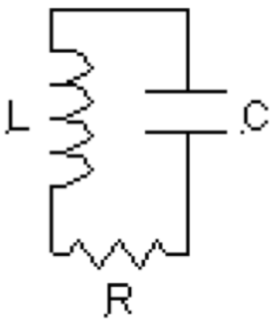
Appendix A: LRC Electrical Harmonic Oscillator Equation Derivation



Kirchoff's Voltage Rule: $\sum_{\text{loop}} V = 0$

Voltages across capacitor, resistor, and inductor:

$$V_C - V_R - V_L = 0$$

Substitute:

$$\frac{q}{C} + IR - \left(-L\frac{dI}{dt}\right) = 0$$

Rearrange:

$$L\frac{dI}{dt} + IR + \frac{q}{C} = 0$$

Divide by L:

$$\frac{dI}{dt} + \frac{R}{L}I + \frac{1}{LC}q = 0$$

Substitute: $I = \frac{dq}{dt}$ and $\frac{dI}{dt} = \frac{d^2q}{dt^2}$

$$\frac{d^2q}{dt^2} + \left(\frac{R}{L}\right)\frac{dq}{dt} + \frac{1}{LC}q = 0$$

Say $\frac{R}{L} = \gamma$, $\frac{1}{LC} = \omega_0^2$:

$$\underbrace{\frac{d^2q}{dt^2} + \gamma\frac{dq}{dt} + \omega_0^2 q = 0}_{\text{second order differential equation}}$$

Propose solution $q(t) = q_0 e^{rt}$

$$\frac{dq(t)}{dt} = q_0 r e^{rt} , \quad \frac{d^2q(t)}{dt^2} = q_0 r^2 e^{rt}$$

Plug in to diff eq:

$$q_0 r^2 e^{rt} + q_0 r e^{rt}\gamma + \omega_0^2 q_0 e^{rt} = 0$$

Simplify:

$$q_0 e^{rt}\underbrace{(r^2 + \gamma r + \omega_0^2)}_{\text{Characteristic equation}} = 0$$

make true by solving Characteristic eqn with quadratic formula:

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}$$

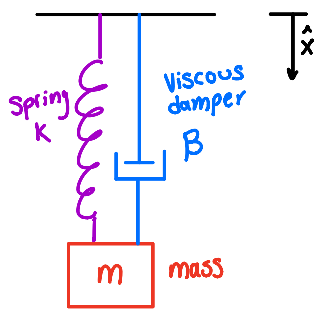
For underdamped,

$$r = \gamma \pm \omega_0 i$$

From eulers Formula

$$\boxed{q(t) = q_0 e^{\left(-\frac{\gamma t}{2}\right)} \cos(\omega_0 t)}$$

Appendix B: Damped Hanging Mass Spring Oscillator Equation Derivation



Force Balance of damper and spring force

$$ma = -\beta V + KX$$

write derivatives and rearrange:

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + KX = 0$$

divide by m:

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{K}{m}x = 0$$

Say $\gamma = \frac{\beta}{m}$ and $\omega_0 = \frac{K}{m}$:

$$\underbrace{\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0}_{\text{second order differential equation}}$$

Propose solution $X(t) = X_0 e^{rt}$

$$\frac{dX(t)}{dt} = X_0 r e^{rt}, \quad \frac{d^2X(t)}{dt^2} = X_0 r^2 e^{rt}$$

Plug in to diff eq:

$$X_0 r^2 e^{rt} + X_0 r e^{rt}\gamma + \omega_0^2 X_0 e^{rt} = 0$$

simplify:

$$X_0 e^{rt}\underbrace{(r^2 + \gamma r + \omega_0^2)}_{\text{Characteristic equation}} = 0$$

make true by solving characteristic eqn with quadratic formula:

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}$$

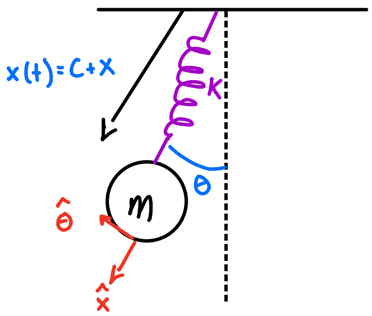
For underdamped,

$$r = \gamma \pm \omega_0 i$$

From eulers Formula

$$\boxed{X(t) = X_0 e^{\left(-\frac{\gamma t}{2}\right)}\cos(\omega_0 t)}$$

Appendix C: Equations of Motion of Elastic Pendulum Derivation



Total Kinetic Energy

$$T = \frac{1}{2} m \left(v_r^2 + v_\theta^2\right) = \frac{1}{2} m \left(\dot{x}^2 + \left(x(t)\dot{\theta}\right)^2\right) = \underline{\frac{1}{2} m \left(\dot{x}^2 + (c+x)^2 \dot{\theta}^2\right)}$$

Total Potential Energy

$$\underline{U = \frac{1}{2} k\, x^2 - mg\,(c+x)\cos\theta}$$

Define Lagrangian:

$$\mathcal{L} = T - U = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m (c+x)^2 \dot{\theta}^2 + mg\,(c+x)\cos\theta - \frac{1}{2} k x^2$$

x equation: $\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$

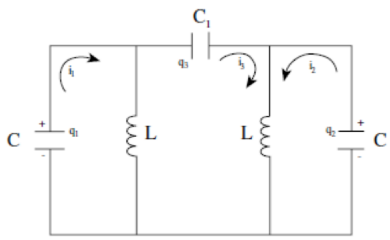
$$m\ddot{x} - \left(m(c+x)\dot{\theta}^2 + mg\cos\theta - kx\right) = 0$$

$$\cancel{m}\ddot{x} - \cancel{m}(c+x)\dot{\theta}^2 + \frac{kx}{m} - \cancel{m}g\cos\theta = 0$$

θ equation: $2\cancel{m}(c+x)\dot{x}\dot{\theta} + \cancel{m}(c+x)^2\ddot{\theta} + \cancel{m}g(c+x)\sin\theta = 0$

$$\boxed{\begin{aligned} \ddot{x} &= (c+x)\dot{\theta}^2 - \frac{k}{m}x + g\cos\theta \\ \ddot{\theta} &= -\frac{2}{c+x}\dot{x}\dot{\theta} - \frac{g}{c+x}\sin\theta \end{aligned}}$$

Appendix D: Coupled LRC Circuit derivation



Coupled LRC circuits without resistance

Equations describing circuit from Kirchoffs Laws:

$$\frac{q_1}{C} = L\frac{d}{dt}(i_1 - i_3)$$

$$\frac{q_2}{C} = L\frac{d}{dt}(i_2 + i_3)$$

$$\frac{q_1}{C} = \frac{q_2}{C} - \frac{q_3}{C_1}, \quad \text{so} \quad i_3 = \frac{C_1}{C}(i_2 - i_1)$$

Combining above:

$$\frac{q_1}{C} = L\frac{d}{dt}\left[i_1\left(1+\frac{C_1}{C}\right) - \frac{C_1}{C}i_2\right]$$

$$\frac{q_2}{C} = L\frac{d}{dt}\left[i_2\left(1+\frac{C_1}{C}\right) - \frac{C_1}{C}i_1\right]$$

Take sum and difference of above

$$(q_1+q_2)\frac{1}{C} = L\frac{d}{dt}(i_1+i_2)$$

$$(q_1-q_2)\frac{1}{C} = \frac{L}{C}(C+2C_1)\frac{d}{dt}(i_1-i_2)$$

Define $q_+ = q_1 + q_2$ and $q_- = q_1 - q_2$ and substitute

$$\frac{d^2}{dt^2}q_+ + \frac{1}{LC}q_+ = 0 \qquad \frac{d^2}{dt^2}q_- + \frac{1}{L(C+2C_1)}q_- = 0$$

Second order diff. eq   Second order diff. eq

Solve differential equations using methods in above appendicies

$$\boxed{\begin{array}{l} q_+ = q_{+0}\sin(\omega_+ t) \qquad q_- = q_{-0}\sin(\omega_- t) \\ \text{Where } \omega_+ = \frac{1}{\sqrt{LC}} \quad \text{and} \quad \omega_- = \frac{1}{\sqrt{L(C+2C_1)}} \end{array}}$$