Week 8

$f(z)$ - function on $\mathbb{C}$

z_0: seed - initial point

iterates of f: $f(f(z_0))$

$$z_0 \to f(z_0) \to f(f(z_0)) \to \ldots$$

$$z_n = f^{\circ n}(z_0) = f \circ f \circ f \circ \ldots \circ f(z_0)$$

$$z_0, z_1, z_2, \ldots = z_0, f(z_0), f(f(z_0)), \ldots$$

↖ the orbit of z_0 under f

define the linear function $L: \mathbb{C} \to \mathbb{C}$

by $L(z) = az$ for some $a \in \mathbb{C}$

Caution $f(z) = az+b$ is not linear in this context

↳ f is an affine map

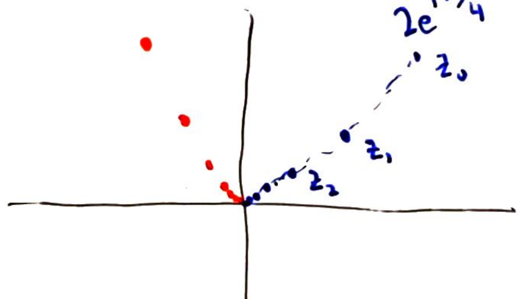
the constant a is called a parameter

a describes a particular system

$$L(z) = \tfrac{1}{2} z$$

$$L(2e^{i\pi/4}) = e^{i\pi/4} = z_1$$

$$L^{\circ 2}(2e^{i\pi/4}) = \tfrac{1}{2} e^{i\pi/4} = z_2$$

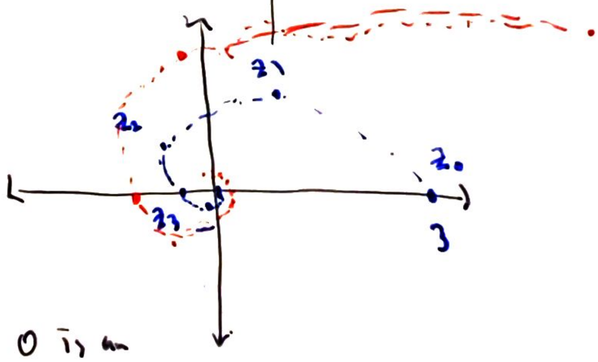


if we looked at $L(z) = \frac{1}{3} e^{i\pi/3} z$

$$L(3) = 1e^{i\pi/3}$$

$$3 \to e^{i\pi/3} \to \tfrac{1}{3} e^{i2\pi/3} \to \tfrac{1}{9} e^{i\pi}$$

$$9e^{i\pi/4} \to 3e^{i\frac{7\pi}{12}} \to e^{i11\pi/12} \to \tfrac{1}{3} e^{i\frac{15\pi}{12}}$$



$L(z) = az$ and $|a| < 1$ then 0 is an attracting point. when orbits spiral to 0

$L(z) = az$ and $|a| > 1$, then orbits spiral away from 0

$L(z) = az$ and $|a| = 1$, points rotate in a circle note that $L(0)=0$ so $L^{\circ n}(0)=0$

orbit of 0 is $0 \to 0 \to 0 \to 0 \to \ldots$

so 0 is a fixed point of L

Normally, in dynamics, different behavior based on different initial points

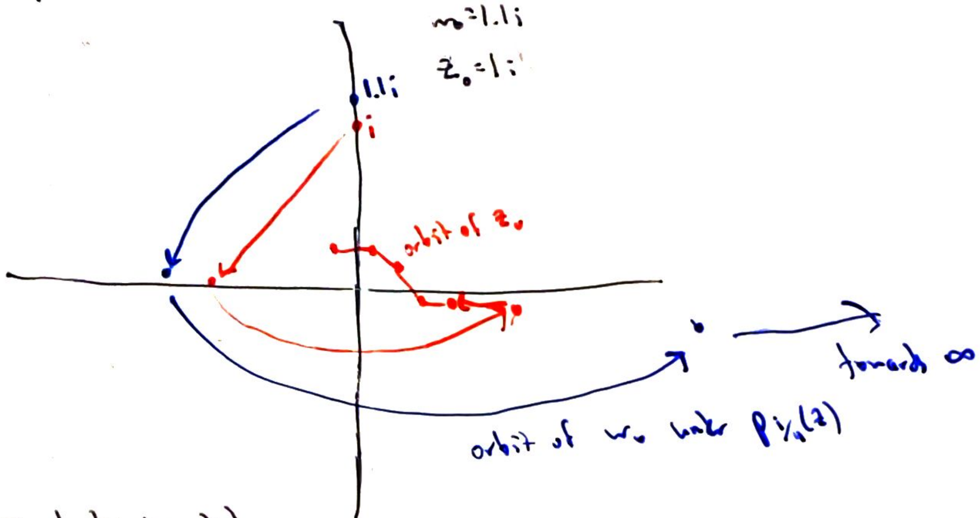
Linear maps, parameter a is all that controlls the behavior of orbits

Quadratic maps $f(z)=z^2+c$ c - parameter

initial point dictate long term behavior

$p_c(z)=z^2+c$

$p_{\frac{i}{4}}(z)=z^2+\frac{i}{4}$



orbit of z_0 looks bounded

orbit of w_0 looks unbounded

What is the long term behavior of a point z under $p_c(z)$?

how does the parameter affect the areas of convergence of different initial values?

$p_{i/4}(z)=z^2+i/4$

Ex 2 in Roeder notes

$p_{i/4}(z)=z$ fixed point

orbit of a fixed point $z \to z \to z \to \ldots$

$z^2+i/4=z \Rightarrow z^2-z=-i/4$

$(z-\frac{1}{2})^2=\frac{1}{4}-\frac{i}{4}$

$z=\frac{1}{2}\pm\sqrt{\frac{1}{4}-i/4}=-0.5+0.288i=z$

$1.05-0.288i$

recall $L(z) = az$ $\quad |a|<1$, spiral in to zero

$\quad |a|>1$, spiral out from 0

$-0.05 + 0.288i = z_0$

$1.05 - 0.288i = z_*$

idea $p_{i/4}(z) \to$ compute linear approximation of f near z_0

compute approx. at z_*

$p'_{i/4}(z) = 2z \qquad p'_{i/4}(z_0) = 2z_0 = 2(-0.05 + 0.288i)$

$|p'_{i/4}(z_0)| \approx 0.58 < 1$

locally near z_0 $\quad p_{i/4}(z) \approx \underbrace{0.58}_{<1}(z - z_0) + p_{i/4}(z_0)$

near z_0, we anticipate spiraling in orbits

$p'_{i/4}(z) = 2z \qquad |p'_{i/4}(z_*)| = |2(1.05 - 0.288i)|$

≈ 2.2

so near z_* $\quad p_{i/4}(z) \approx 2.2(z - z_*) + p(z_*)$

near z_* we expect points to be spiraling out

z_0 is called an attracting fixed point

z_* is called a repelling fixed point

given a value of c, construct $p_c(z) = z^2 + c$

now $p_c(z)$ $\quad c = -1$

$p(z) = z^2 - 1$

$0 \xrightarrow{z^2-1} -1 \xrightarrow{z^2-1} 0 \xrightarrow{z^2-1} -1 \to \ldots$

attracting periodic orbit (period 2)

points reasonably close to any point in an attracting orbit get pulled into it

and $p_c(z) = z^2 + c$ $\quad$ with $c = \frac{1}{2}$

$z^2 + \frac{1}{2}$ $\quad$ does anything converge?

Exercise: for different values of c, can you find different sorts of orbit behavior?