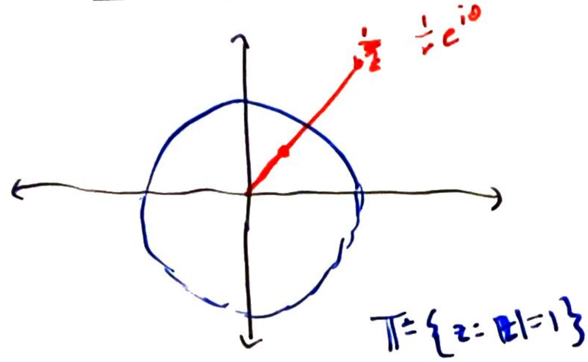
Week 4

Conformal maps

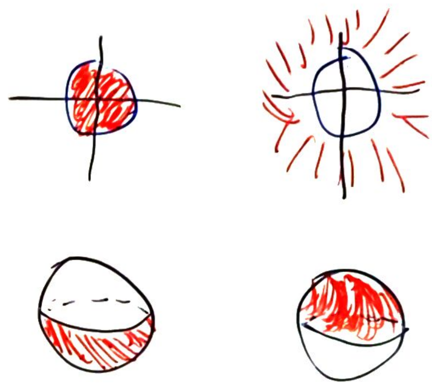
- inversion across a circle
- applications to physical systems
- Linear algebra
  LFT $\longleftrightarrow GL_2(\mathbb{C})$
- bonus content

reflection on the circle



$z \to \frac{1}{\bar z}$

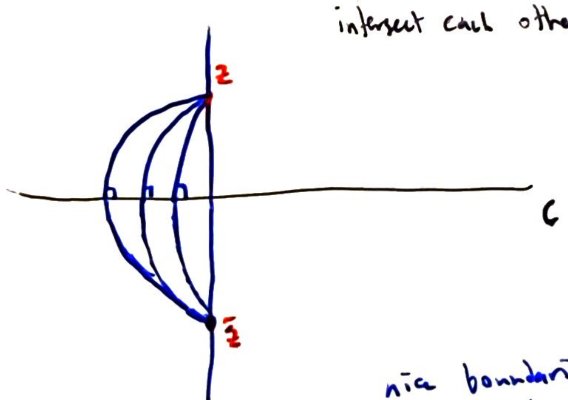
$\mathbb{D} \to \hat{\mathbb{D}}$ (outer disk)



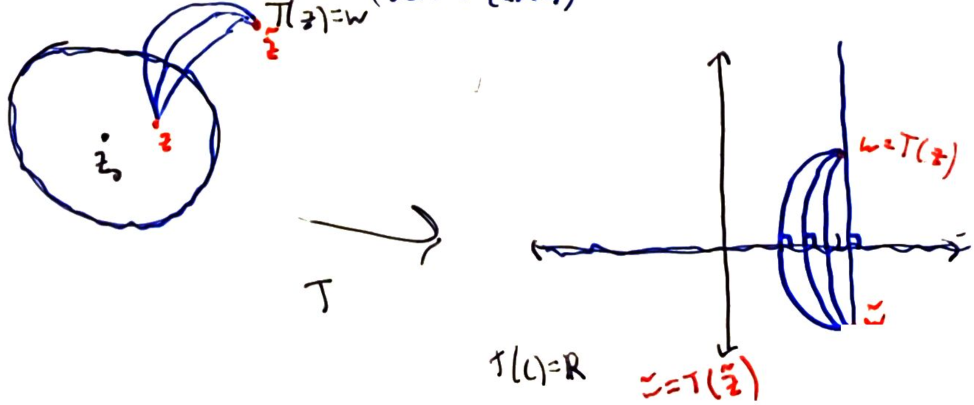
Let C be a general circle centered at z_0

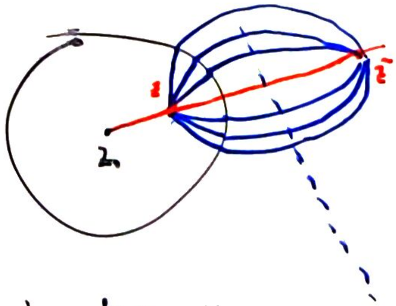
then all circles that pass through a point $z \notin C$ and intersect at right angles, intersect each other at a unique point $\tilde z$

happens when C is a line



nice boundaries can map to other nice boundaries (Jordan curves)





how do we find $\tilde z$

$C = \mathbb{T}$, $\tilde z = \frac{1}{z}$

$$\tilde z = \overline{\left(\frac{\bar z_0 z + R^2 - |z_0|^2}{z - z_0}\right)}$$

the map $z \to \tilde z$ is <u>not</u> analytic because $\bar z$

$z \to \tilde z$ is a combination of LFTs and conjugation

If T is a linear fractional transformation and C is a circle then $T(\tilde z) = \widetilde{T}(z)$

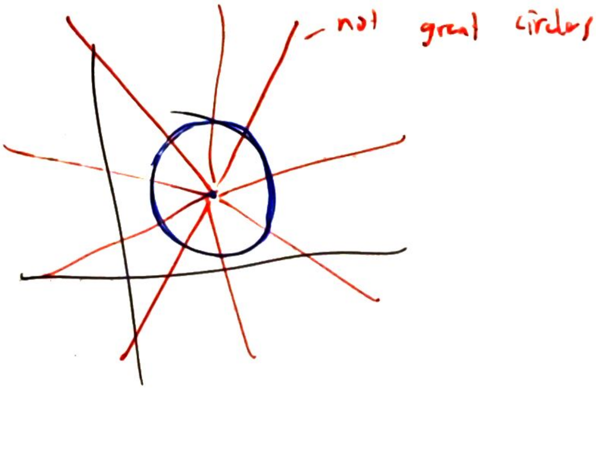
so LFTs preserve reflection across the circle

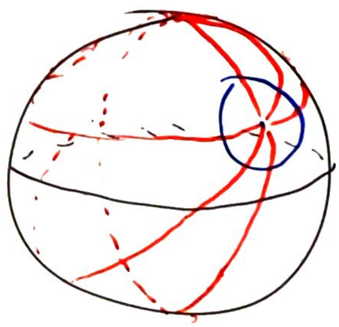
<u>prop</u> 1. $\tilde{\tilde z} = z$

2. $z \to \tilde z$ is not conformal but angles are preserved in magnitude and reversed in orientation

3. If C is a line, $\tilde z$ is the reflection in the perpendicular reflection across the line

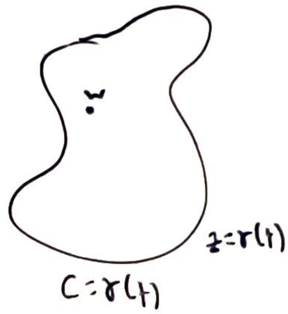
4. $z \to \tilde z$ maps circles to circles





Cauchy integral formula

$$f(w) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{z-w} dz$$



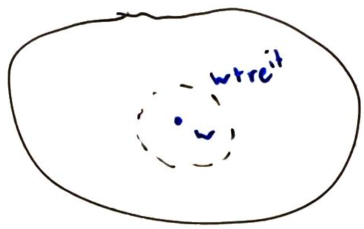
f-analytic

C nice: simple, piecewise cont. and closed

interior values are determined by the boundary values

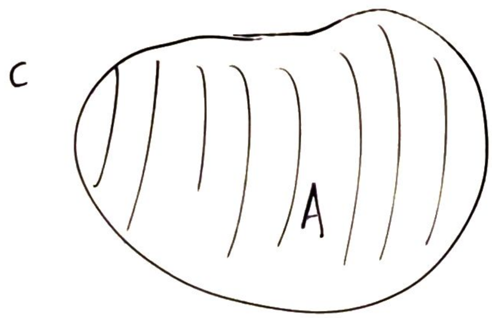
Gauss' mean value theorem

$$f(w) = \frac{1}{2\pi}\int_0^{2\pi} f(w+re^{it})dt$$



Dirichlet Problem

Given a region in $\mathbb{R}^2$, A with boundary C that is bounded, simple closed, find a harmonic function



cont. function u_0 on C

Given $A \subseteq \mathbb{R}^3$, boundary C and boundary values of the function on C, u_0
want to find $u(x,y)$ on A $(\nabla^2 u = 0)$

so that $u = u_0$ on C

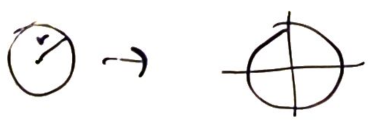
Original motivation

$u_t = \nabla^2 u = u_{xx} + u_{yy}$ equilibrium equations $u_{xx} + u_{yy} = 0$

potential theory

If A is a disk so that C is a circle: u is defined and continuous on C

WLOG



assume A is unit disk
C is unit circle

then $u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\phi}) \, \frac{\rho - |z|^2}{\quad}$

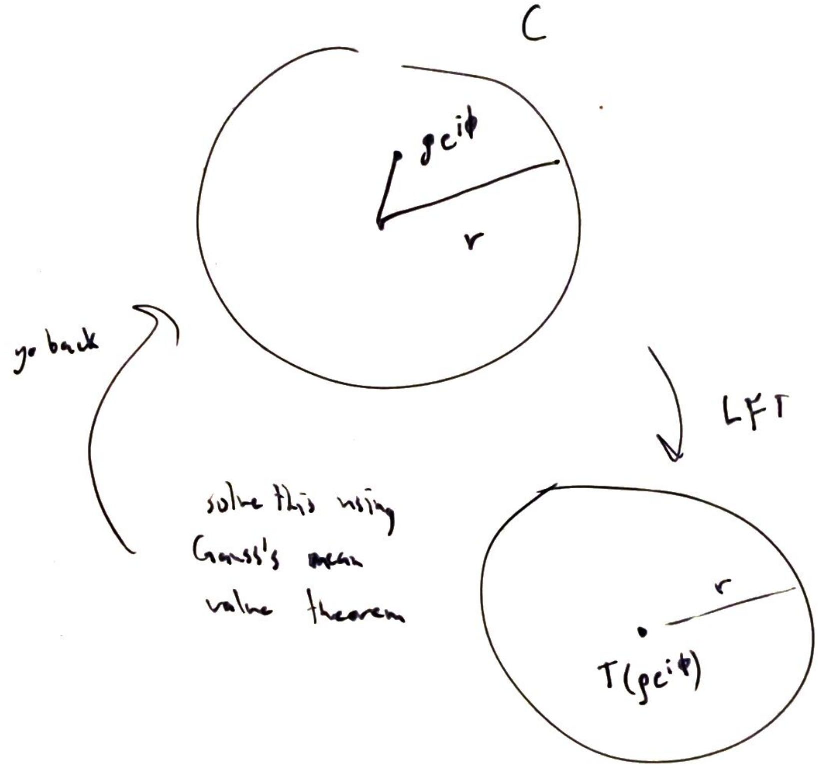
<u>ignore the above</u>

$D(0,r)$, for $\rho < r$

outside of the circle

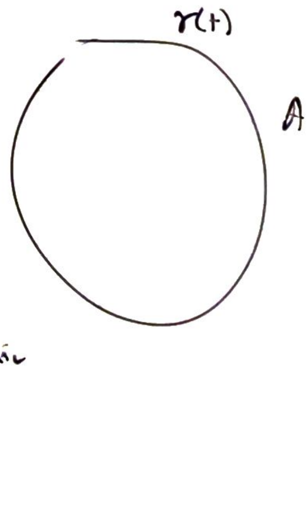
$$u(\rho e^{i\phi}) = \frac{r^2-\rho^2}{2\pi}\int \frac{u(re^{i\theta})}{r^2 - 2r\rho\cos(\phi-\theta) - \rho^2}\, d\theta$$

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\, \frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\, d\theta$$



$$\int_\gamma \vec{X}\cdot\vec{n}\, dS = \int_A \operatorname{div} X \, dx\, dy$$

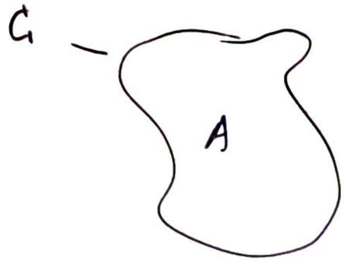
vector field over A



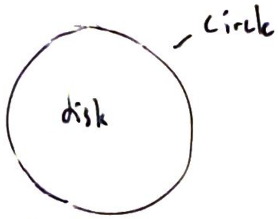
if you have $f(z) = f(x+iy) = u(x+iy) + iv(x+iy)$

then both u and v are harmonic

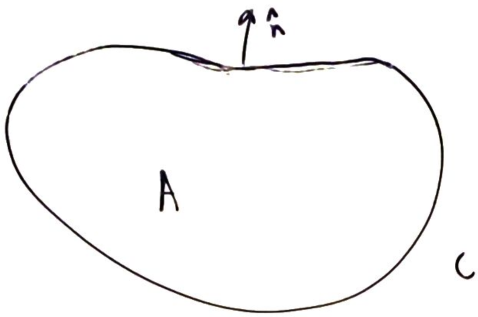
u is given on continuous



Riemann mapping theorem tells you that this transform is possible



Given a domain A



need direction of $\hat{n}$ to vary continuously

Perscribe boundary derivatives

Find u harmonic on A with $\frac{\partial u}{\partial n}$ specified along the boundary

$$\frac{\partial u}{\partial n} = \nabla u \cdot \hat{n}$$

<u>Neumann Problem</u>

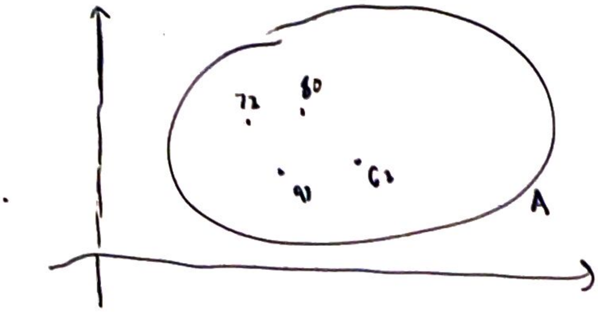
If u existed then $\int_{\gamma} \frac{\partial u}{\partial n} = 0$

if $\vec{X}$ = grad u then $\int \vec{X} \cdot \hat{n} \, ds = \int \operatorname{div} X \, dx dy$

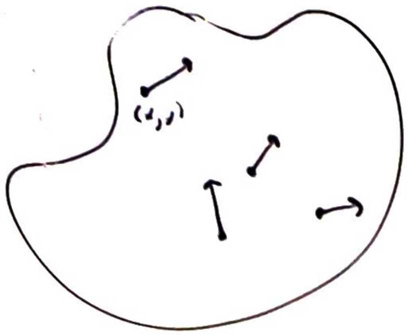
but div(grad u) = 0 always

A scalar field on domain $A \subseteq \mathbb{R}^2$ ## Week 4 pt 2 ##

is a function $u(x,y)$ that asigns to each point in A a scalar value



A vector field is a function $V(x,y)$ that asigns a vector to each point in A



$$V = (v_1(x,y), v_2(x,y))$$

u is a scalar field

$$\nabla u = \text{grad}\, u = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

grad u is a vector pointing in the direction of greatist change in u.

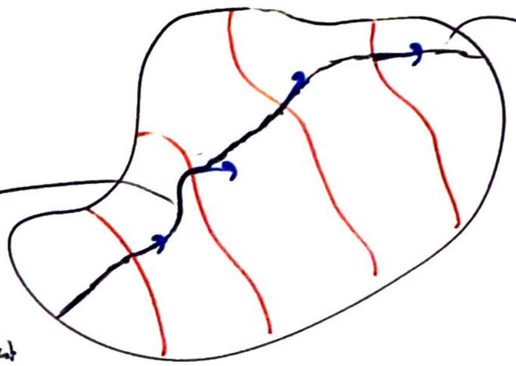
$u \longrightarrow \text{grad } u$

potential — vector field

C^1 — continuous

$u(x,y) = c$ (level curve)

grad u is orthogonal to equipotentia



flow lines / stream lines

we want to find a curve that has these grad vectors as tangent lines

$V = \text{grad } u$

↳ line integrals are path independent

↳ curl is 0 : irrotational fields

curl $V = 0$

$$V = \text{grad } u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, 0\right)$$

$$\text{curl } V = \nabla \times V = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, 0\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u_x & u_y & 0 \end{vmatrix} = (u_{xy} - u_{yx})\hat{k} = 0$$

because $u_{xy} = u_{yx}$ by Clairaut's theorem

this works when u is C^2

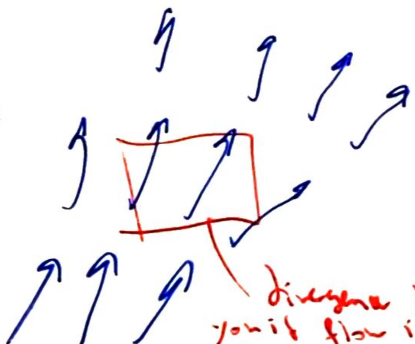
u is harmonic: u is C^2 and $\nabla^2 u = 0$

$\nabla^2 u = \text{div}(\text{grad } u) = 0$

if u is harmonic, $\text{div}(\text{grad } u) = 0$

so grad u is a divergence-free field

divergence-free means flow in = flow out
(flux) (flux)



divergence tells you if flow in equals flow out

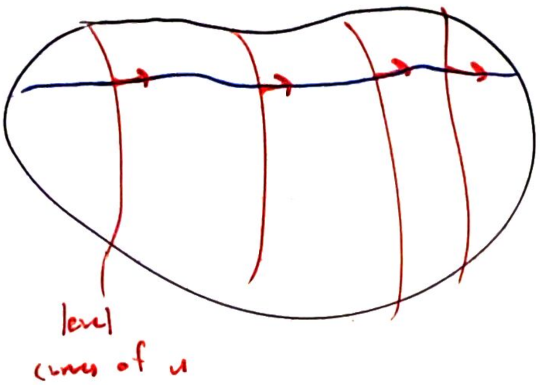
u is harmonic: grad u is irrotational

grad u is divergence free

- heat flow
- fluid flow
- electrostatics

u = harmonic

flow lines are level curves of the harmonic conjugate $u^* = C$



If $u = C$ are equipotentials and u^* is the harmonic conjugate of u,

$u^* = C$ are the stream lines

$f(z) = f(x+iy) = u(x,y) + iu^*(x,y)$

f is analytic, usually called the complex potential

given $f(z) = u + iv$ analytic

$Re(f) = u$ is harmonic

$Im(f) = v$ is the harmonic conjugate

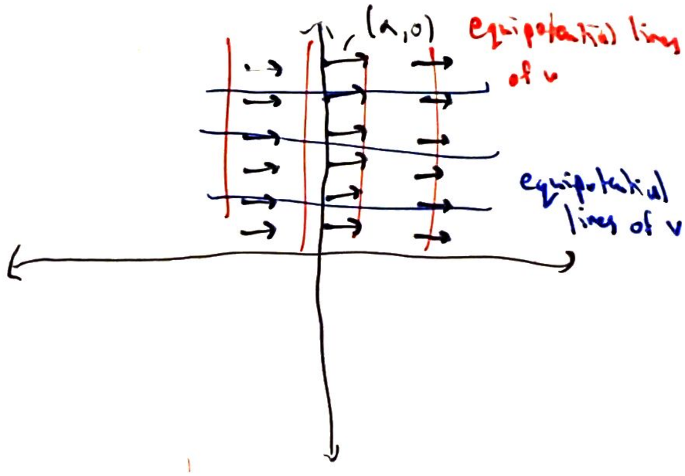
here a complex potential $f(z) = \alpha z$

$Re(f) = \alpha x$

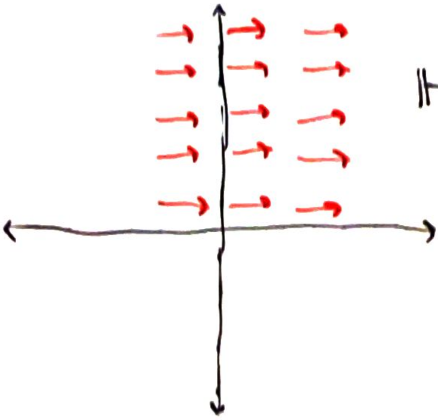
$Im(f) = \alpha y$

$u(x,y) = \alpha x$ $\quad$ $v(x,y) = \alpha y$

grad $u = (\alpha, 0)$



Week 4 pt 3



$\mathbb{H}$

$V(x,y) = (\alpha, 0)$

$u(x,y) = \alpha x$

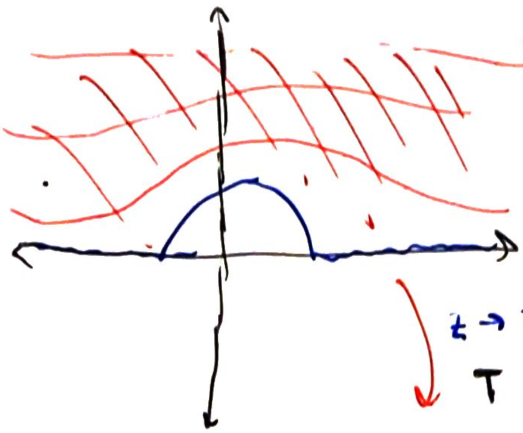
$u = Re[\alpha z]$

complex potential is

$f(z) = \alpha z = \alpha x + i(\alpha y)$

if u_1 is harmonic and u_2 is harmonic then $a u_1 + b u_2$ is still harmonic

$\mathbb{W}$



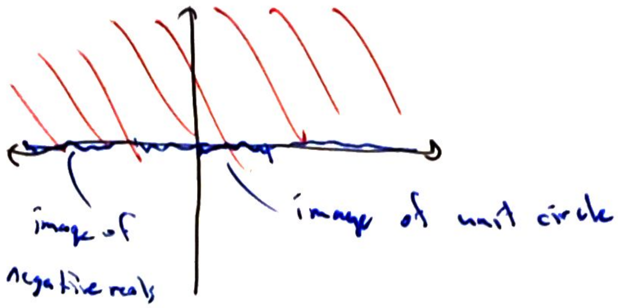
to get this conformally from the first case use the transform

$z \to z + \frac{1}{z}$

$z \to z + \frac{1}{z}$

$\mathbb{T}$

real number a goes to $a + \frac{1}{a}$



then if you're on the unit circle you get twice the real part

$\mathbb{W} \quad \mathbb{H} \quad \mathbb{R}$

$u : \mathbb{H} \to \mathbb{R}$

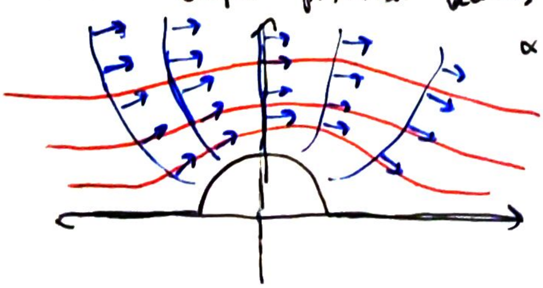
$w \to T(w) \to u(T(w))$

$\phi = u \circ T$

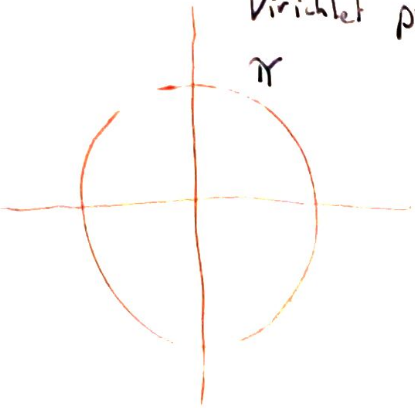
If u is harmonic and T is conformal then $u \circ T$ is harmonic

so the complex potential becomes $f(z) = \alpha\left(z + \frac{1}{z}\right)$

$$\alpha\left(x+iy + \frac{1}{x+iy}\right) = \underbrace{\alpha\left(x + \frac{x}{x^2+y^2}\right)}_{\boxed{\phi}} + \underbrace{i\alpha\left(y + \frac{y}{x^2+y^2}\right)}_{\boxed{\psi}}$$



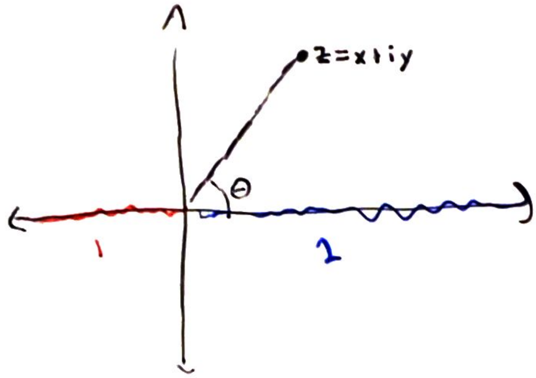
Dirichlet problem



prescribe u on $\mathbb{T}$, u cont.

u = poisson integral

half-plane standard solution



$u = \frac{1}{\pi}\theta$

recall $\log z = \ln(r) + i\arg(z)$

$\frac{1}{\pi i}\mathrm{Log}(z)$

$\frac{1}{\pi}\theta = \mathrm{Re}\left[\frac{1}{i\pi}\mathrm{Log}(z)\right]$