Topics in Combinatorics - Kazhdan-Lusztig theory Spring 2023

Presentation on "KL polynomials for 321-hexagon avoiding perm.s" by Sara Billey, Gregory Warrington

<u>Goal</u>: Prove the following (part of) Thm 1:

Let $w = s_1 s_2 \cdots s_r$ be a reduced word for $w \in \mathfrak{S}_n$. Then w is 321-hexagon-avoiding if and only if for $x \leq w$ we have

$$P_{x,w} = \sum q^{d(\sigma)}$$

where $d(\sigma)$ is the "defect" of a mask σ, and the the sum is over all masks on w whose product is x.

<u>Ex</u> Let $w = s_1 s_2$ (know by degree consideration $P_{id,w} = 1$)

Masks with product $= id$: $\underline{\sigma} = (0,0)$. $\leftarrow$ single summand

$d(\underline{\sigma}) = 0$ because $\ell(s_2) \geq \ell(id)$. $\leftarrow$ q^0

<u>Note</u>: BW also prove

$$\text{321-hex-avoid} \iff C'_w = C'_{s_1} C'_{s_2} \cdots C'_{s_r}$$

$$\iff \mathrm{Poin}(IH^{2i}(\mathrm{Schub}(w)); q^{1/2}) = (1+q)^{\ell(w)}$$

<u>Recall</u> hex.-avoid. means avoids $u = s_5 s_6 s_7 s_3 s_4 s_5 s_6 s_2 s_3 s_4 s_5 s_1 s_2 s_3$ or $u s_4$ or $s_4 u$ or $s_4 u s_4$.

321-avoid. $\iff$ $s_i s_{i\pm 1} s_i$ avoid

Why "hex-avoid"? B/c of shape of heap.

<u>Computational note</u>: pattern avoidance is polynomial, red. wd. is exp.

7243165 in 1-line has 3256 red. wds.

Idea of proof: Use thm of Deodhar '90 (in a restricted setting) to get a polynomial with the property that if it satisfies the degree bound, then it is $P_{x,w}$ KL poly.

$\Rightarrow$ Create a graph for each mask to show the degree bound.

$\Leftarrow$ find an explicit mask which breaks degree bound.

Let $\mathcal{P}_x(w_1 w_2 \cdots w_r) = \{$ all subwords of $w_1 w_2 \cdots w_r$ with product $x\}$

THM (D., '90) Let W a fin. Weyl group. $w = w_1 w_2 \cdots w_r$ reduced.

$$P_x(w) := \sum_{\sigma \in \mathcal{P}_x(w)} q^{|D(\sigma)|}.$$

If $\deg P_x(w) \le (\ell(w) - \ell(x) - 1)/2$, then $P_x(w) = P_{x,w}$ $\forall x \in W$.

Def'n Let $\underline{\sigma} \in \{0,1\}^r$ be a mask. $a = a_1 a_2 \cdots a_r = w^{\sigma}$

$D(\sigma) = \{ j \ge 2 : s_j$ is right descent of $a_1 \cdots a_{j-1}\}$

$D^0(\sigma) = \{ j \in D(\sigma) : \sigma_j = 0\}$

$D^1(\sigma) = \{ j \in D(\sigma) : \sigma_j = 1\}$

$d_\sigma = |D_\sigma|$.

Ex $w = 321432543$, $\sigma = (1,1,0,1,0,1,0,1,0)$ $D(\sigma) = \{6,8,9\}$.

$D^1(\sigma)$ $D^0(\sigma)$

Def $\Delta_\sigma = \dfrac{\ell(w) - \ell(x) - 1}{2} - d_\sigma$

Why? If $\Delta_\sigma \ge 0$ for all $\sigma \in \mathcal{P}_x(w)$, we are done.

Technical lemma: (BW Lemma 2) If $\sigma \ne (1,1,\dots,1)$, then

$$\Delta_\sigma \ge 0 \iff \#\{0\text{'s in } \sigma\} \ge 2d^0(\sigma) + 1$$

Pf Cases $\Rightarrow$ omitted.

The graph: $G_\sigma = (V, E)$ with $V = D^0(\sigma)$.

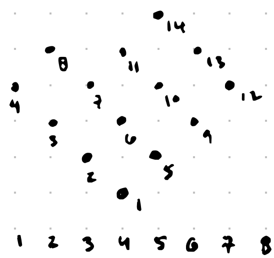
Edges are more complicated. Start with string dgm.

)(if 0 or X if 1

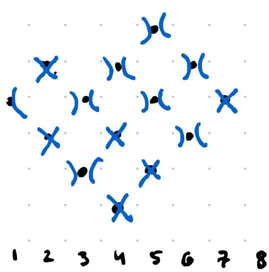
"bounce" "cross"

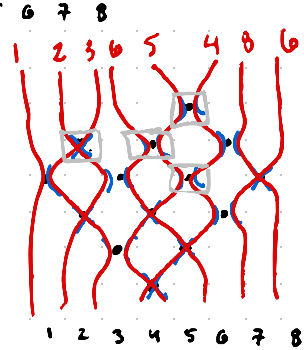
pt(j) in heap := the order in which you put them down,

Ex $w = 43215432654765$, $\sigma = (10101101000100)$



From the mask:

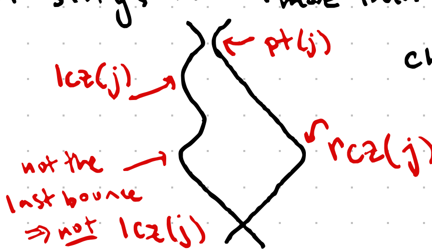




Strings

Defects

If strings interact more than 1x, they had to have changed direction

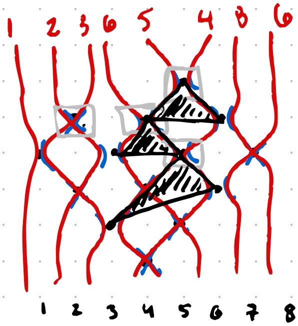


change of direction ⟺ bounce)(so def. "critical zeros".

for $j \in D^0(\sigma)$, say

$\{pt(j), lcz(j), rcz(j)\} =:$ "critical 0's of j" $= C(j)$

Edges of G_σ: (i,j) st $C(i) \cap C(j) \neq \emptyset$.



Strings

Defects

$G_\sigma =$ 

State w/o proof:

__Lemma__ (BW Lemma 5.2 ⇒) If w is 321-hex no point is a critical 0 for 3 distinct defects.

__Prop__: (BW Prop 1) If w is 321-hex-avoid, then G_σ a forest.

Proof

$\Rightarrow$ Let w be 321-hex.-avoid., $P_x(w)$ as before.

Enough to show $\#\{0\text{'s in } \sigma\} \geq 2d^\circ(\sigma)+1$

$\#\{0\text{'s in } \sigma\} \geq \#\{\text{critical } 0\text{'s in } \sigma\}$ b/c 2

An edge in $G_\sigma \leftrightarrow (i,j)$ sharing a critical 0.

$\Rightarrow \#\{\text{critical } 0\text{'s}\} = \underset{\{pt,\, lcz,\, rcz\}}{\underset{\uparrow}{3\cdot d^\circ(\sigma)}} - \underset{\text{correction term}}{\underset{\uparrow}{\#E}}$ (P.I.E.)

Trees have $|V|-1$ edges, so Forests have $\leq |V|-1$ edges.
Thus

$\#\{0\text{'s}\} \geq 3d^\circ(\sigma) - \#E \geq 3d^\circ(\sigma) - (d^\circ - 1) = 2d^\circ(\sigma)+1.$

DONE.

$\Leftarrow$ (contrapositive)

case i) w not 321-avoid.

$\exists$ red. word $w = v s_i s_{i\pm1} s_i v'$ so $\ell(w) = \ell(v)+\ell(v')+3$.

pick $\sigma = (\underbrace{1,1,\ldots,1}_{\ell(v)},\underbrace{1,0,0}_{\uparrow},\underbrace{1,1,\ldots,1}_{\ell(v')})$ ← could maybe still have defect, but not $D^\circ(\sigma)$.

$\Rightarrow d^\circ(\sigma)=1$

$\ell(vs_is_{i+1}) > \ell(vs_i)$, and $\ell(vs_is_i) < \ell(vs_i)$.

$\underset{\underset{2}{\parallel}}{\#0\text{'s}} \geq \underset{\underset{3}{\parallel}}{2d^\circ(\sigma)+1} \overset{\text{Lemma 2}}{\Longleftrightarrow} \Delta_\sigma \geq 0$

$\Rightarrow \Delta_\sigma < 0. \Rightarrow$ Find σ with $\deg(g^{|D(\sigma)|})$ too large.

case ii) 321-avoid but not hex-avoid.

Let u be the red word s.t.

hex-avoid $\iff$ no subword like u, us_4, s_4u, s_4us_4.

then $\ell(u)=14$ so $w = vuv'$ has $\ell(w) = \ell(v)+\ell(v')+14$.

pick $\sigma = (\underbrace{1,1,\dots,1}_{\ell(v)}, 1,1,0,1,0,1,\overset{7}{0},1,1,0,\overset{11}{0},0,\overset{13}{0},\overset{14}{0}, \underbrace{1,1,\dots,1}_{\ell(v')})$

$d^\circ = 4$, $\#\,0\text{'s} = 8$

check these are $D^\circ(\sigma)$

$$8 \not\geq 2\cdot d^\circ + 1 \implies \Delta_\sigma < 0.$$

$\blacksquare$