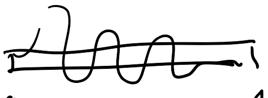
$u_t = u_{xx}$ $\quad u(0,t) = u(1,t) = 0;$

$u(x,0) = f(x);$

$u_{tt} = u_{xx}$ $\quad u(0,t) = u(1,t) = 0$

$u(x,0) = f(x) \quad u_t(x,0) = g(x).$



DISPLACEMENT (u_{tt})

TEMPERATURE (u_t)

SEPARATION OF VARIABLES $\quad u(x,t) = X(x)T(t)$

(H) $\quad \dfrac{T'(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = C \to X''(x) = C X(x)$

$X(0) = X(1) = 0;$

!!

(W) $\quad \dfrac{T''(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = C$

SAME !

START DISCUSSING RELATED 2-D

PROBLEMS:

$u_t = u_{xx} + u_{yy}$ (H)

$u_{tt} = u_{xx} + u_{yy}$ (W).

$u = 0$ ON BOUNDARY

!!

LINE SEGMENTS EASY

SHAPES HARDER.

(H) $\quad u(x,y,0) = f(x,y)$

(W) $\quad u(x,y,0) = f(x,y)$ (D)

$u_t(x,y,0) = g(x,y).$ (V)

START WITH HEAT $u_t = u_{xx} + u_{yy}$. (A)

GUESS: $u = \boxed{T(t)\,V(x,y)}$??

$$\left.\begin{aligned} u_t &= T'V \\ u_{xx} &= T\,V_{xx} \\ u_{yy} &= T\,V_{yy} \end{aligned}\right\}$$

PLUG IN (YOU CHECK!)

~~SEPERATE~~ SEPARATE VARIABLES (B)

$$\frac{T'(t)}{T(t)} = \frac{V_{xx} + V_{yy}}{V} = \lambda$$

DIVIDE BY TV

$X'' = \lambda X$

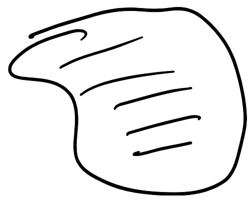
$$V_{xx} + V_{yy} = \lambda V; \quad V(x,y) = 0$$

ON BOUNDARY.

(I) CAN ELIMINATE $\lambda > 0$, $\lambda = 0$;

(II) BOUNDARY CONDITION FOR $u(t,x,y)$ GIVES BOUNDARY CONDITION FOR V:

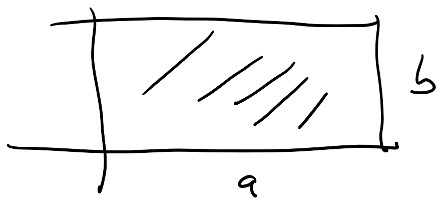
$V(x,y) = 0$ ON BOUNDARY



(III) FOR ARBITRARY SHAPE, CANNOT FIND FORMULAS FOR λ's!

DEPENDS ON GEOMETRY OF SHAPE.

("EASY SHAPES" VS "HARD SHAPES")



TRY RECTANGLE:

$\underline{V}_{xx} + \underline{V}_{yy} = \lambda \underline{V}$ HELMHOLTZ EQUATION.

SEPARATION OF VARIABLES, AGAIN!

$\underline{L}\cdot\underline{V} = \underline{V}_{xx} + \underline{V}_{yy}$

$$\underline{V} = \underline{X}(x)\cdot\underline{Y}(y)$$

$$\underline{X}''\,\underline{Y} + \underline{X}\,\underline{Y}'' = \lambda\,\underline{X}\,\underline{Y};$$

$$= \lambda\,\underline{V}$$

x-DERIV! y-DERIV!

$$\frac{\underline{X}''}{\underline{X}} + \frac{\underline{Y}''}{\underline{Y}} = \lambda$$

$\underline{L} \leftrightarrow$ MATRICES.
$\underline{V} \leftrightarrow$ vectors.
$\lambda \leftrightarrow$ eigenvalue.

$$\frac{\underline{X}''(x)}{\underline{X}(x)} = \lambda - \frac{\underline{Y}''(y)}{\underline{Y}(y)} = d \quad (??)$$

(1) $\underline{X}'' = d\,\underline{X}(x)$; $\underline{X}(0) = \underline{X}(a) = 0$!!

$d=0$? NO
$d>0$ NO
$d<0$. $d=-k^2$

$$\underline{X}(x) = \sin(kx); \quad \underline{X}(a) = 0$$

$\Rightarrow$ $\sin(ka) = 0 \rightarrow ka = N\pi$, OR $k = \frac{N\pi}{a}$

$$\underline{X}_N(x) = \sin\left(\frac{N\pi}{a}x\right); \quad d = d_N = \frac{-N^2\pi^2}{a^2}$$

$$\lambda - \frac{\underline{Y}''(y)}{\underline{Y}(y)} = d = d_N = \frac{-N^2\pi^2}{a^2}$$

$$\ast \ (\lambda - d_N) = \frac{\Psi''(y)}{\Psi(y)} \qquad \Psi(0) = \Psi(b) = 0.$$

HAVE TO SOLVE THIS FOR EACH N!

$$\frac{\Psi''(y)}{\Psi(y)} = -e^2 \rightarrow \Psi(0) = \Psi(b) = 0.$$

$$\Psi(y) = \sin(ey) \qquad \sin(eb) = 0$$

$$eb = m\pi \ !!$$

$$e_m = \frac{m\pi}{b}.$$

$$\Psi(y) = \sin\left(\frac{m\pi}{b} y\right) \ !!$$

$$\frac{\Psi''(y)}{\Psi(y)} = -\frac{m^2\pi^2}{b^2}$$

$$(\lambda - d_N) = \frac{-m^2\pi^2}{b^2} \leftarrow -e^2$$

$$\lambda = -\frac{m^2\pi^2}{b^2} + -\frac{N^2\pi^2}{a^2} = \underline{\underline{\lambda_{m,N}}}$$

DOUBLY INFINITE LIST OF λ's !!!!

$$V(x,y) = V_{m,N} = \underbrace{\sin\left(\frac{N\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)}$$

$$V_{xx} + V_{yy} = \lambda_{m,N} V_{m,N}$$

PRETTY EASY TO CHECK!

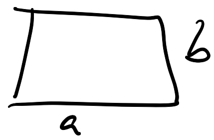
$$\frac{T'(t)}{T(t)} = \lambda = \lambda_{m,N} = -\left[\frac{N^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right]$$

$$T_{m,n}(t) = e^{-C\left[\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right]t}$$

$$u_{m,n}(x,y,t) = e^{-C\left[\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right]t} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{m,n}\, e^{-C\left[\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right]t} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

(W) $\quad u_{tt} = u_{xx} + u_{yy}$ $\quad$ $u = 0$ ON BNDRY.



$U(x,y,t) = T(t)V(x,y)$

$$\frac{T''(t)}{T(t)} = \underbrace{\frac{V_{xx} + V_{yy}}{V} = \lambda}$$

SAME EQN!!!

SO $\quad \dfrac{T''(t)}{T(t)} = \lambda = \lambda_{m,n} = -\pi^2\left[\dfrac{m^2}{b^2} + \dfrac{n^2}{a^2}\right]$

$$T''(t) = T\left[-\pi^2\right]\left[\frac{m^2}{b^2} + \frac{n^2}{a^2}\right],$$

$T'' = -T$

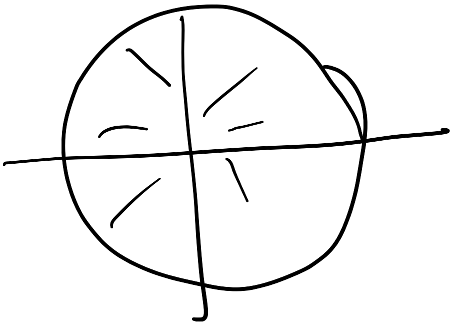
$T = A\cos(t) + B\sin(t)$

$$T(t) = T_{m,n}(t) = \left[a_{m,n}\cos\left(\pi\sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2}}\, t\right) + b_{m,n}\sin\left(\pi\sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2}}\, t\right)\right]$$

FINALLY:

$$u_{m,n} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(a_{m,n}\cos\left(\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\, t\right) + b_{m,n}\sin\left(\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\, t\right)\right)\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)$$

SHOW + TELL:



$u_t(x,y) = u_{xx} + u_{yy}$

$u(x,y,0)$ ON CIRCLE.

SHOW & TELL:

$u_t = T(t)V(x,y)$

$$\frac{T'(t)}{T(t)} = \frac{V_{xx}+V_{yy}}{V} = c$$

$$\frac{V_{xx}+V_{yy}}{V} = c \qquad V_{xx}+V_{yy} = cV.$$

$V = 0$ ON CIRCLE. (?)

EXPECT (?) $c = -k^2$ (NEGATIVE). NOT SQUARE.

THIS TIME: CHANGE ~~TO~~ POLAR COORDINATES

(OCU!) $h_{tt} - h_{xx} = 0 \xrightarrow{COV} h_{uv} = 0.$

$x = u + v$
$y = u - v$

SIMILARLY $V_{xx} + V_{yy} = -k^2 V$

$x = r\cos\theta$
$y = r\sin\theta.$

POLAR COORDINATES:

⋮

$$V_{rr} + \frac{1}{r}V_r + \frac{1}{r^2}V_{\theta\theta} = -k^2 V.$$

HELMHOLTZ IN POLAR COORDINATES.

SO.. $V(r,\theta) = R(r)\Theta(\theta)$

$$R''(r)\Theta + \frac{1}{r}R'(r)\Theta + \frac{1}{r^2}R\Theta'' = -k^2 R\Theta$$

$$\frac{R''\cancel{\Theta}}{R\cancel{\Theta}} + \frac{1}{r}\frac{R'(r)\cancel{\Theta}}{R\cancel{\Theta}} + \frac{1}{r^2}\frac{\cancel{R}\Theta''}{\cancel{R}\Theta} = -k^2$$

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -k^2$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + k^2 = -\frac{1}{r^2}\frac{\Theta''}{\Theta}$$

$$\frac{R'' r^2}{R} + r\frac{R'}{R} + k^2 r^2 = -\frac{\Theta''}{\Theta} = d$$

(1) $R'' r^2 + rR' + k^2 r^2 R = dR$

(2) $\Theta'' = -d\,\Theta$

$\Theta(\theta)$ is 2π periodic (temperature is temperature)

$$\Theta'' = -e^2\Theta \qquad (e \text{ is not } 2.7\ldots)$$

$$\Theta = a\sin(e\theta) + b\sin(e\theta);$$

$$\Theta_m = a_m \sin(m\theta) + b_m\cos(m\theta)$$

$$e = e_m = m \Rightarrow d = m^2$$

$$-d = -d_m = -e_m^2 = -m^2$$

(1) $R'' r^2 + rR' + k^2 r^2 R = +m^2 R$

$$\boxed{R'' r^2 + rR' + (k^2 r^2 - m^2)R = 0}$$

$R(1) = 0$!!! BOUNDARY CONDITION (?)

SO FAR: SINES, COSINES, $\left\{ \text{(H) FUNCTIONS!} \quad \sum e^{inx} e^{-4\pi^2 x} \right\}$

$R(r)$ IS DEFINED AS SOL'N TO ODE.

R IS NOT "KNOWN" FUNCTION —

NEW TRANSCENDENTAL FUNCTION.

BESSEL'S EQN — BESSEL FUNCTIONS.

WILL TAKE SOME TIME TO STUDY THEM!

WARM UP: CAUCHY-EULER EQN:

$$R''(r)r^2 + ArR' + BR = 0.$$

{ LIKE $y'' + ay' + by = 0$ }

{ $y = e^{\lambda x}$ } ?

GUESS: $R(r) = r^{\lambda}$

$R'(r) = \lambda r^{\lambda-1}$

$R''(r) = \lambda(\lambda-1) r^{\lambda-2}$

$$\lambda(\lambda-1) r^{\lambda-2} \, r^2 + Ar\lambda(r^{\lambda-1}) + Br^{\lambda} = 0$$

$$r^{\lambda}\left[\lambda(\lambda-1) + A\lambda + B\right] = 0$$

λ IS SOL'N TO

$$\underbrace{\lambda(\lambda-1) + A\lambda + B = 0}_{\text{CHARACTERISTIC EQN}}.$$

$\lambda_1, \lambda_2,$