"OLD PROBLEM

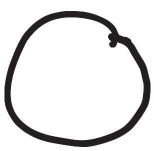
$$U_t = U_{xx} \quad u(x,0) = f(x)$$

$$u(0,t) = u(1,t) = 0 \ ;$$

$$C_N = 2\int_0^1 f(x)\sin(N\pi x)\,dx$$

$$u(x,t) = \sum_{N=1}^{\infty} C_N e^{-\pi^2 N^2 t} \sin(N\pi x)$$

FOURIER SINE SERIES.



CIRCULAR BAR. (COMPLEX FOURIER SERIES)

$$U_t = U_{xx} \quad u(x,0) = f(x)$$

$$u(0,t) = u(1,t); \quad u_x(0,t) = u_x(1,t)$$

PERIODIC BOUNDARY CONDITIONS.

$$u(x,t) = X(x)\,T(t)$$

$$\frac{T'}{T} = \frac{X''}{X} = c$$

$$X''(x) = c\,X(x); \quad X(0) = X(1)$$
$$X'(0) = X'(1)$$

(1) $c = 0$; $\quad X'' = 0 \quad X(x) = Ax + B$

CONSTANT B (SAY 1) WORKS!

(II) $c > 0 = k^2$

$$X(x) = Ae^{kx} + Be^{-kx}$$

$$X'(x) = kAe^{kx} - kBe^{-kx}$$

$$X(0) = A+B = X(1) = Ae + Be^{-1}$$

$$X'(0) = kA - kB = X(1) = kAe^{1} - k.Be^{-1}$$

$$A(1-e) + B(1-e^{-1}) = 0$$

$$A(k)(1-e) + B(k)(-1+e^{-1}) = 0$$

$$\begin{pmatrix} 1-e & 1-e^{-1} \\ (1-e)k & -k(1-e^{-1}) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1-e & 1-e^{-1} & 0 \\ 0 & -2k(1-e^{-1}) & 0 \end{array}\right) \Rightarrow A, B = 0.$$

(III) $c < 0 = -k^2$ $\qquad X(0) = X(1)$

$$X''(x) = -k^2 X(x) \qquad X'(0) = X'(1)$$

IF $X(x)$ IS PERIODIC, WORKS!

$$X = e^{ikx} = (\cos kx) + i\sin(kx)$$

$$X' = (ik)e^{ikx} \qquad e^{ik} = e^{0} = \underline{1}$$

$$X'' = -k^2 e^{ikx} = -k^2 X(x); \qquad k = 2\pi N$$

$$e^{2\pi N i} = \cos(2\pi N) + i\sin(2\pi N) = 1.$$

$$\ldots\ e^{-2\pi i x},\ e^{0 i x},\ e^{2\pi i x},\ e^{4\pi i x},\ e^{6\pi i x},\ \ldots$$

!!

$$e^{i(2\pi N)x} \qquad N = -3, -2, -1, 0, 1, 2, \ldots$$

$$c = -k^2 = -4\pi^2 N^2 \qquad X_N = e^{i(2\pi N)x}$$

$$T'(t) = cT; \qquad T_N = e^{-4\pi^2 N^2 t}$$

$$u_N(x,t) = e^{i(2\pi N)x}\, e^{-4\pi^2 N^2 t}$$

$$N = -3, -2, -1, 0, \ldots$$

$$\boxed{u(x,t) = \sum_{N=-\infty}^{\infty} c_N\, e^{i(2\pi N)x}\, e^{-4\pi^2 N^2 t};}$$

$$f(x) = u(x,0) = \sum_{N=-\infty}^{\infty} c_N\, e^{i(2\pi N x)}$$

$$c_N = ?$$

$$f(x)\, e^{-i(2\pi\cdot 17)x} = \sum_{-\infty} c_N\, e^{i(2\pi N x)}\, e^{-i(2\pi 17)x}$$

$$\int_0^1 f(x)\, e^{-i(2\pi\cdot 17)x}\, dx = \sum_{-\infty}^{\infty} c_N \underbrace{\int_0^1 e^{2\pi i x(N-17)}\, dx}$$

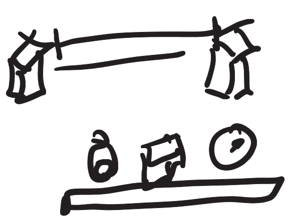
$$\int_0^1 e^{2\pi i l x}\,dx = 1, \text{ IF } l=0;$$

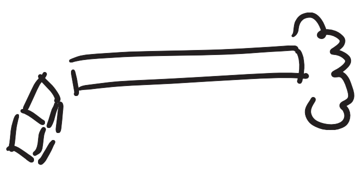
$$= \frac{e^{2\pi i l x}}{2\pi i l}\Big|_0^1 \quad l \neq 0;$$

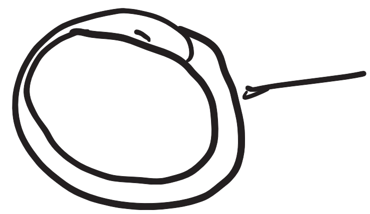
$$= \frac{e^{2\pi i l} - e^{0}}{2\pi i l} = 0!$$

$$\sum_{N=-\infty}^{\infty} c_N \int_0^1 e^{2\pi i (N-17)x}\,dx$$

$$= c_{17}; \quad \text{SO} \quad \boxed{c_N = \int_0^1 f(x)\, e^{-i 2\pi N x}\,dx}$$







NOW "INFINITE BAR" + FOURIER TRANSFORM

INTEGRAL TRANSFORMS: FUNCTION $\longrightarrow$ FUNCTION.

$$f(t) \xrightarrow{t>0} \int_0^\infty f(t)\, e^{-st}\,dt. = \mathcal{L}(f)(s)$$

LAPLACE TRANSFORM (MORE SOON).

SIMILARLY

$$\boxed{F(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ix\omega}\, dx}$$
$\hat{f}(\omega)$ FOURIER TRANSFORM!

$$F^{-1}(g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega)\, e^{i\omega x}\, d\omega$$

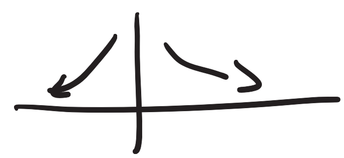
INVERSE FOURIER TRANSFORM.
(ALMOST THE SAME!)

COMPUTING $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ix\omega}\, dx = \hat{f}(\omega)$.

CAN REQUIRE SOME TRICKY INTEGRATION (COMPLEX VARIABLES).

SOME "SIMPLER" EXAMPLES:

$f(x) = e^{-|x|}$



$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ix\omega}\, dx$$
{ we'll put $\frac{1}{\sqrt{2\pi}}$ back in end }

$$= \int_{-\infty}^{0} f(x)\, e^{-i\omega x}\, dx + \int_{0}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \int_{-\infty}^{0} e^{x} e^{-i\omega x}\, dx + \int_{0}^{\infty} e^{-x} e^{-i\omega x}\, dx$$

$$= \int_{-\infty}^{0} e^{x(1-i\omega)}\, dx + \int_{0}^{\infty} e^{x(-1-i\omega)}\, dx$$

$$= \left.\frac{e^{x(1-i\omega)}}{1-i\omega}\right|_{-\infty}^{0} + \left.\frac{e^{x(-1-i\omega)}}{-1-i\omega}\right|_{0}^{\infty}$$

SIDE NOTE: $\left. \dfrac{e^{-(x)(1+i\omega)}}{-1-i\omega} \right|_{0}^{\infty}$

$$= \lim_{R \to \infty} \frac{e^{-R(1+i\omega)}}{-1-i\omega} - \frac{e^{0}}{-1-i\omega}$$

FTC

$$= \lim_{R \to \infty} e^{-R(1+i\omega)} + \frac{1}{1+i\omega}$$

$$=$$

$$\left| e^{-R(1+i\omega)} \right| = \left| e^{-R}\, e^{i(-R\omega)} \right|$$

$$= \left| e^{-R} \left[\cos(-R\omega) + i \sin(-R\omega) \right] \right|$$

$= e^{-R} \cdot \underline{1}$; AS $R \to \infty$, THIS GOES TO <u>ZERO</u>

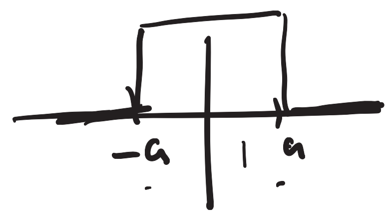
$$= \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{1+i\omega+1-i\omega}{1+\omega^2}$$

6 !!

!!

$$= \frac{2}{1+\omega^2}\left(\frac{1}{\sqrt{2\pi}}\right)$$

$$f(x) = \begin{cases} 1 & |x| \le a \\ 0 & \text{OTHERWISE} \end{cases}$$



$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x} f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-i\omega x}\, 1\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \left.\frac{e^{-i\omega x}}{-i\omega}\right|_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{i}{\omega}\left[e^{-i\omega a} - e^{i\omega a}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\frac{i}{\omega}(2i)\left[\frac{e^{-i\omega a} - e^{i\omega a}}{2i}\right]$$

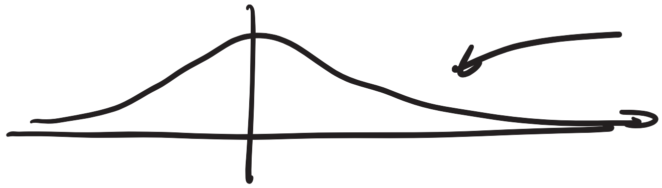
$$= \frac{1}{\sqrt{2\pi}}\frac{-i}{\omega}(2i)\left[\frac{e^{i\omega a} - e^{-i\omega a}}{2i}\right]$$

$$= \boxed{\frac{\sqrt{2}}{\sqrt{\pi}}\,\frac{\sin(a\omega)}{\omega}}$$

$$\left\{ \operatorname{sinc}(x) = \frac{\sin x}{x} \right\}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}; \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$f(x) = e^{-a^2x^2} \quad ; \quad \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{-ixw}\, dx = ?$$



GAUSSIAN (BELL CURVE)

TWO VERY CLEVER TRICKS FOR $\hat{f}$.

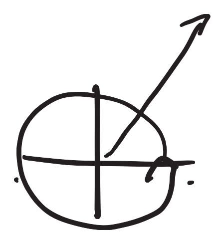
(1) $$\int_{-\infty}^{\infty} e^{-x^2}\, dx = ? = I$$ {CANNOT FIND ANTI-DERIV.}

$$\int_{-\infty}^{\infty} e^{-y^2}\, dy = I$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \int_{-\infty}^{\infty} e^{-y^2} \int_{-\infty}^{\infty} e^{-x^2} dx \; dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx\, dy$$



$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\, dy$$

$$\overset{!!!}{=} \int_0^{\infty} \int_0^{2\pi} e^{-r^2}\, r\, dr\, d\theta \quad !! \, !!$$

$$= \int_0^{2\pi} \left(\int_0^{\infty} e^{-r^2} r\, dr \right) d\theta$$

$u = r^2$
$du = 2r\,dr$

$$= \int_0^{2\pi} \frac{1}{2}\, d\theta = \frac{2\pi}{2} = \boxed{\pi}$$

$I^2 = \pi$

$\boxed{I = \sqrt{\pi}}$

(11) $g(\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{-ix\omega}\, dx$ 18

$$g'(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} (-ix)\, e^{-ix\omega}\, dx$$

$$= \frac{(-i)}{-2a^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} (-2a^2x)\, e^{-ix\omega}\, dx$$

$$= \frac{i}{2a^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(e^{-a^2x^2}\right)' e^{-ix\omega}\, dx$$

IBP $\frac{-i}{2a^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} \left(e^{-ix\omega}\right)' dx \leftarrow \frac{d}{dx}$!!

$$= \frac{-i}{2a^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} (-i\omega)\, e^{-ix\omega}\, dx$$

$$= \frac{-\omega}{2a^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty} e^{-a^2x^2} e^{-i\omega x}\, dx$$

$$= \frac{-\omega}{2a^2}\, g(\omega)$$

$$\boxed{g'(\omega) = \frac{-\omega}{2a^2}\, g(\omega)}$$

FIRST ORDER LINEAR!

$$\frac{dg}{d\omega} = \frac{-\omega}{2a^2}\, g$$

(SEPARATION OF VARIABLES)

$$\int \frac{dg}{g} = \int -\frac{\omega\, d\omega}{2a^2}$$

$$= \frac{-1}{2a^2} \int \omega\, d\omega$$

$$\ln g = \frac{-1}{2a^2}\frac{w^2}{2} + c$$

$$\ln g = \frac{-w^2}{4a^2} + c$$

$$f(w) = g = e^c e^{-\frac{w^2}{4a^2}} = D e^{-w^2/4a^2}$$

$$f(x) = e^{-a^2x^2}$$

$$g(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}dx = D = \frac{\sqrt{\pi}}{a\sqrt{2\pi}} = \frac{1}{\sqrt{2}a}$$

$$\frac{1}{a}\int_{-\infty}^{\infty} e^{-a^2x^2} a\,dx = ? \qquad \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

$u = ax$

$du = a\,dx$

$$= \frac{1}{a}\int_{-\infty}^{\infty} e^{-u^2}du = \frac{\sqrt{\pi}}{a}$$

FOURIER TRANSFORM OF $e^{-a^2x^2}$

$$= \frac{1}{\sqrt{2}a} e^{-w^2/4a^2}$$

$a^2 = \dfrac{1}{4a^2}$

$a^4 = \dfrac{1}{4}$

$a^2 = \dfrac{1}{2}$

$e^{-x^2/2}$ IS ITS OWN FOURIER TRANSFORM