Complex Analysis Prelim Solutions Fall 2017
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1. Write three terms of the Laurent expansion of $f(z) = \dfrac{e^z}{z-1}$ centered at 0 and convergent in $|z| > 1$.

   Solution: Using the geometric series, we write

   $$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + O(z^{-4}) \text{ for } |z| > 1.$$

   The power series for the exponential converges everywhere in $\mathbb{C}$, so we have

   $$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + O(z^4) \text{ for all } z \in \mathbb{C}.$$

   We can multiply these two power series in their common domain of convergence to obtain

   $$f(z) = (1 + z + \frac{z^2}{2} + \frac{z^3}{6} + O(z^4))\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + O(z^{-4})\right) \text{ for } |z| > 1.$$

   The coefficient of $\frac{1}{z}$ in this product is

   $$a_{-1} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + ... = \sum_{n=0}^{\infty}\frac{1}{n!} = e.$$

   The constant term in this product is

   $$a_0 = 1 + \frac{1}{2} + \frac{1}{6} + ... = \sum_{n=1}^{\infty}\frac{1}{n!} = e - 1.$$

   The coefficient of z in this product is

   $$a_1 = \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + ... = \sum_{n=2}^{\infty}\frac{1}{n!} = e - 2.$$

   So, the Laurent series of $f(z)$ centered at zero convergent in $|z| > 1$ looks like

   $$f(z) = ... + \frac{e}{z} + e - 1 + (e-2)z + ...$$

2. Determine the radius of convergence of the power series for $\log z$ centered at $z_0 = -3+4i$.

   Solution: Let R denote the desired radius of convergence. We use the characterization that a power series for a given function will converge on the largest disk on which there is a holomorphic function agreeing with the given function.

   There cannot be holomorphic logarithm defined at $z = 0$, since such a function would have to satisfy $e^w = 0$ for some $w \in \mathbb{C}$, but there is no such w. The distance from z_0 to 0 is 5, so we obtain that R is bounded above by 5.

   There exists a holomorphic logarithm on any simply connected domain which does not include zero. In particular, we can define a holomorphic logarithm on $\mathbb{C}$ with the ray from 0 to $3 - 4i$ removed. This logarithm is holomorphic in the disk centered at z_0 with radius 5, so we conclude that $R = 5$.

3. Show that $\frac{\sin\sqrt{z}}{\sqrt{z}}$ is *entire*.

   Proof. Note that if we formally expand $\sin\sqrt{z}$ near zero, we obtain

$$\begin{aligned}\frac{\sin\sqrt{z}}{\sqrt{z}} &= \frac{z^{1/2} - \frac{z^{3/2}}{3!} + \frac{z^{5/2}}{5!} - ...}{z^{1/2}} \\ &= 1 - \frac{z}{3!} + \frac{z^2}{5!} - ... \\ &:= g(z).\end{aligned}$$

   This power series has infinite radius of convergence (for the same reason that the power series for sin does, since the coefficients are inherited from the coefficients for sin), so defining $g : \mathbb{C} \to \mathbb{C}$ by this power series, we see that g is entire. Our goal is then to make sense of this formal expansion, so that we may identify $\frac{\sin\sqrt{z}}{\sqrt{z}}$ with the entire function $g(z)$.

   Fix any nonzero point $z_0 \in \mathbb{C}$. Define a square root function with branch cut away from z_0, so that $\sqrt{z_0}$ is a fixed non-zero complex number. We can plug this number into the power series for sin to obtain

$$\frac{\sin(\sqrt{z_0})}{\sqrt{z_0}} = 1 - \frac{z_0}{3!} + \frac{z_0^2}{5!} + ... = g(z_0)$$

   as above. Since the expression on the right hand side does not contain any square roots, we see that the result is independent of our choice of the branch of the square root function. Thus, for any nonzero $z_0 \in \mathbb{C}$ and *any* subsequent choice of a square root function so that $\sqrt{z_0}$ is defined, we have

$$\frac{\sin\sqrt{z_0}}{\sqrt{z_0}} = g(z_0).$$

   We may extend this agreement to $z_0 = 0$ by defining the left hand side to be equal to $g(0) = 1$ there, and by the identity principle we therefore identify $\frac{\sin(\sqrt{z})}{\sqrt{z}}$ with the entire function $g(z)$, as desired. □

4. Show that a holomorphic function f on a non-empty open set with $|f|$ constant is itself constant.

   Proof. Let $|f| \equiv C$ for some real constant C, and let Ω denote the domain of f. Then the image of f is a subset of the circle of radius C. The image of Ω under f is not open, since the circle of radius R contains no open disks of positive radius. Therefore, f does not map the open set Ω to an open set. If f were non-constant, this would contradict the open mapping theorem, so we conclude that f must be constant. □

5. Evaluate

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx.$$

Solution: First we make a real change of variables in the integral, so that we have only integer powers involved and do not have to worry about taking a branch cut of the logarithm. Let $u = x^{1/3}$, so $du = \frac{1}{3}x^{-2/3}\,dx$ and thus $dx = 3x^{2/3}du = 3u^2du$. With this change of variables, we have

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx = \int_0^\infty \frac{3u^3}{1+u^6}\,du.$$

Our next step is to relate this integral to one along another ray from the origin in the complex plane, so that when we can "close our contour" with a circular arc. At this point, we fix $R > 0$ and consider the integral from 0 to R. Notice that the integrand is an odd function of u, so

$$\int_0^R \frac{3u^3}{1+u^6}\,du = -\int_{-R}^0 \frac{3u^3}{1+u^6}\,du = \int_{-R}^0 \frac{3(e^{i\pi/3}u)^3}{1+u^6}\,du.$$

Now, we view this as an integral in the complex plane over the line segment from $-R$ to 0 along the negative real axis. Making the change of variables $y = -e^{i\pi/3}u$, we have

$$\int_{-R}^0 \frac{3(e^{i\pi/3}u)^3}{1+u^6}\,du = \int_{L_R} \frac{3(-y)^3}{1+y^6}(-e^{-i\pi/3})\,dy = e^{-i\pi/3}\int_{L_R} \frac{3y^3}{1+y^6}\,dy,$$

where L_R is the line segment in the complex plane starting at the point $Re^{i\pi/3}$ and ending at 0. Therefore, we have

$$\int_0^R \frac{3u^3}{1+u^6}\,dy = e^{-i\pi/3}\int_{L_R} \frac{3y^3}{1+y^6}\,dy. \tag{1}$$

Let γ be the contour in the complex plane obtained by joining the line segment from the origin to the point R on the real axis to the circular arc Γ_R of radius R covering the angular sector $[0, \frac{\pi}{3}]$, and then connecting L_R to this arc, oriented counterclockwise. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \frac{3z^3}{1+z^6}$. We therefore have

$$\int_\gamma f(z)\,dz = \int_0^R f(u)\,du + \int_{\Gamma_R} f(z)\,dz + \int_{L_R} f(z)\,dz \tag{2}$$

$$= \left(1 + e^{i\pi/3}\right)\int_0^R f(u)\,du + \int_{\Gamma_R} f(z)\,dz \tag{3}$$

by (1). The poles of f are located at the sixth roots of -1, which are $\omega := e^{i\pi/6}, \omega^3, \omega^5, \omega^7, \omega^9$, and ω^{11}. The only one of these poles contained inside γ is $\omega = e^{i\pi/6}$. So, by the residue theorem, we have

$$\begin{aligned}\int_\gamma f(z)\,dz = 2\pi i\,\mathrm{Res}_\omega(f) &= 2\pi i\left(\lim_{z\to\omega}\frac{3z^3(z-\omega)}{1+z^6}\right)\\ &= 2\pi i\left(\lim_{z\to\omega}\frac{12z^3 - 9z^2\omega}{6z^5}\right)\\ &= 2\pi i\left(3\frac{\omega^3}{6\omega^5}\right)\\ &= \frac{\pi i}{\omega^2},\end{aligned}$$

where we used L'Hopital's rule to compute the limit.

We now estimate the integral over the circular arc. We have

$$\begin{aligned}\left|\int_{\Gamma_R} f(z)\,dz\right| &\le \sup_{z\in\Gamma_R} |f(z)|\ \text{length}(\Gamma_R)\\ &\le \frac{3R^3}{R^6-1}\frac{\pi R}{3} \to 0 \text{ as } R\to\infty.\end{aligned}$$

Thus, taking the limit $R\to\infty$ in (2), we obtain

$$\frac{\pi i}{\omega^2} = (1+e^{i\pi/3})\int_0^\infty f(u)\,du,$$

so

$$\int_0^\infty f(u)\,du = \pi\frac{e^{i\pi/2}}{e^{i\pi/3}+e^{2\pi i/3}} = \frac{\pi}{e^{-i\pi/6}+e^{i\pi/6}} = \frac{\pi}{2\cos\left(\frac{\pi}{6}\right)} = \frac{\pi}{\sqrt{3}}.$$

So, we obtain

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx = \int_0^\infty f(u)\,du = \frac{\pi}{\sqrt{3}}.$$

6. Let f be holomorphic in the open upper half-plane, with $|f(x+iy)|\le y$ for $y\ge 0$. Show that f is a constant.

Proof. Fix $x\in\mathbb{R}$, and pick any sequence $y_n, n=1,2,...$ of positive real numbers converging to zero. Then we have

$$|f(x+iy_n)|\le y_n.$$

The right hand-side converges to zero as $n\to\infty$. Since this holds for any such sequence y_n, we conclude

$$\lim_{y\to 0^+}|f(x+iy)| = 0.$$

Therefore, f can be continuously extended to the real line by assigned its value to be zero along the real line. Thus, by the Schwarz reflection principle, the function $F:\mathbb{C}\to\mathbb{C}$ defined by

$$F(x+iy) = \begin{cases} f(x+iy), & y>0,\\ 0, & y=0\\ \overline{f(x-iy)}, & y<0\end{cases}$$

is entire. But F is zero along the entire real line, which has accumulation points, so by an analytic continuation argument, F must be identically zero. Since F agrees with f on the upper half plane, we must also have $f\equiv 0$, as desired. □

7. Show that $\frac{\sin\pi z}{\pi z} = \prod_{n\ge 1}\left(1-\frac{z^2}{n^2}\right)$.

Proof. We take for granted the infinite sum expansion

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}.$$

We will use this to derive the infinite product, by first writing an expansion for $\pi \cot \pi z$. We claim that

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n\geq 1} \frac{1}{z-n} + \frac{1}{z+n} := g(z). \tag{4}$$

First, notice that

$$\frac{d}{dz}(\cot \pi z) = \frac{d}{dz}\left(\frac{\cos \pi z}{\sin \pi z}\right) = -\frac{\pi}{\sin^2 \pi z},$$

so

$$\frac{d}{dz}(-\pi \cot \pi z) = \frac{\pi^2}{\sin^2 \pi z}.$$

Our goal is to show that g has the same derivative as $\pi \cot \pi z$, and conclude that their difference is constant, then show the constant is zero. Rewriting the infinite sum in the definition of g, we see

$$\sum_{n\geq 1} \frac{1}{z-n} + \frac{1}{z+n} = \sum_{n\geq 1} \frac{2z}{z^2-n^2},$$

from which it is clear that the sum converges uniformly on compact subsets of $\mathbb{C}$ which do not include any integers. This is sufficient to conclude convergence of the termwise derivative, so we obtain

$$g'(z) = -\frac{1}{z^2} + \sum_{n\geq 1} -\frac{1}{(z-n)^2} - \frac{1}{(z+n)^2} = -\frac{\pi^2}{\sin^2 \pi z} = \frac{d}{dz}(\pi \cot \pi z).$$

Thus, we have

$$\frac{d}{dz}(g(z) - \pi \cot \pi z) = 0,$$

and so

$$g(z) = \pi \cot \pi z + C.$$

Both $\pi \cot \pi z$ and $g(z)$ are odd as functions of z, so the constant C must be zero, and we have proved the claim.

Now, notice that

$$\frac{d}{dz}(\log(\sin \pi z)) = \frac{\pi \cos \pi z}{\sin \pi z} = \pi \cot \pi z.$$

Our goal is to use this fact to integrate (4) and thereby convert the infinite sum into an infinite product. We also have

$$\frac{d}{dz} \log\left(1 \pm \frac{z}{n}\right) = \frac{\pm\frac{1}{n}}{1 \pm \frac{z}{n}} = \frac{1}{z \pm n}.$$

We claim that

$$\log(\sin \pi z) = C + \log z + \sum_{n\geq 1} \log\left(1 - \frac{z}{n}\right) + \log\left(1 + \frac{z}{n}\right). \tag{5}$$

The right hand side converges uniformly on compact subsets of $\mathbb{C}$ which do not include the integers (one can see this using the limit comparison test and the fact that $\sum \frac{z^2}{n^2}$ converges uniformly on compact sets), so we can differentiate term by term to get

$$\begin{aligned}
\frac{d}{dz}\left(C + \log z + \sum_{n\geq 1} \log\left(1 - \frac{z}{n}\right) + \log\left(1 + \frac{z}{n}\right)\right) &= \frac{1}{z} + \sum_{n\geq 1} \frac{1}{z-n} + \frac{1}{z+n} \\
&= \pi \cot \pi z \\
&= \frac{d}{dz}(\log(\sin \pi z)),
\end{aligned}$$

which proves the claim. Exponentiating (5) (and using the fact that the exponential is continuous to exponentiate the infinite sum), we obtain

$$\begin{aligned}
\sin \pi z &= e^C z \prod_{n\geq 1} \left(1 - \frac{z}{n}\right)\left(1 + \frac{z}{n}\right) \\
&= e^C z \prod_{n\geq 1} \left(1 - \frac{z^2}{n^2}\right).
\end{aligned}$$

By matching the linear term in the power series for sin, we conclude that $e^C = \pi$, and so we have the desired result. □

8. Make a change of coordinates to put the elliptic curve $w^2 = z^4 + 1$ into the (essentially) Weierstrass form $y^2 = x^3 + bx + c$.

   Solution: Define $\tilde{z} = e^{-i\pi/4} z$, so that $z^4 = -\tilde{z}^4$, and let $\tilde{w} = iw$, so that $w^2 = -\tilde{w}^2$. (Note: I made this change of variables because I messed up and factored z^4+1 as z^4-1; making this change of variables at the beginning fixes the rest of the solution, but is not necessary if one factors correctly in the first place. The following method applies directly to $w^2 = z^4 + 1$ as well, with the roots replaced appropriately.)

   The equation for the curve then becomes

   $$\tilde{w}^2 = \tilde{z}^4 - 1.$$

   Dropping tildes, we factor

   $$z^4 - 1 = (z+i)(z-i)(z+1)(z-1).$$

   Our first step is to send one of the roots of $z^4 + 1$ to infinity with a fractional linear transformation. This ends up having the effect of reducing the degree of the polynomial from 4 to 3. To do this, we let

   $$x = \frac{z}{1-z}$$

   so that

   $$z = \frac{x}{1+x}.$$

Substituting this into the equation $w^2 = z^4 + 1$, we obtain

$$\begin{aligned} w^2 &= \left(\frac{x}{1+x} + \frac{i(1+x)}{1+x}\right)\left(\frac{x}{1+x} - \frac{i(1+x)}{1+x}\right)\left(\frac{x}{1+x} + \frac{1+x}{1+x}\right)\left(\frac{x}{1+x} - \frac{1+x}{1+x}\right) \\ &= \left(\frac{(1+i)x+i}{1+x}\right)\left(\frac{(1-i)x-i}{1+x}\right)\left(\frac{2x+1}{1+x}\right)\left(-\frac{1}{1+x}\right). \end{aligned}$$

Now, let $y = (1+x)^2 w$, so that $w^2 = \frac{y^2}{(1+x)^4}$. Then, multiplying through by $(1+x)^4$, we obtain

$$\begin{aligned} y^2 &= ((1+i)x+i)((1-i)x-i)(2x+1)(-1) \\ &= -(2x^2+2x+1)(2x+1) \\ &= -4x^3 - 6x^2 - 4x - 1. \end{aligned}$$

Now, we still must clear the leading coefficient and then eliminate the quadratic term. First, let $\tilde{y} = \frac{iy}{2}$, so that

$$\tilde{y}^2 = x^3 + \frac{3}{2}x^2 + x + \frac{1}{4}.$$

We can eliminate the quadratic term by letting $\tilde{x} = x + b$ and choosing b appropriately. We obtain

$$\tilde{y}^2 = (\tilde{x}-b)^3 + \frac{3}{2}(\tilde{x}-b)^2 + (\tilde{x}-b) + \frac{1}{4}.$$

The only quadratic terms in $\tilde{x}$ are $-3b\tilde{x}^2$ and $\frac{3}{2}\tilde{x}^2$, so choosing $b = \frac{1}{2}$ makes these terms cancel. Expanding the right hand side, and dropping $\sim$'s in our notation, we obtain

$$\begin{aligned} y^2 &= x^3 - \frac{3}{2}x^2 + \frac{3}{4}x + \frac{1}{4} + \frac{3}{2}\left(x^2 - x - \frac{1}{4}\right) + x - \frac{1}{2} + \frac{1}{4} \\ &= x^3 + \frac{1}{4}x - \frac{3}{8}, \end{aligned}$$

which is exactly the desired form with $b = \frac{1}{4}$ and $c = -\frac{3}{8}$.