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Phys 432 HW1

① Local Effects of Cosmological Constant?

a) Compute the equivalent mass density of the cosmological constant in SI units.

$$\rho = \frac{3H_0^2}{8\pi G}\Omega_{vac} = \frac{3(70\,km/s/Mpc)^2}{8\pi(6.67\times10^{-11}\frac{m^3}{kg\cdot s^2})}(0.7)$$

$$\rightarrow \rho = 6.1\times10^{12}\frac{km^2}{Mpc^2}\frac{kg}{m^3}\times\left(\frac{3.09\times10^{19}\,km}{Mpc}\right)^{-2} = \boxed{6.4\times10^{-27}\,kg/m^3}$$

b) For the Solar System, assume the Sun dominates the mass and use 1 AU as the characteristic radius. Compute the avg. density.

Approximate SS as sphere:

$$\rightarrow \rho V = m \rightarrow \rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3}{4}\frac{M_\odot}{\pi R_{AU}^3}$$

$$\rho = \frac{3}{4}\frac{2\times10^{30}\,kg}{\pi(1.5\times10^{11}\,m)^3} = \boxed{1.4\times10^{-4}\,kg/m^3}$$

c) For the Galaxy, assume that the mass is $10^{11}M_\odot$ within the Sun's orbit of 8.5 kpc

$$\rightarrow \rho = \frac{3}{4}\frac{10^{11}M_\odot}{\pi(8.5\,kpc^3)}\times\left(\frac{1\,kpc}{3.09\times10^{19}\,m}\right)^3\left(\frac{2\times10^{30}\,kg}{1\,M_\odot}\right) = \boxed{2.6\times10^{-21}\,kg/m^3}$$

d) Applying Birkhoff's theorem, compute the ratios a_{vac}/a_{SS} and a_{vac}/a_{MW}

$$a_{vac} = \frac{GM_{vac}}{R^2} \; ; \; \rho = \frac{M}{\frac{4}{3}\pi R^3} \rightarrow a_{vac} = \frac{3}{4\pi} \rho R G$$

$$\rightarrow \frac{a_{vac}}{a_{SS}} = \frac{\rho_{vac}}{\rho_{SS}} = \frac{6.4\times10^{-27}\,kg/m^3}{1.4\times10^{-4}\,kg/m^3} \approx \boxed{5\times10^{-23}}$$

$$\frac{a_{vac}}{a_{MW}} = \frac{\rho_{vac}}{\rho_{MW}} = \frac{6.4\times10^{-27}\,kg/m^3}{2.6\times10^{-21}\,kg/m^3} \approx \boxed{2\times10^{-6}}$$

$\Rightarrow$ acceleration due to vacuum density is neglible, so we can't (locally) detect the effects of vacuum energy.

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① Robertson-Walker metric

Show that the RW metric

$$c^2 ds^2 = -c^2 dt^2 + a^2(t)\left[dr^2 + S_k^2(r)\,d\psi^2\right]$$

can also be written in the form:

$$c^2 ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dx^2}{1-k\left(\frac{x}{R}\right)^2} + x^2 d\psi^2\right]$$

The above gives $x \equiv S_k(r)$

$k=0$: $S_0(r) = r$

$$\rightarrow c^2 ds^2 = -c^2 dt^2 + a^2(t)\left[dr^2 + r^2 d\psi^2\right]$$

$S_k^2(r)$

✓

$k=-1: S_{-1}(r) = R\sinh(r/R)$

$$dr^2 \overset{?}{=} \frac{dx^2}{1+\left(\frac{x}{R}\right)^2}$$

$$\rightarrow \int dr = \int \frac{dx}{\sqrt{1+x^2/R^2}} = R\int \frac{dx}{\sqrt{R^2+x^2}}$$

$$\rightarrow r = R\sinh^{-1}\left(\frac{x}{R}\right) = R\sinh^{-1}\left(\frac{\cancel{R}\sinh(r/R)}{\cancel{R}}\right)$$

$$\rightarrow r = R(r/R)$$

$$r = r \checkmark \Longrightarrow dr^2 = \frac{dx^2}{1+\left(\frac{x}{R}\right)^2}$$

$$c^2ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dx^2}{1+x^2/R^2} + x^2 d\psi^2\right] \checkmark$$

$k=1: S_1(r) = R\sin(r/R)$

$$dr^2 \overset{?}{=} \frac{dx^2}{1-(x/R)^2}$$

$$\rightarrow \int dr = R\int \frac{dx}{\sqrt{R^2-x^2}}$$

$$\rightarrow r = R\sin^{-1}(x/R)$$

$$r = R\sin^{-1}\left(\frac{\cancel{R}\sin(r/R)}{\cancel{R}}\right)$$

$$\rightarrow r = \cancel{R}(r/\cancel{R})$$

$$r = r \Longleftrightarrow dr^2 = \frac{dx^2}{1-(x/R)^2} \checkmark$$

$$c^2ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dx^2}{1-x^2/R^2} + x^2 d\psi^2\right]$$

$$\Longrightarrow RW: \quad c^2ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dx^2}{1-kx^2/R^2} + x^2 d\psi^2\right] \quad \text{for all } k=0,\pm1 \checkmark$$

3) Particle Horizon

a) Starting from the fact that photons travel on null geodesics with $ds=0$ and $d\psi=0$, use the spacetime metric to write an integral equation for the comoving distance r that a photon travels between two times in the universe.

$$\cancel{ds^2} = -c^2dt^2 + a^2(t)\left[dr^2 + \cancel{S_k(r)\,d\psi^2}\right]$$

$$c^2dt^2 = a^2(t)\,dr^2$$

$$\rightarrow \boxed{\int_{t_1}^{t_2} c\,dt = \int_{r_1}^{r_2} a(t)\,dr}$$

b) Make a change of variables from $t \to a$ and use the Friedmann Equation with $k=0$ and with density $\rho \propto a^{-\alpha}$ to rewrite the integral so the limits are the scale factors $a(t)$ of the universe at two times and the integral depends on α.

$$\int_{t_1}^{t_2} c\,dt = \int_{r_1}^{r_2} a(t)\,dr$$

$$H = \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt}$$

$$\rightarrow dt = \frac{1}{Ha}\,da$$

$$\int_{a(t_1)}^{a(t_2)} c\,\frac{1}{Ha}\,da = \int_{r_1}^{r_2} a\,dr$$

$$c\int_{a(t_1)}^{a(t_2)} \frac{da}{Ha^2} = \int_{r_1}^{r_2} dr$$

Friedmann: $H^2 - \frac{8\pi G\rho}{3} = \cancel{\frac{kc^2}{R^2}}$

$$\rightarrow H = \sqrt{\frac{8\pi G\rho}{3}}$$

$$\Longrightarrow c\int \frac{1}{a^2}\sqrt{\frac{3}{8\pi G\rho}}\,da = \int dr$$

$$\frac{c}{2}\sqrt{\frac{3}{2\pi G}}\int_{a(t_1)}^{a(t_2)} \frac{1}{a^2\sqrt{\rho}}\,da = \int_{r_1}^{r_2} dr \qquad \left(\rho \propto a^{-\alpha} \rightarrow \rho = \rho_0 a^{-\alpha}\right)$$

$$\frac{c}{2}\sqrt{\frac{3}{2\pi G\rho_0}}\int \frac{1}{a^2 a^{-\alpha/2}}\,da = \int dr$$

$$\Longrightarrow \boxed{\frac{c}{2}\sqrt{\frac{3}{2\pi G\rho_0}}\int_{a(t_1)}^{a(t_2)} \frac{da}{a^{2-\alpha/2}} = \Delta r}$$

c) Integral converges if $2-\alpha/2 > 1 \rightarrow \underline{\underline{\alpha > 2}}$

• Is this condition satisfied in a universe that contains matter, radiation, and dark energy?

$\rho_{rad} \propto a^{-4}$, $4>2$ ✓ ; $\rho_{matter} \propto a^{-3}$, $3>2$ ✓ ; $\rho_{\Lambda} \propto a^{0}$, $0>2$ ✗

$\Rightarrow$ Condition only satisfied in universe that contains radiation and matter. Hence it holds in the early radiation-dominated Universe.

(5) Baseball Universe $m = 0.145\,kg$ $r = 0.0369\,m$

a) If non-relativistic baseballs are the only form of mass-energy, with critical density $\Omega = 1$, what is the number density of baseballs in SI units? AU^{-3}

$$\rho_0 \approx 2\times10^{-26}\,\Omega h^2\ kg\ m^{-3}$$

$$\approx 2\times10^{-26}(.68)^2\ kg/m^3 \times \left(\frac{1\ baseball}{0.145\,kg}\right)$$

$$\longrightarrow \boxed{n_\star \approx 6.4\times10^{-26}\ m^{-3}}$$

$$n_\star \times \left(\frac{1\,AU}{1.5\times10^{11}\,m}\right)^{-3} = \boxed{2.1\times10^{8}\ AU^{-3}}$$

b) At that density, what is the average distance (in meters) at which your line of sight intercepts a baseball?

$$\lambda = \frac{1}{n_\star \pi R_\star^2} = \frac{1}{(6.4\times10^{-26}\,m^{-3})\,\pi(.0369\,m)^2} = \boxed{3.7\times10^{27}\ m}$$

c) Re-express your answer in terms of a Hubble length $c/H_0 \sim 4000\,Mpc$. Does the transparency of the Universe set a useful limit on the # density of baseballs?

$$3.7\times10^{27}\,m\cdot\left(\frac{3.09\times10^{22}\,m}{Mpc}\right)^{-1}\left(\frac{c/H_0}{4000\,Mpc}\right) \sim \boxed{29.7\,\frac{c}{H_0}}$$

No – on average, we wouldn't be able to see any!

⑥ Einstein's Static Universe

a) Write equations that express both the matter density and the cosmological constant as energy densities (see pg. 64 Ryden)

$$\varepsilon_m = \rho c^2 \longrightarrow \boxed{\rho = \varepsilon_m / c^2}$$

$$\varepsilon_\Lambda = \Lambda c^2 / 8\pi G \longrightarrow \boxed{\Lambda = 8\pi G \varepsilon_\Lambda / c^2}$$

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b) Write equations for the Pressures of the matter, radiation, and cosmological constant.

$P = w\varepsilon$ Non-relativistic matter: $w = 0$

Radiation: $w = 1/3$

Cosmo Const.: $w = -1$

$$\boxed{\begin{aligned} P_{matter} &= 0 \\ P_{radiation} &= \tfrac{1}{3}\varepsilon_r \\ P_\Lambda &= -\varepsilon_\Lambda \end{aligned}}$$

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c) Write an equation for the acceleration of the Universe in terms of the energy densities and pressures of matter, radiation, and cosmo constant.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P)$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\varepsilon_{matter} + \varepsilon_{rad} + \varepsilon_\Lambda + 3[P_{mat} + P_{rad} + P_\Lambda]\right)}$$

d) Converting a fraction, f, of the matter to radiation, rewrite (c) in terms of f. Does the universe expand or contract?

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\varepsilon_m + \varepsilon_r + \varepsilon_\Lambda + 0 + \varepsilon_r - 3\varepsilon_\Lambda\right)$$

$$= -\frac{4\pi G}{3c^2}\left(\varepsilon_m + 2\varepsilon_r - 3\varepsilon_\Lambda\right)$$

$$= -\frac{4\pi G}{3c^2}\left((1-f)\varepsilon_m + 2(f\varepsilon_m) - 3\varepsilon_\Lambda\right)$$

$$= -\frac{4\pi G}{3c^2}\left(\varepsilon_m - f\varepsilon_m + 2f\varepsilon_m - 3\varepsilon_\Lambda\right)$$

$$= -\frac{4\pi G}{3c^2}\left(f\varepsilon_m + \varepsilon_m - 3\varepsilon_\Lambda\right)$$

We have: $\varepsilon_m = \rho c^2$, $\varepsilon_\Lambda = \frac{\Lambda c^2}{8\pi G}$, and $\Lambda = 4\pi G\rho$

$$\longrightarrow \varepsilon_\Lambda = \frac{1}{2}\rho c^2$$

$$\Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(f\rho c^2 + \rho c^2 - \frac{3}{2}\rho c^2\right)$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G\rho}{3}\underbrace{\left(f - \frac{1}{2}\right)}_{<0}} \longrightarrow > 0$$

The new radiation will provide positive pressure supporting an initial expansion $\dot{a} > 0$, then $f < \frac{1}{2}$ tells us $\ddot{a}/a > 1 \longrightarrow \ddot{a} > 0$, so the universe will continue to expand.