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# ANALYTICAL FIT TO THE LUMINOSITY DISTANCE FOR FLAT COSMOLOGIES WITH A COSMOLOGICAL CONSTANT

UE-LI PEN
Harvard-Smithsonian Center for Astrophysics, 60 Garden Street, Cambridge, MA 02138
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## ABSTRACT

We present a fitting formula for the luminosity and angular diameter distances in cosmological models with pressureless matter, a cosmological constant, and zero spatial curvature. The formula has a relative error of less than 0.4% for $0.2 < \Omega_0 < 1$ for any redshift and a global relative error of less than 4% for any choice of parameters.

*Subject headings:* cosmology: theory — distance scale

## 1. INTRODUCTION

It is presently fashionable to consider spatially flat cosmological models with a cosmological constant (Ostriker & Steinhardt 1995). Several projects are underway to measure the magnitude of the cosmological constant, usually based on the luminosity distance (Perlmutter et al. 1997), the angular diameter distance (Pen 1997), or the volume distance (Kochanek 1996). Unfortunately, these distances are only expressible in terms of elliptic functions (Eisenstein 1997). In order to simplify the repeated computation of difficult transcendental functions or numerical integrals, we present a fitting formula with the following properties:

1. It is exact for $\Omega_0 \to 1^-$ and $\Omega_0 \to 0^+$ at all redshifts.
2. The relative error tends to zero as $z \to \infty$ for any value of $\Omega_0$.
3. For the range $0.2 \leq \Omega_0 \leq 1$, the relative error is less than 0.4%.
4. For any choice of parameters, the relative error is always less than 4%.

Without further ado, the luminosity distance is given as

$$d_L = \frac{c}{H_0}(1+z)\left[\eta(1, \Omega_0) - \eta\left(\frac{1}{1+z}, \Omega_0\right)\right],$$

$$\eta(a, \Omega_0) = 2\sqrt{s^3+1}\left[\frac{1}{a^4} - 0.1540\frac{s}{a^3} + 0.4304\frac{s^2}{a^2} + 0.19097\frac{s^3}{a} + 0.066941 s^4\right]^{-1/8},$$

$$s^3 = \frac{1-\Omega_0}{\Omega_0}. \tag{1}$$

We have used the Hubble constant $H_0$ and the pressureless matter content $\Omega_0$. We recall that an object of luminosity $L$ has flux $F = L/(4\pi d_L^2)$.

## 2. APPROXIMATION

Spatial flatness allows us to rewrite the Friedmann-Robertson-Walker metric as a conformally flat spacetime

$$ds^2 = a^2(-d\eta^2 + dr^2 + r^2\, d\Omega), \tag{2}$$

where $\eta$ is the conformal time and $r$ is the comoving distance. We have set the speed of light $c = 1$. The scale factor $a$ can be normalized in terms of the redshift $a \equiv 1/(1+z)$. The Friedmann equations determine

$$\left(\frac{da}{d\eta}\right)^2 = a\Omega_0 + a^4\Omega_\Lambda, \tag{3}$$

where spatial flatness requires $\Omega_0 + \Omega_\Lambda = 1$. A change of variables to $u = as$, where $s^3 = (1-\Omega_0)/\Omega_0$, allows us to express equation (3) parameter free:

$$\eta = \sqrt{\frac{s^3+1}{s}}\int_0^{as}\frac{du}{\sqrt{u^4+u}}. \tag{4}$$

We can asymptotically approximate equation (4) as

$$\eta_1 = 2\sqrt{\frac{s^3+1}{s}}\left[u^{-n} + n(2X)^{2n+1}u^{-1} + (2X)^{2n}\right]^{-1/2n} \tag{5}$$

where $X \equiv [\int_0^\infty du/(u^4+u)]^{1/2} = 3\pi^{1/2}/[\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})] \simeq 0.3566$; $n$ is a free parameter, and at this stage a choice of $n = 3$ approximates all distances to better than 14% relative accuracy. Equation (5) satisfies the following conditions: First, it converges to equation (4) as $u \to 0$ and $u \to \infty$. Second, its derivative converges to the derivative of $\eta$ as $u \to \infty$. To improve on equation (5) we consider a polynomial expansion of $u^{-1}$ in the denominator with two free parameters by setting $n = 4$:

$$\tilde{\eta} = 2\sqrt{\frac{s^3+1}{s}}\left[u^{-4} + c_1 u^{-3} + c_2 u^{-2} + 4(2X)^9 u^{-1} + (2X)^8\right]^{-1/8}. \tag{6}$$

We will now choose the coefficients $c_1$, $c_2$ to minimize the relative error in the approximate luminosity distance $\tilde{d}_L$,

$$e_\infty \equiv \|(\tilde{d}_L - d_L)/d_L\|_\infty, \tag{7}$$

where the subscript $\infty$ indicates the infinity norm, i.e., the maximal value over the domain. The error in equation (7) tends to be dominated by $z = 0$ (see, for example, Fig. 1), for which we can express

$$e_\infty = \left\|\sqrt{u^4+u}\,\frac{d\tilde{\eta}}{du} - 1\right\|_\infty. \tag{8}$$

Globally optimizing equation (8) allows us to reduce the error to about 2%. But we choose the following trade-off for current cosmological parameters: We want to minimize equation (8) over the range $0.2 \leq \Omega_0 \leq 1$, which covers the

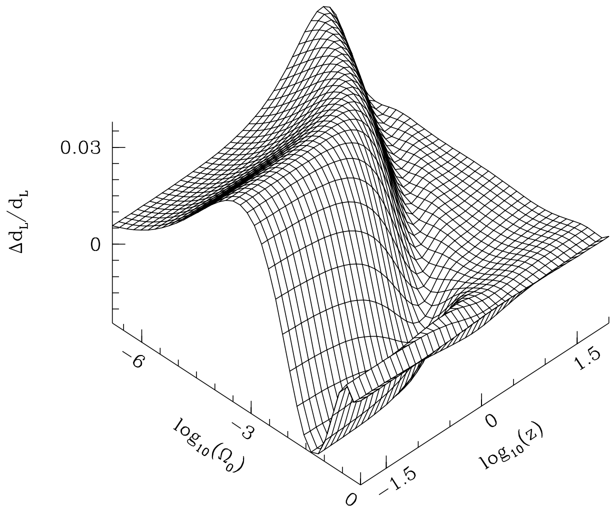


FIG. 1.—Global error surface. Our minmax fit minimized all errors for $\log_{10}\Omega_0 \geq -0.7$, where the error is less than 0.4% for all redshifts.

popularly considered parameter space. In that range, we find through a nonlinear equation solver that $c_1 = -0.1540$ and $c_2 = 0.4304$. This allows us to trade the global error of 2% for a global error of 4%, while reducing the error in the range of interest to 0.4%. The global error surface plot is shown in Figure 1. The error at $z \to 0$ is shown in Figure 2. At small $z$, any errors in $\tilde{\eta}$ are amplified, even though the global errors in $\eta(z)$ are generally significantly smaller. Figure 3 shows the fit for the conformal time $\tilde{\eta}$ and its residual, which is accurate to 0.2% globally.

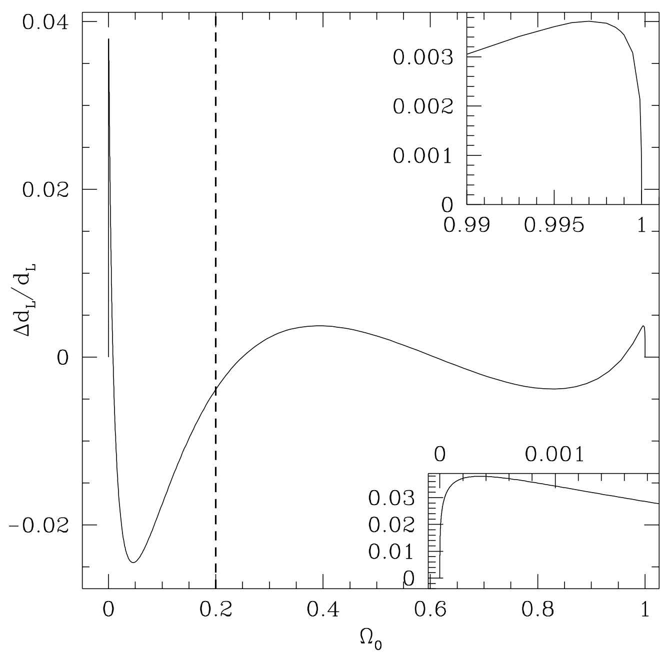


FIG. 2.—Relative error at $z \to 0$, which dominates the global error budget. The minmax errors were optimized in the range $0.2 \leq \Omega_0 \leq 1$, which is the area to the right of the vertical dashed curve. The error goes to zero as a power law at both ends of the graph, which is shown in the two insets.

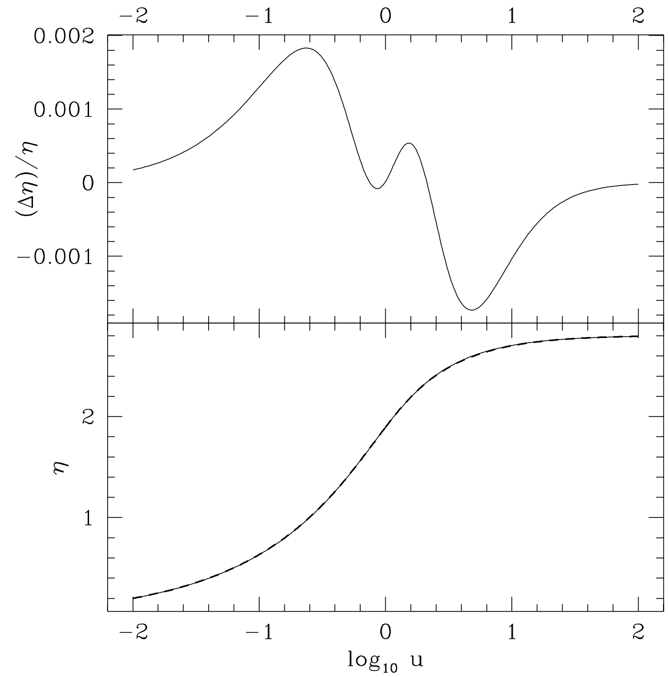


FIG. 3.—Exact and approximate conformal time. *Lower*: conformal time as a function of $u = [(1-\Omega_0)/\Omega_0]^{1/3}/(1+z)$ for the exact integral (*solid line*) and the approximation of eq. (1) (*dashed line*). These two lines are practically indiscernible. *Upper*: the relative error $(\tilde{\eta} - \eta)/\eta$.

One can always express the luminosity distance as a power series expansion around $z = 0$ using equations (1) and (4). For a flat universe, one obtains

$$\frac{H_0}{c}\, d_L = z + \left(1 - \frac{3}{4}\,\Omega_0\right) z^2 + (9\Omega_0 - 10)\,\frac{\Omega_0}{8}\, z^3 + \cdots\;; \tag{9}$$

The series converges only slowly for $z \sim 1$: even when expanded to sixth order in $z$, the maximal relative error in the interval $0 < z < 1$ and $0.2 < \Omega_0 < 1$ using the series expansion in equation (9) is 37%.

## 3. CONCLUSION

We have presented a simple algebraic approximation to the luminosity distance $d_L$ and the proper angular diameter distance $d_A = d_L/(1+z)^2$ in a flat universe with pressureless matter and a cosmological constant.

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