Week 7

9.4 Absorption and Dispersion

(1) $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon}\rho_F$ (3) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(2) $\vec{\nabla}\cdot\vec{B} = 0$ (4) $\vec{\nabla}\times\vec{B} = \mu\vec{J}_F + \mu\varepsilon\frac{\partial \vec{E}}{\partial t}$

And, Ohm's law $\vec{J}_F = \sigma\vec{E}$

And, the continuity equation (charge conservation)

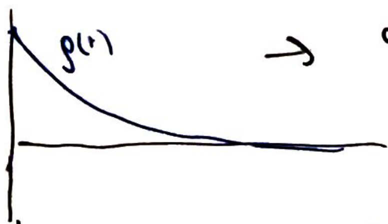
$$\frac{\partial \rho_F}{\partial t} = -\vec{\nabla}\cdot\vec{J}_F$$

$$\hookrightarrow \frac{\partial \rho_F}{\partial t} = -\sigma\vec{\nabla}\cdot\vec{E} = \frac{\sigma\rho_F}{\varepsilon} \Rightarrow \int\frac{\partial \rho_F(t)}{\rho_F(t)} = \int -\frac{1}{\tau}\,\partial t$$

$$\ln(\rho) = -\frac{t}{\tau} + c_1$$

so $\rho(t) = Ce^{-t/\tau}$ at $t=0$, $\rho_F(0) = \rho_0 = C$

$$\rho(t) = \rho_0 e^{-t/\tau}$$



→ excess charges go to the surface

so $\tau \sim \frac{1}{\sigma}$ thus $\sigma\uparrow = \tau\downarrow$

we'll consider only the processes that are slower than τ

$\Rightarrow \rho_F \approx 0$

so we get (1) $\vec{\nabla}\cdot\vec{E} = 0$ (2) $\vec{\nabla}\cdot\vec{B} = 0$

(3) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (4) $\vec{\nabla}\times\vec{B} = \mu\sigma\vec{E} + \mu\varepsilon\frac{\partial \vec{E}}{\partial t}$

$$\vec{\nabla}\times(3) \Rightarrow \underbrace{\vec{\nabla}(\vec{\nabla}\cdot\vec{E})}_{0} - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\underbrace{(\vec{\nabla}\times\vec{B})}_{(4)}$$

$$\nabla^2\vec{E} = \frac{\partial}{\partial t}\left(\mu\sigma\vec{E} + \mu\varepsilon\frac{\partial \vec{E}}{\partial t}\right)$$

$$\boxed{\nabla^2\vec{E} = \mu\sigma\frac{\partial \vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2 \vec{E}}{\partial t^2}} \quad (5) \quad \text{modified wave equation}$$

similarly acting on B gives

$$\boxed{\nabla^2 \vec{B} = \mu\sigma \frac{\partial \vec{B}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2}} \quad (6)$$

$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{i(\tilde{k}z - \omega t)}$ here k is complex

$\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_0 e^{i(\tilde{k}z - \omega t)}$

(5) $\Rightarrow$ $\quad -\tilde{k}^2 \tilde{\vec{E}}_0 = \mu\sigma(i\omega)\cancel{\tilde{\vec{E}}_0} - \omega^2\mu\varepsilon\cancel{\tilde{\vec{E}}_0}$

(6) $\Rightarrow$ same condition

$$\tilde{k}^2 = i\mu\sigma\omega + \omega^2\mu\varepsilon$$

let $\tilde{k} = k + i\kappa$ (κ, k are real numbers)

subbing this in gives $\quad k = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[1 + \sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2}\right]^{1/2}$

$$\kappa = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[-1 + \sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2}\right]^{1/2}$$

note if $\sigma \to 0$ (insolator) $\quad k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{v}$

$\kappa = 0$

So far $\quad \tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{-\kappa z} e^{i(kz - \omega t)}$

$\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_0 e^{-\kappa z} e^{i(kz - \omega t)}$

waves decay over distance $\delta = \frac{1}{\kappa}$ "skin depth"

for bad conductor (water) $\sigma \ll 1$

$\delta \approx 10^4$ m

Good conductors (metal) $\delta \sim \frac{\lambda}{2\pi}$ not really oscillating wave $\Rightarrow$ already decayed

In a vacuum (or linear medium)

$\delta_E = \delta_B = 0$

$\Rightarrow$ Find if theres a phase shift in conducting matter

Extra constraints from Maxwell's equations

$\vec{\nabla}\cdot\vec{E} = 0$

$\vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\times\vec{E} = -\frac{\partial \tilde{\vec{B}}}{\partial t}$

$$\vec{\nabla}\cdot\vec{E} = -\tilde{E}_{0,z}\, \chi k\, e^{\circ} e^{\circ} = 0$$

so $\tilde{E}_{0,z} = 0$

similarly, $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \tilde{B}_{0,z} = 0$

so both are transverse

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_{0x} & E_{0y} & 0 \end{vmatrix}$$

$$= \hat{i}\left(-i\chi k E_{0y} e^{-}\right) - \hat{j}\left(-i\chi k E_{0x} e^{-}\right)$$
$$= \omega \vec{B}_0 e^{-}$$

so $\tilde{k}\,(z\times\vec{E}_0) = \omega\vec{B}_0$ then let $k = Ke^{i\phi}$

so $|\vec{B}_0| = \frac{|k|}{\omega}|\vec{E}_0|$ if $\chi \neq 0$ then the waves are not in phase

goal is to find ϕ $\tilde{k} = k + i\chi$ $K = \sqrt{k^2 + \chi^2}$

$\phi = \text{atan}\left(\frac{\chi}{k}\right)$

$$Ke^{i\phi}\underbrace{e^{i\delta_E}}_{\text{phase of } E}\left(\hat{z}\times\vec{E}_0\right) = \omega e^{i\delta_B}\vec{B}_0$$

compare the phases of the complex #'s on the RHS and LHS

$$\phi + \delta_E = \delta_B$$

$$\phi = \tan^{-1}\left[\frac{\sigma/\omega\epsilon}{1+\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}}\right]$$

if $\sigma \gg 1$ (good conductor)

then $\phi \to 45°$ or $\frac{\pi}{4}$

back to $Ke^{i\phi}e^{i\delta_E}(\hat{z}\times\vec{E}_0) = \omega e^{i\delta_B}\vec{B}_0$

compare amplitudes

$$\frac{K}{\omega}|\vec{E}_0| = |\vec{B}_0| \Rightarrow \sqrt{\mu\epsilon\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}}\; E_0$$

for a poor conductor $\sigma \to 0 \Rightarrow B_0 = \frac{1}{v}E_0$

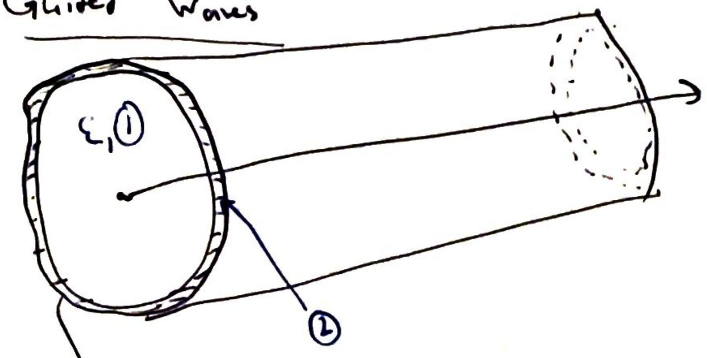
how is energy distributed between E and B waves

$$u = \frac{1}{2}\epsilon E^2 + \frac{1}{2\mu}B^2$$

$$\frac{\langle u_{mag}\rangle}{\langle u_E\rangle} = \frac{\frac{1}{2}\frac{1}{\mu}\mu\epsilon\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}\,E^2}{\frac{1}{2}\epsilon E^2} = \sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} > 1$$

So what's happening to the electric field?
the energy of the electric field goes into moving the free charges

Guided Waves



metal (good conductor)

Perfect conductor: $\Rightarrow \vec{E} = 0$ inside

(or beyond skin depth for finite σ)

$$\frac{\partial \vec{B}}{\partial t} = 0 \Rightarrow \text{B-field inside the metal part is constant}$$

Assume $B = 0$ initially $\Rightarrow$ stay at 0

Previously (for linear media)

B.C. $\varepsilon_1 E_{\perp 1} = \varepsilon_2 E_{\perp 2}$ $\qquad B_{\perp 1} = B_{\perp 2}$

$E_{\parallel 1} = E_{\parallel 2}$ $\qquad \frac{1}{\mu_1} B_{\parallel 1} = \frac{1}{\mu_2} B_{\parallel 2}$

adjust for conductor in contact with dielectric

Dielectric in contact with conductor

(1) $\varepsilon_1 E_{\perp 1} - \varepsilon_2 E_{\perp 2} = \sigma_f$

(2) $B_{\perp 1} - B_{\perp 2} = 0$
(3) $\vec{E}_{\parallel, 1} - \vec{E}_{\parallel 2} = 0$
$\Big\}$ $E_1^{\parallel} = 0$ ($\vec{E} \perp$ to metal part when wave travels)
$B_{\perp 1} = 0$

(4) $\frac{1}{\mu_1} \vec{B}_1^{\parallel} - \frac{1}{\mu_2} \vec{B}_2^{\parallel} = \vec{K}_f \times \hat{n}$