Exercise 1.5

Let $\mathbb{F} = \mathbb{R}$. Consider the matrix-vector equation $A\vec{x} = \vec{b}$:

$$\begin{pmatrix} 1 & 2 & 2 \\ 1 & \alpha & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$$

Determine $\alpha \in \mathbb{R}$ so that $A$ is invertible and $x_1 = x_2$

Cramer's Rule: $x_i = \frac{|A_i|}{|A|}$

$\Rightarrow$ for $x_1 = x_2$: $|A_1| = |A_2|$

also need $|A| \neq 0$ so that it's invertible

$|A| = a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23}$

$|A| = 1(-1)(2) + \alpha(1) + 0$

$|A| = -2 + \alpha \longrightarrow \underline{\alpha \neq 2}$

$$A_1 = \begin{pmatrix} 1 & 2 & 2 \\ 0 & \alpha & 0 \\ 5 & 2 & 3 \end{pmatrix} \rightarrow |A_1| = \alpha(-7) = -7\alpha$$

$$A_2 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 5 & 3 \end{pmatrix} \rightarrow |A_2| = -1(-7) = 7$$

$\rightarrow$ $x_1 = x_2$

$\frac{|A_1|}{|A|} = \frac{|A_2|}{|A|}$

$\rightarrow -7\alpha = 7 \rightarrow \boxed{\alpha = -1}$