Phys 327 HW 1

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① Griffiths 4.4

Use equations:

$$P_l^m \equiv (-1)^m (1-x^2)^{m/2} \left(\frac{d}{dx}\right)^m P_l(x)$$

$$P_l(x) \equiv \frac{1}{2^l l!} \left(\frac{d}{dx}\right)^l (x^2-1)^l$$

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\, e^{im\phi} P_l^m(\cos\theta)$$

to construct the spherical harmonics Y_0^0 and Y_2^1. Check that they are normalized and orthogonal.

$$Y_0^0 = \sqrt{\frac{1}{4\pi}} P_0^0(\cos\theta) = \sqrt{\frac{1}{4\pi}} P_0(\cos\theta) = \boxed{\sqrt{\frac{1}{4\pi}}}$$

$$\int_\Omega |Y_0^0|^2 d\Omega = \frac{1}{4\pi}\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{1}{4\pi}(2)(2\pi) = 1$$

$\longrightarrow Y_0^0$ normalized ✓

$$Y_{l=2}^{m=1} = \sqrt{\frac{[2(2)+1]}{4\pi}\frac{(2-1)!}{(2+1)!}}\, e^{i\phi} P_{l=2}^{m=1}(\cos\theta) = \sqrt{\frac{5}{4\pi}\left(\frac{1}{6}\right)}\, e^{i\phi} P_{l=2}^{m=1}$$

$$= \sqrt{\frac{5}{24\pi}}\, e^{i\phi} (-1)^1 (1-\cos^2\theta)^{1/2} \left(\frac{d}{d\cos\theta}\right) P_2(\cos\theta)$$

$$= -\sqrt{\frac{5}{24\pi}}\, e^{i\phi} (1-\cos^2\theta)^{1/2} \frac{d}{d(\cos\theta)} \frac{1}{2^2 2!} \left(\frac{d}{d\cos\theta}\right)^2 (\cos^2\theta - 1)^2$$

$4\cos^3\theta - 4\cos\theta$

$$= -\frac{1}{8}\sqrt{\frac{5}{24\pi}}\, e^{i\phi} \sqrt{1-\cos^2\theta}\, \frac{d}{d(\cos\theta)}\left[\frac{d}{d(\cos\theta)} 4\cos\theta(\cos^2\theta - 1)\right]$$

$$= -\frac{1}{8}\sqrt{\frac{5}{24\pi}}\, e^{i\phi} \sqrt{1-\cos^2\theta}\, \frac{d}{d(\cos\theta)}\left[12\cos^2\theta - 4\right]$$

$$= -\frac{1}{8}\sqrt{\frac{5}{24\pi}}\, e^{i\phi} \sqrt{1-\cos^2\theta}\; 24\cos\theta$$

$$\longrightarrow Y_{l=2}^{m=1}(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\, e^{i\phi}\sqrt{1-\cos^2\theta}\,\cos\theta = \boxed{-\sqrt{\frac{15}{8\pi}}\, e^{i\phi}\sin\theta\cos\theta}$$

$$\iint_{\Omega} |Y_2^1|^2 d\Omega = \left(\frac{15}{8\pi}\right) \int_0^{2\pi} e^{i\phi} e^{-i\phi} d\phi \int_0^{\pi} \sin^3\theta \cos^2\theta \, d\theta$$

$u = \cos\theta$

$du = -\sin\theta \, d\theta$

$$= \left(\frac{15}{8\pi}\right)(2\pi) \int_0^{\pi} (1-\cos^2\theta) \cos^2\theta \sin\theta \, d\theta$$

$$= \frac{15}{4} \int_{u=1}^{u=-1} -(1-u^2)u^2 du = \frac{15}{4} \int_{-1}^{1} u^2 - u^4 \, du$$

$$= \frac{15}{4} \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_{-1}^{1} = \frac{15}{4} \left\{ \underbrace{\left[\frac{1}{3} - \frac{1}{5}\right]}_{2/15} - \underbrace{\left[\frac{-1}{3} + \frac{+1}{5}\right]}_{-2/15} \right\}$$

$$= \frac{15}{4}\left(\frac{4}{15}\right) = 1 \checkmark \longrightarrow Y_2^1 \text{ is normalized.}$$

<u>Orthogonality</u>:

$$\iint Y_0^{0*} Y_2^1 \, d\Omega = \frac{-1}{\sqrt{4\pi}} \sqrt{\frac{15}{8\pi}} \int_0^{2\pi} e^{i\phi} d\phi \underbrace{\int_0^{\pi} \sin^2\theta \cos\theta \, d\theta}_{\frac{\sin^3\theta}{3} \rightarrow 0}$$

$$= 0 \longrightarrow Y_0^0 \; Y_2^1 \text{ orthogonal}$$

(2) Griffiths 4.11

A particle of mass m is placed in a finite spherical well:

$$V(r) = \begin{cases} -V_0, & r < a \\ 0, & r > a \end{cases}$$

Find the ground state by solving the radial equation with $l = 0$. Show that there is no bound state if $V_0 a^2 < \pi^2 \hbar^2 / 8m$.

Inside the well, the radial equation says:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = (E+V_0)u \qquad \left(\text{define } \gamma \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}\right)$$

$E > V_0$ for bound

$$\rightarrow \frac{d^2u}{dr^2} = -\gamma^2 u$$

$$(1) \Longrightarrow u(r) = A\sin\gamma r + B\cos\gamma r \quad (r < a)$$

Outside the well, the radial equation says:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = Eu \rightarrow \frac{d^2u}{dr^2} = \kappa^2 u \qquad \left(\kappa \equiv \frac{\sqrt{-2mE}}{\hbar}, \; E < 0\right)$$

$$\Rightarrow u(r) = Ce^{-\kappa r} + De^{\kappa r}$$

Blows up as $r \to \infty \Rightarrow D = 0$

$$(2) \Rightarrow u(r) = Ce^{-\kappa r}$$

Now impose boundary conditions on (1) + (2) to find $A, B,$ and C.

①: Note that $\frac{u(r)}{r} = R(r)$

$$\rightarrow R(r) = \frac{A\sin(\gamma r)}{r} + \frac{B\cos(\gamma r)}{r}$$

Second term blows up at $r=0 \rightarrow B = 0$

$$\rightarrow u(r) = A\sin(\gamma r) \qquad \text{(Inside, } r<a)$$

Now impose boundary conditions: $u_{inside}(a) = u_{outside}(a)$

Continuity of du/dr

$$u: \quad A\sin\gamma a = Ce^{-\varkappa a}$$
$$du/dr: \quad \gamma A\cos\gamma a = -\varkappa C e^{-\varkappa a}$$
$$\rightarrow \underline{\underline{\varkappa = -\gamma\cot(\gamma a)}}$$

or $\tan(\gamma a) = -\gamma/\varkappa$

Let $z \equiv \gamma a$ and $z_0 \equiv \frac{a}{\hbar}\sqrt{2mV_0}$

$$\rightarrow \varkappa^2 + \gamma^2 = \cancel{-\frac{2mE}{\hbar^2}} + \cancel{\frac{2mE}{\hbar^2}} + \frac{2mV_0}{\hbar^2} = 2mV_0/\hbar^2$$

$$z_0^2 - z^2 = \cancel{\frac{2mV_0a^2}{\hbar^2}} - \cancel{\frac{2mV_0a^2}{\hbar^2}} - \frac{2mEa^2}{\hbar^2} = \varkappa^2 a^2$$

$$\rightarrow \varkappa a = \sqrt{z_0^2 - z^2}$$

$$\rightarrow \frac{\varkappa}{\gamma} = -\cot z$$

$$\frac{\sqrt{z_0^2 - z^2}}{\gamma a} = -\cot z$$

$\gamma a = z$

$$\rightarrow \underline{\underline{-\cot z = \sqrt{(z_0/z)^2 - 1}}}$$

Transcendental equation for energies

$$\cot z_0 = 0 \rightarrow z_0 = \frac{n\pi}{2}$$

Ground state: $z_0 = \pi/2$

Note that if $z_0 < \pi/2$, there will be no intersection in the transcendental equation and hence there are no bound energies:

No bound state if: $\frac{a}{\hbar}\sqrt{2mV_0} < \frac{\pi}{2} \rightarrow \frac{\pi\hbar}{2a} > \sqrt{2mV_0}$

$$\rightarrow 2mV_0 < \frac{\pi^2\hbar^2}{4a^2} \Rightarrow \boxed{V_0 a^2 < \frac{\pi^2\hbar^2}{8m}}$$

If there is a bound state energy, then z is between $\pi/2$ and π: $(z = \gamma a)$

$$z = \gamma a \rightarrow \frac{\pi}{2} = \frac{\sqrt{2m(E+V_0)}}{\hbar}a \rightarrow E = \frac{\pi^2\hbar^2}{8ma^2} - V_0$$

$$\pi = \frac{\sqrt{2m(E+V_0)}}{\hbar}a \rightarrow E = \frac{\pi^2\hbar^2}{2ma^2} - V_0$$

$\Rightarrow$ The ground state energies fall in the range:

$$\boxed{\frac{\pi^2\hbar^2}{8ma^2} - V_0 < E < \frac{\pi^2\hbar^2}{2ma^2} - V_0}$$

3) Griffiths 4.46

Consider the 3-D harmonic oscillator with the potential

$$V(r) = \frac{1}{2}m\omega^2 r^2$$

a) Show that separation of variables in cartesian coordinates turns this into three one-dimensional harmonic oscillators, and show that the allowed energies are then $E_n = (n+\frac{3}{2})\hbar\omega$

TISE: $-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$

Guess solutions: $\psi(x,y,z) = X(x)Y(y)Z(z)$

$$\rightarrow -\frac{\hbar^2}{2m}\left(YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2}\right) + VXYZ = EXYZ$$

$$\rightarrow -\frac{\hbar^2}{2m}\left(\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}\right) + V = E$$

$$\Rightarrow -\frac{\hbar^2}{2m}(\ldots) + \frac{1}{2}m\omega^2(x^2+y^2+z^2) = E$$

$$\Rightarrow \underbrace{\left[-\frac{\hbar^2}{2m}\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{2}m\omega^2 x^2\right]}_{\text{constant}} + \underbrace{\left[-\frac{\hbar^2}{2m}\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{2}m\omega^2 y^2\right]}_{\text{constant}} - \underbrace{\left[\frac{\hbar^2}{2m}\frac{1}{Z}\frac{\partial^2 Z}{\partial z} + \frac{1}{2}m\omega^2 z^2\right]}_{\text{constant}} = E$$

* Each term constant:

$$\underbrace{-\frac{\hbar^2}{2m}\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 = E_x}_{\text{1-D harmonic oscillator}} \longrightarrow E_x = (n_x+\tfrac{1}{2})\hbar\omega$$

Similarly for y and z:

$$-\frac{\hbar^2}{2m}\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{2}m\omega^2y^2 = E_y \rightarrow E_y = (n_y + \tfrac{1}{2})\hbar\omega$$

$$-\frac{\hbar^2}{2m}\frac{1}{Z}\frac{d^2Z}{dz^2} + \frac{1}{2}m\omega^2z^2 = E_z \rightarrow E_z = (n_z + \tfrac{1}{2})\hbar\omega$$

$$\Rightarrow E = E_x + E_y + E_z = (\underbrace{n_x + n_y + n_z}_{\equiv n} + \tfrac{3}{2})\hbar\omega$$

$$\Rightarrow \boxed{E = (n + \tfrac{1}{2})\hbar\omega}$$

b) Determine the degeneracy $d(n)$ of the energies E_n.

$d(0) = 1$

$d(1) = 3$

$d(2) =$

n_x	n_y	n_z
1	1	0
1	0	1
0	1	1
2	0	0
0	2	0
0	0	2

$\Rightarrow d(2) = 6$

$d(3)$:

n_x	n_y	n_z
3	0	0
0	3	0
0	0	3
2	1	0
2	0	1
1	2	0
0	2	1
1	0	2
0	1	2
1	1	1

1 $d(n)$: $n_x = n \rightarrow n_y = n_z = 0$

2 - $n_x = n-1 \rightarrow n_y = 1$ and $n_z = 0$ or $n_y = 0$ and $n_z = 1$

3 $n_x = n-2 \rightarrow$

n_y	n_z
2	0
0	2
1	1

$n+1$ $n_x = 0 \rightarrow$

n_y	n_z
n	0
$n-1$	1
$\vdots$	$\vdots$
0	n

$$\rightarrow d(n) = 1 + 2 + \ldots + n+1 \Rightarrow \boxed{d(n) = \frac{(n+1)(n+2)}{2}}$$