Superspace coinvariants and hyperplane arrangements

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by Trevor K. Karn (U. Minnesota)
(joint with Robert Angarone, Patricia Commins, Satoshi Murai, and Brendon Rhoades)
on Monday, 4 November, 2024

Main problem:

find a linear basis for the algebra SR_n.

Approach:

$\mathcal{ST}$ algebras of southwest arrangements

What is SR_n?

The symmetric group $\mathfrak{S}_n$ acts on

$$\mathbb{C}[\underline{x}] = \mathbb{C}[x_1, x_2, \ldots, x_n].$$

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Let

$$p_k = x_1^k + x_2^k + \cdots + x_n^k.$$

and let

$$I_n^+ = (p_1, p_2, \ldots, p_n)$$

Definition

$R_n = \mathbb{C}[\underline{x}]/I_n^+$ is the coinvariant ring.

Example

Let $n = 3$, $\mathbb{C}[\underline{x}] = \mathbb{C}[x, y, z]$,

$$p_1 = x + y + z$$

$$p_2 = x^2 + y^2 + z^2$$

$$p_3 = x^3 + y^3 + z^3$$

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so

$$x^4 + xy^3 + xz^3 \in I_3^+$$

and

$$x^2 + y^2 \notin I_3^+$$

In $\mathbb{C}[\underline{x}]/I_3^+$,

$$x + y + z \equiv 0$$

and

$$x^2 + y^2 + z^2 \equiv (-y - z)^2 + y^2 + z^2.$$

so

$$y^2 \equiv -yz - z^2.$$

Similar computation shows that

$$z^3 \equiv 0.$$

Observation: x_i^i can always be rewritten “nicely”

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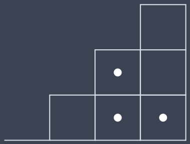
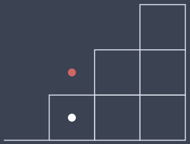
Definition

The staircase monomials are $\underline{x}^\alpha$ where $\alpha_i < i$.

$x_3^2 x_4 \longleftrightarrow$



but not $x_2^2 \longleftrightarrow$



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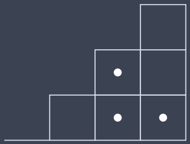

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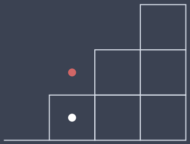

Theorem (E. Artin)

The staircase monomials are a basis for R_n.

What is SR_n?

Superspace is

$$\mathbb{C}[\underline{x}, \underline{\theta}] = \mathbb{C}[x_1, x_2, \ldots, x_n, \theta_1, \theta_2, \ldots, \theta_n]$$

where $\theta_i\theta_j = -\theta_j\theta_i$. Write†

$$\underline{\theta}^J = \prod_{j \in J} \theta_j.$$

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Let $SI_n^+ = (p_1, \ldots, p_n, sp_0, \ldots, sp_{n-1})$.

Definition

The <u>superspace coinvariant ring</u> is

$$SR_n = \mathbb{C}[\underline{x}, \underline{\theta}]/SI_n^+.$$

Sagan and Swanson [SS24] introduced

$$\mathcal{M} = \bigcup_{J \subseteq [n]} \{\underline{\mathsf{x}}^{\alpha} \underline{\theta}^{J} : \alpha < (J\text{-staircase})\}$$

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$$\mathcal{M} = \bigcup_{J \subseteq [n]} \{\underline{x}^{\alpha}\underline{\theta}^{J} : \alpha < (J\text{-staircase})\}$$

Definition

Let $J \subseteq [n]$. A J-staircase is $(\mathrm{st}(J)_1, \mathrm{st}(J)_2, \ldots, \mathrm{st}(J)_n)$ where

$$\mathrm{st}(J)_1 = \begin{cases} 0 & 1 \in J \\ 1 & 1 \notin J \end{cases}$$

and

$$\mathrm{st}(J)_i = \begin{cases} \mathrm{st}(J)_{i-1} & i \in J \\ \mathrm{st}(J)_{i-1} + 1 & i \notin J \end{cases}$$

» Example

Let $J = \{2, 4, 5\} \subseteq [6]$. Then the J-staircase is $(1, 1, 2, 2, 2, 3)$.

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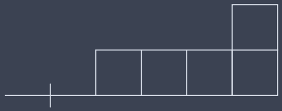
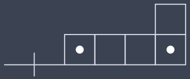
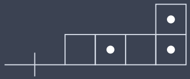


So $\mathcal{M}$ contains monomials

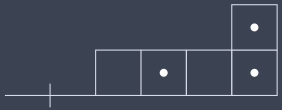
$x_3x_6\theta_2\theta_4\theta_5$ and $x_4x_6^2\theta_2\theta_4\theta_5$ but not $x_5^2\theta_2\theta_4\theta_5$





Punchline

Elements of $\mathcal{M}$ correspond to filled diagrams like



$$\text{shape} \quad \longleftrightarrow \quad \text{skew-commutative } \theta \text{ factor}$$

$$\text{filling} \quad \longleftrightarrow \quad \text{commutative } x \text{ factor}$$

Conjecture [SS24]/Theorem [ACK+24]

$$\mathcal{M} = \bigcup_{J \subseteq [n]} \{\mathsf{x}^\alpha \theta_J : \alpha \leq (J\text{-staircase})\}$$

is a basis for SR_n.

What are SW arrangements and $\mathcal{ST}$ algebras?

For the rest of the talk $S = \mathbb{C}[x_1, x_2, \ldots, x_n]$.

A hyperplane H is a codimension-1 affine linear subspace of $\mathbb{K}^n$.

A (hyperplane) arrangement $\mathcal{A}$ is a union of hyperplanes.

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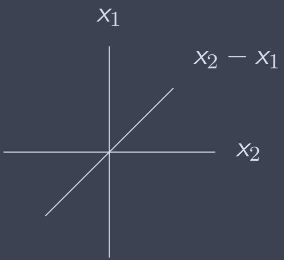
Geometrically, H is a variety cut out by a degree-1 polynomial.

$\mathcal{A}$ is a variety cut out by a product of degree-1 polynomials.

» Examples

An arrangement in $\mathbb{R}^2$:

$x_1 x_2 (x_2 - x_1) \longleftrightarrow$



"Definition"

Define by example the diagram $\tilde{\Phi}_n$ for $n = 5$:

$$
\begin{array}{ccccccccc}
 & & & & \circ & & & & \\
 & & & & x_5 & & & & \\
 & & & \circ & & \circ & & & \\
 & & & x_4 & & x_1 - x_5 & & & \\
 & & \circ & & \circ & & \circ & & \\
 & & x_3 & & x_1 - x_4 & & x_2 - x_5 & & \\
 & \circ & & \circ & & \circ & & \circ & \\
 & x_2 & & x_1 - x_3 & & x_2 - x_4 & & x_3 - x_5 & \\
\circ & & \circ & & \circ & & \circ & & \circ \\
x_1 & & x_1 - x_2 & & x_2 - x_3 & & x_3 - x_4 & & x_4 - x_5
\end{array}
$$

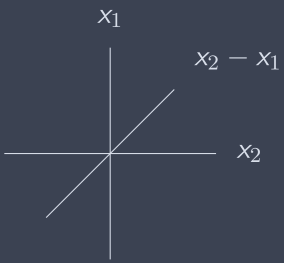
Definition/example

An arrangement $\mathcal{A}$ is called a *southwest arrangement* if its defining polynomial $Q(\mathcal{A})$ is a product of terms of a southwest-closed subset of $\tilde{\Phi}_n$.

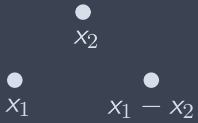
The h-function of a southwest arrangement is the number of hyperplanes on each southeast diagonal.

» Example

$x_1 x_2 (x_2 - x_1) \longleftrightarrow$

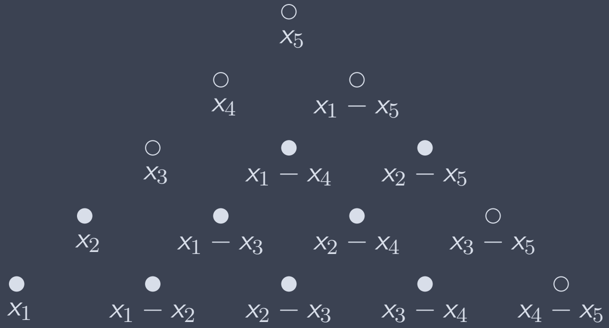


$\longleftrightarrow$



is a southwest arrangement with h-function $(1, 2)$.

» Example



$$x_1 x_2 (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_2 - x_5)(x_3 - x_4)$$

is southwest. It has h-function $(1, 2, 2, 3, 1)$.

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Definition

There is a free S-module $\mathrm{Der}(S)$ with basis $\{\partial_i\}_{i=1}^n$. The module of derivations of $\mathcal{A}$ is

$$\mathrm{Der}(\mathcal{A}) = \{d \in \mathrm{Der}(S) : d(H) \in \mathrm{span}_S H \, \forall H \in \mathcal{A}\}.$$

An arrangement is called free if $\mathrm{Der}(\mathcal{A})$ is a free S-module.

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Example

The arrangement $x_1x_2(x_2 - x_1)$ is free with basis

$$\{x_1\partial_1 + x_2\partial_2,\ x_2(x_2 - x_1)\partial_2\}$$

If $\mathcal{A}$ is a free arrangement, then $\mathrm{Der}(\mathcal{A})$ has a homogeneous basis.

The degrees of polynomials in the homogeneous basis are called the exponents of $\mathcal{A}$.

Exponents are independent of choice of basis and give combinatorial information about $\mathcal{A}$.

Theorem [ACK+24]

Let $\mathcal{A}$ be a southwest arrangement. Then $\mathcal{A}$ is free with exponents given by the h-function.

Definition [AMMN19]

Let $\mathfrak{a} : \mathrm{Der}(\mathcal{A}) \to S$ be an S-module homomorphism. Define the Solomon-Terao algebra to be

$$\mathcal{ST}(\mathcal{A}; \mathfrak{a}) = S/\operatorname{im}\mathfrak{a}.$$

» Example

Define $\mathfrak{i}$ to be the map that sends $\partial_i \mapsto 1$.

Recall that the arrangement $\mathcal{A}$ defined by $x_1x_2(x_2 - x_1)$ is free with basis

$$\{x_1\partial_1 + x_2\partial_2,\ x_2(x_2 - x_1)\partial_2\}.$$

Then

$$\mathcal{ST}(\mathcal{A}, \mathfrak{i}) = \mathbb{C}[x_1, x_2]/(x_1 + x_2, x_2^2 - x_1x_2) \cong \mathbb{C}[x_2]/(x_2^2).$$

Proof ideas

The colon ideal $(I : f)$ is the kernel of $\times f$ so that

$$0 \to S/(I : f) \xrightarrow{\times f} S/I$$

is exact.

Geometrically, for a variety X,

$$(I(X) : f) = I(X - V(f)))$$

» Transfer principle

Rhoades and Wilson [RW23] showed that in order to show $\mathcal{M}$ is a basis for SR_n, it suffices to show

$$\mathcal{M}(J) = \{\mathsf{x}^\alpha : \alpha \leq (J\text{-staircase})\}$$

is a basis for

$$S/(I^+ : f_J)$$

where

$$f_J = \prod_{j \in J} x_j \prod_{i > j} (x_j - x_i).$$

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Upshot

Trade a skew-commutative problem for a family of commutative problems.

The arrangement defined by f_J is not a southwest arrangement.
E.g. $J = \{2, 3\}$:

○
x_5

○ ○
x_4 $x_1 - x_5$

● ○ ●
x_3 $x_1 - x_4$ $x_2 - x_5$

● ○ ● ●
x_2 $x_1 - x_3$ $x_2 - x_4$ $x_3 - x_5$

○ ○ ● ● ○
x_1 $x_1 - x_2$ $x_2 - x_3$ $x_3 - x_4$ $x_4 - x_5$

Theorem [ACK+24]

Let $\mathcal{A}$ be an essential southwest arrangement in $\mathbb{C}^n$ with h-function $h(\mathcal{A})$. Let $\mathfrak{i} : \mathrm{Der}(\mathcal{A}) \to S$ be defined by $\partial_i \mapsto 1$. Then the monomials

$$\{\mathsf{x}^\alpha : \alpha < h(\mathcal{A})\}$$

descend to a basis for $\mathcal{ST}(\mathcal{A}; \mathfrak{i})$.

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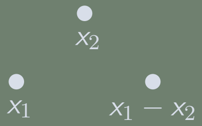
Example

We saw $\{1, x_2\}$ is a basis for $\mathcal{ST}(\mathcal{A}; \mathfrak{i})$ of

$\mathcal{A} \leftrightarrow$



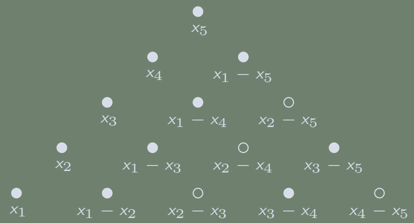
Definition

Let $J \subseteq [n]$. Let $\mathcal{A}_J$ denote the southwest arrangement defined by

$$x_1 x_2 \cdots x_n \prod_{j \notin J} \prod_{i > j} (x_j - x_i)$$

Example

Let $J = \{2, 4\}$, then $\mathcal{A}_J$ is



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Let

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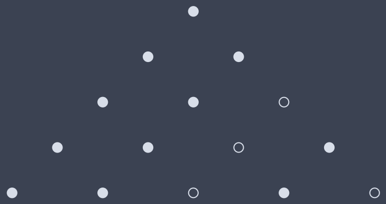
Lemma [ACK+24]

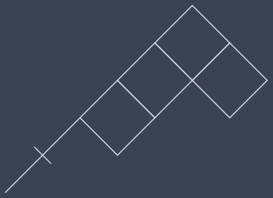
$S/(I_n^+ : \tilde{f}_J) \cong \mathcal{ST}(\mathcal{A}_J; t)$

Lemma

The J-staircase is bounded above by the h-function of $\mathcal{A}_J$. In particular

$$h_k = \begin{cases} \mathrm{st}(J)_k & k \notin J \\ \mathrm{st}(J)_k + 1 & k \in J \end{cases}$$





Theorem [ACK+24]

$\mathcal{M}(J)$ is a basis for $S/(I^+ : f_J)$

Proof: A general fact about colon ideals tells us

$$(I^+ : \tilde{f}_J) : \underline{x}^J = (I^+ : \tilde{f}_J \underline{x}^J) = (I^+ : f_J).$$

Thus,

$$0 \to S/(I^+ : f_J) \xrightarrow{\times x^J} S/(I^+ : \tilde{f}_J) = \mathcal{ST}(\mathcal{A}_J, \mathfrak{i})$$

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Corollary

$\mathcal{M}$ is a basis for SR_n, resolving conjecture of [SS24]

What next?

THANK YOU!

References

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In case Sarah asks “What about type-B?”: