REVIEW

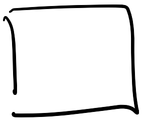
(1) $u_t = u_{xx}$ $\{$ OR $u_{tt} = u_{xx} \}$

$u(0,t) = u(1,t) = 0$

SEPARATION OF VARIABLES $u = T(t)X(x)$

$X''(x) = \lambda X$ $\quad (\lambda = \lambda_n = -\pi^2 n^2)$ $\quad \boxed{\begin{array}{l} X(0)=0 \\ X(1)=0 \end{array}}$

(2) $u_t = u_{xx} + u_{yy}$



$u=0$ ON BOUNDARY.

SEPARATION OF VARIABLE (SOV)

$u = T(t)\,S(x,y)$

HELMHOLTZ

$S_{xx} + S_{yy} = \lambda S$

$S = X(x)Y(y)$

SOV AGAIN!

SQUARE

$\lambda_{m,n} = -\pi^2\left[m^2 + n^2\right]$

$S = X(x)\,Y(y)$ ...

(3)

$u_t = u_{xx} + u_{yy}$

$u=0$ ON BNDRY. $u(1,\theta) = 0$

$= r^2 u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$!!

POLAR HELMHOLTZ

$u = T(t)\,S(r,\theta) \Rightarrow$

$r^2 S_{rr} + \frac{1}{r} S_r + \frac{1}{r^2} S_{\theta\theta} = -\lambda S$ 2

$S = R(r)\,\Theta(\theta)$ $\quad \boxed{S = 0 \text{ ON BOUNDARY}}$ $\quad (\lambda > 0)$ 2

$$\frac{r^2 R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda.$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \lambda = \frac{-\Theta''}{\Theta r^2}$$

$$\frac{r^2 R''}{R} + \frac{rR'}{R} + \lambda r^2 = \frac{-\Theta''}{\Theta} = \ell$$

$\Theta(\theta) = \Theta(\theta + 2\pi)$

BOUNDARY

$0 \le \theta \le 2\pi$

$$\frac{\Theta''}{\Theta} = -\ell$$

$\ell = m^2$

m INTEGER

$\Theta(\theta) = A\sin(m\theta) + B\cos(m\theta)$.

$$\frac{r^2 R''}{R} + \frac{rR'}{R} + \lambda r^2 = m^2$$

$$r^2 R'' + rR' + \lambda r^2 R = m^2 R$$

$$r^2 R'' + rR' + (\lambda r^2 - m^2) R = 0$$

$\lambda > 0$;

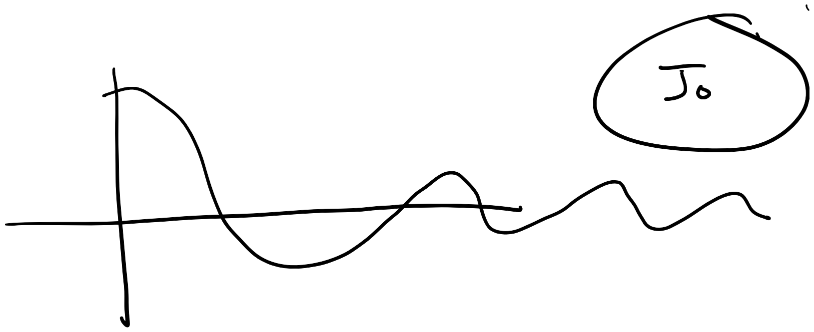
$\lambda = k^2$;

$$(**) \quad \boxed{r^2 R'' + rR' + (k^2 r^2 - m^2) R = 0}$$

TURNS OUT, SOL'N WITH $k = 1$ IS IMPORTANT START.

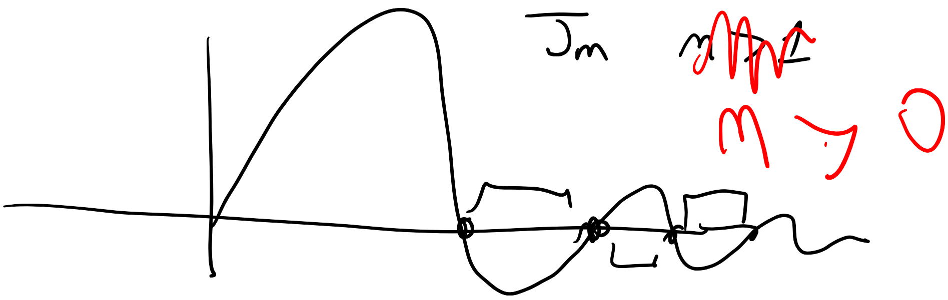
$$(*) \quad r^2 R'' + rR' + (r^2 - m^2) R = 0$$

BESSEL'S EQN. WITH SOL'N {!} $J_m(r)$



ex: $J_0(r)$

1. OSCILLATES
2. $\to 0$ AT ∞
3. SPACINGS OF ROOTS IS NOT QUITE UNIFORM



(★) $r^2 J_m''(r) + r J_m'(r) + (r^2 - m^2) J_m(r) = 0$

$$\underbrace{k^2 r^2 J_m''(kr) + kr J_m'(kr) + (k^2 r^2 - m^2) J_m(kr) = 0}$$

(CALL IT, SAY R)

$R(r) = J_m(kr)$, $R'(r) = J_m'(kr)k$, $R''(r) = k^2 J_m''(kr)$

$$r^2 R'' + r R' + (k^2 r^2 - m^2) R = 0$$

SO SOL'N TO (★★) IS $J_m(kr)$

POLAR HELMHOLTZ

$$S_{rr} + \frac{1}{r} S_r + \frac{1}{r^2} S_{\theta\theta} = -\lambda S$$
$$= -k^2 S.$$

$$S = R(r)\, \Theta(\theta)$$
$$= J_m(kr)\{A \sin(m\theta) + B\cos(m\theta)\}$$

BUT! $S = 0$ ON $\overbrace{\text{BOUNDARY}}$! SO $\underline{\underline{r=1}}$!

$$S(1,\theta) = 0 = J_m(k)\left[A \sin(m\theta) + B\cos(m\theta)\right]$$

SO MUST BE THAT k IS A ROOT OF $J_m(x)$

FOR EACH m, AN INFINITE # OF ROOTS.

$r_{m,n}$ $\quad m = 0, 1, \ldots \quad n = 1, \ldots$

$J_m(r_{m,n}\, r)\sin(m\theta)$ ARE

SOL'NS; EIGENVALUES OF HELMHOLTZ

$= -\lambda = -K^2 = -(r_{m,N})^2$

M = 0 SIMPLEST CASE: RADIALLY SYMMETRIC.

$J_0(r_{0,N} \cdot r)$, $N = 1, 2, \ldots$

SO.. IF YOU WANT TO KNOW SOL'NS TO $\left\{ \begin{matrix} \text{HEAT} \\ \text{WAVE} \end{matrix} \right\}$ EQN, YOU NEED TO SOLVE HELMHOLTZ; TO SOLVE POLAR HELMHOLTZ YOU NEED TO UNDERSTAND BESSEL FUNCS. (= "TRIG FUNCTIONS ON STEROIDS")

DIFFERENTIAL EQUATIONS + POWER SERIES ("INFINITE POLYNOMIALS") & ASIDE

EX: $y'(x) = y(x) \quad y(0) = 1$ $\quad y = e^x$

"DON'T KNOW SOL'N".

GUESS: $y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ldots$

$y(0) = 1 = a_0$

$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \ldots$

$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$a_1 = a_0 \quad 2a_2 = a_1 \quad 3a_3 = a_2 \quad 4a_4 = a_3 \ldots$

$a_0 = 1 \quad \underline{a_1 = 1} \quad a_2 = \frac{a_1}{2} \quad a_3 = \frac{a_2}{3} \quad a_4 = \frac{a_3}{4}$

$a_1 = 1, \quad a_2 = \frac{1}{2} \quad a_3 = \frac{1}{3} \cdot \frac{1}{2} \quad a_4 = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2}$

ASIDE POWER SERIES ("INFINITE POLYNOMIALS").

$1+2+3 \ldots N = \frac{N(N+1)}{2}$.

$$S = 1 + x + x^2 - - - - + x^N$$

$$xS = x + x^2 + - - + x^N + x^{N+1}.$$

$$xS - S = x^{N+1} - 1$$

$$S(x-1) = x^{N+1} - 1$$

GEOMETRIC SUM

$$\boxed{S = \frac{x^{N+1} - 1}{x - 1} = 1 + x + \cdots + x^N}$$

SUPPOSE $|x| < 1$ THEN $|x|^N$ (N HIGH POWER) $\to 0$;

$$\lim_{N \to \infty} \frac{x^{N+1} - 1}{x - 1} = \frac{-1}{x-1} = \lim_{x \to \infty} 1 + x + x^2 \cdots$$

$$= \frac{1}{1-x}.$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 \cdots\cdots$$

IN CALC 3, LEARN HOW TO EXPRESS MORE USEFUL FUNCTIONS THIS WAY:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots\cdots$$

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!}$$

END ASIDE

$a_N = \frac{1}{N!}$; $y(x) = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots$ (5

$= e^x$, OF COURSE.

REWRITE CALCULATION:

$y'(x) = y(x)$ $\quad y(0) = 1$ $\quad y(x) = \sum_{N=0}^{\infty} a_N x^N$

$$y'(x) = \sum_{N=1}^{\infty} N a_N x^{N-1} = \sum_{l=0}^{\infty} (l+1) x^l \cdot a_{l+1}$$

$1!$

COV!!
$l = N-1$
$N = l+1$
$N=1,\ l=0$

$a_0 = y(0) = 1$

l = 'DUMMY VARIABLE'

$$\sum_{l=0}^{\infty} a_l x^l = \sum_{l=0}^{\infty} (l+1) x^l a_{l+1} \quad y'$$

$$\sum_{l=0}^{\infty} x^l \left[a_l - (l+1) a_{l+1} \right] =$$

$$a_l = (l+1) a_{l+1} \qquad \boxed{a_{l+1} = \frac{a_l}{l+1}}$$

RECURSION RELATIONS

RECURSION RELATION, THEN
EXPERIMENT/GUESS FORM OF a_l;
PROVE IT (BY INDUCTION).
FOR MOST IMPORTANT FUNCTIONS, a_l's
HAVE LOTS OF FACTORIALS. ASIDE 2

TECHNICAL ASIDE (CAUCHY, EULER, FROBENIUS)

$$x^2 J_m'' + x J_m' + (x^2 - m^2) J_m = 0 :$$

SIMPLER, RELATED EQN(S)

ASIDE 2: $y'(x) = \frac{1}{1+x}$ $\quad y(0)=0$:

$$(1+x)\,y'(x) = 1 = 1 + 0x + 0x^2 \dots$$

$$y(x) = \sum_{N=0}^{\infty} a_N x^N$$

$$= \cancel{a_0} + a_1 x + a_2 x^2 \dots$$

$$y(0) = 0 = a_0$$

$$y(x) = \sum_{N=1}^{\infty} a_N x^N$$

$$y'(x) = \sum_{N=1}^{\infty} N a_N x^{N-1}$$

$$(1+x)\,y' = \sum_{N=1}^{\infty} a_N \cdot N \left[x^{N-1} + x^N\right]$$

$$= \underbrace{\sum_{N=1}^{\infty} N a_N x^{N-1}} + \sum_{N=1}^{\infty} N a_N x^N$$

$\ell = N - 1$

$N = \ell + 1$

$$\sum_{\ell=0}^{\infty} (\ell+1)\, a_{\ell+1} x^{\ell} + \sum_{N=1}^{\infty} N a_N x^N$$

$$= \sum_{K=0}^{\infty} (K+1)\, a_{K+1} x^K + \sum_{K=1}^{\infty} K a_K x^K = 1 + 0x + 0x^2 \dots$$

$$1 a_1 x^0 + \sum_{K=1}^{\infty} (K+1)\, a_{K+1} x^K + \sum_{K=1}^{\infty} K a_K x^K = 1 + 0x \dots$$

$$a_1 + \sum_{k=1}^{\infty} x^k \left[(k+1)a_{k+1} + k a_k\right] = 1 + 0x + 0x^2 \cdots$$

$$a_1 = 1; \quad (k+1)a_{k+1} + k a_k = 0$$

OR $(k+1)a_{k+1} = -k a_k$

$$a_1 = 1 \qquad \boxed{a_{k+1} = \frac{-k}{(k+1)} a_k}$$

$k=1$ $\quad a_2 = \frac{-1}{2} a_1 = -\frac{1}{2}$

$k=2$ $\quad a_3 = \frac{-2}{3} a_2 = \frac{-2}{3}\left(\frac{-1}{2}\right) = \frac{1}{3}$

$k=3$ $\quad a_4 = \frac{-3}{4} a_3 = \frac{-3}{4} \cdot \frac{1}{3} = -\frac{1}{4}$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 \cdots$$

$$\boxed{+x + -\frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots}$$

$$y' = \frac{1}{1+x} \qquad y(0) = 0;$$

$$y(x) = \int_0^x \frac{dx}{1+x} + \cancel{C}$$

$$y(0) = 0 \qquad \boxed{y(x) = \ln(1+x)}$$

END. {ASIDE}

(CE) $Ax^2y'' + Bxy' + Cy = 0$;

CAUCHY-EULER (LIKE SLCH)

($Ay'' + By' + Cy = 0$, GUESS e^{rx}, $ar^2 + br + c = 0$)

(CE): $y = x^r \quad y' = rx^{r-1} \quad y'' = r(r-1)x^{r-2}$

$$Ax^2(r)(r-1)x^{r-2} + Bx\,rx^{r-1} + Cx^r = 0$$

$$x^r\left[A(r)(r-1) + Br + C\right] = 0.$$

CHARACTERISTIC EQN FOR r.

2 SOL'NS (REAL DISTINCT; DOUBLE; COMPLEX;)

$A = 1$
$B = -1$
$C = -3$

EX: $x^2y'' - xy - 3y = 0.$

$$r(r-1) - r - 3 = 0$$

$$r^2 - 2r - 3 = 0$$

$$(r-3)(r+1) = 0.$$

$y = x^3$

$y = x^{-1}$

$$x^2J'' + xJ' + (x^2 - m^2)J = 0.$$

POWER SERIES METHOD DOES NOT ALWAYS WORK; FROBENIUS MODIFICATION

$$x^2a(x)\,y'' + xb(x)\,y' + c(x)\,y = 0.$$

$a(x) = \sum_{k=0}^{\infty} a_k x^k$; ~~$a_0 = 0$~~; !!! $a_0 \neq 0$!!!

$$b(x) = \sum_{k=0}^{\infty} b_k x^k \qquad c(x) = \sum_{k=0}^{\infty} c_k x^k.$$

THEN

(I) LOOK AT (EASIER) ASSOCIATED CAUCHY-EULER EQN:

$$x^2(a_0)\, y'' + x(b_0) y' + c_0 y = 0$$

("THROW AWAY HIGHER ORDER TERMS")

(II) GET ROOTS TO $\underbrace{a_0 (r)(r-1) + b_0 r + c_0 = 0}_{\text{INDICIAL EQN}}$.

SOL'NS (?) x^{r_1}, x^{r_2}.

$r_1 = 1/2$
$d_0 x^{1/2} + d_1 x^{3/2} + d_2 x^{5/2}$

(III) GUESS SOLUTIONS.

$$x^{r_1} \sum_{k=0}^{\infty} d_k x^k, \quad x^{r_2} \sum_{k=0}^{\infty} e_k x^k;$$

(IV) PLUG INTO DIFFERENTIAL EQN, HAVE FUN!

EX: $x^2 J_m'' + x J_m' + (x^2 - m^2) J = 0$.

RELATED CAUCHY-EULER:

$$x^2 y'' + x y' + (-m^2) y = 0.$$

$y = x^r$

INDICIAL EQN:

$$r(r-1) + r - m^2 = 0$$

$$r^2 - m^2 = 0 \qquad r = \pm m\,!$$

SO FROBENIUS SAYS!

$$y_1(x) = x^m \sum_{k=0}^{\infty} d_k x^k \quad , \quad x^{-m} \sum_{k=0}^{\infty} e_k x^k$$

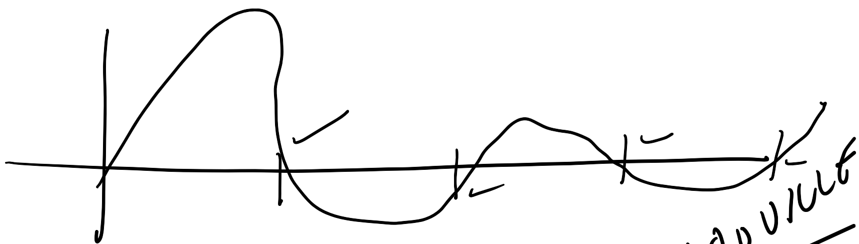
$\{ d_0 \neq 0, \ e_0 \neq 0$ "START WITH" x^m & $x^{-m} \}$

FOR US, $-m$ GIVES UNPHYSICAL SOLUTION.

SO $$\boxed{J_m(x) = x^m \sum_{k=0}^{\infty} d_k x^k}$$

FACT: $$J_m(x) = \frac{x^m}{2^m} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k! \, (k+m)!}$$

PROBLEM: THIS FORMULA IS VERY ACCURATE FOR x NEAR 0. FOR PDE'S, WE NEED ROOTS OF J_m FAR AWAY FROM 0:



$$x^2 J_m'' + x J_m' + (x^2 - m^2) J_m = 0.$$

LIOUVILLE

$J_m = L x^{-1/2}$ [NORMALLY $L(x) = y(x)$] $\cancel{L = x^{1/2} J}$

$$J' = L' x^{-1/2} + L x^{-3/2} \left(-\tfrac{1}{2}\right)$$

$$J'' = L'' x^{-1/2} + 2L' \left(-\tfrac{1}{2}\right) x^{-3/2} + L \left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right) x^{-5/2}$$

$$x^2\left[L'' x^{-1/2} - L' x^{-3/2} + \frac{3}{4} x^{-5/2} L\right]$$

$$+ x\left[L' x^{-1/2} - \frac{1}{2} x^{-3/2} L\right] + L x^{-1/2}(x^2 - m^2) = 0$$

$$L'' x^{3/2} + \frac{3}{4} x^{-1/2} L - \frac{1}{2} x^{-1/2} L + L\left[x^{3/2} - m^2 x^{-1/2}\right] = 0$$

$$L'' + \frac{3}{4}\frac{1}{x^2} L - \frac{1}{2}\frac{1}{x^2} L + L\left[1 - \frac{m^2}{x^2}\right] = 0$$

$$L'' + L\left[1 + \frac{1}{x^2}\left[\frac{1}{4} - m^2\right]\right] = 0$$

$$\boxed{\approx L'' + L = 0 \quad \text{IF } x >> 0.}$$

$$L(x) \cong A\cos(x) + B\sin(x) \quad x >> 0.$$

$$J_m(x) \cong \frac{A\cos(x) + B\sin(x)}{\sqrt{x}} \quad x >> 0.$$

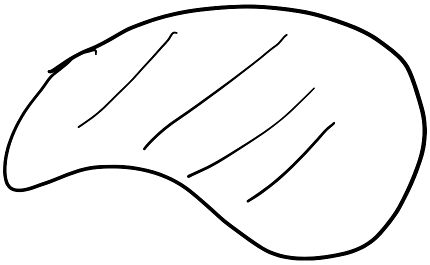
$$A\cos(x) + B\sin(x) = R\cos(x + a)$$

$$R = \sqrt{A^2 + B^2}$$

$$a = \tan^{-1}\left(\frac{B}{A}\right)$$

2π PERIODIC.

SPACING APPROXIMATELY π.

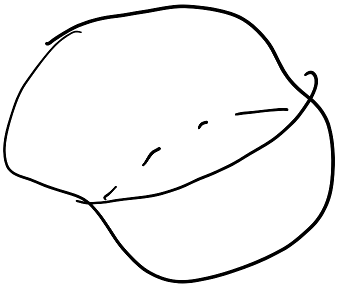


$$u_{tt} = u_{xx} + u_{yy}$$

$u = 0$ ON BOUNDARY,

NEED TO SOLVE RELATED

$$S_{xx} + S_{yy} = \lambda S, \quad S = 0 \text{ ON BNDRY.}$$



VIBRATING MASS (EARTH!)

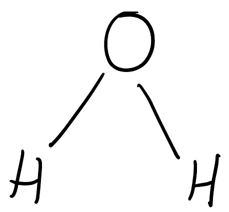
$$S_{xx} + S_{yy} + S_{zz} = \lambda S;$$

RELATED MODIFICATION:

$$\left[\begin{array}{l} S_{xx} + S_{yy} + S_{zz} + \underline{V(x,y,z)} S = \lambda S. \\ \{ T.I.S.E. \} \end{array} \right.$$

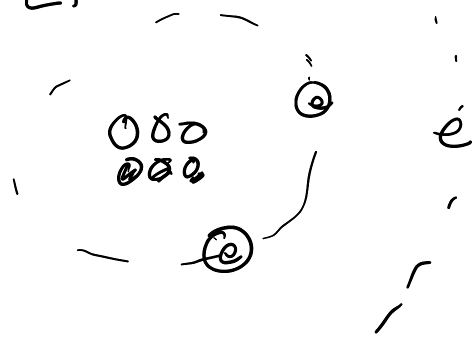
potential

$\underline{V(x,y,z)}$ CAN HAVE PHYSICAL MEANING: FOR EXAMPLE, $V(x,y,z)$ IN QUANTUM CHEMISTRY RELATES TO STRUCTURE OF MOLECULES, ATOMS:



;

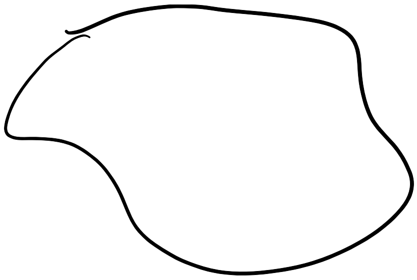
^{6}Li



PROBLEM 1: GIVEN V FIND λ'S.

PROBLEM 2: (INVERSE) GIVEN λ'S, "FIND V". SPECTROSCOPY!

TO UNDERSTAND RELATIONSHIP BETWEEN ① & ② LOOK AT SIMPLER PROBLEM!



$$S_{xx} + S_{yy} = \lambda S$$

$S = 0$ BNDRY.

$$\lambda = -k_1^2, -k_2^2, \ldots$$

CAN USE S's TO SOLVE WAVE EQN;

k_i's GIVE FREQUENCIES OF VIBRATIONS

$$\left\{ \frac{T''(t)}{T(t)} = -k_i^2 \qquad T(t) = A\cos(k_i t) + B\sin(k_i t) \right\}$$

SO IF A DRUM IS HIT, YOU CAN "HEAR" k_i's. CAN YOU TELL WHAT THE DRUM LOOKS LIKE?

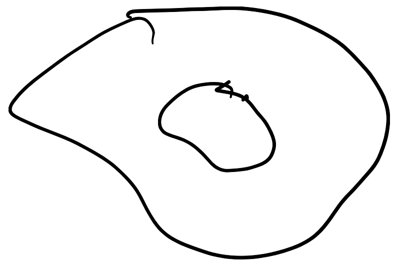
"CAN YOU HEAR THE SHAPE OF A DRUM?"

IF YOU HAVE <u>ALL</u> k_i's $\{??\}$, YOU CAN FIGURE OUT:

① AREA

② LENGTH OF BOUNDARY

③ # OF HOLES !



BUT! YOU CANNOT

HEAR THE SHAPE! (ISOSPECTRAL DOMAINS)