Fall 2020 Complex Analysis Preliminary Exam

University of Minnesota

This is a verbatim transcription of an exam which received a score of 89/100. Mistakes are intentionally included.

1. Give a conformal mapping from the half-disk $H = \{z : |z| < 1 \text{ and } \operatorname{Im}(z) > 0\}$ to the upper half-plane $\mathfrak{h} = \{z : \operatorname{Im}(z) > 0\}$.

   Proof. Define $f : H \to D$ by $z \mapsto z^2$ where D is the unit disk, slit along $\mathbb{R}_{\geq 0}$. Define $g : D \to \mathfrak{h}$ by the Cayley map $\frac{iz+i}{-z+1}$. We clam [sic] $g \circ f$ is the desired mapping. Since $\arg(z) \in (0, \pi)$ for all $z \in H$ and $|z| < 1$, then $\arg(z) \in (2 \cdot 0, 2\pi)$ and $|z^2| < 1^2$, so f is inded [sic]. Note $f'(z) = 2z$ is only zero at $z = 0$, but $0 \notin H$, so it is indeed conformal. The derivatve [sic] of g is $g' = \frac{2i}{(1-z)^2}$, which is nonzero on its domain, which is $\mathbb{C} \backslash \{1\}$. Moreover $f(H) \not\ni 1$. Thus, $g'(f) \neq 0$, and so the composition $g \circ f$ is conformal.

   To see f is onto, let $z \in D$. Then $\arg(z) \in (0, 2\pi)$ and $|z| \leq 1$ so $|\sqrt{z}| \leq 1$ (choosing the principal branch of $\sqrt{\bullet}$) and $\arg(\sqrt{z})$ is in $(0, \pi)$ so f is onto. To see g is onto, note that its explicit inverse is $\frac{z-i}{z+i}$, which takes points closer to i than to $-i$ (i.e. the upper half plane) to a point in the slit disk. $\square$

2. Write three (non-zero) terms of the Laurent expansion of $f(z) = \frac{1}{z(z-1)(z-2)}$ centered at 1 and convergent in $0 < |z - 1| < 1$.

   Proof. First, apply the change of coordinates $w = z - 1$. Then we can write $f(z) = f(w + 1) = \frac{1}{w(w+1)(w-1)}$ and we seek the Laurent expansion of $f(w + 1)$ centered at $w = 0$. Using geometric series, this becomes

   $$\frac{1}{w(w+1)(w-1)} = \frac{1}{w}\left(-\sum_{n\geq 0} w^n\right)\left(\sum_{n\geq 0}(-w)^n\right).$$

   Since $\frac{1}{w-1} = -\sum_{n\geq 0} w^n$ for $|w| < 1$, and $\frac{1}{w+1} = \frac{1}{1-(-w)} = \sum_{n\geq 0}(-w)^n$ for $|w| < 1$. Then multiplying, we get

   $$\begin{aligned} f(w+1) &= \frac{1}{w}(-1)(1) + \frac{1}{w}(-w + (+w)) \\ &+ \frac{1}{w}(-w^2 + (-w)(-w) + (-w)^2) \\ &+ \frac{1}{w}(-w^3 + (-w^2)(-w) + (-w)(-w)^2 + (-w)^3) \\ &+ \frac{1}{w}(-w^4 + (-w^3)(-w) + (-w^2)((-w)^2) + (-w)(-w)^3 + (-w)^4) \\ &+ \cdots \\ &= \frac{-1}{w} + 0 + w + 0 + (-1)w^3 + \cdots \end{aligned}$$

   Undoing our change of coordinates, we get

   $$f(z) = \frac{-1}{z-1} + (z-1) - (z-1)^3 + \cdots$$

   Since it converged for $|w| < 1$, it also converges for $|z - 1| < 1$. $\square$

3. Let f be an entire function taking real values on the real line. Show that, for all complex z, $\overline{f(z)} = f(\overline{z})$.

   Proof. The Schwarz reflection principle states that if Ω is a region which is symmetric about the real-axis, and f is a function hollomorphic [sic] in $\Omega \cap \{z : \operatorname{Im}(z) > 0\}$ which has a continuation onto $\mathbb{R} \cap \Omega$ and that continuation is real-valued, then there is a holomorphic function F on Ω st $F = f$ on $\Omega \cap \{z : \operatorname{Im}(z) > 0\}$ and moreover, $F(z) = \overline{f(\overline{z})}$. Note that we may take $\Omega = \mathbb{C}$ and the continuation to be just the real values on $\mathbb{R}$ which we know it takes. Then $\overline{F(z)} = \overline{f(z)} = \overline{\overline{f(\overline{z})}} = f(\overline{z})$. $\square$

4. Classify entire functions f such that there is a constant (possibly depending on f) such that $|f(z)| \leq C \cdot \log(1+|z|)$.

Proof. If f is entire it admits a powder [sic] series representaton [sic] centered at 0, so

$$f(z) = \sum_{n \geq 0} \alpha_n z^n$$

Cauchy's inequality tells us that on a circle of radius R about z_0, call it γ_R, that

$$|f^{(n)}(z_0)| \leq \frac{n! \max_{z \in \gamma_R} |f(z)|}{R^n}$$

to extract the coefficients α_n from f note that

$$\alpha_n = f^{(n)}(0)/n!$$

so taking $z_0 = 0$, we get

$$|\frac{f^{(n)}(0)}{n!}| = |\alpha_n| = \leq \frac{\max_{z \in \gamma_R} |f(z)|}{R^n}$$

The provided bound tells us that $\max_{C_R} |f(z)| \leq C \cdot \log(1+R)$ and so

$$|\alpha_n| \leq \frac{C \log(1+R)}{R^n}$$

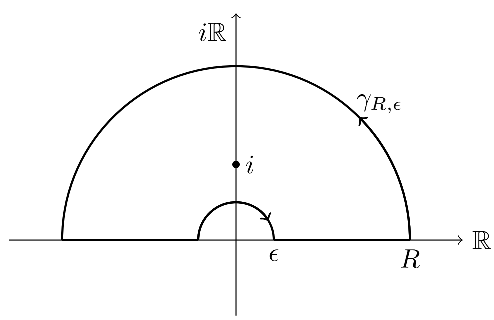
This is independent of R, so we take the limit

$$\lim_{R \to \infty} \frac{Clog(1+R)}{R^n} = \lim_{R \to \infty} \frac{C \frac{1}{1+R}}{nR^{n-1}} \quad \text{(by L'Hôpitals [sic] rule)}$$

which vanishes for $n \geq 1$. So f is constant, but we can do better. If f is constant, $f(z) \equiv f(0)$. So $|f| \leq C \log(1+|0|) = 0$, so f is identically 0 on $\mathbb{C}$. $\square$

5. Evaluate $\int_0^\infty \frac{x^{1/2}}{1+x^2} dx$

Proof. We compute by passing to $\mathbb{C}$ and defing [sic] $f(x) = \frac{z^{(1/4)}}{1+z^2}$. Choose $z^{1/4}$ to be defined using the branch cut from the origin through $-i$. In particular f is not defined at zero. Define the curve $\gamma_{R,\epsilon}$ for $R > 1$, $0 < \epsilon < 1$ as the union of the circles of radii R, ϵ with counter clockwise and clockwise orientations respectively, and connect them along $[-R, \epsilon]$[1] and $[\epsilon, R]$, to get



The residue theorem states that

$$\int_{\gamma_{R,\epsilon}} f dz = 2\pi i \sum_{\substack{\text{poles} \\ \text{inside} \\ \gamma_{R,\epsilon}}} \operatorname{res}_{z_0}(f)$$

The only pole of f inside $\gamma_{R,\epsilon}$ is the one at $z = i$ which is simple. So we may compute

$$\begin{aligned} \operatorname{res}_i(f) &= \lim_{z \to i} (z-i) f(z) \\ &= \frac{i^{1/4}}{2i} \end{aligned}$$

and so $\int_{\gamma_{R,\epsilon}} f dz = \pi i^{1/4}$. Now observe that, if C_R^+ represents the circle of radius R in the upper half plane, then

[1] This should be $-\epsilon$.

$$\int_{\gamma_{R,\epsilon}} f dz = \int_{C_R^+} f dz - \int_{C_\epsilon^+} f dz + \int_{-R}^{-\epsilon} f dz + \int_{\epsilon}^{R} f dz$$

where we subtract the integral on C_ϵ^+ because of the choice of orientation. We will show that the circular parts vanish in the limit, since $\int_{\gamma_{R,\epsilon}} f dz$ was independent of R, ϵ.

The estimation lemma gives the bound

$$\begin{aligned} || &\leq \text{Length}(C_R^+) \max_{|z|=R} |f(z)| \\ &= \pi R \max_{|z|=R} \left| \frac{z^{1/4}}{z^2+1} \right| \\ &= \pi R^{5/4} \max_{|z|=R} \left| \frac{1}{z^2+1} \right| \end{aligned} \tag{1}$$

Note that $\left|\frac{1}{z^2+1}\right|$ is maximized when $|z^2 - (-1)|$ is minimized. Since $|z| = R$ is closes to -1 at $R = -z$, we can write

$$(1) = \pi R^{5/4} \frac{1}{(-R)^2 + 1}$$

Since $5/4 < 2$, (1) vanishes upon taking $R \to \infty$.

The same computation shows that

$$\left| \int_{C_\epsilon^+} f(z) dz \right| \leq \pi \epsilon^{5/4} / (\epsilon^2 + 1)$$

as $\epsilon \to 0$, the numerator vanishes and the denominator $\to 1$. So

$$\pi i^{1/4} = \int_{\gamma_{\infty,0}} f(z) dz = \int_{-\infty}^{0} f dz + \int_{0}^{\infty} f dz$$

A change of variables gives us

$$\int_{-\infty}^{0} f(z) dz = \int_{\infty}^{0} f(-z)(-1) dz = \int_{0}^{\infty} f(-z) dz.$$

But $f(-z) = (-1)^{1/4} f(z)$ so we get

$$\pi i^{1/4} = \left((-1)_1^{1/4}\right) \int_0^\infty f(z) dz$$

so

$$\begin{aligned} \int_0^\infty f(z) dz = \pi i^{1/4} / ((-1)^1/4 + 1) &= \pi \frac{i^{1/4}}{i^{1/2} + 1} \\ &= \pi / (i^{1/4} + i^{-1/4}) \end{aligned}$$

But note that

$$\begin{aligned} i^{1/4} &= \cos(\pi/8) + i \sin(\frac{\pi}{8}) \\ i^{-1/4} &= \cos(\pi/8) - i \sin(\pi/8) \end{aligned}$$

so

$$\int_0^\infty f(z) dz = \boxed{\frac{\pi}{2\cos(\pi/8)}}$$

□

6. Let f, g be holomorphic functions on $\{z : |z| < 2\}$ with f nonvanishing on $|z| - 1$. Show that for all sufficiently small $\epsilon > 0$ the function $f + \epsilon g$ has the same number of zeros inside $|z| = 1$ as does f.

Proof. Rouche's theorem tells us that if f, h holomorphic on and inside (eg) the unit disk, and $|f| > |h|$ on all of the boundary $|z| = 1$, then $f, f + h$ have the same number of zeros inside the unit disk. Since the boundary is compact (closed and bounded), $|f|, |g|$ achieve both a maximum and a minimum on $|z| = 1$. Let $m := \min_{|z|=1} |f|$ and $M := \max_{|z|=1} |g|$. Then $|\frac{g}{M}| \leq 1$ and so

$$\left| \frac{mg}{M} \right| \leq |f| \quad \text{on } |z| = 1.$$

then take $\epsilon < m/M$ and $h = \epsilon g$ in the statement of Rouche's theorem. □

7. Describe all harmonic functions on the punctured unit disk $\{z : 0 < |z| < 1\}$, continuous on the punctured closed disk $\{z : 0 < |z| \leq 1\}$ whose restriction to $\{z : |z| = 1\}$ is the zero function.

   WARNING: The folllowing solution is incorrect. See this StackExchange post:

   https://math.stackexchange.com/questions/376677/characterization-of-harmonic-functions-on-the-punctured-disk

   Proof. The maximum modulus principle tells us that f is holomorphic on $|z| < 1$, then $|f|$ does not attain a maximum on $|z| < 1$, and so any maximum must occur on $|z| = 1$. If a function f is harmonic on $\{z : 0 < |z| < 1\}$, it admits an analytic continuation on $\{z : |z| \leq 1\}$ call this continuation F. Then $|F|$ attains its maximum on $|z| = 1$. Since $F = 0$ on $|z| = 1$, $|F| \leq 0$ on the whole punctured disk, and so f must be identically 0. □

8. Show that the curve $z^3 + w^3 = 1$ has genus 1.

   Proof. The degree-genus formula tells us that if we have a smooth, irreducible plane curve, then its genus is

   $$\frac{(d-1)(d-2)}{2}$$

   where d is the degree of the curve. To see that the given curve is smooth, homogenize so we are dealing with $p = z^3 + w^3 - u^3 = 0$. The partial derivatives $\frac{\partial}{\partial z}p = 3z^2$, $\frac{\partial}{\partial w}p = 3w^2$, $\frac{\partial}{\partial u}p = -3u^2$ are only simultaneously 0 when (in homogeneous coordinates) we are at the point $[0 : 0 : 0]$. But that point is never on our given curve which looks like $[* : * : 1]$. To see it is irreducible, note that $-w^3 + 1$ has zeros at the 3 third roots of unity, which are distinct, so it is squarefree. Hence, we apply the formula with $d = 3$ to get d$\frac{(3-1)(3-2)}{2} = 1$ □