CHARACTERISTICS FOR FIRST ORDER PDE'S

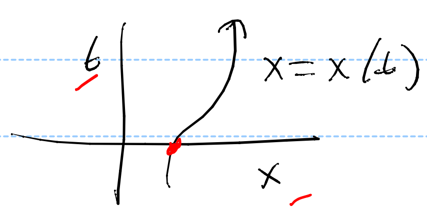
CONSIDER A QUASI-LINEAR PDE

$$u_t(x,t) + a(x,t,u(x,t))\,u_x = b(x,t,u).$$

CHARACTERISTICS ALLOW YOU TO STUDY (AND IN FACT SOLVE) THE PDE BY SOLVING A RELATED SET OF DIFFERENTIAL EQUATIONS – THE CHARACTERISTIC EQUATIONS.

FIRST CONSIDER $\quad u_t + a(x,t,u)\,u_x = 0$

$u = x^2+y^2$



LET'S SEE IF WE CAN GET $u(x,t) = C$, INFORMATION ABOUT LEVEL CURVES OF u. ASSUME

WHO CARES?

THAT A LEVEL CURVE CAN BE PARAMETERIZED AS $(x(t), t)$!! THEN $u(x(t),t)$ IS A CONSTANT, SO $\dfrac{d}{dt}\,u(x(t),t) = 0$

$$= \underbrace{\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}}\overset{CR}{=} 0$$

$u_x a + u_t = 0$

see insert

IF $\frac{dx}{dt} = a(x(t), t, u(x(t),t)) = a(x, t, u)$ THEN

trick 1! but! need to know u (it seems)

WE HAVE AN ODE WHOSE SOL'N IS A LEVEL

CURVE FOR u; HENCE $u(x(t), t) = u(x(0), 0)$

see insert

INITIAL CONDITION FOR u; SO IF $u(x,0) = f(x)$,

$u(x(t), t) = u(x(0), 0) = u(x_0, 0) = f(x_0)$

$$\frac{dx}{dt} = a(x(t), t, f(x_0)) \quad \text{WITH} \quad x(0) = x_0$$

actual ode for x! u is not here! trick 2 !!!!

GIVES LEVEL CURVE FOR $u(x,t)$. CONVERSELY,

characteristic equation.

THE ODE GIVES A WAY OF CONSTRUCTING SOL'NS

TO THE PDE VIA SOLVING DIFFERENTIAL EQNS.

EX: $u_t = u u_x$; $u_t - u u_x = 0$

$u_t + a u_x = 0$

$a(x, t, u) = -u$. CONSIDER

$x(0) = x_0$

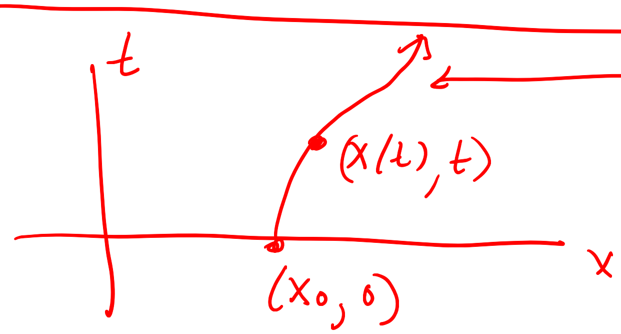
$$\frac{dx}{dt} = a(x, t, f(x_0)) = -f(x_0) \qquad (u(x_0, 0) = f(x_0))$$

— characteristic eqn —

$$x(t) = -f(x_0)t + x_0 \qquad x(0) = x_0$$

If $x(t)$ is level curve for $u(x,t)$ (any u!) then $u_x \frac{dx}{dt} + u_t = 0$. For our u (sol'n to pde) $u_x a + u_t = 0$

subtract: $u_x \left(\frac{dx}{dt} - a \right) = 0$. Unless u_x is always 0, this says $\boxed{\frac{dx}{dt} = a}$



① level curve of u

② sol'n to ode

$u(x(t),t) = u(x(0),0)$

same value.

NOW, WE HAVE TO UNRAVEL THIS. $u(x_0,0)=f(x_0)$

CONSIDER $f(x_0) = \lambda x_0$ (LINEAR INITIAL

$x(t) = -f(x_0)t + x_0$

CONDITIONS) THEN $x(t) = -\lambda x_0 t + x_0$

$= x_0(1-\lambda t)$

where you are

SO IF $\bar{x} = x(t)$, $\bar{x} = x_0(1-\lambda t)$ OR

where you started.

$x_0 = \dfrac{\bar{x}}{1-\lambda t}$; AND $u(\bar{x},t) = u(x_0,0)$

u "here" = u at start

$= \lambda x_0 = \boxed{\dfrac{\lambda \bar{x}}{1-\lambda t} = u(\bar{x},t)}$

OR, MORE SIMPLY, $u(x,t) = \dfrac{\lambda x}{1-\lambda t}$.

NOTICE, IF $\lambda > 0$, YOU GET INTO PROBLEMS AT

$t = \dfrac{1}{\lambda}$.

see page 8

NEXT, CONSIDER $u_t + a(x,t,u)u_x = b(x,t,u)$

USING CHARACTERISTICS, WE WILL OBTAIN CURVES

WHICH ARE NOT LEVEL CURVES OF u, BUT

CURVES ALONG WHICH u CHANGES IN A

"PREDICTABLE" MANNER. LETS LOOK AT A SLIGHT MODIFICATION OF THE EQUATIONS WE HAD

BEFORE: ② $u(x(t), t) = s(t)$

Given u, just write down "old" de. (not saying gives level curve;) but useful curve?

$b = b(x, t, u)$

① $$\boxed{\begin{aligned}\frac{dx}{dt} &= a(x(t), t, u(x(t), t)) \\ &= a(x(t), t, s(t))\end{aligned}}$$

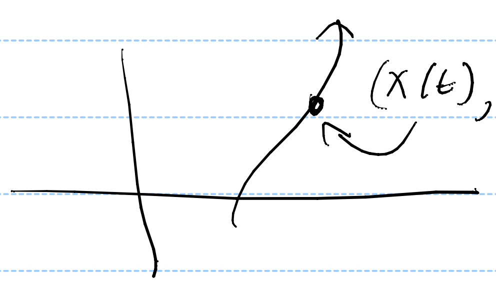
$$\frac{ds}{dt} = \underbrace{\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}}_{CR} = \frac{\partial u}{\partial x}\underbrace{a}_{\text{ode}} + \frac{\partial u}{\partial t} = \underbrace{b}_{\text{from pde!}}$$

SO WE HAVE THE COUPLED SYSTEM

$$\boxed{\begin{aligned}\frac{dx}{dt} &= a(x(t), t, s(t)) \\ \frac{ds}{dt} &= b(x(t), t, u(x(t), t)) = b(x, t, s)\end{aligned}}$$

$\frac{dx}{dt} = a \quad \frac{ds}{dt} = b$

CONVERSELY, IF YOU CAN SOLVE THIS SYSTEM,



, YOU CAN USE IT TO COMPUTE VALUES OF u TO SOLVE THE PDE.

EX: fluid gas flow, mechanics, control theory.

(Charpit de Villecourt 1780?)

EX: $\boxed{u_t - u u_x = \alpha}$

plug in s for ode

$a(x,t,u) = -u$

$b(x,t,u) = \alpha$

$$\frac{dx}{dt} = -s$$

$$\frac{ds}{dt} = \alpha$$

$$s(t) = u(x(t),t) = \alpha t + c \qquad u(x(0),0) = f(x_0)$$

$$s(0) = u(x(0),0) = u(x_0,0) = f(x_0)$$

$$s(t) = \alpha t + f(x_0)$$

$$\frac{dx}{dt} = -s = -\alpha t - f(x_0) \Rightarrow x(t) = \frac{-\alpha t^2}{2} - f(x_0)t + x_0$$

AGAIN, SUPPOSE $f(x_0) = \lambda x_0$;

where we are

where we started.

$$x(t) = \bar{x} = \frac{-\alpha t^2}{2} - \lambda x_0 t + x_0$$

$$= \frac{-\alpha t^2}{2} + x_0(1 - \lambda t)$$

$$\frac{\bar{x} + \frac{\alpha t^2}{2}}{1 - \lambda t} = x_0$$

$$u(\overset{\bar{x}}{x(t)}, t) = s(t) = \alpha t + f(x_0) = \alpha t + \lambda x_0$$

$$= \alpha t + \lambda \left[\frac{\bar{x} + \frac{\alpha t^2}{2}}{1 - \lambda t} \right] = u(\bar{x}, t)$$

$b(x,t,u)$

EX: $\boxed{u_t - u u_x = \alpha u}$

$$\boxed{\begin{aligned}\frac{dx}{dt} &= -S \\ \frac{ds}{dt} &= \alpha S\end{aligned}}$$

$\frac{dx}{dt} = a$

$\frac{ds}{dt} = b$

$$s(t) = C e^{\alpha t} \qquad s(0) = f(x_0) \qquad s(t) = f(x_0) e^{\alpha t}$$

$$\frac{dx}{dt} = -S = -f(x_0) e^{\alpha t}$$

$$x(t) = \frac{-f(x_0)}{\alpha} e^{\alpha t} + C$$

check!

$$x(0) = \frac{-f(x_0)}{\alpha} + \left(x_0 + \frac{f(x_0)}{\alpha}\right)$$

$$x(t) = \frac{f(x_0)}{\alpha}\left(1 - e^{\alpha t}\right) + x_0$$

AGAIN, LET $f(x_0) = \lambda x_0$

(T.I.A. hard to do unless f is simple)

$$X(t) = \bar{X} = x_0 \left[1 + \frac{\lambda}{\alpha}\left(1 - e^{\alpha t}\right)\right]$$

where we are — where we start

$$\text{OR} \quad x_0 = \frac{\bar{x}}{\left(1 + \frac{\lambda}{\alpha}\left(1 - e^{\alpha t}\right)\right)}$$

$s(t) = u(x(t), t)$.

$$u(\bar{x}, t) \overset{= s(t)}{=} \lambda x_0 e^{\alpha t}$$

$s(t) = f(x_0) e^{\alpha t}$

$= \lambda x_0 e^{\alpha t}$

$$= \frac{\lambda \bar{x}}{\left(1 + \frac{\lambda}{\alpha}\left(1 - e^{\alpha t}\right)\right)} e^{\alpha t}$$

$$u_t + u^2 u_x = 0$$

$$u(x,0) = \cancel{\lambda x} = \lambda\sqrt{x} = f(x).$$

$$u_t + a\,u_x = 0$$

$$a(x,t,u)$$

$$a(x,t,u) = u^2$$

CHARACTERISTIC EQN

$$\frac{dx}{dt} = a(x,t,f(x_0))$$

$$x(0) = x_0$$

$$\frac{dx}{dt} = (f(x_0))^2$$

$$x(t) = (f(x_0))t +$$

$$x(t) = (f(x_0))^2 t + x_0$$

$$\bar{x} = (f(x_0))^2 t + x_0$$

$$\bar{x} = (\lambda\sqrt{x_0})^2 t + x_0$$

$$= \lambda^2 x_0 t + x_0$$

$$\bar{x} = x_0(\lambda^2 t + 1)$$

$$x_0 = \frac{\bar{x}}{\lambda^2 t + 1}$$

$$u(\bar{x},t) = u(x_0,0) = f(x_0) = \lambda\sqrt{x_0}$$

$$u(\bar{x},t) = \lambda\sqrt{x_0} = \lambda\sqrt{\frac{\bar{x}}{\lambda^2 t + 1}}$$

$$u(\bar{x},t) = \lambda\sqrt{\frac{\bar{x}}{\lambda^2 t + 1}}$$

$$u(x,t) = \left(\lambda\sqrt{\frac{x}{\lambda^2 t + 1}}\right)$$

$$u_t + t\,u_x = 0 \quad ; \quad u(x_0, 0) = f(x_0)$$

$$a(x, t, u) = t$$

$$\frac{dx}{dt} = a(x, t, f(x_0)) = t$$

$$x(t) = \frac{t^2}{2} + c \qquad x(0) = +c = x_0$$

$$\bar{x} = x(t) = \frac{t^2}{2} + x_0 \qquad \bar{x} - \frac{t^2}{2} = x_0 \; ;$$

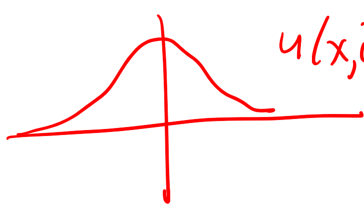
$$u(\bar{x}, t) = u(x_0, 0) = f(x_0)$$

$$= f\left(\bar{x} - \frac{t^2}{2}\right) = u(\bar{x}, t).$$

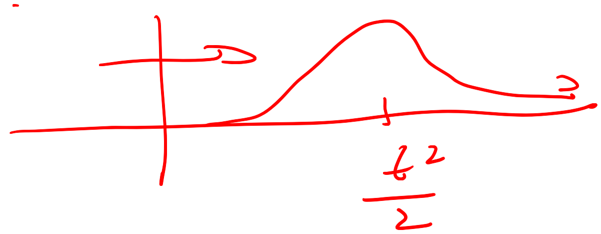
$$u(x, t) = f\left(x - \frac{t^2}{2}\right) \quad !!!$$

$$\boxed{f(x) = \lambda X}$$

$t = 0$



$u(x,0) = f(x).$



$$u_t + t\,u_x = 0 \qquad u(x, 0) = f(X)$$

$$u_t(x, t) + t\,u_x(x, t) = 0. \qquad \text{try another way.}$$

Fourier transform with respect to x!

$$\hat{u}_t + t\,\widehat{(u_x)} = 0 \qquad \widehat{f'(x)} = i\omega \hat{f}(\omega)$$

$$\hat{u}_t + t(i\omega)\,\hat{u}(\omega,t) = 0.$$

$$\hat{u}_t = -t(i\omega)\,\hat{u}(t,\omega). \quad \text{diff eqn.}$$

$$\hat{u}(\omega,t) = c(\omega)\, e^{-i\omega t^2/2}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(\omega)\, e^{-i\omega t^2/2}\, e^{i\omega x}\, d\omega$$

$$u(x,0) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{c(\omega)}_{\hat{f}(\omega)}\, e^{i\omega x}\, d\omega$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-i\omega t^2/2}\, e^{i\omega x}\, d\omega$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega(x - t^2/2)}\, d\omega.$$

$$= f(x - t^2/2) \quad !!$$

$$\hat{u}_t = -i\omega t\,\hat{u}$$

$$\hat{u}(\omega,t)$$

$$g'(t) = -i\omega t\, g.$$

$$\frac{dg}{dt} = -i\omega t g \qquad \int \frac{dg}{g} = \int -i\omega t\, dt$$

$$\ln g = \frac{-i\omega t^2}{2} + C \qquad \hat{u}(\omega,t) = g(t).$$

$$y = e^{-\frac{i\omega t^2}{2} + C} = D e^{-\frac{i\omega t^2}{2}}$$