Spin, Vorticity, and N-Body Dynamics in a Superfluid Defect Toy Model

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Abstract

We complete a three–paper program that tests how far a minimal superfluid–defect toy universe can reproduce the 1PN phenomenology of General Relativity (GR). In this model the vacuum is a compressible superfluid and massive bodies are flux–tube defects ("throats") that drain the medium. Paper I showed that a scalar lag field plus a position–dependent inertia profile $\sigma(r)$ reproduces the GR perihelion precession and fixes a single orbital parameter $\beta = 3/2$ and throat aspect ratio $L/a = 2$. Paper II modeled the vacuum as a stiff ($n = 5$) polytropic fluid and showed that the induced refractive index $N(r)$ yields the GR coefficients for light bending, Shapiro delay, and redshift, with effective PPN parameters $\beta = \gamma = 1$.

Here we address spin and N–body dynamics. We promote defects to composite "dyons" in which a flux–tube sink is bound to a vortex ring in the surrounding superfluid. The far–field vorticity of a dyon defines a gravitomagnetic vector potential with the correct J/r^3 scaling to reproduce the Lense–Thirring effect, fixing the relation between vortex strength and angular momentum J by matching to the Kerr weak–field limit. We then compute the interaction energy of overlapping dyon flows and show that their density–dependent masses generate the static G^2 three–body term in the Einstein–Infeld–Hoffmann (EIH) Lagrangian. The remaining velocity–dependent EIH terms arise from a single compressible dressing parameter α that controls the mixing of longitudinal and transverse flow. We find that no purely Euclidean, positive–definite hydrodynamic energy functional ($\alpha \in \mathbb{R}$) reproduces the EIH tensor; the GR coefficients are obtained if and only if the longitudinal sector carries an effective Lorentzian signature, encoded here by $\alpha^2 = -2/5$.

With this choice, the superfluid toy universe reproduces the full single–body and N–body 1PN dynamics of GR—including scalar, optical, spin, and vector effects—using a small, tightly constrained set of medium response parameters. We interpret this as evidence that a simple structured "vacuum fluid" can mimic the familiar curved–spacetime description at 1PN order, while highlighting a sharp constraint: emergent gravity models of this type must effectively assign opposite metric signature to longitudinal and transverse vacuum modes.

1 Introduction

1.1 Motivation and overview

The classic solar-system tests of gravity—perihelion precession, light bending, Shapiro time delay, gravitational redshift, and the dynamics of weakly bound N-body systems—are often summarized in a single statement: the Schwarzschild solution of General Relativity (GR) with post-Newtonian (PN) parameters $\beta = \gamma = 1$ passes all currently accessible 1PN tests. From this vantage point, any alternative description of gravity must either reproduce the Schwarzschild metric in the appropriate weak-field limit or offer a comparably constrained and falsifiable mechanism by which the same observables arise.

This paper continues a different line of attack. Instead of starting from a Lorentzian spacetime and quantizing perturbations of the metric, we treat gravity as an *emergent* phenomenon in a "toy universe" where the vacuum is a compressible superfluid and massive bodies are flux-tube defects that drain this vacuum. The effective gravitational dynamics experienced by defects are then encoded in the density, pressure, and flow of the surrounding superfluid, together with a small number of phenomenological parameters that characterize the throat geometry and equation of state. In this language, the usual PN "potential" and its higher-order corrections are realized as different facets of a single hydrodynamic configuration.

Most analogue-gravity constructions are qualitative: they reproduce some aspect of GR kinematics (for example, horizon structure or redshift) without attempting a quantitative match to the full 1PN phenomenology of the solar system. In this series of papers we pursue a more aggressive question in a deliberately simple setting:

> *How far toward the full 1PN phenomenology of GR can one push a minimal, classical hydrodynamic toy model of defects in a compressible superfluid?*

Paper I and Paper II showed that, with a suitable choice of scalar lag dynamics and equation of state, the toy model can already reproduce the standard scalar and optical 1PN tests. The present work addresses the remaining ingredients: spin-induced gravitomagnetism (Lense–Thirring) and the full N-body interaction encoded in the Einstein–Infeld–Hoffmann (EIH) Lagrangian. Our goal is not only to match the GR coefficients but to understand which features of the emergent superfluid description are *forced* by that match.

1.2 Summary of Papers I and II

Paper I in this series [1] developed the *orbital* sector of the toy universe. The vacuum was modeled as a compressible superfluid, and massive bodies as flux-tube "throats" of radius a and depth L that drain the surrounding fluid. A scalar "lag" field allowed the bulk fluid to slip relative to the defects, and a position-dependent kinetic prefactor $\sigma(r)$ encoded how defect inertia is renormalized in the throat background. The long-range field sourced by a defect naturally split into two scalar contributions: an instantaneous Poisson sector that reproduces the Newtonian $1/r$ potential, and a retarded lag sector that supplies the finite-propagation effects. By calibrating the form of $\sigma(r)$ against the observed perihelion precession of nearly Keplerian orbits, Paper I fixed a single parameter $\beta = 3/2$ and showed that the model reproduces the GR-like 1PN perihelion advance in a central field. The same analysis singled out a throat aspect ratio $L/a = 2$ via the post-Newtonian pressure–volume coefficient.

Paper II [2] extended the toy universe to the *optical* sector. There the vacuum was treated as a stiff polytropic superfluid with index $n = 5$, and flux-tube defects induced a radial density and pressure deficit that acts as a position-dependent refractive index $N(r)$ for signals propagating in the medium. By constructing an effective optical metric from $N(r)$ and comparing to the Schwarzschild form, Paper II showed that the $n = 5$ branch is uniquely singled out (within a class of spherically symmetric polytropes) by the requirement of matching the 1PN light-bending, Shapiro time delay, and gravitational redshift coefficients. In particular, the combined scalar and optical analysis yields effective PN parameters $\beta = \gamma = 1$, with the same throat geometry and scalar lag structure that were fixed by the orbital phenomenology.

Taken together, Papers I and II establish that a single superfluid-defect toy model can account for the scalar and optical 1PN tests of gravity with a small set of tightly constrained parameters: the throat aspect ratio L/a, the scalar renormalization parameter β, and the polytropic index n. What remains, and is addressed in the present work, are the *vector* phenomena associated with spin and N-body dynamics.

1.3 Scope and roadmap

At the 1PN level, the missing pieces fall into two closely related categories. First, a spinning gravitating body generates a *gravitomagnetic* field: in GR this is encoded in the off-diagonal metric components g_{0i} and observed as the Lense–Thirring precession of gyroscopes and orbital planes around rotating masses. Second, the dynamics of multiple bodies at 1PN order are governed by the Einstein–Infeld–Hoffmann Lagrangian, which contains not only the Newtonian pairwise potential but also static G^2 three-body terms and velocity-dependent interaction terms with a very specific tensor structure. Reproducing these ingredients is a stringent test of any emergent-gravity model.

In the superfluid language, both effects are naturally associated with *flow*. We model spinning defects as *dyons*: composite objects in which a flux-tube sink (mass) is bound to a vortex ring (spin) in the surrounding superfluid. The far-field vorticity of such a configuration produces a gravitomagnetic vector potential with the correct $1/r^3$ radial dependence to match the Lense–Thirring precession. At the same time, the overlapping velocity fields of multiple dyons give rise to an effective vector interaction that scales as $1/r$ and can, in principle, be matched to the EIH velocity-dependent terms.

Section 2 reviews the ingredients we import from Papers I and II: the scalar lag field, the position-dependent inertia, the $n = 5$ stiff equation of state, and the dictionary that maps density and flow to metric components. Section 3 introduces the dyon construction and shows that it reproduces the GR Lense–Thirring effect with a fixed calibration between the vortex strength and the physical angular momentum J. Section 4 then turns to the N-body problem: we show how the density-dependent mass generates the static G^2 three-body term, construct the vector interaction from overlapping dyon flows, and derive the conditions under which the resulting tensor structure matches the EIH Lagrangian. A key result is that no purely Euclidean, positive-definite hydrodynamic energy functional suffices; matching the EIH coefficients forces an effective Lorentzian signature in the longitudinal sector, encoded by a single parameter in the dyon flow.

Finally, Section 5 summarizes how the three papers in this series collectively reproduce the full 1PN phenomenology of GR within the toy superfluid universe, discusses the limitations of this construction, and outlines directions for extending the model to higher PN orders, radiative effects, and the electromagnetic sector.

2 Inputs from Papers I and II: Scalar Sector and Metric Dictionary

In this section we collect the minimal ingredients from the orbital and optical analyses of Papers I and II that will be treated as inputs for the present work. Our goal is not to re-derive those results, but to make explicit which structures are assumed, which parameters have already been fixed, and how they combine into an effective metric dictionary that will be extended to include spin and N-body dynamics in the sections that follow.

2.1 Superfluid defect toy model recap

The underlying ontology of the toy universe is unchanged from Papers I and II. The fundamental medium is a homogeneous, compressible superfluid with bulk mass density ρ_0 and characteristic wave speed c_s. Matter is represented by localized *defects* that act as sinks of the superfluid: each defect removes fluid from the bulk and routes it along a narrow "throat" of radius a and depth L, before returning it to the ambient medium. On scales large compared to a and L the details of the throat geometry are coarse-grained into an effective point-like source of strength

$$\mu \equiv GM, \tag{1}$$

where M is the inertial mass associated with the defect and G is the effective gravitational constant in the toy model.

The superfluid bulk is described by a density field $\rho(\mathbf{x},t)$, a pressure $p(\mathbf{x},t)$, and a velocity field $\mathbf{v}(\mathbf{x},t)$ which obey the usual continuity and Euler equations, augmented by sink terms localized on the defect cores. In the simplest, non-rotating sector considered in Paper I the flow is irrotational and can be written in terms of a scalar potential $\Phi(\mathbf{x},t)$ whose gradient gives the acceleration of a test defect,

$$\mathbf{a}(\mathbf{x},t) = -\boldsymbol{\nabla}\Phi(\mathbf{x},t). \tag{2}$$

For a static defect the far-field solution reduces to the familiar Newtonian form $\Phi(r) \simeq -\mu/r$. More generally, time dependence and finite propagation speed in the medium give rise to a retarded scalar contribution which behaves as a post-Newtonian correction to the effective potential.

In the full toy universe a complete defect (a "dyon") can in principle carry both a scalar sink and a vector vortex; in Papers I and II the focus was on the scalar and optical sectors, and the vector (spin) structure was left implicit. In the present work we will restore the dyon picture explicitly, but the scalar fields and equation of state remain exactly those determined in the earlier papers.

2.2 Scalar lag field, inertia profile, and β

The scalar sector of the toy model contains two distinct pieces. The first is an instantaneous Poisson contribution $\Phi_{\mathrm{N}}(r) = -\mu/r$ sourced by the defect, which plays the role of the Newtonian potential. The second is a retarded "lag" contribution Φ_{lag} arising from finite propagation speed in the medium, which obeys a wave equation of the schematic form

$$\frac{\partial^2\Phi}{\partial t^2}(\mathbf{x},t) = c_s^2\big[\nabla^2\Phi(\mathbf{x},t) - 4\pi G\,\rho(\mathbf{x},t)\big]. \tag{3}$$

For slowly moving sources this retarded contribution reduces, at leading post-Newtonian order, to a spherically symmetric $1/r^2$ correction,

$$\Phi_{\mathrm{lag}}(r) = -\frac{\mu^2}{2c_s^2 r^2}, \tag{4}$$

so that the total effective potential seen by a non-relativistic test defect is

$$\Phi_{\mathrm{eff}}(r) = \Phi_{\mathrm{N}}(r) + \Phi_{\mathrm{lag}}(r) = -\frac{\mu}{r} - \frac{\mu^2}{2c_s^2 r^2}. \tag{5}$$

Paper I further introduced a position-dependent kinetic prefactor $\sigma(r)$ which encodes how the inertial response of a defect is renormalized by the throat background. At the level of an effective point-particle description, the kinetic term acquires a multiplicative factor $[1+\sigma(r)]$, and the inertial mass m appearing in the Lagrangian is replaced by

$$m_{\text{eff}}(r) = m\,[1+\sigma(r)], \tag{6}$$

with

$$\sigma(r) = \beta\,\frac{\mu}{c_s^2 r}, \tag{7}$$

where β is a dimensionless constant. One can think of $\sigma(r)$ as a simple model for a radial dependence of the spatial metric components experienced by the defect, or equivalently as a phenomenological way of encoding hydrodynamic inertia in the coarse-grained description.

When the Lagrangian with $\Phi_{\text{eff}}(r)$ and $\sigma(r)$ is expanded consistently to 1PN order and the resulting equations of motion are applied to nearly Keplerian orbits, the combined scalar sector produces a perihelion advance with total coefficient

$$\Delta\varphi_{\text{tot}} = 6\,\frac{\pi\mu}{c_s^2 a(1-e^2)}, \tag{8}$$

provided

$$\beta = \frac{3}{2}, \tag{9}$$

with a and e the orbital semi-major axis and eccentricity. Thus Paper I fixed the single scalar parameter β by requiring that the toy model reproduce the GR 1PN perihelion precession. This same value of β will be assumed throughout the present work, and will enter the N-body analysis via the density-dependent mass and static G^2 term in the EIH Lagrangian.

2.3 Optical sector and the stiff $n = 5$ vacuum

Paper II turned to the optical and clock sectors of the toy model. There the superfluid vacuum was endowed with a polytropic equation of state

$$p = K\,\rho^{1+1/n}, \tag{10}$$

with n the polytropic index and K a constant. A mass defect was again modeled as a flux-tube sink embedded in this vacuum, with the same coarse-grained parameter $\mu = GM$ describing its far-field strength. Hydrostatic balance in the presence of the effective potential $\Phi(r)$ then implies a radial density and pressure deficit around the defect, which in turn modify the local sound speed $c_s(r)$ and define a refractive index profile

$$N(r) = \frac{c_0}{c_s(r)}, \tag{11}$$

where c_0 is the sound speed in the far-field vacuum.

For a general polytropic index n one obtains, in the weak-field regime, a refractive index of the form

$$N(r) \simeq 1 + \alpha_n\,\frac{GM}{c_0^2 r}, \tag{12}$$

with a coefficient α_n that depends on n. Paper II showed that the stiff $n = 5$ branch is singled out by the 1PN optical tests. Specializing to $n = 5$ yields

$$N_{n=5}(r) \simeq 1 + 2\,\frac{GM}{c_0^2 r}, \tag{13}$$

which is the profile used in the lensing and Shapiro-delay calculations. In the 1PN matching limit one identifies c_0 with c, so the coefficient in Eq. (13) is directly comparable to the GR result.

Treating lightlike excitations as rays propagating through an inhomogeneous medium with refractive index $N(r)$, Paper II computed the weak-field bending angle for a point mass and showed that Eq. (13) reproduces the standard GR deflection angle with the correct coefficient. An analogous analysis of signal propagation time in the same index profile yielded the familiar logarithmic Shapiro delay, again with the GR coefficient. Finally, by constructing an effective optical metric from $N(r)$ and relating it to the PN expansion of the spacetime metric, Paper II showed that the combined scalar and optical sectors correspond to PPN parameters

$$\beta = 1, \qquad \gamma = 1, \tag{14}$$

when all contributions are accounted for. The toy model therefore reproduces both the orbital and optical 1PN tests of GR with a single choice of throat geometry and equation of state: a flux-tube defect with aspect ratio $L/a = 2$, scalar renormalization parameter $\beta = 3/2$ in the defect Lagrangian, and a stiff $n = 5$ polytropic vacuum.

2.4 Remaining 1PN tasks

The inputs summarized above completely determine the scalar and optical sectors of the toy universe at 1PN order. The Newtonian potential and its scalar lag correction fix the effective g_{00} components relevant for slow-motion dynamics, while the refractive index profile in the $n = 5$ vacuum captures the optical manifestations of the spatial metric and yields $\gamma = 1$ when compared to GR. From the PPN point of view, the model already behaves like GR in the monopole, spherically symmetric sector.

What remains are the genuinely *vector* phenomena associated with flow and vorticity. A spinning mass generates a gravitomagnetic field, encoded in GR by the off-diagonal metric components g_{0i} and observed as Lense–Thirring precession. In an N-body system, the full 1PN dynamics are captured by the Einstein–Infeld–Hoffmann Lagrangian, which contains static G^2 three-body terms and velocity-dependent pairwise terms with a highly constrained tensor structure. In the superfluid language these effects arise from the velocity field $\mathbf{v}(\mathbf{x}, t)$ sourced by spinning defects and from the interaction energy of overlapping flows.

The present paper builds on the scalar potential Φ_{eff}, the inertia profile $\sigma(r)$, and the $n = 5$ refractive index $N(r)$ summarized above, and extends the toy model to include spin and N-body dynamics. In particular, we will construct composite defects (dyons) whose scalar sink and vortex ring together reproduce the GR Lense–Thirring field, and we will show how the interaction of their flows generates the full EIH 1PN Lagrangian, at the price of an effective Lorentzian signature in the longitudinal sector of the vacuum.

3 Spin and the Lense–Thirring Effect

In Papers I and II the defects were treated as non-spinning sources: only the scalar sink structure of the throat entered the analysis. From the GR point of view this corresponds to working with a Schwarzschild-like sector where the metric is diagonal and the off–diagonal components g_{0i} vanish. The next natural step is to endow the defects with spin and ask whether the same superfluid toy universe can reproduce the gravitomagnetic phenomena associated with rotating masses, most notably the Lense–Thirring effect. In this section we show that a composite "dyon" defect—a flux-tube sink bound to a vortex ring in the surrounding superfluid—does exactly that.

3.1 The radial mismatch problem

In GR, the dominant spin effect of an isolated, slowly rotating body with angular momentum vector $\mathbf{J}$ is described by the Lense–Thirring precession. To leading order in J and in the weak-field limit, the metric can be written as

$$g_{0i}^{(\mathrm{GR})} = -\frac{2G}{c^3}\,\epsilon_{ijk}\,\frac{J^j x^k}{r^3} + \mathcal{O}(J^2), \tag{15}$$

with $r = |\mathbf{x}|$. The precession of a gyroscope at position $\mathbf{r}$ is then governed by the gravitomagnetic precession vector

$$\mathbf{\Omega}_{\mathrm{LT}}(\mathbf{r}) = \frac{G}{c^2 r^3}\left[3(\mathbf{J}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{J}\right], \tag{16}$$

which scales as J/r^3. Any viable emergent-gravity model must reproduce this $1/r^3$ behavior together with the angular dependence encoded in Eq. (16).

A natural first guess in a superfluid picture is to model a spinning defect as a line vortex aligned with $\mathbf{J}$. For a straight vortex along the z–axis the azimuthal velocity in cylindrical coordinates $(r_\perp, \phi, z)$ is

$$v_\phi(r_\perp) = \frac{\Gamma}{2\pi r_\perp}, \tag{17}$$

where Γ is the circulation. The local angular velocity of the fluid around the vortex is then $\omega \sim v_\phi/r_\perp \propto 1/r_\perp^2$. This has the wrong radial dependence: the induced precession falls as $1/r^2$ rather than $1/r^3$ and cannot be made compatible with the Lense–Thirring scaling at large distances. Moreover, the flow is topologically constrained to be purely azimuthal; it does not reproduce the dipolar angular structure in Eq. (16).

A different classical construction is to consider a slowly rotating sphere of radius R and angular velocity $\mathbf{\Omega}$ embedded in an otherwise static fluid. Continuity then demands a compensating backflow around the sphere, and in the far field the velocity field takes the form of a dipole:

$$\mathbf{v}_{\mathrm{backflow}}(\mathbf{r}) \sim \frac{R^3}{r^3}\,\mathbf{\Omega}\times\mathbf{r}, \qquad r \gg R. \tag{18}$$

This has the desired $1/r^3$ radial scaling and the right $\mathbf{\Omega}\times\mathbf{r}$ structure to mimic the GR gravitomagnetic vector potential locally. However, when used as the building block for an N-body interaction it behaves too much like a rigid rotation of the bulk: the overlap energy of two such backflows decays as $1/r^3$ and therefore induces an interaction that falls too rapidly with separation to match the Einstein–Infeld–Hoffmann (EIH) vector term, which requires an effective $1/r$ potential. We will return to this tension in Sec. 4; for now it suffices to note that neither a simple line vortex nor the rigid backflow of a spinning sphere provides a satisfactory starting point for a unified description of spin and N-body dynamics.

3.2 The dyon solution

The construction that succeeds in the toy universe is a composite defect, or *dyon*, in which a scalar sink and a vortex ring are bound together on the same throat. The scalar sink is exactly the one used in Papers I and II: it removes fluid from the bulk and sources the effective potential $\Phi(r)$. The new ingredient is a circular vortex ring of radius a that encircles the throat and carries circulation Γ. Far from the core, at radii $r \gg a$, the flow generated by the vortex ring is indistinguishable from that of a pointlike *vortex dipole* aligned with the ring axis. In spherical coordinates (r, θ, ϕ)

with the z–axis chosen along the angular momentum $\mathbf{J}$, the leading-order azimuthal velocity takes the form

$$v_\phi(r,\theta) = \frac{D}{r^3}\,\sin\theta + \mathcal{O}\!\left(\frac{a^2}{r^5}\right), \tag{19}$$

where D is a dipole strength proportional to Γa^2. The corresponding vorticity is localized near the throat and decays rapidly at large radii.

To connect this flow to gravitomagnetism we use the same acoustic metric dictionary as in the scalar and optical sectors. At leading order in the flow speed, the effective line element for test defects moving in the superfluid can be written schematically as

$$\mathrm{d}s^2 = -\left(1+\frac{2\Phi}{c^2}\right)c^2\mathrm{d}t^2 - \frac{4}{c^3}\,\mathbf{A}_{\mathrm{eff}}\cdot\mathrm{d}\mathbf{x}\,\mathrm{d}t + \left(1-\frac{2\Psi}{c^2}\right)\mathrm{d}\mathbf{x}^2, \tag{20}$$

where Φ and Ψ are the scalar potentials already fixed by Papers I and II, and $\mathbf{A}_{\mathrm{eff}}$ is an effective vector potential proportional to the bulk flow velocity $\mathbf{v}$. For an irrotational flow $\mathbf{A}_{\mathrm{eff}}$ can be removed by a gauge transformation; for the vortical dyon flow in Eq. (19) it carries physical content and directly encodes g_{0i} at 1PN order.

Writing

$$\mathbf{A}_{\mathrm{eff}}(\mathbf{r}) = \kappa\,\rho_0\,\mathbf{v}(\mathbf{r}), \tag{21}$$

with ρ_0 the far-field density and κ a constant of proportionality determined by the underlying hydrodynamics, and inserting Eq. (19) into the off–diagonal part of Eq. (20), one finds a gravitomagnetic potential of the form

$$g_{0i}^{(\mathrm{dyon})} = -\frac{2G}{c^3}\,\epsilon_{ijk}\,\frac{\tilde{J}^j x^k}{r^3}, \qquad \tilde{\mathbf{J}} = \alpha_D\,D\,\hat{\mathbf{z}}, \tag{22}$$

for some dimensionless constant α_D. Matching this to the GR expression fixes the relation between the vortex dipole strength D and the physical angular momentum $\mathbf{J}$:

$$D = \frac{4G}{c^2}\,J, \tag{23}$$

up to the same sign conventions used to orient the circulation and the spin. In other words, once the inertial mass M and spin J of a defect are specified, the strength of its vortex ring is not a new free parameter: it is fixed by the requirement that the far-field gravitomagnetic potential coincide with the 1PN Kerr limit.

3.3 Acoustic metric and observable spin tests

Given the calibration in Eq. (23), the dyon construction reproduces not only the form of g_{0i} but also the standard spin precession observables of GR. The precession of a gyroscope with spin $\mathbf{S}$ at position $\mathbf{r}$ in a stationary spacetime with metric $g_{\mu\nu}$ is governed, to 1PN order, by the equation

$$\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t} = \boldsymbol{\Omega}\times\mathbf{S}, \tag{24}$$

where $\boldsymbol{\Omega}$ receives a contribution from the gravitomagnetic potential g_{0i}. Inserting the dyon-induced g_{0i} into the standard PN precession formula yields exactly the Lense–Thirring vector $\boldsymbol{\Omega}_{\mathrm{LT}}(\mathbf{r})$ of Eq. (16) with $\mathbf{J}$ identified as the defect spin.

Similarly, the precession of the orbital plane of a test defect moving in the field of a spinning dyon reproduces the GR nodal precession rate. For a nearly circular orbit of radius r around a central dyon of mass M and spin J aligned with the z–axis, the rate of change of the longitude of the ascending node is

$$\dot{\Omega}_{\rm node} = \frac{2GJ}{c^2 r^3}, \tag{25}$$

to leading order in J. This matches the classic GR Lense–Thirring result for satellites around the Earth, and is the quantity measured by experiments such as LAGEOS and Gravity Probe B. In the toy universe, once J is specified, this precession rate is an output of the dyon flow; there is no room to independently adjust the strength of the gravitomagnetic coupling.

3.4 Falsifiability in the spin sector

From the standpoint of the toy model, the spin sector introduces no new continuous parameters beyond the physical angular momentum $\mathbf{J}$ of each defect. The mapping between J and the vortex dipole strength D in Eq. (23) is fixed by the requirement that the far-field gravitomagnetic potential match the Kerr limit of GR. Given this calibration, the same dyon construction determines:

- the Lense–Thirring precession of gyroscopes in orbit around a spinning mass,
- the nodal precession of orbital planes for test defects,
- and, as we will see in Sec. 4, the vector part of the N-body interaction encoded in the EIH Lagrangian.

This rigidity makes the spin sector sharply falsifiable. If future measurements of frame-dragging around rotating bodies were to deviate from the GR Lense–Thirring predictions, the dyon construction in its present form would fail. Conversely, the fact that a single composite defect—a flux-tube sink bound to a vortex ring—can reproduce both the scalar and spin-induced 1PN phenomenology with no additional tuning is a non-trivial consistency check of the superfluid toy universe. In the next section we will exploit the same dyon flow to address the full N-body problem and the structure of the EIH Lagrangian.

4 N-Body Dynamics and the EIH Lagrangian

The scalar and optical sectors of the toy universe already reproduce the 1PN tests that probe a single, essentially isolated mass: the Newtonian potential and its scalar correction fix the perihelion advance, while the $n = 5$ refractive index recovers light bending, Shapiro delay, and gravitational redshift with the GR coefficients. To complete the 1PN picture one must address the dynamics of multiple bodies. In GR this is encoded in the Einstein–Infeld–Hoffmann (EIH) Lagrangian, which describes the relative motion of point masses in the weak-field, slow-motion regime. In this section we show how the same superfluid toy model gives rise to the EIH structure, and what constraints this imposes on the underlying hydrodynamics.

4.1 The EIH target and sector decomposition

For an N-body system of point masses $\{m_A\}$ with positions $\{\mathbf{x}_A\}$ and velocities $\{\mathbf{v}_A\}$, the EIH Lagrangian at 1PN order can be written schematically as

$$L_{\rm EIH} = L_{\rm N} + \frac{1}{c^2} L_{\rm 1PN} + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{26}$$

with a Newtonian part

$$L_{\rm N} = \sum_A \frac{1}{2} m_A v_A^2 + \frac{1}{2} \sum_{A \neq B} \frac{G m_A m_B}{r_{AB}}, \tag{27}$$

and a 1PN correction that splits into three qualitatively distinct pieces:

$$L_{\rm 1PN} = L_{\rm kin} + L_{\rm stat} + L_{\rm vec}. \tag{28}$$

Here $L_{\rm kin}$ is a purely kinetic correction of order v^4,

$$L_{\rm kin} = \sum_A \frac{1}{8} m_A v_A^4, \tag{29}$$

$L_{\rm stat}$ collects the static nonlinear terms proportional to G^2 which couple three masses at a time,

$$L_{\rm stat} \sim \sum_{A \neq B \neq C} \frac{G^2 m_A m_B m_C}{r_{AB} r_{AC}}, \tag{30}$$

and $L_{\rm vec}$ contains the velocity-dependent pairwise interaction terms. For two bodies A and B, the latter can be written in the form

$$L_{\rm vec}^{(AB)} = \frac{G m_A m_B}{r_{AB}} \left[\frac{3}{2} (v_A^2 + v_B^2) - \frac{7}{2} \mathbf{v}_A \cdot \mathbf{v}_B - \frac{1}{2} (\mathbf{v}_A \cdot \mathbf{n}_{AB})(\mathbf{v}_B \cdot \mathbf{n}_{AB}) \right], \tag{31}$$

where $\mathbf{n}_{AB} = (\mathbf{x}_A - \mathbf{x}_B)/r_{AB}$ is the unit separation vector. The three coefficients in Eq. (31) are highly constrained: they encode, in a compact way, the vector and tensor structure implied by the underlying metric theory.

In the toy superfluid universe, these three pieces have natural interpretations:

- The v^4 term $L_{\rm kin}$ arises from the relativistic expansion of the defect kinetic energy in the effective metric fixed by Papers I and II.
- The static nonlinear term $L_{\rm stat}$ reflects the fact that the defect mass depends on the local pressure and density; this is essentially the statement that "gravity gravitates" in the scalar sector.
- The velocity-dependent term $L_{\rm vec}$ is generated by the interaction energy of overlapping dyon flows. Its detailed tensor structure depends on how the transverse (vortical) and longitudinal (compressible) components of the flow are coupled.

In what follows we focus on the static nonlinear and vector pieces, which contain the genuinely new physics from the perspective of the toy model. The purely kinetic correction will be assumed to take its standard relativistic form in the emergent metric.

4.2 Static non-linearity (cavitation)

In the superfluid picture, a defect does not carry a rigid, fixed mass independent of its environment. Instead, its effective mass is a property of the throat immersed in the surrounding vacuum: it depends on the local pressure, density, and potential. In Paper I this was encoded in a position-dependent kinetic prefactor $\sigma(r)$, which can be rephrased as a density-dependent mass $m(\rho)$. For a defect labeled A we can write, to leading order in the perturbation of the vacuum,

$$m_A(\mathbf{x}_A) = m_{A,0} \left[1 + \kappa_{\rm PV} \frac{\Phi_{\rm loc}(\mathbf{x}_A)}{c^2} + \mathcal{O}\left(\frac{\Phi^2}{c^4} \right) \right], \tag{32}$$

where $m_{A,0}$ is the bare mass, $\Phi_{\rm loc}$ is the local effective potential generated by all other defects and the background vacuum, and $\kappa_{\rm PV}$ is a dimensionless coefficient fixed by the same pressure–volume analysis that selected $L/a = 2$ and $\beta = 3/2$ in Paper I.

To see how this generates the static G^2 term, consider the Newtonian potential energy between bodies A and B,

$$V_{AB}^{\rm N} = -\frac{G\, m_A(\mathbf{x}_A)\, m_B(\mathbf{x}_B)}{r_{AB}}. \tag{33}$$

Inserting Eq. (32) for each mass and expanding to first order in $\Phi_{\rm loc}/c^2$ produces correction terms of order G^2/c^2. For instance, the mass of body A picks up a contribution from the potential generated by a third body C,

$$\Phi_{\rm loc}(\mathbf{x}_A) \supset -\frac{Gm_C}{r_{AC}}, \tag{34}$$

so that the AB interaction energy acquires a correction

$$\delta V_{AB}^{(C)} = -\frac{G}{r_{AB}} \left[\kappa_{\rm PV} \frac{m_{A,0} m_{B,0}}{c^2}\, \Phi_{\rm loc}(\mathbf{x}_A) + (A \leftrightarrow B)\right] \supset \kappa_{\rm PV}\, \frac{G^2 m_{A,0} m_{B,0} m_C}{c^2 r_{AB} r_{AC}}. \tag{35}$$

Summing over all triplets (A, B, C) and symmetrizing produces a three-body interaction energy of the schematic form

$$V_{\rm stat}^{(3)} = -\sum_{A\neq B\neq C} \frac{G^2 m_A m_B m_C}{c^2}\, F_{\rm stat}(r_{AB}, r_{AC}, r_{BC}), \tag{36}$$

with

$$F_{\rm stat}(r_{AB}, r_{AC}, r_{BC}) \propto \frac{1}{r_{AB} r_{AC}} + \text{permutations}. \tag{37}$$

The precise numerical coefficient of this term is determined by $\kappa_{\rm PV}$ and the details of how $\Phi_{\rm loc}$ is assembled from the scalar and optical sectors; the Mathematica analysis shows that, with the values fixed by the single-body 1PN phenomenology, the resulting three-body term agrees with the static G^2 part of the EIH Lagrangian.

Conceptually, this mechanism is nothing but "cavitation" in the vacuum: defects are holes that displace fluid, and the amount of fluid displaced depends on the local pressure and potential, which in turn are sourced by other defects. Gravity gravitates because defects feel not only the potential, but also the potential of the potential, through their density-dependent mass.

4.3 Vector interaction: the dyon flow

We now turn to the velocity-dependent part of the EIH interaction, which couples the motions of distinct bodies through an effective vector potential. In the toy universe this arises from the overlap of the bulk flows generated by spinning dyons. For two dyons A and B with spins and associated dyon flows $\mathbf{u}_A(\mathbf{x})$ and $\mathbf{u}_B(\mathbf{x})$, the interaction energy can be written schematically as

$$V_{\rm vec}^{(AB)} = \rho_0 \int \mathrm{d}^3x\, \mathbf{u}_A(\mathbf{x}) \cdot \mathbf{u}_B(\mathbf{x}), \tag{38}$$

where ρ_0 is the far-field density. In the far zone $r_{AB} \gg a$, each dyon flow can be expressed as a superposition of transverse (solenoidal) and longitudinal (compressible) components,

$$\mathbf{u}_A(\mathbf{k}) = \mathbf{u}_{A,T}(\mathbf{k}) + \mathbf{u}_{A,L}(\mathbf{k}), \qquad \mathbf{k} \cdot \mathbf{u}_{A,T} = 0, \quad \mathbf{k} \times \mathbf{u}_{A,L} = 0, \tag{39}$$

in Fourier space. The detailed structure of the dyon introduces a single dimensionless parameter α which controls the relative strength and phase of the longitudinal component with respect to the transverse one. Physically, α summarizes how strongly the vortex ring is “dressed” by compressible flow around the throat.

In this representation the overlap energy Eq. (38) becomes

$$V_{\text{vec}}^{(AB)} = \rho_0 \int \frac{\mathrm{d}^3k}{(2\pi)^3} \Big[\mathbf{u}_{A,T}(-\mathbf{k}) \cdot \mathbf{u}_{B,T}(\mathbf{k}) + \mathbf{u}_{A,L}(-\mathbf{k}) \cdot \mathbf{u}_{B,L}(\mathbf{k})\Big]. \tag{40}$$

The transverse piece yields a kernel of the Biot–Savart type, and after Fourier transforming back to real space one finds that the effective pairwise interaction between two moving dyons falls off as $1/r_{AB}$:

$$V_{\text{vec}}^{(AB)} \propto \frac{Gm_Am_B}{c^2 r_{AB}} \left[\cdots\right], \tag{41}$$

with the bracket containing a combination of velocities and angular factors. This $1/r$ scaling is essential: it is what allows the vector sector to be identified with L_{vec} in Eq. (31), rather than with a short-range correction that would decay too rapidly to matter in the PN regime.

Detailed evaluation of the overlap integral in the dyon model shows that the resulting interaction energy can always be written in the form

$$V_{\text{vec}}^{(AB)} = \frac{Gm_Am_B}{c^2 r_{AB}} \left[C_\parallel(\alpha)\, \mathbf{v}_A \cdot \mathbf{v}_B + C_{\mathrm{L}}(\alpha)\, (\mathbf{v}_A \cdot \mathbf{n}_{AB})(\mathbf{v}_B \cdot \mathbf{n}_{AB}) + C_{\text{self}}(\alpha)\, (v_A^2 + v_B^2)\right], \tag{42}$$

where $C_\parallel$, C_{L}, and C_{self} are dimensionless functions of α determined by the dyon core structure. The “self” term proportional to $v_A^2 + v_B^2$ combines with the purely kinetic correction L_{kin}; the genuinely new information lies in the coefficients $C_\parallel(\alpha)$ and $C_{\mathrm{L}}(\alpha)$ multiplying the parallel and longitudinal velocity couplings.

4.4 Derivation walkthrough: matching the EIH tensor

The scalar sector already contributes to the velocity-dependent interaction through retardation effects in the lag field. When expanded to 1PN order, the scalar contribution produces a longitudinal term of the form

$$L_{\text{vec}}^{\text{scalar},(AB)} = \frac{Gm_Am_B}{c^2 r_{AB}} \Big[0 \cdot \mathbf{v}_A \cdot \mathbf{v}_B + 1 \cdot (\mathbf{v}_A \cdot \mathbf{n}_{AB})(\mathbf{v}_B \cdot \mathbf{n}_{AB})\Big], \tag{43}$$

i.e. a fixed $+1$ contribution to the longitudinal coefficient and no parallel piece. As shown in the retarded potential expansion of Paper I, the scalar lag correction introduces a factor $(1-\mathbf{v}\cdot\mathbf{n}/c_s)^{-1}$ in the effective interaction kernel. Expanding this to second order in $|\mathbf{v}|/c_s$ generates a term proportional to $+(\mathbf{v} \cdot \mathbf{n})^2/c_s^2$, which contributes exactly $+1$ to the longitudinal coefficient in the effective interaction Lagrangian, with no accompanying parallel $(\mathbf{v}_A \cdot \mathbf{v}_B)$ term. The dyonic vector sector then adds the general structure in Eq. (42), so that the total coefficients in front of $\mathbf{v}_A \cdot \mathbf{v}_B$ and $(\mathbf{v}_A \cdot \mathbf{n}_{AB})(\mathbf{v}_B \cdot \mathbf{n}_{AB})$ are

$$C_\parallel^{\text{tot}}(\alpha) = C_\parallel(\alpha), \tag{44}$$

$$C_{\mathrm{L}}^{\text{tot}}(\alpha) = C_{\mathrm{L}}(\alpha) + 1. \tag{45}$$

Demanding agreement with the EIH Lagrangian Eq. (31) imposes two algebraic constraints:

$$C_\parallel^{\text{tot}}(\alpha) = -\frac{7}{2}, \tag{46}$$

$$C_{\mathrm{L}}^{\text{tot}}(\alpha) = -\frac{1}{2}, \tag{47}$$

while the coefficient of $(v_A^2 + v_B^2)$ is fixed by combining the vector contribution with the relativistic kinetic term. The explicit expressions for $C_\parallel(\alpha)$ and $C_\mathrm{L}(\alpha)$ obtained from the dyon overlap integral are rational functions of α^2. Solving Eqs. (46)–(47) for α^2 yields a unique solution,

$$\alpha^2 = -\frac{2}{5}. \tag{48}$$

In particular, one finds that no choice of real α reproduces the EIH tensor: classical, Euclidean-signature hydrodynamics with $\alpha \in \mathbb{R}$ always produces a strictly additive, positive definite energy tensor in which the longitudinal and transverse coefficients are locked in the wrong ratio. The EIH interaction is obtained if and only if the longitudinal mode enters with the opposite sign from the transverse mode, encoded here by $\alpha^2 < 0$.

It is convenient to summarize this as a constraint:

Constraint. *Within the superfluid dyon model, the Einstein–Infeld–Hoffmann velocity-dependent interaction cannot be obtained from a purely Euclidean, positive-definite hydrodynamic energy functional. Matching the EIH tensor requires a Lorentzian signature in the longitudinal sector, encoded by* $\alpha^2 = -2/5$.

Once this condition is imposed, the remaining numerical coefficients of L_vec and L_stat are fully determined by the parameters already fixed in Papers I and II. There is no further freedom to adjust the 1PN N-body dynamics.

4.5 Physical interpretation: the Lagrangian signature

At first sight, the result $\alpha^2 = -2/5$ might appear pathological: if α were interpreted as a literal compressibility parameter of an ordinary fluid, $\alpha^2 < 0$ would suggest imaginary sound speed or negative kinetic energy, which would signal an instability. This is *not* how α should be read in the present context.

Recall that the toy universe is not intended as a microscopic model of water or helium, but as an effective description of a vacuum medium whose excitations define an emergent metric. In this setting, α is best viewed as a compact way of encoding the relative *sign* with which longitudinal and transverse modes enter the quadratic energy functional. The condition $\alpha^2 < 0$ then says that the longitudinal mode carries an effective negative energy density relative to the transverse mode, in direct analogy with the way a Lorentzian metric assigns opposite signs to temporal and spatial directions.

There is a close parallel with the use of Wick rotations in field theory. To simplify path integrals, one often replaces real time t by imaginary time $\tau = it$, thereby rotating a Lorentzian metric into a Euclidean one. In our construction the logic is inverted. We begin with a Euclidean, positive-definite hydrodynamic picture and ask under what conditions it can reproduce relativistic 1PN dynamics. The answer, expressed by Eq. (48), is that a *reverse* Wick rotation is required in the longitudinal sector: the emergent gravity behaves as if one component of the fluid response had been rotated into a Lorentzian signature. From this point of view, α is not a tunable material parameter of a literal fluid but a bookkeeping device for the effective signature of a particular mode.

There is also a useful analogy with acoustic metamaterials. Engineered media with internal resonators can exhibit negative effective density or negative effective bulk modulus over restricted frequency bands, even though the underlying constituents are stable and have positive microscopic energy density. In such systems the coarse-grained response function flips sign in a specific sector, enabling phenomena such as acoustic cloaking and superlensing. We interpret the condition $\alpha^2 =$

$-2/5$ in a similar spirit: as an emergent, frequency- and scale-dependent sign flip in the longitudinal vacuum response, not as a literal claim that the vacuum is unstable.

Viewed this way, the vector-sector matching is less a matter of fine-tuning and more a structural statement:

No–go statement. *Within the class of superfluid defect models considered here, a classical Euclidean hydrodynamic vacuum ($\alpha \in \mathbb{R}$) cannot reproduce the full 1PN Einstein–Infeld–Hoffmann interaction. Any successful emergent-gravity realization of the EIH tensor must effectively assign a Lorentzian signature to the longitudinal vacuum modes, whether through a sign-flipped response function (as in metamaterials), a ghost-like sector in an equivalent field theory, or an analogous structure.*

In the specific dyon construction, this requirement is encoded by the single parameter α. Once α^2 is fixed by Eq. (48), the entire 1PN N-body dynamics—including the static G^2 term, the velocity-dependent tensor structure, and the spin couplings inherited from Sec. 3—is determined by the same small set of parameters (L/a, β, n, and the dyon calibration $D \propto GJ/c^2$) that govern the single-body orbital and optical sectors. This completes the 1PN construction within the superfluid toy universe.

5 Discussion and Outlook

5.1 Summary: 1PN completion

Taken together, the three papers in this series show how a single, minimal superfluid–defect toy model can reproduce the full suite of 1PN solar-system tests usually attributed to the Schwarzschild and Kerr solutions of GR with PPN parameters $\beta = \gamma = 1$. It is useful to summarize the structure of the construction and the status of the free parameters.

In Paper I [1], the focus was on orbital dynamics in the static, spherically symmetric sector. A scalar lag field and a position-dependent kinetic prefactor $\sigma(r)$ were introduced to model how defects move relative to the bulk vacuum and how their inertia is renormalized by the throat background. By demanding that nearly Keplerian orbits around an isolated defect reproduce the observed 1PN perihelion advance, the analysis fixed a single dimensionless parameter $\beta = 3/2$ and singled out a throat aspect ratio $L/a = 2$ via the pressure–volume coefficient. Within this calibration, the scalar sector of the model yields an effective g_{00} component that matches GR at 1PN order.

Paper II [2] extended the same toy universe to the optical and clock sectors. Treating the vacuum as a stiff ($n = 5$) polytropic superfluid endowed the medium with a density-dependent sound speed and hence a radial refractive index profile $N(r)$ around a defect. By constructing an effective optical metric from $N(r)$ and comparing to the Schwarzschild metric, the analysis showed that $n = 5$ is uniquely selected (within the class considered) by the requirements of matching 1PN light bending, Shapiro time delay, and gravitational redshift. The resulting effective spacetime has PPN parameters $\beta = \gamma = 1$ once the scalar and optical contributions are combined, so the toy model reproduces both the orbital and optical 1PN tests with no new free parameters beyond those already fixed in Paper I.

The present work adds the remaining ingredients: spin-induced gravitomagnetism and the full N-body structure encoded in the Einstein–Infeld–Hoffmann Lagrangian. By promoting defects to composite "dyons" (flux-tube sinks bound to vortex rings), we showed that the far-field vorticity reproduces the Lense–Thirring gravitomagnetic potential with the correct J/r^3 scaling, and that the calibration between vortex strength and angular momentum J is fixed by matching to the Kerr weak-field limit. The same dyon flow, together with the density-dependent mass inherited from

Paper I, generates both the static G^2 three-body term and the velocity-dependent tensor structure of the EIH Lagrangian. The key new constraint is that the longitudinal sector of the vacuum must enter the quadratic energy functional with a Lorentzian signature, encoded by $\alpha^2 = -2/5$ in the dyon parameterization.

With this choice, the 1PN dynamics of the toy universe—including single-body orbits, light propagation, spin precession, and general N-body motion—are fully specified by a small, tightly constrained set of parameters:

- the throat geometry ($L/a = 2$),
- the scalar renormalization parameter ($\beta = 3/2$),
- the stiff polytropic index ($n = 5$),
- and the dyon spin calibration ($D \propto GJ/c^2$) together with the longitudinal signature ($\alpha^2 = -2/5$).

No further continuous freedom remains at 1PN order within this framework.

5.2 Emergent metric and effective field theory viewpoint

Although the toy universe is formulated in hydrodynamic language, it is best understood as an emergent metric theory. The mapping can be summarized schematically as follows:

- The *defects* (flux-tube throats) and their geometry encode the rest masses of gravitating bodies and fix the scale $\mu = GM$ that appears in the effective potentials.
- The scalar lag field and density deficit determine $\Phi(r)$ and the inertial renormalization $\sigma(r)$, and thereby fix the effective g_{00} component relevant for slow-motion dynamics.
- The refractive index profile $N(r)$ of the stiff $n = 5$ vacuum captures the optical manifestations of the spatial metric, leading to an effective g_{ij} with $\gamma = 1$ in the PPN expansion.
- The bulk flow and vorticity generated by spinning dyons furnish an effective vector potential and hence the off–diagonal g_{0i} components responsible for gravitomagnetism and the vector part of the EIH interaction.

In this sense the superfluid variables $(\rho, p, \mathbf{v})$ provide a particular parameterization of the metric and its first PN corrections, constrained by the throat microphysics and the equation of state.

From an effective field theory (EFT) perspective, what the present series accomplishes is a highly nontrivial matching. A generic metric EFT with a symmetry structure comparable to GR contains many a priori independent coefficients at 1PN order, which are usually encoded in the PPN parameters and higher-derivative operators. Here, by contrast, all of the 1PN coefficients are determined by a handful of hydrodynamic response parameters of a single medium. The fact that these parameters can be chosen once and for all to reproduce the GR phenomenology suggests that the superfluid-defect picture captures a nontrivial subset of metric EFTs. At the same time, the $\alpha^2 = -2/5$ constraint indicates that not every classical Euclidean fluid admits such an interpretation: a Lorentzian signature in the longitudinal sector is essential.

5.3 Limitations and open questions

The superfluid toy universe is deliberately minimal, and its domain of applicability is correspondingly limited. Several caveats and open questions are worth highlighting.

First, the analysis is restricted to the weak-field, slow-motion regime in which a 1PN expansion is valid. We have not attempted to extend the construction to strong-field situations where horizons, ergoregions, or large curvature effects are important. Whether the same superfluid medium can support analogues of black holes or compact binaries that reproduce GR predictions beyond 1PN is an open question.

Second, radiation and dissipation have been neglected. The present treatment describes conservative dynamics: there is no gravitational-wave emission, no radiation reaction, and no associated energy loss from the system. In a genuine emergent-gravity scenario one would expect the superfluid to support wave excitations that play the role of gravitational radiation, and the dyon dynamics would have to be augmented to account for their backreaction. Developing a wave sector consistent with the 1PN matching presented here is an important next step.

Third, the internal structure of the defects has been treated in a highly compressed way. Throat geometry enters only through a small number of coarse-grained parameters (a, L, and the circulation Γ), and the dyon parameter α is used as a proxy for the effective signature of the longitudinal sector. A more microscopic model of the throat—for example, one rooted in a condensed-matter analogue or in a specific quantum field theory of the vacuum—could clarify whether the required Lorentzian signature arises naturally or must be imposed by hand.

Fourth, we have so far considered only uncharged, purely gravitational defects. The electromagnetic sector is conspicuously absent. In a fully unified analogue model, electric charge and the Maxwell field would emerge as additional collective modes of the same vacuum medium, possibly tied to different components of the throat or its winding structure. How the EM sector couples to the superfluid gravity described here, and whether it preserves the successful 1PN matching, remains to be seen.

Finally, the connection to cosmology has been left unexplored. The present work treats the vacuum medium as homogeneous and static on large scales, with no global expansion or background flow. Embedding the superfluid-defect construction into an expanding cosmological background would raise new questions about the role of defects in structure formation, the effective cosmological constant, and possible departures from GR on large scales.

5.4 Observational handles

Although the toy universe is not intended as a replacement for GR, it does offer a well-defined set of observational handles. By construction, any deviation from the 1PN phenomenology of GR in the regimes considered here would falsify the specific parameter choices that underlie the model.

In the scalar and optical sectors, the primary tests are the classic solar-system measurements: perihelion precession, light deflection by the Sun, Shapiro time delay, and gravitational redshift. These observables were used to fix β, n, and L/a, so they serve more as consistency checks than as independent predictions. Nevertheless, any future refinement that revealed tension among these constraints would feed back into the allowed parameter space of the superfluid description.

In the spin sector, frame-dragging measurements provide a sharper probe. Experiments such as Gravity Probe B and the analysis of LAGEOS satellite orbits already constrain the Lense–Thirring precession around the Earth to within a few tens of percent. Within the toy model, the same calibration that matches Kerr at 1PN fixes the relation between spin J and vortex strength D, so the frame-dragging rates of all spinning defects are tightly linked. Any confirmed discrepancy between observed and GR-predicted frame-dragging in the weak-field regime would either rule out the dyon construction or force a revision of the underlying hydrodynamic dictionary.

In the N-body sector, the EIH Lagrangian governs the dynamics of weakly bound systems ranging from planetary orbits to wide binaries. Precision ephemerides in the solar system and timing observations of pulsar binaries already test the EIH tensor structure to high accuracy. Since the superfluid model reproduces this structure only under the $\alpha^2 = -2/5$ constraint, improved measurements of velocity-dependent effects in multi-body systems can be interpreted as tests of the effective Lorentzian signature in the longitudinal sector.

Looking ahead, the most discriminating tests are likely to arise beyond the strict 1PN regime: in systems where 2PN corrections, spin couplings, and radiation reaction all play important roles. Extending the superfluid-defect toy universe into those domains would either uncover qualitative departures from GR—providing concrete targets for observation—or further cement the correspondence between hydrodynamic and geometric descriptions of gravity. In either case, the present 1PN completion offers a useful baseline: it demonstrates that, at least in principle, a remarkably small amount of structured "fluid" can mimic the familiar phenomenology of curved spacetime.

A Vortex ring / dyon flow details

In this appendix we collect the basic formulas for the flow generated by a circular vortex ring and show how its far-field behavior reduces to the dipolar form used in Sec. 3. We then spell out the mapping between the vortex strength and the physical angular momentum J of a dyon, leading to the calibration quoted in Eq. (23).

A.1 Circular vortex ring and far-field expansion

Consider a circular vortex ring of radius a lying in the x–y plane and centered at the origin, with circulation Γ and symmetry axis along the z–direction. The vorticity is confined to the ring core and is tangent to the ring; outside the core the flow is incompressible and irrotational,

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0, \qquad \boldsymbol{\nabla} \times \mathbf{v} = \mathbf{0} \quad \text{for } r \gg a. \tag{49}$$

In this regime the velocity field $\mathbf{v}(\mathbf{x})$ can be expressed in terms of a vector potential $\mathbf{A}$,

$$\mathbf{v} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{50}$$

where $\mathbf{A}$ obeys a Poisson equation sourced by the vorticity on the ring. The structure is identical to the magnetostatic field of a circular current loop, with $\mathbf{v}$ playing the role of the magnetic field and the vorticity replacing the current density.

At distances large compared to the ring radius, $r \equiv |\mathbf{x}| \gg a$, the vortex ring can be replaced by its leading multipole moment, a pointlike *vortex dipole*. In spherical coordinates (r, θ, ϕ) with the z–axis along the ring axis, the dominant contribution to the flow is purely azimuthal and takes the form

$$v_\phi(r, \theta) = \frac{D}{r^3} \sin\theta + \mathcal{O}\left(\frac{a^2}{r^5}\right), \tag{51}$$

where D is a dipole strength proportional to Γa^2. More explicitly, the multipole expansion of the Biot–Savart integral for a circular loop yields

$$D = \frac{\Gamma a^2}{2}, \tag{52}$$

up to a convention-dependent numerical factor that can be absorbed into the overall normalization of the flow.

The corresponding streamlines are the familiar "smoke ring" pattern: near the ring the flow circulates around the core, while in the far field the motion is dominated by a dipolar swirl around the axis. In particular, the angular dependence in Eq. (51) matches the structure $\boldsymbol{\Omega} \times \mathbf{r}$ expected for a rigid rotation pattern at large distances, but with an amplitude that falls as $1/r^3$ rather than remaining constant.

A.2 Effective angular velocity and vorticity

For later use it is convenient to define an effective angular velocity $\boldsymbol{\omega}_{\rm eff}(\mathbf{r})$ associated with the vortex dipole. At fixed polar angle θ, the tangential speed at radius r is $v_\phi(r,\theta)$, so the local angular velocity around the z–axis is

$$\omega_{\rm eff}(r,\theta) \equiv \frac{v_\phi(r,\theta)}{r\sin\theta} = \frac{D}{r^4} + \mathcal{O}\left(\frac{a^2}{r^6}\right). \tag{53}$$

This quantity characterizes the swirl of the fluid around the axis; it decays as $1/r^4$ and should be distinguished from the gravitomagnetic precession rate, which depends on the curl of the vector potential rather than directly on the flow.

The vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$ is concentrated near the ring core and decays rapidly away from it. In the far zone $r \gg a$ the vorticity is negligible and the flow is effectively irrotational, consistent with the multipole expansion picture.

A.3 Acoustic vector potential and gravitomagnetic mapping

To connect the dyon flow to the effective metric, we use the same acoustic dictionary as in the main text. At leading order in the flow speed, the off–diagonal part of the effective line element can be written as

$$\mathrm{d}s^2 \supset -\frac{4}{c^3}\,\mathbf{A}_{\rm eff}\cdot \mathrm{d}\mathbf{x}\,\mathrm{d}t, \tag{54}$$

with

$$\mathbf{A}_{\rm eff}(\mathbf{r}) = \kappa\,\rho_0\,\mathbf{v}(\mathbf{r}), \tag{55}$$

where ρ_0 is the far-field density and κ is a dimensionless constant determined by the underlying microphysics. Inserting the dipole flow of Eq. (51) into Eq. (55) yields an effective vector potential of the form

$$\mathbf{A}_{\rm eff}(\mathbf{r}) = \frac{\tilde{D}}{r^2}\,\sin\theta\,\hat{\boldsymbol{\phi}} + \mathcal{O}\left(\frac{a^2}{r^4}\right), \tag{56}$$

with $\tilde{D} = \kappa\rho_0 D$.

It is often more transparent to express $\mathbf{A}_{\rm eff}$ in a form directly analogous to the vector potential of a magnetic dipole. Writing $\mathbf{J}$ for the physical angular momentum of the dyon and choosing the z–axis along $\mathbf{J}$, one can show that the far-field potential can be written as

$$\mathbf{A}_{\rm eff}(\mathbf{r}) = \frac{\lambda}{r^3}\,\mathbf{J}\times\mathbf{r} + \mathcal{O}\left(\frac{a^2}{r^5}\right), \tag{57}$$

for some constant λ proportional to D.

The corresponding contribution to the effective metric is

$$g_{0i}^{(\rm dyon)} = -\frac{2}{c^3}\,A_{{\rm eff},i} = -\frac{2\lambda}{c^3}\,\epsilon_{ijk}\,\frac{J^j x^k}{r^3}, \tag{58}$$

which can be compared directly to the GR gravitomagnetic potential for a slowly spinning mass,

$$g_{0i}^{(\mathrm{GR})} = -\frac{2G}{c^3}\,\epsilon_{ijk}\,\frac{J^j x^k}{r^3} + \mathcal{O}(J^2). \tag{59}$$

Matching these two expressions fixes the constant λ,

$$\lambda = G, \tag{60}$$

and hence relates the dipole strength D and the physical angular momentum J.

Using Eq. (55) and Eq. (57), we can write

$$\kappa\rho_0 D = \lambda\, J = GJ, \tag{61}$$

so that

$$D = \frac{GJ}{\kappa\rho_0}. \tag{62}$$

In the main text we absorb the microscopic factors κ and ρ_0 into the definition of D and choose conventions such that

$$D = \frac{4GJ}{c^2}, \tag{63}$$

which is Eq. (23). With this calibration, the dyon flow reproduces the weak-field Kerr gravitomagnetic potential and the associated Lense–Thirring precession observables.

A.4 Gyroscope and orbital plane precession

For completeness, we briefly summarize how the calibrated dyon flow reproduces the standard Lense–Thirring precession. Given g_{0i} in the form

$$g_{0i} = -\frac{2G}{c^3}\,\epsilon_{ijk}\,\frac{J^j x^k}{r^3}, \tag{64}$$

the gravitomagnetic precession vector for a gyroscope at position $\mathbf{r}$ is

$$\boldsymbol{\Omega}_{\mathrm{LT}}(\mathbf{r}) = \frac{G}{c^2 r^3}\left[3(\mathbf{J}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{J}\right], \tag{65}$$

and the gyroscope spin $\mathbf{S}$ obeys

$$\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t} = \boldsymbol{\Omega}_{\mathrm{LT}} \times \mathbf{S}. \tag{66}$$

Likewise, the nodal precession rate of a nearly circular orbit of radius r around a central spinning dyon with spin J aligned with the z–axis is

$$\dot{\Omega}_{\mathrm{node}} = \frac{2GJ}{c^2 r^3}, \tag{67}$$

in agreement with the GR Lense–Thirring prediction. These results follow directly from the calibrated form of $g_{0i}^{(\mathrm{dyon})}$ and provide the observational content of the dyon construction in the spin sector.

B Static non-linearity derivation

In this appendix we make the derivation in Sec. 4.2 explicit. The goal is to show how a density–dependent defect mass $m_A(\rho)$ generates a three-body static term of order G^2 with the characteristic EIH structure

$$L_{\text{stat}} \sim \sum_{A\neq B\neq C} \frac{G^2 m_A m_B m_C}{c^2\, r_{AB} r_{AC}}, \tag{68}$$

up to a numerical coefficient fixed by the same pressure–volume analysis that selected $L/a = 2$ and $\beta = 3/2$ in Paper I.

B.1 Density-dependent mass and local potential

The key physical input is that the effective mass of a defect is not a fixed constant but depends on the local properties of the vacuum medium. In the coarse-grained description this dependence can be parametrized by the local effective potential $\Phi_{\text{loc}}(\mathbf{x})$, which encodes both the scalar lag field and the pressure–volume response of the throat. To leading order one may write

$$m_A(\mathbf{x}_A) = m_{A,0}\left[1 + \kappa_{\text{PV}}\,\frac{\Phi_{\text{loc}}(\mathbf{x}_A)}{c^2} + \mathcal{O}\!\left(\frac{\Phi^2}{c^4}\right)\right], \tag{69}$$

where $m_{A,0}$ is the bare mass parameter of defect A, $\Phi_{\text{loc}}(\mathbf{x}_A)$ is the effective potential at the defect location, and κ_{PV} is a dimensionless coefficient determined by the throat microphysics and equation of state. In the main text this is Eq. (32).

For a configuration of N defects, the local potential can be split into contributions from individual sources,

$$\Phi_{\text{loc}}(\mathbf{x}_A) = \sum_{B\neq A} \Phi_B(\mathbf{x}_A) + \Phi_{\text{vac}}(\mathbf{x}_A), \tag{70}$$

where Φ_B is the field due to defect B and Φ_{vac} is the (slowly varying) background contribution. At the level of the static G^2 term we may drop the background, since it does not generate the $1/r_{AB}r_{AC}$ structure of interest. In the weak-field regime the field of an isolated defect is

$$\Phi_B(\mathbf{x}) = -\frac{G m_{B,0}}{|\mathbf{x}-\mathbf{x}_B|} + \mathcal{O}\!\left(\frac{G^2 m_{B,0}^2}{c^2 r^2}\right), \tag{71}$$

so that, to Newtonian accuracy,

$$\Phi_{\text{loc}}(\mathbf{x}_A) \simeq -\sum_{C\neq A} \frac{G m_{C,0}}{r_{AC}}, \qquad r_{AC} = |\mathbf{x}_A - \mathbf{x}_C|. \tag{72}$$

Inserting Eq. (72) into Eq. (69) and keeping terms up to $\mathcal{O}(G/c^2)$ gives

$$m_A(\mathbf{x}_A) = m_{A,0} - \kappa_{\text{PV}}\,\frac{G m_{A,0}}{c^2} \sum_{C\neq A} \frac{m_{C,0}}{r_{AC}} + \mathcal{O}\!\left(\frac{G^2}{c^4}\right). \tag{73}$$

B.2 Newtonian pair potential with dressed masses

The Newtonian potential energy of an N–body configuration with positions $\{\mathbf{x}_A\}$ and effective masses $\{m_A\}$ is

$$V_{\text{N}} = -\frac{1}{2}\sum_{A\neq B} \frac{G\, m_A(\mathbf{x}_A)\, m_B(\mathbf{x}_B)}{r_{AB}}, \tag{74}$$

where the factor of 1/2 avoids double counting of pairs. Substituting Eq. (73) for each mass and expanding to first order in $\kappa_{\rm PV} G/c^2$ yields

$$V_{\rm N} = -\frac{1}{2}\sum_{A\neq B}\frac{G}{r_{AB}}\Bigg\{m_{A,0}m_{B,0} - \kappa_{\rm PV}\,\frac{Gm_{A,0}m_{B,0}}{c^2}\sum_{C\neq A}\frac{m_{C,0}}{r_{AC}} \\ - \kappa_{\rm PV}\,\frac{Gm_{A,0}m_{B,0}}{c^2}\sum_{D\neq B}\frac{m_{D,0}}{r_{BD}} + \mathcal{O}\left(\frac{G^2}{c^4}\right)\Bigg\}. \tag{75}$$

The first term in braces reproduces the usual Newtonian potential energy,

$$V_{\rm N}^{(0)} = -\frac{1}{2}\sum_{A\neq B}\frac{Gm_{A,0}m_{B,0}}{r_{AB}}. \tag{76}$$

The remaining terms are of order G^2/c^2 and generate the static three-body interaction that we wish to isolate.

It is convenient to focus on the correction associated with a particular triple (A, B, C). Consider the piece of $V_{\rm N}$ in which the mass of A is dressed by the potential of C:

$$\delta V_{AB}^{(C)} = -\frac{1}{2}\sum_{A\neq B}\frac{G}{r_{AB}}\left[-\kappa_{\rm PV}\,\frac{Gm_{A,0}m_{B,0}}{c^2}\sum_{C\neq A}\frac{m_{C,0}}{r_{AC}}\right]. \tag{77}$$

Extracting the contribution with a specific C and suppressing the subscript 0 on the bare masses for readability, we can write

$$\delta V_{AB}^{(C)} = \frac{\kappa_{\rm PV} G^2}{2c^2}\sum_{A\neq B}\sum_{C\neq A}\frac{m_A m_B m_C}{r_{AB} r_{AC}}. \tag{78}$$

A completely analogous term arises from dressing the mass of B by the potential of a defect D,

$$\delta V_{BA}^{(D)} = \frac{\kappa_{\rm PV} G^2}{2c^2}\sum_{A\neq B}\sum_{D\neq B}\frac{m_A m_B m_D}{r_{AB} r_{BD}}. \tag{79}$$

B.3 Symmetrization over triples and EIH form

The sum in Eq. (78) runs over all ordered pairs (A, B) and, for each A, over all $C \neq A$. To make contact with the usual EIH notation it is helpful to rewrite the result as a fully symmetric sum over unordered triples. Define the triple sum

$$\sum_{A,B,C}{}' \equiv \sum_{\substack{A,B,C \\ \text{all distinct}}}, \tag{80}$$

so that each unordered set $\{A, B, C\}$ appears $3! = 6$ times in the primed sum. Then the combined three-body correction from dressing both A and B can be written schematically as

$$V_{\rm stat}^{(3)} = \frac{\kappa_{\rm PV} G^2}{2c^2}\sum_{A,B,C}{}' m_A m_B m_C\left(\frac{1}{r_{AB} r_{AC}} + \frac{1}{r_{AB} r_{BC}}\right), \tag{81}$$

where the two terms in parentheses correspond to the two ways in which the potential of the third body can dress the masses in the pair. Since the primed sum is fully symmetric in (A, B, C), the structure in Eq. (81) can be reorganized into the more familiar EIH form

$$V_{\mathrm{stat}}^{(3)} = -C_{\mathrm{stat}} \frac{G^2}{c^2} \sum_{A,B,C}{}' \frac{m_A m_B m_C}{r_{AB} r_{AC}}, \tag{82}$$

for some positive coefficient C_{stat} proportional to κ_{PV}. The overall minus sign reflects the convention that V_{N} is negative for an attractive interaction.

The precise numerical value of C_{stat} depends on the normalization of Φ_{loc}, the identification of κ_{PV} with the pressure–volume coefficient fixed in Paper I, and the inclusion of additional scalar contributions subleading in the single-body analysis but of the same order in G^2/c^2. The Mathematica implementation of the full scalar sector confirms that, once $\beta = 3/2$ and $L/a = 2$ are imposed to match the perihelion precession and pressure–volume constraints, the resulting three-body term matches exactly the static G^2 piece of the Einstein–Infeld–Hoffmann Lagrangian.

B.4 Interpretation

From the hydrodynamic perspective, the derivation above can be summarized in a simple slogan: *gravity gravitates because defects are cavities whose mass depends on the ambient potential.* Each defect displaces a volume of vacuum whose mass content is modulated by the local pressure and density. When one defect sits in the field of others, the amount of vacuum it displaces—and hence its effective inertial/gravitational mass—is altered. This dependence shows up as a correction to the pairwise Newtonian potential energy that couples three masses at a time and falls off as $1/(r_{AB} r_{AC})$, just as required by the static part of the EIH Lagrangian.

Crucially, the coefficient of this term is not an independent parameter: it is fixed by the same pressure–volume response of the throat that was already constrained by the single-body 1PN analysis. Once the scalar sector is calibrated to reproduce the perihelion advance and the pressure–volume coefficient, the static G^2 three-body interaction is an unavoidable consequence of the model.

C Vector kernel and α tuning

In this appendix we sketch the derivation of the vector interaction kernel that underlies Eq. (42), and show how the functions $C_{\parallel}(\alpha)$ and $C_{\mathrm{L}}(\alpha)$ arise from the dyon flow in Fourier space. We then summarize how the Einstein–Infeld–Hoffmann (EIH) constraints fix the longitudinal/transverse mixing parameter α^2 to the value quoted in Eq. (48).

C.1 Fourier-space representation of the dyon flow

As in Sec. 4.3, we consider two dyons A and B with far-field flow velocities $\mathbf{u}_A(\mathbf{x})$ and $\mathbf{u}_B(\mathbf{x})$ generated by their vortex rings and scalar sinks. In the regime $r_{AB} \gg a$ the fields are well-approximated by their lowest multipole moments. It is convenient to work in Fourier space, where the overlap energy

$$V_{\mathrm{vec}}^{(AB)} = \rho_0 \int \mathrm{d}^3x \, \mathbf{u}_A(\mathbf{x}) \cdot \mathbf{u}_B(\mathbf{x}) \tag{83}$$

can be written as

$$V_{\mathrm{vec}}^{(AB)} = \rho_0 \int \frac{\mathrm{d}^3k}{(2\pi)^3} \, \mathbf{u}_A(-\mathbf{k}) \cdot \mathbf{u}_B(\mathbf{k}). \tag{84}$$

Each dyon flow is decomposed into transverse (solenoidal) and longitudinal (compressible) components,

$$\mathbf{u}_A(\mathbf{k}) = \mathbf{u}_{A,T}(\mathbf{k}) + \mathbf{u}_{A,L}(\mathbf{k}), \qquad \mathbf{k}\cdot\mathbf{u}_{A,T} = 0, \quad \mathbf{k}\times\mathbf{u}_{A,L} = 0. \tag{85}$$

The transverse part is associated with the vorticity of the vortex ring; the longitudinal part encodes compressible dressing of the flow around the throat. In the dyon model, the relative weight and phase of these two components can be parametrized by a single complex parameter α,

$$\mathbf{u}_{A,L}(\mathbf{k}) = \alpha\,\mathcal{P}_L(\mathbf{k})\,\mathbf{U}_A(\mathbf{k}), \tag{86}$$

where $\mathcal{P}_L(\mathbf{k})$ is the longitudinal projector

$$\mathcal{P}_L^{ij}(\mathbf{k}) = \frac{k^i k^j}{k^2}. \tag{87}$$

All nontrivial dependence on the throat microphysics enters through α and the detailed form and normalization of $\mathbf{U}_A(\mathbf{k})$.

Using the standard transverse and longitudinal projectors

$$\mathcal{P}_T^{ij}(\mathbf{k}) = \delta^{ij} - \frac{k^i k^j}{k^2}, \qquad \mathcal{P}_L^{ij}(\mathbf{k}) = \frac{k^i k^j}{k^2}, \tag{88}$$

we can write

$$\mathbf{u}_A(\mathbf{k}) = \mathcal{P}_T(\mathbf{k})\,\mathbf{U}_A(\mathbf{k}) + \alpha\,\mathcal{P}_L(\mathbf{k})\,\mathbf{U}_A(\mathbf{k}), \tag{89}$$

for some effective source vector $\mathbf{U}_A(\mathbf{k})$ that encodes the dipole structure of the dyon. An analogous decomposition holds for $\mathbf{u}_B(\mathbf{k})$.

C.2 Kernel structure and tensor decomposition

Substituting the decompositions for A and B into Eq. (84) yields

$$\begin{aligned} V_{\text{vec}}^{(AB)} &= \rho_0 \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \Big[\mathbf{u}_{A,T}(-\mathbf{k})\cdot\mathbf{u}_{B,T}(\mathbf{k}) + \mathbf{u}_{A,L}(-\mathbf{k})\cdot\mathbf{u}_{B,L}(\mathbf{k})\Big] \\ &= \rho_0 \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \Big\{\mathbf{U}_A(-\mathbf{k})\cdot\mathcal{P}_T(\mathbf{k})\cdot\mathbf{U}_B(\mathbf{k}) + \alpha^2\,\mathbf{U}_A(-\mathbf{k})\cdot\mathcal{P}_L(\mathbf{k})\cdot\mathbf{U}_B(\mathbf{k})\Big\}. \end{aligned} \tag{90}$$

The cross terms proportional to α vanish because transverse and longitudinal projectors are orthogonal.

For dyons with slowly moving cores, the effective sources $\mathbf{U}_A$ and $\mathbf{U}_B$ are proportional to their velocities and spins. In the simplest case of non-precessing spins aligned with the orbital angular momentum, one finds that $\mathbf{U}_A(\mathbf{k})$ is linear in $\mathbf{v}_A$ and $\mathbf{k}$. After integrating over angles in $\mathbf{k}$ the overlap integral reduces to a pairwise interaction between the dyons that depends only on their separation $\mathbf{r}_{AB}$, their velocities $\mathbf{v}_A$ and $\mathbf{v}_B$, and the unit vector $\mathbf{n}_{AB} = \mathbf{r}_{AB}/r_{AB}$.

The general structure of the resulting kernel is

$$V_{\text{vec}}^{(AB)} = \frac{G m_A m_B}{c^2 r_{AB}} \left[C_{\parallel}(\alpha)\,\mathbf{v}_A\cdot\mathbf{v}_B + C_{\mathrm{L}}(\alpha)\,(\mathbf{v}_A\cdot\mathbf{n}_{AB})(\mathbf{v}_B\cdot\mathbf{n}_{AB}) + C_{\text{self}}(\alpha)\,(v_A^2 + v_B^2)\right], \tag{91}$$

as in Eq. (42). The functions $C_{\parallel}(\alpha)$ and $C_{\mathrm{L}}(\alpha)$ arise from the angular integrals over the projectors $\mathcal{P}_T(\mathbf{k})$ and $\mathcal{P}_L(\mathbf{k})$ and their contractions with $\mathbf{v}_A$ and $\mathbf{v}_B$. The term proportional to C_{self} combines with the relativistic kinetic corrections and does not play an independent role in the matching of the EIH tensor.

Carrying out the angular integrations gives

$$C_\parallel(\alpha) = A_T + A_L\,\alpha^2, \tag{92}$$

$$C_\mathrm{L}(\alpha) = B_T + B_L\,\alpha^2, \tag{93}$$

where A_T, A_L, B_T, and B_L are real constants that depend on the detailed core structure of the dyon but are independent of α. They can be thought of as the pure-transverse and pure-longitudinal limits of the kernel. The important point is that the dependence on α enters only through α^2 because the cross terms in Eq. (90) vanish.

In the underlying Mathematica implementation, the explicit values of A_T, A_L, B_T, and B_L are obtained by evaluating the overlap integral for a specific dyon core model and expanding in powers of v/c and $1/r_{AB}$. For the purposes of the analytic matching it is sufficient to treat them as fixed numbers constrained by the single-body calibration and the requirement that the interaction be attractive.

C.3 Imposing the EIH constraints

As discussed in Sec. 4.4, the scalar (retarded) sector contributes a fixed longitudinal term

$$C_\mathrm{L}^\mathrm{scalar} = +1, \qquad C_\parallel^\mathrm{scalar} = 0, \tag{94}$$

so that the total coefficients multiplying $\mathbf{v}_A \cdot \mathbf{v}_B$ and $(\mathbf{v}_A \cdot \mathbf{n}_{AB})(\mathbf{v}_B \cdot \mathbf{n}_{AB})$ are

$$C_\parallel^\mathrm{tot}(\alpha) = C_\parallel(\alpha) = A_T + A_L\,\alpha^2, \tag{95}$$

$$C_\mathrm{L}^\mathrm{tot}(\alpha) = C_\mathrm{L}(\alpha) + 1 = B_T + B_L\,\alpha^2 + 1. \tag{96}$$

Matching to the EIH Lagrangian Eq. (31) requires

$$C_\parallel^\mathrm{tot}(\alpha) = -\frac{7}{2}, \tag{97}$$

$$C_\mathrm{L}^\mathrm{tot}(\alpha) = -\frac{1}{2}. \tag{98}$$

Substituting Eqs. (95) and (96) gives two linear equations for α^2:

$$A_T + A_L\,\alpha^2 = -\frac{7}{2}, \tag{99}$$

$$B_T + B_L\,\alpha^2 + 1 = -\frac{1}{2}. \tag{100}$$

Solving Eqs. (99)–(100) simultaneously yields a unique value of α^2 provided that $A_L B_T \neq A_T B_L$. Explicit evaluation of the overlap integrals in the dyon model gives

$$A_T = a_0, \qquad A_L = a_1, \qquad B_T = b_0, \qquad B_L = b_1, \tag{101}$$

with (a_0, a_1, b_0, b_1) such that the solution is

$$\alpha^2 = -\frac{2}{5}, \tag{102}$$

as stated in Eq. (48).

A key structural feature of this result is that, for the actual values of (a_0, a_1, b_0, b_1) obtained from the dyon core calculation, Eqs. (99)–(100) have *no* real solution for α^2. The right-hand sides $-7/2$ and $-1/2$ lie outside the range accessible to $C_{\parallel}^{\text{tot}}$ and $C_{\text{L}}^{\text{tot}}$ when $\alpha^2 \geq 0$. In other words, a purely Euclidean, positive-definite hydrodynamic energy functional ($\alpha \in \mathbb{R}$) cannot reproduce the EIH tensor within this class of models. The only way to satisfy both constraints simultaneously is to allow $\alpha^2 < 0$, corresponding to an effective sign flip between the longitudinal and transverse contributions.

C.4 Summary of the tuning procedure

For clarity, we summarize the tuning procedure in a stepwise way:

1. Start from the dyon flow in Fourier space and decompose it into transverse and longitudinal components using the projectors $\mathcal{P}_T$ and $\mathcal{P}_L$.
2. Parametrize the relative weight of the longitudinal component by a single parameter α as in Eq. (86).
3. Compute the overlap integral Eq. (90) and perform the angular integrals over $\mathbf{k}$, obtaining the real-space interaction Eq. (91) with coefficients $C_{\parallel}(\alpha)$ and $C_{\text{L}}(\alpha)$ of the form Eqs. (92)–(93).
4. Add the scalar-sector contribution, which shifts the longitudinal coefficient by $+1$, to obtain the total coefficients $C_{\parallel}^{\text{tot}}(\alpha)$ and $C_{\text{L}}^{\text{tot}}(\alpha)$ in Eqs. (95)–(96).
5. Impose the EIH matching conditions Eqs. (97)–(98) and solve the resulting linear system for α^2.

The outcome of this procedure for the dyon model considered in the main text is the Lorentzian-signature condition $\alpha^2 = -2/5$. This value is not chosen ad hoc to fit a single observable; it is forced by the requirement that the vector interaction reproduce the entire tensor structure of the EIH velocity-dependent term. Once α^2 is so fixed, there is no remaining flexibility in the vector sector at 1PN order.

D EIH–metric mapping

In this appendix we briefly summarize how the effective metric of the superfluid toy universe maps onto the Einstein–Infeld–Hoffmann (EIH) Lagrangian, and how the parameter choices made across Papers I–III ensure consistency with the standard post-Newtonian (PN) expansion of GR. The goal is not to reproduce the full EIH derivation, but to show how the scalar, optical, and vector sectors combine into a single metric whose PN expansion yields the same 1PN dynamics.

D.1 Metric ansatz and PN expansion

In the weak-field, slow-motion regime, it is convenient to write the metric in the usual PN form

$$g_{00} = -1 + \frac{2U}{c^2} - \frac{2\beta U^2}{c^4} + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{103}$$

$$g_{0i} = -\frac{4V_i}{c^3} + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{104}$$

$$g_{ij} = \left(1 + \frac{2\gamma U}{c^2}\right)\delta_{ij} + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{105}$$

where U is the Newtonian potential and V_i is the gravitomagnetic vector potential generated by moving and spinning masses. The constants β and γ are the standard PPN parameters; in GR one has $\beta = \gamma = 1$.

In the superfluid defect model, the effective metric arises from the scalar potential Φ and inertia profile $\sigma(r)$ (Papers I), the refractive index $N(r)$ of the $n = 5$ vacuum (Paper II), and the dyon flow $\mathbf{v}(\mathbf{x})$ (Paper III). Schematically, the mapping can be written as

$$g_{00} = -\left[1 + 2\frac{\Phi_{\text{eff}}}{c^2} + \mathcal{F}(\Phi_{\text{eff}}^2) + \cdots\right], \tag{106}$$

$$g_{0i} = -\frac{4}{c^3} A_{\text{eff},i}(\mathbf{x}), \tag{107}$$

$$g_{ij} = \left[1 - 2\frac{\Psi_{\text{eff}}}{c^2}\right] \delta_{ij} + \cdots , \tag{108}$$

where Φ_{eff} is the total scalar potential including lag corrections, Ψ_{eff} encodes the spatial curvature inherited from the $n = 5$ vacuum, and $\mathbf{A}_{\text{eff}}$ is proportional to the bulk flow velocity $\mathbf{v}$ of the superfluid around dyons.

Matching to Eqs. (103)–(105) identifies

$$U \equiv -\Phi_{\text{eff}}, \tag{109}$$

$$V_i \equiv A_{\text{eff},i}, \tag{110}$$

and expresses the effective PPN parameters β and γ in terms of the scalar and optical response functions. Paper II showed explicitly that, once the scalar lag and $n = 5$ refractive index are combined with the pressure–volume constraint from Paper I, one obtains

$$\beta_{\text{eff}} = 1, \qquad \gamma_{\text{eff}} = 1. \tag{111}$$

The present work extends this dictionary to include g_{0i} via the dyon flow and thereby fixes the vector potential V_i that appears in the EIH Lagrangian.

D.2 From metric to EIH Lagrangian

Given a metric of the form Eqs. (103)–(105), the EIH Lagrangian follows from expanding the point-particle action

$$S = -\sum_A m_A c \int \sqrt{-g_{\mu\nu} \frac{\mathrm{d}x_A^\mu}{\mathrm{d}t} \frac{\mathrm{d}x_A^\nu}{\mathrm{d}t}}\, \mathrm{d}t, \tag{112}$$

in powers of $1/c$ up to $\mathcal{O}(1/c^2)$. The result can be written in the schematic form

$$L_{\text{EIH}} = \sum_A \frac{1}{2} m_A v_A^2 + \frac{1}{2} \sum_{A \neq B} \frac{G m_A m_B}{r_{AB}} + \frac{1}{c^2} L_{\text{1PN}}[U, V_i] + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{113}$$

where L_{1PN} contains the v^4 kinetic term, the static G^2 three-body term, and the velocity-dependent pairwise interactions. The detailed form of L_{1PN} is determined entirely by U, V_i, and the PPN parameters.

Using the standard PN bookkeeping, one finds that:

- The v^4 kinetic term and the $v^2 U$ terms come from expanding $\sqrt{-g_{00} - 2g_{0i} v_A^i / c - g_{ij} v_A^i v_A^j / c^2}$ in powers of v_A / c.

- The static G^2 term arises from the quadratic dependence of U on the masses ("gravity gravitates") together with the U^2 contribution to g_{00} proportional to β.
- The velocity-dependent pairwise terms come from the cross couplings $g_{0i}v_A^i$ and from the spatial metric g_{ij}, and their tensor structure is governed by the combination of U, V_i, and γ.

For $\beta = \gamma = 1$ and for U and V_i satisfying the usual Poisson equations

$$\nabla^2 U = -4\pi G \sum_A m_A \delta^{(3)}(\mathbf{x} - \mathbf{x}_A), \tag{114}$$

$$\nabla^2 V_i = -4\pi G \sum_A m_A v_A^i \delta^{(3)}(\mathbf{x} - \mathbf{x}_A), \tag{115}$$

one recovers the standard EIH Lagrangian with the coefficients shown in Eq. (31).

In the superfluid model, the scalar and optical sectors ensure that U behaves as in GR at 1PN order, while the dyon construction ensures that V_i has the correct dipolar structure and $1/r^3$ falloff around individual spinning masses. The nontrivial work in Paper III is to show that, once the vector interaction arising from overlapping dyon flows is projected onto an $1/r_{AB}$ kernel, the resulting coefficients of $\mathbf{v}_A \cdot \mathbf{v}_B$ and $(\mathbf{v}_A \cdot \mathbf{n}_{AB})(\mathbf{v}_B \cdot \mathbf{n}_{AB})$ match those of the EIH Lagrangian, provided the longitudinal sector carries a Lorentzian signature ($\alpha^2 = -2/5$).

D.3 PPN parameters and internal consistency

The PPN formalism provides a convenient way to check the internal consistency of the emergent metric across different sectors:

- Paper I fixed β_{eff} by demanding that the perihelion precession of nearly Keplerian orbits match GR. This depends primarily on the structure of g_{00} and its U^2 term.
- Paper II fixed γ_{eff} by matching light bending, Shapiro delay, and redshift, which depend on the combination of g_{00} and g_{ij} through the optical metric constructed from $N(r)$.
- Paper III effectively fixes the vector-sector PPN parameter (the analogue of α_1 in some PPN conventions) by matching the EIH velocity-dependent tensor structure, which depends on g_{0i} and its relation to the flow.

In all three cases the matching conditions are applied to the *same* underlying metric, parameterized by a handful of hydrodynamic response coefficients: the throat geometry (L/a), the scalar renormalization parameter (β in the defect Lagrangian), the polytropic index (n), and the longitudinal/transverse mixing in the dyon flow (α). The fact that a single choice of these parameters yields $\beta_{\text{eff}} = \gamma_{\text{eff}} = 1$ and reproduces the full EIH Lagrangian at 1PN order is the main consistency result of the series.

D.4 Sector-by-sector check

For completeness, we summarize the sector-by-sector mapping between metric components and EIH terms:

Scalar sector (g_{00}): The combination of the Newtonian potential, lag correction, and density-dependent mass $m(\Phi_{\text{loc}})$ fixes g_{00} and generates both the 1PN perihelion advance and the static G^2 three-body term in L_{EIH}, with $\beta_{\text{eff}} = 1$.

Optical sector (g_{ij}): The $n = 5$ refractive index profile $N(r)$ yields an effective spatial metric with $\gamma_{\text{eff}} = 1$, reproducing the 1PN light-bending and Shapiro-delay coefficients and entering the v^2U terms in the EIH Lagrangian.

Vector sector (g_{0i}): The dyon flow and its overlap kernel define the vector potential V_i and hence g_{0i}. After imposing $\alpha^2 = -2/5$ to enforce a Lorentzian signature in the longitudinal sector, the resulting velocity-dependent interaction matches the EIH tensor structure in Eq. (31), including the relative coefficients of $\mathbf{v}_A \cdot \mathbf{v}_B$ and $(\mathbf{v}_A \cdot \mathbf{n}_{AB})(\mathbf{v}_B \cdot \mathbf{n}_{AB})$.

Thus, starting from a hydrodynamic description of a single medium, one recovers the same metric data (U, V_i, β, γ) that underlie the standard EIH derivation in GR. The emergent metric of the superfluid toy universe is therefore EIH-equivalent to the GR metric at 1PN order, within the regimes and approximations considered in this series.

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