Week 9

$f(z) = z^2 + c$ $\qquad$ $c = 0.19 + 0.54i$

attracting fixed point $f(z) = z$

near fixed point $\boxed{f(z) \approx a(z - z_0)} + a_2(z - z_0)^2 + O(z - z_0)^3$

and $a = f'(z_0)$ so if $|a| < 1$, z_0 is attracting

the basins of attracting solutions have boundaries that are Julia sets, fractal

the mandlebrot set comes up when you always use 0 as the i.c. and vary c in $f(z) = z^2 + c$

$$M = \{c : 0 \text{ stays bounded under iteration of } f\} \qquad f = z^2 + c$$

the main boundary of M is a cardioid, it can be found from solving for attracting solutions to $z^2 + c = z$

periodic orbit of $f(z) = z^2 + c$ is a sequence

$$z_0, z_1, z_2, \dots, z_m$$

$$z_0 \xrightarrow{f} z_1 \xrightarrow{f} z_2 \xrightarrow{f} \dots \xrightarrow{f} z_m = z_0$$

a period orbit of period m

Ex: $f(z) = z^2 - 1$ $\qquad$ $f(0) = -1$ $\qquad$ $f(f(0)) = f(-1) = 1 - 1 = 0$

$0 \to 1 \to 0$ $\qquad$ period 2

points in a periodic orbit are called <u>period points</u> of period m for f

$f(z) = z^2 - 1$ $\qquad$ $f(f(z)) = (z^2 - 1)^2 - 1 = z^4 - 2z^2 + 1 - 1 = z^4 - 2z^2$

$z^4 - 2z^2 = z$ when $\quad z^4 - 2z^2 - z = 0$

$$z(z^3 - 2z - 1) = 0$$

$z = 0$, $z = -1$ $\qquad$ fixed points of $f(f(z))$

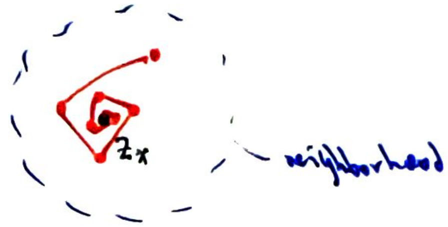
period 2 periodic points for $f(z)$

upshot: fixed point for $f^{\circ n}(z)$

$$\Updownarrow$$

periodic point of $f(z)$ of period j, $j | n$

A fixed point z_* is called <u>attracting</u> if there is a neighborhood of z_* that $\lim_{n\to\infty} f^{\circ n}(z) = z_*$ for any $z \in D(z_*, r)$



A periodic orbit of period m $z_0, \dots, z_m = z_0$ is attracting if z_n is an attracting fixed point for $f^{\circ n}$

$L(z) = az+b$ $L'(z) = a$ $|a|<1$, attracting
$|a|>1$, repelling

The <u>multiplier</u> of a periodic orbit

$$z_0 \to z_1 \to \dots \to z_m = z_0$$

is $\lambda = f'(z_0) f'(z_1) \dots f'(z_{m-1})$

λ is the <u>multiplier</u>

<u>Theorem:</u> Attracting periodic orbit lemma:
A periodic orbit is attracting if the multiplier $\lambda = f'(z_0) f'(z_1) \dots f'(z_m)$ has $|\lambda|<1$

<u>Basic idea:</u> z_i

$$f^{\circ m}(z_i) = z_i$$

$$g(z) = f^{\circ n}(z) - z_i \quad \text{so} \quad g(z_i) = 0$$

$$g(z) = \lambda(z-z_i) + \underbrace{a_2(z-z_i)^2 + \dots}_{\text{all small}}$$

$$\approx \lambda(z-z_i)$$

<u>Basin of attraction</u>: $\mathcal{O}: z_0 \to z_1 \to \dots \to z_m = z_0$ (attracting periodic orbit)

$A(\mathcal{O})$ = basin of attraction

$$= \{ z \in \mathbb{C} : f^{\circ n}(z_0) \to z_j \text{ as } n\to\infty \text{ for some } 0 \le j \le m-1 \}$$

<u>Immediate</u> basin of attraction: union of connected components of $A(\mathcal{O})$ that contain $z_0, z_1, \dots, z_{m-1}$

Koenig's Theorem:

Suppose z_0 is an attracting fixed point of polynomial with multiplier $\lambda \neq 0$, then there exists a neighborhood U of z_0 and conformal map $\phi: U \to \phi(U)$ so that for any $w \in \phi(U)$ we have $\phi \circ f \circ \phi^{-1}(w) = \lambda w$

How to <u>find</u> periodic orbits, how do you know they're attracting

$f(z) = z^2 + c$ can be shown to make

$z_0 \to z_1 \to \dots \to z_n = z_0$ orbit of period n

near an attracting fixed point, z_0, $f(z) \approx a(z - z_0)$

$|a|$ $a = r_a e^{i\theta_a}$

so repeated iteration looks like repeated angle addition and repeated radius reduction

solving $z^2 + c = z$

$z^2 - z = -c$

$z^2 - z + \frac{1}{4} = -c + \frac{1}{4}$

$z = \frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$

so we get 2 fixed points

$z_0 = \frac{1}{2} + \sqrt{\frac{1}{4} - c}$

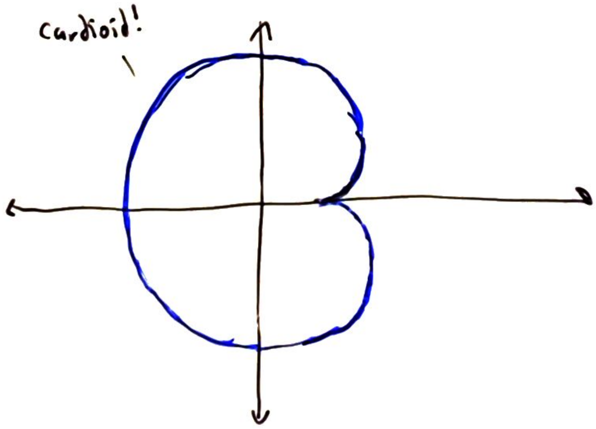
$z_* = \frac{1}{2} - \sqrt{\frac{1}{4} - c}$

so we need to find a to see if they're attracting

$a = f'$

$|f'(z_0)| < 1$

$|f'(z_*)| < 1$ the set of c values that satisfy these



Upshot: If you can find a fixed point, you can test attraction with $|f'(z_0)|$

Period 2 fixed points

$$z_0 \xrightarrow{f} z_1 \xrightarrow{f} z_0$$

assume that $f(z) = z^2 + c$ has c chosen so that f has a period 2 orbit

$f(z_0) = z_1$, $f(z_1) = z_0$

$f(f(z_0)) = f(z_1) = z_0$

so z_0 is a fixed point for $f^{\circ 2} = f \circ f$

$$(z^2+c)^2 + c = z$$

$$z^4 + 2z^2c + c^2 + c = z$$

$$z^4 + 2cz^2 - z + c^2 + c = 0$$

$$p(z) = 0$$

roots of $p(z)$, solutions of $p(z)=0$, fixed points of $f \circ f$

z_1, z_2, z_3, z_4

candidates for our period 2 orbit

$f(z) = z^2 + c$ $\quad$ $f(r_i)$ if this is any of the other points

test $f(r_i) = r_j$ if you found 1 of these, you're done

<u>Fatou-Julia lemma</u> (2.1 in Roeder, pg. 20)

If f is degree d, then f has at most d-1 attracting orbits

Test an orbit, λ = multiplier

$\lambda = f'(z_0) f'(z_1)$ $\quad$ (for 2 orbit)

$$\lambda = \frac{d}{dz}\left(f^{\circ 2}(z)\right)\Big|_{z=z_1}$$

period 2 orbit is attracting if $|\lambda| < 1$

$$|f'(z_0) f'(z_1)| < 1$$

$$\left|\frac{d}{dz}\left(f^{\circ 2}(z)\right)\Big|_{z=z_0}\right| < 1$$

start with <u>$f(z) = z^2 + c$</u> $\quad$ (WTK: dynamics)

- fixed points
- orbit (period)
- for which initial value does $f^{\circ n}(z)$ stay bounded

Given has a period 3 orbit

$$f(f(f(z_0))) = z_0$$

$$z_0 \to z_1 \to z_2 \to z_0 \quad (z_0 \xrightarrow{f^{\circ 3}(z)} z_0)$$

$\deg f^{\circ 3} = 2^3 = 8$ (8 roots)

this gives us $r_1, r_2, \dots, r_8$

the fixed points of f are useless to us, throw them out!

$z^2 + c = z$

$\underbrace{r_1, \dots, r_6,}$ new list

then everything in this list is in <u>some</u> orbit of period 3

lets look at <u>multiplier</u>

$$\lambda = \frac{d}{dz} f^{\circ 3}(z_i) \Big|_{z = r_i}$$

if I knew the actual orbit $r_1 \to r_6 \to r_5 \to r_1$ for ex

Then $\lambda = f'(r_1) f'(r_5) f'(r_6)$

In mathematica, the blue plot is the boundary for the basin of attraction

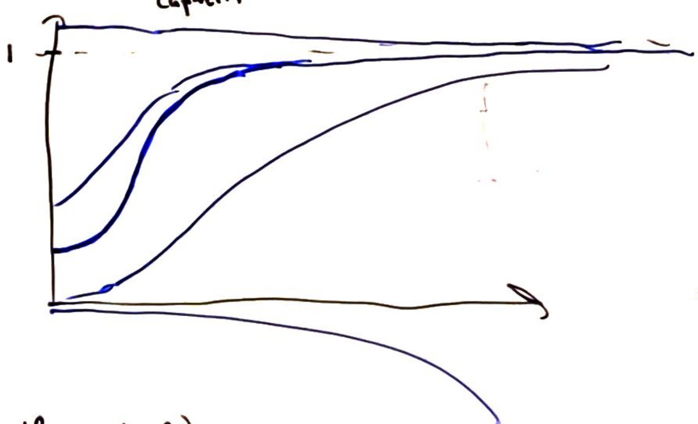
J_c = filled julia set for $z^2 + c$

$= \{ z_0 : f^{\circ n}(z_0)$ stays bounded as $n \to \infty \}$

you can then think of ∞ as a fixed point of a polynomial

$\frac{dP}{dt} = kP$ (leads to exponential growth)

$\frac{dP}{dt} = kP \underbrace{(1-P)}_{\text{carrying capacity}}$ (exponential growth with a limit on P ($P=1$))



stable equilibrium $P=1$

unstable equilibrium $P=0$

$\frac{dP}{dt} = kP(1-P)$

$\frac{P(t+\Delta t) - P(t)}{\Delta t} = kP(t)(1-P(t)) \Rightarrow P(t+\Delta t) - P(t) = kP(t)(1-P(t))$

let $\Delta t = 1$ and $P(t) = P_n$

$P_{t+1} - P_t = kP_t(1-P_t)$

$P_{n+1} - P_n = kP_n(1-P_n) \Rightarrow P_{n+1} = P_n + kP_n(1-P_n)$

$P_{n+1} = kP_n\left(\left(1+\frac{1}{k}\right)-P_n\right)$

$= k\left(1+\frac{1}{k}\right)P_n\left(1-\frac{k}{1+k}P_n\right)$

$P_{n+1} = (k+1)P_n(1-cP_n)$

$P_{n+1} = AP_n(1-cP_n)$ $\quad S_n = cP_n$

$\boxed{S_{n+1} = \alpha S_n(1-S_n)}$ Discrete version of the logistic growth equation

Discrete dynamical system

this is a quadratic map!

$x_{n+1} = cx_n(1-x_n)$ is $\underline{\text{conjugate}}$ to

$f(z) = cz(1-z)$

you can change it by a LFT

$x_{n+1} = x_n^2 + c$

so this is the same thing as the f's that we'd be studying

$cz(1-z) = z$ $\quad z=0$ (fixed)

assume c is possitive, $c>0$

$cz - cz^2 = z \Rightarrow cz^2 + (1-c)z = 0$ so $z=0$ or $\underbrace{z = 1-\frac{1}{c}}$

$z=0$ and if $c>1$, $z = 1-\frac{1}{c}$ are fixed points $\quad$ exists when $c>1$

$f(z) = cz - cz^2$ $\quad f'(z) = c - 2zc$

$z=0$ $\quad f'(0) = c$ $\quad$ so 0 is attractive if $|f'(0)| = c < 1$

for $0<c<1$, 0 is an attractive fixed point

$f'\left(1-\frac{1}{c}\right) = 2-c$ $\quad$ so $\left|f'\left(1-\frac{1}{c}\right)\right| = |2-c| < 1$ when $-1<2-c<1$

$1<c<3$

so $z = 1-\frac{1}{c}$ is attractive for $1<c<3$