Week 2

Maxwell's equations in free space $\rho = \vec{J} = 0$

1) $\vec{\nabla} \cdot \vec{E} = 0$
2) $\vec{\nabla} \cdot \vec{B} = 0$
3) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
4) $\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

$$\underbrace{\vec{\nabla} \times (\vec{\nabla} \times \vec{E})} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 E = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}\right)$$

$\overset{\parallel}{0}$ (1) $\Rightarrow \boxed{\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$ (wave equation)

Similarly, $\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})$

$\hookrightarrow \boxed{\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{B}}$

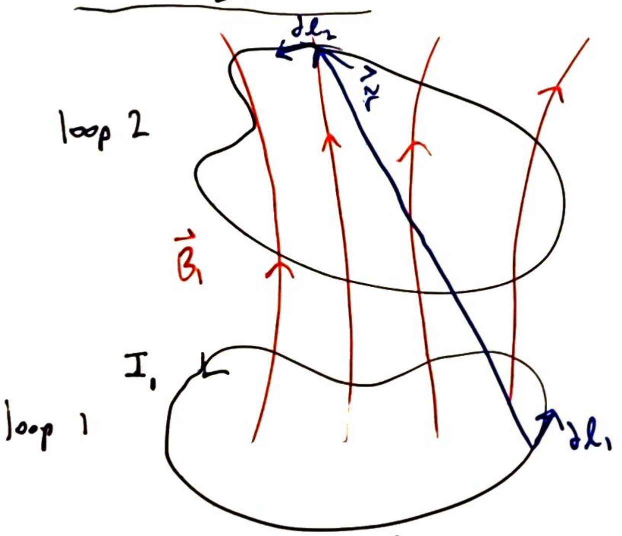
units of $\mu_0 \varepsilon_0$

$\frac{T}{m^2} = \mu_0 \varepsilon_0 \frac{T}{s^2}$

then $\boxed{\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{m}{s}}$ $\frac{1}{\sqrt{8.85 \times 10^{-12} \cdot 4\pi \cdot 10^{-7}}} = 3 \times 10^8 \frac{m}{s} = c$

$\mu_0 \varepsilon_0 = \frac{s^2}{m^2} = \left(\frac{1}{v}\right)^2$

7.2.3 Inductance



I_1 creates $\vec{B}_1$ - penetrates loop 2

$\hookrightarrow$ find Φ_2 through loop 2

Biot-Savart law $B_1 = \frac{\mu_0}{4\pi} I \oint \frac{d\vec{l}_1 \times \hat{r}}{r^2}$

$\hookrightarrow B_1 \propto I_1$

$\Phi_2 \propto B_1$ $\Phi_2 \propto I_1$

we can write $\boxed{\Phi_2 = M_{21} I_1}$

M_{21} is the constant of proportionality

$\hookrightarrow M_{21}$: mutual inductance

$M_{21} = M_{12}$ ← transformer action

proof: $\Phi_2 = \int \vec{B}_1 \cdot d\vec{a}_2 = \int (\vec{\nabla} \times A_1) \cdot d\vec{a}_2 = \oint \vec{A}_1 \cdot d\vec{l}_2$

Stokes theorem along loop 2

5.66 → $\vec{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\vec{l}_1}{r}$

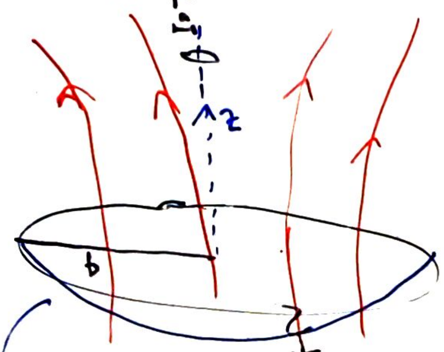
$= \boxed{\frac{\mu_0 I_1}{4\pi} \oint \oint \frac{d\vec{l}_1}{r}}$

so $M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r}$ (Neumann's formula) so: ① M_{12} is purely geometrical

② $M_{12} = M_{21} = M$

whatever the shapes and positions of the loops, the flux through 2 when we run a current I around 1 it is identical to the flux through 1 when we run the same current I around 2.

HW



spherical surface

goal: find flux through loop a, Φ_a ($a \ll b$)

$$\Phi_a \approx B_z \cdot \pi a^2$$

$$\Phi_a \propto \underbrace{M}_{(\text{or } M_{12})} \cdot I$$

for b) current flows in little loop (baby)

assuming $b \gg a$ → treat loop a as a magnetic dipole

$$\vec{B}_{dip} = \ldots (2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$

$$\Phi_b = \oint_{\text{area of loop b}} \vec{B}_{dip} \cdot d\vec{a}$$

(any area enclosed by loop b)

If everything goes according to plan, you'll get $M_{12} = M_{21}$

Just one loop: vary the current

Faraday's law: $|\mathcal{E}| = \frac{d\Phi}{dt}$ $\quad \Phi(t) \propto I$ and $\Phi = LI$

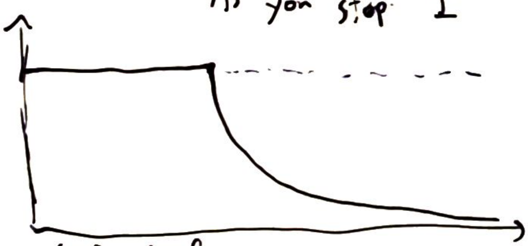
flux of B-field thru loop

L is called the self-inductance

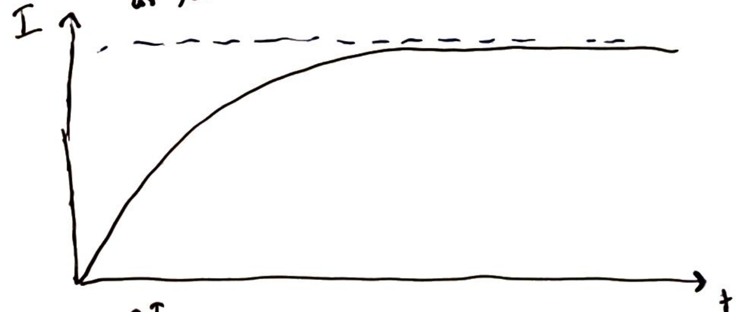
so $\boxed{|\mathcal{E}| = L\frac{dI}{dt}}$ (back emf)

Lenz' law: emf is in such a direction as to oppose any change in current

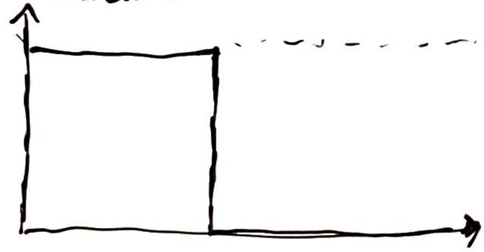
As you stop I



as you start I



instead of



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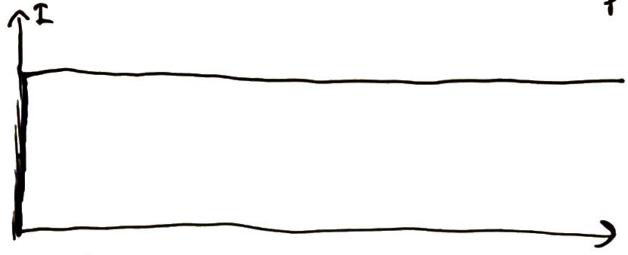
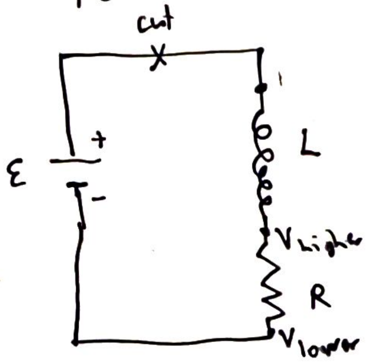


similar to inertia in mechanics: $m\uparrow$ → harder it is to change objects velocities

$L\uparrow$ → harder to change a current example



the large $\left|\frac{dI}{dt}\right|$ will create a large $\mathcal{E}_{ind}$.

if we first turn on the bat, we first see

$$I\uparrow \Rightarrow B_{coils}\uparrow = \Phi_B\uparrow$$

$\mathcal{E}_{ind}$ would oppose the change in flux

going around the circuit clockwise

$$\mathcal{E} - L\frac{\partial I}{\partial t} - IR = 0$$

$$\Rightarrow L\frac{dI}{dt} = \mathcal{E} - IR$$

$$\Rightarrow \int \frac{L\,dI}{\mathcal{E}-IR} = \int dt \;:\; -\frac{1}{R}\ln(\mathcal{E}-IR)\Big|_0^I = t$$

so $I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{R}{L}t}\right)$ $\qquad \left[\frac{L}{R}\right] = s$ $\qquad$ if we wait for $t = \frac{L}{R}$

$$I = \frac{\mathcal{E}}{R}\cdot\left(1 - e^{-1}\right) = 0.63\frac{\mathcal{E}}{R} = 0.63\, I_0$$

$L\uparrow$, the longer it takes to reach 63% of max

It takes energy to start current flowing

↳ work done by battery $\quad dW = -\mathcal{E}_{induced}\cdot dq$

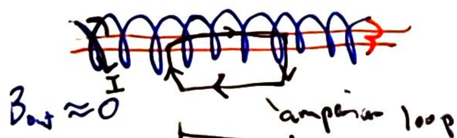
$$= -\left(-L\frac{\partial I}{\partial t}\right)\cdot I\,dt$$ cancel out differentials

$$dW = L\,\frac{dI^2}{2}$$

$$\Rightarrow W = L\frac{I^2}{2}$$

this ends up being equivalent to $\quad W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\, d\tau$

Find self-inductance of a long solenoid



# of turns per unit length is n, $\quad [n] = \frac{1}{m}$

Use ampere's law $\quad \oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl} + \mu_0\varepsilon_0\int \frac{\partial \vec{E}}{\partial t}\, d\vec{a}$

ignore cause no $\vec{E}$

$$B\cdot a = \mu_0 I_{encl} = \mu_0 I N$$ (number of turns in black loop)

$$= \mu_0 I n a$$

$$Ba = \mu_0 I n a \Rightarrow B = \mu_0 I n$$

then $\Phi_{\substack{\text{one}\\\text{coil}}} = BA = (\mu_0 I n)(\pi r^2)$

$$\Phi_{net} = \mu_0 I n \pi r^2 (n l) = \mu_0 I n^2 \pi r^2 l$$

and $\Phi_{net} = LI \Rightarrow L = \mu_0 l \pi r^2 n^2$

you can check that if you put it into $W = \frac{LI^2}{2}$ you get the same thing as $W = \frac{1}{2\mu_0}\int B^2 d\tau$

Ch 8. Conservation Laws

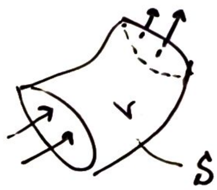
Main idea: not only material particles (that have mass) can carry energy, momentum, and angular momentum, but E and B - fields can carry energy, momentum, and angular momentum

particles and fields can exchange energy, $\vec{p}$, $\vec{L}$

Continuity equation for charge

Notice: in steady state $\frac{dQ}{dt} = 0$

$\Rightarrow \oint \vec{J}\cdot d\vec{a} = 0$



$$\frac{dQ}{dt} = -\oint \vec{J}\cdot d\vec{a}$$

(charge balance)

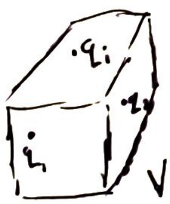
$\Rightarrow \vec{J}_{in} = \vec{J}_{out}$

Differential form: $\frac{d}{dt}\int_V \rho\, d\tau = -\oint \vec{J}\cdot d\vec{a} = -\int_V \vec{\nabla}\cdot\vec{J}\, d\tau$

and we get $\boxed{\frac{\partial \rho}{\partial t} = -\nabla\cdot\vec{J}}$ for a small volume $d\tau$

Given $\underbrace{\vec{B} \text{ and } \vec{E}}$ (due to currents and charges)

act on charges inside V



Work-energy theorem: $\underbrace{dK}_{\substack{\text{change}\\\text{in KE}\\\text{of charge}}} = \underbrace{dW}_{\substack{\text{work done}\\\text{by all forces}\\\text{(net force)}}}$

$dW_q = \vec{F}_q \cdot \underbrace{d\vec{l}}_{\vec{v}\cdot dt}$ $\qquad F_q = q(\vec{E} + \vec{v}\times\vec{B})$

$$\text{so } dW = q(\vec{E} + \underbrace{\vec{v}\times\vec{B})\cdot\vec{v}}_{=0} dt = q\vec{E}\cdot\vec{v}\, dt$$

b/c $\vec{v}\times\vec{B} \perp \vec{v}$ $\qquad \hookrightarrow$ let $q = \rho\, d\tau$

$\hookrightarrow$ total charge in volume $d\tau$

$$\frac{dW_{d\tau}}{dt} = (\rho\, d\tau)\vec{E}\cdot\vec{v} = \vec{E}\cdot\underbrace{\rho\vec{v}}_{\vec{J}} = \vec{E}\cdot\vec{J}\, d\tau$$