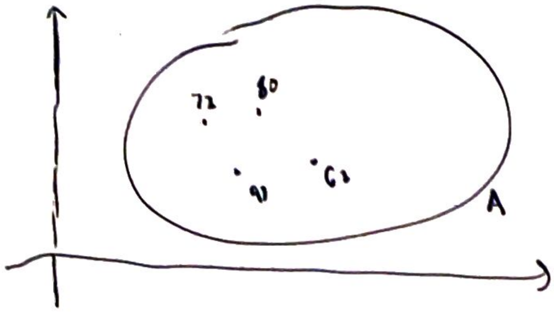
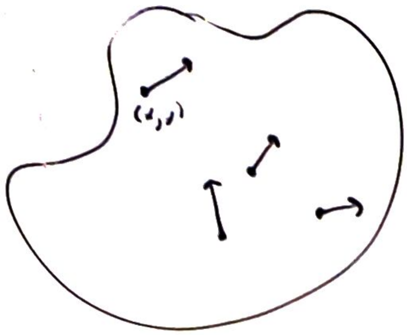
Week 4 pt 2

A scalar field on domain $A \subseteq \mathbb{R}^2$
is a function $u(x,y)$ that asigns to each point in A a scalar value



A vector field is a function $V(x,y)$ that asigns a vector to each point in A



$$v = (v_1(x,y), v_2(x,y))$$

u is a scalar field

$$\nabla u = \text{grad}\, u = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

grad u is a vector pointing in the direction of greatest change in u.

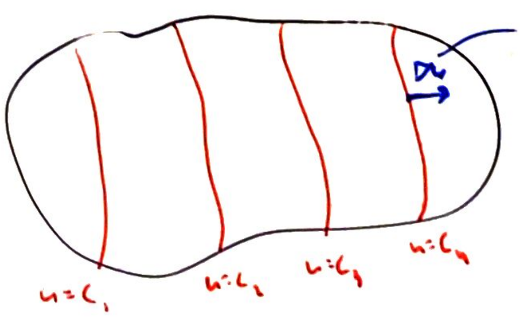
$u \longrightarrow$ grad u

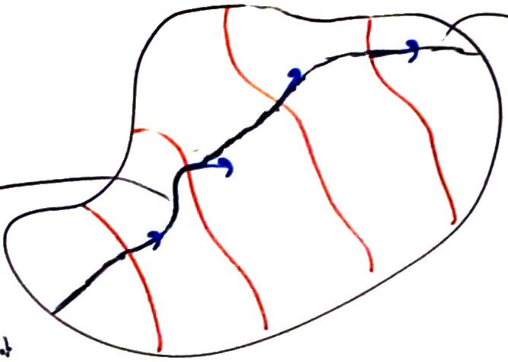
potential $\quad$ vector field

C^1 $\quad$ continuous

$u(x,y) = c$ (level curve)

grad u is orthogonal to equipotential





flow lines / stream lines

we want to find a curve that has these grad vectors as tangent lines

$V = \text{grad } u$

↳ line integrals are path independent

↳ curl is 0 : irrotational fields

curl $V = 0$

$$V = \text{grad } u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, 0\right)$$

$$\text{curl } V = \nabla \times V = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, 0\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u_x & u_y & 0 \end{vmatrix} = (u_{xy} - u_{yx})\hat{k} = 0$$

because $u_{xy} = u_{yx}$ by Clairaut's theorem

this works when u is C^2

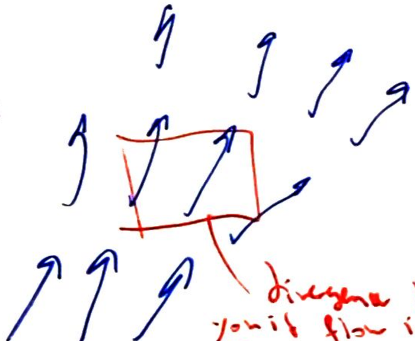
u is harmonic: u is C^2 and $\nabla^2 u = 0$

$\nabla^2 u = \text{div}(\text{grad } u) = 0$

if u is harmonic, $\text{div}(\text{grad } u) = 0$

so grad u is a divergence-free field

divergence-free means flow in = flow out
(flux) (flux)



divergence tells you if flow in equals flow out

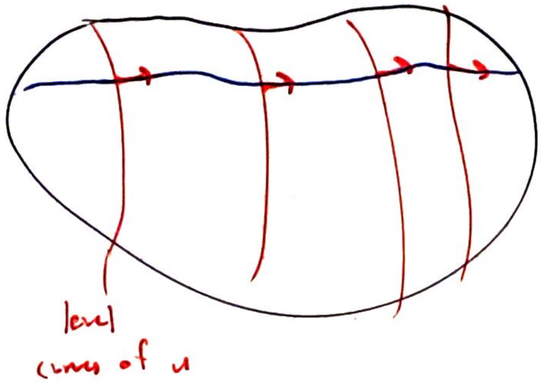
u is harmonic: grad u is irrotational

grad u is divergence free

- heat flow
- fluid flow
- electrostatics

u = harmonic

flow lines are level curves of the harmonic conjugate $u^* = c$



If $u = c$ are equipotentials and u^* is the harmonic conjugate of u,

$u^* = c$ are the stream lines

$f(z) = f(x+iy) = u(x,y) + iu^*(x,y)$

f is analytic, usually called the complex potential

given $f(z) = u + iv$ analytic

$Re(f) = u$ is harmonic

$Im(f) = v$ is the harmonic conjugate

here a complex potential $f(z) = \alpha z$

$Re(f) = \alpha x$

$Im(f) = \alpha y$

$u(x,y) = \alpha x$ $\quad$ $v(x,y) = \alpha y$

grad $u = (\alpha, 0)$