HMWK 4

1. In class, we discussed the wave equation and the heat equation in variables x, y, t. This is called two space, one time variable. I am not happy with the presentation of the material and I want to edit it substantially for the next time I teach the course. You will do it for me. The material is notes 8a and 8b. I strongly suggest you print out the notes to help you while you do this problem. Simply put, I WANT TO SKIP THE SECTION ON POWER SERIES. One part of this, the Liouville trick, was in last week's homework. Still, I want a complete set of lecture notes, so put this all together.

   1) Start off by explaining that, when you solve the wave and heat equation on the interval, the hard work is the same for both problems : you need to solve $X''(x) = \lambda X(x)$, with appropriate boundary conditions. Show this by doing separation of variables for both problems (ONLY up to the point where you see the separated equations).

   2) State what the wave equation and heat equation look like when there are the three variables (two space, one time). Show, using separation of variables again, that this leads to the Helmholtz equation $V_{xx} + V_{yy} = \lambda V$; it does not matter if you solve heat or wave equation, the Helmholtz equation must be solved. Let the reader know that advanced theory tells us that the number λ is always negative.

   3) Remind the reader that this is a complicated problem for a general region, but we will look at a rectangle and a disk.

   4) I am fairly happy with the presentation of the rectangle. YOU CAN SKIP THIS.

   5) For the disk (use radius equal 1), explain that the guess $u = T(t)R(r)\Theta(\theta)$ is more appropriate, and remind the reader that the boundary condition has two pieces: $R(1) = 0$, and $\Theta(\theta)$ must be 2π periodic.

   6) Quote the formula for $u_{xx} + u_{yy}$ in polar coordinates; we did not prove it in class and you do not have to prove it either. Show that, when you try to solve Helmholtz equation in polar coordinates, with the guess $V = R(r)\Theta(\theta)$, you get two separate ODE's for R and Θ.

   7) The ODE for Θ looks very simple. We have boundary conditions that Θ must be 2π periodic in order to give a physically/geometrically valid solution. Show that this leads to $\Theta = A\cos(m\theta) + B\sin(m\theta)$, $m = 1, 2, 3 \ldots$. Note that even $m = 0$ makes sense (in which case, $\Theta = A$ a constant).

   8) Warn the reader that the equation for R is completely different. Tell them we are boldly go exploring a new equation leading to new functions that the reader of your notes has probably never seen (with the exception of me, of course). There are two numbers in the equation for R, they are k and m. m is much MORE important. What can we say about the solutions to this differential equation THE DISCUSSION ABOUT CAUCHY-EULER AND POWER SERIES CAN BE SKIPPED.

   9) The functions which satisfy the R equation with $k = 1$ are called Bessel functions; and a calculation shows that the R function that we want is actually $R = J_n(kr)$. Show this calculation. So we need to understand the functions $J_n(r)$ (or x, the variable is not important)

   10) What IS important is that the boundary condition for $S(r, \theta)$ the solution to Helmholtz (note

that I was calling it V before, I accidently changed the name from a previous lecture) is $S(1,\theta)=0$, and that means that $R(1)=0=J_n(k)$. SO WE NEED TO KNOW WHERE THE ROOTS ARE FOR THE FUNCTIONS $J_n(r)$. The numbers $r_{n,m}$ $n=0,1,2,\dots,m=1,2,3,\dots$ gives the mth root of J_n.

11) So what do these functions look like? A quick reference to find info about the Bessel functions (and almost anything mathematical) is Mathworld, here is a link:

https://mathworld.wolfram.com/BesselFunctionZeros.html

12) Sketch the graphs of a couple of of J_n so the reader does not have to look them up (the reader does not have internet service). Tell the reader to observe that the roots almost seem to have regular spacing. In fact, for J_n, $n=0,1,2,3,4,5$, compute the spacing between $J_n(r_{n,4})$ and $J_n(r_{n,5})$ (Use data from the given web page) and ask the reader to guess what number the distance between the roots is getting closer to. Maybe they are not a good guesser, so tell them; it is π

Also, ask the reader to note that as x gets larger, $J_n(x)$ seems to go to zero. So pose the questions: can we explain why there are the spacings of these roots like they are? Can we explain why the Bessel functions go to zero? Would we ask these questions if we cannot answer them? Yes, Yes, No.

13) To see how to answer these questions, SKIP ALL THE MATERIAL ABOUT POWER SERIES AND GO DIRECTLY TO THE LIOUVILLE TRICK. L is related to J, so it satisfies a differential equation related to the differential equation for J. Follow (and recopy) the notes to obtain the differential equation for L. Then state the equation for L is "almost" the same as the simpler equation $L''+L=0$. Tell the reader than OFTEN, equations that are close to each other have close solutions, and also warn them this is NOT always the case. Now that they are warned, you ask them to believe that L is in fact close to a combination of sin and cos, AND THESE HAVE SPACINGS BETWEEN ROOTS EXACTLY EQUAL TO π. (Fact (which you need not prove): any expression of the form $A\cos(x)+B\sin(x)$ can be rewritten as $M cos(x+\phi)$, M is some positive number and ϕ is some number. Thus for sure our function $A\cos(x)+B\sin(x)$ has root spacings of π so it is plausible that L has root spacings that are approximately π.

14) Also, we now know that J goes to zero at infinity. Why?

Tell the reader that if they want to see solutions to wave equation coming from Bessel functions, there is the youtube link:

https://www.youtube.com/watch?v=mPlCXR6NI8I

and if they want to see real physical models, they can look at:

https://www.youtube.com/watch?v=PMH3U5mYjSU

2. On this problem, after a couple of warm up parts, you will use Laplace transforms to compute the fundamental solution to a r ODE problem.Typically, in Laplace transform problems, the problem is stated as y as a function of t. For the first couple of parts, we will do that.

   1) Solve the differential equation $y''(t)=t^3, y(0)=0, y'(0)=1$, USING LAPLACE TRANSFORMS (Once again, we are practicing a technique which is rediculously complicated for such a simple problem.)

   2) Solve the differential equation $y''(t)=\delta(t-1/2), y(0)=0, y'(0)=1$, USING LAPLACE TRANSFORMS.

   3) Solve the differential equation $y''(t)=\delta(t-t_0), y(0)=0, y'(0)=1$, where $\delta(t-t_0)$ is the delta

function located at some random point t_0. By the way, the table uses $u_c(t) = u(t-c)$ to represent what we call $H(t-c)$, the shifted Heaviside function. This is almost a fundamental solution to the problem with boundary conditions $y(0) = y(1) = 0$.

4) Look at the solution that you get from part 3). Compute $y(1)$ and $y'(1)$ for your solution, The answers depend on t_0. In particular, call $A(t_0) = y(1)$.

5) Your answer from part 3) is ALMOST the fundamental solution for $y''(t) = f(t), y(0) = 0, y(1) = 0$. It satisfies $y(0) = 0$, but as you will have seen in part 4), it does not satisfy $y(1) = 0$. But, if you take your solution $y = y(t, t_0)$ from part 3) (I write it this way to emphasize that the answer depends on t_0) and substract $v(t) = A(t_0)t$ from it (note that v satisfies $v''(t) = 0$) then it DOES satisfy the ODE with $y(0) = y(1) = 0$ (You do not need to check this, I am just telling you). Your answer will have the Heaviside function in the formula. In the notes on fundamental solutions, there is a formula for the fundamental solution for this problem. Show the two answers are equivalent.

Tell the reader that you know that this is a PDE course, not an ODE course, but the notion of fundamental solution is easier to understand in the ODE context.