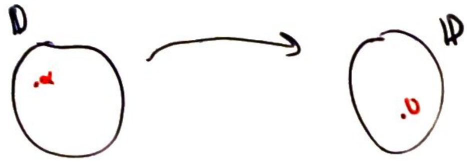
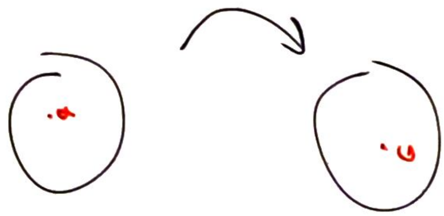
Useful conformal maps: ## Week 2 pt 2 ##

conformal automorphisms of $\mathbb{D}$



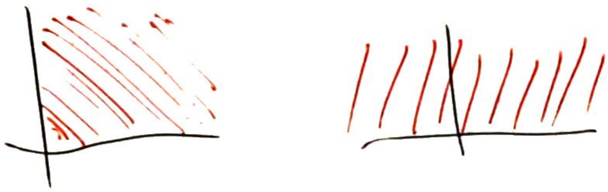
$T(z) = e^{i\theta}\dfrac{z-\alpha}{1-\bar{\alpha}z}$ $\alpha \in \mathbb{D}$

$z \to \dfrac{z-\alpha}{1-\bar{\alpha}z}$



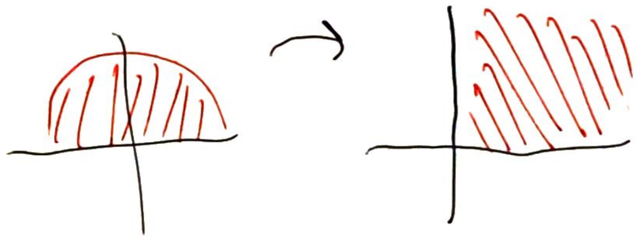
$z \to z^2$

$\sqrt{z} \leftarrow z$



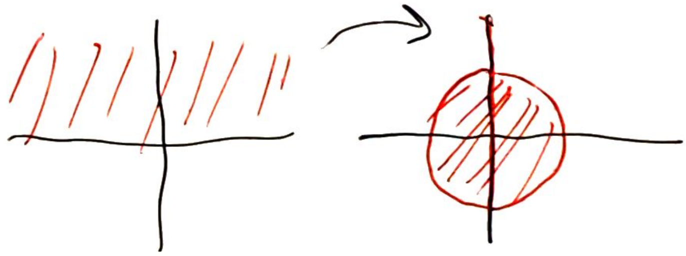
$z \to \dfrac{1+z}{1-z}$

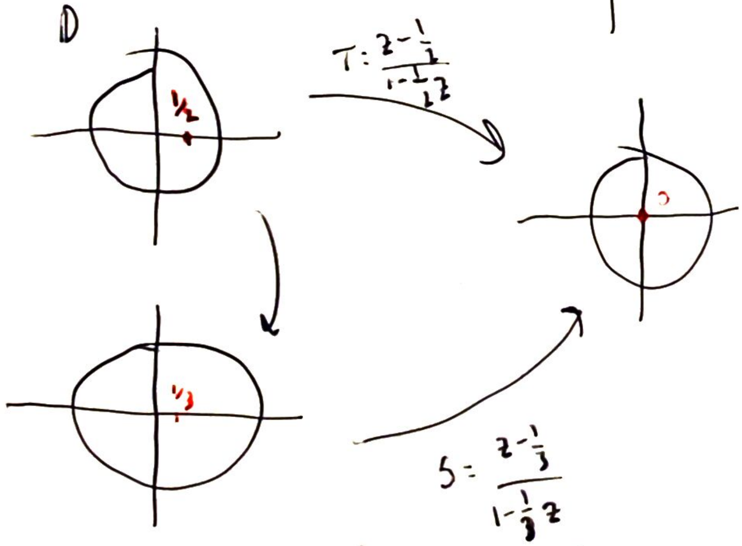
$\dfrac{-(1-z)}{1+z} \leftarrow z$



$z \to \dfrac{z-i}{z+i}$

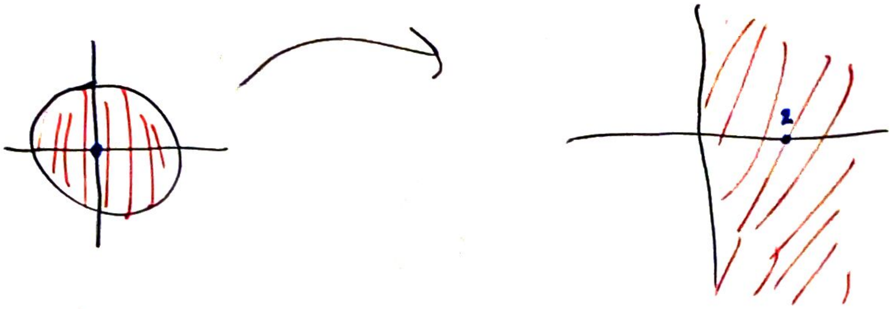
$-i\left(\dfrac{z+1}{z-1}\right) \leftarrow z$





so $S^{-1} \circ T$ sends $\frac{1}{2}$ to $\frac{1}{3}$ and keeps disk hw 2:

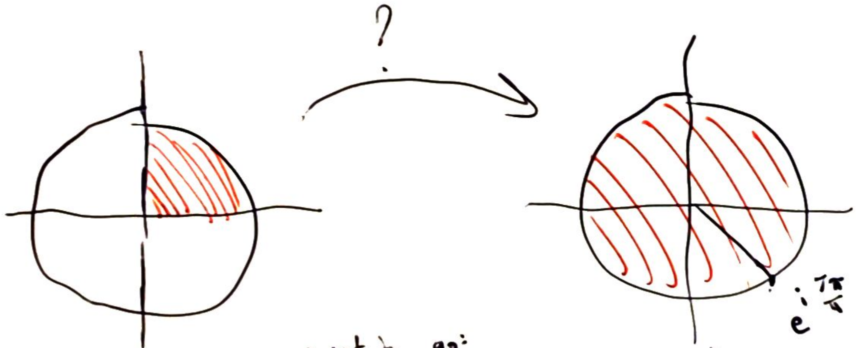
4. want



2. a triangle on the sphere is 3 points connected by parts of great circles

a triangle on the plane can never be a triangle on the sphere

7.



$\left(e^{i\frac{\pi}{4}}\right)^4 = e^{i\pi}$

we want to go:

8.

$f(z) = \frac{az+b}{cz+d} \qquad g(z) = \frac{ez+f}{hz+j}$

$$f(g(z)) = \frac{a\left(\frac{ez+f}{hz+j}\right)+b}{c\left(\frac{ez+f}{hz+j}\right)+d} = \frac{a(ez+f)+b(hz+j)}{c(ez+f)+d(hz+j)}$$

$$= \frac{(ae+bh)z+af+bj}{(ce+dh)z+cf+dj} \quad \text{- LFT}$$

LFT : $T(z) = \dfrac{az+b}{cz+d}$ $\quad ad-bc \neq 0$

Claim : (1) $T(z) = e^{i\theta} \dfrac{z-\alpha}{1-\bar{\alpha}z}$ $\quad \alpha \in \mathbb{D}$

conformal automorphism

(2) If $R(z)$ is a conformal automorphism $\mathbb{D} \to \mathbb{D}$

$R(z) = e^{i\theta} \dfrac{z-\alpha}{1-\bar{\alpha}z}$ (uniqueness)

usefulness:

$T(\alpha) = 0 \qquad T(0) = e^{i\theta}(-\alpha) = -e^{i\theta}\alpha$

Build an S so that $-e^{i\theta}\alpha \to 0$

$S(-e^{i\theta}\alpha) = 0$. $\quad S(z) = \dfrac{z-(-e^{i\theta}\alpha)}{1-(\overline{-e^{i\theta}\alpha})z} = \dfrac{z+e^{i\theta}\alpha}{1+e^{-i\theta}\bar{\alpha}z}$

$S(0) = e^{it}\dfrac{(e^{i\theta}\alpha)}{1} = e^{it}e^{i\theta}\alpha \Rightarrow t = -\theta$

so $S(z) = e^{i\theta} \dfrac{z+e^{i\theta}\alpha}{1+e^{i\theta}\bar{\alpha}z}$

heinous algebra

↳ these are inverse functions

conformal automorphism stuff to check

(1) analytic $\qquad$ zero at $\alpha \in \mathbb{D}$ and pole at $\frac{1}{\bar{\alpha}}$

(2) $T: \mathbb{D} \to \mathbb{D} \qquad T(\mathbb{D}) \subseteq \mathbb{D}$

let ~~$z \in \mathbb{D}$~~ be $|z| = 1$

$|T(z)| = |e^{i\theta}| \left| \dfrac{z-\alpha}{1-\bar{\alpha}z} \right| = \dfrac{|z-\alpha|}{\underbrace{|z|}_{=1}|z^{-1}-\bar{\alpha}|} = \dfrac{|z-\alpha|}{|z^{-1}-\bar{\alpha}|}$

since $|z| = 1 \quad z^{-1} = \bar{z}$

so $= \dfrac{|z-\alpha|}{|z^{-1}-\bar{\alpha}|} = \dfrac{|z-\alpha|}{|\bar{z}-\bar{\alpha}|} = \dfrac{|z-\alpha|}{|\overline{z-\alpha}|} = 1$

so for $z \in \mathbb{D}$, $|T(z)| < 1$ by the maximum modulus principle

so $T(\mathbb{D}) \subseteq \mathbb{D}$

because $S(z) = T^{-1}(z) \qquad S(\mathbb{D}) \subseteq \mathbb{D} \qquad T$ is invertible on disk

T is a bijection

so we have T analytic on $\mathbb{D}$

T bijective on $\mathbb{D}$

just need nonzero derivative

so $T'(z) = e^{i\theta}\left[\frac{1-|\alpha|^2}{(1-\bar{\alpha}z)^2}\right]$

this is not zero

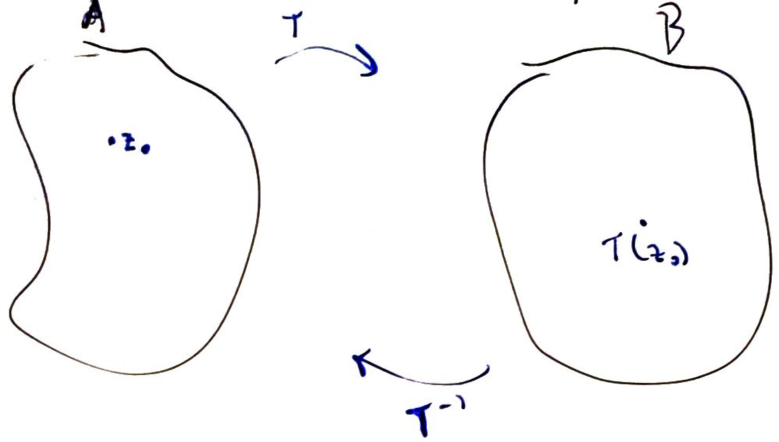
$T(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$

why does the other direction work?

$R(z)$ where R is any conformal automorphism on $\mathbb{D}$

any 2 simply connected domains that aren't all of $\mathbb{C}$ are conformally equivalent

this map is unique if pick $z_0 \in A$, $T(z_0) \in B$ with $T'(z_0) > 0$



start with $R(0)$. Call $R(0)$ α. So $R^{-1}(\alpha) = 0$

Let $T(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$ $\quad T(\alpha) = 0$ also!

~~Let $\theta = \arg(R'(\alpha))$ $R, R', R'(\alpha)$~~

start with R given. Construct $T(z)$ that matches the behavior of R

what do we need? to specify $T(z)$

$T(z) = e^{i\theta}\frac{z-\boxed{\alpha}}{1-\boxed{\bar{\alpha}}z}$

let $\alpha = R^{-1}(0)$

so $T(\alpha) = 0$ $\quad R(\alpha) = 0$

Compute $R'(\alpha)$, let $\theta = \arg(R'(\alpha))$

so $T'(z) = e^{i\theta}\frac{1-|\alpha|^2}{(1-\bar{\alpha}z)^2}$ so $T'(\alpha) = e^{i\theta}$

so $\arg T'(\alpha) = \theta$