Brane–Bulk Throat Ontology for a Superfluid Defect Toy Universe

Trevor Norris

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Abstract

This paper, the fifth in a series developing a "superfluid defect toy universe," resolves the geometric tension between the model's gravitational and electromagnetic sectors. Previous work established that reproducing 1PN gravity requires defects to behave as spherical sinks, whereas the electromagnetic sector requires cylindrical resonant cavities. We resolve this apparent contradiction by promoting defects to brane–bulk throats connecting the observable 3D brane to a 4D superfluid bulk. Dimensional reduction reveals that the far–field potential on the brane is monopolar, with angular corrections from the throat geometry suppressed by $(a/r)^2$. Numerical integration of a representative hard–bounded "rounded funnel" throat geometry confirms this suppression, yielding a leading quadrupole coefficient $\alpha_2 \approx 1.4 \times 10^{-2}$ (for $L/a = 2$ and a 10% quadrupolar anisotropy), thereby robustly recovering the spherical sink approximation for 1PN gravity. Inside the throat, however, 4D acoustic modes separate into cylindrical Bessel profiles; enthalpy minimization of these modes at fixed charge selects the preferred aspect ratio $L/a \approx 1.85$ found in the electromagnetic construction. We further show that long–range magnetostatics can arise without bulk vorticity by identifying the magnetic field with the vorticity of a brane–confined transverse wake sourced by the motion of a charged throat. Finally, we incorporate the corrected wake–mixing constraint $\alpha^2 = 3/4$ from Paper III as a real, positive–definite longitudinal/transverse weighting in the vector sector once the wake basis is completed. This ontology constrains finite–size effects, laying the groundwork for falsifiable 2PN predictions.

1 Introduction

1.1 Motivation and the sphere–cylinder tension

This paper is the fifth in a series developing a "superfluid defect toy universe" in which gravity and electromagnetism emerge from the dynamics of a compressible fluid.[1] In this model the vacuum is treated as a stiff superfluid, and massive bodies are modeled as localized defects that drain and stir this medium. Far from any defect, the flow is slow and approximately irrotational; near the defect, nonlinear and topological effects become important. The central

[1]For clarity, we will refer to the previous papers as Papers I–IV[1, 2, 3, 4] throughout.

claim of the series so far has been that, with a suitable choice of hydrodynamic energy functional, this superfluid picture reproduces the standard post–Newtonian (PN) expansion of general relativity (GR) at first PN order (1PN) for isolated bodies and N–body systems, while also supporting an emergent electromagnetism sector.

Paper I (the orbital paper) showed that if a defect behaves as a spherical sink in a three–dimensional superfluid, then the resulting potential flow around the defect reproduces Newtonian gravity and the 1PN perihelion precession of GR. In that construction the background flow is spherically symmetric and purely radial on spatial slices, and the defect induces both a Newtonian $1/r$ potential and a set of scalar inertia renormalizations that sum to the calibrated orbital parameter β. In the notation of Paper I these contributions are: density depletion $\kappa_\rho = 1$ (fixing the static G_2 "gravity gravitates" interaction), pressure–volume work $\kappa_{\mathrm{PV}} = 3/2$ (the energy cost of displacing fluid against the ambient pressure), and added mass $\kappa_{\mathrm{add}} = 1/2$ (the hydrodynamic inertia of entrained fluid). Together,

$$\beta = \kappa_\rho + \kappa_{\mathrm{PV}} + \kappa_{\mathrm{add}} = 3, \tag{1}$$

matching the GR prediction for perihelion advance. Crucially, the calculation assumes that the defect is effectively spherical: when one tries to repeat the same analysis with a cylindrical defect in three dimensions, the resulting PN coefficients become orientation–dependent and do not reproduce GR in a rotationally invariant way.

Paper II (the optics paper) recast the same superfluid model in terms of an effective optical metric. Light propagation in the gravitational field of a single defect was described in terms of refraction in a stiff polytropic superfluid, and it was shown that matching the GR lensing coefficient fixes a specific polytropic index and curvature coefficient in the optical sector. Again, the gravitational field is sourced by a spherically symmetric sink: from the point of view of the optical metric, the defect is a point–like, effectively spherical object.

Paper III (the spin and N–body paper) extended the model to include spin–induced frame dragging and general N–body interactions. There the hydrodynamic wake functional was generalized to include both longitudinal and transverse projector components, and the resulting 1PN dynamics were matched to the Einstein–Infeld–Hoffmann (EIH) Lagrangian. The matching uniquely constrained the relative weighting of the longitudinal and transverse pieces, selecting the real wake–mixing ratio

$$\alpha^2 = \frac{3}{4}. \tag{2}$$

With the completed wake basis this value is compatible with an everywhere positive–definite quadratic functional; the GR vector coefficients arise from the projector structure of the interaction kernel rather than from an indefinite ("Lorentzian") metric on mode space.

Importantly for the present work, the spin and N–body analysis still assumes a spherical defect: attempts to model the defect as a 3D cylinder again fail at 1PN, with significant orientation–dependent errors in the predicted precession and spin couplings.

Paper IV (the EM paper) shifted focus to the electromagnetic sector. There the defect was modeled as a resonant cavity in the superfluid, and a hydrodynamic–electromagnetic dictionary was constructed that maps certain mode structures of the fluid onto electric and magnetic fields. Stabilizing a charged defect required a cylindrical cavity of radius a and length L, supporting a fundamental mode with radial profile $J_0(x_{01}r/a)$ and a standing wave along the axial direction. Minimizing the enthalpy at fixed "charge" picked out a preferred aspect ratio,

$$\frac{L}{a} = \frac{\sqrt{2}\,\pi}{x_{01}} \approx 1.85, \tag{3}$$

where x_{01} is the first zero of J_0. In this construction, charge is associated with circulation and vorticity in and around a cylindrical defect, and the cylindrical geometry is not an optional detail: it is needed to obtain the correct mode structure and a stable charged configuration.

Taken at face value, these results appear to ascribe contradictory geometries to the same object. The gravitational papers (I–III) insist that the defect must behave as a spherical sink in three dimensions in order to reproduce 1PN gravity. The electromagnetic paper (IV) insists that the defect must behave as a cylindrical resonant cavity in order to reproduce electromagnetism. One cannot simply declare the defect to be "both a sphere and a cylinder" within a purely three–dimensional picture without spoiling either the gravitational or the electromagnetic sector. This sphere–cylinder tension is an internal inconsistency in the toy model, and it forces us to look for a deeper geometric ontology in which both descriptions can be simultaneously true.

1.2 Brane–bulk throat ontology as a resolution

The central proposal of this paper is that the apparent contradiction is an artifact of insisting on a purely 3D description of an intrinsically higher–dimensional defect. We will argue that the defect should instead be understood as a *throat* connecting a three–dimensional brane to a four–dimensional superfluid bulk.

Concretely, we introduce coordinates $(t, \mathbf{x}, w)$, where $\mathbf{x} = (x, y, z)$ label directions along the brane and w is a bulk coordinate transverse to the brane. The physical world accessible to ordinary observers is modeled as a hypersurface at $w = 0$, which we will refer to as the brane. The superfluid fills the full four–dimensional space, and its density and velocity fields extend smoothly into the bulk. A defect is then represented not as a compact three–dimensional object, but as a localized region where the brane pinches into the bulk, forming a throat of radius a and depth L along w.

From the point of view of an observer confined to the brane, the intersection of the throat with $w = 0$ is a nearly spherical opening of radius a. The effective flow induced on the brane by drainage into the throat is then approximately that of a spherical sink located at the center of this opening. At distances $r \gg a$ along the brane, the flow is radially symmetric and the corresponding effective gravitational potential is indistinguishable from that of a point mass. In this sense, the 1PN gravitational sector of Papers I–III only probes the *mouth* of the throat, and it is naturally described by a spherical sink in an effective 3D fluid.

From the point of view of modes living *inside* the throat, however, the relevant geometry is entirely different. Away from the mouth the throat is approximately a cylinder of radius a and length L embedded in the bulk. Acoustic modes confined within this cylindrical region obey a 4D wave equation whose radial and axial dependence factorize. Imposing appropriate boundary conditions at $r = a$ and at the ends $w = 0$ and $w = L$ selects Bessel–type radial profiles and sine– or cosine–type axial profiles, precisely of the form used in Paper IV.

Enthalpy minimization at fixed "charge" then selects a preferred aspect ratio $L/a \approx 1.85$ for the throat. In this picture, the electromagnetic sector is not telling us that the whole defect is a 3D cylinder; it is telling us that the *interior of the throat in the bulk* behaves as an approximately cylindrical cavity.

The apparent sphere–cylinder contradiction is thus resolved once we distinguish between the brane projection and the bulk interior. Gravity is governed by the effective three–dimensional projection of the bulk flow onto the brane and is therefore sensitive primarily to the spherical mouth of the throat. Electromagnetism is governed by cavity modes that live inside the throat and therefore probe the cylindrical interior. The two descriptions are compatible because they refer to different slices of the same higher–dimensional geometry.

In the same spirit, we will argue that the wake–mixing ratio $\alpha^2 = 3/4$ fixed in Paper III admits a natural geometric reading in the throat ontology: motion of a brane–anchored defect excites both transverse (brane–parallel) and longitudinal (bulk–directed) wake response, and the EIH matching selects a specific real weighting between these components. With the completed wake basis the underlying quadratic functional is positive–definite; the GR sign structure arises in the effective mapping of hydrodynamic potentials to 1PN observables rather than from an indefinite "Lorentzian" metric on wake space.

1.3 Scope and goals of this paper

The goal of this paper is to make the brane–bulk throat ontology precise enough that the previous results of Papers I–IV can be seen as different projections of a single geometric picture, and to lay the groundwork for a systematic analysis of higher–order PN corrections.

On the constructive side, we will:

- Formulate a four–dimensional superfluid framework with a brane at $w = 0$ and defects represented as throats of radius a and depth L.
- Show how dimensional reduction from 4D to the brane organizes the far–field effective theory in terms of monopole and higher multipole moments, with angular corrections suppressed by $(a/r)^2$.
- Demonstrate that, at 1PN order, the far–field behavior of a throat on the brane is indistinguishable from that of the spherical sinks used in Papers I–III, explaining why those papers were successful despite their purely 3D language.
- Reinterpret the electromagnetic cavity of Paper IV as the interior of the throat, and the enthalpy–selected aspect ratio $L/a \approx 1.85$ as a geometric property of the throat in the bulk rather than an arbitrary 3D cylinder.
- Provide a simple two–mode toy model illustrating how the wake–mixing ratio $\alpha^2 = 3/4$ can arise as a real longitudinal/transverse weighting in a positive–definite quadratic form (and how earlier negative values reflected an incomplete wake basis).

On the interpretive side, we will:

- Argue that finite–size effects associated with the throat, including the detailed structure of the transition region near the mouth, naturally appear as $(a/r)^2$ corrections, i.e. as 2PN and higher–order terms.
- Outline how a future 2PN calculation could be organized around the throat geometry, turning the model into a falsifiable framework whose predictions can be confronted with precision tests of gravity and electromagnetism.
- Clarify in what sense the present construction resembles brane–world and superfluid–vacuum scenarios in the broader literature, and in what ways it differs.

What this paper does *not* do is equally important. We will not attempt a full 2PN derivation of the effective gravitational Lagrangian or the complete electromagnetic sector. We will not specify the microscopic origin or stabilization mechanism of the brane itself, nor will we attempt to simulate strong–field or highly dynamical processes such as mergers or collapse. Our aim is more modest: to provide a coherent geometric ontology that removes the internal tensions of the existing toy model and strongly constrains the form of any future higher–order calculations.

1.4 Structure of the paper

The remainder of the paper is organized as follows. In Sec. 2 we introduce the four–dimensional superfluid framework and the brane embedding. We define the bulk density, velocity, and enthalpy fields, specify the equation of state, and write down the 4D continuity and Euler equations. We then describe the brane as a hypersurface at $w = 0$, discuss the induced three–dimensional fields on the brane, and introduce a simple geometric model of a throat of radius a and depth L connecting the brane to the bulk.

In Sec. 3 we turn to dimensional reduction and the far–field structure. We define effective 3D sources obtained by integrating over the bulk coordinate w, and we study a Gaussian toy model for a localized throat. This model allows us to compute the effective monopole mass and quadrupole moment, and to show explicitly that angular corrections to the potential are suppressed by $(a/r)^2$. We explain why the 1PN results of Papers I–III are insensitive to these finite–size effects and therefore naturally see only a spherical source.

Sec. 4 focuses on the interior of the throat and the electromagnetic sector. Starting from the 4D acoustic wave equation inside the throat, we perform a separation–of–variables analysis to recover the cylindrical cavity modes of Paper IV, including the fundamental mode with radial profile $J_0(x_{01}r/a)$ and appropriate axial dependence along w. We then show how enthalpy minimization at fixed "charge" selects the aspect ratio $L/a \approx 1.85$, and we reinterpret this result as a geometric constraint on the throat. We also review how charge and mass can be viewed as different projections of the same throat geometry: charge as vorticity flux through the throat cross–section, and mass as volume flux into the throat.

In Sec. 5 we examine the near–field transition region where streamlines bend from radial inflow on the brane to axial flow into the bulk. We describe the resulting stress patterns and construct a multipole expansion of the effective 3D source associated with this region. We argue that the leading corrections are quadrupolar and scale as $(a/r)^2$, providing a geometric interpretation of expected 2PN finite–size effects and a link to the shape–sensitivity tests performed in Paper III.

Sec. 6 revisits the wake mixing $\alpha^2 = 3/4$ in the 1PN vector sector. We briefly recap the relevant structure of the EIH matching from Paper III and then introduce a simple two–mode toy model in which transverse (brane–parallel) and longitudinal (bulk–directed) modes are coupled. We show how an effective mixing between these modes arises and interpret this as an effective projector signature in mode space induced by the brane–bulk geometry of the throat.

In Sec. 7 we outline how a full 2PN analysis might proceed within this ontology. We review the standard PN counting, identify the classes of corrections that are most sensitive to the throat geometry, and discuss possible observational channels through which the model could be tested or ruled out. We emphasize that the ontology developed here sharply constrains the allowed structure of 2PN terms, turning the toy model into a framework that is, in principle, falsifiable.

Finally, in Sec. 8 we summarize the main elements of the brane–bulk throat picture and its relation to Papers I–IV, discuss possible scenarios for what happens at the bottom of the throat and for the stabilization of the brane, and speculate about the behavior of multiple defects interacting through the bulk. Several technical details, including explicit Gaussian multipole integrals, mode separation in the throat, and the two–mode quadratic form underlying $\alpha^2 = 3/4$, are collected in the appendices.

2 4D Superfluid Framework and Brane Embedding

In this section we set up the higher–dimensional fluid framework that will underlie the rest of the paper. We treat the vacuum as a compressible superfluid living in four spatial dimensions plus time, with the observable universe represented as a three–dimensional brane embedded in this bulk. The goal is not to derive everything from first principles, but to lay out a minimal, consistent kinematic and thermodynamic setup that can be used to interpret the results of Papers I–IV in a unified geometric language.

2.1 Bulk fields, equation of state, and hydrodynamic equations

We work with coordinates

$$(t, \mathbf{X}) = (t, \mathbf{x}, w), \qquad \mathbf{x} = (x, y, z), \tag{4}$$

where $\mathbf{x}$ are the spatial coordinates along the brane and w is a single bulk coordinate transverse to the brane. The full superfluid lives in the four–dimensional spatial manifold parametrized by $\mathbf{X} = (x, y, z, w)$.

The state of the bulk fluid is described by

- a mass density $\rho(\mathbf{x}, w, t)$,
- a velocity field $\mathbf{v}_4(\mathbf{x}, w, t)$ with components

$$\mathbf{v}_4 = (v_x, v_y, v_z, v_w) \equiv (\mathbf{v}, v_w), \tag{5}$$

where $\mathbf{v}$ denotes the three components tangent to the brane and v_w is the bulk component, and

- a pressure $P(\mathbf{x}, w, t)$.

We assume a barotropic equation of state of polytropic form,

$$P = K\rho^n, \tag{6}$$

with $K > 0$ and polytropic index $n = 5$. This is the same stiff superfluid equation of state that was singled out in Paper II by matching gravitational lensing and Shapiro delay; here we simply adopt it as part of the bulk ontology. The background is taken to be homogeneous with density ρ_0 and pressure $P_0 = K\rho_0^n$ in the absence of defects, and we will work with small perturbations around this background,

$$\rho = \rho_0 + \delta\rho, \qquad P = P_0 + \delta P, \qquad |\delta\rho| \ll \rho_0, \quad |\delta P| \ll P_0. \tag{7}$$

The bulk dynamics are governed by the usual continuity and Euler equations generalized to four spatial dimensions. In Cartesian coordinates the continuity equation reads

$$\partial_t\rho + \nabla_4 \cdot \big(\rho\, \mathbf{v}_4\big) = 0, \tag{8}$$

where

$$\nabla_4 = (\partial_x, \partial_y, \partial_z, \partial_w), \qquad \nabla_4 \cdot \mathbf{v}_4 = \partial_x v_x + \partial_y v_y + \partial_z v_z + \partial_w v_w. \tag{9}$$

The Euler equation for an inviscid barotropic fluid is

$$\rho\left(\partial_t \mathbf{v}_4 + (\mathbf{v}_4 \cdot \nabla_4)\mathbf{v}_4\right) = -\nabla_4 P + \mathbf{f}_{\text{ext}}, \tag{10}$$

where $\mathbf{f}_{\text{ext}}$ denotes any external or defect–induced body forces. In the far field of a localized defect the flow is slow and nearly irrotational, the nonlinear advective term can be treated perturbatively, and we may take $\mathbf{f}_{\text{ext}} \simeq 0$ for the purpose of deriving the acoustic modes.

It is convenient to introduce the specific enthalpy

$$h(\rho) \equiv \int^{\rho} \frac{\mathrm{d}P}{\rho'} = \frac{nK}{n-1}\,\rho^{n-1}, \tag{11}$$

for the polytropic equation of state (6). Linearizing $h(\rho)$ around ρ_0 yields

$$h(\rho_0 + \delta\rho) \simeq h_0 + \left.\frac{\mathrm{d}h}{\mathrm{d}\rho}\right|_{\rho_0} \delta\rho = h_0 + \frac{c_s^2}{\rho_0}\,\delta\rho, \tag{12}$$

where

$$c_s^2 \equiv \left.\frac{\mathrm{d}P}{\mathrm{d}\rho}\right|_{\rho_0} = nK\rho_0^{n-1} \tag{13}$$

is the squared sound speed. We will henceforth drop the constant offset h_0 and use h to denote the enthalpy perturbation,

$$h(\mathbf{x}, w, t) \simeq \frac{c_s^2}{\rho_0}\,\delta\rho(\mathbf{x}, w, t). \tag{14}$$

In the far field of a defect and for small–amplitude motions we may linearize Eqs. (8)–(10) about the homogeneous background. Writing $\rho = \rho_0 + \delta\rho$ and $\mathbf{v}_4 = \delta\mathbf{v}_4$ and neglecting quadratic terms in the perturbations, the continuity equation becomes

$$\partial_t \delta\rho + \rho_0 \, \nabla_4 \cdot \delta\mathbf{v}_4 = 0, \tag{15}$$

while the Euler equation reduces to

$$\rho_0 \, \partial_t \delta\mathbf{v}_4 = -\nabla_4 \delta P = -\rho_0 \, \nabla_4 h. \tag{16}$$

Taking the divergence of Eq. (16) and using Eq. (15) to eliminate $\nabla_4 \cdot \delta\mathbf{v}_4$ gives a wave equation for h,

$$\partial_t^2 h - c_s^2 \, \nabla_4^2 h = 0, \qquad \nabla_4^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 + \partial_w^2. \tag{17}$$

Equivalently,

$$-\frac{1}{c_s^2} \, \partial_t^2 h + \nabla_4^2 h = 0. \tag{18}$$

This 4D acoustic wave equation is the master equation governing linear perturbations in the bulk. In later sections we will specialize it to the interior of a throat and use it to derive the cavity modes relevant for the electromagnetic sector.

2.2 The brane as a hypersurface at $w = 0$

We now embed a three–dimensional brane in the 4D spatial bulk to represent the observable universe. The brane is defined as the hypersurface

$$\mathcal{B}: \quad w = 0. \tag{19}$$

All of the effective 3D physics described in Papers I–IV is understood as arising from the behavior of the bulk fields restricted to, or projected onto, this hypersurface. In particular:

- The mouths of defects are localized regions on $\mathcal{B}$.
- Test bodies and light rays follow trajectories $(t, \mathbf{x}(t))$ that lie on or very near $w = 0$.

To make contact with the previous 3D description it is useful to define effective brane fields obtained by integrating or sampling the bulk fields in the w–direction. One simple construction is to introduce a weighting kernel $K(w)$ peaked around $w = 0$ and define

$$\rho_{3\mathrm{D}}(\mathbf{x}, t) \equiv \int_{-\infty}^{+\infty} \mathrm{d}w \, K(w) \, \rho(\mathbf{x}, w, t), \tag{20}$$

$$\mathbf{v}_{3\mathrm{D}}(\mathbf{x}, t) \equiv \frac{1}{N} \int_{-\infty}^{+\infty} \mathrm{d}w \, K(w) \, \Pi_\parallel \, \mathbf{v}_4(\mathbf{x}, w, t), \tag{21}$$

where $\Pi_\parallel$ projects onto the x, y, z components and $N = \int \mathrm{d}w \, K(w)$ is a normalization factor. For the purposes of this paper we will not commit to a specific functional form for $K(w)$; one may think of it as either a narrow bump of width comparable to the microscopic brane thickness, or simply as $K(w) = 1$ with the understanding that only the behavior near $w = 0$ contributes significantly.

The fields ρ_{3D} and $\mathbf{v}_{3D}$ play the role of the effective density and velocity fields in the three–dimensional toy models of Papers I–III. When we speak of a "spherical sink" on the brane we are referring to the behavior of ρ_{3D} and $\mathbf{v}_{3D}$ as functions of $\mathbf{x}$, induced by the presence of a throat in the bulk.

At the level of equations of motion, the projections (20)–(21) induce an effective 3D continuity equation and a Poisson–like equation for a Newtonian potential $\Phi(\mathbf{x}, t)$ on the brane. Schematically,

$$\partial_t \rho_{3D} + \nabla \cdot (\rho_{3D} \mathbf{v}_{3D}) = 0, \tag{22}$$

$$\nabla^2 \Phi = 4\pi G_{\text{eff}}\, \rho_{3D}, \tag{23}$$

where ∇ is the 3D gradient with respect to $\mathbf{x}$ and G_{eff} is an effective gravitational constant determined by the bulk parameters. We will not attempt to derive G_{eff} from first principles here; instead we treat it as calibrated by the matching to Newtonian gravity and 1PN corrections in Papers I–III.

Crucially, the effective brane velocity need not be purely longitudinal. On $\mathcal{B}$ we can decompose

$$\mathbf{v}_{3D} = \nabla\Phi + \mathbf{v}_T, \qquad \nabla \cdot \mathbf{v}_T = 0, \tag{24}$$

into an irrotational component that controls the scalar potential sector used in the orbital and optical matching, and a divergence–free transverse component confined to the brane. In the gravitational regime we may take $\mathbf{v}_T$ negligible so that $\mathbf{v}_{3D} \approx \nabla\Phi$, but in the electromagnetic sector the transverse mode provides the natural carrier of a vector potential; in Sec. 4.4 we identify $\mathbf{A} \equiv \kappa_A \mathbf{v}_T$ and $\mathbf{B} = \nabla \times \mathbf{A}$.

The brane also provides natural boundary conditions on the bulk flow. Physically, the brane may be thought of as a locus where some microscopic mechanism pins the fluid or changes its properties, but in this toy model we capture that information simply by specifying the behavior of ρ, $\mathbf{v}_4$, and h at $w = 0$. For example, in the absence of a defect we may assume that the normal velocity vanishes at the brane,

$$v_w(\mathbf{x}, w = 0, t) = 0 \tag{25}$$

so that there is no net flux of fluid across $\mathcal{B}$. In the presence of a defect mouth, by contrast, there is a localized region on the brane where v_w is nonzero and negative, corresponding to drainage into the throat. The details of this boundary condition will matter for the near–field structure and for the relationship between the defect's mass and its drainage rate, but the broad picture is simple: the brane is a distinguished hypersurface whose intersection with a throat appears as a localized sink to brane–bound observers.

2.3 Throat topology and geometry

We are now ready to formalize the geometric picture of a defect as a brane–bulk throat. Intuitively, the brane at $w = 0$ is locally deformed and pinched into the bulk, forming a tube–like region filled with the same superfluid. We model this region as a cylindrical throat of radius a and depth L embedded in the four–dimensional spatial bulk.

Let the throat domain $\mathcal{T}$ be a connected region of space defined approximately by

$$\mathcal{T} \simeq \left\{ (\mathbf{x}, w) \,\middle|\, 0 \le w \le L,\ \sqrt{x^2 + y^2 + z^2} \lesssim a \right\}, \tag{26}$$

with the understanding that near $w = 0$ and $w = L$ the geometry may deviate from an exact cylinder. For most of the paper we will only need a coarse–grained description at scales larger than any microscopic brane thickness, so we treat a and L as effective parameters characterizing the radius and depth of the throat.

From the perspective of an observer living on the brane $\mathcal{B}$, the intersection of the throat with $w = 0$ is a two–sphere of radius a,

$$\partial\mathcal{T} \cap \mathcal{B} \simeq \{\mathbf{x} \,|\, |\mathbf{x}| = a\}. \tag{27}$$

This spherical boundary is what appears, in the effective 3D description, as the "surface" of a spherical defect. The region $|\mathbf{x}| < a$ on the brane is not filled with solid matter, but is instead the mouth of the throat opening into the bulk. This is the geometry implicitly assumed in Papers I–III when modeling a defect as a spherical sink: at distances $r = |\mathbf{x}| \gg a$, the details of the throat interior are invisible and only the total drainage rate through this mouth matters.

From the perspective of the bulk, by contrast, the region $0 < w < L$ with $|\mathbf{x}| \lesssim a$ is approximately a straight 4D cylinder of cross–sectional area πa^2. This cylindrical interior is the domain in which the cavity modes of the electromagnetic sector live. In Sec. 4 we will impose boundary conditions on h at $r = |\mathbf{x}| = a$ and at $w = 0$ and $w = L$ and solve the 4D acoustic wave equation (17) inside $\mathcal{T}$ to recover the Bessel–type radial modes and the preferred aspect ratio L/a found in Paper IV.

At the bottom of the throat, around $w = L$, we deliberately leave the geometry unspecified. Several possibilities are conceivable:

- the throat could close off smoothly, forming a finite cavity;
- it could open into a larger bulk region, allowing radiation or flow into the deep bulk;
- or it could connect to other throats, forming a network of defects linked through the bulk.

The choice among these possibilities affects the detailed mode spectrum and the global topology of the model, but it will not play a direct role in the 1PN considerations of this paper. We return to these questions in Sec. 8.

Finally, it is worth noting that the throat geometry naturally supports topological quantities associated with circulation and vorticity. Loops encircling the throat mouth on the brane can carry quantized circulation, and vorticity lines can thread the throat interior, connecting the brane to the bulk. In the electromagnetic sector these quantities will be related to electric charge and magnetic flux; for now we simply flag the fact that the topology of $\mathcal{T}$ provides the right sort of structure to encode such conserved charges.

With this 4D superfluid framework and throat geometry in place, we can now turn to dimensional reduction and the far–field structure of the effective 3D theory on the brane.

3 Dimensional Reduction and Far-Field Structure

In this section we show how a localized throat in the 4D bulk generates an effective three–dimensional source on the brane. The key points are: (i) integrating over the bulk coordinate w produces an effective 3D density whose leading contribution is a monopole mass $M \sim \rho_0 L a^3$, and (ii) angular deviations from spherical symmetry are naturally suppressed by $(a/r)^2$ at large radii r on the brane. We illustrate these ideas with a simple Gaussian toy model and then connect the resulting far–field potential to the 1PN structure used in Papers I–III.

3.1 General dimensional reduction from 4D to 3D

Given a bulk density $\rho(\mathbf{x}, w, t)$ and velocity $\mathbf{v}_4(\mathbf{x}, w, t)$, the effective 3D fields seen by brane–bound observers are obtained by integrating over the bulk coordinate w with some weighting kernel $K(w)$ localized near the brane:

$$\rho_{3\mathrm{D}}(\mathbf{x}, t) \equiv \int_{-\infty}^{+\infty} \mathrm{d}w\, K(w)\, \rho(\mathbf{x}, w, t), \tag{28}$$

$$\mathbf{v}_{3\mathrm{D}}(\mathbf{x}, t) \equiv \frac{1}{N} \int_{-\infty}^{+\infty} \mathrm{d}w\, K(w)\, \Pi_{\parallel}\, \mathbf{v}_4(\mathbf{x}, w, t), \tag{29}$$

with $N = \int \mathrm{d}w\, K(w)$ and $\Pi_{\parallel}$ projecting onto the brane–parallel components (v_x, v_y, v_z). For the purposes of this paper we may take $K(w) = 1$ and understand that the integrals are dominated by the region where the throat lives; more refined choices would only change numerical prefactors.

Once $\rho_{3\mathrm{D}}$ and $\mathbf{v}_{3\mathrm{D}}$ are defined, their dynamics on the brane are governed by an effective 3D continuity equation and a Poisson–like equation for a Newtonian potential $\Phi(\mathbf{x}, t)$,

$$\partial_t \rho_{3\mathrm{D}} + \nabla \cdot (\rho_{3\mathrm{D}} \mathbf{v}_{3\mathrm{D}}) = 0, \tag{30}$$

$$\nabla^2 \Phi = 4\pi G_{\mathrm{eff}}\, \rho_{3\mathrm{D}}, \tag{31}$$

where ∇ is the 3D gradient with respect to $\mathbf{x}$ and G_{eff} is an effective gravitational constant determined by the bulk parameters and the details of the projection. In practice G_{eff} is fixed by matching Eq. (31) to Newtonian gravity at large distances, as done implicitly in Papers I–III.

For a localized defect represented as a throat of radius a and depth L, $\rho(\mathbf{x}, w, t)$ is strongly peaked near $|\mathbf{x}| \lesssim a$ and $0 < w < L$. At distances $r = |\mathbf{x}| \gg a, L$ on the brane, $\rho_{3\mathrm{D}}$ therefore looks like a nearly point–like source plus small angular corrections. The total effective mass is

$$M \equiv \int \rho_{3\mathrm{D}}(\mathbf{x}, t)\, \mathrm{d}^3x = \int \mathrm{d}^3x \int \mathrm{d}w\, K(w)\, \rho(\mathbf{x}, w, t), \tag{32}$$

and the higher multipole moments of $\rho_{3\mathrm{D}}$ encode finite–size effects associated with the throat geometry and the transition region near its mouth.

3.2 Gaussian toy model for a localized throat

To make these statements concrete, we now introduce a simple toy model for the bulk density perturbation associated with a single throat. We work in spherical coordinates (r, θ, ϕ) on the brane and use w for the bulk coordinate. The toy 4D density profile is taken to be

$$\rho_4(r, \theta, w) = \rho_0 \exp\left(-\frac{r^2}{a^2}\right) \exp\left(-\frac{w^2}{L^2}\right) \left[1 + \varepsilon P_2(\cos\theta)\right], \tag{33}$$

where $P_2(\cos\theta) = (3\cos^2\theta - 1)/2$ is the $\ell = 2$ Legendre polynomial, a is the throat radius, L is the throat depth, and ε is a small dimensionless parameter controlling the degree of angular asymmetry. The exponential factors localize the defect within a region of size $\sim a$ on the brane and $\sim L$ in the bulk; the εP_2 term models the mild asphericity associated with the transition region where streamlines bend from radial to axial flow.

For simplicity we choose $K(w) = 1$ in the projection onto the brane. Integrating Eq. (33) over w gives the effective 3D density

$$\rho_{3\mathrm{D}}(r, \theta) = \int_{-\infty}^{+\infty} \rho_4(r, \theta, w)\, \mathrm{d}w = \sqrt{\pi}\, L\rho_0\, \exp\left(-\frac{r^2}{a^2}\right) \left[1 + \varepsilon P_2(\cos\theta)\right]. \tag{34}$$

The total mass of the defect is then

$$M = \int \rho_{3\mathrm{D}}(r, \theta)\, \mathrm{d}^3x = \pi^2 L a^3 \rho_0, \tag{35}$$

where the ε term integrates to zero by angular symmetry. This scaling is exactly what we expect for a throat of radius a and depth L: M is proportional to the background density times the effective throat volume.

To quantify the leading angular deviation from sphericity we consider the standard quadrupole–like scalar moment

$$Q_{20} \propto \int \rho_{3\mathrm{D}}(r, \theta)\, r^2 P_2(\cos\theta)\, \mathrm{d}^3x. \tag{36}$$

Carrying out the angular integrals and the Gaussian radial integral (details are relegated to Appendix A), we obtain

$$Q_{20} = \frac{3}{10}\, \pi^2 L a^5 \varepsilon \rho_0, \tag{37}$$

and hence

$$\frac{Q_{20}}{M} = \frac{3}{10}\, \varepsilon a^2. \tag{38}$$

The precise numerical prefactor 3/10 is not important for our purposes; the crucial point is the scaling

$$\frac{Q}{M} \sim \varepsilon a^2, \tag{39}$$

which is generic for a localized, mildly aspherical source of size a on the brane.

3.3 Far-field potential and $(a/r)^2$ corrections

The effective Newtonian potential $\Phi(r,\theta)$ generated on the brane by a localized mass distribution $\rho_{\rm 3D}$ admits the usual multipole expansion at large radii $r \gg a, L$:

$$\Phi(r,\theta) \approx -\frac{GM}{r} - G\,\frac{Q}{r^3}\,P_2(\cos\theta) + \cdots\,, \tag{40}$$

where M is the monopole mass, Q is the quadrupole–like scalar defined in Eq. (36) (up to conventional normalization factors), and the ellipsis denotes higher multipoles suppressed by further powers of a/r.

Substituting the scaling $Q/M \sim \varepsilon a^2$ from the Gaussian toy model, Eq. (38), into Eq. (40) gives

$$\Phi(r,\theta) \approx -\frac{GM}{r}\left[1 + \mathcal{O}\!\left(\varepsilon\frac{a^2}{r^2}\right) P_2(\cos\theta) + \cdots\right]. \tag{41}$$

Thus the leading anisotropic correction to the Newtonian potential is suppressed by $\varepsilon(a/r)^2$. In the language of the throat ontology, this correction encodes the imprint of the transition region near the mouth of the throat, where the flow is neither purely radial on the brane nor purely axial in the bulk, but bends from one into the other.

The main takeaway of this section is therefore:

- Dimensional reduction of a localized 4D throat–like density produces an effective 3D source with a monopole mass $M \sim \rho_0 L a^3$ and a tower of finite–size corrections.
- The leading angular distortion is quadrupolar and suppressed by $(a/r)^2$ at large radii.
- These are precisely the types of terms that, in a full post–Newtonian treatment, would be expected to appear at 2PN order as finite–size corrections associated with the physical size a of the body.

3.4 Connection to Papers I–III and why 1PN sees a sphere

We now connect this dimensional reduction picture to the 1PN results of Papers I–III. Those papers effectively treat each defect as a point–like or perfectly spherical source on the brane, described by a monopolar potential $\Phi \sim -GM/r$ plus velocity–dependent 1PN corrections. Finite–size structure of the defect is either neglected or, in the case of the shape–sensitivity tests in Paper III, shown to have only a weak effect on the 1PN precession coefficient for modest deformations.

From the standpoint of the throat ontology, this is exactly what one should expect. At distances $r \gg a, L$, the effective 3D density $\rho_{\rm 3D}$ produced by dimensional reduction is dominated by its monopole component. Angular structure such as the quadrupole is suppressed by $(a/r)^2$, and higher multipoles by even higher powers of a/r. As a result:

- The leading Newtonian potential and the 1PN corrections derived in Papers I–III are insensitive to the detailed throat geometry, as long as the mass M is held fixed.
- The defect therefore looks like a spherical sink at 1PN order, even though its interior is cylindrically structured in the bulk.

- The modest shape sensitivity found in Paper III (e.g. $\sim 10\%$ oblateness leading to $\sim 2\%$ shifts in the precession coefficient) is naturally interpreted as a small leakage of these $(a/r)^2$ corrections into the 1PN observables used in that analysis, and as a hint that the model has a geometrically rich interior that will matter more at 2PN and beyond.

This perspective also clarifies how the gravitational and electromagnetic sectors can coexist without contradiction. Papers I–III operate in the far–field regime on the brane, where only the monopole and velocity–dependent interaction terms matter, so the defect appears spherical. Paper IV probes modes confined inside the throat, where the cylindrical interior and the aspect ratio L/a are crucial. Dimensional reduction shows that these descriptions are simply different projections of the same 4D geometry: the spherical mouth seen by gravity and the cylindrical interior seen by electromagnetism are both encoded in the same throat.

In the next section we turn from this far–field point of view to the interior of the throat itself, deriving the cavity modes of the 4D acoustic equation and reinterpreting the enthalpy–selected aspect ratio $L/a \approx 1.85$ as a geometric property of the throat in the bulk.

4 Throat Geometry and the Electromagnetic Sector

We now turn from the far–field effective description on the brane to the interior of the throat itself. The goal is to show how the cylindrical cavity picture of Paper IV arises naturally once we treat the defect as a brane–bulk throat, and to reinterpret the enthalpy–selected aspect ratio L/a and the notion of "charge" in purely geometric terms. Throughout this section we work with the linear 4D acoustic equation

$$\partial_t^2 h - c_s^2\, \nabla_4^2 h = 0, \tag{42}$$

introduced in Sec. 2.1, but now restricted to the throat domain $\mathcal{T}$.

4.1 4D cavity modes in the throat

Inside the throat, the geometry is approximately that of a straight tube of radius a and depth L extending into the bulk. It is convenient to adopt coordinates adapted to this tube: we take w along the throat and introduce a radial coordinate r measuring distance from the center of the throat within the brane directions, together with angular coordinates on the (x, y, z) directions. For the lowest–lying modes we are interested in, there is no dependence on the azimuthal angles or on any internal structure within the cross–section; the modes are effectively axisymmetric about the throat center.

In this approximation the 4D Laplacian inside the throat separates as

$$\nabla_4^2 \simeq \nabla_\perp^2 + \partial_w^2, \tag{43}$$

where $\nabla_\perp^2$ is the Laplacian in the radial direction r (and its associated angular coordinate), restricted to axisymmetric configurations. We seek separated solutions of the form

$$h(t, r, w) = \mathrm{Re}\big\{ H(r)\, W(w)\, e^{-i\omega t} \big\}\,. \tag{44}$$

Substituting Eq. (44) into Eq. (42) yields

$$\left[\omega^2 - c_s^2 \left(\frac{1}{H}\nabla_\perp^2 H + \frac{1}{W}\partial_w^2 W\right)\right] HW = 0. \tag{45}$$

Dividing by HW and rearranging gives

$$\frac{1}{H}\nabla_\perp^2 H + \frac{1}{W}\partial_w^2 W = \frac{\omega^2}{c_s^2}, \tag{46}$$

which we can separate by setting each side equal to a constant. Introducing separation constants $-k_r^2$ and $-k_w^2$, we obtain

$$\nabla_\perp^2 H + k_r^2 H = 0, \tag{47}$$

$$\partial_w^2 W + k_w^2 W = 0, \tag{48}$$

with the dispersion relation

$$\omega^2 = c_s^2\,(k_r^2 + k_w^2). \tag{49}$$

For axisymmetric modes, $\nabla_\perp^2$ reduces to the radial part of the Laplacian in cylindrical–like coordinates,

$$\nabla_\perp^2 H = \frac{1}{r^2}\partial_r\left(r^2 \partial_r H\right), \tag{50}$$

or, in the thin–throat limit where the cross–section is effectively two–dimensional, to the standard Bessel form

$$\nabla_\perp^2 H = \frac{1}{r}\partial_r(r\partial_r H)\,. \tag{51}$$

In either case the regular, axisymmetric solutions of Eq. (47) at $r = 0$ are Bessel functions of the first kind:

$$H(r) \propto J_0(k_r r), \tag{52}$$

with k_r quantized by boundary conditions at the throat wall $r = a$.

In keeping with the cavity analysis of Paper IV, we impose a boundary condition that the enthalpy fluctuation vanishes at the wall,

$$h(r = a, w, t) = 0 \quad \Rightarrow \quad H(a) = 0, \tag{53}$$

corresponding physically to a *pinned phase boundary* in the stiff superfluid vacuum. The order parameter is topologically fixed at the throat wall, so small enthalpy (pressure) perturbations are forced to zero there. Mathematically, this Dirichlet condition selects the zeros of J_0,

$$k_r = \frac{x_{0n}}{a}, \tag{54}$$

where x_{0n} is the nth zero of J_0. We will be primarily interested in the fundamental radial mode,

$$k_r = \frac{x_{01}}{a}, \tag{55}$$

with radial profile $J_0(x_{01} r/a)$.

Along the throat direction w we impose boundary conditions at $w = 0$ and $w = L$. The simplest idealization is to take h to vanish at both ends,

$$h(r, w = 0, t) = h(r, w = L, t) = 0 \quad \Rightarrow \quad W(0) = W(L) = 0, \tag{56}$$

corresponding to nodes at the mouth and bottom of the throat. This leads to standing–wave solutions

$$W_n(w) \propto \sin\left(\frac{n\pi w}{L}\right), \qquad k_w = \frac{n\pi}{L}, \quad n = 1, 2, \ldots \tag{57}$$

The fundamental axial mode has $n = 1$, so the full fundamental throat mode is

$$h_1(t, r, w) \propto J_0\left(\frac{x_{01} r}{a}\right) \sin\left(\frac{\pi w}{L}\right) \cos(\omega t), \tag{58}$$

with

$$\omega^2 = c_s^2 \left(\frac{x_{01}^2}{a^2} + \frac{\pi^2}{L^2}\right). \tag{59}$$

This is the same J_0–Bessel / standing–wave structure used in the cylindrical cavity analysis of Paper IV, now understood as the fundamental mode of the brane–bulk throat.

4.2 Enthalpy minimization and the aspect ratio L/a

In Paper IV the key electromagnetic result was that minimizing the enthalpy of the cavity at fixed "charge" picked out a preferred aspect ratio

$$\frac{L}{a} = \frac{\sqrt{2}\,\pi}{x_{01}} \approx 1.85. \tag{60}$$

We now sketch how this emerges from the throat picture.

The relevant functional is the total perturbation energy (or enthalpy) of the mode in the throat. For a linear acoustic mode $h(t, r, w)$ with frequency ω, the time–averaged energy stored in the perturbation can be written schematically as

$$\mathcal{E}[h] \sim \int_{\mathcal{T}} \mathrm{d}^3x \, \mathrm{d}w \; \rho_0 \left[\frac{1}{2c_s^2} (\partial_t h)^2 + \frac{1}{2} (\nabla_4 h)^2\right], \tag{61}$$

where we have suppressed numerical prefactors that do not depend on a or L. For a separated mode of the form $h(t, r, w) = AH(r)W(w)\cos(\omega t)$ the time average of $(\partial_t h)^2$ contributes ω^2, while the spatial gradients contribute k_r^2 and k_w^2:

$$\langle(\partial_t h)^2\rangle \propto \omega^2 A^2 H^2 W^2, \qquad \langle(\nabla_4 h)^2\rangle \propto (k_r^2 + k_w^2) A^2 H^2 W^2. \tag{62}$$

Using the dispersion relation (59), the total time–averaged energy carried by the fundamental mode is therefore proportional to

$$\mathcal{E} \propto A^2 (k_r^2 + k_w^2) \int_{\mathcal{T}} H^2 W^2, \tag{63}$$

where the integral over $H^2 W^2$ supplies a factor of order $a^2 L$ for the fundamental mode.

To define a variational problem we must specify which quantity is held fixed when minimizing $\mathcal{E}$. Following Paper IV, we identify a conserved "charge" $\mathcal{Q}$ associated with the mode amplitude, which depends on the same integral of H^2W^2 but not on $k_r^2 + k_w^2$. Schematically,

$$\mathcal{Q} \propto A^2 \int_{\mathcal{T}} H^2 W^2, \tag{64}$$

so that, at fixed $\mathcal{Q}$, the energy scales as

$$\mathcal{E} \propto (k_r^2 + k_w^2)\, \mathcal{Q}. \tag{65}$$

Minimizing $\mathcal{E}$ at fixed $\mathcal{Q}$ therefore amounts to minimizing $k_r^2 + k_w^2$.

For the fundamental throat mode we have

$$k_r^2 + k_w^2 = \frac{x_{01}^2}{a^2} + \frac{\pi^2}{L^2}. \tag{66}$$

Varying a and L while holding the effective cross–sectional area πa^2 and depth L in a constrained way (reflecting the fact that the total mass $M \sim \rho_0 L a^3$ is fixed) leads to a minimum of $k_r^2 + k_w^2$ at a particular ratio L/a. The detailed calculation follows the same steps as in Paper IV; here we simply quote the result:

$$\frac{L}{a} = \frac{\sqrt{2}\,\pi}{x_{01}}. \tag{67}$$

At this aspect ratio, the contributions of radial and axial gradients are in a particular balance that extremizes the energy per unit charge stored in the cavity. In the throat ontology, Eq. (60) is not an arbitrary parameter choice for a 3D cylinder, but a statement about the geometry of the 4D throat: for a given mass and charge, the throat relaxes to a preferred ratio of depth to radius.

4.3 Charge as circulation and vorticity flux through the throat

So far we have treated the "charge" $\mathcal{Q}$ as an abstract conserved quantity associated with the cavity mode amplitude. In the superfluid picture, $\mathcal{Q}$ has a more concrete interpretation in terms of circulation and vorticity.

Consider a closed loop $\mathcal{C}$ on the brane encircling the throat mouth, for example a circle of radius $r > a$ in a plane intersecting the brane. The circulation of the superfluid velocity around this loop is

$$\Gamma = \oint_{\mathcal{C}} \mathbf{v} \cdot \mathrm{d}\boldsymbol{\ell}. \tag{68}$$

In a quantum fluid, Γ would be quantized; in the present classical toy model we treat it as an integer–valued conserved input label, $\Gamma \in \mathbb{Z}$, in the absence of vorticity creation or reconnection events. In the hydrodynamic–electromagnetic dictionary of Paper IV the effective charge is proportional to this circulation (up to a fixed normalization), $\mathcal{Q} \propto \Gamma$. By Stokes' theorem, Γ is equivalently the flux of vorticity through any surface $\mathcal{S}$ bounded by $\mathcal{C}$,

$$\Gamma = \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot \mathrm{d}\mathbf{S}. \tag{69}$$

If the only significant vorticity near $\mathcal{C}$ is carried by the throat, then this flux is, to a good approximation, the vorticity threading the throat cross–section.

In the hydrodynamic–electromagnetic dictionary of Paper IV, the electric charge of the defect is identified (up to a proportionality constant) with this circulation or vorticity flux. The field variables that play the role of the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, are constructed from the enthalpy gradients and the velocity field in such a way that Maxwell's equations emerge as effective equations in the near field of the defect. In that language, $\mathcal{Q}$ is essentially the integral of the mode amplitude over the throat cross–section, which is in turn proportional to the vorticity flux through that cross–section.

The throat picture makes this geometric: the vorticity lines thread the throat like the flux lines of a solenoid threading a torus. The conserved charge is the flux of vorticity through the "hole" of the throat, i.e. through the surface of area πa^2 that spans the spherical mouth on the brane. Different values of charge correspond to different amounts of vorticity threading the throat, and the cavity mode adjusts its amplitude to accommodate this flux while keeping the enthalpy as low as possible. The preferred aspect ratio L/a then reflects how the throat geometry arranges itself to support a given amount of vorticity flux at minimum energetic cost.

4.4 Magnetostatics from brane transverse wakes

A persistent obstruction in potential–flow pictures is that a translating defect in an irrotational bulk produces a dipolar velocity potential $\phi \sim (\mathbf{u}\cdot\mathbf{r})/r^3$, whose curl vanishes identically outside the core. If one attempts to identify the magnetic field directly with bulk vorticity, this would imply $\mathbf{B} = 0$ in the far zone, in conflict with Maxwell magnetostatics.

The brane projection resolves this: the same defect that sources the longitudinal gravitational sector can also excite a brane–confined transverse mode $\mathbf{v}_T$ in the decomposition (24). We treat $\mathbf{v}_T$ as a surface–current (vortex–sheet) degree of freedom whose governing equation on $\mathcal{B}$ is a vector Poisson problem in Coulomb gauge. For a localized moving throat centered at $\mathbf{x}_0(t)$, define $\mathbf{r} \equiv \mathbf{x} - \mathbf{x}_0$. We write

$$\nabla \cdot \mathbf{A} = 0, \qquad \nabla^2 \mathbf{A} = -\kappa_A \, \mathcal{Q} \, \mathbf{u} \, \delta^{(3)}(\mathbf{r}), \tag{70}$$

where $\mathbf{u}$ is the (slow) brane–parallel velocity of the throat and κ_A is a fixed calibration constant. Identifying the vector potential with the brane transverse wake,

$$\mathbf{A} \equiv \kappa_A \, \mathbf{v}_T, \tag{71}$$

the solution of (70) is

$$\mathbf{A}(\mathbf{r}) = \frac{\kappa_A \, \mathcal{Q}}{4\pi} \, \frac{\mathbf{u}}{r}, \tag{72}$$

and therefore

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A} = \frac{\kappa_A \, \mathcal{Q}}{4\pi} \, \frac{\mathbf{u} \times \mathbf{r}}{r^3}, \tag{73}$$

which is the Biot–Savart form for the field of a moving point charge (or, equivalently, a localized current element) up to the overall normalization.

Two points are essential. First, (72) is *not* attributed to viscous drag (a Stokeslet); the $1/r$ kernel arises because $\mathbf{v}_T$ is a brane–localized transverse mode with its own Green's function. Second, this construction leaves the bulk flow used for gravity irrotational in the far zone: magnetism lives in the brane's transverse sector, while gravity lives in the bulk–induced longitudinal sector. A time–dependent completion (radiation and induction) will require promoting (70) to a dynamical wave equation on the brane, which we defer to future work.

4.5 Mass as drainage and volume flux

In contrast to charge, which is associated with circulation and vorticity flux, the mass of the defect is associated with drainage of fluid into the throat and the resulting volume deficit in the surrounding superfluid. From the brane point of view, a defect appears as a localized sink: the normal component of the bulk velocity at the mouth is nonzero and directed into the throat,

$$v_w(\mathbf{x}, w = 0, t) < 0 \tag{74}$$

within the mouth region $|\mathbf{x}| \lesssim a$. The net volumetric flux into the throat is

$$\dot{V} = \int_{\text{mouth}} v_w(\mathbf{x}, w = 0, t)\, \mathrm{d}^2 S, \tag{75}$$

where the integral is over the effective mouth area on the brane. In a steady configuration this flux is balanced by either compression within the throat, circulation of fluid in closed streamline patterns, or outflow at the bottom of the throat, depending on the global topology. The resulting density deficit in the near field is what sources the effective Newtonian potential on the brane.

At the level of scaling, the mass associated with the throat is

$$M \sim \rho_0\, (\text{effective throat volume}) \sim \rho_0\, \pi a^2 L, \tag{76}$$

modulo order–unity geometric factors and corrections from the detailed profile of $\rho(\mathbf{x}, w)$. This is consistent with the Gaussian toy model of Sec. 3.2, where $M \propto \rho_0 L a^3$ up to numerical factors, and with the interpretation of M as the total mass deficit localized within the throat and its immediate surroundings.

Putting these pieces together, the throat ontology suggests a simple geometric picture:

- Mass is associated with the *volume* of the throat and the associated drainage of fluid into it.
- Charge is associated with the *vorticity flux* threading the throat cross–section.

Both are properties of the same geometric object—a brane–bulk throat of radius a and depth L—viewed through different hydrodynamic projections. The gravitational sector is sensitive mainly to the integrated mass deficit and the resulting monopolar potential on the brane, while the electromagnetic sector is sensitive to the internal cavity modes and the vorticity flux they support.

In this way, the sphere–cylinder tension that appeared when Papers I–III and IV were viewed in isolation is naturally resolved: the same throat can appear as a spherical sink on the brane for gravity and as a cylindrical cavity in the bulk for electromagnetism. In the next section we move back toward the brane and examine the transition region near the throat mouth, where finite–size effects and 2PN corrections originate.

5 Near-Field Transition and Finite-Size Effects

So far we have treated the throat in two complementary limits: the far field on the brane, where only the monopole mass and velocity–dependent 1PN interactions matter, and the deep interior of the throat, where cylindrical cavity modes control the electromagnetic sector. Between these lies a transition region near the throat mouth where the flow is strongly curved and neither limit applies cleanly. In this section we give a qualitative description of this near–field region, argue that its imprint on the effective 3D theory is naturally multipolar and suppressed by $(a/r)^2$, and connect this picture to the shape–sensitivity tests of the spin and N–body paper (Paper III).

5.1 Flow regimes: far field, throat interior, and transition region

It is useful to distinguish three qualitatively different flow regimes associated with a single throat:

1. *Far field on the brane* ($r \gg a, L$): Here the flow is almost purely radial in the brane directions and nearly independent of w. The effective density $\rho_{3D}(\mathbf{x})$ obtained by integrating over w is well approximated by a spherically symmetric monopole plus small multipole corrections, as in Sec. 3.2. The velocity field $\mathbf{v}_{3D}(\mathbf{x})$ is essentially that of a point–like sink at the origin.

2. *Throat interior* ($r \lesssim a$, $0 < w < L$): Inside the throat the flow is predominantly along the w direction, with relatively weak dependence on the brane coordinates except through the radial cavity profile $J_0(x_{01}r/a)$ of the fundamental mode. The relevant modes are the separated solutions of the 4D acoustic equation (42), and the leading dynamics are controlled by the discrete set of (k_r, k_w) pairs and the aspect ratio L/a.

3. *Transition region* ($r \sim a$, $w \sim 0$): Near the mouth of the throat the flow lines bend from radial inflow on the brane to axial flow into the bulk. The geometry here is strongly curved; the brane deviates from a flat $w = 0$ hypersurface, and the fluid experiences both significant shear and pressure gradients. This is the region where departures from perfect spherical symmetry on the brane originate and where the throat geometry imprints itself most strongly on the effective 3D source ρ_{3D}.

In the previous sections we treated the far field and the throat interior in controlled approximations: the far field through multipole expansion and the interior through mode separation. The transition region does not admit such a simple analytic treatment, but we can still characterize its effects on the brane in terms of symmetry and scaling. Its characteristic size is set by a; its angular structure is determined by how the brane is deformed into the bulk and by how streamlines reorient from brane–parallel to bulk–directed. These features naturally generate higher multipoles in $\rho_{3\mathrm{D}}$ and $\mathbf{v}_{3\mathrm{D}}$ that are suppressed by powers of (a/r) at large distances.

5.2 Streamline bending and induced stresses

To get a feel for the transition region, consider streamlines of the fluid in a quasi–steady configuration. Far from the defect, on the brane, the streamlines are nearly straight and radial, converging toward the origin with speed $v_r(r) \propto 1/r^2$ for an incompressible sink. Deep inside the throat, by contrast, streamlines are nearly parallel to the w axis, with a roughly uniform axial velocity v_w that carries fluid along the throat.

In order to connect these two regimes, streamlines must bend sharply in a neighborhood of the mouth. The curvature of a streamline is controlled by the local acceleration,

$$\frac{\mathrm{D}\mathbf{v}_4}{\mathrm{D}t} = \partial_t \mathbf{v}_4 + (\mathbf{v}_4 \cdot \nabla_4)\mathbf{v}_4 = -\frac{1}{\rho}\nabla_4 P + \frac{\mathbf{f}_{\mathrm{ext}}}{\rho}, \tag{77}$$

so large curvature implies large pressure gradients in the transition region. Even in a quasi–steady state where $\partial_t \mathbf{v}_4 \simeq 0$, the advective term $(\mathbf{v}_4 \cdot \nabla_4)\mathbf{v}_4$ is enhanced where streamlines squeeze together and turn into the throat, and this must be balanced by correspondingly strong gradients in P and h.

From the brane point of view, these pressure gradients generate stresses in the effective 3D fluid. In particular, the stress tensor constructed from the velocity field and the enthalpy,

$$T_{ij} \sim \rho v_i v_j + \delta_{ij} P + \cdots, \tag{78}$$

has nontrivial angular dependence in the transition region. The components tangent to the brane ($i, j \in \{x, y, z\}$) feel both the radial inflow and the shear associated with bending into the w direction. When this structure is projected onto the brane and coarse–grained over scales larger than a, it manifests as effective multipole moments in the 3D source terms that enter the gravitational and electromagnetic sectors.

Although a full solution of the transition–region flow would require solving the nonlinear 4D Euler equations with a deformed brane geometry, symmetry already constrains the leading corrections. For an isolated defect with no preferred direction on the brane, the transition region must be axisymmetric about the throat axis and reflection–symmetric under $w \to -w$ if we consider a mirrored continuation into the bulk. These symmetries forbid dipolar contributions in $\rho_{3\mathrm{D}}$ and $\mathbf{v}_{3\mathrm{D}}$ and make the quadrupole ($\ell = 2$) the leading nontrivial multipole.

5.3 Multipole expansion of the transition region

We can formalize this intuition by writing the effective 3D density on the brane as a sum of a purely radial piece and angular corrections sourced by the transition region. In spherical coordinates (r, θ, ϕ) on the brane, with the throat axis taken as the polar axis, we expand

$$\rho_{3\mathrm{D}}(r, \theta) = \rho_0(r) + \sum_{\ell \geq 1} \rho_\ell(r) P_\ell(\cos\theta), \tag{79}$$

where $\rho_0(r)$ is the spherically symmetric part and P_ℓ are Legendre polynomials. Axisymmetry removes any dependence on ϕ, and reflection symmetry around the throat axis suppresses odd ℓ for the density; the leading correction is thus the quadrupole term $\ell = 2$.

The Gaussian toy model of Sec. 3.2 provides an explicit realization of this structure, with

$$\rho_{3\mathrm{D}}(r,\theta) \propto e^{-r^2/a^2}\Big[1 + \varepsilon P_2(\cos\theta)\Big], \tag{80}$$

so that $\rho_0(r)$ and $\rho_2(r)$ are both localized near $r \lesssim a$. Integrating $\rho_{3\mathrm{D}}$ against $r^2 P_\ell(\cos\theta)$ over a volume enclosing the throat then yields the multipole moments, and we saw explicitly that $Q/M \sim \varepsilon a^2$ for the quadrupole. Higher multipoles would involve higher powers of a and smaller dimensionless coefficients.

The same logic applies, at least at the level of scaling, to any reasonable model of the transition region: as long as the region is localized within a radius of order a and preserves axisymmetry, all multipole moments higher than the monopole are suppressed by powers of a and decay with appropriate powers of $1/r$ in the far field. The resulting potential can be written as

$$\Phi(r,\theta) = -\frac{GM}{r}\left[1 + \sum_{\ell\geq 2} \alpha_\ell \left(\frac{a}{r}\right)^\ell P_\ell(\cos\theta)\right], \tag{81}$$

with dimensionless coefficients α_ℓ that encode the detailed structure of the throat mouth and transition region. For $\ell = 2$ we recover the $(a/r)^2$ suppression emphasized earlier. In the absence of fine tuning, the α_ℓ are expected to be of order one times small shape parameters characterizing how much the throat deviates from an ideal spherical mouth.

5.4 Quantitative verification: The rounded funnel model

While the Gaussian model of Appendix A provides useful analytic scaling relations, it is a simplified caricature of the throat. To quantitatively assess finite–size corrections for a physically bounded defect, we numerically integrate the effective density for a geometry that explicitly connects the cylindrical bulk interior to the planar brane.

We model this transition using a hard–bounded "rounded funnel" geometry. Deep in the bulk ($w \gg a$), the throat is a cylinder of radius a. Near the brane ($w \to 0$), the throat flares outward and smoothly joins the bulk tube to the brane mouth. We parameterize the throat radius $R(w)$ at depth w as

$$R(w) = a\left[1 + \frac{1}{2}\exp\left(-\frac{5w}{a}\right)\right], \tag{82}$$

so that the mouth radius is $R(0) = 1.5a$ and the profile relaxes exponentially to $R(w) \to a$ for $w \gtrsim a$. The effective density inside this region is taken to be uniform, $\rho = \rho_0$, modulated by a small quadrupolar deformation $\varepsilon P_2(\cos\theta)$ to represent the stress anisotropy discussed in Sec. 5.2.

We performed a numerical integration of the resulting effective 3D multipole moments for a throat with aspect ratio $L/a = 2$ and deformation $\varepsilon = 0.1$ (details in Appendix B). The calculation yields the dimensionless quadrupole coefficient

$$\alpha_2 \simeq 1.4 \times 10^{-2}. \tag{83}$$

This result is significant for two reasons. First, it confirms the sign of the correction in our convention: $\alpha_2 > 0$ means the far–field potential is slightly deeper along the throat axis ($P_2 > 0$) and slightly shallower near the equator ($P_2 < 0$), consistent with taking a positive quadrupolar anisotropy ε in the effective density ansatz. Second, the magnitude is small: for the same ε the Gaussian toy model gives $\alpha_2 = (3/10)\varepsilon = 0.03$ (cf. Eq. (38)), whereas the hard–bounded rounded funnel yields $\alpha_2 \simeq 0.014$, about a factor of two smaller. This suggests that physically bounded throat geometries can "hide" finite–size multipoles more efficiently than soft Gaussian tails.

Consequently, finite–size corrections from the throat geometry enter the brane potential at the level $\sim 10^{-2}(a/r)^2$, providing quantitative support for the spherical sink approximation used in the 1PN orbital sector (Paper I).

5.5 Interpretation as 2PN finite-size corrections

From the point of view of post–Newtonian theory, the multipole corrections generated by the transition region are naturally interpreted as finite–size effects that first appear at 2PN order and beyond. In standard GR language, the small parameter controlling the PN expansion is

$$\epsilon \sim \frac{GM}{rc^2} \sim \left(\frac{v}{c}\right)^2, \tag{84}$$

with 1PN terms scaling as ϵ and 2PN terms as ϵ^2. For a compact object of radius R in GR, finite–size effects typically scale as $(R/r)^2$ or higher powers, and in many situations they enter observables at 2PN order or later.

In the throat ontology, the role of the body's size is played by the throat radius a, and the relevant dimensionless parameter is a/r. For well–separated bodies we have $a \ll r$, so $(a/r)^2$ is parametrically small compared to unity. The multipole expansion (81) then suggests that the leading corrections to the monopole potential scale as

$$\delta\Phi \sim \Phi_{\rm N}\,\alpha_2 \left(\frac{a}{r}\right)^2 \sim -\frac{GM}{r}\alpha_2 \left(\frac{a}{r}\right)^2. \tag{85}$$

If a is itself of order the gravitational radius $r_g = GM/c^2$ of the object, then $(a/r)^2 \sim (GM/rc^2)^2 \sim \epsilon^2$, and these corrections would indeed be 2PN in the usual counting. Even if a is somewhat larger than r_g, as might be natural in a superfluid context, the $(a/r)^2$ suppression still pushes these effects beyond the leading 1PN terms for sufficiently large separations.

This scaling argument underlies our claim in the Introduction that the throat ontology provides a geometric roadmap for 2PN corrections: once a is fixed for each object (e.g. by its mass and charge through the throat geometry), the coefficients α_ℓ in Eq. (81) are not arbitrary; they are determined by the way the brane deforms into the bulk and by the structure of the transition region. A future 2PN calculation in this model would therefore not introduce an unrestricted zoo of new parameters, but a constrained set of finite–size couplings tied to the same throat geometry that already controls the 1PN and electromagnetic sectors.

5.6 Relation to shape sensitivity in the spin/N-body paper

The spin and N–body paper (Paper III) already provided a first glimpse of these geometric finite–size effects. There, the authors replaced the spherical defects of the orbital paper with mildly oblate spheroids and computed how the 1PN perihelion precession coefficient changed as a function of the oblateness. The result was that even a relatively large shape distortion at the level of $\sim 10\%$ in the equatorial radius produced only a modest change of order a few percent in the precession coefficient.

In a purely three–dimensional description, this shape sensitivity might seem puzzling or arbitrary: why does a substantial deformation of the body's shape have only a small effect on the orbit? In the throat ontology the answer is straightforward. Changing the apparent 3D shape of the defect on the brane corresponds to modifying the geometry of the throat mouth and the transition region, while holding fixed the deeper throat interior and its overall mass content. The resulting changes in the effective 3D density are encoded primarily in the multipole moments of ρ_{3D}, which are suppressed by powers of (a/r) and thus have only a small effect on the far–field potential and on 1PN observables.

More concretely, a modest oblateness changes the quadrupole coefficient α_2 in Eq. (81) by some amount of order the oblateness parameter. But because the quadrupole contribution is already suppressed by $(a/r)^2$, the net fractional change in the potential and in the 1PN precession coefficient is doubly suppressed: once by the small shape parameter and once by $(a/r)^2$. This is entirely consistent with the $\sim 2\%$ shifts reported in Paper III for $\sim 10\%$ geometric distortions.

From this perspective, the shape–sensitivity experiment in Paper III is not an ad hoc curiosity but an indirect probe of the transition region and the throat geometry. It shows that the toy model already behaves like a finite–size object in GR: small departures from spherical symmetry induce small corrections to 1PN observables, in a way that is naturally understood as a preview of the 2PN finite–size structure. The throat ontology turns this qualitative observation into a quantitative program: in a full 2PN analysis, one would compute the coefficients α_ℓ and the associated finite–size couplings directly from the geometry and dynamics of the transition region.

In the next section we will see how a different, but related, aspect of the throat geometry—the mixing of brane–parallel and bulk–directed modes in the vector sector—provides a natural interpretation of the longitudinal mixing $\alpha^2 = 3/4$ required to match the Einstein–Infeld–Hoffmann Lagrangian at 1PN order.

6 Wake Mixing Constraint and $\alpha^2 = 3/4$

In Paper III the 1PN *vector* sector was obtained by decomposing the translational wake of a moving defect into isotropic longitudinal and transverse projector components and matching the resulting velocity–dependent interaction terms to the Einstein–Infeld–Hoffmann (EIH) Lagrangian. With the completed wake basis, the matching fixes a single real mixing parameter,

$$\alpha^2 = \frac{3}{4}, \tag{86}$$

which may be viewed as the relative longitudinal/transverse weighting in the effective vector kernel. Importantly, $\alpha^2 > 0$ is compatible with an everywhere positive–definite quadratic wake functional; the wake energy remains bounded below.

The throat ontology provides a natural geometric interpretation of Eq. (86). A defect anchored to the brane is not a purely three–dimensional object: its motion excites both (i) brane–parallel transverse response in the surrounding superfluid and (ii) longitudinal response associated with flow into (and out of) the bulk through the throat. In the far zone, only the brane projection of this combined response enters the 1PN observables, but the bulk degrees of freedom renormalize the *relative* strength of the longitudinal and transverse projector pieces. The EIH match therefore constrains the throat's effective mode content rather than requiring any indefinite "Lorentzian" signature in mode space.

Earlier exploratory versions of the wake decomposition (which truncated the basis of allowed wake fields) can spuriously suggest a negative longitudinal weight. The corrected value (86) should be regarded as the consistent outcome once the full isotropic projector basis is used and the scalar, optical, and spin sectors are simultaneously respected.

7 Toward 2PN: Predictions and Falsifiability

Up to this point we have used the throat ontology primarily to reinterpret existing 1PN and electromagnetic results. In this section we look forward: we outline how a full 2PN analysis would be organized in this framework, and we sketch how the resulting finite–size corrections could, in principle, be confronted with observations. The aim is not to perform a 2PN calculation, but to make clear that such a calculation is both well–posed and constrained, rather than an open invitation to add arbitrary new terms.

7.1 What 2PN means in this toy model

In standard post–Newtonian theory, the expansion parameter is

$$\epsilon \sim \frac{GM}{rc^2} \sim \left(\frac{v}{c}\right)^2, \tag{87}$$

where M is a characteristic mass, r is a characteristic separation, and v is a characteristic orbital speed. Newtonian gravity is $\mathcal{O}(\epsilon^0)$, 1PN corrections are $\mathcal{O}(\epsilon)$, 2PN corrections are $\mathcal{O}(\epsilon^2)$, and so on. In GR this hierarchy is built into the Einstein equations via the expansion of the metric components in powers of $1/c$.

In the superfluid defect toy model, the same small parameter appears, but there is an additional geometric scale: the throat radius a (and, to a lesser extent, the depth L). We can therefore construct two independent dimensionless quantities:

$$\epsilon \sim \frac{GM}{rc^2}, \qquad \delta \sim \frac{a}{r}. \tag{88}$$

The 1PN analysis in Papers I–III effectively assumes that δ is small enough that finite–size effects can be neglected at leading order, and that all relevant corrections scale with ϵ alone. The throat ontology makes this assumption explicit: the far–field theory on the brane is controlled by the monopole mass M and the velocity–dependent interactions encoded in the vector kernel, while the structure of the throat and transition region enters only through multipole corrections suppressed by powers of δ.

In a full 2PN analysis of the toy model, one would therefore organize terms in a double expansion in ϵ and δ, keeping track of how powers of a/r enter alongside powers of $GM/(rc^2)$. Schematically, the effective action or Lagrangian for a binary system would contain terms of the form

$$L_{\text{eff}} = L_{\text{N}} + \epsilon L_{\text{1PN}} + \epsilon^2 L_{\text{2PN}} + \delta^2 \tilde{L}_{\text{2PN}} + \cdots , \tag{89}$$

where L_{2PN} contains the usual point–particle 2PN corrections and $\tilde{L}_{\text{2PN}}$ contains finite–size corrections tied to the throat geometry. The key point is that the throat ontology predicts the *structure* of $\tilde{L}_{\text{2PN}}$ in terms of a small set of geometric parameters, rather than leaving it arbitrary.

7.2 Organizing 2PN corrections via the throat geometry

Within the throat ontology, the most natural way to organize 2PN corrections is to group them according to which part of the geometry they probe:

1. *Far–field multipoles on the brane*: These are corrections to the scalar potential and its velocity–dependent companions arising from the multipole structure of ρ_{3D} and $\mathbf{v}_{3D}$ on the brane. They are controlled by the coefficients α_ℓ in the expansion

$$\Phi(r,\theta) = -\frac{GM}{r}\left[1 + \sum_{\ell\geq 2} \alpha_\ell \left(\frac{a}{r}\right)^\ell P_\ell(\cos\theta)\right], \tag{90}$$

   with α_ℓ determined by the geometry of the throat mouth and the transition region. The leading quadrupole term ($\ell = 2$) is naturally associated with 2PN finite–size effects in the orbital dynamics.

2. *Velocity–dependent couplings*: Just as the 1PN vector sector is controlled by the transverse and longitudinal parts of the vector kernel and the parameter $\alpha^2 = 3/4$, the 2PN vector and tensor sectors will receive corrections from the same kernel but with additional structure induced by the throat. For example, the effective kernel could acquire mild dependence on a and L through higher–derivative terms, leading to corrections that scale as $(a/r)^2\epsilon$ or ϵ^2. These terms would modify the coefficients of velocity–dependent structures in the many–body Lagrangian, but in a way that is tied to the same geometric data $(M, a, L, \ldots)$.

3. *Electromagnetic backreaction*: In the EM paper, charge is associated with vorticity flux threading the throat and the electromagnetic field is sourced by cavity modes inside the throat. At 2PN order, the energy stored in these modes and in the surrounding electromagnetic field will backreact on the gravitational sector. This will generate corrections coupling mass, charge, and throat geometry, schematically of the form

$$L_{\text{eff}} \supset \epsilon^2 \left(\beta_1 \frac{Q^2}{M^2} + \beta_2 \frac{Q^2}{M^2}\frac{a^2}{r^2} + \cdots\right), \tag{91}$$

with coefficients β_i determined by the cavity spectrum. Thus 2PN corrections in this model are expected to carry signatures of both mass and charge in a way that reflects the common throat origin of the gravitational and electromagnetic sectors.

A practical 2PN calculation in this framework would proceed roughly as follows:

- Choose a parametrization of the throat geometry (e.g. a family of smooth embeddings of the brane into the bulk near each defect, characterized by a, L, and a small number of shape parameters).
- Solve the 4D Euler and continuity equations perturbatively in the near field of each throat, matching onto the far–field multipole expansions on the brane and onto the cavity modes in the interior.
- Compute the induced interaction Lagrangian for a system of moving throats by integrating out the fluid degrees of freedom to the appropriate order in ϵ and δ.
- Match the resulting effective Lagrangian onto a PN–style basis of scalars built from positions, velocities, and spins, and read off the finite–size coefficients as functions of $(M, a, L, Q, \ldots)$.

While technically demanding, this program is sharply constrained: once M, a, L, and Q are fixed for each object by low–order data (Newtonian mass, EM charge, and perhaps one additional observable), there is little room to adjust higher–order coefficients without spoiling the consistency of the throat geometry.

7.3 Falsifiability and observational channels

The existence of a geometrically constrained 2PN sector raises a natural question: how could the toy model be tested or ruled out? While a full phenomenological analysis lies beyond the scope of this paper, we can identify several promising observational channels.

Binary pulsars and compact binaries. Timing measurements of binary pulsars and phase evolution of compact binaries (including black hole and neutron star mergers) are sensitive to 2PN–order corrections in the orbital dynamics. In GR, these corrections are controlled by the masses, spins, and, in some cases, tidal deformability parameters of the bodies. In the throat model, there would be additional finite–size contributions determined by the throat radii a_a and a_b and by the way the throats deform under mutual tidal fields. If the model predicts a specific relation between a and M (or between a, M, and Q), then the pattern of 2PN corrections in different systems is highly constrained. Significant inconsistencies between the inferred 2PN coefficients across different binaries would rule out the model.

Solar–system tests. Precision measurements of planetary ephemerides, light deflection, and Shapiro delay in the solar system already probe some aspects of 1PN and 2PN physics. The throat model is designed to match the 1PN sector by construction; deviations are expected to appear, if at all, in subtle finite–size effects associated with the Sun's throat geometry. For instance, small angular dependence in the effective potential or small modifications to perihelion precession beyond the standard 1PN terms could signal a non–GR finite–size structure. Any 2PN calculation in the throat model would need to be checked against the tight solar–system bounds.

Short–range gravity experiments. At laboratory scales, tests of Newton's law probe potential deviations from the $1/r^2$ force at distances comparable to or smaller than the effective size of the gravitating bodies. In the throat ontology, if the throat radius a for ordinary matter is not vanishingly small compared to these scales, one might expect measurable finite–size corrections to the force law at short range. Conversely, the absence of such deviations can be used to place upper bounds on a for laboratory masses, which must be consistent with any a–values inferred from astrophysical systems.

Electromagnetic–gravitational cross–checks. Because mass and charge are different projections of the same throat geometry, the model predicts relationships between gravitational and electromagnetic observables that have no analogue in GR. For example, if a is related to Q via the enthalpy–selected aspect ratio and the cavity structure, then objects with different charge–to–mass ratios should have different patterns of finite–size gravitational corrections at 2PN order. If observations show no such dependence (within experimental uncertainties), or if the required $a(Q, M)$ relations are inconsistent across systems, the model would be disfavored.

In all of these channels, the logical structure is the same: the throat ontology fixes how 2PN coefficients depend on a small set of geometric parameters; observations overdetermine those parameters; inconsistency rules out the model. In this sense, the model is, at least in principle, falsifiable.

7.4 Why the ontology paper must come before a full 2PN analysis

It is natural to ask why we have not simply proceeded directly to a 2PN calculation. The answer is that, without the throat ontology, the space of possible 2PN corrections in the superfluid defect model is too large and too poorly organized to be meaningfully constrained.

Viewed purely as a 3D fluid model with defects, one could write down an enormous number of higher–order terms in the effective action, including arbitrary multipole couplings, higher–derivative corrections to the vector kernel, and a variety of nonlocal interactions. Without additional structure, there is no principled way to decide which of these terms should be present, how large their coefficients should be, or how they should be related across different sectors (gravity vs electromagnetism) and different systems.

The throat ontology changes this situation in three important ways:

- It *geometrizes* finite–size effects, tying them to the geometry of brane–bulk throats (radius a, depth L, shape of the mouth, mode spectrum in the interior), rather than treating them as arbitrary parameters in a 3D effective theory.

- It *links* the gravitational and electromagnetic sectors: the same throat that determines the monopole mass and the 1PN vector couplings also determines the cavity structure and the enthalpy selection of L/a in the EM sector. Any 2PN corrections must be compatible with this shared origin.

- It *explains* the effective projector signature in the 1PN vector sector as a consequence of brane–bulk mode mixing, rather than as an ad hoc choice in the sign of the longitudinal kernel. This strongly constrains how higher–order vector and tensor corrections can appear.

With these ingredients in place, a 2PN calculation is no longer a fishing expedition in an enormous parameter space but a targeted computation in a geometric setting with clear boundary conditions. The present paper is therefore a necessary precursor: it defines the ontology in which the 2PN problem is posed, identifies the small set of geometric parameters that control the finite–size structure, and clarifies the relationships among the different sectors of the model.

In the next and final section we step back from technicalities and discuss the broader implications of the throat ontology, including open questions about the bottom of the throat, the stabilization of the brane, and the behavior of multiple defects interacting through the bulk.

8 Discussion and Outlook

We have proposed a brane–bulk throat ontology for the superfluid defect toy universe developed in Papers I–IV and shown how it resolves several apparent tensions in the earlier work. In this final section we summarize the main elements of this picture, highlight the open questions it raises, and outline directions for future work.

8.1 Summary of the ontology and main results

The central idea of this paper is that the defects representing massive, possibly charged bodies in the superfluid toy model should be understood as *throats* connecting a three–dimensional brane to a four–dimensional superfluid bulk, rather than as purely three–dimensional objects. Each defect corresponds to a localized region where the brane at $w = 0$ pinches into the bulk, forming a tube–like region of radius a and depth L.

From the perspective of a brane–bound observer, the intersection of a throat with the brane is a nearly spherical mouth of radius a. The effective 3D density and velocity fields induced on the brane by drainage into this mouth reproduce the spherical sink picture used in the orbital, optical, and spin/N–body papers: at distances $r \gg a, L$, the defect appears as a point mass with potential $\Phi \sim -GM/r$ and 1PN corrections controlled by the same hydrodynamic energy functional that was matched to the EIH Lagrangian. Dimensional reduction from 4D to 3D shows that finite–size effects from the throat interior and transition region enter the far field only through multipole corrections suppressed by $(a/r)^2$ and higher powers.

From the perspective of the bulk, the region $0 < w < L$ with $|\mathbf{x}| \lesssim a$ is approximately a cylindrical cavity of cross–sectional area πa^2 and length L. The 4D acoustic wave equation separated inside this cavity yields Bessel–type radial modes and standing waves along w; imposing boundary conditions at $r = a$ and at $w = 0, L$ picks out a discrete mode spectrum. Minimizing the enthalpy at fixed "charge" selects a preferred aspect ratio $L/a = \sqrt{2}\pi/x_{01} \approx 1.85$ for the fundamental mode, exactly as found in the electromagnetic paper. In the throat ontology, this result is reinterpreted as a geometric property of the throat, not as an arbitrary parameter choice for a 3D cylinder.

The throat picture also clarifies the physical meaning of mass and charge in the model. Mass is associated with drainage of fluid into the throat and the resulting volume deficit in the surrounding superfluid, with $M \sim \rho_0 \pi a^2 L$ up to order–unity factors. Charge is associated with circulation and vorticity flux threading the throat cross–section; in the hydrodynamic–electromagnetic dictionary, this vorticity flux plays the role of electric charge, and the cavity modes inside the throat generate the effective electromagnetic fields. Thus mass and charge are different projections of the same geometric object—a brane–bulk throat characterized by (a, L) and its internal mode structure.

In the near–field region around the throat mouth, where streamlines bend from radial inflow on the brane to axial flow into the bulk, the geometry is strongly curved and the flow experiences significant shear and pressure gradients. Coarse–graining this region onto the brane generates a tower of multipole corrections to the effective 3D density and potential, with a leading quadrupole term suppressed by $(a/r)^2$. This provides a natural geometric interpretation of finite–size effects and an organizing principle for 2PN corrections: once the throat radius a is fixed for a given defect, the coefficients of these multipoles are determined by the geometry of the throat mouth and its deformation under external fields.

Finally, the throat ontology offers a new perspective on the wake mixing $\alpha^2 = 3/4$ fixed in the 1PN vector sector. In the vector kernel that mediates velocity–dependent interactions, transverse (brane–parallel) and longitudinal (bulk–directed) response enter with different *weights*. The EIH matching selects the specific real weighting encoded by $\alpha^2 = 3/4$, consistent with a positive–definite quadratic wake functional. In the throat picture this parameter quantifies how brane motion couples to bulk–directed flow through the throat and thereby renormalizes the effective longitudinal/transverse projector content seen on the brane.

8.2 What happens at the bottom of the throat?

Throughout this paper we have treated the throat as a finite tube of depth L whose bottom at $w = L$ is left deliberately unspecified. This is sufficient for the 1PN and near–field considerations we have focused on, but it leaves open an important question about the global structure of the model: what happens at the bottom of the throat?

Several possibilities present themselves:

- *Closed cavities.* The throat could close off smoothly, with the brane folding back into itself and the superfluid forming a finite cavity. In this case the mode spectrum inside the throat is purely discrete, and there is no leakage of energy or vorticity into the deep bulk. Charge and mass would then be strictly localized objects, with their values determined entirely by the geometry and mode content of the closed throat.

- *Open throats.* The throat could open into a larger bulk region at $w = L$, allowing acoustic energy and vorticity to escape into the bulk. This would introduce damping and radiation channels not present in the closed–cavity picture. For instance, highly dynamical processes such as mergers could excite bulk modes that carry energy away from the brane, potentially modifying the gravitational wave signal seen by brane–bound observers.

- *Connected throats.* Different defects could be connected through a common bulk region, with throats that meet or merge away from the brane. This would allow for bulk–mediated interactions between defects, beyond those encoded in the effective 3D theory on the brane. In extreme cases, one could imagine networks of throats whose connectivity changes dynamically, perhaps in analogy with reconnection events in vortex tangles.

Each of these scenarios would leave a characteristic imprint on the mode spectrum in the throat, the time dependence of charge and mass, and the way energy is exchanged between defects and the bulk. For example, an open throat might lead to slow leakage of vorticity and hence slow evolution of charge, while a closed cavity would make charge strictly conserved. The present paper does not attempt to choose among these options, but the framework we have developed is flexible enough to incorporate any of them once additional microscopic input is specified.

8.3 What stabilizes the brane?

Another major open question is the origin and stability of the brane itself. In our construction, the brane at $w = 0$ is treated as a given: a distinguished hypersurface on which observers live and on which the effective 3D physics of Papers I–IV is realized. However, in a more complete model the brane should arise as a dynamical object, stabilized by some underlying mechanism.

Several possibilities are familiar from other contexts:

- *Domain walls or defects.* The brane might be a domain wall in an underlying order parameter of the superfluid, separating regions with different phases or densities in the bulk. The tension of the domain wall and the coupling of the superfluid to this order parameter would determine the brane's rigidity and its response to the presence of throats.

- *Potential minima.* The brane could be a locus where some effective potential $V(w)$ for the fluid has a minimum, pinning density and pressure at $w = 0$ and making deviations in the w direction energetically costly. Throats would then be localized regions where this pinning is overcome or modified.

- *Defect condensation.* The brane could represent a condensate of lower–dimensional defects in the bulk superfluid, with throats corresponding to higher–dimensional defects that connect different regions of the condensate.

Any such mechanism would feed back into the throat ontology by determining the allowed shapes of the brane near defects, the cost of bending the brane into the bulk, and the boundary conditions satisfied by the fluid at $w = 0$. In particular, the spectrum of small fluctuations of the brane could interact with the cavity modes in the throats, potentially modifying the relation between L/a, M, and Q. Exploring specific brane stabilization mechanisms and their consequences is an important direction for future work.

8.4 Multiple defects and bulk interactions

So far we have focused on a single isolated throat. In reality, the toy universe contains many defects, representing stars, planets, compact objects, and possibly charged bodies. The throat ontology provides a conceptual framework for thinking about how these defects interact through the bulk as well as through the effective 3D fields on the brane.

At the level of the effective 3D theory, multiple throats interact via the superposition of their monopole and multipole fields on the brane, as captured by the 1PN and (future) 2PN Lagrangians. This is the regime directly comparable to GR. However, the throats also interact through the bulk superfluid: their interior modes can scatter off each other, their transition regions can overlap when defects come close, and their bottom regions at $w = L$ can come into contact if throats are sufficiently deep.

In a merger of two compact objects, for example, one might imagine the following sequence: the throats approach each other on the brane; their transition regions deform and eventually overlap; the interior cavities interact and exchange energy; finally, a single, larger throat emerges, possibly with a different L/a ratio and a different internal mode spectrum. From the brane perspective this would be seen as a change in the effective mass, spin, and charge of the merged object, together with the emission of gravitational and electromagnetic waves. From the bulk perspective there would also be emission of acoustic radiation and rearrangements of vorticity in the deep bulk.

While these speculations are necessarily qualitative at this stage, they point to a rich dynamical phenomenology that goes beyond what is captured in a purely 3D effective theory. A long–term goal of the program initiated by this toy model would be to simulate such multi–throat dynamics directly in the 4D superfluid and compare the emergent signals (both on the brane and in the bulk) with those predicted by GR and standard electromagnetism.

8.5 Relation to brane-world and emergent gravity scenarios

The brane–bulk throat ontology developed here sits conceptually between several strands of work in the broader theoretical physics literature.

On the one hand, it shares with brane–world scenarios the idea that the observable universe is a lower–dimensional hypersurface embedded in a higher–dimensional space, and that localized objects may be associated with tubes or funnels connecting the brane to the bulk. On the other hand, it shares with emergent gravity and superfluid vacuum models the idea that gravitational and electromagnetic phenomena can arise as effective excitations of an underlying medium, with metric structures and gauge fields emerging from the dynamics of collective modes.

However, the present construction is more concrete and mechanical than many of these frameworks. The underlying degrees of freedom are those of a compressible superfluid obeying continuity and Euler equations in a specified number of spatial dimensions; the brane is a geometric hypersurface in this fluid; and defects are explicit deformations of the brane into the bulk. The emergent metric and electromagnetic fields are not fundamental fields in a gravitational action, but effective descriptors of the collective motion of the fluid. This keeps the model firmly in the realm of a toy universe: it is not proposed as a literal description of our universe, but as a playground in which questions about emergence, locality, and finite–size structure can be posed sharply and answered with explicit calculations.

Within this playground, the throat ontology unifies several seemingly disparate features of the earlier papers: the need for spherical sinks in the 1PN gravitational sector, the cylindrical cavity in the EM sector, the wake mixing $\alpha^2 = 3/4$, and the small but nonzero shape–sensitivity of orbital dynamics. All of these phenomena emerge as different aspects of the same geometric structure: brane–bulk throats of radius a and depth L embedded in a 4D superfluid.

To move beyond the toy universe, at least conceptually, one would need to address the microscopic origin of the superfluid, the dynamical nature of the brane, the behavior of the model in strong–field and high–curvature regimes, and the relationship between the emergent effective metric and a fundamental description of spacetime. These are ambitious questions, and we do not attempt to answer them here. Our more modest claim is that, within the limited scope of weak–field, slow–motion phenomena, the throat ontology provides a coherent and falsifiable framework in which gravity and electromagnetism emerge from a single mechanical medium, and in which the next steps—a full 2PN analysis and detailed comparisons with observations —are clearly defined.

If that program succeeds, the superfluid defect toy universe will have served its purpose: not as a rival to GR or Maxwell theory, but as a concrete example of how a higher–dimensional, mechanical substrate can give rise to metric gravity, gauge fields, and finite–size structure in a controlled effective description. If it fails, it will fail in an instructive way, highlighting which aspects of our current theories of gravity and electromagnetism are hardest to reproduce in any emergent, medium–based picture.

A Gaussian 4D Density and 3D Multipole Structure

In this appendix we work out the Gaussian toy model of Sec. 3.2 in detail. The goal is to make explicit how a localized, mildly aspherical 4D density profile produces an effective 3D source with monopole mass $M \propto \rho_0 L a^3$ and quadrupole moment $Q/M \sim \varepsilon a^2$.

A.1 Setup: 4D density and projection to the brane

We start from the 4D density profile introduced in Eq. (33), written here for convenience as

$$\rho_4(r,\theta,w) = \rho_0 \exp\left(-\frac{r^2}{a^2}\right) \exp\left(-\frac{w^2}{L^2}\right) [1 + \varepsilon P_2(\cos\theta)] , \tag{92}$$

where (r, θ, ϕ) are spherical coordinates on the brane, w is the bulk coordinate, a is the characteristic brane–radius of the throat, L is its depth along w, ρ_0 is a reference density, and ε is a small dimensionless parameter controlling the angular asymmetry. The $\ell = 2$ Legendre polynomial is

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1). \tag{93}$$

For $\varepsilon = 0$ the density is spherically symmetric on the brane and separable in r and w; for $\varepsilon \neq 0$ the density has a mild quadrupolar distortion localized within $r \lesssim a$.

To obtain the effective 3D density on the brane we integrate over w, taking the projection kernel $K(w) = 1$ for simplicity:

$$\rho_{\mathrm{3D}}(r, \theta) \equiv \int_{-\infty}^{+\infty} \rho_4(r, \theta, w)\,\mathrm{d}w. \tag{94}$$

Using the Gaussian integral

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{w^2}{L^2}\right)\mathrm{d}w = \sqrt{\pi}\,L, \tag{95}$$

we obtain

$$\rho_{\mathrm{3D}}(r, \theta) = \sqrt{\pi}\,L\rho_0\,\exp\left(-\frac{r^2}{a^2}\right)\left[1 + \varepsilon P_2(\cos\theta)\right], \tag{96}$$

which is Eq. (34) in the main text.

A.2 Total mass of the effective 3D source

The total effective mass M associated with this density is

$$M = \int \rho_{\mathrm{3D}}(r, \theta)\,\mathrm{d}^3x. \tag{97}$$

In spherical coordinates (r, θ, ϕ) this becomes

$$M = \int_0^{2\pi}\mathrm{d}\phi \int_0^{\pi}\mathrm{d}\theta \int_0^{\infty}\mathrm{d}r\ \rho_{\mathrm{3D}}(r, \theta)\,r^2\sin\theta. \tag{98}$$

Substituting Eq. (96) yields

$$M = \sqrt{\pi}\,L\rho_0 \int_0^{\infty}\mathrm{d}r\ r^2\exp\left(-\frac{r^2}{a^2}\right)\int_0^{\pi}\mathrm{d}\theta\ \sin\theta\left[1 + \varepsilon P_2(\cos\theta)\right]\int_0^{2\pi}\mathrm{d}\phi. \tag{99}$$

The ϕ integral simply yields a factor of 2π. The angular integrals are

$$\int_0^{\pi}\sin\theta\,\mathrm{d}\theta = 2, \tag{100}$$

$$\int_0^{\pi}\sin\theta\,P_2(\cos\theta)\,\mathrm{d}\theta = 0, \tag{101}$$

where the second equality follows from the orthogonality of Legendre polynomials. Thus the εP_2 piece does not contribute to the total mass, as expected for a pure quadrupole distortion. We are left with

$$M = \sqrt{\pi}\, L\rho_0\, (2\pi) \left(\int_0^\infty r^2 e^{-r^2/a^2} \mathrm{d}r \right) (2). \tag{102}$$

Using the standard Gaussian integral

$$\int_0^\infty r^2 e^{-r^2/a^2}\, \mathrm{d}r = \frac{\sqrt{\pi}}{4} a^3, \tag{103}$$

we find

$$M = \sqrt{\pi}\, L\rho_0\, (2\pi) \left(\frac{\sqrt{\pi}}{4} a^3 \right) (2) = \pi^2 L a^3 \rho_0. \tag{104}$$

This is Eq. (35) in the main text. Up to the overall numerical factor π^2, the mass scales as

$$M \sim \rho_0 L a^3, \tag{105}$$

as expected for a throat–like object with size a and depth L.

A.3 Quadrupole moment and the scaling $Q/M \sim \varepsilon a^2$

To compute the leading angular deviation from spherical symmetry, we consider a quadrupole–like scalar moment defined by

$$Q_{20} \equiv \int \rho_{\mathrm{3D}}(r,\theta)\, r^2 P_2(\cos\theta)\, \mathrm{d}^3x, \tag{106}$$

up to an overall normalization convention. Substituting Eq. (96) and writing out the volume element gives

$$Q_{20} = \sqrt{\pi}\, L\rho_0 \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \int_0^\infty \mathrm{d}r\ \exp\!\left(-\frac{r^2}{a^2}\right) [1 + \varepsilon P_2(\cos\theta)]\, r^4 P_2(\cos\theta) \sin\theta. \tag{107}$$

Again the ϕ integral gives 2π. Because P_2 is orthogonal to the constant function on the unit sphere, only the term proportional to ε survives:

$$Q_{20} = \sqrt{\pi}\, L\rho_0 (2\pi)\varepsilon \left(\int_0^\infty r^4 e^{-r^2/a^2}\, \mathrm{d}r \right) \left(\int_0^\pi \sin\theta\, [P_2(\cos\theta)]^2 \mathrm{d}\theta \right). \tag{108}$$

We now evaluate the radial and angular factors separately.

For the radial integral, we use the standard formula

$$\int_0^\infty r^{2n} e^{-\lambda r^2}\, \mathrm{d}r = \frac{(2n-1)!!}{2^{n+1}} \sqrt{\frac{\pi}{\lambda^{2n+1}}}, \tag{109}$$

with $n = 2$ and $\lambda = 1/a^2$, giving

$$\int_0^\infty r^4 e^{-r^2/a^2}\, \mathrm{d}r = \frac{3\sqrt{\pi}}{8} a^5. \tag{110}$$

For the angular integral, we exploit orthogonality of Legendre polynomials:

$$\int_0^\pi \sin\theta\,[P_\ell(\cos\theta)]^2 \mathrm{d}\theta = \frac{2}{2\ell+1}. \tag{111}$$

Setting $\ell = 2$ yields

$$\int_0^\pi \sin\theta\,[P_2(\cos\theta)]^2 \mathrm{d}\theta = \frac{2}{5}. \tag{112}$$

Putting these pieces together, we find

$$Q_{20} = \sqrt{\pi}\,L\rho_0(2\pi)\varepsilon\left(\frac{3\sqrt{\pi}}{8}a^5\right)\left(\frac{2}{5}\right) \tag{113}$$

$$= \frac{3}{10}\,\pi^2 La^5\varepsilon\rho_0. \tag{114}$$

This is Eq. (37) in the main text.

Dividing Eq. (114) by the mass Eq. (104) we obtain

$$\frac{Q_{20}}{M} = \frac{\frac{3}{10}\pi^2 La^5\varepsilon\rho_0}{\pi^2 La^3\rho_0} = \frac{3}{10}\,\varepsilon a^2. \tag{115}$$

Up to the numerical factor 3/10, this confirms the scaling

$$\frac{Q}{M} \sim \varepsilon a^2, \tag{116}$$

used in Sec. 3.2. The quadrupole moment per unit mass is proportional to the square of the source size a and to the dimensionless asymmetry parameter ε.

A.4 Remarks and generalizations

A few comments are in order:

- The precise numerical coefficients (π^2, 3/10, etc.) depend on our choice of Gaussian profile and normalization conventions for the multipole moments. Different smooth, localized profiles with the same characteristic size a and small quadrupolar distortion εP_2 would yield different order–unity factors but the same scaling $M \sim \rho_0 La^3$ and $Q/M \sim \varepsilon a^2$.

- The fact that the ε–dependent term drops out of the total mass but dominates the quadrupole is a direct consequence of the orthogonality of Legendre polynomials. This is generic: pure multipole distortions do not alter the monopole mass but do control the higher multipole moments.

- Higher multipoles can be generated by replacing $P_2(\cos\theta)$ in Eq. (92) with $P_\ell(\cos\theta)$ for $\ell > 2$, or by adding a sum $\sum_\ell \varepsilon_\ell P_\ell(\cos\theta)$. A similar calculation then yields

$$\frac{Q_\ell}{M} \sim \varepsilon_\ell a^\ell, \tag{117}$$

up to order–unity coefficients, leading to potential corrections of order $(a/r)^\ell$ in the far field.

- The exponential factors in r and w are chosen for analytic convenience. In the context of the throat ontology, one should view this Gaussian as a caricature of the true density profile near the throat mouth: it captures the localization at scales $r \sim a$, $w \sim L$ and the leading angular structure without pretending to be a faithful solution of the full 4D Euler equations.

These calculations justify the scaling arguments used in the main text: dimensional reduction of a localized 4D throat–like density produces an effective 3D source whose finite–size corrections to the potential are naturally organized as a multipole expansion with coefficients suppressed by powers of (a/r) and controlled by a small set of geometric parameters $(a, L, \varepsilon_\ell, \ldots)$.

B Numerical Evaluation of a Physical Throat Model

In this appendix we detail the numerical evaluation of the finite–size coefficient α_2 quoted in Sec. 5.4. Unlike the analytic Gaussian model of Appendix A, this calculation uses a hard–bounded "rounded funnel" geometry that explicitly connects the cylindrical bulk interior to the brane mouth.

B.1 Geometry and Integration

We use spherical coordinates (r, θ, ϕ) on the brane and a depth coordinate w into the bulk. The throat domain $\mathcal{T}$ is bounded by $0 \le w \le L$ and $0 \le r \le R(w)$, where the radius function flares near the brane according to

$$R(w) = a\left[1 + \frac{1}{2}\exp\left(-\frac{5w}{a}\right)\right]. \tag{118}$$

This profile describes a cylinder of radius a that widens to $1.5a$ at the mouth ($w = 0$), providing a simple proxy for the smooth bending of streamlines in the transition region.

The effective density within this volume is taken to be

$$\rho(r, \theta, w) = \rho_0\big[1 + \varepsilon P_2(\cos\theta)\big], \tag{119}$$

where ε parametrizes a small quadrupolar anisotropy. The monopole mass M and quadrupole moment Q entering the far–field potential (81) are obtained by integrating over $\mathcal{T}$:

$$M = \int_{\mathcal{T}} \rho\, \mathrm{d}^3x\, \mathrm{d}w = \int_0^L \mathrm{d}w \int \mathrm{d}\Omega \int_0^{R(w)} r^2\, \mathrm{d}r\, \rho(r, \theta, w), \tag{120}$$

$$Q = \int_{\mathcal{T}} r^2 P_2(\cos\theta)\, \rho\, \mathrm{d}^3x\, \mathrm{d}w = \int_0^L \mathrm{d}w \int \mathrm{d}\Omega \int_0^{R(w)} r^2\, \mathrm{d}r\, r^2 P_2(\cos\theta)\, \rho(r, \theta, w), \tag{121}$$

with $\mathrm{d}\Omega = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$. Using $\int \mathrm{d}\Omega\, P_2 = 0$ and $\int \mathrm{d}\Omega\, P_2^2 = 4\pi/5$, the angular integrals reduce these expressions to one–dimensional integrals over the funnel profile:

$$M = \frac{4\pi\rho_0}{3} \int_0^L R(w)^3\, \mathrm{d}w, \qquad Q = \frac{4\pi\rho_0\, \varepsilon}{25} \int_0^L R(w)^5\, \mathrm{d}w. \tag{122}$$

We evaluate these integrals numerically for the parameter choices quoted below.

B.2 Results

For the representative parameter set $a = 1.0$, $L = 2.0$, $\rho_0 = 1.0$, and anisotropy $\varepsilon = 0.1$, numerical quadrature yields

$$M \approx 9.98, \qquad Q \approx 0.143. \tag{123}$$

The dimensionless finite–size coefficient α_2 is defined by $\alpha_2 = Q/(Ma^2)$, so the above values imply

$$\alpha_2 = \frac{0.143}{9.98 \times (1.0)^2} \approx 0.014. \tag{124}$$

Thus, for a physically bounded throat geometry with modest anisotropy, the leading multipole coefficient is small ($\alpha_2 \ll 1$), ensuring that the defect behaves as an effectively spherical source in the far field.

C 4D Mode Separation in the Cylindrical Throat

In this appendix we record the basic mode structure of the 4D acoustic equation inside the cylindrical throat. The aim is to make explicit the separation of variables leading to Bessel–type radial profiles and standing waves along the bulk direction w, together with the corresponding eigenfrequencies and orthogonality relations. This is the mode structure used implicitly in Sec. 4 and in the electromagnetic paper when discussing cavity modes and the enthalpy–selected aspect ratio L/a.

C.1 Acoustic equation and throat geometry

We begin from the linear 4D acoustic equation for the enthalpy perturbation h,

$$\partial_t^2 h - c_s^2 \nabla_4^2 h = 0, \qquad \nabla_4^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 + \partial_w^2, \tag{125}$$

as derived in Sec. 2.1. Inside the throat we approximate the geometry as a straight cylinder of radius a in the brane directions and depth L along the bulk direction w. We choose coordinates adapted to this geometry:

$$(r, \phi, w), \tag{126}$$

where (r, ϕ) are polar coordinates in a cross–section through the throat mouth on the brane (with r measured from the throat center), and $w \in [0, L]$ measures distance along the throat into the bulk. For the lowest–lying modes of interest we assume axisymmetry in ϕ and no dependence on the third brane direction orthogonal to the throat axis.

In this approximation the Laplacian restricted to axisymmetric configurations becomes

$$\nabla_4^2 h = \frac{1}{r}\partial_r(r\partial_r h) + \partial_w^2 h, \tag{127}$$

so the wave equation reads

$$\partial_t^2 h - c_s^2 \left[\frac{1}{r}\partial_r(r\partial_r h) + \partial_w^2 h\right] = 0. \tag{128}$$

C.2 Separation of variables and eigenvalue problem

We look for separated, time–harmonic solutions of the form

$$h(t, r, w) = \mathrm{Re}\left\{H(r)\, W(w)\, e^{-i\omega t}\right\}, \tag{129}$$

with real frequency ω. Substituting Eq. (129) into Eq. (128) and dividing by $H(r)W(w)e^{-i\omega t}$ yields

$$-\omega^2 - c_s^2 \left[\frac{1}{H}\frac{1}{r}\partial_r(r\partial_r H) + \frac{1}{W}\partial_w^2 W\right] = 0. \tag{130}$$

Rearranging gives

$$\frac{1}{H}\frac{1}{r}\partial_r(r\partial_r H) + \frac{1}{W}\partial_w^2 W = -\frac{\omega^2}{c_s^2}. \tag{131}$$

The left–hand side is a sum of a function of r and a function of w, while the right–hand side is a constant, so each term must separately be equal to a constant. Introducing separation constants $-k_r^2$ and $-k_w^2$, we write

$$\frac{1}{H}\frac{1}{r}\partial_r(r\partial_r H) = -k_r^2, \tag{132}$$

$$\frac{1}{W}\partial_w^2 W = -k_w^2, \tag{133}$$

and the dispersion relation

$$\omega^2 = c_s^2(k_r^2 + k_w^2). \tag{134}$$

Equations (132)–(133) constitute a pair of ordinary differential equations:

$$\frac{1}{r}\partial_r(r\partial_r H) + k_r^2 H = 0, \tag{135}$$

$$\partial_w^2 W + k_w^2 W = 0. \tag{136}$$

The allowed values of k_r and k_w are determined by boundary conditions on $H(r)$ and $W(w)$ inside the throat.

C.3 Radial modes and Bessel functions

The radial equation (135) is the standard Bessel equation of order zero. Its general solution is

$$H(r) = AJ_0(k_r r) + BY_0(k_r r), \tag{137}$$

where J_0 and Y_0 are Bessel functions of the first and second kind, respectively, and A, B are constants. Regularity at $r = 0$ requires $B = 0$, since $Y_0(x)$ is singular at the origin. We therefore take

$$H(r) = AJ_0(k_r r). \tag{138}$$

At the throat wall $r = a$ we impose a boundary condition on h corresponding to a pinned phase boundary of the stiff superfluid vacuum. Concretely, we take a Dirichlet condition on the enthalpy perturbation,

$$h(r = a, w, t) = 0 \quad \Rightarrow \quad H(a) = 0, \tag{139}$$

This pinning stabilizes the cylindrical throat against radial collapse: expansions or contractions of the wall necessarily excite bulk modes that raise the vacuum enthalpy. This holds for all w and t and implies

$$J_0(k_r a) = 0. \tag{140}$$

Let x_{0n} denote the nth zero of $J_0(x)$, with

$$0 < x_{01} < x_{02} < \cdots . \tag{141}$$

Then the allowed radial wavenumbers are

$$k_r^{(n)} = \frac{x_{0n}}{a}, \qquad n = 1, 2, \ldots, \tag{142}$$

and the corresponding radial eigenfunctions are

$$H_n(r) = A_n J_0\left(\frac{x_{0n} r}{a}\right). \tag{143}$$

These eigenfunctions satisfy an orthogonality relation with respect to the measure $r\,\mathrm{d}r$:

$$\int_0^a r J_0\left(\frac{x_{0m} r}{a}\right) J_0\left(\frac{x_{0n} r}{a}\right) \mathrm{d}r = \frac{a^2}{2} \left[J_1(x_{0n})\right]^2 \delta_{mn}, \tag{144}$$

where J_1 is the Bessel function of order one and δ_{mn} is the Kronecker delta. One can therefore choose the normalization constants A_n so that the $H_n(r)$ form an orthonormal basis on $[0, a]$.

C.4 Axial modes and standing waves along w

The axial equation (136) has the general solution

$$W(w) = C\cos(k_w w) + D\sin(k_w w), \tag{145}$$

with C, D constants. The appropriate boundary conditions depend on the microscopic physics at the mouth ($w = 0$) and bottom ($w = L$) of the throat. For definiteness, and to match the cavity analysis in the main text, we impose Dirichlet conditions at both ends:

$$W(0) = W(L) = 0. \tag{146}$$

The condition $W(0) = 0$ implies $C = 0$, so $W(w) = D\sin(k_w w)$. The condition $W(L) = 0$ then requires

$$\sin(k_w L) = 0 \quad \Rightarrow \quad k_w L = n\pi, \qquad n = 1, 2, \ldots, \tag{147}$$

so the allowed axial wavenumbers are

$$k_w^{(n)} = \frac{n\pi}{L}, \qquad n = 1, 2, \ldots, \tag{148}$$

and the axial eigenfunctions are

$$W_n(w) = D_n \sin\left(\frac{n\pi w}{L}\right). \tag{149}$$

These functions are orthogonal on $[0, L]$:

$$\int_0^L \sin\Big(\frac{m\pi w}{L}\Big) \sin\Big(\frac{n\pi w}{L}\Big)\, \mathrm{d}w = \frac{L}{2}\, \delta_{mn}, \tag{150}$$

so the constants D_n can be chosen to yield an orthonormal set.

Other choices of boundary conditions (Neumann or mixed) are possible and would lead to cosine or sine–cosine combinations with shifted eigenvalues, but the qualitative structure—a discrete tower of axial modes with $k_w \sim n\pi/L$—is the same.

C.5 Mode spectrum and fundamental throat mode

Combining the radial and axial modes, the general separated solution inside the throat is a superposition of eigenmodes labeled by a pair of integers (m, n):

$$h_{mn}(t, r, w) = A_{mn} J_0\Big(\frac{x_{0m} r}{a}\Big) \sin\Big(\frac{n\pi w}{L}\Big) \cos(\omega_{mn} t + \varphi_{mn}), \tag{151}$$

where A_{mn} and φ_{mn} are real amplitude and phase, and the eigenfrequencies are given by

$$\omega_{mn}^2 = c_s^2 \left[\Big(\frac{x_{0m}}{a}\Big)^2 + \Big(\frac{n\pi}{L}\Big)^2\right], \qquad m, n = 1, 2, \ldots. \tag{152}$$

The lowest–lying mode (apart from any zero modes associated with trivial symmetries) is the fundamental $(m, n) = (1, 1)$ mode:

$$h_{11}(t, r, w) \propto J_0\Big(\frac{x_{01} r}{a}\Big) \sin\Big(\frac{\pi w}{L}\Big) \cos(\omega_{11} t), \qquad \omega_{11}^2 = c_s^2 \left[\Big(\frac{x_{01}}{a}\Big)^2 + \Big(\frac{\pi}{L}\Big)^2\right]. \tag{153}$$

This is the mode used in Sec. 4.1 and in the enthalpy–minimization argument for the preferred aspect ratio L/a.

C.6 Normalization and enthalpy functional (sketch)

For completeness, we briefly record how the normalization of the modes enters the enthalpy functional, leaving detailed variations to the main text and to the electromagnetic paper.

The time–averaged perturbation energy (or enthalpy) associated with a mode h can be written schematically as

$$\mathcal{E}[h] = \frac{1}{2} \int_{\mathcal{T}} \mathrm{d}V_4\, \rho_0 \left[\frac{1}{c_s^2} \langle(\partial_t h)^2\rangle + \langle(\nabla_4 h)^2\rangle\right], \tag{154}$$

where $\mathrm{d}V_4 = r\, \mathrm{d}r\, \mathrm{d}\phi\, \mathrm{d}w$ is the 4D volume element restricted to the throat and $\langle\cdot\rangle$ denotes an average over one oscillation period.

For a single eigenmode of the form (151) with frequency ω_{mn}, the time average gives

$$\langle(\partial_t h_{mn})^2\rangle = \frac{1}{2} \omega_{mn}^2 A_{mn}^2 J_0^2\Big(\frac{x_{0m} r}{a}\Big) \sin^2\Big(\frac{n\pi w}{L}\Big), \tag{155}$$

$$\langle(\nabla_4 h_{mn})^2\rangle = \frac{1}{2} A_{mn}^2 \left[\Big(\frac{x_{0m}}{a}\Big)^2 + \Big(\frac{n\pi}{L}\Big)^2\right] J_0^2\Big(\frac{x_{0m} r}{a}\Big) \sin^2\Big(\frac{n\pi w}{L}\Big), \tag{156}$$

so that the integrand in Eq. (154) is proportional to $(k_r^2 + k_w^2) A_{mn}^2 J_0^2 \sin^2$, with $k_r = x_{0m}/a$ and $k_w = n\pi/L$. Using the orthogonality relations (144) and (150), one finds that the total energy stored in the mode scales as

$$\mathcal{E}_{mn} \propto A_{mn}^2 (k_r^2 + k_w^2)\, a^2 L, \tag{157}$$

up to order–unity constants coming from the angular integrals and normalization choices.

A "charge"–like quantity associated with the mode amplitude, such as the vorticity flux through the throat cross–section, typically scales as

$$\mathcal{Q}_{mn} \propto A_{mn}^2 a^2 L, \tag{158}$$

again up to order–unity factors. At fixed $\mathcal{Q}_{mn}$, the energy is therefore proportional to $k_r^2 + k_w^2$:

$$\mathcal{E}_{mn} \propto (k_r^2 + k_w^2)\, \mathcal{Q}_{mn}. \tag{159}$$

For the fundamental mode $(m, n) = (1, 1)$, this reduces to

$$\mathcal{E}_{11} \propto \left[\left(\frac{x_{01}}{a} \right)^2 + \left(\frac{\pi}{L} \right)^2 \right] \mathcal{Q}_{11}. \tag{160}$$

The enthalpy–minimization problem at fixed "charge" discussed in Sec. 4.2 and in the electromagnetic paper is equivalent to minimizing this combination with respect to a and L, subject to appropriate constraints (e.g. fixed mass and/or other geometric quantities). The resulting optimum selects a particular aspect ratio L/a, which in the toy model of Paper IV and in the present ontology is

$$\frac{L}{a} = \frac{\sqrt{2}\,\pi}{x_{01}}, \tag{161}$$

interpreted as a geometric property of the throat rather than an externally imposed parameter.

We will not repeat the full variational calculation here; the purpose of this appendix is to show how the basic mode structure and its dependence on a and L arise from the 4D acoustic equation inside the cylindrical throat.

D Two-Mode Toy Model for $\alpha^2 = 3/4$

It is useful to keep a minimal algebraic picture in mind for the meaning of the wake–mixing parameter α^2 that appears in the isotropic longitudinal / transverse projector decomposition of the vector kernel.

Consider two quadratic degrees of freedom, a transverse amplitude u_T and a longitudinal amplitude u_L, with a positive–definite energy

$$E = \frac{1}{2} \left(A_T u_T^2 + A_L u_L^2 \right), \tag{162}$$

where $A_T > 0$ and $A_L > 0$. Suppose that the physical (EIH–mediating) combination excited by a moving throat involves a fixed mixing $u_L = \alpha\, u_T$. Substituting into Eq. (162) gives an effective single–mode energy

$$E = \frac{1}{2} \left(A_T + \alpha^2 A_L \right) u_T^2. \tag{163}$$

For the calibrated value $\alpha^2 = 3/4$,

$$E = \frac{1}{8}\,(4A_T + 3A_L)\,u_T^2, \tag{164}$$

which is manifestly positive if $A_T, A_L > 0$.

In matrix language the quadratic form in (162) has eigenvalues $\{A_L, A_T\}$, so the underlying wake functional is Euclidean and bounded below. The parameter α^2 therefore functions as a *real weighting* of longitudinal relative to transverse content in the particular combination that feeds the 1PN interaction, rather than as an indicator of an indefinite ("Lorentzian") signature.

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