# Math 323 HW 1

Trever McCaffrey
1/13/2021

## ① ODE Review

a) Find the general solution to $y' = -4y$ and then solve the initial value problem with $y(0) = 2$

$$\frac{dy}{dx} = -4y \rightarrow \int \frac{dy}{y} = \int -4 dx$$

$$\ln y = -4x + C$$

$$y = e^{-4x} e^{C} \longrightarrow \boxed{y = Ae^{-4x}}$$

$$2 = Ae^{-4(0)} \rightarrow A = 2 \rightarrow \boxed{y = 2e^{-4x}}$$

—//—

b) Find the general solution to $y' = -4y + 1$ then solve the initial value problem with $y(0) = 2$.

$$\frac{dy}{dx} = -4y+1 \rightarrow \int \frac{dy}{4y-1} = \int -dx \rightarrow \frac{1}{4}\ln 4y-1 = -x + C \rightarrow \ln 4y-1 = -4x + C$$

$$4y - 1 = Ae^{-4x} \rightarrow y = \frac{Ae^{-4x}+1}{4} \rightarrow \boxed{y = Ae^{-4x} + \frac{1}{4}}$$

$$2 = A + \frac{1}{4} \rightarrow A = \frac{7}{4} \rightarrow \boxed{y = \frac{1}{4}(7e^{-4x} + 1)}$$

—//—

c) Find the gen. soln to $y'' = 9y$ then solve the IVP $y(0) = 1$, $y'(0) = 0$

$$\boxed{y = Ae^{3x} + Be^{-3x}} \rightarrow y' = 9Ae^{9x} - 9Be^{-9x}$$

$$1 = A + B \rightarrow A = 1 - B, \quad 0 = 9(1-B) - 9B = 9 - 18B \Rightarrow B = \frac{1}{2}$$

$$A = 1 - \frac{1}{2} = \frac{1}{2} \rightarrow y = \frac{1}{2}\left(e^{3x} + e^{-3x}\right)$$

d) Find the general solution to $y'' = -9y$ and solve the IVP $y(0) = 1$, $y'(0) = 0$

$\boxed{y = A\sin 3x + B\cos 3x}$ $\quad y' = 3A\cos 3x - 3B\sin 3x$

$1 = B$ $\quad 0 = 3A \rightarrow A = 0$

$\rightarrow \boxed{y = \cos 3x}$

② Partial Derivatives Review

a) For the function $f(x,y) = x^2e^y - y\sin(x+y)$, compute $f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$

$$\boxed{\begin{aligned} f_x &= 2xe^y - y\cos(x+y) \\ f_{xx} &= 2e^y + y\sin(x+y) \\ f_{xy} &= 2xe^y - \cos(x+y) + y\sin(x+y) \end{aligned}}$$

$\boxed{f_y = x^2e^y - \sin(x+y) - y\cos(x+y)}$

$f_{yy} = x^2e^y - \cos(x+y) + y\sin(x+y) - \cos(x+y)$

$\rightarrow \boxed{f_{yy} = x^2e^y + y\sin(x+y) - 2\cos(x+y)}$

$\boxed{f_{yx} = 2xe^y + y\sin(x+y) - \cos(x+y)}$

b) $f_{yx} = f_{xy}$ ✓

③ Differential Equation System

Consider the system of differential equations:

$$u_1'(t) = 11u_1(t) - 30u_2(t),$$
$$u_2'(t) = 4u_1(t) - 11u_2(t)$$

$$\vec{u}' = A\vec{u}, \text{ where}$$

$$\vec{u}' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix}, \quad A = \begin{pmatrix} 11 & -30 \\ 4 & -11 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$A\vec{v} = \lambda\vec{v}$$
$$(A - \lambda I)\vec{v} = 0$$
$$\begin{pmatrix} 11-\lambda & -30 \\ 4 & -11-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det A = (11-\lambda)(-11-\lambda) + 120 = 0 = -121 + \lambda^2 + 120$$
$$\lambda^2 = 1$$
$$\rightarrow \lambda = \pm 1$$

$\lambda = 1$: $\left(\begin{array}{cc|c} 10 & -30 & 0 \\ 4 & -12 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 1 & -3 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$v_1 = 3v_2 \rightarrow \vec{v} = v_2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$\lambda = -1$: $\left(\begin{array}{cc|c} 12 & -30 & 0 \\ 4 & -10 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$2v_1 = 5v_2 \rightarrow v_2\begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\rightarrow \boxed{\vec{u}(t) = c_1 e^{t}\begin{pmatrix}3\\1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}5\\2\end{pmatrix}}$$

Initial Conditions: $u_1(0) = 11,\ u_2(0) = 4$

$$\begin{pmatrix}11\\4\end{pmatrix} = \begin{pmatrix}3c_1 + 5c_2\\ c_1 + 2c_2\end{pmatrix} \rightarrow c_1 = 4 - 2c_2$$

$$11 = 3(4 - 2c_2) + 5c_2 = 12 - 6c_2 + 5c_2$$

$$11 = 12 - c_2 \rightarrow \boxed{c_2 = 1}$$

$$c_1 = 4 - 2(1) = 2$$

$$\Rightarrow \boxed{\vec{u}(t) = 2e^{t}\begin{pmatrix}3\\1\end{pmatrix} + e^{-t}\begin{pmatrix}5\\2\end{pmatrix}}$$

④ Differential Equation System 2

$$u_1'(t) = -7u_1(t) - 24u_2(t) \qquad u_1(0) = 17$$
$$u_2'(t) = -24u_1(t) + 7u_2(t) \qquad u_2(0) = -6$$

$$\vec{u}' = A\vec{u} \quad \text{with} \quad A = \begin{pmatrix} -7 & -24 \\ -24 & 7 \end{pmatrix}$$

$$\det(A - \lambda I) = (-7-\lambda)(7-\lambda) - 24^2 = -49 + \lambda^2 - 576 = 0$$
$$\lambda^2 = 625 \rightarrow \underline{\lambda = \pm 25}$$

$$\lambda = +25: \left(\begin{array}{cc|c} -32 & -24 & 0 \\ -24 & -18 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} -4 & -3 & 0 \\ -4 & -3 & 0 \end{array}\right) \rightarrow V_1 = -\tfrac{3}{4}V_2$$

$$\lambda = -25: \left(\begin{array}{cc|c} 18 & -24 & 0 \\ -24 & 32 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 3 & -4 & 0 \\ -3 & 4 & 0 \end{array}\right) \rightarrow V_1 = \tfrac{4}{3}V_2$$

$$\rightarrow \boxed{\vec{u}(t) = c_1 e^{25t}\begin{pmatrix} -3 \\ 4 \end{pmatrix} + c_2 e^{-25t}\begin{pmatrix} 4 \\ 3 \end{pmatrix}}$$

$$-3c_1 + 4c_2 = 17 \rightarrow c_2 = \tfrac{3}{4}c_1 + \tfrac{17}{4}$$
$$4c_1 + 3\left(\tfrac{3}{4}c_1 + \tfrac{17}{4}\right) = -6 = 4c_1 + \tfrac{9}{4}c_1 + \tfrac{51}{4}$$

$$-24 = 16c_1 + 9c_1 + 51 \rightarrow 25c_1 = -75 \rightarrow c_1 = -3$$
$$\rightarrow c_2 = \tfrac{3}{4}(-3) + \tfrac{17}{4} = -\tfrac{9}{4} + \tfrac{17}{4} = 2$$

$$\rightarrow \boxed{u_2(t) = -3e^{25t}\begin{pmatrix} -3 \\ 4 \end{pmatrix} + 2e^{-25t}\begin{pmatrix} 4 \\ 3 \end{pmatrix}}$$

(5) By substituting into the equation, determine whether these functions are solutions to the PDEs.

a) Is $v(x,t) = x+t$ a solution to $v_t + v_x = 0$

$v_x = 1, \; v_t = 1 \longrightarrow 1+1 \neq 0 \Rightarrow \boxed{NO}$

b) $v(x,t) = x+t$ solution to $v_t - v_x = 0$?

$1 - 1 = 0$ ✓ $\Rightarrow \boxed{YES}$

c) Is $v(x,t) = \sin(x+t) - \cos(x+t)$ a solution to $v_{tt} - v_{xx} = 0$?

$v_t = \cos(x+t) + \sin(x+t)$

$v_{tt} = -\sin(x+t) + \cos(x+t)$

$v_x = \cos(x+t) + \sin(x+t)$

$v_{xx} = -\sin(x+t) + \cos(x+t)$

$v_{tt} = v_{xx} \longrightarrow v_{tt} - v_{xx} = 0$ ✓ $\Rightarrow \boxed{YES}$

d) Is $u(x,y) = x^2 - y^2$ a solution to $u_{xx} + u_{yy} = 0$

$u_x = 2x \qquad u_{xx} = 2$

$u_y = -2y \qquad u_{yy} = -2$

$u_{xx} + u_{yy} = 2 - 2 = 0$ ✓ $\Rightarrow \boxed{YES}$

(6) We have the heat equation $u_t = \alpha^2 u_{xx}$ subject to the boundary condition $u(0,t) = u(L,t) = 0$

Now, <u>guess</u> that there are solutions of the form
$$u(x,t) = X(x)T(t)$$

This gives $u_t = X(x)T'(t)$ and $u_{xx} = X''(x)T(t)$
$$\rightarrow \quad X(x)T'(t) = \alpha^2 X''(x)T(t)$$

Divide by XT: $\dfrac{T'(t)}{T(t)} = \alpha^2 \dfrac{X''(x)}{X(x)}$

t only   x only $\Rightarrow$ both sides must be constant

First x: $\alpha^2 \dfrac{X''}{X} = c \rightarrow X'' = \frac{c}{\alpha^2} X$

Three cases: $\begin{cases} c < 0 & \rightarrow c = k^2 \\ c = 0 & \\ c > 0 & \rightarrow c = -k^2 \end{cases}$

Remember: $u(x,t) = T(t)X(x) \Rightarrow u(0,t) = T(t)X(x=0)$
$$\rightarrow 0 = T(t)X(x=0) \rightarrow X(0) = X(L) = 0$$

i) $c = k^2$   $X''(x) = \frac{k^2}{\alpha^2} X(x) \rightarrow X'' - \frac{k^2}{\alpha^2} X = 0$
$$\left[\lambda^2 - \left(\tfrac{k}{\alpha}\right)^2\right] e^{\lambda x} = 0 \; ; \quad \lambda^2 - (k/\alpha)^2 = 0 \rightarrow \lambda = \pm \frac{k}{\alpha}$$

$$X(x) = c_1 e^{kx/\alpha} + c_2 e^{-kx/\alpha}$$
$X(0) = 0 = c_1 + c_2 \rightarrow c_1 = -c_2$, $X(L) = 0 = c_1\left(e^{kL/\alpha} - e^{-kL/\alpha}\right)$
$\rightarrow e^{2kL/\alpha} = 1 \rightarrow k = 0$ X inconsistent with $c > 0$

ii) $c=0$ $\quad X''(x) = 0 \rightarrow X(x) = ax+b$

$X(0)=0=b$

$X(L) = 0 = aL+0 \longrightarrow a=0 \rightarrow X(x)=0$

Not true.

—"—

iii) $c=-k^2$

$$X'' = \frac{-k^2}{\alpha^2}X$$

$$X'' + \frac{k^2}{\alpha^2}X = 0$$

$$\lambda^2 + (k/\alpha)^2 = 0 \rightarrow \lambda^2 = -\frac{k^2}{\alpha^2} \rightarrow \lambda = \pm\frac{ik}{\alpha}$$

$$\rightarrow X(x) = c_1 e^{ikx/\alpha} + c_2 e^{-ikx/\alpha}$$

$$\Rightarrow X(x) = c_1 \cos\frac{kx}{\alpha} + c_2 \sin\frac{kx}{\alpha}$$

$X(0) = 0 = c_1$

$X(L) = 0 = c_2 \sin\frac{kL}{\alpha}$

$c_2 \neq 0 \rightarrow \sin\frac{kL}{\alpha} = 0 \rightarrow \frac{kL}{\alpha} = n\pi \quad (n=1,2,3,\dots)$

$k = n\pi\alpha/L$

$\Rightarrow X_n(x) = \sin\left(\frac{\alpha\pi n x}{L}\right)$ $\quad$ $c_2$ arbitrary

—"—

Also: $\frac{T'(t)}{T(t)} = c \rightarrow T' = -k^2 T$

$$\rightarrow T_n(t) = e^{-(\alpha n\pi/L)^2 t}$$

$$u(x,t) = u_n(x,t) = T_n(t)X_n(x)$$

$$\rightarrow u(x,t) = e^{-(\alpha n\pi/L)^2 t}\sin\left(\frac{\pi n x}{L}\right) \qquad n=1,2,3,\dots$$

* $u_t = u_{xx}$ or $u_t - u_{xx} = 0$ is linear; if $v(x,t)$, $w(x,t)$ are solns, so are $c_1 v(x,t) + c_2 w(x,t)$

Therefore:

$$u(x,t) = c_1 e^{-\alpha^2\pi^2 t/L^2} \sin\left(\frac{\alpha\pi x}{L}\right) + c_2 e^{-4\alpha^2\pi^2 t/L^2} \sin\left(\frac{2\alpha\pi x}{L}\right) + c_3 e^{-9\alpha^2\pi^2 t/L^2} \sin\left(\frac{3\alpha\pi x}{L}\right) + \cdots$$

Is a solution for arbitrary choices of $c_n$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(\alpha n\pi/L)^2 t} \sin\left(\frac{\alpha n\pi x}{L}\right)}$$