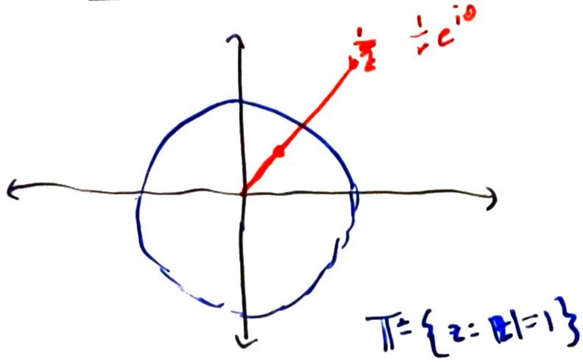
Week 4

Conformal maps

- inversion across a circle
- applications to physical systems
- Linear algebra

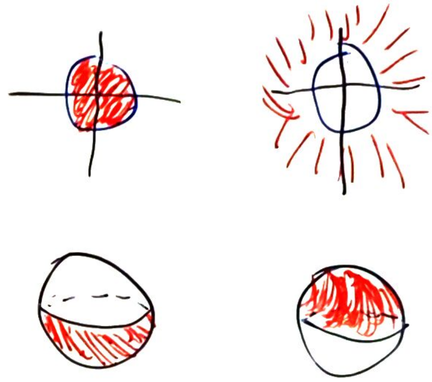
  LFT $\longleftrightarrow GL_2(\mathbb{C})$
- bonus content

reflection on the circle



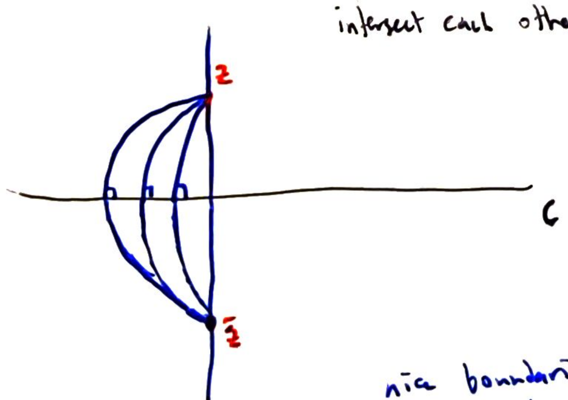
$z \to \frac{1}{\bar{z}}$

$\mathbb{D} \to \hat{\mathbb{D}}$ (outer disk)



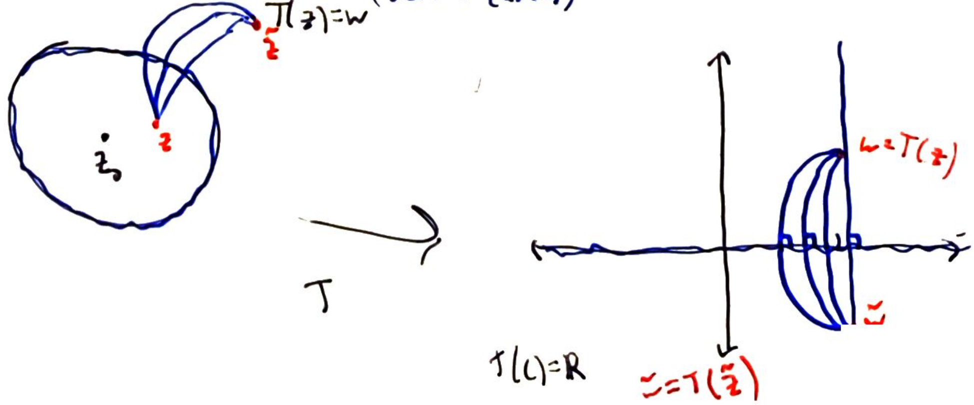
Let C be a general circle centered at z_0

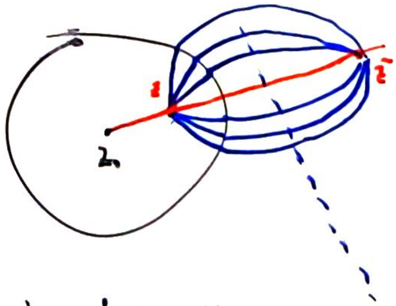
then all circles that pass through a point $z \notin C$ and intersect at right angles, intersect each other at a unique point $\tilde{z}$



happens when C is a line

nice boundaries can map to other nice boundaries (Jordan curves)





how do we find $\tilde z$

$C = \mathbb{T}$, $\tilde z = \frac{1}{\bar z}$

$$\tilde z = \overline{\left(\frac{\bar z_0 z + R^2 - |z_0|^2}{z - z_0}\right)}$$

the map $z \to \tilde z$ is <u>not</u> analytic because $\bar z$

$z \to \tilde z$ is a combination of LFTs and conjugation

If T is a linear fractional transformation and C is a circle
then $T(\tilde z) = \widetilde{T(z)}$

so LFTs preserve reflection across the circle

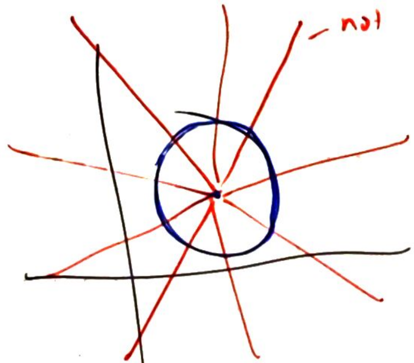
<u>prop</u> 1. $\tilde{\tilde z} = z$

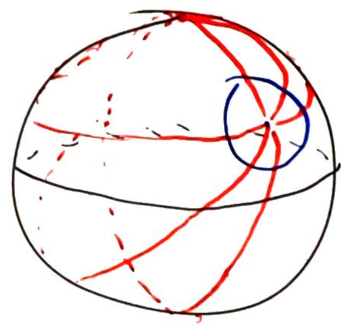
2. $z \to \tilde z$ is not conformal but angles are preserved in magnitude and reversed in orientation

3. If C is a line, $\tilde z$ is the reflection in the perpendicular
reflection across the line

4. $z \to \tilde z$ maps circles to circles

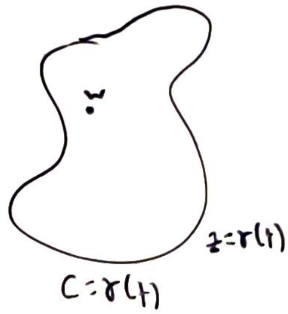
not great circles





Cauchy integral formula

$$f(w)=\frac{1}{2\pi i}\oint_C \frac{f(w)}{z-w}\,dz$$



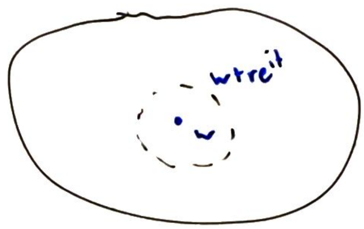
f-analytic

C nice: simple, piecewise cont. and closed

interior values are determined by the boundary values

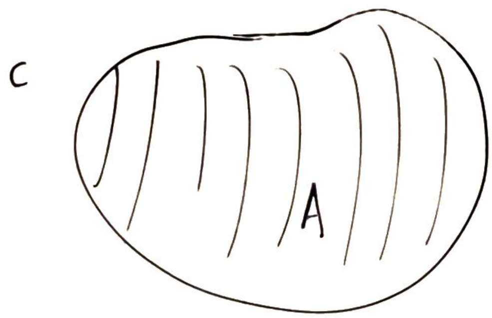
Gauss' mean value theorem

$$f(w)=\frac{1}{2\pi}\int_0^{2\pi} f(w+re^{it})\,dt$$



Dirichlet Problem

Given a region in $\mathbb{R}^2$, A with boundary C that is bounded, simple closed, find a harmonic function



cont. function u_0 on C

Given $A \subseteq \mathbb{R}^2$, boundary C and boundary values of the function on C, u_0

want to find $u(x,y)$ on A $(\nabla^2 u = 0)$

so that $u = u_0$ on C

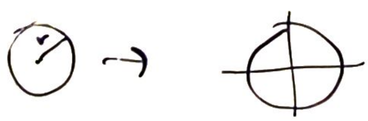
Original motivation

$u_t = \nabla^2 u = u_{xx} + u_{yy}$ equilibrium equations $u_{xx}+u_{yy}=0$

potential theory

If A is a disk so that C is a circle: u is defined and continuous on C

WLOG



assume A is unit disk

C is unit circle

then $u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\phi})\ \underline{\rho - |z|^2}$

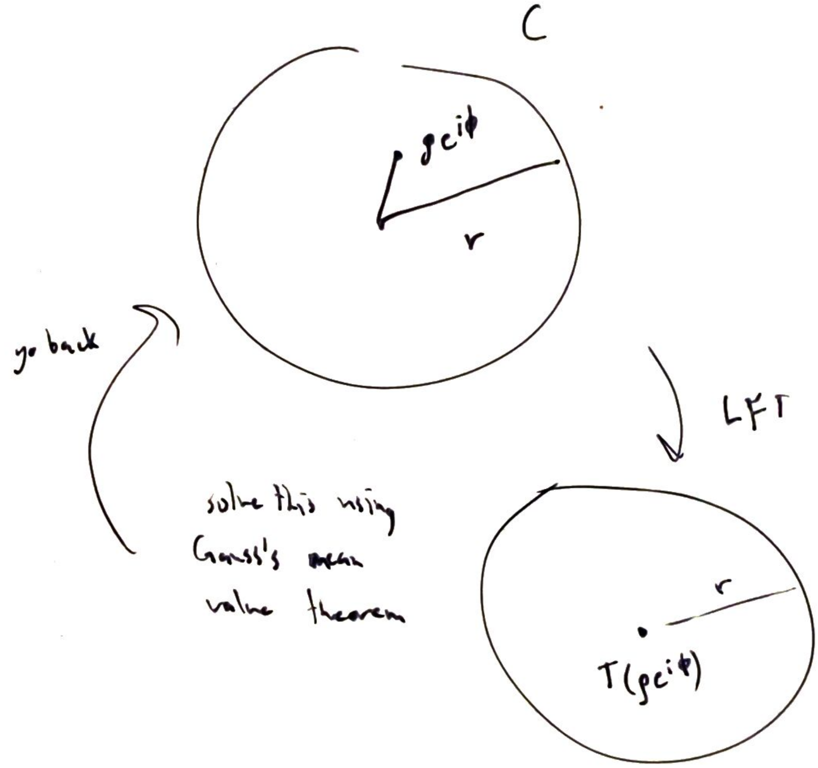
ignore the above

$D(0,r)$, for $\rho < r$

outside of the circle

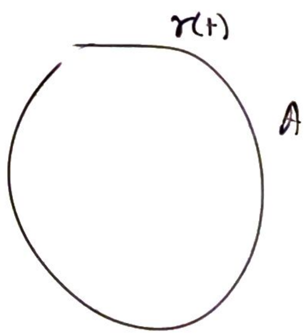
$$u(\rho e^{i\phi}) = \frac{r^2-\rho^2}{2\pi}\int \frac{u(re^{i\theta})}{r^2 - 2r\rho\cos(\phi-\theta) - \rho^2}\,d\theta$$

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\,d\theta$$



$$\int_\gamma \vec{X}\cdot\vec{n}\,dS = \int_A \operatorname{div} X\,dx\,dy$$

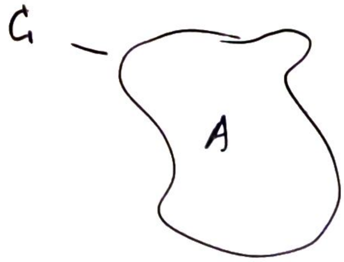
vector field over A



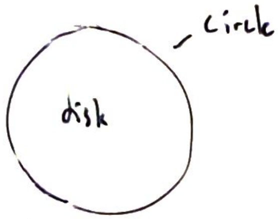
if you have $f(z) = f(x+iy) = u(x+iy) + iv(x+iy)$

then both u and v are harmonic

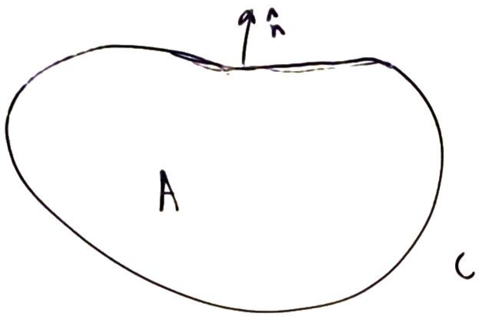
u is given on continuous



Riemann mapping theorem tells you that this transform is possible



Given a domain A



need direction of $\hat{n}$ to vary continuously

Perscribe boundary derivatives

Find u harmonic on A with $\frac{du}{dn}$ specified along the boundary

$\frac{du}{dn} = \nabla u \cdot \hat{n}$

<u>Neumann Problem</u>

If u existed then $\int_\gamma \frac{du}{dn} = 0$

if $\vec{X} = \text{grad } u$ then $\int \vec{X} \cdot \hat{n}\, ds = \int \text{div} X \, dx dy$

but $\text{div}(\text{grad } u) = 0$ always