Unit 2: The road to the zeta function ## Week 2 pt 2

- interpolation
- gamma function
- analytic continuation
- zeta function
- $1+2+3+\ldots = -\frac{1}{12}$

Interpolation

Given initial data $\lambda_1, \ldots \lambda_n$

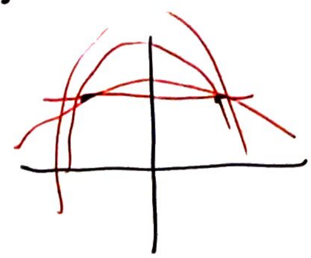
and target data $z_1, \ldots, z_n$

we want a function f such that

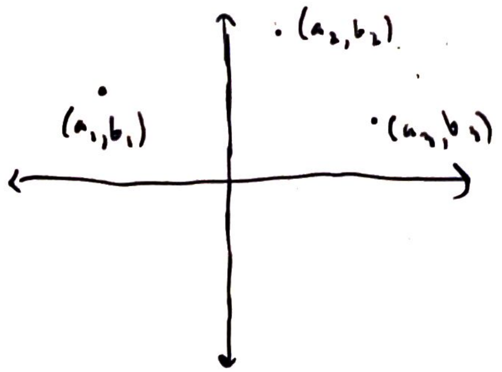
$$f(\lambda_i) = z_i \quad \text{for } i=1,\ldots,n$$

specifically we want a continuous, differentiable bounded function

if $f(\lambda)$ does this, it is called an interpolating function



classic 3-point problem:



how do we find an interpolating function?

lets try a 2^{nd}-order polynomial: $f(\lambda) = c_0 + c_1\lambda + c_2\lambda^2$

$b_1 = c_0 + c_1 a_1 + c_2 a_1^2$

$b_2 = c_0 + c_1 a_2 + c_2 a_2^2$

$b_3 = c_0 + c_1 b_3 + c_3 b_3$

elbow-greese version

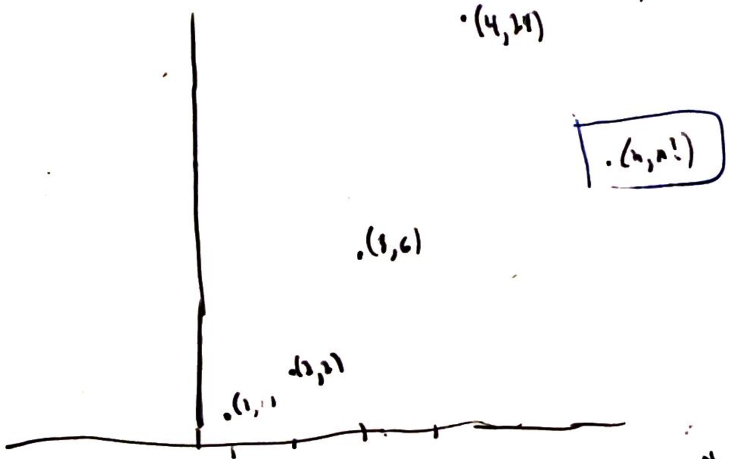
Lagrange interpolation

lets find $f_{a_1}(a_1) = 1$ and $f_{a_1}(a_2) = f_{a_1}(a_3) = 0$

$$\Rightarrow \quad \frac{(x-a_2)(x-a_3)}{(a_1-a_2)(a_1-a_3)} = f_{a_1}$$

so we can construct an interpolating function out of these guys

$$f(x) = \frac{(x-a_2)(x-a_3)}{(a_1-a_2)(a_1-a_3)} b_1 + \frac{(x-a_1)(x-a_3)}{(a_2-a_1)(a_2-a_3)} b_2 + \frac{(x-a_1)(x-a_2)}{(a_3-a_1)(a_3-a_2)} b_3$$



lets call it

$\gamma(n) = n!$

interpolate the factorials, continuous

lets think about $\int_0^\infty t e^{-t}$ $\qquad \lim_{N\to\infty} \int_0^N t e^{-t}\,dt$

$u = t \quad dv = e^{-t}$

$du = 1 \quad v = -e^{-t}$

$$\lim_{N\to\infty} \cancel{-te^{-t}}^{\,0} + \int_0^N e^{-t}\,dt = 1$$

$$\lim_{N\to\infty} \int_0^\infty t^2 e^{-t}\,dt = \lim_{N\to\infty} \left[-t^2 e^{-t} \Big|_0^N + \underbrace{2\int_0 t e^{-t}\,dt}_{=2\cdot(1)} \right]$$

so lets call $\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\,dt$

so $\Gamma(n) = (n-1)!$ $\qquad$ so $\Gamma(n+1) = n\Gamma(n)$ $\quad$ this can be made to be $x \geq 0$

what about $\Gamma(x)$ for $\boxed{x \geq 1}$ just a real number

so lets throw some complex numbers into them

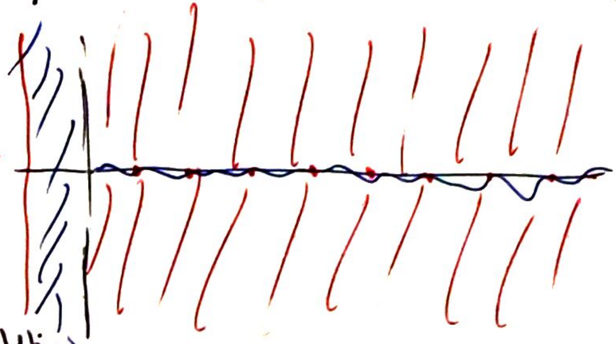
$$\boxed{\begin{aligned} \Gamma(z) &= \int_0^\infty t^{z-1} e^{-t}\,dt \\ \Gamma(x+iy) &= \int_0^\infty t^{x+iy-1} e^{-t}\,dt = \int_0^\infty t^{x-1} t^{iy} e^{-t}\,dt \end{aligned}}$$

so this guy converges when $x \geq 1$

we have a nice analog of the factorial from the reals

$\Gamma(z+1) = z\Gamma(z)$

we're good on everywhere over to the right



analytic continuation

↳ extending the domain of analytic functions

we can just define $\Gamma(z)$ here by using the fact that $\Gamma(z+1) = z\Gamma(z)$

so you could say something stupid like $(-2+3i)!$ is something