Numerical Analysis Qualifying Exam Study Sheet

Trevor Loe

January 10, 2025

This document contains a brief overview of all the topics that I have found may appear on the numerical analysis qualifying exam. I have tried to organize the topics by similarlity. The list of topics was started based off the UCLA qual website `https://ww3.math.ucla.edu/qualifying-exam-dates/` and appended as past qual problems were done.

Past numerical analysis qualifying exams can be found at `https://ww3.math.ucla.edu/past-qualifying-exams/`

Much of the material on undergraduate numerical analysis was taken from [1].

1 Interpolation

1.1 Lagrange Interpolation

If $x_0, ..., x_n$ are $n+1$ distinct real numbers and f is defined on those numbers, then there is a unique, degree-n polynomial P such that

$$P(x_i) = f(x_i)$$

for all $i = 0, ..., n$. The **n'th Lagrange Interpolating Polynomial** is given by

$$\begin{aligned} L_{n,k}(x) &= \frac{(x-x_0)(x-x_1)...(x-x_{k-1})(x-x_{k+1})...(x-x_n)}{(x_k-x_0)...(x_k-x_{k-1})(x_k-x_{k+1})...(x_k-x_n)} \\ &= \prod_{i=0, i\neq k} \frac{(x-x_i)}{(x_k-x_i)} \end{aligned}$$

The unique polynomial P is then given by

$$P(x) = f(x_0)L_{n,0}(x) + f(x_1)L_{n,1}(x) + ... + f(x_n)L_{n,n}(x)$$

1.2 Divided Differences

The divided differences for a function f and real numbers $x_0, ..., x_n$ is defined by

$$f[x_i] = f(x_i)$$

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

and the kth divided difference is given by

$$f[x_i, x_{i+1}, ..., x_{i+k}] = \frac{f[x_{i+1}, ..., x_{i+k}] - f[x_i, ..., x_{i+k-1}]}{x_{i+k} - x_i}$$

If P is the polynomial made from the Lagrange interpolating polynomials above, then (Newton's Divided Difference Formula)

$$P(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, ..., x_k](x-x_0)...(x-x_{k-1})$$

If $f \in C^n[a,b]$ then there exists a number $\xi \in (a,b)$ such that

$$f[x_0, ..., x_n] = \frac{f^{(n)}(\xi)}{n!}$$

1.3 Cubic Spline Interpolation

For a function f defined on $[a, b]$ and a set of notes $a = x_0 < ... < x_n = b$ the **cubic spline** interpolation is the piecewise-cubic function S such that

1. On each sub-interval $[x_j, x_{j+1}]$, S is a cubic function, denoted S_j
2. $S_j(x_j) = f(x_j)$, $S_j(x_{j+1}) = f(x_{j+1})$
3. $S_j(x_{j+1}) = S_{j+1}(x_{j+1})$
4. $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$
5. $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$

If $S''(x_0) = S''(x_n) = 0$, it is a natural or free boundary. If $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$, it is a clamped boundary.

There is always a unique natural spline interpolating between n data points and a unique clamped spline interpolating between n data points (assuming that $f'(x_0)$ and $f'(x_n)$ are given).

If $f \in C^4[a, b]$ with $\max |f^{(4)}(x)| \leq M$ then there will be some constant C such that

$$|f(x) - S(x)| \leq C \max_j (x_j - x_{j+1})^4$$

for a clamped spline $C = \frac{5M}{384}$

2 Numerical Differentiation

Forward difference formula

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

Backward difference formula

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$

$(n+1)$-point formula

$$f'(x_j) = \sum_{k=0}^{n} f(x_k) L'_k(x_j) + \frac{f^{(n+1)}(\xi(x_j))}{(n+1)!} \prod_{k=0, k\neq j} (x_j - x_k)$$

which comes from the error bound for the Lagrange interpolating polynomials.

Second derivative midpoint formula

$$f''(x_0) = \frac{1}{h^2}(f(x_0 - h) - 2f(x_0) + f(x_0 + h)) - \frac{h^2}{12} f^{(4)}(\xi)$$

3 Richardson Extrapolation

If your error is of the form

$$M = N(h) + K_1 h + K_2 h^2 + ...$$

then you can define $N_2(h) = N(\frac{h}{2}) + \big(N(\frac{h}{2}) - N(h)\big)$ to get

$$M = N_2(h) - \frac{K_2}{2} h^2 - \frac{3K_3}{4} h^3 + ...$$

increasing the order of the approximation by 1.

References

[1] R.L. Burden and J.D. Faires. *Numerical Analysis*. Cengage Learning, 2010. ISBN: 9780538733519. URL: `https://books.google.com/books?id=zXnSxY9G2JgC`.