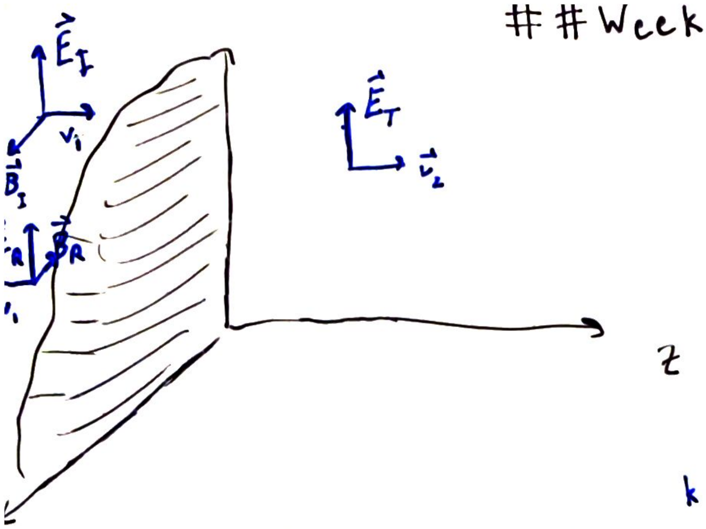
Week 6



Normal incitence

$\vec{E}_I$-incident

$\omega_1 = \omega_2 = \omega$

But $k_1 = \frac{\omega_1}{v_1} = \frac{\omega}{v_1}$

$k_2 = \frac{\omega_2}{v_2} = \frac{\omega}{v_2}$

$k_2 = \frac{n_2}{n_1} k_1$

$v_1 = \frac{c}{n_1}$; $v_2 = \frac{c}{n_2}$

Goal: find any phase shift and energy distribution amoung the waves

$$\tilde{\vec{E}}_I(z,t) = \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{x} \qquad (1)$$

direction of prop

our choice

$$\tilde{\vec{B}}_I = \frac{1}{v_1}(\hat{k} \times \tilde{\vec{E}}_1) = \frac{1}{v_1}(\hat{z} \times \hat{x}) \tilde{E}_I = \frac{\hat{y}}{v_1} \tilde{E}_I$$

$$\tilde{\vec{B}}_I(z,t) = \frac{\tilde{E}_{0I}}{v_1} e^{i(k_1 z - \omega t)} \hat{y} \qquad (2)$$

similarly, $$\tilde{\vec{E}}_R(z,t) = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{x} \qquad (3)$$

$$\tilde{\vec{B}}_R(z,t) = \frac{\tilde{E}_{0R}}{v_1} e^{i(-k_1 z - \omega t)} (-\hat{y}) \qquad (4)$$

$$\tilde{\vec{E}}_T(z,t) = \tilde{E}_{0T} e^{i(k_2 z - \omega t)} \hat{x} \qquad (5)$$

$$\tilde{\vec{B}}_T(z,t) = \frac{\tilde{E}_{0T}}{v_2} e^{i(k_2 z - \omega t)} \hat{y} \qquad (6)$$

Apply the boundary conditions

(A) $\hat{n}$ no (trivially satisfied)

(B) $\tilde{E}_{0I} \hat{x} + \tilde{E}_{0R} \hat{x} = \tilde{E}_{0T} \hat{x}$

(C) $\frac{\tilde{E}_{0I}}{\mu_1 v_1} \hat{y} - \frac{\tilde{E}_{0R}}{\mu_1 v_1} \hat{y} = \frac{\tilde{E}_{0T}}{\mu_2 v_2} y$

(D) $0 = 0$ $\hat{n}$

(A) $\epsilon_1 \tilde{E}_{\perp,1} = \epsilon_2 \tilde{E}_{\perp,2}$

(B) $\tilde{E}_{\parallel,1} = \tilde{E}_{\parallel,2}$

(C) $\frac{1}{\mu_1} \tilde{B}_{\parallel,1} = \frac{1}{\mu_2} \tilde{B}_{\parallel,2}$

(D) $\tilde{B}_{\perp,1} = \tilde{B}_{\perp,2}$

so $\boxed{\frac{\mu_2 v_2}{\mu_1 v_1}} (\tilde{E}_{0I} - \tilde{E}_{0R}) = \tilde{E}_{0T}$

$\frac{1}{\beta}$

$$\boxed{\frac{1}{\beta}(E_{0I} - E_{0R}) = E_{0T}}$$

$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$

$$E_{0R} = \left(\frac{1-\beta}{1+\beta}\right) E_{0I}$$

$$E_{0T} = \left(\frac{2}{1+\beta}\right) E_{0I}$$

with complex phases

$$E_{0R} e^{i\delta_R} = \frac{1-\beta}{1+\beta} e^{i\delta_I}$$

if $\beta < 1$ $\quad \delta_R = \delta_I$

if $\beta > 1$ $\quad \delta_R = \delta_I + \pi$

if $\beta < 1$, $\frac{1-\beta}{1+\beta} > 0$

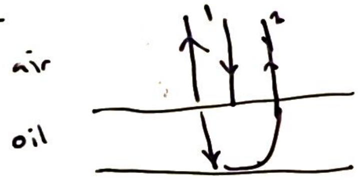
if $\beta > 1$ $\quad \frac{1-\beta}{1+\beta} < 0$

$$\beta \equiv \frac{\mu_1 n_2}{\mu_2 n_1}$$

for most media $\mu_1 = \mu_2$ approx.

then $\beta \simeq \frac{n_2}{n_1} = \frac{v_1}{v_2}$

in physics II



1 is shifted by π in phase compared with 2

No phase shift between incident and transmitted waves

$$\Rightarrow E_{0R} = \left|\frac{1-\beta}{1+\beta}\right| E_{0I} \quad \text{for amplitudes}$$

$$E_{0T} = \frac{2}{1+\beta} E_{0I}$$

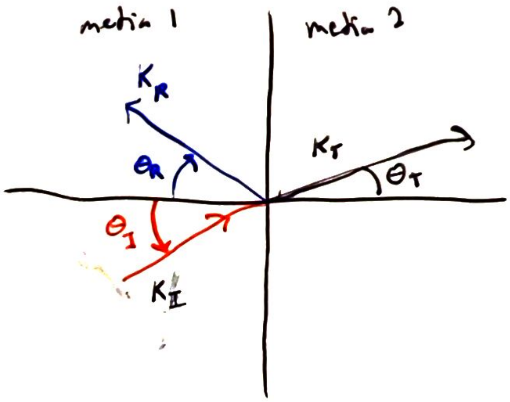
when $\mu_1 \approx \mu_2$

$$\Rightarrow \quad E_{0R} = \left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{0I}$$

$$E_{0T} = \frac{2n_1}{n_1+n_2} E_{0I}$$

if $n_1 \approx n_2$ there is basically no reflection

Oblique Incidence



incoming hits boundary at arbitrary angle θ_I

↳ if $\theta_I = 0$ → get normal incidence

$\vec{E}_I = \vec{E}_{0I}\, e^{i(\vec{k}_I\cdot\vec{r} - \omega t)}$

$\vec{B}_I = \frac{1}{v_1}(\hat{k}_I \times \vec{E}_I)$

$\vec{E}_R = \vec{E}_{0R}\, e^{i(\vec{k}_R\cdot\vec{r} - \omega t)}$

$\vec{B}_R = \frac{1}{v_1}(\hat{k}_R \times \vec{E}_R)$

$\vec{E}_T = \vec{E}_{0T}\, e^{i(\vec{k}_T\cdot\vec{r} - \omega t)}$

$\vec{B}_T = \frac{1}{v_2}(\hat{k}_T \times \vec{E}_T)$

frequency is determined by the source → doesn't change

↳ all waves have same ω

→ $k_1 v_1 = k_2 v_2$

→ $k_I = k_R = k_T \left(\frac{n_1}{n_2}\right)$

combine the fields at the boundary ($z=0$)

$$\left(\quad\right)_{\text{amp } I} e^{i(\vec{k}_I\cdot\vec{r} - \omega t)} + \left(\quad\right)_{\text{amp } R} e^{i(\vec{k}_R\cdot\vec{r} - \omega t)} = \left(\quad\right)_{\text{amp } T} e^{i(\vec{k}_T\cdot\vec{r} - \omega t)}$$

at $\underline{z=0}$

Since the x, y, t dependance at the boundary is in the exponential. You can remove the t-dependance as well, because all of this happens at the same time

You balance the exponentials, so

$$\vec{k}_I\cdot\vec{r} = \vec{k}_R\cdot\vec{r} = \vec{k}_T\cdot\vec{r}$$

$$k_{I,x}x + k_{I,y}y = k_{R,x}x + k_{R,y}y = k_{T,x}x + k_{T,y}y$$

then $k_{I,y} = k_{R,y} = k_{T,y} = 0$

That means

$\vec{k}_I, \vec{k}_R, \vec{k}_T$ are all coplanar (1st law)

This means that $k_{I,x} = k_{R,x} = k_{T,x}$

so $k_I \sin\theta_I = k_R \sin\theta_R$

since $k_I = k_R \Rightarrow \sin\theta_I = \sin\theta_R$ (Law of reflection)

and $k_{I,x} = k_{T,x} \Rightarrow k_I \sin\theta_I = k_T \sin\theta_T$

$\Rightarrow n_2 \sin\theta_T = n_1 \sin\theta_I$ (Snell's Law)

B.C.'s ① Gauss's law

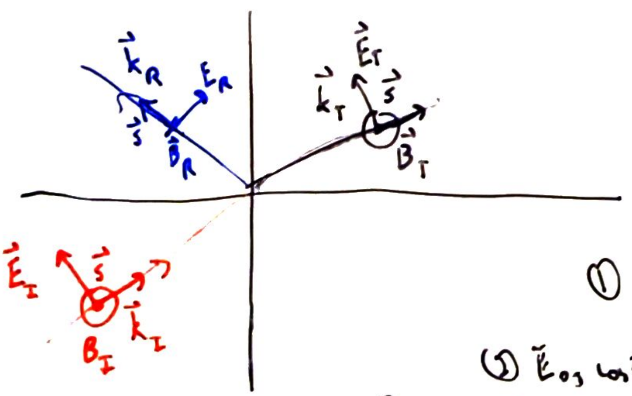
$$\epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp \rightarrow \epsilon_1(\tilde{\vec E}_{0,I} + \tilde{\vec E}_{0,R})_z = \epsilon_2(\tilde{\vec E}_{0,T})_z$$

② Faraday's law

$$E_1^\parallel = E_2^\parallel \qquad (\tilde{\vec E}_{0I} + \tilde{\vec E}_{0R})_{x,y} = (\tilde{\vec E}_{0,T})_{x,y}$$

③ No-name $\quad B_1^\perp = B_2^\perp \qquad (\tilde{\vec B}_{0,I} + \tilde{\vec B}_{0,R})_z = (\tilde{\vec B}_{0,T})_z$

④ Ampere's $\quad \frac{1}{\mu_1}B_1^\parallel = \frac{1}{\mu_2}B_2^\parallel \Rightarrow \frac{1}{\mu_1}(\tilde{\vec B}_{0I} + \tilde{\vec B}_{0,R})_{x,y} = \frac{1}{\mu_2}(\tilde{\vec B}_{0,T})_{x,y}$



Pick the polarization in the x-z plane so the incident wave is ∥ to the plane of incidence

① $\epsilon_1(-\tilde E_{0I}\sin\theta_I + \tilde E_0 \sin\theta_R) = \epsilon_2(-\tilde E_{0T}\sin\theta_T)$

② $\tilde E_{0I}\cos\theta_I + \tilde E_{0,R}\cos\theta_R = \tilde E_{0T}\cos\theta_T$

③ No information ∴ $\vec B \perp \hat z$

④ $\frac{1}{\mu_1}(\tilde B_{0I} - \tilde B_{0R}) = \frac{1}{\mu_2}\tilde B_{0T} \qquad B = \frac{E}{v}$

$$\Rightarrow \frac{1}{\mu_1}\left(\frac{\tilde E_{0I}}{v_1} - \frac{\tilde E_{0R}}{v_1}\right) = \frac{1}{\mu_2}\left(\frac{\tilde E_{0T}}{v_2}\right)$$

we can use $\theta_I = \theta_R \rightarrow$ ① becomes

$$E_{0,I} - E_{0,R} = \frac{\epsilon_2}{\epsilon_1}E_{0,T}\frac{\sin\theta_T}{\sin\theta_I} = \frac{\epsilon_2 n_1}{\epsilon_1 n_2}\tilde E_{0,T} \qquad (1)$$

for ④ $\tilde E_{0I} - \tilde E_{0R} = \frac{\mu_1 v_1}{\mu_2 v_2}\tilde E_{0T} = \beta \tilde E_{0T}$

$$\frac{\epsilon_2 n_1}{\epsilon_1 n_2} = \underbrace{\frac{\mu_1 v_1}{\mu_2 v_2}}_{\beta} = \frac{\epsilon_2 \mu_2}{\epsilon_1 \mu_1} = \frac{v_2^2}{v_1^2}$$

so $\tilde E_{0I} - \tilde E_{0R} = \beta \tilde E_{0T}$ (from both 1 and 2) (5)

$$\tilde E_{0I} + \tilde E_{0R} = \frac{\cos\theta_T}{\cos\theta_I}\tilde E_{0T} = \alpha \tilde E_{0T} \qquad (6)$$

add 5 & 6 to get $2\tilde E_{0,I} = (\alpha + \beta)\tilde E_{0T}$

$$\frac{\tilde E_{0,T}}{\tilde E_{0,I}} = \frac{2}{\alpha + \beta}$$

From 6 $\quad \tilde{E}_{0R} = -\tilde{E}_{0I} + \alpha \tilde{E}_{0T} = \left(-1 + \frac{2\alpha}{\alpha+\beta}\right) \tilde{E}_{0I}$

so
$$\boxed{\begin{aligned} \frac{\tilde{E}_{0,R}}{\tilde{E}_{0,I}} &= \left(\frac{\alpha-\beta}{\alpha+\beta}\right) \\ \frac{\tilde{E}_{0,T}}{\tilde{E}_{0,I}} &= \frac{2}{\alpha+\beta} \end{aligned}}$$

Physics ① E_T is always in phase with E_I

② E_R can change directions (out of phase) with respect to E_I

$\alpha > \beta$

$\frac{\mu_2}{n_2}\cos\theta_T > \frac{\mu_1}{n_1}\cos\theta_I$ they are in phase

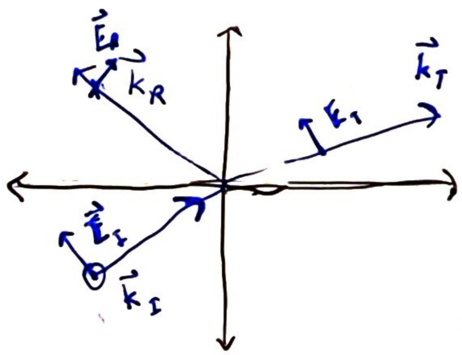
$\alpha < \beta$ $\quad \left(\frac{\mu_2}{n_2}\cos\theta_T < \frac{\mu_1}{n}\cos\theta_I\right)$ out of phase

③ Amplitudes $\tilde{E}_R$ and $\tilde{E}$ depend on θ_I

$\theta_I = 0 \Rightarrow$ gives the same equations as normal incidence

$\frac{\tilde{E}_{0,R}}{\tilde{E}_{0I}} = \frac{n_1-n_2}{n_1+n_2} \qquad \frac{\tilde{E}_{0T}}{\tilde{E}_{0I}} = \frac{2}{n_1+n_2}$

if $\theta = 90^\circ$ $\quad E_{0,T} = 0 \quad E_{0,R} = 1$ (normal grazing)



$\alpha = \frac{\cos\theta_T}{\cos\theta_I} \qquad \beta = \frac{\mu_1 v_1}{\mu_2 v_2}$

E_R and E_I are in phase if $\alpha > \beta$

$E_T = E_I$ are always in phase

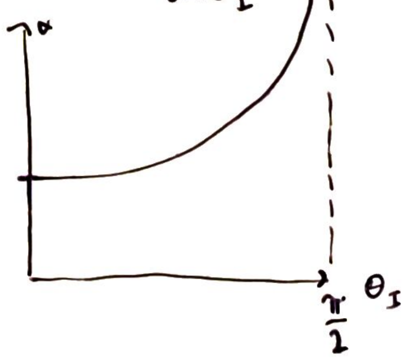
$\alpha = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_I\right)^2}}{\cos\theta_I}$

$\theta_I = 0$ (normal incidence) $\alpha = 1$

$\theta_I = \frac{\pi}{2}$ (wave glides along boundary)

$\Rightarrow \alpha \to \infty \quad \tilde{E}_{0,T} = 0$

$\tilde{E}_{0,R} = \tilde{E}_{0I}$



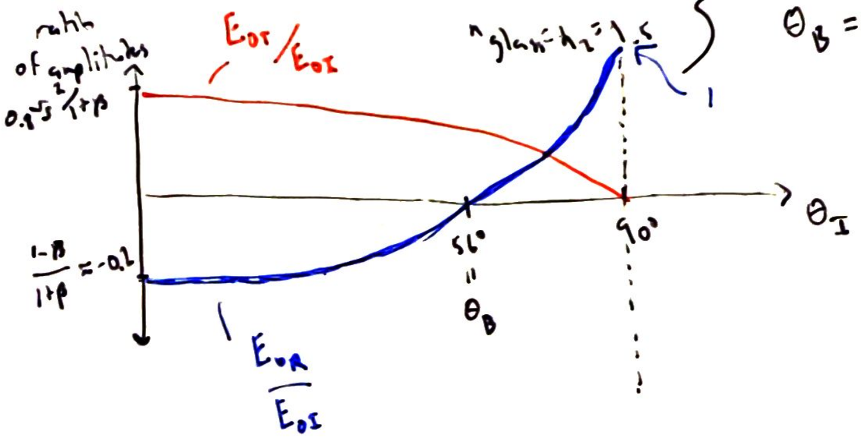
Note: for $\alpha = \beta$ no reflection at all (only refraction)

For $\mu_1 \approx \mu_2$, $\beta = \frac{n_2}{n_1}$ denote the angle that corresponds to this as θ_B "Brewster's angle"

$$\sin\theta_B = \frac{\beta}{\sqrt{1+\beta^2}}$$ or using

$$\tan\theta_B = \beta \approx \frac{n_2}{n_1}$$

For air/glass with $n_{air} = n_1 = 1$, $n_{glass} = n_2 = 1.5$ } $\theta_B = 56°$



Energy balance

$$I_I = \frac{1}{2}\epsilon_1 v_1 E_{0I}^2 \cos\theta_I$$ power delivered per interface area

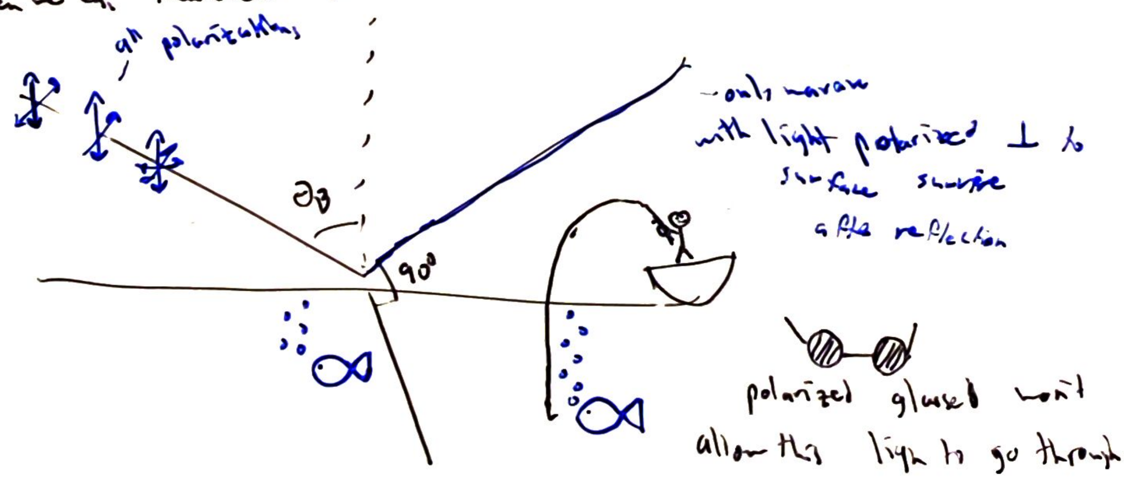
$$I_R = \frac{1}{2}\epsilon_1 v_1 E_{0R}^2 \cos\theta_R$$

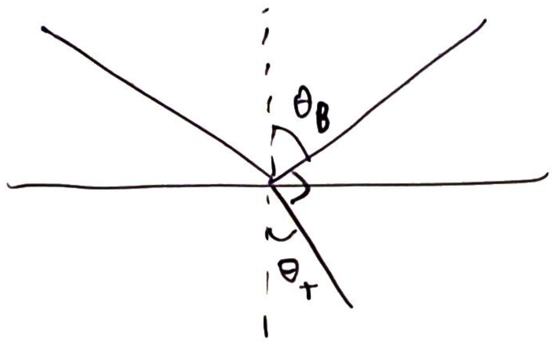
$$I_T = \frac{1}{2}\epsilon_2 v_2 E_{0T}^2 \cos\theta_T$$

$$\frac{I_R}{I_I} = \frac{E_{0R}^2}{E_{0I}^2} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 = R$$

$$\frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\frac{\cos\theta_T}{\cos\theta_I}\frac{E_{0T}^2}{E_{0I}^2} = \beta\alpha\left(\frac{2}{\alpha+\beta}\right)^2 = \frac{4\alpha\beta}{(\alpha+\beta)^2} = T$$

then we can then check $R + T = 1$





$\sin\theta_B \quad \frac{\beta}{\sqrt{1+\beta^2}}$ $\qquad \beta = \frac{n_2}{n_1}$

WTS $\quad \theta_B + 90 + \theta_T = 180$

$\theta_B + \theta_T = 90$

$\cos\theta_T = \sqrt{1-\sin^2\theta_T}$

$= \sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_B\right)^2}$

$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_B$

$= \frac{\sin\theta_B}{\beta}$

$n_1 \sin\theta_B = n_2 \sin\theta_T$

$n_1 \quad \frac{\beta}{\sqrt{1+\beta^2}}$

$\sin(\theta_B+\theta_T) = \sin\theta_B \cos\theta_T + \cos\theta_B \sin\theta_T$

$\frac{\beta}{\sqrt{1+\beta^2}}\cos\theta_T + \frac{\sin\theta_T}{\sqrt{1+\beta^2}}$

$\frac{\beta\sqrt{1-\sin^2\theta_T}}{\sqrt{1+\beta^2}} + \frac{\beta}{1+\beta^2}$

$\frac{\beta\sqrt{1-\frac{\beta^2}{1+\beta^2}}}{\sqrt{1+\beta^2}} + \frac{\beta}{1+\beta^2}$

$\sqrt{\frac{\beta^2}{1+\beta^2} - \frac{\beta^2}{(1+\beta^2)^2}} + \frac{\beta}{1+\beta^2}$

$\sqrt{\frac{\beta^2+\beta^4-\beta^2}{(1+\beta^2)^2}} + \frac{\beta^2}{1+\beta^2}$

$\frac{\beta^2}{1+\beta^2} + \frac{\beta^2}{1+\beta^2}$