Phys 327 HW4

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① Griffiths 4.37

a) Apply S_- to $|1\,0\rangle$, and confirm that you get $\sqrt{2}\hbar|1\,-1\rangle$

$$\underset{s\,m}{|1\,0\rangle} = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \; ; \quad |1\,-1\rangle = |\downarrow\downarrow\rangle$$

$$S_-|1\,0\rangle = \frac{1}{\sqrt{2}}\left(S_-^{(1)} + S_-^{(2)}\right)\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$
$$= \frac{1}{\sqrt{2}}\left[S_-^{(1)}|\uparrow\downarrow\rangle + S_-^{(1)}|\downarrow\uparrow\rangle + S_-^{(2)}|\uparrow\downarrow\rangle + S_-^{(2)}|\downarrow\uparrow\rangle\right]$$

Using the property $S_-\chi_- = 0$ and $S_-\chi_+ = \hbar\chi_-$

$$S_-|1\,0\rangle = \frac{1}{\sqrt{2}}\Big[\left(S_-^{(1)}|\uparrow\rangle\right)|\downarrow\rangle + |\uparrow\rangle\left(S_-^{(2)}|\downarrow\rangle\right)$$
$$+ \left(S_-^{(1)}|\downarrow\rangle\right)|\uparrow\rangle + |\downarrow\rangle\left(S_-^{(2)}|\uparrow\rangle\right)\Big]$$

$$= \frac{\hbar}{\sqrt{2}}\left[|\downarrow\downarrow\rangle + 0 + 0 + |\downarrow\downarrow\rangle\right]$$

$$\rightarrow \quad S_-|1\,0\rangle = \sqrt{2}\hbar\,|\downarrow\downarrow\rangle = \sqrt{2}\,|1\,-1\rangle \checkmark$$

b) Apply S_+ to the state $|0\,0\rangle$ and confirm you get 0

$$|0\,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \; ; \quad S_+ = S_+^{(1)} + S_+^{(2)}$$

$$S_+|0\,0\rangle = \frac{1}{\sqrt{2}}\left(S_+^{(1)} + S_+^{(2)}\right)\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$
$$= \frac{1}{\sqrt{2}}\left[(S_+|\uparrow\rangle)|\downarrow\rangle + |\uparrow\rangle(S_+|\downarrow\rangle) - (S_+|\downarrow\rangle)|\uparrow\rangle - |\downarrow\rangle(S_+|\uparrow\rangle)\right]$$
$$= \frac{1}{\sqrt{2}}\left[0 + |\uparrow\uparrow\rangle - |\uparrow\uparrow\rangle - 0\right]$$
$$= 0 \checkmark$$

c) Show that $|11\rangle$ and $|1-1\rangle$ are eigenstates of S^2 with the appropriate eigenvalue.

i) $S^2|11\rangle \stackrel{?}{=} 2\hbar^2|11\rangle$; $|11\rangle = |\uparrow\uparrow\rangle$

We have $S^2 = S_1^2 + S_2^2 + 2\vec{S}_1 \cdot \vec{S}_2$

It follows that

$$\vec{S}_1 \cdot \vec{S}_2 |\uparrow\uparrow\rangle = (S_{x_1}|\uparrow\rangle)(S_{x_2}|\uparrow\rangle) + (S_{y_1}|\uparrow\rangle)(S_{y_2}|\uparrow\rangle) + (S_{z_1}|\uparrow\rangle)(S_{z_2}|\uparrow\rangle)$$
$$= (\tfrac{\hbar}{2}|\downarrow\rangle)(\tfrac{\hbar}{2}|\downarrow\rangle) + (-\tfrac{i\hbar}{2}|\downarrow\rangle)(\tfrac{i\hbar}{2}|\downarrow\rangle) + (\tfrac{\hbar}{2}|\uparrow\rangle)(\tfrac{\hbar}{2}|\uparrow\rangle)$$
$$= \frac{\hbar^2}{4}|\uparrow\uparrow\rangle + \cancel{\frac{\hbar^2}{4}|\downarrow\downarrow\rangle} - \cancel{\frac{\hbar^2}{4}|\downarrow\downarrow\rangle}$$
$$= \frac{\hbar^2}{4}|\uparrow\uparrow\rangle$$

We now have the equation [Since eigenvalues of $S_1^2, S_2^2 = \frac{3\hbar^2}{4}$

$$S^2|11\rangle = \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\frac{\hbar^2}{4}\right)|11\rangle$$
$$= 2\hbar^2|11\rangle \checkmark \implies 2\hbar^2 \text{ eigenvalue for } S^2$$

in eigenstate $|11\rangle$

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ii) $S^2|1-1\rangle \stackrel{?}{=} 2\hbar^2|1-1\rangle$

This is the same process as before:

$$\vec{S}_1 \cdot \vec{S}_2|\downarrow\downarrow\rangle = \frac{\hbar^2}{4}|\downarrow\downarrow\rangle + \cancel{\frac{\hbar^2}{4}|\uparrow\uparrow\rangle} - \cancel{\frac{\hbar^2}{4}|\uparrow\uparrow\rangle} = \frac{\hbar^2}{4}|\downarrow\downarrow\rangle$$

And we have the same solution:

$$S^2|1-1\rangle = \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\frac{\hbar^2}{4}\right)|1-1\rangle = 2\hbar^2|1-1\rangle \checkmark$$

$\implies 2\hbar^2$ is also an eigenvalue for eigenstate $|1-1\rangle$.

② Griffiths 4.58

An electron is in the spin state:

$$\chi = A \begin{pmatrix} 1-2i \\ 2 \end{pmatrix}$$

a) Determine the normalization constant by normalizing χ

$$1 = A^2|\chi_1|^2 + A^2|\chi_2|^2 = A^2 (\underbrace{1^2+2^2+2^2}_{9})$$

$$1 = 9A^2 \longrightarrow \boxed{A = \frac{1}{3}}$$

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b) If you measured S_z of this electron, what values could you get, and what are their probabilities?

The possible values are $\pm\frac{\hbar}{2}$

In terms of the eigenspinors of S_z

$$\chi = \left(\tfrac{1}{3} - \tfrac{2i}{3}\right)\chi_+ + \tfrac{2}{3}\chi_-$$

$$\Rightarrow P\left(\tfrac{\hbar}{2}\right) = \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{2}{3}\right)^2 = \tfrac{1}{9} + \tfrac{4}{9} = \boxed{\tfrac{5}{9}}$$

$$P\left(-\tfrac{\hbar}{2}\right) = \left(\tfrac{2}{3}\right)^2 = \boxed{\tfrac{4}{9}}$$

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c) Same for S_x? Again, possible values are $\pm\frac{\hbar}{2}$

The generic eigenspinor $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$ can be expressed as a linear combination of the eigenspinors of S_x:

$$\chi = \left(\frac{a+b}{\sqrt{2}}\right)\chi_+^{(x)} + \left(\frac{a-b}{\sqrt{2}}\right)\chi_-^{(x)}$$

Thus our spinor can be expressed as

$$3\chi = \left(\frac{3-2i}{\sqrt{2}}\right)\chi_+^{(x)} + \left(\frac{-1-2i}{\sqrt{2}}\right)\chi_-^{(x)}$$

$$\Rightarrow P\left(+\tfrac{\hbar}{2}\right) = \frac{3^2+2^2}{18} = \boxed{\frac{13}{18}}, \quad P\left(-\tfrac{\hbar}{2}\right) = \frac{1^2+2^2}{18} = \boxed{\frac{5}{18}}$$

d) What about S_y?

We need to know the generic form of an eigenspinor $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$ in terms of the eigenspinors of S_y

First determine the characteristic equation:

$$\begin{vmatrix} -\lambda & -i\hbar/2 \\ i\hbar/2 & -\lambda \end{vmatrix} = 0 \longrightarrow \lambda^2 - \hbar^2/4 \longrightarrow \lambda = \pm \hbar/2$$

* Again, we can measure $\pm \hbar/2$ for S_y.

An eigenvector v will follow: $S_y V = \lambda v$

$$\frac{i\hbar}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$\longrightarrow -iv_2 = v_1$ and $iv_1 = v_2$

$\Rightarrow \chi_+^{(y)'} = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$ $\quad \sqrt{1^2+1^2} = \sqrt{2}$

The other eigenvector:

$$\frac{i\hbar}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$\longrightarrow -iv_2 = -v_1 \qquad iv_1 = -v_2$

$\Rightarrow v_1 = iv_2 \qquad \Rightarrow v_2 = -iv_1$

$$\chi_-^{(y)'} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \Rightarrow \chi_-^{(y)} = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$$

We now have the eigenspinors of S_y:

$$\chi_+^{(y)} = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} \quad \text{and} \quad \chi_-^{(y)} = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$$

The generic spinor $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$ can be represented as a linear combination of each of them:

$$\begin{pmatrix} a \\ b \end{pmatrix} = A \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} + B \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$$

$$a = \frac{1}{\sqrt{2}}(A+B) \qquad b = \frac{i}{\sqrt{2}}(A-B)$$

$$-ib = \frac{1}{\sqrt{2}}(A-B)$$

$$\rightarrow a-ib = \frac{1}{\sqrt{2}}(2A) = \sqrt{2}A \Rightarrow A = \frac{a-ib}{\sqrt{2}}$$

$$a+ib = \frac{1}{\sqrt{2}}(2B) = \sqrt{2}B \Rightarrow B = \frac{a+ib}{\sqrt{2}}$$

Thus $\chi = \left(\frac{a-ib}{\sqrt{2}}\right)\chi_+^{(y)} + \left(\frac{a+ib}{\sqrt{2}}\right)\chi_-^{(y)}$

Our spinor $\frac{1}{3}\begin{pmatrix} 1-2i \\ 2 \end{pmatrix}$ can be express as

$$3\chi = \left(\frac{1-4i}{\sqrt{2}}\right)\chi_+^{(y)} + \left(\frac{1}{\sqrt{2}}\right)\chi_-^{(y)}$$

$$\Rightarrow P\left(+\frac{\hbar}{2}\right) = \frac{1^2+4^2}{18} = \boxed{\frac{17}{18}}$$

$$P\left(-\frac{\hbar}{2}\right) = \frac{1^2}{18} = \boxed{\frac{1}{18}}$$

③ Griffiths 4.61

Find the matrix representing S_x for a particle of spin $s = 3/2$ (using the eigenstates of S_z as a basis). Solve the characteristic equation to determine the eigenvalues of S_x.

For $s = \frac{3}{2}$, possible values of m are $-\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}$

Thus there are four possible eigenstates $|\frac{3}{2}\,\frac{-3}{2}\rangle, |\frac{3}{2}\,\frac{-1}{2}\rangle, |\frac{3}{2}\,\frac{1}{2}\rangle, |\frac{3}{2}\,\frac{3}{2}\rangle$
and the generic spinor is then $\chi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a\chi_1 + b\chi_2 + c\chi_3 + d\chi_4$

where $|\frac{3}{2}\,\frac{+3}{2}\rangle : \chi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\qquad |\frac{3}{2}\,\frac{+1}{2}\rangle : \chi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$|\frac{3}{2}\,\frac{-1}{2}\rangle : \chi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\qquad |\frac{3}{2}\,\frac{-3}{2}\rangle : \chi_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

The spin operators will thus be 4x4 matrices

Using the equation $S_z|sm\rangle = \hbar m|sm\rangle$

$S_z|\frac{3}{2}\,\frac{3}{2}\rangle = \frac{3\hbar}{2}|\frac{3}{2}\,\frac{3}{2}\rangle$ $\qquad S_z|\frac{3}{2}\,\frac{1}{2}\rangle = \frac{\hbar}{2}|\frac{3}{2}\,\frac{1}{2}\rangle$

$S_z|\frac{3}{2}\,\frac{-1}{2}\rangle = -\frac{\hbar}{2}|\frac{3}{2}\,\frac{-1}{2}\rangle$ $\qquad S_z|\frac{3}{2}\,\frac{-3}{2}\rangle = \frac{-3\hbar}{2}|\frac{3}{2}\,\frac{-3}{2}\rangle$

It follows that

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Using the equation

$$S_\pm |\tfrac{3}{2} m\rangle = \hbar\sqrt{\underbrace{\tfrac{3}{2}(\tfrac{3}{2}+1)}_{15/4} - m(m\pm 1)}\ |s\,(m\pm 1)\rangle$$

$$S_+|\tfrac{3}{2}\,\tfrac{3}{2}\rangle = 0$$

$$S_+|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \hbar\sqrt{15/4 - \tfrac{1}{2}(\tfrac{3}{2})} = \hbar\sqrt{3}\,|\tfrac{3}{2}\,\tfrac{3}{2}\rangle$$

$$S_+|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle = \hbar\sqrt{15/4 + \tfrac{1}{2}(\tfrac{1}{2})} = 2\hbar\,|\tfrac{3}{2}\,\tfrac{1}{2}\rangle$$

$$S_+|\tfrac{3}{2}\,-\tfrac{3}{2}\rangle = \hbar\sqrt{15/4 + \tfrac{3}{2}(-\tfrac{1}{2})} = \hbar\sqrt{3}\,|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle$$

It then follows that

$$S_+ = \hbar\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S_-|\tfrac{3}{2}\,\tfrac{3}{2}\rangle = \hbar\sqrt{15/4 - \tfrac{3}{2}(\tfrac{1}{2})}\,|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \hbar\sqrt{3}\,|\tfrac{3}{2}\,\tfrac{1}{2}\rangle$$

$$S_-|\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \hbar\sqrt{15/4 - \tfrac{1}{2}(-\tfrac{1}{2})}\,|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle = 2\hbar\,|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle$$

$$S_-|\tfrac{3}{2}\,-\tfrac{1}{2}\rangle = \hbar\sqrt{15/4 + \tfrac{1}{2}(-\tfrac{3}{2})}\,|\tfrac{3}{2}\,-\tfrac{3}{2}\rangle = \hbar\sqrt{3}\,|\tfrac{3}{2}\,-\tfrac{3}{2}\rangle$$

$$S_-|\tfrac{3}{2}\,-\tfrac{3}{2}\rangle = 0$$

Therefore

$$S_- = \hbar\begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

We know that $S_x = \frac{1}{2}(S_+ + S_-)$

$$\Rightarrow \boxed{S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}}$$

The usual characteristic equation is $\det(S_x - \lambda I) = 0$

$$S_x - \lambda I = \frac{\hbar}{2}\begin{pmatrix} -\lambda & \sqrt{3} & 0 & 0 \\ \sqrt{3} & -\lambda & 2 & 0 \\ 0 & 2 & -\lambda & \sqrt{3} \\ 0 & 0 & \sqrt{3} & -\lambda \end{pmatrix}$$

Along first column:

$$|S_x - \lambda I| = -\lambda \underbrace{\begin{vmatrix} -\lambda & 2 & 0 \\ 2 & -\lambda & \sqrt{3} \\ 0 & \sqrt{3} & -\lambda \end{vmatrix}}_{D_1} - \sqrt{3} \underbrace{\begin{vmatrix} \sqrt{3} & 0 & 0 \\ 2 & -\lambda & \sqrt{3} \\ 0 & \sqrt{3} & -\lambda \end{vmatrix}}_{D_2}$$

$$D_1 = -\lambda(\lambda^2 - 3) - 2(-2\lambda) = -\lambda^3 + 7\lambda$$
$$D_2 = \sqrt{3}(\lambda^2 - 3) = \sqrt{3}\lambda^2 - \sqrt{3}3$$

$$\rightarrow |S_x - \lambda I| = \lambda^4 - 7\lambda^2 - 3\lambda^2 + 9 = \lambda^4 - 10\lambda^2 + 9 = 0$$

$$\Rightarrow (\lambda^2 - 9)(\lambda^2 - 1) = 0$$

Thus the eigenvalues of S_x are

$$\boxed{\lambda = \pm\frac{3\hbar}{2} \text{ and } \lambda = \pm\frac{\hbar}{2}}$$