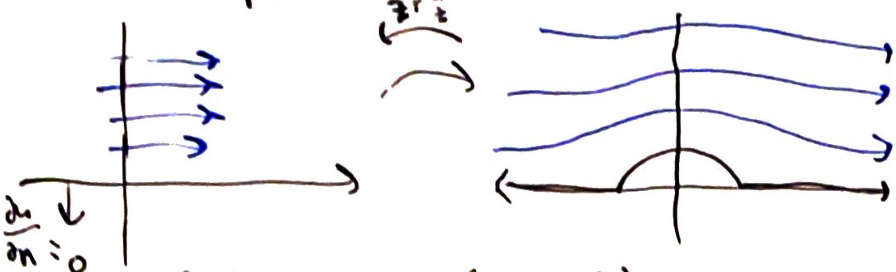
Week 5

Neumann problem (standard)



$\frac{\partial u}{\partial n} = 0$

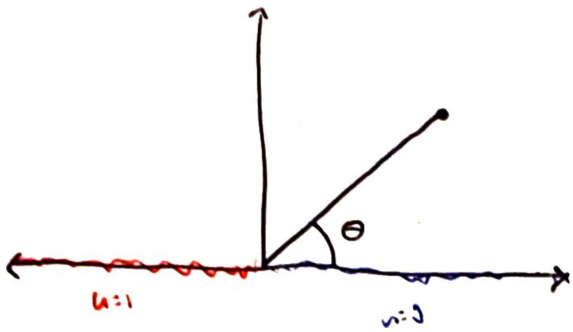
$\nabla u \cdot \hat{n}$ "

$f(z) = \alpha z$

$F = f \circ \left(z + \frac{1}{z}\right)$

Dirichlet Problem

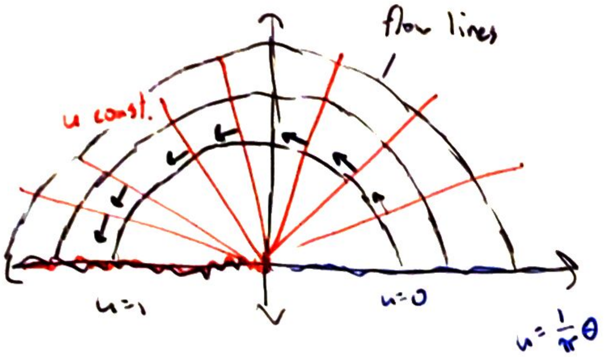
u harmonic on upper half plane



$f(z) = \frac{1}{\pi i} \log(z)$

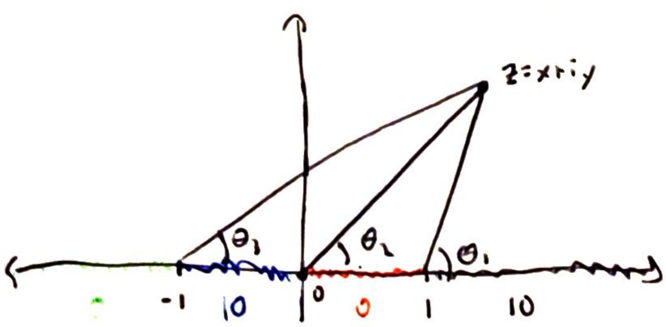
$u = \frac{1}{\pi}\theta = \mathrm{Re}(f)$ is harmonic

$$u(x,0) = \begin{cases} 0 & x > 0 \\ 1 & x < 0 \end{cases}$$



$u = \frac{1}{\pi}\theta$

This solution is useful because u is bounded



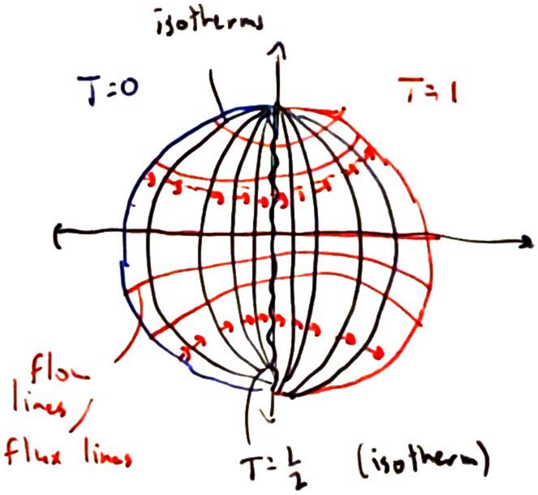
$$u = 10 + \frac{-10}{\pi}\arctan\left(\frac{y}{x-1}\right) + \frac{10}{\pi}\arctan\left(\frac{y}{x}\right) + \frac{-10}{\pi}\arctan\left(\frac{y}{x+1}\right)$$

more generally for c_0 c_1 c_2 c_3 we have

$$u = c_3 + \left(\frac{c_2 - c_3}{\pi}\right)\theta_3 + \left(\frac{c_1 - c_2}{\pi}\right)\theta_2 + \frac{c_0 - c_1}{\pi}\theta_1$$

Standard upper half plane solution

$u = \mathrm{Re}(f)$ $\quad f = c_3 + \frac{c_2 - c_3}{\pi i}\log(z-c) + \frac{c_1 - c_2}{\pi i}\log(z-b) + \frac{c_0 - c_1}{\pi}\log(z-a)$



Temperature = scalar

heat flow

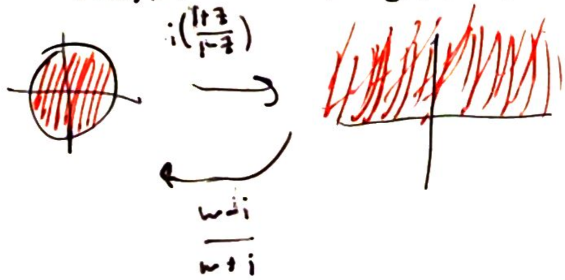
grad T = vector field

(points in direction of greatest change)

Find the function T describing this picture

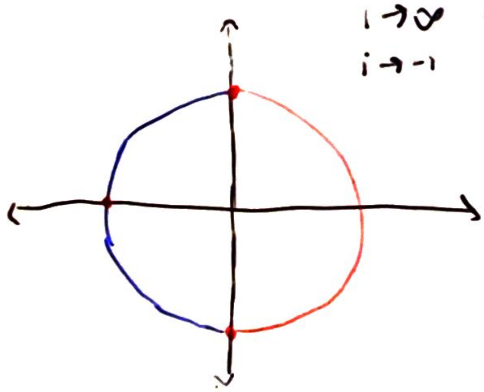
harmonic T ($\nabla^2 T = 0$) or $T = \mathrm{Re}(f)$ for f analytic

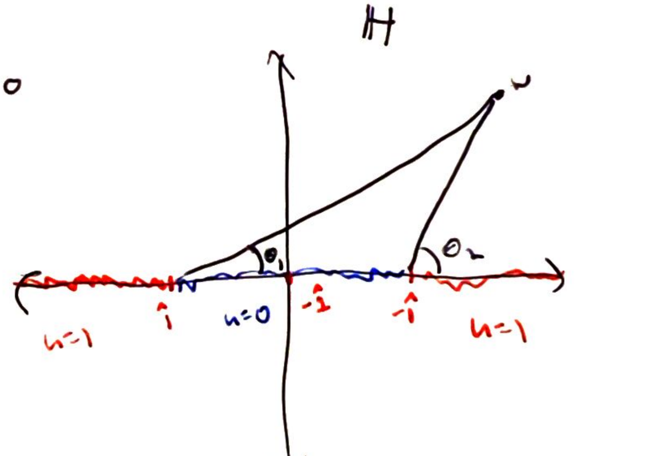
T satisfies boundary conditions



$1 \to \infty \quad -1 \to 0$

$i \to -1$





standard solution in upper half plane

$u(w) = 1 - \frac{1}{\pi}\theta_2 + \frac{1}{\pi}\theta_1$

$f(z) = 1 - \frac{1}{\pi i}\log(w-1) + \frac{1}{\pi i}\log(w+1)$

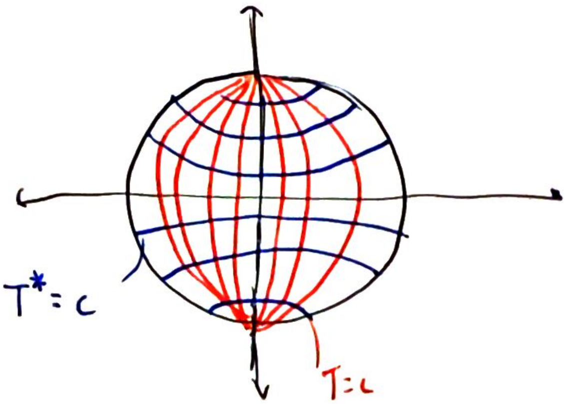
f analytic so u is harmonic

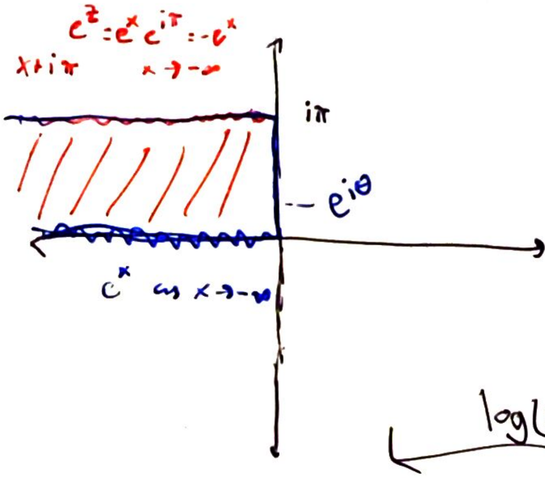
f is the complex potential on $\mathbb{H}$

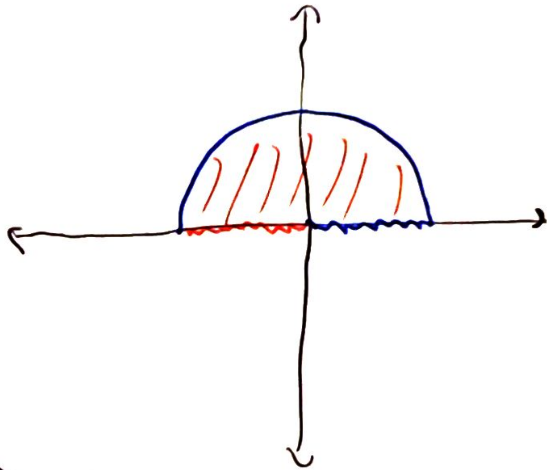
$\mathbb{D} \xrightarrow[i\left(\frac{1+z}{1-z}\right)]{} \mathbb{H} \xrightarrow[f(z)]{}$ complex potential so $\hat{f} = f\left(i\left(\frac{1+z}{1-z}\right)\right)$ then $T = Re(\hat{f})$

$$T = Re\left[1 - \frac{1}{\pi i}\log\left(i\left(\frac{1+z}{1-z}\right) - 1\right) + \frac{1}{\pi i}\log\left(i\left(\frac{1+z}{1-z}\right) + 1\right)\right]$$

$T^* = Im(\hat{f})$

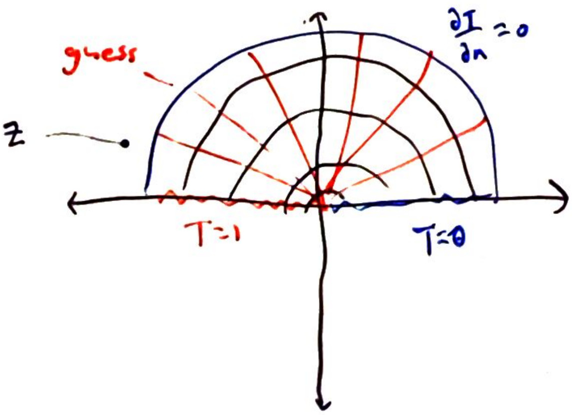


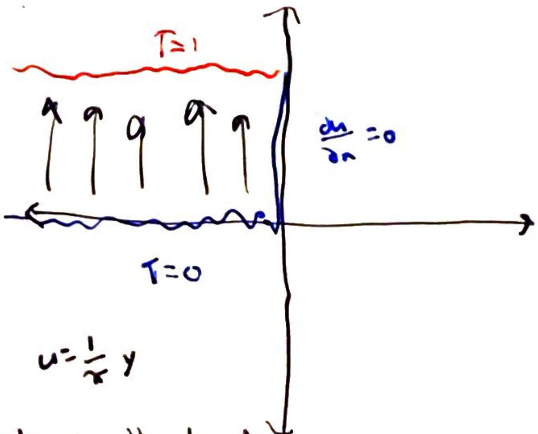




$\log(z)$

mixed boundary value problem





$u = \frac{1}{\pi} y$

u works at the boundary

$u = Re\left[\frac{1}{i\pi} z\right]$

$\hat{f} = \frac{1}{i\pi}\log(z)$

$T = \frac{1}{\pi}\arctan\left(\frac{y}{x}\right)$

Unit 2: The road to the zeta function ## Week 2 pt 2

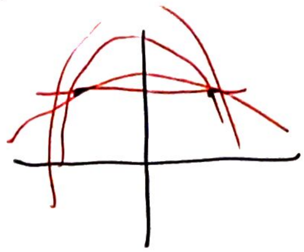
- interpolation
- gamma function
- analytic continuation
- zeta function
- $1+2+3+\ldots = -\frac{1}{12}$

Interpolation

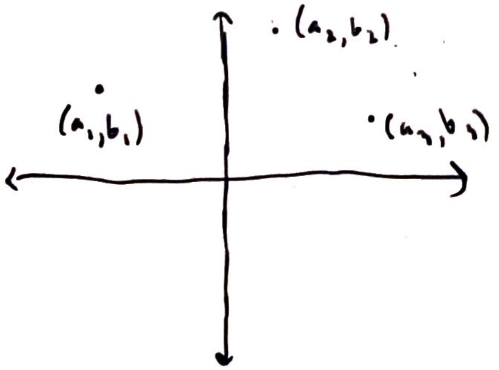
Given initial data $x_1, \ldots x_n$
and target data $z_1, \ldots, z_n$
we want a function f such that
$$f(x_i) = z_i \quad \text{for } i=1,\ldots,n$$

specifically we want a continuous, differentiable bounded function

if $f(x)$ does this, it is called an interpolating function



classic 3-point problem:



how do we find an interpolating function?
lets try a 2nd-order polynomial: $f(x) = c_0 + c_1 x + c_2 x^2$

$b_1 = c_0 + c_1 a_1 + c_2 a_1^2$

$b_2 = c_0 + c_1 a_2 + c_2 a_2^2$

$b_3 = c_0 + c_1 b_3 + c_3 b_3$

elbow-grease version

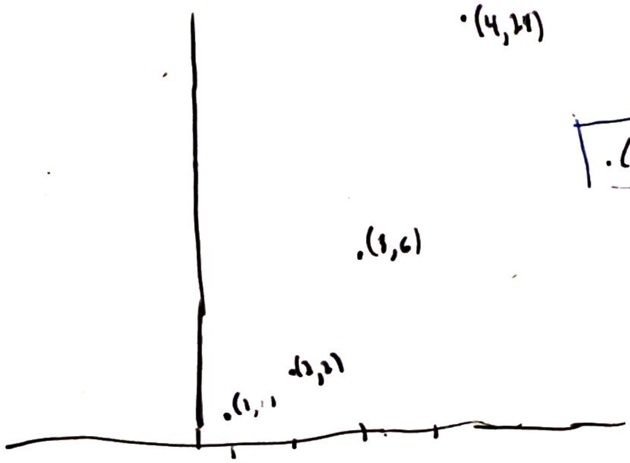
Lagrange interpolation

lets find $f_{a_1}(a_1) = 1$ and $f_{a_1}(a_2) = f_{a_1}(a_3) = 0$

$$\Rightarrow \frac{(x-a_2)(x-a_3)}{(a_1-a_2)(a_1-a_3)} = f_{a_1}$$

so we can construct an interpolating function out of these guys

$$f(x) = \frac{(x-a_2)(x-a_3)}{(a_1-a_2)(a_1-a_3)} b_1 + \frac{(x-a_1)(x-a_3)}{(a_2-a_1)(a_2-a_3)} b_2 + \frac{(x-a_1)(x-a_2)}{(a_3-a_1)(a_3-a_2)} b_3$$



lets call it

$\gamma(n) = n!$

interpolate the factorials,

; continuous

lets think about $\int_0^\infty te^{-t}$ $\quad \lim_{N\to\infty} \int_0^N te^{-t}\,dt$

$u = t \quad dv = e^{-t}$

$du = 1 \quad v = -e^{-t}$

$$\lim_{N\to\infty} \cancel{-te^{-t}}^{\,0} + \int_0^N e^{-t}\,dt = 1$$

$$\lim_{N\to\infty} \int_0^\infty t^2 e^{-t}\,dt = \lim_{N\to\infty} \left[-t^2 e^{-t} \Big|_0^N + \underbrace{2\int_0^N te^{-t}\,dt}_{=2\cdot(1)} \right]$$

so lets call $\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\,dt$

so $\Gamma(n) = (n-1)!$ so $\Gamma(n+1) = n\Gamma(n)$ this can be made to be $x \geq 0$

what about $\Gamma(x)$ for $\boxed{x \geq 1}$ just a real number

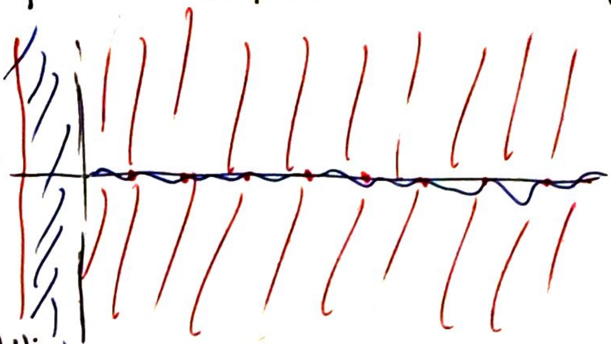
so lets throw some complex numbers into there

$$\boxed{\begin{aligned} \Gamma(z) &= \int_0^\infty t^{z-1} e^{-t}\,dt \\ \Gamma(x+iy) &= \int_0^\infty t^{x+iy-1} e^{-t}\,dt = \int_0^\infty t^{x-1} t^{iy} e^{-t}\,dt \end{aligned}}$$

so this guy converges when $x \geq 1$

we have a nice analog of the factorial from the reals

$\Gamma(z+1) = z\Gamma(z)$



analytic continuation

↳ extending the domain of analytic functions

we can just define $\Gamma(z)$ here by using the fact that $\Gamma(z+1) = z\Gamma(z)$

so you could say something stupid like $(-2+3i)!$ is something