1) CONFORMAL MAPPING $H \to \mathbb{H}$

DEFINE $f : H \to D$ BY $z \mapsto z^2$, WHERE D IS THE UNIT DISK, SLIT ALONG $\mathbb{R}_{\geq 0}$

DEFINE $g : D \to \mathbb{H}$ BY THE CAYLEY MAP $\frac{iz+i}{-z+1}$

WE CLAIM $g \circ f$ IS THE DESIRED MAPPING. SINCE $\arg(z) \in (0, \pi)$ FOR ALL $z \in H$ AND $|z| < 1$, THEN $\arg(z) \in (2 \cdot 0, 2\pi)$ AND $|z^2| < 1^2$, SO f IS INDEED. NOTE $f'(z) = 2z$ IS ONLY ZERO AT $z = 0$, BUT $0 \notin H$, SO IT IS INDEED CONFORMAL. THE DERIVATIVE OF g IS $g' = 2i/(1-z)^2$, WHICH IS NONZERO ON ITS DOMAIN, WHICH IS $\mathbb{C} \setminus \{1\}$. BUT $1 \notin D$, SO g IS HOLOMORPHIC FROM $D \to \mathbb{H}$. THE CHAIN RULE TELLS US THAT $(g \circ f)' = g'(f) f'$. WE KNOW $f' \neq 0$ ON THE DOMAIN. MOREOVER, $f(H) \not\ni 1$. THUS, $g'(f) \neq 0$, AND SO THE COMPOSITION $g \circ f$ IS CONFORMAL.

TO SEE f IS ONTO, LET $z \in D$. THEN $\arg(z) \in (0, 2\pi)$ AND $|z| \leq 1$ SO $|\sqrt{z}| \leq 1$ (CHOOSING THE PRINCIPAL BRANCH OF $\sqrt{\cdot}$) AND $\arg(\sqrt{z})$ IS IN $(0, \pi)$, SO f IS ONTO. TO SEE g IS ONTO, NOTE THAT ITS EXPLICIT INVERSE IS $\frac{z-i}{z+i}$, WHICH TAKES POINTS CLOSER TO i THAN TO $-i$ (ie. THE UPPER HALF PLANE) TO A POINT IN THE SLIT DISK.

2 NONZERO TERMS OF $f(z) = \frac{1}{z(z-1)(z-2)}$ CENTERED AT 1.

FIRST, APPLY THE CHANGE OF COORDINATES $w = z-1$. THEN WE CAN WRITE $f(z) = f(w+1) = \frac{1}{w(w+1)(w-1)}$, AND WE SEEK THE LAURENT EXPANSION OF $f(w+1)$ CENTERED AT $w=0$. USING GEOMETRIC SERIES, THIS BECOMES

$$\frac{1}{w(w+1)(w-1)} = \frac{1}{w}\left(-\sum_{n\geq 0} w^n\right)\left(\sum_{n\geq 0}(-w)^n\right)$$

SINCE $\frac{1}{w-1} = -\sum_{n\geq 0} w^n$ FOR $|w|<1$, AND $\frac{1}{w+1} = \frac{1}{1-(-w)} = \sum_{n\geq 0}(-w)^n$ FOR $|w|<1$. THEN MULTIPLYING, WE GET

$$\begin{aligned} f(w+1) = {} & \frac{1}{w}(-1)(1) + \frac{1}{w}\left(-w + (+w)\right) \\ & + \frac{1}{w}\left(-w^2 + (-w)(-w) + (-w)^2\right) \\ & + \frac{1}{w}\left(-w^3 + (-w^2)(-w) + (-w)(-w)^2 + (-w)^3\right) \\ & + \frac{1}{w}\left(-w^4 + (-w^3)(-w) + (-w^2)((-w)^2) + (-w)(-w)^3 + (-w)^4\right) \\ & + \dots \\ = {} & -\frac{1}{w} + 0 + w + 0 + (-1)w^3 + \dots \end{aligned}$$

UNDOING OUR CHANGE OF COORDINATES, WE GET

$$f(z) = \frac{-1}{z-1} + (z-1) - (z-1)^3 + \dots$$

SINCE IT CONVERGED FOR $|w|<1$, IT ALSO CONVERGES FOR $|z-1|<1$.

COMPLEX ANALYSIS

3. SHOW (UNDER CONDITIONS) $\overline{f(z)} = f(\bar{z})$.

THE SCHWARZ REFLECTION PRINCIPLE STATES THAT IF Ω IS A REGION WHICH IS SYMMETRIC ABOUT THE REAL-AXIS, AND f IS A FUNCTION HOLOMORPHIC IN $\Omega \cap \{z : \mathrm{Im}(z) > 0\}$ WHICH HAS A CONTINUATION ONTO $\mathbb{R}^{\cap \Omega}$ AND THAT CONTINUATION IS REAL-VALUED, THEN THERE IS A HOLOMORPHIC FUNCTION F ON Ω ST $F = f$ ON $\Omega \cap \{z : \mathrm{Im}(z) > 0\}$, AND MOREOVER, $F(z) = \overline{f(\bar{z})}$.

NOTE THAT WE MAY TAKE $\Omega = \mathbb{C}$, AND THE CONTINUATION TO BE JUST THE REAL VALUES ON $\mathbb{R}$ WHICH WE KNOW IT TAKES.

THEN $\overline{F(z)} = \overline{f(z)} = \overline{\overline{f(\bar{z})}} = f(\bar{z})$. $\boxtimes$

4. CLASSIFY ENTIRE f SUCH THAT $|f(z)| \leq C \log(1+|z|)$

IF f IS ENTIRE IT ADMITS A POWER SERIES REPRESENTATION CENTERED AT 0, SO

$$f(z) = \sum_{n \geq 0} \alpha_n z^n$$

CAUCHY'S INEQUALITY TELLS US THAT ON A CIRCLE OF RADIUS R ABOUT z_0, CALL IT γ_R, THAT

$$|f^{(n)}(z_0)| \leq \frac{n! \max_{z \in \gamma_R} |f(z)|}{R^n}$$

TO EXTRACT THE COEFFICIENTS α_n FROM f, NOTE THAT

$$\alpha_n = f^{(n)}(0)/n!$$

SO TAKING $z_0 = 0$, WE GET

$$\left|\frac{f^{(n)}(0)}{n!}\right| = |\alpha_n| \leq \frac{\max_{z \in \gamma_R} |f(z)|}{R^n}$$

THE PROVIDED BOUND TELLS US $\max_{C_R} |f(z)| \leq C \cdot \log(1+R)$ AND SO

$$|\alpha_n| \leq \frac{C \log(1+R)}{R^n}.$$

THIS IS INDEPENDENT OF R, SO WE TAKE THE LIMIT

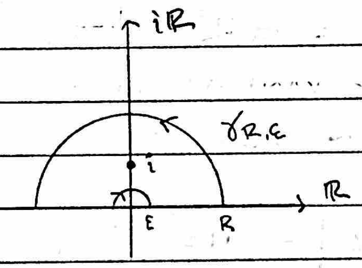
$$\lim_{R \to \infty} \frac{C \log(1+R)}{R^n} = \lim_{R \to \infty} \frac{C \frac{1}{1+R}}{n R^{n-1}} \qquad \text{(BY L'HÔPITAL'S RULE)}$$

WHICH VANISHES FOR $n \geq 1$. SO f IS CONSTANT, BUT WE CAN DO BETTER. IF f IS CONSTANT $f(z) \equiv f(0)$, SO $|f| \leq C \log(1+|0|) = 0$, SO f IS IDENTICALLY 0 ON $\mathbb{C}$.

$\square$

5. EVALUATE $\int_0^\infty \frac{x^{1/4}}{1+x^2}\,dx$

WE COMPUTE BY PASSING TO $\mathbb{C}$ AND DEFINING $f(z) = \frac{z^{1/4}}{1+z^2}$. CHOOSE $z^{1/4}$ TO BE DEFINED USING THE BRANCH CUT FROM THE ORIGIN THROUGH $-i$. IN PARTICULAR f IS NOT DEFINED AT ZERO. DEFINE THE CURVE $\gamma_{R,\varepsilon}$ FOR $R>1$, $0<\varepsilon<1$ AS THE UNION OF THE CIRCLES OF RADII R, ε WITH COUNTER CLOCKWISE AND CLOCKWISE ORIENTATIONS RESPECTIVELY, AND CONNECT THEM ALONG $[-R,\varepsilon]$ AND $[\varepsilon,R]$, TO GET



THE RESIDUE THEOREM STATES THAT

$$\int_{\gamma_{R,\varepsilon}} f\,dz = 2\pi i \sum_{\substack{\text{POLES} \\ \text{INSIDE} \\ \gamma_{R,\varepsilon}}} \operatorname{res}_{z_0}(f)$$

THE ONLY POLE OF f INSIDE $\gamma_{R,\varepsilon}$ IS THE ONE AT $z=i$, WHICH IS SIMPLE. SO WE MAY COMPUTE

$$\operatorname{res}_i(f) = \lim_{z\to i}(z-i)f(z)$$

$$= \frac{i^{1/4}}{2i}$$

AND SO $\int_{\gamma_{R,\varepsilon}} f\,dz = \pi i^{1/4}$. NOW OBSERVE THAT, IF C_R^+ REPRESENTS THE CIRCLE OF RADIUS R IN THE UPPER HALF PLANE, THEN

$$\int_{\gamma_{R,\varepsilon}} f\,dz = \int_{C_R^+} f\,dz - \int_{C_\varepsilon^+} f\,dz + \int_{-R}^{\varepsilon} f\,dz + \int_{\varepsilon}^{R} f\,dz$$

WHERE WE SUBTRACT THE INTEGRAL ON C_ε^+ BECAUSE OF THE CHOICE OF ORIENTATION. WE WILL SHOW THE CIRCULAR PARTS VANISH IN THE LIMIT, SINCE $\int_{\gamma_{R,\varepsilon}} f\,dz$ WAS INDEPENDENT OF R, ε.

CONTINUED →

THE ESTIMATION LEMMA GIVES THE BOUND

$$\left|\int_{C_R^+} f(z)\,dz\right| \le \text{LENGTH}(C_R^+) \max_{|z|=R} |f(z)|$$

$$= \pi R \max_{|z|=R}\left|\frac{z^{1/4}}{z^2+1}\right|$$

$$= \pi R^{5/4} \max_{|z|=R}\left|\frac{1}{z^2+1}\right| \qquad (*)$$

NOTE THAT $\left|\frac{1}{z^2+1}\right|$ IS MAXIMIZED WHEN $|z^2-(-1)|$ IS MINIMIZED.
SINCE $|z|=R$ IS CLOSEST TO -1 AT $z=-z$, WE CAN WRITE

$$(*) = \pi R^{5/4}\,\frac{1}{(-R)^2+1}.$$

SINCE $5/4 < 2$, $*$ VANISHES UPON TAKING $R \to \infty$.
THE SAME COMPUTATION SHOWS THAT

$$\left|\int_{C_\varepsilon^+} f(z)\,dz\right| \le \pi \varepsilon^{5/4}/(\varepsilon^2+1)$$

AS $\varepsilon \to 0$, THE NUMERATOR VANISHES AND THE DENOMINATOR $\to 1$,
SO

$$\pi i^{1/4} = \int_{\gamma_{\infty,0}} f(z)\,dz = \int_{-\infty}^{0} f\,dz + \int_0^{\infty} f\,dz$$

A CHANGE OF VARIABLES GIVES US

$$\int_{-\infty}^0 f(z)\,dz = \int_{\infty}^0 f(-z)(-1)\,dz = \int_0^\infty f(-z)\,dz.$$

BUT $f(-z) = (-1)^{1/4} f(z)$, SO WE GET

$$\pi i^{1/4} = \left((-1)^{1/4}+1\right)\int_0^\infty f(z)\,dz$$

SO

$$\int_0^\infty f(z)\,dz = \pi i^{1/4}/\left((-1)^{1/4}+1\right) = \pi\,\frac{i^{1/4}}{i^{1/2}+1}$$

$$= \pi/\left(i^{1/4}+i^{-1/4}\right)$$

CONTINUED →

5 CONTINUED FROM PG (6) BUT NOTE THAT

$$i^{1/4} = \cos(\pi/8) + i\sin(\pi/8)$$
$$i^{-1/4} = \cos(\pi/8) - i\sin(\pi/8)$$

SO

$$\int_0^\infty f(z)\,dz = \boxed{\pi/2\cos(\pi/8)}$$

∎

6. SHOW f, $f+\varepsilon g$ HAVE SAME # OF ZEROS.

ROUCHE'S THEOREM TELLS US THAT IF f, h HOLOMORPHIC ON AND INSIDE (eg) THE UNIT DISK, AND $|f| > |h|$ ON ALL OF THE BOUNDARY $|z|=1$, THEN f, $f+h$ HAVE THE SAME NUMBER OF ZEROS INSIDE THE UNIT DISK. SINCE THE BOUNDARY IS COMPACT (CLOSED AND BOUNDED), $|f|, |g|$ ACHIEVE BOTH A MAXIMUM AND A MINIMUM ON $|z|=1$. LET $m := \min_{|z|=1} |f|$ AND $M := \max_{|z|=1} |g|$. THEN $|g/M| \leq 1$ AND SO

$$\left|\frac{mg}{M}\right| \leq |f| \quad \text{on } |z|=1.$$

THEN TAKE $\varepsilon < m/M$, AND $h = \varepsilon g$ IN THE STATEMENT OF ROUCHE'S THEOREM. ∎

7. THE MAXIMUM MODULUS PRINCIPLE TELLS US THAT f IS HOLOMORPHIC ON $|z|<1$, THEN $|f|$ DOES NOT ATTAIN A MAXIMUM ON $|z|<1$, AND SO ANY MAXIMUM MUST OCCUR ON $|z|=1$. IF A FUNCTION f IS HARMONIC ON $\{z: 0<|z|<1\}$ IT ADMITS AN ANALYTIC CONTINUATION ON $\{z: |z|\leq 1\}$ CALL THIS CONTINUATION F. THEN $|F|$ ATTAINS ITS MAXIMUM ON $|z|=1$. SINCE $F=0$ ON $|z|=1$, $|F|\leq 0$ ON THE WHOLE PUNCTURED DISK, AND SO f MUST BE IDENTICALLY 0.

8. SHOW GENUS OF $z^3+w^3=1$ IS 1

THE DEGREE-GENUS FORMULA TELLS US THAT IF WE HAVE A SMOOTH IRREDUCIBLE PLANE CURVE, THEN ITS GENUS IS

$$\frac{(d-1)(d-2)}{2}$$

WHERE d IS THE DEGREE OF THE CURVE. TO SEE THAT THE GIVEN CURVE IS SMOOTH, HOMOGENIZE SO WE ARE DEALING WITH $P=z^3+w^3-u^3=0$. THE PARTIAL DERIVATIVES $\frac{\partial}{\partial z}P=3z^2$, $\frac{\partial}{\partial w}P=3w^2$, $\frac{\partial}{\partial u}=-3u^2$ ARE ONLY SIMULTANEOUSLY 0 WHEN (IN HOMOGENEOUS COORDINATES) WE ARE AT THE POINT $[0:0:0]$. BUT THAT POINT IS NEVER ON OUR GIVEN CURVE, WHICH ALWAYS LOOK LIKE $[*:*:1]$. TO SEE IT IS IRREDUCIBLE, NOTE THAT $-w^3+1$ HAS ZEROS AT THE 3 THIRD ROOTS OF UNITY, WHICH ARE DISTINCT, SO IT IS SQUARE FREE. HENCE, WE APPLY THE FORMULA WITH $d=3$ TO GET $\frac{(3-1)(3-2)}{2}=1$. ■

SCRATCH $\log(1+|z|)$

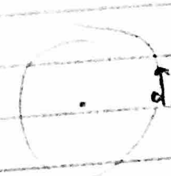
Cauchy

$$|f^{(n)}(0)| \le \frac{n!\, \max|f|}{R^n}$$

$$|a_n| \le \frac{\max|f|}{R^n}$$

$$\le \frac{C \log(1+R)}{R^n}$$

$$\lim_{R\to\infty} \frac{C \frac{1}{1+R}}{nR^{n-1}} = 0 \qquad n \ge 1$$



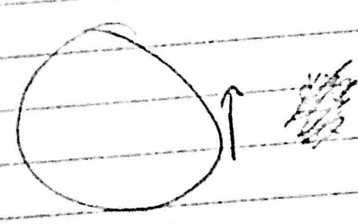
$d \nexists$ WANT TO SHOW FUNCTION IS CONSTANT.

CONTRADICTION

$\nexists$ LIOUVILLE NOT APPLICABLE b/c NOT ENTIRE.

$u = x$ POSITION $\quad \frac{\partial u}{\partial x} = 0$?

$v = y$ POSITION.



MAX MODULUS

$$i^{1/4} = e^{(i\pi/2)^{1/4}} = e^{i\pi/8}$$

$\perp$ 1, 2, 3, 4, 5, 6, 8

SCRATCH

$f \quad z \xrightarrow{f} z^2$

g CAUCHY

$g \circ f$

$$g' = \frac{(\cancel{-z}+1)i + (\cancel{iz}+i)}{(-z+1)^2} = \frac{2i}{\ }$$

2

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{(z-1)} \cdot \frac{1}{(z-1)+1} \cdot \frac{1}{(z-1)-1}$$

LET $w = z-1$ THEN $= \frac{1}{w} \cdot \frac{1}{w+1} \cdot \frac{1}{w-1}$
CENTERED AT 0.

$$\frac{1}{w-1} = -\frac{1}{1-w} = -\sum_{n=0} w^n$$

$$1$$

$$(-w^3)(-w) + (-w^2)(-w)^2 + (-w)(-w)^3$$

$$w^4 + -w^4 + -w^4$$

$$(i^{2})^{1/4} = i^{1/2} + 1$$

$$i^{1/4} = (e^{i\pi/2})^{1/4} = e^{i\pi/8} = \cos\left(\tfrac{\pi}{8}\right) + i\sin\left(\tfrac{\pi}{8}\right)$$

$$i^{1/2} = (e^{i\pi/2})^{1/2} = e^{i\pi/4} = \cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)$$

$$\frac{i^{1/4}}{i^{1/2}+1} = \frac{i^{1/4}}{i^{1/4}} \cdot \frac{1}{i^{1/4} + i^{-1/4}}$$

$$i^{1/4} = \cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)$$
$$= e^{i\pi/4}$$

$$f + \varepsilon g$$

$$\min |g| < |f|$$
$$\max g$$

Harmonic functions satisfy   Acs un VB = 1

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$