Discrete Mixture Models

1 Introduction

Probability mass function for first bonomial:

$$\phi_1(k_1(i)) = p(k_1(i)|p) = \binom{n}{k_1(i)} p^{k_1(i)}(1-p)^{n-k_1(i)}$$

Probability mass function for second binomial:

$$\phi_0(k_1(i)) = p(k_1(i)|q) = \binom{n}{k_1(i)} q^{k_1(i)}(1-q)^{n-k_1(i)}$$

Probability mass function for mixture model:

$$\begin{aligned} p(k_1(1), \cdots, k_1(N)|\alpha, p, q) &= \prod_{i=1}^{N} [\alpha\phi_1(k_1(i)) + (1-\alpha)\phi_0(k_1(i))] \\ &= \prod_{i=1}^{N} \left[\alpha\binom{n}{k_1(i)} p^{k_1(i)}(1-p)^{n-k_1(i)} + (1-\alpha)\binom{n}{k_1(i)} q^{k_1(i)}(1-q)^{n-k_1(i)}\right] \end{aligned}$$

2 FIM and CRLB

3 Maximum Likelihood and Expectation-Maximization

3.1 Maximum Likelihood Equations

Want to do: $\frac{\delta \log p}{\delta\theta} = \mathbf{0}$

$$\log p(k_1(1), \cdots, k_1(N)|\alpha, p, q) = \sum_{i=1}^{N} \log\left[\alpha\binom{n}{k_1(i)} p^{k_1(i)}(1-p)^{n-k_1(i)} + (1-\alpha)\binom{n}{k_1(i)} q^{k_1(i)}(1-q)^{n-k_1(i)}\right]$$

$$\frac{\delta \log p}{\delta\theta} = \begin{bmatrix} \sum_{i=1}^{N} \frac{\phi_1(k_1(i))-\phi_0(k_1(i))}{\alpha\phi_1(k_1(i))+(1-\alpha)\phi_0(k_1(i))} \\ \sum_{i=1}^{N} \frac{\alpha\binom{n}{k_1(i)}\left(k_1(i)p^{k_1(i)-1}(1-p)^{n-k_1(i)} - p^{k_1(i)}(n-k_1(i))(1-p)^{n-k_1(i)-1}\right)}{\alpha\phi_1(k_1(i))+(1-\alpha)\phi_0(k_1(i))} \\ \sum_{i=1}^{N} \frac{(1-\alpha)\binom{n}{k_1(i)}\left(k_1(i)q^{k_1(i)-1}(1-q)^{n-k_1(i)} - q^{k_1(i)}(n-k_1(i))(1-q)^{n-k_1(i)-1}\right)}{\alpha\phi_1(k_1(i))+(1-\alpha)\phi_0(k_1(i))} \end{bmatrix} = \mathbf{0}$$

3.2 EM Algorithm

Let $k_1(1), \cdots, k_1(N)$ be the observed data. We introduce membership variables $y(i), \cdots, y(N)$ (hidden data) such that $p(y(i) = c) = \alpha_c$. Because we only have two clases, then

$$
\begin{aligned}
&p(y(i) = 1) = \alpha \\
&p(y(i) = 0) = 1 - p(y(i) = 1) = 1 - \alpha
\end{aligned}
$$

E-step is given by:

$$
p(y(i) = 1|k_1(i), \theta) = \frac{\alpha\binom{n}{k_1(i)}p^{k_1(i)}(1-p)^{n-k_1(i)}}{\alpha\binom{n}{k_1(i)}p^{k_1(i)}(1-p)^{n-k_1(i)} + (1-\alpha)\binom{n}{k_1(i)}q^{k_1(i)}(1-q)^{n-k_1(i)}}
$$

$$
p(y(i) = 0|k_1(i), \theta) = \frac{(1-\alpha)\binom{n}{k_1(i)}q^{k_1(i)}(1-q)^{n-k_1(i)}}{\alpha\binom{n}{k_1(i)}p^{k_1(i)}(1-p)^{n-k_1(i)} + (1-\alpha)\binom{n}{k_1(i)}q^{k_1(i)}(1-q)^{n-k_1(i)}}
$$

M-step:
$\alpha_1^{k+1} = \alpha^{k+1} = \frac{1}{N}\sum_{i=1}^{N} p(y(i) = 1|k_1(i), \theta^{\mathbf{k}})$
$\alpha_0^{k+1} = 1 - \alpha_1^{k+1} = 1 - \alpha^{k+1}$

4 Method of Moments