Week 3 #

we currently have $\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}$ charge conservation

$$\frac{\partial u}{\partial t} = -\vec{\nabla}\cdot\vec{S} \quad \text{(for energy)}$$

u → energy density

$\vec{S}$ → Poynting vector

Phys 141: $\underbrace{\vec{F}_{1 on 2} = -\vec{F}_{2 on 1}}$

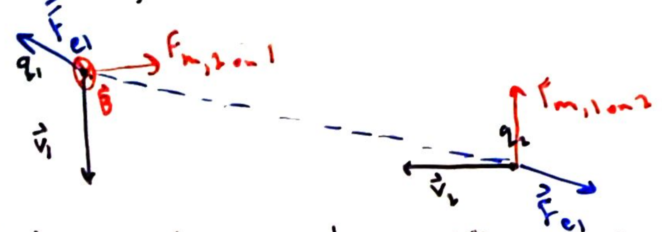
⇓

total momentum is conserved

$$\vec{F}_{1 on 2} = \frac{\partial \vec{p}_2}{\partial t} \qquad \vec{F}_{2 on 1} = \frac{\partial \vec{p}_1}{\partial t}$$

$$\frac{\partial}{\partial t}(\vec{p}_1 + \vec{p}_2) = \vec{F}_{2 on 1} + \vec{F}_{1 on 2} = 0 \quad \text{so} \quad \vec{p}_1 + \vec{p}_2 = \vec{P}_{tot} \text{ is conserved}$$

Now 2 moving charged particles (both positive)

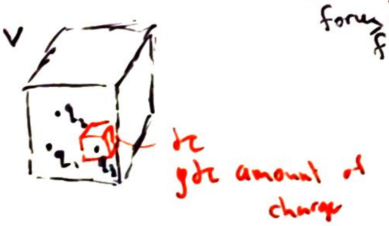


$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$$

Good news, the momentum is <u>still</u> conserved

↳ particles and fields interact

$$\vec{F} = \int_V (\vec{E} + \vec{v}\times\vec{B})\rho = \int_V \underbrace{(\rho\vec{E} + \vec{J}\times\vec{B})}_{\text{force density } \vec{f}} d\tau \qquad (\vec{v}\rho = \vec{J})$$



so $\vec{f} = \rho E + \vec{J}\times\vec{B}$

- get rid of $\rho, \vec{J}$ (express in terms of fields)
- $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ • $\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times B - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

After a lot of algebra:

$$\vec{f} = \varepsilon_0\left[(\vec{\nabla}\cdot\vec{E})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{E} + \frac{1}{\mu_0}\left[(\vec{\nabla}\cdot\vec{B})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{B}\right]\right] - \frac{1}{2}\nabla\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$\vec{f} = \vec{\nabla}\cdot\overleftrightarrow{T} - \varepsilon_0\mu_0\frac{\partial\vec{S}}{\partial t}$$

$\overleftrightarrow{T}$ → tensor

where $\overleftrightarrow{T}$ is the maxwell stress tensor

$$\overleftrightarrow{T}_{ij} = \varepsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$$

$$\vec{\nabla}\cdot\overleftrightarrow{T} = \sum_k \nabla_k T_{ki} = \sum_{k=x,y,z}\frac{\partial}{\partial x_k}\overleftrightarrow{T}_{ki} = \frac{\partial}{\partial x}\overleftrightarrow{T}_{xi} + \frac{\partial}{\partial y}\overleftrightarrow{T}_{yi} + \frac{\partial}{\partial z}\overleftrightarrow{T}_{zi}$$

note $\overleftrightarrow{T}$ is symmetric so $T_{ij} = T_{ji}$

$$\vec{F} = \underbrace{\oint \overleftrightarrow{T}\cdot d\vec{a}}_{\text{divergence theorem}} - \varepsilon_0\mu_0\frac{d}{dt}\int_V \vec{S}\, d\tau$$

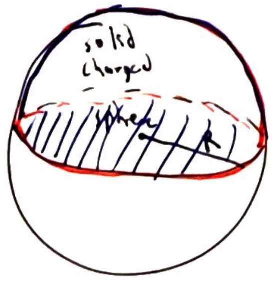
Special case: steady state

$\Rightarrow \frac{\partial}{\partial t}(\ldots)=0$

$\Rightarrow \vec{F} = \underbrace{\oint \overleftrightarrow{T} \cdot d\vec{a}}$ notice: the force on q's inside V can be found if we know E, B on the surface of V

$\frac{N}{m^2} = \frac{\text{force}}{\text{area}}$, diagonal terms represent pressure

off-diagonal terms represent shear



Find the force on the upper sphere due to the lower sphere

sphere is uniformly charged

Symmetry $\rightarrow$ only $F_z \neq 0$

$$\left(\overleftrightarrow{T} \cdot d\vec{a}\right)_z = \sum_{j=x,y,z} T_{zj}\, da_j = T_{zx}\, da_x + T_{zy}\, da_y + T_{zz}\, da_z$$

$$\left(\overleftrightarrow{T} \cdot d\vec{a}\right)_i = \sum_j T_{ij}\, da_j$$

$d\vec{a} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

Momentum conservation:

Newton's law: $\vec{F} = \frac{d\vec{p}_{mech}}{dt}$

total force due to E and B fields on all charges in volume V

$$\vec{F} = \oint_S \overleftrightarrow{T} \cdot d\vec{a} - \varepsilon_0 \mu_0 \frac{d}{dt} \int_V \vec{S}\, d\tau$$

conservation of momentum:

$$\frac{d\vec{p}_{mech}}{dt} = -\varepsilon_0 \mu_0 \frac{d}{dt} \int_V \vec{S}\, d\tau + \oint \overleftrightarrow{T} \cdot d\vec{a}$$

① ②

① Rate of change of momentum stored in the fields $\vec{p} = \mu_0 \varepsilon_0 \int_V \vec{S}\, d\tau$

② momentum per unit time flowing in (out) through the surface

the momentum density in the fields

$$\vec{g} = \mu_0 \varepsilon_0 \vec{S} = \varepsilon_0 (\vec{E} \times \vec{B})$$

special case: no charges in V $\Rightarrow \frac{d\vec{P}_{mech}}{dt} = \int \frac{\partial g}{\partial t} d\tau = \oint \overleftrightarrow{T} \cdot d\vec{a} = \int (\vec{\nabla} \cdot \overleftrightarrow{T}) d\tau$

V

no charges

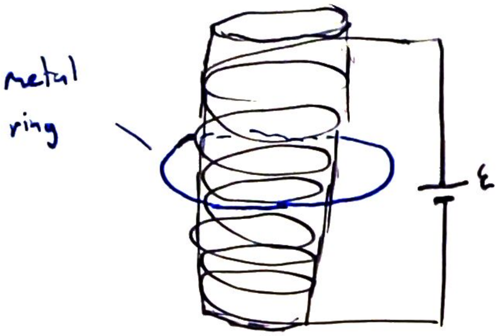
$$\Rightarrow \frac{\partial \vec{g}}{\partial t} = \vec{\nabla} \cdot \overleftrightarrow{T} \quad \text{continuity equation}$$

<u>Angular</u> <u>momentum</u>: $\vec{\ell} = \vec{r} \times \vec{g}$

↳ due to EM fields per unit volume

idea: EM can store angular momentum

Feynmann disk paradox:



I constant $\Rightarrow$ nothing happens

$I\downarrow = \frac{d\vec{B}}{dt} \neq 0$

$\Rightarrow \vec{\nabla} \times \vec{E} \neq 0 \rightarrow \vec{E} \propto \hat{\phi}$

charges inside the ring experience a torque

↳ ring rotates

Physics 141 can't explain the rotation

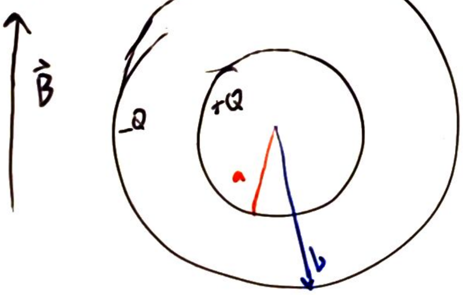
$\vec{\tau} = I\vec{\alpha}$

net torque, angular accel

but you can explain it with

<u>hw 8.9</u>

spherical shells



external B-field $\vec{B} = B_0 \hat{z}$

a) find the angular momentum $\vec{\ell}$ due to fields and total $\vec{L} = \int \vec{\ell}$

initially present

b) $B\downarrow$ gradually $\Rightarrow \vec{L}_{in\ field} \downarrow$

$\Rightarrow \vec{L}_{in\ field} \rightarrow \vec{L}_{spheres}$

check: $\vec{L}_a + \vec{L}_b = \vec{L}_{spheres} = \vec{L}_{in\ field}$

a) i) find $\vec{\ell} = \vec{r} \times \vec{g} = \varepsilon_0 [\vec{r} \times [\vec{E} \times \vec{B}]]$

ii) $\vec{E} \rightarrow$ gauss's law, $\vec{B}$ given $\quad (A \times (B \times C)) = B(A \cdot C) - C(A \cdot B)$

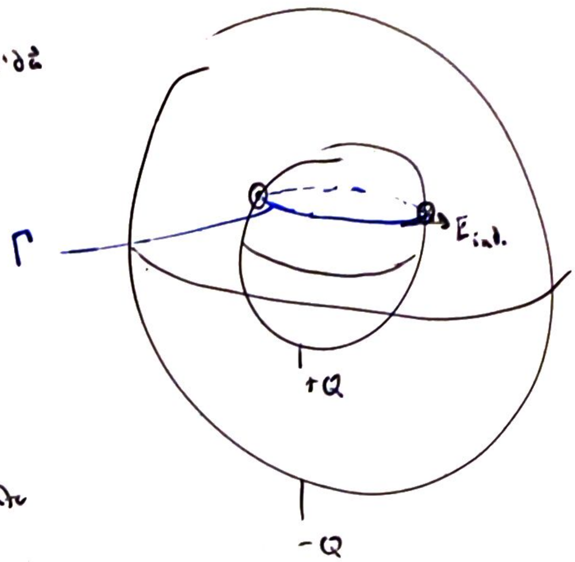
iii) $\vec{L} = \int \vec{\ell}\, d\tau \quad d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$

for L_{sphere}: $\vec{\tau} = \vec{r} \times F_{el}$. so need $\vec{E}$ due to change in $\vec{B}$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

in integral form: $\oint_{\Gamma} \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int_{S} \vec{B} \cdot d\vec{a}$

Γ (contour)



keep $-\frac{\partial B}{\partial t}$ as given in this step

then to get τ consider $d\vec{\tau} = \vec{r} \times (dq \vec{E})$

$dq = \sigma da$ for spherical da

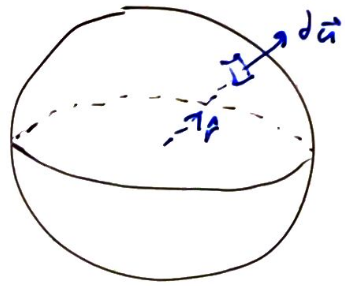
then once you find τ_{net} you integrate

$$L_{sphere} = \int_0^\infty \tau_{net} \, dt$$

Continuing a problem from last time

Solid sphere of radius R, charge Q

Find force that lower half exerts on top half



$$\vec{F} = \oint_S \overleftrightarrow{T} \cdot d\vec{a}$$

here S is surface that bounds the top 1/2 of sphere

↳ disk @ equator + upper half of ball

$d\vec{a} = \hat{r} R^2 \sin\theta \, d\theta \, d\phi$

$$(\overleftrightarrow{T})_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right)$$

→ 0 no B-field here

Symmetry ⇒ net force is upper

$F_z = \oint (\overleftrightarrow{T} \cdot d\vec{a})_z \Rightarrow (\overleftrightarrow{T} \cdot d\vec{a})_z = \sum_{x,y,z} T_{zi} \, da_i$

$\hat{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{r}$ $\quad \hat{r} \cdot \hat{x} = \frac{x}{r} = \sin\theta\cos\phi$

$= T_{zx} da_x + T_{zy} da_y + T_{zz} da_z$

$da_x = \sin\theta\cos\phi \, R^2 \sin\theta \, d\theta \, d\phi$ $\quad da_z = R^2 \cos\theta \sin\theta \, d\theta \, d\phi$

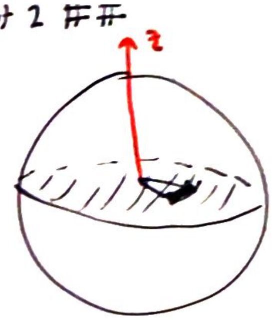
E on surface is $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$ $\quad E_x = \frac{\sin\theta\cos\phi \, Q}{4\pi\epsilon_0 r^2}$ $\quad E_z = \frac{\cos\theta \, Q}{4\pi\epsilon_0 r^2}$

$T_{zx} = \epsilon_0 E_z E_x$ $\quad T_{zy} = \epsilon_0 E_z E_y$ $\quad T_{zz} = \epsilon_0 \left(E_z^2 - \frac{1}{2}E^2\right)$

↳ algebra

$$(\overleftrightarrow{T} \cdot d\vec{a})_z = \frac{\epsilon_0}{2} \left(\frac{Q}{4\pi\epsilon_0 R}\right)^2 \sin\theta \cos\theta \, d\theta \, d\phi$$

Week 2 pt 2



the $F_{bowl} = \int (\overleftrightarrow{T} \cdot d\vec{a})_z = \frac{1}{4\pi\epsilon_0} \frac{Q^2}{8R^2}$

the disk part

only a z part of $d\vec{a}$

$d\vec{a} = s\, d\phi\, ds$

so $(\overleftrightarrow{T} \cdot d\vec{a})_z = T_{zz}\, da_z$

$T_{zz} = \frac{\epsilon_0}{2}(\cancel{E_z^2} - E_x^2 - E_y^2)$

0

b/c $\theta = 90$

$E \, 4\pi s^2 = \frac{Q s^3}{\epsilon_0 R^3} \Rightarrow E = \frac{Q s}{4\pi\epsilon_0 R^3}$

$-E_x^2 - E_y^2 = \frac{Q s}{4\pi\epsilon_0 R^3}$

$T_{zz} = -\frac{\epsilon_0}{2}\left(\frac{Q s}{4\pi\epsilon_0 R^3}\right)^2$

so $(\overleftrightarrow{T} \cdot d\vec{a})_z = \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)^2 s^2 \cdot s\, d\phi\, ds$

$F_{disk} = \int_{disk} (\overleftrightarrow{T} \cdot d\vec{a})_z = \frac{\epsilon_0 Q^2}{2(4\pi\epsilon_0 R^3)^2} \int_0^R s^3\, ds \int_0^{2\pi} d\phi$

$= \frac{1}{4\pi\epsilon_0} \frac{Q^2}{16R^2}$

So $|F_{net}| = (F_{net})_z = F_{disk} + F_{bowl} = \frac{1}{4\pi\epsilon_0} \frac{3Q}{16R^2}$

Alternatively $\vec{F} = \int d\vec{F} = \int E\, dq = \int_V \vec{E} \rho\, d\tau$

$E = \frac{Q s}{4\pi\epsilon_0 R^3} \hat{s}$ $\quad E_z = \frac{Q s \cos\theta}{4\pi\epsilon_0 R^3}$

$d\tau = s^2 \sin\theta\, ds\, d\theta\, d\phi$

do this for extra credit