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1 Lecture #1

1.1 Syllabus

1.1.1 Why Modern Physics?

Already learned about mechanics, electromagnetism, and thermodynamics, which forms the basis of "classical physics", however, classical physics fails to explain many observed phenomena, for example, black-body radiation, stability of atoms/radioactivity.

Modern Physics refers to developments that began 1900, leading to relativity and quantum theories and their applications. Is not irrelevant - many very important applications i.e. GPS, semiconductors, medical imaging, lasers.

1.1.2 Course Logistics

- Office hours Mon/Wed @ 1 - 2:30 pm
- Krane's Modern Physics 3rd or 4th edition, electronic versions are fine
- Weekly reading assignments posted on Canvas
- Homework posted on Canvas typically on Wednesdays, due the following Thursday at start of class (over canvas)
- Late assignments are not accepted, but lowest HW score will be dropped
- Professor is teaching from CERN. You can go to the classroom but the teacher won't be there.

There are no stupid questions, please ask questions. Being able to ask questions is an important skill. Questions outside of class can be answered during office hours or on Canvas discussion boards. Physics workshop is good for homework help, and there is also tutoring.

There will be at least one TA, assigned at some point.

1.2 Introduction to Special Relativity

Relativity - How are events observed in different reference frames?

Nothing new – Galileo & Newton knew the Relativity Principle applied to mechanics, that the laws of mechanics are the same in all inertial reference frames.

Inertial reference frame?

Reference frame in which newton's 1st law is valid, i.e. a non-accelerating frame.

Newton's laws are the same as seen in different inertial reference frames (invariant)

1.2.1 Galilean Transformations

Consider two different reference frames S and S', where S' is moving with constant uniform velocity u . Assume that time is the same for all observers ($t = t'$)

If there is an Object P at coordinates x' , y' , z' , the Observer O would observe object at $x = x' + ut$, $y = y'$, $z = z'$.

Galilean Transformations

Coordinate Transformation:

$$x' = x - ut, y' = y, z' = z$$

Velocity Transformation:

$$\begin{aligned}\frac{d}{dt}(x') &= \frac{dx}{dt} - u \rightarrow V'_x = V_x - u \\ V'_y &= V_y \\ V'_z &= V_z\end{aligned}$$

Acceleration Transformation:

$$\begin{aligned}\frac{d}{dt}(V'_x) &= \frac{dV_x}{dt} a'_x = a_x \\ a'_y &= a_y \\ a'_z &= a_z\end{aligned}$$

Newton's second law holds for all observers (it obeys the relativity principle), and that no inertial reference frame is special.

1.2.2 The need for the ether

What about Maxwell's equations of electromagnetism?

Electromagnetic waves (light) propagate with a velocity $c = 3 \times 10^8 m/s$.

Instead of thinking of P as a point, think of it as a light source. Would $V_x = V'x + u$ imply $V_x = c + u$??

Maxwell's equations are not invariant under Galilean transformation (evident from how Maxwell's equations predicted speed of light!)

Speed of something depends on which inertial frame you are observing it from - are Maxwell's equations only correct in a certain frame?

1.2.3 Michelson-Moorley experiment

Physicists in 19th century postulated that light traveled through a medium (ether).

Speed of light predicted by Maxwell's equations was interpreted as that relative to the ether, i.e. ether proposed as an absolute reference system from which c was constant.

This would mean that the laws of physics is not different, because you're simply moving relative to the ether!

Michelson and Morley used an interferometer mounted on a 1.5m stone slab floating in mercury, to reduce the effect of vibrations and isolate the setup.

1.2.4 Michelson interferometer

Monochromatic beam of light split in two, travel different paths, then recombined

Phase difference between combined beams cause fringes to appear

Two contributions

1. Phase difference AB-AC, one beam might travel longer distance
2. Time difference when moving parallel to ether direction vs perpendicular to it - - rotating by 90 degrees should isolate this!

Expectation: Fringe shift as interferometer was rotated with respect to the ether velocity. Michelson-Morley could resolve 0.005 periods and expected 0.4 periods of shift.

This was not observed

OK, so maybe it was the rotation of the earth around the sun that canceled this out. So they repeated half a year later, and still a null result. Speed of light is always c .

1.2.5 Modern-day interferometers

LIGO: Laser Interferometer Gravitational-Wave Observatory

September 14, 2015: First direct detection of gravitational waves and the first observation of a binary black hole merger.

Measuring $\frac{\Delta L}{L}$ on the order of 10^{-21} , based on the difference between the two sites.

This is a direct development of the interferometer Michelson and Morley used.

Einstein's Postulates

- **The principle of relativity:** The laws of physics are the same in all inertial reference frames. There is no way to detect absolute motion, and no preferred inertial system exists.
- **The constancy of the speed of light:** The speed of light in free space has the same value c in all inertial reference frames.

$V'_x = V_x + u$ is **not** true.

2 Lecture #2

2.1 Relativity of Time

Consequence of Posulates

What is the time interval measured by a person on a vehicle?

Person in vehicle sees $\Delta t = \frac{2d}{c}$

Person outside vehicle sees the light travel along a triangle.

total x distance : $v\Delta t$

total light distance : $c\frac{\Delta t}{2}$

Pythagorean theorem gives : $d^2 + (\frac{v\Delta t}{2})^2 = (\frac{c\Delta t}{2})^2$

$$\Delta t = \frac{2d}{\sqrt{c^2 + v^2}}$$

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Gamma Factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\gamma > 1 \text{ for } v < c$$

$$\Delta t = \gamma \Delta t_0$$

... Time dilation!

2.1.1 Cosmic Muons

Lifetime of muons is $2.2 \mu s$

Produced in the atmosphere (atmosphere 100km high)

Average max distance:

$$L = 2.2 \cdot 10^{-6} s \cdot 3 \cdot 10^8 m/s = 660 m$$

Time dilation is a real effect, even if not noticeable at normal speeds!

μ lifetime according to observer on Earth:

$$\tau = \gamma_0$$

$$\tau_0 = 2.2 \mu s$$

$$\text{if } v = 0.99999c, \gamma \approx 224$$

$\tau \approx 490 \mu s$! Length travelled:

$$L = \tau \cdot c = 490 \cdot 10^{-6} \cdot 3 \cdot 10^8 = 147,000 m$$

Two stars

We have an observer on earth, and two stars in space. There is a spaceship travelling from star 1 to star 2 at velocity v .

The observer at rest on Earth w.r.t the two stars, measures:

Distance between stars : L_0

Time for spaceship to travel : $\Delta t = \frac{L_0}{v}$

Observer on spaceship traveling with speed v from star to star, measures:

Smaller time of travel :

$$\Delta t_0 = \frac{\Delta t}{\gamma}$$

Distance measure :

$$L = v\Delta t_0 = \frac{v\Delta t}{\gamma}$$

$$L = \frac{v\Delta t}{\gamma} = \frac{L_0}{\gamma}$$

.. Length contraction!

Length contraction takes place in direction of motion. Objects do not "shrink" - there is just a difference in measured length by different observers.

2.2 Relativistic Velocity Addition

$$\frac{v + u}{1 + \frac{vu}{c^2}}$$

2.3 Relativistic Doppler Shift

Classical Doppler effect for sound waves, $f' = f * \frac{v+v_0}{v-v_s}$

For light, with no medium, and no preferred reference frame...

3 Lecture #3

3.1 Doppler Shift

Classical Doppler Shift

v = speed of waves in medium

v_0 = speed of observer w.r.t. medium

v_s = speed of source w.r.t. medium

$$f' = f \cdot \frac{v + v_0}{v - v_s}$$

Relativistic Doppler Shift

f = frequency in source rest frame

f' = frequency measured by some observer moving away from source w/ speed u .

$$f = \frac{N}{\Delta t_0}$$

$$f' = \frac{c}{\lambda'}$$

$$\lambda' = \frac{u\Delta t' + c}{N}$$

$$f' = \frac{cN}{u\Delta t' + c\Delta t'}$$

$$f' = \frac{cf\Delta t_0}{(u + c)\Delta t'}$$

$$f' = f \sqrt{\frac{1 - u/c}{1 + u/c}}$$

No dependence on whether the source is moving or the observer is moving.

Note: If O' is moving toward the source, u is replaced with $-u$.

Distant galaxy moving away from earth. Blue hydrogen line expected at wavelength 434 nm measured at 600 nm. What is the speed of the galaxy relative to earth?

$$f = c/\lambda$$

$$c/\lambda' = c/\lambda \sqrt{\frac{1 - u/c}{1 + u/c}}$$

$$(\lambda/\lambda')^2 = \frac{1 - u/c}{1 + u/c}$$

$$u = c \frac{1 - (\frac{\lambda}{\lambda'})^2}{1 + (\frac{\lambda}{\lambda'})^2} = .31c$$

3.2 Lorentz transformations

Galilean coordinate transformations are not consistent with Einstein's postulates, and don't agree with experiments at high speeds.

New coordinate transformation must fulfill... linearity in space/time, reduce for Galilean version for $v \ll c$, consistent with Einstein's postulates.

$$\begin{aligned}x' &= \frac{x - ut}{\sqrt{1 - u^2/c^2}} \\y' &= y, z' = z, \\t' &= \frac{t - u/c^2 x}{\sqrt{1 - u^2/c^2}}\end{aligned}$$

Lorentz transformation is just a generalized way to write what happens w/in a coordinate system, and the other rules can follow from these.

3.2.1 Velocity transformation

We can derive velocity transformation by differentiating the Lorentz coordinate transformation (space & time!)

$$\begin{aligned}V'_x &= \frac{V_x - u}{1 - V_x u / c^2} \\V'_y &= \frac{V_y \sqrt{1 - u^2/c^2}}{1 - V_y u / c^2} \\V'_z &= \frac{V_z \sqrt{1 - u^2/c^2}}{1 - V_z u / c^2}\end{aligned}$$

Velocity transformation in three dimensions. If we assume movement in the x direction only, you still have this complicated stuff in other dimensions because you're also deriving with respect to time.

3.2.2 What would the photon see?

Infinitely length contracted & if it had a clock, that clock would not be moving. BUT it is not an inertial reference frame in which measurements mean anything. Mathematical singularity – not defined – can't divide by zero!

3.2.3 Simultaneity

Time according to O for light to reach two clocks the same distance from a light source:

$$t = \frac{L}{2c} \text{ clock 1 \& 2 are synchronized}$$

From point of view of moving observer O'? Use Lorentz transformations!

Clock 1 receives signal at :

$$t'_1 = \frac{t_1 - (u/c^2)x_1}{\sqrt{1 - u^2/c^2}} = \frac{L/2c}{\sqrt{1 - u^2/c^2}}$$

Clock 2 receives signal at :

$$t'_2 = \frac{t_2 - (u/c^2)x_2}{\sqrt{1 - u^2/c^2}} = \frac{L/2c - (u/c^2)L}{\sqrt{1 - u^2/c^2}}$$

Difference:

$$\Delta t' = t'_1 - t'_2 = \frac{uL/c^2}{\sqrt{1-u^2/c^2}}$$

3.3 Relativistic Kinematics

3.3.1 Relativistic Momentum

Consider an elastic collision as observed in the rest frame of observer O', where a particle of mass $2m$, moving to the left at $-0.750c$, collides with a particle of mass $2m$ at rest with respect to O'.

Classical Interpretation

$$\begin{aligned} p'_i &= m_i v'_{1i} + m'_2 v'_{2i} \\ &= 2m \cdot 0 + m(-0.750c) \\ &= -0.75mc \end{aligned}$$

Final:

$$\begin{aligned} p'_f &= m'_1 v'_{1f} + m'_2 v'_{2f} \\ &= 2mv'_{1f} + mv'_{2f} \\ p'_i = p'_f &\rightarrow v'_{1f} = -0.5c, v'_{2f} = 0.25c \end{aligned}$$

Momentum is conserved.

Now assume that reference frame of O' is moving with velocity $0.550c$ in the positive x direction relative to observer O. What does O think about momentum conservation?

$$v_{1i} = 0 + 0.55c = 0.55c, \quad v_{2i} = \frac{v'_{2i} + u}{1 + v'_{2i}u/c^2} = -0.340c$$

Using the same steps as above:

$$v_{1f} = 0.069c, \quad v_{2f} = 0.703c$$

Momentum:

$$\begin{aligned} p_i &= m_1 v_{1i} + m_2 v_{2i} = 2m(0.550c) + m(-0.340c) = 0.76mc \\ p_f &= m_1 v_{1f} + m_2 v_{2f} = 2m(0.069c) + m(0.703c) = 0.841mc \end{aligned}$$

Momentum is not conserved?

Need a new definition of momentum that:

1. Reduces to classical version $p = mv$ at low speeds ($v \ll c$)
2. Conserves momentum such that principle of relativity holds, if momentum is conserved at one reference frame it must be conserved at all reference frames.

Relativistic Interpretation

m: mass of particle

v: velocity of particle as measured in a particular reference frame

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}} (= \gamma mv)$$

From above, for a stationary observer:

$$p'_i = \frac{m_1 v'_{1i}}{\sqrt{1 - v'_{1i}^2/c^2}} + \frac{m_2 v'_{2i}}{\sqrt{1 - v'_{2i}^2/c^2}} = -1.134mc$$

Note, the total initial momentum is different than in above classical interpretation.

Now, after finding final velocity $v'_{1f} = 0.585c$, $v'_{2f} = 0.294c$

$$p'_f = \frac{m_1 v'_{1f}}{\sqrt{1 - v'_{1f}^2/c^2}} + \frac{m_2 v'_{2f}}{\sqrt{1 - v'_{2f}^2/c^2}} = -1.134mc$$

Using the same steps as above to find the initial and final velocities in reference frame O:

$$v_{1i} = 0.55c, v_{2i} = -0.340c$$

$$v_{1f} = 0.069c, v_{2f} = 0.703c$$

Initial momentum $p_i = 0.956mc$ Final momentum $p_f = 0.956mc$

Momentum is conserved in all reference frames

Comparison of Classical and Relativistic Momentum

Consider the momentum of an electron ($m=9.11 \times 10^{-31} kg$) moving with speed $v = 0.75c$. How does the relativistic momentum compare with the classical result?

Answer

$$p = mv = 2.05 \cdot 10^{-22} kgm/s$$

$$p = \gamma mv = 3.1 \cdot 10^{-22} kgm/s$$

Relativistic energy is about 50% larger!

3.3.2 Relativistic Kinetic Energy

Similar as momentum, the classical definition of kinetic energy would lead to an apparent lack of energy conservation.

$$KE = \frac{1}{2}mv^2$$

We need a relativistic definition for kinetic energy.

Consider the work-energy theorem (change in kinetic energy is equal to the integral of the force):

$$W = \int F dx = \text{int}\left(\frac{dp}{dt}\right) dx = \int \frac{dx}{dt} dp = \int v dp$$

Using integration by parts:

$$W = pv - \int p dv$$

Therefore,

$$\begin{aligned} KE &= \gamma mv \cdot v - \int_0^v \gamma mv \, dv \\ &= \gamma mv^2 + mc^2\gamma - mc^2 \\ KE &= \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2 \end{aligned}$$

3.3.3 Rest Energy

We can write relativistic kinetic energy in terms of relativistic total energy and rest energy, E and E_0 .

$$K = E - E_0, \quad E = \gamma mc^2, \quad E_0 = mc^2$$

m: rest mass, mass as measured in rest frame of particle.

Rest Energy

What is the rest energy of an electron ($M = 9.11 \times 10^{-31} kg$)? What is the momentum of an electron moving with speed $v = 0.75c$?

Answer

$$\begin{aligned} E_0 &= mc^2 = (9.11 \times 10^{-31} kg)(3.00 \times 10^8 m/s)^2 \\ &= 9.19 \cdot 10^{-14} kgm^2/s^2 \end{aligned}$$

Unit: eV: $1 \text{ eV} = 1.609 \cdot 10^{-19}$

$$E_0 = 5.11 \cdot 10^5 \text{ eV} = 0.511 \text{ MeV}$$

Electron mass: $m_e = 0.511 \text{ MeV}/c^2$

$$p = \gamma mv = \frac{0.511 \text{ MeV}/c^2 \cdot 0.75c}{1 - 0.75^2} = 0.58 \text{ MeV}/c^2 c = 0.58 \text{ MeV}/c$$

Units:

These units are very convenient when working with particles at relativistic speeds.

Energy:	MeV
Momentum:	MeV/c
Mass:	MeV/c^2

3.3.4 Energy-Momentum Relationship

It is often useful to write the relativistic total energy of a particle in terms of its momentum:

$$\begin{aligned} p &= mv\gamma \\ E &= mc^2\gamma \\ E^2 &= \frac{(mc^2)^2}{1-v^2/c^2} \\ (pc)^2 &= \frac{m^2v^2c^2}{1-v^2/c^2} \\ E^2 &= \frac{(mc^2)^2 + m^2v^2c^2 - m^2v^2c^2}{1-v^2/c^2} \\ &= \frac{(mc^2)^2(1-v^2/c^2)}{(1-v^2/c^2)} + (pc)^2 \end{aligned}$$

Energy-Momentum Relationship

$$E^2 = (pc)^2 + (mc^2)^2$$

4 Lecture #4

4.1 Examples

LHC Protons

At CERN's Large Hadron Collider, protons are accelerated to a total energy of $E = 6.5 \text{ TeV}$. What speed are the protons moving with?

Answer

$$m_p = 938 \text{ MeV}/c^2$$

$$E = 6.5 \text{ TeV} = 6.5 \cdot 10^6 \text{ MeV}$$

Note that $m_p \ll E$

$$E = mc^2(\gamma - 1) \rightarrow E = mc^2\gamma$$

$$V = c\sqrt{1 - \frac{mc^2}{E}}$$

$$v = c \cdot \sqrt{1 - \frac{938 \text{ MeV}}{6.5 \cdot 10^6 \text{ MeV}}^2}$$

$$v = 99.99999c$$

Neutral K Meson

A neutral K meson (mass $497.7 \text{ MeV}/c^2$) has a kinetic energy of 77 MeV . It decays into a pi meson (mass $139.6 \text{ MeV}/c^2$), moving in the direction of the original K meson with momentum $381.6 \text{ MeV}/c$, and another particle of unknown mass.

- What is the momentum total relativistic energy of the unknown particle?

1

$$E^2 = (pc)^2 + (mc^2)^2$$

Momentum?

$$p_x = p_k - p_\pi, p_\pi = 381.6 \text{ MeV}/c$$

$$p_k = 1/c \sqrt{E_k^2 - (m_k c^2)^2} \rightarrow E_k = K_k + M_k c^2 = 66 \text{ MeV} + 497.7 \text{ MeV}/c^2 c^2 = 574.7 \text{ MeV}$$

$$p_k = 287.4 \text{ MeV}/c$$

$$p_x = 287.4 \text{ MeV}/c - 381.6 \text{ MeV}/c = -94.2 \text{ MeV}/c$$

The unknown particle is moving in the opposite direction.

Total energy?

$$E_x = E_k - E_\pi = \sqrt{(381.6 \text{ MeV}/c \cdot c)^2 + (139.6 \text{ MeV})^2} = 406.3 \text{ MeV}$$

$$E_x = 574.7 - 406.3 \text{ MeV} = 168.5 \text{ MeV}$$

- What is the mass of the unknown particle?

2

$$\begin{aligned} m_x c^2 &= \sqrt{E_x^2 - (pc)^2} \\ &= \sqrt{(168.4 \text{ MeV})^2 - (-94.2 \text{ MeV}/c \cdot c)^2} \approx 139.6 \text{ MeV} \end{aligned}$$

The Mass is $139.6 \text{ MeV}/c^2$, which is another pion!

Example of mass-energy equivalence – fission

Decay of heavy radioactive nucleus at rest into several lighter particles, emitted with large kinetic energies (fission). Generally element : ${}^A_Z X$

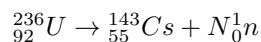
Where:

X = element name

Z = atomic number (# of protons)

A = mass number (# of protons + # neutrons)

decay of an unexcited uranium nucleus at rest



How much energy is released per fission?

Use:

$$M({}^{236}U) = 236.045563u$$

$$M({}^{90}Rb) = 89.314811u$$

$$M({}^{143}Cs) = 142.927220u$$

$$m_n = 1.008665u$$

Total re. energy: $E = E_0 + k$

Initial: $E(U) = E_0(U) + K(U) = E_0(U)$

Final: $E_{final} = E_0(Rb) + E_0(Cs) + 3 \cdot E_o(U) + K(Rb) + K(Cs) + 3 \cdot K(U)$ "released energy" = Q

$$Q = M(U)c^2 - M(Rb)c^2 - M(Cs)c^2 - 3m(n)c^2 = 0.177536uc^2$$

$$= 165.4 \text{ MeV}$$

This means that the rest energy of the Uranium nucleus is greater than the rest energy of all the elements it decays into. This difference in energy is released as kinetic energy.

Energy released when 1.0 kg unarium undergoes fission at 40 % efficiency

Determine of nuclei in 1kg ${}^{236}U$?

$$N = \frac{6.022 \cdot 10^{23} \text{ nuclei/mol}}{236 \text{ g/mol}} \cdot 1000 \text{ g} = 2.55 \cdot 10^{24} \text{ nuclei}$$

Total energy produced:

$$E = \text{efficiency} \cdot N \cdot Q$$

$$E = 0.40 \cdot 2.55 \cdot 10^{24} \text{ nuclei} \cdot 165.4 \text{ MeV/nuclei}$$

$$E = 7.5 \cdot 10^6 \text{ kWh}$$

Enough energy to keep a lightbulb on for tens of thousands of years!

4.2 Conservation Laws

- E: Total relativistic energy
 K: Relativistic kinetic energy

Momentum Conservation

In an isolated system of particles, the total linear momentum remains constant.

Energy Conservation

In an isolated system of particles, the relativistic total energy (kinetic energy plus rest energy) remains constant.

5 Lecture #5

Here we will go into the early work that lead up to the development of **Quantum Mechanics**
 Three concepts showed that electromagnetic waves have particle-like properties:

1. The photoelectric effect
2. Thermal radiation (blackbody radiation)
3. The Compton effect

Energy is delivered not smoothly or continuously over a wave front, but rather in discrete bundles ("quanta") known as "photons".

5.1 The Photoelectric Effect

5.1.1 Electromagnetic Waves

EM Waves are predicted by Maxwell's equations, the electric and magnetic field each satisfy a wave equation.
 EM waves travel through vacuum with speed of light c:

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

μ_0 = magnetic permeability, ϵ_0 = electric permittivity

The electric and magnetic fields are perpendicular to each other, and perpendicular to the direction of the wave propagation. For a plane EM wave traveling in the position z direction:

$$E = E_0 \sin(kz - \omega t), \quad B = B_0 \sin(kz - \omega t)$$

k = wave number, $k = \frac{2\pi}{\lambda}$, ω = angular frequency = $2\pi f$

Magnitudes of an EM wave are related as $\frac{E_0}{B_0} = c$. EM waves carry energy, the energy flux is specified by the Poynting vector:

$$S = 1/\mu_0 E \times B$$

For a plane wave:

$$S = 1/\mu_0 \cdot E_0 \cdot B_0 \cdot \sin^2(kz - \omega t) \hat{k}$$

Where \hat{k} is the unit vector in z-direction.

Note:

Power = energy per unit time

Energy flux = power per unit area

Intensity = average power per unit area (average energy per unit time and unit area)

Assume we have a receiver to determine the EM power delivered by plane wave to received with sensitive area A:

$$P = SA$$

If we can assume the receiver is perpendicular, we can ignore the vector component.

$$P = 1/\mu_0 \cdot E_0 \cdot B_0 \cdot A \cdot \sin^2(kz - \omega t) = \frac{1}{\mu_0 c} E_0^2 \cdot A \cdot \sin^2(kz - \omega t)$$

Important takeaway is that intensity is proportional to the square of the amplitude and that something something time.

5.1.2 Interference & Diffraction

Key property of waves is the principle of superposition.

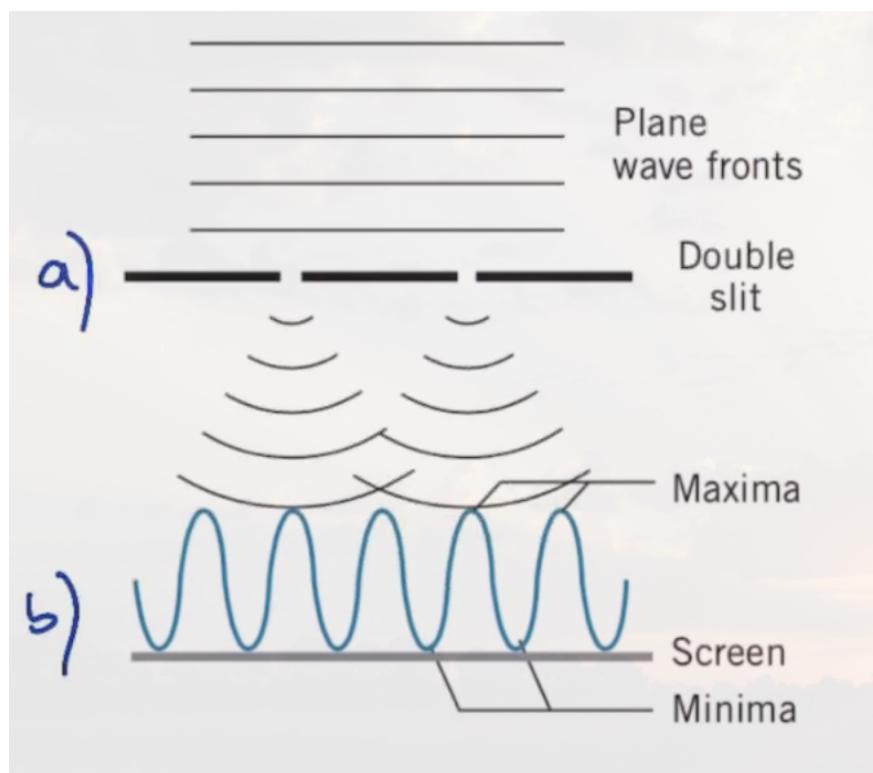


Figure 1: Setup 1

Wave theory of light is insufficient though!

5.1.3 Photoelectric Effect

What is it?

Illuminate a metal surface with light - electrons can be emitted from the surface (1887). Work function: Minimum quantity of energy needed to remove an electron (material dependent).

When emitted in this way, they're referred to as "photoelectrons".

Measure:

Rate of electron emission, measured as an electric current by an ammeter.

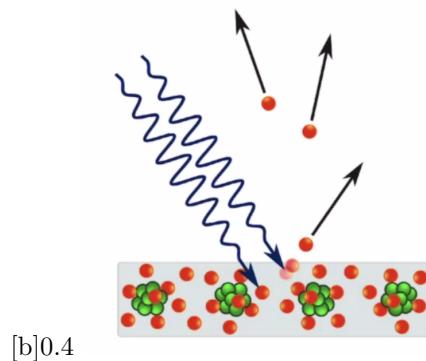


Figure 2: Setup 1

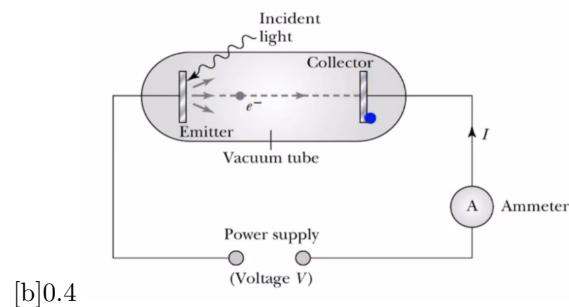
Experimental setup

Figure 3: Setup 1

Maximum kinetic energy of the photoelectrons, measure by applying negative potential difference.
Electrons travelling from emitter to collector gain potential energy:

$$\Delta U = q\Delta V = -e\Delta V (> 0, \Delta V \text{ is negative})$$

and lose the same amount of kinetic energy.

We can now determine the Maximum Kinetic Energy (k_{max}), by measuring the stopping potential (V_s).

$$K_{max} = eV_s$$

Just large enough to stop the electrons from reaching the collector.

What are the expected results?

1. For a given wavelength of light, as the intensity increases, the maximum kinetic energy of the photoelectrons should also increase.
2. For a given wavelength of light, at low enough intensities there should be a time delay in electron emission.
3. Electrons should be emitted for all wavelengths of light

None of these expectations were met by the experimental results.

Actual Experimental results:

1. For a given wavelength of light, the maximum kinetic energy is independent of the intensity of the light source
2. First photoelectrons are emitted virtually instantaneously after the light source is turned on
3. The photoelectric effect does not occur if the frequency of light is below a certain value

This lead Einstein to formulate the quantum theory of the photoelectric effect

Einstein's quantum theory of the photoelectric effect

Energy of an EM radiation is not continuously distributed across wave but concentrated in bundles: "quanta" (photons).

Energy (E) of a photon associated with an EM wave:

$$E = hf$$

f = frequency of EM wave

h = Planck's constant = $6.626 \cdot 10^{-34} J \cdot s$

Because $f = c/\lambda \rightarrow E = \frac{hc}{\lambda}$

Because photons travel at the speed of light:

$$E = pc$$

Where E is the energy and p is the momentum.

$$p = \frac{E}{c} = \frac{\frac{hc}{\lambda}}{c} = \frac{h}{\lambda}$$

5.1.4 Resulting interpretation of photoelectric effect

One photoelectron released is due to encounter with a single photon. The entire energy of the photon is delivered instantaneously to the electron. If the photon energy is greater than the work function of a material, then a photoelectron is released. Otherwise, there is no photoelectric effect.

If a photon energy exceeds work function, excess energy is the kinetic energy of the electron:

$$K_{max} = E_{photon} - \phi = hf - \phi$$

Where ϕ = work function (depends on material).

This now explains why the maximum kinetic energy is independent of the intensity of the light source. I.e. Higher intensity means more photons striking surface, and more photoelectrons released, but all with the same maximum kinetic energy! There is no time delay, because the entire energy of the photon is delivered to the photoelectron. The photoelectric effect only occurs if the frequency of light is above a certain value, because the energy has to be greater than the work function, $E_{photon} > \phi$, $hf > \phi$.

Detailed tests of the photon theory were done by Millikan in 1915. He measured maximum kinetic energy (stopping potential) for different frequencies of light using sodium, and extracted Planck's constant (h).

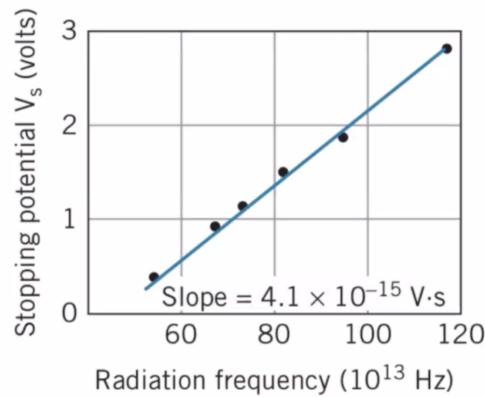


Figure 4: Millikan's results

Energy of a photon

What is energy and momentum of a photon of red light with wavelength 650 nm?

Energy?

$$E = hf = \frac{hc}{\lambda}$$

Often convenient to use "eV".

$$hc = 1240 \text{ eV} \cdot \text{nm}$$

$$E = \frac{1240}{650} \approx 1.9 \text{ eV}$$

Momentum?

$$p = \frac{E}{c} = 1.9 \text{ eV}/c$$

Very convenient units!

Example

What happens if frequency is doubled while keeping photon emission rate unchanged?

Frequency

Doubling the frequency will double the energy of the photons re. $e = hf$. Because $K_{max} = hf - \phi$ we will have a greater stopping potential, but current will be unchanged.

What happens if the metal surface was replaced with one having a larger work function?

Work Function

If we have a greater work function this means that the maximum kinetic energy will be lower. If the work function exceeds the photon energy, then there will be no photoelectric effect.

6 Thermal Radiation

Thermal Radiation - Emission of electromagnetic waves by all objects with temperature greater than absolute zero. Lightbulb, the sun, a human being, etc. However, a human body would emit light in the infrared range.

6.0.1 Experimanetal results

1. Total intensity (I) radiated across all wavelengths is proportional to temperature (T) as:

Stefan's Law

$$I = \sigma T^4$$

Where

$$T$$

$= K$, and the Stefan-Boltzmann constant $\sigma = 5.67 \cdot 10^{-8} \frac{W}{m^2 K^4}$ and u refers to unit area.

2. Wavelength where the emitted intensity is maximal:

$$\lambda_{max} \propto 1/T$$

Wien's Displacement Law

$$\lambda_{max} \cdot T = 2.8978 \cdot 10^{-3} m \cdot K$$

6.0.2 Examples

Example

At what wavelength does a human body emit maximum thermal radiation?

Answer

$$T \approx 310K$$

$$\lambda_{max} = \frac{2.8978 \cdot 10^{-3} m \cdot K}{310k} \approx 9.3\mu m$$

Indeed, the infrared range.

Example

To what temperature must we heat an object to see its peak thermal radiation as visible light?

Answer

if $\lambda_{max} = 700nm$:

$$T = \frac{2.8978 \cdot 10^{-3} m \cdot K}{700 \cdot 10^{-9}m}$$

$$\approx 4000K$$

6.1 Black-Body

To simplify further analysis, we consider a "black-body":

Black-body

Absorbs all radiation incident on it and reflects none of the incident radiation.

(Note: A black-body can absorb energy which heats it up, and it will then emit its own radiation. But only its temperature determines the intensity and wavelength.)

We want to know how the intensity of the emitted radiation depends on wavelength.

Classical Picture

Lord Rayleigh used classical theories of electromagnetism and thermodynamics to show that black-body intensity vs wavelength should be

$$I(\lambda) = \frac{2\pi c}{\lambda^4} kT$$

This "Rayleigh-Jeans" formula is not accurate.

One major problem is that it fails badly to match experimental data for low wavelengths.

6.2 Quantum Theory of thermal radiation

Planck assumed that the radiation in the blackbody was emitted by some sort of "oscillators", and used statistical methods recently developed by Boltzmann.

Planck's modification:

The cavity radiation originates from oscillations inside the cavy walls. These oscillations can only have discrete values of energy:

$$E_n = n \cdot \epsilon, \quad n = 1, 2, 3, 4\dots$$

These oscillators can absorb or emit energy only in discrete bundles ("quanta") of the fundamental quantum energy given by:

$$\sigma = hf$$

$$h = \text{Planck's constant} = 6.63 \cdot 10^{-34} J \cdot s$$

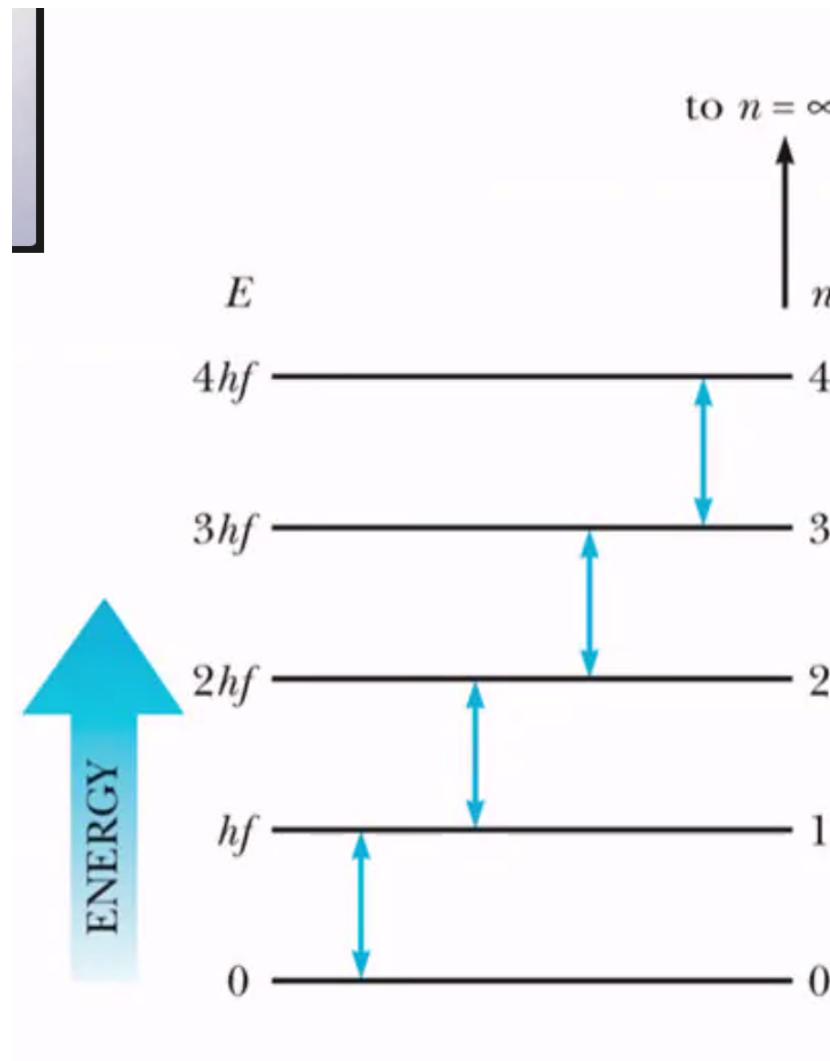


Figure 5: Setup 3

Intensity as a Function of Wavelength

$$I(\lambda) = \frac{2\pi hc^2}{\lambda^5} \cdot \frac{1}{e^{hc/\lambda kT} - 1}$$

- I: Intensity of radiation
- h: Planck's constant
- λ : Wavelength
- k: Boltzmann's constant
- T: Temperature

Stefan's law from Planck's law gives a relationship between Stefan-Boltzmann's constant (σ) and Planck's constant (h):

$$\sigma = \frac{2\pi^5 k^4}{15c^2 h^3}$$

Found by integrating across all wavelengths.

Millikan obtained the value of Planck's constant using the photoelectric effect, measuring the maximum kinetic energy as a function of frequency(absorption of EM radiation). Planck determined the value of h using blackbody intensity data (emission of EM radiation).

These agreeing implies it is not an accident but a property of the EM field.

6.2.1 Cosmic Microwave Background (CMB)

CMB is very close to a perfect black-body.

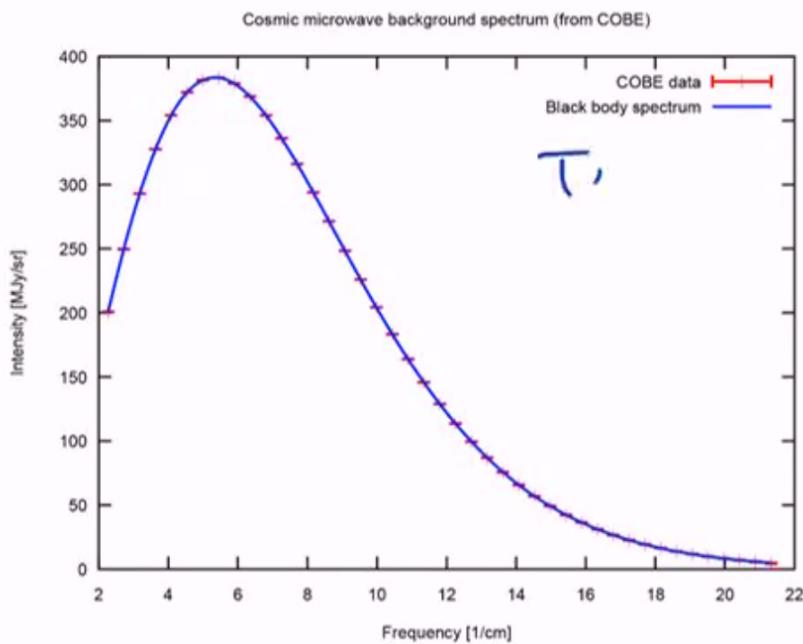


Figure 6: Setup 3

Origin:

- In very early times, the Universe was filled with hot plasma of particles. These photons were constantly interacting with free electrons, "stopping" them.
- After some 380,000 years, Universe had expanded and cooled to 3000K, where electrons and protons could form hydrogen atoms
- Photons now free to roam!
- These photons fill the Universe today and are detected as the CMB!
- Blackbody spectra observed is that of $T \approx 2.7\text{K}$ because photon frequency we detect is greatly Doppler shifted (wavelength is "redshifted").

Example: The sun as a blackbody

Treating the Sun as a blackbody (good approximation), and using that its radius is $7 \cdot 10^8\text{m}$ (R), the average Earth-Sun distance is $1.5 \cdot 10^{11}\text{m}$ (R_s), and the power per unit area from the Sun as measured on Earth is W/m^2 . Estimate the surface temperature of the Sun.

Answer

Need: connect $I(R_s)$ to $I(R)$

Conservation of energy:

$$I(R_s) \cdot 4\pi R_s^2 = I(R)4\pi R^2$$

$$I(R_s) = I(R) \cdot \frac{R^2}{R_s^2}$$

Can deduce that the surface temp $T = I = \sigma T^4$

$$\sigma T^4 = I(R) \cdot \frac{R^2}{R_s^2}$$

$$T = \left(\frac{I(R)}{\sigma} \cdot \frac{R^2}{R_s^2} \right)^{1/4} = \frac{1400\text{W/m}^2}{5.67 \cdot 10^{-8}} \cdot \frac{1.5 \cdot 10^{11}}{7.0 \cdot 10^8}$$

$T \approx 5800\text{K}$

7 The Compton Effect

Finally shifted public support for Einstein's explanation of black-body radiation.

Scattering of radiation from loosely bound, nearly free electrons.

The classical wave picture would predict that scattered radiation is less energetic than incident radiation, and would have the same wavelength. *the latter point failed in experiments*

Consider this process instead as the scattering of a photon -

An interaction between two particles, incident photon and electron off which it scatters. In the classical sense an interaction is a collision.

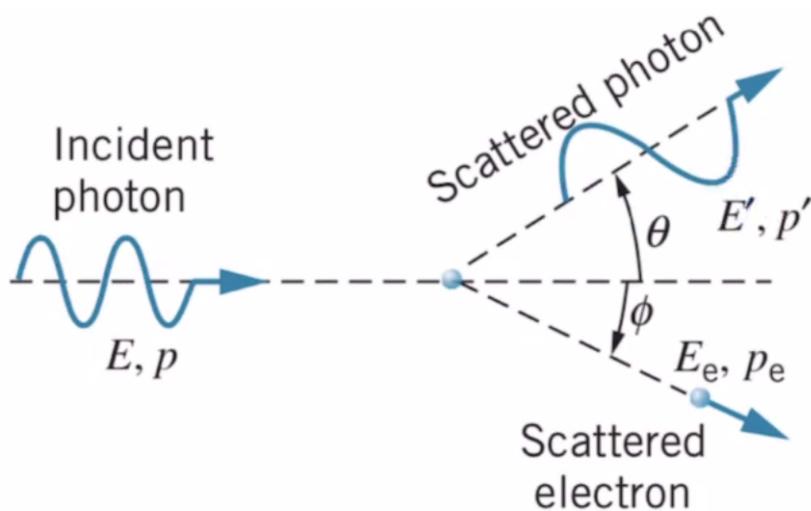


Figure 7: Compton Effect

Conservation laws for relativistic momentum and total energy

$$\text{Energy: } E + m_e c^2 = E' + E_e$$

Momentum:

$$\text{Horizontal : } p = p' \cos(\theta) + p_e \cos(\phi)$$

$$\text{Vertical : } 0 = p' \sin(\theta) - p_e \sin(\phi)$$

Eliminate ϕ

$$p_e \cos \phi = p - p' \cos \theta$$

$$p_e \sin \phi = p' \sin \theta$$

$$p_e^2 (\cos^2 \phi + \sin^2 \theta) = p^2 + p'^2 \cos^2 \theta - 2pp' \cos \theta p'$$

$$p_e^2 = p^2 + p'^2 - 2pp' \cos \theta$$

Eliminate E_e

$$E_e = E - E' + mc^2$$

Note: $E^2 = (mc^2)^2 + (pc)^2$

$$E_e^2 = (E - E')^2 + (m_e c^2)^2 + 2(E - E')m_e c^2$$

$$(m_e c^2)^2 + (p_e c)^2 = (E - E')^2 + (m_e c^2)^2 + 2(E - E')m_e c^2$$

$$(p_e c)^2 = (E - E')^2 + 2(E - E')m_e c^2$$

Plug part a into the expression.

$$p^2 c^2 + p'^2 c^2 - 2pp' c^2 \cos\theta = E^2 + E'^2 - 2EE' + 2(E - E')m_e c^2$$

Note: $E = pc$ for a photon. $p^2 c^2 = E^2, p'^2 c^2 = E'^2$.

$$-2EE' \cos\theta = -2EE' + 2(E - E')m_e c^2$$

Compton Scattering

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m_e c^2} (1 - \cos\theta)$$

Wavelength

$$E = \frac{hc}{\lambda}$$

$$\lambda' - \lambda = \frac{h}{m_e c} (1 - \cos\theta)$$

λ' : wavelength of scattered photon

λ : wavelength of incident photon

The difference between those two values depends on the angle. If the angle is 0.

Compton found that light scattered from a particle at rest is shifted in wavelength! This was viewed as a direct and incontrovertible experimental evidence that light behaves as a particle (on a subatomic scale!) This particle is the photon (symbol γ).

Kinetic energy gained by the electron?

$$E + m_e c^2 = E' + m_e c^2 + K_e \rightarrow E' + E$$

Examples (Groups)

Why are x-ray photons used in the Compton experiment, rather than visible light photons?

Three cases

1. High energy rays from cobalt, $\lambda = 0.00106 \text{ nm}$.

$$\lambda' - \lambda = \frac{h}{m_e c} (1 - \cos\theta)$$

$$\lambda' - \lambda = \frac{6.63 \cdot 10^{-34} \text{ Js}}{9.1093835610 \cdot 31 \text{ kg}} (1) = 7.274 \cdot 10^{-4} = 2.4277 \text{ nm}$$

The shift is substantially larger than the incident wavelength.

2. Shift = 2.4277 nm (same regardless of incident wavelength), 0.0712 nm is the incident.
This one is just right.
3. Shift = 2.4277 nm (same regardless of incident wavelength), 546 nm is the incident.
The shift in wavelength is very small compared to the incident wavelength.

Binding energy

The binding energy will be very small compared to the energy of the x-ray.

$$hc/\lambda = \text{a lot}$$

Photoelectric effect in Zinc

Photoelectrons released from zinc by ultraviolet light were stopped by a voltage of 4.3 V. What is the maximum kinetic energy and speed of the photoelectrons?

7.1 What is a photon?

- Just like an EM wave, photons move at the speed of light
- Photons have zero mass and zero rest energy
- Photons carry energy and momentum, related to the frequency/wavelength of EM waves

$$E = hf = \frac{hc}{\lambda} = pc \rightarrow p = \frac{h}{\lambda}$$

- Photons can be created or destroyed when radiation is emitted or absorbed
- They can collide with other particles, like electrons

Wave-particle duality, light is neither a particle nor a wave, it is both. Our failure to classify light as one or the other is more a failure of our vocabulary, as light is a "quantum particle".

7.1.1 Connection between wave behavior and particle behavior of light

Probability to observe photons is proportional to the electric field amplitude squared

8 Atoms

Around year 1900, evidence indicating that the atom was not a fundamental unit, but instead composed of smaller particles...

8.1 Knowledge of atoms at the time

- Atoms are very small
- Atoms are stable (usually)
- Atoms are electrically neutral, but since they contain electrons, there must be a corresponding net positive charge
 - The electron was discovered, protons were not, but it was understood that some positive charge had to be present
- Atoms can emit and absorb electromagnetic radiation

8.1.1 Atomic Model 1904

An early model of the atom was proposed by J.J. Thomson. It was known as the Thomson model (alternatively plum-pudding model), where protons and electrons were all mixed up together. Thomson thought an electron could vibrate about its equilibrium position when the atom was heated and produce electromagnetic radiation.

This would predict that light would scatter in a way inconsistent with observations.

Scattering in the Thomson model

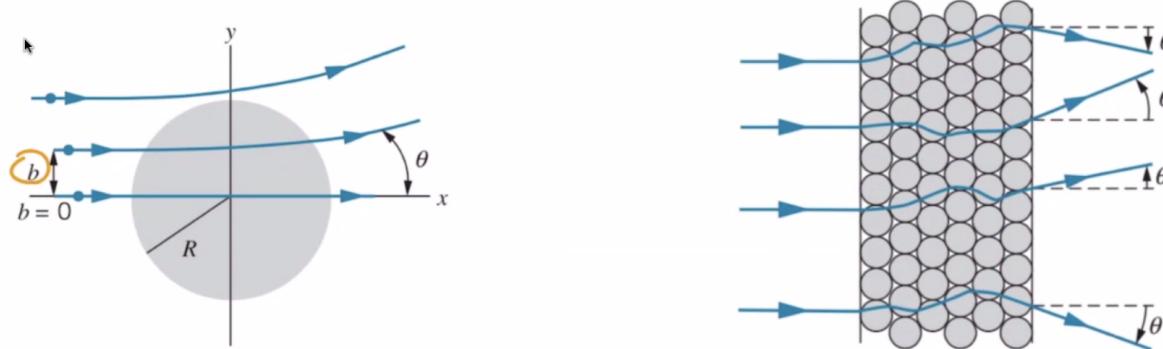


Figure 8: Thomson

Tested with micrometer thick foil ($n \approx 10^4$). Expected scattering angle would be about a degree, no scattering at large angles.

8.2 Rutherford Scattering

1910 performed important scattering experiment in the laboratory of E. Rutherford, in order to study the structure of matter. Alpha particles ($\text{He nucleus } (+2e)$) scattering by a thin gold foil. Observed a non-negligible scattering at very large angles.

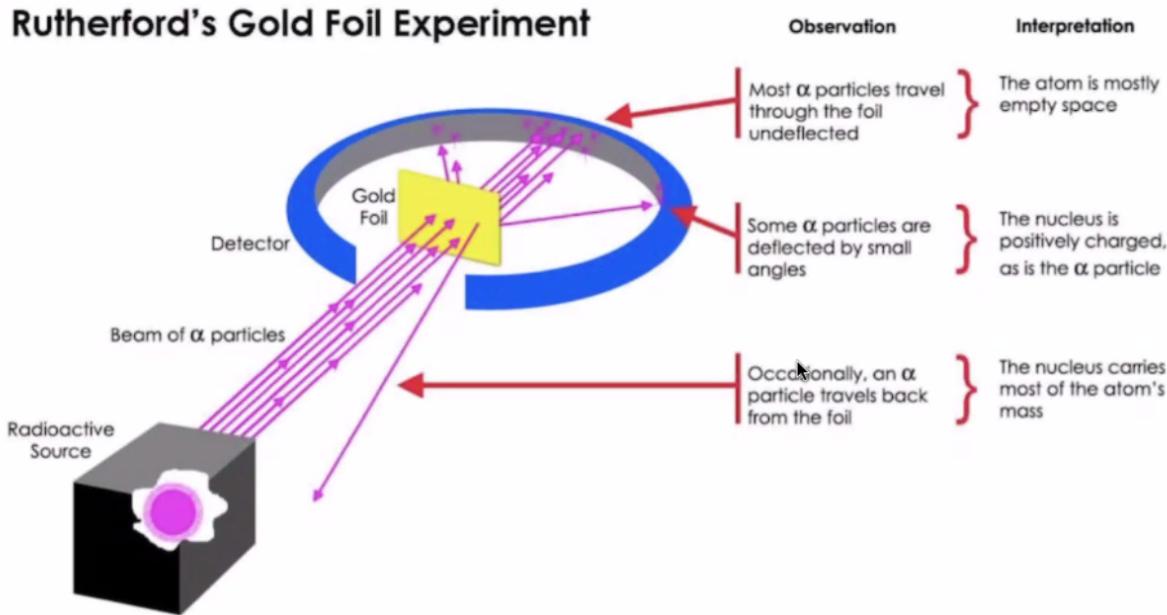


Figure 9: Rutherford

Lead to the realization that atoms were mostly empty space, with positive charge and mass concentrated in a very small region. This is known as the nucleus.

8.2.1 Analysis

Repulsive force experienced by projectile:

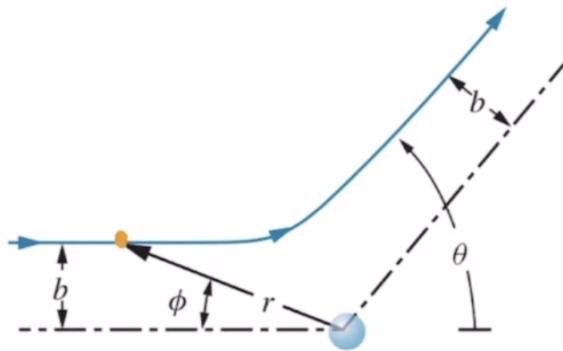


Figure 10: Rutherford

$$F = \frac{1}{4\pi\epsilon_0} \frac{|q_1||q_2|}{r^2} = \frac{(ze)(Ze)}{4\pi\epsilon_0 r^2}$$

Assumptions:

1. Electrons in the atom do not have any sizeable contribution to the scattering due to their small mass.
2. Nucleus much more massive than the projectile such that it doesn't noticeably move during scattering process

It can be shown that projectile follows a hyperbolic path:

$$1/r = 1/b \sin\phi + \frac{zZe^2}{8\pi\epsilon_0 b^2 k} (\cos\phi - 1)$$

z , Z : electric charge of particle, nucleus

K : kinetic energy of particle

We can relate the impact parameter (b) to the scattering angle (θ):

$$b = \frac{zZ}{2k} \frac{e^2}{4\pi\epsilon_0} \cot\left(\frac{\theta}{2}\right)$$

Where

$$\frac{e^2}{4\pi\epsilon_0} = 1.44 eV * nm$$

What fraction of projectiles are scattered at angles greater than some particular value? Some impact parameter b_0 corresponds to a given angle θ_0 . A projectile scattered at $\theta > \theta_0$ means it must have $b < b_0$. What fraction has that?

1. Single layer of atoms:

$$f = \frac{\pi b_0^2}{\pi R^2}$$

2. Multiple layers of atoms: (Due to technological constraints)

Volume of the target :

$$V = At$$

Atoms in this volume:

$$\rho = \text{density } N = n_{\text{moles}} \cdot N_A = \frac{m}{M} \cdot N_A$$

$$= \frac{\rho V}{M} \cdot N_A = \frac{\rho At}{M} \cdot N_A \# \text{ atoms per unit volume:}$$

$$n = \frac{N}{V} = \frac{\rho N_A}{M}$$

Incident projectile sees: n^*t nuclei/unit area

Effective area that each nuclei contributes will be $\frac{1}{n \cdot t}$

Fractions of projectiles that fall within πb_0^2 :

$$f = \frac{\pi b_0^2}{1/(ut)} = nt\pi b_0^2$$

Fraction of projectiles that scatter with angle $> \theta_0$:

$$f(\theta > \theta_0) = f(b < b_0) = nt\pi b_0^2$$

Example

We have a thin gold foil ($2.0 \cdot 10^{-4}$) off which alpha particles with kinetic energy of 8 MeV are scattered. What fraction are scattered at angles larger than 90°?

Scattering at 90°

$$b = \frac{zZ}{2K} \frac{e^2}{4\pi\epsilon_0} \cot(\theta/2)$$

z: 2

Z: 79

K = 8 MeV

 $\theta = 90^\circ$

$$b \approx 14 fm$$

Nuclei per unit volume

$$n = \frac{\rho N_A}{M} = 5.9 \cdot 10^{28} nuclei/m^3$$

Fraction scatters $\theta > 90^\circ$

$$f = nt\pi b^2 = 7.5 \cdot 10^{-5}$$

Note that this is small but compared to the Thomson model it's massive and actually observable.

Distance of closest approach to nucleus

As a positively charged projectile approaches a nucleus, it will slow down - part of its kinetic energy is exchanged for electrostatic potential energy due to the charged nucleus:

Answer

Potential energy:

$$U = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r} = \frac{zZe^2}{4\pi\epsilon_0}$$

Total energy:

$$E = K + U$$

When $U = 0$ ($r \gg 0$) :

$$E = K_{max} = 1/2mv^2$$

When $U = U_{max}$:

$$E = K_{min} + U_{max} = 1/2mv^2 + \frac{zZe^2}{4\pi\epsilon_0}$$

Note that the last two are the same, angular momentum is also conserved!

Far from nucleus: $L = mvb$

At r_{min} : $L = mv_{min}r_{min}$

$$v_{min} = \frac{vb}{r_{min}}$$

Distance of closest approach: Set $b = 0$ and solve for r_{min}

Distance of closest approach

$$d = \frac{1}{4\pi\epsilon_0} \frac{zZe^2}{K}$$

9 Lecture #6

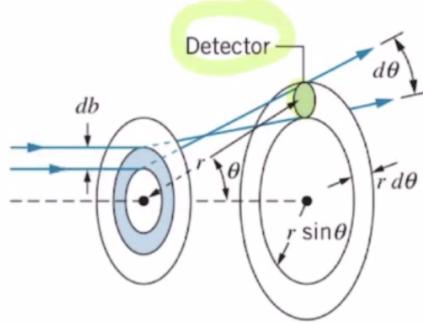


Figure 11: Setup 3

Rutherford Scattering Formula

Determine the probability for a projectile to be scattered at an angle θ .

From the fraction of projectiles scattered at angles greater than angle θ , we can determine the fraction scattered into a small angular region from θ to $d\theta$.

$$f = nt\pi b^2$$

Fraction scattered into region:

$$df = nt\pi bdb$$

Impact parameter (b) related to scattering angle θ :

$$b = \frac{zZ}{2K} \cdot \frac{e^2}{4\pi\epsilon_0} \cot(\theta/2)$$

Then we can differentiate b to express df with respect to d θ .

The fraction scattered into a small angular region (df) is scattered into a ring with area dA:

$$dA = (2\pi r \sin\theta) * rd\theta$$

The probability per unit area is the fraction scattered per unit area:

$$N(\theta) = \frac{|df|}{dA} = \frac{nt}{4r^2} \cdot \left(\frac{zZ}{2k}\right)^2 \cdot \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \cdot \frac{1}{\sin^4(\theta/2)}$$

Where

n: number of nuclei per unit volume

t: thickness of film

r: distance between detector and nucleus

z, Z: electric charges of projectile, nucleus

K: kinetic energy

θ : scattering angle

These predictions were extensively tested by his Ph.D students, and confirmed.

6.1 Line Spectra

In the late 19th century, pure chemical elements were observed to produce unique wavelengths of light when burned. But not *why*.

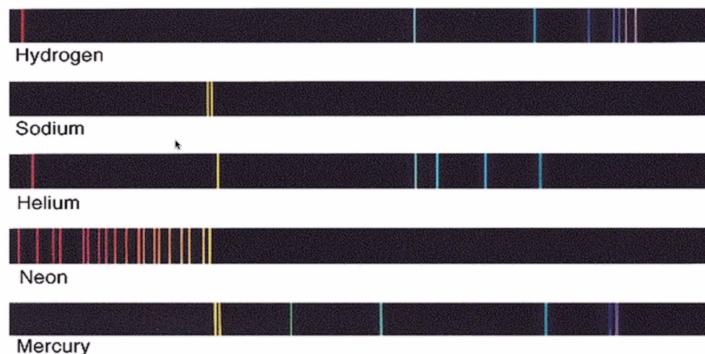


Figure 12: Setup 3

Atoms can emit and absorb electromagnetic radiation. There can be either a continuous or discrete spectra (line spectra).

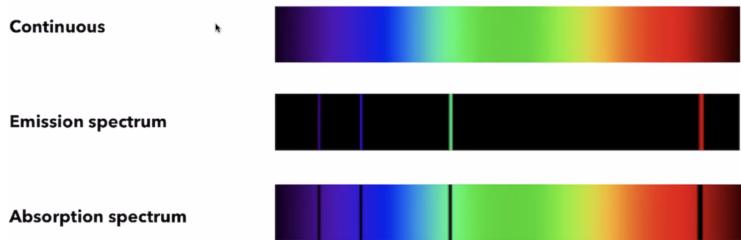


Figure 13: Setup 3

Absorption spectra generally correspond to many but *not all* lines seen in emission spectra!

6.1.1 Experimentally

Emission spectra - Electric discharge created inside a tube containing a vaporized element.
Absorption spectra - Continuous spectrum of light sent through a vaporized element.

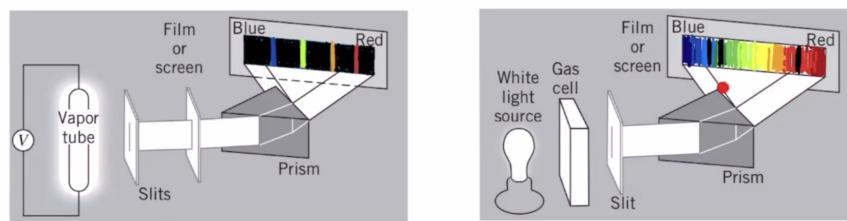


Figure 14: Setup 3

6.1.2 Balmer Series

1885 school teacher showed (basically from trial and error) that the wavelengths of the hydrogen emission lines in the visible region could be calculated as:

$$\lambda = (364.5\text{nm}) \cdot \frac{n^2}{n^2 - 4}$$

where $n = 3, 4, 5, 6\dots$

General Hydrogen Spectra

Soon a more general formula describing all lines was discovered.

$$\lambda = \lambda_{limit} \frac{n^2}{n^2 - n_0^2}$$

For $n = n_0 + 1, n_0 + 2, n_0 + 4, n_0 + 4\dots$ λ_{limit} : wavelength of each series limit ($n \rightarrow \infty$). $n_1 = 1$: Lyman Series $n_2 = 1$: Balmer Series $n_3 = 1$: Paschen Series

$$f = \frac{c}{\lambda}$$

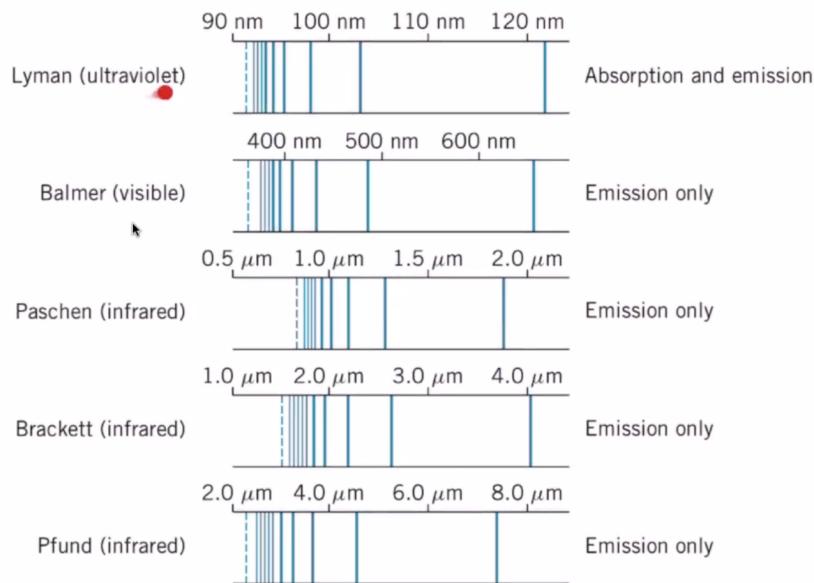


Figure 15: Wavelengths of various spectra

Ritz Combination Principle

Peculiar property: Frequencies corresponding to certain wavelength can be added to yield other frequencies on the spectrum

Example of Ritz

Show that the two longest wavelengths of the Lyman series and the longest wavelength of the Balmer series fulfill the Ritz combination principle.

Answer

Longest λ of Balmer series ($n=3$, $n_0 = 2$)

$$\lambda = 364.5\text{nm} \cdot \frac{n^2}{n^2 - 4} = 651\text{nm}$$

$$f = \frac{c}{\lambda} = 4.57 \cdot 10^{14}\text{Hz}$$

Longest λ of Lyman series ($n = 2, 3$), $n_0 = 1$)

$n = 2$:

$$\lambda = 91.13\text{nm} \cdot \frac{n^2}{n^2 - 1} = 121.5\text{nm}$$

$$f = \frac{c}{\lambda} = 24.67 \cdot 10^{14}\text{Hz}$$

$n = 3$:

$$\lambda = 91.13\text{nm} \cdot \frac{n^2}{n^2 - 1} = 102.5\text{nm}$$

$$f = \frac{c}{\lambda} = 29.24 \cdot 10^{14}\text{Hz}$$

If we add:

$$24.67 \cdot 10^{14}\text{Hz} + 4.57 \cdot 10^{14}\text{Hz} = 29.24 \cdot 10^{14}\text{Hz}$$

6.2 Bohr Model

Builds on Rutherford's description, proposed in 1913 that the atom resembled a mini-planetary system, with electrons orbiting the positively charged nucleus.

The atom doesn't collapse for the same reason the solar system doesn't collapse, i.e. orbit.

Consider a model of an atom with e- moving at v with centripetal force F, orbiting +Ze at radius r.
Coulomb force:

$$F = \frac{1}{4\pi\epsilon_0} \frac{|q_1||q_2|}{r^2} = k \frac{e^2}{r^2}$$

Centripetal acceleration: $a = v^2/r$.

Newton's second law: $F = ma$.

$$k \frac{e^2}{r^2} = \frac{mv^2}{r}$$

Total energy will be $E = U + K$

$$E = -k \frac{e^2}{r} + 1/2 mv^2$$

$$E = -k \frac{e^2}{r} + 1/2k \frac{e^2}{r} = -1/2k \frac{e^2}{r}$$

However, in classical physics an accelerating electric charge should be radiating EM energy, and will eventually crash into the nucleus!

6.2.1 Bohr's Proposal

There are certain states in which the electron can exist without radiating electromagnetic energy called "stationary states". In these stationary states, the electron angular momentum (L) have only certain values:

$$L = n\bar{h}, \quad n = 1, 2, 3, 4\dots$$

$$\bar{h} = \frac{h}{2\pi}$$

Where h is Planck's constant.

Going back to the earlier example, $\vec{L} = \vec{r} \times \vec{p}$, if r and p are perpendicular, magnitude $L = rp = rmv$.

$$K = 1/2m \left(\frac{n\bar{h}}{rm} \right)^2 = 1/2k \frac{e^2}{r}$$

Solving for r :

$$r_n = \frac{4\pi\epsilon_0\bar{h}^2}{me^2} n^2, \quad n = 1, 2, 3, 4\dots$$

Predicted only certain radii are valid for electrons in hydrogen atoms.

Allowed levels in atomic hydrogen

In the ground state ($n=1$), the orbital radius is the "Bohr radius":

$$s_0 = \frac{4\pi\epsilon_0\bar{h}^2}{me^2} = 0.0529nm$$

$$r_n = a_0 n^2$$

Total electron energy:

$$\begin{aligned} E &= -\frac{e^2}{8\pi\epsilon_0} \frac{me^2}{4\pi\epsilon_0\bar{h}^2} \frac{1}{h^2} \\ E &= -\frac{me^4}{32\pi^2\epsilon_0^2\bar{h}^2} \frac{1}{h^2} \quad n = 1, 2, 3, 4\dots \\ &= -13.60eV \cdot 1/h^2 \end{aligned}$$

$n=1$: Ground state:

$$E_1 = -13.60eV$$

something here

Excitation energy is the energy above the ground state ($n=1$)

$$\Delta E = E_2 - E_1 = -3.4eV + 13.60eV = 10.2eV$$

Magnitude of electron's energy is called its binding energy

$$|E_2| = |3.4eV| = 3.4eV$$

Note, the energy is negative because it is bounded. To free an electron from the atom requires the atom to absorb a certain amount of energy equal to the binding energy, also called "ionization energy". If more energy than the minimum necessary is absorbed, this excess is kinetic energy of the free electron. Binding energy is also the energy that is released as photons when the atom is formed from a nucleus and free electron.

6.3 Connection of Bohr's model with line spectra

Bohn posulated that electron can exist in stationary states without emitting radiation, but it can emit radiation when it moves to a lower energy state!

If the electron jumps from the $n = n_i$ to $n = n_f$ state

The energy difference will be released as a photon

$$hf = E_{ni} - E_{nf}$$

Frequency of photon

$$f = 1/h(E_{ni} - E_{nf}) = \frac{1}{2\pi\hbar} \frac{me^4}{32\pi^2\epsilon_0^2\hbar^2} \left(-\frac{1}{h_i^2} - \left(-\frac{1}{h_f} \right)^2 \right)$$

$$f = \frac{me^2}{64\pi^3\epsilon_0^2\hbar^3} \left(\frac{1}{h_f^2} - \frac{1}{h_i^2} \right)$$

Wavelength of photon:

$$\lambda = \frac{c}{f} = \frac{1}{R_\infty} \left(\frac{h_i^2 h_f^2}{h_i^2 - h_f^2} \right)$$

Where $R_\infty =$ that mes up there $= 1.093737 \cdot 10^7 m^{-1}$

This agrees with something we calculated yesterday.

If you have an excited energy, there are many possible emissions possible. However, if you excite an atom it is generally going to be completely in the ground state, so the absorption spectra has only one possibility.

6.3.1 Bohr model also explains

1. Ritz combination principle

- Since frequency of an emitted photon is $E=hf$, summing
-
-

1. Ritz combination principle

-
-
-

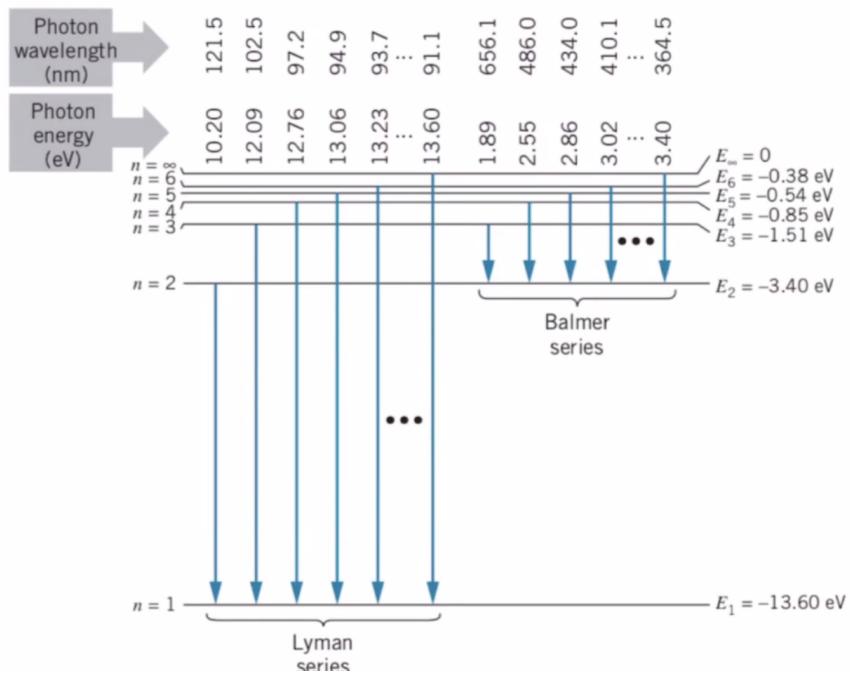


Figure 16: This explains the difference between the emission and absorption spectra

6.3.2 Caveats

Bohr can only work for atoms with a single electron.

$$\text{Allowed radii : } r_n = \frac{e_0 n^2}{Z}$$

Z: # positive charges,
He : Z = 2, ...

Allowed energies:

$$E_k = (-13.60 \text{ eV}) \frac{Z^2}{n^2}$$

6.4 Franck-Hertz Experiment

Illustrates the transition between different levels well.

Filled the tube with mercury vapor.

7 Wave-like properties of particles

A natural extension of the behaviour of photons to other particles.

7.1 De Broglie Wavelength

Proposed that there is a wave with wavelength λ associated to any matter particle moving with momentum p:

$$\lambda = \frac{h}{p}$$

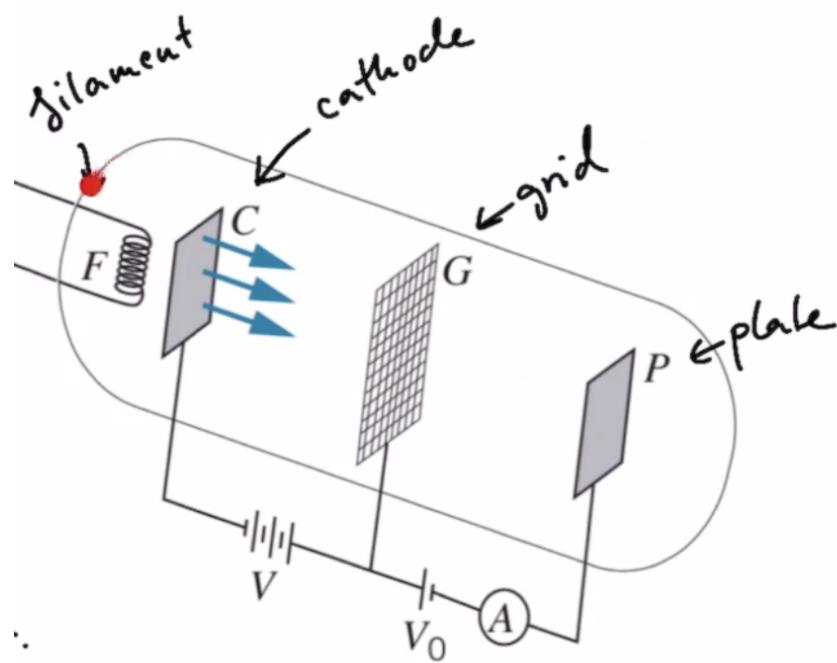


Figure 17: Filament heats cathode, emits electrons that are accelerated using a voltage difference V . After passing grid, they are slowed by a retarding potential V_0

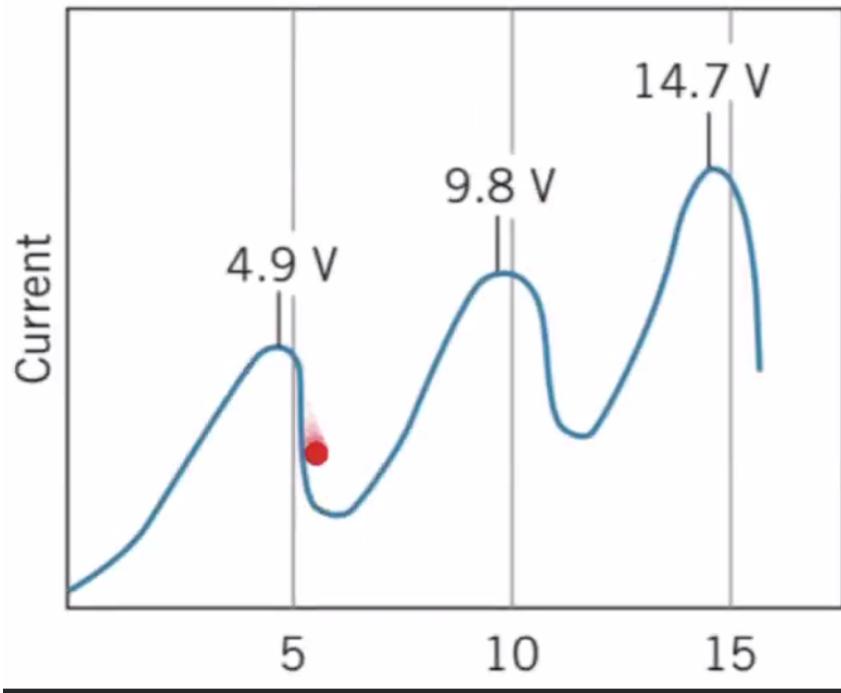


Figure 18: Results meant that at some point the electron will have enough energy to excite one atom, and then eventually another atom.

Where

h : Planck's constant

λ : de Broglie wavelength

Note that because Planck's constant is so small, this is only really observable in tiny particles.

This kind of came out of nowhere, but it turned out to be supported by evidence.

Example

10g bullet travelling at 100 m/s

Macroscopic object

$$\lambda = \frac{h}{p} = \frac{6.6 \cdot 10^{-34} J \cdot s}{0.010 kg \cdot 100 m/s} = 6.6 \cdot 10^{-34} m$$

An electron ($m = 0.511 MeV/c^2$) with kinetic energy 100 MeV

Subatomic object

Kinetic energy $>>$ rest energy, we are in the extreme relativistic regime, so we can assume that the rest energy doesn't matter.

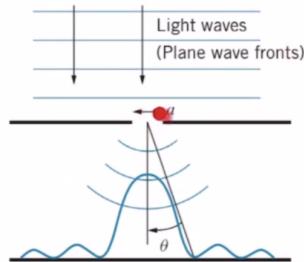
$$\lambda = \frac{h}{p} = \frac{hc}{pc} = \frac{1240 MeV \cdot fm}{100 MeV} = 12.4 fm = 1.24 \cdot 10^{-14} m$$

Why were wave-like properties of particle not observed previously?

- For macroscopic objects, the wavelength is minuscule
- No experiment can be done to measure

7.2 Diffraction Experiments

Diffraction by a single narrow slit



Diffraction by atoms of a crystal

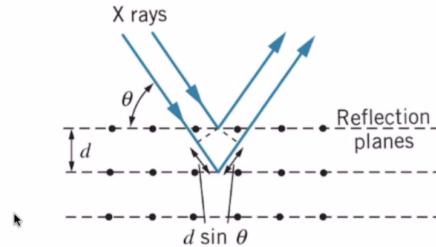


Figure 19:

7.2.1 Davisson-Germer Experiment

First experiment verifying de Broglie's hypothesis. Investigated pattern of scattering electrons by surface of a crystal - which were found to display a typical diffraction pattern.

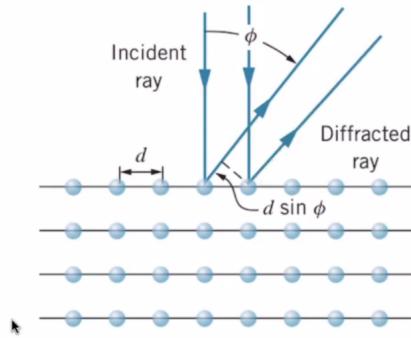
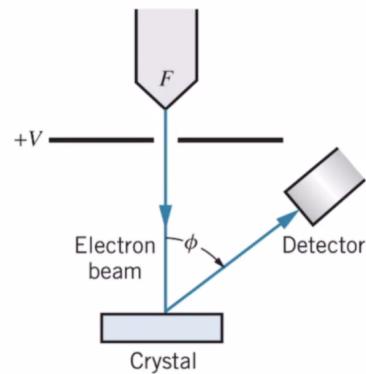


Figure 20: Similar patterns were found regardless of x-rays or electrons were used!

These similarities between diffraction patterns strongly suggested electrons are behaving like waves in these experiments. However, not just electrons can be used - any particle with a momentum has a de Broglie wavelength.

7.3 Double-Slit Experiments

First double-slit experiment with electrons was done first in 1961 (as numerous technological challenges had to be overcome).

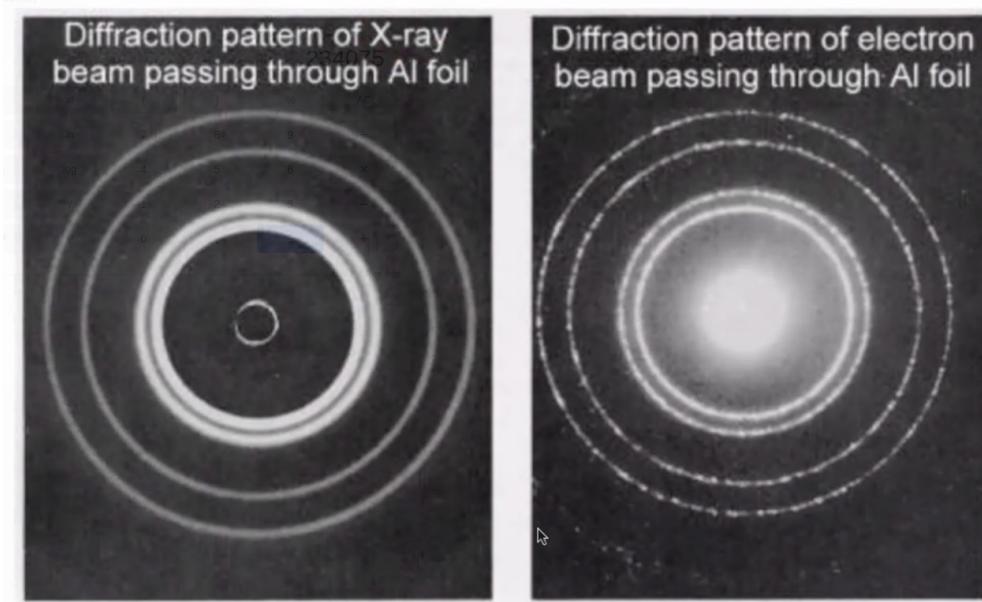


Figure 21: Polycrystalline material, shown for both photons and electrons

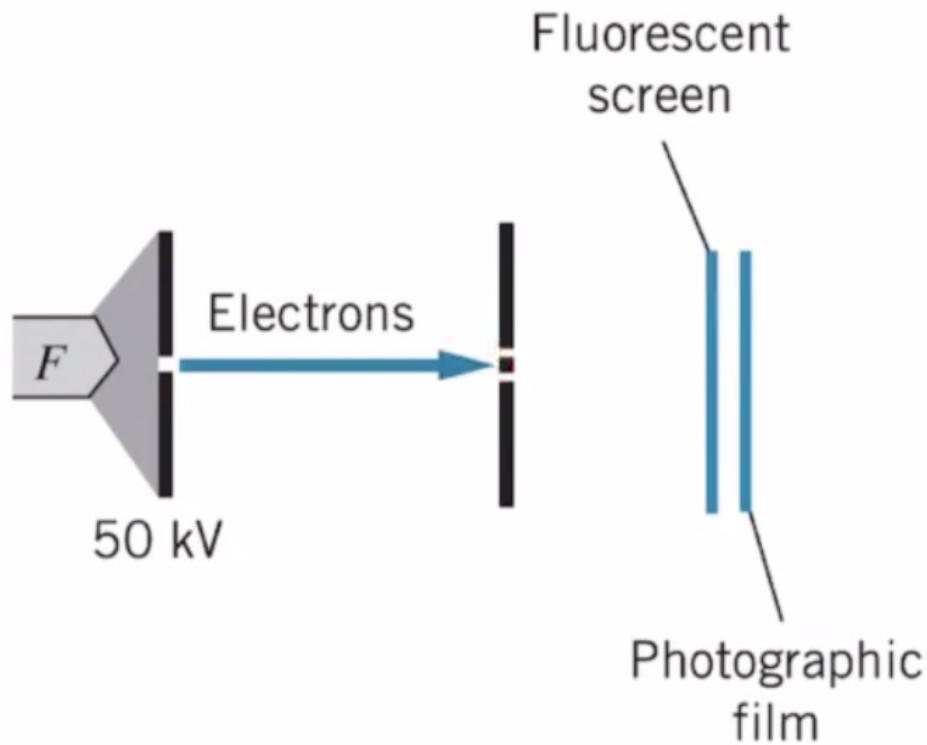


Figure 22: Slit spacing of $2 \mu\text{m}$, width $0.5 \mu\text{m}$. An interference pattern was observed!

7.3.1 Electron Microscope

A major advancement that this caused was the electron microscope.
It uses electron waves rather than visible light to illuminate and make images of objects. *Why?* The wavelength can be made much shorter than those of visible light (and shorter than x-rays).

They can therefore be used for many purposes, e.g. in biomedical research.

7.4 More Double-Slit

Experiment with electrons passing through double-slits one at a time, slowly building up an interference image.



Figure 23: 11 electrons

Even though the electrons are passing through one at a time, it seems somehow as though each electron is passing through both slits and is interfering with itself???

Principle of Complementarity

The electron (and other particles) behaves as a particle or a wave, but we cannot observe both aspects of its behavior simultaneously.

A complete description of a photon/electron/particle cannot be made in terms of only particle properties or only wave properties. Both must be considered. The type of behavior we observe depends on the experiment we do!

7.5 Uncertainty Principle

7.5.1 Classical Case

For a particle, we will want to know its position and momentum.

For a quantum particle, we will see that there are tradeoffs between these.

Waves and wave packets -

A pure sinusoidal wave extends from negative to positive infinity. This cannot describe a particle. Real waves are rather represented by “wave packets”, disturbance localized into a small region of space.

Uncertainty around measuring wavelength for a classical wave:

Uncertainty:

$$\Delta\lambda \approx \epsilon\lambda (\epsilon \text{ around } 10\%)$$

Size of disturbance:

$$\Delta x \approx \lambda$$

Product of these:

$$\Delta\lambda\Delta x \approx \epsilon\lambda^2$$

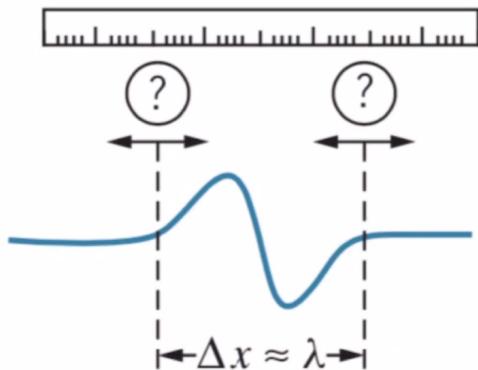


Figure 24: Uncertainty regarding where the wave begins and ends.

Try to make a larger wave packet:

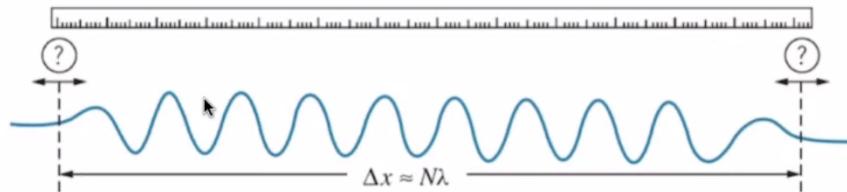


Figure 25: Uncertainty regarding where the wave begins and ends.

We now have N cycles of wave $\Delta x \approx N\lambda$. But we still have some uncertainty for start/end: ϵx . If we want to find the wavelength:

$$\Delta x \approx \frac{\epsilon\lambda}{N}$$

Same product as before:

$$\Delta x\Delta\lambda \approx N\lambda \frac{\epsilon\lambda}{N} = \epsilon\lambda^2$$

Water waves

We are measuring wavelength of water waves and count 10 waves in a distance of 196 cm. What is the minimum uncertainty on the wavelength from this measurement?

Answer

$$\text{Wavelength: } \lambda = \frac{196\text{cm}}{N} = 19.6\text{cm}$$

$$\text{Uncertainty: } \epsilon = 1\% < \epsilon < 100\% \rightarrow \epsilon \approx 10\%$$

Uncertainty in measured wavelength:

$$\Delta\lambda \approx \frac{\epsilon\lambda^2}{\Delta x} = \frac{0.1 \cdot (19.6)^2}{196\text{cm}} = 0.196\text{cm}$$

7.5.2 Heisenberg Uncertainty Relationships

We can apply the uncertainty relationships (which apply to all waves) specifically to de Broglie waves.

$$\text{de Broglie wavelength: } \lambda = \frac{h}{p}$$

It turns out that:

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}$$

Uncertainty in position times uncertainty in momentum is greater than pi times Planck's constant.

Similarly:

$$\Delta y \Delta p_y \geq \frac{\hbar}{2}$$

$$\Delta z \Delta p_z \geq \frac{\hbar}{2}$$

We sometimes assume that this product is just equal to hbar.

Electron moves

An electron moves in the x direction with a speed of $3.6 \cdot 10^6 \text{ m/s}$. We can measure its speed with a precision of 1%. With what precision can we simultaneously measure its x coordinate?

Answer

Electron's momentum:

$$p_x = mv_x$$

$$p_x = (9.11 \cdot 10^{-31} \text{ kg})(3.6 \cdot 10^6 \text{ m/s}) = 3.3 \cdot 10^{-24} \text{ kg m/s}$$

Uncertainty on the momentum:

$$\Delta p_x = 0.01 \cdot p_x$$

Uncertainty on the position:

$$\Delta x \geq \frac{\bar{h}}{2\Delta p_x} = \frac{1.05 \cdot 10^{-34} \text{ J s}}{2 \cdot 3.3 \cdot 10^{-26}} \text{ kg m/s}$$

$$\Delta x \geq 1.6 \text{ nm}$$

This uncertainty is actually several atoms large, which is pretty massive compared to the size of an electron itself.

7.6 Heisenberg's Uncertainty Principles

The quantum mechanical limit to the uncertainty of certain measurements, which results from

- Wave-particle duality
- The interaction between what is observed and the observing instrument

Uncertainty Principle

1.

$$\Delta x \Delta p_x \geq \frac{\bar{h}}{2}$$

$$\begin{aligned} \bar{h} &= 1.05 \cdot 10^{-34} \text{ J s} \\ &= 6.58 \cdot 10^{-16} \text{ eV s} \end{aligned}$$

2.

$$\Delta E \Delta t \geq \frac{\bar{h}}{2}$$

It is not possible to make a simultaneous determination of the energy and the time coordinate of a particle with unlimited precision.

There is a second, similar relationship associated with the energy of the wave packet and the time taken to measure that energy. Consider that we have an uncertainty in the measured position of an object of Δx , which has a corresponding uncertainty for time traveled by the photon $\Delta t \approx \frac{\Delta x}{c}$. The uncertainty in energy is therefore

$$\Delta E \approx \Delta p \cdot c$$

Example

- A) A charged pi meson has a rest energy of 140 MeV and a lifetime of 26 ns. What is the energy uncertainty of the pion?
 B) What about for a rho meson (rest energy of 765 MeV, lifetime of $4.4 \cdot 10^{-24} s$)?

A

Using Heisenberg's second uncertainty principle, since we only have a limited amount of time to measure the energy.

$$\Delta E \geq \frac{\bar{h}}{2\Delta t} = \frac{6.58 \cdot 10^{-16} eV \cdot s}{2 \cdot 26 \cdot 10^{-9} s} = 1.3 \cdot 10^{-8} eV$$

$$\frac{\Delta E}{E} = \frac{1.3 \cdot 10^{-14} MeV}{140 MeV} \approx 10^{-16}$$

This is a very small fractional error.

B

$$\Delta E \geq \frac{\bar{h}}{2\Delta t} = \frac{6.58 \cdot 10^{-16} eV \cdot s}{2 \cdot 4.4 \cdot 10^{-24} s} \approx 7.5 \cdot 10^7 eV$$

$$\frac{\Delta E}{E} = \frac{7.5 \cdot MeV}{765 MeV} \approx 10\%$$

For very short-lived particles, we cannot directly measure this lifetime. Instead, we can deduce the lifetime from the width/spread in the distribution of its rest energy.

Example

A ball of mass 50 g moves with a speed of 30 m/s. If its speed is measured to an accuracy of 0.1%, what is the minimum uncertainty in its position?

Answer

Momentum of ball:

$$p_x = mv_x = 0.05\text{kg} \cdot 30\text{m/s} = 1.5\text{kg} \cdot \text{m/s}$$

Uncertainty in momentum:

$$\Delta p_x = 0.0015\text{kg} \cdot \text{m/s}$$

Minimum uncertainty in position:

$$\Delta x \geq \frac{\bar{h}}{2\Delta p_x} = \frac{1.05 \cdot 10^{-34}\text{J} \cdot \text{s}}{2 \cdot 0.0015\text{kg} \cdot \text{m/s}} = 3.5 \cdot 10^{-30}\text{m}$$

What does this mean?

A typical nucleus of an atom is -15 to -14 m in the order of magnitude.
The uncertainty here is roughly 16 orders of magnitude smaller.

7.7 Difraction Experiment and Uncertainty Principle (and you!)

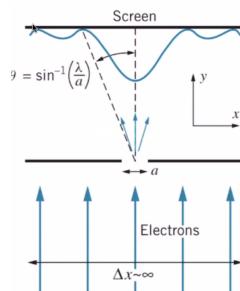


Figure 26: Incident electrons pass through a slit

Incident electrons moving in the y-direction with no component of momentum in the x-direction:

$$p_x = 0, \Delta p_x = 0 \rightarrow \Delta x = \infty$$

Once electrons pass through the slit:

$$\Delta x = a \rightarrow \Delta p_x \geq \frac{\bar{h}}{2\Delta x} = \frac{\bar{h}}{2a}$$

Passing through slit means they acquire momentum component in x-direction!?

Where on the screen would electron with this minimum momentum end up?

$$p_y = \frac{h}{\lambda}$$

For small angles $\tan\theta \approx \sin\theta \approx \theta$, so

$$\theta \approx \frac{p_x}{p_y} = \frac{\frac{h}{2\pi a}}{h/\lambda} = \frac{\lambda}{2\pi a}$$

For this diffraction experiment, minima described as:

$$a \cdot \sin\theta = h\lambda \quad (n = 1, 2, 3)$$

First minimum larger than spread of angles into which most particles diffracted. The distribution of transverse momentum roughly similar to the spread of the beam into the central diffraction peak.

The narrower the slit, the larger the uncertainty in momentum in x-direction is and the more the beam is spread out.

In a sense, understanding the diffraction experiment is a matter of the wave-like nature of particles, but looking at the central distribution is a matter of the particle nature.

7.8 Wave Packets

A wave associated with a particle is going to be a localized wave, a wave pulse or group, of limited spatial extent.

7.8.1 Wave Group

A wave group is a superposition of waves with different λ s, adjusted such that waves interfere constructively over a small region of space. Outside of this, they interfere destructively to zero amplitude everywhere. A physical phenomena to think of which is similar in concept is a beat.

7.8.2 Forming Wave Packets

Key to forming wave packets is adding waves of different wavelength. Adding a small number gives more pronounced regions of large amplitude, but would still repeat from negative to positive infinity.

Indeed, *any finite combination of waves with discrete wavelengths will produce patterns that repeat from negative to positive infinity.*

Side note: Static wave vs traveling wave.

Static wave: $y(x) = A \cos(kx)$

Traveling wave: $y(x, t) = A \cos(kx - \omega t)$

Where k is the wave number, $k = \frac{2\pi}{\lambda}$, and ω is the angular frequency = $2\pi f$.

If we add two waves with a different wave number $k + \Delta k/2$ and $k - \Delta k/2$, the resulting wave:

$$\begin{aligned} y(x, t) &= A \cos[(kt + \Delta k/2)x - (\omega + \Delta\omega/2)t] + A \cos[(kt - \Delta k/2)x - (\omega - \Delta\omega/2)t] \\ &= 2A \cos(\Delta k/2x - \Delta\omega/2) \cos(kx - \omega t) \end{aligned}$$

We define the speed of the wave packet, the group speed:

$$v_g = \frac{\Delta\omega}{\Delta k}$$

And the speed of the higher frequency wave within the group, phase speed (generally, phase speed is speed of a point of constant phase on the wave):

$$v_p = \omega/k$$

We can generalize the group speed for more complicated wave packets as:

$$v_g = \frac{d\omega}{dk}$$

Group Velocity

Phase velocity of deep water waves with a wavelength λ is given by:

$$v_p = \sqrt{\frac{g\lambda}{2\pi}}$$

Where g is the acceleration due to gravity.

Also, $v_p = \omega/k$, $\omega/k = \sqrt{g/k} \rightarrow \omega = \sqrt{gk}$
 Group velocity $v_g = \frac{d\omega}{dk} = \frac{d}{dk}(gk)^{1/2} = 1/2\sqrt{g/k}$

7.9 For de Broglie waves

Energy of particle related to its frequency $E = hf = \bar{h}\omega$

Momentum of particle related to its wavelength $p = \frac{h}{\lambda} = \bar{h}k$

Group speed of de Broglie wave:

$$v_g = \frac{d\omega}{dk} = \frac{dE/\bar{h}}{dp/\bar{h}} = \frac{dE}{dp}$$

For classical particles with only kinetic energy:

$$E = K + \frac{p^2}{2m}$$

$$v_g = \frac{p}{m} = v$$

Which brings us back to where we started. The speed of a particle is equal to the group speed of the corresponding wave packet!

8 Probabilities

If we flip a coin can we predict the outcome? Similarly, in terms of systems governed by the laws of quantum mechanics, we cannot predict the outcome of a single measurement of, e.g., position of an electron in an atom. However, if we do enough experiments we can find the statistical distribution. This is a fundamental aspect of nature and not a result of limited knowledge of the system we are studying.

8.1 In matter particles, what is waving?

The probability of finding the particle at any point depends on the amplitude of its de Broglie wave at that point: Probability to observe particles $\propto |\text{de Broglie wave amplitude}|^2$.

8.2 Wave function

de Broglie waves are represented by wave functions: $\Psi(x, t)$, $\Psi(x, y, z, t)$

Wave functions are generally represented with complex numbers and used to calculate the probability of finding the particle at a given time in a small volume of space.

The probability that a particle will be found in the infinitesimal interval dx about the point x ($P(x)dx$) is:
 $P(x)$: Probability density function

$$P(x) \cdot dx = |\Psi(x, t)|^2 dx$$

Complex wave function can be written as:

$$\Psi = \text{Re}(\Psi) + i \text{Im}(\Psi)$$

The complex conjugate:

$$z^* = x - iy$$

$$\Psi^* = \text{Re}(\Psi) - i \text{Im}(\Psi)$$

Squared magnitude of a complex number (or wave function) is the product of the number and its complex conjugate:

$$|z|^2 = zz^* = x^2 + y^2$$

$$|\Psi|^2 = \Psi\Psi^* = \text{Re}(\Psi)^2 + \text{Im}(\Psi)^2$$

Complex numbers are very useful in describing wave mechanics, in terms of amplitude and phases. Euler's formula expresses complex exponentials in terms of real trigonometric functions:

$$e^{i\theta} = \cos\theta + i \sin\theta$$

$$e^{-i\theta} = \cos\theta - i \sin\theta$$

Squared magnitude:

$$|e^{i\theta}|^2 = e^{i\theta}e^{-i\theta} = e^0 = 1$$

$$|e^{i\theta}|^2 = (\cos\theta + i \sin\theta)(\cos\theta - i \sin\theta) = 1$$

Trigonometric functions can be written in terms of complex functions as, e.g.:

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

$$\cos\theta = \frac{1}{2i}(e^{i\theta} + e^{-i\theta})$$

Wave function of a free particle

$$\Psi(x, t) = A e^{i(kx - \omega t)} = A (\cos(kx - \omega t) + i \sin(kx - \omega t))$$

k : wave number

ω : angular frequency

$$k = \frac{2\pi}{\lambda} = \frac{p}{\hbar}$$

$$\omega = 2\pi f = \frac{E}{\hbar}$$

The probability function is given by

$$|\Psi(x, t)|^2 = A^2$$

8.3 Schrödinger Equation

Basic equation of (non-relativistic) quantum mechanics.

Classical equivalent is Newton's second law:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Schrödinger equation similarly written for a particle interacting with its environment but in terms of the potential energy rather than the force.

8.4 Waves at Boundaries

What happens to the wavelength and amplitude of a wave crossing from one region/medium to another?

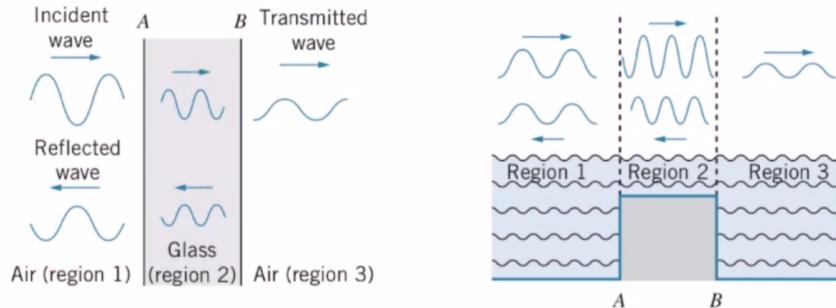


Figure 27: Similar behavior occurs for de Broglie waves - transmitted and reflected components!

Penetration of reflected wave:

- For a fully reflected light wave, an “evanescent wave” (exponentially decreasing) penetrates into the second medium.
- For de Broglie waves, these can also penetrate a short distance into the forbidden region though cannot be directly observed in that region.
- More on this (“tunneling”) next week!

8.4.1 Continuity at Boundaries

When a wave crosses a boundary, two boundary conditions must be fulfilled.

1. Wave function is continuous at the boundary
2. Slope of wave function is continuous at the boundary, except when the boundary height is infinite (such as dropping a small metal ball onto a steel surface)

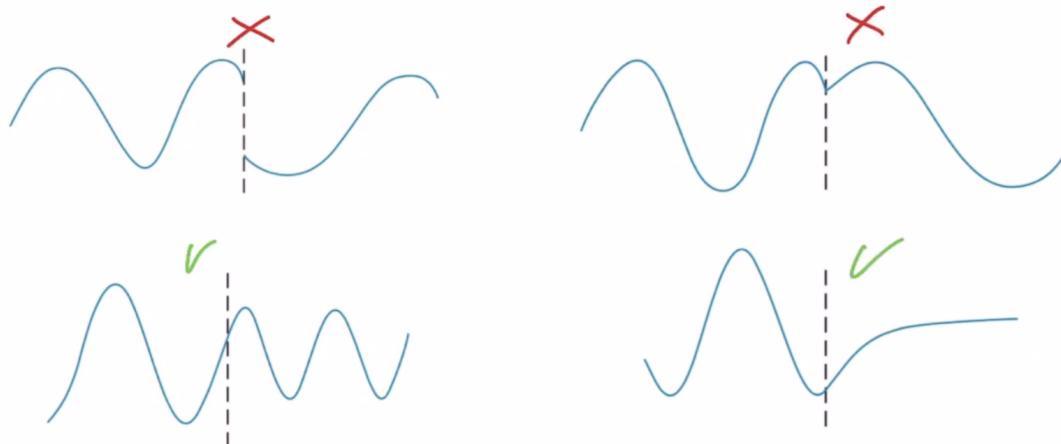


Figure 28: Examples of invalid and valid wave phenomena at boundaries.

8.5 Particle in a Potential Energy Well

A free particle is by definition not confined - can be located anywhere, has definite wavelength, momentum, and energy.

A confined particle is instead represented by a wave packet and can be found within a small region of space Δx (e.g. electron confined in an atom)

8.5.1 Finite Potential Barrier

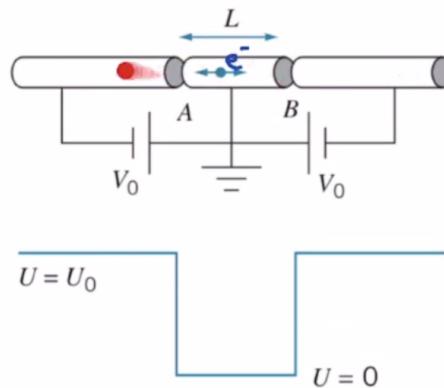


Figure 29: Potential well

Potential energy:

$$U_0 = qV = (-e)(-V_0) = eV_0$$

If electron kinetic energy K is smaller than the potential energy $K < U_0$, the electron cannot overcome the potential confined in that region.

8.5.2 Infinite Potential Barrier

Consider simplified scenario where the potential energy barrier at points A and B is infinitely high.

- Penetration into “forbidden” region cannot occur
- Probability of waves find electron is exactly zero in those regions
- Wave function must be zero at boundaries A and B to be continuous

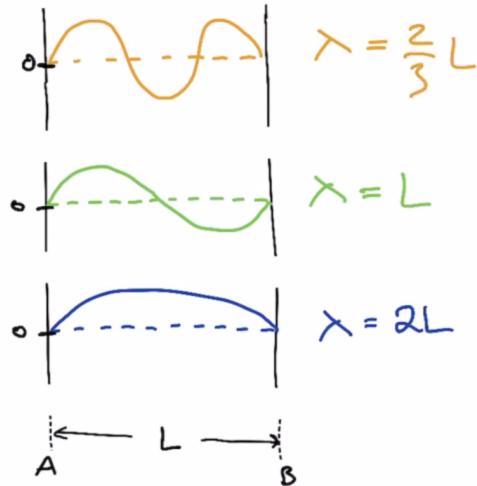


Figure 30: Possible waves and wavelengths that could describe this scenario

We see that only certain wavelengths are allowed.

$$\lambda_n = \frac{2L}{n}, \quad n = 1, 2, 3, \dots$$

This also means that only certain momenta are allowed:

$$p_n = n \frac{\hbar}{2L} \quad n = 1, 2, 3, \dots$$

This also means that only certain energies are allowed:

$K = \frac{p^2}{2m} = E$ given that potential energy is zero.

$$E_n = n^2 \frac{\hbar^2}{8mL^2} \quad n = 1, 2, 3, \dots$$

Energies are quantized!

The classical version of this is, essentially guitars.

On energy quantization

Any attempt to confine a particle to a finite region of space will result in quantization of energy.

This is a core feature of quantum mechanics

Model of an atom

We can approximate an atomic electron as confined in an infinite square well. Use this model to approximate the energy required to raise an atomic electron from the state $n=1$ to $n=2$, assuming the atom has a radius of 1 Å.

Answer

Allowed energy

8.6 Normalization of Wave Function

For a particle along the x-axis (one-dimensional), the particle must be *somewhere* along the x-axis and the probabilities summed over all values must add to exactly 1.

$$\int_{-\infty}^{\infty} P(x, t) dx = \int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx = 1$$

Any wave function satisfying this is said to be "normalized".

Similarly, probability of finding the particle in a finite interval $a \leq x \leq b$:

$$\text{Probability } P = \int_a^b |\Psi(x, t)|^2 dx$$

Example

The initial ($t=0$) wave function for a particle is given by:

$$\Psi(x, 0) = ce^{-|x|/x_0}$$

Normalize the wave function

$$1 = \int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx = c^2 \int_{-\infty}^{\infty} e^{-2|x|/x_0} dx$$

This integral is symmetric about the y axis. Simplify as 2x the integral from 0 to infinity.

$$1 = 2c^2 \int_0^{\infty} e^{-2|x|/x_0} dx = 2c^2 \left[-\frac{x_0}{2} e^{-2x/x_0} \right]_0^{\infty} = 2c^2(0 + \frac{x_0}{2}) = c^2 x_0$$

$$c = \sqrt{1/x_0}$$

Probability of finding the particle on an interval

$$P = \int_{-x_0}^{x_0} |\Psi(x, 0)|^2 dx = c^2 \int_{-\infty}^{\infty} e^{-2|x|/x_0} dx$$

$$P = 2 \frac{1}{x_0} \left(-\frac{x_0^0}{2} \right) \left[e^{-2/x_0} \right] = (-1)(e^{-2} - 1)$$

Approximately 0.865. Apparently has nothing to do with x_0 .

9 The Schrödinger Equation

The differential equation that describes the wave behavior of particles for non-relativistic motion.

Time-Independent Version

$$-\frac{\bar{h}^2}{2m} \frac{d^2\Psi}{dx^2} + U(x)\Psi(x) = E\Psi(x)$$

$\Psi(x)$: Wave function describing particle at a particular time

$U(x)$: potential energy

E : total non-relativistic energy of the particle

m : mass of the particle

$\bar{h} : \frac{h}{2\pi}$ Note, this cannot be derived, it is motivated.

9.1 Motivation of the time-independent form of the Schrödinger equation

$$-\frac{\bar{h}^2}{2m} \frac{d^2\psi}{dx^2} + U(x)\psi(x) = E\psi(x)$$

If the latter two terms are potential and total energy, is the first term corresponding to kinetic energy?

$$K = \frac{p^2}{2m} = \frac{(h/\lambda)^2}{2m} = \frac{(\bar{h}k)^2}{2m}$$

Note that this is sort of similar to the first part, the difference is if we divide this by $-k^2$ then it's the same as the first fraction in the first term of the Schrödinger equation.

For a wave function: $\Psi(x) = A\sin(kx)$

$$\frac{d\psi}{dx} = kA\cos(kx)$$

$$\frac{d^2\psi}{dx^2} = -k^2A\cos(kx)$$

Think of the first term as kinetic energy:

$$k\psi(x) = -\frac{\bar{h}^2}{2m} \frac{d^2\psi}{dx^2}$$

If we take potential energy as used in deriving the Bohr model:

$$U(x) \rightarrow U(r) = -\frac{1}{4\pi\epsilon_0}$$

9.2 Time-dependent Schrödinger equation

This is sorta beyond the scope of the class.... but:

Wave function:

$$\begin{aligned}\psi(x) & \text{ vs } \Psi(x, t) \\ -\frac{\bar{h}^2}{2m} \frac{\delta^2 \Psi}{\delta x^2} + U(x)\Psi(x, t) & = i\bar{h} \cdot \frac{\delta \Psi}{\delta t} \\ \Psi(x, t) & = Ae^{i(kx-\omega t)}\end{aligned}$$

This new part:

$$i\bar{h} \frac{\delta \Psi}{\delta t} = i\bar{h}A(-i\omega)e^{i(kx-\omega t)} = \bar{h}w\Psi(x, t)$$

Note the extra hbar omega.

$$\Psi(x, t) = \psi(x)e^{-i\omega t}$$

The last part is the time-independent part.

9.3 Solving the Schrödinger Equation?

1. Write down time-independent Schrödinger equation with the appropriate potential $U(x)$, may need different expressions for different regions if $U(x)$ changes discontinuously.
2. Find wave function solution $\psi(x)$
3. Apply boundary conditions (often more than one solution is found - applying boundary conditions help eliminate solutions), and continuity conditions (if applicable).
4. Normalize the wave function.
5. Since wave function (squared) represents a probability, any solution that becomes infinite must be discarded.

From the wave function, we can calculate:

- Probability to find the particle in an interval
- Outcome of a single measurement is not guaranteed, but we can determine the average of many measurements, e.g. of the position coordinate x of a particle: *We measure : $x_1, n_1, \text{ times}, x_2, n_2, \text{ time}....$*
 Average: $x_{av} = \frac{x_1n_1+x_2n_2+...}{n_1+n_2+...} = \sum \frac{x_i n_i}{n_i}$
 In terms of probabilities: $P_1 = \frac{n_1}{N}, \dots$

$$x_{av} = \sum x_i P_i$$

Possible values are distributed continuously - probability of finding particle in an infinitesimal range dx about point x is:

$$\begin{aligned}P(x)dx \\ \text{Average : } < x > = \int_{-\infty}^{\infty} xP(x) dx = \int_{-\infty}^{\infty} x|\Psi(x)|^2 dx\end{aligned}$$

These $< x >$ are called expectation values.

9.4 Back to that Infinite Potential Well

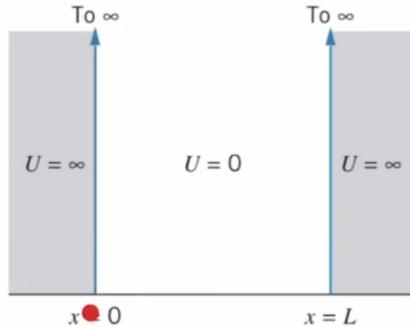


Figure 31: Infinite potential well

Potential energy can be expressed as:

$$U(x) = 0 \text{ for } 0 \leq x \leq L$$

$$U(x) = \infty \text{ for } x < 0, x > L$$

Wave function for region outside of well?

Probability of finding particle then must be zero everywhere:

$$si(x) = 0$$

for $x < 0, x > L$

Schrödinger equation for region inside of well ($0 < x < L$):

$$\frac{\bar{h}^2}{2m} \frac{d^2\psi}{dx^2} = E\psi(x)$$

General solution $\left(\frac{d^2\psi}{dx^2}\right) = \text{constant} \cdot \psi(x)$

$$\psi(x) = A\sin(kx) + B\cos(ks)$$

(for $0 \leq x \leq L$)

$$\frac{d^2\psi}{dx^2} = -k^2\psi(x)$$

$$k^2 = \frac{2mE}{\bar{h}^2}$$

or:

$$k = \frac{\sqrt{2mE}}{\bar{h}}$$

So what's up with those random constants A and B?

It has to do with the constraint that the wave function at the boundaries must be 0.

$$0 = \psi(0) = A\sin(0) + B\cos(0) = B \rightarrow B = 0$$

$$0 = \psi(L) = A\sin(kL) + B\cos(kL) = A\sin(kL)$$

$$A\sin(kL) = 0$$

So A can't be 0 because then our wave function isn't useful. Therefore, $\sin(kL) = 0$

$$kL = n\pi, \quad (n = 1, 2, 3, \dots)$$

$$k = \frac{n\pi}{L}$$

Wavelength?

$$\lambda = \frac{2\pi}{k} = \frac{2L}{n}$$

Same as yesterday!

We now have:

$$\psi(x) = A \sin\left(\frac{n\pi}{L}x\right) \text{ for } 0 \leq x \leq L$$

What is A? Normalize!

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = A^2 \int_0^L \sin^2\left(\frac{n\pi}{L}x\right) dx$$

Use $\sin^2 x = 1/2(1 - \cos(2x))$.

$$1 = A^2 1/2 \int_0^L (1 - \cos\left(\frac{n\pi}{L}x\right)) dx$$

$$1 = A^2 1/2 \left[x - \frac{L}{2n\pi} \sin\left(\frac{2n\pi}{L}x\right) \right]_0^L = \frac{A^2 L}{2}$$

$$A = \sqrt{\frac{2}{L}}$$

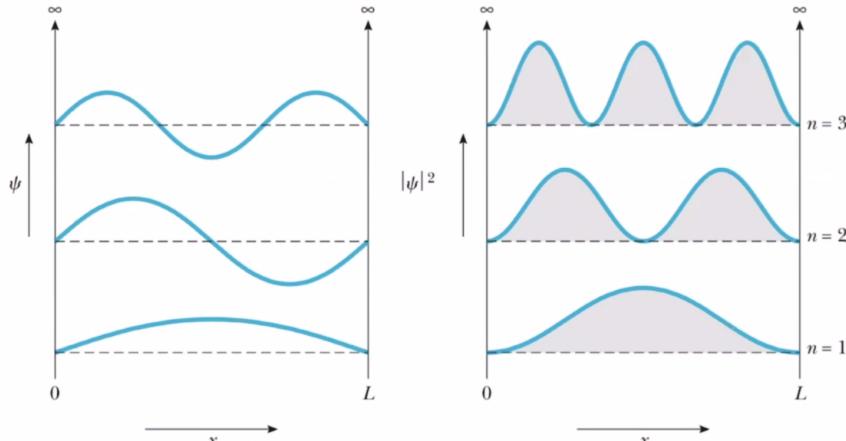


Figure 32: We can look at the wave function and probability density for $n=1,2,3$

9.5 Stationary States

Solutions to the time-dependent Schrödinger equation that can be written as above are called **stationary states**. Their probability density does not depend on time, since:

$$|\Psi(x, t)|^2 = |\psi(x)|^2 |e^{-i\omega t}|^2$$

$$|e^{-i\omega t}|^2 = e^{-i\omega t} e^{+i\omega t} = 1$$

In our course, we will restrict our discussion to stationary states and the time-independendnt Schrödinger equation.

Schrödinger Equation for Constant Potential Energy

Let's examine the Schrödinger equation when $U(x)$ is constant. $U(x) = U_0$.

$$\frac{-\bar{h}^2}{2m} \frac{d^2\psi}{dx^2} + U_0\psi(x) = E\psi(x)$$

or:

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\bar{h}^2}(E - U_0)\psi(x)$$

Right side of the equation is constant. Given $E > U_0$

$$\frac{d^2\psi}{dx^2} = -k^2\psi(x)$$

$$k = \sqrt{\frac{2m(E - U_0)}{\bar{h}^2}}$$

General Solution:

$$\psi(x) = A\sin(kx) + B\cos(kx)$$

Since:

$$\frac{d\psi}{dx} = Ak\cos(kx) - Bk\sin(kx)$$

$$\frac{d\psi^2}{dx^2} = -Ak^2\cos(kx) - Bk^2\sin(kx) = -k^2\psi(x)$$

Given $E < U_0$

$$\frac{d^2\psi}{dx^2} = (k')^2\psi(x)$$

$$k' = \sqrt{\frac{2m(U_0 - E)}{\bar{h}^2}}$$

General Solution:

$$\psi(x) = Ae^{k'x} + Be^{-k'x}$$

Since:

$$\frac{d\psi}{dx} = Ak'e^{-k'x} - Bk'e^{k'x}$$

$$\frac{d\psi}{dx} = A(k')^2e^{-k'x} - B(k')^2e^{k'x} = (k')^2\psi(x)$$

This is the forbidden region

9.6 Schrödinger Equation for Free Particle

Free particle implies no force acting upon it implies potential energy is constant. We can choose any value for that constant, but for simplicity we choose 0.

Schrödinger equation:

$$-\frac{\bar{h}^2}{2m} \frac{d^2\psi}{dx^2} = E\psi(x)$$

We already determined the general solutions!

$$\psi(x) = A\sin(kx) + B\cos(kx)$$

$$k = \sqrt{\frac{2m(E - U_0)}{\bar{h}^2}}$$

Or expressing energies (E) in terms of wave numbers (k):

$$E = \frac{\hbar^2 k^2}{2m} \left(= \frac{p^2}{2m} \right) \leftarrow \text{Energy not quantized}$$

Useful to express wave function as a complex exponential:

$$\text{Euler's formula : } e^{ikx} = \cos(kx) + i \sin(kx)$$

$$\cos(xk) = \frac{e^{ikx} + e^{-ikx}}{2}, \quad \sin(xk) = \frac{e^{ikx} - e^{-ikx}}{2i}$$

Substitute these in the expression for the wave function:

$$\begin{aligned} \psi(x) &= A \left(\frac{e^{ikx} + e^{-ikx}}{2} \right) + B \left(\frac{e^{ikx} - e^{-ikx}}{2i} \right) \\ &= A'e^{ikx} + B'e^{-ikx} \end{aligned}$$

Where A' and B' are constants.

This was the time-independent part, we write the complete time dependent wave function as:

$$\Psi(x, t) = \psi(x)e^{-i\omega t} = A'e^{i(kx-\omega t)} + B'e^{-i(kx-\omega t)}$$

A term is a wave moving in the positive x direction, and the B term is a wave moving in the negative x direction.

A Free Particle

Consider a free particle moving in the positive x direction - must set B' = 0.

$$\Psi(x, t) = A'e^{i(kx-\omega t)}$$

Associated Probability Density

$$P(x) = |\Psi(x, t)|^2 = |A'|^2 e^{i(kx-\omega t)} e^{-i(kx-\omega t)} \rightarrow \text{The } e \text{ terms cancel out} \rightarrow |A'|^2$$

This is a constant!

9.7 Finite Potential Energy Well

Previously we considered separately the solutions for inside/outside an infinite potential well and the resulting boundary conditions. This lead to quantization of energy.

This infinite potential well rests on assumptions we can't live with!

Potential energy:

$$U(x) = 0 \text{ for } 0 \leq x \leq L$$

$$U(x) = U_0 \text{ for } x < 0 \text{ or } x > L$$

We will consider separately regions inside and outside.

Inside well, the same general wave function solution:

$$\psi(x) = A\sin(kx) + B\cos(kx)$$

Where

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

We need those A and B constants at some point.

Now, outside of the well, particle is confined (energy is less than the potential energy):

$$(E < U_0) : \psi(x) = Ce^{k'x} + De^{-k'x}$$

$$k' = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}}$$

We need those C and D constants at some point.

Now, let's consider separately the regions $x < 0$ and $x > L$:

$$x < 0 : \text{As } x \rightarrow -\infty = \psi(X) = Ce^{k'x}$$

As D would approach infinity so we can't have that.

$$x > L : \text{As } x \rightarrow \infty = \psi(X) = De^{-k'x}$$

As C would approach infinity so we can't have that.

In summary, we have found for the finite potential energy well:

$$\psi(x) = Ce^{k'x} \text{ for } x < 0$$

$$\psi(x) = Asin(kx) + Bcos(kx) \text{ for } 0 \leq x \leq L$$

$$\psi(x) = De^{-k'x} \text{ for } x > L$$

We have a lot of equations for a lot of unknowns, and solving is quite hard.

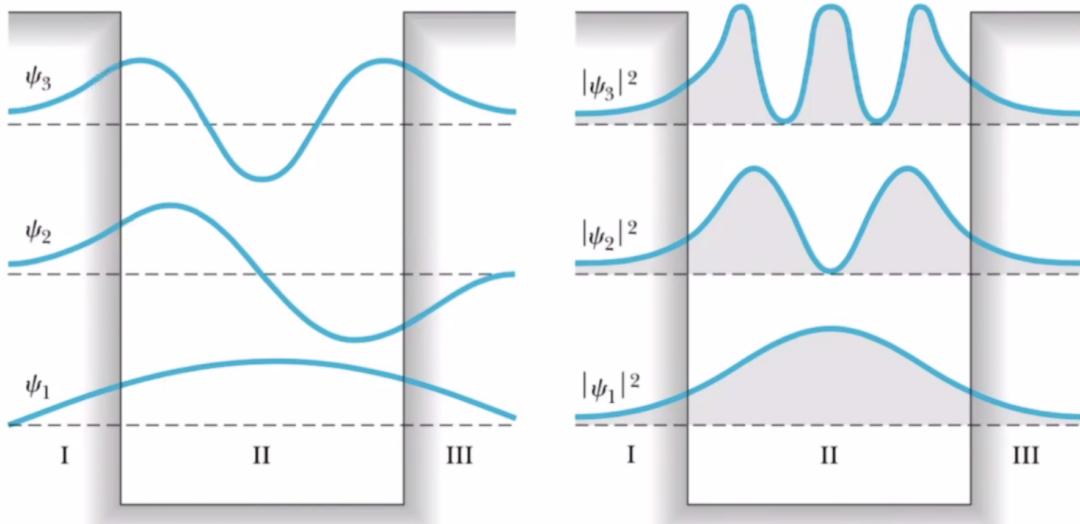


Figure 33: Note how the probability outside the finite potential energy well is NOT zero!!

10 Lecture #7

8 Quantum Harmonic Oscillator

Consider a classical simple harmonic oscillator. For example, a block with mass m, attached to a spring with constant k, oscillates about its equilibrium on a frictionless surface. The force exerted by the spring is $F =$

$-kx$. The elastic potential energy is $U = 1/2 kx^2$.

From Newton's laws, the motion of the classical oscillator is shown to have an angular frequency ω_0 and period T given by:

$$\omega_0 = \sqrt{\frac{k}{m}} \quad T = 2\pi\sqrt{\frac{m}{k}}$$

The quantum mechanical version of this system

In nature many systems behave *approximately* as a quantum oscillator, for example a vibrating diatomic molecule.

The Schrödinger equation:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + 1/2 kx^2\psi(x) = E\psi(x)$$

Here, no boundaries between regions of different potential energy apply. For x approaches positive or negative infinity, the wave function must vanish. Simplest function satisfying this (which will be the ground state!):

$$\psi(x) = Ae^{-ax^2}$$

Where a, A are constants.

Let's find the constants a, A :

$$\begin{aligned} \frac{d\psi}{dx} &= A(-2ax)e^{-ax^2} \\ \frac{d^2\psi}{dx^2} &= A(-2a)e^{-ax^2} + A(-2ax)e^{-ax^2} \\ \frac{d^2\psi}{dx^2} &= A(-2a)e^{-ax^2} + A(-2ax)^2e^{-ax^2} \end{aligned}$$

Substitute into Schrödinger equation:

$$-\frac{\hbar^2}{2m}2a(-2ax^2 - 1)Ae^{-ax^2} + 1/2 kx^2 A(-2ax)e^{-ax^2} = E\psi(x)$$

The exponentials cancel out on both sides of the equation.

$$-\frac{a\hbar^2}{m}(2ax^2 - 1) + 1/2kx^2 = E$$

We want a solution that is valid for ANY value of x :

$$\left(\frac{k}{2} - \frac{2a^2\hbar^2}{m}\right)x^2 = E - \frac{a\hbar^2}{2}$$

Each side must be equal to zero

$$\begin{aligned} \frac{k}{2} - \frac{2a^2\hbar^2}{m} &= 0 \rightarrow a = \frac{\sqrt{km}}{2\hbar} \\ E - \frac{a\hbar^2}{m} &= 0 \rightarrow E - \frac{a\hbar}{m} = \frac{\hbar}{2}\sqrt{\frac{k}{m}} \end{aligned}$$

Write energy in terms of classical angular frequency:

$$\omega_0 = \text{blah}$$

The ground state wave function for our quantum oscillator is now:

$$\psi(x) = Ae^{-ax^2} \text{ for } a = \frac{\sqrt{km}}{2\hbar}$$

We can find the constant A by normalization, yielding:

$$\psi(x) = \left(\frac{m\omega_0}{\hbar\pi} \right)^{1/3} e^{-\sqrt{km}/2\hbar x^2}$$

The *general* solutions is more involved:

$$\psi_n(x) = A f_n(x) e^{-ax^2}$$

Where $f_n(x)$: a polynomial with highest power of x is x^n .

The corresponding energies will be:

$$E_n = \left(n + \frac{1}{2} \right) \hbar\omega_0 \quad n = 0, 1, 2, \dots$$

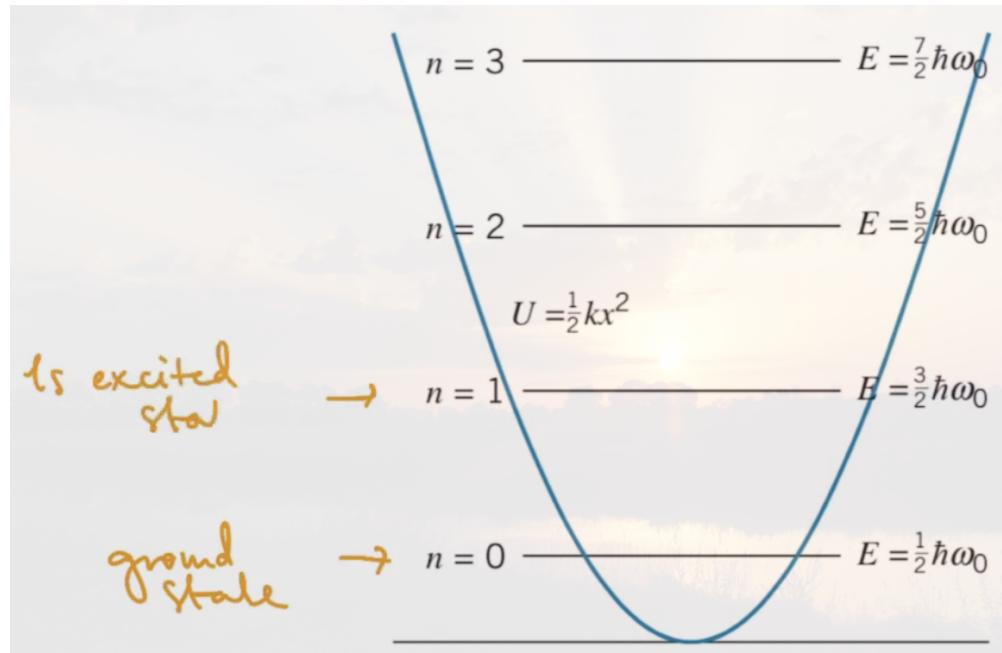


Figure 34: This describes the ground state, 1st excited state, etc.

8.1 Harmonic Oscillator & Probabilities

The below shows the probability density for a few different states for the quantum oscillator. The dashed lines show the classical probabilities for the equivalent energies.

Note: Probabilities extend beyond the classical region! And, distance between the classical turning points increases with increasing energy.

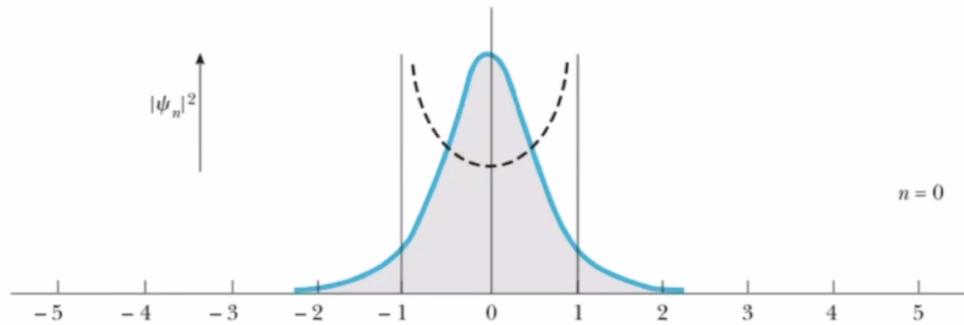


Figure 35: We have a non-zero probability to find the particle in the forbidden region

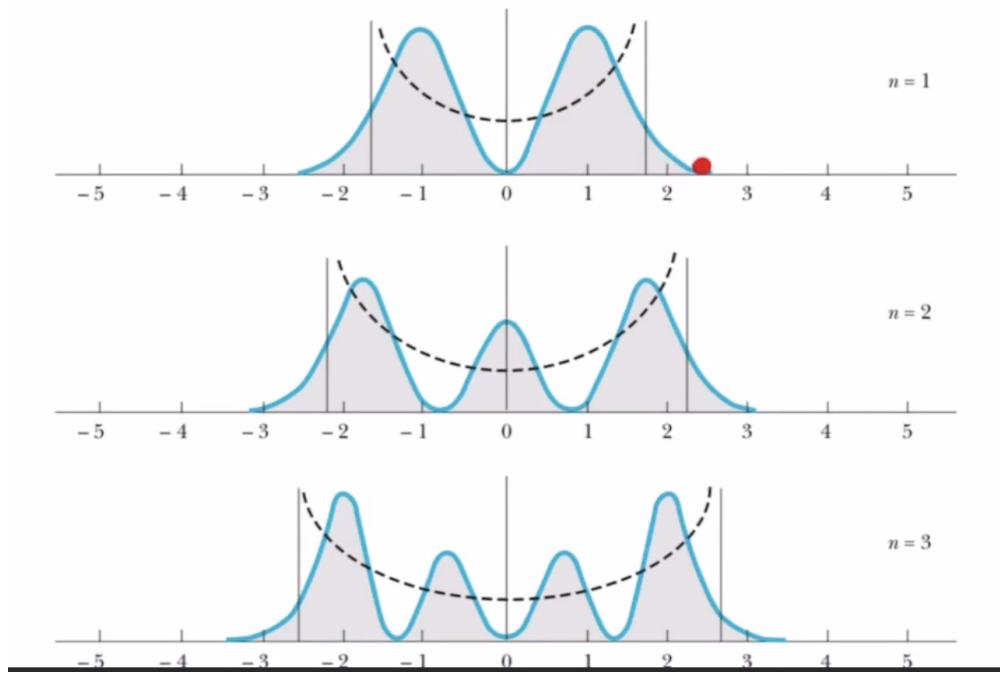


Figure 36: We have a non-zero probability to find the particle in the forbidden region

9 Potential Energy Steps

9.1 Where $E > U$

In the following we will see what happens when a particle moves (in one dimension) from a region of constant potential energy to a region with a different potential energy.

General solutions:

$$\text{For } x < 0 : \psi_0(x) = A \sin(k_0 x) + B \cos(k_0 x) \quad k_0 = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\text{For } x \geq 0 : \psi_1(x) = A' \sin(k_1 x) + B' \cos(k_1 x) \quad k_1 = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}}$$

It is convenient to write these in terms of Complex Exponentials instead. We also want to introduce time-dependence!

$$x < 0 : \Psi_0(x, t) = A' e^{i(k_0 x - \omega t)} + B' e^{-i(k_0 x + \omega t)}$$

$$x \geq 0 : \Psi_1(x, t) = C' e^{i(k_1 x - \omega t)} + D' e^{-i(k_1 x + \omega t)}$$

(1) Potential energy step where $E > U_0$

$$\begin{aligned} U(x) &= 0 && \text{for } x < 0 \\ U(x) &= U_0 && \text{for } x \geq 0 \end{aligned}$$

General solutions

(Note: Compare our discussion of general solutions to Schrödinger equation for constant potential energy)

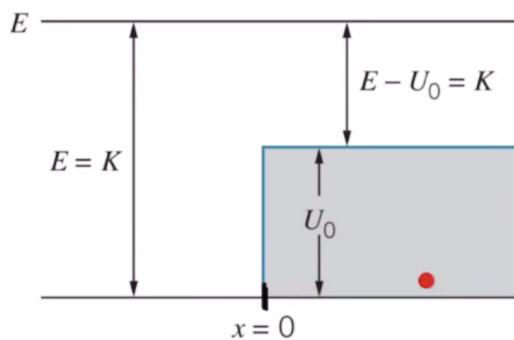


Figure 37: Greater potential on the right side.

Where A' , B' , C' , D' are constants that we would need to determine.

We can think of this as a sum of waves; positive e term in the positive x-direction and the negative e term in the negative x-direction. The squared magnitude of each coefficient gives the intensity of the corresponding component wave.

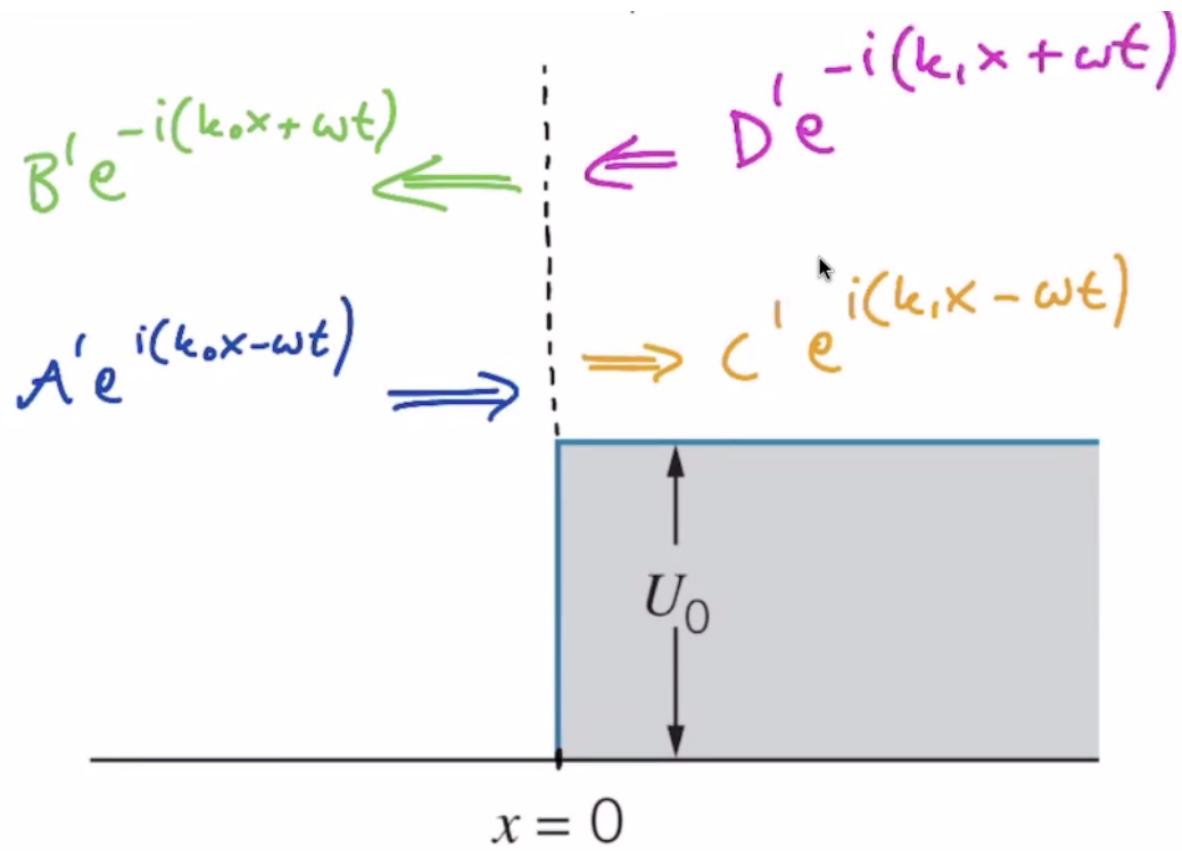


Figure 38: We can think of this as reflected and transmitted components. The blue wave is the incident wave, with the green wave the component that is reflected. The orange wave is the wave that is transmitted, and the purple wave must be zero.

If we are describing particles moving from the left to the potential energy step: $D' = 0$
 $|A'|^2$ = intensity of incident wave

$|B'|^2$ = intensity of reflected wave
 $|C'|^2$ = intensity of transmitted wave

From this we can say something about the fraction that is reflected vs transmitted:
Fraction reflected :

$$\frac{|B'|^2}{|A'|^2}$$

Fraction transmitted :

$$\frac{|C'|^2}{|A'|^2}$$

The sum must be 1.

9.1.1 Probability Density

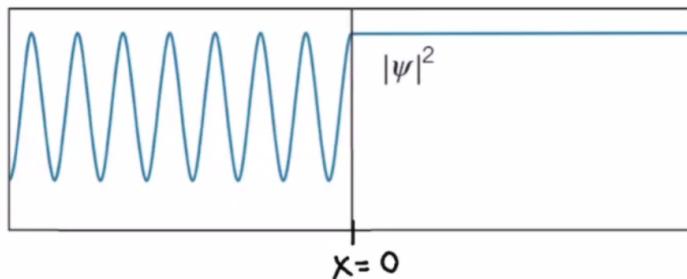


Figure 39: There is an oscillating behavior to the left of the boundary, which is caused by an interference between the incident and reflected wave.

9.2 Where $E < U$

(2) Potential energy step where $E < U_0$

General solutions

(Note: Compare again our discussion of general solutions to Schrödinger equation for constant potential energy)

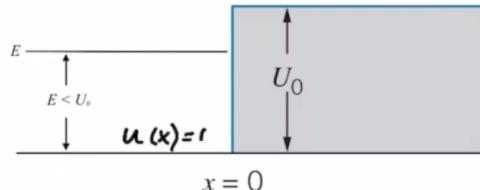


Figure 40:

General solutions:

$$\text{For } x < 0 : \psi_0(x) = A \sin(k_0 x) + B \cos(k_0 x) \quad k_0 = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\text{For } x \geq 0 : \Psi_1(x, t) = C e^{k_1 x} + D e^{-i k_1 x} \quad k_1 = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}}$$

As x approaches infinity, the C term approaches infinity, so it must be set to 0.

All particles are reflected from the barrier. We must have:

$$x < 0 : \Psi_0(x, t) = A' e^{i(k_0 x - \omega t)} + B' e^{-i(k_0 x + \omega t)}$$

With: $|A'|^2 = |B'|^2$

9.2.1 Probability Density

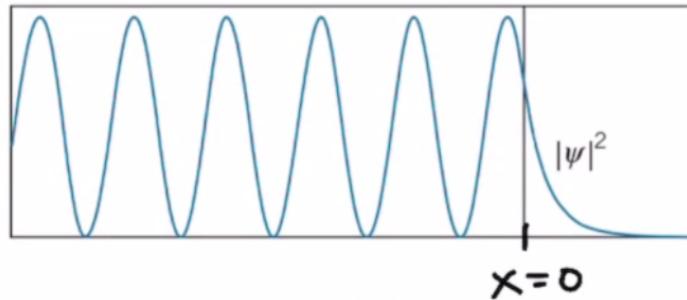


Figure 41: Probability Density ($E < U_0$). The region right of $x=0$ would be referred to the classically forbidden region. The probability density is nonzero.

9.3 Energy Conservation & Penetration into Forbidden Region

Penetration into the forbidden region is associated with the wave nature of the particle as well as the uncertainty in the particle's energy or location (Heisenberg).

For probability density ($x > 0$):

$$P(x) = |\psi(x)|^2 \propto |e^{-k_1 x}|^2 = e^{-2k_1 x} > 0$$

To enter the forbidden region, the particle must temporarily gain energy to get over the potential energy step:

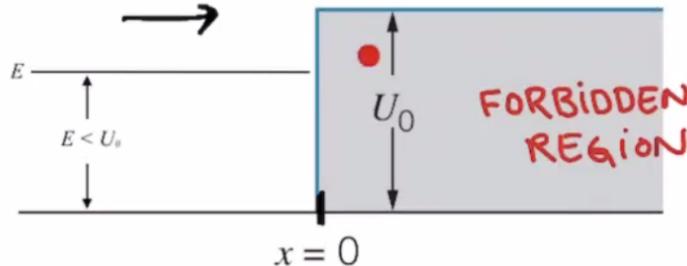


Figure 42: Must gain energy corresponding to $(U_0 - E)$

Suppose that the particle "borrows" energy sufficient to overcome the potential energy and gain a kinetic energy K:

$$(U_0 - E) + K$$

This is OK according to Heisenberg's Uncertainty Principle!

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Because conservation of energy doesn't apply at times smaller than Δt except within an amount:

$$\Delta E \approx \frac{\hbar}{\Delta t}$$

Put another way, the particle can violate classical physics for a very short amount of time!

From this, one can find the maximum penetration distance (Δx_{max}) into the forbidden region:

$$\Delta x_{max} = \frac{\hbar}{1\sqrt{2m(U_0 - E)}}$$

9.4 Potential Energy Barrier

Next we consider a potential energy barrier:

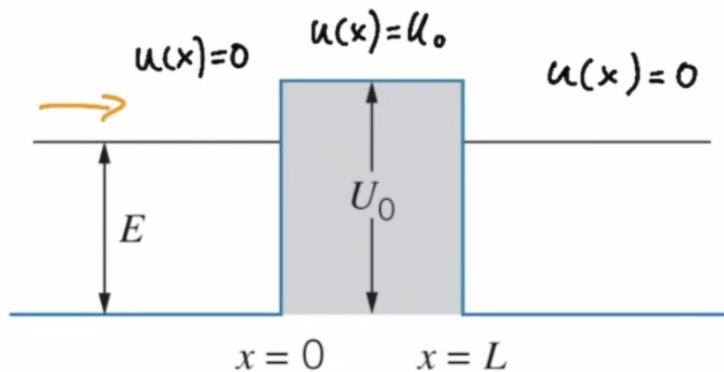


Figure 43: Must gain energy corresponding to $(U_0 - E)$

Particles are incident from the left side of the barrier.

- **Classically**, we would never expect to find particles on the right side of the barrier ($x > L$), since their energy is insufficient to overcome the barrier.
- **Quantum Mechanically**, we do have particles transmitted!

This is known as quantum mechanical tunneling. Particles can never be observed in the classically forbidden region, but they can “tunnel” through it and be observed on the other side!

This is not a hypothetical but a REAL effect that can be observed

9.4.1 Analogy with Wave Optics

If light passing through a glass prism reflects from an internal surface with an angle greater than the critical angle, total internal reflection occurs.

However, the electromagnetic field actually decays exponentially just outside the prism.

If we bring another prism very close to the first one, light passes into the second prism.

This is analogous to quantum-mechanical tunneling.

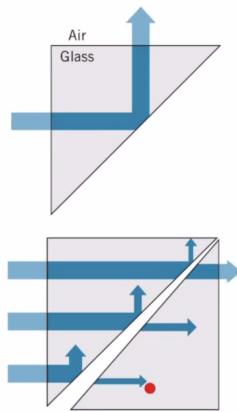


Figure 44:

10 Practical Applications of Quantum Tunneling

10.1 Alpha decay

Nuclei contain protons and neutrons, constantly moving. One type of radioactive decay is emittance of an alpha particle from the nuclei.

However, alpha particles must penetrate a potential energy barrier to escape the nucleus! Measurements of this probability agree very well with quantum mechanical prediction.

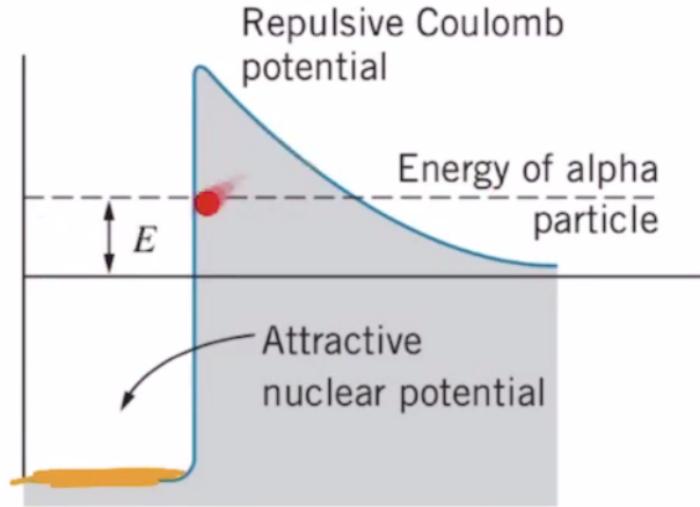


Figure 45:

10.2 Scanning Tunneling Microscope

Electrons trapped in a surface by a potential energy barrier (the work function of the material). When a needle like probe is placed very close to the surface, electrons can tunnel through the barrier between surface & probe - with a current that can be recorded.

This type of scanning tunneling microscope gives details corresponding to 1/100 of an atom.

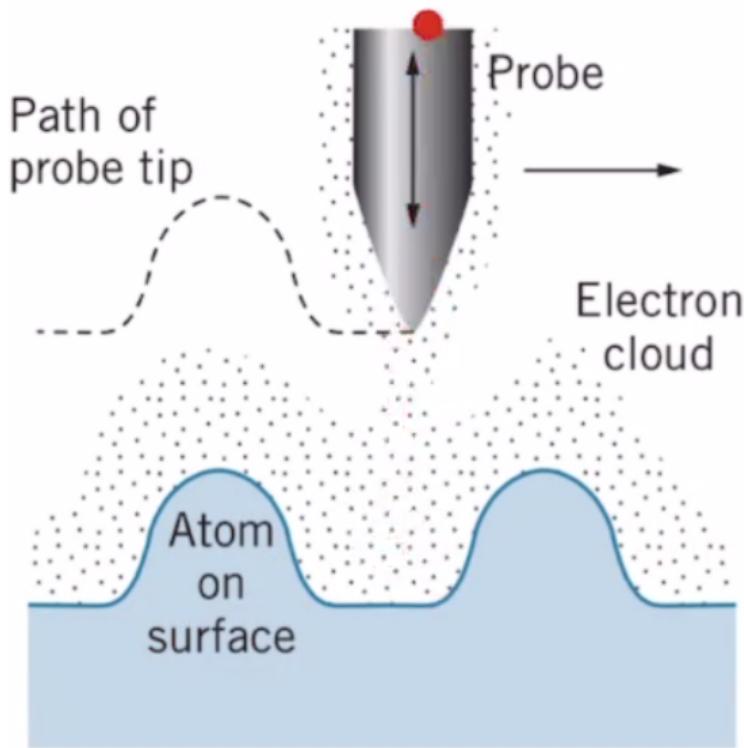


Figure 46:

11 The Hydrogen Atom

Ultimately, we must study the hydrogen atom in three dimensions. However, we'll start with just one dimension.

Note: When talking about a hydrogen atom, we assume no neutrons.

1. Energy levels are the same as the Bohr's model
2. Position of electron is NOT the same
 - (a) Bohr's model gives fixed radii for the electron, where they revolve around the proton in a circular orbit.
 - (b) Quantum mechanics gives there's no "fixed" radius or orbital plane, instead we must talk about probabilities

Coulomb potential energy:

$$U(r) = -\frac{e^2}{4\pi\epsilon_0} * \frac{1}{r}$$

Schrödinger equation:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} - \frac{e^2}{4\pi\epsilon_0} * \frac{1}{x} \psi(x) = E\psi(x)$$

The simplest wave function solution:

$$x > 0$$

Bound state: $\psi(x) \rightarrow 0$ when $x \rightarrow \infty$

$$\psi(x) = Axe^{-bx}$$

If we substitute into the Schrödinger equation:

$$b = \frac{me^2}{4\pi\epsilon_0\hbar^2} \leftarrow \text{This} := \frac{1}{a_0}$$

$$a_0 = 0.0529$$

Corresponding energy is identical to energy of ground state in the Bohr model!

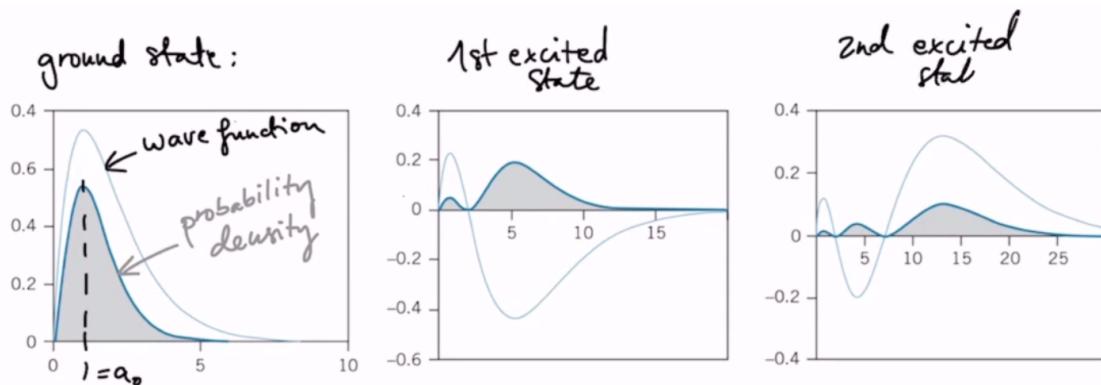


Figure 47: Simplified 1-dimensional version

Example

What is the normalization constant for the ground-state wave function for the one-dimensional Coulomb potential?

$$\psi(x) = Axe^{-bx}, \quad b = \frac{1}{a_0}, \quad A = ?$$

Answer

$$\begin{aligned} 1 &= \int_0^\infty |\psi(x)|^2 dx = \int_0^\infty |Axe^{-bx}|^2 dx \\ &= A^2 \int_0^\infty x^2 e^{-2bx} dx \\ &= \int_0^\infty x^n e^{-cx} dx = \frac{n!}{c^{n+1}} \end{aligned}$$

$$n = 2, \quad c = /a_0$$

$$1 = A^2 \frac{2!}{(2/a_0)^3} = A^2 \frac{2a_0^3}{2^3}$$

Solve for A:

$$A = \sqrt{\frac{2^3}{a_0^3}} = (2a_0)^{-3/2}$$

11.1 Angular Momentum and the Hydrogen Atom (and you!)

Remember: Bohr model used the idea that angular momentum was quantized

Classical orbits

Angular momentum of a particle:

$$\vec{L} = \vec{r} \times \vec{p}$$

where:

\vec{L} : angular momentum vector

\vec{r} : position vector

\vec{p} : linear momentum vector

11.2 Angular Momentum in Quantum Mechanics

Angular momentum in quantum mechanics looks a bit different... For a three-dimensional wave function there are two associated quantum numbers:

- Angular momentum quantum number (l) determines length of angular momentum vector (\vec{L}):

$$|\vec{L}| = \sqrt{l(l+1)}\hbar \quad l = 0, 1, 2, \dots$$

Born said $|\vec{L}| = n\hbar$ —

- Magnetic quantum number (m_l) tells us about the z-component of the angular momentum vector:

$$L_z = m_l \hbar \quad m_l = 0, +1, -1, +2, -2, \dots, +l$$

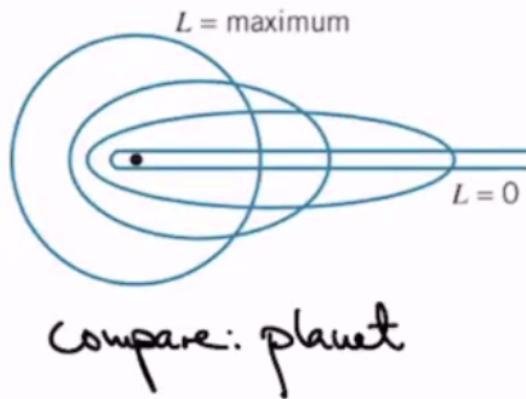


Figure 48: Angular momentum remains constant

Example

Compare the length and z-component of angular momentum vector that represents the orbital motion of an electron in quantum states: $l = 1$ vs $l = 2$ $l = 1$:

$$|\vec{L}| = \sqrt{l(l+1)}\hbar = \frac{2}{\hbar}$$

m_l if $l = 1$? $m_l = 0, + - 1$

$L_z = m_l \hbar$, possible values: $L_z = -\hbar, 0, \hbar$ $l = 2$:

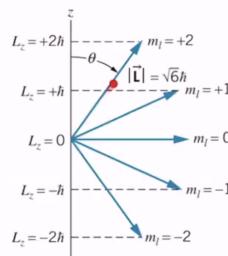
$$|\vec{L}| = \sqrt{l(l+1)}\hbar = \frac{6}{\hbar}$$

m_l if $l = 2$? $m_l = 0, + - 1, + - 2$

$L_z = m_l \hbar$, possible values: $L_z = -2\hbar, -\hbar, 0, \hbar, 2\hbar$

Polar angle θ that vector L makes with z-axis:

$$\cos(\theta) = \frac{L_z}{|\vec{L}|} = \frac{m_l}{\sqrt{l(l+1)}}$$

Figure 49: Graphical representation for the $l=2$ scenario

“Spacial quantization” - only certain orientations of angular momentum vectors are allowed. Number of allowed orientations? $2L + 1$

11.3 The Angular Momentum Uncertainty Relationships

- In quantum mechanics, the maximum information known about the angular momentum vector is the length and z-component.
- To fully describe a three-dimensional vector requires three numbers....
- There is an uncertainty (indeterminacy) in specifying the angular momentum vector

$$\Delta L_z \Delta \phi \geq \hbar$$

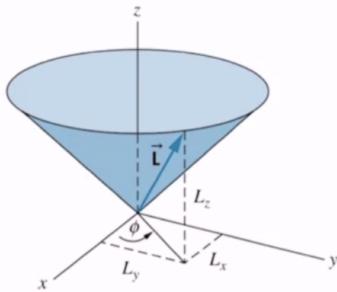


Figure 50: We don't have enough components to determine all dimensions - thus we need *phi* to find the angle

When one component of L_x, L_y, L_z is determined, we know nothing about the other two.

11.3.1 Spherical Polar Coordinates

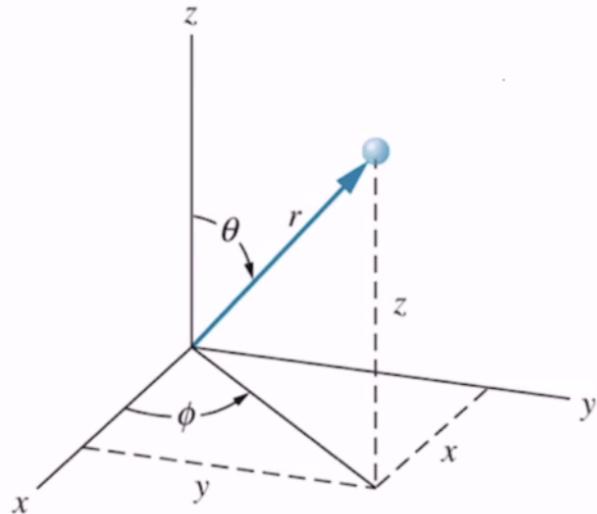


Figure 51: $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$

$$\begin{aligned}x^2 + y^2 + z^2 &= r^2 \\ \cos\theta &= \frac{z}{r} \\ \tan\phi &= \frac{y}{x}\end{aligned}$$

11.4 Schrödinger Equation in Three Dimensions

In Cartesian coordinates:

$$-\frac{\hbar^2}{2m} \left(\frac{\delta^2 \psi}{\delta x^2} + \frac{\delta^2 \psi}{\delta y^2} + \frac{\delta^2 \psi}{\delta z^2} \right) + U(x, y, z)\psi(x, y, z) = E\psi(x, y, z)$$

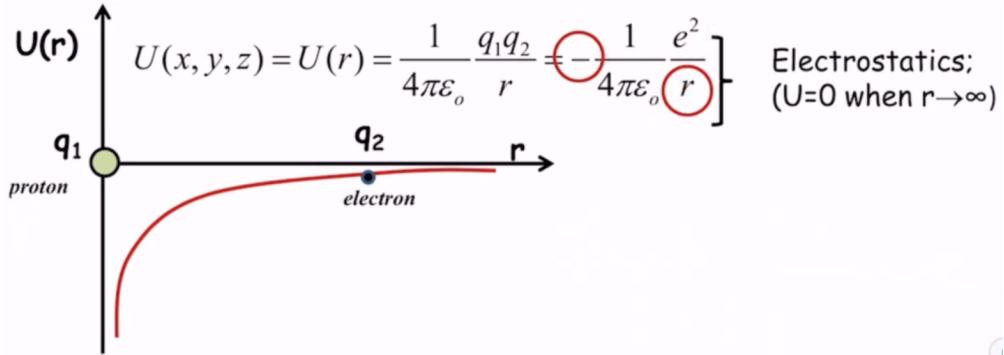


Figure 52: This would get very messy if we tried to solve it

It's more convenient to work in spherical coordinates (r, θ, ϕ) to obtain a separable solution:

$$\psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$$

Radial function, polar function, azimuthal function.

Reminder – spherical polar coordinates:

$$x = rsin(\theta)cos(\phi)$$

$$y = rsin(\theta)sin(\phi)$$

$$z = rcos(\theta)$$

Our penalty for this change of coordinates is a more complex Schrödinger equation:

$$-\frac{\hbar^2}{2m} \left(\frac{\delta^2 \phi}{\delta r^2} + \frac{2 \delta \phi}{r \delta r} + \frac{1}{r^2 sin(\theta)} \frac{\delta}{\delta \theta} \left(sin\theta \frac{\delta \phi}{\delta \theta} \right) + \left(sin^2 \theta \frac{\delta^2 \phi}{\delta \phi^2} \right) \right) + U(r, \theta, \phi)\psi(r, \theta, \phi) = E\psi(r, \theta, \phi)$$

Hydrogen the U function is just $U(r)$.

With our separabel function written as above, this allows us to split the equation into three separate differential equations:

- Azimuthal:

$$\frac{d^2 \Phi}{d\phi^2} + m_2^2 \Phi(\phi) = 0$$

- Polar:

$$\frac{1}{sin\theta} \frac{d}{d\theta} \left(sin\theta \frac{d\Theta}{d\theta} \right) + \left(l(l+1) - \frac{m_l^2}{sin^2 \theta} \right) \Theta(\theta) = 0$$

- Radial:

$$-\frac{\hbar}{2m} \left(\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} \right) + \left(\frac{-e^2}{4\pi\epsilon_0 r} + \frac{l(l+1)\hbar^2}{2mr^2} \right) R(r) = ER(r)$$

Where:

m: electron mass

(r, θ, ϕ) : electron position in polar coordinates

E: Total energy

and some quantum numbers.

11.4.1 More on these Quantum Numbers

When we solved the one-dimensional Schrödinger equation for the infinite potential energy well, we found that our solutions for allowed wavelength and energies were quantized. The index, or quantum number, “n” appeared in our solutions.

Similarly when solving the three-dimensional Schrödinger equation, **three** parameters emerge as indices. Only certain values for these are allowed to ensure that solutions are physical and don’t diverge. These are *three quantum numbers* that describe the solutions:

- Principle quantum number: n of n = 1,2,3 fame
- l where ($l = 0, 1, 2, 3, \dots, n-1$)
- m_l where ($m_l = 0, +1, -1, \dots, +l$)

Same as discussed yesterday!

Quantized Energy Levels

$$E_n = -\frac{me^4}{32\pi^2\epsilon_0^2\hbar^2} = \frac{-13.60 \text{ eV}}{n^2}$$

Same as in the Bohr’s model, sweet

Wave Function Solutions

$$\psi_{n,l,m_l}(r, \theta, \phi) = R_{n,l}(r)\Theta_{l,m_l}(\theta)\Phi_{m_l}(\phi)$$

11.4.2 Degeneracy

Different sets of quantum numbers can correspond to exactly the same energy. This is known as “degeneracy”, and such an energy level is said to be “degenerate”.

Ground state of hydrogen atom:

$$E_n = \frac{-13.60 \text{ eV}}{n^2} \rightarrow n = 1 : \rightarrow E_1 = -13.60 \text{ eV}$$

$n, l, m_l : 1, 0, 0$

First excited state:

$n = 2$:

$$E_2 = \frac{-13.60 \text{ eV}}{2^2} = -3.40 \text{ eV}$$

$[n, l, m_l] : [2, 0, 0], [2, 1, 0], [2, 1, -1], [2, 1, 1]$

Generally: State $n \rightarrow n^2$ degenerate

Why do we still list these separately?

1. Levels are not precisely degenerate (1 e-5 eV separations)
2. We will see that transitions between energy levels depend on quantum numbers of energy level that transition starts from
3. Each set of quantum numbers correspond to a different *wave function*, and thus a different *state of motion* for an electron (different probability densities)

11.5 Probability Densities

Probability density for one-dimensional wave function: $P(x) = |\psi(x)|^2$

Note: wave function can be complex, so this is wave function times its complex conjugate $|\psi|^2 = \psi\psi^*$

We are now working in THREE dimensions, so we need volume probability density:

$$P(r, \theta, \phi) = |\psi(r, \theta, \phi)|^2$$

Probability to find electron in a small volume centered at (r, θ, ϕ) is:

$$P(r, \theta, \phi)dV \quad (\text{volume element})$$

In cartesian coordinates $dV : dx dy dz$. But, in polar:

$$dV = r^2 \sin\theta dr d\theta d\phi$$

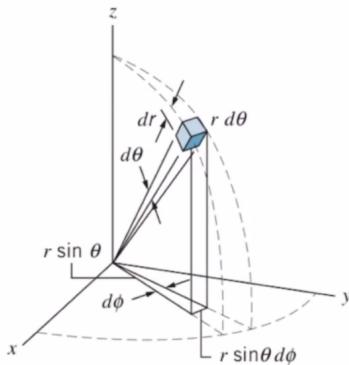


Figure 53: Finding the probability of finding the electron in some tiny volume element

Or, written out in terms of the separable wave function:

$$|\psi(r, \theta, \phi)|^2 dV = |R_{n,l}(r)|^2 |\Theta_{l,m_l}(\theta)|^2 |\Phi_{m_l}(\phi)|^2 r^2 \sin\theta dr d\theta d\phi$$

We can think of these as smeared out distributions of electric charge in the atom, due to uncertainty in locating the electron. Or, the statistical result of large numbers of measurements of the electron's position.

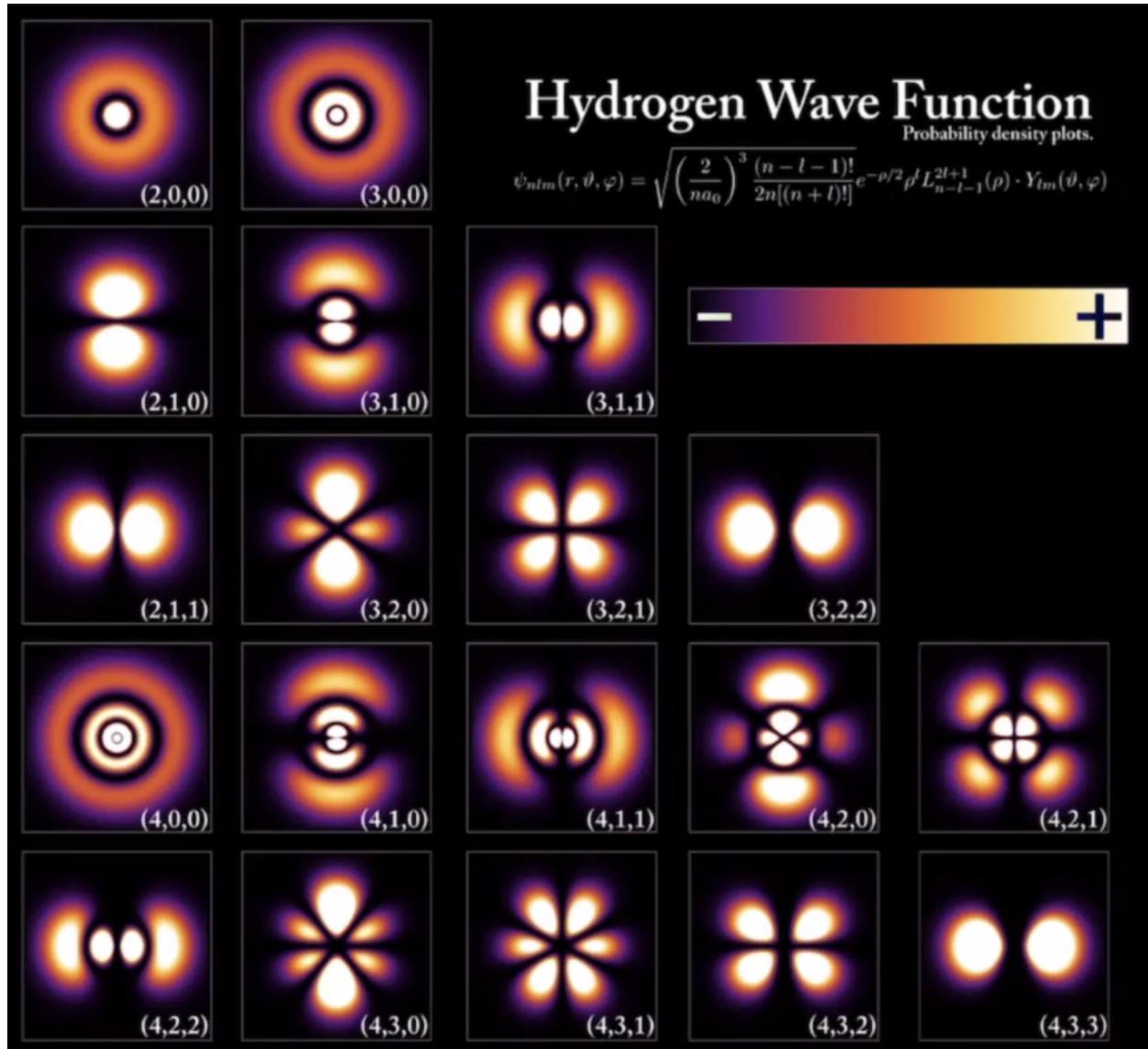


Figure 54: Finding the probability of finding the electron in some tiny volume element

11.5.1 Radial Probability Density

Consider just the probability to locate the electron at a particular radial distance from the nucleus. You can just integrate over the θ, ϕ coordinates!

$$P(r)dr = |R_{n,l}(r)|^2 r^2 dr \int_0^\pi |\Theta_{l,m_l}(\theta)|^2 \sin\theta d\theta \int_0^{2\pi} |\Phi_{m_l}(\phi)|^2 d\phi$$

But hey, if you integrate over all valid values of the angular components, they just end up being 1 because the wave function is normalized! So we get:

$$P(r) = r^2 |R_{n,l}(r)|^2$$

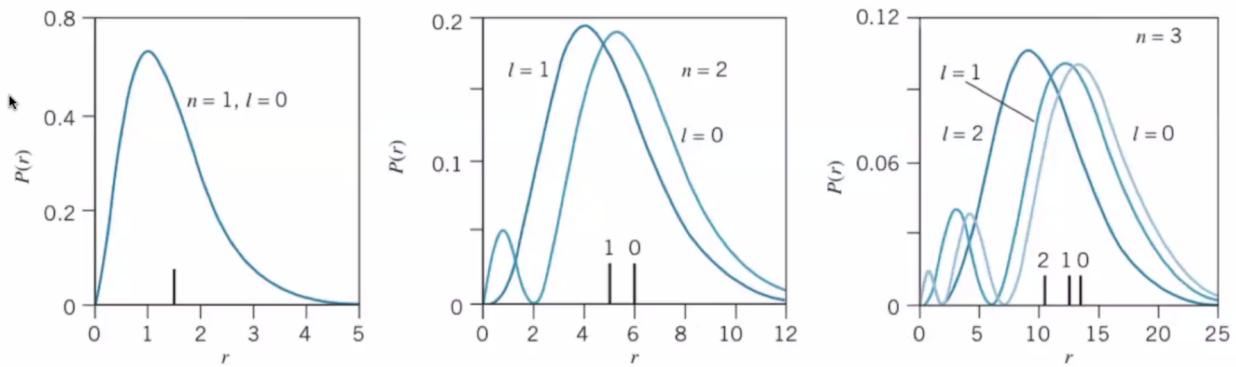


Figure 55: The probability density of r alone. The highest point won't necessarily line up with the radius predicted by Bohr? The lower you go for l the more little peaks you get.

11.5.2 Angular Probability Density

Consider instead the angular probability density:

$$P(\theta, \phi) = |\Theta_{l,m_l}(\theta)\Phi_{m_l}(\phi)|^2$$

Note: These are all cylindrically symmetric.

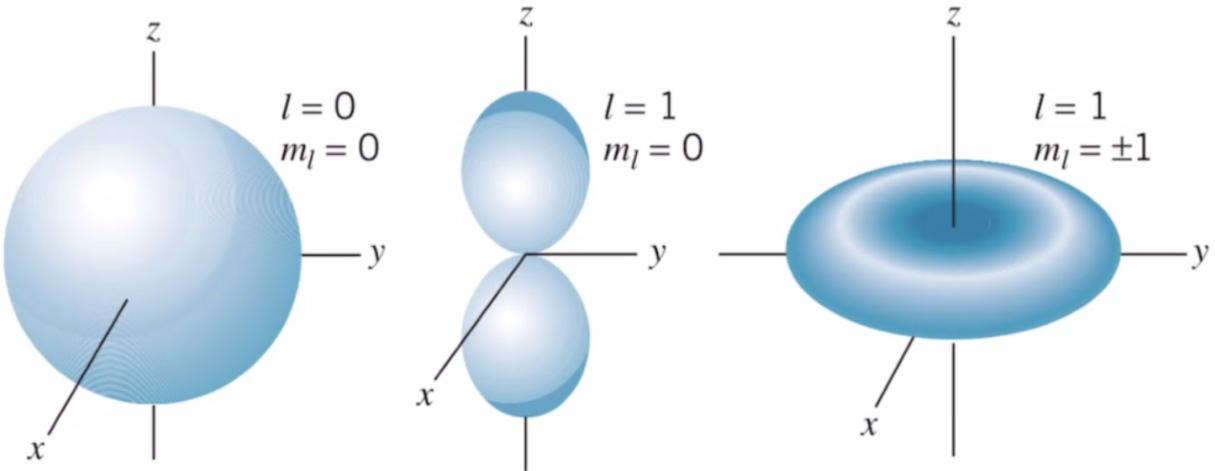


Figure 56:

11.6 Time-Dependence

We are working with stationary states which won't change with time.

I missed this

Example

For the n=2 states ($l=0, l=1$), compare the radial probabilities of finding the electron inside the Bohr radius?

$$R_{2,0}(r) = \frac{1}{(2a_0)^{3/2}} \left(2 - \frac{r}{a_0} \right) e^{-r/2a_0}$$

Radial Probability:

$$P(r)dr = r^2 |R_{n,l}(r)|^2 dr \quad 0 \leq r < a_0$$

for n=2, l=0:

$$\begin{aligned} P(0 : a_0) &= \int_0^a r^2 |R_{2,0}(r)|^2 dr = \int_0^a r^2 \frac{1}{(2a_0)^{3/2}} \left(2 - \frac{r}{a_0} \right)^2 e^{-r/a_0} dr \\ &= \frac{1}{8a_0^3} \int_0^a \left(4r^2 - \frac{4r^3}{a_0} + \frac{r^4}{a_0^2} \right) e^{-r/a_0} dr \end{aligned}$$

To solve these types of integrals:

$$\int x^n e^{-cx} dx = -\frac{e^{-cx}}{c} \left(x^n + \frac{nx^{n-1}}{c} + \frac{n(n-1)x^{n-2}}{c^2} \dots + \frac{n!}{c^n} \right)$$

$$P(0 : a_0) = 0.034$$

If we do the same for the n=2, l=1 state:

$$P(0 : a_0) = 0.0037$$

11.7 Magnetic Moments

The orbital angular momentum of the electron is related to its magnetic dipole moment. Electron's orbital

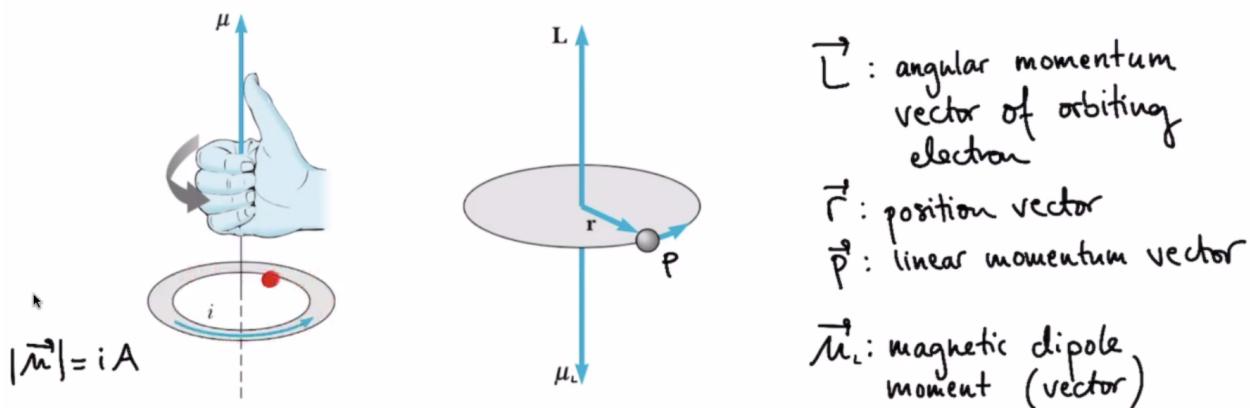


Figure 57: The direction of current is by convention that of positive charge, while the electron is negatively charged. Note the subscript "L" on \vec{u}

magnetic dipole moment (follows from considering orbiting electron as a circular loop of current):

$$\vec{u}_L = -\frac{e}{2m} \vec{L}$$

The z-component of the magnetic moment:

$$\mu_{Lz} = -\frac{e}{2m} L_z = -\frac{e}{2m} m_l \bar{h}$$

Where m_l is the magnetic quantum number and m is the electron mass.
Bohr magneton defined as:

$$\mu_B = \frac{e\bar{h}}{2m} \approx 9.274 \cdot 10^{-24} J/T \rightarrow \mu_{Lz} = -m_l \mu_B$$

What happens to a magnetic dipole moment ($\vec{\mu}$) in an external magnetic field (\vec{B})?

It experiences a torque:

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

Analogous to an electric dipole moment in an external electric field:

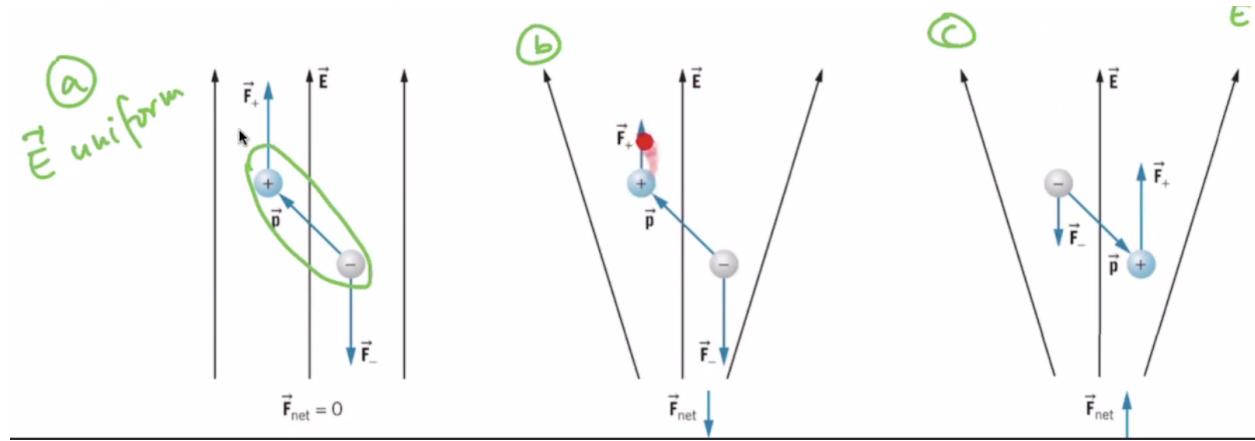


Figure 58: B + C non-uniform E-field.

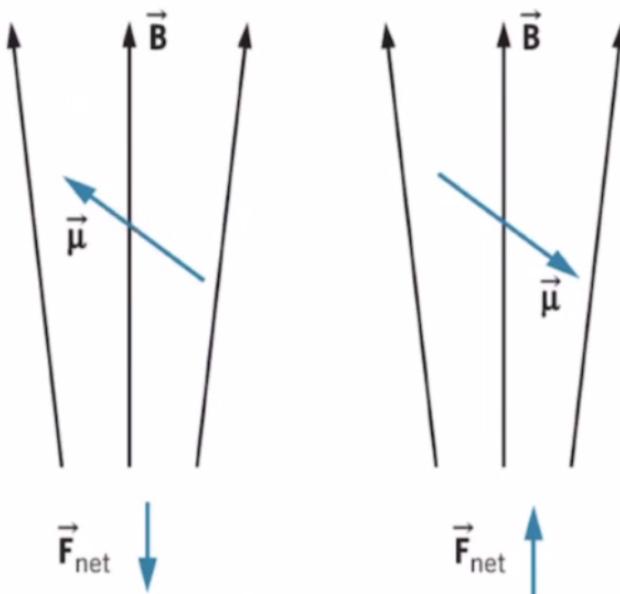


Figure 59: Magnetic dipoles in nonuniform magnetic field experience a net force (direction depending on direction dipole is pointing in)

What should happen to atoms passing through a region of nonuniform magnetic field?

For hydrogen: $\mu_{L,z} = -m_l \mu_B$

if $n = 1, l = 0 \rightarrow m_l = 0 \rightarrow \mu_{Lz} = 0 \rightarrow$ no deflection

if $n = 2, l = 1 \rightarrow m_l = 0, \pm 1 \rightarrow \mu_{Lz} = 0, \pm \mu_b$

11.7.1 Stern-Gerlach Experiment

Initially done in 1921 using silver atoms – more complicated structure than hydrogen atom but same general principle. Repeated in 1927 using hydrogen atoms.

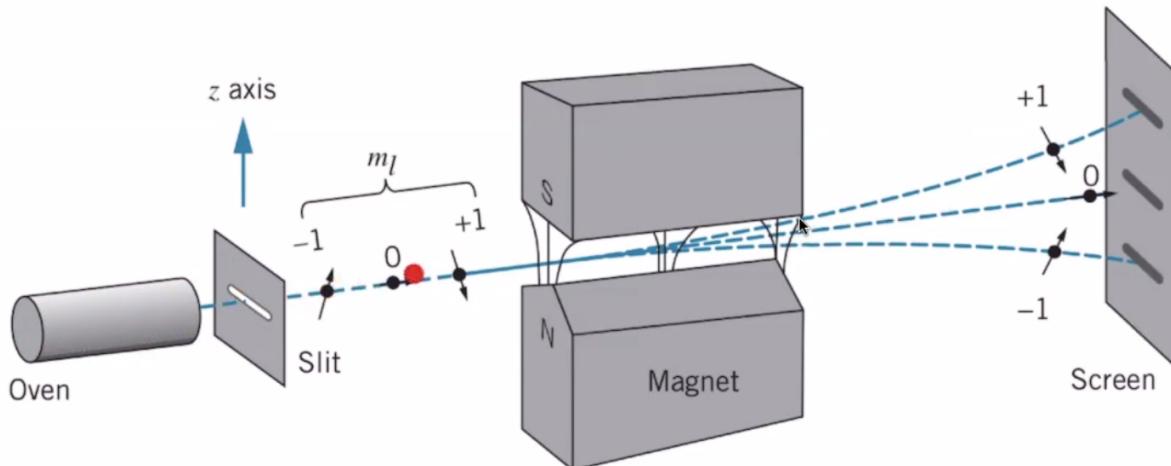


Figure 60: $n=2, l=1, m_l = -1, 0, +1$. Expectation was that the beam will be split in $(2l + 1)$, the number of quantum numbers.

Experimental result: Beams were split in **TWO**?!?

$$2 \cdot S + 1 = 2 \cdot 1/2 + 1 = 2$$

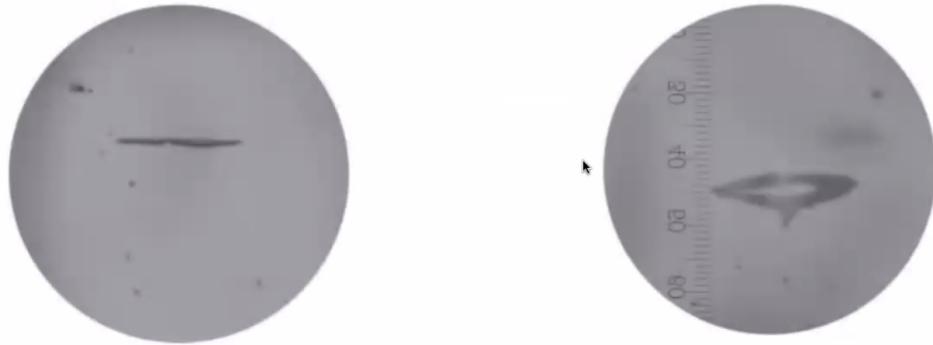


Figure 61: Left: Magnetic field OFF. Right: Magnetic field ON

Explanation: There is an additional component to the electron's angular momentum, with an associated magnetic moment. This turns out to be an intrinsic property of the electron (like its mass or charge), called intrinsic **spin**.

An electron has an **intrinsic spin quantum number $s = 1/2$** (it has “spin 1/2”).

Spin is a property of all elementary particles: photon has “spin 1”, Higgs boson has a “spin 0”.

11.8 Intrinsic Spin

Spin ends up being super important when dealing with multiple electrons.

- Electron has orbital angular momentum: \vec{L}
- Electron has intrinsic angular momentum: \vec{S}

	Orbital	Spin
Quantum number	$l = 0, 1, 2, \dots$	$s = 1/2$
Length of vector	$ \vec{L} = \sqrt{l(l+1)}\hbar$	$ \vec{S} = \sqrt{s(s+1)}\hbar = \sqrt{3/4}\hbar$
z component	$L_z = m_l\hbar$	$S_z = m_s\hbar$
Magnetic quantum number	$m_l = 0, \pm 1, \pm 2, \dots, \pm l$	$m_s = \pm 1/2$
Magnetic moment	$\vec{\mu}_L = -(e/2m)\vec{L}$	$\vec{\mu}_S = -(e/m)\vec{S}$

Figure 62: These are very very similar.

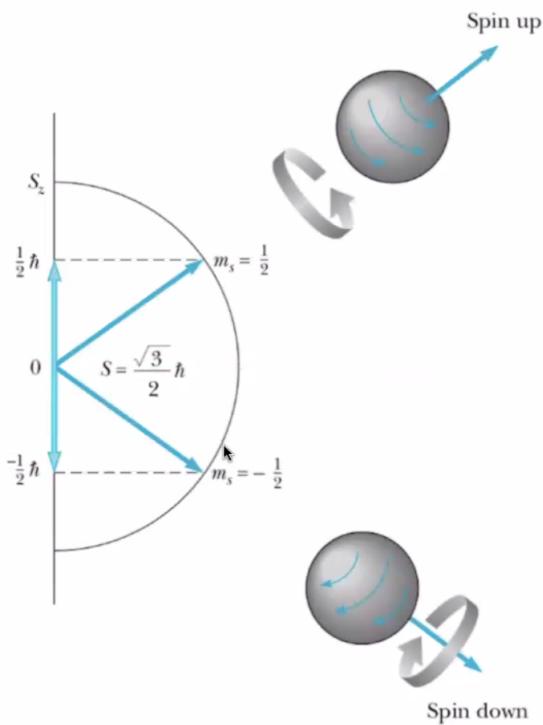


Figure 63: We talk about “spin up” vs “spin down”. Note: While thinking about intrinsic spin as the electron spinning about its own axis, in the same way as the Earth spins about its own axis – “spin angular momentum” + orbits about the sun “orbital angular momentum” is a useful image, but it is physically incorrect as the electron is a point particle with no physics size.

11.8.1 Energy Levels (Reloaded)

We in fact need to use FOUR quantum numbers to fully describe the electronic states in hydrogen:

$$n, l, m_l, m_s \quad (s = 1/2 \text{ for electrons})$$

Degeneracy of a given energy level “n” is: $2n^2$

Usually it is sufficient to specify (n, l) for these energy levels: **spectroscopic notation**:

Value of l	0	1	2	3	4	5	6
Designation	s	p	d	f	g	h	i

$$\text{Q.S.: } \begin{cases} n = Z \\ l = 0 \end{cases}$$

Figure 64:

11.8.2 Fine Structure (bonus)

We are now equipped to discuss why energy levels are not really 100% degenerate as mentioned yesterday... this is called “fine structure”.

From electron’s perspective, it is being orbited by a positively charged proton. This “proton current” causes a magnetic field at the electron, which interacts with the electron’s spin magnetic moment. This results in a small energy difference between “spin up” and “spin down” states:

$$\Delta E = 2\mu_B B$$

For the bohr magneton and effective induced magnetic field.

To fully determine this quantitatively, one must consider relativistic quantum mechanics (way beyond the

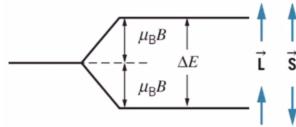


Figure 65:

scope of this course). But the difference is on the order of 0.00001 eV.

11.9 Probability, again?

Example

From the $n=2, l=1, m_l = 0$ wave function, find the directions in space at which the maximum probability occurs.

Answer

$$\begin{aligned} P(\theta, \phi) &= |\Theta_{l,0}(\theta)\Phi_0(\phi)|^2 \\ &= \left| \sqrt{\frac{3}{2}} \cos\theta \cdot \frac{1}{\sqrt{2\pi}} \right|^2 = \frac{3}{4\pi} \cos^2\theta \end{aligned}$$

Note that there is no dependence on phi.

Maximum probability?

$$\begin{aligned} \frac{dP}{d\theta} &= \frac{3}{4\pi} (-2\cos\theta \cdot \sin\theta) = 0 \\ &= \cos\theta = 0 \rightarrow \theta = \frac{\pi}{2} \\ &= \sin\theta = 0 \rightarrow \theta = 0, \pi \end{aligned}$$

Max or min?

$$\frac{d^2P}{d\theta^2} = \frac{3}{2\pi} (\sin^2\theta - \cos^2\theta)$$

if $\theta = \pi/2$, second derivative is greater than zero, so it's a minimum. if $\theta = 0, \pi$, second derivative is less than zero, so it's a maximum.

12 Many-Electron Atoms

- When we begin to consider atoms with more than one electron, potential energy now depends both on:
 - An attractive Coulomb potential between the electrons and nucleus
 - A repulsive Coulomb potential between different electrons
- We quickly fail to solve exactly the Schrödinger's equation
- Instead, solutions are obtained numerically

12.1 Pauli's Exclusion Principle

We have seen that an atomic electron is described by four quantum numbers:

$$n, l, m_l, m_s \text{ where } s = 1/2 \text{ always for electrons}$$

How many electrons in an atom can have the same four quantum numbers? I.e. how many electrons in an atom can be in the same state?

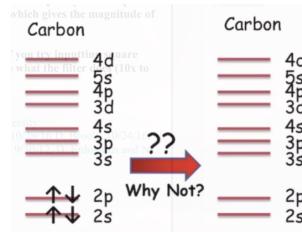


Figure 66:

Pauli's exclusion principle

No two electrons in an atom can have the same set of quantum numbers!

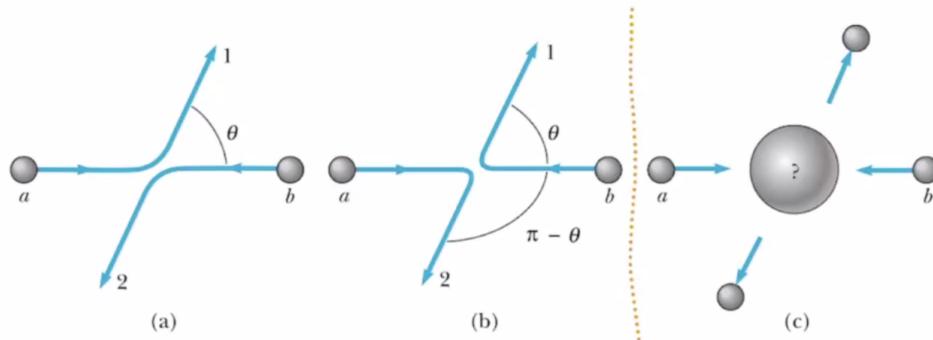


Figure 67: Picture a collision of two electrons. Classical on the left. On the right, in the quantum we can't do this anymore. You don't know which outcome is the right one, because when they are interacting we can't tell which one is which due to their wave properties.

Probability density describing the probability of finding the two electrons:

Two-electron wave function: $\Psi(\vec{r}_1, \vec{r}_2)$, where r_x is the position of electron x.

$$|\Psi(\vec{r}_1, \vec{r}_2)|^2 = |\Psi(\vec{r}_2, \vec{r}_1)|^2$$

There are two options for the wave function now:

$$\Psi(\vec{r}_1, \vec{r}_2) = \Psi(\vec{r}_2, \vec{r}_1) \rightarrow \text{Bosons}$$

$$\Psi(\vec{r}_1, \vec{r}_2) = -\Psi(\vec{r}_2, \vec{r}_1) \rightarrow \text{Fermions}$$

“Fermions” are particles with spin 1/2: electrons, protons, neutrons, are all fermions.

“Bosons” are particles with integer spin: photon, Higgs boson, gluons ...

Pauli's exclusion principle applies to all "spin 1/2" particles!

Variations of chemical and physical properties of different elements stems from how their electrons are structured:

Element: Electrons, (n, l, m_l, m_s)

Hydrogen: 1 electron, ground state: $(1, 0, 0, \pm 1/2)$

Helium: 2 electrons: $(1, 0, 0, +1/2), (1, 0, 0, -1/2)$

Lithium: 3 electrons: $(1, 0, 0, 1/2), (1, 0, 0, -1/2), (2, 0, 0, \pm 1/2)$

Electronic configurations follow from this principle. Hydrogen: $1s$

Value of l	0	1	2	3	4	5	6
Designation	s	p	d	f	g	h	i

Figure 68: In a given state: max $2(2l+1)$ electrons. Where the first 2 is nbr m_s , second term nvm of m_l .

Helium: $1s^2$

Lithium: $1s^2 2s$

.

Flourine: $1s^2 2s^2 2p^5$

12.2 Shells & Subshells

Electronic "shells" are labeled according to their principle quantum number:

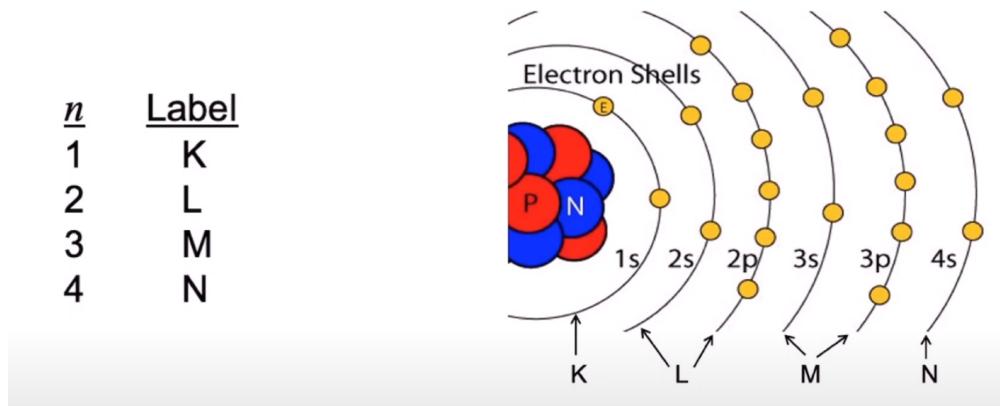


Figure 69:

Shell: Described by principle quantum number (n)

Subshell: Described by principle quantum number (n) + angular momentum quantum number (l) ($s, 2p, ..$)

12.2.1 Periodic Table and Subshells (and You!)

Approximate filling of energy levels in many-electron systems, as the atomic number Z for the atom increases:
 Different subshells correspond to different energy levels.

(Note: for hydrogen atoms with a single electron, only principle quantum number “n” determines the energy level. In multi-electron systems, this is not the case.)

Filling “rules”:

- Capacity of each subshell is $2(2l+1)$
- Electrons occupy the lowest energy states available
- An electron in the 2s level has a greater probability to be found at a small radii compared to an electron in the 2p level.
- Impacts “effective” net charge that the electron “sees” ergo. tighter binding of 2s electrons compared to 2p.
- More dominant effect for larger principle quantum number, e.g. n=4
 - Tighter binding of 4s and 4p electrons results in these electrons’ energy levels being almost as low as 3d level
 - Exact energy levels depends on the given atom!

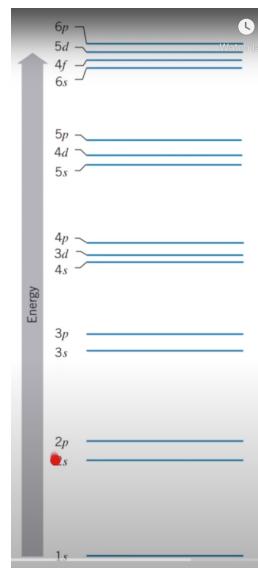


Figure 70:

12.3 Electron Screening

There's no simple way to calculate orbital radius for electrons in most atoms. Generally it's a reasonable assumption that e.g. a 2s electron is farther from the nucleus than 1s electron.

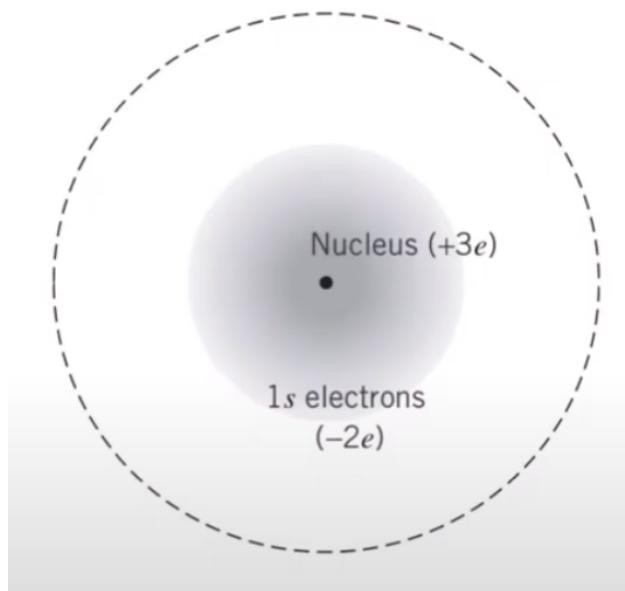


Figure 71:

Effective nuclear charge

For an electron in 2s state:

+3e in nucleus, -2e from 1s state electrons

$\rightarrow Z_{eff.} = +1e$ Approximate estimate of energy of this level:

$$E_n = (-13.60\text{eV}) \frac{Z_{eff}^2}{n^2}$$

Li, 2s electron: $E = -3.40\text{eV}$

Ionization of neutral lithium atom?

5.39 eV (not bad).

This effect is called electron screening – to an outer electron, the charge of the nucleus is “screened” or “shielded” by the electrons in the inner shells.

The less penetrating an orbit is for the outer electron, the more accurate the above approximation is.

12.4 Optical Transitions

If an outer electron is excited to a higher energy level or removed from the atom, “vacancy” is created and can be filled by electrons dropping in to this state.

Energy lost appears as emitted photons, usually have wavelengths in the visible range of the spectrum “optical transitions”.

Note:

$$\Delta l = \pm 1$$

13 Molecular Structure

When atoms brought together to form molecules, atomic states of the electrons become molecular states. Probability density of occupied molecular states determine the nature of the molecular bonds and the structure/properties of molecules.

Why should molecules form at all? Coulomb potential of electrons in one atom repels electrons in another atom?

No, probability densities of atomic orbits are not always spherically symmetrical!

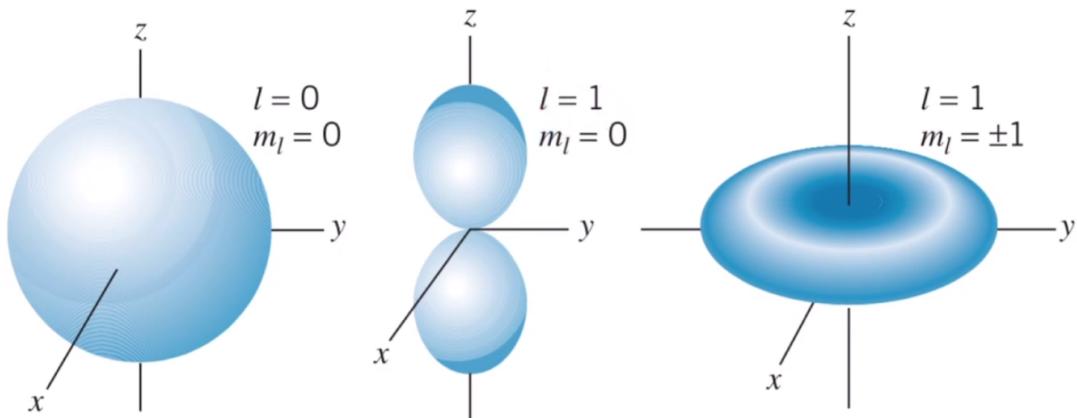


Figure 72:

13.1 Hydrogen Molecule

Lets start even simpler: Hydrogen Molecule Ion! H_2^+

- Electron is shared between the two!
- When the electrons are close, the wave functions begin to overlap, and we must combine them according to the rules of quantum mechanics:
 - First add the wave functions
 - Then square the wave functions to get the probability density
- When combining two wave functions, relative sign matters!



Figure 73: Wave function for two electrons separated by a large distance

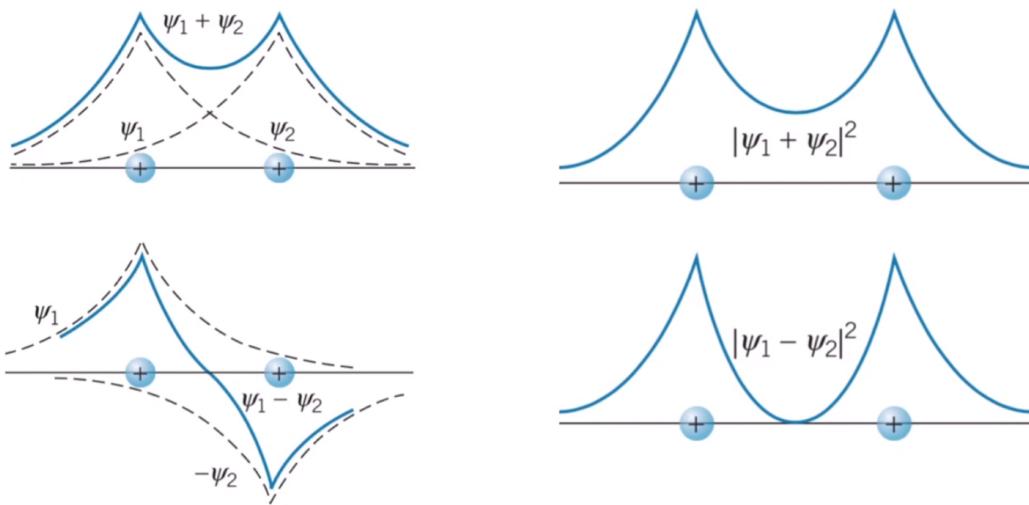


Figure 74: Left: Overlap of two wave functions. Right: Corresponding probability density

Two contributions to energy of hydrogen molecule ion:

1. Coulomb repulsion between the two positively charged protons
2. Coulomb attraction between the protons and the electrons

For stable molecules to form, total energy must be negative and the electron must provide enough negative energy of attraction to overcome the positive repulsion.

Generally we define molecular binding energy as energy difference between the separate components (H and H^+) and the combined system, e.g.:

$$B = E(H + H^+) - E(H_2)$$

Molecular bonds involve sharing of a single electron between two atoms of the molecule. Electron spends most of its time between the two protons!

13.1.1 Continuing with hydrogen molecule...

Molecular state that leads to a stable molecule: “bonding state” One state that does not lead to a stable molecule: “anti-bonding state”

Note: for two electrons to occupy the same molecular state, their spins must be in opposite directions ergo pauli exclusion principle

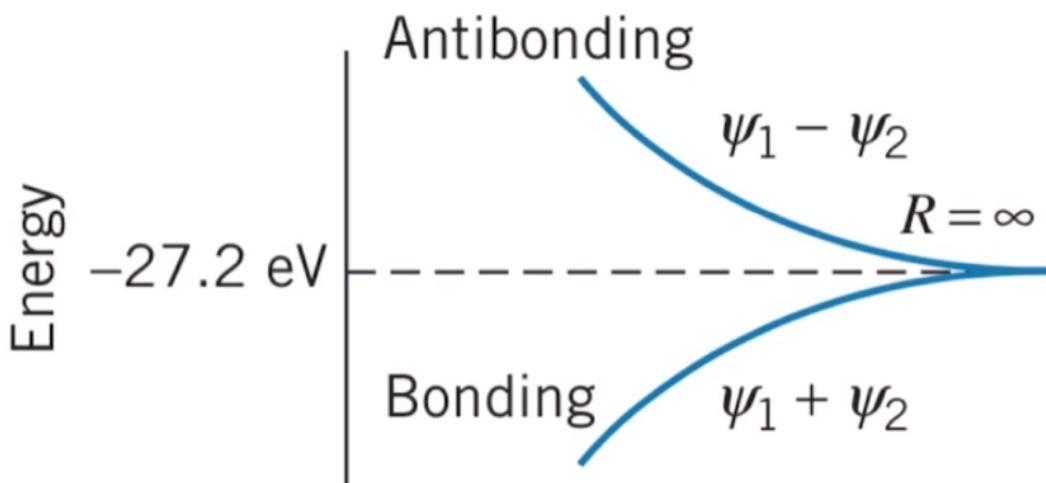


Figure 75:

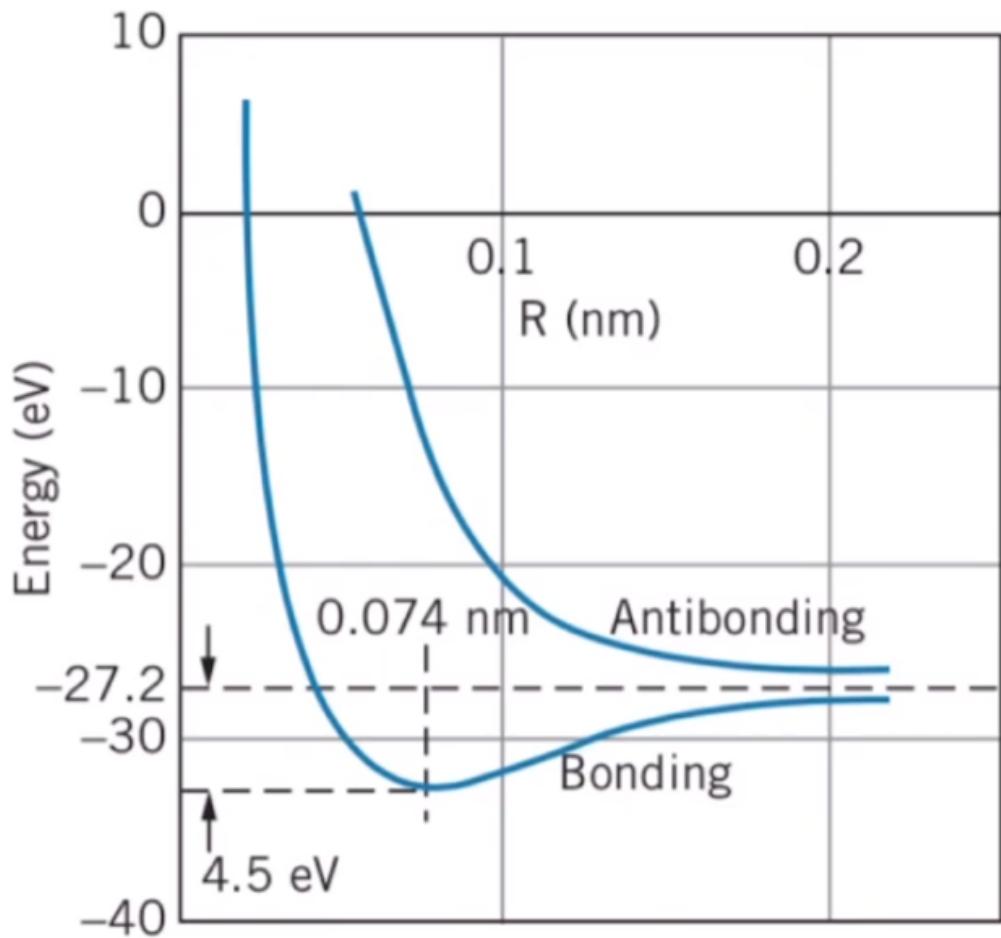


Figure 76: Molecular Binding Energy

Molecular Binding Energy

$$\begin{aligned}
 B &= E(H + H^+) - E(H_2) \\
 &= 2(-13.60 \text{ eV}) - (-31.7 \text{ eV}) \\
 &= 4.5 \text{ eV}
 \end{aligned}$$

13.1.2 Molecules

Forming of molecules is a tradeoff between attractive and repulsive Coulomb force. Charge is usually distributed unevenly in atoms.

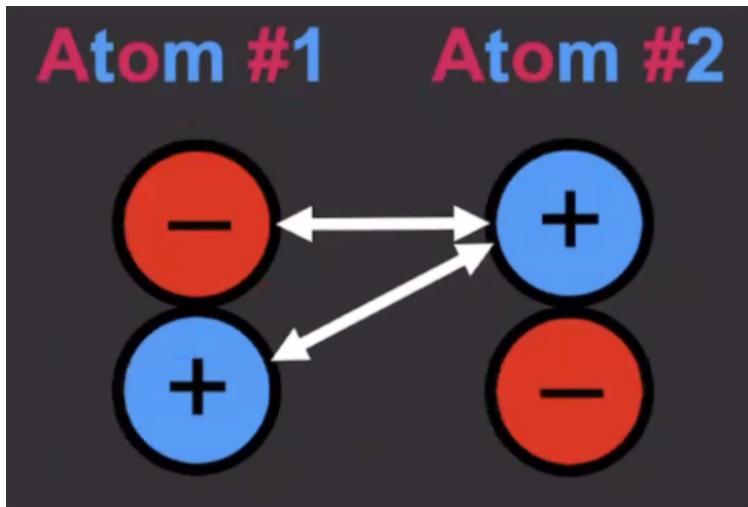


Figure 77:

13.2 Covalent Bonding

Common for molecules containing two identical atoms.

As two atoms are brought together, the electrons interact and form molecular states rather than separate atomic states:

For one of these, the overlap of the electron wave functions results in a lower energy than that of the separate atoms

For the other, anti-bonding state has an increased energy compared to the separate atoms, ergo no stable molecule.

Note: Pauli's principle applies to molecular states as well!

13.3 Ionic bonding

You remember this stuff, it's whatever.

13.4 Molecular Vibrations and Rotations

Like atoms, molecules can absorb or emit energy by changing the configuration of their electrons. Molecules can also emit or absorb energy by vibrating about their equilibrium position or rotating about their center of mass.

13.4.1 Molecular Vibrations

Remember quantum harmonic oscillator, for oscillator in one-dimension:

$$E_N = (N + 1/2)\hbar\omega \quad N : \text{quantum number}, \omega = \sqrt{\frac{k}{m}}$$

These are vibrational energies for molecules.

Allowed transitions for vibrational transitions (absorb or emit electromagnetic radiation):

$$\Delta N = \pm 1$$

13.4.2 Vibrating Diatomic Molecules

What is the effective mass m?

Consider point of motion (equilibrium point) where total energy is only kinetic energy:

$$E_{tot} = \frac{p_1^2}{2m_1}$$

Where p_1, p_2 are the linear momentums of atom 1, 2.

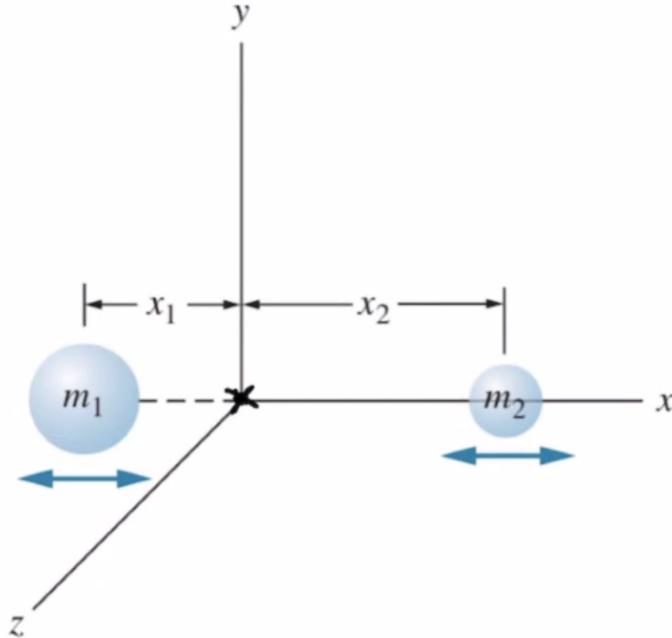


Figure 78:

The total momentum must be zero in the reference frame of the molecule:

$$\vec{p}_1 + \vec{p}_2 = 0, \quad p_1 = p_2 = p$$

$$E_{tot} = \frac{p^2}{2m_1} + \frac{p^2}{2m_2} = \frac{p^2}{2} \left(\frac{m_1 + m_2}{m_1 m_2} \right)$$

Reduce "effective" mass: $m = \frac{m_1 + m_2}{m_1 m_2}$

What is the effective force constant k ?

Can be approximated by considering the motion about equilibrium position. Tabulated for different molecules (in units of eV/m^2 or N/m).

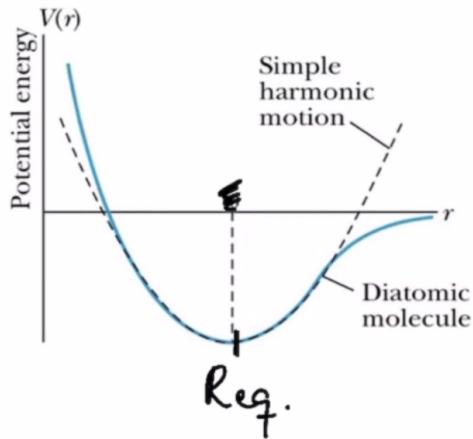


Figure 79:

13.4.3 Molecular Rotations

A diatomic molecule can be thought of as two atoms held together with a massless, rigid rod.

In a purely rotational system, kinetic energy is expressed in terms of the rotational angular momentum (L) and rotational inertia (I):

$$K = \frac{1}{2}I\omega^2 = \frac{L^2}{2I} \quad (\text{From classical physics})$$

In quantum mechanics angular momentum is quantized!

$$E_L = \frac{L(L+1)\hbar^2}{2I}$$

Using upper case L to distinguish this angular momentum quantum number of a molecular system as opposed to lower case l representing that of a particle.

Allowed transitions for rotational transitions (absorb or emit electromagnetic radiation):

$$\Delta L = \pm 1$$

13.4.4 Rotating Diatomic Molecules

Rotational inertia of the molecule:

$$I = m_1 x_1^2 + m_2 x_2^2$$

In terms of reduced mass:

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

and equilibrium separation:

$$R_{eq} = x_1 + x_2$$

Ergo, rotational inertia:

$$I = m R_{eq}^2$$

Rotational energies:

$$E_L = \frac{L(L+1)\hbar^2}{2mR_{eq}^2} \quad l = 0, 1, ..$$

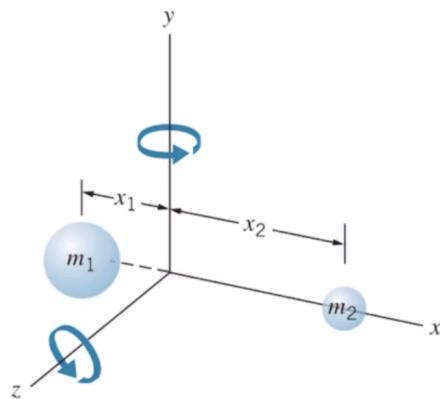


Figure 80:

Typical size of transitions

- Vibrational $\approx 0.1 - 1 \text{ eV}$
- Rotational $\approx 0.01 - 0.1 \text{ eV}$

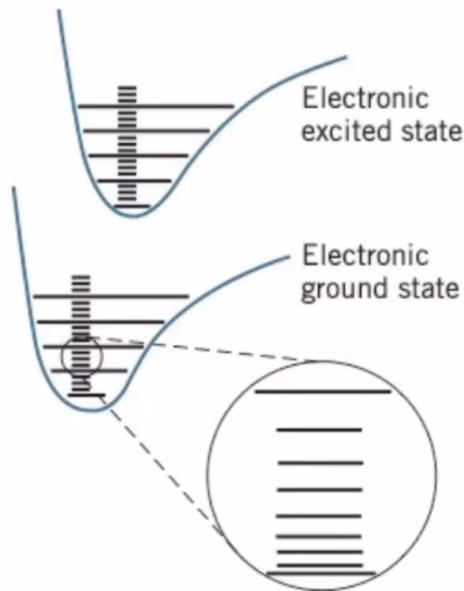


Figure 81: In practice, these all combine to form different energy levels for molecules

14 Light Amplification by Stimulated Emission of Radiation

Lasers were first proposed by Einstein in 1917. It took nearly half a century for the first prototype to work.

Different ways in which radiation interact with atomic energy levels:

- Spontaneous emission: When an atom in an excited state falls into a lower energy level, it emits a photon of light:

$$\text{atom}^* \rightarrow \text{atom} + \text{photon}$$

- When an atom encounters a photon, it can absorb the photon's energy and jump to an excited state:

$$\text{atom} + \text{photon} \rightarrow \text{atom}^*$$

- Stimulated emission: When a photon encounters an atom in an excited state, the photon can induce the atom to emit its energy as another photon of light, resulting in two identical photons:

$$\text{atom}^* + \text{photon} \rightarrow \text{atom} + 2 \text{ photons}$$

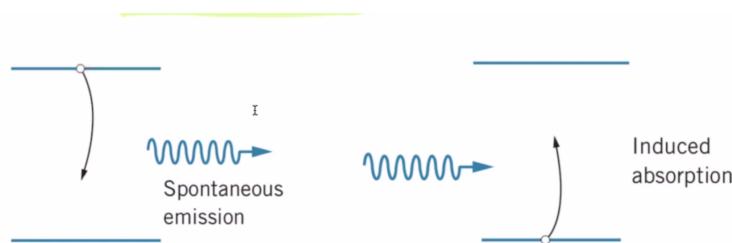


Figure 82: In practice, these all combine to form different energy levels for molecules

Problems with this picture:

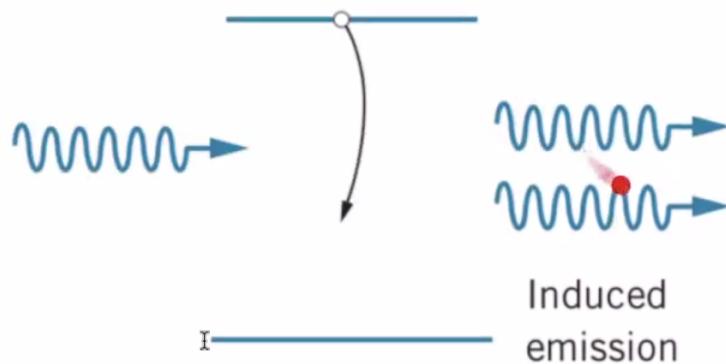


Figure 83: The two photons emerging from this process will travel in exactly the same direction, have exactly the same frequency, and be perfectly in phase.

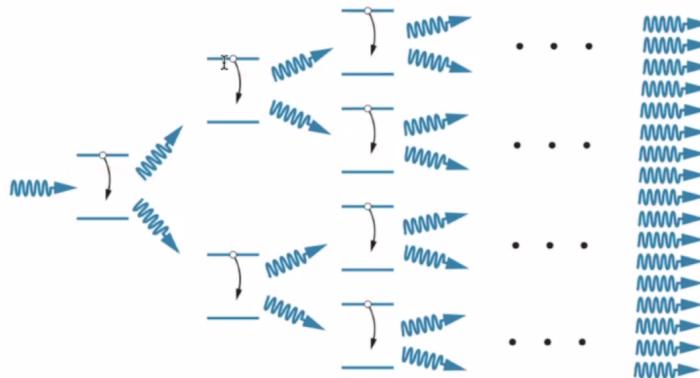


Figure 84: Ideal picture of the build-up of a laser beam.

- Difficult to keep a collection of atoms in an excited state
- Atoms back in the ground-state would undergo absorption and *remove* photons

Solution to 1: Population inversion

Atoms are “pumped” into a very short-lived excited state which decays to a “metastable” state (atom lives in this state for 1e-3 compared to 1e-8 in the short-lived state)

By this pumping we can have more atoms in the metastable state than in the ground state.
Ergo, population inversion!

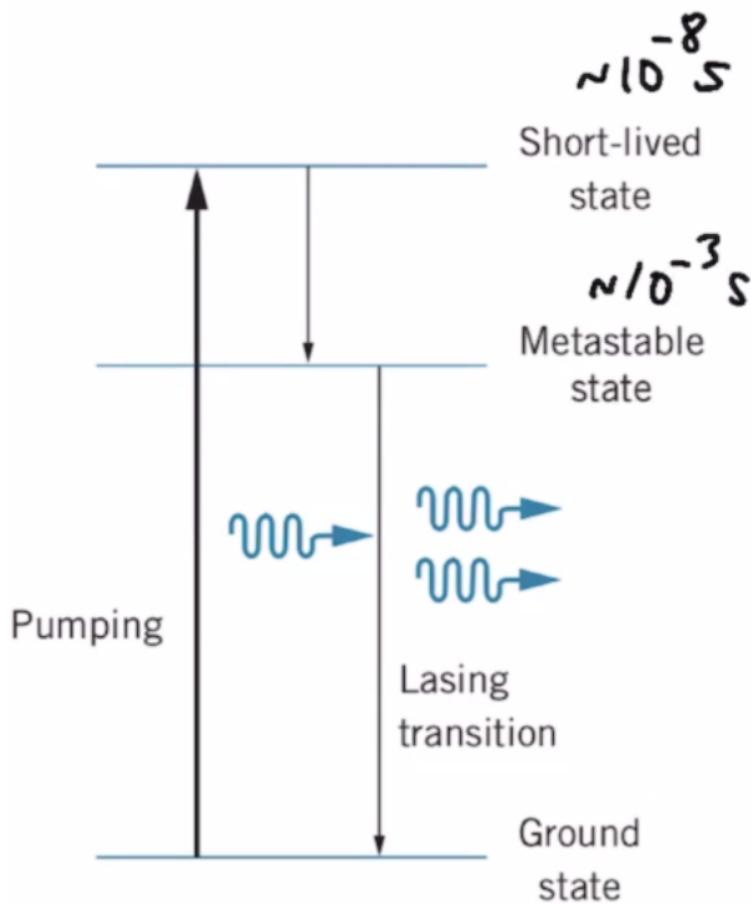


Figure 85: Transition rule: $\Delta l = \pm 1$. A 2s electron can't decay to 1s easily.

Solution to 2: Four-level Laser

Atoms in the ground state are pumped to an excited state, then rapidly decay to a metastable state.

The “lasing” transition proceeds from the metastable state to another intermediate short-lived state, before the atom finally decays to the ground state.

Atoms in the ground state cannot absorb photons with energy corresponding to the lasing transition!

14.1 Helium-neon Laser

Common example of a four-level laser. There is a gas mixture of 90% helium to 10% neon.

Electric current pumps helium from its ground state $1s^2$ to the excited state ($1s^1 2s^1$). This is metastable, since transition is forbidden.

Occasionally, excited helium atom will collide with ground-state neon atom. Excited atom yadda yadda. Typical helium-neon lasers have a very low efficiency. However, it is useful because it has high coherence and directionality and its energy density is very high.

There are many applications of lasers but of course, bar-code scanners, laser-printers, LiDAR.

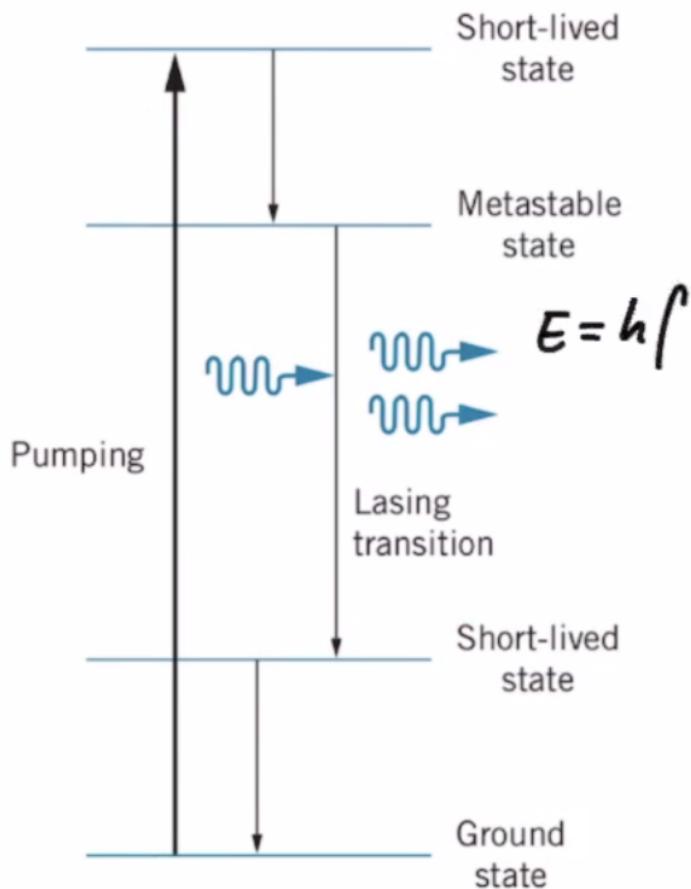


Figure 86: T

15 Bonding in solids

A crystalline solid consists of a large number of atoms arranged in a regular array, forming a periodic structure. This structure is called a lattice.

Bonding schemes for molecules are also appropriate for describing the bonding mechanisms in solids, e.g.

- Ions in the NaCl are ionically bonded
- Carbon atoms in the diamond structure form covalent bonds
- Van der Waals bonding is responsible for cohesion of organic solids in yadda fml

15.1 Ionic Solids

Dominant effect is the Coulomb interaction, creating a net attractive potential energy due to all the interactions:

$$U_c = -\alpha \frac{e^2}{4\pi\epsilon_0} \frac{1}{R}$$

α : Madelung constant

R: Distance between two adjacent ions

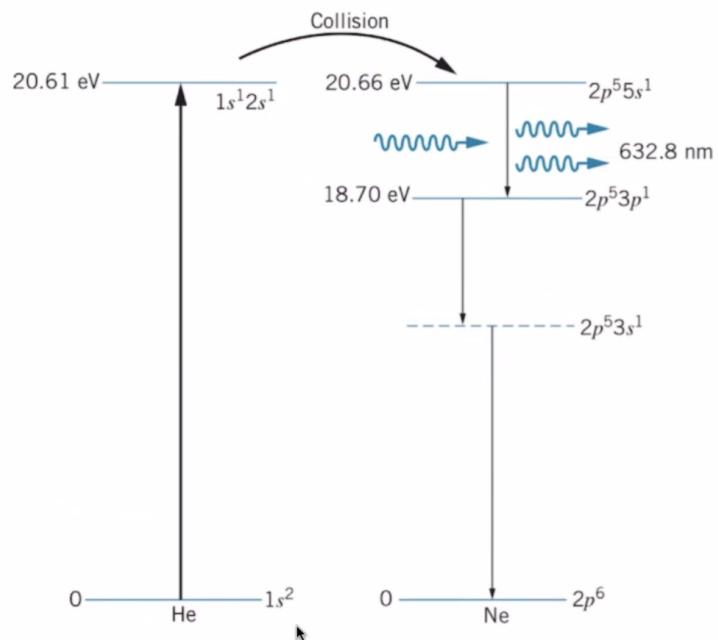


Figure 87: T

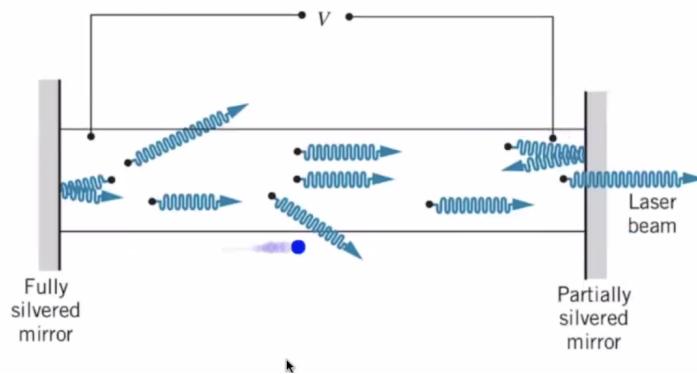


Figure 88: T

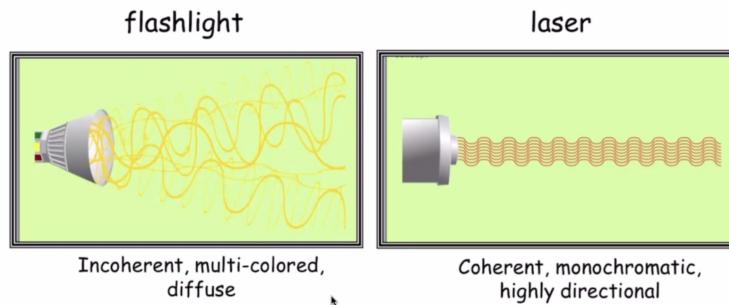


Figure 89: T

Madelung constant depends on the geometry of the lattice.

For NaCl is an fcc: $\alpha = 1.7476$

For CsCl is an bcc: $\alpha = 1.7627$

There is also a repulsive potential energy due to Pauli exclusion principle:

$$U_R = AR^{-n}$$

A: Strength of potential energy

n: How rapidly it increases for smaller R

The total potential energy per ion pair of the crystal:

$$U_{total} = U_c + U_R = -\alpha \frac{e^2}{4\pi\epsilon_0 R} \frac{1}{R} + AR^{-n}$$

The potential energy has a minimum (U_0) called ionic cohesive energy at the equilibrium separation (R_0).

$$U_0 = \frac{-\alpha e^2}{4\pi\epsilon_0 R_0} \left(1 - \frac{1}{n}\right)$$

Binding energy: $B = -U_0$.

	Nearest-Neighbor Separation (n)	Cohesive Energy (kJ/mol)	n	Structure
LiF	0.201	1030	6	fcc
LiCl	0.257	834	7	fcc
NaCl	0.281	769	9	fcc
NaI	0.324	682	9.5	fcc
KCl	0.315	701	9	fcc
KBr	0.330	671	9.5	fcc
RbF	0.282	774	8.5	fcc
RbCl	0.329	680	9.5	fcc
CsCl	0.356	657	10.5	bcc
CsI	0.395	600	12	bcc
MgO	0.210	3795	7	fcc
BaO	0.275	3029	9.5	fcc

Figure 90: Tabulated measured values for different ionic crystals. Listed molar cohesive energy related to above minimum potential energy as: $E_{coh.} = -U_0 \cdot N_A = B \cdot N_A$

15.2 Covalent Solids

Specifically, diamond.

- Atomic carbon electron configuration: $1s^2 2s^2 2p^2$
- Each carbon atom in diamond bonds covalently to four other carbon atoms to form a stable closed-shell structure - basic structure of diamond is called tetrahedral
- Atomic cohesive energies are generally greater than in ionic solids

Crystal	Nearest-Neighbor Distance (nm)	Cohesive Energy (kJ/mol)
ZnS	0.235	609
C (diamond)	0.154	710
Si	0.234	447
Ge	0.244	372
Sn	0.280	303
CuCl	0.236	921
GaSb	0.265	580
InAs	0.262	549
SiC	0.189	1185

Figure 91:

15.3 Metallic Solids

Metallic bonds are generally weaker than ionic or covalent bonds. The valence electrons in a metal are relatively free to move throughout the material. The binding mechanism in a metal is the attractive force between yadda fml.

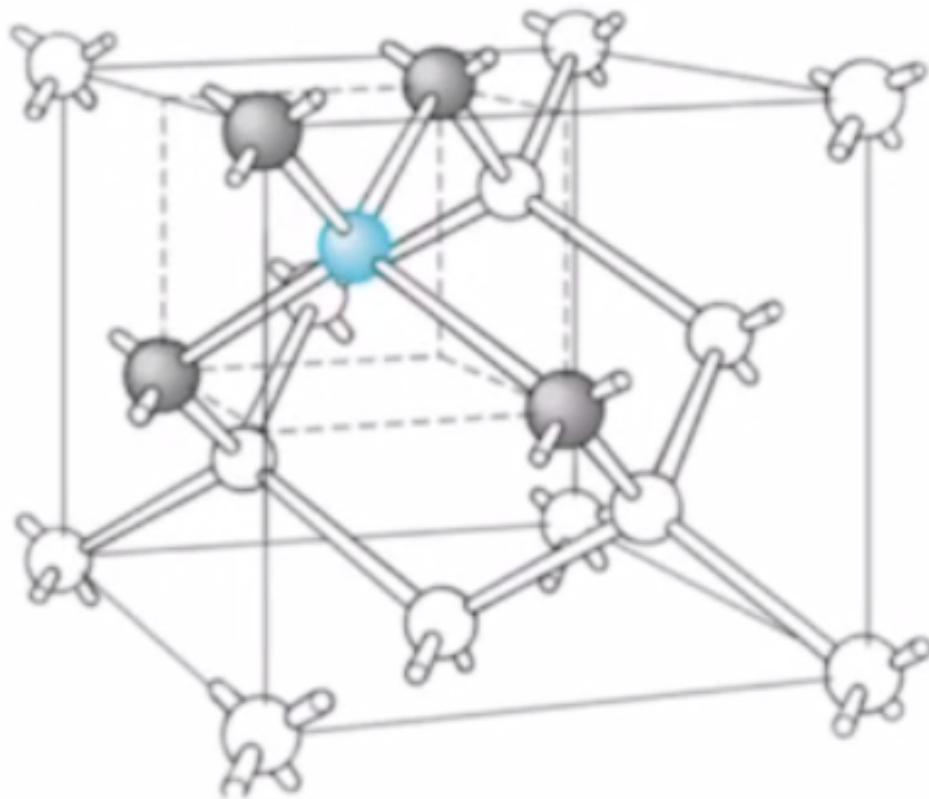


Figure 92:

Ionic Solids

Determine the experimental value of the binding energy of an ion pair in the NaCl lattice from the cohesive energy.

A

$$\begin{aligned} \text{Binding energy } B &= \frac{E_{coh}}{N_A} \\ &= \frac{769 \cdot 10^3 \text{ J/mol}}{6.02 \cdot 10^{23} \text{ ions/mol}} \approx 7.98 \text{ eV} \end{aligned}$$

Find the expected value of the binding energy based on the lattice parameters.

B

Binding energy from minimum potential energy:

$$B = -U_0 = \frac{\alpha e^2}{4\pi\epsilon_0 R_0} \left(1 - \frac{1}{n}\right)$$

The constants are 1.44 eV nm and 1.7476. So we get

$$1.44 \text{ eV} \cdot \text{nm} \cdot 1.7476 / 0.281 \text{ nm} \cdot (1 - 1/9)$$

$$\approx 7.96 \text{ eV}$$

These two answers end up being almost exactly identical!

16 Statistical Physics Detour

Statistical physics deals with the distribution of a fixed amount of energy among a number of particles that are identical and indistinguishable in any way (quantum particles). The probability of finding particles in a given energy state is called a “distribution function”. For quantum particles, these probabilities are very different for “fermions” (particles of spin 1/2 like electrons), and “bosons” (particles with integer spin, like photons).

- Bose-Einstein distribution - Describes bosons. No limit on number of particles in a given energy state.
- Fermi-Dirac distribution - Describes fermions. No more than one particle per quantum state allowed.

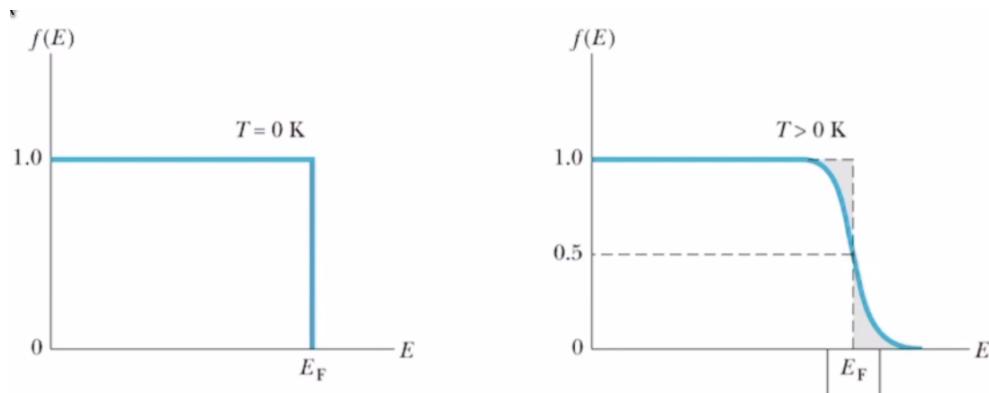


Figure 93: $f(E)$. E_F : Fermi energy

16.1 Electrons in Metals and Conduction

In metals, each atom contributes one or more loosely bound electrons to an “electron gas”.

- Shortly after discovery of the electron, Drude and Thomson proposed the “free electron theory” of metals to explain the physical properties of a metal.
- Models metal as a classical gas of conduction electrons through a fixed lattice of positive ions (used to explain electrical and thermal conductivities).

Ohm's Law

$$\vec{j} = \sigma \vec{E}$$

Where:

E is electric field (V/m)

σ is electrical conductivity (1/Ω m)

j is current density (A/m²)

This was first an experimental result, applicable to different metals and semiconductors. It states that current density in a material is directly proportional to the applied electric field.

One can actually derive Ohm's law by considering a conductor to consist of a gas of classical particles moving through a background of immobile, heavy ions.

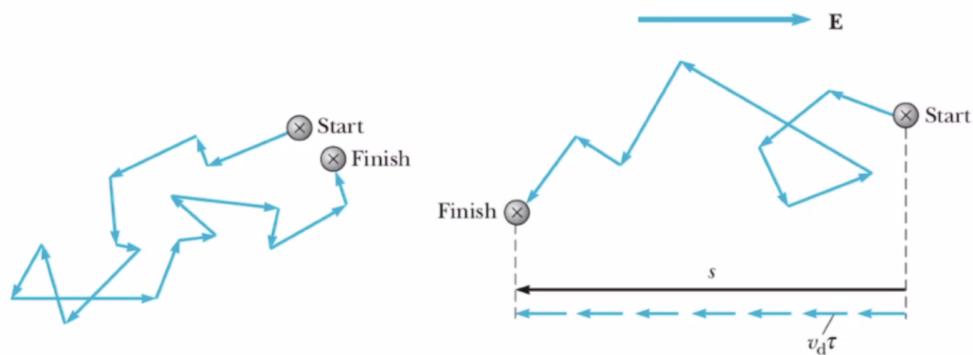


Figure 94:

Free electrons in the electron gas experience a force:

$$\vec{F} = -e\vec{E}$$

... and corresponding acceleration:

$$\vec{a} = \frac{\vec{F}}{m_e} = \frac{-e\vec{E}}{m_e}$$

Acceleration of electrons is countered by collisions, net result is that electrons acquire a steady "drift velocity" (v_d), given by acceleration \times average time between collisions (τ):

$$\vec{v}_d = \vec{a}\tau = \frac{-e\vec{E}\tau}{m_e}$$

Magnitude of current density is determined by number of charge carriers and their average speed:

$$\vec{j} = -nev_d = \frac{ne^2\tau}{m_e}\vec{E}$$

Note: $\tau = l/v_{av}$

Where l = mean free path and v_{av} = average speed of electrons in the lattice.

Nice! Electron gas model predicts Ohm's law. However, it results in conductivity values that differ from measured values by an order of magnitude... **quantum effects were ignored!**

Must use Fermi-Dirac distribution for the conduction electrons in the metal.

Must take the wave nature of the electron into account in calculating the electron “mean free path”.

16.2 Band Theory of Solids

What happens if you have two atoms that you bring together?

- Two sodium atoms, each having an outermost 3s electron with a specific energy
- As the two sodium atoms are brought closer together, their wave functions overlap, and the two degenerate, isolated 3s energy levels are split into two different levels.

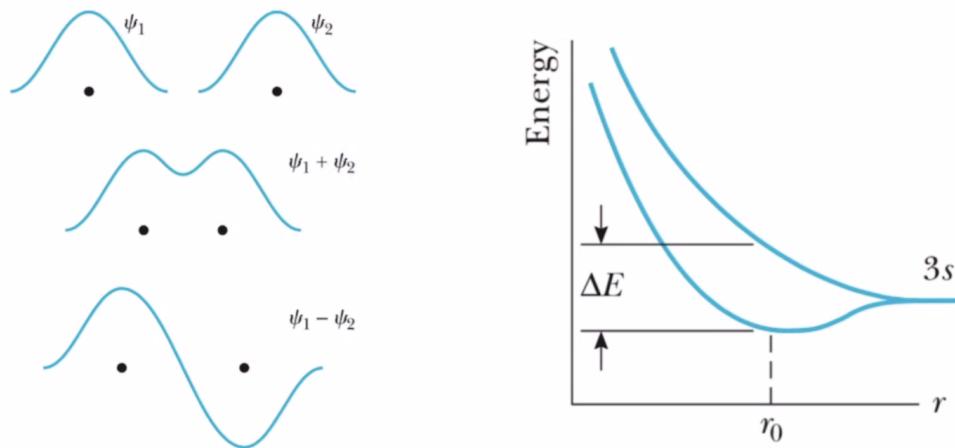


Figure 95: When the wave functions add, we have a sizeable probability between the two atoms, whereas when they subtract we have a low probability between the two atoms.

When a large number of atoms are brought together, a similar phenomenon occurs. As the atoms are brought close together, the various isolated-atom energy levels begin to split.

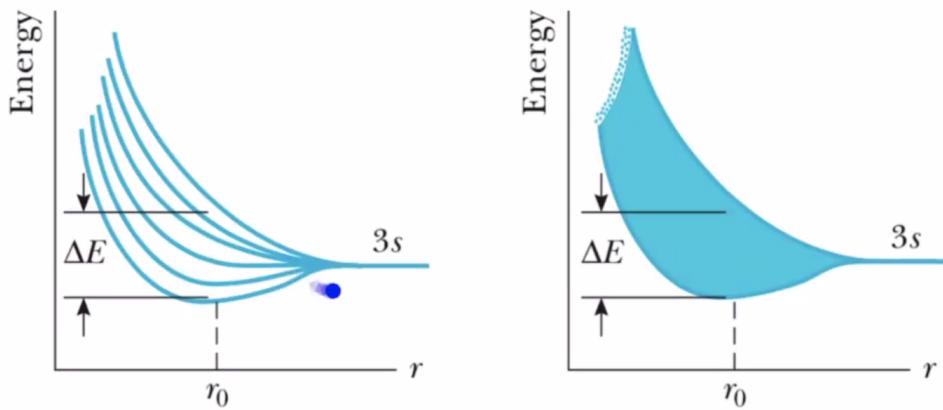


Figure 96:

All atoms in a solid:

Considering all atoms in a solid gives a very large number of levels (determined by N) spaced within the

width ΔE . These levels can be regarded as a *continuous* band of energy levels.

Energy bands in sodium metal: Sodium is a good electrical conductor, the 3s band is only half-filled. Large variation in electrical conductivity of metals, insulators, and semiconductors can be explained qualitatively in terms of energy bands.

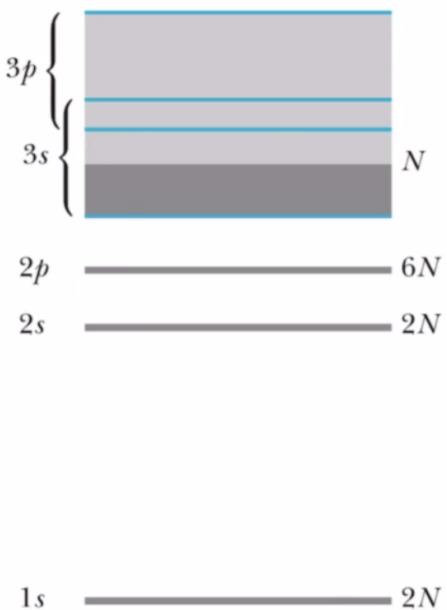


Figure 97:

16.3 Conduction in Metals vs Insulators vs Semiconductors

16.3.1 Conduction in metals

If an electric field is applied to the metal... electrons with energies near the Fermi energy can gain a small amount of additional energy from the field and reach nearby empty energy states.

i.e. Electrons are free to move only a small applied field in a metal due to many unoccupied energy states very close to occupied energy states == good conductor.

16.3.2 Conduction in Insulators

Although an insulator has many vacant states in the conduction band that can accept electrons, there are so few electrons actually occupying conduction band that can accept electrons, there are so few electrons actually occupying conduction-band states at room temperature.

Separation between outermost filled and empty bands referred to as the “energy gap” of the material. Lower band filled with electrons is the “valence band” and upper, empty band is the “conduction band”.

16.3.3 Conduction in Semiconductors

Semiconductors are materials that have a smaller energy gap (< 1 eV). At T = 0 K all electrons are in the valence band. At ordinary temperatures, a sizable number of electrons are thermally excited from the valence to the conduction band.

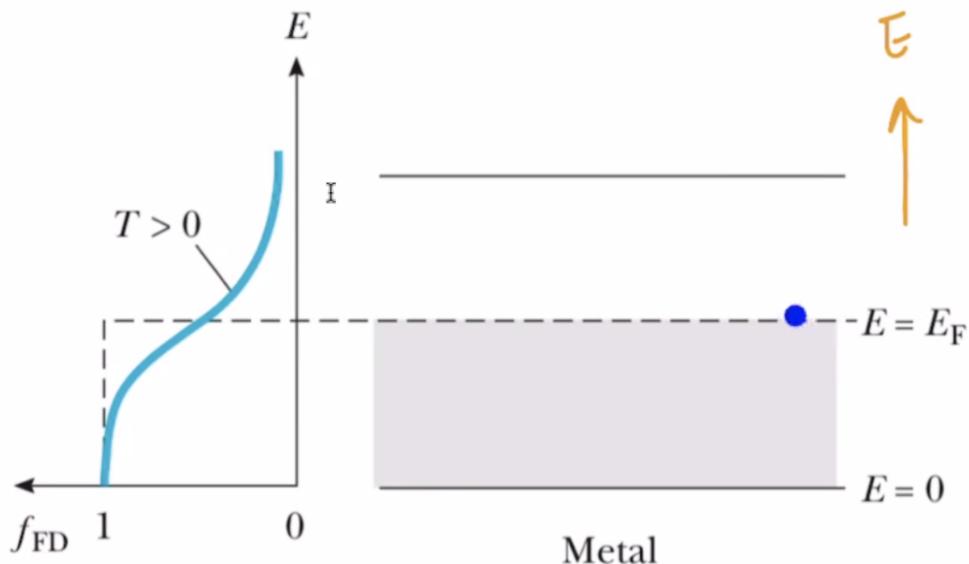


Figure 98:

- Many empty nearby states in the conduction band implies small applied potential can easily raise the energy of the electrons in the conduction band, resulting in a moderate current.
- Because thermal excitation across the narrow gap is more probable at higher temperatures, the conductivity of semiconductors depends strongly on temperature.

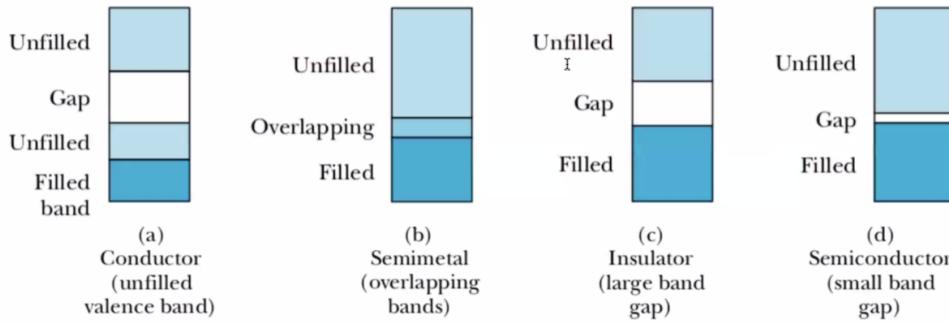


Figure 99:

16.4 Back to Sodium

For a single sodium atom: 3s level is half full (1 electron, max 2 allowed). For a sodium solid: 3s band is half full (N electrons in the band, max $2N$ allowed).

16.5 Bands

Large variation in electrical conductivity of metals, insulators, and semiconductors can be explained qualitatively in terms of energy bands.

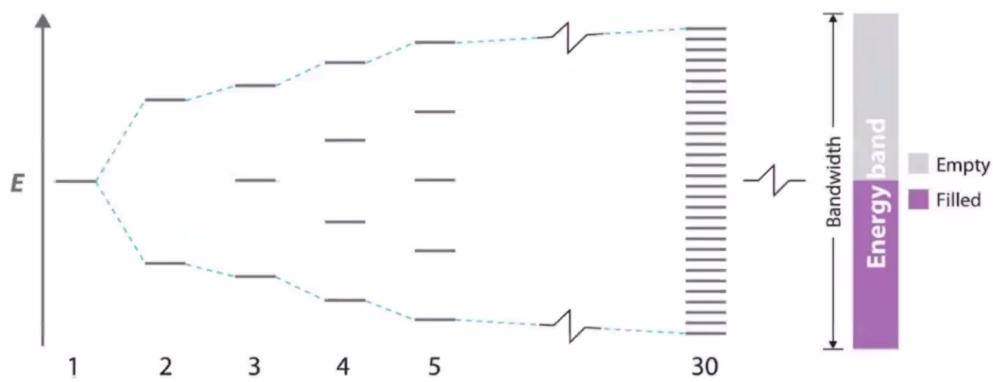


Figure 100: As we add more and more sodium atoms together, energy levels of the combined system split further and further, until the energy difference between distinct levels is so small it cannot be distinguished.

16.5.1 Metals

Metals have partially filled energy bands. As a result, there are many unoccupied energy states very close to occupied energy states. Electrons can therefore easily move through the metal, with only a small applied electric field. They are good conductors.

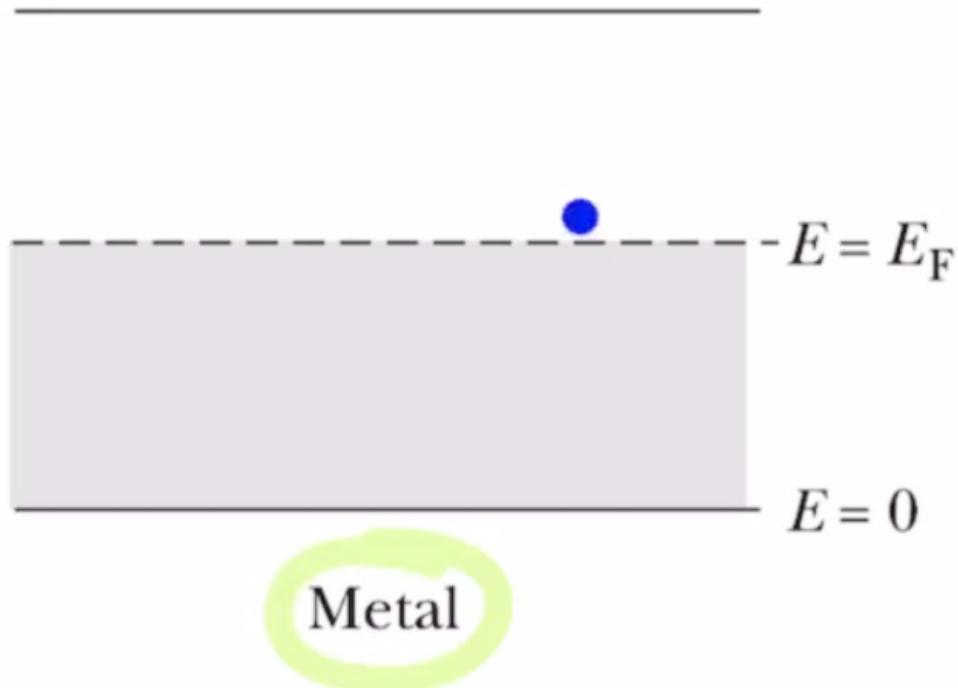


Figure 101:

16.5.2 Insulators

Have a lower completely filled band (valence band) and an upper completely empty band (conduction band). These are separated by a large energy gap, which is so large that electrons effectively cannot get thermally excited to the conduction band. The electrons cannot move, and it does not conduct.

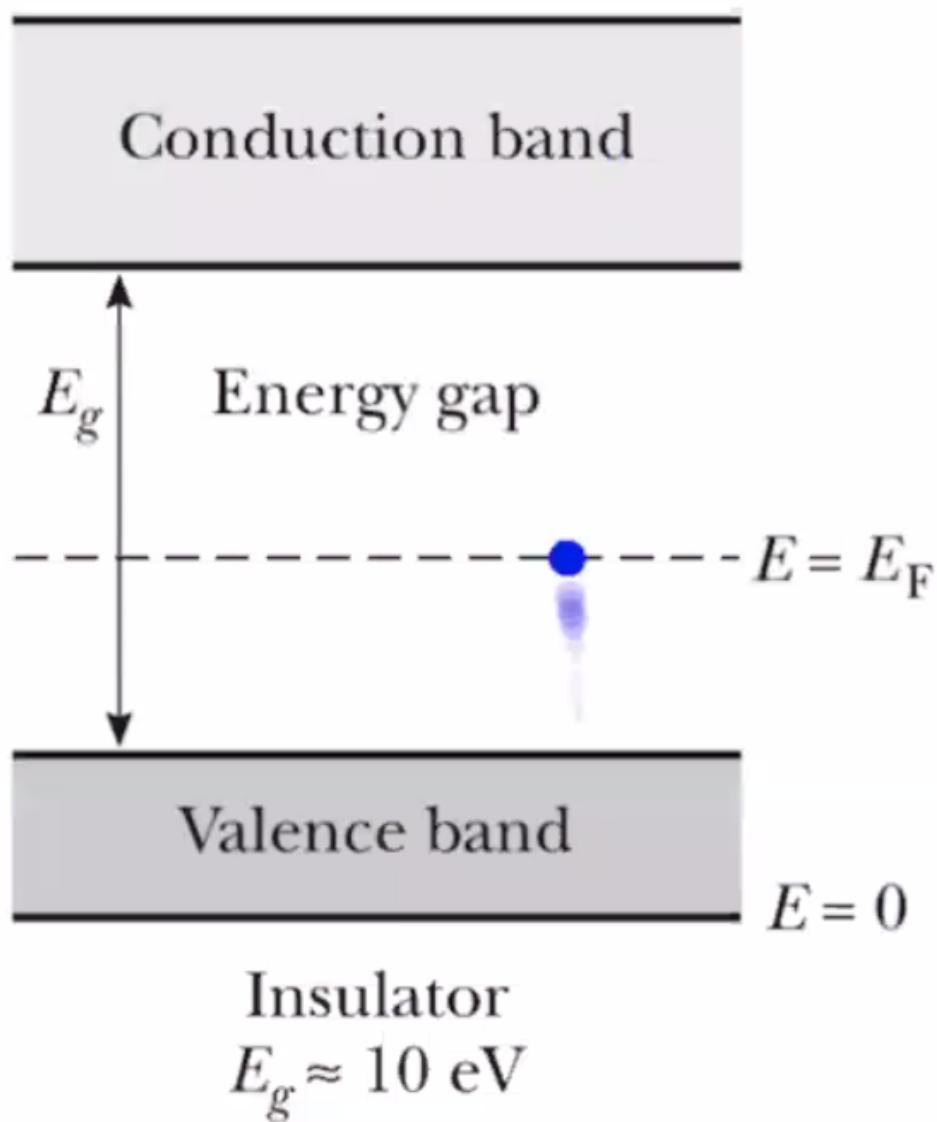


Figure 102:

16.5.3 Semiconductors

SMaterial with a small energy gap, such that some electrons are thermally excited to the conduction band where they can move. Many empty nearby states, therefore small applied electric field can easily raise the energy of electrons inthe conduction band, resulting in a moderate current. They are decent conductors at room temperatures.

16.5.4 More Complicated Band Structures

Why is magnesium a good conductor? Magnesium atom: $1s^2 2s^2 2p^6 3s^2$ (filled outer shell).
Why is carbon a good conductor? Magnesium atom: $1s^2 2s^2 2p^2$ (filled outer shell).

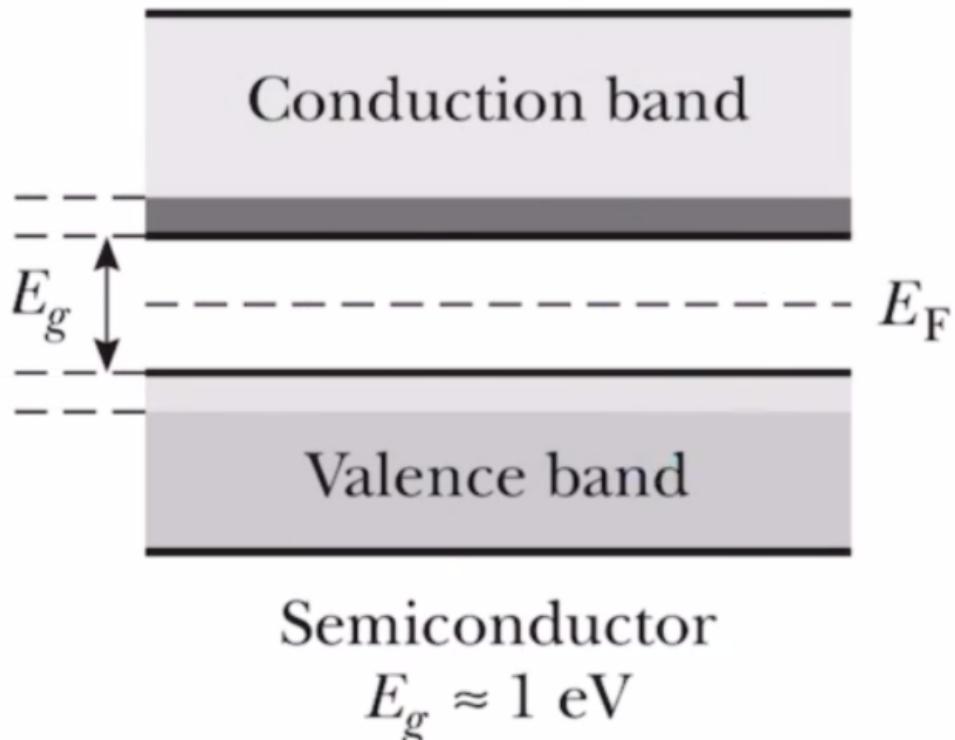


Figure 103:

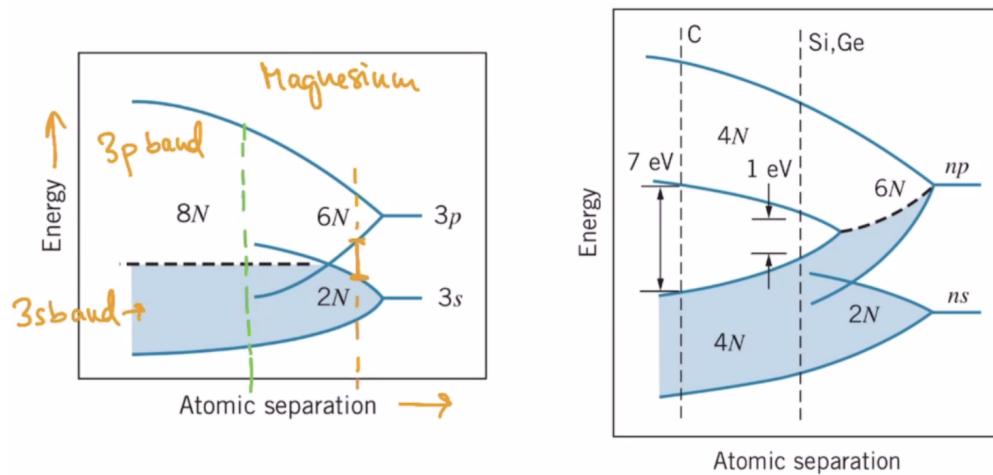


Figure 104: In Magnesium, the 3p and 3s levels overlap so it's easy to excite. In carbon, there's actually a full band and a large energy gap up to the next four levels, because crazy reasons??

16.6 Conduction in Semiconductors

16.6.1 Holes

When electrons move into the conduction band, they leave behind vacancies in the valence band. These vacancies are called holes.

Because holes represent the absence of negative charges, it is useful to think of them as positive charges.

Electrons move in a direction opposite to the applied electric field, the holes move in the direction of the electric field.

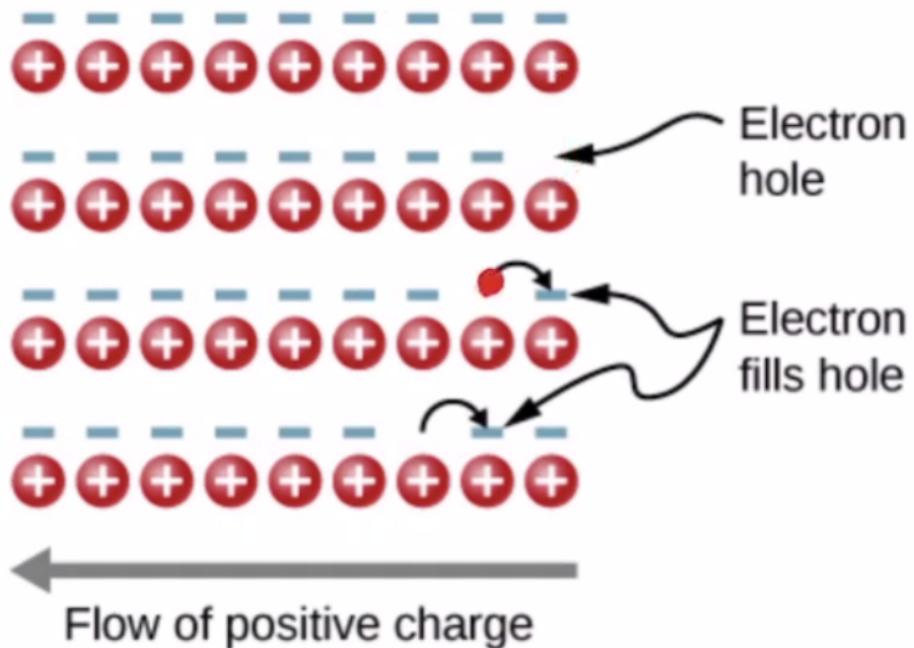


Figure 105:

16.6.2 Intrinsic Semiconductors

A semiconductor in which there is a balance between the... number of electrons in the conduction band, and the number of holes in the valance band.. is called an intrinsic semiconductor.

16.6.3 Impurities and Doping

Both the type and number of carriers in a semiconductor may be tailored to the needs of a particular device by the addition of specific impurities in a process called doping. For example, conductivity can be increased through small amount of impurity added. These materials are called “extrinsic” semiconductors or “impurity” semiconductors (or “doped” semiconductors).

Resulting energy levels formed by valence-5 impurities are known as “donor states” and a semiconductor that has been doped with donor impurities is a “n-type” semiconductor. (Conductivity mostly due to negative electrons).

Resulting energy levels formed by valence-3 impurities are known as “acceptor states” and a semiconductor that has been doped with donor impurities is a “p-type” semiconductor. (Conductivity mostly due to positively charged holes).

16.7 Diodes

Little energy is needed for eelctrons in donor states to enter the conduction band, or for electrons from valence bands to fill acceptor states and give positively charged holes in the valence band.

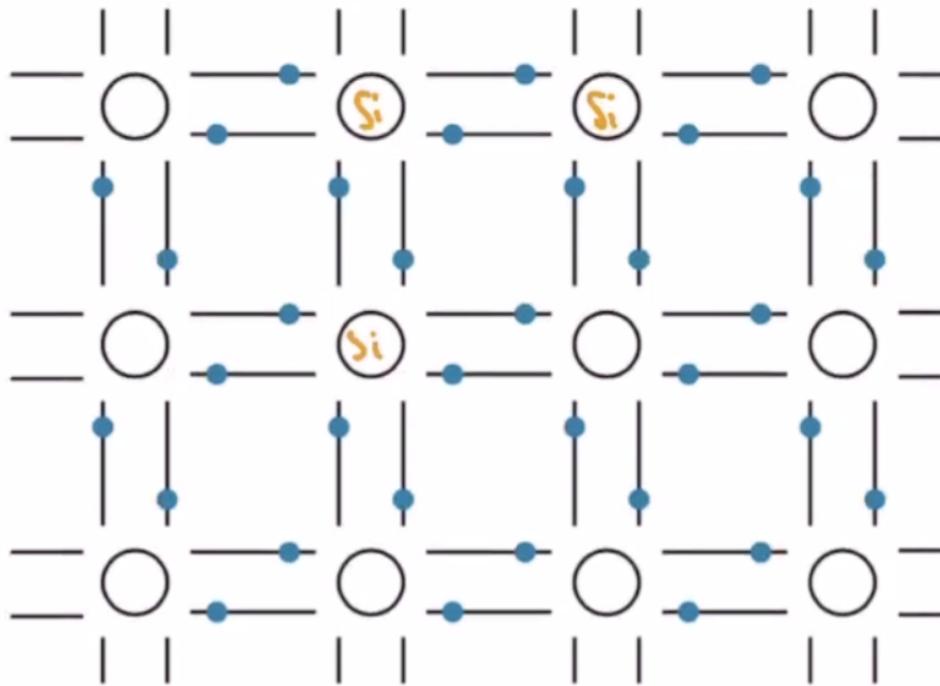


Figure 106: Covalent bonding in Si or Ge, with each atom providing four electrons for covalent bonds with its neighbors.

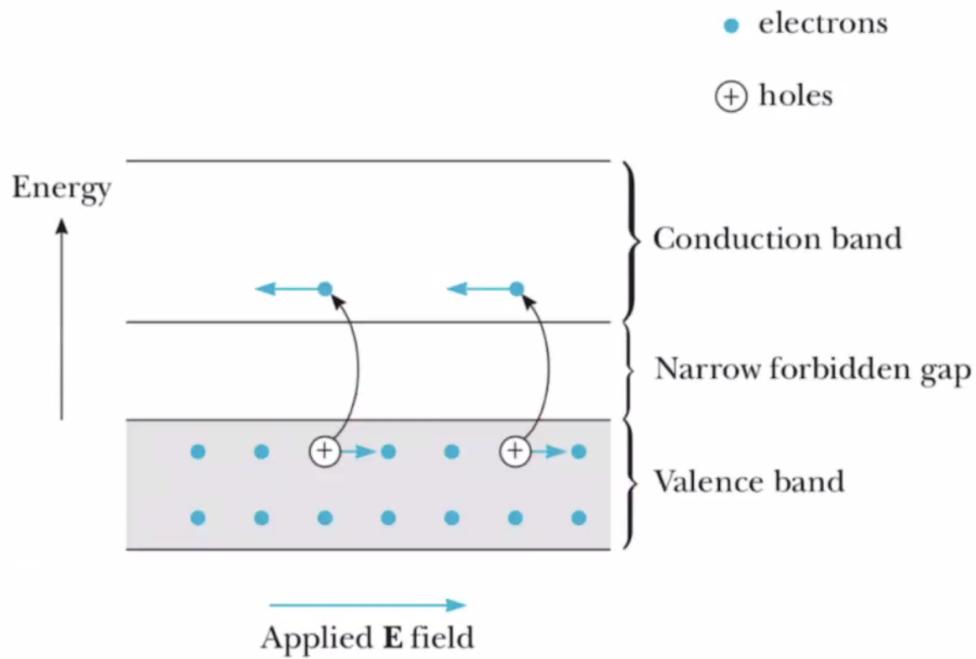


Figure 107: Another illustration of conduction in an intrinsic semiconductor.

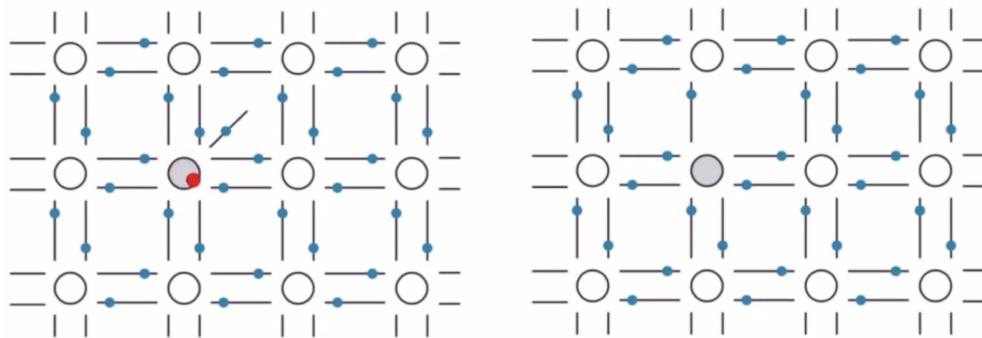


Figure 108: Example: Replace a Si/Ge atom with a valence-5 atom(left) or a valence-3 atom(right)

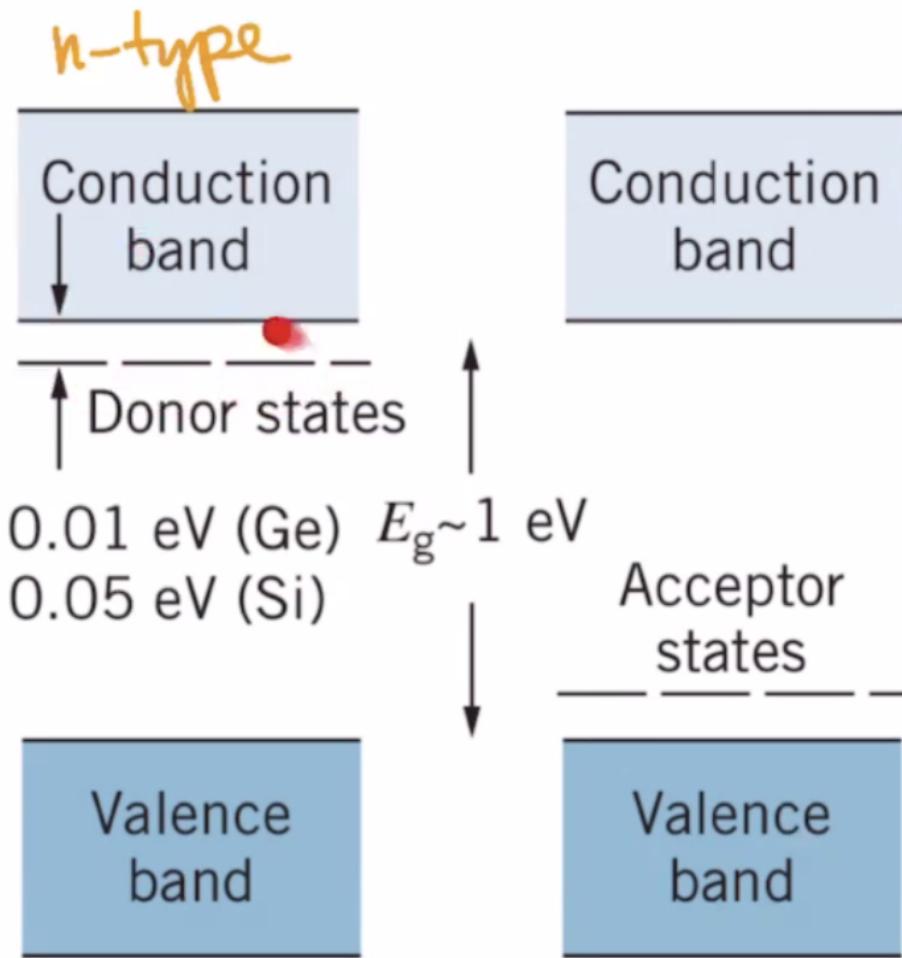


Figure 109: Note: both are electrically neutral, since made of neutral atoms!

Region between the two material is known as depletion region as it is largely depleted of charge carriers. In equilibrium, enough negative charge builds up in the p-type region to stop further any flow of electrons. There is a net electric field blah.

16.7.1 Diode: Forward-bias configuration

Connect positive side of battery to p-type material. Depletion region is narrowed and potential energy difference across p-n junction is reduced. Electrons (holes) flow easily.

16.7.2 Diode: Reverse-bias configuration

Connect positive side of battery to n-type material. Depletion region is widened and potential energy difference across p-n junction is increased. Few or none of the electrons can climb the potential barrier - no current.

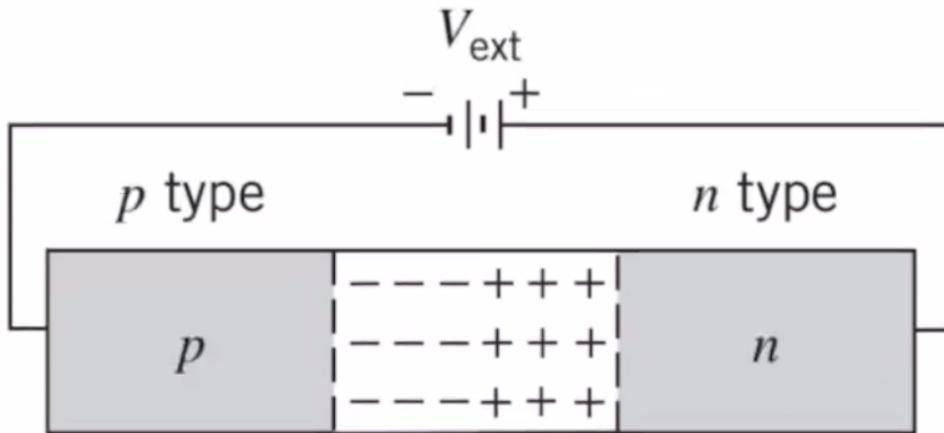


Figure 110:

16.8 Transistor

Transistors are devices that amplify voltages or currents in many kinds of circuits. The first transistor was developed in 1948 and revolutionized society. An example of a junction transistor consists of a semiconducting material with a very narrow n-type region sandwiched between two p-type regions, called the pnp transistor.

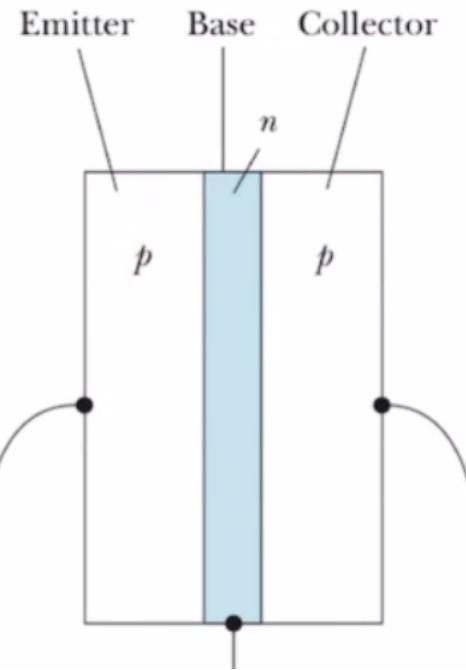


Figure 111: Junction Transistor

17 Nuclear Physics

17.1 Nuclear Size and Shape

Nuclei lack a hard surface or easily definable radius. The density of the nucleus is rather uniform, and does not depend much on the mass number A.

$$\frac{\text{#neutrons} + \text{protons}}{\text{volume of nucleus}} = \frac{A}{4/3 \pi R^3} \approx \text{constant}$$

Nuclear Radius

$$R = R_0 \cdot A^{1/3}$$

R_0 is a constant from experiments, $\approx 1.2 \text{ fm}$.

What is the density of a typical nucleus?

$$\begin{aligned} \sigma &= \frac{m}{V} \approx \frac{A \cdot m_p}{4/3 \pi r^3} = \frac{A \cdot m_p}{4/3 \pi R_0^3 (A^{1/3})^3} = \frac{m_p}{4/3 \pi R_0^3} \\ &\approx \frac{1.67 \cdot 10^{-27} \text{ kg}}{4/3 \pi (1.2 \cdot 10^{-15} \text{ m})^3} \\ \sigma &\approx 2 \cdot 10^{17} \text{ kg/m}^3 \end{aligned}$$

Compare to density of water, 1000 kg/m^3 14 orders of magnitude larger!!!

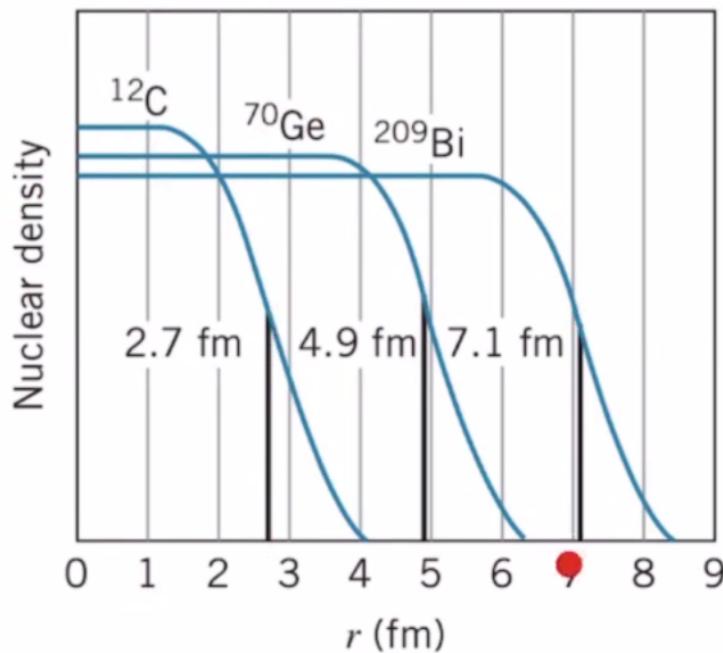


Figure 112:

17.2 Nuclear Mass and Binding Energy

The nuclear binding energy is calculated as the difference between the rest energy of the constituents and the rest energy of their combination.

For a nucleus X with mass number A and Z protons + N neutrons: ${}^A_Z X_N$

$$B = (N \cdot m_N + Z \cdot m({}_1^1 H_0) - m({}^A_Z X_N)) c^2$$

We use atomic masses rather than nuclear masses (allows using tabulated atomic masses)! Electron masses cancel out in this expression! Binding energy of electrons is in comparison very, very small – can safely be ignored.

17.2.1 Binding Energy Per Nucleon

B/A = Total binding energy/number of nucleons (protons + neutrons ($A = Z + N$))

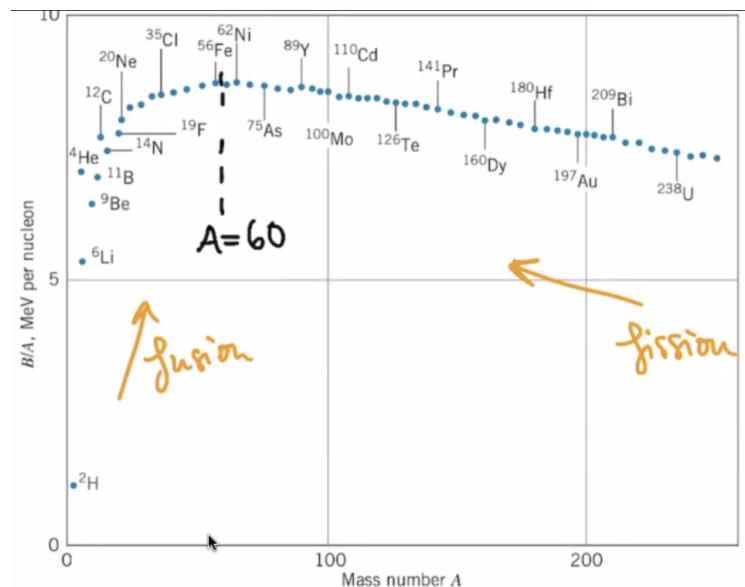


Figure 113: Iron is the most stable nucleus. After which, the binding energy per nucleon begins to decrease, due to increasing Coulomb repulsion. The decrease we observe for low mass number elements can be explained by surface area - more balls are sticking out relative to the size of the nucleus itself.

17.2.2 Separation Energies

Ionization energy is the smallest amount of energy necessary to remove an electron from an atom. This typical scale is in the order of eV.

The Proton/neutron separation energy by contrast is the energy required to remove the least bound proton/neutron from the nucleus. This typical scale is in the order of MeV.

17.3 Nuclear Force

The nuclear force is the strongest of known forces, also called the “strong force”. (The four fundamental forces of nature are):

1. Gravitational interaction
2. Weak interaction
3. Electromagnetic interaction
4. Strong interaction

It has a very short range – acts over distances of about 1 fm. Compare how nuclear density was about constant - it's like a nuclear force is acting between nucleons that are touching each other.

Nuclear force does not depend on whether nucleons are protons or neutrons.

It's like a spring - at a certain point it snaps then does nothing.

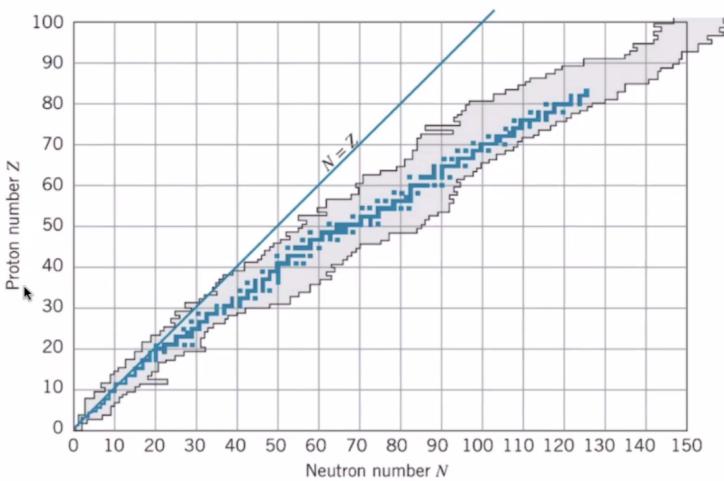


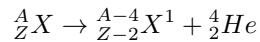
Figure 114: The blue line is the line where $N = Z$. The blue band is made up of different stable nuclei, while the larger grey band is made up of all known nuclei, which are radioactive. The known nuclei does not follow the $N=Z$ line - only at low N . As we go higher, due to increasing Coulomb repulsion, we need more neutrons.

17.4 Stable Nuclei and Radioactive Decays

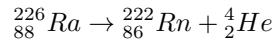
Most nuclei are unstable – they decay into lighter, more stable nuclei.

17.4.1 Different Decay Processes

- **Alpha Decay:** Emission of a helium nucleus.



What is the alpha decay chain of 226-radium?



- **Beta Decay:** they do stuff too??

- **Gamma Decay:**

17.4.2 Activity and Decay Probabilities

Rate at which unstable nuclei decay is called the “activity”.

We measure activity in units of 1 becquerel (Bq) = 1 decay/second
 We measure activity in units of 1 curie (Ci) = $3.7 \cdot 10^{10}$ decays/second

Decay probability per nucleus per second is called the “decay constant” (λ). It is assumed to be constant with time for a particular material, i.e. the probability is independent of age of sample.

Relationship between activity and decay constant

$a = \lambda N$ Where N=number of nuclei.

Exponential law of radioactive decay

Activity $a = -\frac{dN}{dt}$

$$\lambda N = -\frac{dN}{dt} \text{ or } \frac{dN}{N} = -\lambda dt$$

$$N = N_0 e^{-\lambda t}$$

Where N_0 : number of nuclei at $t=0$. Usually we cannot measure N, but we can express this in terms of activity instead, which is easier to measure.

$$a = a_0 e^{-\lambda t}$$

Where a_0 is the original activity at $t=0$.

Also talk about the “half-life” of the decay, the time for activity to be reduced by half:

$$t_{1/2} = \frac{1}{\lambda} \ln(2) \approx \frac{0.693}{\lambda}$$

The mean lifetime:

$$\tau = \frac{1}{\lambda}$$

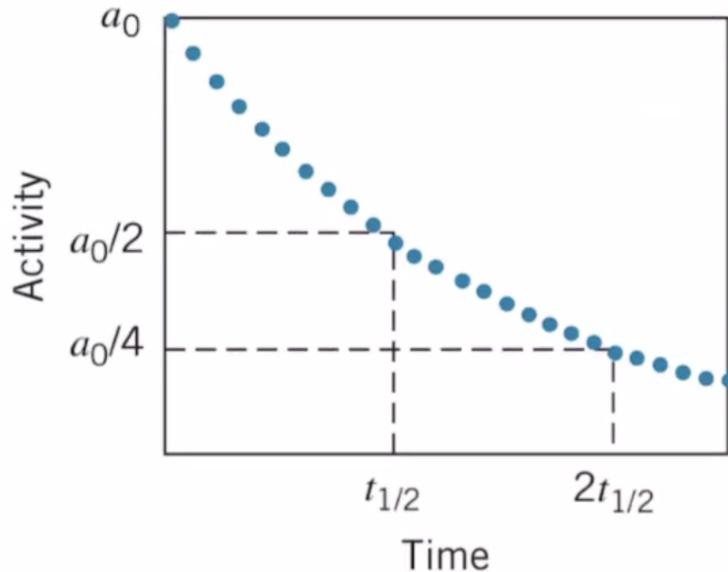


Figure 115:

17.5 Alpha Decay

Alpha decay involves disintegrating an unstable nucleus into a lighter nucleus and alpha particle.

The energy released (Q) through this process appears as kinetic energy (K) of the alpha particles and “daughter” nucleus (assume initial atom X at rest):

$$Q = (m(x) - m(x') - m({}_2^4He)) c^2$$

$$Q = K_{x'} + K_\alpha$$

Also:

$$P_{x'} = P_{alpha}$$

Where

$$K = \frac{P^2}{2m}$$

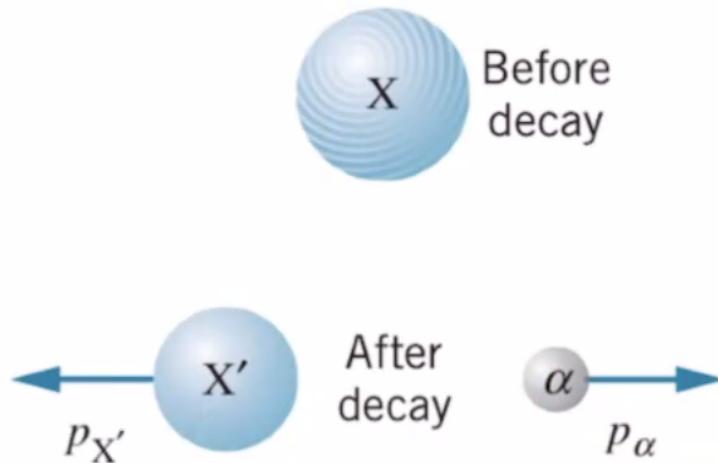


Figure 116:

Both energy and linear momentum must be conserved in this process:

$$K_\alpha \approx \frac{A-4}{A} Q$$

17.6 Beta Decay

Beta decay involves a neutron transforming into a proton or vice-versa. One may think that the process as $n \rightarrow p + e^-$, but this has problems.

This would violate conservation of angular momentum (spin!)

Measurements of energy of emitted electrons showed a continuous spectrum of emitted energies rather than the expected value.

$$Q = (m_n - m_p - m_e) c^2 \approx 0.78 MeV$$

Solution: There is a third particle involved!!

Neutrino : ν

Antineutrino : $\bar{\nu}$

Neutrinos: zero electric charge, teensy mass, spin 1/2.

Neutron decay: $n \rightarrow p + e^- + \bar{\nu}$, $\tau(n) \approx 14 \text{ min } 40 \text{ sec.}$

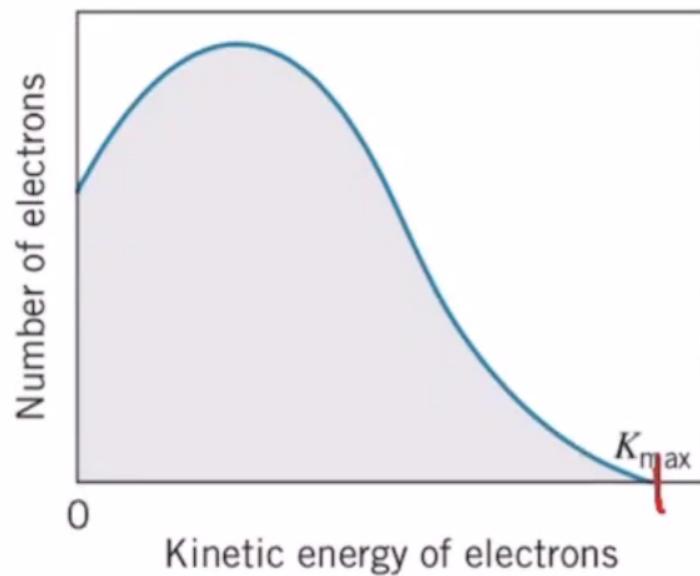
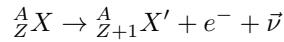


Figure 117: There is a distribution instead of a single value as expected.

Beta decay of neutron in a nucleus:



Protons in some nuclei can also undergo beta decay:

