(REVIEW) LAPLACE TRANSFORM

+ ODE'S

$$f(t) \longrightarrow F(s) = \mathcal{L}(f)(s)$$

$$= \int_0^\infty f(t)\, e^{-st}\, dt.$$

EX! $f(t) = e^{at}$

$$F(s) = \int_0^\infty e^{-st}\, e^{at}\, dt$$

$$= \int_{t=0}^{t=\infty} e^{t(a-s)}\, dt = \left. \frac{e^{t(a-s)}}{(a-s)} \right|_{t=0}^{t=\infty}$$

$$= \left\{ \begin{matrix} \text{IF} \\ s > a \end{matrix} \right\} \quad 0 - \frac{1}{(a-s)} = \frac{1}{(s-a)}$$

f	$\mathcal{L}(f)$
e^{at}	$\frac{1}{(s-a)}$

EX! $f(t) = t^N$

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st}\, t^N\, dt$$

$$\left\{ \begin{matrix} u = st \\ du = s\,dt \end{matrix} \right\} = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^N \frac{du}{s}$$

$$= \frac{1}{s^{N+1}} \int_0^\infty e^{-u}\, u^N\, du. = \frac{N!}{s^{N+1}}$$

$$\left\{ \text{FACT:} \quad \int_0^\infty e^{-u} u^N \, du = (N!) \right\}$$

f	$\mathcal{L}(f)$
e^{at}	$\frac{1}{(s-a)}$
t^N	$\frac{N!}{s^{N+1}}$

LAPLACE & DERIVATIVES.

$$\mathcal{L}(f')(s) = \int_0^\infty f'(t) e^{-st} \, dt$$

$$\overset{IBP}{=} \cancel{f(t) e^{-st} \Big|_0^\infty}^{s>0} - \int_0^\infty f(t)(-s) e^{-st} \, dt$$

$$= -f(0) + s \int_0^\infty f(t) e^{-st}$$

$$= s \mathcal{L}(f)(s) - f(0)$$

f	$\mathcal{L}(f)$
e^{at}	$\frac{1}{(s-a)}$
t^N	$\frac{N!}{s^{N+1}}$
$f'(t)$	$s \mathcal{L}(f)(s) - f(0)$
$f''(t)$	(CHECK!) $s^2 \mathcal{L}(f)(s) - s f(0) - f'(0)$

"BABY" EXAMPLE:

$$y'(t) = y(t) - e^{2t} \qquad y(0) = 3$$

"DO $\mathcal{L}$":

$$\mathcal{L}(y') = \mathcal{L}(y - e^{2t})$$
$$\overset{*}{=} \mathcal{L}(y) - \mathcal{L}(e^{2t})$$

(* LINEAR!)
$$= \mathcal{L}(y) + \frac{-1}{s-2}$$

$$s\mathcal{L}(y) - y(0) = \mathcal{L}(y) + \frac{-1}{s-2}$$
$$(s-1)\mathcal{L}(y) = 3 + \frac{-1}{(s-2)} = \frac{3s-7}{(s-2)}$$
$$\mathcal{L}(y) = \frac{3s-7}{(s-1)(s-2)}$$

PARTIAL FRACTIONS!

$$\left\{ \frac{3s-7}{(s-1)(s-2)} \overset{PFG}{=} \frac{A}{(s-1)} + \frac{B}{(s-2)} \right.$$
$$\overset{!}{=} \frac{A(s-2) + B(s-1)}{(s-1)(s-2)}$$

SO $3s - 7 = A(s-2) + B(s-1)$

TRICK! PLUG IN MAGIC NUMBERS - $s=2$, $s=1$

$s=1$ $\quad -4 = A(-1) \quad A = 4$

$s=2$ $\quad -1 = B(1) \quad B = -1$

$$\left. \frac{3s-7}{(s-1)(s-2)} = \frac{4}{(s-1)} + \frac{-1}{(s-2)} \right\}$$

"WE LEAVE IT TO THE READER TO SHOW THAT"

$$\frac{3s-7}{(s-1)(s-2)} = \frac{4}{(s-1)} - \frac{1}{(s-2)}$$

$$\| \qquad\qquad \|$$

$$\mathcal{L}(y) = 4\,\mathcal{L}(e^{t}) - \mathcal{L}(e^{2t})$$

$$y(t) = 4e^{t} - e^{2t} \qquad \| \quad \mathcal{L}(4e^{t} - e^{2t})$$

CAN DO MORE COMPLICATED EXAMPLES, MORE COMPLICATED PARTIAL FRACTION ALGEBRA:

EX $\quad y''(t) + 3y'(t) + 2y(t) = e^{4t} \qquad y(0) = 1, \quad y'(0) = 2$

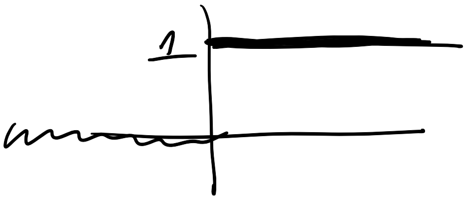
IDEAS ARE THE SAME.

NOTICE: $\quad \mathcal{L}(f)(s) = \int_0^\infty f(t)\, e^{-st}\, dt$ ONLY

USES VALUES OF $f(t)$ FOR $t \geq 0$

DON'T CARE (ASSUME 0).

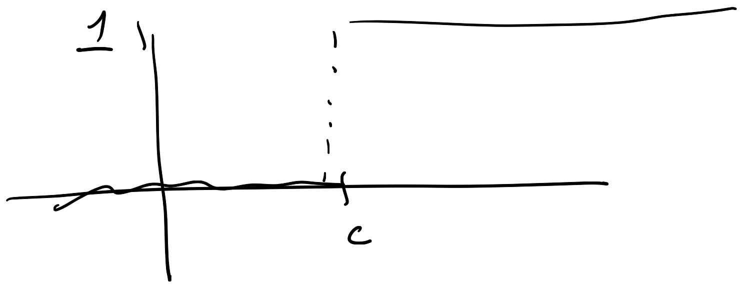
L5



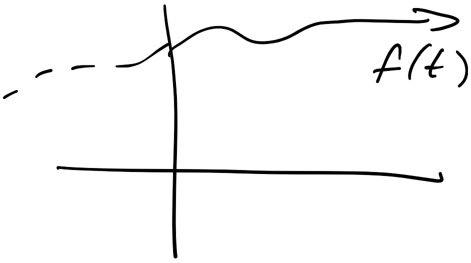
$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

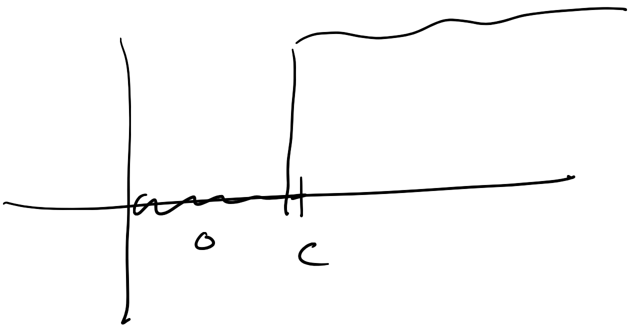
HEAVISIDE STEP FUNCTION.

$H(t-c)$ RIGHT SHIFT:



SHIFT TRUNCATE FORMULA: $H(t-c) \cdot f(t-c)$





$$\boxed{\begin{aligned} g(t) &= H(t-c)\, f(t-c) \\ \mathcal{L}(g)(s) &= \mathcal{L}(f)(s)\, e^{-cs} \end{aligned}}$$

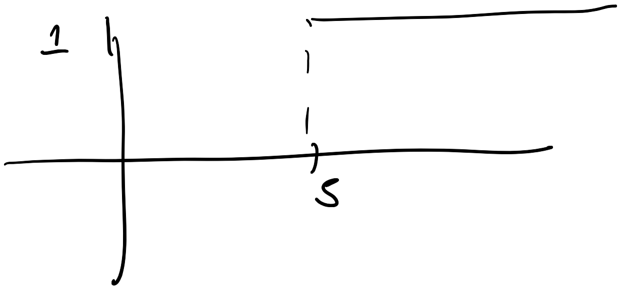
$$\begin{aligned} \mathcal{L}(g) &= \int_0^\infty H(t-c)\, f(t-c)\, e^{-st}\, dt. \\ &= \int_c^\infty f(t-c)\, e^{-st}\, dt \qquad u = t-c,\quad t = u+c \\ &= \int_0^\infty f(u)\, e^{-s(u+c)}\, du \\ &= e^{-sc} \int_0^\infty f(u)\, e^{-su}\, du \end{aligned}$$

$$= e^{-sc}\mathcal{L}(f)(s)$$

LAPLACE TRANSFORMS ARE GOOD FOR DEALING WITH DISCONTINUITIES

EX: $y'(t) = y(t) + H(t-c)$ $\quad y(0)=1$

$H(t-5)$



$H_c(t) = H(t-c)$

$$\mathcal{L}(y') = \mathcal{L}(y) + \mathcal{L}(H(t-c))$$

$$\mathcal{L}(H_c) = \int_0^\infty H(t-c)\,e^{-st}\,dt$$

$u = t-c$

$$= \int_c^\infty H(t-c)\,e^{-st}\,dt = \int_0^\infty e^{-s(u+c)}\,du$$

$$= e^{-sc}\int_0^\infty e^{-su}\,du$$

$$= e^{-sc}\int_0^\infty u^0 e^{-su}\,du$$

$$= e^{-sc}\frac{(0)!}{s^{0+1}} = \frac{e^{-sc}}{s}$$

$$\mathcal{L}(H_c) = \frac{e^{-sc}}{s}$$

$$s\mathcal{L}(y) - y(0) = \mathcal{L}(y) + \frac{e^{-sc}}{s}$$

$$(s-1)\mathcal{L}(y) = 1 + \frac{e^{-sc}}{s}$$

$$\mathcal{L}(y) = \frac{1}{(s-1)} + \frac{e^{-sc}}{s(s-1)}$$

$$= \frac{1}{(s-1)} + e^{-sc}\frac{1}{s(s-1)}$$

$$\left\{ \begin{array}{l} \text{WLITTRTOST :} \\ \frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s} \end{array} \right\}$$

$$\mathcal{L}(y) = \underbrace{\frac{1}{(s-1)}}_{1} + \underbrace{e^{-sc}\left(\frac{1}{s-1} - \frac{1}{s}\right)}_{2}$$

$$\mathcal{L}(e^t) = \frac{1}{(s-1)}$$

$$\mathcal{L}(1) = \frac{1}{s}$$

$$\mathcal{L}(\underbrace{e^t - 1}_{f(t)}) = \left(\frac{1}{(s-1)} - \frac{1}{s}\right)$$

$$\mathcal{L}(H_c(t)f(t-c)) = e^{-sc}\left(\frac{1}{(s-1)} - \frac{1}{s}\right)$$

$$(e^{(t-c)} - 1)H(t-c)$$

$$y = e^t + (e^{(t-c)} - 1)H(t-c)$$

$$y(t) = \begin{cases} e^t & 0 \le t \le c \\ e^t + (e^{(t-c)} - 1) & t \ge c \end{cases}$$

$$= e^t\left(1 + e^{-c}\right) - 1$$

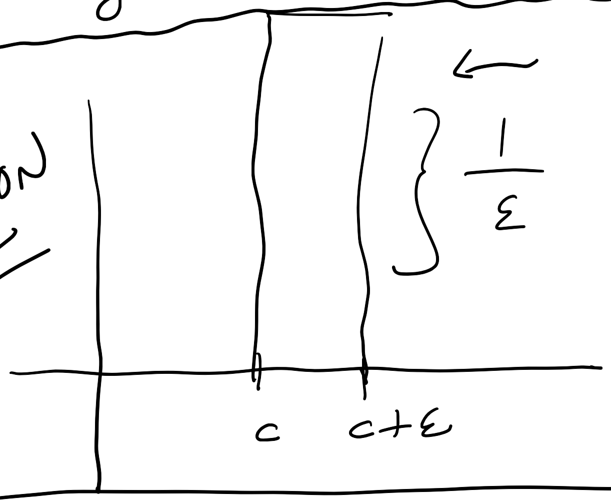
$$y' = y \qquad t \le c$$

$$y' = y + 1 \qquad t \ge c$$

$$y' = y + H_c(t)$$

EX!!

2) QUESTION



$$g(t) = g(t,\varepsilon) \qquad \frac{1}{\varepsilon}$$

$$= \frac{1}{\varepsilon}\left[H_c(t) - H_{c+\varepsilon}(t)\right]$$

$$\boxed{y' = y + g(t) \qquad y(0) = 0}$$

$$s\mathcal{L}(y) - \cancel{y(0)} = \mathcal{L}(y) + \frac{1}{\varepsilon}\left[\frac{e^{-cs}}{s} - \frac{e^{-(c+\varepsilon)(s)}}{s}\right]$$

$$(s-1)\mathcal{L}(y) = \frac{1}{\varepsilon}\left[\frac{e^{-cs}}{s} - \frac{e^{-(c+\varepsilon)(s)}}{s}\right]$$

$$\mathcal{L}(y) = \frac{1}{\varepsilon} e^{-cs} \cdot \frac{1}{s(s-1)} - \frac{1}{\varepsilon} e^{-(c+\varepsilon)s} \frac{1}{s(s-1)}$$

$$\frac{1}{s(s-1)} = \frac{1}{(s-1)} - \frac{1}{s}$$

$$\downarrow \mathcal{L}^{-1}$$

$$e^t - 1$$

$\mathcal{L}^{-1}$

INVERSE LAPLACE TRANSFORM

NO FORMULA!

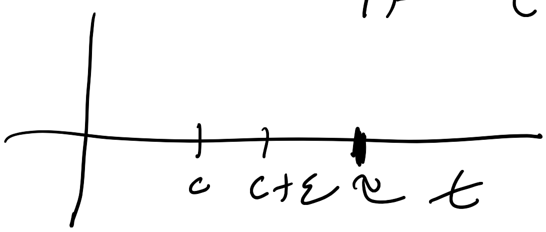
$$e^{-cs}\frac{1}{s(s-1)} \xrightarrow{\mathcal{L}^{-1}} H(t-c)\left[e^{(t-c)}-1\right]$$

$$e^{-(c+\varepsilon)s}\frac{1}{s(s-1)} \xrightarrow{\mathcal{L}^{-1}} H\big(t-(c+\varepsilon)\big)\left[e^{+t-(c+\varepsilon)}-1\right]$$

$$\frac{1}{\varepsilon}\left[\frac{e^{-cs}}{s(s-1)}-\frac{e^{-(c+\varepsilon)s}}{s(s-1)}\right] \to$$

$$\frac{1}{\varepsilon}\left[H(t-c)\left(e^{(t-c)}-1\right)-H\big(t-(c+\varepsilon)\big)\left(e^{t-(c+\varepsilon)}-1\right)\right.$$

IF $t > c+\varepsilon$



$$\frac{1}{\varepsilon}\left[\left(e^{(t-c)}-\cancel{1}\right)-\left(e^{t-(c+\varepsilon)}-\cancel{1}\right)\right]$$

$$= e^{t}\left[\frac{e^{-c}-e^{-(c+\varepsilon)}}{\varepsilon}\right]$$

$$= e^{t-c}\left[\frac{1-e^{-\varepsilon}}{\varepsilon}\right]$$

LOOK AT LIMIT AS $\varepsilon \to 0$

$$= e^{(t-c)}\lim_{\varepsilon\to 0}\left[\frac{1-e^{-\varepsilon}}{\varepsilon}\right]$$

$$\overset{L'H!}{=} e^{(t-c)}\lim_{\varepsilon\to 0}\left[\frac{e^{-\varepsilon}}{1}\right]$$

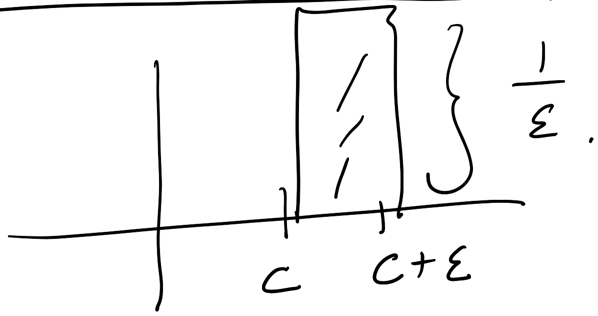
$= e^{(t-c)}$ FOR $t > c+\varepsilon$

BUT $\varepsilon \to 0$ $\quad e^{(t-c)}$ FOR $t \geq c$.

$\quad 0 \quad t < c$.

ANSWER $\boxed{H(t-c)\, e^{(t-c)}}$

$g(t,\varepsilon)$



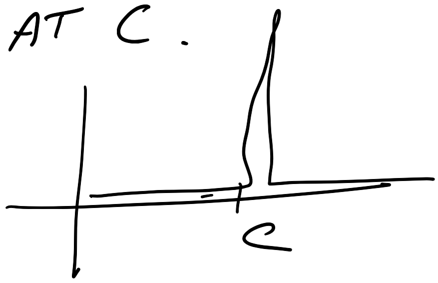
$$y' = y + g \qquad \{ y(0) = 0 \}$$

$$s\mathcal{L}(y) - y(0) = \mathcal{L}(y) + \mathcal{L}(g)$$

$$\mathcal{L}(g) = \int_0^\infty e^{-st} \frac{[H(t-(c+\varepsilon)) - H(t-c)]}{\varepsilon}\, dt$$

FACT: $\displaystyle \lim_{\varepsilon \to 0} \int_a^b f(t)\, g(t,\varepsilon)\, dt = f(c)$.

DIRAC FAMILY: $g(t,\varepsilon)$. CENTERED AT c.



(I) $g(t,\varepsilon) \geq 0$ (∞)

(II) $\displaystyle \int_{-\infty}^{\infty} g(t,\varepsilon)\, dt = 1$

(III) $\displaystyle \lim_{\varepsilon \to 0} g(t,\varepsilon) = 0$ IF $t \neq c$

1 IF g IS A DIRAC FAMILY, THEN

$$\lim_{\varepsilon \to 0} \int_a^b g(t,\varepsilon)\, f(t)\, dt = f(c)$$

"DIRAC δ-FUNCTION" AT c:

$$\delta_c(t) \begin{cases} 0 & t \neq c \\ \infty & t = c \end{cases}$$

$$\int_a^b \delta_c(t)\, dt = 1;$$

AND $\displaystyle\int_a^b f(t)\, \delta_c(t)\, dt = f(c)$

(i) $f(t) = \delta_c(t)$

$$\mathcal{L}(f) = ? = \int_0^\infty e^{-st} \delta_c(t)\, dt$$

$$= e^{-sc}.$$

(NOT $y' = y + g$!)

BEGIN $y' = y + \delta_c \qquad y(0) = 0.$

$$s\mathcal{L}(y) = \mathcal{L}(y) + e^{-sc}$$

$$(s-1)\mathcal{L}(y) = e^{-sc}$$

$$\mathcal{L}(y) = e^{-sc}\left(\frac{1}{(s-1)}\right)$$

$$\frac{1}{(s-1)} \xrightarrow{\mathcal{L}^{-1}} e^{t}$$

$$\frac{e^{-sc}}{(s-1)} \xrightarrow{\mathcal{L}^{-1}} H(t-c)\, e^{(t-c)}$$

$$= \begin{cases} 0 & t < c \\ e^{(t-c)} & t \geq c \end{cases}$$

SAME ANSWER AS (A)

END!

FORMAL CALCULATIONS WITH δ-FUNCTION CAN BE MADE LEGITIMATE, BUT HARDER.
