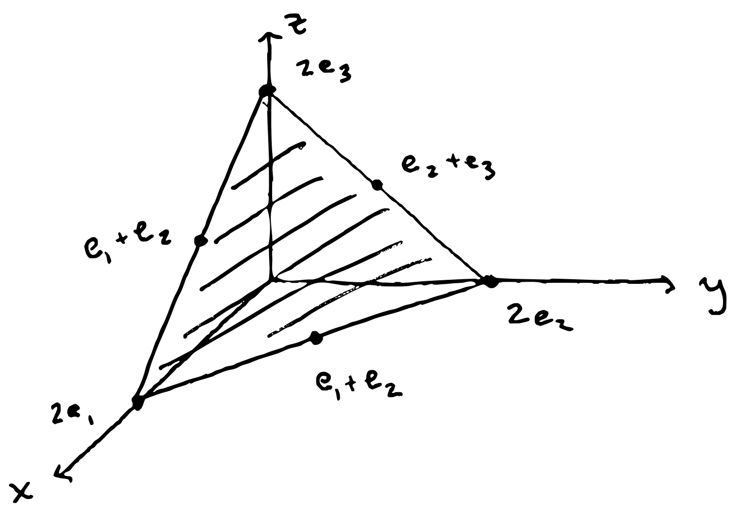
Consider $\mathfrak{S}_4$ acting on coordinates of $\mathbb{R}^4$ by permutation.

The convex hull of the orbit of $(1,1,0,0)$ defines a shape. What is it?

All points live in the hyperplane $\{x+y+z+w=2\}$ so we may project into $\mathbb{R}^3$

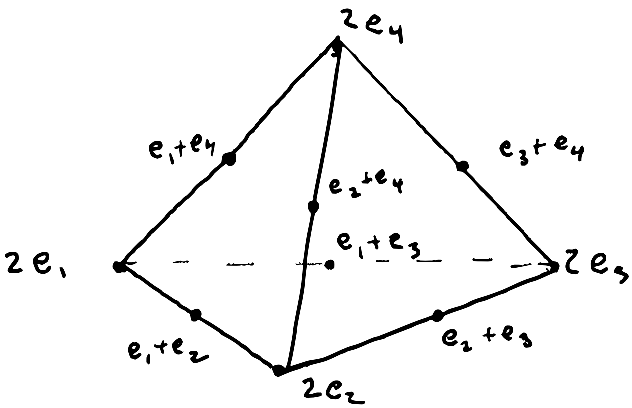
By analogy, we first consider a lower dimensional space. For a moment consider $\{x+y+z=2\} \subseteq \mathbb{R}^3$. This intersects the positive quadrant in a simplex.



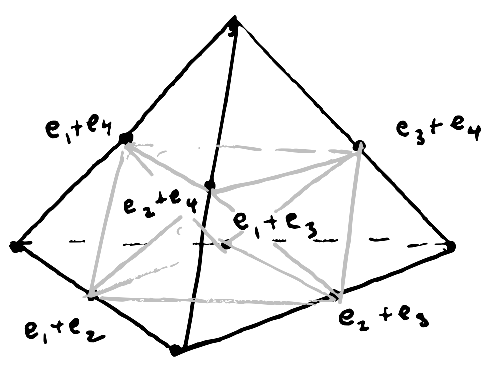
Now consider the points $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$ they are e_1+e_2, e_1+e_3, and e_2+e_3.

The point of this analogy is to give intuition for the $\mathbb{R}^4$ case.

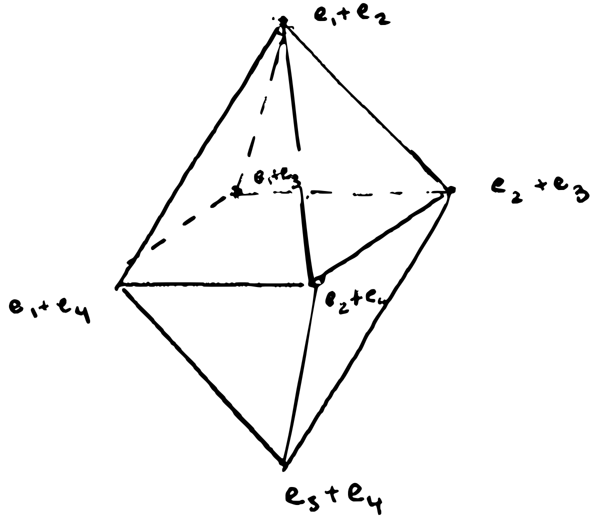
As previously stated, the points live in the hyperplane $\{x+y+z+w=2\}$. Notice that since the shape we consider is the orbit of a point with non-negative coordinates, we may restrict our attention to the intersection of the hyperplane with the positive orthant. As before this intersection is a simplex:



Now note that $(1,1,0,0) = e_1+e_2$ and so the orbit of the point is all e_i+e_j where $i \neq j$. These points are drawn below.



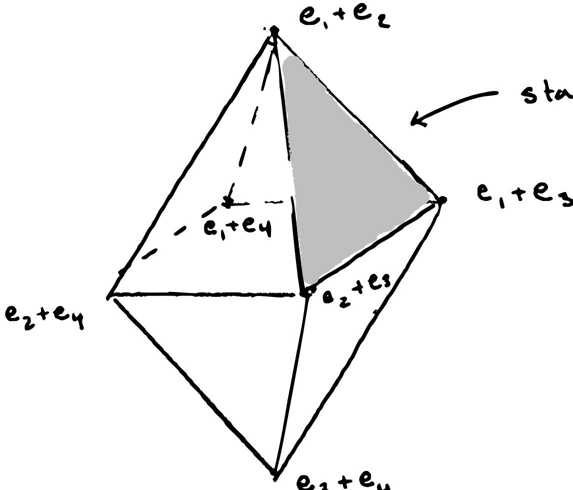
Removing the simplex we see that the shape is an octahedron.



Now, the action of S_4 on the vertices induces an action on the faces. The action is a permutation representation on each orbit. There are two distinct orbits, each of which consists of a checkerboard pattern of faces, which can be seen on the next page. The permutation representation carried by a single orbit can be found by inducing the trivial representation on the stabilizer of a single face:

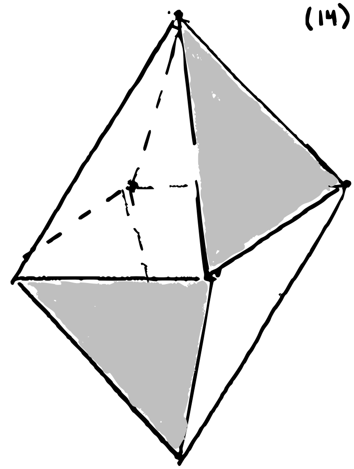
$$\mathrm{Ind}_{S_3 \times S_1}^{S_4} \left(\mathbb{1}_{S_3 \times S_1} \right).$$

The trivial representation $\mathbb{1}_{S_3 \times S_1}$ may be written as the tensor product $\mathbb{1}_{S_3} \otimes \mathbb{1}_{S_1}$. The induction corresponds to multiplication of Schur functions: $s_3 \cdot s_1 = s_4 + s_{3,1}$. Since there are two distinct orbits, and the stabilizer of any face is isomorphic to $S_3 \times S_1$, the rep'n is $(V^4 \oplus V^{3,1})^{\oplus 2}$.

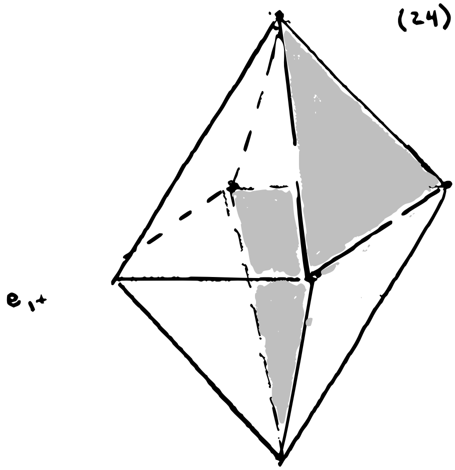


stabilized by $\mathfrak{S}_{\{1,2,3\}} \times \mathfrak{S}_4$

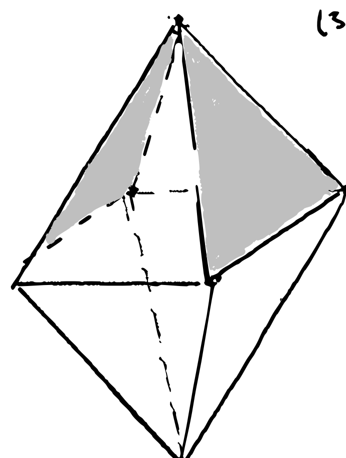
(14) · face



(24) · face



(34) face



<u>Orbit:</u>