Stirling Numbers of the Second Kind and Their Generating Functions

DEF The Stirling numbers of the second kind $S(n,k)$ count the number of set partitions of $\{1,2,\dots,n\}$ into k blocks.

The point of this note is to give both the ordinary and exponential generating functions for $S(n,k)$.

Both the ordinary and exponential generating functions rely on the the following recusion:

$$(1) \qquad S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$$

To see the recusion holds notice that there are two ways to find a set partition of $\{1,\dots,n\}$ from a set partition of $\{1,\dots,n-1\}$. The first way is to have the singleton $\{n\}$ and a partition of $\{1,\dots,n-1\}$ into $(k-1)$-many blocks. The second is to start with a partition of $\{1,\dots,n-1\}$ into k-many blocks and add n into one of those blocks.

Ex LET $n = 5$. The partition

$$\{\{1,3\},\{2,4\},\{5\}\}$$

contributes to $S(5,3)$, and comes from the partition

$$\{\{1,3\},\{2,4\}\}$$

by adding $\{5\}$. It contributes to $S(4,2)$. Now, the partitions

$$\{\{1,5\},\{2,3\},\{4\}\}$$
$$\{\{1\},\{2,3,5\},\{4\}\}$$
$$\{\{1\},\{2,3\},\{4,5\}\}$$

all come from

$$\{\{1\},\{2,3\},\{4\}\}$$

by adding 5 to one of the 3 existing blocks, so we see

$$S(5,3) = S(4,2) + 3S(4,3).$$

To go from a recusion to a generating function, we apply what Herbert Wilf calls in "generating functionology":

THE METHOD

The method is to turn the recursion into an equation of generating functions by summing over both sides and multiplying by x^n. So we get

$$A_k(x) = \sum_{n \geq 0} S(n,k)\, x^n = \sum_{n \geq 0} \left(S(n-1,k-1) + kS(n-1,k)\right) x^n$$

$$B_k(x) = \sum_{n \geq 0} S(n,k) \frac{x^n}{n!} = \sum_{n \geq 0} \left(S(n-1,k-1) + kS(n-1,k)\right) \frac{x^n}{n!}$$

OGF

$$A_k(x) = \sum_{n \geq 0} S(n-1,k-1)\, x^n + \sum_{n \geq 0} k\, S(n-1,k)\, x^n$$

$$= x \sum_{n \geq 0} S(n-1,k-1)\, x^{n-1} + kx \sum_{n \geq 0} S(n-1,k)\, x^{n-1}$$

$$= x\, A_{k-1}(x) + kx\, A_k(x)$$

We can solve for A_k:

$$A_k(x) = \frac{x\, A_{k-1}(x)}{1 - kx}$$

But at this point, we only have a relationship between A_k and A_{k-1}.

So, to find a closed form for A_k, let's think about A_0, A_1 and then induct on k.

A_0 is the generating function for portions of n into zero parts. The only number which can be partitioned into zero parts is zero itself. So,

$$A_0 = S(0,0) + S(1,0)x + S(2,0)x^2 + \cdots$$

$$= 1 + 0x + 0x^2 + \cdots = 1$$

Similarly, there is only one way to partition a positive number into one part, so

$$A_1 = S(0,1) + S(1,1)x + S(2,1)x^2 + \cdots$$

$$= 0 + x + x^2 + \cdots = x \cdot \frac{1}{1-x},$$

since $\emptyset$ cannot be partitioned into one part and

$$\sum_{n \geq 0} x^n = \frac{1}{1-x}.$$

If

$$A_k(x) = \frac{x A_{k-1}(x)}{1-kx}$$

and $A_0 = 1$, then

$$A_1(x) = \frac{x}{1-x}.$$

applying induction, we see:

$$A_k(x) = \prod_{j=1}^{k} \frac{x}{1-jx}.$$

EGF Now we consider the situation of exponential generating functions. Recall that

$$B_k(x) = \sum_{n\geq 0} S(n,k)\frac{x^n}{n!} = \sum_{n\geq 0} (S(n-1,k-1)+kS(n-1,k))x^n$$

and so

$$B_k(x) = \sum_{n\geq 0} S(n-1,k-1)\frac{x^n}{n!} + k\sum_{n\geq 0} S(n-1,k)\frac{x^n}{n!}$$

In order to get both the exponent and denominator of

$$\sum_{n\geq 0} S(n-1,k-1)\frac{x^n}{n!}$$

to agree with the first argument of $S(n-1,k-1)$, we take the derivative with respect to x:

$$\frac{\partial}{\partial x} B_k = \sum_{n\geq 0} S(n-1,k-1)\frac{x^{n-1}}{(n-1)!} + k\sum_{n\geq 0} S(n-1,k)\frac{x^{n-1}}{(n-1)!}$$

which yields the functional equation

(2) $$\frac{\partial}{\partial x} B_k(x) = B_{k-1}(x) + kB_k(x)$$

(since $S(-1,k)=0$ for all k)

Now we want to find a solution to the differential equation (2). Since the equation depends on k, lets first look at small values of k. This will allow us to guess a formula for $B_k(x)$, which we will then prove both satisfies (2) and is the unique solution which makes sense in this situation.

First, take $k=1$. Then

$$\frac{\partial}{\partial x} B_1(x) = B_0(x) + B_1(x)$$

Similarly to our analysis of $A_0(x)$, we find that $B_0(x) = 1$. So we conclude that

$$B_1'(x) - B_1(x) = 1.$$

This is a differential equation of the form

$$(3) \qquad P(x)\,y(x) + y'(x) = Q(x).$$

To solve these, multiply both sides by the integrating factor

$$I = e^{\int P(x)\,dx}$$

taking $P(x) = -1$.
So we get

$$-e^{-x} B_1(x) + e^{-x} B_1'(x) = e^{-x}$$

$$\frac{\partial}{\partial x}\left[e^{-x} B_1(x)\right] = e^{-x}$$

or in other words:

$$e^{-x} B_1(x) = \int e^{-x} dx = -e^{-x} + c$$

so

$$B_1(x) = -1 + ce^{x}.$$

We can determine the value of c by recognising that $B_1(0) = S(0,1) = 0$, and so

$$B_1(x) = -1 + e^{x}$$

This base-case is helpful, but not quite what we need yet, so lets try the $k=2$ case.

$$B_2'(x) = B_1(x) + 2B_2(x)$$

This again fits the form (3), so we may apply the same approach to finding its solution.

$$B_2'(x) - 2B_2(x) = e^{x} - 1$$

$$\frac{\partial}{\partial x}\left(e^{-2x} B_2(x)\right) = e^{-x} - e^{-2x}$$

$$e^{-2x} B_2(x) = \int e^{-x} - e^{-2x} dx$$

$$= -e^{-x} + \frac{1}{2} e^{-2x} + c$$

We have the initial condition $B_2(0) = S(0,2) = 0$, and so we see that

$$B_2(0) = 0 = -e^0 + \tfrac{1}{2} + ce^{2\cdot 0} = -1 + \tfrac{1}{2} + c$$

hence $c = \frac{1}{2}$. This is a perfectly fine answer, but with the benefit of knowing the answer, I'll write it like this:

$$B_2(x) = \tfrac{1}{2}(e^x - 1)^2.$$

Repeating the process for $k=3$, we see

$$B_3(x) = \tfrac{1}{6}(e^x - 1)^3.$$

This leads us to take a guess that

$$B_k(x) = \frac{1}{k!}(e^x - 1)^k.$$

Now we can check that this solution holds and is correct by

i) showing it satisfies the differential equation (2)

ii) applying the existence and uniqueness theorem for ODEs.

First, taking $B_k(x) = \frac{1}{k!}(e^x - 1)^k$, we see

$$\frac{\partial}{\partial x} B_k(x) = \frac{1}{(k-1)!}(e^x - 1)^{k-1} e^x$$

and then

$$B_{k-1}(x) + k B_k(x) = \frac{1}{(k-1)!}(e^x-1)^{k-1} + \frac{k}{k!}(e^x-1)^k$$
$$= \frac{1}{(k-1)!}(e^x-1)^{k-1}(1+e^x-1)$$
$$= \frac{1}{(k-1)!}(e^x-1)^{k-1}e^x$$
$$= \frac{\partial}{\partial x} B_k(x).$$

So $B_k(x) = \frac{1}{k!}(e^x-1)^k$ satisfies (2)

<u>Thm</u>: (Existence and uniqueness for an IVP).

Consider the IVP

$$(*) = \begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

If f is continuous in a neighborhood of (x_0,y_0), then there is a solution to $(*)$ and if moreover $\frac{\partial}{\partial y} f$ is continuous in a neighborhood of (x_0,y_0), then the solution is unique.

So we are considering the IVP

$$(4) = \begin{cases} y' = ky + \frac{1}{(k-1)!}(e^x-1)^{k-1} \\ y(0) = S(0,k) \end{cases}$$

where $S(0,k) = \begin{cases} 0 & k \neq 0 \\ 1 & k = 0 \end{cases}$. Thus, the fact that $\frac{\partial}{\partial y}\left(ky + \frac{1}{(k-1)!}(e^x-1)^{k-1}\right) = k$ is continuous on all of $\mathbb{R}^2$ implies that $B_k(x)$ is the unique solution to (4).

At this point we have the two equations

$$A_k(x) = \prod_{j=1}^{k} \frac{x}{1-jx}$$

$$B_k(x) = \frac{1}{k!}(e^x - 1)^k,$$

which was our goal.