Physics 432/532: Cosmology
Winter 2022
Prof. Michael S. Vogeley

Problem Set 1

1. Local Effects of Cosmological Constant?

If there really is a non-zero cosmological constant with $\Omega_{vac} = 0.7$, what is the effect of this component of mass-energy density on motions in the Solar system? On our orbit around the Galaxy? Proceed by comparing this density with the average densities and accelerations in these systems.

(a) Compute the equivalent mass density of the cosmological constant in SI units.

(b) For the solar system, assume that the Sun dominates the mass (because it does) and use 1 AU as the characteristic radius. Compute the average density.

(c) For the Galaxy, assume that the mass is 10^{11} solar masses within the Sun's orbit of 8.5kpc. Calculate the average mass density.

(d) Applying Birkhoff's theorem (we can ignore gravitational acceleration from outer mass shells in a homogeneous medium), compute the ratios a_{vac}/a_{SS} and a_{vac}/a_{MW}. Could we detect the effects of vacuum energy? (Compare to the precision with which we know G.)

2. Robertson-Walker metric

Show that the Robertson-Walker metric

$$c^2ds^2 = -c^2dt^2 + a^2(t)[dr^2 + S_k^2(r)d\psi^2]$$

can also be written in the form

$$c^2ds^2 = -c^2dt^2 - a^2(t)[dx^2/(1 - k(x/R)^2) + x^2d\psi^2]$$

where x is a new radial coordinate, not the Cartesian coordinate. (Hint: You're allowed to work backwards from the second form.)

3. Particle Horizon

Show that the dominant form of mass-energy at early times must scale as $\rho \propto a^{-\alpha}$ with $\alpha > 2$ for a particle horizon to exist (light travels a finite distance since the beginning of the universe). Follow the steps.

(a) Starting from the fact that photons travel on null geodesics with $ds = 0$ and $d\Psi = 0$, using the spactime metric to write an integral equation for the comoving distance r that a photon travels between two times in the universe.

(b) Make a change of variables of that integral from t to a and use the Friedmann equation with $k = 0$ (justified because the universe is nearly flat at early times) and with density $\rho \propto a^{-\alpha}$ to rewrite the integral so the limits are the scale factors $a(t)$ of the universe at two times and the integral depends on α.

(c) For what conditions on α does this integral converge as $a(t_0) \to 0$?. Is this condition satisfied in a universe that contains matter, radiation, and vacuum energy? Explain by reference to how these components evolve with scale factor.

4. Evolution of $\Omega_{matter}, \Omega_{vac}$
Compute how the density parameters Ω_{matter} and Ω_{vac} evolve with time in different cosmologies and plot the results on a figure of Ω_{matter} vs. Ω_{vac} (like Fig. 5.6 in Ryden). In other words, for each choice of $\Omega_{matter}, \Omega_{vac}$ today, plot the trajectory of the model backwards in time to where it would lie in that same diagram at $t \approx 0$. On the plot, clearly indicate which end of each curve is now and which is at $t = 0$. Pick models from all regions of the diagram and be sure to include the following (in other words, do more than just these):
$\Omega_{matter} = 1, \Omega_{vac} = 0$
$\Omega_{matter} = 0.2, \Omega_{vac} = 0$
$\Omega_{matter} = 0.3, \Omega_{vac} = 0.7$
Comment about what these results tell you about what the universe looks like at early times. (Hint for how to compute evolution of the Ω's: Note how Ω depends on H and see eq. 5.81 in Ryden to see how H depends on the scale factor a. Thus, a mass-energy component Ω_i implicitly evolves with time because $a = a(t)$ and so $H = H(t)$.)

5. Baseball Universe
I'm in heaven; the universe is filled with baseballs! A regulation baseball has mass $m = 0.145$kg and radius $r = 0.0369$m.

(a) If non-relativistic baseballs are the only form of mass-energy, with critical density $\Omega = 1$, what is the number density of baseballs in SI units (m^{-3})? In number per AU^3?

(b) At that density, what is the average distance (in meters) at which your line of sight intercepts a baseball?

(c) Re-express your answer to (b) in units of the "Hubble distance" c/H_0. Suppose that in this baseball universe our telescopes can see baseballs at a distance $c/H_0 \sim 4000$Mpc (we would have to explain why they emit enough light to see them...). Does the transparency of the universe set a useful limit on the number density of baseballs?

6. Einstein's static universe
Consider Einstein's static universe, in which the attractive force of the matter density ρ is exactly balanced by the repulsive force of the cosmological constant, $\Lambda = 4\pi G\rho$.. Suppose that some fraction, f, of the matter is converted into radiation (by stars, for instance). Will the universe start to expand or contract? Follow the steps.

(a) Write equations that xpress both the matter density and the cosmological constant as energy densities. Hint: Look at discussion at end of p. 64 in Ryden.

(b) Write equations for the pressures of the matter, radiation, and cosmological constant.

(c) Write an equation for the acceleration of the universe in terms of the energy densities and pressures of matter, radiation, and cosmological constant.

(d) Converting a fraction, f, of the matter to radiation, rewrite (c) in terms of f. Does the universe expand or contract? Explain physically why this happens.

EXTRA CREDIT (e) Given an energy density $\epsilon_m = \rho c^2$ in matter and energy density ϵ_r in radiation, what value of Λ would be required to make the universe static?