MATH 323, PARTIAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS (LECTURE SUMMARY-PRELIMINARY SCHEDULE) WINTER 2021-22

Note: If a topic appears twice it means that it may be included in the notes on lecture N, but not covered in lecture N; but it will covered in lecture $N + 1$

1. Intro

- Just as study of ode's looks as their different types and solution techniques, so partial differential equations is the study of the different types, and their solution techniques
- A list is given of important example equations we will study
- Example: heat equation. If you are given an alleged solution, you should be able to check that it is a solution. Finding solutions is harder.
- Solving PDE's uses both ode techniques and ideas and techniques from linear algebra.
- The main technique for going from PDE's to ODE's is SEPARATION OF VARIABLES. It will produce many "seed" solutions.
- To get all solutions, we will need the concept of Fourier series.

2. linear algebra and systems of differential equations

- Review linear algebra - eigenvalues, eigenvectors, what they are and how to find them. Characteristic equation
- Second order linear constant coefficient ode's; solution techniques, the three cases.
- Systems of equations: Example - go from second order equation to first order system.
- "Product theorem". Eigenvector v and eigenvalue λ give solution to linear system.

3. Intro to the heat equation

- Statement of the problem: the PDE, the boundary conditions, initial condition
- Separation of variables guess, calculation leads to two ODE's for $T(t), X(x)$
- Boundary conditions for u give boundary conditions for X; get boundary value problem for X.
- The three cases for the constant c, elimination of positive and zero cases.
- Values of $c = -k^2$ for which there exist solutions to the boundary value problem.
- Solve related equation for T; produce infinite list of product solutions to PDE, and any sum of such is a solution
- Certain finite sum initial conditions are easy to solve; you can just "write down" the related solutions.

4. Fourier series and linear algebra

- To satisfy boundary conditions for non-simple, non-finite term examples, we use Fourier trick to compute the constants c_k
- Review of some topics from linear algebra; bases, orthonormal bases; analogy with Fourier series.

5. More Linear Algebra

Review of some topics from linear algebra; bases, orthonormal bases; analogy with Fourier series.

6. "Ice and insulation" boundary conditions"

Review of standard case; Simple initial conditions restated. Computation of "Ice-left , Insulation-right" case.

7. Heat eqn cont

Driven heat equation and decoupling

8. Heat equation and Fourier transforms

Circular bar. Introduction to Fourier transforms. Exponential, step, Gaussian

9. More Fourier transform

- Inverse transform of transform is original function - check for Gaussian
- Fourier transform and differentiation
- 1-d Fourier transform gives Fourier solution to heat equation on the line. Important example: Initial condition is a Gaussian.

10. Fourier transform, cont.

- Review; differentiation of Gaussian solution gives polynomial times Gaussian solution
- Convolution form of solution
- direct check (exercise) that convolution is a solution
- Limiting value as t goes to zero leads to discussion of δ-function.
- Heat kernel is Dirac family

11. Laplace transform and ODE's

Laplace transform; using to solve ODE's; δ-function calculations.

12. Convolutions

- Recap solution to heat equation on line; Gaussians as initial conditions; differentiation gives new solutions.
- Convolution form of solution
- Initial condition from convolution form question leads to appearance of δ-function from heat kernel.

13. Wave equation

Wave equation with 0 boundary conditions on a finite interval. Wave equation on the line. Fourier version. D'Alembert's solution to the wave equation, derivation using change of variables. Example of step function.
Convolution interpretation of D'Alembert's formula.

14. Two space dimensions

Wave equation and heat equation in two space dimensions; Helmholtz equation. Helmholtz for the rectangle; solution to heat equation and wave equation on the rectangle. Movies! Wave and heat equation on the disk - polar coordinates and Helmholtz equation for disk. Separation of variables leads to new differential equation - Bessel's equation. Warm up - Cauchy-Euler equation.

15. Helmholtz

Review: Helmholtz for square, separation of variables for the disc. Gives a differential equation (with k and m). If $k = 1$, you get exactly Bessel's equation. $J_n(kr)$ satisfies the equation with k in it. Roots of J_n are $r_{n,m}, n = 0 \ldots, m = 1 \ldots.$ To satisfy boundary conditions on disk, we need to look at $J_n(r_{m,n}r)$ The J_n are new functions, various ways to understand them. Graph indicates decreasing behavior, and nearly periodic behavior. BRIEFLY review power series; the geometric series. Power series technique for finding solutions to diffferential equations. A couple of examples. (Not covered - change of variables, implies behavior indicated by graph. Modifications - three dimensional Helmholtz, Helmholtz with potential - time indepedent Schrodinger equation. Inverse problem for Helmholtz.)

16. Review

- Separation of variables for the heat equation (or wave equation) on the disk leads to Helmholtz equation for the disk, best written in polar coordinates. Separation of variables for Helmholtz on the disk leads to two equations for $R(r), \Theta(\theta)$, a simple one for $\Theta(\theta)$ leading to sines and cosines; and a new and harder one for $R(r)$.
- This equation has two numbers in it k and m. m is an integer. If $k = 1$, we have Bessel's equation with solution $J_m(r)$. The solution to the original equation is just $J_m(kr)$.
- Graphing $J_m(r)$ shows a function which oscillates, and goes to zero. How do we understand J_m? ROOTS OF $J_m(r)$ give solutions to polar Helmholtz equation, So we need to understand J_m.
- Review power series; power series for $1/(1 - x)$, also called the geometric series. Quote formulas for $e^x, \ln(1 + x)$.
- Power series solution for differential equations: $y' = y, y(0) =$. Solution written two ways (second way uses Σ notation.) Second example $y' = 1/(1 + x), y(0) = 0$, using Σ notation.
- Frobenius says "simple" power series method does not always work. Frobenius method: look at related Cauchy-Euler equations, find solutions x^{r_1}, x^{r_2}. Then $y(x) = x^r \sum_0^\infty a_i x^i$ works. After some magic, get explicit formula for J_m.
- But to find roots, need behavior of J_m for large r. (now new material). Liouville's trick gives information about $L = \sqrt(r)J_m$; it looks like it should be sines and cosines. (Liouville trick details were not discussed in class). Brief discussion of time independent Schrodinger equation; introduction to inverse problem for planar shapes: can you hear the shape of a drum? Area, length, number of holes - but there exist isospectral domains.
- Integral operators have been used to solve PDE problems (non-Fourier formulas). Heat equation, for example. Integral operators are closely related to matrix equations. Eigenvector method of solving $Ax = b$.
- Poisson's equation. Solution using eigenvector technique.
- Non-eigenvector technique for "baby problem" $u'' = f, u(0) = u(1) = 0$. Goal: write solution as integral operator. Need explicit formula for $K(x, y)$ (TBC).

17. Review

18. First order equations

Characteristics of first order equations.

- If equation is "homogeneous", then characteristics give level curves for solution u.
- If equation is non-homogeneous, then characteristics give curves (which are not level curves) along which the solution values are "predictable"

19. Calculus of Variations

Euler-Lagrange equations

20. CAT scans

CAT scans and Fourier transform