SUMMARY:

U

$u_t = u_{xx}$ $\quad \left\{ u_t = \alpha^2 u_{xx} \right\}$ PDE

$u(0,t) = u(1,t) = 0$ BC

$u(x,0) = f(x)$ INITIAL COND. IC

"BIG IDEA": IGNORE IC FOR A MOMENT: FIND "BUILDING BLOCKS" OF THE FORM (CLEVER GUESS!)

$u(x,t) = X(x)\,T(t)$ $\quad$ $\boxed{\text{SEPARATION OF VARIABLES}}$

"ANSATZ" = GUESS

⋮

ODES: $X''(x) = c\,X(x)$ (1), $T'(t) = c\,T(t)$ (2)

$BC \Rightarrow X''(x) = c\,X(x)$, $\quad X(0) = 0$, $X(1) = 0$

⋮ $c < 0$; $\quad c = -\pi^2 N^2$, $N = 1, 2, \ldots$

$X_N(x) = \sin(N\pi x)$

$T_N(t) = e^{-\pi^2 N^2 t}$

SO $u_N(x,t) = e^{-\pi^2 N^2 t} \sin(N\pi x)$

LINEARITY $\Rightarrow$ $\boxed{u(x,t) = \sum_{N=1}^{\infty} C_N e^{-\pi^2 N^2 t} \sin(N\pi X)}$ L^2

SOL'N, FOR ANY $C_1, C_2, \ldots$

<u>NOW</u> USE INITIAL CONDITION:

$$f(x) = u(x,0) = \sum_{N=1}^{\infty} C_N \sin(N\pi x)$$

$\{t = 0\,!\}$

FOURIER SAYS ONE MORE "TRICK" THEN SOME TRIG INTEGRALS

$$C_N = 2\int_0^1 f(x)\sin(\pi N X)\,dx$$

THAT'S IT!

COMPUTATIONALLY, THIS CAN LEAD TO ANNOYING INTEGRALS.

BUT! SOMETIMES EASY. (FINITE SUMS)

EX: $$C_2 e^{-4\pi^2 t}\sin(2\pi X) + C_5 e^{-25\pi^2 t}\sin(5\pi X) + C_7 e^{-49\pi^2 t}\sin(7\pi X)$$

ALL OTHER $C_K = 0$

TRY $C_2 = 3$, $C_5 = -3$, $C_7 = 4$

(3)

THEN $u(x,t) =$

$$3e^{-4\pi^2 t}\sin(2\pi x)$$
$$-3e^{-25\pi^2 t}\sin(5\pi x)$$
$$+4e^{-49\pi^2 t}\sin(7\pi x).$$

$$u(x,0) = 3\sin(2\pi x) - 3\sin(5\pi x) + 4\sin(7\pi x) = f(x);$$

CONVERSELY, $u_t = u_{xx}$ $u(0,t) = u(1,t) = 0$

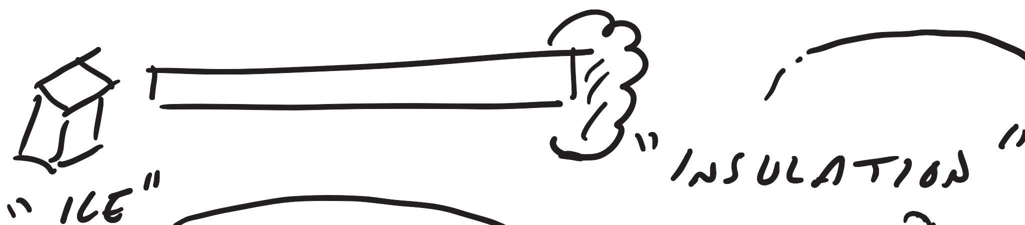
$$u(x,0) = f(x) = 3\sin(2\pi x) - 3\sin(5\pi x) + 4\sin(7\pi x)$$

YOU CAN IMMEDIATELY WRITE DOWN SOLN

$$u(x,t) = 3e^{-4\pi^2 t}\sin(2\pi x) - 3e^{-25\pi^2 t}\sin(5\pi x) + 4.\ e^{-49\pi^2 t}\sin(7\pi x)$$

YOU CAN FILL IN MISSING $T_N(t)$ FACTORS WITH NO COMPUTATION!

MANY MODIFICATIONS OF INCREASING COMPLEXITY. WE WILL JUST LOOK AT A COUPLE:



$$u_t = u_{xx}, \quad u(0,t) = 0 \quad u_x(1,t) = 0$$

$$u(x,0) = f(x)$$

① SEPARATION OF VARIABLES (SOV) GUESS

$$u(x,t) = X(x)\,T(t)$$

$$\vdots$$

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} \overset{!!}{=} C$$

$$X''(x) = C\,X(x) \qquad \boxed{\underline{\text{AND}} \quad \text{(i)} \quad X(0) = X'(1) = 0}$$

{ LIKE $Av = \lambda v$! EIGENVECTOR! }

(i) $c > 0$? (ii) $c = 0$? (iii) $c < 0$?

NO NO

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(iii) $c < 0$ $\quad c = -k^2$

$$X''(x) = -k^2 X(x)$$

$$X(x) = A\cos(kx) + B\sin(kx).$$

$$X(0) = 0 = A\cos(0) + B\sin(0)$$
$$= A \longrightarrow A = 0$$

$$X(x) = B\sin(kx)$$

$$X'(x) = kB\cos(kx)$$

$$X'(1) = 0 = kB\cos(k);$$

$$\cos\left(\frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = \cos\left(\frac{(2N+1)\pi}{2}\right) \cdots$$
$$= 0$$

SO $\quad X_N(x) = \sin\left(\frac{(2N+1)\pi x}{2}\right);$

$$c = -k^2 = -\frac{(2N+1)^2\pi^2}{4}$$

$\left\{T_N' = c_N T\right\}$ $\quad T_N(t) = e^{-\frac{(2N+1)^2\pi^2}{4}t}$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{(2n+1)^2\pi^2 t}{4}} \sin\left(\frac{(2n+1)\pi}{2}x\right)$$

$$f(x) = u(x,0) = \sum_{n=1} c_n \sin\left(\frac{(2n+1)\pi}{2}x\right) \quad (1)$$

$$c_n = 2\int_0^1 f(x)\sin\left(\frac{(2n+1)\pi}{2}x\right)dx .$$