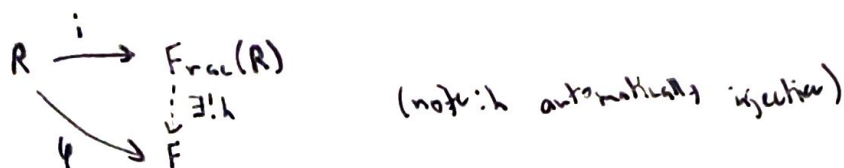


Today: Field of fractions

Inspiration: From \mathbb{Z} to \mathbb{Q}

Goal: Start with integral domain R , find a "smallest" field "containing" R
 More precisely: given an integral domain R , find a field, called the field of fractions (denoted $\text{Frac}(R)$) with an injective ring hom. $i: R \rightarrow \text{Frac}(R)$ such that ... if $\phi: R \rightarrow F$ is an injection, then ϕ factors through this injection.



What about non-domains? Bad things happen...

Today: we'll construct this and verify it has all these properties
 Exactly, modelled on the construction of the rationals

Example: If F is a field then $\text{Frac}(F) = F$

soon: $\text{Frac}(\mathbb{Z}) = \mathbb{Q}$

The construction: idea: Make the set of fractions $\frac{a}{b}$ $b \neq 0$

Careful method: look at ordered pairs $(a, b) \in R \times R$

$$\text{then let } T = \{(a, b) \in R \times R : b \neq 0\} \subseteq R \times R$$

Issue: in \mathbb{Q} : $\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc$ in \mathbb{Z}

then let's define a relation \sim on T

by $(a, b) \sim (c, d)$ iff $ad = bc$ in R

Claim: \sim is an equivalence relation

reflexive $(a, b) \sim (a, b) \Rightarrow ab = ba$ (integral domain)

symmetric if $(a, b) \sim (c, d) \Rightarrow ad = bc \Rightarrow da = cb \Rightarrow (c, d) \sim (a, b)$

transitive (exercise)

So we can define $\text{Frac}(R) := T / \sim$

then for our notation $\frac{a}{b}$ is the class of (a, b)

Operations:

$$\text{addition: } \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$(a, b) + (c, d) = (ad+bc, bd)$$

$$\text{multiplication: } \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

$$(a, b) \cdot (c, d) = (ac, bd)$$

$$\text{if } \frac{a}{b} = \frac{a'}{b'} \quad \text{and } \frac{c}{d} = \frac{c'}{d'} \quad \text{is } \frac{a'}{b'} + \frac{c'}{d'} = \frac{a}{b} + \frac{c}{d}$$

$$\text{and } \frac{a'}{b'} \cdot \frac{c'}{d'} = \frac{a}{b} \cdot \frac{c}{d}$$

exp!

Verify $\text{Frac}(R)$ is a ring

identity elements: 0 is $\frac{0}{1}$

1 is $\frac{1}{1}$

inverses for nonzero elements

suppose $\frac{a}{b} \in \text{Frac}(R)$ is nonzero ($a \neq 0$)

so $\frac{b}{a} \in \text{Frac}(R)$ then $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = 1$ (because $ab \cdot 1 = 1 \cdot ba$)
(so it's a field)

injective map $i: R \rightarrow \text{Frac}(R)$
 $r \mapsto \frac{r}{1}$

Universal property: Suppose $\varphi: R \rightarrow F$

is an injective ring hom. to a field

why is there a unique homomorphism $h: \text{Frac}(R) \rightarrow F$?

$$\begin{array}{ccc} R & \xrightarrow{i} & \text{Frac}(R) \\ & \searrow \varphi & \downarrow h \\ & & F \end{array}$$

Suppose $h: \text{Frac}(R) \rightarrow F$ is a homo.

for any $\frac{a}{b} \in \text{Frac}(R)$:

$$\begin{aligned} h\left(\frac{a}{b}\right) &= h\left(\frac{a}{1} \cdot \frac{1}{b}\right) = h\left(\frac{a}{1}\right) \cdot h\left(\frac{1}{b}\right) \\ &= h(i(a)) \cdot h\left(\frac{1}{b}\right) \\ &= \varphi(a) [h\left(\frac{1}{b}\right)]^{-1} \\ &= \varphi(a) [\varphi(b)]^{-1} \end{aligned}$$

then define $h: \text{Frac}(R) \rightarrow F$

$$\text{by } h\left(\frac{a}{b}\right) = \varphi(a) [\varphi(b)]^{-1}$$

it is well defined and unique s.t. $\varphi = h \circ i$

Example: ② $R = \mathbb{Q}[x]$

$$\text{Frac}(\mathbb{Q}[x]) = \left\{ \frac{p(x)}{q(x)} : p, q \in \mathbb{Q} : q(x) \neq 0 \right\}$$

we call this $= \mathbb{Q}(x)$

$$\textcircled{3} R = \mathbb{Z}[x] \quad (\text{claim: } \text{Frac}(\mathbb{Z}[x]) = \mathbb{Q}(x))$$

In general: if R is an integral domain

$$\text{Frac}(R[x]) = \text{Frac}(R)(x)$$

④ Previously we encountered $\mathbb{Q}[\sqrt{2}] \subseteq \mathbb{R}$

$$\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} \subseteq \mathbb{R}$$

original notation $\text{ev}_{\sqrt{2}}: \mathbb{Q}[x] \rightarrow \mathbb{R}$

$$\text{so } \text{ev}_{\sqrt{2}}(\mathbb{Q}[x]) = \mathbb{Q}[\sqrt{2}] \quad f(x) \mapsto f(\sqrt{2})$$

Note that $\mathbb{Q}[\sqrt{2}]$ is already a field

$$\text{reason: } \left(\text{in } \text{Frac}(\mathbb{Q}[\sqrt{2}]) \right) \frac{1}{a+b\sqrt{2}} \cdot \frac{a-b\sqrt{2}}{a-b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2 - 2b^2} = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2} \sqrt{2} \in \mathbb{Q}[\sqrt{2}]$$

\Rightarrow inverses exist in $\mathbb{Q}[\sqrt{2}]$

$$\Rightarrow \text{Frac}(\mathbb{Q}[\sqrt{2}]) = \mathbb{Q}[\sqrt{2}] = \mathbb{Q}(\sqrt{2})$$

$$\text{so } \text{Frac}(\mathbb{Z}[\sqrt{2}]) = \mathbb{Q}(\sqrt{2})$$

for any integral domain R ,

we constructed the field of fractions, $\text{Frac}(R)$

universal prop. $R \rightarrow \text{Frac}(R)$

$$\begin{array}{ccc} & \downarrow \exists! h & \\ \varphi & \searrow & F(\text{field}) \end{array}$$

$$\text{Frac}(\mathbb{Z}) = \mathbb{Q}$$

$$\text{Frac}(F[x]) = F(x)$$

$$\text{Frac}(\mathbb{Z}[\sqrt{2}]) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\} = \mathbb{Q}[\sqrt{2}] = \mathbb{Q}(\sqrt{2})$$

Next: Understand fields,



understand homomorphisms between fields

part 1: Understand existing hom. $F \xrightarrow{\varphi} E$

part 2: Construct hom. $F \xrightarrow{\varphi} E$

① Every hom. from a field is injective

So given $\varphi: F \rightarrow E$ hom. between fields

φ injective and $F \cong \text{im}(\varphi) \subseteq E$

Def: A field extension is an inclusion of fields $F \subseteq E$

F is the base field

E is called the extension field

By our observation, there's a correspondence between field extensions and field homomorphisms

$$F \subseteq E \longrightarrow F \hookrightarrow E$$

$$\varphi(F) \subseteq E \longleftarrow F \xrightarrow{\varphi} E$$

② Every field has a characteristic

Intuitive def:

$$\text{char}(F) = p > 0 \Leftrightarrow \underbrace{1 + 1 + \dots + 1}_{p \text{ times}} = 0$$

$$\text{char}(F) = 0 \Rightarrow \underbrace{1 + 1 + \dots + 1}_{n \text{ times}} \neq 0 \quad \forall n$$

formal def: have a unique hom.

$$\varphi: \mathbb{Z} \rightarrow F$$

$$1 \mapsto 1_F$$

$$\text{then } \text{Ker}(\varphi) = \{n \in \mathbb{Z} : n \cdot 1_F = 0\}$$

Then $\text{Ker}(\varphi) = (d)$ for some $d \in \mathbb{Z}$ (\mathbb{Z} is a PID)

so $\mathbb{Z}/\text{Ker}(\varphi) = \text{im}(\varphi) \subseteq F \Rightarrow \mathbb{Z}/\text{Ker}(\varphi)$ is an integral domain

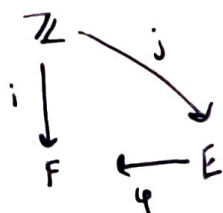
$\hookrightarrow \text{Ker}(\varphi)$ is a prime ideal

$$\hookrightarrow \text{Ker}(\varphi) = \begin{cases} (0) \\ (p), p \text{ prime} \end{cases}$$

$\hookrightarrow \text{char}(F) = d$ (0 or prime)

(3) Fields with different characteristic have no hom. between them

Reason: If $\varphi: F \rightarrow E$ is a field hom.



$$\ker(i) = \ker(j)$$

Next: Create new fields

Example ①: Start with the rationals \mathbb{Q}

Know there is no rational $r \in \mathbb{Q}$ with $r^2 = 2$

Idea: Want a new element α that satisfies the relation

$\alpha^2 = 2 \rightarrow$ want a bigger field E that "extends" \mathbb{Q}

and contains $\alpha \in E$ that satisfies $\alpha^2 = 2$ in E (minimally such)

How to do this:

step 1: introduce new free element, while still remaining a ring

$$\begin{array}{c} \mathbb{Q}[x] \\ \uparrow \\ \mathbb{Q} \end{array}$$

Step 2: Want to force x to satisfy the relation $x^2 = 2$

$$\Leftrightarrow \underbrace{x^2 - 2}_{p(x)} = 0$$

\Rightarrow Use quotient ring!

Quotient by the ideal generated by $I = (x^2 - 2) \subseteq \mathbb{Q}[x]$

$$\hookrightarrow \mathbb{Q}[x] / (x^2 - 2)$$

Step 3: Did this work?

Q1: Is this even a field? A1: $\mathbb{Q}[x]/I$ is a field when I is maximal

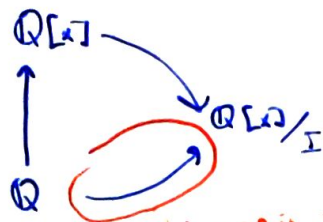
$$\text{here } I = \underbrace{(x^2 - 2)}_{p(x)}$$

note that we're in $\mathbb{Q}[x]$ which is a Euclidean domain (also a PID)

so $\Rightarrow p(x) = x^2 - 2$ is irreducible in $\mathbb{Q}[x] \Rightarrow (x^2 - 2)$ is a maximal ideal
we good $\ddot{\smile}$

Q2: Let $E = \mathbb{Q}[x]/I$, does it "extend" the rationals?

A2:



this map is induced by the other one automatically, injective

Q3: how do we work with the field E ?

A3: Elements of E are cosets of an ideal $I = (x^2 - 2) \subseteq \mathbb{Q}[x]$

Example: $f(x) = x^3 - 2x + 1 \in \mathbb{Q}[x]$

Then the image of f in E is the coset

$$f(x) + I = \{f(x) + g(x)(x^2 - 2) : g(x) \in \mathbb{Q}[x]\} = \{f(x) + g(x)(x^2 - 2) : g(x) \in \mathbb{Q}[x]\}$$

A nice representative for this coset is its member of smallest degree which we can compute by division:

Divide $f(x) = x^3 - 2x + 1$ by $p(x) = x^2 - 2$

$$\begin{array}{r} x \\ x^2 - 2 \overline{) x^3 + 0x^2 - 2x + 1} \\ \underline{x^3 + 0x^2 - 2x} \\ 1 \end{array}$$

← Sane representative for this coset

Magi- step: Let $\alpha = x + I \in E$

(coset for x , image of x in E)

$$\begin{aligned} \text{Then notice: } f(x) + I &= x^3 - 2x + 1 + I = (x + I)^3 - 2(x + I) + (1 + I) \\ &= \alpha^3 - 2\alpha + 1 \end{aligned}$$

← identifying \mathbb{Q} with its image in E

$$\text{But also } x^3 - x + 1 + I = 1 + I$$

$$\text{so } \alpha^3 - 2\alpha + 1 = 1 \quad (?)$$

notice α satisfies

$$\alpha^2 - 2 = (x + I)^2 - (2 + I)$$

$$= x^2 - 2 + I = 0 + I$$

Final conclusion $\cdot \mathbb{Q} \rightarrow E$ a field

$\cdot \alpha \in E$ satisfying $\alpha^2 - 2 = 0$ in E

\cdot every element in E corresponds to $c_0 + c_1 \alpha$, $c_1, c_0 \in \mathbb{Q}$

$$\hookrightarrow c_0 + c_1 \alpha \in E$$

$$\text{so } E = \{c_0 + c_1 \alpha : c_1, c_0 \in \mathbb{Q}\}$$