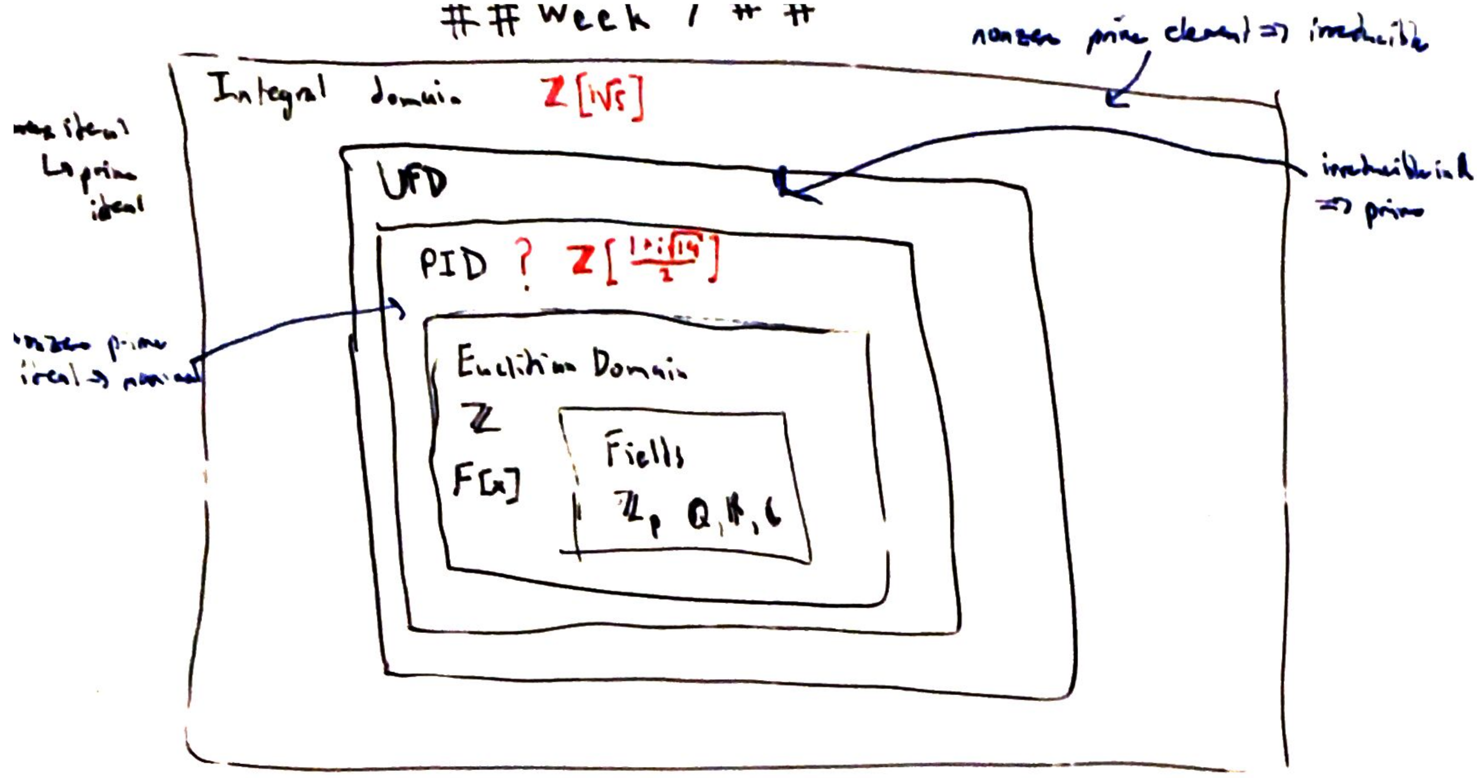
Week 1



Claim 1: In a PID, every nonzero prime ideal is maximal

<u>proof</u>: Suppose I is a nonzero prime ideal in a PID, R. (I is proper)

Suppose $I \subseteq J$

R a PID so $I = (a)$ and $J = (b)$ for $a, b \in R$

Then $I = (a) \subseteq (b) = J \Rightarrow a \in (b)$ so $a = rb$ for some $r \in R$

Then $rb = a \in I$ so by I being prime $\Rightarrow r \in I$ or $b \in I$.

Suppose $r \in I$. Then $r = ka$ for some $k \in R$

$a = rb = kab$ $(a \neq 0)$ so cancel a

$1 = kb$ so $1 \in (b) = J$ and $J = R$

Next suppose $b \in I$ then $b = ka$ for some $k \in R$

$a = rka$ $1 = rk$

Then $J = (b) \subseteq I \Rightarrow J = I$

Claim: Every PID is a UFD

<u>proof</u> (sketch): Suppose R is a PID

(Existence of factorization into irreducibles in R)

Suppose $r \in R$ is a nonzero, nonunit element

If r is irreducible in R, then done (factorization $r = r$)

If r is reducible in R $r = s_1 t_1$ for nonunits s_1, t_1

If s_1, t_1 are irreducible, then you're done

Now suppose s_1 is reducible in R

↳ $s_1 = s_2 t_2$ continue...

Q: Why must this process terminate

Elements: $r = s_1 t_1$ $s_1 = s_2 t_2$ $s_2 = s_3 t_3$, ...

ideals: $(r) \subsetneq (s_1)$ $s_1 \subsetneq (s_2) \subsetneq (s_3) \subsetneq (\dots$

can this happen infinitely in a PID?

No, PID's are Noetherian

Aside: in a general ring R a chain of ideals can be 1) finite or infinite

ascending $I_1 \subseteq I_2 \subseteq \dots$

descending or $\dots \subseteq I_3 \subseteq I_2 \subseteq I_1$

A ring is Noetherian if there are no strictly ascending infinite chains of ideals

A ring is Artinian if there are no strictly descending infinite chains

Ex: in $\mathbb{Z}$ $(60) \subseteq (15) \subseteq (5) \subseteq (5) \subseteq \dots$

Claim 3: Every PID is Noetherian

proof: Suppose $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ is an ascending chain of ideals in a PID, R

Let $I = \bigcup_{k=1}^{\infty} I_k$

(Exercise: this is an ideal)

R is a PID $\Rightarrow$ $I = (a)$ for some $a \in I$

Then $a \in \bigcup_{k=1}^{\infty} I_k$ so it's in I_N for some $N \in \mathbb{N}$

$\Rightarrow a \in I_n$ for all $n \geq N$

but then $I = (a) \subseteq I_n \subseteq I$ so $I_n = I$ for all $n \geq N$

so it must stabilize