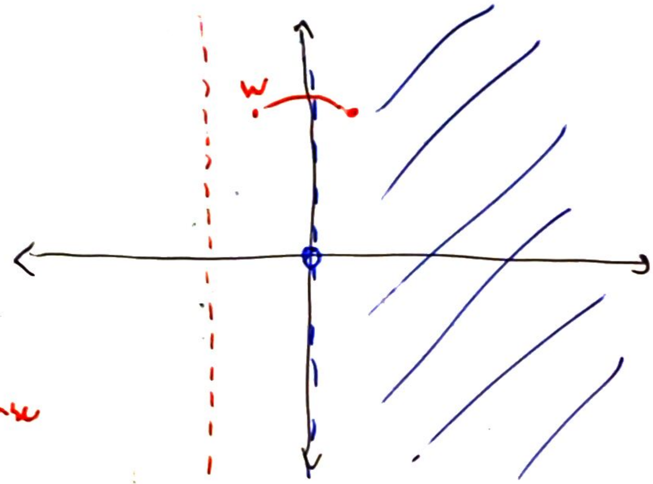
Week 6

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \qquad \text{for } Re(z) > 0$$

$\Gamma(n+1) = n!$ or $\Gamma(n+1) = n\Gamma(n)$

this can be extended to get the functional equation

$$\boxed{\Gamma(z+1) = z\Gamma(z)}$$



$-1 < Re(w) < 0$

$\Gamma(w)$ can be defined by

$(w-1)+1 = w$

$\Gamma(w)$

ex: $\Gamma(-\frac{1}{2}+i)$ doesn't make sense

but $\Gamma(\frac{1}{2}+i)$ is defined

so we can define $\Gamma(-\frac{1}{2}+i)(-\frac{1}{2}+i) = \Gamma(\frac{1}{2}+i)$

generally $\frac{\Gamma(w+1)}{w} = \Gamma(w)$ this gives you $-1 < Re(w) < 0$

Then we can define the next strip over, and so on and so on

what about $f(z) = \sin(\pi z)$

for any integer $\sin(k\pi) = 0$

so note $\Gamma(3) = 2!$ and $\Gamma(3) + \sin(3\pi) = 2!$

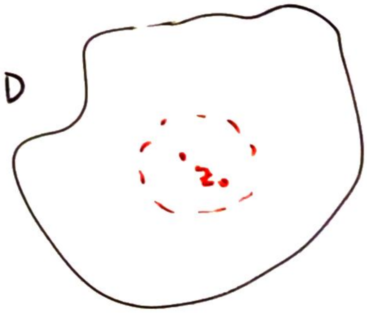
$\Phi(z) = \Gamma(z) + \sin \pi z$

$\Phi(k+1) = \Gamma(k+1) + \sin(k\pi) = k! = \Gamma(k+1)$ (Pseudo-gamma function)

Fact: If f is nonconstant analytic on domain D and z_0 is a zero of f

$f(z_0) = 0$

then $\exists$ a neighborhood N of z_0 so that $f(w) \neq 0$ for any $w \in N \setminus \{z_0\}$



$f(z_0) = 0$

but $f(z) \neq 0$ anywhere around there

let z_i be a sequence of zeros for an analytic function

$f(z_i)=0 \ \forall i \in \mathbb{Z}$ what if $z_i \to w$?

example: $f(z)=(z-1)(z-\frac{1}{2})(z-\frac{1}{4})(z-\frac{1}{8})\ldots$

$f(1)=0$, $f(\frac{1}{2})=0$, $f(\frac{1}{4})=0$, ...

f, g are analytic on the same set

suppose $z_1, z_2, \ldots \in D$ so $f(z_i)=g(z_i) \ \forall i$

$z_i \to z \in D$ then $f=g$ on D

this has to be true because $f-g$ is analytic and has to be the zero function because its analytic at z and $(f-g)(z_i)=0$

Q: can you use complex analysis to find holes in a domain

Theorem: Suppose f and g are analytic on a domain D (open and path connected)

Let $z_1, z_2, \ldots$ be a sequence of distinct points with $z_i \to z \in D$ and $f(z_i)=g(z_i)$ $i=1,2,3,\ldots$

then $f=g$ on D

why? f,g analytic $\Rightarrow$ $f-g$ analytic and $(f-g)(z_i)=0 \ \forall i$

and $(f-g)(z)=0$ but this isn't isolated

so $f-g=0$ on D so $f=g$ on D

If $f=g$ on a convergent sequence, then $f=g$ on the whole domain

If $f=g$ on a continuous curve also

If $f=g$ on an open subset of the domain

If there is a single point $\hat{z}_0$ in D and a neighborhood $\{|z-z_0|<\varepsilon\}$ where f and g agree, then they agree on all of D

Theorem: Let $f: A \to \mathbb{C}$ and $g: B \to \mathbb{C}$ be analytic on $A, B \subseteq \mathbb{C}$

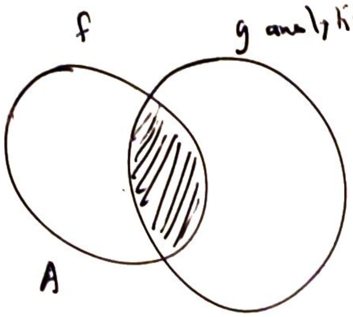
Suppose $A \cap B \neq \emptyset$ and $f=g$ on $A \cap B$

let $h = f(z)$ on A and $g(z)$ on B

then h is analytic on $A \cup B$ <u>and</u> h is the only analytic function on $A \cup B$ that has $h|_A = f$ and $h|_B = g$

h is called the <u>analytic</u> <u>continuation</u> of f onto B

analytic continuation of f onto B



f analytic on A

$f = g$ on $A \cap B$

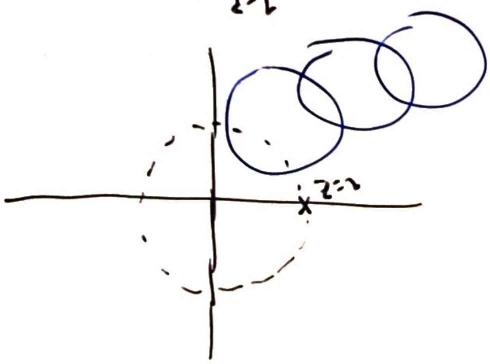
then define $h = f$ on A

$h = f$ on B

then h is analytic on $A \cup B$

h is the unique analytic function extending f onto B

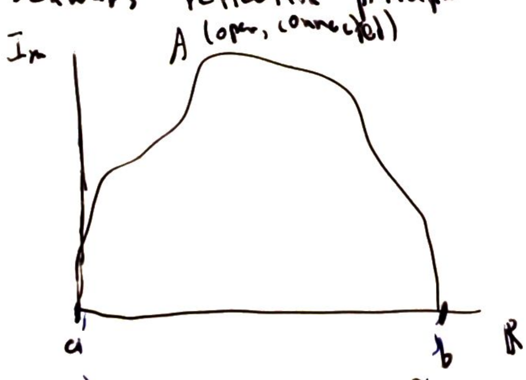
ex: $f(z) = \frac{1}{z-2}$



taylor series for f has radius 2

goal: need some methods to extend the domain of analytic functions

Schwarz reflection principle



if f is analytic on A

and f continuous on (a,b)

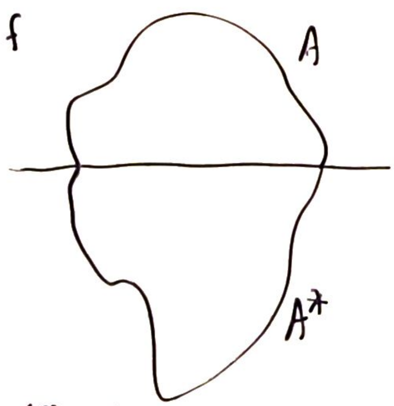
and f is real valued, for real inputs

Then let $A^* = \{z : \bar{z} \in A\}$

A^*

define $g(z) = \overline{f(z)}$ on A^* (the reflection of f), then g is analytic and is the unique analytic continuation of f to the set $A \cup (a,b) \cup A^*$

In the proof, the key step is Morera's theorem (converse to Cauchy-Goursat)

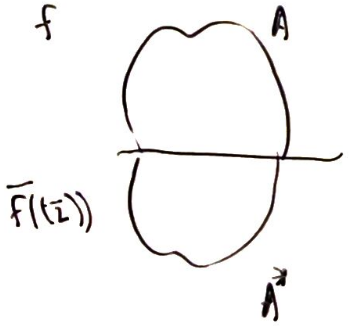


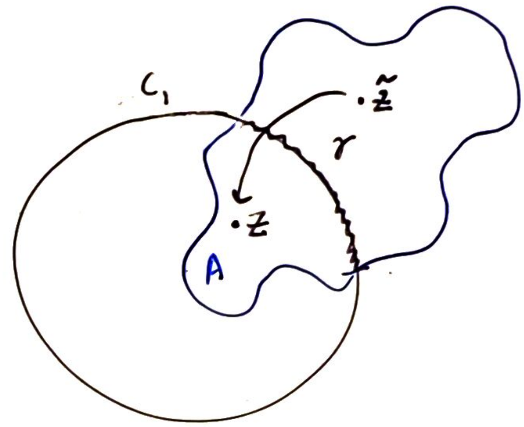
x real $\Rightarrow$ $f(x)$ is real

$g(x) = \overline{f(\bar{x})} = \overline{f(x)} = f(x)$

(because conjugate does nothing to a real number)

$g(z) = \overline{f(\bar{z})}$





let A be the region in the interior or exterior of a circle

part of the boundary of A is on an arc of the circle (γ)

Suppose f is analytic on A and continuous on γ

let $f(\gamma)$ be on arc of a different circle C_2

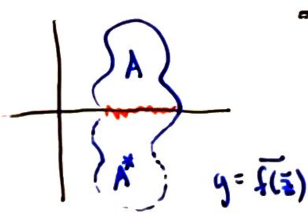
let $g(z) = \widetilde{f(\tilde{z})}$ where $\sim$ is the reflection across the circle

on $\tilde{A} = \{z : \tilde{z} \in A\}$

if $C_1 = Re(z)$ then $\tilde{z} = \bar{z}$

<u>How to analytically continue a function</u>

1. formula + conformal maps
2. reflection
   - automatic analyticity



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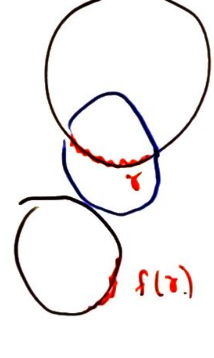
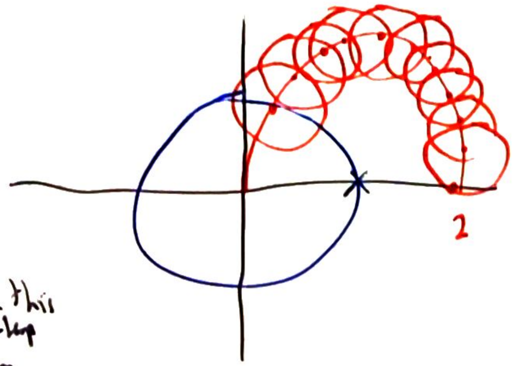


3. Draw a curve, make power series

   f analytic on a disk centered at z_0

consider $f(z) = \sum_{k=0} z^k$

then we want to know $f(2)$



each circle has f analytic so we just keep picking points in each radius

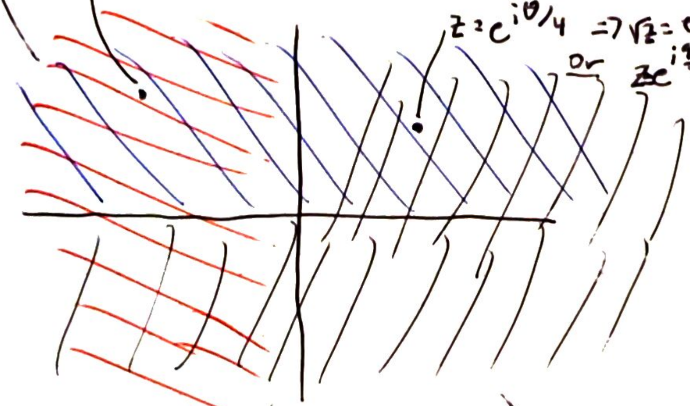
$\sqrt{z}$ agrees on this overlap

$z = e^{i3\pi/4}$

$f(z) = \sqrt{z} = \sqrt{re^{i\theta}} = \sqrt{r}\,e^{i\theta/2}$ what if the curve intersects itself

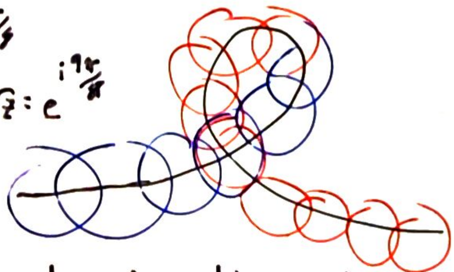
$z = e^{i\theta/4} \Rightarrow \sqrt{z} = e^{i\pi/8}$

or $z = e^{i\frac{9\pi}{4}}$ $\sqrt{z} = e^{i\frac{9\pi}{8}}$



$\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$

$\pi < \arg z < \frac{5\pi}{2}$



are the red and blue circles going to have f that agrees on their overlap

$\sqrt{z}$ agrees on the overlap of black and red

but $\sqrt{z}$ does not agree on the overlap of black and blue