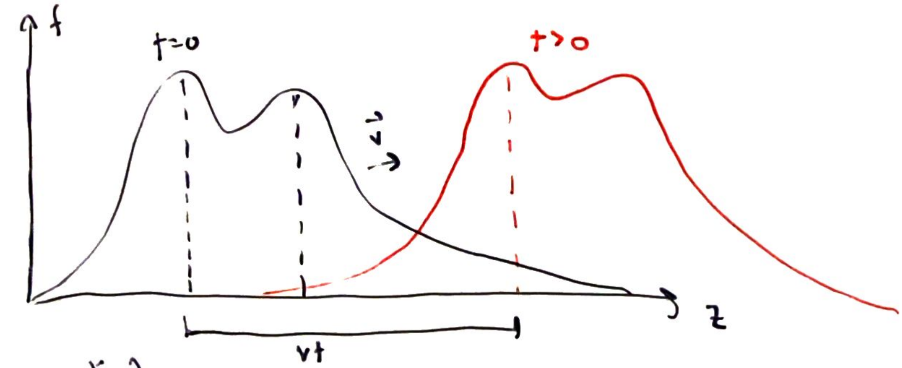
Week 4

9.1 Wave basics



1D (no spreading)

Fixed shape (no dispersion or absorbtion)

$$f(z,t) = f(z-vt, 0)$$

↳ z,t must appear together as combination z-vt

Wave equation in 1D (see Griffiths or Knight → derivation of wave equation)

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

v = wave speed

$v = \sqrt{\frac{T}{\mu}}$ on a string

↳ solutions are of the form $f(z,t) = f(z-vt, 0) = g(v-zt)$

↳ velocity enters as v^2 → we can have a solution $f(z,t) = f(z+vt, 0) \equiv h(z+vt)$

travels to the left, negative z direction

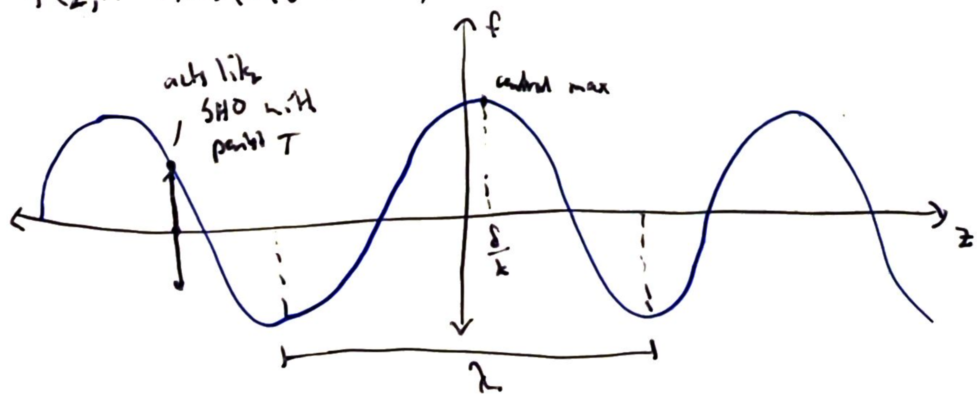
solutions $f(z,t) = g(z-vt) + h(z+vt)$

Note: g and h are not necessarily the same

if same, then we get standing waves

specific type of f

$$f(z,t) = A\cos(k(z-vt)+\delta)$$



phase $= k(z-vt)+\delta$

phase constant $= \delta$

Notice $f(z,t) = \max(\text{peak})$ when phase $= 0 \Rightarrow k(z_{max} - vt) + \delta = 0$

suppose $t=0 \Rightarrow$ $kz_{max} = -\delta$

$$z_{max} = -\frac{\delta}{k}$$

if $\delta > 0 \rightarrow$ step to the left by $\left|\frac{\delta}{k}\right|$

if $\delta < 0 \rightarrow$ step to the right by $\left|\frac{\delta}{k}\right|$

$\lambda = \frac{2\pi}{k}$ $\qquad T = \frac{\lambda}{v} = \frac{2\pi}{kv}$

wave number

$\nu = \frac{1}{T} = \frac{kv}{2\pi}$ (frequency) $\qquad \omega = 2\pi\nu = kv$

$(z - vt)$ must show up in this combination

to the right: $f(z,t) = A\cos(kz - kvt + \delta) = A\cos(kz - \omega t + \delta)$

wave travelling to the right

to the left: $f(z,t) = A\cos(k(z+vt) - \delta) = A\cos(kz + \omega t - \delta)$

$= A\cos(-kz - \omega t + \delta)$

*

In general, $f(z,t) = A\cos(kz - \omega t + \delta)$

travelling right $\Rightarrow k > 0$

travelling left $\Rightarrow k < 0$

$\lambda = \frac{2\pi}{|k|}$

$\omega = |k| v$

Complex notations! (we'll use them a lot)

Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

$Re(e^{i\theta}) = \cos\theta$

$Re\left[Ae^{i(kz-\omega t+\delta)}\right] = A\cos(kz - \omega t + \delta) \rightarrow$ same old *

$$Ae^{i(kz-\omega t+\delta)} = \underbrace{Ae^{i\delta}}_{\tilde{A} \text{ (complex amplitude)}} e^{i(kz-\omega t)}$$

The wave equation is linear

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \Rightarrow f_1, f_2 \text{ are possible solutions}$$

$f = c_1 f_1 + c_2 f_2 + c_3 f_3 + \dots + c_n f_n$ is also a solution of the differential equation

we can create a superior wave by superimposing a bunch of different waves

What makes waves 'different'?

$k, \delta, A, \cancel{\omega}$

$\omega = kv \rightarrow$ determined by medium

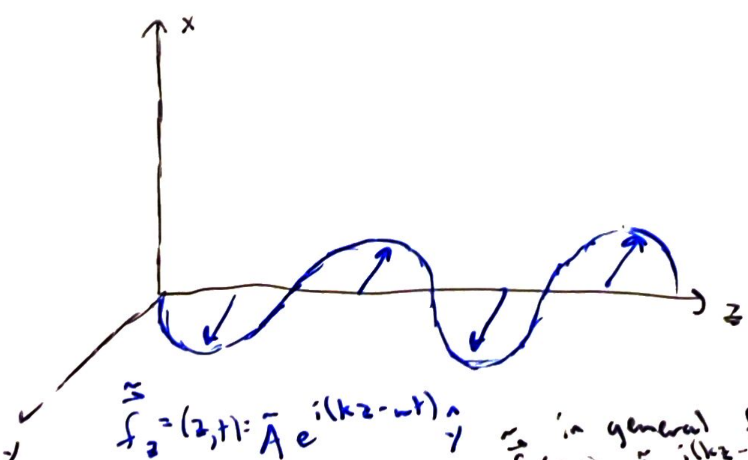
so ω and k are not independent

$$\text{so } f(z,t) = \sum_{\text{all } k} A_k \cos(kz - \omega t + \delta_k)$$

$$= \mathrm{Re}\left[\sum_{k=-\infty}^{\infty} \tilde{A}_k e^{i(kz-\omega t)}\right] = \mathrm{Re}\int_{-\infty}^{\infty} \tilde{A}(k) e^{i(kz-\omega t)}\, dk$$

Polarization

$\tilde{\vec{f}}(z,t) = \tilde{A} e^{i(kz-\omega t)}\,\hat{x}$ — oscillates along $\hat{x}$ direction

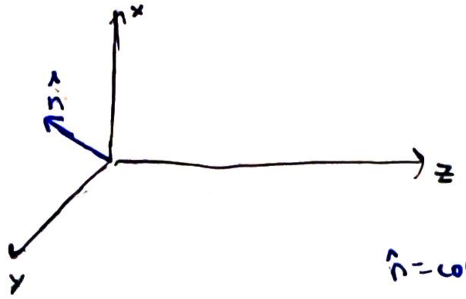


$\tilde{\vec{f}}_2(z,t) = \tilde{A} e^{i(kz-\omega t)}\,\hat{y}$

in general for transverse waves,

$\tilde{\vec{f}}(z,t) = \tilde{A} e^{i(kz-\omega t)}\,\hat{n}$

Week 4 pt 2



$\hat{n}$ - Polarization axis

for transverse waves here

$\hat{n} \cdot \hat{z} = 0$

$\hat{n} = \cos\theta \hat{x} + \sin\theta \hat{y}$

In ch. 7 we derived (no ρ, no J, free space)

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E}$$

$$\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{B}$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \frac{m}{s}$$

Monochromatic Plane waves

↳ single wavelength (single color)

↳ only dependance on one coordinate

will choose z

$$\nabla^2 = \cancel{\frac{\partial^2}{\partial x^2}}^{0} + \cancel{\frac{\partial^2}{\partial y^2}}^{0} + \frac{\partial^2}{\partial z^2}$$

so we get $\frac{\partial^2}{\partial z^2} \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$

$$\frac{\partial^2 \vec{B}}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

$\Rightarrow$ solutions to above equations is $\vec{E} = \vec{E}_0 \cos(kz - \omega t + \delta)$

(3) $\vec{E} = \mathrm{Re}\left[\tilde{\vec{E}}_0 e^{i(kz - \omega t)}\right] = \mathrm{Re}\left[\tilde{\vec{E}}\right]$

(4) $B = \mathrm{Re}\left[\tilde{\vec{B}}_0 e^{i(kz - \omega t)}\right] = \mathrm{Re}\left[\tilde{\vec{B}}\right]$

complex amplitudes

Goal: find additional information about 3 and 4

sub into maxwell's equations

$$\left.\begin{array}{l} \vec{\nabla} \cdot \vec{E} = 0 \quad (\text{b/c } \rho = 0) \\ \vec{\nabla} \cdot \vec{B} = 0 \quad (\text{always}) \end{array}\right\}$$

$\vec{\nabla} \cdot \tilde{\vec{E}} = \frac{\partial}{\partial z} \tilde{E}_z = \tilde{E}_{0,z} (ik) e^{i(kz - \omega t)} = 0$

Similarly $\vec{\nabla} \cdot \tilde{B}(z,t) = \tilde{B}_{0,z} (ik) e^{i(kz - \omega t)} = 0$

so $\tilde{E}_{0,z} = \tilde{B}_{0,z} = 0$

so EM waves are not longitudinal, they are transverse

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \text{(Faraday's law)}$$

$$\vec{\nabla}\times\tilde{\vec{E}} = -\frac{\partial\tilde{\vec{B}}}{\partial t}$$

$$-\frac{\partial\vec{B}}{\partial t} = -i\omega\tilde{\vec{B}}_0 e^{i(kz-\omega t)}$$

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \partial/\partial z \\ \tilde{E}_x & \tilde{E}_y & 0 \end{vmatrix} = \hat{x}\left(-\frac{\partial}{\partial z}\tilde{E}_y\right) - \hat{y}\left(-\frac{\partial}{\partial z}\tilde{E}_x\right)$$

$$= \hat{y}\frac{\partial}{\partial z}\tilde{E}_x - \hat{x}\frac{\partial}{\partial z}\tilde{E}_y$$

so $\frac{\partial}{\partial z}\tilde{E}_x\hat{y} - \frac{\partial}{\partial z}\tilde{E}_y\hat{x} = -i\omega\tilde{\vec{B}}_0 e^{i(kz-\omega t)}$

$= -i\omega\tilde{\vec{B}}$

so $-i\omega\tilde{B}_y = \frac{\partial}{\partial z}\tilde{E}_x$ and $-i\omega\tilde{B}_x = \frac{\partial}{\partial z}\tilde{E}_y$

$\Rightarrow$ $-\omega\tilde{B}_{0,y} = k\tilde{E}_{0,x}$

$-\omega\tilde{B}_{0,x} = -k\tilde{E}_{0,y}$

$\hookrightarrow$ $\tilde{\vec{B}}_0 = \frac{k}{\omega}\underbrace{(\hat{z}\times\tilde{\vec{E}}_0)}$ so $\boxed{|\vec{B}_0| = \frac{1}{c}|\vec{E}_0|}$

no phase shift between E and B waves

$\delta_E = \delta_B$ (in vacuum)

Week 4 pt 3

In electromagnetic wave in a vacuum

1) $\vec{B} \perp \vec{E}$
2) Transverse
3) $|\vec{B}| = \frac{|\vec{E}|}{c}$
4) $\delta_E = \delta_B$ (waves are in phase)

$k(\hat{z} \times \vec{E}_0) = \omega \vec{B}_0$ (from Faraday's)

$|\vec{B}_0| = \frac{k}{\omega} |\hat{z} \times \vec{E}_0|$

$= \frac{k}{\omega} |\hat{z}| |\vec{E}_0| \sin\left(\frac{\pi}{2}\right)$

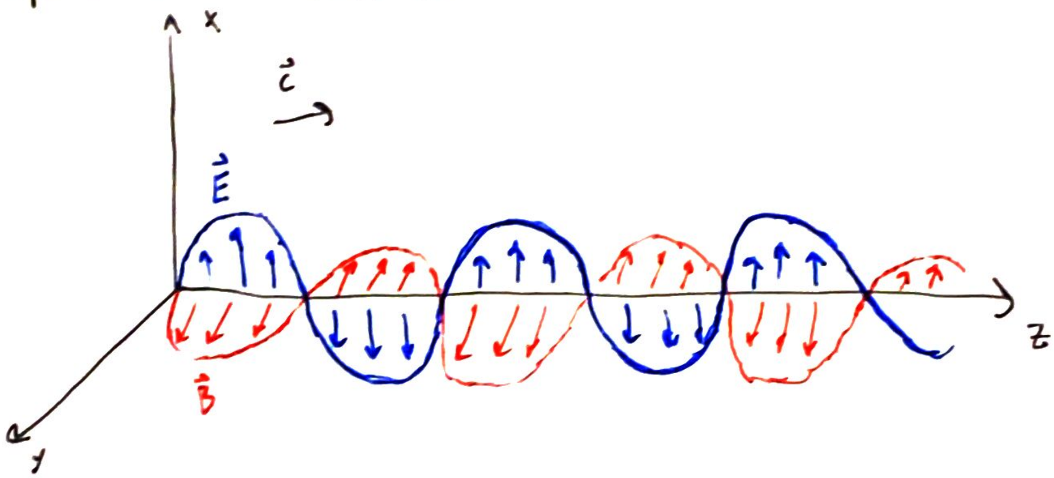
$= \frac{k}{\omega} |\vec{E}_0|$

$\vec{B}_0 = \frac{k}{\omega} \vec{E}_0$

$B_0 e^{i\delta_B} = \frac{k}{\omega} E_0 e^{i\delta_E}$

$\Rightarrow \delta_B = \delta_E$

Ampere's law $\rightarrow$ same shit



$\vec{E} \times \vec{B}$ shows direction of propagation

Energy and Momentum in EM waves

* $\vec{E}(z,t) = E_0 \cos(kz - \omega t + \delta)\hat{x}$

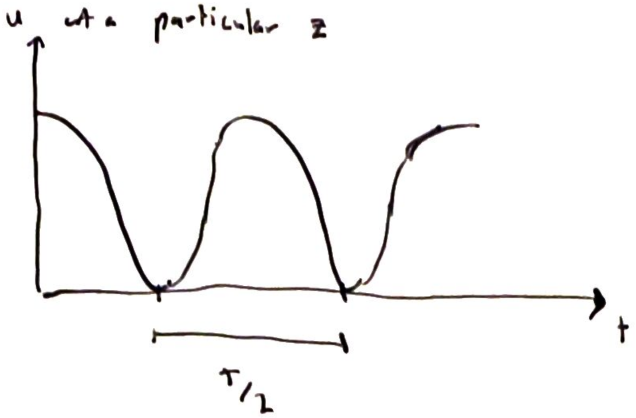
** $\vec{B}(z,t) = \sqrt{\mu_0 \varepsilon_0}\, E_0 \cos(kz - \omega t + \delta)\hat{y}$ $\quad B_0 = \frac{1}{c} E_0 = \sqrt{\mu_0 \varepsilon_0}\, E_0$

Recall: energy density: $u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{1}{\mu_0} B^2$

equal contribution due to E and B fields

$= \varepsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) = \frac{1}{\mu_0} B_0^2 \cos^2(kz - \omega t + \delta)$

u at a particular z $\qquad T = \frac{2\pi}{\omega}$



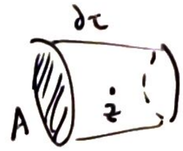
$$\vec{S} = \frac{\text{energy}}{\text{time}\cdot\text{area}} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0 \epsilon_0}\frac{1}{c}\epsilon_0 \cos^2(kz - \omega t + \delta)\,\hat{z}$$

* *

we can check $\vec{\nabla}\cdot\vec{S} = -\frac{\partial}{\partial z}cu = \frac{\partial}{\partial t}u$

energy balance equation

wave carries energy $u\,d\tau = u\,A\,dz = uAc\,dt$



$$\frac{\text{energy}}{\text{area}\cdot\text{time}} = uc$$

$\vec{g} = \epsilon_0(\vec{E}\times\vec{B})$ (momentum density in EM fields)

$$= \epsilon_0\mu_0\underbrace{\left(\frac{1}{\mu_0}\vec{E}\times\vec{B}\right)}_{\vec{S}} = \frac{1}{c^2}\vec{S} = \boxed{\frac{1}{c}u\hat{z}}$$

In experiments, only avg. values can be measured

Time-averaged value of <u>anything</u>

$$\langle x \rangle = \int_0^T x(t)\,dt\left(\frac{1}{T}\right)$$

$$\langle u \rangle = u_{max}\underbrace{\frac{1}{T}\int_0^T \cos^2(kz - \omega t + \delta)}_{\frac{1}{2}} = \frac{u_{max}}{2}$$

Trick: $\cos^2 x + \sin^2 x = 1$

integrate over 1 period $\Rightarrow$ $\underbrace{\frac{1}{2\pi}\int_0^{2\pi}\cos^2 x\,dx}+\underbrace{\frac{1}{2\pi}\int_0^{2\pi}\sin^2(x)\,dx} = 1$

these should be the same so each is $\frac{1}{2}$

so $\langle u \rangle = \frac{1}{2}\varepsilon_0 E_0^2$

$$\langle \vec{S} \rangle = \langle c u \hat{z} \rangle = c\hat{z}\langle u \rangle = \frac{1}{2} c \varepsilon_0 E_0^2 \hat{z}$$

$$\langle \vec{g} \rangle = \langle \frac{1}{c} u \hat{z} \rangle = \frac{1}{c}\hat{z}\langle u \rangle = \frac{1}{2c}\varepsilon_0 E_0^2 \hat{z}$$

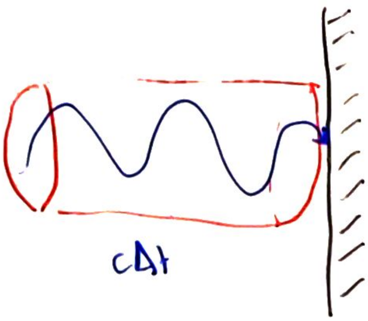
$$\text{Intensity} = \frac{\text{Avg. energy}}{(\text{area})\cdot\text{time}} = \frac{\text{Avg. power}}{\text{area}}$$

$$= |\langle \vec{S} \rangle|$$

$$I = \frac{1}{2} c \varepsilon_0 E_0^2$$

perfect absorber

Light falling on a surface exerts a force



$$F = \frac{|\Delta p|}{\Delta t}$$

$$\text{pressure} = \frac{F}{A} = \frac{1}{A}\frac{|\cancel{p_f}^{\,0} - p_i|}{\Delta t}$$

$$\vec{p}_i = \vec{g}\,(\text{volume}) = \vec{g} A c \Delta t$$

$$\text{pressure} = \frac{1}{A}\frac{\langle g \rangle c}{\Delta t} = c\langle g \rangle = \frac{1}{2}\varepsilon_0 E_0^2 = \frac{I}{c} \quad (\text{Radiation pressure})$$

compare with I