Superspace coinvariants and hyperplane arrangements

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Main problem:

find a linear basis for the ring SR_n.

Approach:

$\mathcal{ST}$ algebras of SW arrangements

What is SR_n?

The coinvariant ring:

$$R_n = \mathbb{Q}[x_1, x_2, \ldots, x_n]/I^+$$

Superspace:

$$\mathbb{Q}[\mathsf{x}, \theta] = \mathbb{Q}[x_1, x_2, \ldots, x_n, \theta_1, \theta_2, \ldots, \theta_n]$$

where $\theta_i \theta_j = -\theta_j \theta_i$

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The superspace coinvariant ring:

$$SR_n = \mathbb{Q}[\mathsf{x}, \theta] / SI^+$$

Sagan and Swanson [SS24] conjectured the following $\mathbb{Q}$-basis of monomials for SR_n:

$$\mathcal{M} = \bigcup_{J \subseteq [n]} \{\mathsf{x}^{\alpha}\theta_J : \alpha \leq (J\text{-staircase})\}$$

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Definition

A J-staircase is $(\mathrm{st}(J)_1, \mathrm{st}(J)_2, \ldots, \mathrm{st}(J)_n)$ where

$$\mathrm{st}(J)_1 = \begin{cases} -1 & 1 \in J \\ 0 & 1 \notin J \end{cases}$$

and

$$\mathrm{st}(J)_{i+1} = \begin{cases} \mathrm{st}(J)_i & i \in J \\ \mathrm{st}(J)_i + 1 & i \notin J \end{cases}$$

Example

Let $J = \{2, 4, 5\} \subseteq [6]$. Then the J-staircase is $(0, 0, 1, 1, 1, 2)$.

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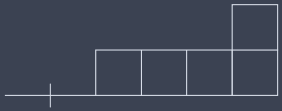

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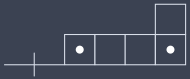
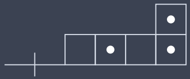
Let $J = \{2,4,5\} \subseteq [6]$. Then the J-staircase is $(0,0,1,1,1,2)$.

So $\mathcal{M}$ contains monomials

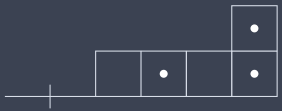
$x_3x_6\theta_2\theta_4\theta_5$ and $x_4x_6^2\theta_2\theta_4\theta_5$ NOT $x_5^2\theta_2\theta_4\theta_5$





» Punchline

Elements of $\mathcal{M}$ correspond to filled diagrams like



Staircase shape $\longleftrightarrow$ skew-commutative θ factor

Staircase filling $\longleftrightarrow$ commutative x factor

Conjecture [SS24]/Theorem [ACK+24]

$$\bigcup_{J\subseteq[n]} \{\mathsf{x}^\alpha \theta_J : \alpha \leq (J\text{-staircase})\}$$

is a basis for SR_n.

What are SW arrangements and $\mathcal{ST}$ algebras?

For the rest of the talk $S = \mathbb{Q}[x_1, x_2, \ldots, x_n]$.

The colon ideal $(I : f)$ is the kernel of $\times f$. So

$$0 \to S/(I : f) \xrightarrow{\times f} S/I$$

is exact.

» Transfer principle

Rhoades and Wilson [RW23] showed that it suffices to show

$$\mathcal{M}(J) = \{\mathsf{x}^\alpha : \alpha \leq (J\text{-staircase})\}$$

is a basis for

$$S/(I^+ : f_J)$$

where

$$f_J = \prod_{j \in J} x_j \prod_{i > j} (x_j - x_i)$$

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Upshot

Trade a skew-commutative problem for a family of commutative problems.

Consider the diagram $\tilde{\Phi}_5$:

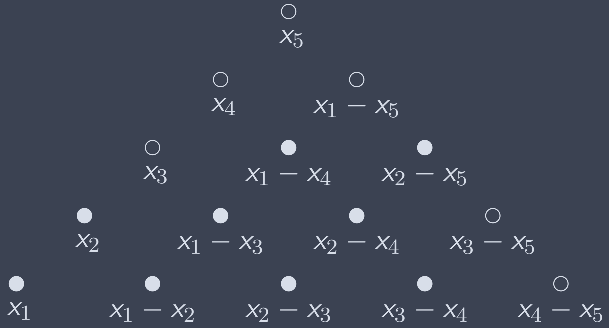
$$
\begin{array}{ccccccccc}
 & & & & \underset{x_5}{\circ} & & & & \\
 & & & \underset{x_4}{\circ} & & \underset{x_1 - x_5}{\circ} & & & \\
 & & \underset{x_3}{\circ} & & \underset{x_1 - x_4}{\circ} & & \underset{x_2 - x_5}{\circ} & & \\
 & \underset{x_2}{\circ} & & \underset{x_1 - x_3}{\circ} & & \underset{x_2 - x_4}{\circ} & & \underset{x_3 - x_5}{\circ} & \\
\underset{x_1}{\circ} & & \underset{x_1 - x_2}{\circ} & & \underset{x_2 - x_3}{\circ} & & \underset{x_3 - x_4}{\circ} & & \underset{x_4 - x_5}{\circ}
\end{array}
$$

Definition

An arrangement $\mathcal{A}$ is called a *southwest arrangement* if its defining polynomial $Q(\mathcal{A})$ is a product of terms of a southwest-closed subset of $\tilde{\Phi}_n$.

The h-function of a southwest arrangement is the number of hyperplanes on each southeast diagonal.

» Example



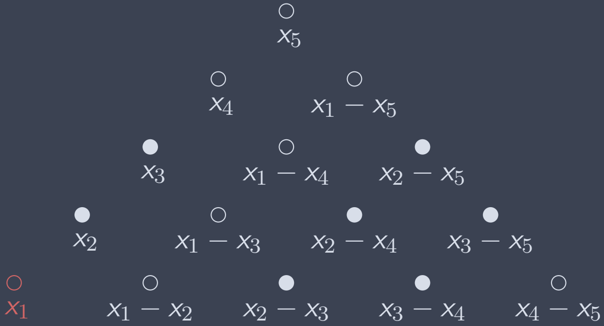
$\mathcal{A}$ defined by

$$x_1x_2(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_2 - x_5)(x_3 - x_4)$$

is southwest. It has *h*-function $(1, 2, 2, 3, 1)$.

» Non-example

The arrangement defined by f_J is not a southwest arrangement.
E.g. $J = \{2, 3\}$:



Definition

Let
$$\tilde{f}_J = \prod_{j \in J} \prod_{i > j} (x_j - x_i)$$

Note that $\tilde{f}_J$ defines a southwest arrangement.

$$f_J = \prod_{j \in J} x_j \prod_{i > j} (x_j - x_i)$$

$$\tilde{f}_J = \prod_{j \in J} \prod_{i > j} (x_j - x_i)$$

» Example

If $J = \{2, 3\}$, then $\tilde{f}_J$ corresponds to the southwest arrangement

$$
\begin{array}{ccccccccc}
 & & & & \circ & & & & \\
 & & & & x_5 & & & & \\
 & & & \circ & & \circ & & & \\
 & & & x_4 & & x_1 - x_5 & & & \\
 & & \circ & & \circ & & \bullet & & \\
 & & x_3 & & x_1 - x_4 & & x_2 - x_5 & & \\
 & \circ & & \circ & & \bullet & & \bullet & \\
 & x_2 & & x_1 - x_3 & & x_2 - x_4 & & x_3 - x_5 & \\
\circ & & \circ & & \bullet & & \bullet & & \circ \\
x_1 & & x_1 - x_2 & & x_2 - x_3 & & x_3 - x_4 & & x_4 - x_5
\end{array}
$$

Definition [AMMN19]

Let $\mathfrak{a} : \mathrm{Der}(\mathcal{A}) \to S$ be an S-module homomorphism. Define the Solomon-Terao algebra to be

$$\mathcal{ST}(\mathcal{A}; \mathfrak{a}) = S / \operatorname{im} \mathfrak{a}$$

Theorem [ACK+24]

Let $\mathcal{A}$ be an essential southwest arrangement in $\mathbb{Q}^n$ with h-function $h(\mathcal{A})$. Let $\mathfrak{i} : \mathrm{Der}(\mathcal{A}) \to S$ be defined by $\partial_i \mapsto 1$. Then the monomials

$$\{\mathsf{x}^\alpha : \alpha < h(\mathcal{A})\}$$

descend to a basis for $\mathcal{ST}(\mathcal{A}; \mathfrak{i})$.

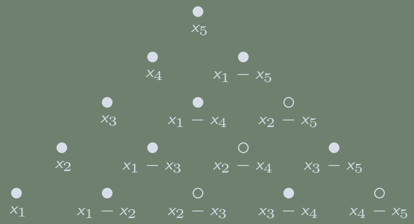
Definition

Let $J \subseteq [n]$ Let $\mathcal{A}_J$ denote the hyperplane arrangement defined by

$$x_1 x_2 \cdots x_n \prod_{j \notin J} \prod_{i > j} (x_j - x_i)$$

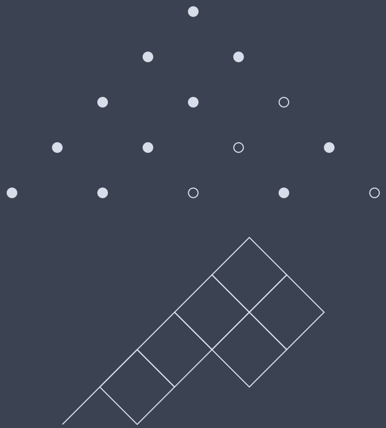
Example

Let $J = \{2, 4\}$, then $\mathcal{A}_J$ is



Lemma

The J-staircase is bounded above by the h-function of $\mathcal{A}_J$



Theorem [ACK+24]

$\mathcal{M}(J)$ is a basis for $S/(I^+ : f_J)$

Proof:

$$0 \to S/(I^+ : f_J) \xrightarrow{\times x^J} S/(I^+ : \tilde{f}_J) = \mathcal{ST}(\mathcal{A}_J, \mathfrak{i})$$

is exact.

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Corollary

$\mathcal{M}$ is a basis for SR_n, resolving conjecture of [SS24]

THANK YOU!

References

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