Combinatorics in Kazhdan–Lusztig polynomials of paving matroids

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(joint with George Nasr, Nick Proudfoot, and Lorenzo Vecchi)
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A classical story

Our story

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Euclid stated five postulates for rigorous geometry.

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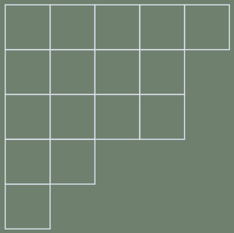
We can drop the fifth and still do geometry.

A partition $\lambda \vdash n$ is a weakly decreasing sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ summing to n.

Example

$\lambda = (5, 4, 4, 2, 1) \vdash 16$ has Ferrers diagram



A Young tableau T is a filling of a Ferrers diagram by positive integers. T is standard if it is filled by $\{1, 2, \ldots, n\}$ and increasing in rows and columns. Define f^{λ} as the number of standard tableaux of shape λ.

Example

One of $f^{(5,4,4,2,1)} = 549120$ standard Young tableaux:

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 6 & 10 & 13 & 16 \\ \hline 2 & 7 & 11 & 14 \\ \cline{1-4} 3 & 8 & 12 & 15 \\ \cline{1-4} 4 & 9 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$$

Fact

Fix n. Then

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

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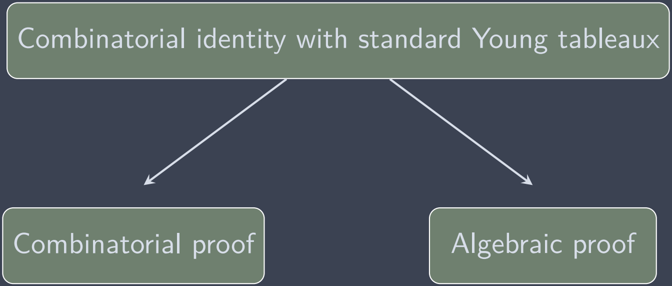
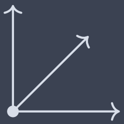
Proof 2:
Uses that irreducible representations of $\mathfrak{S}_n$ are indexed by $\lambda \vdash n$ and have dimension f^λ


Combinatorial identity with standard Young tableaux
Combinatorial proof
Algebraic proof

Our story

Let $v_1, \ldots, v_n$ be vectors in a vector space V. Then any two bases A, B for the span of $v_1, \ldots v_n$ satisfy the following requirements:

1) There must be at least one basis
2) If $a \in A - B$ then there is a $b \in B$ with $(A - a) \cup b$ a basis.



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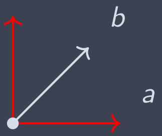

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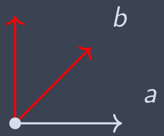

Definition

A **matroid** $M = (E, \mathcal{B})$ is a set E with $\emptyset \neq \mathcal{B} \subseteq 2^E$ such that if $A, B \in \mathcal{B}$ and $a \in A$, there exists $b \in B$ such that

$$(A - a) \cup b \in \mathcal{B}$$

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Matroids have the combinatorics of vectors without the "zeroth postulate".

» Vocab

Finite set of vectors	Matroids
Maximally independent sets	Bases $\mathcal{B}$
Minimally dependent sets	Circuits $\mathcal{C}$
Dimension of span	Rank
Codimension 1 subspaces	Hyperplanes $\mathcal{H}$

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Definition

$\mathcal{CH} = C \cap H$ is the set of circuit hyperplanes.

A hyperplane H is stressed if every subset of H of size $\mathrm{rk}(E)$ is in $\mathcal{C}$. Denote the set of (nontrivial) stressed hyperplanes by $\mathcal{SH}$.

Fact

There is a very large class of matroids called (sparse) paving matroids.

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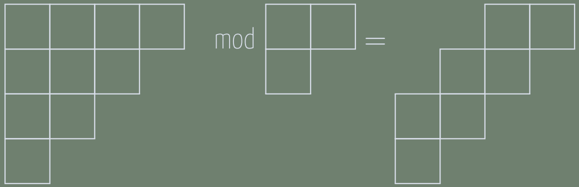
Fact

The matroid Kazhdan–Lusztig polynomial $P_M(t)$ is an interesting polynomial invariant of a matroid M, introduced by Elias, Proudfoot, and Wakefield in 2016.

A skew partition λ/μ is a pair of partitions where the diagram of μ is contained in the diagram of λ

Example

If $\lambda = (4,3,2,1)$ and $\mu = (2,1)$ then λ/μ has diagram



A <u>skew tableau</u> T is a filling of a skew diagram by positive integers. T is <u>standard</u> if it is filled by $\{1, 2, \ldots, |\lambda| - |\mu|\}$ and increasing in rows and columns. Define $f^{\lambda/\mu}$ as the number of standard skew tableaux of shape λ/μ.

Example

Two of $f^{(4,3,2,1)/(2,1)} = 272$ standard skew tableaux:

		1	2
	3	4	
5	6		
7			

		3	7
	2	6	
1	5		
4			

Theorem (Lee, Nasr, Radcliffe '21)

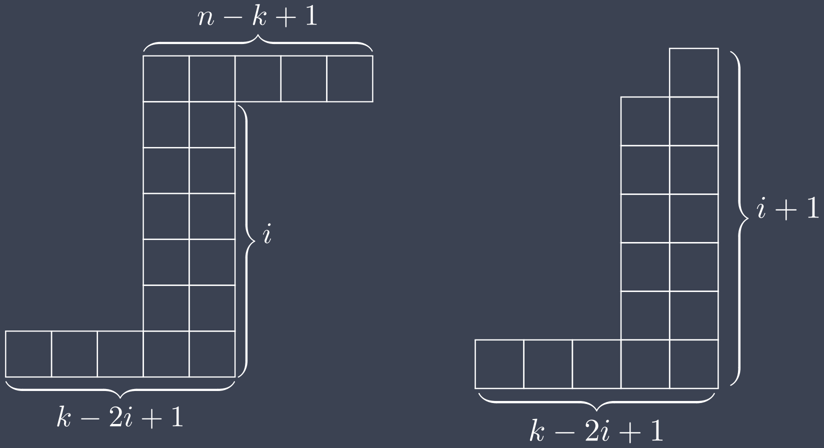
Let M be a rank-k, sparse paving matroid with $E = [n]$ and circuit hyperplanes $\mathcal{CH}$. The t^i coefficient in $P_M(t)$ is

$$f^{\lambda/\mu} - |\mathcal{CH}| f^{\lambda'/\mu'}$$

where

$$\lambda = \left[n - 2i, (k - 2i + 1)^i\right], \mu = \left[(k - 2i - 1)^i\right]$$

$$\lambda' = \left[(k - 2i + 1)^{i+1}\right], \mu' = \left[k - 2i, (k - 2i - 1)^{i-1}\right]$$


$n-k+1$
i
$k-2i+1$
$i+1$
$k-2i+1$

Theorem (Lee, Nasr, Radcliffe '21)

For a sparse paving matroid M, the t^i coefficient in $P_M(t)$ is
$$f^{\lambda/\mu} - |\mathcal{CH}| f^{\lambda'/\mu'}$$

Proof 1 (LNR '21): Combinatorial argument with recursion.

Fact

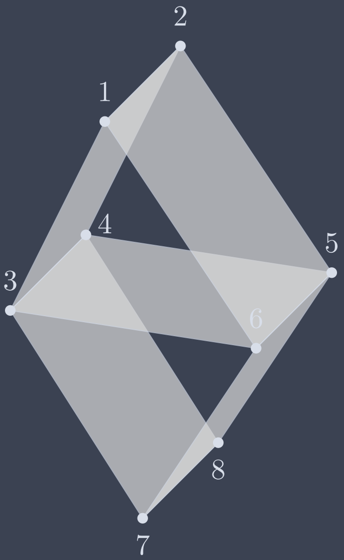
There is a (reducible) $\mathfrak{S}_n$ representation $S^{\lambda/\mu}$ of dimension $f^{\lambda/\mu}$.

Theorem (Lee, Nasr, Radcliffe '21)

For a sparse paving matroid M, the t^i coefficient in $P_M(t)$ is
$$f^{\lambda/\mu} - |\mathcal{CH}| f^{\lambda'/\mu'}$$

Proof 1 (LNR '21): Combinatorial argument with recursion.
Proof 2 (KNPV '22): $\dim$(some $S^{\lambda/\mu}$ coming from M).

» Example: Vámos matroid



$\lambda = [6,3],\ \mu = [1] \longrightarrow$ 

$\lambda' = [3,3]\ , \mu' = [2] \longrightarrow$ 

$$|\mathcal{CH}| = 5$$

$$f^{\lambda/\mu} - 5f^{\lambda'/\mu'} = 48 - 15 = 33$$

$$\boxed{P_V(t) = 1 + 33t}$$

» Example: Projective plane over $\mathbb{F}_3$

$$\lambda/\mu = \ydiagram{11,2}$$

$$\mathcal{CH} = \emptyset$$

$$f^{\lambda/\mu} \qquad = 65 \qquad \neq 0$$

$$\boxed{P_M(t) = 1}$$

» Example: Projective plane over $\mathbb{F}_3$

$\lambda/\mu =$ 

$$|\mathcal{SH}| = 13$$

$\lambda'/\mu' =$ 

$$f^{\lambda/\mu} - 13 f^{\lambda'/\mu'} = 65 - 13 * 5 = 0$$

$$\boxed{P_M(t) = 1}$$

Theorem

For a (arbitrary!) paving matroid M, the t^i coefficient in $P_M(t)$ is

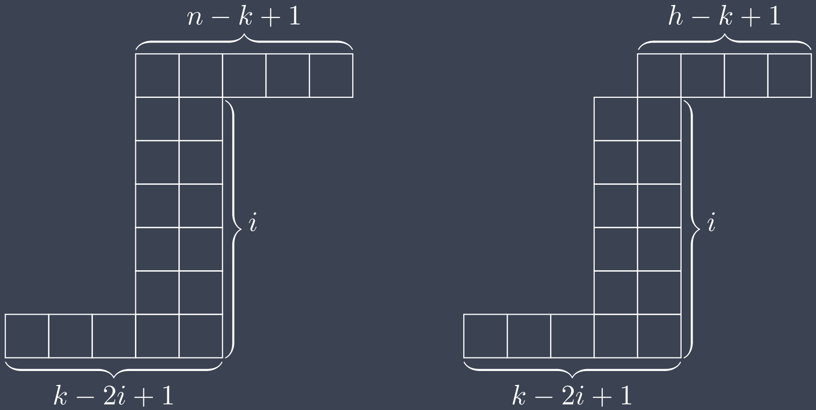
$$f^{\lambda/\mu} - \sum_{H \in \mathcal{SH}} f^{\lambda'(|H|)/\mu'}$$

where

$$\lambda = \left[n-2i, (k-2i+1)^i\right], \mu = \left[(k-2i-1)^i\right]$$

$$\lambda'(h) = \left[h-2i+1, (k-2i+1)^i\right], \mu' = \left[h-2i, (k-2i-1)^{i-1}\right]$$

Proof: Our proof of LNR's theorem implies this more general result


$n-k+1$
i
$k-2i+1$
$h-k+1$
i
$k-2i+1$

Certain $P_M(t)$ in terms of standard skew Young tableaux

Combinatorial proof [LNR21]

Algebraic proof [KNPV22]

THANK YOU!

» References

- Ben Elias, Nicholas Proudfoot, and Max Wakefield, The Kazhdan–Lusztig polynomial of a matroid, Advances in Mathematics **299** (2016), 36–70.
- Luisa Ferroni, George D. Nasr, and Lorenzo Vecchi, Stressed hyperplanes and Kazhdan–Lusztig gamma-positivity for matroids, 2021.
- Katie R. Gedeon, Nicholas Proudfoot, and Benjamin Young, The equivariant Kazhdan–Lusztig polynomial of a matroid, J. Comb. Theory, Ser. A **150** (2017), 267–294.
- Alice L. L. Gao and Matthew H. Y. Xie, The inverse Kazhdan-Lusztig polynomial of a matroid, J. Combin. Theory Ser. B **151** (2021), 375–392. MR 4294228

» References (cont.)

- Alice L. L. Gao, Matthew H.Y. Xie, and Arthur L. B. Yang, The equivariant inverse Kazhdan–Lusztig polynomials of uniform matroids, 2021.
- Trevor Karn, George Nasr, Nicholas Proudfoot, and Lorenzo Vecchi, Equivariant Kazhdan-Lusztig theory of paving matroids, 2022.
- Kyungyong Lee, George D. Nasr, and Jamie Radcliffe, A combinatorial formula for Kazhdan–Lusztig polynomials of sparse paving matroids, Electron. J. Comb. **28** (2021).
- Dillon Mayhew, Mike Newman, Dominic Welsh, and Geoff Whittle, On the asymptotic proportion of connected matroids, European J. Combin. **32** (2011), no. 6, 882–890. MR 2821559

» References (cont.)

- Peter Orlik and Louis Solomon, Combinatorics and topology of complements of hyperplanes, Inventiones mathematicae **56** (1980), no. 2, 167–189 (eng).
- Rudi Pendavingh and Jorn van der Pol, On the number of matroids compared to the number of sparse paving matroids, Electron. J. Combin. **22** (2015), no. 2, Paper 2.51, 17. MR 3367294
- Nicholas Proudfoot, Yuan Xu, and Benjamin Young, The Z-polynomial of a matroid, Electron. J. Comb. **25** (2018), P1.26.