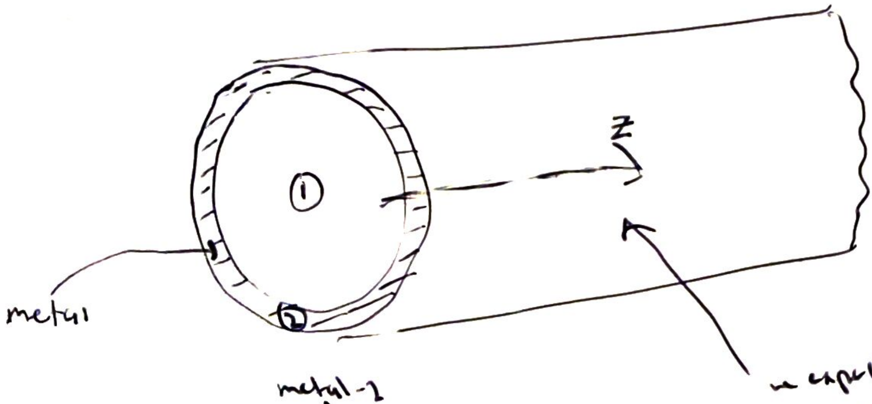
Week 8



we expect translational invariance along the z axis

trial solution: $\vec{E}(x,y,z,t) = \tilde{\vec{E}}_0(x,y)\, e^{i(kz-\omega t)}$ (*)

$\underbrace{\tilde{\vec{E}}_0(x,y)}_{\text{no dependance on } z}$; $e^{i(kz-\omega t)}$ ← propogation direction (along z)

$$\tilde{\vec{B}}(x,y,z,t) = \tilde{\vec{B}}_0(x,y)\, e^{i(kz-\omega t)} \quad (**)$$

$$\boxed{\begin{array}{l}\text{B.C. on metal surface} \\ \vec{E}^{\parallel} = 0 \;;\quad B^{\perp} = 0\end{array}}$$

Also maxwell's equations (medium 1)

1. $\vec{\nabla}\cdot\vec{E} = 0$
2. $\vec{\nabla}\cdot\vec{B} = 0$
3. $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$
4. $\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$

Goal: use maxwell's equations + BC's to find the wavenumber, $\tilde{\vec{E}}_0(x,y)$ and $\tilde{\vec{B}}_0(x,y)$

Sub. in equation (3) → find 3 differential equations for x, y, z components

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_{0,x}(x,y)e^{i(kz-\omega t)} & E_{0,y}e^{i(kz-\omega t)} & E_{0,z}e^{i(kz-\omega t)} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial \tilde{E}_{0,z}}{\partial y} - ikE_{0,y}\right)e^{i(kz-\omega t)} - \hat{j}\left(\frac{\partial \tilde{E}_{0,z}}{\partial x} - \tilde{E}_{0,x}ik\right)e^{i(kz-\omega t)}$$

$$+\hat{k}\left(\frac{\partial \tilde{E}_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y}\right)e^{i(kz-\omega t)}$$

and $\frac{\partial \vec{B}}{\partial t} = -i\omega\,\tilde{B}_0(x,y)\,e^{i(kz-\omega t)}$

so we get

$$\partial_x E_y - \partial_y E_x = i\omega B_z \qquad (a)$$

$$\partial_y E_z - ikE_y = i\omega B_x \qquad (b)$$

$$ikE_x - \partial_x E_z = i\omega B_y \qquad (c)$$

from ④ we get $\partial_x B_y - \partial_y B_x = -\frac{i\omega}{c^2} E_z$ (a)

$\partial_y B_z - ikB_y = -\frac{i\omega}{c^2} E_x$ (b)

$ikB_x - \partial_x B_z = -\frac{i\omega}{c^2} E_y$ (c)

solve for E_x, E_y, B_x, B_y in terms of E_z, B_z

④b $\Rightarrow$ $-i\frac{\omega}{c^2} E_x = \partial_y B_z - ikB_y$

③c $B_y = \frac{1}{i\omega}(ikE_x - \partial_x E_z)$

$$\boxed{E_x = \frac{i}{(\omega/c)^2 - k^2}\left(\omega \frac{\partial B_z}{\partial y} + k\frac{\partial E_z}{\partial x}\right)} \quad ⑤$$

similarly, $E_y = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left[k\frac{\partial}{\partial y} E_z - \omega\frac{\partial}{\partial x} B_z\right]$ ⑥

$$B_x = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left[k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right] \quad ⑦$$

$$B_y = \frac{i}{(\frac{\omega}{c})^2 - k^2}\left[k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right] \quad ⑧$$

Terminology: TE = transverse electric wave $\Rightarrow$ $E_{0,z} = 0$

TM = transverse magnetic wave $\Rightarrow$ $B_{0,z} = 0$

TEM = both E and B waves are transverse $\Rightarrow$ $E_{0,z} = B_{0,z} = 0$

Is it possible to send a perfectly transverse electric and magnetic wave along the waveguide? → No!

① $\vec{\nabla}\cdot\vec{E} = 0$ $\partial_x E_x + \partial_y E_y + \partial_z E_z = 0$

$\partial_x ⑤ + \partial_y ⑥ + ikE_z = 0$ one line of algebra

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] E_z = 0 \quad ⑨$$

and $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] B_z = 0$ ⑩

Next apply B.C. → TEM wave → then $E_z = 0$; $B_z = 0$

then Gauss's law gives us $\partial_x E_x + \partial_y E_y = 0$ $\vec{\nabla}_\perp \cdot \vec{E}_\perp = 0$

the $B_z = 0 \Rightarrow \partial_y E_x - \partial_x E_y = 0$ $\qquad \vec{\nabla}_\perp \times \vec{E}_\perp = 0$

Rem. satisfied if $\vec{E}_\perp = -\nabla_\perp V$

$\Rightarrow -\nabla_\perp^2 V = 0$ (Laplace's equation)

if TEM → $\nabla_\perp^2 V = 0$ has no minima or maxima

but B.C. tell us that $E^{\parallel} = 0$ (E must be ⊥ to metal)

↳ metal surface is equipotential

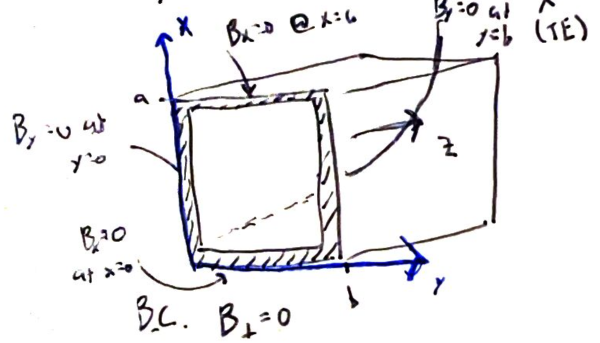
↳ V must be a constant inside the waveguide

↳ $\vec{E} = 0 = \vec{\nabla} V$

and $B_x = 0$, and $B_y = 0$

so we can't have TEM in a hollow waveguide

Today: transverse electric wave in a rectangular metal waveguide (TE)



(10) need to solve for $B_z(x,y)$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] B_z = 0$$

Subject to B.C. $B_\perp = 0$ on boundaries and solution satisfying maxwell's eq. (1, 2, 3, 4)

Separation of variables: $B_z(x,y) = X(x)\,Y(y)$

$$\Rightarrow \underbrace{\frac{1}{X}\frac{d^2 X}{dx^2}}_{=\text{const.} \,\text{"}-k_x^2\text{"}} + \underbrace{\frac{1}{Y}\frac{d^2 Y}{dy^2}}_{=\text{const.}\,\text{"}-k_y^2\text{"}} + \left(\frac{\omega}{c}\right)^2 - k^2 = 0$$

$$\Rightarrow -k_x^2 - k_y^2 + \left(\frac{\omega}{c}\right)^2 - k^2 = 0$$

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2 - k_y^2}$$ BTW this is k_z

note $k_x = k_y = 0 \rightarrow k = \frac{\omega}{c}$ (plane wave)

At $x=0$ and $x=a$, $B_x = 0$

from eq (7) $\quad B_x = \dfrac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left[k\dfrac{\partial B_z}{\partial x} - \dfrac{\omega}{c^2}\cancel{\dfrac{\partial E_z}{\partial y}}\right]$ $=0$ because $\vec{E}$ is transverse

↳ $\dfrac{\partial B_z}{\partial x} = 0$ on boundaries as well ($x=0$, $x=a$)

$\sim \quad \frac{\partial (XY)}{\partial x} = 0 \implies YX' = 0$

$Y = 0$ is impossible b/c it would mean $B_z = 0$

so $X' = 0 = \frac{\partial X}{\partial x}$ { remember from * $\frac{\partial^2 X}{\partial x^2} = -k_x^2 X$

$\hookrightarrow X(x) = A\sin(k_x x) + B\cos(k_x x)$ so $\frac{\partial X}{\partial x} = k_x A\cos(k_x x) + k_x B\sin(k_x x)$

$\implies A = 0$ $\left(\frac{\partial x}{\partial x} = 0 \text{ @ } 0\right)$ $\implies k_x = \frac{m\pi}{a}$ $\left(\frac{\partial X}{\partial x} = 0 \text{ @ } x = a\right)$

$m = 0, 1, 2, \ldots$

Similarly, $Y(y) = C\sin(k_y y) + D\cos(k_y y)$

$B_y(y=0) = 0$ and $B_y(y=b) = 0 \implies \frac{\partial Y}{\partial y} = 0$ @ boundaries

$\implies C = 0 \implies k_y = \frac{n\pi}{b}$

$B_{0,z}(x,y) = \underbrace{BD}_{B_0} \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$

Full solution $B_z(x,y,z,t) = B_0 \underbrace{\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)}_{\text{standing waves}} \underbrace{e^{i(kz-\omega t)}}_{\text{travelling part}}$

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

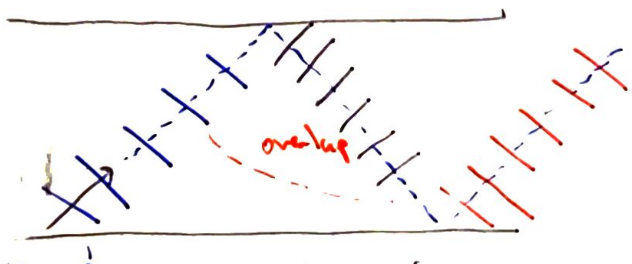
$m = n = 0$ is not allowed $\rightarrow k = \frac{\omega}{c} \rightarrow B_z = 0$ TEM not allowed

k is imaginary if $\frac{\omega}{c} < \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}\,\pi$

or if $\omega < \omega_{mn}$ $\hookrightarrow$ cutoff frequency $\omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$

you send down the waveguide, it will not excite modes higher than ω_{mn}

lowest possible ω for a given waveguide $\omega_{mn} = \frac{c\pi}{a}$ ↖ largest dimension



blue ; black overlap to form standing waves

blue ; red overlap to transmit the wave the down the waveguide