Phys 327 HW 6

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2/21/2022

① Griffiths 5.9, modified

In example 5.1 & Problem 5.5b we ignored spin. Do it now for particles of spin 1/2. Construct the four lowest-energy configurations, and specify their energies and degeneracies. Suggestion: use notation $\psi_{n_1 n_2}|s\,m\rangle$, where

$$\psi_{n_1 n_2}(x_1, x_2) = \psi_{n_1}(x_1)\psi_{n_2}(x_2)$$

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With spin, two particles occupy the state:

$$\psi_{n_1 n_2}(x_1, x_2)\,\chi(1,2)$$

or

$$\psi_{n_1 n_2}|s\,m\rangle$$

The state $\psi_{11}$ is symmetric, so it must be paired with the antisymmetric singlet configuration:

$1^{st}$ State: $\boxed{\psi_{11} |0\,0\rangle}$

It has energy $(1^2+1^2)K = \boxed{\frac{\pi^2\hbar^2}{ma^2}}$

The first excited state $\psi_{12}$ is antisymmetric and so may be paired with any of the three symmetric triplet configurations:

Next 3 states:

$$\boxed{\begin{array}{l} \psi_{12} |1\,1\rangle \\ \psi_{12} |1\,0\rangle \\ \psi_{12} |1\,{-1}\rangle \end{array}}$$

Each state has energy $(1^2+2^2)K = \boxed{\frac{5\pi^2\hbar^2}{2ma^2}}$

② Griffiths, 5.29

Suppose you have three particles in a potential such that the wave function is seperable into products of one-particle states. Also suppose that there are only three distinct one-particle states $\psi_a(x_1)$, $\psi_b(x_2)$, and $\psi_c(x_3)$.

- How many different three-particle states can you construct if the particles are:
  a) distinguishable, b) identical bosons, c) identical fermions?

The particles do not necessarily have to be in different states: if the particles are distinguishable, one possible state is $\psi_a(x_1)\psi_a(x_2)\psi_a(x_3)$

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aaa

abc

a) For distinguishable particles, just count the # of possible permutations of

$$\psi_a(x_1)\psi_b(x_2)\psi_c(x_3)$$

$$\rightarrow \quad 3 \cdot 3 \cdot 3 = \boxed{27}$$

b) For identical bosons, the ordering of states doesn't matter.

$\psi_a\psi_a\psi_a$ $\quad$ $\psi_c\psi_c\psi_a$

$\psi_b\psi_b\psi_b$ $\quad$ $\psi_c\psi_c\psi_b$

$\psi_c\psi_c\psi_c$ $\quad$ $\psi_a\psi_b\psi_c$

$\psi_a\psi_a\psi_b$ $\quad \Longrightarrow \quad$ 10

$\psi_a\psi_a\psi_c$

$\psi_b\psi_b\psi_a$

$\psi_b\psi_b\psi_c$

c) No two identical fermions can occupy the same state, so the only possible state is $\psi_a \psi_b \psi_c \rightarrow \boxed{1}$

③ Griffiths 5.21

The density of copper is $8.96\,g/cm^3$ and its atomic weight is $63.5\,g/mol$.

a) Calculate the Fermi energy for copper. Assume $d=1$ and give answer in eV.

$$E_F = \frac{\hbar^2}{2m}\left(3\rho\pi^2\right)^{2/3}$$

$$\rho \equiv \frac{Nd}{V} = \frac{M_{tot}\,d}{\mu V} = \frac{\rho_m}{\mu} = \frac{8.96\,g/cm^3}{63.5\,g/mol} = 0.141\,mol/cm^3$$

$$= 1.41 \times 10^5\,mol/m^3$$

$$= 8.5 \times 10^{28}\,atoms/m^3$$

$$\rightarrow E_F = \frac{(1.05 \times 10^{-34}\,Js)^2}{2(9.11 \times 10^{-31}\,kg)}\left[3\left(8.5 \times 10^{28}\,atoms/m^3\right)\pi^2\right]^{2/3}$$

$$E_F = 1.1 \times 10^{-18}\,J \times \frac{1\,eV}{1.6 \times 10^{-19}\,J} = \boxed{7\,eV}$$

b) What is the correspond electron velocity?

$$E_F = \frac{1}{2}mv^2 \rightarrow v = \sqrt{2E_F/m} = \sqrt{2(1.1 \times 10^{-18}\,J)/(9.11 \times 10^{-31}\,kg)}$$

$$\rightarrow \boxed{v \approx 1561\,km/s}$$

$$\frac{v}{c} = \frac{1561}{300000} = .005 \rightarrow \text{electrons non-relativistic} \checkmark$$

c) At what temperature does the Fermi energy equal its characteristic thermal energy?

$$E_F = k_B T$$

$$\rightarrow T = \frac{1.1 \times 10^{-18} J}{1.38 \times 10^{-23} J/K} = \boxed{79710 K}$$

d) Calculate the degeneracy pressure of copper in the free electron gas model.

$$P = \frac{(3\pi^2)^{2/3} \hbar^2}{5m} \rho^{5/3}$$

$$= \frac{(3\pi^2)^{2/3} (1.05 \times 10^{-34} Js)^2}{5(9.11 \times 10^{-31} kg)} (8.5 \times 10^{28} \text{ atoms}/m^3)^{5/3}$$

$$\rightarrow \boxed{P = 3.8 \times 10^{10} N/m^2}$$

(4) Griffiths, Problem 5.30

Calculate the Fermi Energy for non-interacting electrons in a two-dimensional infinite square well. Let $\sigma$ be the number of free electrons per unit area.

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In two dimensions, Eq. 5.50 reduces to:

$$E_{n_x n_y} = \frac{\hbar^2 \pi^2}{2m}\left(\frac{n_x^2}{l_x^2} + \frac{n_y^2}{l_y^2}\right) = \frac{\hbar^2 k^2}{2m}, \quad \vec{k} = (k_x, k_y)$$

For a two-dimensional $k_x$-$k_y$ space, with lines at $k_x = \pi/l_x, 2\pi/l_x$ and $k_y = \pi/l_y, 2\pi/l_y$, each intersection point represents a distinct one-particle stationary state. Each state occupies an area

$$\frac{\pi^2}{l_x l_y} = \frac{\pi^2}{A}$$

If we have N atoms, each has d electrons; no two electrons can occupy the same state. They will fill one quadrant of a circle in $k_x$-$k_y$ space, with radius $k_F$ given by:

$$\frac{1}{4}(\pi k_F^2) = \frac{2Nd}{2}\left(\frac{\pi^2}{A}\right) \longrightarrow k_F = \left(\frac{2Nd\pi}{A}\right)^{1/2}$$

or $k_F = \sqrt{2\pi\sigma}$

$$\Longrightarrow \quad E_F = \frac{\hbar^2 k_F^2}{2m} = \boxed{\frac{\hbar^2 \pi \sigma}{m}}$$