7A

$u(x,t)$

= DISPLACEMENT AT LOCATION x, TIME t

$u_{tt} = u_{xx}$ $\quad u(0,t) = u(1,t) = 0$;

$u(x,0) = f(x)$ INITIAL DISPLACEMENT

$u_t(x,0) = g(x)$ INITIAL VELOCITY

GUESS: $u(x,t) = X(x)\,T(t)$

$\vdots$

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = C$$

(1) $X''(x) = C\,X(x)$ EXACTLY AS BEFORE! $\Rightarrow$ $X_N(x) = \sin(N\pi x)$;

$X_N'' = -N^2\pi^2 X_N$. WHAT IS NEW IS

$$\frac{T''(t)}{T(t)} = -N^2\pi^2, \text{ so } T'' = -N^2\pi^2 T$$

$$\rightarrow T_N(t) = A_N \cos(N\pi t) + B_N \sin(N\pi t)$$

$$u_N(x,t) = \left[A_N \cos(N\pi t) + B_N \sin(N\pi t)\right] \sin(N\pi x);$$

SO AS WITH HEAT EQN:

$$u(x,t) = \sum_{N=1}^{\infty} \left(A_N \cos(N\pi t) + B_N \sin(N\pi t)\right) \sin(N\pi x)$$

USE INITIAL CONDITIONS TO FIND A_N, B_N:

$$u(x,0) = f(x) = \sum_{N=1}^{\infty} A_N \sin(N\pi x) \; ;$$

SO $\boxed{A_N = 2\int_0^1 f(x) \sin(N\pi x)}$ NO SURPRISE

BUT SECOND INITIAL CONDITION INFORMATION IS NEW:

$$u_t(x,t) = \sum_{N=1}^{\infty} N\pi \left[-A_N \sin(N\pi t) + B_N \cos(N\pi t)\right] \sin(N\pi x)$$

$$u_t(x,0) = g(x) = \sum_{N=1}^{\infty} \underbrace{(N\pi B_N)}_{d_N} \sin(N\pi x)$$

d_N = NTH FOURIER COEFFICIENT OF g

$$= 2 \int_0^1 g(x) \sin(N\pi x)\, dx$$

$$= N\pi B_N \, ,$$ OR

$$\boxed{B_N = \frac{2}{N\pi} \int_0^1 g(x) \sin(N\pi x)\, dx}$$

JUST AS WITH HEAT EQN, THERE ARE SIMPLE INITIAL CONDITIONS THAT YOU CAN SOLVE BY INSPECTION

$$f(x) = 3\sin(2\pi x) + 11\sin(5\pi x) - 2\sin(7\pi x);$$

$$A_2 = 3, \quad A_5 = 11, \quad A_7 = -2;$$

$$g(x) = 2\sin(\pi x) - 10\sin(4\pi x);$$

$$d_1 = 2 \quad d_4 = -10, \text{ so}$$

$$\underbrace{\left\{ B_N = \frac{d_N}{N\pi} \right\}}_{\text{FOURIER REPRESENTATION OF SOL'N TO WAVE EQN ON } [0,1]} \quad B_1 = \frac{2}{1\pi} \quad B_4 = \frac{-10}{4\pi} = \frac{-5}{2\pi}$$

FOURIER REPRESENTATION OF SOL'N TO WAVE EQUATION ON REAL LINE:

$$u_{tt} = u_{xx}; \quad u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

$$\{ f, g \to 0 \text{ AT } \pm\infty \}$$

RATHER THAN DERIVE FOURIER SOL'N, I WILL SUMMARIZE (SIMILAR TO PREVIOUS CALCULATIONS)

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw} \left[\cos(wt)\hat{f}(w) + \frac{\sin(wt)}{w}\hat{g}(w)\right] dw$$

$$u(x,0) = f(x) \stackrel{?}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw} \hat{f}(w)\, dw \quad \checkmark$$

$$u_t(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw} \left[-w\sin(wt)\hat{f}(w) + \cos(wt)\hat{g}(w)\right] dw$$

$$u(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw} \hat{g}(w)\, dw \quad \checkmark$$

? PDE?

$$\underline{u_{tt}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw} \left[-w^2\cos(wt)\hat{f}(w) - \frac{w^2}{w}\sin(wt)\hat{g}(w)\right] dw$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw} \underline{\left[-w^2\right]} \left[\cos(wt)\hat{f}(w) + \frac{\sin(wt)}{w}\hat{g}(w)\right] dw$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw} \left[\cos(wt)\hat{f}(w) + \frac{\sin(wt)}{w}\hat{g}(w)\right] dw$$

$$\underline{u_{xx}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw} \underline{(iw)^2} \left[\cos(wt)\hat{f}(w) + \frac{\sin(wt)}{w}\hat{g}(w)\right] dw$$

HOW TO DERIVE SOLUTION?
$u_{tt} = u_{xx}$; TAKE FOURIER TRANSFORM WITH RESPECT TO $\underline{x}$; GET ODE FOR EACH FIXED w – SECOND ORDER IN t

2ND SOL'N TO WAVE EQN:

$h_{tt} - h_{xx} = 0$; CHANGE OF VARIABLE!

$u = x+t$, $v = x-t$

$x = \frac{u+v}{2}$ $t = \frac{u-v}{2}$.

$$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial h}{\partial v} \frac{\partial v}{\partial x}$$

$\boxed{\text{CHAIN RULE}}$

$$= \frac{\partial h}{\partial u} + \frac{\partial h}{\partial v}$$

$$\frac{\partial^2 h}{\partial x^2} = \left(\frac{\partial^2 h}{\partial u^2} \frac{\partial u}{\partial x} + \frac{\partial^2 h}{\partial u \partial v} \frac{\partial v}{\partial x}\right) + \left(\frac{\partial^2 h}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^2 h}{\partial v^2} \frac{\partial v}{\partial x}\right)$$

$$= (h_{uu} + 2h_{uv} + h_{vv}).$$

$$\frac{\partial h}{\partial t} = \frac{\partial h}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial h}{\partial v} \frac{\partial v}{\partial t}$$

$$= \frac{\partial h}{\partial u} - \frac{\partial h}{\partial v}$$

!!

$$= \left(\frac{\partial h}{\partial u} - \frac{\partial h}{\partial v}\right)$$

$$\frac{\partial^2 h}{\partial t^2} = \left[\frac{\partial^2 h}{\partial u^2} \frac{\partial u}{\partial t} + \frac{\partial^2 h}{\partial u \partial v}\left(\frac{\partial v}{\partial t}\right)\right] - \left[\frac{\partial^2 h}{\partial u \partial v} \frac{\partial u}{\partial t} + \frac{\partial^2 h}{\partial v^2} \frac{\partial v}{\partial t}\right]$$

$$= [h_{uu} - h_{uv}]$$

$$- [h_{uv} - h_{vv}]$$

$$h_{tt} = [h_{uu} + h_{vv} - 2h_{uv}]$$

$$h_{xx} = [h_{uu} + h_{vv} + 2h_{uv}]$$

$$h_{tt} - h_{xx} = (-4)h_{uv} = 0.$$

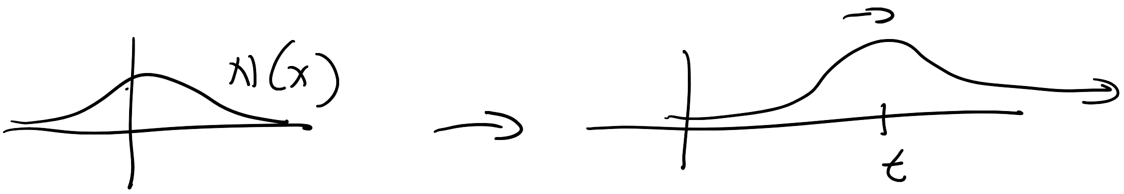
!! $\boxed{h_{uv} = 0}$ 2ND FORM OF WAVE EQUATION.

$$(h_u)_v = 0 \rightarrow (h_u) = \text{"}c\text{"} = m(u)$$

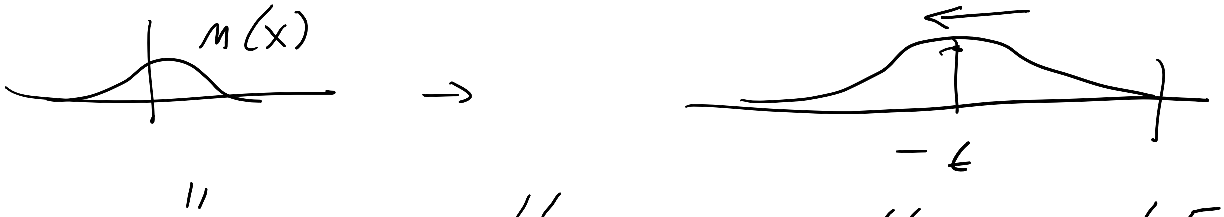
$$h = \int m(u)\,du + \text{"}c\text{"} = \underbrace{\int m(u)\,du}_{M} + N(v) \;!!$$

$$h(u,v) = M(u) + N(v)$$

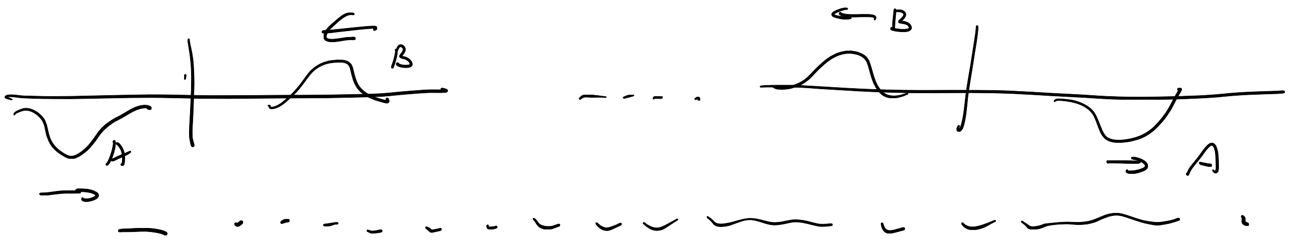
$$h(x,y) = M(x+t) + N(x-t) \quad !!!$$



"WAVE" MOVING "UNCHANGED" RIGHT.



" " " LEFT.



$$u(x,t) = m(x+t) + N(x-t)$$

$$u(x,0) = f(x) = m(x) + N(x)$$

$$u_t(x,t) = m'(x+t) - N'(x-t).$$

$$u_t(x,0) = g(x) = m'(x) - N'(x)$$

$$G'(x) = g(x) \qquad G'(x) = m'(x) - N'(x)$$

$$(?) \quad G(x) = m(x) - N(x)$$

$$m(x) = \frac{f(x) + G(x)}{2} \qquad N(x) = \frac{f(x) - G(x)}{2}$$

$$u(x,t) = \frac{f(x+t) + G(x+t)}{2} + \frac{f(x-t) - G(x-t)}{2}$$

$$= \frac{f(x+t) + f(x-t)}{2} + \frac{G(x+t) - G(x-t)}{2}$$

$$= \boxed{\frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(r)\,dr}$$

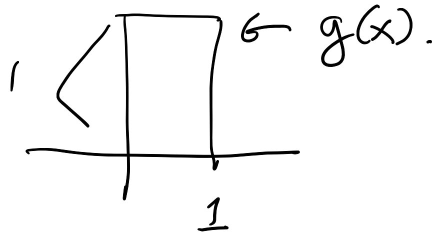
$$u(7,0) = 0$$

$$u(7,1) = \frac{1}{2}\int_{7-1}^{7+1} g = 0 !$$

$$u(7,5) = \frac{1}{2}\int_{7-5}^{7+5} g = 0 !$$

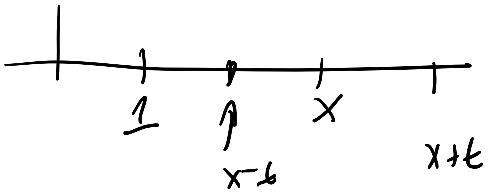
FOR $t < 6$, $u(7,t) = 0$

FINITE SPEED OF PROPAGATION OF WAVE-DISTURBANCE - SINGULARITIES.



$$u(x,t) = \begin{cases} & x-t \geq 1 \quad ① \\ 0 \leq x-t \leq 1 & ② \\ & x-t \leq 0 \quad ③ \end{cases}$$

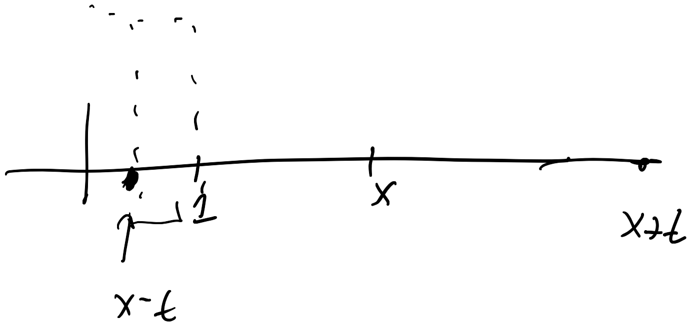
①



③

③ $\frac{1}{2}\int_{x-t}^{x+t} g \, du = \frac{1}{2}$

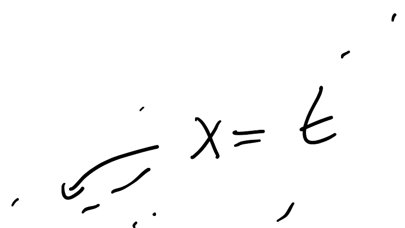
②



① $\frac{1}{2}\int_{x-t}^{x+t} g \, du = 0$

② $\frac{1}{2}\int_{x-t}^{x+t} g \, du = \frac{1}{2}\left(1-(x-t)\right) = \frac{t-x}{2}$

$$u = \begin{cases} 0 & x \geq t+1 \\ \frac{t-x}{2} & t \leq x \leq t+1 \\ \frac{1}{2} & x \leq t \end{cases}$$



$x = t+1$

$t = x-1$