[redacted]

$$f(x) \rightarrow \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixw} dx$$

FOURIER TRANSFORM

$$g(w) \rightarrow \check{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w) e^{iwx} dw$$

INVERSE FOURIER TRANSFORM

ALMOST THE SAME!

RELATION TO HEAT EQN: (FOURIER SOL'N)

$$u_t = u_{xx} \qquad u(x,0) = f(x); \quad -\infty < x < \infty$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw} e^{-tw^2} \hat{f}(w) \, dw$$

SOL'N REQUIRES "DOABILITY" OF 2 INTEGRALS.

WE SAW ONE VERY IMPORTANT SPECIAL CASE! $f(x) = e^{-a^2x^2}$

"GAUSSIAN"

① IF $f(x)$ IS GAUSSIAN THEN $\hat{f}(w)$ IS GAUSSIAN.

② $e^{-tw^2} \hat{f}(w)$ IS GAUSSIAN

(A GAUSSIAN TIMES A GAUSSIAN IS GAUSSIAN)

③ $u(x,t) = \frac{1}{\sqrt{2\pi}} \int e^{ix\omega} \underbrace{e^{-t\omega^2} \hat{f}(\omega)}_{\text{"G"}} d\omega$

IS (FOR EACH t), GAUSSIAN

(INVERSE TRANSFORM OF GAUSSIAN IS GAUSSIAN)

DETAILS LAST TIME; SOL'N:

$$u(x,t) = \frac{1}{\sqrt{4a^2t+1}} \, e^{-\frac{x^2a^2}{4a^2t+1}}$$

(I) $u_t = u_{xx}$ CHECK!

(II) $u(x,0) = e^{-a^2x^2}$ PLUG IN $t=0$!

MAKE NEW, "CHEAP" SOL'NS.

<u>IF $v(x,t) = u_x(x,t)$, THEN</u>

<u>$v(x,t)$ IS A SOL'N.</u>

CHECK

$$v_t = (u_x)_t = (u_t)_x$$

$$v_{xx} = (u_x)_{xx} = \cancel{(u_x)_{xx}} \; (u_{xx})_x$$

$\underline{u_t = u_{xx}}$ SO $(u_t)_x = (u_{xx})_x$

$$v_t = v_{xx} \; ✓$$

EX: $u(x,t) = \frac{1}{\sqrt{4a^2t+2}} \, e^{-\frac{a^2x^2}{4a^2t+1}}$

$$v(x,t) = u_x$$

$$= \frac{-2a^2 x \, e^{-a^2x^2/(4a^2t+1)}}{(4a^2t+1)^{3/2}}$$

$$w(x,t) = v_x$$

$$= \frac{-8a^2 e^{-\left(\frac{a^2x^2}{4a^2t+1}\right)}\left(\frac{1}{4} + \left(\frac{-x^2}{2}+t\right)a^2\right)}{(4a^2t+1)^{5/2}}$$

$$u(x,0) = f_0(x) = e^{-a^2x^2}$$

$$v(x,0) = f_1(x) = -2a^2 x \, e^{-a^2x^2}$$

$$w(x,0) = f_2(x) = (-2a^2 + 4a^4x^2)$$

TIMES $e^{-a^2x^2}$

SO, $c_1 u + c_2 v + c_3 w$

SOLVES HEAT FOR ANY

$f(x) = P_2(x) \, e^{-a^2x^2}$ P_2 QUAD.

2ND FORM FOR SOLUTION:

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw} e^{-tw^2} \hat{f}(w)\, dw$$

FOURIER FORM OF SOL'N

$u(x,0) = f(x)$

$u_t = u_{xx}$.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw} e^{-tw^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-iyw}\, dy\, dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \int_{-\infty}^{\infty} e^{ixw} e^{-tw^2} e^{-iyw}\, dw\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(y-x)w} e^{-tw^2}\, dw\, dy$$

$$g(x) = e^{-a^2x^2} \qquad \hat{g}(w) = \frac{1}{\sqrt{2}\, a} e^{-\frac{w^2}{4a^2}}$$

$t = a^2$

$a = \sqrt{t}$

$$= \frac{1}{\sqrt{2}\sqrt{t}} e^{-(y-x)^2/4t}$$

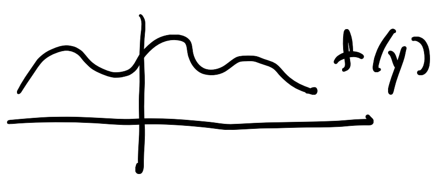
$$= \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2}\sqrt{t}} e^{-(y-x)^2/4t}\, dy.$$

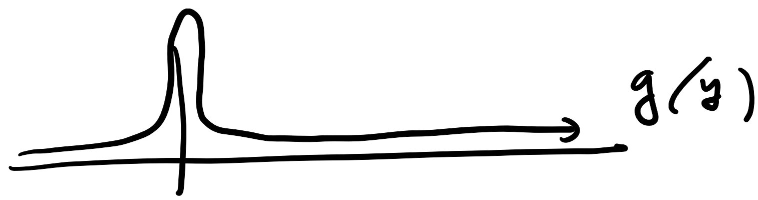
$$\boxed{= \int_{-\infty}^{\infty} f(y) \underbrace{\frac{1}{\sqrt{4\pi t}} e^{-(y-x)^2/4t}}\, dy. = u(x,t).}$$

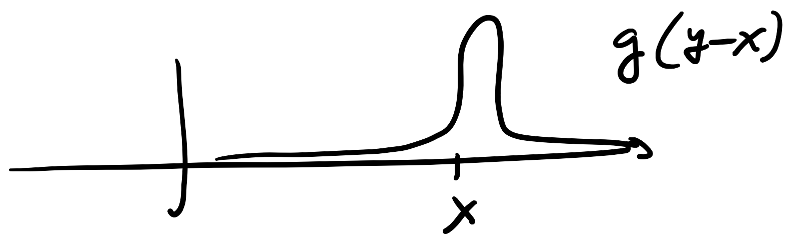
<u>CONVOLUTION</u> FORM OF SOL'N

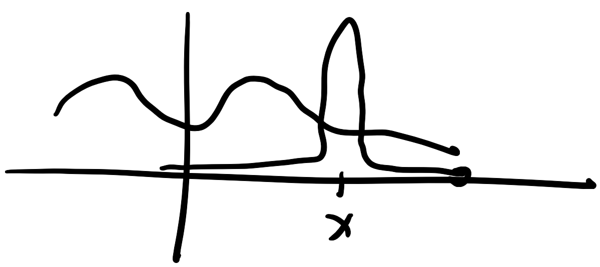
DEF: GIVEN TWO FUNCTIONS $f(x)$, $g(x)$, $(f * g)(x)$ = THE CONVOLUTION OF f AND g

$$= \int_{-\infty}^{\infty} f(y)\, g(y-x)\, dy$$



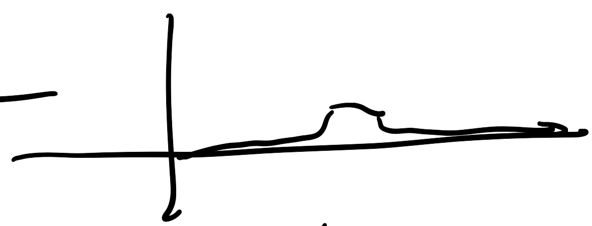






$f(y) \cdot g(y-x)$

"CUTS OFF f INFO AWAY FROM x" $\leftarrow$



$$K(x,t) = \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}.$$

THEN

$$u(x,t) = \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{(y-x)^2}{4t}}\, dy$$

$= f \circledast K$ CONVOLUTION FORM OF SOL'N.

SUPPOSE FORGET ALL PREVIOUS CALCULATIONS, AND ARE

GIVEN $u(x,t) = \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{4\pi t}} e^{-\frac{(y-x)^2}{4t}} dy.$

(i) $u_t = u_{xx}$? (ii) $u(x,0) = f(x)$?

$$u_t = \int_{-\infty}^{\infty} f(y) \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{4\pi t}} e^{-\frac{(y-x)^2}{4t}}\right] dy.$$

$$u_{xx} = \int_{-\infty}^{\infty} f(y) \frac{\partial^2}{\partial x^2}\left[\frac{1}{\sqrt{4\pi t}} e^{-\frac{(y-x)^2}{4t}}\right] dy$$

CHECK $u_t = u_{xx}$. (i) ✓

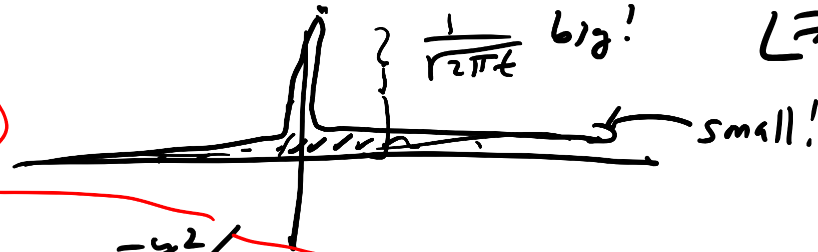
BUT! CANNOT SIMPLY "PLUG IN" $t=0$.

BUT! CAN ASK IF $\lim_{t\to 0} u(x,t) = f(x)$. ?

SUPPOSE t IS SMALL:

$$u(x,t) = \int_{-\infty}^{\infty} f(y) \underbrace{\frac{1}{\sqrt{4\pi t}} e^{-\frac{(y-x)^2}{4t}}}\, dy$$

(I) $\frac{1}{\sqrt{2\pi t}} e^{-y^2/4t}$ (L'H!)



(II) $$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-y^2/4t}\, dy$$

$y = 2\sqrt{t}\, u$

$dy = 2\sqrt{t}\, du$

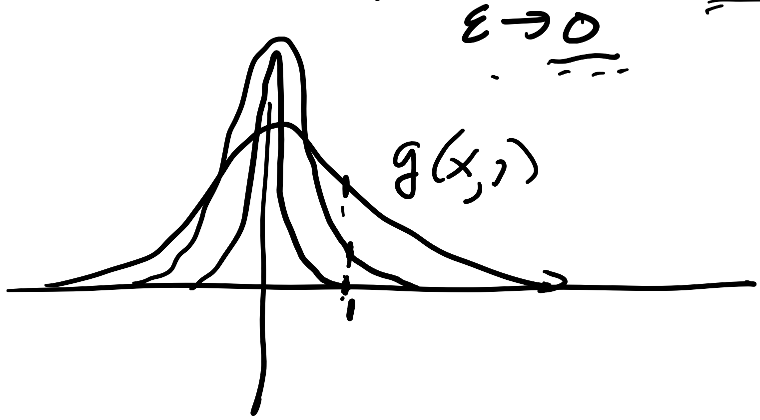
$$= \frac{2\sqrt{t}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-u^2}\, du$$

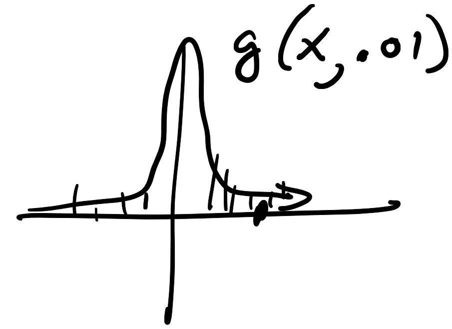
$$= \frac{2\sqrt{t}}{\sqrt{4\pi t}} \sqrt{\pi} = \boxed{1}$$

DEF: A DIRAC FAMILY IS A FAMILY OF FUNCTIONS $g(x,\varepsilon)$ SO THAT. $g > 0$.

(I) FOR EVERY ε, $\int_{-\infty}^{\infty} g(x,\varepsilon)\, dx = 1$

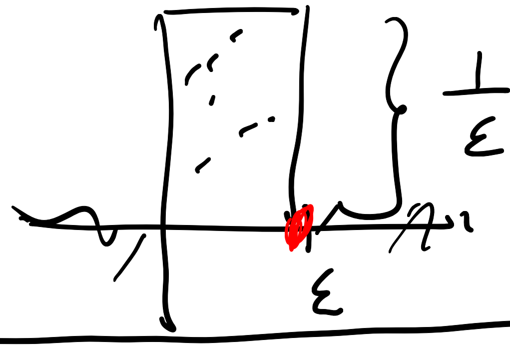
(II) IF $x \neq 0$, $\lim_{\varepsilon \to 0} g(x,\varepsilon) = 0$.





$$\left\{ \lim_{\varepsilon \to 0} g(0, \varepsilon) = \infty \, ! \right\}$$

EX!



$$g(x, \varepsilon) = \begin{cases} \dfrac{1}{\varepsilon} & 0 \le x \le \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

$$\lim_{\varepsilon \to 0} f * g = f.$$

$$f * g = \int_{-\infty}^{\infty} f(y)\, g(y - x, \varepsilon)\, dy$$

$z = y - x$

$$\equiv g(z, \varepsilon)$$

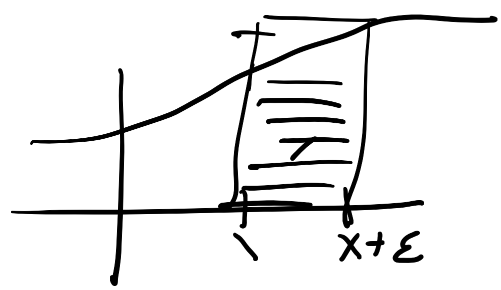
$y = z + x$

$dy = dz$

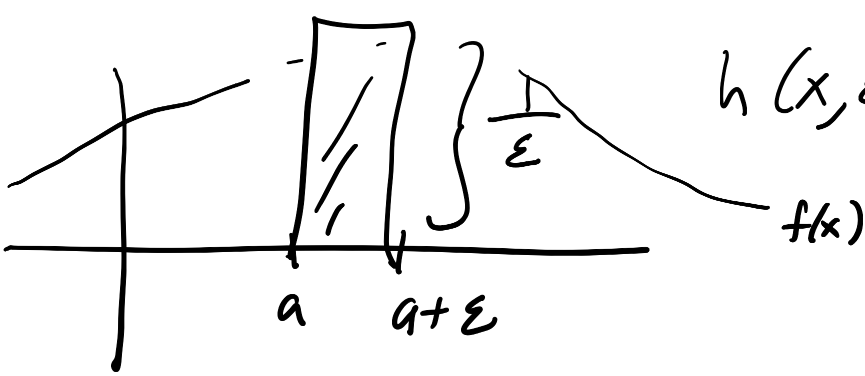
$$= \int_{-\infty}^{\infty} f(z + x)\, g(z)\, dz$$

$$= \int_{z=0}^{z=\varepsilon} \frac{1}{\varepsilon} f(z + x)\, dz$$

$$= \frac{1}{\varepsilon} \int_{y=x}^{y=x+\varepsilon} f(y)\, dy.$$



$$\approx \frac{1}{\varepsilon} f(x) \left[(x + \varepsilon) - x \right] = f(x).$$



$h(x, \varepsilon)$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,h(x,\varepsilon)\,dx$$

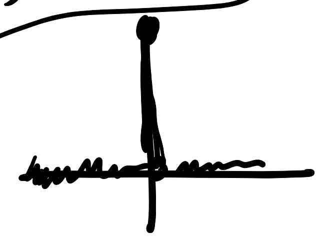
$$= \frac{1}{\varepsilon}\int_{a}^{a+\varepsilon} f(x)\,dx \approx \frac{1}{\varepsilon} f(a)\left[a+\varepsilon-a\right]$$

$$= f(a).$$

$$\boxed{\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} f(x)h(x,\varepsilon)\,dx = \cancel{f(a)}}\ \underline{\underline{f(a)}}.$$

"DIRAC δ-FUNCTION"

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x)\,dx = \underline{1}$$



AND $\int_{-\infty}^{\infty} \underline{\delta(x)\, f(x)}\,dx = f(0).$

__MEANS__ TAKE A DIRAC FAMILY,

$\underline{h(x,\varepsilon)}$, THEN COMPUTE

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} f(x)\,h(x,\varepsilon)\,dx = f(a)$$

~~FRENCH STORY~~

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REPEAT: IF $h(x,\varepsilon)$ IS A DIRAC FAMILY, THEN $\lim_{\varepsilon\to 0} f * h = f.$

$$h(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \text{ IS A}$$

DIRAC FAMILY!

SO
$$\lim_{t \to 0} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} f(y)\, dy$$

$$= f(x).$$

WAVE EQUATION "VIBRATING STRING".

TWO PHYSICAL PROBLEMS:

(I)

"RUBBER BAND"

$u(x,t)$

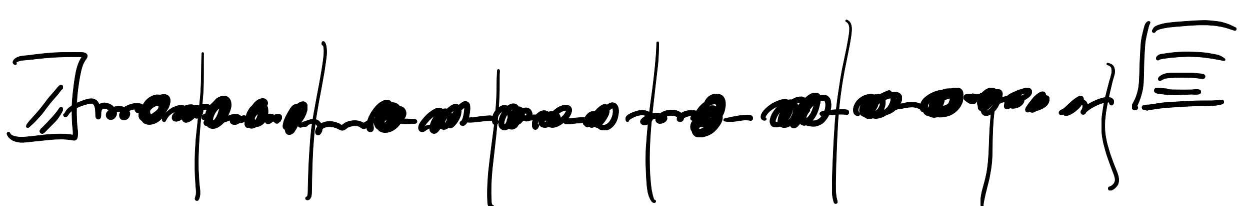
TRANSVERSE WAVES.

$u(0,t) = u(1,t) = 0$; $\boxed{u_{tt} = u_{xx}}$

$u(x,0) = f(x)$ INITIAL DISPLACEMENT;

$u_t(x,0) = g(x)$ INITIAL VELOCITY.

(II) (LIMIT OF DENSE SPRING SYSTEM)



SAY TOTAL MASS = 1; EACH MASS $m_k = \frac{1}{N}$

$k = 1, \dots N.$

CAN TALK ABOUT TOTAL MASS IN EACH INTERVAL; AS $N \to \infty$

AS # OF INT $\to \infty$

COMPRESSION WAVES

$u_{tt} = u_{xx}$; BUT $u_{tt} = c^2 u_{xx}$

↑ WAVE SPEED

$u(0,t) = u(1,t)$ $\quad$ $u(x,0) = f(x)$

$u_t(x,0) = g(x)$

$$u(x,t) = X(x)\,T(t)$$

$$u_{tt} = X(x)\,T''(t)$$

$$u_{xx} = X''(x)\,T(t).$$

$$\frac{X(x)\,T''(t)}{(X \cdot T)} = \frac{X''(x)\,T(t)}{(X \cdot T)}.$$

$$= \frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = K \;!!$$

$$X''(x) = K\,X(x) \qquad X(0) = 0, \quad X(1) = 0.$$

HEY! $$X_N(x) = \sin(N\pi x) \qquad N = 1, \dots$$

$$X_N'' = -N^2\pi^2 X \qquad K \neq 0, \quad K \not> 0$$

$$= K;$$

$$\frac{T''(t)}{T(t)} = K \qquad T''(t) = K\,T.$$

$$T_N''(t) = -N^2\pi^2 T_N.$$

$- \dots$