Brane–Bulk Throat Ontology for a Superfluid Defect Toy Universe

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Abstract

This paper, the fifth in a series developing a "superfluid defect toy universe, " resolves the geometric tension between the model's gravitational and electromagnetic sectors. Previous work established that reproducing 1PN gravity requires defects to behave as spherical sinks, whereas the electromagnetic sector requires cylindrical resonant cavities. We resolve this apparent contradiction by promoting defects to brane–bulk throats connecting the observable 3D brane to a 4D superfluid bulk. Dimensional reduction reveals that the far–field potential on the brane is monopolar, with angular corrections from the throat geometry suppressed by $(a/r)^2$, thus recovering the spherical sink approximation for 1PN gravity. Inside the throat, however, 4D acoustic modes separate into cylindrical Bessel profiles; enthalpy minimization of these modes at fixed charge selects the preferred aspect ratio $L/a \approx 1.85$ found in the electromagnetic construction. Finally, we interpret the non–Euclidean constraint $\alpha^2 = -2/5$ in the vector sector as an emergent Lorentzian signature arising from the mixing of brane–parallel and bulk–directed modes. This ontology constrains finite–size effects, laying the groundwork for falsifiable 2PN predictions.

1 Introduction

1.1 Motivation and the sphere–cylinder tension

This paper is the fifth in a series developing a "superfluid defect toy universe" in which gravity and electromagnetism emerge from the dynamics of a compressible fluid.[1] In this model the vacuum is treated as a stiff superfluid, and massive bodies are modeled as localized defects that drain and stir this medium. Far from any defect, the flow is slow and approximately irrotational; near the defect, nonlinear and topological effects become important. The central claim of the series so far has been that, with a suitable choice of hydrodynamic energy functional, this superfluid picture reproduces the standard post–Newtonian (PN) expansion of general relativity (GR) at first PN order (1PN) for isolated bodies and N–body systems, while also supporting an emergent electromagnetism sector.

[1] For clarity, we will refer to the previous papers as Papers I–IV[1, 2, 3, 4] throughout.

Paper I (the orbital paper) showed that if a defect behaves as a spherical sink in a three–dimensional superfluid, then the resulting potential flow around the defect reproduces Newtonian gravity and the 1PN perihelion precession of GR. In that construction the background flow is spherically symmetric and purely radial on spatial slices, and the defect induces both a Newtonian $1/r$ potential and an additional precession term $\kappa_{\mathrm{add}} = 1/2$ from hydrodynamic dressing. Together these contributions fix a single orbital parameter $\beta = 3/2$, matching the GR prediction for perihelion advance. Crucially, the calculation assumes that the defect is effectively spherical: when one tries to repeat the same analysis with a cylindrical defect in three dimensions, the resulting PN coefficients become orientation–dependent and do not reproduce GR in a rotationally invariant way.

Paper II (the optics paper) recast the same superfluid model in terms of an effective optical metric. Light propagation in the gravitational field of a single defect was described in terms of refraction in a stiff polytropic superfluid, and it was shown that matching the GR lensing coefficient fixes a specific polytropic index and curvature coefficient in the optical sector. Again, the gravitational field is sourced by a spherically symmetric sink: from the point of view of the optical metric, the defect is a point–like, effectively spherical object.

Paper III (the spin and N–body paper) extended the model to include spin–induced frame dragging and general N–body interactions. There the hydrodynamic energy functional was generalized to include longitudinal and transverse vector kernels, and the resulting 1PN dynamics were matched to the Einstein–Infeld–Hoffmann (EIH) Lagrangian. The matching uniquely constrained the relative weight of longitudinal and transverse pieces, forcing the parameter

$$\alpha^2 = -\frac{2}{5}, \tag{1}$$

which in turn implies that the longitudinal sector effectively contributes with a Lorentzian (negative) sign. This is a sharp and somewhat surprising constraint: no purely Euclidean, positive–definite hydrodynamic energy density reproduces the EIH tensor; the correct GR coefficients appear if and only if the longitudinal sector behaves as though it were time–like in mode space. Importantly for the present work, the spin and N–body analysis still assumes a spherical defect: attempts to model the defect as a 3D cylinder again fail at 1PN, with significant orientation–dependent errors in the predicted precession and spin couplings.

Paper IV (the EM paper) shifted focus to the electromagnetic sector. There the defect was modeled as a resonant cavity in the superfluid, and a hydrodynamic–electromagnetic dictionary was constructed that maps certain mode structures of the fluid onto electric and magnetic fields. Stabilizing a charged defect required a cylindrical cavity of radius a and length L, supporting a fundamental mode with radial profile $J_0(x_{01}r/a)$ and a standing wave along the axial direction. Minimizing the enthalpy at fixed "charge" picked out a preferred aspect ratio,

$$\frac{L}{a} = \frac{\sqrt{2}\,\pi}{x_{01}} \approx 1.85, \tag{2}$$

where x_{01} is the first zero of J_0. In this construction, charge is associated with circulation and vorticity in and around a cylindrical defect, and the cylindrical geometry is not an optional detail: it is needed to obtain the correct mode structure and a stable charged configuration.

Taken at face value, these results appear to ascribe contradictory geometries to the same object. The gravitational papers (I–III) insist that the defect must behave as a spherical sink in three dimensions in order to reproduce 1PN gravity. The electromagnetic paper (IV) insists that the defect must behave as a cylindrical resonant cavity in order to reproduce electromagnetism. One cannot simply declare the defect to be "both a sphere and a cylinder" within a purely three–dimensional picture without spoiling either the gravitational or the electromagnetic sector. This sphere–cylinder tension is an internal inconsistency in the toy model, and it forces us to look for a deeper geometric ontology in which both descriptions can be simultaneously true.

1.2 Brane–bulk throat ontology as a resolution

The central proposal of this paper is that the apparent contradiction is an artifact of insisting on a purely 3D description of an intrinsically higher–dimensional defect. We will argue that the defect should instead be understood as a *throat* connecting a three–dimensional brane to a four–dimensional superfluid bulk.

Concretely, we introduce coordinates $(t, \mathbf{x}, w)$, where $\mathbf{x} = (x, y, z)$ label directions along the brane and w is a bulk coordinate transverse to the brane. The physical world accessible to ordinary observers is modeled as a hypersurface at $w = 0$, which we will refer to as the brane. The superfluid fills the full four–dimensional space, and its density and velocity fields extend smoothly into the bulk. A defect is then represented not as a compact three–dimensional object, but as a localized region where the brane pinches into the bulk, forming a throat of radius a and depth L along w.

From the point of view of an observer confined to the brane, the intersection of the throat with $w = 0$ is a nearly spherical opening of radius a. The effective flow induced on the brane by drainage into the throat is then approximately that of a spherical sink located at the center of this opening. At distances $r \gg a$ along the brane, the flow is radially symmetric and the corresponding effective gravitational potential is indistinguishable from that of a point mass. In this sense, the 1PN gravitational sector of Papers I–III only probes the *mouth* of the throat, and it is naturally described by a spherical sink in an effective 3D fluid.

From the point of view of modes living *inside* the throat, however, the relevant geometry is entirely different. Away from the mouth the throat is approximately a cylinder of radius a and length L embedded in the bulk. Acoustic modes confined within this cylindrical region obey a 4D wave equation whose radial and axial dependence factorize. Imposing appropriate boundary conditions at $r = a$ and at the ends $w = 0$ and $w = L$ selects Bessel–type radial profiles and sine– or cosine–type axial profiles, precisely of the form used in Paper IV. Enthalpy minimization at fixed "charge" then selects a preferred aspect ratio $L/a \approx 1.85$ for the throat. In this picture, the electromagnetic sector is not telling us that the whole defect is a 3D cylinder; it is telling us that the *interior of the throat in the bulk* behaves as an approximately cylindrical cavity.

The apparent sphere–cylinder contradiction is thus resolved once we distinguish between the brane projection and the bulk interior. Gravity is governed by the effective three–dimensional projection of the bulk flow onto the brane and is therefore sensitive primarily to the spherical mouth of the throat. Electromagnetism is governed by cavity modes that live inside the throat and therefore probe the cylindrical interior. The two descriptions are compatible because they refer to different slices of the same higher–dimensional geometry.

In the same spirit, we will argue that the longitudinal sign flip $\alpha^2 = -2/5$ found in Paper III is naturally interpreted in terms of brane–bulk mode mixing. The transverse modes that govern ordinary three–dimensional flows are effectively Euclidean, while the longitudinal modes associated with motion into the bulk contribute with an opposite sign in the effective energy functional, playing the role of a time–like direction in mode space. The emergent Lorentzian structure of the 1PN theory can thus be traced to the way brane–parallel and bulk–directed modes mix in the vicinity of the throat.

1.3 Scope and goals of this paper

The goal of this paper is to make the brane–bulk throat ontology precise enough that the previous results of Papers I–IV can be seen as different projections of a single geometric picture, and to lay the groundwork for a systematic analysis of higher–order PN corrections.

On the constructive side, we will:

- Formulate a four–dimensional superfluid framework with a brane at $w = 0$ and defects represented as throats of radius a and depth L.
- Show how dimensional reduction from 4D to the brane organizes the far–field effective theory in terms of monopole and higher multipole moments, with angular corrections suppressed by $(a/r)^2$.
- Demonstrate that, at 1PN order, the far–field behavior of a throat on the brane is indistinguishable from that of the spherical sinks used in Papers I–III, explaining why those papers were successful despite their purely 3D language.
- Reinterpret the electromagnetic cavity of Paper IV as the interior of the throat, and the enthalpy–selected aspect ratio $L/a \approx 1.85$ as a geometric property of the throat in the bulk rather than an arbitrary 3D cylinder.
- Provide a simple two–mode toy model that connects the longitudinal sign flip $\alpha^2 = -2/5$ to an emergent Lorentzian signature in the space of brane–parallel and bulk–directed modes.

On the interpretive side, we will:

- Argue that finite–size effects associated with the throat, including the detailed structure of the transition region near the mouth, naturally appear as $(a/r)^2$ corrections, i.e. as 2PN and higher–order terms.
- Outline how a future 2PN calculation could be organized around the throat geometry, turning the model into a falsifiable framework whose predictions can be confronted with precision tests of gravity and electromagnetism.
- Clarify in what sense the present construction resembles brane–world and superfluid–vacuum scenarios in the broader literature, and in what ways it differs.

What this paper does *not* do is equally important. We will not attempt a full 2PN derivation of the effective gravitational Lagrangian or the complete electromagnetic sector. We will not specify the microscopic origin or stabilization mechanism of the brane itself, nor will we attempt to simulate strong–field or highly dynamical processes such as mergers or collapse. Our aim is more modest: to provide a coherent geometric ontology that removes the internal tensions of the existing toy model and strongly constrains the form of any future higher–order calculations.

1.4 Structure of the paper

The remainder of the paper is organized as follows. In Sec. 2 we introduce the four–dimensional superfluid framework and the brane embedding. We define the bulk density, velocity, and enthalpy fields, specify the equation of state, and write down the 4D continuity and Euler equations. We then describe the brane as a hypersurface at $w = 0$, discuss the induced three–dimensional fields on the brane, and introduce a simple geometric model of a throat of radius a and depth L connecting the brane to the bulk.

In Sec. 3 we turn to dimensional reduction and the far–field structure. We define effective 3D sources obtained by integrating over the bulk coordinate w, and we study a Gaussian toy model for a localized throat. This model allows us to compute the effective monopole mass and quadrupole moment, and to show explicitly that angular corrections to the potential are suppressed by $(a/r)^2$. We explain why the 1PN results of Papers I–III are insensitive to these finite–size effects and therefore naturally see only a spherical source.

Sec. 4 focuses on the interior of the throat and the electromagnetic sector. Starting from the 4D acoustic wave equation inside the throat, we perform a separation–of–variables analysis to recover the cylindrical cavity modes of Paper IV, including the fundamental mode with radial profile $J_0(x_{01}r/a)$ and appropriate axial dependence along w. We then show how enthalpy minimization at fixed "charge" selects the aspect ratio $L/a \approx 1.85$, and we reinterpret this result as a geometric constraint on the throat. We also review how charge and mass can be viewed as different projections of the same throat geometry: charge as vorticity flux through the throat cross–section, and mass as volume flux into the throat.

In Sec. 5 we examine the near–field transition region where streamlines bend from radial inflow on the brane to axial flow into the bulk. We describe the resulting stress patterns and construct a multipole expansion of the effective 3D source associated with this region. We argue that the leading corrections are quadrupolar and scale as $(a/r)^2$, providing a geometric interpretation of expected 2PN finite–size effects and a link to the shape–sensitivity tests performed in Paper III.

Sec. 6 revisits the longitudinal sign flip $\alpha^2 = -2/5$ in the 1PN vector sector. We briefly recap the relevant structure of the EIH matching from Paper III and then introduce a simple two–mode toy model in which transverse (brane–parallel) and longitudinal (bulk–directed) modes are coupled. We show how an effective sign flip between these modes arises and interpret this as an emergent Lorentzian signature in mode space induced by the brane–bulk geometry of the throat.

In Sec. 7 we outline how a full 2PN analysis might proceed within this ontology. We review the standard PN counting, identify the classes of corrections that are most sensitive to the throat geometry, and discuss possible observational channels through which the model could be tested or ruled out. We emphasize that the ontology developed here sharply constrains the allowed structure of 2PN terms, turning the toy model into a framework that is, in principle, falsifiable.

Finally, in Sec. 8 we summarize the main elements of the brane–bulk throat picture and its relation to Papers I–IV, discuss possible scenarios for what happens at the bottom of the throat and for the stabilization of the brane, and speculate about the behavior of multiple defects interacting through the bulk. Several technical details, including explicit Gaussian multipole integrals, mode separation in the throat, and the two–mode quadratic form underlying $\alpha^2 = -2/5$, are collected in the appendices.

2 4D Superfluid Framework and Brane Embedding

In this section we set up the higher–dimensional fluid framework that will underlie the rest of the paper. We treat the vacuum as a compressible superfluid living in four spatial dimensions plus time, with the observable universe represented as a three–dimensional brane embedded in this bulk. The goal is not to derive everything from first principles, but to lay out a minimal, consistent kinematic and thermodynamic setup that can be used to interpret the results of Papers I–IV in a unified geometric language.

2.1 Bulk fields, equation of state, and hydrodynamic equations

We work with coordinates

$$(t, \mathbf{X}) = (t, \mathbf{x}, w), \qquad \mathbf{x} = (x, y, z), \tag{3}$$

where $\mathbf{x}$ are the spatial coordinates along the brane and w is a single bulk coordinate transverse to the brane. The full superfluid lives in the four–dimensional spatial manifold parametrized by $\mathbf{X} = (x, y, z, w)$.

The state of the bulk fluid is described by

- a mass density $\rho(\mathbf{x}, w, t)$,
- a velocity field $\mathbf{v}_4(\mathbf{x}, w, t)$ with components

$$\mathbf{v}_4 = (v_x, v_y, v_z, v_w) \equiv (\mathbf{v}, v_w), \tag{4}$$

  where $\mathbf{v}$ denotes the three components tangent to the brane and v_w is the bulk component, and
- a pressure $P(\mathbf{x}, w, t)$.

We assume a barotropic equation of state of polytropic form,

$$P = K\rho^n, \tag{5}$$

with $K > 0$ and polytropic index $n = 5$. This is the same stiff superfluid equation of state that was singled out in Paper II by matching gravitational lensing and Shapiro delay; here we simply adopt it as part of the bulk ontology. The background is taken to be homogeneous with density ρ_0 and pressure $P_0 = K\rho_0^n$ in the absence of defects, and we will work with small perturbations around this background,

$$\rho = \rho_0 + \delta\rho, \qquad P = P_0 + \delta P, \qquad |\delta\rho| \ll \rho_0, \quad |\delta P| \ll P_0. \tag{6}$$

The bulk dynamics are governed by the usual continuity and Euler equations generalized to four spatial dimensions. In Cartesian coordinates the continuity equation reads

$$\partial_t\rho + \nabla_4 \cdot \big(\rho\, \mathbf{v}_4\big) = 0, \tag{7}$$

where

$$\nabla_4 = (\partial_x, \partial_y, \partial_z, \partial_w), \qquad \nabla_4 \cdot \mathbf{v}_4 = \partial_x v_x + \partial_y v_y + \partial_z v_z + \partial_w v_w. \tag{8}$$

The Euler equation for an inviscid barotropic fluid is

$$\rho\left(\partial_t\mathbf{v}_4 + (\mathbf{v}_4 \cdot \nabla_4)\mathbf{v}_4\right) = -\nabla_4 P + \mathbf{f}_{\text{ext}}, \tag{9}$$

where $\mathbf{f}_{\text{ext}}$ denotes any external or defect–induced body forces. In the far field of a localized defect the flow is slow and nearly irrotational, the nonlinear advective term can be treated perturbatively, and we may take $\mathbf{f}_{\text{ext}} \simeq 0$ for the purpose of deriving the acoustic modes.

It is convenient to introduce the specific enthalpy

$$h(\rho) \equiv \int^{\rho} \frac{\mathrm{d}P}{\rho'} = \frac{nK}{n-1}\,\rho^{n-1}, \tag{10}$$

for the polytropic equation of state (5). Linearizing $h(\rho)$ around ρ_0 yields

$$h(\rho_0 + \delta\rho) \simeq h_0 + \left.\frac{\mathrm{d}h}{\mathrm{d}\rho}\right|_{\rho_0} \delta\rho = h_0 + \frac{c_s^2}{\rho_0}\,\delta\rho, \tag{11}$$

where

$$c_s^2 \equiv \left.\frac{\mathrm{d}P}{\mathrm{d}\rho}\right|_{\rho_0} = nK\rho_0^{n-1} \tag{12}$$

is the squared sound speed. We will henceforth drop the constant offset h_0 and use h to denote the enthalpy perturbation,

$$h(\mathbf{x}, w, t) \simeq \frac{c_s^2}{\rho_0}\,\delta\rho(\mathbf{x}, w, t). \tag{13}$$

In the far field of a defect and for small–amplitude motions we may linearize Eqs. (7)–(9) about the homogeneous background. Writing $\rho = \rho_0 + \delta\rho$ and $\mathbf{v}_4 = \delta\mathbf{v}_4$ and neglecting quadratic terms in the perturbations, the continuity equation becomes

$$\partial_t\delta\rho + \rho_0\,\nabla_4 \cdot \delta\mathbf{v}_4 = 0, \tag{14}$$

while the Euler equation reduces to

$$\rho_0\,\partial_t\delta\mathbf{v}_4 = -\nabla_4\delta P = -\rho_0\,\nabla_4 h. \tag{15}$$

Taking the divergence of Eq. (15) and using Eq. (14) to eliminate $\nabla_4 \cdot \delta \mathbf{v}_4$ gives a wave equation for h,

$$\partial_t^2 h - c_s^2 \nabla_4^2 h = 0, \qquad \nabla_4^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 + \partial_w^2. \tag{16}$$

Equivalently,

$$-\frac{1}{c_s^2} \partial_t^2 h + \nabla_4^2 h = 0. \tag{17}$$

This 4D acoustic wave equation is the master equation governing linear perturbations in the bulk. In later sections we will specialize it to the interior of a throat and use it to derive the cavity modes relevant for the electromagnetic sector.

2.2 The brane as a hypersurface at $w = 0$

We now embed a three–dimensional brane in the 4D spatial bulk to represent the observable universe. The brane is defined as the hypersurface

$$\mathcal{B}: \quad w = 0. \tag{18}$$

All of the effective 3D physics described in Papers I–IV is understood as arising from the behavior of the bulk fields restricted to, or projected onto, this hypersurface. In particular:

- The mouths of defects are localized regions on $\mathcal{B}$.
- Test bodies and light rays follow trajectories $(t, \mathbf{x}(t))$ that lie on or very near $w = 0$.

To make contact with the previous 3D description it is useful to define effective brane fields obtained by integrating or sampling the bulk fields in the w–direction. One simple construction is to introduce a weighting kernel $K(w)$ peaked around $w = 0$ and define

$$\rho_{3\mathrm{D}}(\mathbf{x}, t) \equiv \int_{-\infty}^{+\infty} \mathrm{d}w\, K(w)\, \rho(\mathbf{x}, w, t), \tag{19}$$

$$\mathbf{v}_{3\mathrm{D}}(\mathbf{x}, t) \equiv \frac{1}{N} \int_{-\infty}^{+\infty} \mathrm{d}w\, K(w)\, \Pi_{\parallel}\, \mathbf{v}_4(\mathbf{x}, w, t), \tag{20}$$

where $\Pi_{\parallel}$ projects onto the x, y, z components and $N = \int \mathrm{d}w\, K(w)$ is a normalization factor. For the purposes of this paper we will not commit to a specific functional form for $K(w)$; one may think of it as either a narrow bump of width comparable to the microscopic brane thickness, or simply as $K(w) = 1$ with the understanding that only the behavior near $w = 0$ contributes significantly.

The fields $\rho_{3\mathrm{D}}$ and $\mathbf{v}_{3\mathrm{D}}$ play the role of the effective density and velocity fields in the three–dimensional toy models of Papers I–III. When we speak of a "spherical sink" on the brane we are referring to the behavior of $\rho_{3\mathrm{D}}$ and $\mathbf{v}_{3\mathrm{D}}$ as functions of $\mathbf{x}$, induced by the presence of a throat in the bulk.

At the level of equations of motion, the projections (19)–(20) induce an effective 3D continuity equation and a Poisson–like equation for a Newtonian potential $\Phi(\mathbf{x}, t)$ on the brane. Schematically,

$$\partial_t \rho_{3\mathrm{D}} + \nabla \cdot (\rho_{3\mathrm{D}} \mathbf{v}_{3\mathrm{D}}) = 0, \tag{21}$$

$$\nabla^2 \Phi = 4\pi G_{\mathrm{eff}}\, \rho_{3\mathrm{D}}, \tag{22}$$

where ∇ is the 3D gradient with respect to $\mathbf{x}$ and G_{eff} is an effective gravitational constant determined by the bulk parameters. We will not attempt to derive G_{eff} from first principles here; instead we treat it as calibrated by the matching to Newtonian gravity and 1PN corrections in Papers I–III.

The brane also provides natural boundary conditions on the bulk flow. Physically, the brane may be thought of as a locus where some microscopic mechanism pins the fluid or changes its properties, but in this toy model we capture that information simply by specifying the behavior of ρ, $\mathbf{v}_4$, and h at $w = 0$. For example, in the absence of a defect we may assume that the normal velocity vanishes at the brane,

$$v_w(\mathbf{x}, w = 0, t) = 0 \tag{23}$$

so that there is no net flux of fluid across $\mathcal{B}$. In the presence of a defect mouth, by contrast, there is a localized region on the brane where v_w is nonzero and negative, corresponding to drainage into the throat. The details of this boundary condition will matter for the near–field structure and for the relationship between the defect's mass and its drainage rate, but the broad picture is simple: the brane is a distinguished hypersurface whose intersection with a throat appears as a localized sink to brane–bound observers.

2.3 Throat topology and geometry

We are now ready to formalize the geometric picture of a defect as a brane–bulk throat. Intuitively, the brane at $w = 0$ is locally deformed and pinched into the bulk, forming a tube–like region filled with the same superfluid. We model this region as a cylindrical throat of radius a and depth L embedded in the four–dimensional spatial bulk.

Let the throat domain $\mathcal{T}$ be a connected region of space defined approximately by

$$\mathcal{T} \simeq \left\{ (\mathbf{x}, w) \,\middle|\, 0 \le w \le L,\ \sqrt{x^2 + y^2 + z^2} \lesssim a \right\}, \tag{24}$$

with the understanding that near $w = 0$ and $w = L$ the geometry may deviate from an exact cylinder. For most of the paper we will only need a coarse–grained description at scales larger than any microscopic brane thickness, so we treat a and L as effective parameters characterizing the radius and depth of the throat.

From the perspective of an observer living on the brane $\mathcal{B}$, the intersection of the throat with $w = 0$ is a two–sphere of radius a,

$$\partial\mathcal{T} \cap \mathcal{B} \simeq \{\mathbf{x} \,|\, |\mathbf{x}| = a\}\,. \tag{25}$$

This spherical boundary is what appears, in the effective 3D description, as the "surface" of a spherical defect. The region $|\mathbf{x}| < a$ on the brane is not filled with solid matter, but is instead the mouth of the throat opening into the bulk. This is the geometry implicitly assumed in Papers I–III when modeling a defect as a spherical sink: at distances $r = |\mathbf{x}| \gg a$, the details of the throat interior are invisible and only the total drainage rate through this mouth matters.

From the perspective of the bulk, by contrast, the region $0 < w < L$ with $|\mathbf{x}| \lesssim a$ is approximately a straight 4D cylinder of cross–sectional area πa^2. This cylindrical interior is the domain in which the cavity modes of the electromagnetic sector live. In Sec. 4 we will impose boundary conditions on h at $r = |\mathbf{x}| = a$ and at $w = 0$ and $w = L$ and solve the 4D acoustic wave equation (16) inside $\mathcal{T}$ to recover the Bessel–type radial modes and the preferred aspect ratio L/a found in Paper IV.

At the bottom of the throat, around $w = L$, we deliberately leave the geometry unspecified. Several possibilities are conceivable:

- the throat could close off smoothly, forming a finite cavity;
- it could open into a larger bulk region, allowing radiation or flow into the deep bulk;
- or it could connect to other throats, forming a network of defects linked through the bulk.

The choice among these possibilities affects the detailed mode spectrum and the global topology of the model, but it will not play a direct role in the 1PN considerations of this paper. We return to these questions in Sec. 8.

Finally, it is worth noting that the throat geometry naturally supports topological quantities associated with circulation and vorticity. Loops encircling the throat mouth on the brane can carry quantized circulation, and vorticity lines can thread the throat interior, connecting the brane to the bulk. In the electromagnetic sector these quantities will be related to electric charge and magnetic flux; for now we simply flag the fact that the topology of $\mathcal{T}$ provides the right sort of structure to encode such conserved charges.

With this 4D superfluid framework and throat geometry in place, we can now turn to dimensional reduction and the far–field structure of the effective 3D theory on the brane.

3 Dimensional Reduction and Far-Field Structure

In this section we show how a localized throat in the 4D bulk generates an effective three–dimensional source on the brane. The key points are: (i) integrating over the bulk coordinate w produces an effective 3D density whose leading contribution is a monopole mass $M \sim \rho_0 L a^3$, and (ii) angular deviations from spherical symmetry are naturally suppressed by $(a/r)^2$ at large radii r on the brane. We illustrate these ideas with a simple Gaussian toy model and then connect the resulting far–field potential to the 1PN structure used in Papers I–III.

3.1 General dimensional reduction from 4D to 3D

Given a bulk density $\rho(\mathbf{x}, w, t)$ and velocity $\mathbf{v}_4(\mathbf{x}, w, t)$, the effective 3D fields seen by brane–bound observers are obtained by integrating over the bulk coordinate w with some weighting kernel $K(w)$ localized near the brane:

$$\rho_{3\mathrm{D}}(\mathbf{x}, t) \equiv \int_{-\infty}^{+\infty} \mathrm{d}w\, K(w)\, \rho(\mathbf{x}, w, t), \tag{26}$$

$$\mathbf{v}_{3\mathrm{D}}(\mathbf{x}, t) \equiv \frac{1}{N} \int_{-\infty}^{+\infty} \mathrm{d}w\, K(w)\, \Pi_\parallel\, \mathbf{v}_4(\mathbf{x}, w, t), \tag{27}$$

with $N = \int \mathrm{d}w\, K(w)$ and $\Pi_\parallel$ projecting onto the brane–parallel components (v_x, v_y, v_z). For the purposes of this paper we may take $K(w) = 1$ and understand that the integrals are dominated by the region where the throat lives; more refined choices would only change numerical prefactors.

Once $\rho_{3\mathrm{D}}$ and $\mathbf{v}_{3\mathrm{D}}$ are defined, their dynamics on the brane are governed by an effective 3D continuity equation and a Poisson–like equation for a Newtonian potential $\Phi(\mathbf{x}, t)$,

$$\partial_t \rho_{3\mathrm{D}} + \nabla \cdot (\rho_{3\mathrm{D}} \mathbf{v}_{3\mathrm{D}}) = 0, \tag{28}$$

$$\nabla^2 \Phi = 4\pi G_{\mathrm{eff}}\, \rho_{3\mathrm{D}}, \tag{29}$$

where ∇ is the 3D gradient with respect to $\mathbf{x}$ and G_{eff} is an effective gravitational constant determined by the bulk parameters and the details of the projection. In practice G_{eff} is fixed by matching Eq. (29) to Newtonian gravity at large distances, as done implicitly in Papers I–III.

For a localized defect represented as a throat of radius a and depth L, $\rho(\mathbf{x}, w, t)$ is strongly peaked near $|\mathbf{x}| \lesssim a$ and $0 < w < L$. At distances $r = |\mathbf{x}| \gg a, L$ on the brane, $\rho_{3\mathrm{D}}$ therefore looks like a nearly point–like source plus small angular corrections. The total effective mass is

$$M \equiv \int \rho_{3\mathrm{D}}(\mathbf{x}, t)\, \mathrm{d}^3x = \int \mathrm{d}^3x \int \mathrm{d}w\, K(w)\, \rho(\mathbf{x}, w, t), \tag{30}$$

and the higher multipole moments of $\rho_{3\mathrm{D}}$ encode finite–size effects associated with the throat geometry and the transition region near its mouth.

3.2 Gaussian toy model for a localized throat

To make these statements concrete, we now introduce a simple toy model for the bulk density perturbation associated with a single throat. We work in spherical coordinates (r, θ, ϕ) on the brane and use w for the bulk coordinate. The toy 4D density profile is taken to be

$$\rho_4(r, \theta, w) = \rho_0 \exp\left(-\frac{r^2}{a^2}\right) \exp\left(-\frac{w^2}{L^2}\right) \big[1 + \varepsilon P_2(\cos\theta)\big], \tag{31}$$

where $P_2(\cos\theta) = (3\cos^2\theta - 1)/2$ is the $\ell = 2$ Legendre polynomial, a is the throat radius, L is the throat depth, and ε is a small dimensionless parameter controlling the degree of angular asymmetry. The exponential factors localize the defect within a region of size $\sim a$ on the brane and $\sim L$ in the bulk; the εP_2 term models the mild asphericity associated with the transition region where streamlines bend from radial to axial flow.

For simplicity we choose $K(w) = 1$ in the projection onto the brane. Integrating Eq. (31) over w gives the effective 3D density

$$\rho_{3\mathrm{D}}(r, \theta) = \int_{-\infty}^{+\infty} \rho_4(r, \theta, w)\, \mathrm{d}w = \sqrt{\pi}\, L\rho_0\, \exp\left(-\frac{r^2}{a^2}\right) \big[1 + \varepsilon P_2(\cos\theta)\big]. \tag{32}$$

The total mass of the defect is then

$$M = \int \rho_{3\mathrm{D}}(r, \theta)\, \mathrm{d}^3x = \pi^2 L a^3 \rho_0, \tag{33}$$

where the ε term integrates to zero by angular symmetry. This scaling is exactly what we expect for a throat of radius a and depth L: M is proportional to the background density times the effective throat volume.

To quantify the leading angular deviation from sphericity we consider the standard quadrupole–like scalar moment

$$Q_{20} \propto \int \rho_{3\mathrm{D}}(r,\theta)\, r^2 P_2(\cos\theta)\, \mathrm{d}^3x. \tag{34}$$

Carrying out the angular integrals and the Gaussian radial integral (details are relegated to Appendix A), we obtain

$$Q_{20} = \frac{3}{10}\, \pi^2 L a^5 \varepsilon \rho_0, \tag{35}$$

and hence

$$\frac{Q_{20}}{M} = \frac{3}{10}\, \varepsilon a^2. \tag{36}$$

The precise numerical prefactor 3/10 is not important for our purposes; the crucial point is the scaling

$$\frac{Q}{M} \sim \varepsilon a^2, \tag{37}$$

which is generic for a localized, mildly aspherical source of size a on the brane.

3.3 Far-field potential and $(a/r)^2$ corrections

The effective Newtonian potential $\Phi(r,\theta)$ generated on the brane by a localized mass distribution $\rho_{3\mathrm{D}}$ admits the usual multipole expansion at large radii $r \gg a, L$:

$$\Phi(r,\theta) \approx -\frac{GM}{r} - G\,\frac{Q}{r^3}\, P_2(\cos\theta) + \cdots, \tag{38}$$

where M is the monopole mass, Q is the quadrupole–like scalar defined in Eq. (34) (up to conventional normalization factors), and the ellipsis denotes higher multipoles suppressed by further powers of a/r.

Substituting the scaling $Q/M \sim \varepsilon a^2$ from the Gaussian toy model, Eq. (36), into Eq. (38) gives

$$\Phi(r,\theta) \approx -\frac{GM}{r}\left[1 + \mathcal{O}\!\left(\varepsilon \frac{a^2}{r^2}\right) P_2(\cos\theta) + \cdots\right]. \tag{39}$$

Thus the leading anisotropic correction to the Newtonian potential is suppressed by $\varepsilon(a/r)^2$. In the language of the throat ontology, this correction encodes the imprint of the transition region near the mouth of the throat, where the flow is neither purely radial on the brane nor purely axial in the bulk, but bends from one into the other.

The main takeaway of this section is therefore:

- Dimensional reduction of a localized 4D throat–like density produces an effective 3D source with a monopole mass $M \sim \rho_0 L a^3$ and a tower of finite–size corrections.
- The leading angular distortion is quadrupolar and suppressed by $(a/r)^2$ at large radii.

- These are precisely the types of terms that, in a full post–Newtonian treatment, would be expected to appear at 2PN order as finite–size corrections associated with the physical size a of the body.

3.4 Connection to Papers I–III and why 1PN sees a sphere

We now connect this dimensional reduction picture to the 1PN results of Papers I–III. Those papers effectively treat each defect as a point–like or perfectly spherical source on the brane, described by a monopolar potential $\Phi \sim -GM/r$ plus velocity–dependent 1PN corrections. Finite–size structure of the defect is either neglected or, in the case of the shape–sensitivity tests in Paper III, shown to have only a weak effect on the 1PN precession coefficient for modest deformations.

From the standpoint of the throat ontology, this is exactly what one should expect. At distances $r \gg a, L$, the effective 3D density ρ_{3D} produced by dimensional reduction is dominated by its monopole component. Angular structure such as the quadrupole is suppressed by $(a/r)^2$, and higher multipoles by even higher powers of a/r. As a result:

- The leading Newtonian potential and the 1PN corrections derived in Papers I–III are insensitive to the detailed throat geometry, as long as the mass M is held fixed.
- The defect therefore looks like a spherical sink at 1PN order, even though its interior is cylindrically structured in the bulk.
- The modest shape sensitivity found in Paper III (e.g. $\sim$ 10% oblateness leading to $\sim$ 2% shifts in the precession coefficient) is naturally interpreted as a small leakage of these $(a/r)^2$ corrections into the 1PN observables used in that analysis, and as a hint that the model has a geometrically rich interior that will matter more at 2PN and beyond.

This perspective also clarifies how the gravitational and electromagnetic sectors can co-exist without contradiction. Papers I–III operate in the far–field regime on the brane, where only the monopole and velocity–dependent interaction terms matter, so the defect appears spherical. Paper IV probes modes confined inside the throat, where the cylindrical interior and the aspect ratio L/a are crucial. Dimensional reduction shows that these descriptions are simply different projections of the same 4D geometry: the spherical mouth seen by gravity and the cylindrical interior seen by electromagnetism are both encoded in the same throat.

In the next section we turn from this far–field point of view to the interior of the throat itself, deriving the cavity modes of the 4D acoustic equation and reinterpreting the enthalpy–selected aspect ratio $L/a \approx 1.85$ as a geometric property of the throat in the bulk.

4 Throat Geometry and the Electromagnetic Sector

We now turn from the far–field effective description on the brane to the interior of the throat itself. The goal is to show how the cylindrical cavity picture of Paper IV arises naturally once we treat the defect as a brane–bulk throat, and to reinterpret the enthalpy–selected aspect ratio L/a and the notion of "charge" in purely geometric terms. Throughout this section we work with the linear 4D acoustic equation

$$\partial_t^2 h - c_s^2\, \nabla_4^2 h = 0, \tag{40}$$

introduced in Sec. 2.1, but now restricted to the throat domain $\mathcal{T}$.

4.1 4D cavity modes in the throat

Inside the throat, the geometry is approximately that of a straight tube of radius a and depth L extending into the bulk. It is convenient to adopt coordinates adapted to this tube: we take w along the throat and introduce a radial coordinate r measuring distance from the center of the throat within the brane directions, together with angular coordinates on the (x, y, z) directions. For the lowest–lying modes we are interested in, there is no dependence on the azimuthal angles or on any internal structure within the cross–section; the modes are effectively axisymmetric about the throat center.

In this approximation the 4D Laplacian inside the throat separates as

$$\nabla_4^2 \simeq \nabla_\perp^2 + \partial_w^2, \tag{41}$$

where $\nabla_\perp^2$ is the Laplacian in the radial direction r (and its associated angular coordinate), restricted to axisymmetric configurations. We seek separated solutions of the form

$$h(t, r, w) = \mathrm{Re}\big\{H(r)\, W(w)\, e^{-i\omega t}\big\}\,. \tag{42}$$

Substituting Eq. (42) into Eq. (40) yields

$$\left[\omega^2 - c_s^2 \left(\frac{1}{H}\nabla_\perp^2 H + \frac{1}{W}\partial_w^2 W\right)\right] HW = 0. \tag{43}$$

Dividing by HW and rearranging gives

$$\frac{1}{H}\nabla_\perp^2 H + \frac{1}{W}\partial_w^2 W = \frac{\omega^2}{c_s^2}, \tag{44}$$

which we can separate by setting each side equal to a constant. Introducing separation constants $-k_r^2$ and $-k_w^2$, we obtain

$$\nabla_\perp^2 H + k_r^2 H = 0, \tag{45}$$
$$\partial_w^2 W + k_w^2 W = 0, \tag{46}$$

with the dispersion relation

$$\omega^2 = c_s^2\, (k_r^2 + k_w^2). \tag{47}$$

For axisymmetric modes, $\nabla_\perp^2$ reduces to the radial part of the Laplacian in cylindrical–like coordinates,

$$\nabla_\perp^2 H = \frac{1}{r^2}\partial_r\big(r^2 \partial_r H\big)\,, \tag{48}$$

or, in the thin–throat limit where the cross–section is effectively two–dimensional, to the standard Bessel form

$$\nabla_\perp^2 H = \frac{1}{r}\partial_r(r\partial_r H)\,. \tag{49}$$

In either case the regular, axisymmetric solutions of Eq. (45) at $r = 0$ are Bessel functions of the first kind:

$$H(r) \propto J_0(k_r r), \tag{50}$$

with k_r quantized by boundary conditions at the throat wall $r = a$.

In keeping with the cavity analysis of Paper IV, we impose a boundary condition that the enthalpy fluctuation vanishes at the wall,

$$h(r = a, w, t) = 0 \quad \Rightarrow \quad H(a) = 0, \tag{51}$$

corresponding physically to a *pinned phase boundary* in the stiff superfluid vacuum. The order parameter is topologically fixed at the throat wall, so small enthalpy (pressure) perturbations are forced to zero there. Mathematically, this Dirichlet condition selects the zeros of J_0,

$$k_r = \frac{x_{0n}}{a}, \tag{52}$$

where x_{0n} is the nth zero of J_0. We will be primarily interested in the fundamental radial mode,

$$k_r = \frac{x_{01}}{a}, \tag{53}$$

with radial profile $J_0(x_{01}r/a)$.

Along the throat direction w we impose boundary conditions at $w = 0$ and $w = L$. The simplest idealization is to take h to vanish at both ends,

$$h(r, w = 0, t) = h(r, w = L, t) = 0 \quad \Rightarrow \quad W(0) = W(L) = 0, \tag{54}$$

corresponding to nodes at the mouth and bottom of the throat. This leads to standing–wave solutions

$$W_n(w) \propto \sin\!\left(\frac{n\pi w}{L}\right), \qquad k_w = \frac{n\pi}{L}, \quad n = 1, 2, \ldots \tag{55}$$

The fundamental axial mode has $n = 1$, so the full fundamental throat mode is

$$h_1(t, r, w) \propto J_0\!\left(\frac{x_{01}r}{a}\right)\sin\!\left(\frac{\pi w}{L}\right)\cos(\omega t), \tag{56}$$

with

$$\omega^2 = c_s^2\left(\frac{x_{01}^2}{a^2} + \frac{\pi^2}{L^2}\right). \tag{57}$$

This is the same J_0–Bessel / standing–wave structure used in the cylindrical cavity analysis of Paper IV, now understood as the fundamental mode of the brane–bulk throat.

4.2 Enthalpy minimization and the aspect ratio L/a

In Paper IV the key electromagnetic result was that minimizing the enthalpy of the cavity at fixed "charge" picked out a preferred aspect ratio

$$\frac{L}{a} = \frac{\sqrt{2}\,\pi}{x_{01}} \approx 1.85. \tag{58}$$

We now sketch how this emerges from the throat picture.

The relevant functional is the total perturbation energy (or enthalpy) of the mode in the throat. For a linear acoustic mode $h(t, r, w)$ with frequency ω, the time–averaged energy stored in the perturbation can be written schematically as

$$\mathcal{E}[h] \sim \int_{\mathcal{T}} \mathrm{d}^3x\,\mathrm{d}w\;\rho_0 \left[\frac{1}{2c_s^2}\,(\partial_t h)^2 + \frac{1}{2}\,(\nabla_4 h)^2\right], \tag{59}$$

where we have suppressed numerical prefactors that do not depend on a or L. For a separated mode of the form $h(t, r, w) = AH(r)W(w)\cos(\omega t)$ the time average of $(\partial_t h)^2$ contributes ω^2, while the spatial gradients contribute k_r^2 and k_w^2:

$$\langle(\partial_t h)^2\rangle \propto \omega^2 A^2 H^2 W^2, \qquad \langle(\nabla_4 h)^2\rangle \propto (k_r^2 + k_w^2) A^2 H^2 W^2. \tag{60}$$

Using the dispersion relation (57), the total time–averaged energy carried by the fundamental mode is therefore proportional to

$$\mathcal{E} \propto A^2 (k_r^2 + k_w^2) \int_{\mathcal{T}} H^2 W^2, \tag{61}$$

where the integral over H^2W^2 supplies a factor of order a^2L for the fundamental mode.

To define a variational problem we must specify which quantity is held fixed when minimizing $\mathcal{E}$. Following Paper IV, we identify a conserved "charge" $\mathcal{Q}$ associated with the mode amplitude, which depends on the same integral of H^2W^2 but not on $k_r^2 + k_w^2$. Schematically,

$$\mathcal{Q} \propto A^2 \int_{\mathcal{T}} H^2 W^2, \tag{62}$$

so that, at fixed $\mathcal{Q}$, the energy scales as

$$\mathcal{E} \propto (k_r^2 + k_w^2)\,\mathcal{Q}. \tag{63}$$

Minimizing $\mathcal{E}$ at fixed $\mathcal{Q}$ therefore amounts to minimizing $k_r^2 + k_w^2$.

For the fundamental throat mode we have

$$k_r^2 + k_w^2 = \frac{x_{01}^2}{a^2} + \frac{\pi^2}{L^2}. \tag{64}$$

Varying a and L while holding the effective cross–sectional area πa^2 and depth L in a constrained way (reflecting the fact that the total mass $M \sim \rho_0 L a^3$ is fixed) leads to a minimum of $k_r^2 + k_w^2$ at a particular ratio L/a. The detailed calculation follows the same steps as in Paper IV; here we simply quote the result:

$$\frac{L}{a} = \frac{\sqrt{2}\,\pi}{x_{01}}. \tag{65}$$

At this aspect ratio, the contributions of radial and axial gradients are in a particular balance that extremizes the energy per unit charge stored in the cavity. In the throat ontology, Eq. (58) is not an arbitrary parameter choice for a 3D cylinder, but a statement about the geometry of the 4D throat: for a given mass and charge, the throat relaxes to a preferred ratio of depth to radius.

4.3 Charge as circulation and vorticity flux through the throat

So far we have treated the "charge" $\mathcal{Q}$ as an abstract conserved quantity associated with the cavity mode amplitude. In the superfluid picture, $\mathcal{Q}$ has a more concrete interpretation in terms of circulation and vorticity.

Consider a closed loop $\mathcal{C}$ on the brane encircling the throat mouth, for example a circle of radius $r > a$ in a plane intersecting the brane. The circulation of the superfluid velocity around this loop is

$$\Gamma = \oint_{\mathcal{C}} \mathbf{v} \cdot \mathrm{d}\boldsymbol{\ell}. \tag{66}$$

In a quantum fluid, Γ would be quantized; in the present classical toy model we simply treat it as a conserved quantity in the absence of vorticity creation or reconnection events. By Stokes' theorem, Γ is equivalently the flux of vorticity through any surface $\mathcal{S}$ bounded by $\mathcal{C}$,

$$\Gamma = \int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot \mathrm{d}\mathbf{S}. \tag{67}$$

If the only significant vorticity near $\mathcal{C}$ is carried by the throat, then this flux is, to a good approximation, the vorticity threading the throat cross–section.

In the hydrodynamic–electromagnetic dictionary of Paper IV, the electric charge of the defect is identified (up to a proportionality constant) with this circulation or vorticity flux. The field variables that play the role of the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, are constructed from the enthalpy gradients and the velocity field in such a way that Maxwell's equations emerge as effective equations in the near field of the defect. In that language, $\mathcal{Q}$ is essentially the integral of the mode amplitude over the throat cross–section, which is in turn proportional to the vorticity flux through that cross–section.

The throat picture makes this geometric: the vorticity lines thread the throat like the flux lines of a solenoid threading a torus. The conserved charge is the flux of vorticity through the "hole" of the throat, i.e. through the surface of area πa^2 that spans the spherical mouth on the brane. Different values of charge correspond to different amounts of vorticity threading the throat, and the cavity mode adjusts its amplitude to accommodate this flux while keeping the enthalpy as low as possible. The preferred aspect ratio L/a then reflects how the throat geometry arranges itself to support a given amount of vorticity flux at minimum energetic cost.

4.4 Mass as density starvation and volume flux

In contrast to charge, which is associated with circulation and vorticity flux, the mass of the defect is associated with a *scalar* density depletion anchored on the throat. The drainage of fluid into the throat provides the kinematic mechanism that establishes the throat and fixes a boundary condition at its mouth, but the long–range Newtonian field on the brane is sourced by the resulting hydrostatic density "starvation" field, not by the short–range kinetic energy of the inflow itself.

From the brane point of view, a defect appears as a localized sink: the normal component of the bulk velocity at the mouth is nonzero and directed into the throat,

$$v_w(\mathbf{x}, w = 0, t) < 0 \tag{68}$$

within the mouth region $|\mathbf{x}| \lesssim a$. The net volumetric flux into the throat is

$$\dot{V} = \int_{\text{mouth}} v_w(\mathbf{x}, w = 0, t)\, \mathrm{d}^2 S, \tag{69}$$

where the integral is over the effective mouth area on the brane. In a steady configuration this flux is balanced by compression within the throat, recirculation along closed streamlines, or outflow at the far end of the throat, depending on the global topology.

If the mass were sourced purely by the kinetic energy associated with this flow, the induced density deficit would be strongly localized near the mouth and would fall off too rapidly with radius to reproduce a Newtonian $1/r$ potential; this is precisely what is seen in explicit lensing calculations when one takes $\rho \propto v^2$ as the source. In the toy model, the superfluid vacuum instead responds to the throat as a stiff, self–gravitating medium: the drainage and geometry at the mouth impose a boundary condition on the scalar enthalpy and density fields, and the vacuum relaxes into a hydrostatic configuration with a long–range density depletion $\delta\rho(r)$ extending far from the throat. It is this scalar density–starvation profile, rather than the local kinetic energy of the inflow, that sources the effective Newtonian potential on the brane.

At the level of scaling, the throat geometry still sets the core mass scale,

$$M \sim \rho_0\,(\text{effective throat volume}) \sim \rho_0\, \pi a^2 L, \tag{70}$$

modulo order–unity geometric factors and corrections from the detailed profile of $\rho(\mathbf{x}, w)$. This is consistent with the Gaussian toy model of Sec. 3.2, where $M \propto \rho_0 L a^3$ up to numerical factors. In the hybrid scalar–vector picture developed in the companion Paper VI, this core contribution is supplemented by an extended scalar depletion halo falling roughly as $1/r$, so that the total Newtonian mass includes both the localized throat deficit and the global hydrostatic response of the vacuum.

Putting these pieces together, the throat ontology suggests a simple geometric picture:

- Mass is associated with the *density starvation field* anchored on the throat: drainage and throat geometry fix a boundary condition, and the stiff superfluid vacuum responds with a long–range scalar depletion profile whose core scale is set by the effective throat volume.

- Charge is associated with the *vorticity flux* threading the throat cross–section, which controls the internal cavity modes and the effective electromagnetic fields on the brane.

Both are properties of the same geometric object—a brane–bulk throat of radius a and depth L—viewed through different hydrodynamic projections. The gravitational sector is sensitive mainly to the integrated density deficit and the resulting monopolar potential on the brane, while the electromagnetic sector is sensitive to the internal cavity modes and the vorticity flux they support.

In this way, the sphere–cylinder tension that appeared when Papers I–III and IV were viewed in isolation is naturally resolved: the same throat can appear as a spherical sink on the brane for gravity, supported by a scalar density–starvation halo, and as a cylindrical cavity in the bulk for electromagnetism, with its own preferred aspect ratio and cavity spectrum. In the next section we move back toward the brane and examine the transition region near the throat mouth, where finite–size effects and 2PN corrections originate.

5 Near-Field Transition and Finite-Size Effects

So far we have treated the throat in two complementary limits: the far field on the brane, where only the monopole mass and velocity–dependent 1PN interactions matter, and the deep interior of the throat, where cylindrical cavity modes control the electromagnetic sector. Between these lies a transition region near the throat mouth where the flow is strongly curved and neither limit applies cleanly. In this section we give a qualitative description of this near–field region, argue that its imprint on the effective 3D theory is naturally multipolar and suppressed by $(a/r)^2$, and connect this picture to the shape–sensitivity tests of the spin and N–body paper (Paper III).

5.1 Flow regimes: far field, throat interior, and transition region

It is useful to distinguish three qualitatively different flow regimes associated with a single throat:

1. *Far field on the brane* ($r \gg a, L$): Here the flow is almost purely radial in the brane directions and nearly independent of w. The effective density $\rho_{3D}(\mathbf{x})$ obtained by integrating over w is well approximated by a spherically symmetric monopole plus small multipole corrections, as in Sec. 3.2. The velocity field $\mathbf{v}_{3D}(\mathbf{x})$ is essentially that of a point–like sink at the origin.

2. *Throat interior* ($r \lesssim a$, $0 < w < L$): Inside the throat the flow is predominantly along the w direction, with relatively weak dependence on the brane coordinates except through the radial cavity profile $J_0(x_{01}r/a)$ of the fundamental mode. The relevant modes are the separated solutions of the 4D acoustic equation (40), and the leading dynamics are controlled by the discrete set of (k_r, k_w) pairs and the aspect ratio L/a.

3. *Transition region* ($r \sim a$, $w \sim 0$): Near the mouth of the throat the flow lines bend from radial inflow on the brane to axial flow into the bulk. The geometry here is

strongly curved; the brane deviates from a flat $w = 0$ hypersurface, and the fluid experiences both significant shear and pressure gradients. This is the region where departures from perfect spherical symmetry on the brane originate and where the throat geometry imprints itself most strongly on the effective 3D source ρ_{3D}.

In the previous sections we treated the far field and the throat interior in controlled approximations: the far field through multipole expansion and the interior through mode separation. The transition region does not admit such a simple analytic treatment, but we can still characterize its effects on the brane in terms of symmetry and scaling. Its characteristic size is set by a; its angular structure is determined by how the brane is deformed into the bulk and by how streamlines reorient from brane–parallel to bulk–directed. These features naturally generate higher multipoles in ρ_{3D} and $\mathbf{v}_{3D}$ that are suppressed by powers of (a/r) at large distances.

5.2 Streamline bending and induced stresses

To get a feel for the transition region, consider streamlines of the fluid in a quasi–steady configuration. Far from the defect, on the brane, the streamlines are nearly straight and radial, converging toward the origin with speed $v_r(r) \propto 1/r^2$ for an incompressible sink. Deep inside the throat, by contrast, streamlines are nearly parallel to the w axis, with a roughly uniform axial velocity v_w that carries fluid along the throat.

In order to connect these two regimes, streamlines must bend sharply in a neighborhood of the mouth. The curvature of a streamline is controlled by the local acceleration,

$$\frac{\mathrm{D}\mathbf{v}_4}{\mathrm{D}t} = \partial_t \mathbf{v}_4 + (\mathbf{v}_4 \cdot \nabla_4)\mathbf{v}_4 = -\frac{1}{\rho}\,\nabla_4 P + \frac{\mathbf{f}_{\text{ext}}}{\rho}, \tag{71}$$

so large curvature implies large pressure gradients in the transition region. Even in a quasi–steady state where $\partial_t \mathbf{v}_4 \simeq 0$, the advective term $(\mathbf{v}_4 \cdot \nabla_4)\mathbf{v}_4$ is enhanced where streamlines squeeze together and turn into the throat, and this must be balanced by correspondingly strong gradients in P and h.

From the brane point of view, these pressure gradients generate stresses in the effective 3D fluid. In particular, the stress tensor constructed from the velocity field and the enthalpy,

$$T_{ij} \sim \rho v_i v_j + \delta_{ij} P + \cdots \,, \tag{72}$$

has nontrivial angular dependence in the transition region. The components tangent to the brane ($i, j \in \{x, y, z\}$) feel both the radial inflow and the shear associated with bending into the w direction. When this structure is projected onto the brane and coarse–grained over scales larger than a, it manifests as effective multipole moments in the 3D source terms that enter the gravitational and electromagnetic sectors.

Although a full solution of the transition–region flow would require solving the nonlinear 4D Euler equations with a deformed brane geometry, symmetry already constrains the leading corrections. For an isolated defect with no preferred direction on the brane, the transition region must be axisymmetric about the throat axis and reflection–symmetric under $w \to -w$ if we consider a mirrored continuation into the bulk. These symmetries forbid dipolar contributions in ρ_{3D} and $\mathbf{v}_{3D}$ and make the quadrupole ($\ell = 2$) the leading nontrivial multipole.

5.3 Multipole expansion of the transition region

We can formalize this intuition by writing the effective 3D density on the brane as a sum of a purely radial piece and angular corrections sourced by the transition region. In spherical coordinates (r, θ, ϕ) on the brane, with the throat axis taken as the polar axis, we expand

$$\rho_{\mathrm{3D}}(r, \theta) = \rho_0(r) + \sum_{\ell \geq 1} \rho_\ell(r) P_\ell(\cos\theta), \tag{73}$$

where $\rho_0(r)$ is the spherically symmetric part and P_ℓ are Legendre polynomials. Axisymmetry removes any dependence on ϕ, and reflection symmetry around the throat axis suppresses odd ℓ for the density; the leading correction is thus the quadrupole term $\ell = 2$.

The Gaussian toy model of Sec. 3.2 provides an explicit realization of this structure, with

$$\rho_{\mathrm{3D}}(r, \theta) \propto e^{-r^2/a^2} \Big[1 + \varepsilon P_2(\cos\theta) \Big], \tag{74}$$

so that $\rho_0(r)$ and $\rho_2(r)$ are both localized near $r \lesssim a$. Integrating ρ_{3D} against $r^2 P_\ell(\cos\theta)$ over a volume enclosing the throat then yields the multipole moments, and we saw explicitly that $Q/M \sim \varepsilon a^2$ for the quadrupole. Higher multipoles would involve higher powers of a and smaller dimensionless coefficients.

The same logic applies, at least at the level of scaling, to any reasonable model of the transition region: as long as the region is localized within a radius of order a and preserves axisymmetry, all multipole moments higher than the monopole are suppressed by powers of a and decay with appropriate powers of $1/r$ in the far field. The resulting potential can be written as

$$\Phi(r, \theta) = -\frac{GM}{r} \left[1 + \sum_{\ell \geq 2} \alpha_\ell \left(\frac{a}{r} \right)^\ell P_\ell(\cos\theta) \right], \tag{75}$$

with dimensionless coefficients α_ℓ that encode the detailed structure of the throat mouth and transition region. For $\ell = 2$ we recover the $(a/r)^2$ suppression emphasized earlier. In the absence of fine tuning, the α_ℓ are expected to be of order one times small shape parameters characterizing how much the throat deviates from an ideal spherical mouth.

5.4 Interpretation as 2PN finite-size corrections

From the point of view of post–Newtonian theory, the multipole corrections generated by the transition region are naturally interpreted as finite–size effects that first appear at 2PN order and beyond. In standard GR language, the small parameter controlling the PN expansion is

$$\epsilon \sim \frac{GM}{rc^2} \sim \left(\frac{v}{c} \right)^2, \tag{76}$$

with 1PN terms scaling as ϵ and 2PN terms as ϵ^2. For a compact object of radius R in GR, finite–size effects typically scale as $(R/r)^2$ or higher powers, and in many situations they enter observables at 2PN order or later.

In the throat ontology, the role of the body's size is played by the throat radius a, and the relevant dimensionless parameter is a/r. For well–separated bodies we have $a \ll r$, so $(a/r)^2$

is parametrically small compared to unity. The multipole expansion (75) then suggests that the leading corrections to the monopole potential scale as

$$\delta\Phi \sim \Phi_{\rm N}\, \alpha_2 \left(\frac{a}{r}\right)^2 \sim -\frac{GM}{r}\alpha_2 \left(\frac{a}{r}\right)^2 . \tag{77}$$

If a is itself of order the gravitational radius $r_g = GM/c^2$ of the object, then $(a/r)^2 \sim (GM/rc^2)^2 \sim \epsilon^2$, and these corrections would indeed be 2PN in the usual counting. Even if a is somewhat larger than r_g, as might be natural in a superfluid context, the $(a/r)^2$ suppression still pushes these effects beyond the leading 1PN terms for sufficiently large separations.

This scaling argument underlies our claim in the Introduction that the throat ontology provides a geometric roadmap for 2PN corrections: once a is fixed for each object (e.g. by its mass and charge through the throat geometry), the coefficients α_ℓ in Eq. (75) are not arbitrary; they are determined by the way the brane deforms into the bulk and by the structure of the transition region. A future 2PN calculation in this model would therefore not introduce an unrestricted zoo of new parameters, but a constrained set of finite–size couplings tied to the same throat geometry that already controls the 1PN and electromagnetic sectors.

5.5 Relation to shape sensitivity in the spin/N-body paper

The spin and N–body paper (Paper III) already provided a first glimpse of these geometric finite–size effects. There, the authors replaced the spherical defects of the orbital paper with mildly oblate spheroids and computed how the 1PN perihelion precession coefficient changed as a function of the oblateness. The result was that even a relatively large shape distortion at the level of $\sim 10\%$ in the equatorial radius produced only a modest change of order a few percent in the precession coefficient.

In a purely three–dimensional description, this shape sensitivity might seem puzzling or arbitrary: why does a substantial deformation of the body's shape have only a small effect on the orbit? In the throat ontology the answer is straightforward. Changing the apparent 3D shape of the defect on the brane corresponds to modifying the geometry of the throat mouth and the transition region, while holding fixed the deeper throat interior and its overall mass content. The resulting changes in the effective 3D density are encoded primarily in the multipole moments of $\rho_{3\rm D}$, which are suppressed by powers of (a/r) and thus have only a small effect on the far–field potential and on 1PN observables.

More concretely, a modest oblateness changes the quadrupole coefficient α_2 in Eq. (75) by some amount of order the oblateness parameter. But because the quadrupole contribution is already suppressed by $(a/r)^2$, the net fractional change in the potential and in the 1PN precession coefficient is doubly suppressed: once by the small shape parameter and once by $(a/r)^2$. This is entirely consistent with the $\sim 2\%$ shifts reported in Paper III for $\sim 10\%$ geometric distortions.

From this perspective, the shape–sensitivity experiment in Paper III is not an ad hoc curiosity but an indirect probe of the transition region and the throat geometry. It shows that the toy model already behaves like a finite–size object in GR: small departures from spherical symmetry induce small corrections to 1PN observables, in a way that is naturally understood as a preview of the 2PN finite–size structure. The throat ontology turns this qualitative observation into a quantitative program: in a full 2PN analysis, one would compute the coefficients α_ℓ and the associated finite–size couplings directly from the geometry and dynamics of the transition region.

In the next section we will see how a different, but related, aspect of the throat geometry—the mixing of brane–parallel and bulk–directed modes in the vector sector—provides a natural interpretation of the longitudinal sign flip $\alpha^2 = -2/5$ required to match the Einstein–Infeld–Hoffmann Lagrangian at 1PN order.

6 Lorentzian Constraint and $\alpha^2 = -2/5$

In Papers I–III the effective 1PN dynamics on the brane were obtained by matching a hydrodynamic defect model to the Einstein–Infeld–Hoffmann (EIH) Lagrangian. A striking outcome of this matching, emphasized in Paper III, was that the velocity–dependent interaction terms could only be made to agree with GR if the longitudinal component of a certain vector kernel entered with an effective negative sign. In the parameterization used there, this requirement took the form

$$\alpha^2 = -\frac{2}{5}, \tag{78}$$

i.e. the longitudinal weight α must be purely imaginary. No choice of real α in an everywhere positive–definite quadratic form reproduces the EIH coefficients.

In this section we first recap, at a schematic level, how this constraint arises in the 1PN vector sector. We then show how a simple two–mode toy model captures the essence of the calculation and makes the sign flip transparent. Finally, we interpret the result in the brane–bulk throat ontology, where it appears naturally as an emergent Lorentzian signature in mode space associated with the mixing of brane–parallel and bulk–directed flows.

6.1 Recap of the 1PN vector sector from Paper III

The EIH Lagrangian for a system of N point masses m_a at 1PN order can be written schematically as

$$L_{\mathrm{EIH}} = L_{\mathrm{N}} + \frac{1}{c^2} L_{\mathrm{1PN}}, \tag{79}$$

where L_{N} is the Newtonian Lagrangian and L_{1PN} contains terms quadratic in velocities and linear in the Newtonian potential. For a pair of bodies a, b, the velocity–dependent part of L_{1PN} includes structures of the form

$$L_{\mathrm{1PN}}^{(ab)} \supset \frac{G m_a m_b}{r_{ab}} \Big[c_1\, \mathbf{v}_a^2 + c_2\, \mathbf{v}_b^2 + c_3\, \mathbf{v}_a \cdot \mathbf{v}_b + c_4\, (\mathbf{v}_a \cdot \hat{\mathbf{n}}_{ab})(\mathbf{v}_b \cdot \hat{\mathbf{n}}_{ab}) \Big], \tag{80}$$

with fixed numerical coefficients c_i determined by GR, and where $\hat{\mathbf{n}}_{ab}$ is the unit vector from b to a.

In the superfluid defect model, these velocity–dependent interactions arise from nonlocal couplings mediated by the fluid velocity field and its propagator. The relevant object is a vector kernel $K_{ij}(\mathbf{x})$ appearing in the quadratic part of the hydrodynamic energy functional,

$$E_{\mathrm{vec}} = \frac{1}{2} \int \mathrm{d}^3x\, \mathrm{d}^3x'\; v_i(\mathbf{x})\, K_{ij}(\mathbf{x} - \mathbf{x}')\, v_j(\mathbf{x}'), \tag{81}$$

where repeated indices are summed. In Fourier space and in the absence of dissipation, rotational invariance implies that $K_{ij}(\mathbf{k})$ can be decomposed into transverse and longitudinal projectors,

$$K_{ij}(\mathbf{k}) = C_T(\alpha)\, P^{\mathrm{T}}_{ij}(\mathbf{k}) + C_L(\alpha)\, P^{\mathrm{L}}_{ij}(\mathbf{k}), \tag{82}$$

with

$$P^{\mathrm{T}}_{ij}(\mathbf{k}) = \delta_{ij} - \hat{k}_i\hat{k}_j, \qquad P^{\mathrm{L}}_{ij}(\mathbf{k}) = \hat{k}_i\hat{k}_j, \qquad \hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|}. \tag{83}$$

The coefficients $C_T(\alpha)$ and $C_L(\alpha)$ are functions of a dimensionless parameter α that controls the relative weight of longitudinal and transverse pieces in the underlying energy functional. Schematically, one can think of them as coming from a combination

$$E_{\mathrm{vec}} \sim \int \mathrm{d}^3x \,\left[A_T\, |\mathbf{v}_{\mathrm{T}}|^2 + A_L\, |\mathbf{v}_{\mathrm{L}}|^2\right], \qquad \mathbf{v} = \mathbf{v}_{\mathrm{T}} + \mathbf{v}_{\mathrm{L}}, \tag{84}$$

with $A_T > 0$ and $A_L > 0$ and the parameter α encoding how these components mix into the physical modes that mediate interactions between defects.

When one computes the induced interaction Lagrangian between two moving defects by integrating out the fluid degrees of freedom, the result contains the same structures as in Eq. (80), but with coefficients that depend on $C_T(\alpha)$ and $C_L(\alpha)$ instead of the fixed GR coefficients. Matching to the EIH Lagrangian imposes a set of linear constraints on C_T and C_L, and hence on α. Paper III showed that these constraints have no solution with $\alpha^2 > 0$ and $A_T, A_L > 0$, but do have a unique solution if α^2 is allowed to be negative:

$$\alpha^2 = -\frac{2}{5}. \tag{85}$$

In other words, the effective weight of the longitudinal contribution must enter with the opposite sign relative to what one would obtain from a strictly Euclidean, positive–definite quadratic form. This is what we mean when we say that the longitudinal sector carries an effective Lorentzian signature.

6.2 Two-mode toy model: transverse vs longitudinal

The full calculation in Paper III involves convolutions with the vector kernel, integration over sources, and careful identification of the resulting terms in the many–body Lagrangian. To understand the origin of the sign flip and the role of α^2, however, it is enough to consider a simple two–mode toy model.

Let u_T and u_L denote amplitudes of a transverse–like and a longitudinal–like mode that can be excited by the motion of a defect. Think of u_T as representing brane–parallel shear patterns in the fluid velocity and u_L as representing compressional or bulk–directed patterns. We start with a positive–definite quadratic energy in these amplitudes,

$$E[u_T, u_L] = A_T\, |u_T|^2 + A_L\, |u_L|^2, \qquad A_T > 0,\ A_L > 0. \tag{86}$$

Suppose further that the defect couples not to u_T and u_L separately, but to a linear combination

$$u_{\mathrm{phys}} = u_T + \alpha u_L, \tag{87}$$

where α is a (so far arbitrary) complex weight. For example, this could represent the fact that a moving defect sources both transverse and longitudinal components of the fluid velocity, with relative amplitude controlled by α.

We now imagine integrating out u_T and u_L to obtain an effective interaction energy between defects mediated by u_{phys}. At the level of this toy model, that procedure simply replaces u_T and u_L by their combination in Eq. (87) and yields an effective quadratic form in u_{phys}:

$$E_{\text{eff}}[u_{\text{phys}}] = A_{\text{eff}}(\alpha)\,|u_{\text{phys}}|^2, \tag{88}$$

with

$$A_{\text{eff}}(\alpha) = \frac{A_T + \alpha^2 A_L}{(1+\alpha^2)}, \tag{89}$$

where, for simplicity, we have taken u_T and u_L to be aligned in phase and used the relation $u_L = \alpha^{-1}(u_{\text{phys}} - u_T)$ to eliminate u_L. The precise algebra is not important; the crucial feature is that the effective coefficient controlling the physical mode is a *linear* combination $A_T + \alpha^2 A_L$ of the original transverse and longitudinal coefficients.

If α is real, and $A_T, A_L > 0$, then $A_{\text{eff}}(\alpha)$ is necessarily positive for all $\alpha^2 > 0$. There is no way to obtain an effective coefficient with the opposite sign of either A_T or A_L from a linear combination like Eq. (89). This is the toy model analogue of the statement that a purely Euclidean, positive–definite energy functional cannot reproduce the EIH coefficients: all of the induced interactions remain "of the same sign" as the underlying quadratic form.

If we now allow α to be purely imaginary, $\alpha^2 < 0$, the situation changes. The combination $A_T + \alpha^2 A_L$ can have either sign, depending on the relative sizes of A_T and A_L and the value of α^2. In particular, there may exist a value of $\alpha^2 < 0$ for which

$$A_T + \alpha^2 A_L = \lambda A_T, \tag{90}$$

with λ taking the specific value required by the EIH matching. In the full theory, the role of A_T and A_L is played by the coefficients $C_T(\alpha)$ and $C_L(\alpha)$ in Eq. (82), and the parameter α controls how the longitudinal projector enters relative to the transverse one. The EIH calculation then effectively picks out the unique value

$$\alpha^2 = -\frac{2}{5} \tag{91}$$

for which the combination of transverse and longitudinal contributions yields the correct coefficients of $\mathbf{v}_a^2$, $\mathbf{v}_a \cdot \mathbf{v}_b$, and $(\mathbf{v}_a \cdot \hat{\mathbf{n}}_{ab})(\mathbf{v}_b \cdot \hat{\mathbf{n}}_{ab})$ in Eq. (80).

From the point of view of the toy energy (86), allowing $\alpha^2 < 0$ means that the direction in the (u_T, u_L) space picked out by u_{phys} has an *indefinite* norm: its quadratic form is not purely positive but has a minus sign built into the relative weight of u_T and u_L. Put differently, the mode space spanned by u_T and u_L acquires a Lorentzian signature when expressed in terms of the physical combinations that mediate interactions between defects.

6.3 Brane–bulk interpretation: emergent Lorentzian signature

The throat ontology provides a natural geometric interpretation of this mode–space Lorentzian structure. The transverse–like and longitudinal–like modes in the vector kernel can be viewed as encoding fluctuations that are predominantly tangential to the brane and fluctuations that are predominantly directed into the bulk.

In the 4D superfluid picture, the fluid velocity field has components both along the brane, $\mathbf{v}(\mathbf{x}, w, t)$, and in the bulk, $v_w(\mathbf{x}, w, t)$. Small perturbations can be decomposed into patterns that, at each point, are divergence–free along the brane (transverse) and patterns that involve compressional or normal motion (longitudinal). The brane–parallel modes govern how fluid shears and circulates within the brane; the bulk–directed modes govern how fluid drains into or out of the throat and how the brane is deformed into the bulk. Both types of motion are excited when defects move.

From this perspective, the parameter α controls how strongly the bulk–directed component participates in the mode that mediates velocity–dependent interactions on the brane. The requirement $\alpha^2 = -2/5$ then says that this bulk component enters with an effective minus sign in the quadratic form that defines the interaction energy. The space of modes relevant for the 1PN dynamics thus has an effective metric with one direction (associated with bulk–directed motion) carrying a sign opposite to the brane–parallel directions. This is precisely the hallmark of a Lorentzian signature.

There are two complementary ways to think about this:

- In configuration space, the 4D acoustic equation

$$-\frac{1}{c_s^2}\partial_t^2 h + \nabla_4^2 h = 0 \tag{92}$$

  already singles out time with a minus sign relative to the spatial derivatives. The throat geometry adds another distinguished direction—the bulk direction w—which is treated differently from the brane directions in the boundary conditions and in the mode structure. When one integrates out the bulk to obtain an effective 3D theory on the brane, the combination of these features leads to an effective mode space in which one direction (a mixture of bulk–directed flow and temporal variation) behaves as time–like, while the brane–parallel modes behave as space–like.

- In the effective interaction Lagrangian, the velocity–dependent terms can be viewed as arising from an inner product on the space of 1PN "gravitational potentials" (scalar, vector, and tensor). The constraint $\alpha^2 = -2/5$ ensures that this inner product has the same indefinite structure as the EIH mapping to the GR metric: the longitudinal/bulk combination of modes plays the role of the time–like component of the metric, while the transverse/brane combinations play the role of the spatial components. The nontrivial sign in the longitudinal sector is what allows the effective theory to reproduce the geodesic structure of GR at 1PN order.

In either view, the message is the same: the unusual value $\alpha^2 = -2/5$ is not a mysterious fine–tuning, but a reflection of the brane–bulk structure of the defect. The same throat that reconciles the sphere–cylinder tension between gravity and electromagnetism also enforces a Lorentzian signature in the mode space controlling the 1PN vector sector. The longitudinal "direction" in this mode space is associated with motion into the bulk and with the deformation of the brane at the throat mouth, and it is this direction that must carry the negative sign in order for the effective 3D dynamics on the brane to match GR.

In the next section we step back to a more global view and discuss how this brane–bulk, Lorentzian mode structure constrains the form of higher–order post–Newtonian corrections, and how the resulting 2PN finite–size corrections could, in principle, be used to test or falsify the toy model.

7 Toward 2PN: Predictions and Falsifiability

Up to this point we have used the throat ontology primarily to reinterpret existing 1PN and electromagnetic results. In this section we look forward: we outline how a full 2PN analysis would be organized in this framework, and we sketch how the resulting finite–size corrections could, in principle, be confronted with observations. The aim is not to perform a 2PN calculation, but to make clear that such a calculation is both well–posed and constrained, rather than an open invitation to add arbitrary new terms.

7.1 What 2PN means in this toy model

In standard post–Newtonian theory, the expansion parameter is

$$\epsilon \sim \frac{GM}{rc^2} \sim \left(\frac{v}{c}\right)^2, \tag{93}$$

where M is a characteristic mass, r is a characteristic separation, and v is a characteristic orbital speed. Newtonian gravity is $\mathcal{O}(\epsilon^0)$, 1PN corrections are $\mathcal{O}(\epsilon)$, 2PN corrections are $\mathcal{O}(\epsilon^2)$, and so on. In GR this hierarchy is built into the Einstein equations via the expansion of the metric components in powers of $1/c$.

In the superfluid defect toy model, the same small parameter appears, but there is an additional geometric scale: the throat radius a (and, to a lesser extent, the depth L). We can therefore construct two independent dimensionless quantities:

$$\epsilon \sim \frac{GM}{rc^2}, \qquad \delta \sim \frac{a}{r}. \tag{94}$$

The 1PN analysis in Papers I–III effectively assumes that δ is small enough that finite–size effects can be neglected at leading order, and that all relevant corrections scale with ϵ alone. The throat ontology makes this assumption explicit: the far–field theory on the brane is controlled by the monopole mass M and the velocity–dependent interactions encoded in the vector kernel, while the structure of the throat and transition region enters only through multipole corrections suppressed by powers of δ.

In a full 2PN analysis of the toy model, one would therefore organize terms in a double expansion in ϵ and δ, keeping track of how powers of a/r enter alongside powers of $GM/(rc^2)$. Schematically, the effective action or Lagrangian for a binary system would contain terms of the form

$$L_{\text{eff}} = L_{\text{N}} + \epsilon L_{\text{1PN}} + \epsilon^2 L_{\text{2PN}} + \delta^2 \tilde{L}_{\text{2PN}} + \cdots, \tag{95}$$

where L_{2PN} contains the usual point–particle 2PN corrections and $\tilde{L}_{\text{2PN}}$ contains finite–size corrections tied to the throat geometry. The key point is that the throat ontology predicts the *structure* of $\tilde{L}_{\text{2PN}}$ in terms of a small set of geometric parameters, rather than leaving it arbitrary.

7.2 Organizing 2PN corrections via the throat geometry

Within the throat ontology, the most natural way to organize 2PN corrections is to group them according to which part of the geometry they probe:

1. *Far–field multipoles on the brane*: These are corrections to the scalar potential and its velocity–dependent companions arising from the multipole structure of ρ_{3D} and $\mathbf{v}_{3D}$ on the brane. They are controlled by the coefficients α_ℓ in the expansion

$$\Phi(r,\theta) = -\frac{GM}{r}\left[1 + \sum_{\ell\geq 2}\alpha_\ell\left(\frac{a}{r}\right)^\ell P_\ell(\cos\theta)\right], \tag{96}$$

   with α_ℓ determined by the geometry of the throat mouth and the transition region. The leading quadrupole term ($\ell = 2$) is naturally associated with 2PN finite–size effects in the orbital dynamics.

2. *Velocity–dependent couplings*: Just as the 1PN vector sector is controlled by the transverse and longitudinal parts of the vector kernel and the parameter $\alpha^2 = -2/5$, the 2PN vector and tensor sectors will receive corrections from the same kernel but with additional structure induced by the throat. For example, the effective kernel could acquire mild dependence on a and L through higher–derivative terms, leading to corrections that scale as $(a/r)^2\epsilon$ or ϵ^2. These terms would modify the coefficients of velocity–dependent structures in the many–body Lagrangian, but in a way that is tied to the same geometric data $(M, a, L, \ldots)$.

3. *Electromagnetic backreaction*: In the EM paper, charge is associated with vorticity flux threading the throat and the electromagnetic field is sourced by cavity modes inside the throat. At 2PN order, the energy stored in these modes and in the surrounding electromagnetic field will backreact on the gravitational sector. This will generate corrections coupling mass, charge, and throat geometry, schematically of the form

$$L_{\text{eff}} \supset \epsilon^2\left(\beta_1\frac{Q^2}{M^2} + \beta_2\frac{Q^2}{M^2}\frac{a^2}{r^2} + \cdots\right), \tag{97}$$

   with coefficients β_i determined by the cavity spectrum. Thus 2PN corrections in this model are expected to carry signatures of both mass and charge in a way that reflects the common throat origin of the gravitational and electromagnetic sectors.

A practical 2PN calculation in this framework would proceed roughly as follows:

- Choose a parametrization of the throat geometry (e.g. a family of smooth embeddings of the brane into the bulk near each defect, characterized by a, L, and a small number of shape parameters).

- Solve the 4D Euler and continuity equations perturbatively in the near field of each throat, matching onto the far–field multipole expansions on the brane and onto the cavity modes in the interior.

- Compute the induced interaction Lagrangian for a system of moving throats by integrating out the fluid degrees of freedom to the appropriate order in ϵ and δ.
- Match the resulting effective Lagrangian onto a PN–style basis of scalars built from positions, velocities, and spins, and read off the finite–size coefficients as functions of $(M, a, L, Q, \ldots)$.

While technically demanding, this program is sharply constrained: once M, a, L, and Q are fixed for each object by low–order data (Newtonian mass, EM charge, and perhaps one additional observable), there is little room to adjust higher–order coefficients without spoiling the consistency of the throat geometry.

7.3 Falsifiability and observational channels

The existence of a geometrically constrained 2PN sector raises a natural question: how could the toy model be tested or ruled out? While a full phenomenological analysis lies beyond the scope of this paper, we can identify several promising observational channels.

Binary pulsars and compact binaries. Timing measurements of binary pulsars and phase evolution of compact binaries (including black hole and neutron star mergers) are sensitive to 2PN–order corrections in the orbital dynamics. In GR, these corrections are controlled by the masses, spins, and, in some cases, tidal deformability parameters of the bodies. In the throat model, there would be additional finite–size contributions determined by the throat radii a_a and a_b and by the way the throats deform under mutual tidal fields. If the model predicts a specific relation between a and M (or between a, M, and Q), then the pattern of 2PN corrections in different systems is highly constrained. Significant inconsistencies between the inferred 2PN coefficients across different binaries would rule out the model.

Solar–system tests. Precision measurements of planetary ephemerides, light deflection, and Shapiro delay in the solar system already probe some aspects of 1PN and 2PN physics. The throat model is designed to match the 1PN sector by construction; deviations are expected to appear, if at all, in subtle finite–size effects associated with the Sun's throat geometry. For instance, small angular dependence in the effective potential or small modifications to perihelion precession beyond the standard 1PN terms could signal a non–GR finite–size structure. Any 2PN calculation in the throat model would need to be checked against the tight solar–system bounds.

Short–range gravity experiments. At laboratory scales, tests of Newton's law probe potential deviations from the $1/r^2$ force at distances comparable to or smaller than the effective size of the gravitating bodies. In the throat ontology, if the throat radius a for ordinary matter is not vanishingly small compared to these scales, one might expect measurable finite–size corrections to the force law at short range. Conversely, the absence of such deviations can be used to place upper bounds on a for laboratory masses, which must be consistent with any a–values inferred from astrophysical systems.

Electromagnetic–gravitational cross–checks. Because mass and charge are different projections of the same throat geometry, the model predicts relationships between gravitational and electromagnetic observables that have no analogue in GR. For example, if a is related to Q via the enthalpy–selected aspect ratio and the cavity structure, then objects with different charge–to–mass ratios should have different patterns of finite–size gravitational corrections at 2PN order. If observations show no such dependence (within experimental uncertainties), or if the required $a(Q, M)$ relations are inconsistent across systems, the model would be disfavored.

In all of these channels, the logical structure is the same: the throat ontology fixes how 2PN coefficients depend on a small set of geometric parameters; observations overdetermine those parameters; inconsistency rules out the model. In this sense, the model is, at least in principle, falsifiable.

7.4 Why the ontology paper must come before a full 2PN analysis

It is natural to ask why we have not simply proceeded directly to a 2PN calculation. The answer is that, without the throat ontology, the space of possible 2PN corrections in the superfluid defect model is too large and too poorly organized to be meaningfully constrained.

Viewed purely as a 3D fluid model with defects, one could write down an enormous number of higher–order terms in the effective action, including arbitrary multipole couplings, higher–derivative corrections to the vector kernel, and a variety of nonlocal interactions. Without additional structure, there is no principled way to decide which of these terms should be present, how large their coefficients should be, or how they should be related across different sectors (gravity vs electromagnetism) and different systems.

The throat ontology changes this situation in three important ways:

- It *geometrizes* finite–size effects, tying them to the geometry of brane–bulk throats (radius a, depth L, shape of the mouth, mode spectrum in the interior), rather than treating them as arbitrary parameters in a 3D effective theory.
- It *links* the gravitational and electromagnetic sectors: the same throat that determines the monopole mass and the 1PN vector couplings also determines the cavity structure and the enthalpy selection of L/a in the EM sector. Any 2PN corrections must be compatible with this shared origin.
- It *explains* the emergent Lorentzian signature in the 1PN vector sector as a consequence of brane–bulk mode mixing, rather than as an ad hoc choice in the sign of the longitudinal kernel. This strongly constrains how higher–order vector and tensor corrections can appear.

With these ingredients in place, a 2PN calculation is no longer a fishing expedition in an enormous parameter space but a targeted computation in a geometric setting with clear boundary conditions. The present paper is therefore a necessary precursor: it defines the ontology in which the 2PN problem is posed, identifies the small set of geometric parameters that control the finite–size structure, and clarifies the relationships among the different sectors of the model.

In the next and final section we step back from technicalities and discuss the broader implications of the throat ontology, including open questions about the bottom of the throat, the stabilization of the brane, and the behavior of multiple defects interacting through the bulk.

8 Discussion and Outlook

We have proposed a brane–bulk throat ontology for the superfluid defect toy universe developed in Papers I–IV and shown how it resolves several apparent tensions in the earlier work. In this final section we summarize the main elements of this picture, highlight the open questions it raises, and outline directions for future work.

8.1 Summary of the ontology and main results

The central idea of this paper is that the defects representing massive, possibly charged bodies in the superfluid toy model should be understood as *throats* connecting a three–dimensional brane to a four–dimensional superfluid bulk, rather than as purely three–dimensional objects. Each defect corresponds to a localized region where the brane at $w = 0$ pinches into the bulk, forming a tube–like region of radius a and depth L.

From the perspective of a brane–bound observer, the intersection of a throat with the brane is a nearly spherical mouth of radius a. The effective 3D density and velocity fields induced on the brane by drainage into this mouth reproduce the spherical sink picture used in the orbital, optical, and spin/N–body papers: at distances $r \gg a, L$, the defect appears as a point mass with potential $\Phi \sim -GM/r$ and 1PN corrections controlled by the same hydrodynamic energy functional that was matched to the EIH Lagrangian. In the hybrid scalar–vector description, the Newtonian $1/r$ potential is sourced by a scalar density–starvation field on the brane, while the 1PN corrections are governed by the vector sector encoded in the Lorentzian energy functional. Dimensional reduction from 4D to 3D shows that finite–size effects from the throat interior and transition region enter the far field only through multipole corrections suppressed by $(a/r)^2$ and higher powers.

From the perspective of the bulk, the region $0 < w < L$ with $|\mathbf{x}| \lesssim a$ is approximately a cylindrical cavity of cross–sectional area πa^2 and length L. The 4D acoustic wave equation separated inside this cavity yields Bessel–type radial modes and standing waves along w; imposing boundary conditions at $r = a$ and at $w = 0, L$ picks out a discrete mode spectrum. Minimizing the enthalpy at fixed "charge" selects a preferred aspect ratio $L/a = \sqrt{2}\pi/x_{01} \approx 1.85$ for the fundamental mode, exactly as found in the electromagnetic paper. In the throat ontology, this result is reinterpreted as a geometric property of the throat, not as an arbitrary parameter choice for a 3D cylinder.

The throat picture also clarifies the physical meaning of mass and charge in the model. Mass is associated with the scalar density–starvation field anchored on the throat: drainage into the throat and the throat geometry fix a boundary condition for the enthalpy and density fields, and the stiff superfluid vacuum responds with a long–range depletion halo. The core scale of the Newtonian mass is still set by the effective throat volume, with $M \sim \rho_0 \pi a^2 L$ up to order–unity factors, but the full gravitational mass measured on the brane includes both this localized core and the extended scalar depletion profile. Charge is associated with circulation and vorticity flux threading the throat cross–section; in the hydrodynamic–electromagnetic dictionary, this vorticity flux plays the role of electric charge, and the cavity modes inside the throat generate the effective electromagnetic fields. Thus mass and charge are different projections of the same geometric object—a brane–bulk throat characterized by (a, L) and its internal mode structure.

In the near–field region around the throat mouth, where streamlines bend from radial inflow on the brane to axial flow into the bulk, the geometry is strongly curved and the flow experiences significant shear and pressure gradients. Coarse–graining this region onto the brane generates a tower of multipole corrections to the effective 3D density and potential, with a leading quadrupole term suppressed by $(a/r)^2$. This provides a natural geometric interpretation of finite–size effects and an organizing principle for 2PN corrections: once the throat radius a is fixed for a given defect, the coefficients of these multipoles are determined by the geometry of the throat mouth and its deformation under external fields.

Finally, the throat ontology offers a new perspective on the longitudinal sign flip $\alpha^2 = -2/5$ required in the 1PN vector sector. In the vector kernel that mediates velocity–dependent interactions, transverse (brane–parallel) and longitudinal (bulk–directed) modes enter with different weights. The EIH matching demands that the longitudinal contribution effectively carry a negative sign relative to the transverse one, which can be captured by a simple two–mode toy model in which the quadratic form on mode space has an indefinite (Lorentzian) signature. In the throat picture this is interpreted as an emergent Lorentzian structure in the space of modes associated with the mixing of brane–parallel and bulk–directed flows near the throat.

8.2 What happens at the bottom of the throat?

Throughout this paper we have treated the throat as a finite tube of depth L whose bottom at $w = L$ is left deliberately unspecified. This is sufficient for the 1PN and near–field considerations we have focused on, but it leaves open an important question about the global structure of the model: what happens at the bottom of the throat?

Several possibilities present themselves:

- *Closed cavities.* The throat could close off smoothly, with the brane folding back into itself and the superfluid forming a finite cavity. In this case the mode spectrum inside the throat is purely discrete, and there is no leakage of energy or vorticity into the deep bulk. Charge and mass would then be strictly localized objects, with their values determined entirely by the geometry and mode content of the closed throat.

- *Open throats.* The throat could open into a larger bulk region at $w = L$, allowing acoustic energy and vorticity to escape into the bulk. This would introduce damping and radiation channels not present in the closed–cavity picture. For instance, highly dynamical processes such as mergers could excite bulk modes that carry energy away from the brane, potentially modifying the gravitational wave signal seen by brane–bound observers.

- *Connected throats.* Different defects could be connected through a common bulk region, with throats that meet or merge away from the brane. This would allow for bulk–mediated interactions between defects, beyond those encoded in the effective 3D theory on the brane. In extreme cases, one could imagine networks of throats whose connectivity changes dynamically, perhaps in analogy with reconnection events in vortex tangles.

Each of these scenarios would leave a characteristic imprint on the mode spectrum in the throat, the time dependence of charge and mass, and the way energy is exchanged between defects and the bulk. For example, an open throat might lead to slow leakage of vorticity and hence slow evolution of charge, while a closed cavity would make charge strictly conserved. The present paper does not attempt to choose among these options, but the framework we have developed is flexible enough to incorporate any of them once additional microscopic input is specified.

8.3 What stabilizes the brane?

Another major open question is the origin and stability of the brane itself. In our construction, the brane at $w = 0$ is treated as a given: a distinguished hypersurface on which observers live and on which the effective 3D physics of Papers I–IV is realized. However, in a more complete model the brane should arise as a dynamical object, stabilized by some underlying mechanism.

Several possibilities are familiar from other contexts:

- *Domain walls or defects.* The brane might be a domain wall in an underlying order parameter of the superfluid, separating regions with different phases or densities in the bulk. The tension of the domain wall and the coupling of the superfluid to this order parameter would determine the brane's rigidity and its response to the presence of throats.

- *Potential minima.* The brane could be a locus where some effective potential $V(w)$ for the fluid has a minimum, pinning density and pressure at $w = 0$ and making deviations in the w direction energetically costly. Throats would then be localized regions where this pinning is overcome or modified.

- *Defect condensation.* The brane could represent a condensate of lower–dimensional defects in the bulk superfluid, with throats corresponding to higher–dimensional defects that connect different regions of the condensate.

Any such mechanism would feed back into the throat ontology by determining the allowed shapes of the brane near defects, the cost of bending the brane into the bulk, and the boundary conditions satisfied by the fluid at $w = 0$. In particular, the spectrum of small fluctuations of the brane could interact with the cavity modes in the throats, potentially modifying the relation between L/a, M, and Q. Exploring specific brane stabilization mechanisms and their consequences is an important direction for future work.

8.4 Multiple defects and bulk interactions

So far we have focused on a single isolated throat. In reality, the toy universe contains many defects, representing stars, planets, compact objects, and possibly charged bodies. The throat ontology provides a conceptual framework for thinking about how these defects interact through the bulk as well as through the effective 3D fields on the brane.

At the level of the effective 3D theory, multiple throats interact via the superposition of their monopole and multipole fields on the brane, as captured by the 1PN and (future) 2PN Lagrangians. This is the regime directly comparable to GR. However, the throats also interact through the bulk superfluid: their interior modes can scatter off each other, their transition regions can overlap when defects come close, and their bottom regions at $w = L$ can come into contact if throats are sufficiently deep.

In a merger of two compact objects, for example, one might imagine the following sequence: the throats approach each other on the brane; their transition regions deform and eventually overlap; the interior cavities interact and exchange energy; finally, a single, larger throat emerges, possibly with a different L/a ratio and a different internal mode spectrum. From the brane perspective this would be seen as a change in the effective mass, spin, and charge of the merged object, together with the emission of gravitational and electromagnetic waves. From the bulk perspective there would also be emission of acoustic radiation and rearrangements of vorticity in the deep bulk.

While these speculations are necessarily qualitative at this stage, they point to a rich dynamical phenomenology that goes beyond what is captured in a purely 3D effective theory. A long–term goal of the program initiated by this toy model would be to simulate such multi–throat dynamics directly in the 4D superfluid and compare the emergent signals (both on the brane and in the bulk) with those predicted by GR and standard electromagnetism.

8.5 Relation to brane-world and emergent gravity scenarios

The brane–bulk throat ontology developed here sits conceptually between several strands of work in the broader theoretical physics literature.

On the one hand, it shares with brane–world scenarios the idea that the observable universe is a lower–dimensional hypersurface embedded in a higher–dimensional space, and that localized objects may be associated with tubes or funnels connecting the brane to the bulk. On the other hand, it shares with emergent gravity and superfluid vacuum models the idea that gravitational and electromagnetic phenomena can arise as effective excitations of an underlying medium, with metric structures and gauge fields emerging from the dynamics of collective modes.

However, the present construction is more concrete and mechanical than many of these frameworks. The underlying degrees of freedom are those of a compressible superfluid obeying continuity and Euler equations in a specified number of spatial dimensions; the brane is a geometric hypersurface in this fluid; and defects are explicit deformations of the brane into the bulk. The emergent metric and electromagnetic fields are not fundamental fields in a gravitational action, but effective descriptors of the collective motion of the fluid. This keeps the model firmly in the realm of a toy universe: it is not proposed as a literal description of our universe, but as a playground in which questions about emergence, locality, and finite–size structure can be posed sharply and answered with explicit calculations.

Within this playground, the throat ontology unifies several seemingly disparate features of the earlier papers: the need for spherical sinks in the 1PN gravitational sector, the cylindrical cavity in the EM sector, the longitudinal sign flip $\alpha^2 = -2/5$, and the small but nonzero shape–sensitivity of orbital dynamics. All of these phenomena emerge as different aspects of the same geometric structure: brane–bulk throats of radius a and depth L embedded in a 4D superfluid.

To move beyond the toy universe, at least conceptually, one would need to address the microscopic origin of the superfluid, the dynamical nature of the brane, the behavior of the model in strong–field and high–curvature regimes, and the relationship between the emergent effective metric and a fundamental description of spacetime. These are ambitious questions, and we do not attempt to answer them here. Our more modest claim is that, within the limited scope of weak–field, slow–motion phenomena, the throat ontology provides a coherent and falsifiable framework in which gravity and electromagnetism emerge from a single mechanical medium, and in which the next steps—a full 2PN analysis and detailed comparisons with observations —are clearly defined.

If that program succeeds, the superfluid defect toy universe will have served its purpose: not as a rival to GR or Maxwell theory, but as a concrete example of how a higher–dimensional, mechanical substrate can give rise to metric gravity, gauge fields, and finite–size structure in a controlled effective description. If it fails, it will fail in an instructive way, highlighting which aspects of our current theories of gravity and electromagnetism are hardest to reproduce in any emergent, medium–based picture.

A Gaussian 4D Density and 3D Multipole Structure

In this appendix we work out the Gaussian toy model of Sec. 3.2 in detail. The goal is to make explicit how a localized, mildly aspherical 4D density profile produces an effective 3D source with monopole mass $M \propto \rho_0 L a^3$ and quadrupole moment $Q/M \sim \varepsilon a^2$.

A.1 Setup: 4D density and projection to the brane

We start from the 4D density profile introduced in Eq. (31), written here for convenience as

$$\rho_4(r,\theta,w) = \rho_0 \exp\left(-\frac{r^2}{a^2}\right) \exp\left(-\frac{w^2}{L^2}\right) \left[1 + \varepsilon P_2(\cos\theta)\right], \tag{98}$$

where (r,θ,ϕ) are spherical coordinates on the brane, w is the bulk coordinate, a is the characteristic brane–radius of the throat, L is its depth along w, ρ_0 is a reference density, and ε is a small dimensionless parameter controlling the angular asymmetry. The $\ell = 2$ Legendre polynomial is

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1). \tag{99}$$

For $\varepsilon = 0$ the density is spherically symmetric on the brane and separable in r and w; for $\varepsilon \neq 0$ the density has a mild quadrupolar distortion localized within $r \lesssim a$.

To obtain the effective 3D density on the brane we integrate over w, taking the projection kernel $K(w) = 1$ for simplicity:

$$\rho_{\mathrm{3D}}(r,\theta) \equiv \int_{-\infty}^{+\infty} \rho_4(r,\theta,w)\,\mathrm{d}w. \tag{100}$$

Using the Gaussian integral

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{w^2}{L^2}\right)\mathrm{d}w = \sqrt{\pi}\,L, \tag{101}$$

we obtain

$$\rho_{\mathrm{3D}}(r,\theta) = \sqrt{\pi}\,L\rho_0\,\exp\left(-\frac{r^2}{a^2}\right)\left[1+\varepsilon P_2(\cos\theta)\right], \tag{102}$$

which is Eq. (32) in the main text.

A.2 Total mass of the effective 3D source

The total effective mass M associated with this density is

$$M = \int \rho_{\mathrm{3D}}(r,\theta)\,\mathrm{d}^3x. \tag{103}$$

In spherical coordinates (r,θ,ϕ) this becomes

$$M = \int_0^{2\pi}\mathrm{d}\phi \int_0^{\pi}\mathrm{d}\theta \int_0^{\infty}\mathrm{d}r\;\rho_{\mathrm{3D}}(r,\theta)\,r^2\sin\theta. \tag{104}$$

Substituting Eq. (102) yields

$$M = \sqrt{\pi}\,L\rho_0 \int_0^{\infty}\mathrm{d}r\;r^2\exp\left(-\frac{r^2}{a^2}\right)\int_0^{\pi}\mathrm{d}\theta\;\sin\theta\left[1+\varepsilon P_2(\cos\theta)\right]\int_0^{2\pi}\mathrm{d}\phi. \tag{105}$$

The ϕ integral simply yields a factor of 2π. The angular integrals are

$$\int_0^{\pi}\sin\theta\,\mathrm{d}\theta = 2, \tag{106}$$

$$\int_0^{\pi}\sin\theta\,P_2(\cos\theta)\,\mathrm{d}\theta = 0, \tag{107}$$

where the second equality follows from the orthogonality of Legendre polynomials. Thus the εP_2 piece does not contribute to the total mass, as expected for a pure quadrupole distortion. We are left with

$$M = \sqrt{\pi}\,L\rho_0\,(2\pi)\left(\int_0^{\infty} r^2 e^{-r^2/a^2}\mathrm{d}r\right)(2). \tag{108}$$

Using the standard Gaussian integral

$$\int_0^{\infty} r^2 e^{-r^2/a^2}\,\mathrm{d}r = \frac{\sqrt{\pi}}{4}a^3, \tag{109}$$

we find

$$M = \sqrt{\pi}\, L\rho_0\,(2\pi)\left(\frac{\sqrt{\pi}}{4}a^3\right)(2) = \pi^2 L a^3 \rho_0. \tag{110}$$

This is Eq. (33) in the main text. Up to the overall numerical factor π^2, the mass scales as

$$M \sim \rho_0 L a^3, \tag{111}$$

as expected for a throat–like object with size a and depth L.

A.3 Quadrupole moment and the scaling $Q/M \sim \varepsilon a^2$

To compute the leading angular deviation from spherical symmetry, we consider a quadrupole–like scalar moment defined by

$$Q_{20} \equiv \int \rho_{3\mathrm{D}}(r,\theta)\, r^2 P_2(\cos\theta)\, \mathrm{d}^3x, \tag{112}$$

up to an overall normalization convention. Substituting Eq. (102) and writing out the volume element gives

$$Q_{20} = \sqrt{\pi}\, L\rho_0 \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \int_0^{\infty} \mathrm{d}r\ \exp\left(-\frac{r^2}{a^2}\right)[1+\varepsilon P_2(\cos\theta)]\, r^4 P_2(\cos\theta)\sin\theta. \tag{113}$$

Again the ϕ integral gives 2π. Because P_2 is orthogonal to the constant function on the unit sphere, only the term proportional to ε survives:

$$Q_{20} = \sqrt{\pi}\, L\rho_0(2\pi)\varepsilon\left(\int_0^{\infty} r^4 e^{-r^2/a^2}\, \mathrm{d}r\right)\left(\int_0^{\pi} \sin\theta\, [P_2(\cos\theta)]^2 \mathrm{d}\theta\right). \tag{114}$$

We now evaluate the radial and angular factors separately.

For the radial integral, we use the standard formula

$$\int_0^{\infty} r^{2n} e^{-\beta r^2}\, \mathrm{d}r = \frac{(2n-1)!!}{2^{n+1}}\sqrt{\frac{\pi}{\beta^{2n+1}}}, \tag{115}$$

with $n = 2$ and $\beta = 1/a^2$, giving

$$\int_0^{\infty} r^4 e^{-r^2/a^2}\, \mathrm{d}r = \frac{3\sqrt{\pi}}{8} a^5. \tag{116}$$

For the angular integral, we exploit orthogonality of Legendre polynomials:

$$\int_0^{\pi} \sin\theta\, [P_\ell(\cos\theta)]^2 \mathrm{d}\theta = \frac{2}{2\ell+1}. \tag{117}$$

Setting $\ell = 2$ yields

$$\int_0^{\pi} \sin\theta\, [P_2(\cos\theta)]^2 \mathrm{d}\theta = \frac{2}{5}. \tag{118}$$

Putting these pieces together, we find

$$Q_{20} = \sqrt{\pi}\, L\rho_0(2\pi)\varepsilon \left(\frac{3\sqrt{\pi}}{8} a^5\right)\left(\frac{2}{5}\right) \tag{119}$$

$$= \frac{3}{10}\,\pi^2 L a^5 \varepsilon \rho_0. \tag{120}$$

This is Eq. (35) in the main text.

Dividing Eq. (120) by the mass Eq. (110) we obtain

$$\frac{Q_{20}}{M} = \frac{\frac{3}{10}\pi^2 L a^5 \varepsilon \rho_0}{\pi^2 L a^3 \rho_0} = \frac{3}{10}\,\varepsilon a^2. \tag{121}$$

Up to the numerical factor 3/10, this confirms the scaling

$$\frac{Q}{M} \sim \varepsilon a^2, \tag{122}$$

used in Sec. 3.2. The quadrupole moment per unit mass is proportional to the square of the source size a and to the dimensionless asymmetry parameter ε.

A.4 Remarks and generalizations

A few comments are in order:

- The precise numerical coefficients (π^2, 3/10, etc.) depend on our choice of Gaussian profile and normalization conventions for the multipole moments. Different smooth, localized profiles with the same characteristic size a and small quadrupolar distortion εP_2 would yield different order–unity factors but the same scaling $M \sim \rho_0 L a^3$ and $Q/M \sim \varepsilon a^2$.

- The fact that the ε–dependent term drops out of the total mass but dominates the quadrupole is a direct consequence of the orthogonality of Legendre polynomials. This is generic: pure multipole distortions do not alter the monopole mass but do control the higher multipole moments.

- Higher multipoles can be generated by replacing $P_2(\cos\theta)$ in Eq. (98) with $P_\ell(\cos\theta)$ for $\ell > 2$, or by adding a sum $\sum_\ell \varepsilon_\ell P_\ell(\cos\theta)$. A similar calculation then yields

$$\frac{Q_\ell}{M} \sim \varepsilon_\ell a^\ell, \tag{123}$$

  up to order–unity coefficients, leading to potential corrections of order $(a/r)^\ell$ in the far field.

- The exponential factors in r and w are chosen for analytic convenience. In the context of the throat ontology, one should view this Gaussian as a caricature of the true density profile near the throat mouth: it captures the localization at scales $r \sim a$, $w \sim L$ and the leading angular structure without pretending to be a faithful solution of the full 4D Euler equations.

These calculations justify the scaling arguments used in the main text: dimensional reduction of a localized 4D throat–like density produces an effective 3D source whose finite–size corrections to the potential are naturally organized as a multipole expansion with coefficients suppressed by powers of (a/r) and controlled by a small set of geometric parameters $(a, L, \varepsilon_\ell, \ldots)$.

B 4D Mode Separation in the Cylindrical Throat

In this appendix we record the basic mode structure of the 4D acoustic equation inside the cylindrical throat. The aim is to make explicit the separation of variables leading to Bessel–type radial profiles and standing waves along the bulk direction w, together with the corresponding eigenfrequencies and orthogonality relations. This is the mode structure used implicitly in Sec. 4 and in the electromagnetic paper when discussing cavity modes and the enthalpy–selected aspect ratio L/a.

B.1 Acoustic equation and throat geometry

We begin from the linear 4D acoustic equation for the enthalpy perturbation h,

$$\partial_t^2 h - c_s^2 \nabla_4^2 h = 0, \qquad \nabla_4^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 + \partial_w^2, \tag{124}$$

as derived in Sec. 2.1. Inside the throat we approximate the geometry as a straight cylinder of radius a in the brane directions and depth L along the bulk direction w. We choose coordinates adapted to this geometry:

$$(r, \phi, w), \tag{125}$$

where (r, ϕ) are polar coordinates in a cross–section through the throat mouth on the brane (with r measured from the throat center), and $w \in [0, L]$ measures distance along the throat into the bulk. For the lowest–lying modes of interest we assume axisymmetry in ϕ and no dependence on the third brane direction orthogonal to the throat axis.

In this approximation the Laplacian restricted to axisymmetric configurations becomes

$$\nabla_4^2 h = \frac{1}{r}\partial_r (r \partial_r h) + \partial_w^2 h, \tag{126}$$

so the wave equation reads

$$\partial_t^2 h - c_s^2 \left[\frac{1}{r}\partial_r (r \partial_r h) + \partial_w^2 h\right] = 0. \tag{127}$$

B.2 Separation of variables and eigenvalue problem

We look for separated, time–harmonic solutions of the form

$$h(t, r, w) = \mathrm{Re}\{H(r)\, W(w)\, e^{-i\omega t}\}, \tag{128}$$

with real frequency ω. Substituting Eq. (128) into Eq. (127) and dividing by $H(r)W(w)e^{-i\omega t}$ yields

$$-\omega^2 - c_s^2 \left[\frac{1}{H}\frac{1}{r}\partial_r(r\partial_r H) + \frac{1}{W}\partial_w^2 W \right] = 0. \tag{129}$$

Rearranging gives

$$\frac{1}{H}\frac{1}{r}\partial_r(r\partial_r H) + \frac{1}{W}\partial_w^2 W = -\frac{\omega^2}{c_s^2}. \tag{130}$$

The left–hand side is a sum of a function of r and a function of w, while the right–hand side is a constant, so each term must separately be equal to a constant. Introducing separation constants $-k_r^2$ and $-k_w^2$, we write

$$\frac{1}{H}\frac{1}{r}\partial_r(r\partial_r H) = -k_r^2, \tag{131}$$

$$\frac{1}{W}\partial_w^2 W = -k_w^2, \tag{132}$$

and the dispersion relation

$$\omega^2 = c_s^2(k_r^2 + k_w^2). \tag{133}$$

Equations (131)–(132) constitute a pair of ordinary differential equations:

$$\frac{1}{r}\partial_r(r\partial_r H) + k_r^2 H = 0, \tag{134}$$

$$\partial_w^2 W + k_w^2 W = 0. \tag{135}$$

The allowed values of k_r and k_w are determined by boundary conditions on $H(r)$ and $W(w)$ inside the throat.

B.3 Radial modes and Bessel functions

The radial equation (134) is the standard Bessel equation of order zero. Its general solution is

$$H(r) = AJ_0(k_r r) + BY_0(k_r r), \tag{136}$$

where J_0 and Y_0 are Bessel functions of the first and second kind, respectively, and A, B are constants. Regularity at $r = 0$ requires $B = 0$, since $Y_0(x)$ is singular at the origin. We therefore take

$$H(r) = AJ_0(k_r r). \tag{137}$$

At the throat wall $r = a$ we impose a boundary condition on h corresponding to a pinned phase boundary of the stiff superfluid vacuum. Concretely, we take a Dirichlet condition on the enthalpy perturbation,

$$h(r = a, w, t) = 0 \quad \Rightarrow \quad H(a) = 0, \tag{138}$$

This pinning stabilizes the cylindrical throat against radial collapse: expansions or contractions of the wall necessarily excite bulk modes that raise the vacuum enthalpy. This holds for all w and t and implies

$$J_0(k_r a) = 0. \tag{139}$$

Let x_{0n} denote the nth zero of $J_0(x)$, with

$$0 < x_{01} < x_{02} < \cdots . \tag{140}$$

Then the allowed radial wavenumbers are

$$k_r^{(n)} = \frac{x_{0n}}{a}, \qquad n = 1, 2, \ldots, \tag{141}$$

and the corresponding radial eigenfunctions are

$$H_n(r) = A_n J_0\left(\frac{x_{0n} r}{a}\right). \tag{142}$$

These eigenfunctions satisfy an orthogonality relation with respect to the measure $r\,\mathrm{d}r$:

$$\int_0^a r J_0\left(\frac{x_{0m} r}{a}\right) J_0\left(\frac{x_{0n} r}{a}\right) \mathrm{d}r = \frac{a^2}{2} \left[J_1(x_{0n})\right]^2 \delta_{mn}, \tag{143}$$

where J_1 is the Bessel function of order one and δ_{mn} is the Kronecker delta. One can therefore choose the normalization constants A_n so that the $H_n(r)$ form an orthonormal basis on $[0, a]$.

B.4 Axial modes and standing waves along w

The axial equation (135) has the general solution

$$W(w) = C \cos(k_w w) + D \sin(k_w w), \tag{144}$$

with C, D constants. The appropriate boundary conditions depend on the microscopic physics at the mouth ($w = 0$) and bottom ($w = L$) of the throat. For definiteness, and to match the cavity analysis in the main text, we impose Dirichlet conditions at both ends:

$$W(0) = W(L) = 0. \tag{145}$$

The condition $W(0) = 0$ implies $C = 0$, so $W(w) = D \sin(k_w w)$. The condition $W(L) = 0$ then requires

$$\sin(k_w L) = 0 \quad \Rightarrow \quad k_w L = n\pi, \qquad n = 1, 2, \ldots, \tag{146}$$

so the allowed axial wavenumbers are

$$k_w^{(n)} = \frac{n\pi}{L}, \qquad n = 1, 2, \ldots, \tag{147}$$

and the axial eigenfunctions are

$$W_n(w) = D_n \sin\left(\frac{n\pi w}{L}\right). \tag{148}$$

These functions are orthogonal on $[0, L]$:

$$\int_0^L \sin\left(\frac{m\pi w}{L}\right) \sin\left(\frac{n\pi w}{L}\right) \mathrm{d}w = \frac{L}{2}\,\delta_{mn}, \tag{149}$$

so the constants D_n can be chosen to yield an orthonormal set.

Other choices of boundary conditions (Neumann or mixed) are possible and would lead to cosine or sine–cosine combinations with shifted eigenvalues, but the qualitative structure—a discrete tower of axial modes with $k_w \sim n\pi/L$—is the same.

B.5 Mode spectrum and fundamental throat mode

Combining the radial and axial modes, the general separated solution inside the throat is a superposition of eigenmodes labeled by a pair of integers (m, n):

$$h_{mn}(t, r, w) = A_{mn} J_0\Big(\frac{x_{0m} r}{a}\Big) \sin\Big(\frac{n\pi w}{L}\Big) \cos(\omega_{mn} t + \varphi_{mn}), \tag{150}$$

where A_{mn} and φ_{mn} are real amplitude and phase, and the eigenfrequencies are given by

$$\omega_{mn}^2 = c_s^2 \left[\Big(\frac{x_{0m}}{a}\Big)^2 + \Big(\frac{n\pi}{L}\Big)^2\right], \qquad m, n = 1, 2, \ldots . \tag{151}$$

The lowest–lying mode (apart from any zero modes associated with trivial symmetries) is the fundamental $(m, n) = (1, 1)$ mode:

$$h_{11}(t, r, w) \propto J_0\Big(\frac{x_{01} r}{a}\Big) \sin\Big(\frac{\pi w}{L}\Big) \cos(\omega_{11} t), \qquad \omega_{11}^2 = c_s^2 \left[\Big(\frac{x_{01}}{a}\Big)^2 + \Big(\frac{\pi}{L}\Big)^2\right]. \tag{152}$$

This is the mode used in Sec. 4.1 and in the enthalpy–minimization argument for the preferred aspect ratio L/a.

B.6 Normalization and enthalpy functional (sketch)

For completeness, we briefly record how the normalization of the modes enters the enthalpy functional, leaving detailed variations to the main text and to the electromagnetic paper.

The time–averaged perturbation energy (or enthalpy) associated with a mode h can be written schematically as

$$\mathcal{E}[h] = \frac{1}{2} \int_{\mathcal{T}} \mathrm{d}V_4 \, \rho_0 \left[\frac{1}{c_s^2} \langle(\partial_t h)^2\rangle + \langle(\nabla_4 h)^2\rangle\right], \tag{153}$$

where $\mathrm{d}V_4 = r \, \mathrm{d}r \, \mathrm{d}\phi \, \mathrm{d}w$ is the 4D volume element restricted to the throat and $\langle\cdot\rangle$ denotes an average over one oscillation period.

For a single eigenmode of the form (150) with frequency ω_{mn}, the time average gives

$$\langle(\partial_t h_{mn})^2\rangle = \frac{1}{2} \omega_{mn}^2 A_{mn}^2 J_0^2\Big(\frac{x_{0m} r}{a}\Big) \sin^2\Big(\frac{n\pi w}{L}\Big), \tag{154}$$

$$\langle(\nabla_4 h_{mn})^2\rangle = \frac{1}{2} A_{mn}^2 \left[\Big(\frac{x_{0m}}{a}\Big)^2 + \Big(\frac{n\pi}{L}\Big)^2\right] J_0^2\Big(\frac{x_{0m} r}{a}\Big) \sin^2\Big(\frac{n\pi w}{L}\Big), \tag{155}$$

so that the integrand in Eq. (153) is proportional to $(k_r^2 + k_w^2) A_{mn}^2 J_0^2 \sin^2$, with $k_r = x_{0m}/a$ and $k_w = n\pi/L$. Using the orthogonality relations (143) and (149), one finds that the total energy stored in the mode scales as

$$\mathcal{E}_{mn} \propto A_{mn}^2 (k_r^2 + k_w^2) \, a^2 L, \tag{156}$$

up to order–unity constants coming from the angular integrals and normalization choices.

A "charge"–like quantity associated with the mode amplitude, such as the vorticity flux through the throat cross–section, typically scales as

$$\mathcal{Q}_{mn} \propto A_{mn}^2 a^2 L, \tag{157}$$

again up to order–unity factors. At fixed $\mathcal{Q}_{mn}$, the energy is therefore proportional to $k_r^2 + k_w^2$:

$$\mathcal{E}_{mn} \propto (k_r^2 + k_w^2)\, \mathcal{Q}_{mn}. \tag{158}$$

For the fundamental mode $(m, n) = (1, 1)$, this reduces to

$$\mathcal{E}_{11} \propto \left[\left(\frac{x_{01}}{a} \right)^2 + \left(\frac{\pi}{L} \right)^2 \right] \mathcal{Q}_{11}. \tag{159}$$

The enthalpy–minimization problem at fixed "charge" discussed in Sec. 4.2 and in the electromagnetic paper is equivalent to minimizing this combination with respect to a and L, subject to appropriate constraints (e.g. fixed mass and/or other geometric quantities). The resulting optimum selects a particular aspect ratio L/a, which in the toy model of Paper IV and in the present ontology is

$$\frac{L}{a} = \frac{\sqrt{2}\,\pi}{x_{01}}, \tag{160}$$

interpreted as a geometric property of the throat rather than an externally imposed parameter.

We will not repeat the full variational calculation here; the purpose of this appendix is to show how the basic mode structure and its dependence on a and L arise from the 4D acoustic equation inside the cylindrical throat.

C Two-Mode Toy Model for $\alpha^2 = -2/5$

In this appendix we spell out a simple two–mode toy model that captures the effective sign flip in the longitudinal sector required by the Einstein–Infeld–Hoffmann (EIH) matching, and that motivates the condition $\alpha^2 = -2/5$ from a mode–space point of view. The goal is not to reproduce the full calculation of Paper III, but to isolate the algebraic structure behind the statement that the longitudinal contribution to the vector kernel must enter with an effective minus sign.

C.1 Euclidean quadratic form and mode combinations

Consider two real mode amplitudes u_T and u_L representing, respectively, transverse–like (brane–parallel) and longitudinal–like (bulk–directed) patterns in the fluid velocity field. We take the bare energy stored in these modes to be a strictly positive quadratic form,

$$E[u_T, u_L] = \frac{1}{2} \left(A_T u_T^2 + A_L u_L^2 \right), \qquad A_T > 0,\ A_L > 0, \tag{161}$$

which is the simplest representation of a Euclidean mode space with two orthogonal directions. We deliberately omit any off–diagonal $u_T u_L$ term at this stage; such a term can be reintroduced later if desired, but it is not needed to see the essential point.

Suppose now that the motion of a defect couples to a particular linear combination of these modes,

$$u_{\rm phys} = u_T + \alpha u_L, \tag{162}$$

where α is a dimensionless parameter controlling the relative longitudinal weight. This captures, in a toy setting, the idea that a moving defect sources both transverse and longitudinal components of the fluid velocity, with a fixed ratio set by the geometry and the structure of the vector kernel.

We can complete $u_{\rm phys}$ to a basis by introducing an orthogonal combination

$$u_\perp = -\alpha u_T + u_L. \tag{163}$$

The transformation from (u_T, u_L) to $(u_{\rm phys}, u_\perp)$ is linear and invertible as long as $1+\alpha^2 \neq 0$. In matrix form,

$$\begin{pmatrix} u_{\rm phys} \\ u_\perp \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} u_T \\ u_L \end{pmatrix}, \qquad \det \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix} = 1+\alpha^2. \tag{164}$$

The inverse transformation is

$$\begin{pmatrix} u_T \\ u_L \end{pmatrix} = \frac{1}{1+\alpha^2} \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} u_{\rm phys} \\ u_\perp \end{pmatrix}, \tag{165}$$

i.e.

$$u_T = \frac{1}{1+\alpha^2} \left(u_{\rm phys} - \alpha u_\perp\right), \tag{166}$$

$$u_L = \frac{1}{1+\alpha^2} \left(\alpha u_{\rm phys} + u_\perp\right). \tag{167}$$

C.2 Effective quadratic form for the physical mode

We now express the energy (161) in terms of $(u_{\rm phys}, u_\perp)$ using Eqs. (166)–(167). Substituting, we obtain

$$\begin{aligned} E &= \frac{1}{2}\left[A_T u_T^2 + A_L u_L^2\right] \\ &= \frac{1}{2(1+\alpha^2)^2}\Big[A_T \left(u_{\rm phys} - \alpha u_\perp\right)^2 + A_L \left(\alpha u_{\rm phys} + u_\perp\right)^2\Big]. \end{aligned} \tag{168}$$

Expanding the squares,

$$\begin{aligned} E = \frac{1}{2(1+\alpha^2)^2}\Big\{ & A_T \left(u_{\rm phys}^2 - 2\alpha u_{\rm phys} u_\perp + \alpha^2 u_\perp^2\right) \\ & + A_L \left(\alpha^2 u_{\rm phys}^2 + 2\alpha u_{\rm phys} u_\perp + u_\perp^2\right)\Big\}. \end{aligned} \tag{169}$$

Collecting coefficients of $u_{\rm phys}^2$, $u_{\rm phys} u_\perp$, and $u_\perp^2$, we find

$$\begin{aligned} E = \frac{1}{2(1+\alpha^2)^2}\Big[& \left(A_T + \alpha^2 A_L\right) u_{\rm phys}^2 \\ & + 2\alpha(A_L - A_T)\, u_{\rm phys} u_\perp \\ & + \left(\alpha^2 A_T + A_L\right) u_\perp^2\Big]. \end{aligned} \tag{170}$$

In many situations of interest, the defect couples primarily to $u_{\rm phys}$ and only weakly to $u_\perp$, so that $u_\perp$ can be integrated out or treated as a heavy mode that has been set to its (approximate) minimum. As a simple caricature of this situation, we can set $u_\perp = 0$, in which case Eq. (170) reduces to

$$E_{\rm eff}[u_{\rm phys}] = \frac{1}{2} A_{\rm eff}(\alpha)\, u_{\rm phys}^2, \tag{171}$$

with an effective coefficient

$$A_{\rm eff}(\alpha) = \frac{A_T + \alpha^2 A_L}{(1+\alpha^2)^2}. \tag{172}$$

Up to a trivial overall factor $(1+\alpha^2)^{-2}$ (which can be absorbed into a normalization of $u_{\rm phys}$), the relevant combination of A_T and A_L is

$$A_T + \alpha^2 A_L. \tag{173}$$

This is the algebraic structure emphasized in the main text: the effective coefficient governing the physical mode depends linearly on α^2 and interpolates between A_T and A_L as α^2 is varied.

C.3 No sign flip with real α

If α is restricted to be real and $A_T, A_L > 0$, then the combination (173) is strictly positive for all $\alpha^2 \geq 0$:

$$A_T + \alpha^2 A_L \geq A_T > 0. \tag{174}$$

Consequently $A_{\rm eff}(\alpha)$ in Eq. (172) is also positive for all real α, and no choice of α can produce an effective coefficient with the opposite sign of A_T or A_L. In particular, there is no way to obtain a negative effective coefficient starting from a strictly positive–definite quadratic form (161) by merely changing the relative weight of u_T and u_L in the physical combination $u_{\rm phys}$.

This mirrors the situation encountered in the full vector–kernel calculation of Paper III: if the longitudinal contribution to the kernel is weighted by a real parameter in an otherwise Euclidean quadratic form, the resulting velocity–dependent interaction terms in the effective many–body Lagrangian cannot match the EIH coefficients of GR. The longitudinal sector needs to enter with an effective negative sign relative to the transverse sector, which is impossible with $\alpha^2 > 0$ and $A_T, A_L > 0$.

C.4 Allowing $\alpha^2 < 0$ and emergent Lorentzian signature

The situation changes qualitatively if we allow α to be purely imaginary, so that

$$\alpha^2 < 0. \tag{175}$$

We can parametrize this by writing

$$\alpha^2 = -\lambda, \qquad \lambda > 0. \tag{176}$$

Then the relevant combination (173) becomes

$$A_T + \alpha^2 A_L = A_T - \lambda A_L. \tag{177}$$

For λ larger than A_T/A_L, this expression becomes negative even though A_T and A_L are both positive. In other words, the physical direction u_{phys} in (u_T, u_L)–space can acquire an effective *negative* norm with respect to the original quadratic form (161).

From the perspective of the basis $(u_{\text{phys}}, u_\perp)$, this means that the quadratic form in Eq. (170) can develop an *indefinite* signature: one eigenvalue can become negative while the other remains positive. When expressed in a suitable orthonormal basis, the metric on mode space then takes a Lorentzian form

$$E \sim +\frac{1}{2}\tilde{A}_T \tilde{u}_T^2 - \frac{1}{2}\tilde{A}_L \tilde{u}_L^2, \tag{178}$$

with one "time–like" direction and one "space–like" direction in the space of mode amplitudes. The sign choice here is conventional; the key point is that the longitudinal–like direction can carry the opposite sign from the transverse–like direction.

In the full vector–kernel calculation, the roles of A_T and A_L are played by the transverse and longitudinal coefficients C_T and C_L in the kernel decomposition

$$K_{ij}(\mathbf{k}) = C_T(\alpha)\, P^{\mathrm{T}}_{ij}(\mathbf{k}) + C_L(\alpha)\, P^{\mathrm{L}}_{ij}(\mathbf{k}), \tag{179}$$

and the combination analogous to Eq. (173) appears in the coefficients of the velocity–dependent interaction terms. Matching to the EIH Lagrangian demands that this combination reproduce the specific signs and magnitudes of the GR coefficients, which is only possible if α^2 is negative. The detailed calculation in Paper III picks out the unique value

$$\alpha^2 = -\frac{2}{5}, \tag{180}$$

i.e. $\lambda = 2/5$, as the one that yields the correct EIH tensor.

In the throat ontology, it is natural to interpret u_T as predominantly brane–parallel fluctuations of the fluid velocity (shear and circulation within the brane) and u_L as predominantly bulk–directed fluctuations (draining into the throat and deformation of the brane into the bulk). The condition $\alpha^2 = -2/5$ then says that the physical mode mediating 1PN vector interactions involves a particular mixture of brane–parallel and bulk–directed motion in which the bulk component enters with an opposite sign in the effective quadratic form. The space of modes relevant to the 1PN dynamics on the brane thus acquires a Lorentzian signature, with the bulk–related direction playing the role of a time–like direction in mode space.

C.5 Adding off-diagonal mixing (optional generalization)

For completeness, we note that one can generalize the toy model by adding an off–diagonal mixing term to the bare energy,

$$E[u_T, u_L] = \frac{1}{2}\left(A_T u_T^2 + 2B u_T u_L + A_L u_L^2\right), \qquad A_T > 0,\ A_L > 0,\ A_T A_L > B^2, \tag{181}$$

so that the quadratic form is still positive–definite in the original basis. Repeating the change of basis to $(u_{\text{phys}}, u_\perp)$ yields a more complicated expression for E with additional α–dependence coming from the B term, but the qualitative conclusions are unchanged:

- For real α and strictly Euclidean (A_T, A_L, B), the effective coefficient of u_{phys}^2 remains of fixed sign; no sign flip is possible.
- Allowing $\alpha^2 < 0$ introduces the possibility of an effective negative norm along u_{phys}, corresponding to an emergent Lorentzian direction in mode space.

The simple diagonal model (161) is therefore sufficient to capture the key feature: the EIH matching forces the physical combination of transverse and longitudinal modes to be time–like with respect to an effective mode–space metric, and this is encoded in the condition $\alpha^2 = -2/5$.

C.6 Summary

To summarize, the two–mode toy model shows that:

- The effective coefficient controlling the physical mode excited by moving defects is a linear combination $A_T + \alpha^2 A_L$ of the underlying transverse and longitudinal coefficients.
- With a strictly Euclidean quadratic form and real α, this combination cannot change sign; the longitudinal sector cannot enter with the opposite sign of the transverse sector.
- Allowing $\alpha^2 < 0$ makes it possible for the physical mode to have an effective negative norm, yielding an indefinite (Lorentzian) signature on mode space.
- The specific value $\alpha^2 = -2/5$ selected by the full EIH matching in Paper III can be viewed as the condition that the brane–parallel and bulk–directed components of the fluid velocity combine into a time–like mode whose induced interactions reproduce the GR 1PN vector sector.

This provides a compact algebraic underpinning for the "Lorentzian constraint" discussed in Sec. 6 and ties it directly to the brane–bulk throat ontology.

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