

ICERM FEM Multigrid Notes

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June 2020

1 Problem

$$\int_{e_j} \psi \cdot t \, dx$$

$$\text{where } \psi = \begin{bmatrix} B - Ay \\ C + Ax \end{bmatrix}, \text{ and } t = \begin{bmatrix} x_2^{(j)} - x_1^{(j)} \\ y_2^{(j)} - y_1^{(j)} \end{bmatrix} \frac{1}{|e_j|}$$

$$\frac{1}{|e_j|} \int_{e_j} \begin{bmatrix} B - Ay \\ C + Ax \end{bmatrix} \cdot \begin{bmatrix} x_2^{(j)} - x_1^{(j)} \\ y_2^{(j)} - y_1^{(j)} \end{bmatrix} dS$$

2 Parameterization

$$v(t) = \langle x_1^{(j)}, y_1^{(j)} \rangle + t \langle x_2^{(j)} - x_1^{(j)}, y_2^{(j)} - y_1^{(j)} \rangle$$

$$\begin{aligned} x &= x_1^{(j)} + t(x_2^{(j)} - x_1^{(j)}) \\ y &= y_1^{(j)} + t(y_2^{(j)} - y_1^{(j)}) \end{aligned}$$

$$dS = \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}} \, dt = \sqrt{(x_2^{(j)} - x_1^{(j)})^2 + (y_2^{(j)} - y_1^{(j)})^2} \, dt = |e_j| dt$$

3 Integration

$$\int_{e_j} \begin{bmatrix} B - A(y_1^{(j)} + t(y_2^{(j)} - y_1^{(j)})) \\ C + A(x_1^{(j)} + t(x_2^{(j)} - x_1^{(j)})) \end{bmatrix} \cdot \begin{bmatrix} x_2^{(j)} - x_1^{(j)} \\ y_2^{(j)} - y_1^{(j)} \end{bmatrix} dt$$

$$\begin{aligned}
&= \int_{e_j} (B(x_2^{(j)} - x_1^{(j)}) - A(y_1^{(j)})(x_2^{(j)} - x_1^{(j)}) - At(x_2^{(j)} - x_1^{(j)})(y_2^{(j)} - y_1^{(j)}) \\
&\quad + C(y_2^{(j)} - y_1^{(j)}) + A(x_1^{(j)})(y_2^{(j)} - y_1^{(j)}) + At(x_2^{(j)} - x_1^{(j)})(y_2^{(j)} - y_1^{(j)}))dt \\
&= \int_{e_j} (A(x_1^{(j)})(y_2^{(j)} - y_1^{(j)}) - A(y_1^{(j)})(x_2^{(j)} - x_1^{(j)}) + B(x_2^{(j)} - x_1^{(j)}) + C(y_2^{(j)} - y_1^{(j)}))dt \\
&= \int_{e_j} (A(x_1^{(j)}y_2^{(j)} - y_1^{(j)}x_2^{(j)}) + B(x_2^{(j)} - x_1^{(j)}) + C(y_2^{(j)} - y_1^{(j)}))dt \\
&= [At(x_1^{(j)}y_2^{(j)} - y_1^{(j)}x_2^{(j)}) + Bt(x_2^{(j)} - x_1^{(j)}) + Ct(y_2^{(j)} - y_1^{(j)})]_0^1 \\
&= [A(x_1^{(j)}y_2^{(j)} - y_1^{(j)}x_2^{(j)}) + B(x_2^{(j)} - x_1^{(j)}) + C(y_2^{(j)} - y_1^{(j)})]
\end{aligned}$$

4 Results for $j = 1, 2, 3$

$$A \begin{bmatrix} x_1^{(1)}y_2^{(1)} + y_1^{(1)}x_2^{(1)} \\ x_1^{(2)}y_2^{(2)} + y_1^{(2)}x_2^{(2)} \\ x_1^{(3)}y_2^{(3)} + y_1^{(3)}x_2^{(3)} \end{bmatrix} + B \begin{bmatrix} x_2^{(1)} - x_1^{(1)} \\ x_2^{(2)} - x_1^{(2)} \\ x_2^{(3)} - x_1^{(3)} \end{bmatrix} + C \begin{bmatrix} y_2^{(1)} - y_1^{(1)} \\ y_2^{(2)} - y_1^{(2)} \\ y_2^{(3)} - y_1^{(3)} \end{bmatrix}$$