Homework 2, Math 387, Winter 2021

Exercise 1.1 In this exercise we are working in the field $\mathbb{Z}_5$.

(i) Compute $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$.

(ii) Find $\begin{pmatrix} 2 & 2 & 0 \\ 1 & 4 & 3 \\ 1 & 0 & 0 \end{pmatrix}^{-1}$.

Exercise 1.2 In this exercise we are working in the field $\mathbb{C}$. Make sure you write the final answers in the form $a + bi$, with $a, b \in \mathbb{R}$. For instance, $\frac{1+i}{2-i}$ should not be left as a final answer, but be reworked as

$$\frac{1+i}{2-i} = (\frac{1+i}{2-i})(\frac{2+i}{2+i}) = \frac{2+i+2i+i^2}{2^2+1^2} = \frac{1+3i}{5} = \frac{1}{5} + \frac{3i}{5}.$$

Notice that in order to get rid of i in the denominator, we decided to multiply both numerator and denominator with the complex conjugate of the denominator.

(i) Find $\det \begin{pmatrix} 3+i & 1-2i \\ 1+2i & -i \end{pmatrix}$.

(ii) Find $\begin{pmatrix} 2-i & 2+i \\ 4-i & 4 \end{pmatrix}^{-1}$.

Exercise 1.3 Let $\mathbb{F} = \mathbb{Z}_5$. Determine whether $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2$, where

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}.$$

If so, write $\mathbf{b}$ as a linear combination of $\mathbf{a}_1, \mathbf{a}_2$

Exercise 1.4 Let $\mathbb{F} = \mathbb{Z}_3$. Find the set of all solutions to the system of linear equations

$$\begin{cases} x_1 + x_2 & = 2 \\ 2x_1 + x_2 + x_3 = 0 \end{cases}$$

Exercise 1.5 Let $\mathbb{F} = \mathbb{R}$. Consider the matrix vector equation $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{pmatrix} 1 & 2 & 2 \\ 1 & \alpha & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}.$$

Determine $\alpha \in \mathbb{R}$ so that A is invertible and $x_1 = x_2$. (Hint: use Cramer's rule.)

Exercise 1.6 Recall that the *trace* of a square matrix is defined to be the sum of its diagonal entries. Thus $\mathrm{tr}[(a_{ij})_{i,j=1}^n] = a_{11} + \cdots + a_{nn} = \sum_{j=1}^n a_{jj}$.

(a) Show that if $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times n}$, then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(b) Give an example of matrices $A, B, C \in \mathbb{F}^{n \times n}$ so that $\mathrm{tr}(ABC) \neq \mathrm{tr}(BAC)$.