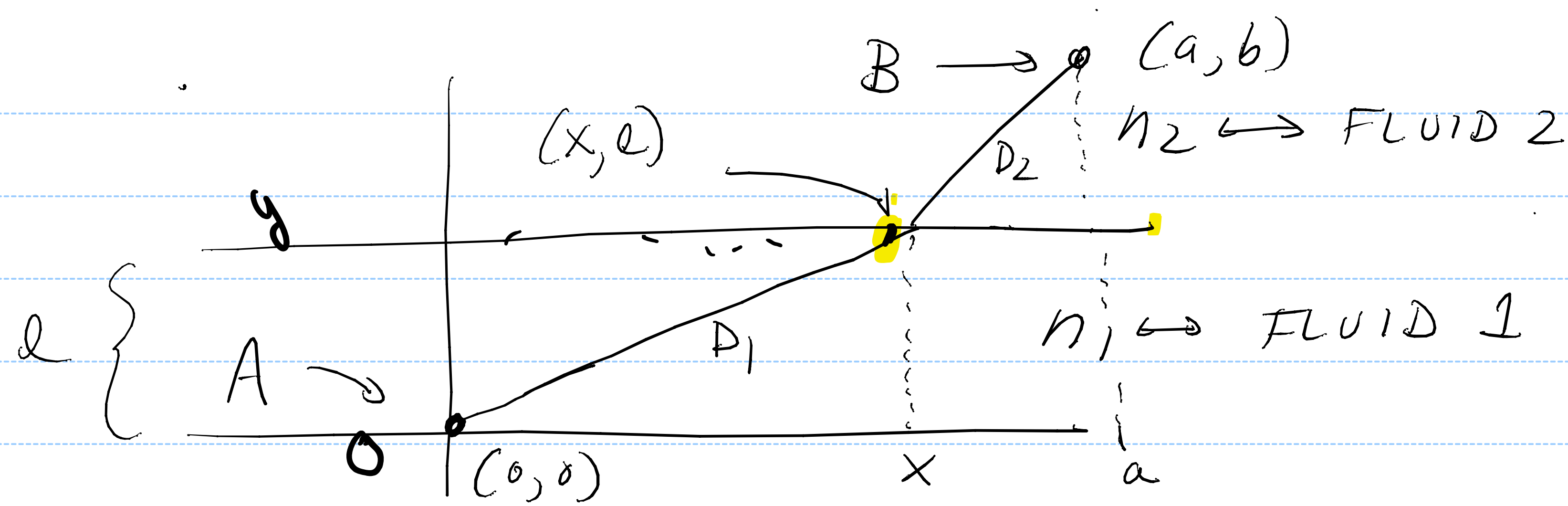
(1)

EX 1

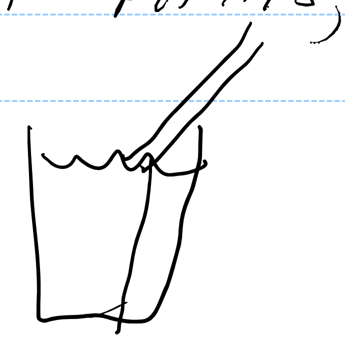


LIGHT PATHS – GEOMETRIC OPTICS

SUPPOSE WE HAVE A TWO-LEVEL FLUID SYSTEM.

FROM FERMATS OBSERVATIONS OF LIGHT PATHS,

HE POSTULATED THAT



(1) TIME OF TRAVEL IN <u>HOMOGENEOUS</u> FLUID

$$= T = \frac{\text{DISTANCE}}{\text{VELOCITY}} = \frac{D}{V};$$ (STRAIGHT LINE)

V IS A CHARACTERISTIC OF THE FLUID;

$n = \frac{1}{V}$ = <u>INDEX</u> <u>OF</u> <u>REFRACTION</u>; $T = Dn$

(2) PATH IN TWO-LEVEL SYSTEM IS PIECE-WISE LINEAR, SUCH THAT <u>TIME OF TRAVEL</u> IS MINIMIZED:

<u>FERMAT</u>

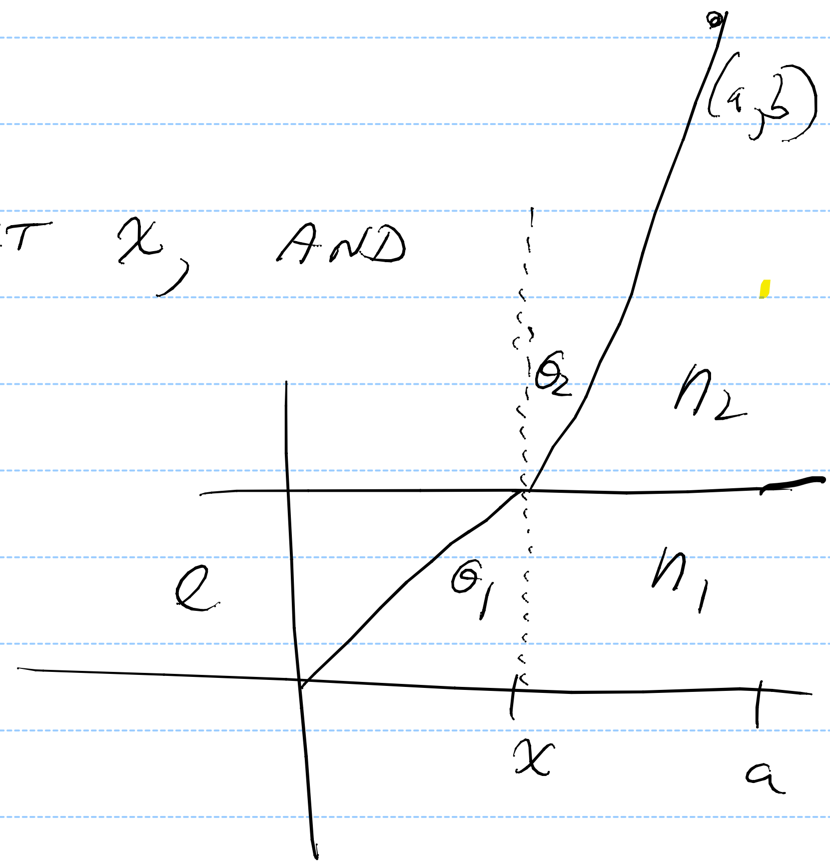
$$T = n_1 \sqrt{x^2 + l^2} + n_2 \sqrt{(a-x)^2 + (b-l)^2}$$

$$= n_1 D_1 + n D_2$$

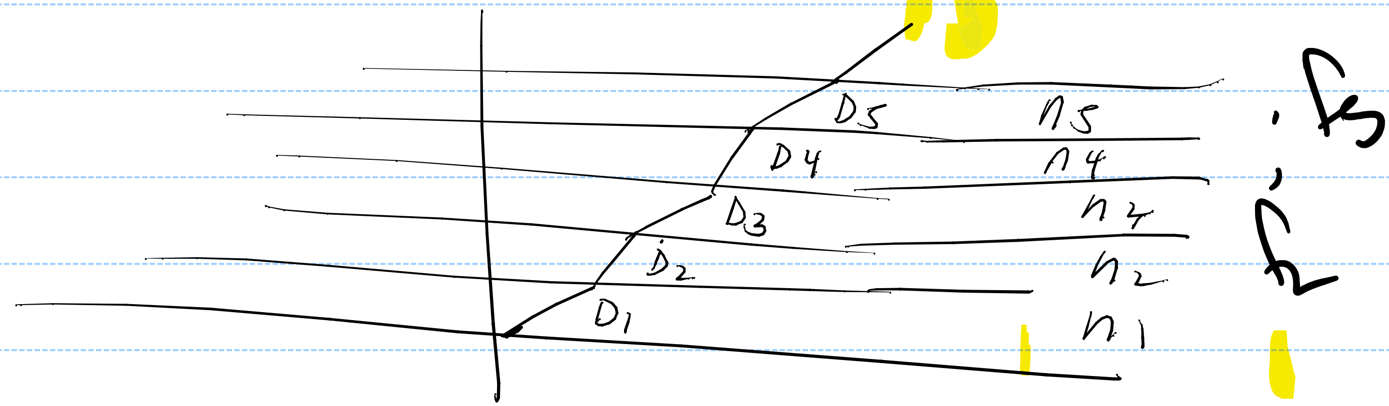
EXERCISE: MINIMIZE THIS WRT x, AND OBTAIN THE CONDITION

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$

SNELL'S LAW.



CAN EXTEND THIS TO MULTI-LEVEL SYSTEM



$$T = \sum D_i n_i$$

IN THE LIMIT OF A CONTINUOUSLY CHANGING INDEX OF REFRACTION:

$$T = T(y(x)) = \int_a^b n(y(x)) \sqrt{1 + y'(x)^2}\, dx$$

WE SEEK $y(x)$ SO THAT T IS AS SMALL AS POSSIBLE (FERMAT'S PRINCIPLE OF LEAST TIME)

NOTE 1: WE ARE IMPLICITLY ASSUMING THAT THE DESIRED PATH IS THE GRAPH OF A FUNCTION.

NOTE 2: THE DISCUSSION ASSUMED THAT n ONLY DEPENDS ON VERTICAL COMPONENT.

MORE GENERALLY, COULD HAVE:

$$\int_a^b n(x, y(x)) \sqrt{1+(y'(x))^2}\, dx$$

(2)
SO OUR PROBLEM REDUCES TO THE FOLLOWING (THE FIRST PROBLEM IN THE CALCULUS OF VARIATIONS):

SUPPOSE $F(u, v, w)$ IS A SUFFICIENTLY SMOOTH FUNCTION OF 3 VARIABLES; AND WE CONSIDER THE QUANTITY

$$\mathcal{L}(y) = \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx$$

DEFINED FOR FUNCTIONS $y(x)$ WHICH ARE (i) C^1 $\{y + y'$ CONTINUOUS ON $[x_0, x_1]\}$ AND FOR

LAGRANGE

!"#$*

(ii) $y(x_0) = y_0$; $y(x_1) = y_1$. FIND FUNCTIONS $y(x)$ SATISFYING THESE TWO CONDITIONS WHICH MINIMIZES $\mathcal{L}$. (OR EXTREMUM)

COMMENTS: 1) EVEN FOR $G: R^N \to R$, THERE CAN OF COURSE BE MULTIPLE MAX/MIN

2) EXTREMA CAN BE LOCAL;

3) CRITICAL POINTS $(\nabla G = 0)$ GIVE CANDIDATE SET FOR LOCATION OF EXTREMA.

DIRECTIONAL DERIV:

$$\left.\frac{d}{dt} G(\vec{p} + t\vec{v})\right|_{t=0} = D_{\vec{v}} G = \boxed{\nabla G \cdot \vec{v}}$$

4) EQUIVALENT TO 3) <u>ALL</u> DIRECTIONAL DERIVATIVES VANISH.

FUNDAMENTAL CALCULATION:

$$\mathcal{L}(y) = \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx \qquad \begin{aligned} y(x_0) &= y_0 \\ y(x_1) &= y_1 \end{aligned}$$

CONSIDER "ANY" SMOOTH FUNCTION $\eta(x)$ WITH $\eta(x_0) = \eta(x_1) = 0$. THEN FOR ANY VALUE ε

$y(x) + \varepsilon\, \eta(x)$ SATISFIES <u>SAME</u> BOUNDARY CONDITIONS

$y+\varepsilon\eta$ (THE <u>VARIATION</u>) IS A "SMALL" PERTURBATION OF y.

NOW CONSIDER THE FUNCTION OF **1** VARIABLE:

$$g(\varepsilon) = \mathcal{L}(y+\varepsilon\eta) = \int_{x_0}^{x_1} F(x, y(x)+\varepsilon\,\eta(x), y'(x)+\varepsilon\,\eta'(x))\,dx$$

IF $y(x)$ IS A LOCAL EXTREMUM FOR $\mathcal{L}$, THEN $\varepsilon=0$ IS A LOCAL EXTREMUM FOR g, AND HENCE $g'(0)=0$. THIS SHOULD BE TRUE FOR ANY $\eta(x)$.

WHAT DOES THIS SAY ABOUT $y(x)$?

$$g'(\varepsilon) = \int_{x_0}^{x_1} \left(\frac{\partial F}{\partial V}\, \eta(x) + \frac{\partial F}{\partial W}\, \eta'(x) \right) dx$$

$$= \underbrace{\int_{x_0}^{x_1} \frac{\partial F}{\partial V}\, \eta(x)\, dx}_{①} + \underbrace{\int_{x_0}^{x_1} \frac{\partial F}{\partial W}\, \eta'(x)\, dx}_{②}$$

$$② \overset{\text{IBP}}{=} \frac{\partial F}{\partial W}\, \eta(x) \Big|_{x=x_0}^{x=x_1} - \int_{x_0}^{x_1} \eta(x) \frac{d}{dx} \frac{\partial F}{\partial W}\, dx$$

$$g'(\varepsilon) = \int_{x_0}^{x_1} \eta(x) \left[F_V(x, y+\varepsilon\eta, y'+\varepsilon\eta') - \frac{d}{dx} F_W(\cdot, \cdot, \cdot) \right] dx$$

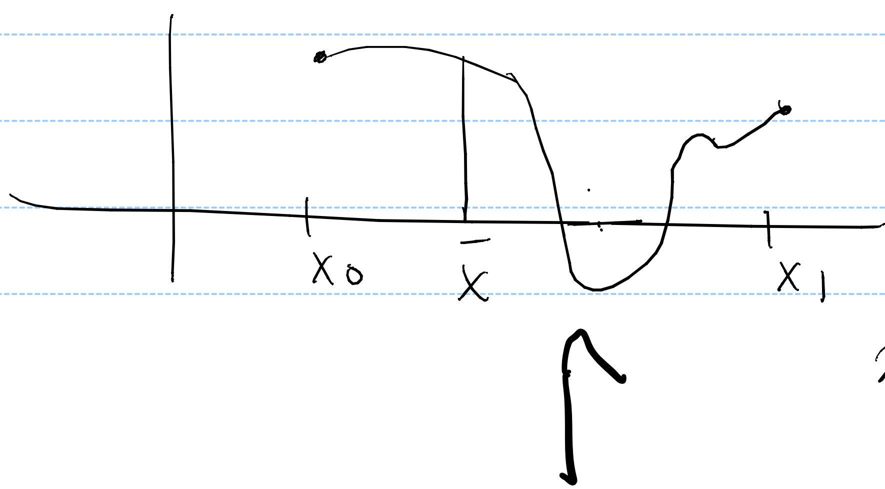
$$g'(0) = \int_{x_0}^{x_1} \eta(x) \left[\frac{\partial F}{\partial V}(x, y(x), y'(x)) - \frac{d}{dx} F_W(x, y(x), y'(x)) \right] dx$$

$= 0$, INDEPENDENT OF $\eta(x)$.

LET $h(x) = \frac{\partial F}{\partial v}(x, y(x), y'(x)) - \frac{d}{dx} F_w(x, y(x), y'(x))$

CLAIM: $h(x) \equiv 0$! IF NOT, THEN ASSUMING $h(x) \not\equiv 0$, THERE IS SOME POINT $\bar{x}$ WITH $h(\bar{x}) \neq 0$

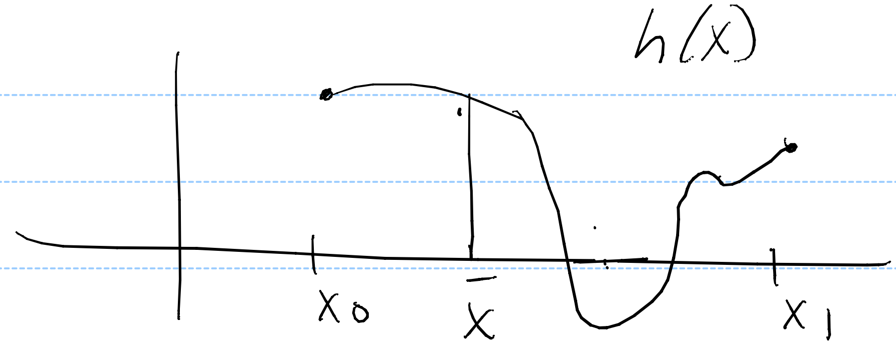
ASSUME WLOG $h(\bar{x}) > 0$. CONSTRUCT $\eta(x)$ WITH

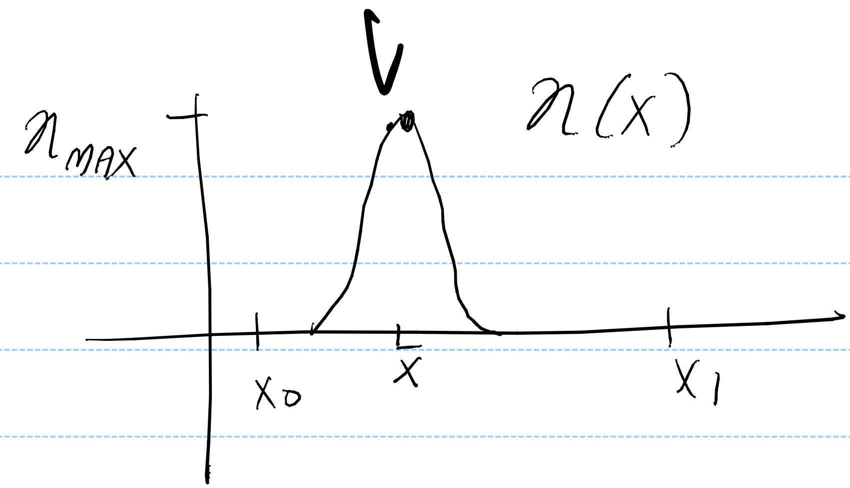


$\eta(x_0) = \eta(x_1) = 0$;

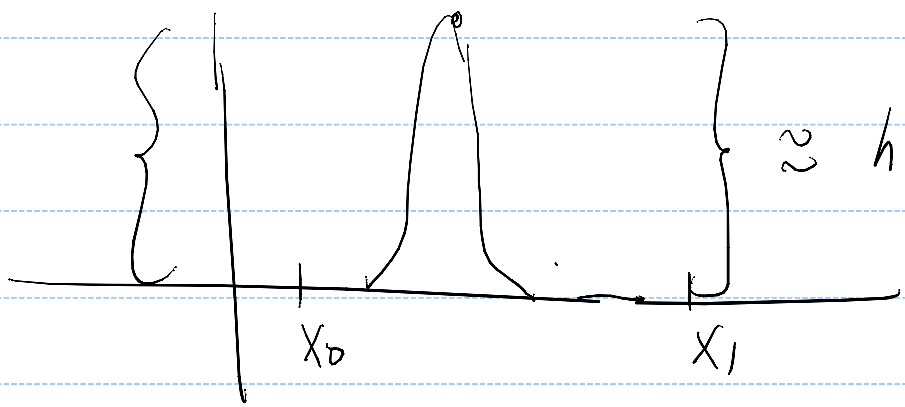
$\eta(x) \geq 0$;

$\eta(x) \equiv 0$ OUTSIDE SMALL NBD $\bar{x}$





$\underline{\underline{n(x)\,h(x)}}$



$\approx h(\bar{x})\, n_{MAX}$

$$\int_{x_0}^{x_1} n(x)\,h(x)\,dx > 0 \quad \not\Longleftrightarrow .$$

THUS, IF $y(x)$ IS AN EXTREMUM FOR $\mathcal{L}$, AND IF y HAS CONTINUOUS FIRST AND 2ND DERIVATIVES, THEN $y(x)$ SATISFIES THE SECOND ORDER DIFFERENTIAL EQUATION

EULER-LAGRANGE EQN !!

$$\boxed{F_v\left(x, y(x), y'(x)\right) - \frac{d}{dx} F_w\left(x, y(x), y'(x)\right) = 0}$$

THIS EQUATION IS ABSOLUTELY <u>ESSENTIAL</u> TO THE SUBJECT!

$$\int_a^b F\left(x, y(x), y'(x)\right) dx \quad ; \quad F(u, v, w).$$

!"#$,'

COMMENTS:

① TECHNICALLY, WE HAVE SHOWN THAT IF $y(x)$ IS AN EXTREMUM FOR $\mathcal{L}$ (WHICH ONLY REQUIRES CONTINUOUS FIRST DERIVATIVE TO BE DEFINED) AND <u>IF</u> $y''(x)$ EXISTS, THEN $y(x)$ SATISFIES E-L. CAN SHOW THAT, FOR SUITABLE F, y AN EXTREMUM $\Rightarrow$ $y''(x)$ EXISTS. WE'LL AVOID SUCH TECHNICAL DETAILS.

② TYPICALLY, FOR SECOND ORDER DE'S, YOU WANT TO WRITE EQN AS $y''(x) = \cdots$; CAN PROVE EXISTENCE THEOREMS IN THIS FORM. BUT E-L IS

$$F_v(x, y(x), y'(x)) - \frac{d}{dx} F_w(x, y(x), y'(x)) = 0$$

OR $F_v(x, y, y') - F_{wu} - F_{wv}\, y'(x) - F_{ww}\, y''(x) = 0$

REQUIRE $F_{ww}(x, y(x), y'(x)) \neq 0$ ALONG CURVE

(MOST EASILY GUARANTEED IF $F_{ww} > 0$ EVERYWHERE

(3) BASIC EXISTENCE THEOREMS FOR DE'S ARE INITIAL VALUE PROBLEMS:

$$y''(x) = f(x, y(x), y'(x))$$

$$, \quad y(x_0) = y_0 \quad y'(x_0) = y_p$$

WE HAVE A <u>BOUNDARY VALUE</u> PROBLEM, FOR WHICH THE FUNDAMENTAL THEOREM DOES NOT APPLY.

(4) THE STANDARD NOTATION (UNFORTUNATELY!)

I HAVE WRITTEN $F(u, v, w) : \mathbb{R}^3 \to \mathbb{R}$;

$$\mathcal{L} = \int F(x, y(x), \underline{y'}(x))\, dx$$

FUNCTION ON FUNCTIONS: FUNCTIONAL

E-L: $$F_v(x, y, y') - \frac{d}{dx} F_w(x, y, y') = 0$$

IN STANDARD NOTATION, WRITE v AS y,

AND w AS y': $F(x, y, y')$:

$$F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') = 0$$

MAXIMAL CONFUSION! NOTATION MEANS PARTIAL DIFFERENTIATION WRT 2ND & 3RD SLOTS. BUT $F_v(u, v, w) = \frac{\partial F}{\partial v}\Big|_{u,\, w \text{ FIXED}}$;

SO F_y "LOOKS" LIKE $\frac{\partial F}{\partial y}\Big|_{x,\, y' \text{ FIXED}}$. HOW CAN YOU FIX y, AND VARY y'? YOU CAN'T!

JUST ANOTHER NOTATION FOR PARTIAL DERIVATIVES WRT THE VARIOUS SLOTS. THIS NOTATION HAS CONFUSED GENERATIONS OF MATH/SCIENCE STUDENTS!

BACK TO LIGHT PATHS: $\mathcal{L}(y) = \int_a^b n(x,y)\sqrt{1+(y'(x))^2}\,dx$

(3)

EX: $n = 1$ $\quad \mathcal{L}(y) = \int_0^1 \sqrt{1+(y'(x))^2}\,dx$

$$F(u, v, w) = \sqrt{1+w^2}$$

$$\mathcal{L}(y) = \int F(x, y(x), y'(x))\,dx$$

$$F_v = 0 \qquad F_w = \frac{w}{\sqrt{1+w^2}} = F_w(u, v, w)$$

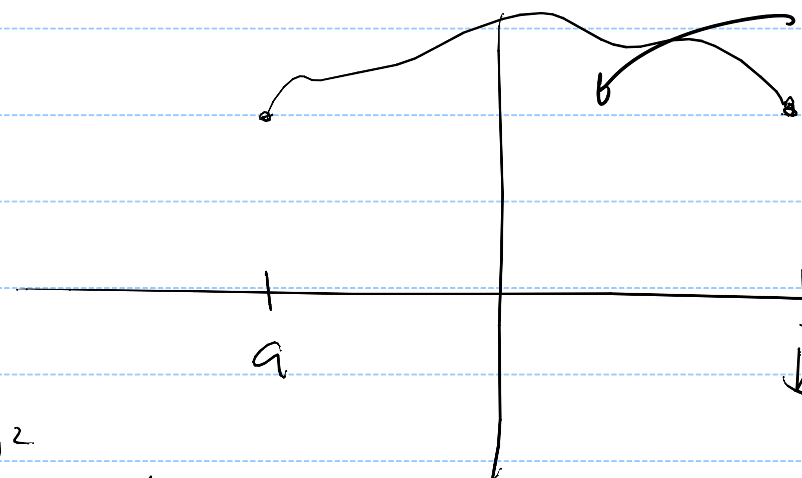
E-L:

$$0 = F_v - \frac{d}{dx} F_w(x, y, y') = \frac{d}{dx} \frac{y'}{\sqrt{1+(y')^2}}$$

$$\rightarrow \frac{y'}{(1+(y')^2)^{1/2}} = C \rightarrow \frac{(y')^2}{1+(y')^2} = C^2$$

$$\rightarrow y' = D \qquad y = Dx + E$$

Ex: $n = \frac{1}{y}$ $\quad y > 0$



~~velo~~ speed $= y$.

$$\mathcal{L}(y) = \int_a^b \frac{\sqrt{1+(y'(x))^2}}{y}\, dx$$

$$F(a, v, w) = \frac{\sqrt{1+w^2}}{v}$$

$$F_v = -\frac{\sqrt{1+w^2}}{v^2} \qquad F_w = \frac{1}{v} \frac{w}{\sqrt{1+w^2}}$$

$$F_v(x, y(x), y'(x)) - \frac{d}{dx} F_w(x, y(x), y'(x)) = 0$$

$$-\frac{\sqrt{1+(y')^2}}{y^2} - \frac{d}{dx} \frac{1}{y} \frac{y'}{\sqrt{1+(y')^2}} = 0$$

$$D = \left(1 + (y')^2\right)^{1/2} \qquad D' = \frac{y' y''}{D}$$

E-L:

$$-\frac{D}{y^2} - \frac{d}{dx}\,\frac{y'}{y\,D} = 0$$

$$-\frac{D}{y^2} - \frac{\left[\, y'' y D - y'\left[y' D + y\,\frac{y' y''}{D}\right] \right]}{y^2 D^2} = 0$$

$$D^3 + y'' y D - (y')^2 D - \frac{(y')^2\, y\, y''}{D} = 0$$

$$D^4 + y'' y D^2 - (y')^2 D^2 - (y')^2\, y\, y'' = 0$$

$$(1+(y')^2)^2 + y''y(1+(y')^2) - (y')^2(1+(y')^2) - (y')^2\, y\, y'' = 0$$

$$1 + 2(y')^2 + \cancel{(y')^4} + y''y + \cancel{y''y(y')^2} - (y')^2 - \cancel{(y')^4} - \cancel{(y')^2 y\, y''} = 0$$

$$\boxed{1 + (y')^2 + y''y = 0}$$

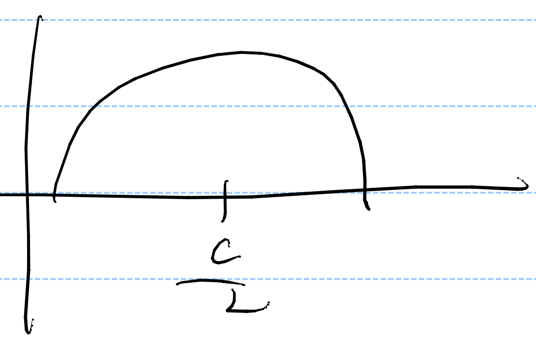
E-L SIMPLIFIES TO $1+(y')^2+y''y=0$

$(yy')' = -1 \rightarrow yy' = -x + C$

$\left(\frac{y^2}{2}\right)' = -x + C \qquad \frac{y^2}{2} = -\frac{x^2}{2} + Cx + D$

$x^2 + y^2 = Cx + D \rightarrow x^2 - Cx + y^2 = D$

$\left(x - \frac{c}{2}\right)^2 + y^2 = D$



CIRCLE!!!!

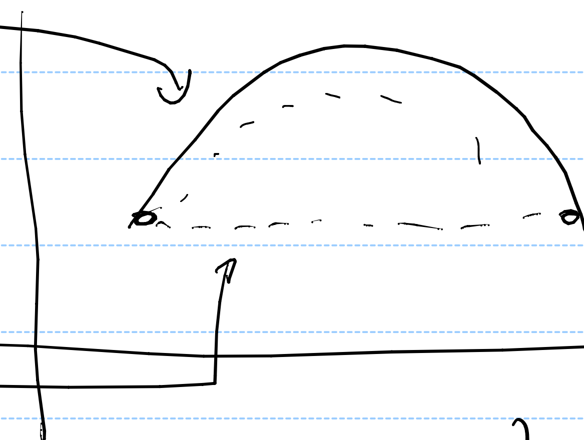
$$\mathcal{L}(y) = \int \frac{\sqrt{1+(y'(x))^2}}{y(x)}\,dx$$

IF YOU SEEK TO MAKE $\mathcal{L}(y)$ AS SMALL AS POSSIBLE, YOU ARE <u>REWARDED</u> IF $y(x)$ IS LARGE

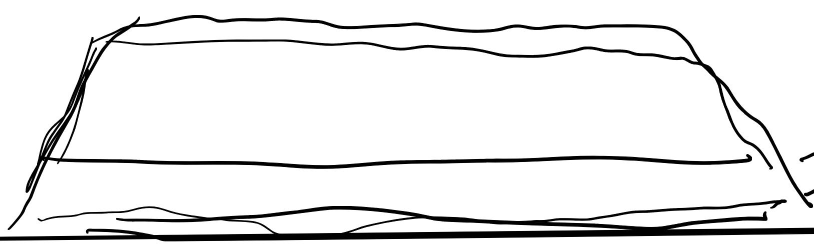
speed. big fast

TIME OF TRANSIT SMALL

ORDINARY LENGTH SMALL



speed slow small



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