Week 2

Maxwell's equations in free space $\rho = \vec{J} = 0$

1) $\vec{\nabla} \cdot \vec{E} = 0$
2) $\vec{\nabla} \cdot \vec{B} = 0$
3) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
4) $\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

$$\underbrace{\vec{\nabla} \times (\vec{\nabla} \times \vec{E})} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

$$\vec{\nabla}(\underset{\overset{\parallel}{0}\ (1)}{\vec{\nabla} \cdot \vec{E}}) - \nabla^2 E = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\Rightarrow \boxed{\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}} \quad \text{(wave equation)}$$

Similarly, $\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})$

$$\hookrightarrow \boxed{\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{B}}$$

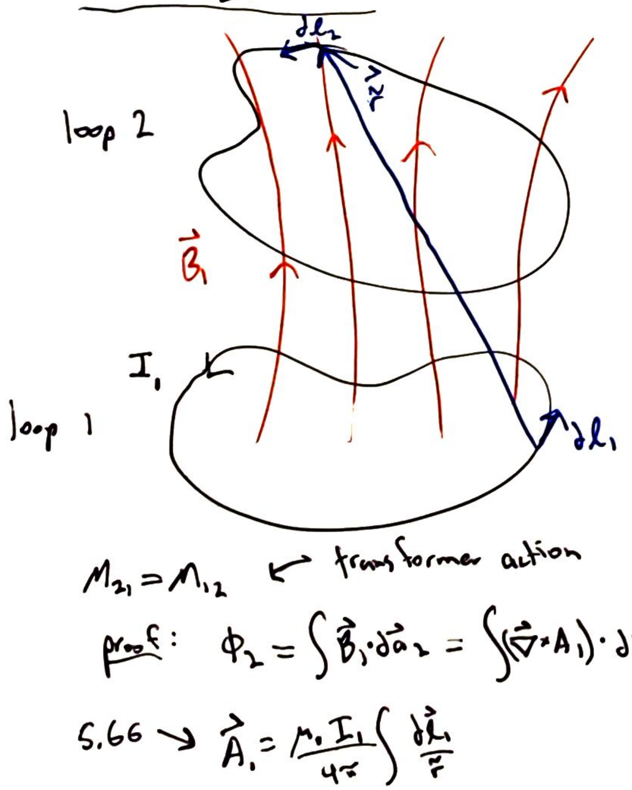
units of $\mu_0 \varepsilon_0$

$$\frac{T}{m^2} = \mu_0 \varepsilon_0 \frac{T}{s^2}$$

$$\mu_0 \varepsilon_0 = \frac{s^2}{m^2} = \left(\frac{1}{v}\right)^2$$

then $\boxed{\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{m}{s}}$ $\quad \frac{1}{\sqrt{8.85 \times 10^{-12} \cdot 4\pi \cdot 10^{-7}}} = 3 \times 10^8 \frac{m}{s}$ $= c$

7.2.3 Inductance



I_1 creates $\vec{B}_1$ – penetrates loop 2

$\hookrightarrow$ find Φ_2 through loop 2

Biot-Savart law $B_1 = \frac{\mu_0}{4\pi} I \oint \frac{d\vec{l}_1 \times \hat{r}}{r^2}$

$\hookrightarrow B_1 \propto I_1$

$\Phi_2 \propto B_1 \qquad \Phi_2 \propto I_1$

we can write $\boxed{\Phi_2 = M_{21} I_1}$

M_{21} is the constant of proportionality

$\hookrightarrow M_{21}$: mutual inductance

$M_{21} = M_{12} \leftarrow$ transformer action

proof: $\Phi_2 = \int \vec{B}_1 \cdot d\vec{a}_2 = \int (\vec{\nabla} \times A_1) \cdot d\vec{a}_2 = \oint \vec{A}_1 \cdot d\vec{l}_2$

Stokes theorem along loop 2

5.66 $\rightarrow \vec{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\vec{l}_1}{r}$

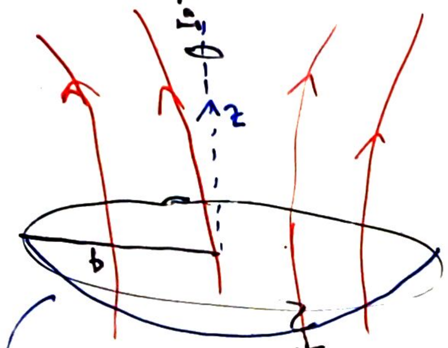
$$= \boxed{\frac{\mu_0 I_1}{4\pi} \oint \oint \frac{d\vec{l}_1}{r}}$$

so $M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r}$ (Neumann's formula) so: (1) M_{12} is purely geometrical

(2) $M_{12} = M_{21} = M$

whatever the shapes and positions of the loops, the flux through 2 when we run a current I around 1 it is identical to the flux through 1 when we run the same current I around 2.

HW



goal: find flux through loop a, ϕ_a ($a \ll b$)

$$\phi_a \approx B_z \cdot \pi a^2$$

$\phi_a \propto M \cdot I$ (or M) — M_{12}

for b) current flows in little loop (baby)

assuming $s \gg a$ → treat loop a as a magnetic dipole

$$\vec{B}_{dip} = \frac{\dots}{\dots}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$\phi_b = \oint_{\text{area of loop b}} \vec{B}_{dip} \cdot d\vec{a}$$ (any area enclosed by loop b)

If everything goes according to plan, you'll get $M_{12} = M_{21}$

<u>Just one loop</u>: vary the current

Faraday's law: $|\mathcal{E}| = \frac{d\phi}{dt}$ $\phi(t) \propto I$ and $\phi = LI$ — flux of B-field thru loop

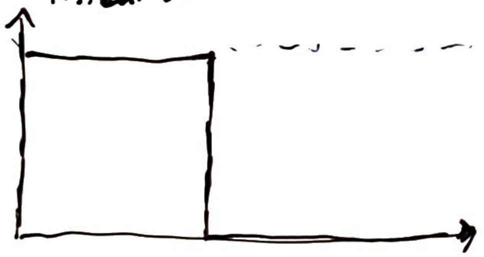
L is called the <u>self-inductance</u>

so $\boxed{|\mathcal{E}| = L\frac{dI}{dt}}$ (back emf)

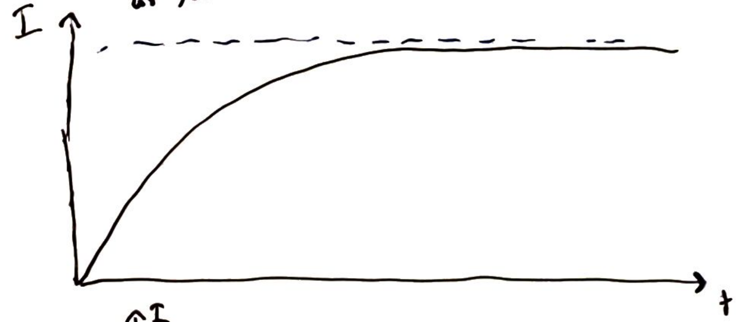
Lenz' law: emf is in such a direction as to oppose any change in current

As you stop I

instead of



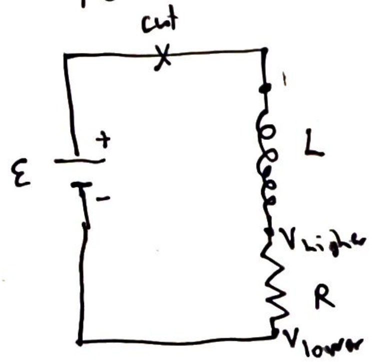
as you start I



instead of

similar to inertia in mechanics: $m\uparrow$ → harder it is to change objects velocities

$L\uparrow$ → harder to change a current example



the large $\left|\frac{dI}{dt}\right|$ will create a large $\mathcal{E}_{ind}$.

if we first turn on the bat, we first see

$$I\uparrow \Rightarrow B_{coils}\uparrow = \Phi_B\uparrow$$

$\mathcal{E}_{ind}$ would oppose the change in flux

going around the circuit clockwise

$$\mathcal{E} - L\frac{\partial I}{\partial t} - IR = 0$$

$$\Rightarrow L\frac{dI}{dt} = \mathcal{E} - IR$$

$$\Rightarrow \int \frac{L\,dI}{\mathcal{E}-IR} = \int dt = -\frac{1}{R}\ln(\mathcal{E}-IR)\Big|_0^I = t$$

so $I(t) = \frac{\mathcal{E}}{R}\left(1-e^{-\frac{R}{L}t}\right)$ $\quad\left[\frac{L}{R}\right] = s$ $\quad$ if we wait for $t=\frac{L}{R}$

$$I = \frac{\mathcal{E}}{R}\left(1-e^{-1}\right) = 0.63\frac{\mathcal{E}}{R} = 0.63 I_0$$

$L\uparrow$, the longer it takes to reach 63% of max

It takes energy to start current flowing

↳ work done by battery $dW = -\mathcal{E}_{induced}\cdot dq$

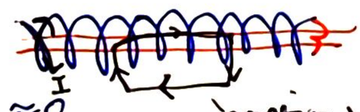
$= -\left(-L\frac{\partial I}{\partial t}\right)\cdot I\,dt$ cancel out differentials

$$dW = L\frac{dI^2}{2}$$

$$\Rightarrow W = L\frac{I^2}{2}$$

this ends up being equivalent to $W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau$

Find self-inductance of a long solenoid



# of turns per unit length is n, $[n] = \frac{1}{m}$

Use ampere's law $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl} + \mu_0\epsilon_0\int \frac{\partial \vec{E}}{\partial t}\,d\vec{a}$

ignore cause no $\vec{E}$

$B\cdot a = \mu_0 I_{encl} = \mu_0 I N$ (number of turns in black loop)

$= \mu_0 I n a$

$$Ba = \mu_0 I n a \Rightarrow B = \mu_0 I n$$

then $\Phi_{\substack{\text{one}\\\text{coil}}} = BA = (\mu_0 I n)(\pi r^2)$

$$\Phi_{net} = \mu_0 I n \pi r^2 (n\ell) = \mu_0 I n^2 \pi r^2 \ell$$

and $\Phi_{net} = LI \Rightarrow L = \mu_0 \ell \pi r^2 n^2$

you can check that if you put it into $W = \frac{LI^2}{2}$ you get the same thing as $W = \frac{1}{2\mu_0}\int B^2 d\tau$

Ch 8. Conservation Laws

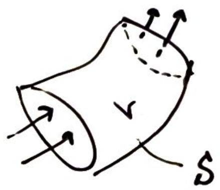
Main idea: not only material particles (that have mass) can carry energy, momentum, and angular momentum, but E and B - fields can carry energy, momentum, and angular momentum

particles and fields can exchange energy, $\vec{p}$, $\vec{L}$

Continuity equation for charge

Notice: in steady state $\frac{dQ}{dt} = 0$

$\Rightarrow \oint \vec{J} \cdot d\vec{a} = 0$



$$\frac{dQ}{dt} = -\oint \vec{J} \cdot d\vec{a}$$

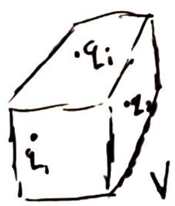
(charge balance)

$\Rightarrow \vec{J}_{in} = \vec{J}_{out}$

Differential form: $\frac{d}{dt}\int_V \rho\, d\tau = -\oint \vec{J} \cdot d\vec{a} = -\int_V \vec{\nabla} \cdot \vec{J}\, d\tau$

and we get $\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{J}}$ for a small volume $d\tau$

Given $\underbrace{\vec{B} \text{ and } \vec{E}}$ (due to currents and charges) act on charges inside V



Work-energy theorem: $\underbrace{dK}_{\substack{\text{change}\\\text{in KE}\\\text{of charge}}} = \underbrace{dW}_{\substack{\text{work done}\\\text{by all forces}\\\text{(net force)}}}$

$dW_q = \vec{F}_q \cdot \underbrace{d\vec{l}}_{\vec{v}\cdot dt}$ $\qquad F_q = q(\vec{E} + \vec{v} \times \vec{B})$

, $dW = q(\vec{E} + \underbrace{\vec{v} \times \vec{B}) \cdot \vec{v}}_{=0} dt = q\vec{E} \cdot \vec{v}\, dt$

b/c $\vec{v} \times \vec{B} \perp \vec{v}$ $\quad \hookrightarrow$ let $q = \rho\, d\tau$

$\hookrightarrow$ total charge in volume $d\tau$

$$\frac{dW_{d\tau}}{dt} = (\rho\, d\tau)\vec{E} \cdot \vec{v} = \vec{E} \cdot \underbrace{\rho\vec{v}}_{\vec{J}} = \vec{E} \cdot \vec{J}\, d\tau$$

$$\frac{dW_{tot}}{dt} = \int_V \vec{E}\cdot\vec{J}\, d\tau$$

Week 2 pt. 2

energy needed per unit time

change of KE of charges inside V per unit time

- get rid of $\vec{J}$ in favor of $\vec{E}$ and $\vec{B}$ (use maxwell's (3)) (painful algebra)

$$\vec{J} = \frac{1}{\mu_0}\left[\vec{\nabla}\times\vec{B} - \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right]$$

- $$\frac{dW}{dt} = \int_V \left\{-\frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{1}{\mu_0}B^2\right]\right\}d\tau - \underbrace{\int_V \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})\,d\tau}_{\text{use divergence theorem}}$$

$\hookrightarrow \oint (\vec{E}\times\vec{B})\,d\vec{a}$ (surface integral)

Finally get $$\frac{dW}{dt} = \underbrace{-\frac{d}{dt}\int_V\left[\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{1}{\mu_0}B^2\right]d\tau}_{①} \underbrace{- \frac{1}{\mu_0}\oint_S (\vec{E}\times\vec{B})\cdot d\vec{a}}_{②}$$

① = decrease of energy already inside volume V per unit time

② = energy transfer through the boundaries, to/from volume V, per unit time

$$\frac{\text{energy}}{\text{volume}} = u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0}$$ (Rem: phys 3, electrical energy per volume in capacitor) magnetic energy in inductor

$\Rightarrow$ Energy stored in fields

Define the Poynting vector

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \quad \left(\frac{\text{energy}}{(\text{time})(\text{area})}\right)$$

$$\boxed{\frac{dW}{dt} = -\frac{d}{dt}\int_V u\, d\tau - \oint \vec{S}\cdot d\vec{a}}$$ Poynting Theorem

What if there are no charges inside V? $\Rightarrow \frac{dW}{dt} = 0$

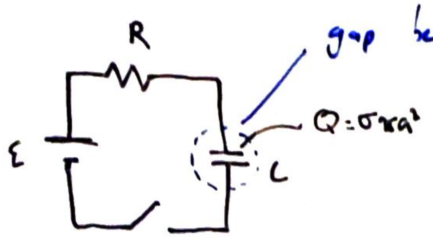
$$\int_V \frac{\partial u}{\partial t}\, d\tau = -\oint_S \vec{S}\cdot d\vec{a}$$

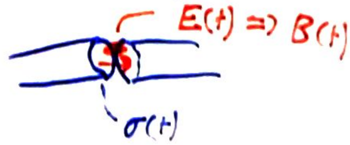
$\hookrightarrow$ divergence theorem $= -\int_V (\vec{\nabla}\cdot\vec{S})\,d\tau$

$\hookrightarrow$ $$\frac{\partial u}{\partial t} = -\vec{\nabla}\cdot\vec{S}$$

HW 8.2

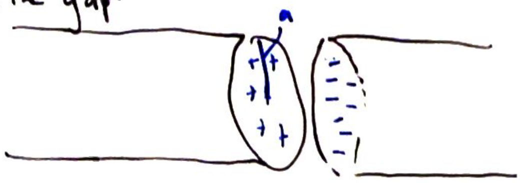
gap between 2 cylindrical wires





at $t=0$, close the switch $\Rightarrow I(t)$

in the gap:



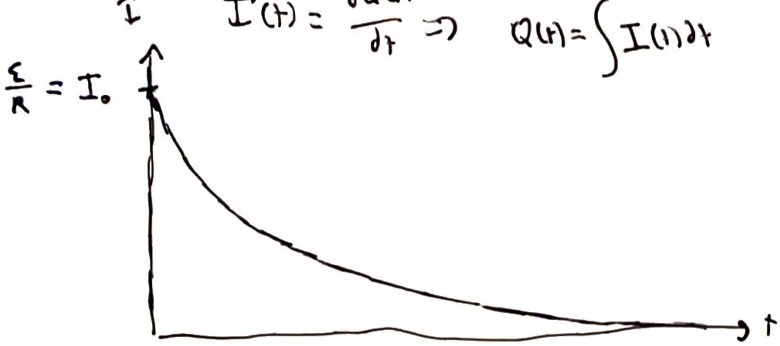
goal: find $E(t)$, $B(t)$ inside the gap

check the continuity equations hold

find $\vec{E}$ and $\vec{B}$

$$\vec{E}_{cap} = \frac{\sigma(t)}{\varepsilon_0} = \frac{Q(t)}{\varepsilon_0 \pi a^2}$$

$$I(t) = \frac{dQ(t)}{dt} \Rightarrow Q(t) = \int I(t)\,dt$$



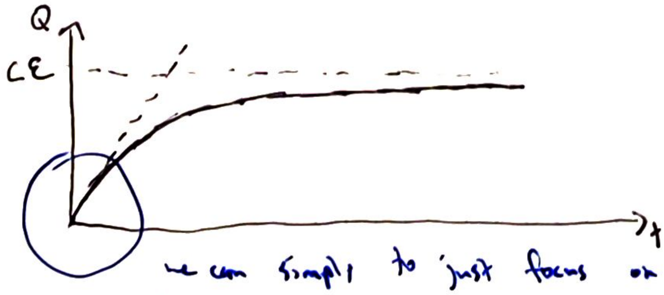
$$\sum_{loop} V = 0$$

$$\varepsilon - IR - \frac{Q}{C} = 0$$

$$\varepsilon - \frac{dQ}{dt}R - \frac{Q}{C} = 0 \quad \text{(first order ODE)}$$

$$\varepsilon - R\frac{dQ}{dt} - \frac{Q}{C} = 0$$

$\Rightarrow$ separation of variables



we can simply to just focus on the linear regime

$Q(t) \sim t$