(Ch. 5.2 in Marsden/Hoff) ## Week 3 ##

Definition: A linear fractional transformation is a degree 1 ration complex function of the form

$$T(z) = \frac{az+b}{cz+d} \quad \text{with } ad-bc \neq 0$$

(why is this excluded?)

LFT: $T(z) = \frac{az+b}{cz+d}$ has a pole of order 1 at $z = -\frac{d}{c}$

zero of order 1 at $z = -\frac{b}{a}$

prop: let $T(z) = \frac{az+b}{cz+d}$ be a LFT, then T is bijective, conformal from $A = \mathbb{C} \setminus \{z = -\frac{d}{c}\}$ to $B = \mathbb{C} \setminus \{\frac{a}{c}\}$

$$T^{-1}(w) = -\frac{dw+b}{cw-a}$$

note: T^{-1} is also conformal and bijective

let $T = \frac{az+b}{cz+d}$ $\quad S = \frac{-dw+b}{cw-a}$

T analytic on A, S analytic on B, now show $T \circ S = S \circ T = z$

conclusion: S, T analytic, $S = T^{-1}$

To show that $T'(z) \neq 0$

observe, $1 = \frac{d}{dz} z = \frac{d}{dz}\left(S(T(z))\right) = S'(T(z))\, T'(z) \neq 0$

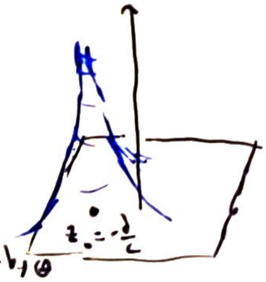
so $T'(z) \neq 0$ and $T^{-1} = S$ is conformal along with T

Well, actually

$$T(z) = \frac{az+b}{cz+d} : A = \mathbb{C} \setminus \{-\frac{d}{c}\} \to \mathbb{C} \setminus \{\frac{a}{c}\}$$

lets just say $T(-\frac{d}{c}) = \infty$ define extended complex plane

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$



with this patch, T is conformal on $\hat{\mathbb{C}}$

Geometrically:

$f(z) = az+b$ $\quad$ if $f(z) = az$ then $f(z) = re^{i\theta}(z)$

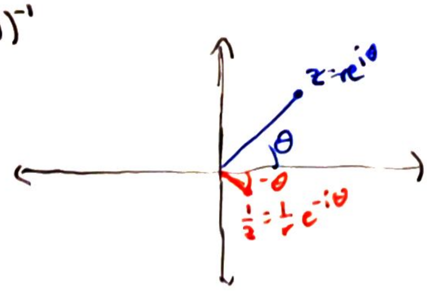
rotation by $i\theta$

magnification by r

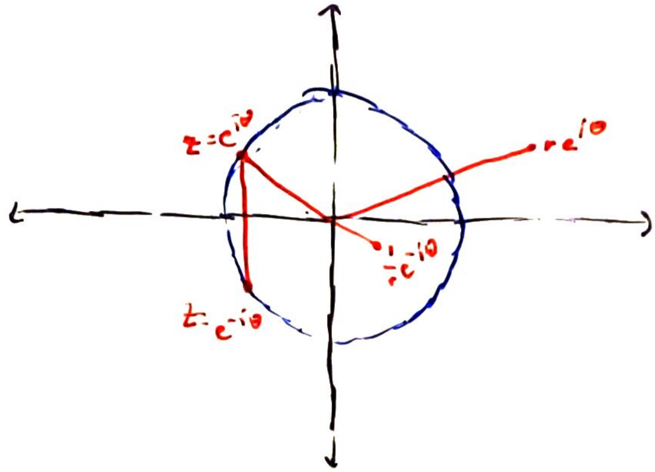
$T(z) = \frac{az+b}{cz+d}$ if $d=1$, $b=0$, $c=0$, $a \neq 0$ you get rotation and scaling

$T(z) = \frac{az+b}{cz+d} = (az+d)(cz+d)^{-1}$

$T(z) = \frac{1}{z} = \frac{1}{r}e^{-i\theta}$



on the Riemann sphere

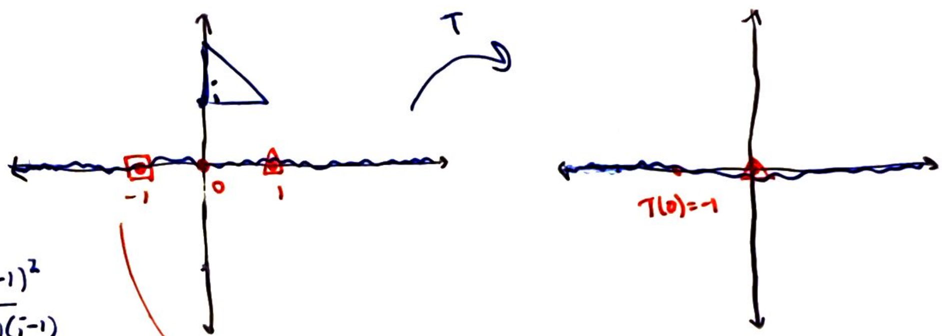


flip the top and bottom of the sphere

(Theorem: LFT's map circles/lines → circles/line)

Any set of 3 distinct points determines a unique circle

$T(z) = \dfrac{z-1}{z+1}$ what does T do to the real line?

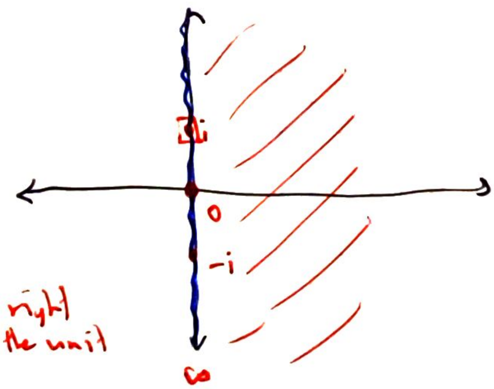


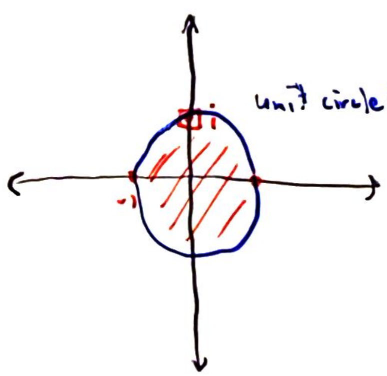
$T(i) = \dfrac{i-1}{i+1} = \dfrac{(i-1)^2}{(i+1)(i-1)}$

$= \dfrac{i^2-2i+1}{-2} = \dfrac{-2i}{-2} = i$

∞ □ goes to $T(-1)=\infty$

$T(\infty) = 1$

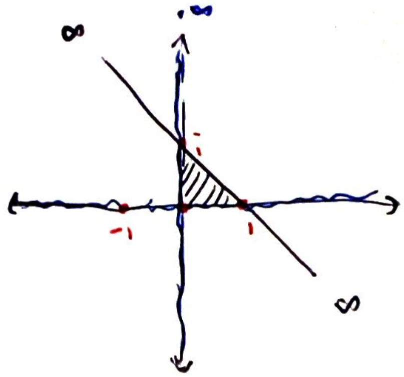


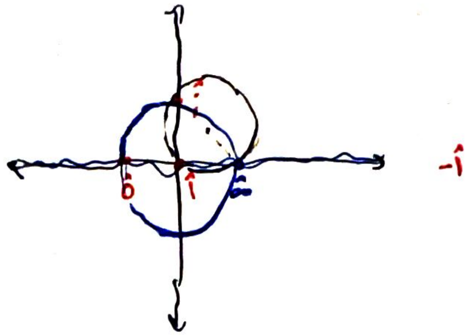


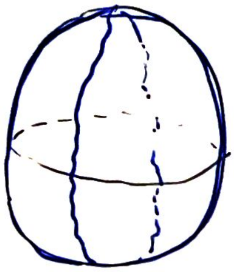
unit circle!

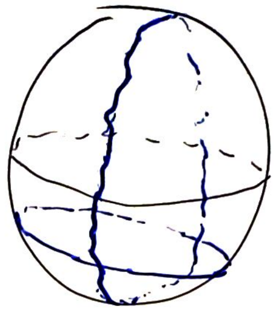
this takes the right half plane to the unit disk

$T(z) = \frac{z-1}{z+1}$ $\quad \infty \to 1$

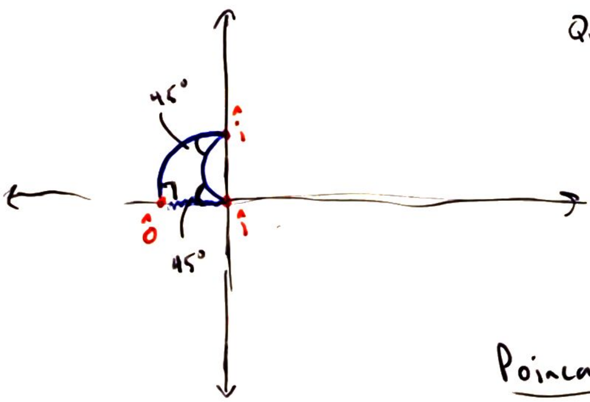




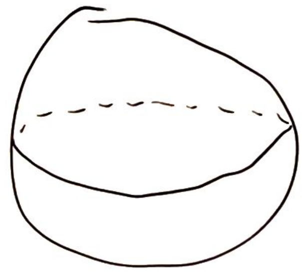




the triangle



Question: is this a legit triangle on the Riemann sphere



Poincare disk model

$T(z) = \frac{az+b}{cz+d}$ shift, magnification, rotation, inversion

for just mag, rot. and shift $T(z) = re^{i\theta}z + b$

a line will get magnified and rotated (still lines)

a circle will get magnified and rotated (still circle

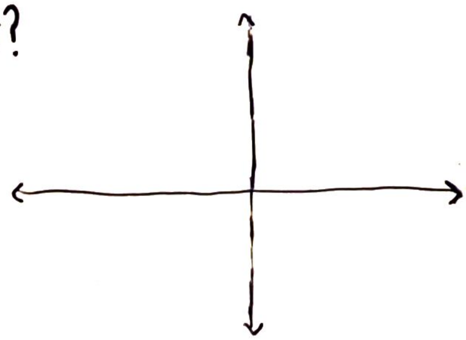
focus: what about $\frac{1}{z}$?

Circle: $Ax + By + C(x^2+y^2) = D$

$\frac{1}{z} = u + iv$

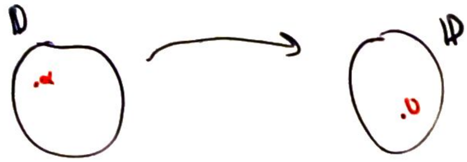
$z = x + iy \quad u = \frac{x}{x^2+y^2} \quad v = \frac{-y}{x^2+y^2}$

$Au - Bv - D(u^2+v^2) = -C$



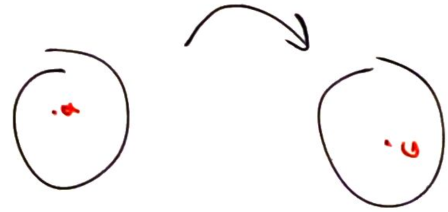
Useful conformal maps: ## Week 2 pt 2 ##

conformal automorphisms of $\mathbb{D}$



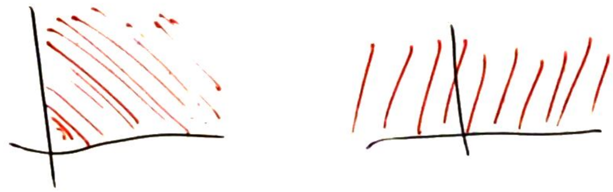
$T(z) = e^{i\theta} \frac{z-\alpha}{1-\bar{\alpha}z}$ $\quad \alpha \in \mathbb{D}$

$z \to \frac{z-\alpha}{1-\bar{\alpha}z}$



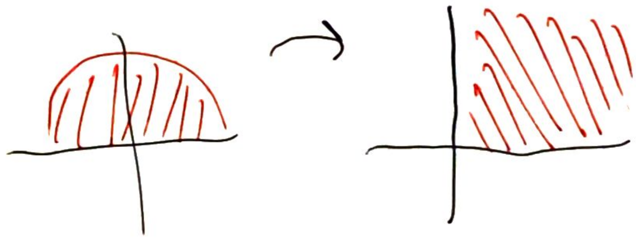
$z \to z^2$

$\sqrt{z} \leftarrow z$



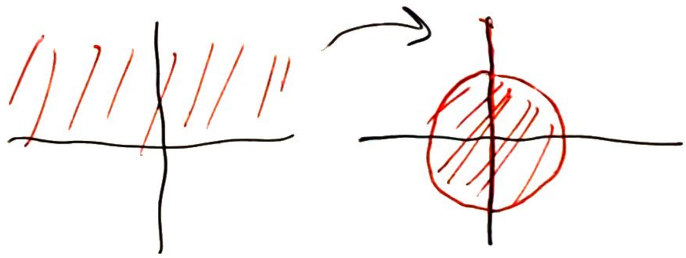
$z \to \frac{1+z}{1-z}$

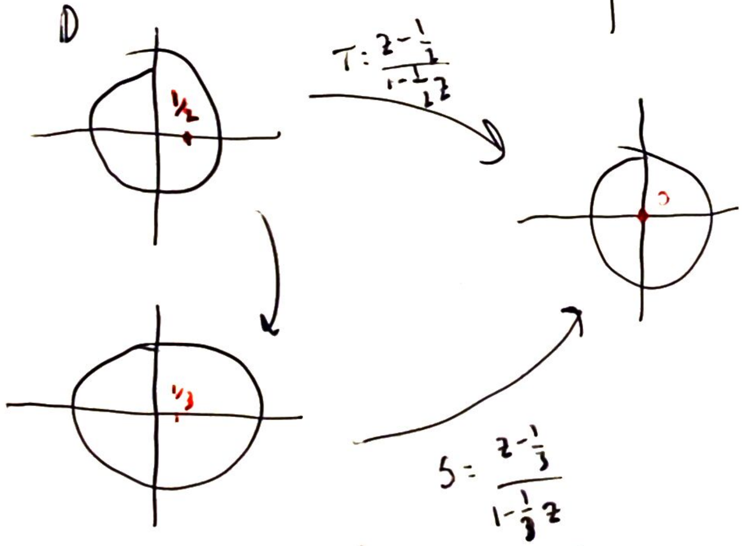
$\frac{-(1-z)}{1+z} \leftarrow z$



$z \to \frac{z-i}{z+i}$

$-i\left(\frac{z+1}{z-1}\right) \leftarrow z$



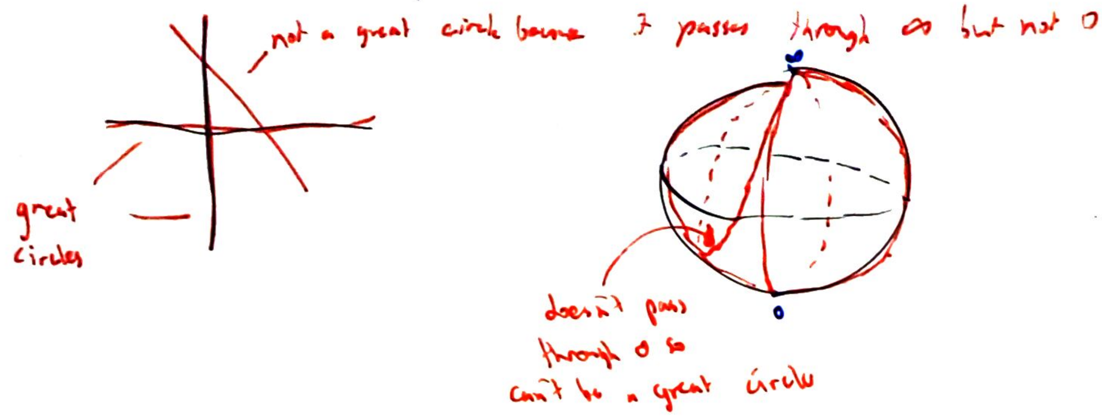


so $S^{-1} \circ T$ sends $1/2$ to $\frac{1}{3}$ and keeps disk hw 2:

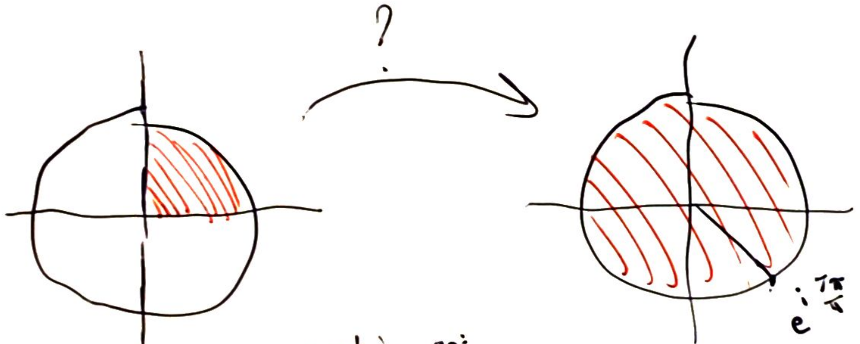
4. want

2. a triangle on the sphere is 3 points connected by parts of great circles

a triangle on the plane can never be a triangle on the sphere



7.



$\left(e^{i\frac{\pi}{4}}\right)^4 = e^{i\pi}$ we want to go:

8.

$f(z) = \frac{az+b}{cz+d} \qquad g(z) = \frac{ez+f}{hz+j}$

$$f(g(z)) = \frac{a\left(\frac{ez+f}{hz+j}\right)+b}{c\left(\frac{ez+f}{hz+j}\right)+d} = \frac{a(ez+f)+b(hz+j)}{c(ez+f)+d(hz+j)}$$

$$= \frac{(ae+bh)z+ad+bj}{(ce+dh)z+cf+dj} \quad - \text{LFT}$$

LFT : $T(z) = \frac{az+b}{cz+d}$ $\qquad ad-bc \neq 0$

Claim : (1) $T(z) = e^{i\theta} \frac{z-\alpha}{1-\bar{\alpha}z}$ $\qquad \alpha \in \mathbb{D}$

conformal automorphism

(2) If $R(z)$ is a conformal automorphism $\mathbb{D} \to \mathbb{D}$

$R(z) = e^{i\theta} \frac{z-\alpha}{1-\bar{\alpha}z}$ $\qquad$ (uniqueness)

usefulness:

$T(\alpha) = 0$ $\qquad T(0) = e^{i\theta}(-\alpha) = -e^{i\theta}\alpha$

Build an S so that $-e^{i\theta}\alpha \to 0$

$S(-e^{i\theta}\alpha) = 0$. $\qquad S(z) = \frac{z-(-e^{i\theta}\alpha)}{1-(\overline{-e^{i\theta}\alpha})z} = \frac{z+e^{i\theta}\alpha}{1+e^{-i\theta}\bar{\alpha}z}$

$S(0) = e^{it}\frac{(e^{i\theta}\alpha)}{1} = e^{it}e^{i\theta}\alpha \implies t = -\theta$

so $S(z) = e^{i\theta} \frac{z+e^{i\theta}\alpha}{1+e^{i\theta}\bar{\alpha}z}$ $\qquad$ heinous algebra

↳ these are inverse functions

conformal automorphism stuff to check

(1) analytic $\qquad$ zero at $\alpha \in \mathbb{D}$ and pole at $\frac{1}{\bar{\alpha}}$

(2) $T : \mathbb{D} \to \mathbb{D}$ $\qquad T(\mathbb{D}) \subseteq \mathbb{D}$

let ~~$z \in \mathbb{D}$~~ be $|z| = 1$

$|T(z)| = |e^{i\theta}| \left| \frac{z-\alpha}{1-\bar{\alpha}z} \right| = \frac{|z-\alpha|}{\underbrace{|z|}_{=1}|z^{-1}-\bar{\alpha}|} = \frac{|z-\alpha|}{|z^{-1}-\bar{\alpha}|}$

since $|z|=1$ $\qquad z^{-1} = \bar{z}$

so $= \frac{|z-\alpha|}{|z^{-1}-\bar{\alpha}|} = \frac{|z-\alpha|}{|\bar{z}-\bar{\alpha}|} = \frac{|z-\alpha|}{|\overline{z-\alpha}|} = 1$

so for $z \in \mathbb{D}$, $|T(z)| < 1$ by the maximum modulus principle

so $T(\mathbb{D}) \subseteq \mathbb{D}$

because $S(z) = T^{-1}(z)$ $\qquad S(\mathbb{D}) \subseteq \mathbb{D}$ $\qquad$ T is invertible on disk

T is a bijection

so we have T analytic on $\mathbb{D}$

T bijective on $\mathbb{D}$

just need nonzero derivative

so $T'(z) = e^{i\theta}\left[\frac{1-|\alpha|^2}{(1-\bar{\alpha}z)^2}\right]$

this is not zero

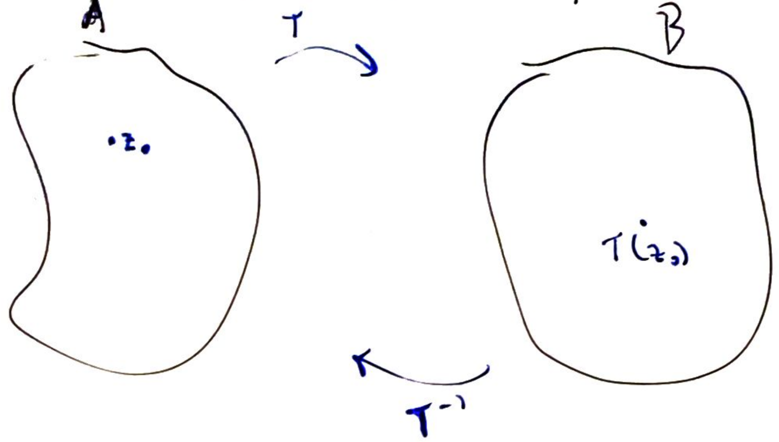
$T(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$

why does the other direction work?

$R(z)$ where R is any conformal automorphism on $\mathbb{D}$

any 2 simply connected domains that aren't all of $\mathbb{C}$ are conformally equivalent

This map is unique if pick $z_0 \in A$, $T(z_0) \in B$ with $T'(z_0) > 0$



start with $R(0)$. Call $R(0)$ α. So $R^{-1}(\alpha) = 0$

Let $T(z) = e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$ $\quad T(\alpha) = 0$ also!

~~Let $\theta = \arg(R'(\alpha))$ $R, R', R'(\alpha)$~~

start with R given. Construct $T(z)$ that matches the behavior of R

what do we need? to specify $T(z)$

$T(z) = e^{i\theta}\frac{z-\boxed{\alpha}}{1-\boxed{\bar{\alpha}}z}$

let $\alpha = R^{-1}(0)$

so $T(\alpha) = 0$ $\quad R(\alpha) = 0$

Compute $R'(\alpha)$, let $\theta = \arg(R'(\alpha))$

so $T'(z) = e^{i\theta}\frac{1-|\alpha|^2}{(1-\bar{\alpha}z)^2}$ so $T'(\alpha) = e^{i\theta}$

So $\arg T'(\alpha) = \theta$