Equivariant Kazhdan–Lusztig theory of paving matroids

by Trevor K. Karn (U. Minnesota)
(joint with George Nasr, Nick Proudfoot, and Lorenzo Vecchi)
on Friday, November 4, 2022

Background and motivation

Show me the matroids!

Main result

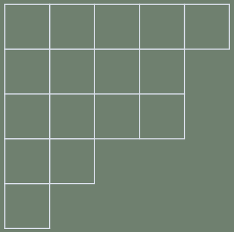
Background and motivation

A partition $\lambda \vdash n$ is a weakly decreasing sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ summing to n.

A partition $\lambda \vdash n$ is a weakly decreasing sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ summing to n.

Example

$\lambda = (5, 4, 4, 2, 1) \vdash 16$ has Ferrers diagram



A Young tableau T is a filling of a Ferrers diagram by positive integers. T is standard if it is filled by $\{1, 2, \ldots, n\}$ and increasing in rows and columns. Define f_λ as the number of standard tableaux of shape λ.

Example

One of $f_{(5,4,4,2,1)} = 549120$ standard Young tableaux:

1	6	10	13	16
2	7	11	14	
3	8	12	15	
4	9			
5				

Fact

Fix n. Then

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!$$

Fact

Fix n. Then

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!$$

Proof 1:

The Robinson-Schensted bijection:

pairs of standard tableaux of same shape $\longleftrightarrow$ symmetric group $\mathfrak{S}_n$

Fact

Irreducible $\mathfrak{S}_n$ representations are indexed by $\lambda \vdash n$ and have dimension f_λ.

Fact

Irreducible $\mathfrak{S}_n$ representations are indexed by $\lambda \vdash n$ and have dimension f_λ.

Fact

Let $d_1, d_2, \ldots, d_r$ be the dimensions of the irreducible complex representations of a finite group. Then

$$\sum_i d_i^2 = |G|.$$

Fact

Fix n. Then
$$\sum_{\lambda \vdash n} f_\lambda^2 = n!$$

Proof 2:

$$\sum_{\lambda} f_\lambda^2 = \sum_i d_i^2 = |G| = |\mathfrak{S}_n| = n!$$

Combinatorial identity with standard Young tableaux

Combinatorial proof

Algebraic proof

Show me the matroids!

Let M be a rank-k matroid with lattice of flats $L(M)$

$$\chi_M(t) := \sum_{F \in L(M)} \mu(\bar{\emptyset}, F) t^{k-r(F)}$$

Let M be a rank-k matroid with lattice of flats $L(M)$

$$\chi_M(t) := \sum_{F \in L(M)} \mu(\bar{\emptyset}, F) t^{k-r(F)}$$

A matroid M on groundset E, has Orlik–Solomon algebra $\mathcal{OS}(M)$, a certain quotient of the exterior algebra $\bigwedge E$

Let M be a rank-k matroid with lattice of flats $L(M)$

$$\chi_M(t) := \sum_{F \in L(M)} \mu(\overline{\emptyset}, F) t^{k-r(F)}$$

A matroid M on groundset E, has Orlik–Solomon algebra $\mathcal{OS}(M)$, a certain quotient of the exterior algebra $\bigwedge E$

Theorem (Orlik, Solomon '80)

$\chi_M(t)$ determines the Poincaré polynomial of $\mathcal{OS}(M)$

Definition/Theorem (Elias, Proudfoot, Wakefield '16)

Fix M. There is a unique polynomial $P_M(t)$ satisfying:

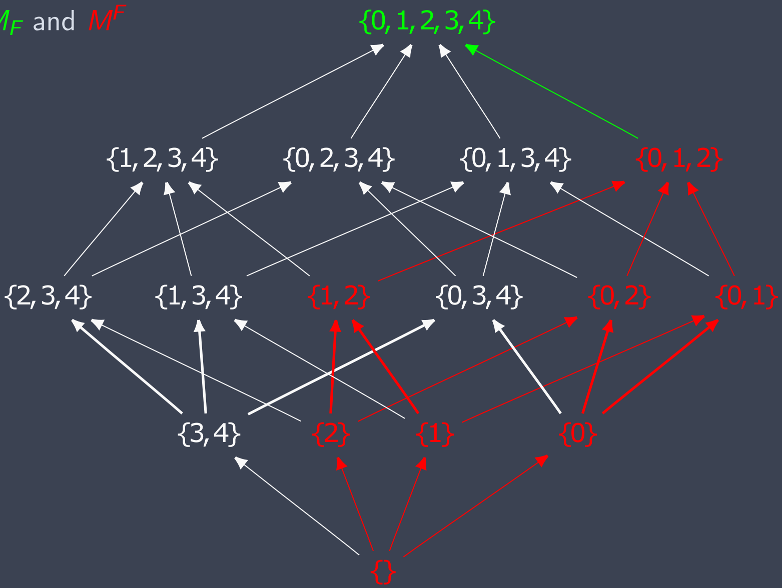
$$P_M(t) = 1 \text{ if } r(M) = 0,$$

$$\deg P_M(t) < r(M)/2 \text{ when } r(m) > 0,$$

$$t^{r(M)} P_M(t^{-1}) = \sum_{F \in L(M)} P_{M_F}(t) \chi_{M^F}(t).$$

$P_M(t)$ is the <u>matroid Kazhdan–Lusztig polynomial</u>

M_F and M^F



Conjecture (Elias, Proudfoot, Wakefield '16)

$P_M(t)$ has positive coefficients

Conjecture (Elias, Proudfoot, Wakefield '16)

$P_M(t)$ has positive coefficients

Theorem (Lee, Nasr, Radcliffe '21)

Conjecture true for sparse paving matroids

Conjecture (Elias, Proudfoot, Wakefield '16)

$P_M(t)$ has positive coefficients

Theorem (Lee, Nasr, Radcliffe '21)

Conjecture true for sparse paving matroids

Theorem (Braden, Huh, Matherne, Proudfoot, Wang '20)

Conjecture true for any M

M is a paving matroid if all circuits have size at least $k = r(M)$

A paving matroid is sparse if the set $\mathcal{CH}$ of circuit hyperplanes satisfies $\binom{E}{k} = \mathcal{CH} \sqcup \mathcal{B}$

M is a paving matroid if all circuits have size at least $k = r(M)$

A paving matroid is sparse if the set $\mathcal{CH}$ of circuit hyperplanes satisfies $\binom{E}{k} = \mathcal{CH} \sqcup \mathcal{B}$

Conjecture (Mayhew, Newman, Welsh, Whittle '11)

Asymptotically almost all matroids are sparse paving

Theorem (Pendavingh, van der Pol '15)

Asymptotically logarithmically almost all matroids are sparse paving

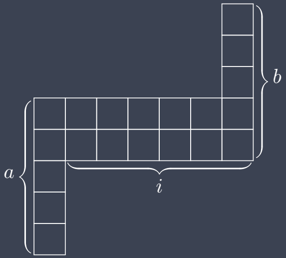
Theorem (Lee, Nasr, Radcliffe '21)

Let M be rank-k, sparse paving, on a groundset of size n, with circuit hyperplanes $\mathcal{CH}$. The t^i coefficient in $P_M(t)$ is

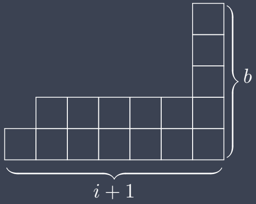
$$\mathrm{SSkYT}(n-k+1, i, k-2i+1) - |\mathcal{CH}| \cdot \overline{\mathrm{SSkYT}}(i, k-2i+1)$$

Proof idea: Combinatorial argument with recursion.

$\mathrm{SSkYT}(a, i, b) = \#$standard fillings of



$\overline{\mathrm{SSkYT}}(i, b) = \#$standard fillings of



[LNR21] combinatorially prove an identity involving Kazhdan-Lusztig polynomials and standard fillings of skew Young tableaux.

[LNR21] combinatorially prove an identity involving Kazhdan-Lusztig polynomials and standard fillings of skew Young tableaux.

Fact

Standard skew Young tableaux count the dimension of certain (reducible) $\mathfrak{S}_n$ representations

[LNR21] combinatorially prove an identity involving Kazhdan-Lusztig polynomials and standard fillings of skew Young tableaux.

Fact

Standard skew Young tableaux count the dimension of certain (reducible) $\mathfrak{S}_n$ representations

Question

Is there representation theory lurking?

K., Nasr, Proudfoot, Vecchi '22

YES!

Main result

Let W be a group. An equivariant matroid $W \curvearrowright M$ is a matroid with a W-action such that $gI \in \mathcal{I}$ for all $g \in W$ and $I \in \mathcal{I}$.

Let W be a group. An equivariant matroid $W \curvearrowright M$ is a matroid with a W-action such that $gI \in \mathcal{I}$ for all $g \in W$ and $I \in \mathcal{I}$.

The action of W induces an action on $\mathcal{OS}(M)$. The equivariant characteristic polynomial of $W \curvearrowright M$ is a graded virtual representation $\chi_M^W(t)$. The coefficient of t^i is determined by $\mathcal{OS}(M)_i$.

Let W be a group. An equivariant matroid $W \curvearrowright M$ is a matroid with a W-action such that $gI \in \mathcal{I}$ for all $g \in W$ and $I \in \mathcal{I}$.

The action of W induces an action on $\mathcal{OS}(M)$. The equivariant characteristic polynomial of $W \curvearrowright M$ is a graded virtual representation $\chi_M^W(t)$. The coefficient of t^i is determined by $\mathcal{OS}(M)_i$.

$$\boxed{\chi_M^W(t)} \xrightarrow{\text{“dimension”}} \boxed{\chi_M(t)}$$

Definition/Theorem (Gedeon, Proudfoot, Young '17)

Let $W \curvearrowright M$ be an equivariant matroid, W_F denote the stabilizer of F. Then there exists $P_M^W(t)$ satisfying

If $r(M) = 0$, then $P_M^W(t)$ is $\mathbb{1}_W t^0$

If $r(M) > 0$, then $\deg P_M^W(t) < r(M)/2$

$$t^{r(M)} P_M^W(t^{-1}) = \sum_{[F] \in L(M)/W} \operatorname{Ind}_{W_F}^W \left(P_{M_F}^{W_F}(t) \otimes \chi_{M^F}^{W_F} \right)$$

$\varphi : W' \to W$ a homom. then $P_M^{W'}(t) = \varphi^* P_M^W(t)$

Compare:

$$t^{r(M)} P_M(t^{-1}) = \sum_{F \in L(M)} P_{M_F}(t) \chi_{M^F}(t)$$

and

$$t^{r(M)} P_M^W(t^{-1}) = \sum_{[F] \in L(M)/W} \mathrm{Ind}_{W_F}^{W} \left(P_{M_F}^{W_F}(t) \otimes \chi_{M^F}^{W_F} \right).$$

$$P_M^W(t) \xrightarrow{\text{“dimension”}} P_M(t)$$

» Relaxation

A stressed hyperplane H of a rank-k matroid $M = (E, B)$ has every k-subset a circuit.

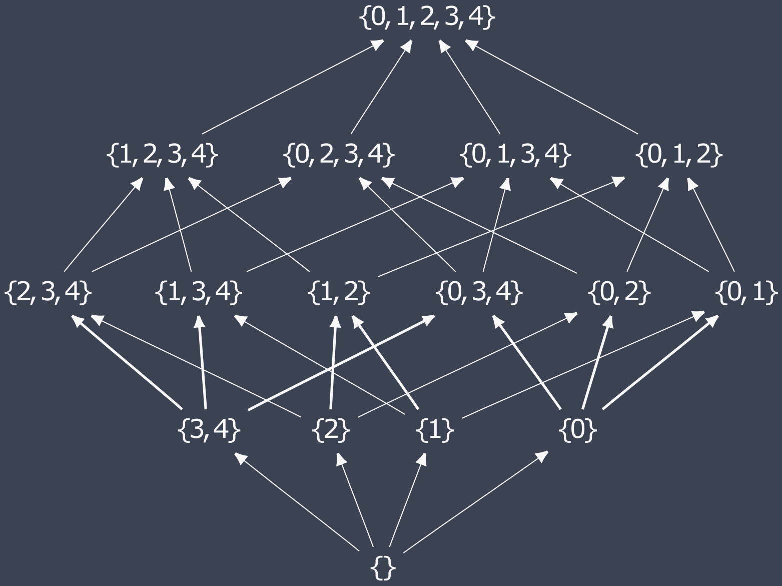
» Relaxation

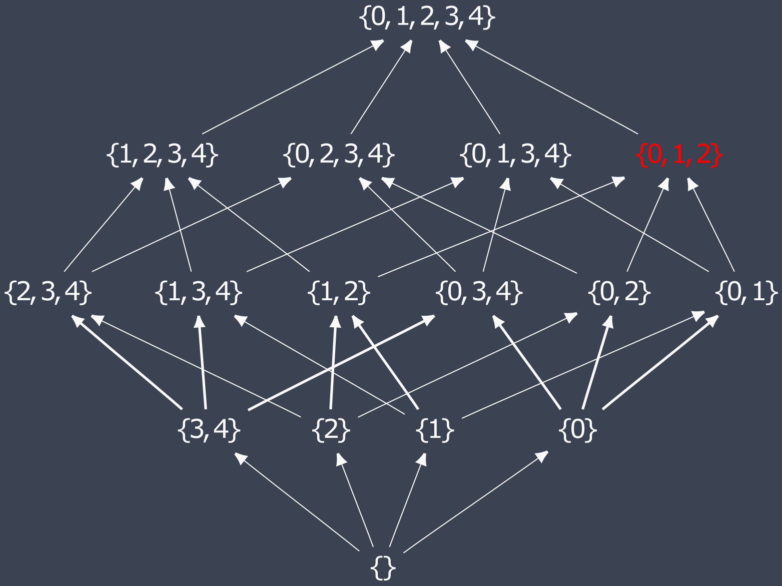
A stressed hyperplane H of a rank-k matroid $M = (E, B)$ has every k-subset a circuit.

Theorem (Ferroni, Nasr, Vecchi '21)

The operation of relaxation at a stressed hyperplane H forms a new matroid $\tilde{M} = (E, \tilde{\mathcal{B}})$ with bases

$$\tilde{\mathcal{B}} = \mathcal{B} \sqcup \{S \subseteq H : |S| = k\}.$$


{0, 1, 2, 3, 4}
{1, 2, 3, 4}
{0, 2, 3, 4}
{0, 1, 3, 4}
{0, 1, 2}
{2, 3, 4}
{1, 3, 4}
{1, 2}
{0, 3, 4}
{0, 2}
{0, 1}
{3, 4}
{2}
{1}
{0}
{}


{0, 1, 2, 3, 4}
{1, 2, 3, 4}
{0, 2, 3, 4}
{0, 1, 3, 4}
{0, 1, 2}
{2, 3, 4}
{1, 3, 4}
{1, 2}
{0, 3, 4}
{0, 2}
{0, 1}
{3, 4}
{2}
{1}
{0}
{}

Theorem (Ferroni, Nasr, Vecchi '21)

There exists a polynomial $p_{k,h}$ such that

$$P_M(t) = P_{\tilde{M}}(t) - p_{k,h}$$

Theorem (Ferroni, Nasr, Vecchi '21)

There exists a polynomial $p_{k,h}$ such that

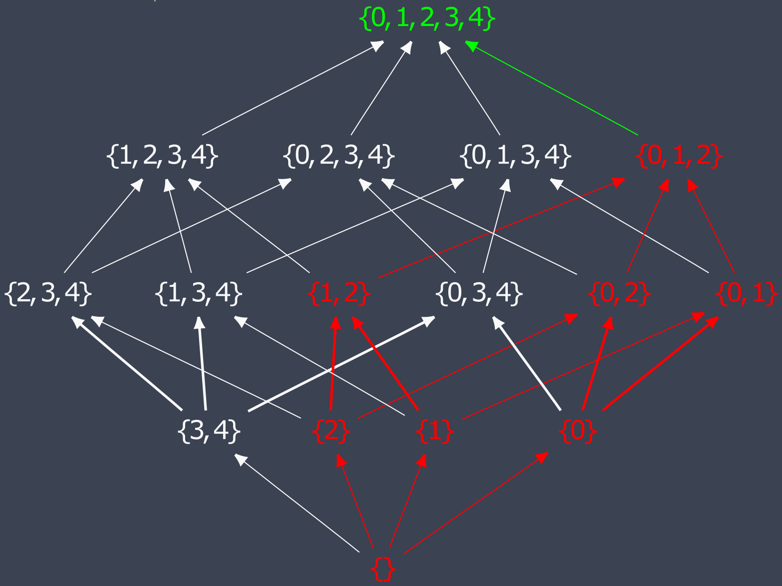
$$P_M(t) = P_{\tilde{M}}(t) - p_{k,h}$$

Theorem (Ferroni, Nasr, Vecchi '21)

If M is a paving matroid with $|E| = n$ and has exactly λ_h-many stressed hyperplanes of size h, then

$$P_M(t) = P_{U_{k,n}}(t) - \sum_{h \geq k} \lambda_h \cdot p_{k,h}.$$

» Idea of the proof



Let $W \curvearrowright M$ be an equivariant matroid with stressed hyperplane H.

Let $W \curvearrowright M$ be an equivariant matroid with stressed hyperplane H.

Let $W \curvearrowright \tilde{M}$ denote the equivariant matroid found by simultaneously relaxing all hyperplanes in $[H]$.

Let $W \curvearrowright M$ be an equivariant matroid with stressed hyperplane H.

Let $W \curvearrowright \tilde{M}$ denote the equivariant matroid found by simultaneously relaxing all hyperplanes in $[H]$.

Theorem (K.-Nasr-Proudfoot-Vecchi '22)

There exists an equivariant polynomial $p_{k,h}^{\mathfrak{S}_h}$ such that

$$P_M^W(t) = P_{\tilde{M}}^W(t) - \operatorname{Ind}_{W_H}^W \operatorname{Res}_{W_H}^{\mathfrak{S}_h} p_{k,h}^{\mathfrak{S}_h}$$

Theorem (K.-Nasr-Proudfoot-Vecchi '22)

The coefficients of t^i are

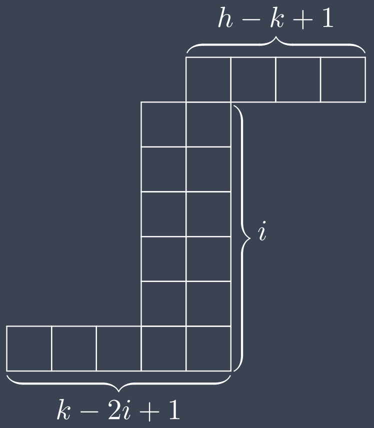
$$\{t^i\}p_{k,h}^{\mathfrak{S}_h} = S^{\mu_i/\lambda_i}$$

where $\mu_i, \lambda_i \vdash h$ are:

$\mu_i = h - 2i + 1, (k - 2i + 1)^i$ and

$\lambda_i = k - 2i, (k - 2i - 1)^{i-1}$

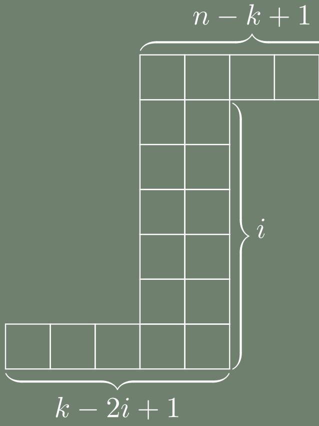
The coefficient of t^i in $p_{k,h}^{\mathfrak{S}_h}$ is



which has dimension equal to the number of standard fillings

Theorem (Gao, Xie, Yang '21)

Every coefficient of t^i in $P^{\mathfrak{S}_n}_{U_{k,n}}(t)$ is given by the skew shape:



» Idea of proof

Relax $U^{\mathfrak{S}_h}_{k-1,h} \oplus U_{1,1}$ to $U^{\mathfrak{S}_{h+1}}_{k,h+1}$.

» Idea of proof

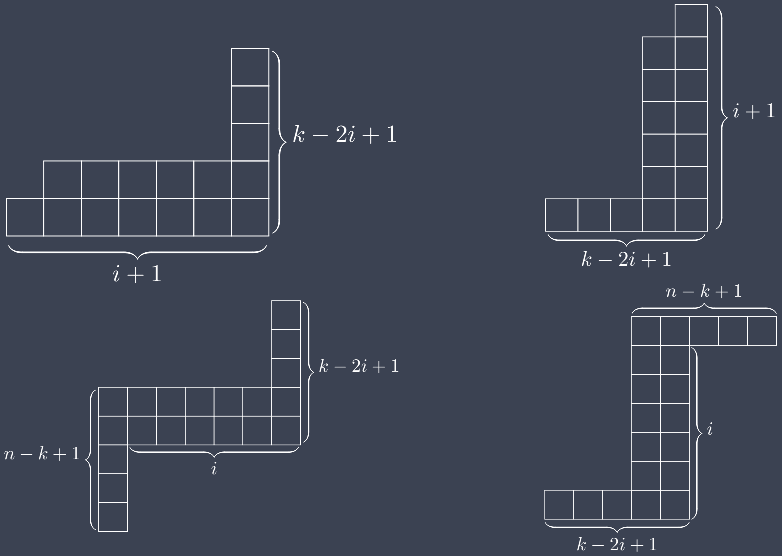
Relax $U^{\mathfrak{S}_h}_{k-1,h} \oplus U_{1,1}$ to $U^{\mathfrak{S}_{h+1}}_{k,h+1}$.

$p^{\mathfrak{S}_h}_{k,h}$ depends only on k, h, so one example is enough.

A series of relaxations can be performed to a sparse paving matroid to obtain the uniform matroid. In other words:

$$P_M^W(t) = P_{U_{k,n}}^W(t) - \sum_{H \in \mathcal{CH}} \operatorname{Ind}_{W_H}^W \operatorname{Res}_{W_H}^{\mathfrak{S}_h} p_{k,h}^{\mathfrak{S}_h}$$

For sparse paving matroids, $h = k$. This provides representation theoretic proof of the Lee–Nasr–Radcliffe formula!



$$\sum_{\lambda} f_\lambda = n!$$

Combinatorial proof

Algebraic proof

$P_M(t)$ in terms of standard skew Young tableaux

Combinatorial proof [LNR21]

Algebraic proof [KNPV22]

THANK YOU!

» References

Ben Elias, Nicholas Proudfoot, and Max Wakefield, The Kazhdan–Lusztig polynomial of a matroid, Advances in Mathematics **299** (2016), 36–70.

Luisa Ferroni, George D. Nasr, and Lorenzo Vecchi, Stressed hyperplanes and Kazhdan–Lusztig gamma-positivity for matroids, 2021.

Katie R. Gedeon, Nicholas Proudfoot, and Benjamin Young, The equivariant Kazhdan–Lusztig polynomial of a matroid, J. Comb. Theory, Ser. A **150** (2017), 267–294.

Alice L. L. Gao and Matthew H. Y. Xie, The inverse Kazhdan-Lusztig polynomial of a matroid, J. Combin. Theory Ser. B **151** (2021), 375–392. MR 4294228

» References (cont.)

- Alice L. L. Gao, Matthew H.Y. Xie, and Arthur L. B. Yang, The equivariant inverse Kazhdan–Lusztig polynomials of uniform matroids, 2021.
- Trevor Karn, George Nasr, Nicholas Proudfoot, and Lorenzo Vecchi, Equivariant kazhdan-lusztig theory of paving matroids, 2022.
- Kyungyong Lee, George D. Nasr, and Jamie Radcliffe, A combinatorial formula for Kazhdan–Lusztig polynomials of sparse paving matroids, Electron. J. Comb. **28** (2021).
- Dillon Mayhew, Mike Newman, Dominic Welsh, and Geoff Whittle, On the asymptotic proportion of connected matroids, European J. Combin. **32** (2011), no. 6, 882–890. MR 2821559

» References (cont.)

- Peter Orlik and Louis Solomon, Combinatorics and topology of complements of hyperplanes, Inventiones mathematicae **56** (1980), no. 2, 167–189 (eng).
- Rudi Pendavingh and Jorn van der Pol, On the number of matroids compared to the number of sparse paving matroids, Electron. J. Combin. **22** (2015), no. 2, Paper 2.51, 17. MR 3367294
- Nicholas Proudfoot, Yuan Xu, and Benjamin Young, The Z-polynomial of a matroid, Electron. J. Comb. **25** (2018), P1.26.