The purpose of this assignment is to write up materials that you could use to present your own lectures in class. Your target audience: "the old you", at the beginning of the quarter, who did not know PDE's.

1. Consider the PDE with boundary conditions

$$u_t = u_{xx}, u_x(0,t) = 0, u(1,t) = 0$$

This is very similar to one of the examples done in class. Show, using the separation of variables process, how you can come up with an infinite list of product solutions of the form:

$$u_n(x,t) = X_n(x)T_n(t)$$

. Use these solutions to solve the PDE with the initial condition $u(x,0) = \cos(\frac{\pi}{2}x) - 7\cos(\frac{5\pi}{2}x)$

2. Write down the definition of the Fourier transform and the inverse transform. These are integrals of course, show how to do those integrals for each of the following examples of $f(x)$:

   (a) $f(x) = e^{-|x|}$. In the notes.

   (b) $e^{-a^2x^2}$. This is also in the notes, but harder. It requires a clever integration trick (and polar coordinates); then a clever differentation trick and solving a simple ordinary differential equation. Notice that this says that the Fourier transform of a Gaussian is a Gaussian.

   (c) e^{x-x^2}. This is not in the notes. It will not be, I think, difficult. Write down the definition of the Fourier transform for this function, then do a "completing the square" for the term $x - x^2$, then a simple change of variable (shift of variable).

3. Find the Fourier transform version of the formula for the heat equation. Start with $u_t(x,t) = u_{xx}$. Take the Fourier transform with respect to the x variable ; $\hat{u}_t(x,t) = \widehat{(u_{xx})}$. If $'$ means differentiation with respect to x, then show that $\widehat{(f')}(w) = iw\hat{f}(w)$ and that $\widehat{(f'')}(w) = -w^2\hat{f}(w)$. Using this (here comes the conceptually tricky part), you get (for each w) an ordinary differential equation for $\hat{u}$. First order! You can easily solve it, and this gives the Fourier formula for the solution to the heat equation.

4. Using the Fourier formula given above, and the fact that Fourier transforms of Gaussians are Gaussians, solve the heat equation with the initial condition $u(x,0) = e^{-a^2x^2}$.