Complex Analysis Prelim Solutions Fall 2016
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1. Write three terms of the Laurent expansion of $f(z) = \dfrac{1}{z(z-1)(z+1)}$ in the annulus $1 < |z|$.

   Solution: Using the geometric series, we have

$$\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n = -1 - z - z^2 - ... \text{ for } |z| < 1,$$

and

$$\frac{1}{z+1} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n = 1 - z + z^2 - z^3 + ... \text{ for } |z| < 1.$$

So, for $0 < |z| < 1$, both series converge, and we can multiply them together to obtain

$$f(z) = \frac{1}{z}(-1 - z - z^2 - ...)(1 - z + z^2 - z^3 + ...).$$

The coefficient of $\frac{1}{z}$ in this product is $a_{-1} = -1$. The constant term in the product is $a_0 = -1 + 1 = 0$. The coefficient of z in the product is $a_1 = (1 - 1 - 1) = -1$. The coefficient of z^2 in the product is also zero. The coefficient of z^3 is $a_3 = (-1 - 1 + 1 - 1 + 1) = -1$. So, the Laurent series for f in $0 < |z| < 1$ looks like

$$f(z) = ... - \frac{1}{z} - z - z^3 + ...$$

2. Show that $\displaystyle\sum_n \frac{z^n}{n}$ converges at all points of the unit circle $|z| = 1$ except $z = 1$.

   Proof. We use Dirichlet's criterion, which says that if $\{a_n\}$ is a non-increasing sequence of real numbers converging to zero and $\{b_n\}$ is a sequence of complex numbers such that the partial sums

$$S_N := \sum_{n=1}^{N} b_n$$

are bounded uniformly in N, then $\displaystyle\sum_{n=1}^{\infty} a_n b_n$ converges.

Fix z with $|z| = 1, z \neq 1$. The sequence $a_n := \frac{1}{n}, n = 1, 2, ...$ is decreasing and converges to zero. Let $\{b_n\}$ be the complex sequence $b_n := z^n$. Then we have

$$\left| \sum_{n=1}^{N} b_n \right| = \left| \sum_{n=1}^{N} z^n \right| = \left| \frac{1 - z^{N+1}}{1 - z} \right| \leq \frac{1 + |z|^{N+1}}{|1 - z|} = \frac{2}{|1 - z|}.$$

Since $z \neq 1$, $|1-z|$ is a finite real number, and so the partial sums corresponding to b_n are bounded uniformly in N. Thus, by Dirichlet's criterion, the series

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

converges, as desired.

If $z = 1$, then

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series, which diverges. □

3. Determine the radius of convergence of the power series for $\frac{z}{\sin z}$ expanded at 0.

   Solution: Let R denote the radius of convergence of the power series. R will be equal to the radius of the largest open disk on which there is a holomorphic function agreeing with $\frac{z}{\sin z}$. Fixing $z \neq 0$ and using the power series for sin, we have

$$\frac{z}{\sin z} = \frac{z}{z - \frac{z^3}{3!} + O(z^5)} = \frac{1}{1 - \frac{z^2}{3!} + O(z^4)} := g(z).$$

   Clearly $g(z) \to 1$ as $z \to 0$, so the singularity of $\frac{z}{\sin z}$ at 0 is removable. The function $1 - \frac{z^2}{3!} + O(z^4)$ is a power series with infinite radius of convergence (for the same reason the power series for sin has infinite radius of convergence), and so defines an entire function. g will therefore be differentiable at any point where the denominator does not vanish, by the quotient rule. The denominator is the power series for the entire function $\frac{\sin z}{z}$, which vanishes when $z = n\pi$ for nonzero integer n. Therefore, we conclude that g is holomorphic on the open disk $\{z : |z| < \pi\}$, and that $\frac{z}{\sin z} \to \infty$ as $z \to \pi$. The former implies $R \geq \pi$, and the latter implies $R \leq \pi$, so we conclude $R = \pi$.

4. Show that a holomorphic function f with $|f(z)| = 1$ for all z is constant.

   Proof. By the open mapping theorem, if f were non-constant, it would be an open map, and hence map any open disk in its domain to an open set. But the image of f is contained in the unit circle, which contains no open sets, so f must be constant. □

5. Evaluate $\displaystyle\int_0^{\infty} \frac{\sqrt[3]{x}}{1+x^2}\,dx$.

   Solution: First we make a real change of variables in the integral, so that we have only integer powers involved and do not have to worry about taking a branch cut of the logarithm. Let $u = x^{1/3}$, so $du = \frac{1}{3}x^{-2/3}\,dx$ and thus $dx = 3x^{2/3}du = 3u^2du$. With this change of variables, we have

$$\int_0^{\infty} \frac{x^{1/3}}{1+x^2}\,dx = \int_0^{\infty} \frac{3u^3}{1+u^6}\,du.$$

   Our next step is to relate this integral to one along another ray from the origin in the complex plane, so that when we can "close our contour" with a circular arc. At this point, we fix $R > 0$ and consider the integral from 0 to R. Notice that the integrand is an odd function of u, so

$$\int_0^R \frac{3u^3}{1+u^6}\,du = -\int_{-R}^0 \frac{3u^3}{1+u^6}\,du = \int_{-R}^0 \frac{3(e^{i\pi/3}u)^3}{1+u^6}\,du.$$

Now, we view this as an integral in the complex plane over the line segment from $-R$ to 0 along the negative real axis. Making the change of variables $y = -e^{i\pi/3}u$, we have

$$\int_{-R}^0 \frac{3(e^{i\pi/3}u)^3}{1+u^6}\,du = \int_{L_R} \frac{3(-y)^3}{1+y^6}(-e^{-i\pi/3})\,dy = e^{-i\pi/3}\int_{L_R} \frac{3y^3}{1+y^6}\,dy,$$

where L_R is the line segment in the complex plane starting at the point $Re^{i\pi/3}$ and ending at 0. Therefore, we have

$$\int_0^R \frac{3u^3}{1+u^6}\,dy = e^{-i\pi/3}\int_{L_R} \frac{3y^3}{1+y^6}\,dy. \tag{1}$$

Let γ be the contour in the complex plane obtained by joining the line segment from the origin to the point R on the real axis to the circular arc Γ_R of radius R covering the angular sector $[0, \frac{\pi}{3}]$, and then connecting L_R to this arc, oriented counterclockwise. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \frac{3z^3}{1+z^6}$. We therefore have

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_0^R f(u)\,du + \int_{\Gamma_R} f(z)\,dz + \int_{L_R} f(z)\,dz && (2)\\ &= \left(1+e^{i\pi/3}\right)\int_0^R f(u)\,du + \int_{\Gamma_R} f(z)\,dz && (3)\end{aligned}$$

by (1). The poles of f are located at the sixth roots of -1 $\omega := e^{i\pi/6}, \omega^3, \omega^5, \omega^7, \omega^9$, and ω^{11}. The only one of these poles contained inside γ is $\omega = e^{i\pi/6}$. So, by the residue theorem, we have

$$\begin{aligned}\int_\gamma f(z)\,dz = 2\pi i\,\mathrm{Res}_\omega(f) &= 2\pi i\left(\lim_{z\to\omega}\frac{3z^3(z-\omega)}{1+z^6}\right)\\ &= 2\pi i\left(\lim_{z\to\omega}\frac{12z^3-9z^2\omega}{6z^5}\right)\\ &= 2\pi i\left(3\frac{\omega^3}{6\omega^5}\right)\\ &= \frac{\pi i}{\omega^2},\end{aligned}$$

where we used L'Hopital's rule to compute the limit.

We now estimate the integral over the circular arc. We have

$$\begin{aligned}\left|\int_{\Gamma_R} f(z)\,dz\right| &\le \sup_{z\in\Gamma_R}|f(z)|\,\mathrm{length}(\Gamma_R)\\ &\le \frac{3R^3}{R^6-1}\frac{\pi R}{3} \to 0 \text{ as } R\to\infty.\end{aligned}$$

Thus, taking the limit $R \to \infty$ in (2), we obtain

$$\frac{\pi i}{\omega^2} = (1 + e^{i\pi/3}) \int_0^\infty f(u)\, du,$$

so

$$\int_0^\infty f(u)\, du = \pi \frac{e^{i\pi/2}}{e^{i\pi/3} + e^{2\pi i/3}} = \frac{\pi}{e^{-i\pi/6} + e^{i\pi/6}} = \frac{\pi}{2\cos\left(\frac{\pi}{6}\right)} = \frac{\pi}{\sqrt{3}}.$$

So, we obtain

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\, dx = \int_0^\infty f(u)\, du = \frac{\pi}{\sqrt{3}}.$$

6. Let f be holomorphic, bounded in the upper half-plane $\mathfrak{H}$, and real-valued on $\mathbb{R}$. Show that f is constant.

   Proof. The idea is that we can restrict f to the upper half plane, then extend it back to an entire function by the Schwarz reflection principle, which will be bounded and hence constant by Liouville's theorem. This extension will be an entire function agreeing with f on an open set, and hence must agree with f everywhere f is defined by the identity principle. More explicitly, the function $F : \mathbb{C} \to \mathbb{C}$ defined by

$$F(x+iy) = \begin{cases} f(x+iy), & y \geq 0, \\ \overline{f(x-iy)}, & y < 0 \end{cases}$$

   is entire by the Schwarz reflection principle, since f is real-valued on the real axis. Since f is bounded on the upper half plane, F is bounded on $\mathbb{C}$, and hence is constant by Liouville's theorem. Since F agrees with f on the upper half-plane (and both functions are holomorphic there), by the identity principle, f must agree with F everywhere on its domain. Thus, since F is constant, f must also be constant. $\square$

7. Show that there is a holomorphic function f on the region $|z| > 2$ such that $f(z)^4 = (z^2-1)(z^2-4)$.

   Proof. There exists a holomorphic branch of the logarithm on any simply connected open set that does not contain the origin. Let L be the holomorphic logarithm defined on $\mathbb{C} \setminus [0, \infty)$ by

$$L(z) = \log|z| + i \arg z,$$

   where $0 < \arg z < 2\pi$. The function $g_2 : \mathbb{C} \setminus [2, \infty)$ defined by $g_2(z) := e^{\frac{1}{4}L(z-2)}$ is therefore holomorphic, and similarly so are the functions $g_1 : \mathbb{C} \setminus [1, \infty)$ such that $g_1(z) := e^{\frac{1}{4}L(z-1)}$, $g_{-1} : \mathbb{C} \setminus [-1, \infty)$ such that $g_{-1}(z) := e^{\frac{1}{4}L(z+1)}$, and $g_{-2} : \mathbb{C} \setminus [-2, \infty)$ such that $g_{-2}(z) := e^{\frac{1}{4}L(z+2)}$. The product $f := g_1 g_2 g_{-1} g_{-2}$ is therefore holomorphic on $\mathbb{C} \setminus [-2, \infty)$ since all four of the functions in the product are holomorphic there. Note that for $|z| > 2$, we have

$$f(z)^4 = e^{L(z-2)} e^{L(z-1)} e^{L(z+1)} e^{L(z+2)} = (z-2)(z-1)(z+1)(z-1) = (z^2-1)(z^2-4).$$

We claim first that f is continuous on $\mathbb{C} \setminus [-2,2]$. Since we know f is continuous on $\mathbb{C} \setminus [-2,\infty)$, it suffices to show that f is continuous across the ray $[2,\infty)$. So, we fix $x \in [2,\infty)$ and $\epsilon > 0$. We have

$$\begin{aligned} f(x+i\epsilon) &= g_2(x+i\epsilon)g_1(x+i\epsilon)g_{-1}(x+i\epsilon)g_{-2}(x+i\epsilon) \\ &= e^{\frac{1}{4}(L(x-2+i\epsilon)+L(x-1+i\epsilon)+L(x+1+i\epsilon)+L(x+2+i\epsilon)} \\ &= h(x)e^{\frac{1}{4}i(\arg(x-2+i\epsilon)+\arg(x-1+i\epsilon)+\arg(x+1+i\epsilon)+\arg(x+2+i\epsilon))}, \end{aligned}$$

where we have collected the terms involving the real-valued logarithm into the function $h(x)$, which is continous in its argument since the real logarithm is. Since $x > 2$, we have

$$\lim_{\epsilon \to 0^+} \arg(x-2+i\epsilon) = \lim_{\epsilon \to 0^+} \arg(x-1+i\epsilon) = \lim_{\epsilon \to 0^+} \arg(x+1+i\epsilon) = \lim_{\epsilon \to 0^+} \arg(x+2+i\epsilon) = 0,$$

and

$$\lim_{\epsilon \to 0^+} \arg(x-2-i\epsilon) = \lim_{\epsilon \to 0^+} \arg(x-1-i\epsilon) = \lim_{\epsilon \to 0^+} \arg(x+1-i\epsilon) = \lim_{\epsilon \to 0^+} \arg(x+2-i\epsilon) = 2\pi.$$

So,

$$\lim_{\epsilon \to 0+} f(x+i\epsilon) = h(x)e^{i0} = h(x),$$

and

$$\lim_{\epsilon \to 0+} f(x-i\epsilon) = h(x)e^{i\frac{1}{4}(2\pi+2\pi+2\pi+2\pi)} = h(x)e^{2\pi i} = h(x) = \lim_{\epsilon \to 0^+} f(x-i\epsilon).$$

Thus, f is continuous across $[2,\infty)$, and hence continuous on $\mathbb{C} \setminus [-2,2]$.

We now show that f is in fact holomorphic on $\mathbb{C} \setminus [-2,2]$. We could argue using the monodromy theorem, but we choose instead to show this using Morera's theorem. Any triangle contained in $\mathbb{C} \setminus [-2,2]$ is either contained in $\mathbb{C} \setminus [-2,\infty)$ or has an interior with non-empty intersection with $[2,\infty)$. Since f is holomorphic on $\mathbb{C} \setminus [-2,\infty)$, we know that the integral of f over any triangle contained in that region is zero by Goursat's theorem. Therefore, by Morera's theorem, to prove that f is holomorphic in $\mathbb{C} \setminus [-2,2]$ it suffices to show that the integral of f over any triangle whose interior intersects $[2,\infty)$ is zero.

Let T_0 be such a triangle, viewed as a contour oriented counterclockwise. Let T_1 be the smaller triangle whose bottom edge is the segment of the real axis contained inside T_0, and whose other edges coincide with segments of the the edges of T_0. Then, drawing line segments from points at which T_0 intersects the real axis to the midpoint of the bottom edge of T_0 divides the part of T_0 below the real axis into three further triangles. So, in total we have divided T_0 into 4 smaller triangles. Let T_2 be the triangle whose top edge is a line segment on the real axis, and let T_3 and T_4 be the triangles which each have a vertex lying along the real axis. We may choose the orientations for the triangles T_j so that

$$\int_{T_0} f(z)\,dz = \int_{T_1} f(z)\,dz + \int_{T_2} f(z)\,dz + \int_{T_3} f(z)\,dz + \int_{T_4} f(z)\,dz.$$

(This is a standard construction often used, for instance, in a proof of the Schwarz reflection principle.)

Fix a small $\epsilon > 0$, and let T_1^ϵ be the triangle obtained by shifting the bottom edge of T_1 by $i\epsilon$, let T_2^ϵ be the triangle obtained by shifting the top edge of T_2 by $-i\epsilon$, let $T_{3,4}^\epsilon$ be the triangle obtained by shifting the top vertex of $T_{3,4}$ by $-i\epsilon$, respectively. The triangles $T_j^\epsilon, j = 1, 2, 3, 4$ are all contained in $\mathbb{C} \setminus [-2, \infty)$, so by Goursat's theorem we have

$$\int_{T_j^\epsilon} f(z)\, dz = 0, j = 1, 2, 3, 4$$

for all ϵ (sufficiently small so that we preserve the orientation of our triangles with these shifts). By continuity of f, we have

$$\int_{T_j} f(z)\, dz = \lim_{\epsilon \to 0} \int_{T_j^\epsilon} f(z)\, dz = 0, j = 1, 2, 3, 4.$$

Therefore, we conclude

$$\int_{T_0} f(z)\, dz = 0,$$

and so f is holomorphic in $\mathbb{C} \setminus [-2, 2]$, and hence on $|z| > 2$, as desired. □

8. Show that $\dfrac{\pi^2}{\sin^2 \pi z} = \displaystyle\sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$.

Proof. Our goal is to show that the difference between these two functions is holormorphic and bounded, hence constant by Liouville's theorem, and that the constant is zero. We do this by showing that these functions have the same poles, so that when we take the difference the poles will cancel and leave us with a holomorphic function.

Let

$$f(z) = \frac{\pi^2}{\sin^2 \pi z}.$$

Recall the power series for $\sin \pi z$:

$$\sin \pi z = \pi z - \frac{\pi^3 z^3}{6} + O(z^5).$$

For sufficiently small non-zero z, we have

$$\begin{aligned} \frac{\pi^2}{\sin^2 \pi z} &= \frac{\pi^2}{\pi^2 z^2 - \frac{\pi^4 z^4}{3} + O(z^6)} \\ &= \frac{1}{z^2} \frac{1}{1 - \frac{\pi^2 z^2}{3} + O(z^4)} \\ &= \frac{1}{z^2} \left(1 + \frac{\pi^2 z^2}{3} + O(z^4) \right) \end{aligned}$$

where the last equality follows from expanding $1/(1 - a(z))$ in the previous line as a geometric series, which is valid for sufficiently small z (which follows from convergence of the power series for $\sin \pi z$). Thus, we have

$$\frac{\pi^2}{\sin^2 \pi z} = \frac{1}{z^2} + g(z)$$

where g is holomorphic in a neighborhood of zero, hence f has a pole of order 2 at $z = 0$. Let u be defined by

$$u(z) = \frac{\pi^2}{\sin^2 \pi z} - \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}.$$

Near $z = 0$, we have

$$\begin{aligned} u(z) &= \frac{1}{z^2} + g(z) - \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2} \\ &= g(z) - \sum_{n \neq 0} \frac{1}{(z-n)^2}, \end{aligned}$$

which is holomorphic in a neighborhood of $z = 0$. By periodicity, $\frac{\pi^2}{\sin^2(\pi z)}$ has a pole of order 2 with coefficient 1 at each $n \in \mathbb{Z}$, exactly matching the pole in the infinite sum, so it follows that u has no poles in $\mathbb{C}$, and hence is entire.

Now we show that u is bounded and hence constant by Liouville's theorem. Since both terms in u are periodic in the sense that they satisfy $f(z+n) = f(z)$ for $n \in \mathbb{Z}$, first we know that u is bounded on the real axis (as a continuous periodic function), and also it suffices to show that u is bounded on the strip $0 \leq x \leq 1, y > 0$, where $z = x + iy$. Furthermore, u is continuous and hence bounded on compact sets, so it suffices to show that for each fixed $x \in [0, 1]$, $\lim_{y \to \infty} u(x + iy)$ is bounded. So, fix $x \in [0, 1]$.

For the second term in u, we have

$$\left| \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2} \right| \leq \sum_{n \in \mathbb{Z}} \frac{1}{|z-n|^2} = \sum_{n \in \mathbb{Z}} \frac{1}{|x-n+iy|^2} = \sum_{n \in \mathbb{Z}} \frac{1}{(x-n)^2 + y^2}.$$

For $y > 0$, the sum on the right hand side is convergent, and goes to zero as $y \to \infty$, for instance by the dominated convergence theorem.

For the first term in u, we have

$$|\sin^2 \pi z| = \left| \frac{e^{-\pi y} e^{i\pi x} - e^{\pi y} e^{-i\pi x}}{2i} \right| \to \infty \text{ as } y \to \infty$$

by continuity of the norm. Therefore,

$$\left| \frac{\pi^2}{\sin^2 \pi z} \right| \to 0 \text{ as } y \to \infty.$$

Combining these results, we obtain

$$\lim_{y \to \infty} |u(x + iy)| = 0.$$

So u is bounded, and therefore constant by Liouville's theorem. Since this limit is zero, this constant must also be zero, so we conclude $u(z) \equiv 0$, and hence

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2},$$

as desired. $\square$

9. Show that the curve described by $w^2 = z^4 + 1$, with points added at infinity as needed, has genus 1, that is, is an elliptic curve.

Proof. (The following is taken more or less directly from email correspondence with Paul Garrett). Equations of the form $y^2 = f(x)$, where f is a polynomial with no repeated roots define *hyper-ellptic curves.* If d is the degree of f, then the genus g of a curve defined by such an equation satisfies $2g + 2 = d$ if d is even and $2g + 1 = d$ if d is odd (this can be proven from the Riemann-Hurwitz formula). In this case, we have $f(z) = z^4 + 1$, which has no repeated roots and degree 4, and hence the curve described by $w^2 = z^4 + 1$ has genus $\frac{4-2}{2} = 1$, as desired. $\square$