Week 3

we already defined ideals

↳ additive subrings that are closed under scaling

Properties of ideals

- trivial, proper, improper
- generators: ⟶ finitely generated

  ⟶ principle

Example 1: R is any ring

trivial ideal: $I = \{0\}$

improper ideal: $I = R$

Proposition: Suppose I is an ideal in a ring with unity then TFAE:

① I is improper, i.e. $I = R$

② I contains a unit, i.e. there is a unit $u \in I$ (with multiplicative inverse)

③ $1 \in I$

proof ① ⟹ ② $I = R \Rightarrow 1 \in I$ which is a unit

② ⟹ ③ Suppose you have a unit

then u has an inverse element $u^{-1} \in R$

I ideal $\Rightarrow 1 = \underbrace{u^{-1}}_{\in R} \cdot \underbrace{u}_{\in I} \in I$

③ ⟶ ① Suppose $1 \in I$, take any $r \in R$, then $r = r \cdot 1$ and $r \cdot 1 \in I$ because I ideal

$\Rightarrow I = R$

Corrolary 1 (Roy's sadness corrolary)

Suppose F is a field and $I \subseteq F$ is an ideal then I is trivial or improper

Corrolary 2: Suppose F is a field

then every homomorphism $\phi : F \to R$ where R is (nontrivial) ring is injective

proof: $\mathrm{Ker}(\phi)$ is an ideal so it's either trivial ($\Rightarrow$ injective) or $\mathrm{Ker}(\phi) = F$ (b/c $\phi(1_F) = 1_R$) so can't happen

Generators for ideals:

Suppose R is a commutative ring (w/1)

Let $A \subseteq R$ be any subset

Then the ideal generated by A is the smallest ideal in R that contains A

Notational options:

1: (A)

2: $\langle A \rangle$

3: RA

Given an ideal $I \subseteq R$, we say:

- I is <u>generated</u> by A if $I = (A)$
- I is <u>finitely</u> <u>generated</u> if $I = (A)$ for some finite set A
- I is principle if $I = (\{a\}) = (a)$ for an element $a \in R$

<u>FACTS</u> ① $(A) = \bigcap\limits_{\substack{I \subseteq R : \text{ideal} \\ A \subseteq I}} I$

② $(A) = \left\{ \sum\limits_{\text{finite}} r_i a_i : r_i \in R,\ a_i \in A \right\}$

Example 2: In $\mathbb{Z}$, all ideals $I \subseteq \mathbb{Z}$ are of the form

$$I = (n) = n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$$

$\Rightarrow$ all ideals principle

Example 3: in $\mathbb{Z}[x]$

$(2) = \{2p(x) : p(x) \in \mathbb{Z}[x]\} = 2\mathbb{Z}[x]$

$(2, x) = \{2p(x) + xq(x) : q(x), p(x) \in \mathbb{Z}[x]\}$

Example 2 (cont.) $(2, 6) = \{2k + 6m : k, m \in \mathbb{Z}\} = \{2(k + 3m) : k, m \in \mathbb{Z}\}$
$= (2)$

claim $(2, x)$ is <u>not</u> a principle ideal ($\Rightarrow$ not every ideal in $\mathbb{Z}[x]$ is principle)

first check $(2, x) = \{2p(x) + xq(x) : p(x), q(x) \in \mathbb{Z}[x]\}$

$= \{ f(x) \in \mathbb{Z}[x] : f(0)$ is even ($\Leftrightarrow f(x)$ has an even constant term) $\}$
($f(x) = c_0 + c_1 x + \dots + c_n x^n$ where c_0 is even)

quick consequence: $(2, x)$ is proper, $1 \notin (2, x)$

Now suppose $(2, x)$ is principle $(2, x) = (g(x))$

then ① $2 \in (2, x) = (g(x)) \Rightarrow 2 = p(x) g(x)$ for some $p(x)$

Using degree, this implies $p(x), g(x)$ are constant

$g(x) = \pm 1$ or $g(x) = \pm 2$

If $g(x) = \pm 1$ are units (so $(g(x)) = R$), so no

$g(x) = \pm 2$

Then $x \in (2, x) = (g(x)) \Rightarrow x = p(x)g(x) = \underbrace{p(x)(\pm 2)}_{x \text{ coefficient even}}$

so not possible

$\Rightarrow (2, x)$ is not principle

Prime and Maximal ideals

Theme: For an ideal $I \subseteq R$

properties of $I \leftrightarrow$ properties of R/I

Recall: For each ideal $I \subseteq R$ we have a "natural projection"

$$\pi : R \to R/I$$
$$r \mapsto r + I = \{r + i : i \in I\}$$

Remember: $a + I = b + I \iff a - b \in I$

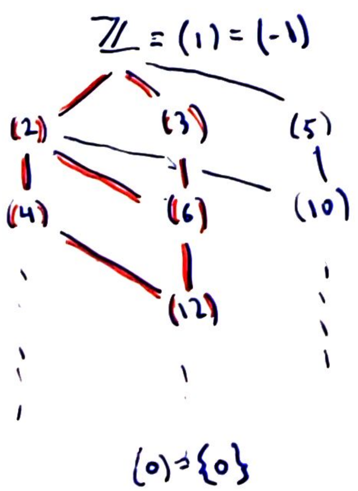
Example: ①

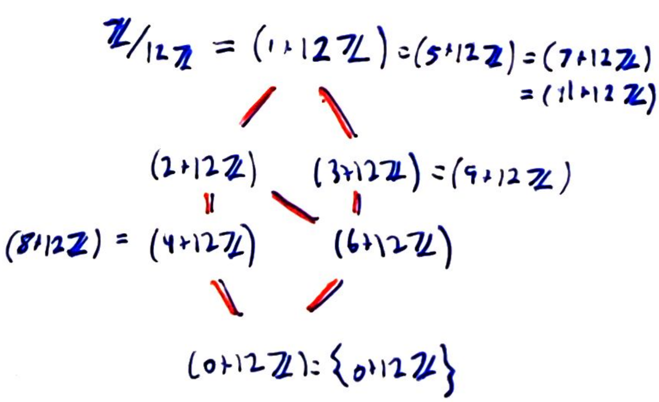
$I \subseteq R$ is proper $\iff R/I$ is nontrivial

Example ②: In $\mathbb{Z}$, look at ideal $(12) = 12\mathbb{Z}$

recall: all ideals in $\mathbb{Z}$ are principle; of the form (n) for $n \in \mathbb{Z}$

2 $(m) \subseteq (n) \iff n$ divides m





FACT Lattice isomorphism theorem for rings

Given an ideal $I \subseteq R$

1. There is a bijection $\{\text{ideals } J \supseteq I \text{ in } R\} \leftrightarrow \{\text{ideals } \bar{J} \text{ in } R/I\}$

$$J \mapsto \pi(J) = \{j + I : j \in J\}$$
$$\pi^{-1}(\bar{J}) \longleftarrow \bar{J}$$

2. This is a bijection of "lattices" i.e. it respects inclusions, intersections

Maximal ideals

<u>Def</u>: A proper ideal $I \subseteq R$ is <u>maximal</u> if it it's maximal among proper ideals (ordered by inclusion) i.e.

if J is an ideal with $I \subseteq J \subseteq R$ then either $J = I$ or $J = R$

in $\mathbb{Z}[x]$ Last time: $J = (2, x)$ is a proper $_\wedge$ ideal
no principle

consider $I = (x) = \{x p(x) : p(x) \in \mathbb{Z}[x]\}$

Definitely: $(x) \subseteq (2, x) \subsetneq \mathbb{Z}[x]$ but note $2 \in J = (2, x)$
but $2 \notin (x)$

$\Rightarrow (x) \subsetneq (2, x) \subsetneq \mathbb{Z}[x]$

$$\begin{array}{c} \mathbb{Z}[x] \\ \\ (2,x) \\ | \\ (x) \end{array}$$ <u>not</u> maximal

<u>Theorem</u>: A proper ideal $I \subseteq R$ is maximal

$\Updownarrow$

R/I is a field

<u>proof</u>: $I \subseteq R$ is maximal $\Leftrightarrow$ only ideals in R that contain I are I and R

$\Leftrightarrow$ lattice iso thm: the only ideals in the quotient ring are I/I = triv. and R/I

$\Updownarrow$

R/I is a field ■

Ex (1) (cont.) $(x) \subseteq \mathbb{Z}[x]$ is <u>not</u> maximal $\Leftrightarrow$ $\mathbb{Z}[x]/(x)$ is not a field

idea: try to involve first isomorphism theorem

Ideal $= (x) = \{x \cdot p(x) : p(x) \in \mathbb{Z}[x]\} = \{f(x) \in \mathbb{Z}[x] : f(0) = 0\}$

$= \mathrm{Ker}(ev_0)$

where $ev_0 : \mathbb{Z}[x] \to \mathbb{Z}$
$f(x) \mapsto f(0)$

image? $\mathbb{Z}$

$\Rightarrow \mathbb{Z}[x]/(x) = \mathbb{Z}[x]/\mathrm{Ker}(ev_0) \cong \mathrm{im}(ev_0) = \mathbb{Z}$
not a field :(

Exercise: $\phi: \mathbb{Z}[x] \to \mathbb{Z}/2\mathbb{Z}$

$f(x) \mapsto f(0) + 2\mathbb{Z}$

claim: $\mathrm{Ker}(\phi) = (2, x)$

$\mathrm{im}(\phi) = \mathbb{Z}/2\mathbb{Z} \Rightarrow \frac{\mathbb{Z}[x]}{(2,x)} \cong \mathbb{Z}/2\mathbb{Z}$ field

so $(2,x)$ is maximal

but $\mathbb{Z}$ is an integral domain

<u>Theorem</u>: Suppose $I \subseteq R$ is a proper ideal

then I is prime

$\Updownarrow$

R/I is an integral domain

<u>proof</u>: R/I integral domain $\Leftrightarrow$ if $(a+I)(b+I) = 0$ then $a+I = 0+I$ or $b+I = 0+I$

$ab + I = 0 + I$

$\Leftrightarrow$ if $ab \in I$, then either $a \in I$ or $b \in I$

$\Leftrightarrow$ I is prime

<u>Def</u>: A proper ideal is prime if whenever $ab \in I$, then $a \in I$ or $b \in I$

Observations / facts:

① All maximal ideals are prime

② Every proper ideal is contained in a maximal ideal

(need: Zorn's lemma)

③ In $\mathbb{Z}$: all the prime ideals are maximal

Intuition for ideals: subspaces of a vector space

special case: principle ideals $(r) = \{cr : c \in R\}$

"span of r"

Claim: r is a unit iff (r) is improper / $(r) = R$

($\Rightarrow$) r is a unit $\Rightarrow$ there is some element $v \in R$ s.t. $rv = 1$

so $1 = vr \in (r) \Rightarrow (r) = R$

($\Leftarrow$) Suppose $(r) = R$

then $1 \in (r) \Rightarrow 1 = cr$ for some $c \in R$

$\Rightarrow r$ is a unit

Ex: · in $\mathbb{Z}$ (all ideals are principle)

$(6) = \{\ldots, -6, 0, 6, \ldots\}$

Claim: this is not a prime ideal

Proof: $2, 3 \notin (6)$ but $2 \cdot 3 = 6 \in (6)$

Proof: Look at $\mathbb{Z}/6\mathbb{Z}$ not a domain

b/c $\bar{2}, \bar{3} \neq 0$ but $\bar{2} \cdot \bar{3} = \bar{0}$

$(2) = \{\ldots, -2, 0, 2, 4, \ldots\} = 2\mathbb{Z}$

claim: (2) is a prime ideal

proof: Look at $\mathbb{Z}/2\mathbb{Z}$ is a field (so an integral domain)

$\Rightarrow$ (2) is maximal (thus prime)

proof: Suppose $ab \in (2)$ for some $a, b \in \mathbb{Z}$

so $2 \mid ab \Leftrightarrow ab = 2k$ for some $k \in \mathbb{Z}$

$\Rightarrow 2 \mid a$ or $2 \mid b$

$\Rightarrow a \in (2)$ or $b \in (2) \Rightarrow (2)$ is prime

note that in $\mathbb{Z}$ $(1) = \mathbb{Z}$ (neither max or prime)

$(0) = \{0\}$ (prime but not maximal)

$\mathbb{Z}/\{0\} \simeq \mathbb{Z}$ (int. domain but not a field)

Ex (3): $\mathbb{R}[x]$

$(2) = \{2f(x) : f(x) \in \mathbb{R}[x]\} = \mathbb{R}[x]$

$(x) = \{x f(x) : f(x) \in \mathbb{R}[x]\}$

$= \{g(x) \in \mathbb{R}[x] : g(x) = c_1 x + c_2 x^2 + \ldots\}$

$= \{g(x) \in \mathbb{R}[x] : g(0) = 0\}$

Claim: x is a prime ideal

proof 1: Suppose $f(x) g(x) \in (x)$ for some $\mathbb{R}[x] \ni f(x), g(x)$

$\Rightarrow f(x) g(x) = x h(x)$ for some $h \in \mathbb{R}[x]$

$x = 0$: $f(0) \cdot g(0) = 0 \Rightarrow$ so $f(0) = 0$ or $g(0) = 0$

so $f \in (x)$ or $g \in (x)$

proof: $\mathbb{R}[x]/(x)$

find homo. $\phi : \mathbb{R}[x] \to S$

with $\ker(\phi) = (x)$