Week 9

Ch. 10: Potentials and Fields

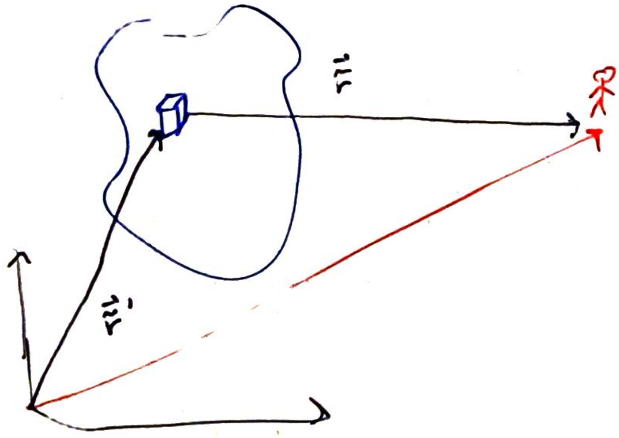
Goals: 1) How to take into account the fact that electromagnetic "information" does not travel instantaneously

2) Finding more general solutions to Maxwell's equations

$$\boxed{\begin{array}{ll} (1)\ \vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho & (3)\ \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ (2)\ \vec{\nabla}\cdot\vec{B} = 0 & (4)\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t} \end{array}}$$

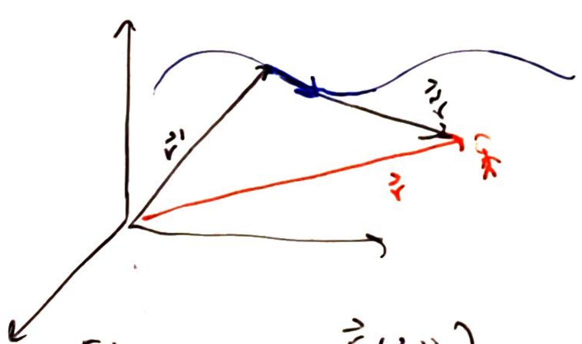
So far $\rho(\vec{r},t) = \rho(\vec{r}) \Rightarrow$ Coulomb's law

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d\tau' \frac{\rho(\vec{r}\,')}{\mathscr{r}^2}\hat{\mathscr{r}}$$



Magnetostatics $\vec{J}(\vec{r},t) = \vec{J}(\vec{r})$

Biot-Savart law



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d\tau' \frac{\vec{J}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

EM waves
$$\left.\begin{array}{l}\vec{E}(\vec{r},t)\\ \vec{B}(\vec{r},t)\end{array}\right\} \propto e^{i\omega t}$$

Next step → simplify the math → it's easier to work with potentials

Recall: Electrostatics → $\vec{\nabla}\times\vec{E}=0$ so we could use $\vec{E}=-\nabla V$

↳ we can't do that if $\frac{\partial \vec{B}}{\partial t}\neq 0$:(

However, $\vec{\nabla}\cdot\vec{B}=0$ always → can still use $\boxed{\vec{B}=\vec{\nabla}\times\vec{A}}$ *

Faraday's law: $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ $\Rightarrow$ $\vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0$

so $\vec{E}+\frac{\partial\vec{A}}{\partial t}=-\vec{\nabla}V$ $\Rightarrow$ $\boxed{\vec{E}=-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t}\ **}$ satisfies Faraday's law

So far: (*) and (**) satisfy eq (2) and (3). Now need to satisfy (1) and (4)

(1) $\vec{\nabla}\cdot\vec{E}=\frac{1}{\varepsilon_0}\rho$ with ** becomes

$$-\nabla^2 V-\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}=\frac{1}{\varepsilon_0}\rho \quad (5)$$

(4) $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A}=\mu_0\vec{J}-\mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{\nabla}V-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\left(\vec{\nabla}^2\vec{A}-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right)-\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}+\mu_0\varepsilon_0\frac{\partial V}{\partial t}\right)=-\mu_0\vec{J} \quad (6)$$

Previously we had electric and magnetic field as unknowns (6 scalars)

Now $\vec{A}$ has 3 components and V so (4 scalar) unknown

Note: in electrostatics $\frac{\partial}{\partial t}\to 0$ get Poisson's equation $\nabla^2 V=-\frac{\rho}{\varepsilon_0}$

in magnetostatics → $\nabla^2\vec{A}-\vec{\nabla}(\vec{\nabla}\cdot\vec{A})=-\mu_0\vec{J}$ (√) choose $\vec{A}$ such that divergence $\vec{\nabla}\cdot\vec{A}=0$

math island: Assume $\vec{A}_0$ satisfies eq. (√)

so $\vec{A}=\vec{A}_0+\vec{\nabla}\lambda$

← allowed since $\vec{\nabla}\times\vec{\nabla}\lambda=0$ so it won't affect $\vec{B}$

Can we adjust λ so that $\vec{\nabla}\cdot\vec{A}=0$? Yes, we can

$\vec{\nabla}\cdot\vec{A}=\vec{\nabla}\cdot\vec{A}_0+\vec{\nabla}^2\lambda=0$ $\quad\nabla^2\lambda=-\vec{\nabla}\cdot\vec{A}_0$ by existance we can find a λ satisfying this

ρ

$\lambda=\frac{1}{4\pi}\int\frac{\rho}{r}d\tau'$ this is possible!

Next: make (5) and (6) more managable

Can make shift: $\vec{A}' = \vec{A} + \vec{\alpha}$ $\quad V' = V + \beta$

Constraint:

we can make choices for $\vec{\alpha}, \beta$ that don't affect physics

$\vec{E} = \vec{E}'$

$\vec{B} = \vec{B}'$

Sub in $\vec{A}'$ in $\vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{B}$

$\vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\alpha} = \vec{\nabla} \times \vec{A}'$

$\Rightarrow \vec{B}' = \vec{B} + \underbrace{\vec{\nabla} \times \vec{\alpha}}_{=0}$ $\quad$ so $\vec{\nabla} \times \vec{\alpha} = 0$

so $\vec{\alpha} = \vec{\nabla} \lambda$

still unknown

Similarly for $\vec{E}$, $\vec{E}' = E$ $\quad \vec{\nabla} V' - \frac{\partial \vec{A}'}{\partial t} = \vec{\nabla} - \nabla \beta - \frac{\partial \vec{A}}{\partial t} - \frac{\partial \vec{\alpha}}{\partial t}$

$\vec{E}' = \vec{E} - \underbrace{\vec{\nabla} \beta - \frac{\partial \vec{\alpha}}{\partial t}}_{=0}$

$\Rightarrow \vec{\nabla}\left(\beta + \frac{\partial \lambda}{\partial t}\right) = 0$

$\beta = -\frac{\partial \lambda}{\partial t}$

Shift $\boxed{\vec{A}' = \vec{A} + \vec{\nabla} \lambda \qquad V' = V - \frac{\partial \lambda}{\partial t}}$ so that $\vec{B}' = \vec{B}$, $\vec{E}' = \vec{E}$

Two possible choices:

Coulomb gauge:

select λ such that $\vec{\nabla} \cdot \vec{A} = 0$

(5) $\vec{\nabla}^2 V = -\frac{1}{\varepsilon_0} \rho \Rightarrow V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r, t)}{r} d\tau'$

note $V(r,t)$ depends on $\rho(\vec{r}', t)$

same time

BUT the information cannot travel instantaneously

Lorentz gauge

$V(r,t)$ is not physical, only $\vec{E}$ is and $\vec{E}(\vec{r}, t) = \underbrace{-\vec{\nabla} V}_{\text{instantaneous}} - \frac{\partial \vec{A}}{\partial t}$ — not instantaneous

Lorentz Gauge $\Rightarrow$ makes 5 & 6 similarly easy / difficult to solve

↳ select λ such that $\vec{\nabla} \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0$

Advantage $\rightarrow$ (6) $\boxed{\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}(\vec{r}, t)}$ (7)

(5) becomes $\boxed{\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}\rho(\vec{r},t)}$ (8)

can we find such a λ → yes! this is really hard to show, but possible

(7) and (8) have the same structure

$$\boxed{\begin{aligned} &* \quad \Box^2 V = -\frac{1}{\epsilon_0}\rho(\vec{r},t) \\ &** \quad \Box^2 \vec{A} = -\mu_0 \vec{J}(\vec{r},t) \end{aligned}}$$

with $\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ "d'Alembertian" (generalized ∇^2)

From electrostatics and magnetostatics

↳ $\nabla^2 V = -\frac{1}{\epsilon_0}\rho(\vec{r},t) \Rightarrow V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t)}{\mathscr{r}} d\tau'$

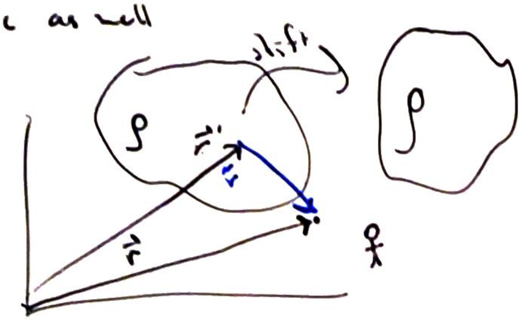
↳ $\nabla^2 \vec{A} = -\mu_0 \vec{J} \Rightarrow \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\, d\tau'}{\mathscr{r}}$

Heuristic Argument: EM waves travel with speed c

⇒ "news" travels with speed c as well

⇒ guess: $V(\vec{r},t)$ determined by $\rho(\vec{r}', t - \frac{\mathscr{r}}{c})$

similarly $\vec{A}(\vec{r},t)$ determined by $\vec{J}(\vec{r}', t - \frac{\mathscr{r}}{c})$



$t_r = t - \frac{\mathscr{r}}{c}$ retarded time

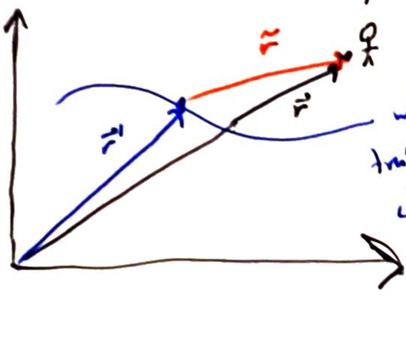
time for information to reach point $\vec{r}$

↳ note: depends on position of observer and charge distribution $\vec{r}, \vec{r}'$

$$\boxed{\begin{aligned} V(\vec{r},t) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)}{\mathscr{r}} d\tau' \\ \vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}} d\tau' \end{aligned}}$$

→ still need to show that the guess solutions satisfy eq. 7 and 8 (derivations in textbooks)

possible to prove that these satisfy * and ** (but we won't do it)



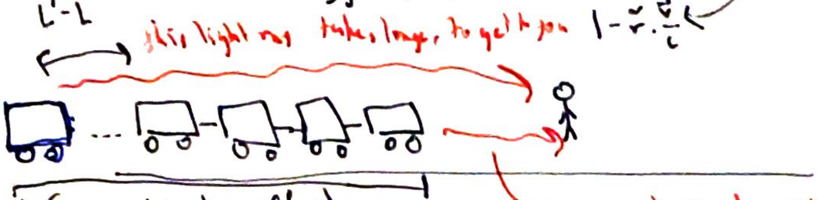
$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t)}{\mathscr{r}} d\tau'$$

pull out $\mathscr{r}$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{\mathscr{r}}\underbrace{\int \rho(\vec{r}',t_r)\, d\tau}_{\text{naively } = q \text{ (total charge)}}$$

✗ no

Actually, $\int \rho(\vec{r}\,', t_r)\, d\tau' = \frac{q}{1 - \hat{r}\cdot\frac{\vec{v}}{c}}$ ← retarded velocity

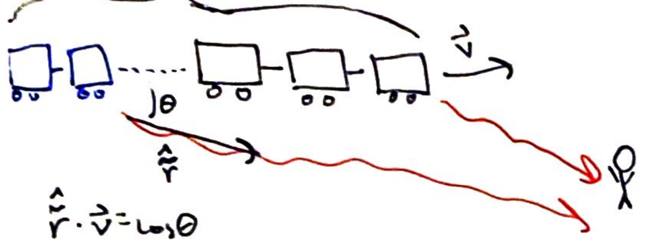


you see the blue train tail rather than the actual length of the train b/c the light rays get to you at the same time

time it took light to go from A to B $= \frac{L'}{c}$

during this same time, the train moved by $L' - L$

$\frac{L'}{c} = \frac{L'-L}{v}$ ⇒ apparent length $\boxed{L' = \frac{L}{1 - \frac{v}{c}}}$ essentially like the doppler



$\hat{r}\cdot\vec{v} = \cos\theta$

$$\boxed{L' = \frac{L}{1 - \frac{v\cos\theta}{c}}}$$

time it takes light to go from A to B

$$\frac{L'\cos\theta}{c} = \frac{L'-L}{v}$$

Note: train does not appear taller or wider, just longer

⇒ apparent volume of the train $\tau' = \frac{\tau}{1 - \hat{r}\cdot\frac{\vec{v}}{c}}$ — actual volume of train

↑ to get the cosine

so we can say $d\tau' = \frac{d\tau}{1 - \hat{r}\cdot\frac{\vec{v}}{c}}$ for an arbitrary infinitesimal volume

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\vec{r}\,', t_r)\, d\tau = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\frac{q}{1 - \hat{r}\cdot\frac{\vec{v}}{c}}$$ ← v at earlier retarded time t_r

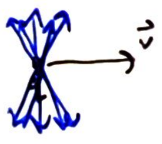
$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\rho(\vec{r}\,', t)\,\vec{v}(t_r)}{r}\, d\tau' = \frac{\mu_0}{4\pi}\frac{qc\vec{v}}{rc - \vec{r}\cdot\vec{v}} = \boxed{\frac{\vec{v}}{c^2}V(\vec{r}, t)}$$

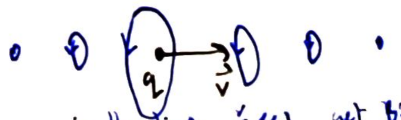
these are the Lienard-Wiechert potentials (for a point charge)

Next: find the observables $\vec{E}(t)$, $\vec{B}(t)$ using the Lienard-Wiechert potentials

↳ compare with Coulomb's law and Biot-Savart Law for a point charge

Looking ahead:





in the limit $\frac{v}{c} \ll 1$ get Biot-Savart Law