

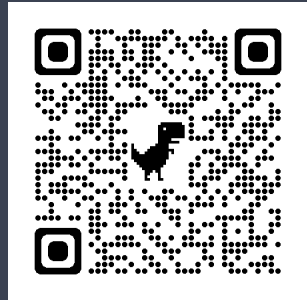
Combinatorics in matroid Kazhdan–Lusztig polynomials

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(joint with George Nasr, Nick Proudfoot, and Lorenzo Vecchi)
on Thursday, March 23, 2023

A classical story

Matroids

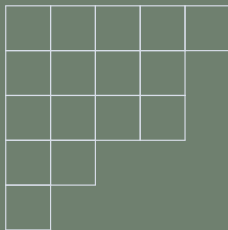
Combinatorial formulas



A partition $\lambda \vdash n$ is a weakly decreasing sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ summing to n .

Example

$\lambda = (5, 4, 4, 2, 1) \vdash 16$ has Ferrers diagram



A Young tableau T is a filling of a Ferrers diagram by positive integers. T is standard if it is filled by $\{1, 2, \dots, n\}$ and increasing in rows and columns. Define f^λ as the number of standard tableaux of shape λ .

Example

One of $f^{(5,4,4,2,1)} = 549120$ standard Young tableaux:

1	6	10	13	16
2	7	11	14	
3	8	12	15	
4	9			
5				

Fact

Fix n . Then

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

Proof 1:

The Robinson-Schensted bijection:

pairs of standard tableaux of same shape \longleftrightarrow permutation in \mathfrak{S}_n

Definition

A representation of a group G is a homomorphism

$$\rho : G \rightarrow \mathrm{GL}_n(\mathbb{C}).$$

A representation is irreducible if there is no G -stable subspace $W \subseteq \mathbb{C}^n$.

Example

There is a representation of \mathfrak{S}_3 defined by sending

$$(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(23) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

This defines an action on \mathbb{C}^3 where $\pi \in \mathfrak{S}_3$ permutes coordinates. So

$$(12) \cdot \langle 1, 2, 3 \rangle = \langle 2, 1, 3 \rangle,$$

and

$$\pi \cdot \langle 1, 1, 1 \rangle = \langle 1, 1, 1 \rangle.$$

Fact

Irreducible \mathfrak{S}_n representations are indexed by $\lambda \vdash n$.
Denote them by S^λ . Then

$$\dim S^\lambda = f^\lambda.$$

Example

The representation on the last slide is $S^{(3)} \oplus S^{(2,1)}$.

Fact

Let d_1, d_2, \dots, d_r be the dimensions of all irreducible representations of a finite group. Then

$$\sum_i d_i^2 = |G|.$$

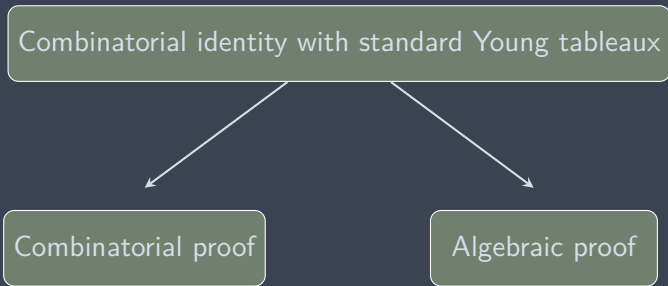
Fact

Fix n . Then

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

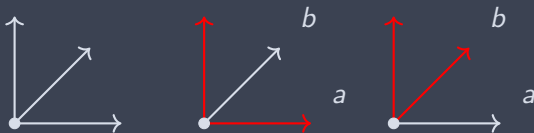
Proof 2:

$$\sum_{\lambda} (f^\lambda)^2 = \sum_i d_i^2 = |G| = |\mathfrak{S}_n| = n!$$



Let v_1, \dots, v_n be vectors in a vector space V . Then any two bases A, B for the span of v_1, \dots, v_n satisfy the following requirements:

- 1) There must be at least one basis
- 2) If $a \in A - B$ then there is a $b \in B$ with $(A - a) \cup b$ a basis.



Definition 1

A **matroid** $M = (E, \mathcal{B})$ is a finite set E with $\emptyset \neq \mathcal{B} \subseteq 2^E$ such that if $A, B \in \mathcal{B}$ and $a \in A$, there exists $b \in B$ such that

$$(A - a) \cup b \in \mathcal{B}$$

Call \mathcal{B} the bases of the matroid.

Matroids have the combinatorics of vectors without the “zeroth postulate”.

Definition 2

A **matroid** $M = (E, \mathcal{C})$ is a set E with $\mathcal{C} \subseteq 2^E$ such that

$$\emptyset \notin \mathcal{C}$$

If $C_1, C_2 \in \mathcal{C}$ with $C_1 \subseteq C_2$, then $C_1 = C_2$.

If $C_1, C_2 \in \mathcal{C}$ are distinct, and $e \in C_1 \cap C_2$, then there is a $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$

Call \mathcal{C} the circuits.

Example

The **uniform matroid** $U_{k,n}$ models n -many k -dimensional vectors in general position

Bases \longleftrightarrow any set of k -many vectors

Circuits \longleftrightarrow any set of $k + 1$ -many vectors

Example of the example

$U_{3,12}$ corresponds to 12 generic vectors in \mathbb{R}^3 . One choice of basis is $\{e_1, e_2, e_3\}$. On the other hand $\{e_1, e_2, e_3, v\}$ is dependent for any $v \in \mathbb{R}^3$.

Example

A graph Γ with edges E forms a matroid:

Bases \longleftrightarrow spanning trees

Circuits \longleftrightarrow cycles



Example

The columns of a matrix form a matroid:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 2 & 4 \end{bmatrix}$$



Example

The protective plane is the set of lines through the origin in \mathbb{F}^3 .

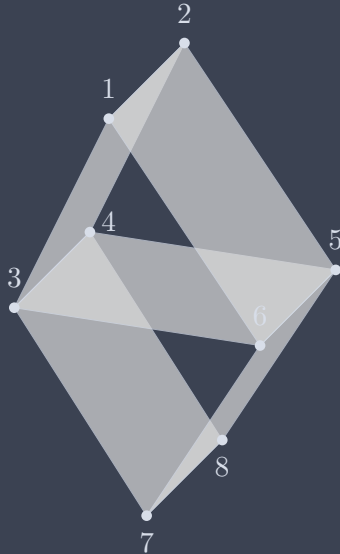
If $\mathbb{F} = \mathbb{F}_2$,

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Theorem (Nelson, 2018)

Almost all matroids cannot be written as a matrix.

» Example: Vámos matroid



» Vocab

Finite set of vectors	Matroids
Maximally independent sets	Bases \mathcal{B}
Minimally dependent sets	Circuits \mathcal{C}
Dimension of span	Rank
Subspaces	Flats \mathcal{F}
Codimension 1 subspaces	Hyperplanes \mathcal{H}

Definition

$\mathcal{CH} = \mathcal{C} \cap \mathcal{H}$ is the set of circuit hyperplanes.

A hyperplane H is stressed if every subset of H of size $\text{rk}(E)$ is in \mathcal{C} . Denote the set of (nontrivial) stressed hyperplanes by \mathcal{SH} .

M is a paving matroid if all circuits have size at least $k = \text{rk}(E)$

A paving matroid is sparse if $\binom{E}{k} = \mathcal{CH} \sqcup \mathcal{B}$

Conjecture (Mayhew, Newman, Welsh, Whittle '11)

Asymptotically almost all matroids are sparse paving

Theorem (Pendavingh, van der Pol '15)

Asymptotically logarithmically almost all matroids are sparse paving

M is a paving matroid if all circuits have size at least $k = \text{rk}(E)$

A paving matroid is sparse if $\binom{E}{k} = \mathcal{CH} \sqcup \mathcal{B}$

Example

$U_{k,n} \leftrightarrow$ sparse paving

$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 2 & 4 \end{bmatrix} \leftrightarrow$ paving

$\mathbb{P}\mathbb{F}_3 \leftrightarrow$ paving

$V \leftrightarrow$ sparse paving

Fact

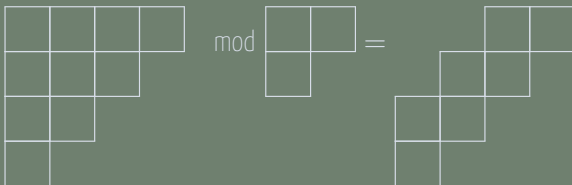
The matroid Kazhdan–Lusztig polynomial $P_M(t)$ is an interesting polynomial invariant of a matroid M , introduced by Elias, Proudfoot, and Wakefield in 2016.

It is defined in terms of \mathcal{F} .

A skew partition λ/μ is a pair of partitions where the diagram of μ is contained in the diagram of λ

Example

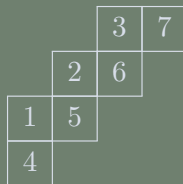
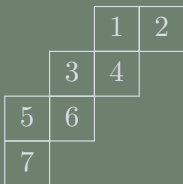
If $\lambda = (4, 3, 2, 1)$ and $\mu = (2, 1)$ then λ/μ has diagram



A skew tableau T is a filling of a skew diagram by positive integers. T is standard if it is filled by $\{1, 2, \dots, |\lambda| - |\mu|\}$ and increasing in rows and columns. Define $f^{\lambda/\mu}$ as the number of standard skew tableaux of shape λ/μ .

Example

Two of $f^{(4,3,2,1)/(2,1)} = 272$ standard skew tableaux:



Theorem (Lee, Nasr, Radcliffe '21)

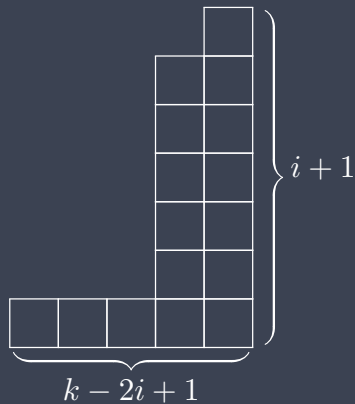
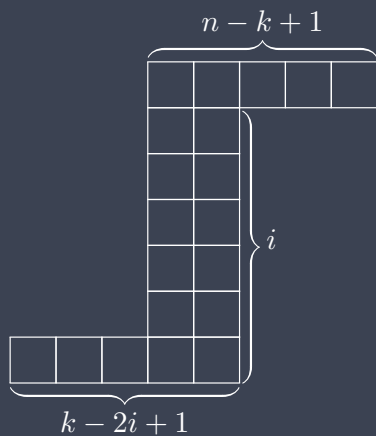
Let M be a rank- k , sparse paving matroid with $E = [n]$ and circuit hyperplanes \mathcal{CH} . The t^i coefficient in $P_M(t)$ is

$$f^{\lambda/\mu} - |\mathcal{CH}| f^{\lambda'/\mu'}$$

where

$$\lambda = [n - 2i, (k - 2i + 1)^i], \mu = [(k - 2i - 1)^i]$$

$$\lambda' = [(k - 2i + 1)^{i+1}], \mu' = [k - 2i, (k - 2i - 1)^{i-1}]$$



Theorem (Lee, Nasr, Radcliffe '21)

For a sparse paving matroid M , the t^i coefficient in $P_M(t)$ is

$$f^{\lambda/\mu} - |\mathcal{CH}| f^{\lambda'/\mu'}$$

Proof 1 (LNR '21): Combinatorial argument with recursion.

Fact

There is a (reducible) \mathfrak{S}_n representation $S^{\lambda/\mu}$ of dimension $f^{\lambda/\mu}$.

Theorem (Lee, Nasr, Radcliffe '21)

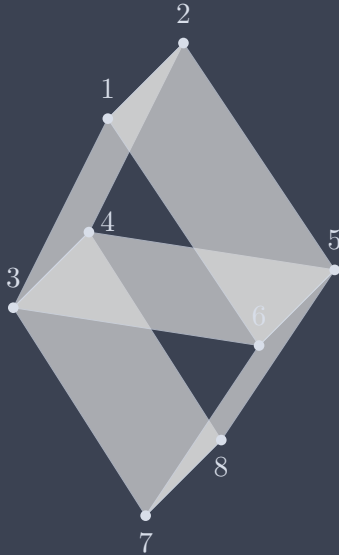
For a sparse paving matroid M , the t^i coefficient in $P_M(t)$ is

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Proof 1 (LNR '21): Combinatorial argument with recursion.

Proof 2 (KNPV '22): $\dim(\text{some } S^{\lambda/\mu} \text{ coming from } M)$.

» Example: Vámos matroid



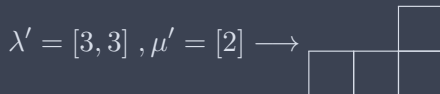
Some general facts:

Know $P_M(t)$ always has constant term 1.

Know $\deg P_M(t) < \frac{\text{rk } E}{2}$.

$\text{rk } V = 4$ so $P_V(t) = 1 + ?t$.

Only need to compute linear coefficient!



$$|\mathcal{CH}| = 5$$

$$f^{\lambda/\mu} - 5f^{\lambda'/\mu'} = 48 - 15 = 33$$

$$P_V(t) = 1 + 33t$$

» Example: Projective plane over \mathbb{F}_3



$$\mathcal{CH} = \emptyset$$

$$|\mathcal{SH}| = 13$$



$$f^{\lambda/\mu} - 13f^{\lambda'/\mu'} = 65 - 13 * 5 = \neq 0$$

From Elias, Proudfoot, and Wakefield, we know

$$P_M(t) = 1$$

Theorem

For a (arbitrary!) paving matroid M , the t^i coefficient in $P_M(t)$ is

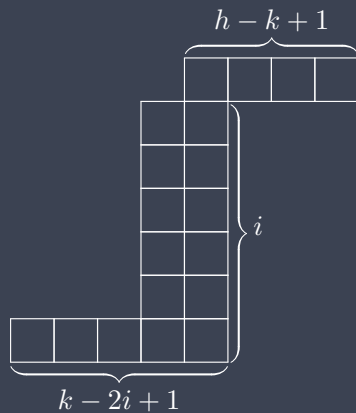
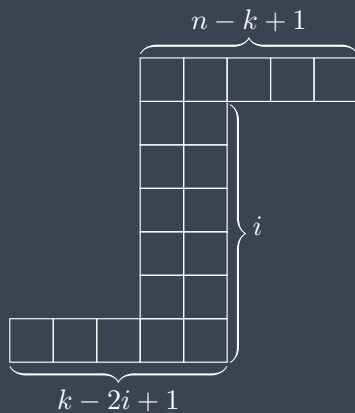
$$f^{\lambda/\mu} = \sum_{H \in \mathcal{SH}} f^{\lambda'(|H|)/\mu'}$$

where

$$\lambda = [n - 2i, (k - 2i + 1)^i], \mu = [(k - 2i - 1)^i]$$

$$\lambda'(h) = [h - 2i + 1, (k - 2i + 1)^i], \mu' = [h - 2i, (k - 2i - 1)^{i-1}]$$

Proof: Our proof of LNR's theorem implies this more general result



» Proof idea

In a stressed hyperplane, all size- k subsets are circuits.
Create a new matroid by turning all of the circuits into bases



turns into

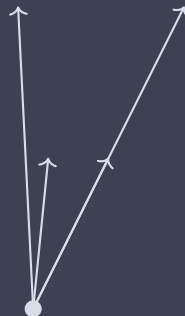


» Proof idea

In a stressed hyperplane, all size- k subsets are circuits.
Create a new matroid by turning all circuits in H into bases

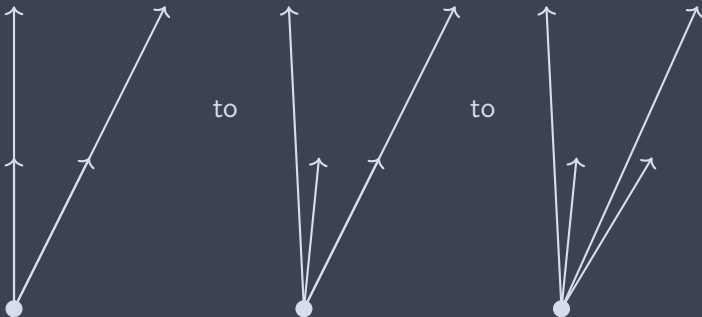


turns into



» Proof idea

Do this until you obtain $U_{k,n}$



Each step accounts for $S^{\lambda'(h)/\mu'}$.

Certain $P_M(t)$ in terms of standard skew Young tableaux



Combinatorial proof [LNR21]

Algebraic proof [KNPV22]
+ extension

THANK YOU!

» References



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