LECTURE 2 (CONT OF LEC 1)

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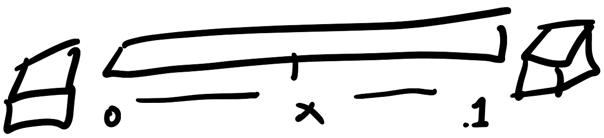

$u(x,t) =$ TEMPERATURE AT TIME t, LOCATION x.

① PDE: $u_t = u_{xx}$ (HEAT EQN)

② BC $u(0,t)=0$; $u(1,t)=0$
(BOUNDARY COND).

③ IC (INITIAL CONDITION) $u(x,0) = f(x)$.

GUESS "SIMPLE SOL'NS"

$u(x,t) = X(x)\,T(t)$

= "PURE x" TIMES "PURE t"

{SEPARATION OF VARIABLES}

⋮

$u_N(x,t) = e^{-\pi^2 N^2 t} \sin(N\pi x)$

BUT: PDE (& BC!) ARE LINEAR.

PDE: $u_t - u_{xx} = 0$; IF $v(x,t)$, $w(x,t)$
ARE SOL'NS, SO ARE $c_1 v(x,t) + c_2 w(x,t)$

AND $c_1 u_1(x,t) + c_2 u_2(x,t) \ldots$

SO $u(x,t) = c_1 u_1(x,t) + c_2 u_2(x,t) + c_3 u_3(x,t) \ldots$ 2

GIVES LOTS OF SOL'NS.

BUT! WE WANT $u(x,0) = f(x)$, FOR ANY CHOICE OF $f(x)$ [INITIAL TEMP. DISTRIBUTION]

$$u(x,0) = c_1 u_1(x,0) + c_2 u_2(x,0) \cdots$$

$$\left\{ u_n(x,t) = e^{-\pi^2 n^2 t} \sin(n\pi x) \to u_n(x,0) = \sin(n\pi x) \right\}$$

$$f(x) \stackrel{?}{=} c_1 \sin(\pi x) + c_2 \sin(2\pi x) + c_3 \sin(3\pi x) \ldots$$

IF $c_1, c_2, \ldots$ CAN BE FOUND SO THAT LHS = RHS, THEN CAN DO FOLLOWING (!!!)

$$\begin{aligned} f(x)\sin(17\pi x) = {} & c_1 \sin(\pi x)\sin(17\pi x) \\ & + c_2 \sin(2\pi x)\sin(17\pi x) \\ & \vdots \\ & + c_{16} \sin(16\pi x)\sin(17\pi x) \\ & + c_{17} \sin(17\pi x)\sin(17\pi x) \\ & + c_{18} \sin(18\pi x)\sin(17\pi x) \\ & \vdots \end{aligned}$$

SO...

$$\int_0^1 f(x) \sin(17\pi x)\, dx$$

$$= \int_0^1 c_1 \sin(\pi x) \sin(17\pi x)\, dx$$
$$\int_0^1 c_2 \sin(2\pi x) \sin(17\pi x)\, dx$$
$$\vdots$$
$$\int_0^1 c_{16} \sin(16\pi x) \sin(17\pi x)\, dx$$
$$\int_0^1 c_{17} \sin(17\pi x) \sin(17\pi x)\, dx$$
$$\int_0^1 c_{18} \sin(18\pi x) \sin(17\pi x)$$
$$\vdots$$

$$\int_0^1 f(x) \sin(17\pi x)\, dx =$$
$$c_1 \int_0^1 \sin(\pi x) \sin(17\pi x)\, dx$$
$$\vdots$$
$$c_{17} \int_0^1 \sin(17\pi x) \sin(17\pi x)\, dx$$
$$\vdots$$

"JUST" INTEGRALS

$$\int_0^1 \sin(m\pi x) \sin(n\pi x)\, dx = 0, \quad m \neq n \quad (\text{INT.})$$

$$\int_0^1 \sin(m\pi x) \sin(m\pi x)\, dx$$

$$= \int_0^1 (\sin(m\pi x))^2\, dx \stackrel{!!}{=} \frac{1}{2}$$

SO $\int_0^1 f(x) \sin(17\pi x)\,dx$

$= 0 \ldots + 0 \ldots$

$+ C_{17}\left(\frac{1}{2}\right) + \cdots$

OR $C_{17} = 2\int_0^1 f(x)\sin(17\pi x)\,dx$

OF COURSE

$$\boxed{C_N = 2\int_0^1 f(x)\sin(N\pi x)\,dx}$$

SO FINALLY, OUR SOL'N TO PDE

IS $u(x,t) = C_1 e^{-\pi^2 t}\sin(\pi x)$

$\vdots$

$+ C_N e^{-\pi^2 N^2 t}\sin(N\pi x)$

$\vdots$

$C_N =$

AND $f(x) = $ I.C.

$= C_1 \sin(\pi x)$

$+ C_2 \sin(2\pi x) \ldots$

MANY QUESTIONS

(i) WHERE DID THIS TRICK COME FROM? (ANSWER: GENIUS). (G.B.)

(ii) $$f(x) = \sum_{N=1}^{\infty} C_N \sin(N\pi x)$$

RHS = INFINITE SERIES; FROM CALC 3, KNOW YOU HAVE TO BE CAREFUL: INFINITE SERIES DO NOT ALWAYS CONVERGE!

"CONCEPTUAL INTERPRETATION":

RECALL IF V IS A VECTOR SPACE ($R^2, R^3, \ldots$) THEN A BASIS FOR V IS A SET OF VECTORS $\{e_1, e_2, \ldots e_N\}$

SO THAT

(i) EVERY VECTOR $\vec{V}$ CAN BE WRITTEN AS

$$\vec{V} = c_1 e_1 + c_2 e_2 \quad -- \quad + c_N e_N$$

(<u>LINEAR COMBINATION</u> OF BASIS VECTORS)

(ii) THE #'S $c_1, c_2, --$ ARE UNIQUE.

EX: $\mathbb{V} = R^3$; $e_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $e_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$e_3 = \begin{pmatrix} 1 \\ 1 \\ 10 \end{pmatrix}$. IS A BASIS.

IF GIVEN A RANDOM VECTOR $\vec{V}$, FINDING c_1, c_2, c_3 SO THAT $\vec{V} = c_1 e_1 + c_2 e_2 + c_3 e_3$ CAN BE HARD [EX: $\vec{V} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$]

!!!

BUT! IF BASIS HAS SPECIAL PROPERTIES, FINDING c'S IS MUCH EASIER.

DEF: A BASIS IS CALLED ORTHOGONAL IF $e_i \cdot e_j = 0$ WHEN $i \neq j$ ($\bullet$ = DOT PRODUCT)

EX: $e_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

"NEW e's"

$e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_3 = 0$.

$e_1 \cdot e_1 = 2 \quad e_2 \cdot e_2 = 2 \quad e_3 \cdot e_3 = 1$

$\left\{ e_k \cdot e_k \text{ IS NEVER } 0, \; e_k \cdot e_k = \|e_k\|^2 ! \right\}$

EX: $\vec{v} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$

$\vec{v} = c_1 e_1 + c_2 e_2 + c_3 e_3$

$\vec{v} \cdot e_1 = (c_1 e_1 + c_2 e_2 + c_3 e_3) \cdot e_1$

$= c_1 \underline{e_1 \cdot e_1} + c_2 \overset{\to 0}{e_2 \cdot e_1} + c_3 \overset{\to 0}{e_3 \cdot e_1}$

$= c_1 \cdot 2 \qquad \vec{v} \cdot e_1 = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 3 + 3 + 0 = 6$

$\boxed{c_1 = 3}$

$$\vec{V} \cdot e_2 = c_2 (e_2 \cdot e_2) = 2 c_2$$

$$\vec{V} \cdot e_2 = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0 \qquad \boxed{c_2 = 0}$$

$$\vec{V} \cdot e_3 = c_3 (e_3 \cdot e_3) = c_3$$

$$\begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1 \qquad \boxed{c_3 = 1}$$

$$\begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

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(1) $\overline{V}$ = FUNCTIONS ON INTERVAL $[0,1]$; $f(x), g(x)$ FUNCTIONS THEN $c_1 f(x) + c_2 g(x)$ IS A FUNCTION

$v = f(x)$

$w = g(x)$

$$V \cdot W = \int_0^1 f(x) g(x) \, dx$$

(2)

(3) $e_1 = \sin(\pi x)$, $e_2 = \sin(2\pi x)$

$e_3 = \sin(3\pi x)$, ...

$$e_m \cdot e_N = \int_0^1 \sin(m\pi x) \sin(N\pi x) \, dx = 0 \quad N \neq m$$

$e_m \cdot e_N = 1/2.$

so $f(x) = c_1 \sin(\pi x)$
$+ c_2 \sin(\pi \cdot 2x)$
$+ c_3 \sin(3\pi x) \ldots$

$$\int_0^1 f(x) \sin(N\pi x)\,dx = \frac{1}{2} c_N, \text{ or}$$

$$c_N = 2\int_0^1 f(x) \sin(N\pi x)\,dx$$

INTERPRETATION OF FOURIER SERIES IN TERMS OF LINEAR ALGEBRA