(Ch. 5.2 in Marsden/Hoff) ##Week 3##

Definition: A linear fractional transformation is a degree 1 ration complex function of the form

$$T(z) = \frac{az+b}{cz+d} \quad \text{with } ad-bc \neq 0$$

(why is this excluded?)

LFT: $T(z) = \frac{az+b}{cz+d}$ has a pole of order 1 at $z = -\frac{d}{c}$

zero of order 1 at $z = -\frac{b}{a}$

prop: let $T(z) = \frac{az+b}{cz+d}$ be a LFT, then T is bijective, conformal from $A = \mathbb{C} \setminus \{z = -\frac{d}{c}\}$ to $B = \mathbb{C} \setminus \{\frac{a}{c}\}$

$$T^{-1}(w) = -\frac{dw+b}{cw-a}$$

note: T^{-1} is also conformal and bijective

let $T = \frac{az+b}{cz+d}$ $\quad S = \frac{-dw+b}{cw-a}$

T analytic on A, S analytic on B, now show $T \circ S = S \circ T = z$

conclusion: S, T analytic, $S = T^{-1}$

To show that $T'(z) \neq 0$

observe, $1 = \frac{d}{dz} z = \frac{d}{dz}\left(S(T(z))\right) = S'(T(z))\,T'(z) \neq 0$

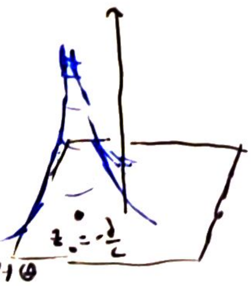
so $T'(z) \neq 0$ and $T^{-1} = S$ is conformal along with T

Well, actually

$$T(z) = \frac{az+b}{cz+d} : A = \mathbb{C} \setminus \{-\tfrac{d}{c}\} \to \mathbb{C} \setminus \{\tfrac{a}{c}\}$$

Lets just say $T(-\frac{d}{c}) = \infty$ $\quad$ define extended complex plane

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$



with this patch, T is conformal on $\hat{\mathbb{C}}$

Geometrically:

$f(z) = az+b$ $\quad$ if $f(z) = az$ then $f(z) = re^{i\theta}(z)$

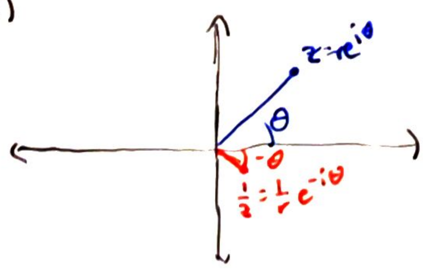
rotation by θ

magnification by r

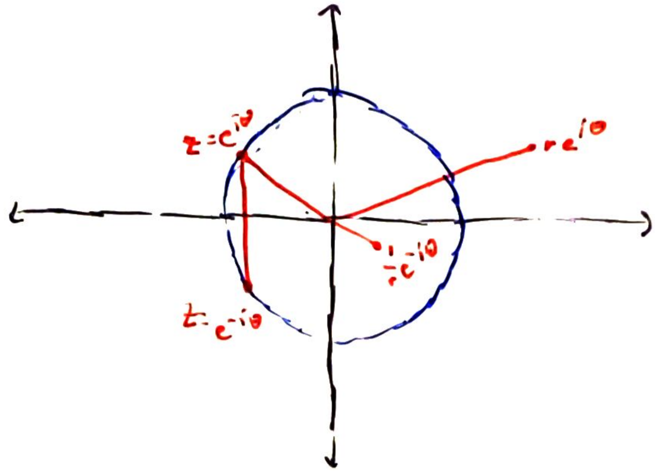
$T(z) = \frac{az+b}{cz+d}$ if $d=1, b=0, c=0$, $a \neq 0$ you get rotation and scaling

$$T(z) = \frac{az+b}{cz+d} = (az+d)(cz+d)^{-1}$$

$$T(z) = \frac{1}{z} = \frac{1}{r} e^{-i\theta}$$



on the Riemann sphere

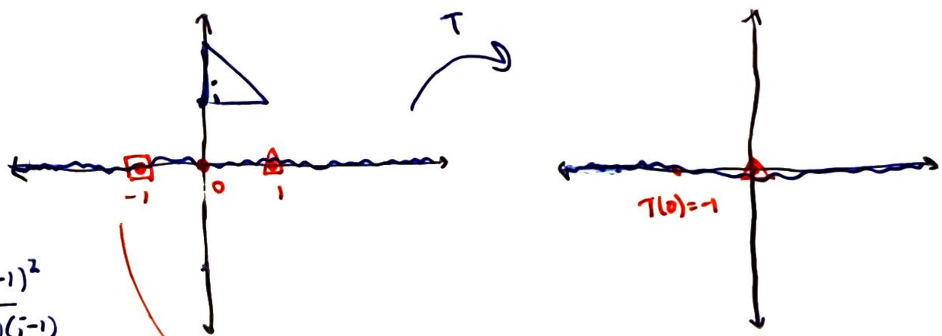


flip the top and bottom of the sphere

(Theorem: LFT's map circles/lines → circles/line)

Any set of 3 distinct points determines a unique circle

$T(z) = \frac{z-1}{z+1}$ what does T do to the real line?

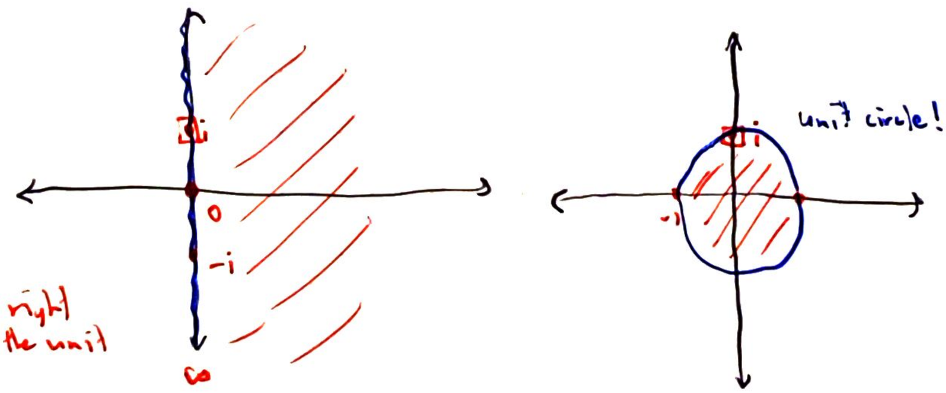


$T(i) = \frac{i-1}{i+1} = \frac{(i-1)^2}{(i+1)(i-1)}$

$= \frac{i^2-2i+1}{-2} = \frac{-2i}{-2} = i$

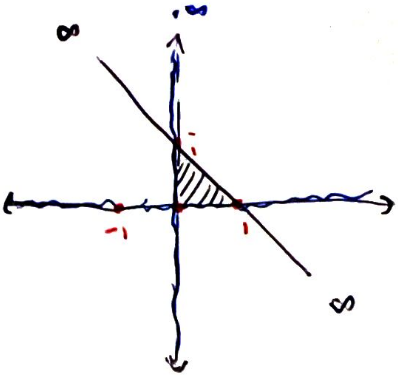
∞ □ goes to $T(-1) = \infty$

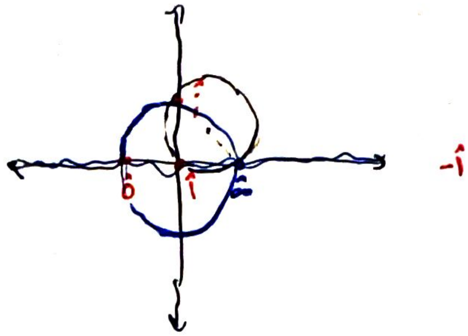
$T(\infty) = 1$

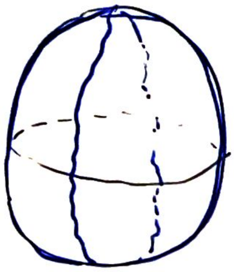


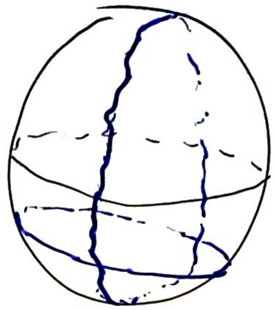
this takes the right half plane to the unit disk

$T(z) = \frac{z-1}{z+1}$ $\infty \to 1$

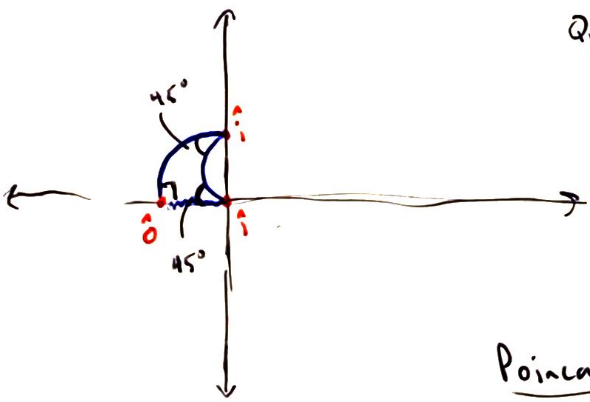




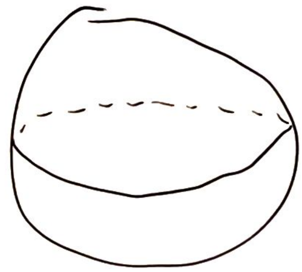




the triangle



Question: is this a legit triangle on the Riemann sphere



Poincare disk model

$T(z) = \frac{az+b}{cz+d}$ shift, magnification, rotation, inversion

for just mag, rot. and shift $T(z) = re^{i\theta}z + b$

a line will get magnified and rotated (still lines)

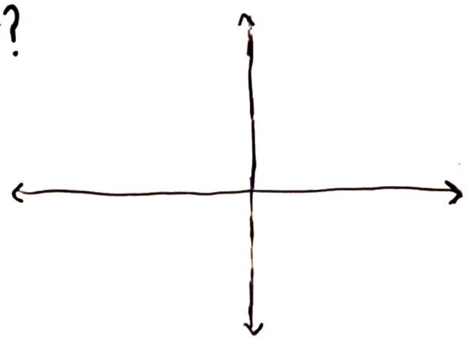
a circle will get magnified and rotated (still circle

focus: what about $\frac{1}{z}$?

Circle: $Ax + By + C(x^2+y^2) = D$

$\frac{1}{z} = u + iv$

$z = x + iy$ $u = \frac{x}{x^2+y^2}$ $v = \frac{-y}{x^2+y^2}$



$Au - Bv - D(u^2+v^2) = -C$