Complex Analysis Prelim Solutions Spring 2017
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1. Write three terms of the Laurent expansion of $f(z) = \dfrac{e^z - 1}{z(z-1)}$ centered at 0 and convergent in $|z| > 1$.

   Solution: Using the geometric series, we write

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ... \text{ for } |z| > 1.$$

   The power series for the exponential has infinite radius of convergence, so we have for all z

$$e^z - 1 = \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...\right) - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...$$

   We can multiply these infinite series together in their mutual domain of convergence to obtain

$$f(z) = \frac{1}{z}\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...\right)\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ...\right) \text{ for } |z| > 1.$$

   The coefficient of $\frac{1}{z}$ in this product is

$$a_{-1} = 1 + \frac{1}{2!} + \frac{1}{3!} + ... = e - 1.$$

   The constant term in this product is

$$a_0 = \frac{1}{2!} + \frac{1}{3!} + ... = e - 2.$$

   The coefficient of z in this product is

$$a_1 = \frac{1}{3!} + \frac{1}{4!} + ... = e - \frac{5}{2}.$$

   So the Laurent series for f centered at 0 and convergent in $|z| > 1$ looks like

$$f(z) = ... + (e-1)\frac{1}{z} + (e-2) + \left(e - \frac{5}{2}\right)z + ...$$

2. Show that an $\mathbb{R}$-valued holomorphic function is constant.

   Proof. Let Ω be an open set contained in the domain of f. Then $f(\Omega)$ is contained in the real line, and hence is not open in the complex plane. If f were non-constant, this would contradict the open mapping theorem, so we conclude that f must be constant. $\square$

3. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx$.

Solution: Let $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) = \frac{e^{iz}}{1+z^2} = \frac{e^{iz}}{(z+i)(z-i)}.$$

Let $R > 0$, and let γ be the contour in $\mathbb{C}$ obtained by joining the line segment from $-R$ to R along the real axis to the semicircle of radius R in the upper half-plane, oriented counterclockwise. Let C_R denote the semicircle itself. We have

$$\int_\gamma f(z)\,dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)\,dx. \tag{1}$$

The function f has poles at $z = \pm i$. For large R, γ encloses the pole at $z = i$, so by the residue theorem we have

$$\begin{aligned}\int_\gamma f(z)\,dz = 2\pi i\,\mathrm{Res}_i(f) &= 2\pi i \left(\lim_{z\to i} \frac{e^{iz}}{(z+i)(z-i)}(z-i) \right) \\ &= 2\pi i \left(\frac{e^{-1}}{2i} \right) \\ &= \frac{\pi}{e}.\end{aligned}$$

For the integral over the semicircle, we have the estimate

$$\left| \int_{C_R} f(z)\,dz \right| \le \sup_{z\in C_R} \left| \frac{e^{iz}}{1+z^2} \right| \mathrm{length}(C_R).$$

Note that, writing $z = x+iy$, we have $|e^{iz}| = |e^{ix}e^{-y}| = e^{-y} \le 1$ for $y \ge 0$. Since C_R is contained in the upper half-plane, we therefore have

$$\left| \int_{C_R} f(z)\,dz \right| \le \frac{1}{R^2-1}\pi R \to 0 \text{ as } R \to \infty$$

(using the reverse triangle inequality to bound the denominator). Thus, taking the limit $R \to \infty$ in (1), we obtain

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = \frac{\pi}{e}.$$

4. Determine the radius of convergence of the power series for $\frac{z^2}{1-\cos z}$ at 0.

Solution: The radius of convergence, which we denote by R, is equal to the radius of the largest open disk on which there is a holomorphic function agreeing with $\frac{z^2}{1-\cos z}$.

Note that $1 - \cos z = 0$ if and only if $z = 2\pi k$ for some integer k. Fix $z_0 \in C$ such that z_0 is not equal to $2\pi k$ for any integer k. Then $1 - \cos z_0 \neq 0$, so, using the power series for cos (which has infinite radius of convergence, and so can be evaluated at z_0), we have

$$\frac{z_0^2}{1-\cos z_0} = \frac{z_0^2}{1 - \left(1 - \frac{z_0^2}{2!} + \frac{z_0^4}{4!} + \ldots\right)} = \frac{1}{\frac{1}{2} - \frac{z_0^2}{4!} + \ldots} := g(z_0).$$

By the quotient rule, the function g is holomorphic at any point where the denominator does not vanish, since the denominator is power series with infinite radius of convergence and hence an entire function. The denominator is the power series for $\frac{1-\cos z}{z^2}$ evaluated at z_0, and so is equal to zero whenever $1 - \cos z = 0$ except for $z = 0$, where we have $g(0) = 2$. Thus, g is holomorphic on $|z| < 2\pi$, since the denominator does not vanish there. Since g agrees with $\frac{z^2}{1-\cos z}$ (once we extend the latter to be defined at $z = 0$ by its limit), we obtain $R \geq 2\pi$. Note that

$$\lim_{z \to 2\pi} \frac{z^2}{1 - \cos z} = \infty,$$

so there cannot be a holomorphic function agreeing with $\frac{z^2}{1-\cos z}$ on a disk centered at 0 of radius larger than 2π, and hence we conclude $R = 2\pi$.

5. Let f be an entire function such that $|f(z)| \leq 1 + \sqrt{|z|}$ for all $z \in \mathbb{C}$. Show f is constant.

Proof. Fix a point $z \in \mathbb{C}$, and let C_R denote the circle of radius R centered at z. By Cauchy's formula, we have

$$\begin{aligned} |f'(z)| &= \frac{1}{2\pi} \left| \int_{C_R} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta \right| \\ &\leq \frac{1}{2\pi} \int_{C_R} \frac{|f(\zeta)|}{|\zeta - z|^2} \, d\zeta. \end{aligned}$$

Since inside the integral ζ is on the boundary of C_R, and z is the center of the circle, we have $|\zeta - z| = R$. Also, for $\zeta \in C_R$, we have $|\zeta| \leq |z| + R$ by the triangle inequality, so we have $|f(\zeta)| \leq |1 + \sqrt{|\zeta|}| \leq 1 + \sqrt{|z| + R}$ inside the integral. Thus,

$$\begin{aligned} |f'(z)| &\leq \frac{1}{2\pi} \sup_{\zeta \in C_R} \frac{|f(\zeta)|}{|\zeta - z|^2} 2\pi R \\ &\leq \frac{1}{R}(1 + \sqrt{|z| + R}) \to 0 \text{ as } R \to \infty. \end{aligned}$$

Therefore, we conclude $f'(z) = 0$, but since z was arbitrary we obtain $f' \equiv 0$ on $\mathbb{C}$, which implies that f is constant. □

6. Show that there is a holomorphic function f on the region $|z| > 2$ such that $f(z)^4 = z^4 + z + 1$.

Proof. Let $p(z) := z^4 + z + 1$. Our strategy is to explicitly construct the fourth root by exponentiating a logarithm-type integral and showing that the resulting function is well-defined and in fact gives the desired fourth root.

First we show that the roots of $p(z)$ are all in $|z| < 2$ using Rouché's theorem. For any z with $|z| = 2$, we have

$$|z^4 + 1| \geq ||z|^4 - 1| = 15 > 2 = |z|.$$

Thus, by Rouché's theorem, z^4+1 and $p(z) = z^4+1+z$ have the same number of roots inside the circle $|z| = 2$. The function z^4+1 has four roots inside this circle, so all 4 of the roots of p also lie in $|z| < 2$. We denote these roots of $p(z)$ by w_1, w_2, w_3, and w_4, so that $p(w) = (w-w_1)(w-w_2)(w-w_3)(w-w_4)$.

Now, we define $f : \{|z| > 2\} \to \mathbb{C}$ by

$$f(z) = C \exp\left(\frac{1}{4}\int_\gamma \frac{p'(w)}{p(w)}\,dw\right),$$

where γ is any piecewise smooth path in $|z| \geq 2$ connecting 2 to z, and C is a real number satisfying $C^4 = p(2) = 19$. First we show that f is well-defined, i.e. we show that the value of f at z is independent of the choice of the path γ.

To do this, we examine the behavior of this integral over closed curves. If Γ is any (sufficiently regular, e.g. piecewise smooth, which we will always assume to be the case from now on) closed path in $\{|z| > 2\}$ which does not enclose the disk $|z| < 2$, then we have

$$\int_\Gamma \frac{p'(w)}{p(w)}\,dw = 0,$$

by the residue theorem, since in this case the integrand is holomorphic in an open set containing Γ and its interior. Now, if Γ is any closed path which does enclose the disk $|z| < 2$, then we have (using the product rule to expand the integrand)

$$\int_\Gamma \frac{p'(w)}{p(w)}\,dw = \int_\Gamma \frac{1}{w-w_1} + \frac{1}{w-w_2} + \frac{1}{w-w_3} + \frac{1}{w-w_4}\,dw.$$

Now, Γ encloses the residues of the terms in the integrand, so by the residue theorem each individual integral in the sum is equal to $2\pi i n$, where n is the winding number of Γ about the origin. Thus, we have

$$\int_\Gamma \frac{p'(w)}{p(w)}\,dw = 8\pi i n.$$

Thus, for any closed curve Γ in $|z| > 2$, we have

$$\int_\Gamma \frac{p'(w)}{p(w)}\,dw = 8\pi i n$$

for some (possibly zero) integer n. Now, suppose γ_1 and γ_2 are two distinct paths in $|z| > 2$ which both connect 2 to z, and consider the integral over the closed curve Γ defined by connecting γ_1 and γ_2 so that we traverse γ_1 with its positive orientation and γ_2 with its negative orientation. We then have, for some n

$$\exp\left(\frac{1}{4}\int_\Gamma \frac{p'(w)}{p(w)}\,dw\right) = \exp\left(\frac{1}{4}8\pi i n\right) = \exp(2\pi i n) = 1.$$

But, on the other hand,

$$\begin{aligned} 1 = \exp\left(\frac{1}{4}\int_\Gamma \frac{p'(w)}{p(w)}\,dw\right) &= \exp\left(\frac{1}{4}\int_{\gamma_1} \frac{p'(w)}{p(w)}\,dw - \frac{1}{4}\int_{\gamma_2} \frac{p'(w)}{p(w)}\,dw\right) \\ &= \exp\left(\frac{1}{4}\int_{\gamma_1} \frac{p'(w)}{p(w)}\,dw\right)\exp\left(-\frac{1}{4}\int_{\gamma_2} \frac{p'(w)}{p(w)}\,dw\right) \end{aligned}$$

so we conclude

$$\exp\left(\frac{1}{4}\int_{\gamma_1}\frac{p'(w)}{p(w)}\,dw\right)=\exp\left(\frac{1}{4}\int_{\gamma_2}\frac{p'(w)}{p(w)}\,dw\right),$$

and thus the expression for f is independent of the choice of path γ, as desired. So f is well-defined.

Now we show that f is holomorphic. We do this by directly looking at difference quotients, mimicking the standard construction of a holomorphic logarithm. However, there is some added subtlety due to the fact that we do not have a holomorphic logarithm in this case but only the fourth root.

Fix a point $z \in \{|z| > 2\}$, and fix a curve γ in $\{|z| \geq 2\}$ connecting 2 to z. For sufficiently small h, let γ_h be the curve obtained by adjoining the straight line segment from z to $z+h$ to the curve γ, and denote the line segment itself by ℓ_h, so that

$$\begin{aligned} f(z+h) = C\exp\left(\frac{1}{4}\int_{\gamma_h}\frac{p'(w)}{p(w)}\,dw\right) &= C\exp\left(\frac{1}{4}\int_{\gamma}\frac{p'(w)}{p(w)}\,dw + \frac{1}{4}\int_{\ell_h}\frac{p'(w)}{p(w)}\,dw\right) \\ &= C\exp\left(\frac{1}{4}\int_{\gamma}\frac{p'(w)}{p(w)}\,dw\right)\exp\left(\frac{1}{4}\int_{\ell_h}\frac{p'(w)}{p(w)}\,dw\right) \\ &= f(z)\exp\left(\frac{1}{4}\int_{\ell_h}\frac{p'(w)}{p(w)}\,dw\right). \end{aligned}$$

Thus, we have

$$\begin{aligned} \frac{1}{h}(f(z+h)-f(z)) &= \frac{1}{h}f(z)\left[\exp\left(\frac{1}{4}\int_{\ell_h}\frac{p'(w)}{p(w)}\,dw\right)-1\right] \\ &= \frac{1}{h}f(z)\left[1+\frac{1}{4}\int_{\ell_h}\frac{p'(w)}{p(w)}\,dw+\psi_h(z)-1\right] \\ &= \frac{1}{h}f(z)\left[\frac{1}{4}\int_{\ell_h}\frac{p'(w)}{p(w)}\,dw+\psi_h(z)\right], \end{aligned}$$

where we have used the Taylor series for the exponential and collected in $\psi_h(z)$ the terms which are quadratic or higher in the integral. By a basic continuity argument (this is essentially a basic version of the Lebesgue differentiation theorem), we have

$$\frac{1}{h}\frac{1}{4}\int_{\ell_h}\frac{p'(w)}{p(w)}\,dw \to \frac{1}{4}\frac{p'(z)}{p(z)} \text{ as } h \to 0,$$

which also implies that $\frac{1}{h}\psi_h(z) \to 0$ as $h \to 0$. Therefore, we conclude that f is holomorphic in $|z| > 2$, and

$$f'(z) = \frac{1}{4}\frac{p'(z)}{p(z)}f(z).$$

Finally, we show that we indeed have $(f(z))^4 = p(z)$. We have, omitting arguments for convenience,

$$\begin{aligned}
\frac{d}{dz}\left[\frac{f^4}{p}\right] &= \frac{p(4f^3 f') - f^4 p'}{p^2} \\
&= \frac{4pf^3 f' - f^4(4pf')/f}{p^2} \\
&= 0.
\end{aligned}$$

So we conclude f^4/p is equal to some constant. Taking the limit $z \to 2$, by construction we have $f(2) = C$ and $p(2) = C^4$, so we conclude that this constant must be 1, and thus we have $(f(z))^4 = p(z)$, as desired. □

7. Show that

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}.$$

Proof. Our goal is to show that the difference between these two functions is holormorphic and bounded, hence constant by Liouville's theorem, and that the constant is zero. We do this by showing that these functions have the same poles, so that when we take the difference the poles will cancel and leave us with a holomorphic function.

Let

$$f(z) = \frac{\pi^2}{\sin^2 \pi z}.$$

Recall the power series for $\sin \pi z$:

$$\sin \pi z = \pi z - \frac{\pi^3 z^3}{6} + O(z^5).$$

For sufficiently small non-zero z, we have

$$\begin{aligned}
\frac{\pi^2}{\sin^2 \pi z} &= \frac{\pi^2}{\pi^2 z^2 - \frac{\pi^4 z^4}{3} + O(z^6)} \\
&= \frac{1}{z^2}\frac{1}{1 - \frac{\pi^2 z^2}{3} + O(z^4)} \\
&= \frac{1}{z^2}\left(1 + \frac{\pi^2 z^2}{3} + O(z^4)\right)
\end{aligned}$$

where the last equality follows from expanding $1/(1 - a(z))$ in the previous line as a geometric series, which is valid for sufficiently small z (which follows from convergence of the power series for $\sin \pi z$). Thus, we have

$$\frac{\pi^2}{\sin^2 \pi z} = \frac{1}{z^2} + g(z)$$

where g is holomorphic in a neighborhood of zero, hence f has a pole of order 2 at $z = 0$. Let u be defined by

$$u(z) = \frac{\pi^2}{\sin^2 \pi z} - \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}.$$

Near $z = 0$, we have

$$\begin{aligned} u(z) &= \frac{1}{z^2} + g(z) - \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2} \\ &= g(z) - \sum_{n \neq 0} \frac{1}{(z-n)^2}, \end{aligned}$$

which is holomorphic in a neighborhood of $z = 0$. By periodicity, $\frac{\pi^2}{\sin^2(\pi z)}$ has a pole of order 2 with coefficient 1 at each $n \in \mathbb{Z}$, exactly matching the pole in the infinite sum, so it follows that u has no poles in $\mathbb{C}$, and hence is entire.

Now we show that u is bounded and hence constant by Liouville's theorem. Since both terms in u are periodic in the sense that they satisfy $f(z+n) = f(z)$ for $n \in \mathbb{Z}$, first we know that u is bounded on the real axis (as a continuous periodic function), and also it suffices to show that u is bounded on the strip $0 \leq x \leq 1, y > 0$, where $z = x + iy$. Furthermore, u is continuous and hence bounded on compact sets, so it suffices to show that for each fixed $x \in [0,1]$, $\lim_{y \to \infty} u(x+iy)$ is bounded. So, fix $x \in [0,1]$.

For the second term in u, we have

$$\left| \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2} \right| \leq \sum_{n \in \mathbb{Z}} \frac{1}{|z-n|^2} = \sum_{n \in \mathbb{Z}} \frac{1}{|x-n+iy|^2} = \sum_{n \in \mathbb{Z}} \frac{1}{(x-n)^2 + y^2}.$$

For $y > 0$, the sum on the right hand side is convergent, and goes to zero as $y \to \infty$, for instance by the dominated convergence theorem.

For the first term in u, we have

$$|\sin^2 \pi z| = \left| \frac{e^{-\pi y} e^{i\pi x} - e^{\pi y} e^{-i\pi x}}{2i} \right| \to \infty \text{ as } y \to \infty$$

by continuity of the norm. Therefore,

$$\left| \frac{\pi^2}{\sin^2 \pi z} \right| \to 0 \text{ as } y \to \infty.$$

Combining these results, we obtain

$$\lim_{y \to \infty} |u(x+iy)| = 0.$$

So u is bounded, and therefore constant by Liouville's theorem. Since this limit is zero, this constant must also be zero, so we conclude $u(z) \equiv 0$, and hence

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2},$$

as desired. $\square$

8. Make a change of coordinates to put the elliptic curve $w^2 = z^4 + 1$ into the (essentially) Weierstrass form $y^2 = x^3 + bx + c$.

Solution: Define $\tilde{z} = e^{-i\pi/4}z$, so that $z^4 = -\tilde{z}^4$, and let $\tilde{w} = iw$, so that $w^2 = -\tilde{w}^2$. (Note: I made this change of variables because I messed up and factored z^4+1 as z^4-1; making this change of variables at the beginning fixes the rest of the solution, but is not necessary if one factors correctly in the first place. The following method applies directly to $w^2 = z^4 + 1$ as well, with the roots replaced appropriately.)

The equation for the curve then becomes

$$\tilde{w}^2 = \tilde{z}^4 - 1.$$

Dropping tildes, we factor

$$z^4 - 1 = (z+i)(z-i)(z+1)(z-1).$$

Our first step is to send one of the roots of $z^4 + 1$ to infinity with a fractional linear transformation. This ends up having the effect of reducing the degree of the polynomial from 4 to 3. To do this, we let

$$x = \frac{z}{1-z}$$

so that

$$z = \frac{x}{1+x}.$$

Substituting this into the equation $w^2 = z^4 + 1$, we obtain

$$\begin{aligned} w^2 &= \left(\frac{x}{1+x} + \frac{i(1+x)}{1+x}\right)\left(\frac{x}{1+x} - \frac{i(1+x)}{1+x}\right)\left(\frac{x}{1+x} + \frac{1+x}{1+x}\right)\left(\frac{x}{1+x} - \frac{1+x}{1+x}\right) \\ &= \left(\frac{(1+i)x+i}{1+x}\right)\left(\frac{(1-i)x-i}{1+x}\right)\left(\frac{2x+1}{1+x}\right)\left(-\frac{1}{1+x}\right). \end{aligned}$$

Now, let $y = (1+x)^2 w$, so that $w^2 = \frac{y^2}{(1+x)^4}$. Then, multiplying through by $(1+x)^4$, we obtain

$$\begin{aligned} y^2 &= ((1+i)x+i)((1-i)x-i)(2x+1)(-1) \\ &= -(2x^2+2x+1)(2x+1) \\ &= -4x^3 - 6x^2 - 4x - 1. \end{aligned}$$

Now, we still must clear the leading coefficient and then eliminate the quadratic term. First, let $\tilde{y} = \frac{iy}{2}$, so that

$$\tilde{y}^2 = x^3 + \frac{3}{2}x^2 + x + \frac{1}{4}.$$

We can eliminate the quadratic term by letting $\tilde{x} = x + b$ and choosing b appropriately. We obtain

$$\tilde{y}^2 = (\tilde{x}-b)^3 + \frac{3}{2}(\tilde{x}-b)^2 + (\tilde{x}-b) + \frac{1}{4}.$$

The only quadratic terms in $\tilde{x}$ are $-3b\tilde{x}^2$ and $\frac{3}{2}\tilde{x}^2$, so choosing $b = \frac{1}{2}$ makes these terms cancel. Expanding the right hand side, and dropping $\sim$'s in our notation, we obtain

$$\begin{aligned} y^2 &= x^3 - \frac{3}{2}x^2 + \frac{3}{4}x + \frac{1}{4} + \frac{3}{2}\left(x^2 - x - \frac{1}{4}\right) + x - \frac{1}{2} + \frac{1}{4} \\ &= x^3 + \frac{1}{4}x - \frac{3}{8}, \end{aligned}$$

which is exactly the desired form with $b = \frac{1}{4}$ and $c = -\frac{3}{8}$.