STANDARD PROBLEM:

$u_t = u_{xx}$; $u(0,t) = u(1,t) = 0$

$u(x,0) = f(x)$

SOL'N:

$$f(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)$$

$$c_k = 2\int_0^1 f(x)\sin(k\pi x);$$

$$u(x,t) = \sum_{k=1}^{\infty} \underbrace{c_k e^{-k^2\pi^2 t}} \sin(k\pi x). \quad !!$$

DEFINE $d_k(t) = e^{-k^2\pi^2 t} c_k$

$d_k(0) = c_k$ AND

$$u(x,t) = \sum_{k=1}^{\infty} d_k(t)\sin(k\pi x).$$

ANOTHER WAY TO THING ABOUT OUR SOL'N TECHNIQUE:

LOOK AT STANDARD PROBLEM.

"ASSOCIATE" TO $u(0,t) = u(1,t) = 0$

BNDRY CONDITIONS FOR ODE:

$X(0) = X(1) = 0$

SOLVE (WORK!) $X''(x) = \lambda X(x)$ {"EIGENVALUE PROBLEM"} [3]

$$X_N(x) = \sin(N\pi x);$$
$$\lambda_N = -N^2\pi^2.$$

GUESS THAT: $u(x,t) = \sum_{k=1}^{\infty} d_k(t) \sin(k\pi x);$

$$u(x,0) = f(x) = \sum_{k=1}^{\infty} d_k(0) \sin(k\pi x)$$

SO $d_k(0) = c_k = 2\int_0^1 f(x)\sin(k\pi x);$

$$u_t = u_{xx}$$

$$\sum_{k=1}^{\infty} d_k'(t) \sin(k\pi x) = \sum_{k=1}^{\infty} d_k(t)(-\pi^2 k^2)\sin(k\pi x)$$

OR $\sum_{k=1}^{\infty} \left[d_k'(t) + \pi^2 k^2 d_k(t)\right] \sin(k\pi x) = 0$

(PLAUSIBLE THAT) THIS IMPLIES.

$$d_k'(t) = -\pi^2 k^2 d_k(t) \quad d_k(0) = c_k$$

SIMPLE ODE!!!

$$d_k(t) = A e^{-\pi^2 k^2 t}$$

$$d_k(0) = A = c_k \qquad d_k(t) = c_k e^{-\pi^2 k^2 t}$$

so $u(x,t) = \sum_{k=1}^{\infty} c_k e^{-\pi^2 k^2 t} \sin(k\pi x)$. L3

(DOESN'T TELL US A LOT MORE.

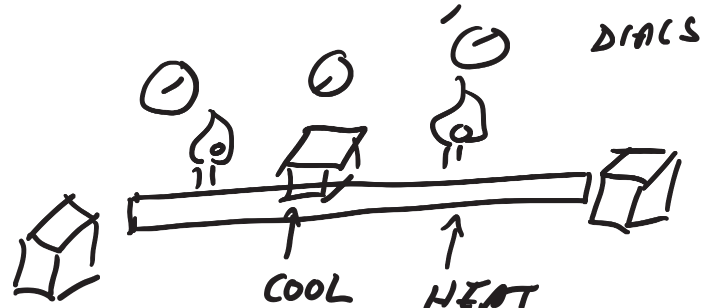
BUT! WILL HELP SOLVE NEW

PROBLEM:

$$u_t = u_{xx} + g(x,t)$$

$u(0,t) = u(1,t) = 0$.

$$u(x,0) = f(x)$$



INHOMOGENEOUS HEAT EQN:

HOW TO SOLVE:

$\left\{ \text{EVEN } g(x,t) = g(x) \text{ IS NEW} \right\}$

FIRST LOOK AT (RELATED) ODE PROBLEM

!!! $$y'(t) = a\,y(t) + h(t)$$

$y(0) = y_0$

$$y' - ay = h(t) \quad \text{I.F.} = e^{-at}$$

$$e^{-at} y' - e^{-at} \cdot a \cdot y = e^{-at} h(t)$$

$$\left(e^{-at} y\right)' = e^{-at} h(t)$$

$$e^{-at}y = C + \int_0^t e^{-au}h(u)\,du$$

$$y(t) = Ce^{at} + e^{at}\int_0^t e^{-au}h(u)\,du$$

$$y(0) = C = y_0$$

$$y(t) = y_0 \cdot e^{at} + e^{at}\int_0^t e^{-au}h(u)\,du$$

$$\rightarrow \quad y' - ay - h = 0$$

WILL USE THIS:

$$\underline{\underline{u_t}} = u_{xx} + g(x,t) \qquad u(x,0) = f(x)$$

$$u(0,t) = u(1,t) = 0;$$

(1) USE B.C. FOR u

TO DETERMINE BC FOR

$$X''(x) = \lambda X(x) \quad \left\{ X(0) = X(1) = 0 \right\}$$

$$X_N(x) = \sin(N\pi x), \quad \lambda_N = -N^2\pi^2.$$

<u>GUESS</u>: $u(x,t) = \sum_{k=1}^{\infty} d_k(t)\sin(k\pi x);$

$$u(x,0) = f(x) = \sum_{k=1}^{\infty} d_k(0)\sin(k\pi x);$$

$$d_k(0) = 2\int_0^1 \sin(k\pi x)f(x)\,dx; \quad = C_k$$

FOURIER COEFF OF $f(x)$

$$g(x,t) = \text{GIVEN FUNCTION}$$

$$= \sum_{k=1}^{\infty} e_k(t)\sin(k\pi x) \quad !!$$

$$u_t = u_{xx} + g(x,t).$$

USE FOURIER SERIES.

$$\sum_{k=1}^{\infty} d_k'(t) \sin(k\pi x) = \sum_{k=1}^{\infty} -k^2\pi^2 d_k(t) \sin(k\pi x) + \sum_{k=1}^{\infty} e_k(t) \sin(k\pi x)$$

SO
$$\sum_{k=1}^{\infty} \left[d_k'(t) + k^2\pi^2 d_k(t) - e_k(t) \right] \sin(k\pi x) = 0.$$

!!!

SO
$$d_k'(t) + k^2\pi^2 d_k(t) - e_k(t) = 0$$

$k = 1, 2, \ldots$

$d_k(0) = c_k$

INFINITE LIST OF (DECOUPLED) ODE'S

$$y' - ay - h = 0$$

$$y(t) = y_0 e^{at} + e^{at} \int_0^t e^{-au} h(u)\,du$$

SO... $\{ a = -k^2\pi^2 \}$

$$d_k(t) = c_k e^{-\pi^2 k^2 t} + e^{-\pi^2 k^2 t} \int_0^t e^{\pi^2 k^2 u} e_k(u)\,du.$$

$$u(x,t) = \sum_{k=1}^{\infty} d_k(t) \sin(n\pi x)$$

IF ALL $e_k = 0$, THEN SAME SOL'N AS BEFORE.

EX: $u_t = u_{xx} + t\sin(5\pi x)$  16

$f(x) = 1\cdot\sin(2\pi x) - 2\sin(4\pi x)$

GUESS $u(x,t) = \sum_{k=1}^{\infty} d_k(t)\sin(k\pi x)$

$$d_k(t) = c_k e^{-\pi^2k^2t} + e^{-\pi^2k^2t}\int_0^t e^{\pi^2k^2u}e_k(u)\,du$$

$c_2 = 1,\ c_4 = -2,\ e_5(t) = t$

ALL OTHERS ARE ZERO

$$d_2(t) = 1\cdot e^{-\pi^2 4t}$$

$$d_4(t) = -2\cdot e^{-\pi^2 16t}$$

$$d_5(t) = e^{-25\pi^2t}\int_0^t e^{25\pi^2u}\,u\,du.$$

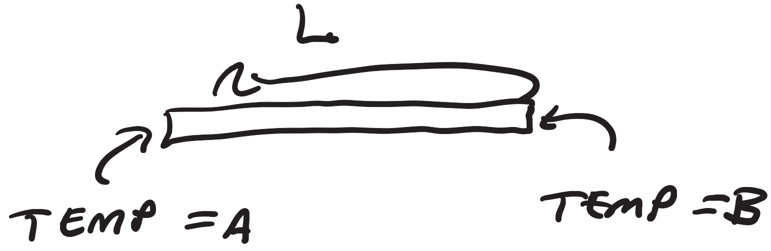
$$= e^{-25\pi^2t}\left[\frac{t^2e^{25\pi^2t}}{25\pi^2} - \frac{te^{25\pi^2t}}{625\pi^4} + \frac{t}{625\pi^4}\right]$$

$$= \frac{t^2}{25\pi^2} - \frac{t}{625\pi^4} + \frac{te^{-25\pi^2t}}{625\pi^4}$$

$$u(x,t) = e^{-\pi^2 4t}\sin(2\pi x) - 2e^{-16\pi^2t}\sin(4\pi x)$$

$$+\left[\frac{t^2}{25\pi^2} - \frac{t}{625\pi^4} + \frac{te^{-25\pi^2t}}{625\pi^4}\right]\sin(5\pi x)$$

TIME INDEPENDENT SOL'NS



$u_t = u_{xx}$

$u(0,t) = A$

$u(L,t) = B$

INTUITIVELY, TEMP WILL "STABILIZE" AS $t \to \infty$.

SO, LOOK FOR TIME INDEPENDENT SOL'NS $u_t = 0 = u_{xx}$

!!

SO u ONLY DEPENDS ON x!

$$u_{xx} = 0 \longrightarrow u = Ax + B;$$

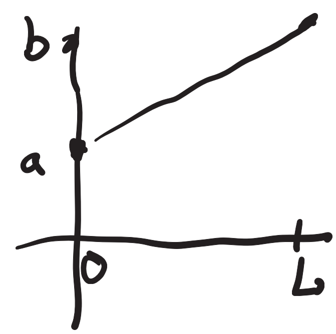
$$u(0) = A \cdot 0 + B = a;$$

$$u(L) = A(L) + B = A(L) + a$$

$$= b$$

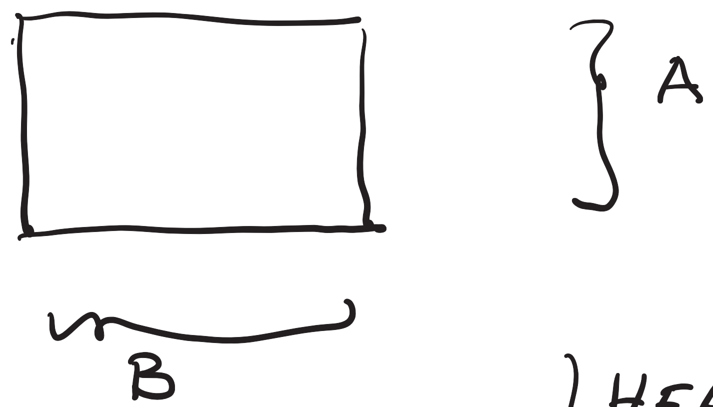
$$A = \frac{b-a}{L}$$

$$u(x) = \boxed{\frac{b-a}{L}x + a}$$



LINEAR TEMPERATURE DISTRIB.

"PREVIEW OF COMING EVENTS"



$$u_t = u_{xx} + u_{yy}$$

HEAT CONDUCTION IN PLATE.

COMPLICATED! WE LOOK FOR TIME INDEPENDENT SOL'NS. $u_t \equiv 0$; $u = u(x,y)$.

(L2) $u_{xx} + u_{yy} = 0$ LAPLACE'S EQN.

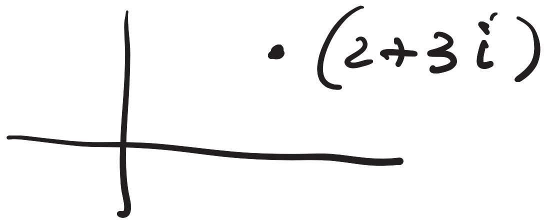
PEOPLE ALSO LOOK AT:

(L3) $u_{xx} + u_{yy} + u_{zz} = 0$

LAPLACE IN 3-D

NOT ONLY IS (L3) MORE COMPLICATED THAN (L2), BUT ALSO THERE ARE TWO WAYS TO SOLVE (L2) {ONE FOR (L3)}

SPECIAL WAY FOR (L2)



USES COMPLEX NUMBERS.

WE TYPICALLY USE x FOR "RANDOM" REAL #; WE USE $x+iy$ FOR "RANDOM" COMPLEX #; $z = x+iy$ ABBREVIATION

TAKE ANY "REGULAR" (!!) FUNCTION, $f(x)$, REPLACE x BY $z = x+iy$.

<u>COMPLEX VALUED FUNCTION</u>

<u>OF A COMPLEX VARIABLE</u>

$\left\{\begin{array}{l} \text{!! : } \underline{\text{REGULAR}} \text{ FUNCTION MEANS} \\ \text{IT HAS POWER SERIES EXPANSION} \end{array}\right\}$

EX: $f(x) = x^2$

$f(z) = z^2 = (x+iy)^2$

$= x^2 + i(2xy) - y^2 = (\underline{x^2-y^2}) + i(\underline{\underline{2xy}})$

REAL PART IMAGINARY PART.

$$f(x) = \frac{1}{x}$$

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy}\underbrace{\frac{x-iy}{x-iy}}_{\text{TRICK!}}$$

$$= \frac{x-iy}{\underset{!!}{x^2+y^2}} = \underbrace{\frac{x}{x^2+y^2}}_{\text{REAL PART}} + i\underbrace{\left(\frac{y}{x^2+y^2}\right)}_{\text{IMAG PART.}}$$

{ ANOTHER WORD FOR REGULAR IS ANALYTIC }

FACT: IF $f(x)$ IS REGULAR,

THEN $f(z) = u(x,y) + i\,v(x,y)$;

THEN $u_{xx} + u_{yy} = 0$

$$v_{xx} + v_{yy} = 0.$$

EX: $f(z) = z^2 = \underbrace{(x^2 - y^2)}_{u} + i\underbrace{(2xy)}_{v}$

$u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$;
OBVIOUSLY.

EX: $f(z) = \frac{1}{z} = \underbrace{\frac{x}{x^2+y^2}}_{u} + i\underbrace{\frac{y}{x^2+y^2}}_{v}$

$u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$
(CHECK!)

EX: $f(x) = e^x$

$f(z) = e^z = e^{x+iy}$

$= e^x e^{iy} = e^x(\cos y + i \sin y)$

$= \underbrace{e^x \cos y}_{u} + i\underbrace{e^x \sin(y)}_{v}$

CHECK: $u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$

¿ PRODUCES MANY SOL'NS; BUT DOES IT PRODUCE ALL SOL'NS ?

YES, BUT IT TAKES WORK TO CONCLUDE THIS.

$+ \; V_2 e^{\lambda_2 t}$