Week 2 pt 2

lets use $ev_0 : \mathbb{R}[x] \to \mathbb{R}$
$f(x) \mapsto f(0)$

then $\operatorname{Ker}(ev_0) = (x)$
$\operatorname{im}(ev_0) = \mathbb{R}$

so $\mathbb{R}[x]/(x) \simeq \mathbb{R}$ is a field

compare: $(x^2-1) \subseteq \mathbb{R}[x]$

claim: (x^2-1) is not a prime ideal

proof: $(x^2-1) = (x-1)(x+1) \in (x^2-1)$

and $(x-1), (x+1) \notin (x^2+1)$

proof: if $x+1 \in (x^2+1)$ then $x+1 = (x^2-1) f(x)$

$\Rightarrow \deg(x+1) = \deg(x^2-1) + \deg(f(x))$

contradiction!

proof. $ev_{1,-1} : \mathbb{R}[x] \to \mathbb{R} \times \mathbb{R}$
$f(x) \mapsto (f(1), f(-1))$

check: $\operatorname{Ker}(ev_{1,-1}) = (x^2-1)$
$\operatorname{im}(ev_{1,-1}) = \mathbb{R} \times \mathbb{R}$

so $\mathbb{R}[x]/(x^2-1) \simeq \underbrace{\mathbb{R} \times \mathbb{R}}_{\text{not an integral domain}}$

$(1,0) \times (0,1) = (0,0)$:(

(Chinese remainder theorem)

proof: $\mathbb{R}[x]/(x^2-1) = \mathbb{R}[x]/(x-1)(x+1) = \mathbb{R}[x]/(x-1) \times \mathbb{R}[x]/(x+1) \simeq \mathbb{R}[x] \times \mathbb{R}[x]$

Ex (4) $\overset{J}{(4)} \subseteq \overset{I}{(2)} \subseteq \overset{R}{\mathbb{Z}}$

$\mathbb{Z}$

$\dots, \boxed{-4}, -3, \boxed{-2}, -1, \boxed{0}, 1, \boxed{2}, \dots$

$2\mathbb{Z} = (2) = \{-4, -2, 0, 2, 4, \dots\}$

$4\mathbb{Z} = (4) = \{-4, 0, 4, \dots\}$

$\longrightarrow$ $\mathbb{Z}/4\mathbb{Z}$

$4\mathbb{Z}/4\mathbb{Z}$ $\boxed{C_0} = \{\dots, -4, 0, 4, \dots\}$

$C_1 = \{\dots, -3, 1, 5, \dots\}$

$2\mathbb{Z}/4\mathbb{Z}$ $\boxed{C_2} = \{\dots, -3, 2, 6, \dots\}$

$C_3 = \{\dots, -1, 3, 7, \dots\}$

$2\mathbb{Z}/4\mathbb{Z} = \{C_0, C_2\}$

$\longrightarrow$ $\dfrac{\mathbb{Z}/4\mathbb{Z}}{(2\mathbb{Z}/4\mathbb{Z})} \simeq \mathbb{Z}/2\mathbb{Z}$

$D_0 = \{C_0, C_2\}$

$D_1 = \{C_1, C_3\}$

Better proof: $\dfrac{R/J}{I/J} \simeq R/I \qquad J \subseteq I \subseteq R$

Find hom. $\phi : R/J \to R/I$

with $\operatorname{Ker}(\phi) = I/J$
$\operatorname{im}(\phi) = R/I$

the $\phi(r+J) = r + I$ will work

why well defined?

suppose $r+J = r'+J$

$r-r' \in J$ so $r-r' \in I$ because $J \subseteq I$

so $r+I = r'+I$