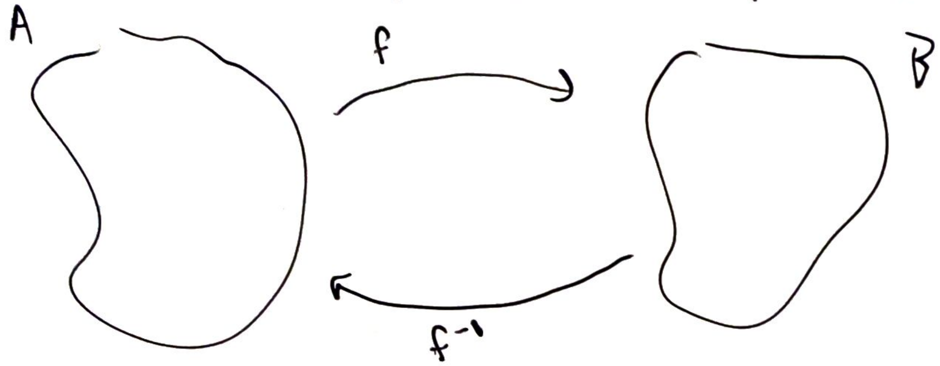
Week 2

Tenative OH: M - 11-12

Th - 3-4

$\underbrace{\text{Bijective}}_{\text{invertible}}$ $\underbrace{\text{conformal maps}}_{\text{perserves local geometry}}$:

Def: Two domains $A, B \subseteq \mathbb{C}$ are called conformally equivalent if there exists a bijective conformormal map $f: A \to B$



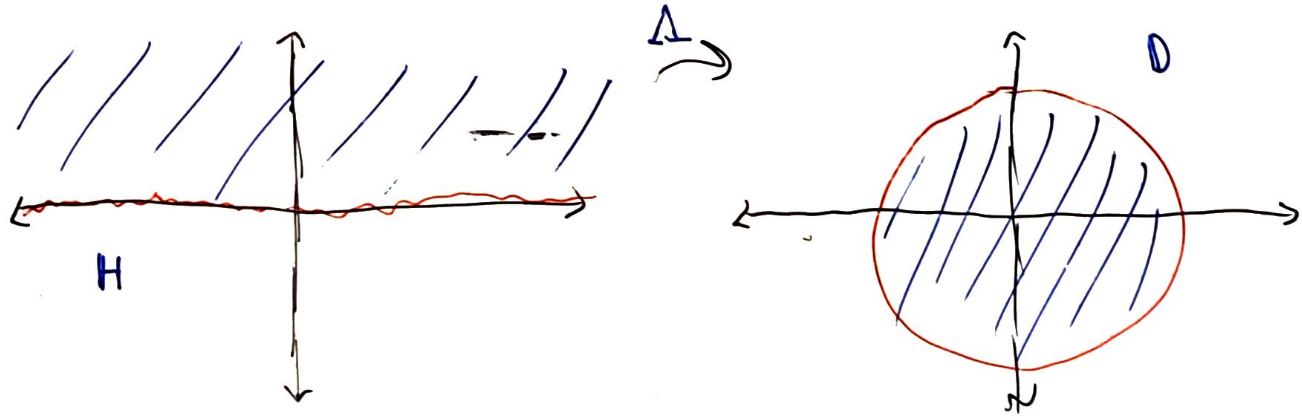
Let $g(z): A \to \mathbb{C}$

if A is "bad" and B is "nice" and A, B are conformaly equivalent, then we define a new function $\hat{g}(z) = g \circ f^{-1}$ (where $f: A \to B$)

$f^{-1}: B \to A$

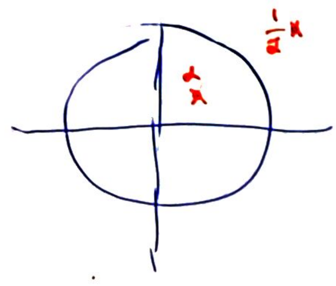
then $\hat{g}$ is on a nice domain

upper-half plan and unit-disk are conformald equivalent



ex: figure out why this is true

Example: $f(z) = \dfrac{z - \alpha}{1 - \bar{\alpha} z}$ $\quad |\alpha| < 1$



claim: $f: \mathbb{D} \to \mathbb{D}$

$\ddot{f} = \Lambda^{-1} \cdot f \circ \Lambda$ this goes from the half plane to itself

$\mathbb{H} \leftarrow \mathbb{D} \leftarrow \mathbb{D} \leftarrow \mathbb{H}$

Theorem: Riemann mapping theorem

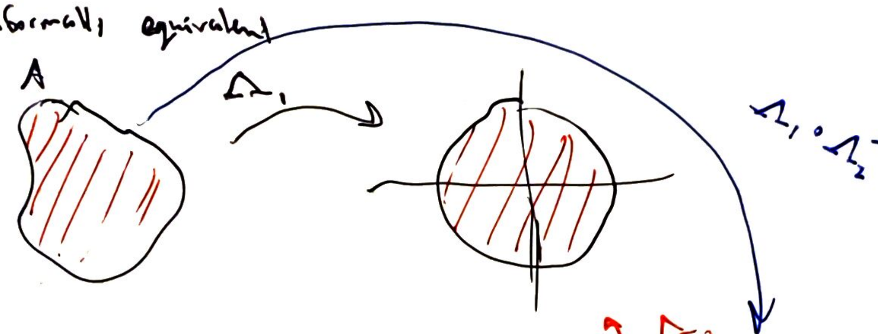
Let A be a simple connecte domain where $A \neq \mathbb{C}$ then there exists a bijective conformal map $f: A \to \mathbb{D}$

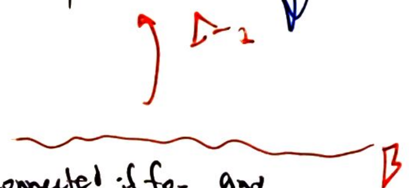
Moreover, for any fixed $z_0 \in A$, we can find a unique such map f with $f(z_0) = 0$ and $f'(z_0) > 0$

Annoyance: this is just an existance theorem

Corrolary: Any simple connected domain A, B, neither one of which is all of $\mathbb{C}$ are conformally equivalent

proof:





So by $\Lambda_1 \circ \Lambda_2$ A is conformally equivalent to B

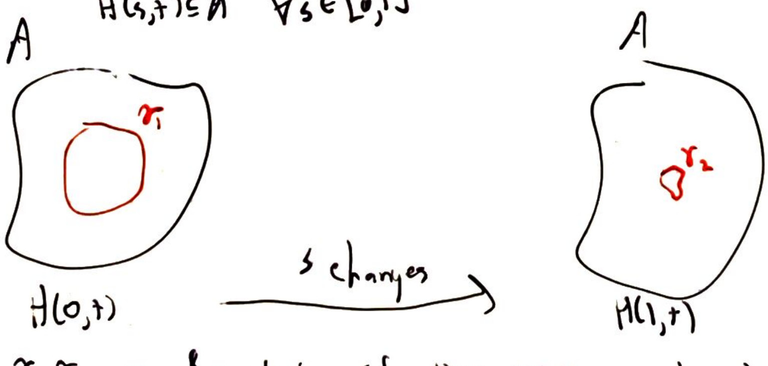
Def: (new version) A domain A is simply connected if for any simple closed curve $\gamma(t) \subseteq A$, $\gamma(t)$ can be continuously deformed to a simple point z_0 in A without leaving A

Give curves $\gamma_1(t), \gamma_2(t) \subseteq A$, a homotopy from γ_1 to γ_2 is a function (continuous)

$$H(s,t) \text{ with } s \in [0,1]$$

$$H(0,t) = \gamma_1(t) \quad \text{and} \quad H(1,t) = \gamma_2(t)$$

$$H(s,t) \subseteq A \quad \forall s \in [0,1]$$

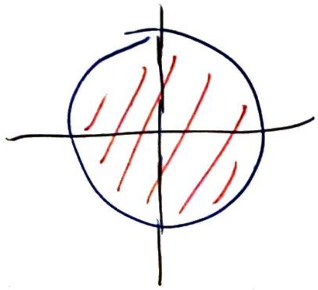


γ_1, γ_2 are homotopic if there exists a homotopy between them

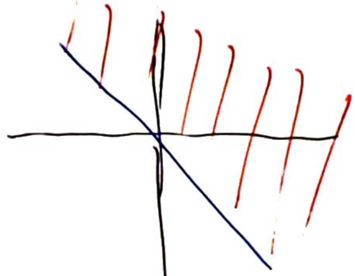
$$H: \gamma_1 \to \gamma_2$$

A domain A is simply connected if every simple closed curve in A is homotopic to a point (if and only if technically)

examples

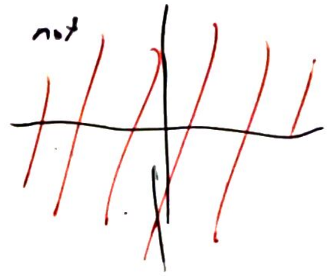


disk



half-plane

why not



Remember the Cauchy integral theorem for derivatives?

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

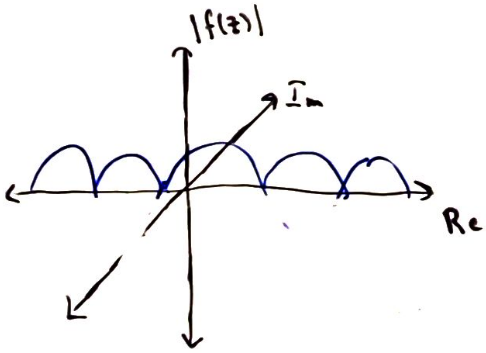
Working with entire functions

↳ analytic on all of $\mathbb{C}$

ex: $e^x = \sum \frac{x^n}{n!}$ converges on all of $\mathbb{R}$ $(R=\infty)$ unbounded on $\mathbb{R}$

$\sin x = \sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ $R=\infty$ bounded on $\mathbb{R}$

By bounded: $\exists M>0$ s.t. $|f(x)|<M$ $\forall x \in \mathbb{R}$



$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

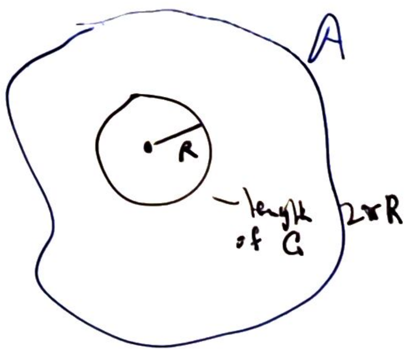
$e^{iz} = e^{i(x+iy)}$

$= e^{ix} e^{-y}$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

assume f is bounded on some domain A containing circle C

$|f(z)| < M$



$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz \right|$$

$$< \frac{n!}{2\pi} \frac{M}{R^{n+1}} 2\pi R \quad \text{(M-L inequality)}$$

(bound for $f(z)/(z-z_0)^{n+1}$)

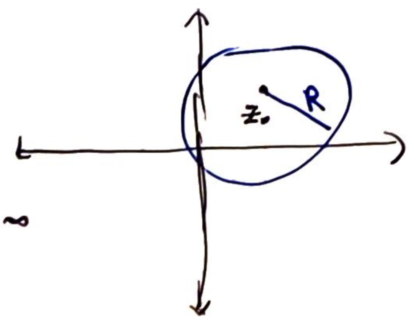
so we get $|f^{(k)}(z_0)| \le \frac{k!}{R^n} M$

<u>Cauchy Inequality</u>: If f is analytic and bounded on A by $|f(z_0)| < M$ and C a circle of radius R in A centered at z_0 then

$$|f^{(n)}(z_0)| \le \frac{n!}{R^n} M$$

Let f be <u>entire</u> and let f be bounded on $\mathbb{C}$

($|f(z_0)| < M$ for all $z_0 \in \mathbb{C}$)



apply cauchy inequality with $n=1$

$$|f'(z_0)| < \frac{M}{R}$$

This works for any circle so if we let $R \to \infty$

we get $|f'(z_0)| = 0 \Rightarrow f'(z_0) = 0$

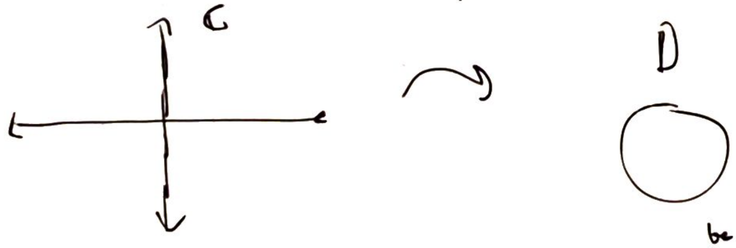
But this is also true for <u>every</u> $z_0 \in \mathbb{C}$

so $f'(z) \equiv 0 \Rightarrow f(z) = c$ for some constant c

so f is a constant function

<u>Liouville's Theorem</u>: Every bounded entire function is necessarily a constant function

Back to question 7: $\mathbb{D} \xrightarrow{\text{conformal bijection}} \mathbb{C}$



suppose $\exists f$ conformal from $\mathbb{C}$ to $\mathbb{D}$

if it existed, it's outputs would be bounted

$\Rightarrow f$ would be bounted

<u>BUT</u> f is entire and bounted

↳ f must be constant

so it can't be one-to-one

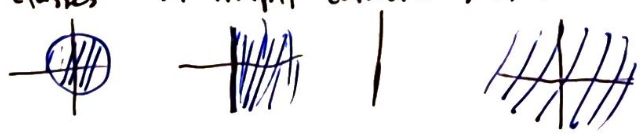
So f cannot exist like this

You <u>also</u> can't go back

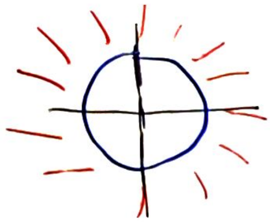
if we found bijective analytic $f: \mathbb{D} \to \mathbb{C}$ because it would be invertible and not work because of the argument above

Riemann mapping theorem: If A, B simple connected and $A, B \neq \mathbb{C}$ then A is conformally equivalent to B

Two "classes" of simply connected sets:



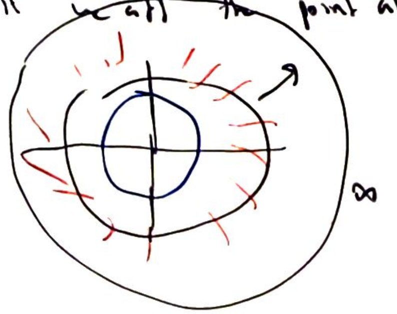
what about:



$\{|z|>1\}$

Not a simply connected domain

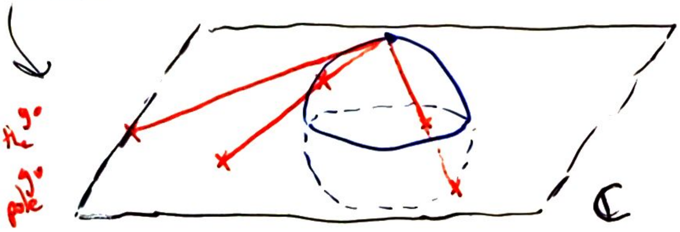
what if we add the point at ∞?



Then a loop can be expanded to the point at ∞

$\mathbb{C} \cup \{\infty\}$ - extended complex plane

as the points go out further to ∞, the points on the sphere go out to the north pole



simply connected?

this area up here is simply connected

Disk-like sets: disks, half planes

$\mathbb{C}$-like sets - just $\mathbb{C}$

Riemann-sphere annuli - outter annuli