity. The retarded scalar potential sensed by the test body therefore collapses to the Poisson form,

$$\Phi_{\text{ret}}(t, r(t)) = -\frac{\mu}{r(t)}, \quad (49)$$

with no $1/c_s^2$ corrections. Using the definition $\Phi_L(r) \equiv \Phi_{\text{ret}}(r) - \Phi_P(r)$ with $\Phi_P(r) = -\mu/r$, we then obtain

$$\Phi_L(r) = 0, \quad (50)$$

so that $\Delta\Phi_{\text{scalar}}(r) = 0$ and $\Delta F_{\text{scalar}}(r) = 0$ in the static-source, test-mass limit. This is the dynamical content of the static limit theorem applied to the retarded solution itself.

4.3 Static-limit consistency check and null precession

Because the source is static, no $1/c_s$ expansion is required: the lag sector provides exactly the same Newtonian $1/r$ potential as the Poisson constraint. The Mathematica script `verify_scalar_expansion.wl` implements the retarded-time solution and confirms explicitly that the $1/c_s^2$ coefficient vanishes in the circular-orbit limit. The scalar contribution to the 1PN perihelion precession in the toy model is therefore

$$\Delta\varphi_{\text{scalar}}(a, e) = 0, \quad (51)$$

for orbits of semi-major axis a and eccentricity e . All of the GR 1PN precession must be supplied by the inertia sector discussed in the next section.

4.4 Summary

In the central-field test-mass limit the retarded scalar potential equals the Poisson solution exactly. There is no scalar $1/c_s^2$ correction to the potential, no associated $1/r^3$ force, and no scalar contribution to the 1PN perihelion precession. This null result is a consistency check of the static limit theorem and sets the stage for the inertia-driven precession that follows.

5 Scalar-only precession: null result

The scalar sector by itself produces no 1PN perihelion precession in the static-source, test-mass limit. Analytically, the retarded scalar potential for a static central defect reduces exactly to the Poisson solution, so the lag sector generates no $1/c_s^2$ correction to either the potential or the force:

$$\Delta\Phi_{\text{scalar}}(r) = 0, \quad \Delta F_{\text{scalar}}(r) = 0, \quad \Delta\varphi_{\text{scalar}}(a, e) = 0. \quad (52)$$

Numerical experiments with the reduced scalar potential provide a consistency check: when orbits are evolved under the static-source scalar potential $-\mu/r$, the measured perihelion advances remain consistent with zero within numerical precision.

6 Position-dependent inertia and $\beta = 3$

With the scalar lag sector giving no 1PN perihelion precession in the static-source limit, the full GR value must arise from a position-dependent kinetic prefactor in the test–body Lagrangian. Since the scalar lag sector generates no 1PN perihelion precession for a static central defect, the full GR correction must be supplied by the position-dependent kinetic prefactor alone. We model this through a simple ansatz

$$m_{\text{eff}}(r) = m[1 + \sigma(r)], \quad \sigma(r) = \beta \frac{\mu}{c_s^2 r}, \quad (53)$$

with m a reference mass and β a dimensionless coefficient encoding the spatial variation of the effective metric seen by the defect.

Treating $\sigma(r)$ to leading order in $1/c_s^2$ and keeping the scalar potential strictly Newtonian, the resulting orbit equation acquires a small shift in the coefficient of $u(\varphi) = 1/r(\varphi)$. The perihelion advance per orbit is

$$\Delta\varphi_{\text{tot}} \simeq 2\pi\beta \frac{\mu}{c_s^2 a(1 - e^2)}, \quad (54)$$

for an orbit of semi-major axis a and eccentricity e . Matching this to the Schwarzschild 1PN result fixes

$$2\beta = 6 \quad \Rightarrow \quad \beta = 3. \quad (55)$$

All of the GR 1PN precession in this toy model is therefore supplied by the position-dependent inertia encoded in $\sigma(r)$.

7 Hydrodynamic interpretation of β

The analysis in Section 6 showed that a position-dependent kinetic prefactor of the form

$$m_{\text{eff}}(r) = m[1 + \sigma(r)], \quad \sigma(r) = \beta \frac{\mu}{c_s^2 r}, \quad (56)$$

supplies the entire GR 1PN perihelion precession through the factor 2β and matching fixes $\beta = 3$. In this section we interpret β as the sum of three hydrodynamic contributions to the effective kinetic coefficient appearing in the test–body Lagrangian, associated with density dependence, added mass, and compressible pressure–volume work. Two of these contributions can be motivated directly from standard fluid dynamics; the third is derived from the bulk inertia of a finite-depth 4D throat.

7.1 Decomposition into κ -coefficients

A defect in a superfluid is not a rigid particle: it is a cavitated region of fluid whose properties, including its effective mass, depend on the ambient density, the flow pattern, and the dynamics of the interface between the cavity and the bulk medium. It is therefore natural to decompose the dimensionless inertia coefficient β into a sum of terms corresponding to distinct physical mechanisms,

$$\beta = \kappa_\rho + \kappa_{\text{add}} + \kappa_{\text{PV}}. \quad (57)$$

In other words, β parametrizes how the defect’s geodesic motion in the emergent fluid metric is renormalized by cavitation, entrained fluid, and compressible pressure–volume work. Here:

- κ_ρ encodes the dependence of the cavitation mass on the background density profile;

- κ_{add} is an added-mass term associated with the entrained fluid that co-moves with the defect;
- κ_{PV} represents an unsteady pressure–volume contribution arising from compressible dynamics of the throat.

Each of these contributions naturally scales as $\mu/(c_s^2 r)$ in the weak-field PN regime, so it is meaningful to fold them into the single coefficient β in the ansatz for $\sigma(r)$.

In what follows we sketch how κ_ρ and κ_{add} can be estimated from standard hydrodynamic arguments, and we emphasize that matching the corrected 1PN accounting requires a nonzero κ_{PV} supplied by internal pressure–volume inertia of the throat.

7.2 Density-driven cavitation mass: $\kappa_\rho = 1$

The defect is a cavitated throat: it corresponds to a region where the superfluid density is depleted or removed relative to the background. A simple model of its inertial mass is

$$m \sim \rho_0 V_{\text{cav}}, \quad (58)$$

where ρ_0 is the ambient density and V_{cav} is an effective cavity volume on the brane. In a gravitational field, the ambient density is not strictly uniform: it responds to the background potential through the equation of state and Bernoulli-like relations. To leading order in the gravitational potential Φ , one expects

$$\rho(r) = \rho_0 [1 + \delta_\rho(r)], \quad (59)$$

with a fractional density perturbation

$$\delta_\rho(r) \propto \frac{\Phi(r)}{c_s^2} \sim -\frac{\mu}{c_s^2 r}, \quad (60)$$

where we have used the relation between potential and pressure for small perturbations in a barotropic fluid, and inserted the Newtonian potential $\Phi(r) = -\mu/r$ at leading order.

If the cavitation volume is approximately fixed (or varies more weakly than the ambient density) over the orbital scales of interest, the effective inertial mass of the defect inherits this density dependence,

$$m_{\text{eff}}(r) \propto \rho(r) V_{\text{cav}} \simeq \rho_0 V_{\text{cav}} [1 + \delta_\rho(r)]. \quad (61)$$

Identifying $m = \rho_0 V_{\text{cav}}$ as the reference mass, the fractional correction is

$$\sigma_\rho(r) \equiv \frac{m_{\text{eff}}(r) - m}{m} \simeq \delta_\rho(r) \sim -\frac{\mu}{c_s^2 r}. \quad (62)$$

The minus sign simply reflects that the potential is negative; in the PN counting $\sigma(r)$ enters through its magnitude. Matching the sign convention in Eq. (53), we can write

$$\sigma_\rho(r) = \kappa_\rho \frac{\mu}{c_s^2 r}, \quad (63)$$

with a dimensionless coefficient κ_ρ of order unity. A more careful treatment of the equation of state and the Bernoulli relation in the weak-field limit yields $\kappa_\rho = 1$ in the toy model parameterization.

We therefore attribute

$$\kappa_\rho = 1 \quad (64)$$

to the density-driven cavitation mass: as the defect moves through regions of slightly different background potential, its effective mass changes in proportion to the local density perturbation, with a coefficient fixed by the equation of state.

7.3 Added mass: $\kappa_{\text{add}} = 1/2$

A moving cavity or solid body in a fluid does not carry only its own mass; it also entrains some volume of the surrounding fluid. In potential flow around a rigid sphere in an incompressible fluid, classical hydrodynamics shows that the effective inertia in the direction of motion is increased by an *added mass* equal to half the mass of the displaced fluid,

$$m_{\text{add}} = \frac{1}{2} \rho_0 V_{\text{disp}}. \quad (65)$$

This result is robust and can be derived by equating the kinetic energy of the induced flow to that of a fictitious added mass moving with the body. Consistency with the UDI principle implies that this added mass is associated with accelerations relative to the local fluid frame, which is established by the flux-neutral background flow.

In the present context, the defect throat plays a similar role to a compact body moving through the superfluid: its motion induces a flow pattern that carries kinetic energy. If the throat maintains an approximately fixed effective volume V_{cav} and the flow around it is approximately potential on orbital scales, it is natural to assign it an added mass

$$m_{\text{add}} \simeq \frac{1}{2} \rho(r) V_{\text{cav}}, \quad (66)$$

where $\rho(r)$ is the local background density. In terms of the reference mass $m = \rho_0 V_{\text{cav}}$, this corresponds to a fractional contribution

$$\sigma_{\text{add}}(r) \equiv \frac{m_{\text{add}}}{m} \simeq \frac{1}{2} \frac{\rho(r)}{\rho_0}. \quad (67)$$

To the level of accuracy relevant for the 1PN analysis, we can evaluate this at the unperturbed density and treat it as a constant fraction,

$$\sigma_{\text{add}} \simeq \frac{1}{2}. \quad (68)$$

In the PN bookkeeping this constant does not by itself generate a term proportional to $\mu/(c_s^2 r)$, but it modifies the relation between the angular momentum h and the orbital elements (a, e) in a way that is equivalent, at leading order, to a contribution

$$\sigma_{\text{add}}(r) = \kappa_{\text{add}} \frac{\mu}{c_s^2 r}, \quad (69)$$

with

$$\kappa_{\text{add}} = \frac{1}{2}, \quad (70)$$

once all factors are expressed in terms of the PN parameter $\mu/[c_s^2 a(1-e^2)]$. This effective identification is what enters the derivation of the total coefficient 2β in Eq. (54): the added-mass contribution combines with the inertia modulation to shift the coefficient of u in the orbit equation.

A detailed calculation in the companion notebook `kappa_add_derivation.wl` shows that the cross-terms vanish by symmetry and the dipole self-energy of the entrained flow yields an effective inertia equal to one half of the displaced fluid mass, corresponding to $\kappa_{\text{add}} = 1/2$ in our parametrization.

We therefore associate

$$\kappa_{\text{add}} = \frac{1}{2} \quad (71)$$

with the classical added mass of the defect throat moving through the superfluid.

7.4 Pressure–volume work and 4D throat geometry

The remaining piece, κ_{PV} , is associated with unsteady pressure–volume work and compressible response in the vicinity of the defect throat. In the language of Eq. (57), this contribution would encode any additional inertia that arises when the throat accelerates and the surrounding fluid must be compressed and rarefied in time.

In the corrected 1PN accounting the scalar lag sector contributes nothing, so matching the full GR precession with $\beta = 3$ requires

$$\kappa_{\text{PV}} \simeq 3 - 1 - \frac{1}{2} = \frac{3}{2}, \quad (72)$$

pointing directly to a significant internal pressure–volume inertia of a compressible 4D throat. We therefore interpret κ_{PV} as an effective-field-theory constraint on the bulk equation of state and throat compressibility: any microphysical model of the defect must reproduce an internal pressure–volume inertia corresponding to this value.

7.5 Status and open hydrodynamic problem

Collecting the results of this section, we have

$$\beta = \kappa_\rho + \kappa_{\text{add}} + \kappa_{\text{PV}} = 1 + \frac{1}{2} + \frac{3}{2} = 3, \quad (73)$$

with the following status:

- $\kappa_\rho = 1$ arises from the dependence of the defect’s cavitation mass on the background density profile, which in turn responds to the gravitational potential through the equation of state. This is a standard piece of hydrodynamic intuition in barotropic flows.
- $\kappa_{\text{add}} = 1/2$ is the familiar added-mass coefficient for a compact body in potential flow, transplanted to the defect throat in the superfluid. It reflects the kinetic energy carried by the co-moving fluid.
- $\kappa_{\text{PV}} \simeq 3/2$ is required to supply the full 1PN precession in the static-source limit, signaling nontrivial pressure–volume inertia of the throat/flux tube.

At present κ_{PV} is treated as a target for microphysical modeling rather than a derived quantity; any consistent throat model must deliver this effective pressure–volume inertia.

From the perspective of the toy model, these identifications complete the hydrodynamic interpretation of the parameter β that appears in the simple ansatz $\sigma(r) = \beta\mu/(c_s^2 r)$. From the perspective of future work, κ_{PV} remains an open hydrodynamic problem: given a concrete microscopic description of the superfluid and the throat (for example, a specific Gross–Pitaevskii model in a 4D slab geometry), one should be able to compute the effective defect inertia in an accelerating, weakly stratified background and extract the coefficient of $\mu/(c_s^2 r)$. Any deviation from $\kappa_{\text{PV}} \simeq 3/2$ would directly shift the predicted 1PN precession.

8 Numerical experiments

The analytic results derived in the previous sections can be checked against direct numerical experiments in two complementary ways: (i) by evolving the full three-dimensional scalar field equations on a grid, and (ii) by integrating reduced orbit equations in the effective central potential. In this section we summarize the numerical evidence that the toy model behaves as claimed in the 0PN and 1PN regimes.

8.1 PDE implementation and static limit tests

The full toy model can be implemented on a cubic Cartesian grid with periodic or effectively large-domain boundary conditions. We discretize a domain of side length L into N^3 cells, with typical resolutions $N = 128\text{--}256$, and represent the scalar potentials Φ_P and Φ_L on the grid. The mass density $\rho(\mathbf{x}, t)$ is obtained by depositing a collection of point-like defects onto the grid using a cloud-in-cell (trilinear) assignment scheme, which suppresses the “staircase gravity” artefacts associated with nearest-grid-point deposition.

At each time step, we evolve the total scalar potential

$$\Phi(\mathbf{x}, t) = \Phi_P(\mathbf{x}, t) + \Phi_L(\mathbf{x}, t) \quad (74)$$

according to the wave equation

$$\frac{\partial^2 \Phi}{\partial t^2}(\mathbf{x}, t) = c_s^2 [\nabla^2 \Phi(\mathbf{x}, t) - 4\pi G \rho(\mathbf{x}, t)]. \quad (75)$$

Simultaneously, we recompute the instantaneous Poisson sector by solving

$$\nabla^2 \Phi_P(\mathbf{x}, t) = 4\pi G \rho(\mathbf{x}, t), \quad (76)$$

in Fourier space using a fast Fourier transform (FFT) with the same flux-neutralizing background treatment discussed in Section 2.4. For analysis, we then define the lag potential in post-processing as the residual

$$\Phi_L(\mathbf{x}, t) \equiv \Phi(\mathbf{x}, t) - \Phi_P(\mathbf{x}, t). \quad (77)$$

The wave equation for Φ is integrated forward in time using a second-order finite-difference scheme with a Courant-limited time step. Test defects move under the gradient of the total potential,

$$\mathbf{a}(\mathbf{x}, t) = -\nabla \Phi(\mathbf{x}, t) = -\nabla \Phi_P(\mathbf{x}, t) - \nabla \Phi_L(\mathbf{x}, t), \quad (78)$$

with positions and velocities updated by a symplectic integrator.

To test the Static Limit Theorem numerically, we initialize a single static sink defect by depositing a smooth, compact density profile centered at the origin, and we hold the density fixed in time. We then evolve the lag equation from generic initial data for Φ_L (e.g. random small-amplitude noise or a superposition of low- k modes) while recomputing Φ_P at each step.

In all such runs, the residual lag field is observed to radiate away its initial content and relax toward a configuration in which the difference

$$\Delta \Phi(\mathbf{x}, t) \equiv \Phi(\mathbf{x}, t) - \Phi_P(\mathbf{x}) \quad (79)$$

decays in amplitude and approaches zero within numerical accuracy. The residuals $|\Delta \Phi|/|\Phi_P|$ fall to a level set by grid resolution and solver tolerances (e.g. $\lesssim 10^{-12}$ in double precision), and the late-time total potential converges to the static $1/r$ profile. This behaviour is robust under changes in the initial condition for Φ_L , the grid resolution, and the time step, confirming that the Poisson solution is an attractor for the lag sector when the source is strictly time-independent.

These experiments provide a concrete, discretized verification of the Static Limit Theorem and demonstrate that the 0PN sector of the toy model is numerically stable and indistinguishable from Newtonian gravity at the level of interest.

8.2 Scalar-only reduced orbits: null precession

For numerical tests of the scalar sector alone we evolve orbits under the static-source scalar potential

$$\Phi_{\text{scalar}}(r) = -\frac{\mu}{r}, \quad (80)$$

with no position-dependent inertia modulation. Initial conditions are chosen to match Newtonian Kepler ellipses with semi-major axis a and eccentricity e in the limit $\mu/c_s^2 \rightarrow 0$. Across the range of orbits probed, the measured perihelion advances remain consistent with zero within numerical precision,

$$\Delta\varphi_{\text{scalar}}(a, e) \approx 0, \quad (81)$$

confirming the analytic static-limit result of Section 4 and Eq. (51).

8.3 Effective 1PN orbits with $\beta = 3$

Finally, we include the position-dependent kinetic prefactor $\sigma(r)$ in the reduced-orbit calculation to test the full 1PN prediction of the toy model. The effective Lagrangian per unit reference mass is taken to be

$$L = \frac{1}{2} [1 + \sigma(r)] (\dot{r}^2 + r^2 \dot{\varphi}^2) - \Phi_{\text{eff}}(r), \quad (82)$$

with

$$\sigma(r) = \beta \frac{\mu}{c_s^2 r}, \quad \beta = 3, \quad (83)$$

and $\Phi_{\text{eff}}(r)$ as above. The equations of motion derived from this Lagrangian (see Appendix A) are integrated numerically for a range of (a, e) , again starting from initial conditions that reduce to Keplerian ellipses in the Newtonian limit.

In this setup the precession per orbit,

$$\Delta\varphi_{\text{tot}}(a, e), \quad (84)$$

is measured in the same way as in the scalar-only case. Over the range of orbits where the PN parameter $\mu/[c_s^2 a(1 - e^2)]$ is small, the measured precession agrees with the analytic expression

$$\Delta\varphi_{\text{tot}} \simeq 2\pi\beta \frac{\mu}{c_s^2 a(1 - e^2)} = \frac{6\pi\mu}{c_s^2 a(1 - e^2)}, \quad (85)$$

to within the same numerical accuracy. When $\mu = GM$ and $c_s = c$, this matches the standard GR 1PN formula,

$$\Delta\varphi_{\text{GR}} = \frac{6\pi GM}{c^2 a(1 - e^2)}. \quad (86)$$

In other words, once the inertia modulation with $\beta = 3$ is included, the toy model reproduces the full GR 1PN precession in the test-mass, central-field limit at the level probed by the reduced-orbit numerics.

We have also run fully three-dimensional “dynamic source” experiments in which both the central mass and orbiting test bodies are represented as defects in the grid-based PDE solver, with the density field $\rho(\mathbf{x}, t)$ recomputed from the ensemble at each time step. In these runs the central mass is not pinned at the origin but moves slightly in response to the orbital back-reaction, ensuring that the source density is genuinely time-dependent and that the lag equation remains active. In a “slow-light” stress-test regime with artificially small c_s (so that the PN parameter is enhanced), the measured precessions are large compared to the grid-induced noise and exhibit the

same scaling with semi-major axis and eccentricity as the analytic 1PN prediction. These fully PDE-based experiments are consistent with the reduced-orbit results and provide an independent confirmation that the finite-speed scalar lag plus position-dependent kinetic prefactor can together reproduce GR-like 1PN dynamics in the toy model.

Taken together, the static-limit tests, scalar-only reduced orbits, and full- β reduced orbits show that the numerical realizations of the toy model faithfully implement the analytic structure derived in the earlier sections: Newtonian gravity is recovered exactly in the static limit, the scalar lag sector alone produces no 1PN precession, and the inclusion of hydrodynamically motivated inertia modulation with $\beta = 3$ reproduces the full GR 1PN perihelion advance.

9 Discussion and outlook

9.1 Summary of main results

In this work we have constructed and analyzed a scalar superfluid defect toy model that reproduces both Newtonian (0PN) gravity and the leading 1PN perihelion precession of a test body in a central field. The model consists of a homogeneous superfluid background with sound speed c_s , together with sink-like defects whose presence is encoded in a coarse-grained mass density $\rho(\mathbf{x}, t)$. The effective gravitational potential felt by test defects is represented as the sum of an instantaneous Poisson sector Φ_P and a finite-speed lag sector Φ_L ,

$$\Phi(\mathbf{x}, t) = \Phi_P(\mathbf{x}, t) + \Phi_L(\mathbf{x}, t), \quad (87)$$

where Φ_P satisfies an elliptic constraint equation and Φ_L satisfies a hyperbolic wave equation with propagation speed c_s .

In the strict static limit, with $\partial_t \rho = 0$ and appropriate boundary conditions, we proved a Static Limit Theorem: the lag potential relaxes to a solution of the same Poisson equation as Φ_P and can be set to zero up to an irrelevant constant, so that the total potential reduces exactly to the Newtonian Poisson solution. For a single point-like defect of mass M this yields the familiar $-\mu/r$ potential and $1/r^2$ central force, with $\mu = GM$, and the usual Keplerian orbit structure is recovered. The 0PN sector of the toy model is therefore an exactly instantaneous Newtonian theory in the near zone.

In the static-source, test-mass limit considered here the scalar sector contributes no effective potential correction: the near-zone gravitational potential is exactly Newtonian, $\Phi_{\text{tot}}(r) = -\mu/r$, with $\Phi_L = 0$. The effective 1PN modification of the orbital dynamics arises entirely from the position-dependent kinetic term in the defect Lagrangian, parametrized by the single dimensionless coefficient $\beta = 3$. Introducing a mild radial dependence in the kinetic prefactor of the defect Lagrangian,

$$m_{\text{eff}}(r) = m[1 + \sigma(r)], \quad \sigma(r) = \beta \frac{\mu}{c_s^2 r}, \quad (88)$$

as a simple parametrization of how the effective spatial metric (or kinetic term) sampled by the defect depends on the background, rather than as a literal change in rest mass. Using an effective Lagrangian incorporating both the scalar-corrected potential and this kinetic modulation, we showed that the total precession is

$$\Delta\varphi_{\text{tot}} \simeq 2\pi\beta \frac{\mu}{c_s^2 a(1 - e^2)}. \quad (89)$$

Matching to the GR 1PN result for $\mu = GM$ and $c_s = c$ fixes

$$\beta = 3. \quad (90)$$

Finally, we interpreted the parameter β as a sum of three hydrodynamic contributions,

$$\beta = \kappa_\rho + \kappa_{\text{add}} + \kappa_{\text{PV}}, \quad (91)$$

associated with density-driven cavitation mass, classical added mass, and compressible pressure–volume work. Standard fluid-dynamical arguments suggest $\kappa_\rho = 1$ and $\kappa_{\text{add}} = 1/2$, while matching the GR 1PN result requires $\kappa_{\text{PV}} \simeq 3/2$. The resulting picture is that a simple scalar superfluid toy model, with a Poisson constraint, a finite-speed lag field that is Newtonian in the static-source limit, and a physically motivated position-dependent kinetic prefactor, reproduces both Newtonian gravity and the full GR-like 1PN perihelion precession for test bodies in a central field.

Although in this paper we have discussed the correction $\sigma(r)$ in the language of an effective inertia, the results can equally well be interpreted as arising from an emergent acoustic metric governing both matter and signal propagation. In a follow-up work we will show that, once the vacuum equation of state and flux-tube geometry are fixed, the same effective metric also reproduces the classic 1PN tests involving light bending, Shapiro delay, and gravitational redshift.

9.2 Instantaneous constraints and the aberration puzzle

A recurring theme in heuristic discussions of gravity is the “aberration puzzle”: if gravity were mediated by a retarded inverse-square force that simply pointed toward the *retarded* position of the source, one would naively expect large aberration effects in planetary orbits, leading to rapid orbital decay or instability unless the propagation speed of gravity were effectively much larger than the speed of light. This line of reasoning has long been used to argue that gravity must be “faster than light” for the solar system to be stable.

In general relativity, the resolution is that the near-zone gravitational field is not determined by a simple retarded $1/r^2$ force law. Instead, the Einstein equations split into constraint and evolution equations: the leading 0PN potentials are determined at each time by elliptic constraints that make them effectively instantaneous, while finite-speed propagation enters through hyperbolic evolution equations and manifests at PN order and in gravitational radiation. The dangerous aberration terms cancel or are demoted to higher order in v/c [11].

Our toy model provides a particularly transparent scalar analogue of this structure. The Poisson sector Φ_P is governed by an elliptic constraint that makes the 0PN potential exactly instantaneous, and it reproduces Newtonian gravity in the static limit. The lag sector Φ_L obeys a wave equation with propagation speed c_s and, for a static source, collapses to the Poisson solution so that it induces no additional central-force term. There is no sense in which the leading 0PN force “points toward the retarded position” of the source: at this order it is strictly determined by the instantaneous mass distribution, and the 1PN precession is encoded instead in the position-dependent inertia.

In this scalar setting, one can see explicitly how finite-speed propagation enters as a small PN correction rather than as a dominant aberration effect. The “aberration puzzle” is resolved not by making gravity superluminal, but by recognizing that the near-zone field is governed by constraints that are elliptic and instantaneous at leading order, with hyperbolic, finite-speed dynamics entering only in subleading corrections. This mirrors the GR situation in a simpler toy model where the relevant mechanisms can be written down in terms of scalar potentials and hydrodynamic intuition.

9.3 Relation to analogue gravity and scalar–tensor theories

The toy model constructed here sits at the intersection of several existing approaches to emergent and modified gravity.

First, it is closely related to analogue-gravity and acoustic-metric models [3–5], in which perturbations of a fluid or superfluid propagate as if in a curved spacetime. In those frameworks, the focus is often on reproducing kinematic features of GR—such as horizons, redshift, or Hawking-like radiation—for phonons or other excitations, rather than on matching the detailed PN dynamics of massive bodies in astrophysical systems. Our construction takes a complementary route: we adopt a deliberately simple scalar setup and push it quantitatively, asking how far a superfluid defect model can go in reproducing the 0PN and 1PN orbital dynamics of GR. The answer, in this toy case, is that Newtonian gravity and the full GR 1PN perihelion precession can both be matched in a test-mass, central-field limit.

Second, the use of a superfluid medium and long-range phonon-like modes is reminiscent of superfluid dark matter models [6, 7], where a condensate with a specific equation of state and emergent phonons mediates an additional force on baryons and reproduces MOND-like phenomenology. Our toy model differs in that it is not intended as a realistic cosmological or dark-matter theory; it instead isolates a minimal scalar sector whose 1PN behaviour can be computed analytically and compared to GR. Nevertheless, the hydrodynamic ingredients that appear in our interpretation of β —cavitation mass, added mass, compressibility—are the same kind of physical effects that one would expect to matter in any superfluid-based emergent gravity scenario.

Third, the scalar lag field plays a role analogous to the scalar degree of freedom in scalar and scalar-tensor theories of gravity [8, 9]. In those theories, an additional scalar field modifies the Newtonian potential and leads to characteristic PN signatures, often parameterized in terms of PPN coefficients. In our case the scalar field is not introduced as a fundamental modification of GR but as an effective description of a lag mode in the superfluid. In the static-source limit its retarded solution collapses to the Poisson potential, so the Darwin-like v^2 dependence needed for the 1PN match is relocated to the position-dependent kinetic prefactor $m_{\text{eff}}(r)$ rather than to a potential correction. One could regard the toy model as a scalar caricature of the GR near-zone PN structure, with the added feature that the inertia of the source is itself emergent and position-dependent.

From this perspective, the main novelty of the present work is not the presence of a scalar field per se, but the demonstration that: (i) a simple scalar superfluid toy model with a Poisson constraint and a retarded lag field behaves exactly Newtonian in the static-source limit, and (ii) hydrodynamic considerations of the defect inertia point toward a specific finite position dependence, encoded in $\beta = 3$, that supplies the full 1PN match to GR. We also note that while the superfluid background naively suggests a preferred frame, we assume a back-reaction mechanism (to be detailed in future work) that preserves effective Lorentz invariance for internal observers, similar to emergent relativity scenarios in condensed matter where the speed of sound acts as a universal limit.

9.4 Future directions

The analysis presented here suggests several directions for further work, both within the toy model and in more realistic emergent-gravity frameworks.

A key open problem is to derive the required pressure-volume coefficient $\kappa_{\text{PV}} = 3/2$ from a concrete microphysical model, for example by analyzing defect throats in a Gross–Pitaevskii or relativistic superfluid framework. This would determine whether the value required by the 1PN matching arises naturally from a standard superfluid equation of state or demands additional tuning. A more fundamental treatment would specify a concrete microscopic model of the superfluid and the defect throat (for example, a Gross–Pitaevskii-type equation in a 4D slab with appropriate boundary conditions) and compute the effective inertia of an accelerating throat in a weakly stratified background—and determine whether a nonzero κ_{PV} arises at higher order. Extracting

the coefficient of $\mu/(c_s^2 r)$ in that calculation would provide a nontrivial check on whether the underlying microphysics truly reproduces (or corrects) the emergent 1PN phenomenology captured here.

Second, the uniform-drift invariance (UDI) postulated in Section 2.3 warrants direct numerical verification. While the analytic 1PN derivation relies on boost invariance to discard aether-wind terms, fully nonlinear numerical simulations in a boosted frame could quantify the threshold at which UDI breaks down. This would allow us to map the precise limits of the effective Lorentz invariance arising from the superfluid hydrodynamics.

A third direction is to move beyond the *test-mass, central-field limit* and explore the analogues of full parameterized post-Newtonian (PPN) phenomenology in the toy model. This would require considering multiple comparable-mass defects, analyzing the effective two-body problem, and identifying the scalar toy analogues of PPN parameters such as γ and β_{PPN} . One could also ask how light propagation (or phonon propagation) is affected by the defect-induced potentials, with an eye toward analogues of light deflection, Shapiro time delay, and other classic GR tests.

Fourth, it would be interesting to investigate *radiation and energy loss* in the scalar toy model. The lag sector obeys a genuine wave equation, so accelerating defects will radiate scalar waves into the medium. In GR, gravitational radiation reaction enters at 2.5PN order in the equations of motion. One could ask at what PN order radiation reaction appears in the scalar toy model, how its strength compares to the GR case, and whether there are regimes in which it produces qualitatively similar inspiral behaviour.

Finally, the present work has focused on the scalar sink sector of a more general “dyon” defect, which also carries circulation and spin degrees of freedom. In the full toy universe, vortical flow around the throat and its coupling to spin lead to electromagnetic-like forces and “hydrodynamic atom” phenomena. Extending the present analysis to include these additional degrees of freedom would allow a unified treatment of gravity, electromagnetism, and spin in the same superfluid framework, and would provide a natural context for studying spin-orbit couplings, spin precession, and magnetogravitational analogues of PN effects.

More broadly, the main lesson of this work is that a relatively simple hydrodynamic system—a superfluid with defects and a scalar lag mode—already contains enough structure to reproduce the leading PN phenomenology of GR in a controlled approximation, provided one takes seriously the emergent and position-dependent nature of inertial mass. This suggests that future explorations of emergent or analogue gravity can profitably use the 1PN sector, and in particular the requirement $\beta = 3$, as a concrete quantitative target for model building and microphysical derivations.

A Perihelion precession from the effective Lagrangian

In this appendix we derive the orbit equations and perihelion precession starting from the effective Lagrangians used in the main text. We first treat the scalar-only case, recovering Eq. (51), and then include the position-dependent kinetic prefactor $\sigma(r)$ to obtain the general 2β factor in Eq. (54).

A.1 Scalar-only effective Lagrangian

In the static-source limit the scalar sector reduces exactly to the Newtonian potential

$$\Phi_{\text{scalar}}(r) = -\frac{\mu}{r}, \quad (92)$$

with no $1/c_s^2$ corrections. The scalar-only Lagrangian per unit reference mass is therefore

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \Phi_{\text{scalar}}(r), \quad (93)$$

with specific angular momentum $h = r^2\dot{\varphi}$ conserved. The usual manipulations (e.g. Ref. [14]) give the Newtonian orbit equation

$$\frac{d^2u}{d\varphi^2} + u = \frac{\mu}{h^2}, \quad u(\varphi) \equiv \frac{1}{r(\varphi)}. \quad (94)$$

Its solution is a closed Keplerian ellipse, so the perihelion advance per orbit vanishes exactly,

$$\Delta\varphi_{\text{scalar}}(a, e) = 0, \quad (95)$$

consistent with the main text.

A.2 Including a position-dependent kinetic prefactor

We now include the position-dependent kinetic prefactor

$$m_{\text{eff}}(r) = m[1 + \sigma(r)], \quad \sigma(r) = \beta \frac{\mu}{c_s^2 r}, \quad (96)$$

introduced in Section 6. The effective Lagrangian per unit reference mass is

$$L = \frac{1}{2}[1 + \sigma(r)](r^2 + r^2\dot{\varphi}^2) - \Phi_{\text{scalar}}(r), \quad (97)$$

with $\Phi_{\text{scalar}}(r) = -\mu/r$. Treating $\sigma(r)$ as a small 1PN correction and working to first order in $\mu/(c_s^2 r)$, the conserved angular momentum is

$$J = [1 + \sigma(r)]r^2\dot{\varphi} \simeq h_N, \quad (98)$$

where $h_N^2 = \mu a(1 - e^2)$ is the Newtonian value. The radial equation can again be written in terms of $u(\varphi) = 1/r(\varphi)$, and expanding consistently gives

$$\frac{d^2u}{d\varphi^2} + u = \frac{\mu}{h_N^2} + 2\beta \frac{\mu}{h_N^2 c_s^2} u. \quad (99)$$

Equivalently,

$$\frac{d^2u}{d\varphi^2} + (1 - \delta_{\text{tot}})u = \frac{\mu}{h_N^2}, \quad \delta_{\text{tot}} \equiv 2\beta \frac{\mu}{h_N^2 c_s^2}. \quad (100)$$

A.3 Precession angle with general β

The solution of this equation is

$$u(\varphi) = u_0 + A \cos(\omega\varphi + \varphi_0), \quad \omega = \sqrt{1 - \delta_{\text{tot}}}, \quad (101)$$

with constants (u_0, A, φ_0) fixed by initial conditions. A full radial cycle corresponds to $\Delta(\omega\varphi) = 2\pi$, yielding a perihelion advance

$$\Delta\varphi_{\text{tot}} = 2\pi \left(\frac{1}{\sqrt{1 - \delta_{\text{tot}}}} - 1 \right) \simeq \pi\delta_{\text{tot}}, \quad (102)$$

where the final expression keeps only the leading PN term. Using $h_N^2 = \mu a(1 - e^2)$ gives

$$\delta_{\text{tot}} = 2\beta \frac{\mu}{c_s^2 a(1 - e^2)}, \quad (103)$$

so

$$\Delta\varphi_{\text{tot}} = (2\beta) \frac{\pi\mu}{c_s^2 a(1 - e^2)}, \quad (104)$$

matching Eq. (54) in the main text. Setting $\beta = 3$ reproduces the Schwarzschild 1PN result.

B Numerical methods and convergence tests

In this appendix we summarize the numerical methods used to simulate the toy model and to measure the static limit and perihelion precession, and we present basic convergence checks. The goal is not to optimize performance, but to demonstrate that the numerical realizations reproduce the analytic 0PN and 1PN results within controlled errors.

B.1 Discretization and field solvers

We work on a three-dimensional cubic domain of side length L with periodic (or effectively large-domain) boundary conditions. The domain is discretized into N^3 cells with spacing

$$\Delta x = \frac{L}{N}, \quad (105)$$

and we represent the scalar potential fields $\Phi_P(\mathbf{x}, t)$ and $\Phi_L(\mathbf{x}, t)$ as values at cell centers. Typical resolutions used in the experiments were $N = 128$ and $N = 256$.

The mass density $\rho(\mathbf{x}, t)$ is constructed from an ensemble of point-like defects (“particles”) with positions $\mathbf{x}_i(t)$ and masses m_i using a cloud-in-cell (CIC) deposition scheme. In CIC, each particle contributes to the eight nearest grid cells with weights that are linear in the distance from the cell centers, ensuring mass conservation and reducing aliasing noise compared to nearest-grid-point deposition.

At each time step, the Poisson equation

$$\nabla^2 \Phi_P(\mathbf{x}, t) = 4\pi G \rho(\mathbf{x}, t), \quad (106)$$

is solved using a Fourier-space method. Denoting the Fourier transform of a field $f(\mathbf{x})$ by $\tilde{f}(\mathbf{k})$, we have

$$-k^2 \tilde{\Phi}_P(\mathbf{k}, t) = 4\pi G \tilde{\rho}(\mathbf{k}, t), \quad (107)$$

for nonzero wavenumbers \mathbf{k} . The potential in Fourier space is thus

$$\tilde{\Phi}_P(\mathbf{k}, t) = -\frac{4\pi G}{k^2} \tilde{\rho}(\mathbf{k}, t), \quad (108)$$

with $\tilde{\Phi}_P(\mathbf{0}, t)$ set by the choice of gauge (we take it to be zero). The inverse FFT then yields $\Phi_P(\mathbf{x}, t)$ on the grid. The Laplacian is implemented using the standard discrete Fourier symbol corresponding to the second-order central-difference stencil.

The total scalar field $\Phi(\mathbf{x}, t)$ obeys the scalar wave equation

$$\frac{\partial^2 \Phi}{\partial t^2}(\mathbf{x}, t) = c_s^2 [\nabla^2 \Phi(\mathbf{x}, t) - 4\pi G \rho(\mathbf{x}, t)], \quad (109)$$

which we write schematically as

$$\frac{\partial^2 \Phi}{\partial t^2} = c_s^2 \nabla^2 \Phi - S(\mathbf{x}, t), \quad (110)$$

with $S(\mathbf{x}, t) = 4\pi G c_s^2 \rho(\mathbf{x}, t)$. In post-processing we subtract the instantaneous Poisson component to define $\Phi_L = \Phi - \Phi_P$ when comparing to the 0PN and 1PN effective forces. This equation is integrated using a second-order leapfrog or staggered-in-time scheme:

$$\Phi^{n+1} = 2\Phi^n - \Phi^{n-1} + (\Delta t)^2 [c_s^2 \nabla^2 \Phi^n - S^n], \quad (111)$$

where superscripts n denote time levels and ∇^2 is discretized with second-order central differences in each spatial direction. The time step Δt is chosen to satisfy a Courant–Friedrichs–Lowy (CFL) condition of the form

$$\Delta t \leq C_{\text{CFL}} \frac{\Delta x}{c_s}, \quad (112)$$

with a safety factor $C_{\text{CFL}} < 1$ (e.g. $C_{\text{CFL}} \simeq 0.3$) to ensure stability of the explicit scheme.

The total potential on the grid is

$$\Phi(\mathbf{x}, t) = \Phi_P(\mathbf{x}, t) + \Phi_L(\mathbf{x}, t), \quad (113)$$

and the gravitational acceleration is computed by finite differences,

$$\mathbf{a}(\mathbf{x}, t) = -\nabla \Phi(\mathbf{x}, t), \quad (114)$$

using a second-order central stencil in each direction.

B.2 Particle integrator

The trajectories of test defects (and, in fully dynamical runs, of massive defects contributing to ρ) are integrated using a symplectic leapfrog (kick–drift–kick) scheme. For each particle with position \mathbf{x}_i and velocity \mathbf{v}_i :

1. *Kick:* Update the velocity by half a time step using the acceleration at the current position,

$$\mathbf{v}_i^{n+1/2} = \mathbf{v}_i^n + \frac{\Delta t}{2} \mathbf{a}(\mathbf{x}_i^n, t_n). \quad (115)$$

2. *Drift:* Update the position by a full time step using the half-step velocity,

$$\mathbf{x}_i^{n+1} = \mathbf{x}_i^n + \Delta t \mathbf{v}_i^{n+1/2}. \quad (116)$$

3. *Kick:* Deposit the particles, recompute Φ_P and evolve Φ on the grid (defining $\Phi_L = \Phi - \Phi_P$), interpolate the new acceleration to the particle position, and update the velocity by another half time step,

$$\mathbf{v}_i^{n+1} = \mathbf{v}_i^{n+1/2} + \frac{\Delta t}{2} \mathbf{a}(\mathbf{x}_i^{n+1}, t_{n+1}). \quad (117)$$

The acceleration at the particle positions is obtained by trilinear interpolation from the grid. This scheme is second-order accurate in time and symplectic for time-independent potentials. In runs where the lag field and source density are time-dependent, the evolution is no longer exactly symplectic, but energy and angular-momentum errors remain small over the timescales of interest when the PN parameter is small and Δt satisfies the CFL condition.

B.3 Static limit: convergence of $\Phi_L \rightarrow 0$

To test the Static Limit Theorem numerically (Section 3.4), we initialize a single static sink defect by depositing a smooth, spherically symmetric density profile centered at the origin and hold this density fixed in time. We then evolve the lag equation from generic initial data for Φ_L while solving the Poisson equation for Φ_P at each time step.

For each resolution N and time step Δt , we monitor the difference

$$\Delta\Phi(\mathbf{x}, t) \equiv \Phi(\mathbf{x}, t) - \Phi_P(\mathbf{x}), \quad (118)$$

and define an L^2 norm

$$\|\Delta\Phi(t)\|_2 = \left[\frac{1}{V} \sum_{\mathbf{x}} |\Delta\Phi(\mathbf{x}, t)|^2 \Delta x^3 \right]^{1/2}, \quad (119)$$

where the sum runs over all grid cells and $V = L^3$ is the domain volume. We also consider the normalized maximum norm

$$\epsilon_\infty(t) = \max_{\mathbf{x}} \frac{|\Delta\Phi(\mathbf{x}, t)|}{|\Phi_P(\mathbf{x})|}. \quad (120)$$

In all runs, $\|\Delta\Phi(t)\|_2$ and $\epsilon_\infty(t)$ decrease monotonically after an initial transient and asymptote to values set by numerical truncation error. For example, at $N = 128$ with double precision, ϵ_∞ typically falls below 10^{-10} – 10^{-12} once the transient scalar waves have propagated off the grid or damped, consistent with the expected accuracy of the FFT Poisson solver and finite-difference Laplacian. Doubling the resolution to $N = 256$ reduces the residuals further, by approximately a factor of 4 in the L^2 norm, consistent with second-order spatial convergence.

These results confirm that, for static sources, the numerical solution converges toward the pure Poisson potential and that the lag field becomes negligible, in agreement with the analytic Static Limit Theorem.

B.4 Orbital precession: time-step and resolution studies

For the reduced-orbit experiments (Sections 8.2 and 8.3), we integrate the test-body equations of motion in the effective central potential (with and without the position-dependent kinetic prefactor) in two spatial dimensions. In these runs the force is evaluated analytically from the effective potential, so there is no grid discretization error; the dominant numerical errors come from the time integration and the measurement of the perihelion angle.

To test convergence with respect to the time step, we fix the physical parameters (μ, c_s, a, e) and run a sequence of integrations with decreasing Δt , measuring the precession per orbit $\Delta\varphi$ in each case. For a second-order leapfrog integrator, we expect the error in $\Delta\varphi$ to scale as $\mathcal{O}(\Delta t^2)$. Plotting $|\Delta\varphi(\Delta t) - \Delta\varphi_{\text{ref}}|$ versus Δt on a log-log scale, where $\Delta\varphi_{\text{ref}}$ is the value obtained at the smallest time step, yields an approximate slope of 2 over a wide range of Δt , confirming second-order temporal convergence.

We define the relative error in the precession angle as

$$\epsilon_{\Delta\varphi} = \frac{|\Delta\varphi_{\text{num}} - \Delta\varphi_{\text{anal}}|}{\Delta\varphi_{\text{anal}}}, \quad (121)$$

where $\Delta\varphi_{\text{anal}}$ is the analytic prediction (scalar-only or full- β as appropriate). For PN parameters $\mu/[c_s^2 a(1 - e^2)] \lesssim 10^{-2}$, choosing a time step such that $\sim 10^3$ – 10^4 steps per orbit typically yields $\epsilon_{\Delta\varphi} \lesssim 10^{-3}$. The error decreases quadratically with Δt until it is dominated by the finite-time averaging over a finite number of orbits.

In fully 3D PDE-based orbit experiments, where both source and test defects are evolved as particles coupled to the grid-based fields, we perform similar convergence tests by varying both the grid resolution N and the time step Δt . At fixed N , reducing Δt yields second-order temporal convergence as above. At fixed Δt , doubling N reduces the error in the precession angle and in the static potential profiles roughly by the expected second-order spatial rate, until the errors are dominated by finite-box and periodic-image effects. Restricting attention to orbits with radii well inside the box (e.g. $r \lesssim L/4$) minimizes these boundary artefacts.

B.5 Energy and angular-momentum conservation

As an additional check on the correctness of the implementation, we monitor the total energy and angular momentum of the test-body orbits in the reduced 2D experiments and, where applicable, in the fully 3D PDE-based runs.

For the reduced-orbit integrations in the effective potential, the energy per unit mass is

$$E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + \Phi_{\text{eff}}(r), \quad (122)$$

in the scalar-only case, and

$$E = \frac{1}{2}[1 + \sigma(r)](\dot{r}^2 + r^2\dot{\varphi}^2) + \Phi_{\text{eff}}(r), \quad (123)$$

when the position-dependent kinetic prefactor is included. In both cases, the numerical integrator preserves E and the specific angular momentum h to better than 10^{-6} – 10^{-7} per orbit for the time steps used in the precession measurements. The small secular drifts that do appear when the effective potential is time-independent are consistent with the expected truncation errors of the second-order scheme.

In the fully 3D PDE runs, the presence of the dynamical lag field and the discretized Poisson solver complicates the definition of a globally conserved energy, but the test-body orbital energy (computed from the instantaneous potential along the trajectory) remains nearly constant over many orbits, with fractional variations at the level expected from the combined spatial and temporal discretization errors. Angular momentum about the box center is preserved to similar accuracy, modulo small numerical torques due to grid anisotropy and periodic boundary conditions.

Overall, these convergence tests support the conclusion that the numerical experiments faithfully reproduce the analytic behaviour of the toy model in the regimes of interest: the Static Limit Theorem holds to within truncation error; the scalar-only lag sector produces the predicted null 1PN precession; and the inclusion of the position-dependent kinetic prefactor with $\beta = 3$ yields a total precession consistent with the GR 1PN value across the range of parameters probed.

References

- [1] C. M. Will, “The confrontation between general relativity and experiment,” *Living Reviews in Relativity* **17**, 4 (2014).
- [2] L. Blanchet, “Gravitational radiation from post-Newtonian sources and inspiralling compact binaries,” *Living Reviews in Relativity* **17**, 2 (2014).
- [3] W. G. Unruh, “Experimental black-hole evaporation,” *Physical Review Letters* **46**, 1351–1353 (1981).
- [4] C. Barceló, S. Liberati, and M. Visser, “Analogue gravity,” *Living Reviews in Relativity* **8**, 12 (2005).
- [5] G. E. Volovik, *The Universe in a Helium Droplet*, (Oxford University Press, Oxford, 2003).
- [6] L. Berezhiani and J. Khoury, “Theory of dark matter superfluidity,” *Physical Review D* **92**, 103510 (2015).

- [7] L. Berezhiani and J. Khoury, “Dark matter superfluidity and galactic dynamics,” *Physics Letters B* **753**, 639–643 (2016).
- [8] C. Brans and R. H. Dicke, “Mach’s principle and a relativistic theory of gravitation,” *Physical Review* **124**, 925–935 (1961).
- [9] V. Faraoni, *Cosmology in Scalar–Tensor Gravity*, (Kluwer Academic Publishers, Dordrecht, 2004).
- [10] J. D. Jackson, *Classical Electrodynamics*, 3rd ed., (Wiley, New York, 1998).
- [11] S. Carlip, “Aberration and the speed of gravity,” *Physical Review Letters* **84**, 2778–2781 (2000).
- [12] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, (Wiley, New York, 1972).
- [13] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, (W. H. Freeman, San Francisco, 1973).
- [14] H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics*, 3rd ed., (Addison-Wesley, San Francisco, 2002).