Spring 2015 Complex Analysis Preliminary Exam

University of Minnesota

1. Describe all values of $(-1)^i$ where $i = \sqrt{-1}$.

   Proof. In general, we use a choice of logarithm to describe exponentiation, i.e. $z^\alpha = e^{\alpha \log z}$. Additionally, on the principle branch, if we have $z = re^{i\theta}$, then $\log z = \log(r) + i\theta$. So we compute

$$
\begin{aligned}
(-1)^i &= e^{i \log(-1)} \\
&= e^{i \log(e^{i(\pi+2\pi k)})} \\
&= e^{i(\log(1)+i(\pi+2\pi k))} \\
&= e^{-(\pi+2\pi k)} \\
&= -1
\end{aligned}
$$

$\square$

2. Write three terms of the Laurent expansion of $f(x) = \frac{1}{z(z-1)(z-2)}$ in the annulus $1 < |z| < 2$.

   Proof.

$$
\begin{aligned}
\frac{1}{z(z-1)(z-2)} &= \frac{1}{z} \cdot \frac{-1}{2(1-\frac{z}{2})} \cdot \frac{1}{z(1-\frac{1}{z})} \\
&= \frac{-1}{2z^2} \cdot \left(\sum_{n \geq 0} (z/2)^n \right) \cdot \left(\sum_{n \geq 0} (1/z)^n \right) \\
&= \frac{-1}{2z^2} \left(\sum_{n=-\infty}^{\infty} \sum_{i-j=n} (z/2)^i (1/z)^j \right) \\
&= \frac{-1}{2z^2} \left(\sum_{n=-\infty}^{\infty} \sum_{i-j=n} (1/2)^i z^{i-j} \right) \\
&= \frac{-1}{2z^2} \left(\sum_{n=-\infty}^{\infty} z^n \sum_{i-j=n} (1/2)^i \right) \\
&= \frac{-1}{2z^2} \left(\cdots + \sum_{i-j=0} \frac{1}{2} + z \sum_{i-j=1} \frac{1}{2} + z^2 \sum_{i-j=2} \frac{1}{2} + \cdots \right) \\
&= \frac{-1}{2z^2} \left(\cdots + \frac{1}{1-1/2} + z \frac{1}{1-1/2} + z^2 \frac{1}{1-1/2} + \cdots \right) \\
&= \frac{-1}{2z^2} \left(\cdots + 2 + 2z + 2z^2 + \cdots \right) \\
&= \frac{-1}{z^2} \left(\cdots + 1 + z + z^2 + \cdots \right) \\
&= \left(\cdots - z^{-2} - z^{-1} - 1 + \cdots \right)
\end{aligned}
$$

□