MATH 323

1 LINEAR ALGEBRA : SOLVE $A\vec{x} = b$

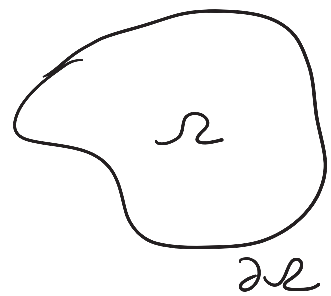
(I) SOLVE $Av = \lambda v$, $\quad v_1, v_2, \dots v_N$

$\lambda_1 \dots \lambda_N$

(II) $b = \sum c_k v_k$

(III) $\vec{x} = \sum \frac{c_k}{\lambda_k} v_k.$

POISSON'S EQUATION:



$u = 0$ ON $\partial\Omega$;

$u_{xx} + u_{yy} = f(x,y).$

OR, SIMPLER PROBLEM $\quad u''(x) = f(x) \quad u(0) = u(1) = 0.$

I) SOLVE $\quad u''(x) = \lambda u(x) \quad u_N = \sin(N\pi x)$

$\lambda_N = -N^2\pi^2$

II) $f(x) = \sum_{N=1}^{\infty} c_N u_N(x)$; $\quad \left\{ c_N = 2\int_0^1 f(x)\sin(N\pi x)\,dx \right\}$

III) $u(x) = \sum_{N=1}^{\infty} \frac{c_N}{\lambda_N} u_N(x)$

"FOURIER METHOD"

2ND METHOD

(I) SOLVE $u_p''(x) = f(x)$

{ANTI-DIFFERENTIATE! WRITE IN SPECIAL FORM}

(II) SOLVE $u_H''(x) = 0$, $\quad u_H(0) = u_p(0)$, $u_H(1) = u_p(1)$

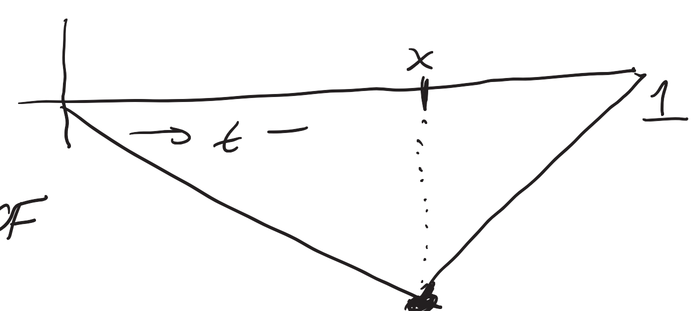
THEN $u(x) = u_p(x) - u_H(x)$ SATISFIES.

D.E. AND BOUNDARY CONDITIONS.

SOL'N $\quad u(x) = \displaystyle\int_{t=0}^{t=1} f(t)\,K(x,t)\,dt$

$$K(x,t) = \begin{cases} x(t-1) & t \geq x \\ t(x-1) & t \leq x \end{cases}$$

FIX X:



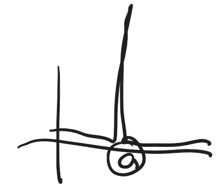
FUNCTION OF t

K(x,t) IS GREEN'S FUNCTION

$Lu = u''$

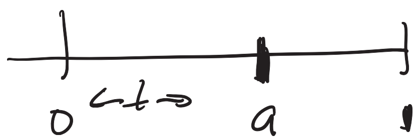
(KERNEL FOR INTEGRAL OPERATOR FOR $\downarrow$)

2ND INTERPRETATION OF GREEN'S FUNCTION.

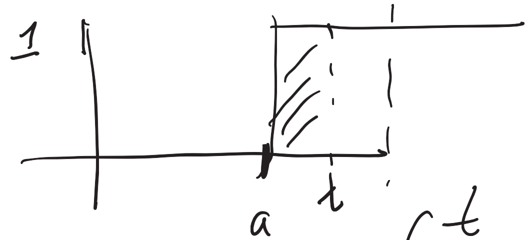


SOLVE $u''(x) = \delta_a(x)$

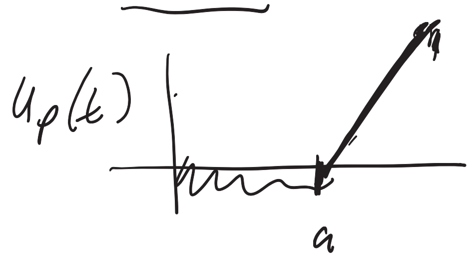
$u(0) = u(1) = 0$;



$$u_p'(t) = \int_0^t \delta_a(s)\,ds = \begin{cases} 0 & t \le a \\ 1 & t \ge a. \end{cases}$$



$$u_p(t) = \int_0^t u_p'(t)\,dt = \begin{cases} 0 & t \le a \\ (t-a) & t \ge a \end{cases}$$



$u_p(0) = 0 \quad u_p(1) = (1-a)$

WANT $u_H(t)$ $\quad u_H(0) = 0 \quad u_H(1) = (1-a)$

$u_H''(t) = 0.$

$$u_H(t) = (1-a)t.$$

$$u(t) = u_p(t) - u_H(t)$$

$$= \begin{cases} -(1-a)t & t \le a \\ (t-a) - (1-a)t & t \ge a. \end{cases}$$

$$u(t,a) = \begin{cases} (a-1)t & t \le a \\ \cancel{at - a} & t \ge a \\ a(t-1). & \end{cases} \qquad u'' = \delta_a.$$

OR:

$$u(t,x) = \begin{cases} (x-1)t & t \le x \\ x(t-1) & t \ge x \end{cases}$$

IS SOL'N TO $\quad u''(t) = \delta_x(t) = \delta(t-x)$;

IMPULSE FUNCTION

THIS IS <u>EXACTLY</u> $K(x,t)$..!!

THE SOLUTION TO THE PROBLEM

$u'' = \delta_a$ (OR $u_{xx} + u_{yy} = \delta_{(a,b)}(x,y)$ ETC.)

IS CALLED A <u>FUNDAMENTAL</u> <u>SOL'N</u>;

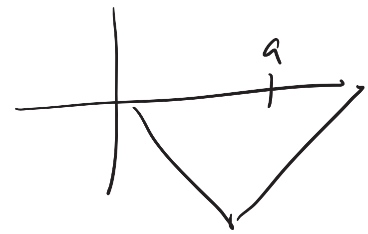
IT IS THE KERNEL = GREEN'S FUNCTION FOR THE PROBLEM.

WAIT! MAYBE IT IS A COINCIDENCE. (OF COURSE NOT).

NOTICE $\underline{u''(x) = \delta_a(x)}$

MEANS $u''(x) = 0$

AWAY FROM a. SO



SOLVES HOMOGENEOUS

PROBLEM ALMOST EVERYWHERE.

$\underline{\underline{u_{xx} + u_{yy} = f}}$

EX! $u_{xx} + u_{yy} = \delta_{(0,0)}(x,y)$.

$$(*) \quad u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = \delta.$$

LOOK FOR SOL'NS

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0;$$

GUESS SOL'N TO $(*)$ JUST

DEPENDS ON r.

$$u_{rr} + \frac{1}{r} u_r = 0 \qquad u_r = \frac{c}{r}$$

$$r u_{rr} + u_r = 0 \qquad u = c \ln(r) + D$$

$$(r u_r)' = 0$$

$$r u_r = c$$

$\left\{ \begin{array}{c} \text{ASSUME} \\ D = 0 \end{array} \right\}$

WHAT SHOULD IT MEAN THAT $\left\{ \begin{array}{c} g(r,\theta) \\ = \ln(r) \end{array} \right\}$

$$\underline{\Delta u} = \underline{\Delta(g\ )} = \underline{\delta_{(0,0)}(x,y)} \ ?$$

6

$$\iint \Delta(g)\, f(x,y)\, dx\, dy = f(0,0).$$

CHECK (INTEGRATION BY PARTS)

$$= \iint g\, (\Delta f)\, dx\, dy$$ (FOR "NORMAL FUNCTIONS")

$$= \int_0^{2\pi} \int_0^{\infty} (\ln r)\left(f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta}\right) r\, dr\, d\theta.$$

$\{$ LET'S TRY $f(r,\theta) = f(r)$ $\}$

$$= 2\pi \int_0^{\infty} (\ln r)\left(f_{rr} + \frac{1}{r} f_r\right) r\, dr\, d\theta.$$

$$= 2\pi \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} r(\ln r) f_{rr} + \ln r)f_r\, dr$$

$$= f_r\, r \ln r) \Big|_{\varepsilon}^{\infty} - \int_{\varepsilon}^{\infty} f_r\, [\ln r + 1]\, dr$$

$$+ \int_{\varepsilon}^{\infty} \ln r\, f_r\, dr$$

$$= -\int_{\varepsilon}^{\infty} f_r\, dr = -f \Big|_{\varepsilon}^{\infty} = f(\varepsilon) \to f(0).$$

DIRICHLET PROBLEM

BEFORE: $u''(x) = f(x)$ $\quad u(0) = u(1) = 0$.

(1) $u_p'' = f(x)$; $\quad u_p(0) = u_0$
$\quad u_p(1) = u_1$

THEN SOLVE $u_H'' = 0$; $\quad u_H(0) = u_p(0)$
$\quad u_H(1) = u_p(1)$.

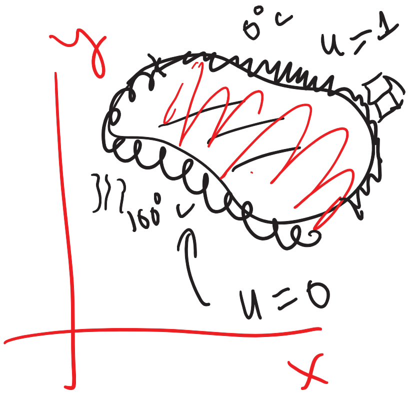
SPECIFIED

THEN $u = u_p - u_H$ IS DESIRED

SOLUTION.

BEGIN MAY 6

IN 2-D, SIMILAR PROBLEM

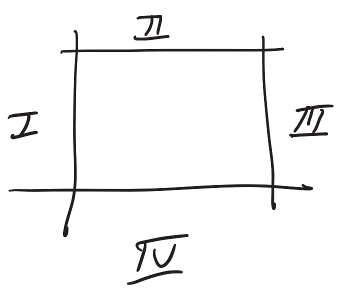


$u_{xx} + u_{yy} = 0$, $\quad$ u SPECIFIED ON BOUNDARY.

LAPLACES EQN.

CALLED THE DIRICHLET PROBLEM, SOL'N DEPENDS ON SHAPE.

EX:



~~RECTANGLE~~. SQUARE

BREAK UP INTO 4 PROBLEMS.

IV: $u_{xx} + u_{yy} = 0$; $\quad u = 0$ ON I, II, III;
$u(x,0) = f(x)$ ON $[0,1]$.

GUESS: $u(x,y) = X(x)\,Y(y)$

$$X''(x)\,Y + X\,Y''(x) = 0$$

$$\frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = \lambda.$$

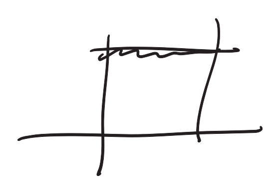
NOTE: $X(0) = X(1) = 0$!

BECAUSE
$$\left.\begin{aligned} X(0)\,Y(y) &= 0 \\ X(1)\,Y(y) &= 0 \end{aligned}\right\}\ \text{ALL } y!$$

$$X''(x) = \lambda X(x) \to X_n(x) = \sin(n\pi x)$$
$$\lambda_n = -n^2\pi^2.$$

$$-\frac{Y''}{Y} = -n^2\pi^2 \qquad Y''(y) = n^2\pi^2\,Y$$

$Y(1) = 0$!



$$Y(y) = A\cosh(n\pi y) + B\sinh(n\pi y)$$

$Y(1) = 0$

$\{\ \sinh(0) = 0$

$\cosh x = \dfrac{e^{x}+e^{-x}}{2}$

$\sinh(x) = \dfrac{e^{x}-e^{-x}}{2}$

$$\Psi(y) = \sinh(\pi N(y-1))$$

$$\Psi(0) = -\sinh(\pi N)$$

$$Y_N(y) = \frac{-\sinh(\pi N(y-1))}{\sinh(\pi N)}. \qquad Y_N(0) = 1$$

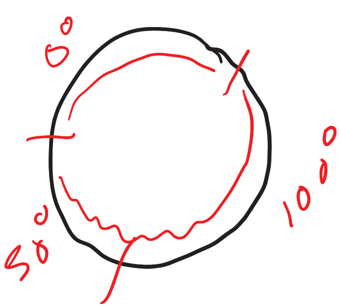
$$u_N(x,y) = \frac{-\sinh(\pi N(y-1))}{\sinh(\pi N)} \sin(N\pi x)$$

$$u(x,y) = \sum_{N=1}^{\infty} c_N u_N(x,y)$$

$$c_N = 2\int_0^1 \sin(N\pi x) f(x)\, dx.$$

AWKWARD FORMULA, AWKWARD BOUNDARY.

(1)



$$u_{xx} + u_{yy} = 0$$

$$u(1,\theta) = f(\theta)$$

SPECIFIED ON BOUNDARY.

SWITCH TO POLAR.

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0;$$

$$u = R(r)\,\Theta(\theta);$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = 0$$

$$\frac{r^2 R''}{R} + r\frac{R'}{R} + \frac{\Theta''(\theta)}{\Theta} = 0$$

$$\frac{r^2 R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = C \;;$$

$$\Theta''(\theta) = -C\,\Theta$$

Θ PERIODIC (2π) SO
C CANNOT BE
NEGATIVE.

$$\underline{\Theta(\theta)} = e^{iN\theta}$$

$N = -2, -1, 0, 1, \ldots$

π

$$-C = -N^2$$
$$C = N^2$$

$\cos(N\pi\theta)$

$\sin(N\pi\theta)$

$\cos(N\pi\theta)$

$i\sin(N\pi\theta)$

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = N^2$$

$$r^2R'' + rR' = N^2 R$$

$$r^2R'' + rR' - N^2R = 0.$$

CAUCHY-EULER; $\quad R = r^{\lambda}$

$$\lambda(\lambda-1)\, r^2 r^{\lambda-2} + \lambda r r^{\lambda-1} - N^2 r^{\lambda} = 0.$$

$$\lambda(\lambda-1) + \lambda - N^2 = 0$$

$$\lambda^2 - N^2 = 0; \qquad \lambda = \pm N.$$

$R = Ar^{N} + Br^{-N}$ BUT! r TO A NEGATIVE POWER MAKES R "BLOW UP" AT ORIGIN. BAD!

$$R_N(r) = r^{|N|}$$

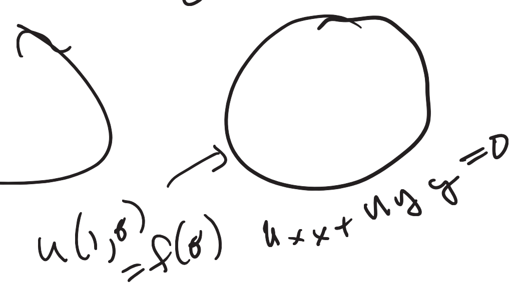
$$u_N(r,\theta) = r^{|N|} e^{iN\theta} \qquad N = -1, -2, 0, 1, \ldots$$

$$u(r,\theta) = \sum_{-\infty}^{\infty} c_N r^{|N|} e^{iN\theta}.$$

$$u(1,\theta) = \left\{ \sum_{-\infty}^{\infty} c_N e^{iN\theta} \right\} = f(\theta)$$

$$c_N = \frac{1}{2\pi}\int_0^{2\pi} e^{-iN\theta} f(\theta)\, d\theta.$$

$\boxed{\text{FOURIER FORM OF SOL'N:}}$



CONVOLUTION FORM:

$$u(r,\theta) = \sum_{-\infty}^{\infty} r^{|N|} e^{iN\theta} c_N$$

$$= \sum_{-\infty}^{\infty} r^{|N|} e^{iN\theta} \left(\frac{1}{2\pi}\right) \int_0^{2\pi} e^{-iNt} f(t)\, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(t) \sum_{N=-\infty}^{N=\infty} r^{|N|} e^{iN(\theta - t)}\, dt$$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, P(r, \theta - t)\, dt$$

$$P(r,\theta) = \sum_{N=-\infty}^{N=\infty} r^{|N|} e^{iN\theta}. = ??$$

= POISSON KERNEL.

$$= 1 + \sum_{1}^{\infty} r^N e^{iN\theta} + \sum_{N=-\infty}^{N=-1} r^{-N} e^{iN\theta}.$$

$$= 1 + \sum_{1}^{\infty} (re^{i\theta})^N, + \sum_{m=1}^{m=\infty} \underbrace{r^{N} e^{-iN\theta}}_{m=-N}$$

$$= 1 + \sum_{1}^{\infty} (re^{i\theta})^n + \sum_{1}^{\infty} (re^{-i\theta})^n$$

$$\left\{ \begin{array}{l} w + w^2 + w^3 \cdots = w(1 + w + w^2 \cdots \\ \underline{|w| < 1} \qquad = \dfrac{w}{1-w} \end{array} \right\} \quad \text{GEOMETRIC SERIES}$$

$$\frac{1}{1-w}$$

$$= 1 + \frac{re^{i\theta}}{1-re^{i\theta}} + \frac{re^{-i\theta}}{1-re^{-i\theta}}$$

DOABLE SUM!

$$= 1 + \frac{re^{i\theta} - r^2 + re^{-i\theta} - r^2}{1 - re^{i\theta} - re^{-i\theta} + r^2}$$

$$= \frac{1 \cancel{- re^{i\theta}} \cancel{- re^{-i\theta}} + r^2 \quad \cancel{+ re^{i\theta}} - r^2 \cancel{+ re^{-i\theta}}}{1 - 2r\cos\theta + r^2}$$

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta$$

$$= \boxed{\frac{1-r^2}{1 - 2r\cos\theta + r^2}} = \boxed{P(r,\theta)}$$

$$\underline{u(r,\theta)} = \frac{1}{2\pi} \int_0^{2\pi} f(t) \frac{1-r^2}{1 - 2r\cos(\theta - t) + r^2} \, dt$$

$$\underline{u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}} = 0 \, ?$$

PROBLEM WHEN $r = 1$.

$$u(1,\theta) = f(\theta) \quad ?$$

$$\lim_{r \to 1} u(r,\theta) = f(\theta) = ?$$

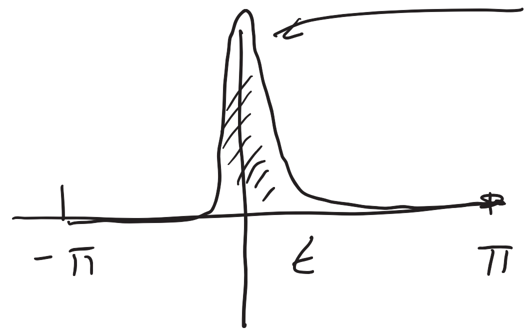
$$\lim_{r \to 1} u(r,\theta) = f(\theta) \; ?$$

$$u(r,\theta) = \int_0^{2\pi} f(t)\left[\frac{1}{2\pi}\,\frac{1-r^2}{1-2r\cos t + r^2}\right] dt.$$

$r \approx 1$ $\quad K(r,t)$.

LOOKS LIKE

A δ-FUNCTION.



$$\approx \cancel{\tfrac{1}{2\pi}} \; f(\theta).$$