Week 8

Today: Field of fractions

Inspiration: From $\mathbb{Z}$ to $\mathbb{Q}$

Goal: start with integral domain R, find a "smallest" field "containing" R
More precisely: given an integral domain R, find a field, called the field of fractions (denoted $\mathrm{Frac}(R)$) with an injective ring hom. $i: R \to \mathrm{Frac}(R)$
such that .... if $\phi: R \to F$ is an injection, then φ factors through this injection

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "\varphi"'] & \mathrm{Frac}(R) \arrow[d, dashed, "\exists! h"] \\ & F \end{tikzcd}$$

(note: h automatically injective)

what about non-domains? Bad things happen...

Today: we'll construct this and verify it has all these properties
Exactly modelled on the construction of the rationals

Example: If F is a field then $\mathrm{Frac}(F) = F$

soon: $\mathrm{Frac}(\mathbb{Z}) = \mathbb{Q}$

The construction: idea: Make the set of fractions $\frac{a}{b}$ $b \neq 0$

careful method: look at ordered pairs $(a,b) \in R \times R$
then let $T = \{(a,b) \in R \times R : b \neq 0\} \subseteq R \times R$

Issue: in $\mathbb{Q}$: $\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc$ in $\mathbb{Z}$

then let's define a relation $\sim$ on T
by $(a,b) \sim (c,d)$ iff $ad = bc$ in R

Claim: $\sim$ is an equivalence relation
reflexive $(a,b) \sim (a,b) \Rightarrow ab = ba$ (integral domain)
symmetric if $(a,b) \sim (c,d) \Rightarrow ad = bc \Rightarrow da = cb \Rightarrow (c,d) \sim (a,b)$
transitive (exercise)

So we can define $\mathrm{Frac}(R) := T/\sim$

then for our notation $\frac{a}{b}$ is the class of (a,b)

Operations:

addition: $\frac{a}{b}+\frac{c}{d} = \frac{ad+bc}{bd}$ $\qquad (a,b)+(c,d) = (ad+bc, bd)$

multiplication: $\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$ $\qquad (a,b)\cdot(c,d) = (ac, bd)$

if $\frac{a}{b} = \frac{a'}{b'}$ and $\frac{c}{d} = \frac{c'}{d'}$ is $\frac{a'}{b'}+\frac{c'}{d'} = \frac{a}{b}+\frac{c}{d}$

and $\frac{a}{b}\cdot\frac{c}{d} = \frac{a'}{b'}\cdot\frac{c'}{d'}$

yep!

Verify $Frac(R)$ is a ring

- identity elements: 0 is $\frac{0}{1}$
  1 is $\frac{1}{1}$
- inverses for nonzero elements

suppose $\frac{a}{b} \in Frac(R)$ is nonzero $(a \neq 0)$

so $\frac{b}{a} \in Frac(R)$ then $\frac{a}{b}\cdot\frac{b}{a} = \frac{ab}{ba} = 1$ (because $ab\cdot 1 = 1\cdot ba$)

(so it's a field)

injective map $i: R \to Frac(R)$
$r \mapsto \frac{r}{1}$

Universal property: Suppose $\varphi: R \to F$ is an injective ring hom. to a field

why is there a unique homomorphism $h: Frac(R) \to F$?

$$\begin{tikzcd}
R \arrow[r, "i"', "r \mapsto r/1"] \arrow[dr, bend right, "\varphi"'] & Frac(R) \arrow[d, "h"] \\
 & F
\end{tikzcd}$$

Suppose $h: Frac(R) \to F$ is a homo.

for any $\frac{a}{b} \in Frac(R)$:

$$\begin{aligned}
h\left(\tfrac{a}{b}\right) = h\left(\tfrac{a}{1}\cdot\tfrac{1}{b}\right) &= h(a/1)\cdot h\left(\tfrac{1}{b}\right) \\
&= h(i(a))\cdot h\left(\tfrac{1}{b}\right) \\
&= \varphi(a)\left[h\left(\tfrac{b}{1}\right)\right]^{-1} \\
&= \varphi(a)[\varphi(b)]^{-1}
\end{aligned}$$

then define $h: \mathrm{Frac}(R) \to F$

by $h(\frac{a}{b}) = \varphi(a)[\varphi(b)]^{-1}$

its well defined and unique s.t. $\varphi = h \circ i$

Example: ② $R = \mathbb{Q}[x]$

$$\mathrm{Frac}(\mathbb{Q}[x]) = \{ \frac{p(x)}{q(x)} : p,q \in \mathbb{Q} : q(x) \neq 0 \}$$

we call this $= \mathbb{Q}(x)$

③ $R = \mathbb{Z}[x]$ Claim: $\mathrm{Frac}(\mathbb{Z}[x]) \cong \mathbb{Q}(x)$

In general: if R is an integral domain

$$\mathrm{Frac}(R[x]) \cong \mathrm{Frac}(R)(x)$$

④ Previously we've encountered $\mathbb{Q}[\sqrt{2}] \subseteq \mathbb{R}$

$\mathbb{Q}[\sqrt{2}] = \{ a + b\sqrt{2} : a,b \in \mathbb{Q} \} \subseteq \mathbb{R}$

origin of notation $ev_{\sqrt{2}} : \mathbb{Q}[x] \to \mathbb{R}$

$f(x) \mapsto f(\sqrt{2})$

so $ev_{\sqrt{2}}(\mathbb{Q}[x]) = \mathbb{Q}[\sqrt{2}]$

Note that $\mathbb{Q}[\sqrt{2}]$ is already a field

reason: (in $\mathrm{Frac}(\mathbb{Q}[\sqrt{2}])$) $\frac{1}{a+b\sqrt{2}} \cdot \frac{a-b\sqrt{2}}{a-b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2-2b^2} = \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$

$\implies$ inverses exist in $\mathbb{Q}[\sqrt{2}]$

$\implies \mathrm{Frac}(\mathbb{Q}[\sqrt{2}]) = \mathbb{Q}[\sqrt{2}] = \mathbb{Q}(\sqrt{2})$

vs $\mathrm{Frac}(\mathbb{Z}[\sqrt{2}]) = \mathbb{Q}(\sqrt{2})$

for any integral domain R,

we constructed the <u>field of fractions</u>, $\mathrm{Frac}(R)$

universal prop.

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[dr, bend right, "\varphi"'] & \mathrm{Frac}(R) \arrow[d, dashed, "\exists! h"] \\ & F \text{ (field)} \end{tikzcd}$$

$\mathrm{Frac}(\mathbb{Z}) = \mathbb{Q}$

$\mathrm{Frac}(F[x]) = F(x)$

$\mathrm{Frac}(\mathbb{Z}[\sqrt{2}]) = \{ a + b\sqrt{2} : a,b \in \mathbb{Q} \} = \mathbb{Q}[\sqrt{2}] = \mathbb{Q}(\sqrt{2})$

Next: Understand fields

$\Updownarrow$

understand homomorphisms between fields

part 1: Understand existing hom. $F \xrightarrow{\varphi} E$

part 2: Construct hom. $F \xrightarrow{\varphi} E$

① Every hom. from a field is injective

So given $\varphi: F \to E$ hom between fields

φ injective and $F \simeq \mathrm{im}(\varphi) \subseteq E$

Def: A field extension is an inclusion of fields $F \subseteq E$

F is the <u>base field</u>

E is called the <u>extension field</u>

By our observation, there's a correspondence between field extensions and field homomorphisms

$$F \subseteq E \longrightarrow F \underset{i}{\hookrightarrow} E$$

$$\varphi(F) \subseteq E \longleftarrow F \xrightarrow{\varphi} E$$

② Every field has a <u>characteristic</u>

Intuitive def:

$\mathrm{char}(F) = p > 0 \iff \underbrace{1+1+\dots+1}_{p \text{ times}} = 0$

$\mathrm{char}(F) = 0 \Rightarrow \underbrace{1+1+\dots}_{n \text{ times}} \neq 0 \quad \forall n$

formal def: have a unique hom.

$\varphi: \mathbb{Z} \to F$ then $\mathrm{Ker}(\varphi) = \{ n \in \mathbb{Z} : n \cdot 1_F = 0 \}$

$1 \mapsto 1_F$

Then $\mathrm{Ker}(\varphi) = (d)$ for some $d \in \mathbb{Z}$ ($\mathbb{Z}$ is a PID)

so $\mathbb{Z}/_{\mathrm{Ker}(\varphi)} \cong \mathrm{im}(\varphi) \subseteq F \Rightarrow \mathbb{Z}/_{\mathrm{Ker}(\varphi)}$ is an integral domain

$\hookrightarrow \mathrm{Ker}(\varphi)$ is a prime ideal

$\hookrightarrow \mathrm{Ker}(\varphi) = \begin{cases} (0) \\ (p), & p \text{ prime} \end{cases}$

$\hookrightarrow \mathrm{char}(F) = d$ (0 or prime)

(3) Fields with different characteristic have no hom. between them

Reason: If $\varphi: F \to E$ is a field hom.

$$\begin{tikzcd} \mathbb{Z} \arrow[d, "i"'] \arrow[dr, bend left, "j"] & \\ F & E \arrow[l, "\varphi"] \end{tikzcd}$$

$\ker(i) = \ker(j)$

Next: Create new fields

Example ①: Start with the rationals $\mathbb{Q}$

Know there is no rational $r \in \mathbb{Q}$ with $r^2 = 2$

Idea: Want a new element α that satisfies the relation $\alpha^2 = 2$ → want a bigger field E that "extends" $\mathbb{Q}$ and contains $\alpha \in E$ that satisfies $\alpha^2 = 2$ in E (minimally such)

How to do this:

step 1: introduce new free element, while still remaining a ring

$$\begin{tikzcd} \mathbb{Q}[x] \\ \mathbb{Q} \arrow[u] \end{tikzcd}$$

Step 2: Want to force x to satisfy the relation $x^2 = 2$

$\iff \underbrace{x^2 - 2}_{p(x)} = 0$

$\Rightarrow$ Use quotient ring!

Quotient by the ideal generated by $I = (x^2 - 2) \subseteq \mathbb{Q}[x]$

↳ $\mathbb{Q}[x] / (x^2 - 2)$

step 3: Did this work?

Q1: Is this even a field? A1: $\mathbb{Q}[x]/I$ is a field when I is maximal

here $I = (\underbrace{x^2 - 2}_{p(x)})$

note that we're in $\mathbb{Q}[x]$ which is a Euclidean domain (also a PID)

so $\Rightarrow$ $p(x) = x^2 - 2$ is irreducible in $\mathbb{Q}[x]$ $\Rightarrow$ $(x^2 - 2)$ is a maximal ideal

we good ☺

Q2: Let $E = \mathbb{Q}[x]/I$, does it "extend" the rationals?

A_2:

$$\begin{tikzcd} \mathbb{Q}[x] \arrow[dr, bend left] & \\ \mathbb{Q} \arrow[u] \arrow[r] & \mathbb{Q}[x]/I \end{tikzcd}$$

this map is induced by the others an automatically injective

$\underline{Q3}$: how do we work with the field E?

$\underline{A3}$: Elements of E are cosets of the ideal $I = (x^2-2) \subseteq \mathbb{Q}[x]$

Example: $f(x) = x^3 - 2x + 1 \in \mathbb{Q}[x]$

then the image of f in E is the coset

$f(x) + I = f(x) + \{g(x)(x^2-2) : g(x) \in \mathbb{Q}[x]\} = \{f(x) + g(x)(x^2-2) : g(x) \in \mathbb{Q}[x]\}$

A nice representative for this coset is its member of smallest degree

which we can compute by division:

Divide $f(x) = x^3 - 2x + 1$ by $p(x) = x^2 - 2$

$$\begin{array}{r|l} & x \\ x^2 + 0x - 2 & x^3 + 0x^2 - 2x + 1 \\ & x^3 + 0x^2 - 2x \\ \hline & \qquad\qquad\qquad 1 \end{array}$$

$1 \leftarrow$ favorite representative for this coset

Magic step: Let $\alpha = x + I \in E$

(coset for x, image of x in E)

Then notice: $f(x) + I = x^3 - 2x + 1 + I = (x+I)^3 - 2(x+I) + (1+I)$

$= \alpha^3 - 2\alpha + 1$

$\nwarrow$ identifying $\mathbb{Q}$ with its image in E

But also $x^3 - x + 1 + I = 1 + I$

so $\alpha^3 - 2\alpha + 1 = 1$ (?) notice α satisfies

$\alpha^2 - 2 = (x+I)^2 - (2+I)$

$= x^2 - 2 + I = 0 + I$

Final conclusion
- $\mathbb{Q} \to E$ a field
- $\alpha \in E$ satisfying $\alpha^2 - 2 = 0$ in E
- every element in E corresponds to $c_0 + c_1 x$, $c_1, c_0 \in \mathbb{Q}$
  $\hookrightarrow c_0 + c_1\alpha \in E$

so $E = \{c_0 + c_1\alpha : c_1, c_0 \in \mathbb{Q}\}$