$h(x,0) = f(x)$

$h_t(x,0) = g(x)$.

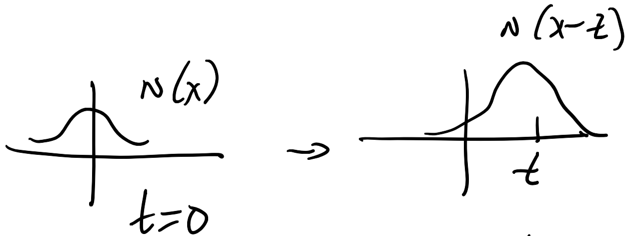
$h_{tt} - h_{xx} = 0$;

$u = x + t \qquad v = x - t$

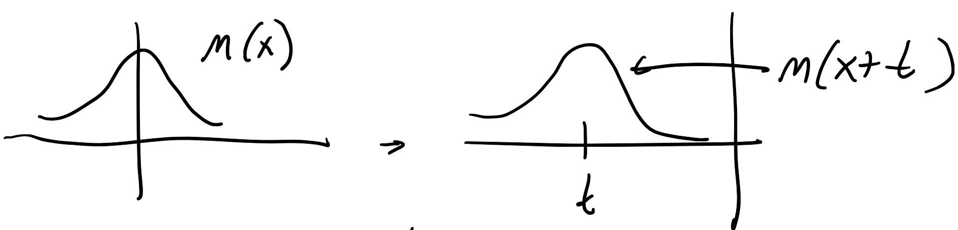
$h_{uv} = 0 \longrightarrow h(u,v) = M(u) + N(v)$

$= \boxed{M(x+t) + N(x-t)}$

M + N ARBITRARY



MOVES RIGHT.



MOVES LEFT.

$h = M(x+t) + N(x-t)$

{USE u AGAIN}

INITIAL CONDITIONS:

$u(x,0) = f(x) = M(x) + N(x)$

$u_t(x,t) = M'(x+t) - N'(x-t)$

$u_t(x,0) = g(x) = M'(x) - N'(x)$

$G(x) = \int g(x)dx = M(x) - N(x)$ ② ?

$2M(x) = G(x) + f(x)$

$2N(x) = f(x) - G(x)$

$$M(x) = \frac{G(x) + f(x)}{2}$$

$$N(x) = \frac{f(x) - G(x)}{2}$$

$$u(x,t) = M(x+t) + N(x-t)$$

$$= \frac{G(x+t) + f(x+t)}{2} + \frac{f(x-t) - G(x-t)}{2}$$

$$= \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\left[G(x+t) - G(x-t)\right]$$

$$= \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\left[\int_a^{x+t} g(u)\,du - \int_a^{x-t} g(u)\,du\right]$$

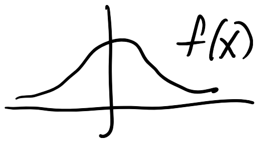
$$= \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\left[\int_a^{x-t} g\,du + \int_{x-t}^{x+t} g - \int_a^{x-t} g\,du\right]$$

$$= \boxed{\frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(u)\,du}$$

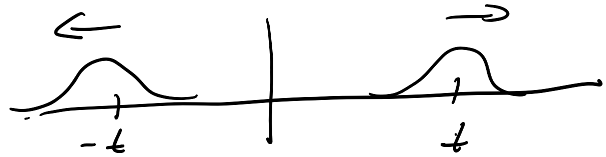
VS

$$\boxed{u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix\omega}\left[\cos(\omega t)\,\hat{f}(\omega) + \frac{\sin(\omega t)}{\omega}\,\hat{g}(\omega)\right] d\omega}$$

① $g \equiv 0$ $\quad u(x,t) = \dfrac{f(x+t) + f(x-t)}{2}$

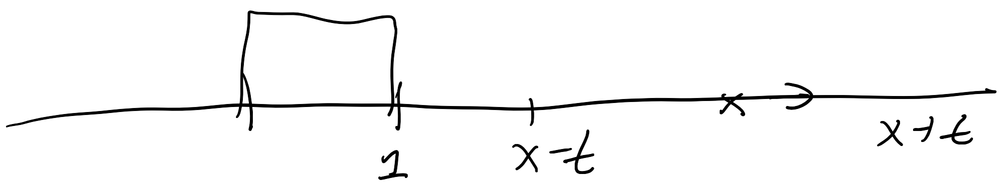


$u(x,t)$

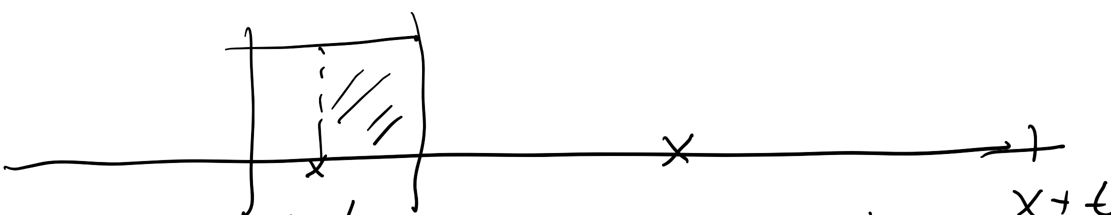


② $f \equiv 0$ $\quad u(x,t) = \dfrac{1}{2} \displaystyle\int_{x-t}^{x+t} g(u)\,du$

EX: $g(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{OTHERWISE} \end{cases}$



(I) $x - t > 1$ $\quad u(x,t) = 0.$

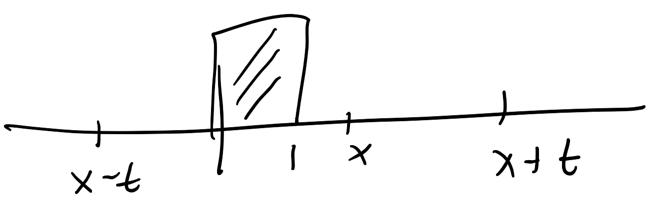


(II)

$1 > x - t > 0$ $\quad \dfrac{1}{2} \displaystyle\int_{x-t}^{x+t} g(u)\,du$

$= \dfrac{1}{2}\left[1 - (x-t)\right] = \dfrac{1}{2}\left[t + 1 - x\right]$

(III)



$x - t < 0$

$u(x,t) = \dfrac{1}{2} \displaystyle\int_{x-t}^{x+t} g(u)\,du = \dfrac{1}{2}$

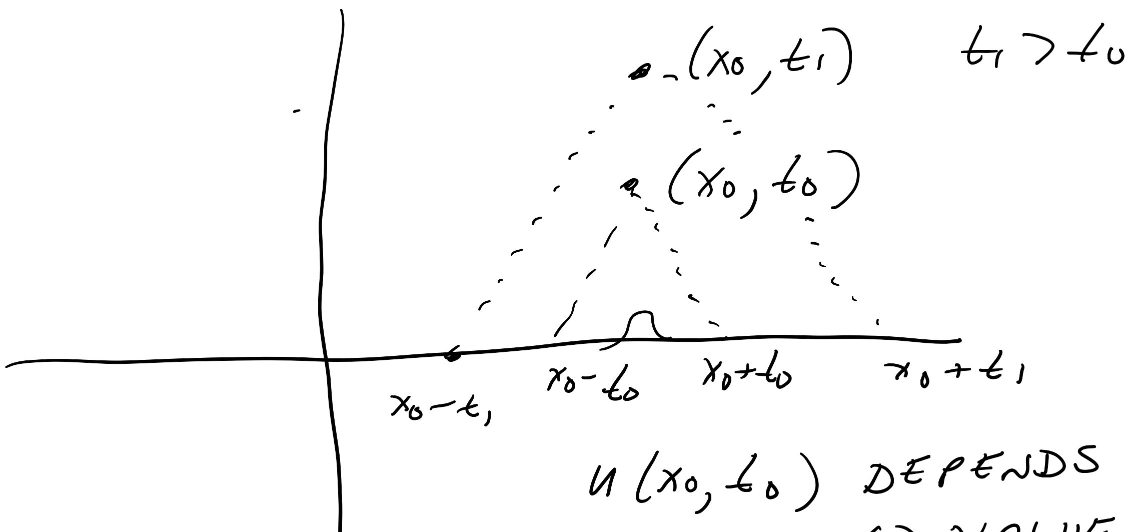
$x - t > 1 \qquad u = 0$

$1 > x - t > 0 \qquad u = \frac{1}{2}\left[t + 1 - x\right]$

$x - t < 0 \qquad u = \frac{1}{2}$

IN GENERAL

$$u(x,t) = \frac{f(x-t) + f(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(u)\,du$$



$u(x_0, t_0)$ DEPENDS UPON ALL $g(x)$ VALUES IN $[x_0 - t_0,\ x_0 + t_0]$

$u(x_0, t_1)$ " " " " " $[x_0 - t_1,\ x_0 + t_1]$; INCLUDES $[x_0 - t_0,\ x_0 + t_0]$

"ONCE A DISTURBANCE AFFECTS $u(x_0, t_0)$ IS CONTINUES TO AFFECT $u(x,t)$, $t > t_0$"

"THERE IS A GREAT DISTURBANCE IN THE FORCE. I HAVE FELT IT"

CONVOLUTION FORM (INTERPRETATION) OF SOL'N TO WAVE EQN:

REVIEW HEAT EQN:

$u_t = u_{xx} \quad u(x,0) = f(x)$

FOURIER SOL'N: $u = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\omega} e^{-t\omega^2} \hat{f}(\omega)\, d\omega$

{BTW: DERIVING TAKES WORK! BUT CHECKING IS EASY!}

ALSO $\quad u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(y)\, e^{-\frac{(x-y)^2}{4t}}\, dy$

$(f * g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy.$

$K_t(x) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} \qquad u(x,t) = (f * K_t)(x).$

(↑ NOT $\frac{\partial}{\partial t}$

$= K(x,t)$)

CONVOLUTION FORM.

WAVE EQN SOL'N

$$u(x,t) = \underbrace{\frac{f(x+t) + f(x-t)}{2}}_{1} + \underbrace{\frac{1}{2}\int_{x-t}^{x+t} g(y)\, dy}_{2}$$

TRY CONVINCE YOU THIS IS ALREADY IN CONVOLUTION FORM:

DEFINE $\chi_t(x) = \begin{cases} 1/2 & -t \le x \le t \\ 0 & \text{OTHER-WISE} \end{cases}$

$\chi(x,t)$

$$(\chi_t * g)(x) = \int_{-\infty}^{\infty} g(y)\, \chi_t(x-y)\, dy$$

$$= \int_{-\infty}^{\infty} g(y)\, \chi_t(y-x)\, dy \qquad (\chi_t \text{ IS EVEN})$$

$z = y - x \qquad y = z + x \qquad \underline{\underline{dz = dy}}\,!$

$$\int_{-\infty}^{\infty} g(z+x)\, \chi_t(z)\, dz$$

$$= \int_{z=-\infty}^{-t} g(z+x)\, \underbrace{\chi_t(z)}_{0}\, dz + \int_{z=-t}^{t} g(z+x)\, \underbrace{\chi_t}_{1/2}\, dz$$

$$+ \int_{t}^{\infty} g(z+x)\, \underbrace{\chi_t(z)}_{0}\, dz$$

$$= \frac{1}{2} \int_{z=-t}^{z=t} g(z+x)\, dz$$

$y = z + x \qquad z = y - x$

$z = -t; \quad y = x - t$

$z = t \quad y = x + t$

$$= \frac{1}{2} \int_{y=x-t}^{y=x+t} g(y)\, dy$$

SO $\frac{1}{2} \int_{x-t}^{x+t} g(y)\, dy = (g * \chi_t)(x)$.

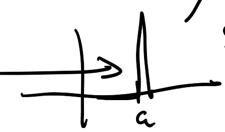
① $\frac{1}{2}\left[f(x+t)+f(x-t)\right]$

REQUIRES "δ-FUNCTION" CALCULATIONS.

$\delta(x)$ "="  $\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$

$\{\delta(x)=\delta_0(x)\}$ $\quad\infty\quad$ $\delta(x-a)$

$$\int_{-\infty}^{\infty} f(y)\,\delta(y-a)\,dy = f(a);$$

$\delta_a(x) \longrightarrow$  SHIFTED δ-FUNCTION

$y = z+a$

$$= \int_{-\infty}^{\infty} f(z+a)\,\delta(z)\,dz = f(a)$$

so $\frac{1}{2}\left[f(x+t)+f(x-t)\right]$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f(y)\left[\delta(y-(x+t)) + \delta(y-(x-t))\right]dy$$

$$\frac{1}{2}\left[\delta(y-(x+t)) + \delta(y-(x-t))\right]$$

$$= \frac{1}{2}\left[\delta((y-t)-x) + \delta((y+t)-x)\right]$$

$$\left\{\int_{-\infty}^{\infty} f(y)\left[g(y-x)\right]dy = (f\circledast g)(x)\right\}$$

$$g(y) = \left[\frac{\delta(y-t)+\delta(y+t)}{2}\right]$$

$$g(x) = \left[\frac{\delta(x-t)+\delta(y+t)}{2}\right]$$

$$= \frac{1}{2}\left[\delta_t(x) + \delta_{-t}(x)\right]$$

$i\psi_t = \psi_{xx}$ SCHRODINGER EQN.

$$\psi(x,0) = f(x)$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\omega} e^{i\omega^2 t} \hat{f}(\omega)\, d\omega$$

CHECK:

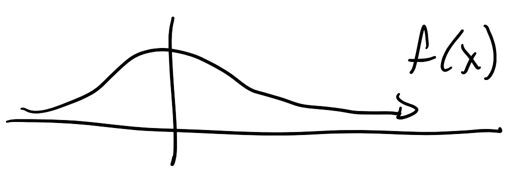
$$\psi_x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\omega} (i\omega) e^{i\omega^2 t} \hat{f}(\omega)\, d\omega$$

$$\psi_{xx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\omega} (-\omega^2) e^{i\omega^2 t} \hat{f}(\omega)\, d\omega$$

$$\psi_t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\omega} (i\omega^2) e^{i\omega^2 t} \hat{f}(\omega)\, d\omega$$

$i\psi_t = \psi_{xx}$ ✓

ASIDE: HOW BIG IS A FUNCTION?



① $\|f\|_{MAX} = \max_x |f(x)|$

$\{\|f\|_\infty\}$

② $\|f\|_1 = \int_{-\infty}^{\infty} |f(x)|\, dx$ "1-NORM"

(3) $\|f\|_2 = \sqrt{\int_{-\infty}^{\infty} |f(x)|^2 \, dx}$ "2-NORM"

"INDEPENDENT" OF EACH OTHER.

FACT: $\|f\|_2 = \|\hat{f}\|_2$.

CONSEQUENCE FOR SCHRODINGER EQN:

$$\psi(x, t_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \underbrace{e^{i\omega^2 t_0} \hat{f}(\omega)} \, d\omega$$

$\psi(x, t_0)$ IS (ESSENTIALLY) FOURIER TRANSFORM OF $e^{i\omega^2 t} \hat{f}(\omega)$)

BUT! $\|\psi(x, t_0)\|_2 = \| e^{i\omega^2 t} \hat{f}(\omega) \|$

$$= \sqrt{\left\{ \int_{-\infty}^{\infty} | \hat{f}(\omega) e^{i\omega^2 t} |^2 \, d\omega \right\}}$$

$$= \sqrt{\int_{-\infty}^{\infty} | \hat{f}(\omega)|^2 \, | e^{i\omega^2 t} |^2 \, d\omega}$$

BUT! $e^{i\omega^2 t} = \cos(\omega^2 t) + i \sin(\omega^2 t)$

$|e^{i\omega^2 t}|^2 = \sqrt{\cos^2 + \sin^2} = \underline{1}$

$$= \sqrt{\int_{-\infty}^{\infty} | \hat{f}(\omega)|^2 \, d\omega} = \|\hat{f}\|_2$$

$$\| \hat{\psi}(x,t) \|_2 = \| \hat{f}(\nu) \|_2$$

INDEPENDENT OF TIME!