Combinatorics in matroid Kazhdan–Lusztig polynomials

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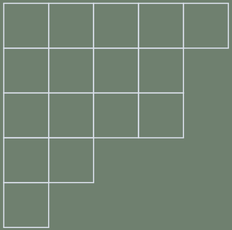
A classical story

A partition $\lambda \vdash n$ is a weakly decreasing sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ summing to n.

Example

$\lambda = (5, 4, 4, 2, 1) \vdash 16$ has Ferrers diagram



A Young tableau T is a filling of a Ferrers diagram by positive integers. T is standard if it is filled by $\{1, 2, \ldots, n\}$ and increasing in rows and columns. Define f^{λ} as the number of standard tableaux of shape λ.

Example

One of $f^{(5,4,4,2,1)} = 549120$ standard Young tableaux:

1	6	10	13	16
2	7	11	14	
3	8	12	15	
4	9			
5				

Fact

Fix n. Then

$$\sum_{\lambda \vdash n} (f^{\lambda})^2 = n!$$

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Proof 1:

The Robinson-Schensted bijection:

pairs of standard tableaux of same shape $\longleftrightarrow$ permutation in $\mathfrak{S}_n$

Definition

A representation of a group G is a homomorphism

$$\rho : G \to \mathbb{GL}_n(\mathbb{C}).$$

A representation is irreducible if there is no G-stable subspace $W \subseteq \mathbb{C}^n$.

Example

There is a representation of $\mathfrak{S}_3$ defined by sending

$$(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(23) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

This defines an action on $\mathbb{C}^3$ where $\pi \in \mathfrak{S}_3$ permutes coordinates. So

$$(12) \cdot \langle 1, 2, 3 \rangle = \langle 2, 1, 3 \rangle,$$

and

$$\pi \cdot \langle 1, 1, 1 \rangle = \langle 1, 1, 1 \rangle.$$

Fact

Irreducible $\mathfrak{S}_n$ representations are indexed by $\lambda \vdash n$. Denote them by S^λ. Then

$$\dim S^\lambda = f^\lambda.$$

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Example

The representation on the last slide is $S^{(3)} \oplus S^{(2,1)}$.

Fact

Let $d_1, d_2, \ldots, d_r$ be the dimensions of all irreducible representations of a finite group. Then

$$\sum_i d_i^2 = |G|.$$

Fact

Fix n. Then

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

Proof 2:

$$\sum_{\lambda} (f^\lambda)^2 = \sum_i d_i^2 = |G| = |\mathfrak{S}_n| = n!$$

Combinatorial identity with standard Young tableaux

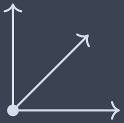
Combinatorial proof

Algebraic proof

Matroids

Let $v_1, \ldots, v_n$ be vectors in a vector space V. Then any two bases A, B for the span of $v_1, \ldots v_n$ satisfy the following requirements:

1) There must be at least one basis
2) If $a \in A - B$ then there is a $b \in B$ with $(A - a) \cup b$ a basis.



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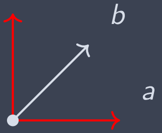

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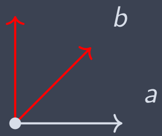

Definition 1

A **matroid** $M = (E, \mathcal{B})$ is a finite set E with $\emptyset \neq \mathcal{B} \subseteq 2^E$ such that if $A, B \in \mathcal{B}$ and $a \in A$, there exists $b \in B$ such that

$$(A - a) \cup b \in \mathcal{B}$$

Call $\mathcal{B}$ the <u>bases</u> of the matroid.

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Matroids have the combinatorics of vectors without the "zeroth postulate".

Definition 2

A **matroid** $M = (E, \mathcal{C})$ is a set E with $\mathcal{C} \subseteq 2^E$ such that

$\emptyset \notin \mathcal{C}$

If $C_1, C_2 \in \mathcal{C}$ with $C_1 \subseteq C_2$, then $C_1 = C_2$.

If $C_1, C_2 \in \mathcal{C}$ are distinct, and $e \in C_1 \cap C_2$, then there is a $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$

Call $\mathcal{C}$ the circuits.

Example

The **uniform matroid** $U_{k,n}$ models n-many k-dimensional vectors in general position

Bases $\longleftrightarrow$ any set of k-many vectors

Circuits $\longleftrightarrow$ any set of $k+1$-many vectors

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Example of the example

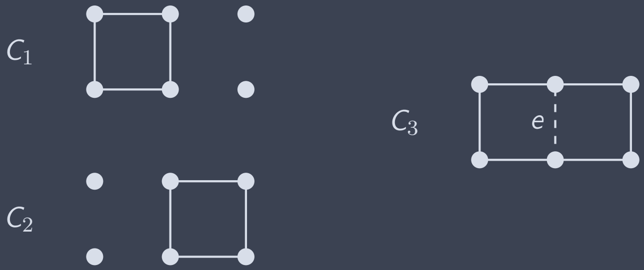
$U_{3,12}$ corresponds to 12 generic vectors in $\mathbb{R}^3$. One choice of basis is $\{e_1, e_2, e_3\}$. On the other hand $\{e_1, e_2, e_3, v\}$ is dependent for any $v \in \mathbb{R}^3$.

Example

A graph Γ with edges E forms a matroid:

Bases $\longleftrightarrow$ spanning trees

Circuits $\longleftrightarrow$ cycles



Example

The columns of a matrix form a matroid:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 2 & 4 \end{bmatrix}$$

Example

The protective plane is the set of lines through the origin in $\mathbb{F}^3$.

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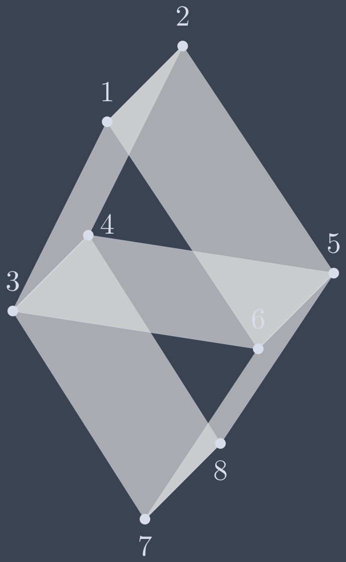
If $\mathbb{F} = \mathbb{F}_2$,

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Theorem (Nelson, 2018)

Almost all matroids cannot be written as a matrix.

» Example: Vámos matroid



» Vocab

Finite set of vectors	Matroids
Maximally independent sets	Bases $\mathcal{B}$
Minimally dependent sets	Circuits $\mathcal{C}$
Dimension of span	Rank
Subspaces	Flats $\mathcal{F}$
Codimension 1 subspaces	Hyperplanes $\mathcal{H}$

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Definition

$\mathcal{CH} = \mathcal{C} \cap \mathcal{H}$ is the set of circuit hyperplanes.

A hyperplane H is stressed if every subset of H of size $\mathrm{rk}(E)$ is in $\mathcal{C}$. Denote the set of (nontrivial) stressed hyperplanes by $\mathcal{SH}$.

M is a paving matroid if all circuits have size at least $k = \operatorname{rk}(E)$

A paving matroid is sparse if $\binom{E}{k} = \mathcal{CH} \sqcup \mathcal{B}$

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Conjecture (Mayhew, Newman, Welsh, Whittle '11)

Asymptotically almost all matroids are sparse paving

Theorem (Pendavingh, van der Pol '15)

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$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 2 & 4 \end{bmatrix} \leftrightarrow$ paving

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$\mathbb{P}\mathbb{F}_3 \leftrightarrow$ paving

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Example

$U_{k,n} \leftrightarrow$ sparse paving

$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 4 & 2 & 4 \end{bmatrix} \leftrightarrow$ paving

$\mathbb{PF}_3 \leftrightarrow$ paving

$V \leftrightarrow$ sparse paving

Fact

The matroid Kazhdan–Lusztig polynomial $P_M(t)$ is an interesting polynomial invariant of a matroid M, introduced by Elias, Proudfoot, and Wakefield in 2016.

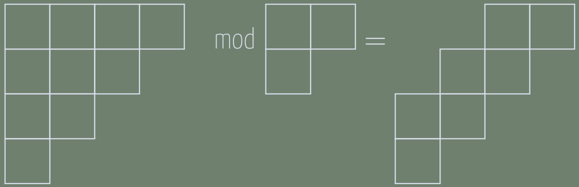
It is defined in terms of $\mathcal{F}$.

Combinatorial formulas

A <u>skew partition</u> λ/μ is a pair of partitions where the diagram of μ is contained in the diagram of λ

Example

If $\lambda = (4, 3, 2, 1)$ and $\mu = (2, 1)$ then λ/μ has diagram



A *skew tableau* T is a filling of a skew diagram by positive integers. T is *standard* if it is filled by $\{1, 2, \ldots, |\lambda| - |\mu|\}$ and increasing in rows and columns. Define $f^{\lambda/\mu}$ as the number of standard skew tableaux of shape λ/μ.

Example

Two of $f^{(4,3,2,1)/(2,1)} = 272$ standard skew tableaux:

$$\begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{2}{c|}{} & 1 & 2 \\ \cline{2-4} \multicolumn{1}{c|}{} & 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-3} 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{2}{c|}{} & 3 & 7 \\ \cline{2-4} \multicolumn{1}{c|}{} & 2 & 6 & \multicolumn{1}{c}{} \\ \cline{1-3} 1 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

Theorem (Lee, Nasr, Radcliffe '21)

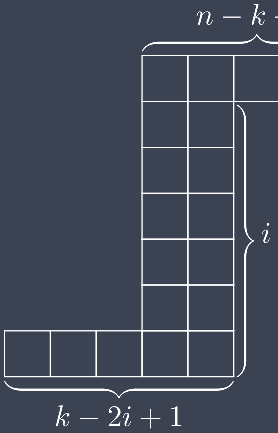
Let M be a rank-k, sparse paving matroid with $E = [n]$ and circuit hyperplanes $\mathcal{CH}$. The t^i coefficient in $P_M(t)$ is

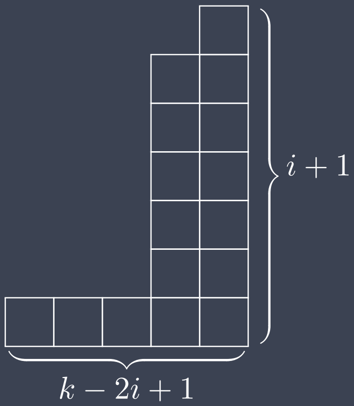
$$f^{\lambda/\mu} - |\mathcal{CH}| f^{\lambda'/\mu'}$$

where

$$\lambda = \left[n - 2i, (k - 2i + 1)^i\right], \mu = \left[(k - 2i - 1)^i\right]$$

$$\lambda' = \left[(k - 2i + 1)^{i+1}\right], \mu' = \left[k - 2i, (k - 2i - 1)^{i-1}\right]$$


$n-k+1$
i
$k-2i+1$


$i+1$
$k-2i+1$

Theorem (Lee, Nasr, Radcliffe '21)

For a sparse paving matroid M, the t^i coefficient in $P_M(t)$ is
$$f^{\lambda/\mu} - |\mathcal{CH}| f^{\lambda'/\mu'}$$

Proof 1 (LNR '21): Combinatorial argument with recursion.

Fact

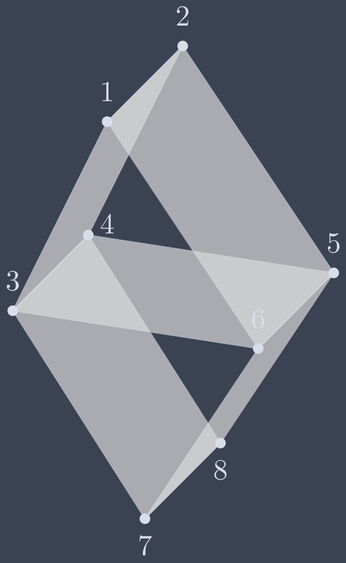
There is a (reducible) $\mathfrak{S}_n$ representation $S^{\lambda/\mu}$ of dimension $f^{\lambda/\mu}$.

Theorem (Lee, Nasr, Radcliffe '21)

For a sparse paving matroid M, the t^i coefficient in $P_M(t)$ is
$$f^{\lambda/\mu} - |\mathcal{CH}| f^{\lambda'/\mu'}$$

Proof 1 (LNR '21): Combinatorial argument with recursion.
Proof 2 (KNPV '22): $\dim$(some $S^{\lambda/\mu}$ coming from M).

» Example: Vámos matroid



Some general facts:

Know $P_M(t)$ always has constant term 1.

Know $\deg P_M(t) < \frac{\operatorname{rk} E}{2}$.

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Know $\deg P_M(t) < \frac{\operatorname{rk} E}{2}$.

$\operatorname{rk} V = 4$ so $P_V(t) = 1 + ?t$.

Only need to compute linear coefficient!

$\lambda = [6,3],\ \mu = [1] \longrightarrow$ 

$\lambda' = [3,3]\ ,\mu' = [2] \longrightarrow$ 

$$|\mathcal{CH}| = 5$$

$$f^{\lambda/\mu} - 5f^{\lambda'/\mu'} = 48 - 15 = 33$$

$$\boxed{P_V(t) = 1 + 33t}$$

» Example: Projective plane over $\mathbb{F}_3$

$$\lambda/\mu = \ydiagram{11,2}$$

$$\mathcal{CH} = \emptyset$$

$$f^{\lambda/\mu} \qquad = 65 \qquad \neq 0$$

From Elias, Proudfoot, and Wakefield, we know

$$\boxed{P_M(t) = 1}$$

» Example: Projective plane over $\mathbb{F}_3$

$\lambda/\mu =$ 

$$|\mathcal{SH}| = 13$$

$\lambda'/\mu' =$ 

$$f^{\lambda/\mu} - 13 f^{\lambda'/\mu'} = 65 - 13 * 5 = 0$$

From Elias, Proudfoot, and Wakefield, we know

$$\boxed{P_M(t) = 1}$$

Theorem

For a (arbitrary!) paving matroid M, the t^i coefficient in $P_M(t)$ is

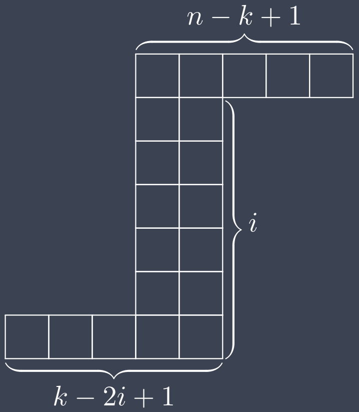
$$f^{\lambda/\mu} - \sum_{H \in \mathcal{SH}} f^{\lambda'(|H|)/\mu'}$$

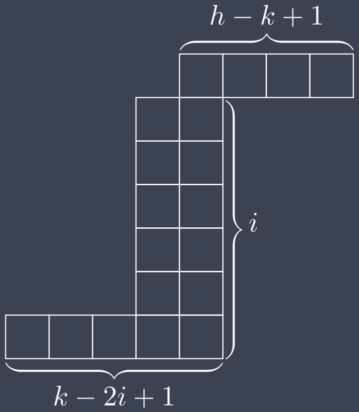
where

$$\lambda = \left[n-2i, (k-2i+1)^i\right], \mu = \left[(k-2i-1)^i\right]$$

$$\lambda'(h) = \left[h-2i+1, (k-2i+1)^i\right], \mu' = \left[h-2i, (k-2i-1)^{i-1}\right]$$

Proof: Our proof of LNR's theorem implies this more general result


$n-k+1$
i
$k-2i+1$


$h-k+1$
i
$k-2i+1$

» Proof idea

In a stressed hyperplane, all size-k subsets are circuits.
Create a new matroid by turning all of the circuits into bases

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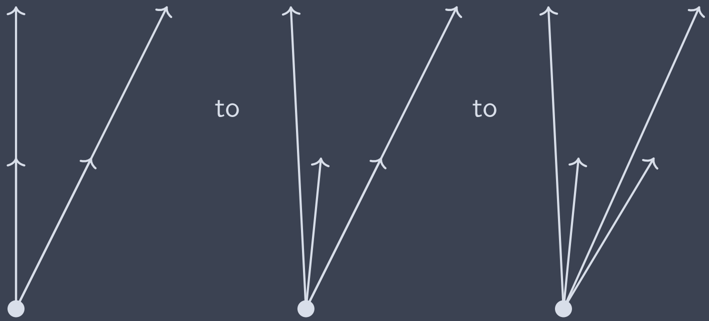
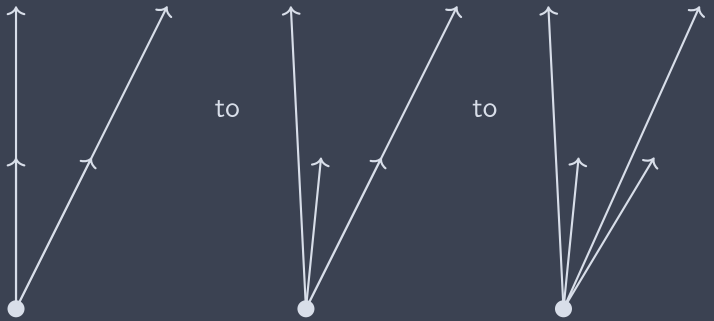


» Proof idea

Do this until you obtain $U_{k,n}$



Each step accounts for $S^{\lambda'(h)/\mu'}$.

Certain $P_M(t)$ in terms of standard skew Young tableaux

Combinatorial proof [LNR21]

Algebraic proof [KNPV22] + extension

THANK YOU!

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