PARTIAL DIFFERENTIAL EQNS.

REVIEW ODE'S - AN EQN WITH AN UNKNOWN FUNCTION OF 1 VARIABLE:

$$y''(x) + 3y'(x) + 2y(x) = 0.$$

SOL'N TECHNIQUES, PROPERTIES OF SOL'NS, APPLICATIONS.

A PARTIAL DIFFERENTIAL EQN IS AN EQUATION INVOLVING A FUNCTION OF TWO OR MORE VARIABLES, & ITS PARTIAL DERIVATIVES. HERE ARE SOME IMPORTANT EXAMPLE.

① $u(x,t)$: $\quad u_t = u_{xx}$ (1-D HEAT EQN)

② $u(x,y,t)$ $\quad u_t = u_{xx} + u_{yy}$ (2-D HEAT EQN)

③ $u(x,t)$: $\quad u_{tt} = u_{xx}$ (1-D WAVE EQN)

④ $u(x,y,t)$: $\quad u_{tt} = u_{xx} + u_{yy}$ (2-D WAVE)

⑤ $u(x,y)$: $\quad u_{xx} + u_{yy} = 0$ (Laplace eqn)

EX: 1-D HEAT EQN: $u_t = u_{xx}$

$u(x,t)$ = TEMPERATURE AT TIME t, LOCATION x.

$$u(x,t) = e^{-t} \sin x$$

$$\underline{u_t = -e^{-t} \sin x} \qquad u_{xx} = \underline{-e^{-t} \sin x}$$

PHYSICS: PHYSICAL PROBLEM → EQN;

EASY PART: CHECKING IF A FUNCTION IS A SOL'N.

HARD PART: <u>FINDING SOL'NS.</u>

SOLVING PDE'S → DIFF EQNS (ODE)

SOLVING PDE'S → LINEAR ALGEBRA

① PDE'S → ODE'S $\boxed{\text{SEPARATION OF VARIABLES}}$

② FINDING ALL SOL'NS ⟷ FOURIER SERIES.

③ <u>CONCEPTUAL</u> INTERPRETATION USING LINEAR ALGEBRA.

LET $\mathbb{V}$ = VECTOR SPACE OF COLUMN VECTORS; $\mathbb{V} = R^2 \begin{pmatrix} a \\ b \end{pmatrix}$; $\mathbb{V} = R^3 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ ETC.

LET L = SQUARE MATRIX (2×2, 3×3, ETC.)

IF $v \in \mathbb{V}$, & $v \neq 0$, THEN v IS CALLED AN EIGENVECTOR IF

$LV = \lambda V$, WHERE λ IS A NUMBER, THE EIGENVALUE.

EX: $L = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$; "GUESS" $V = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\underset{L}{\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}} \underset{V}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

"GUESS" $V = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \lambda = 4.$$

HOW DO YOU FIND λ, V?

IF $LV = \lambda V$, THEN $LV = \lambda I V$

$\left\{ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ETC.}, \quad IV = V \text{ ANY } V \right\}$

SO $LV - \lambda I V = 0$ OR $\underline{(L-\lambda I)V = 0}$. 14

$V \neq 0$; SO $(L-\lambda I) = B$ DOES <u>NOT</u> HAVE AN INVERSE $\left\{ \text{IF } BV = 0,\ \begin{aligned} B^{-1}BV &= B^{-1}0 \\ V &= 0 \end{aligned} \right\}$

SO $\det(L-\lambda I) = 0$;

AN <u>EQUATION</u> FOR λ. <u>CHARACTERISTIC EQN</u>.

EX: $L = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ $\quad L - \lambda I$

$$= \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$$ CHAR. EQN:

$$\det \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = 0 = (3-\lambda)^2 - \underline{1}$$

$$= \lambda^2 - 6\lambda + 9 - \underline{1} = \lambda^2 - 6\lambda + 8$$
$$= (\lambda - 4)(\lambda - 2)$$
$$\lambda = 4 \quad !! \quad \lambda = 2.$$

$(L - \lambda I)V = 0$;

$$\lambda = 4 \quad \left(\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \right) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-v_1 + v_2 = 0$$
$$v_1 - v_2 = 0$$

REDUNDANT.

$$\boxed{v_1 = v_2}$$

$$\begin{pmatrix} v_2 \\ v_2 \end{pmatrix} = v_2 \underline{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \longleftrightarrow 4 \quad ; \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} \longleftrightarrow 2.$$

SECOND ORDER, LINEAR, CONSTANT COEFFICIENT, HOMOGENEOUS ODE'S.

(S L O C C H D E ("SLOCHDEE"))

$$a y''(x) + b y'(x) + c y(x) = 0.$$

A MILLION APPLICATIONS. WE WILL NEED SOL'NS:

GUESS: $y(x) = e^{\lambda x}$, λ SOME #.

THEN $(a\lambda^2 + b\lambda + c)\, e^{\lambda x} = 0$;

SO $a\lambda^2 + b\lambda + c = 0$ CHARACTERISTIC EQN

1. TWO ROOTS, REAL & DISTINCT;
2. TWO EQUAL REAL ROOTS
3. COMPLEX ROOTS.

① $1 \cdot y'' - 3y' + 2 \cdot y = 0$; $\underline{L6}$

$1\lambda^2 - 3\lambda + 2 = 0$; $y = e^{\lambda x}$

$(\lambda - 2)(\lambda - 1) = 0$; $\lambda = 2, \lambda = \underline{1}$.

e^{2x}, e^{x} SOL'NS. LINEARITY

IMPLIES $c_1 e^{2x} + c_2 e^{x}$ IS

SOL'N

② $y'' - 2y' + y = 0$;

$\lambda^2 - 2\lambda + \underline{1} = 0 = (\lambda - 1)^2$;

$\lambda = 1, \underline{1}$ $\underline{\text{DOUBLE}}$ $e^x + e^x$ (????)

e^x $\underline{\text{IS}}$ SOL'N, BUT SO IS xe^x.

③ $y'' - 2y' + 2y = 0$;

$\lambda^2 - 2\lambda + 2 = 0$

$$\lambda = \frac{2 \pm \sqrt{4 - 4 \cdot 2}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$$e^{(1+i)x}, \quad e^{(1-i)x}$$

FACT: IF $e^{(a+ib)x}$ IS A SOL'N, SO ARE $\underline{e^{ax}\cos(bx) \text{ \& } e^{ax}\sin(bx)}$

AND $c_1 e^{ax}\cos(bx) + c_2 e^{ax}\sin(bx)$;

$$\left\{ e^{ix} = \cos(x) + i\sin(x) \right\}$$

SYSTEMS OF DIFFERENTIAL EQNS:

$$y'' - 3y' + 2y = 0;$$

$$\left.\begin{array}{ll} u_1 = y & u_2 = y' \\ u_1' = y' = u_2 & \end{array}\right\} \quad y'' = 3y' - 2y$$

$$u_2' = y'' = 3y' - 2y = 3u_2 - 2u_1$$

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

SYSTEM OF FIRST ORDER EQUATIONS

SWITCH FROM x TO t;

$$\vec{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

$$\vec{u}'(t) = L\vec{u}(t) \qquad L = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$$

FACT: L BE ANY 2×2 ($3 \times 3, \dots$) MATRIX. SUPPOSE $L\vec{V} = \lambda \vec{V}$;

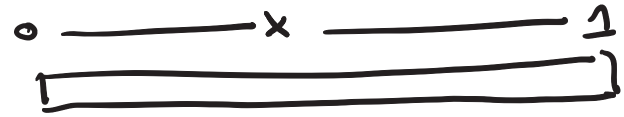
THEN $\vec{u}(t) = \vec{v}\, e^{\lambda t}$ IS A SOL'N TO $\vec{u}'(t) = L\vec{u}(t)$.

CHECK: LHS $u'(t) = \vec{V} \cdot (\lambda) e^{\lambda t}$;

RHS $L(ve^{\lambda t}) = e^{\lambda t} L(V) = \lambda V e^{\lambda t}$

THE HEAT EQN :

$$u_t = u_{xx}$$



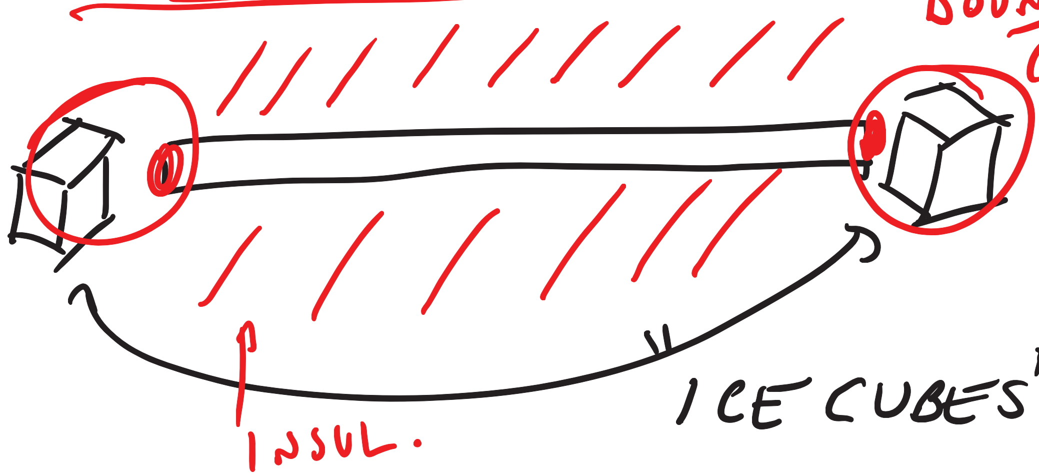
$u(x,t)$ = TEMPERATURE AT TIME t, LOCATION x.

NEED MORE INFO.

① $u(x,0) = f(x)$ INITIAL CONDITION

② $u(0,t) = 0 \qquad u(1,t) = 0$

BOUNDARY CONDITIONS



BOUNDARY CONDITIONS

SEPARATION OF VARIABLES GUESS !!

$$u_t = u_{xx} \qquad u(0,t) = u(1,t) = 0$$

GUESS THAT THERE ARE SIMPLE SOL'NS OF THE FORM.

$$u(x,t) = X(x) \cdot T(t)$$

ANSATZ $= $ "PURE X" . "PURE t"

$$u_t = T'(t)\, X(x) \quad !!!$$

$$u_{xx} = T(t)\, X''(x)$$

$$T'(t)\, X(x) = T(t)\, X''(x)$$

DIVIDE BY $T(t)X(x)$:

$$\frac{T'X}{TX} = \frac{TX''}{TX} \qquad \text{OR}$$

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

PURE t! } PURE x!

→ <u>CONSTANT</u>!

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = C \quad ;$$

$$\boxed{\text{SEPARATION OF VARIABLES TURNS PDE PROBLEM INTO ODE PROBLEM}}$$

LET'S LOOK AT ODE'S.

$$\frac{X''(x)}{X(x)} = c \quad \text{OR} \quad X''(x) = cX$$

i) $c > 0$ $\quad c = k^2$, $k \neq 0$

(ii) $c = 0$

(iii) $c < 0$ $\quad c = -k^2$, $k \neq 0$

S.O.V. GUESS $\boxed{T(t)X(x)}$ $u(x,t)$

$$u(0,t) = 0 = X(0)\,T(t)$$

$X(0) = 0$ (SIMILARLY $X(1) = 0$)

$$X''(x) = cX(x),$$

$$X(0) = X(1) = 0.$$

BOUNDARY VALUE PROBLEM FOR X

(i) $c > 0$ $\quad X''(x) = k^2 X$ $\quad$ 13

$k \neq 0$

$X''(x) - k^2 X(x) = 0.$

(SLOCHEE)

$X(x) = e^{\lambda x}$

$(\lambda^2 - k^2) e^{\lambda x} = 0 ;\quad \lambda^2 - k^2 = 0$

$\lambda^2 = k^2 \quad \lambda = \pm k.$

$X(x) = c_1 e^{kx} + c_2 e^{-kx}$

$X(0) = 0 = c_1 + c_2 = 0 \quad c_2 = -c_1$

$X(x) = c_1 e^{kx} - c_1 e^{-kx}$

$= c_1 \left[e^{kx} - e^{-kx} \right]$

$X(1) = 0 = c_1 \left[e^{k} - e^{-k} \right]$

BUT! $\quad e^{k} - e^{-k} = 0$

$e^{2k} - 1 = 0$

$e^{2k} = 1 \quad k = 0 \quad \not\Leftrightarrow$

(ii) $c=0$. $\quad X''(x) = 0\cdot X = 0$. $c14$

$X(x) = \underline{ax+b}$.

$X(0) = \underline{0 = b}$; $X(x) = ax$

$X(1) = 0$; $X(1) = a\cdot 1 = 0$

$\underline{a = 0}$

$c<0$

(iii) $X''(x) = -k^2 X(x)$;

$X''(x) + k^2 X(x) = 0$;

$\underline{\text{SLOCH}\cancel{\text{EE}}}$

$\lambda^2 + k^2 = 0$;

$\lambda^2 = -k^2$; $\lambda = \pm i k$.

e^{ikx} $\quad ikx = 0 + i\underline{\underline{kx}}$

SO $\cos(kx)$, $\sin(kx)$ SOL'NS.

OR $\quad c_1\cos(kx) + c_2\sin(kx)$

$X''(x)$

$X(0) = 0 = c_1 1 + c_2\cdot 0 \quad \underline{\underline{c_1 = 0}}$

$$X(x) = c_2 \sin(kx)$$

$$X(1) = 0 = c_2 \sin(k)$$

$c_2 = 0$? NO! $\sin(k) = 0$?

$k = \pi \cdot N \qquad N = 1, 2, 3, \ldots$

$$\boxed{X_N(x) = \sin(\pi N x)}$$

c_2 ARBITRARY

$$\frac{T'(t)}{T(t)} = c \quad \left\{ \begin{aligned} T'(t) &= cT \\ &= -k^2 T \\ &= -(\pi N)^2 T \end{aligned} \right.$$

$$T_N(t) = e^{-(\pi N)^2 t}$$

$$u(x,t) = u_N(x,t)$$

$$= T_N(t)\, X_N(x)$$

$$= e^{-\pi^2 N^2 t} \sin(N\pi x)$$

A LOT OF SOL'NS.

$N = 1, 2, 3, \ldots$

$u_t = u_{xx}$, OR $u_t - u_{xx} = 0$

IS LINEAR; IF $v(x,t)$, $w(x,t)$ ARE SOL'NS, SO ARE $c_1 v(x,t) + c_2 w(x,t)$.

SO...

$$
\begin{aligned}
u(x,t) = {} & c_1 e^{-\pi^2 t} \sin(\pi x) \\
& + c_2 e^{-\pi^2 4 t} \sin(2\pi x) \\
& + c_3 e^{-9\pi^2 t} \sin(3\pi x) \\
& \quad \vdots
\end{aligned}
$$

IS A SOL'N, FOR ARBITRARY CHOICE OF c'S.

$u_t = u_{xx}$ (PDE)

$u(0,t) = u(1,t) = 0$ (BOUNDARY COND)

$u(x,0) = f(x)$ (INITIAL CONDITION)

HAVE NOT USED THIS

REMEMBER

$$u_N(x,t) = e^{-\pi^2 N^2 t} \sin(N\pi x);$$

SO $u_N(x,0) = \sin(N\pi x);$

SOLVE $u_t = u_{xx}$ $u(0,t) = u(1,t) = 0$

$$u(x,0) = f(x) = 3\sin(2\pi x) + 2\sin(4\pi x) - \sin(7\pi x)$$

OUR SOL'NS ARE

$$u(x,t) = c_1 u_1(x,t) + c_2 u_2(x,t \ldots$$

SO $u(x,0) = c_1 u_1(x,0) + c_2 u_2(x,0) \ldots$

$$= c_1 \sin(\pi x) + c_2 \sin(2\pi x) + c_3 \sin(3\pi x) \ldots$$

?

$c_1 = 0;$ $c_2 = 3$ $c_3 = 0$ $c_4 = 2$

$c_7 = -1$ 17

so $u(x,t) = 3e^{-4\pi^2 t}\sin(2\pi x) + 2e^{-16\pi^2 t}\sin(4\pi x) - e^{-49\pi^2 t}\sin(7\pi x)$.

IF INITIAL CONDITION $f(x)$ IS A FINITE SUM OF TERMS OF THE FORM ~~$c_n e^{-n^2\pi^2 t}$~~, THEN $u(x,t)$ IS <u>EASY</u> TO WRITE DOWN.

$c_n \sin(n\pi x)$

$C^{C} = \ -1$

$T(t) = e^{-t}$ $\quad X(x) = \sin x$

$$\frac{T'}{T} = \frac{X''}{X} = \frac{-\sin x}{\sin x} = -1$$

$$u = T \cdot X$$

$$\frac{-e^{-t}}{e^{-t}} = -1$$

$$u = e^{-t} \sin x$$