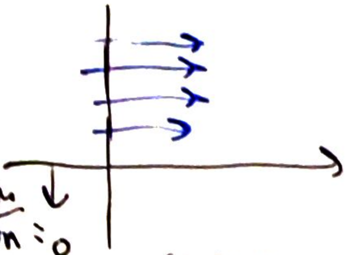
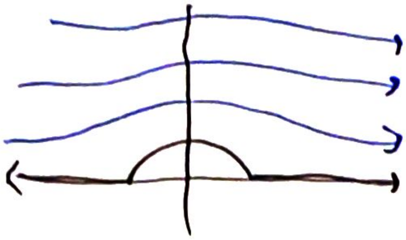
Week 5

Neumann problem (standard)



$\frac{\partial u}{\partial n} = 0$ ("$\nabla u \cdot \hat{n}$")

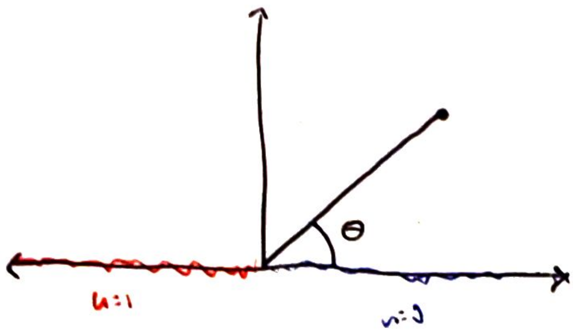
$f(z) = \alpha z$



$F = f \circ \left(z + \frac{1}{z}\right)$

Dirichlet Problem

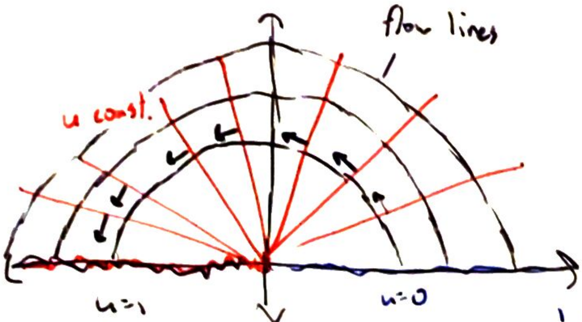
u harmonic on upper half plane



$f(z) = \frac{1}{\pi i} \log(z)$

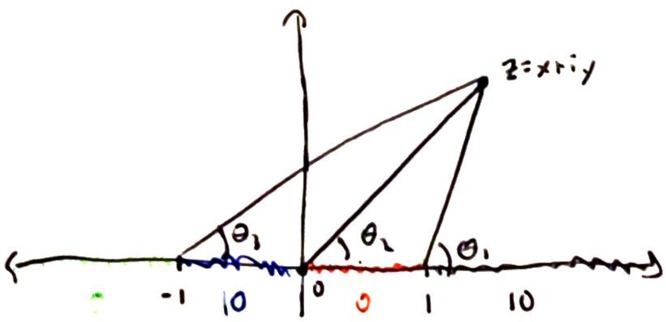
$u = \frac{1}{\pi}\theta = \mathrm{Re}(f)$ is harmonic

$u(x,0) = \begin{cases} 0 & x > 0 \\ 1 & x < 0 \end{cases}$



$u = \frac{1}{\pi}\theta$

This solution is useful because u is bounded



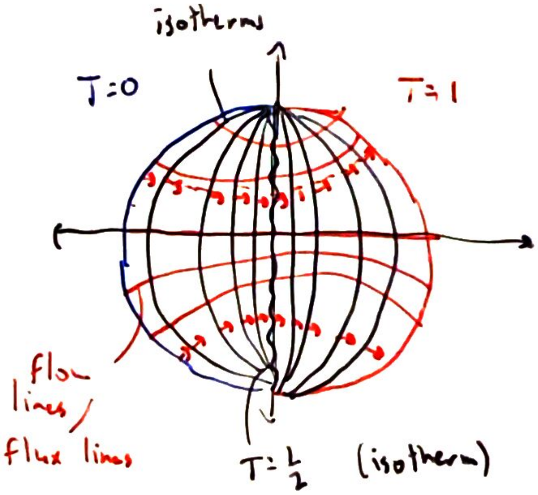
$u = 10 + \frac{-10}{\pi}\arctan\left(\frac{y}{x-1}\right) + \frac{10}{\pi}\arctan\left(\frac{y}{x}\right) + \frac{-10}{\pi}\arctan\left(\frac{y}{x+1}\right)$

more generally for c_0 c_1 c_2 c_3 we have

$u = c_3 + \left(\frac{c_2 - c_3}{\pi}\right)\theta_3 + \left(\frac{c_1 - c_2}{\pi}\right)\theta_2 + \frac{c_0 - c_1}{\pi}\theta_1$

Standard upper half plane solution

$u = \mathrm{Re}(f)$ $\quad f = c_3 + \frac{c_2 - c_3}{\pi i}\log(z-c) + \frac{c_1 - c_2}{\pi i}\log(z-b) + \frac{c_0 - c_1}{\pi}\log(z-a)$



Temperature = scalar

heat flow

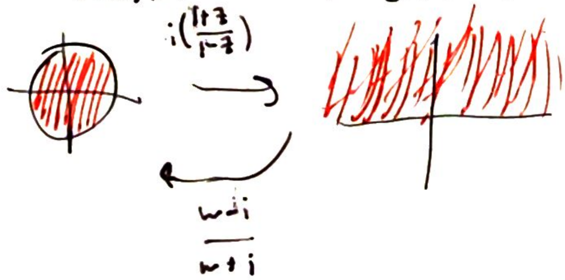
grad T = vector field

(points in direction of greatest change)

Find the function T describing this picture

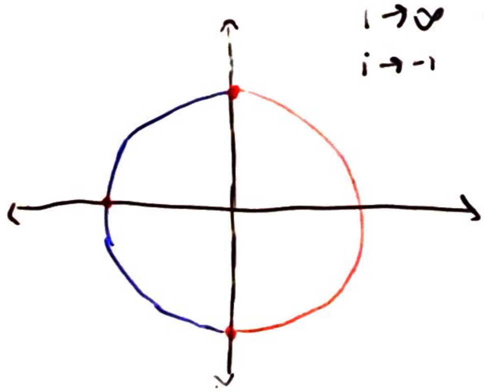
harmonic T $(\nabla^2 T = 0)$ or $T = \mathrm{Re}(f)$ for f analytic

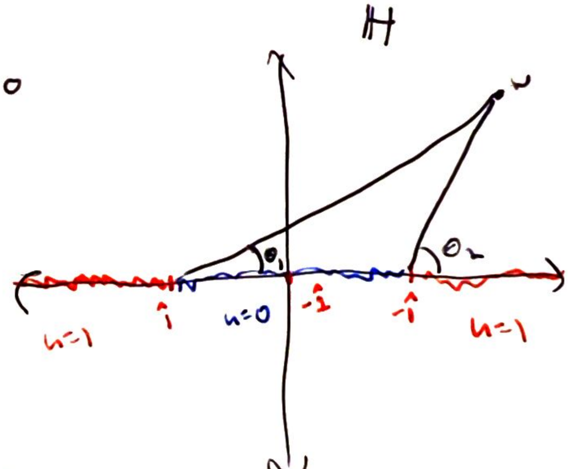
T satisfies boundary conditions



$1 \to \infty \quad -1 \to 0$

$i \to -1$





standard solution in upper half plane

$$u(w) = 1 - \frac{1}{\pi}\theta_2 + \frac{1}{\pi}\theta_1$$

$$f(z) = 1 - \frac{1}{\pi i}\log(w-1) + \frac{1}{\pi i}\log(w+1)$$

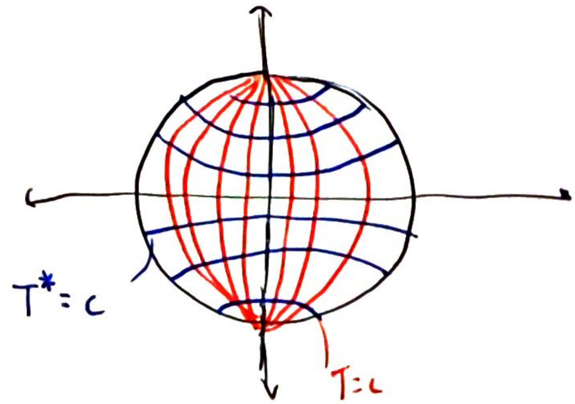
f analytic so u is harmonic

f is the complex potential on $\mathbb{H}$

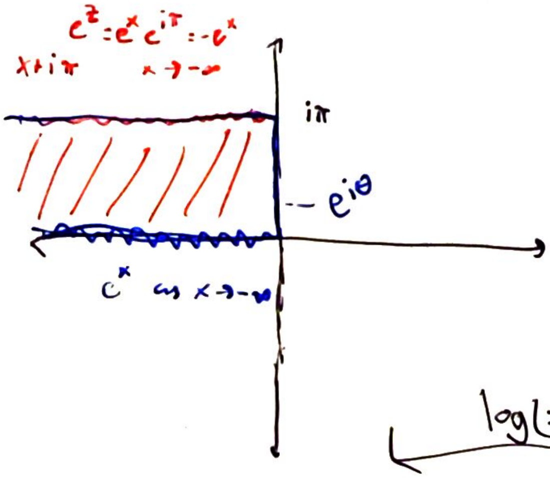
$$\mathbb{D} \xrightarrow[i\left(\frac{1+z}{1-z}\right)]{} \mathbb{H} \xrightarrow[f(z)]{} \text{complex potential}$$

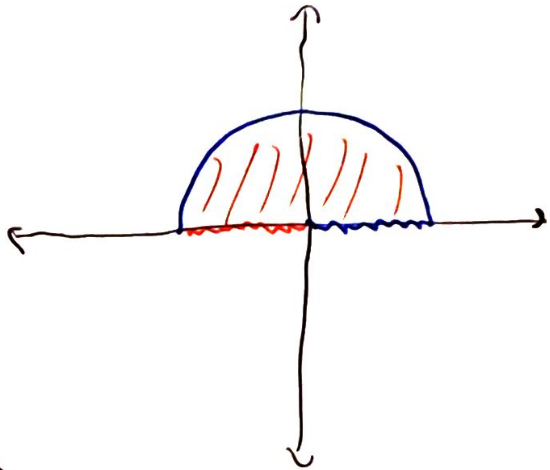
so $\hat{f} = f\left(i\left(\frac{1+z}{1-z}\right)\right)$ then $T = Re(\hat{f})$

$$T = Re\left[1 - \frac{1}{\pi i}\log\left(i\left(\frac{1+z}{1-z}\right) - 1\right) + \frac{1}{\pi i}\log\left(i\left(\frac{1+z}{1-z} + 1\right)\right)\right]$$



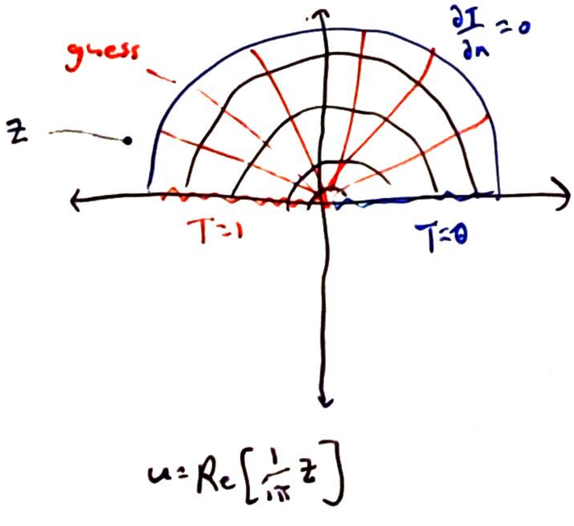
$T^* = Im(\hat{f})$

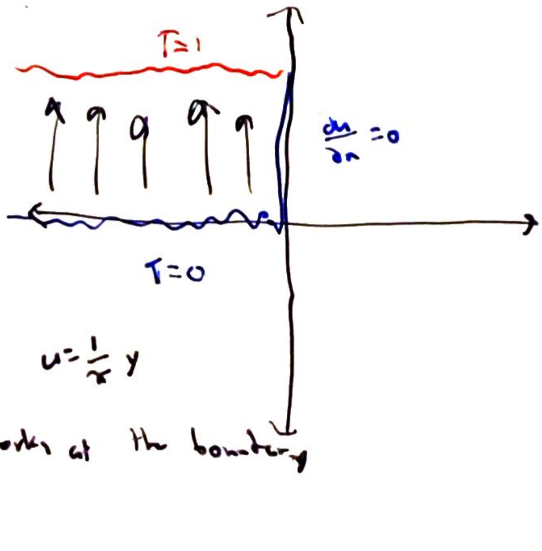




$\log(z)$

mixed boundary value problem





$u = \frac{1}{\pi} y$

u works at the boundary

$u = Re\left[\frac{1}{i\pi} z\right]$

$\hat{f} = \frac{1}{i\pi}\log(z)$

$T = \frac{1}{\pi}\arctan\left(\frac{y}{x}\right)$