HMWK 3: DUE Feb 10, PDF file

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1. Solve the system of differential equations:

$$\mathbf{u}'(t) = A\,\mathbf{u}$$

where

$$\mathbf{u} = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} \frac{593}{25} & \frac{24}{25} & -\frac{144}{5} \\ \frac{24}{25} & \frac{607}{25} & \frac{108}{5} \\ -\frac{144}{5} & \frac{108}{5} & 2 \end{bmatrix}$$

with initial condition

$$\mathbf{u}(\mathbf{0}) = \begin{bmatrix} 66 \\ 13 \\ -65 \end{bmatrix}$$

I am going to save you an enormous amount of work. First of all, I will tell you that the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 25, \lambda_2 = -25, \lambda_3 = 50$; secondly, I will tell you that the eigenvectors are the COLUMNS of the matrix M

$$M = \begin{bmatrix} 3/5 & \frac{12}{25} & \frac{16}{25} \\ 4/5 & -\frac{9}{25} & -\frac{12}{25} \\ 0 & 4/5 & -3/5 \end{bmatrix}$$

IN THAT ORDER.

- Show the the column vectors are ORTHONORMAL; each one "dot producted" with itself equals 1, and each column vector "dot producted" with another column vector equals zero.
- Use the facts that you know the eigenvalues and eigenvectors of $\mathbf{A}$ to write down the general solution for the system of differential equations.
- In your general solution, you will have three arbitrary constants c_1, c_2, c_3. You can use the orthonormal property of the eigenvectors to find c_1, c_2, c_3 with very little work, much easier than solving three equations with three unknowns, row echelon form, etc.

2. In class I solved the wave equation on the real line, using D'Alembert's formula, for a specific example: Initial displacement $f(x) = 0$, and initial velocity $g(x)$ piecewise defined:

$$g(x) = 0, x < 0; \quad g(x) = 1, 0 < x < 1; \quad g(x) = 0, x > 1$$

The formula for the solution $u(x,t)$ was also piecewise defined. I want you to solve the problem with a slightly more complicated initial condition: $u_{tt} = u_{xx}$, with $f(x) = 0$ and

$$g(x) = 0, x < 0; \quad g(x) = x, 0 < x < 1;, \quad g(x) = 0,\ x > 1$$

This is not a very realistic initial condition physically, because it has a discontinuity in the initial velocity $g(x)$. However, more realistic initial conditions require more complicated calculations.

3. In class, I showed two examples of how to use the power series method to solve differential equations. For the two examples, it is MUCH simpler to use other methods you have learned, but these are just test case. The method can be used to solve Bessel's equation, for which there is no easy solution.

   In this problem, I want you to practice the power series on two problems which are nearly the same as the ones in class. Again, this is just to get you to practice with the technique

   - Using power series technique solve the differential equation $y'(x) = \frac{1}{2x+1}$, $y(0) = 0$. Also compute the solution by simple integration!
   - Using the power series technique, solve the differential equation $y'(x) = y(x) + 1$, $y(0) = 0$. Also solve it using the other techniques you know.

4. The Bessel differential equation is $x^2 {J_m}'(x) + x{J_m}'(x) + (x^2 - m^2)J_m(x) = 0$. The behavior of J for large values of x is for sure not obvious from this form. Liouville came up with a clever substitution that is very useful in this context. Define $L_m(x) = \sqrt{x}J_m(x)$, or $J_m(x) = x^{-1/2}L_m(x)$. Since J satisfies a differential equation, and L is closely related to J, maybe it is not surprising that L satisfies a differential equation, too. The calculation to find this differential equation is in the notes, but somewhat sloppily written. I want you to write it up more carefully, to obtain the differential equation for L. For large x, the differential equation for L is pretty close to $L''(x) = -L(x)$, so $L(x)$ is close to a linear combination of $\cos(x)$ and $\sin(x)$. This tells you what J looks like for x large, and explains some of the behavior of the graphs of J that I showed in class.