HOMEWORK 6

Problem 1

> *An electron is trapped inside an infinite potential energy well (one dimensional). The width of the potential energy well is L=0.300 nm.*
> *(a) If the electron is in the ground state (n=1), what is the probability of finding it within 0.100 nm of the left-side wall?*
> *(b) What is the probability if the electron is instead in the 99th excited state (n=100)?*
> *(c) What probability would you expect for a classical particle? How does this compare with your results from (a) and (b)? This is an example of the so-called correspondence principle.*

A)
The wave function for the particle is

$$\psi(x) = \frac{2}{L} sin\left(\frac{n\pi}{L}x\right)$$

Here:
L = 0.3 nm
n = 1
The probability of finding the particle in a given region from x_0 *to* x_1 is

$$\int_{x_0}^{x_1} |\psi(x)|^2 = \frac{2}{L}\int_{x_0}^{x_1} sin^2\left(\frac{n\pi}{L}x\right)\ dx$$

Where $x_0 = 0$ nm and $x_1 = 0.1$ nm. We can use the sine power-reducing formula:

$$\int_{x_0}^{x_1} |\psi(x)|^2 = \frac{2}{2L}\int_{0}^{0.1} 1 - cos\left(\frac{2n\pi}{L}x\right)\ dx$$

$$= \frac{1}{L}\left[\int_0^{0.1} 1\ dx - \int_0^{0.1} cos\left(\frac{2n\pi}{L}x\right)\ dx\right]$$

$$= \frac{1}{L}\left[x - \frac{L}{2n\pi}sin\left(\frac{2n\pi}{L}x\right)\right]_0^{.1}$$

$$= \frac{1}{0.3nm}\left[x - \frac{0.3nm}{2\pi}sin\left(\frac{2\pi}{0.3nm}x\right)\right]_0^{.1}$$

$$= 1/3 - \frac{1}{2\pi}sin\left(\frac{2\pi}{3}\right) = 0.1955$$

B)
We can just change n to 100 at the end.

$$= \frac{1}{0.3}\left[x - \frac{0.3}{2*100\pi}sin\left(\frac{2*100\pi}{0.3}x\right)\right]_0^{.1}$$

$$= \frac{1}{3} - \frac{1}{2*100\pi}sin\left(\frac{2*100\pi}{3}\right) = 0.3320$$

C)
In classical mechanics the probability of finding the particle in a length equal to 1/3 the length of the box is 1/3. The result from part A does not correspond with this probability, but the result from part B does.

Problem 2

> *A particle is confined to a three-dimensional infinite potential energy well. Sketch an energy level diagram that shows the energies (in terms of E), quantum numbers, and degeneracies for the lowest 10 energy levels for this particle.*

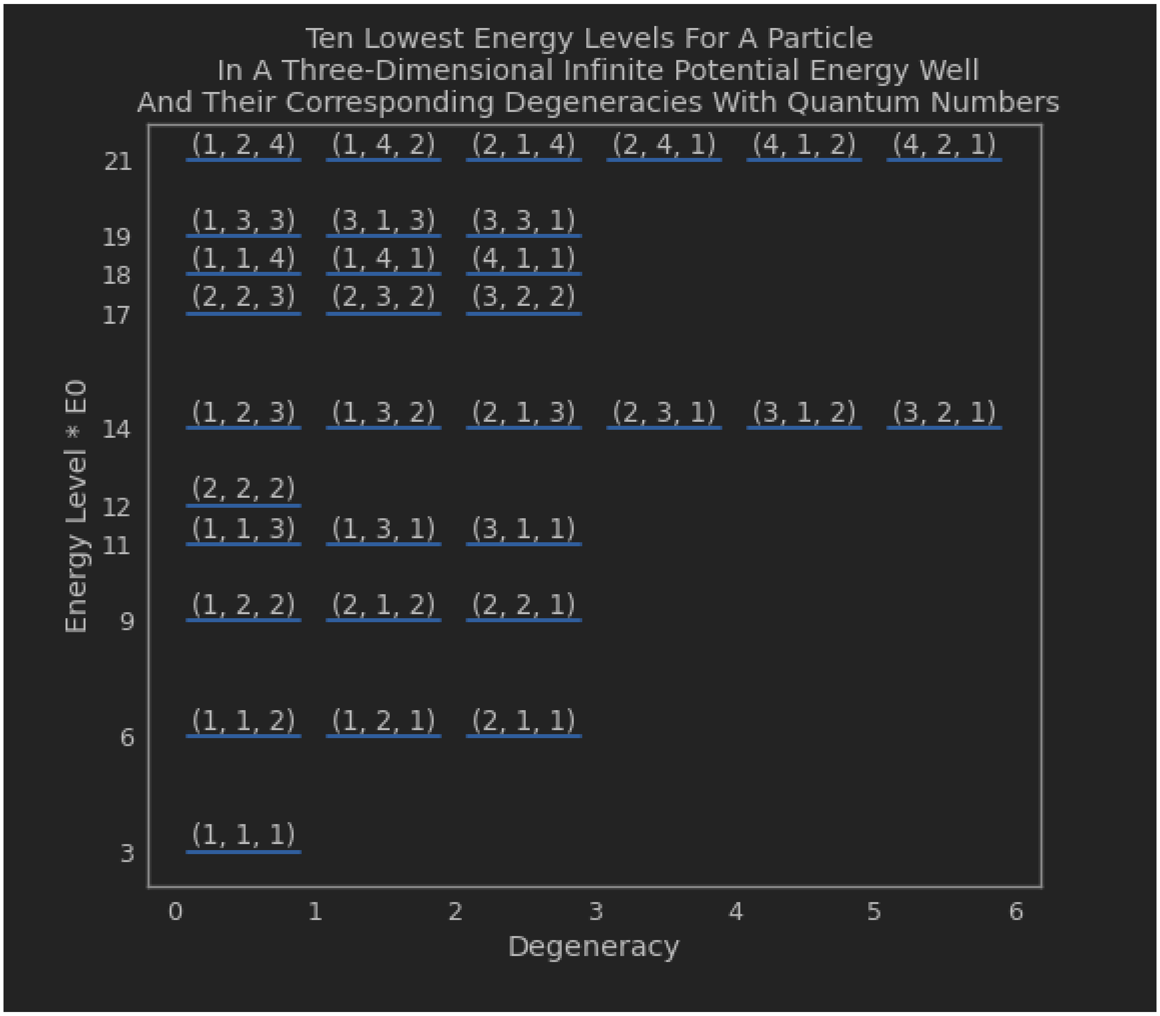

Figure 1: Setup 3

Problem 3

> *Consider the quantum harmonic oscillator and its ground state wave function:*
> *(a) Find the average value $< x >$ and the average value $< x^2 >$.*
> *(b) Find the uncertainty x using the above. (Hint: see also textbook, end of Section 4.4 + "extra" notes for Lecture 17 in Canvas.)*

The wave function of the ground state is

$$\psi(x) = Ae^{-\alpha x^2}$$

Where

$$A = \frac{m\omega_0}{\hbar\pi}^{1/4}$$

$$\alpha = \frac{\omega_0}{2\hbar}$$

THe average value of x is:

$$\int_{-\infty}^{\infty} x\psi(x)^2 \ dx$$

$$\int_{-\infty}^{\infty} xAe^{-2\alpha x^2} \ dx$$

This is an odd function, and the integral will evaluate to zero.

The average value of x^2 is:

$$\int_{-\infty}^{\infty} x^2\psi(x)^2 \ dx$$

$$\int_{-\infty}^{\infty} x^2A^2e^{-2\alpha x^2} \ dx$$

Where $u = \sqrt{2\alpha}x$, $du = \sqrt{2\alpha}dx$
$x = \frac{u}{\sqrt{2\alpha}}$, $dx = \frac{du}{\sqrt{2\alpha}}$

$$\frac{A^2}{(\sqrt{2\alpha})^3}\int_{-\infty}^{\infty} u^2e^{-u^2} \ dx$$

the value of the integral is equal to $\frac{\sqrt{\pi}}{4}$. This gives:

$$\frac{A^2}{(2\alpha)^{3/2}}\frac{\sqrt{\pi}}{4}$$

$$\left(\frac{m\omega_0}{\hbar\pi}\right)^{1/2}\left(\frac{\omega_0}{2\hbar}\right)^{-3/2}\frac{\sqrt{\pi}}{4}$$

$$\left(\frac{m\omega_0}{\hbar\pi}\right)^{1/2}\left(\frac{2\hbar}{\omega_0}\right)^{3/2}\frac{\sqrt{\pi}}{4}$$

$$\frac{\hbar}{\omega_0}\left(\frac{m}{\pi}\right)^{1/2}(2)^{3/2}\frac{\sqrt{\pi}}{4}$$

$$\frac{\hbar}{\omega_0}\sqrt{\frac{m}{2}}$$

B)

$$\Delta x = \sqrt{< x^2 > - < x >^2}$$

$$\Delta x = \sqrt{\frac{\hbar}{\omega_0}\sqrt{\frac{m}{2}} - 0}$$

$$\Delta x = \sqrt{\frac{\hbar}{\omega_0}\sqrt{\frac{m}{2}}}$$

Problem 4

> *Consider again the quantum harmonic oscillator. The first excited state of the harmonic oscillator has a wave function of the form:*
>
> $$\psi(x) = Axe^{-ax^2}$$
>
> *(a) Similarly as we did in class for the ground-state wave function, find the constant a in the exponential, and the energy E for this (first excited state) wave function.*
> *(b) Find the constant A for the wave function using the normalization condition.*

$$\frac{d\psi}{dx} = Ax(-2ax)e^{-ax^2} + Ae^{-ax^2}$$
$$\frac{d\psi}{dx} = -2Aax^2e^{-ax^2} + Ae^{-ax^2}$$

$$\frac{d^2\psi}{dx^2} = -4Aaxe^{-ax^2} + 4Aa^2x^3e^{-ax^2} - 2Aaxe^{-ax^2}$$
$$\frac{d^2\psi}{dx^2} = -6Aaxe^{-ax^2} + 4Aa^2x^3e^{-ax^2}$$
$$\frac{d^2\psi}{dx^2} = A(-6a + 4a^2x^2)xe^{-ax^2}$$

The Schrödinger equation is:

$$-\frac{\bar{h}^2}{2m}\frac{d^2\psi}{dx^2} + 1/2\ kx^2\psi(x) = E\psi(x)$$

Substitute into Schrödinger equation:

$$-\frac{\bar{h}^2}{2m}A(-6a + 4a^2x^2)xe^{-ax^2} + 1/2\ kx^2Axe^{-ax^2} = EAxe^{-ax^2}$$

The exponentials cancel out on both sides of the equation.

$$-\frac{\bar{h}^2}{2m}(-6a + 4a^2x^2) + 1/2\ kx^2 = E$$

We want a solution that is valid for ANY value of x:

$$\frac{\bar{h}^2 6a}{2m} - \frac{\bar{h}^2}{2m}4a^2x^2 + 1/2\ kx^2 = E$$
$$-\frac{\bar{h}^2}{2m}4a^2x^2 + 1/2\ kx^2 = E - \frac{\bar{h}^2 6a}{2m}$$
$$\left(1/2\ k - \frac{\bar{h}^2}{2m}4a^2x^2\right)x^2 = E - \frac{\bar{h}^2 6a}{2m}$$
$$\left(1/2\ k - \frac{\bar{h}^2}{2m}4a^2x^2\right)x^2 = E - \frac{\bar{h}^2 6a}{2m}$$

Each side must be equal to zero

$$\frac{k}{2} = \frac{\bar{h}^2}{2m}4a^2$$
$$\frac{km}{4\bar{h}^2} = a^2$$
$$a = \frac{\sqrt{km}}{2\bar{h}}$$
$$E = \frac{\bar{h}^2 6a}{2m}$$
$$E = \frac{\bar{h}^2 6}{2m}\frac{\sqrt{km}}{2\bar{h}}$$
$$E = 3/2\bar{h}\sqrt{\frac{k}{m}}$$

B)

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 \ dx$$

$$1 = A^2 \int_{-\infty}^{\infty} x^2 e^{-2ax^2} \ dx \quad where \ a = \frac{\sqrt{km}}{2\bar{h}}$$

$$1 = A^2 \int_{-\infty}^{\infty} x^2 e^{-2ax^2} \ dx \quad where \ a = \frac{\sqrt{km}}{2\bar{h}}$$

Where $u = \sqrt{2\alpha}x$, $du = \sqrt{2\alpha}dx$
And $x = \frac{u}{\sqrt{2\alpha}}$, $dx = \frac{du}{\sqrt{2\alpha}}$

$$1 = \frac{A^2}{(\sqrt{2\alpha})^3} \int_{-\infty}^{\infty} u^2 e^{-u^2} \ dx$$

$$1 = \frac{A^2}{(\sqrt{2\alpha})^3} \frac{\sqrt{\pi}}{4}$$

$$A^2 = \frac{4(\sqrt{2\alpha})^3}{\sqrt{\pi}}$$

$$A = \sqrt[4]{\frac{2}{\pi}} (\sqrt[4]{\alpha})^3$$

$$a = \frac{\omega_0}{2\bar{h}}$$

$$A = \sqrt[4]{\frac{2}{\pi}} \left(\sqrt[4]{\frac{\omega_0}{2\bar{h}}} \right)^3$$

$$A = \sqrt{\pi/2} \left(\frac{2}{\pi} \right)^{3/4} \left(\frac{\omega_0}{2\bar{h}} \right)^{3/4}$$

$$A = \sqrt{\pi/2} \left(\frac{\omega_0}{\bar{h}\pi} \right)^{3/4}$$

Problem 5

> *Find the normalization constants B, D in terms of A by applying the boundary conditions at x=0 for the given equations in the situation of $E < U_0$.*

$$\psi_0(0) = B, \quad \psi_1(0) = D$$

B = D

$$\frac{d\psi_0}{dx} = Ak_0cos(k_0x) - Bk_0sin(k_0x)$$
$$\frac{d\psi_1}{dx} = -k_1De^{-k_1x}$$
$$\psi_0'(0) = Ak_0$$
$$\psi_1'(0) = -Dk_1$$

These derivatives must be equivalent

$$D = B = -A\frac{k_0}{k_1}$$

Problem 6

> *A beam of electrons, each with kinetic energy K = 9.0 keV, is incident on a potential energy step of U = 5.0 keV. What fraction of the beam is reflected by the potential?*

$$x < 0: \ \Psi_0(x,t) = A'e^{i(k_0x-\omega t)} + B'e^{-i(k_0x+\omega t)}$$
$$x \geq 0: \ \Psi_1(x,t) = C'e^{i(k_1x-\omega t)}$$

D is 0 because it tends towards infinity.

At x = 0 the equations must be equivalent, therefore:

$$A + B = C$$

The derivatives must also be equivalent:

$$k_0(A - B) = k_1C$$

Plugging in:

$$k_0A - k_0B = k_1A + k_1B$$
$$k_0A - k_1A = k_0B + k_1B$$
$$A(k_0 - k_1) = B(k_0 + k_1)$$

The reflection coefficient will relate k_0 *and* k_1 by $\frac{|A|^2}{|B|^2}$, where $k_1/k_0 = \frac{U_0-E)}{\sqrt{E}}$:

$$R = \left(\frac{\sqrt{E} - \sqrt{E - U_o}}{\sqrt{E} + \sqrt{E - U_o}}\right)^2$$

Where E = 9.0 keV and U = 5.0 keV

$$R = \left(\frac{\sqrt{9.0} - \sqrt{9.0 - 5.0}}{\sqrt{9.0} + \sqrt{9.0 - 5.0}}\right)^2$$
$$R = \frac{3 - 2}{3 + 2}^2$$
$$R = \frac{1}{5}^2 = 1/25$$