MATH 323

WITH PDE'S, WE HAVE SEEN A FEW PROBLEMS THAT CAN BE SOLVED IN A COUPLE OF WAYS (FOURIER VS "NON-FOURIER")

EX: $u_t = u_{xx} \quad -\infty < x < \infty \qquad u(x,0) = f(x)$;

(i) $u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw} e^{-tw^2} \hat{f}(w)\, dw$;

OR $u(x,t) = \int_{-\infty}^{\infty} f(y) \cdot \underbrace{\left(\frac{1}{\sqrt{4t\pi}}\right) e^{-\frac{(x-y)^2}{4t}}}_{K(t,x,y)} dy$

"HEAT KERNEL"

AN INTEGRAL OPERATOR $\qquad \mathcal{L}(f)(s) = \int_0^{\infty} e^{-ts} f(t)\, dt$

$(Lf)(x) = \int_a^b \underbrace{K(x,y)}_{\text{kernel}} f(y)\, dy$

$\approx \sum_{j=1}^{N} K(x, y_j) f(y_j) \Delta y$

$(Lf)(x_i) \approx \sum_{j=1}^{N} K(x_i, y_j) f(y_j)$

IF $V = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} f(y_1) \\ \vdots \\ f(y_N) \end{pmatrix}$

IF $A_{i,j} = K(x_i, y_j)$

$A = (A_{ij})$

$$W = \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix} = \begin{pmatrix} L(f)(x_1) \\ L(f)(x_2) \\ \vdots \\ L(f)(x_N) \end{pmatrix}$$

THEN $w \cong AV$

DISCRETIZATIONS OF INTEGRAL TRANSFORMS GIVE MATRICES (!!)

NEW PROBLEM: <u>POISSON'S EQUATION</u>

FIRST, SOME LINEAR ALGEBRA:

$L = n \times n$ MATRIX, $\vec{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}$ A GIVEN VECTOR

HOW DO YOU SOLVE

$L\vec{u} = \vec{f}$? $\quad \vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$

$\{ Ax = \underline{b} \}$ $\quad \left(A \,\vdots\, \begin{matrix} b_1 \\ \vdots \\ b_N \end{matrix} \right) \leftarrow$

(I) FIND L^{-1} MATRIX

(II) GAUSS-JORDAN ELIMINATION

(III) (NEW) EIGENVECTOR METHOD.

SIMPLE FACT: IF $Lv = \lambda v$;

THEN $(L^{-1})(v) = \frac{1}{\lambda} v$

(PRF) $L^{-1}(LV) = L^{-1}(\lambda V) = \lambda L^{-1}(V)$

$$\left.\begin{aligned} V &= \lambda L^{-1}(V) \\ \frac{V}{\lambda} &= L^{-1}(V) \end{aligned}\right\}$$

HOW TO SOLVE $L\vec{u} = \vec{f}$;

(I) FIND A BASIS OF EIGENVECTORS FOR L: $V_1, V_2, \ldots V_N$. WITH $\lambda_1, \lambda_2, \ldots$

(II) WRITE f AS A LINEAR COMBINATION OF V_i's: $f = C_1 V_1 + C_2 V_2 \cdots + C_N V_N$

$$= \sum_{K=1}^{N} C_K V_K.$$

(III) $L\vec{u} = \vec{f} = \sum_{K=1}^{N} C_K V_K.$

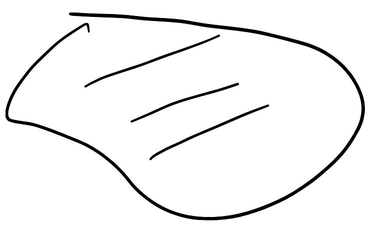
$$\vec{u} = L^{-1}\left(\sum_{K=1}^{N} C_K V_K\right)$$

$$= \sum_{K=1}^{N} C_K L^{-1}(V_K)$$

$$\vec{u} = \sum_{N=1}^{N} \frac{C_K V_K}{\lambda_K} \quad ✓$$

BACK TO PDE:

POISSON'S EQUATION



$u_{xx} + u_{yy} = f(x,y)$;

$u = 0$ ON THE BOUNDARY.

$f(x,y)$ GIVEN.

LET'S REWRITE THIS AS: LAPLACE OPERATOR

$$Lu = f \qquad Lu = u_{xx} + u_{yy}$$

$$= \Delta u \qquad 2$$

TO SOLVE THIS LOOK AT RELATED HELMHOLTZ PROBLEM

$$L(S) = \Delta S = S_{xx} + S_{yy} = \lambda S$$

$S = 0$ ON BOUNDARY.

INFINITE LIST: $L S_i = \lambda_i S_i$ $\quad S_i(x,y)$

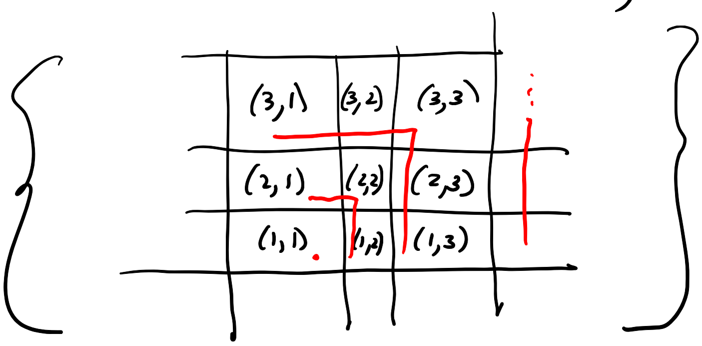
$i = 1, 2, \ldots$

THEN; WRITE $\quad f(x,y) = f = \sum_{i=1}^{\infty} c_i S_i$

THEN $\quad u = \sum c_i \frac{1}{\lambda_i} S_i$

EX: (SQUARE): $S_{m,n} = \sin(m\pi x)\sin(n\pi y)$

$$\lambda_{m,n} = -\pi^2(m^2 + n^2)$$



$$f(x,y) = \sum c_{m,n} \sin(m\pi x)\sin(n\pi y);$$

$$c_{m,n} = 4\int_0^1\int_0^1 f(x,y)\sin(m\pi x)\sin(n\pi y)\,dx\,dy$$

$$u = \sum \frac{-C_{m,n}}{\pi^2(m^2+n^2)} \sin(m\pi x)\sin(n\pi y)$$

"FOURIER METHOD" SOL'N TO $\Delta u = f$.

NON-FOURIER (INTEGRAL/TRANSFORM METHOD)
OPERATOR

HARDER HERE.

"BABY" PROBLEM. $u''(x) = f(x)$ $\quad u(0) = u(1) = 0$.

NOTE: WE WILL WORK REDICULOUSLY HARD ON THIS PROBLEM; IDEAS APPLY TO NOT SO SIMPLE PROBLEMS.

(I) NON-FOURIER METHOD FIRST. INTEGRAL OPERATOR

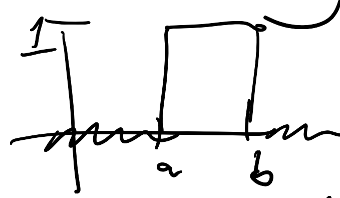
(II) THEN FOURIER METHOD.

(III) COMPARE?

$u''(x) = f(x)$ $\quad u(0) = u(1) = 0$ $\quad$ [AYKM?]

$$u'(x) = \int_0^x f(t)\,dt$$

$$\chi_{[a,b]}(x) = \begin{cases} 1 & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



$$u'(x) = \int_{t=0}^{t=1} \chi_{[0,x]}(t)\, f(t)\,dt.$$

$$\int_{t=0}^{t=x} + \int_{t=x}^{t=1}$$

$$u(x) = \int_0^x u'(s)\,ds$$

$$= \int_0^1 \chi_{[0,x]}(s)\, u'(s)\,ds$$

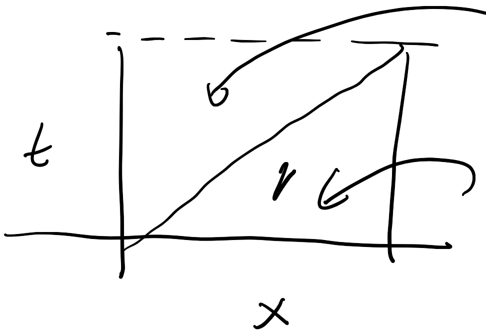
$$= \int_0^1 \chi_{[0,x]}(s) \left\{ \int_0^1 \chi_{[0,s]}(t) f(t)\,dt \right\} ds$$

$$u(x) = \int_0^1 f(t) \left\{ \underbrace{\int_0^1 \chi_{[0,x]}(s)\, \chi_{[0,s]}(t)\, ds}_{K(x,t)\,!} \right\} dt \qquad (AJKM)^2$$

$$= \boxed{\int_0^1 f(t)\, K(x,t)\, dt.}$$

$u''(x) = f(x)$

$$\int_0^1 \chi_{[0,x]}(s)\, \chi_{[0,s]}(t)\, ds.$$

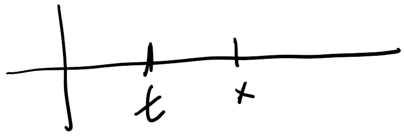


$t \geq x.$

$x \geq t$

(I) $t \geq x$ $\qquad \int_0^1 \underset{s \leq x}{\chi_{[0,x]}(s)}\ \underset{t \leq s}{\chi_{[0,s]}(t)}\, ds.$

$$= 0\,!$$



(II) $t \leq x$ $\qquad \int_0^1 \underset{s \leq x}{\chi_{[0,x]}(s)}\ \underset{t \leq s}{\chi_{[0,s]}(t)}\, ds$

$$= \int_t^x ds = (x-t)$$

$$K(x,t) = \begin{cases} 0, & t \geq x \\ (x-t), & t \leq x \end{cases}$$

CHECK: $u''(x) = x$ (!!) $= f(x)$ AYKM 3.

$$u(x) = \int_{t=0}^{t=1} f(t)\, K(x,t)\, dt$$

$$= \int_{t=0}^{t=x} f(t)\, K(x,t)\, dt + \int_{t=x}^{t=1} f(t)\, K(x,t)\, dt$$

$$= \int_0^x (t)\,(x-t)\, dt + 0 \;!$$

$$= \int_0^x (xt - t^2)\, dt$$

$$= \left[\frac{xt^2}{2} - \frac{t^3}{3} \right]_0^x = \frac{x^3}{2} - \frac{x^3}{3}$$

$$= \frac{x^3}{6} \;!$$

GIVES SOLUTION!

GIVES SOLUTION?

$u(0) = 0, \quad u(1) = \frac{1}{6} \;!!$ $\quad \boxed{\begin{array}{l} u''(x) = f(x) \\ u(0) = u(1) = 0 \end{array}}$

IF $u'' = f$ THEN $\tilde{u}(x)$

$= u(x) + (ax+b)$

IS ALSO A SOL'N.

$\boxed{\begin{array}{l} l(x) = ax+b \\ l'' = 0 \end{array}}$

MODIFY u TO SATISFY BNDRY CONDITION: $l(x) = ax+b$.

$$l(0) = u(0)$$

$$l(1) = u(1).$$

THEN $\tilde{u}(x) = u(x) - l(x)$

SATISFIES DE AND BOUNDARY CONDITIONS. (EX $l(x) = \frac{1}{6}x$)

$$\tilde{u}(x) = u(x) - l(x) = \frac{x^3}{6} - \frac{x}{6};$$

$\tilde{u}(0) = 0 \quad \tilde{u}(1) = 0 \qquad \tilde{u}''(x) = x.$

$$u(x) = \int_0^1 f(t)\, K(x,t)\, dt.$$

$$u(0) = \int_0^1 f(t)\, K(0,t)\, dt$$

$$= 0$$

$$u(1) = \int_0^1 f(t)\, K(1,t)\, dt$$

$$= \int_0^1 f(t)\,(1-t)\, dt$$

$\ell(0)=0$
$\ell(1)=$

$\ell(x)$

$$\tilde{u}(x) = u(x) - \ell(x) \left\{ \int_0^1 f(t)(1-t)\,dt \cdot x \right\}$$

$$= \int_0^1 f(t)\, K(x,t)\, dt - \int_0^1 x\,(1-t)\, f(t)\, dt$$

$$= \int_0^1 f(t) \Big[\underbrace{K(x,t) - x + tx}_{\tilde{K}(x,t)} \Big]\, dt$$

$$K(x,t) = \begin{cases} 0 & t \geq x \\ x - t & t \leq x \end{cases}$$

$$\tilde{K}(x,t) = \begin{cases} -x + tx & t \geq x \\ \cancel{x} - t \cancel{-x} + tx & t \leq x \end{cases}$$

$$\tilde{K} = \begin{cases} x\,(t-1) & t \geq x \\ t\,(x-1) & t \leq x \end{cases}$$

$$\boxed{\tilde{u}(x) = \int_0^1 f(t)\, \tilde{K}(x,t)\, dt}$$

$\tilde{u}'' = f$
$\tilde{u}(0) = \tilde{u}(1) = 0$

SUMMARY: (1) FIND A SOLUTION TO $u_F'' = f$ (INTEGRAL METHOD), BUT DOES NOT SATISFY CORRECT BNDRY. COND'S {0 ON BNDRY}

(2) SOLVE $\ell''(x) = 0$

$\ell(x)$ HAS SAME BOUNDARY CONDITIONS AS u_F;

(3) $\tilde{u} = u_F - \ell$ SATISFIES $(\tilde{u})'' = f$

$\tilde{u} = 0$ ON BOUNDARY.

(AYKM)

FOURIER METHOD.

$u''(x) = f(x)$ $\quad u(0) = 0, \; u(1) = 0,$

$S''(x) = \lambda S$ $\quad S = S_N = \sin(N\pi x)$

$\lambda_N = -\pi^2 N^2.$

$$f(x) = \sum_{N=1}^{\infty} c_N \sin(N\pi x)$$

$u'' = f$

$u'' = x$

$$c_N = 2\int_0^1 f(t)\sin(N\pi t)\,dt.$$

THEN $$u(x) = \sum_{N=1}^{\infty} \frac{c_N}{-\pi^2 N^2}\sin(N\pi x). \quad !!$$

BUT! $$u(x) = \sum_{N=1}^{\infty} 2\int_0^1 f(t)\sin(N\pi t)\,dt\, \frac{\sin(N\pi x)}{-\pi^2 N^2}$$

$$= \int_0^1 f(t)\underbrace{\left\{\left(\frac{-2}{\pi^2}\right)\sum_{N=1}^{\infty}\frac{\sin(N\pi t)\sin(N\pi x)}{N^2}\right\}}_{\tilde{K}(x,t)}dt.$$

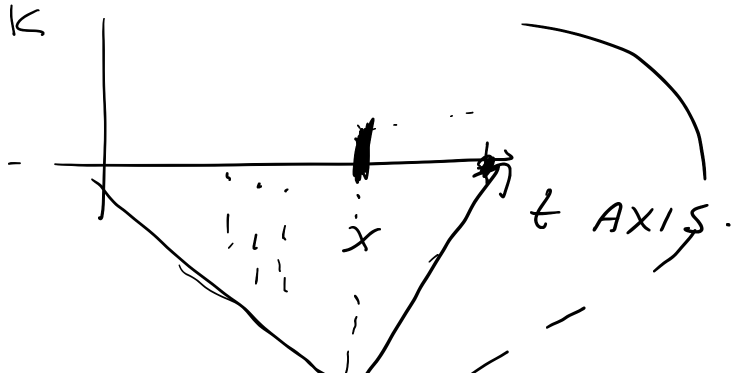
INTEGRAL FORM FOR SOLN

SO DOES $\tilde{K}(x,t) = K(x,t)$?

YES! $$\tilde{K}(x,t) = \begin{cases} x(t-1) & t \geq x \\ t(x-1) & t \leq x \end{cases}$$

FIX X

$\equiv$



FIX X

$$\tilde{K}(x,t) = \frac{-2}{\pi^2} \sum_{N=1}^{\infty} \frac{\sin(N\pi x)}{N^2} \sin(N\pi t)$$

$$= \sum_{N=1}^{\infty} \underbrace{\left(\frac{-2}{\pi^2}\right) \sin(N\pi x)}_{C_N} \sin(N\pi t)$$

$\tilde{K}(x,t)$? //

FOURIER COEFFICIENTS FOR $\tilde{K}(x,t)$.

$$\Delta u = u_{xx} + u_{yy} = f(x,y);$$

$$u(x,y) = \iint f(s,t)\, K(x,y,s,t)\, ds\, dt.$$

KERNEL. GREENS' FUNCTION.