Phys 327 HW3

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① Griffiths 4.21

The raising and lowering operators $L_\pm$ change the value of m_l by one unit,

$$L_\pm f_l^m = A_l^m f_l^{m\pm 1}$$

where A_l^m is some constant. Compute A_l^m assuming the eigenfunctions f_l^m are normalized.

Answer: $A_l^m = \hbar\sqrt{l(l+1) - m(m\pm 1)} = \hbar\sqrt{(l\mp m)(l\pm m+1)}$

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First, show that $L_\mp$ is the Hermitian conjugate of $L_\pm$

$$L_\pm \equiv L_x \pm iL_y$$

$$\langle f | L_\pm g\rangle = \langle f | (L_x \pm iL_y) g\rangle = \langle f | L_x g\rangle \pm \langle f | iL_y g\rangle$$

$$= \langle L_x f | g\rangle \pm \langle iL_y f | g\rangle^* = \langle (L_x \mp iL_y) f | g\rangle$$

$$= \langle L_\mp f | g\rangle$$

$$\langle f | L_\pm g\rangle = \langle L_\mp f | g\rangle \Longleftrightarrow (L_\pm)^\dagger = L_\mp \quad \checkmark$$

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Applying each to the eigenfunction and taking inner product:

$\langle f_l^m | L_\pm L_\mp f_l^m\rangle$, and using

$$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z \rightarrow L_\pm L_\mp = L^2 - L_z^2 \mp \hbar L_z$$

$$\Rightarrow \langle f_l^m | (L^2 - L_z^2 \pm \hbar L_z) f_l^m\rangle = \langle f_l^m | \underbrace{(\hbar^2 l(l+1) - \hbar^2 m^2 \mp \hbar^2 m)}_{\text{constant}} f_l^m\rangle$$

$$= \hbar^2\left[l(l+1) - m(m\pm 1)\right]$$

Also, since $(L_\pm)^\dagger = L_\mp$:

$$\langle f_l^m | L_\pm L_\mp f_l^m\rangle = \langle L_\pm f_l^m | L_\pm f_l^m\rangle = \langle A_l^m f_l^{m\pm 1} | A_l^m f_l^{m\pm 1}\rangle$$

$$= |A_l^m|^2 \langle f_l^{m\pm1} | f_l^{m\pm1} \rangle = |A_l^m|^2$$

$$\Longrightarrow \quad |A_l^m|^2 = \hbar^2 [l(l+1) - m(m\pm1)]$$

$$\Longrightarrow \quad \boxed{A_l^m = \hbar\sqrt{l(l+1) - m(m\pm1)}}$$

At the top rung $m=l$ and $A_l^m = 0$, forcing all eigenfunctions with $m>l$ to $=0$.

At the bottom rung $m=-l$ the same occurs

$$A_l^m = \hbar\sqrt{l(l+1) + l(-l\mp1)} = \hbar\sqrt{l(l+1) - l(l\pm1)} = 0.$$

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② Griffiths 4.25

a) What is $L_+ Y_l^l$? (No calculation allowed!)

$\boxed{0.}$ $\quad \cancel{m > l}$

b) Use the result of (a) together with the equation

$$L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)$$

And the fact that $L_z Y_l^l = \hbar l Y_l^l$ to determine $Y_l^l(\theta,\phi)$ up to a normalization constant

$$L_z Y_l^l = \hbar l Y_l^l$$

$$-i\hbar\frac{\partial Y_l^l}{\partial\phi} = \hbar l Y_l^l \rightarrow \frac{\partial Y_l^l}{\partial\phi} = i l Y_l^l$$

$$\Longrightarrow \quad \underline{\underline{Y_l^l(\theta,\phi) = A(\theta)e^{il\phi}}}$$

may depend on θ, not ϕ

Applying the raising operator:

$$L_+ Y_l^l = 0 = \cancel{\hbar} e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)A(\theta)e^{il\phi}$$

$$e^{il\phi}\frac{\partial A}{\partial\theta} + i\cot\theta A(\theta)\frac{\partial}{\partial\phi}(e^{il\phi}) = 0$$

$$e^{il\phi}\frac{\partial A}{\partial\theta} + i\cot\theta A(\theta)(il)e^{il\phi} = 0$$

$$e^{il\phi}\frac{\partial A}{\partial\theta} - A(\theta)l\cot\theta e^{il\phi} = 0$$

$$\frac{\partial A}{\partial\theta} = Al\cot\theta$$

$$\int\frac{dA}{A} = \int l\cot\theta\, d\theta$$

$$\ln A = l\ln(\sin\theta) + C$$

$$A(\theta) = e^c e^{\ln(\sin^l\theta)} = B\sin^l\theta$$

$$\longrightarrow \boxed{Y_l^l(\theta,\phi) = B\sin^l\theta e^{il\phi}}$$

c) Find B.

$$1 = \int_\Omega |Y_l^l|^2 d\Omega = \int_0^{2\pi}\int_{\theta=0}^{\pi}(Y_l^l)^* Y_l^l \sin\theta\, d\theta\, d\phi$$

$$1 = |B|^2\int_0^{2\pi}\int \sin^{2l}\theta\sin\theta\, d\theta\, d\phi = |B|^2\int_0^{2\pi}d\phi\int_0^{\pi}\sin^{2l+1}\theta\, d\theta$$

$$\frac{1}{2\pi|B|^2} = \int_0^{\pi}\sin^{2l+1}\theta\, d\theta$$

For $n>0$, we have $\displaystyle\int_0^{\pi}\sin^n\theta\, d\theta = \cancel{\frac{\sin^{n-1}\theta\cos\theta}{n}\Big|_0^{\pi}} + \frac{n-1}{n}\int_0^{\pi}\sin^{n-2}\theta\, d\theta$

$$\int_0^\pi \sin^n\theta d\theta = \frac{(n-1)}{n}\int_0^\pi \sin^{n-2} d\theta$$

$$\Longrightarrow \int_0^\pi \sin^{2l+1}\theta\, d\theta = \frac{2l}{2l+1}\underbrace{\int_0^\pi \sin^{2l-1} d\theta}_{I}$$

$$I = \frac{2l-2}{2l-1}\int_0^\pi \sin^{2l-3} d\theta$$

$$\Rightarrow \int_0^\pi \sin^{2l+1} d\theta = \left(\frac{2l}{2l+1}\right)\left(\frac{2l-2}{2l-1}\right)\int_0^\pi \sin^{2l-3} d\theta$$

This repeats until $n = 2l+1 = 1 \rightarrow l = 0$.
But this is forbidden, so $l = 1$

$$\rightarrow \left(\frac{2l}{2l+1}\right)\left(\frac{2l-2}{2l-1}\right)\cdots\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)\underbrace{\int_0^\pi \sin\theta\, d\theta}_{=[-\cos\pi + \cos 0] = 2}$$

$$\ast \Longrightarrow \int_0^\pi \sin^{2l+1}\theta\, d\theta = \frac{2\cdot 4\cdot 6 \cdot \ldots \cdot 2l-2\cdot 2l}{3\cdot 5\cdot 7\cdot \ldots \cdot 2l-1\cdot 2l+1}(2)$$

$$\text{or} \quad \frac{1}{4\pi|B|^2} = \frac{(2\cdot 4\ldots 2l)^2}{1\cdot 2\cdot 3\cdot 4\cdot\ldots 2l-1\cdot 2l\cdot 2l+1} = \frac{(2^l l!)^2}{(2l+1)!}$$

$$\Rightarrow |B|^2 = \frac{(2l+1)!}{(2^l[l!])^2 4\pi}$$

$$\rightarrow B = \sqrt{\frac{(2l+1)!}{\pi}}\cdot\frac{1}{2^l l!}\frac{1}{2}$$

$$\Longrightarrow B = \frac{1}{2^{l+1} l!}\sqrt{\frac{(2l+1)!}{\pi}}$$

$$\boxed{Y_l^l(\theta,\phi) = \frac{1}{2^{l+1} l!}\sqrt{\frac{(2l+1)!}{\pi}}\sin^l\theta\, e^{il\phi}}$$