PHYSICS 432/532: COSMOLOGY
Prof. Michael S. Vogeley

Lecture Notes #1: Isotropic Universe

Important concepts:

- cosmological principle
- isotropy, homogeneity
- Robertson-Walker metric
- dynamics of the expansion
- evolution of matter, radiation density
- stretching of light - redshift
- observed properties

1 Isotropy and homogeneity

Just as Einstein began development of relativity by aiming to write down the simplest possible generalization of gravity, we begin investigations of cosmology by considering the simplest possible model for the distribution of mass in the universe. We know that Earth is not the center of the universe. Neither is the Sun. Next step is to assume that our Galaxy is not at a special place. This is commonly known as the "Cosmological Principle." We assume that the universe looks the same from any vantage point, to within statistical fluctuations. Note that this is obviously an approximation – most of space is NOT near a large galaxy, so the fact that we live in a large spiral galaxy already biases our view of the universe.

Some definitions:
Isotropic = the same in all directions (e.g., brightness of the sky).
Homogeneous = same at all places (e.g., constant density).

From Earth, we see a universe that is, on very large scales, isotropic. Three different kinds of observations show this: (1) distribution of faint galaxies on the sky (faint enough to be far beyond the local group), (2) the isotropic recession velocities of other galaxies which obey the relation

$$v = Hr$$

(as found by Slipher and studied in detail by Hubble), and (3) the near uniformity of the Cosmic Microwave Background radiation.

If we observe isotropy and assume the Cosmological Principle, we infer that the universe is both isotropic AND homogeneous. Can see this by considering what two observers see. Imagine overlapping shells around observers A and B. If A see an isotropic distribution within his shell and B sees an isotropic distribution within hers, then the matter within the shell must be homogeneous.

2 The Spacetime Metric and the geometry of spacetime

Our first goal is to find an appropriate solution to Einstein's equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Here I deliberately include the Λ term, the "cosmological constant." The most general form of these equations includes that term, which arises as a constant of integration. I am of the opinion that one must have a reason to set it to zero, if one so chooses. We shall see momementarily that, if moved to the right side of the equation, Λ behaves like a vacuum energy density. Einstein may have called it his "greatest blunder" but observations indicate that a cosmological constant, or something that behaves quite like one, is necessary to explain the accelerating expansion of the universe.

In other words, Λ appears naturally in these equations as a constant of nature. Because it affects the dynamics of the universe, astronomers can measure it. I find it remarkable that the equations for the relationship between geometry ($G_{\mu\nu}$) and mass-energy ($T_{\mu\nu}$), a purely classical (meaning non-quantum) theory, include a term that seems best explained as a purely quantum phenomenon. How did that happen? More later...

A *spacetime metric* tells us how to compute the spacetime interval. Recall from special relativity that the spacetime interval between two events is Lorentz invariant. Thus, in flat space,

$$ds^2 = -c^2dt^2 + (dx^2 + dy^2 + dz^2)$$

We could also have written this in terms of the proper time, τ, by placing $c^2d\tau^2$ on the left side and swapping the signs on the right side. This is the special case in which the metric tensor is diagonal, with elements $g_{00} = 1, g_{11} = g_{22} = g_{33} = -1$ and zero elsewhere (another convention reverses those signs, so pay attention in a GR class, but don't worry about it here).

In spherical polar coordinates, the angular separation is $d\Psi^2 = d\theta^2 + \sin^2\theta d\phi^2$, and the spacetime interval in flat (Minkowski) space can be written as

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\Psi^2$$

This is more convenient for cosmology, because we are most often concerned with distances along the line of sight r toward a distance object. Note that I am using $d\Psi$ rather than $d\Omega$ (as in Ryden) for the angular increment, to avoid confusion with later use of Ω as dimensionless mass-energy density.

We need to come up with a metric that describes the spacetime interval between events in an isotropic homogeneous universe. Imagine a set of observers spread throughout this universe. They synchronize their clocks by setting their local time when the density of matter reaches some previously-agreed upon value (this procedure works only in an exactly homogeneous universe). This defines a global time coordinate t. One can show that the spacetime metric must have the form

$$ds^2 = -c^2dt^2 + a^2(t)[f^2(r)dr^2 + g^2(r)d\psi^2]$$

Here $a(t)$ is the dimensionless scale factor of the universe, defined so that $a = 1$ now, and r is the comoving radial coordinate, which does not vary with time. The remaining problem is to determine the functions f and g.

Robertson and Walker (1936) found a solution to this metric of the form

$$ds^2 = -c^2dt^2 + a^2(t)[dr^2 + S_k^2(r)d\psi^2]$$

There are three possible types of geometry with solutions in which $S_k(r)$ is $R\sin(r/R)$ ($k = 1$), $R\sinh(r/R)$ ($k = -1$), or r ($k = 0$) for geometries that correspond to closed (hyperspherical), open (hyperbolic), or flat, respectively, and R is the radius of curvature in the non-flat cases or, more generally, a scale size for the universe. We'll discuss geometry below.

As written above, the metric uses a spatial coordinate system (r, θ, ϕ) in spherical polar coordinates where r is the proper distance along the line of sight to an object ($dt = 0, d\Psi = 0$). Another choice is to use a radial coordinate that is, for lack of a better term, an "effective distance," x (I'll use x here to be consistent with Ryden even though I find it annoying to re-use the symbol for a Cartesian coordinate), that depends on the geometry of spacetime (and see discussion below about angular diameter distance). To do so, we could write (and you can verify in a homework) the spacetime metric in the form

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dx^2}{(1 - kx^2/R^2)} + x^2d\psi^2\right]$$

Note the difference between x (not the Cartesian coordinate) in this and r in the the previous metric. The relation between these forms can be seen by substituting $x \equiv S_k(r)$.

What is the geometry of the universe? The three cases correspond to distinct geometries. For $k = 0$, we recover Minkowski space,

$$ds^2 = -c^2dt^2 + a^2(t)[dr^2 + r^2d\psi^2]$$

in which space is Euclidean. The volume of this space is infinite. We call such a universe "flat."

For $k = 1$,

$$ds^2 = -c^2dt^2 + a^2(t)[dr^2 + R^2\sin^2(r/R)d\psi^2]$$

For large r, the spatial separation between points separated by fixed ψ is less than in the Euclidean case because $R\sin(r/R) < r$. We can say that there is an "angle deficit" in this geometry. Space here has the geometry of a 3-sphere, with positive curvature that has radius of curvature $R(t)$. This space has a finite volume. The hidden "fourth spatial dimension", towards the center of the 3-sphere, has no physical meaning whatsoever. We speak of such a universe as being "closed."

For $k = -1$,

$$ds^2 = -c^2dt^2 + a^2(t)[dr^2 + R^2\sinh^2(r)d\psi^2]$$

In this case, there is an "angle excess" – points are further apart than their distance and angular separation would indicate in Euclidean space because $R\sinh(r/R) > R$. As we look further and further away, the circumference of spheres get larger and larger. This space is hyperbolic – with negative curvature. Locally, the curvature is like that of a saddle. Space is infinite in this case - unlike the closed case, there is no natural boundary to this "open" geometry.

Note that in all cases, the very local universe looks flat because

$$\lim_{r\to 0} S_k(r) = r$$

3 Proper Distance

What is the distance between things in the universe? In an evolving universe, this question only makes sense if we're clear about what kind of distance we mean. If I mean "the physical distance as would be measured with a tape measure" (what I colloquially think of as "tape measure distance"), that corresponds exactly to the Proper Distance—the distance at fixed cosmic time, t (where we defined that global time coordinate above). Using the metric above and setting $dt = 0$, an element of distance along the line of sight r to another galaxy ($d\Psi = 0$) is simply $ds = a(t)dr$, thus the proper distance is simply

$$d = a(t)r$$

Again, r is the comoving coordinate distance and $a(now) = 1$.

4 Hubble constant and cosmological redshift

The Hubble "constant" in the famous Hubble law $v = Hr$ parametrizes the expansion rate of the universe at a particular epoch.

$$H = \frac{\dot{a}}{a}$$

Note that H has the units of inverse time and that the Hubble constant is not constant at all. Trivial rewriting of this equation yields

$$v = Hd$$

where v is the rate at which proper distance between galaxies increases and d is that distance. To an observer, this expansion appears as a recession of all other galaxies.

How can we observe the expansion? Photons always experience zero spacetime interval, $ds = 0$, thus $cdt = a(t)dr$, or $c = a(t)dr/dt$. The time between emission of wave crests is so small that we can consider $a(t)$ to be the same for both. The wave moves across the universe to us while the universe expands. The speed of light doesn't change as the universe expands, so dr/dt must decrease. Here dr is an increment of comoving coordinate distance and dt is an increment of local clock time. The comoving coordinate distance between wavecrests does not change, so the local clock time increment between arrival of wave crests must increase by $a(t)$. Of course, this is matched by the increase in *proper* distance, adr, between wavecrests by the same factor $a(t)$, so that the speed of light is constant $c = \nu_{emit}\lambda_{emit} = \nu_{obs}\lambda_{obs}$.

Thus, the expansion of the universe causes the radiation that we observe to be lower in frequency and longer in wavelength,

$$\frac{\nu_{emit}}{\nu_{obs}} = \frac{\lambda_{obs}}{\lambda_{emit}} = \frac{a(t_{obs})}{a(t_{emit})} = 1 + z$$

This defines the "redshift" z as a fractional change in frequency or wavelength. We detect this by comparing the frequency or wavelength of observed spectral lines with their frequency or wavelength in the laboratory on Earth.

For small separations, thus small velocities, this looks like a Doppler shift,

$$1 + z \approx 1 + v/c$$

But it is NOT a Doppler shift. It is caused by the expansion of the universe during the travel of the photons. A wavelength of light stretches during its flight by the same factor that the proper distance between the galaxies changes. We will later discuss the Doppler shift of wavelength or frequency caused by *peculiar velocities'* of galaxies relative to their local rest frame, but for now we are thinking about locally stationary objects that emit light.

Note that $a(t = \mathrm{now}) = 1$, by definition, thus for observations made by Earth observers,

$$a = \frac{1}{1+z}$$

Light from an object that lies at redshift $z = 1$ is observed to have twice the wavelength that was emitted and that light was emitted when the universe was half its current size.

Again, the change in wavelength is NOT a Doppler shift. Do NOT try to convert a large redshift to a recession velocity by using the relativistic Doppler formula

$$1 + z = \sqrt{(1 + v/c)/(1 - v/c)}$$

(I've seen this done by notable physicists!) This turns out to yield the correct result only for an empty universe.

5 Dynamics of the expanding universe, Friedmann's equation, Birkhoff's theorem

Next we consider the dynamics of the expanding universe. We can almost do this with Newtonian mechanics - almost. Proof of this equation requires inserting the Robertson-Walker metric into the GR field equations (which would take more time than I have). Here we heuristically derive the **Friedmann equation**. [As an historical aside, Lemaitre independently derived similar results from General Relativity and was the first to interpret Hubble's observations in terms of these equations.]

Consider a thin spherical shell of matter. [DRAW THIS] If the universe is roughly homogeneous and isotropic, then the gravity that this shell feels is only that due to the mass interior to the shell, which looks like a point source at the center. We need General Relativity to show that the matter at large distance outside the shell does not contribute; this result is known as Birkhoff's theorem.

Look at the motion of a particle of mass m in a shell of radius R that encloses mass M. What is its energy? It has kinetic and potential energies

$$K = \frac{mv^2}{2}$$

$$V = -\frac{GMm}{R}$$

The velocity is just the rate of change of R, $\dot{R}$, thus $v = \dot{R}$. Conservation of energy requires $K + V =$ constant,

$$\frac{m\dot{R}^2}{2} - \frac{GMm}{R} = \text{constant}$$

or, after dividing out m and redefining the constant,

$$\dot{R}^2 = \frac{2GM}{R} + \text{constant}$$

What happens as time passes? Consider only models in which $\dot{R}$ is initially positive, thus R increases with time. Thus, at least initially, the first term on right hand side is getting smaller with time.

If the constant on the right hand side is > 0: The first term on rhs goes to zero as R gets infinitely large, with expansion rate approaching a constant. Energetically, kinetic energy is always greater than potential energy, so the expansion continues forever.

Constant < 0: At some R, right hand side equals zero, thus velocity equals zero. R achieves some maximum size, expansion stops, recollapses. Kinetic energy is too small, thus the shell will begin to fall back - expansion ceases and we have recollapse.

Constant $= 0$: First term on rhs approaches zero, second term is identically zero, thus expansion rate asymptotes to zero as R goes to infinity. Kinetic and potential exactly balance, expansion continues forever, but gets slower and slower.

6 More "Newtonian" cosmology

Consider again the simple equation for expansion of a spherical shell of radius R around mass M:

$$\dot{R}^2 = \frac{2GM}{R} + \text{constant}$$

The sign of the constant on the rhs is related to the fate of this shell: expands forever if > 0, recollapses if < 0, approaches zero expansion rate if $= 0$.

To relate this more directly to cosmology, rewrite this equation, using a homogeneous mass interior to the sphere, $M = 4\pi\rho/R^3$,

$$\dot{R}^2 - \frac{8\pi G\rho R^2}{3} = \text{constant}$$

Divide by R^2, plug in $H = \dot{R}/R$. Rewrite:

$$H^2 - \frac{8\pi G\rho}{3} = \frac{\text{constant}}{R^2}$$

This allows us to define a critical density $\rho_{crit} = 3H^2/(8\pi G)$. At exactly this density, the constant on the rhs above is zero, and we have asymptotic expansion to an infinite universe. For smaller density, the universe expands forever. For larger density, the universe recollapses.

Inserting the R-W metric into the GR field equations allows us to solve for the constant, yielding

$$H^2 - \frac{8\pi G\rho}{3} = -\frac{kc^2}{R^2}$$

Equivalently

$$\dot{R}^2 - \frac{8\pi G\rho R^2}{3} = -kc^2$$

This equation, and variants of it (move around where the R's go), is the Friedmann equation for the dynamics of a homogeneous isotropic universe. Originally, Friedmann derived it to describe a universe that contains only matter but, as we shall discuss, it is generally applicable for models that contain all kinds of mass-energy. The density ρ in this equation includes contributions from all forms of mass-energy, including matter, radiation, and vacuum energy.

Comparing the rhs of the Friedmann equation to our shell model above, it is obvious that the sign of k is related to dynamics of universe, which depends on density of matter. For *negligible vacuum energy only*:
If $k = -1$ (open universe) then universe expands forever.
If $k = +1$ (closed universe) then universe recollapses.
If $k = 0$ (flat universe) then universe expands asymptotically.
However, as we'll discuss shortly, the vacuum energy is probably NOT negligible, so this correspondence between curvature and dynamics is not exact.

Related to the Friedmann equation is its time derivative.

$$\frac{d}{dt}(\dot{R}^2) = \frac{d}{dt}\left[\frac{8\pi G\rho R^2}{3} - kc^2\right]$$

Tricky part is what happens to the density ρ, but we can get at this by using conservation of energy (First Law of Thermodynamics)

$$d(\rho c^2 R^3) = -p d(R^3)$$

which yields

$$\ddot{R} = -\frac{4\pi G R}{3}(\rho c^2 + 3p)$$

In this form, we see that the second derivative of the scale factor, the "acceleration" of the expansion, depends on the equation of state of the contents of the universe (depends on relationship between density and pressure). This will be particularly important for examining what vacuum energy does. A general time-independent equation of state is

$$p = w\rho c^2$$

where $w = 1/3$ for radiation, $w = 0$ for pressureless matter (dust), and $w = -1$ for vacuum energy.

What does this equation say? If today $\dot{R} \geq 0$ and if in the past $\rho c^2 + 3p$ was always positive, then $\ddot{R}$ was always negative. Thus, at some finite time in the past $R = 0$; this singularity in this classical theory is the "big bang" event.

For the case $w = -1$, $\rho c^2 + 3p = -2\rho c^2$, thus

$$\ddot{R} = \frac{8\pi G}{3} R\rho_{vac} c^2 > 0$$

which describes an accelerating universe.

7 Evolution of the universe

So, if we measure H and ρ, we measure the geometry of the universe. We call the ratio of observed density to critical density

$$\Omega \equiv \frac{\rho}{\rho_{crit}} = \frac{8\pi G\rho}{3H^2}$$

The density here includes all contributions: matter, radiation, vacuum. The cases $\Omega > 1, < 1, = 1$ correspond to the $k = +1, -1, 0$, respectively. Again, in the absence of vacuum energy, these are recollapsing, forever expanding, and asymptotically expanding universes.

It is important to note that both ρ and H are functions of time. We sometimes (not often enough?) denote quantities at the present epoch as H_0 and Ω_0. Another convention is that we often write the Hubble constant at the present epoch in terms of a dimensionless number

$$h \equiv \frac{H_0}{100\,\mathrm{km\,s^{-1}Mpc^{-1}}}$$

These units arose because astronomers first measured galaxy velocities in $\mathrm{km\,s^{-1}}$ and distances in Mpc, via $v = Hr$. In these units the current density of the universe is roughly (see the book for exact values)

$$\rho_0 \approx 2\times 10^{-26}\Omega h^2 \mathrm{kg\,m^{-3}}$$

$$\rho_0 \approx 3\times 10^{11}\Omega h^2 M_\odot\,\mathrm{Mpc^{-3}}$$

where $M_\odot$ is one solar mass and $1\,\mathrm{Mpc} \approx 3\times 10^6\,\mathrm{ly} \approx 3\times 10^{22}\,\mathrm{m}$. One parsec is roughly 3 light years. One year is roughly $\pi \times 10^7\,\mathrm{s}$ long. A solar mass $M_\odot = 2.0\times 10^{30}\,\mathrm{kg}$.

Again, the density evolves with time as the universe expands. But matter, radiation and vacuum energy do NOT evolve in the same way. Conservation of mass implies that density of matter is simply the mass divided by volume. Imagine for the moment that all matter is in the form of particles of mass m. The density measured in some volume V simply

$$\rho_{matter} = \frac{mN}{V}$$

where N is the number of particles inside V. Neither m nor N change with the expansion, but $V \propto R^3$ does change. Thus for pressureless matter

$$\rho_{matter} \propto R^{-3} \propto a^{-3}$$

where we used $a = R/R_0 = (1+z)^{-1}$. For photons, the equivalent mass energy density of a volume V filled with photons of frequency ν is

$$\rho_{rad} = \frac{(h\nu/c^2)N}{V}$$

Therefore, density of radiation gets an extra factor of the scale factor, since photons have their energy diluted by the expansion (longer wavelength means lower frequency, thus lower energy $E = h\nu$),

$$\rho_{rad} \propto R^{-4} \propto a^{-4}$$

The vacuum energy density is constant with time,

$$\rho_{vac} = constant$$

The density of the universe at any epoch is related to the Ω's measured now by

$$\rho = \frac{3H_0^2}{8\pi G}(\Omega_{matter}a^{-3} + \Omega_{rad}a^{-4} + \Omega_{vac})$$

[By definition, $a = 1$ right now.] We see that, at early times when $a \to 0$, the universe is radiation dominated. At late times, when $a \to \infty$, the universe is vacuum dominated. Somewhere in between (the middle ages?), the universe is matter dominated and structure forms. Today, we seem to be in the transition between the matter dominated era and a vacuum dominated era. In the latter, the universe expands exponentially, which works to suppress the formation of structure by dragging the matter apart.

8 More Evolution of the Isotropic Universe

Earlier we wrote the Friedmann equation with all the mass-energy terms subsumed within the one density term. Consistent with Einstein's equations, the cosmological constant term is often broken out separately,

$$H^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda c^2}{3} - \frac{kc^2}{R^2}$$

In Einstein's equations,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

we can move the cosmological constant over to the right hand side and treat it like a part of the stress-energy tensor,

$$T^{\mu\nu}_{vac} = \frac{\Lambda c^4}{8\pi G} g^{\mu\nu}$$

Thus, the vacuum energy density, in the same units as other densities, is

$$\rho_{vac} = \frac{\Lambda c^2}{8\pi G}$$

which is obviously constant with time, with $p_{vac} = -\rho_{vac}c^2$ The contribution of the vacuum energy to the density parameter Ω is

$$\Omega_{vac} = \frac{\rho_{vac}}{\rho_{crit}} = \frac{\Lambda c^2}{3H^2}$$

Right now, the radiation density is negligible, thus

$$\Omega_0 = \Omega_{matter} + \Omega_{vac}$$

As we'll see later on in the course, there is a strong theoretical prejudice for spatially flat models, which have $k = 0$. This requires $\Omega_{matter} + \Omega_{vac} = 1$, thus there is only one free parameter.

We talked earlier about the rate of acceleration of the universe. Astronomers historically described this in terms of a dimensionless **deceleration parameter**,

$$q \equiv -\frac{\ddot{R}R}{\dot{R}^2}$$

This is related to the various contributions to the mass-energy density by

$$q = \frac{\Omega_{matter}}{2} + \Omega_{rad} - \Omega_{vac}$$

The minus sign might confuse you; note that $q > 0$ means decelerating and $q < 0$ means accelerating. Note that a universe with only matter has $q = \Omega/2$ but in general it measures a linear combination of mass-energy components.

Note the curious fact that the vacuum energy density contributes in the opposite sense to the matter and radiation density! If $\Lambda > 0$, then the vacuum energy tends to *accelerate* the expansion. $q < 0$ implies an "accelerating universe."

Another quantity of interest is the scale factor at the present epoch. From the Friedmann equation we find that

$$R_0 = \frac{c}{H_0}\left[\frac{\Omega - 1}{k}\right]^{-1/2}$$

For $k \neq 0$, this is the curvature scale of the universe. For zero density (empty universe), the scale factor is simply equal to the so-called **Hubble length**, c/H_0.

The total density parameter varies with the scale size of the universe as

$$\Omega(a) - 1 = \frac{\Omega - 1}{1 - \Omega + \Omega_{vac}a^2 + \Omega_{matter}a^{-1} + \Omega_{rad}a^{-2}}$$

where all the Ω's on the right hand side are quantities measured today. Look what happens at very early times, when a is small (remember, $a = R/R_0$): the Ω_{matter} and Ω_{rad} terms in the denominator blow up, so unless the universe has only vacuum energy, the rhs of this equation goes to zero. This means that, at early times when the universe is small, all models look like $\Omega = 1$, i.e., the universe begins spatially flat. In other words, if $\Omega \neq 1$ today, in the past it only had to differ from unity by a tiny amount. The problem of starting the observed universe flat enough that Ω today is not vastly different from unity is called the "flatness problem."

9 Matter-radiation equality, recombination, and the matter-dominated universe

Running the clock backwards to an earlier denser universe we also find that the relative importance of matter and radiation changes. Today, we can safely ignore radiation as a component of the mass-energy density in the universe. But this was not always the case. Since

$$\rho_{matter} \propto (1+z)^3$$

while

$$\rho_{rad} \propto (1+z)^4$$

the ratio of radiation to matter density rises as we look back to earlier times. Doing the calculation, we find that the densities of radiation and matter were equal at a redshift

$$1 + z_{eq} = 23900\Omega h^2$$

We'll see later in the course how this epoch impacted the formation of structure in the universe. We say the the universe before this time was "radiation dominated" and that it is "matter dominated" thereafter.

Shortly after this epoch, the universe cools down enough for Hydrogen to form. This is the epoch of recombination, so-called because electrons and protons can "recombine" to form Hydrogen (odd terminology, since they were never "combined" before this time!). This occurs roughly at a redshift

$$1 + z_{rec} \approx 1000$$

We cannot see any photons emitted from before this time, because the photons were tightly coupled to the free electrons by Thomson scattering, thus the universe was opaque.

[PLOT LOG-LOG showing evolution of scale factor for different cosmologies.]

10 Special Cosmologies

10.1 Static universe

Again, the Friedmann equation,

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -kc^2$$

relates expansion rate of universe to the average density and curvature.

Recall that Einstein originally thought that the universe should be static. In this case, we require

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2 = 0$$

thus

$$\rho = \frac{3kc^2}{8\pi G R^2}$$

We also need

$$\ddot{R} = -\frac{4\pi G}{3}R(\rho c^2 + 3p) = 0$$

thus we need $\rho c^2 = -3p$. The only way to get this is to have lots of vacuum energy, which has $p = -\rho c^2$, so that $\rho = \rho_{matter} + \rho_{vac} = 3\rho_{vac}$. That is, the mass density is exactly twice the vacuum density. Since ρ is positive, $k = 1$, so this model is closed and static. But it's also highly unstable. Look what happens if we slightly change the scale factor R: the matter density gets bigger or smaller but the vacuum energy density remains the same, so the universe begins to expand or contract.

10.2 de Sitter space

Another interesting special case is where there is only vacuum energy. The Friedmann equation looks like

$$\dot{R}^2 - \frac{8\pi G}{3}\rho_{vac}R^2 = -kc^2$$

The equation of state of the vacuum, $p = -\rho c^2$ implies (substitute in for ρ_{vac}) for the derivative of the Friedmann equation,

$$\ddot{R} = -\frac{4\pi G}{3}R(\rho_{vac}c^2 + 3p) = 8\pi G R \rho_{vac} c^2$$

so $\dot{R}$ increases without limit. Thus, R grows indefinitely making both terms on the lhs of the Friedmann equation huge, thus the rhs is negligible, effectively looking like $k = 0$. The solution to the differential equation for the expansion,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho_{vac}}{3},$$

is obviously

$$R \propto \exp Ht$$

because

$$H = \sqrt{\frac{8\pi G\rho_{vac}}{3}} = \sqrt{\frac{\Lambda c^2}{3}}$$

This result is interesting because it says models with positive vacuum energy eventually end up with exponential expansion, since $\rho_{matter} \propto R^{-3}$, which effectively drives the universe toward $k = 0$. The counterexample is the case of supercritical matter density and small vacuum energy, which results in recollapse before the universe becomes vacuum dominated.

10.3 Flat Universes

The case $k = 0$ (same as $\Omega = 1$) is a very important class of models because inflationary models, which have an early exponential phase of expansion, predict that the universe should be almost exactly (to within many decimals places) flat. We'll discuss these models in two weeks.

It is worth pointing out that for a time there was a strong theoretical prejudice for $\Lambda = 0$, $\Omega_{matter} = 1$, often called the Einstein-de Sitter model.

A universe that is exactly spatially flat always stays that way, thus $\Omega = 1$ forever. The individual contributions to the mass-energy density may change in importance, however. Since the radiation density is negligible today, these models have only one free density parameter because

$$\Omega_{matter} + \Omega_{vac} = 1$$

11 Cosmological Parameters: Ω_{matter} vs. Λ

[Discuss plot of Ω_m vs. Λ]

Recall that $\Omega_{vac} = \Lambda c^2/3H^2$, thus $\Lambda > 0$ and $\Omega_{vac} > 0$ mean the same thing.

Note that the line $\Omega_{tot} = 1$ separates open from closed universes (geometry!).

The line described by

$$q_0 = \frac{\Omega_{matter}}{2} - \Omega_{vac} = 0$$

separates accelerating ($q_0 > 0$) and decelerating ($q_0 < 0$) universes.

The case of $\Lambda = 0$ shows the simple correspondence between Ω_{matter}, curvature, and destiny. Note that all models with $\Lambda = 0, \Omega_{matter} \geq 0$ are decelerating.

Note carefully that the possibility of non-zero Λ makes the relationship between density, spatial curvature, and future of the universe more complex.

$\Lambda < 0$ (below dotted line in figure) always leads to recollapse at some epoch. Look at the acceleration equation,

$$\ddot{R} = -\frac{4\pi G R}{3}(\rho c^2 + 3p) = -\frac{4\pi G R c^2}{3}(\rho_{matter} + \rho_{vac} - 3\rho_{vac}) = -\frac{4\pi G R c^2}{3}(\rho_{matter} - 2\rho_{vac})$$

and remember that $\rho_{matter} \propto R^{-3}$ while $\rho_{vac} = \text{constant} < 0$ in this case. Either the model recollapses before anything happens (large Ω_{matter}) or Ω_{matter} is small enough that Λ eventually dominates. Such models can be open, closed, or flat.

All $\Lambda > 0$, $\Omega_{matter} < 1$ models (white triangle in figure plus area above that triangle) expand to infinity. In other words, these expand forever, regardless of whether they are spatially open, closed, or flat.

If $\Omega_{matter} > 1$, recollapse occurs unless Ω_{vac} is positive and large enough to oppose the collapse. Note the small wedge of $\Lambda > 0$, $\Omega_{matter} > 1$ models that still recollapse.

For large enough Λ (upper left corner), there can be no big bang because the vacuum energy would create a "bounce" if run back in time. For critical values of Λ with small enough Ω_{matter}, the universe "loiters" for an infinite time at some maximum redshift. Near this line, the universe spends extra time at a particular scale factor. In general, models with non-zero Λ have a "coasting" phase in which they expand less quickly than corresponding $\Lambda = 0$ models.

What constraints do we currently have? After considering the data in hand the range of likely models is strongly centered in and around the white triangle. Big bang nucleosynthesis indicates $\Omega_{matter} > 0.05$ or so. The large-scale distribution of galaxies and clusters of galaxies indicate $\Omega_{matter} \sim 0.2 - 0.4$. Cosmic microwave background anisotropy measurements strongly prefer $\Omega_{vac} + \Omega_{matter} \sim 1$. SNe Ia observations prefer $\Omega_{matter} \sim 0.3, \Omega_{vac} \sim 0.7$.

11.1 The Cosmic Triangle

For an excellent review of status in the field c. 1999 that remains relevant today see "The Cosmic Triangle: Assessing the State of the Universe," Science 284, 1481-1488, (1999). I expect that everyone will be able to read this article, if not now, then certainly by the end of the course,

12 Horizons

How far can a photon get in time t? For a photon, the proper time interval $d\tau = 0$, therefore (now look back at the RW metric)

$$c^2 dt^2 = a^2 dr^2$$

and the amount of coordinate distance r that a photon travels is

$$\Delta r = \int_{t_0}^{t_1} \frac{cdt}{a(t)}$$

Substitute

$$dt = \frac{dR}{\dot{R}} \propto \frac{da}{\sqrt{\rho a^2}}$$

from the Friedmann equation. The integral converges if $\rho a^2 \to \infty$ as $t_0 \to 0$, which is true if $\rho \propto a^{-4}$ as in the radiation dominated case, which occurs in our universe. This gives you lots of hints for how to show this in more detail in a homework problem. Thus, light travels a finite distance since the big bang. This is called a "particle horizon." It implies that there is a finite volume of the universe that is in causal contact. When we later say that something or some scale "comes within the horizon" we mean within the size of this observable universe at a particular time.

Another type of horizon is an "event horizon." An event horizon limits the volume of spacetime that is ever observable. If an event is outside our event horizon, that mean it cannot ever affect us. An event can affect us if a light signal can travel from the event to us. Again, the coordinate distance a photon can travel is

$$\Delta r = \int_{t_0}^{t_1} \frac{cdt}{a(t)}$$

In the case where Δr remains finite as $t_1 \to \infty$, there is an event horizon because, although an expanding universe might expand infinitely, light can travel only a finite distance. Examining the integral we see that we get an event horizon if $R(t)$ grows faster than cdt. This sounds like it requires faster than light travel, but remember that we're talking about the scale of the universe increasing, not the proper motion of any object with respect to the space of the universe. Recall that a de Sitter universe expands as $R \propto \exp(Ht)$, thus the distance between galaxies in such a universe does eventually increase at faster than the speed of light and there is an event horizon around the observer, beyond which the universe is forever unobservable. Another instance of an event horizon – due to entirely different physics – is the Schwarzschild radius of a black hole. Events inside the Schwarzschild radius are forever unobservable to those outside the black hole.

13 Observations in cosmology

We have now sketched out the structure of spacetime in an isotropic homogeneous universe. Our next step is to consider how our observations of that universe depend on the cosmological parameters.

13.1 Distance-redshift Relation

We can relate comoving distance, r, to redshift, z, via

$$dr = \frac{c}{H(z)} dz$$

$$dr = \frac{c}{H_0} [(1-\Omega)(1+z)^2 + \Omega_{vac} + \Omega_{matter}(1+z)^3 + \Omega_{rad}(1+z)^4]^{-1/2} dz$$

where Ω is the total density parameter. This equation can be integrated to yield the comoving coordinate distance as a function of observed redshift if we know the density parameters (necessary for mapping the universe!).

For a matter-only universe, the relation above can be integrated to produce Mattig's formula (1958) that relates an "effective distance," sometimes called the "RW coordinate distance" to the redshift, Hubble constant, and density parameter,

$$D(z) = S_k(r) = \frac{2c}{H_0} \frac{\Omega z + (2-\Omega)[1-\sqrt{1+\Omega z}]}{\Omega^2(1+z)}$$

and where $S_k(r)$ is evaluated with $R = R_0$. As we'll see below, this type of distance is related to how large and bright objects appear.

There is no simple generalization for this distance-redshfit relation that includes Ω_{vac}. You'll have to numerically integrate the relation for dr/dz. However, to second order in z, a useful approximation can be written in terms of the deceleration parameter at the present epoch,

$$D(z) = S_k(r) \approx \frac{c}{H_0} \left(z - \frac{1+q_0}{2} z^2 \right)$$

Note that this relation will be accurate only over a limited range of redshift!

13.2 Proper transverse size and angular diameter distance

To relate astronomical observations to physics, we need to relate apparent angular sizes and redshifts to physical sizes.

For small angles, the proper size (size in physical units, like length of a ruler) of an object that appears to have an angular size θ is

$$l = \frac{\theta D(z)}{1+z}$$

This is the formula to use if you see a galaxy whose angular size is θ at redshift z and you want to know how big it is compared to the Milky Way. This defines the "angular diameter distance,"

$$D_A(z) = \frac{D(z)}{1+z}$$

Remember that in curved universes, $S_k(r) \neq r$, thus the dependence of angular size on distance is not Euclidean!

Again remembering the difference between comoving and proper distances, the comoving separation between two objects at the same redshift z, separated by a small angle θ on the sky is simply

$$l = \theta D(z)$$

The factor of $(1+z)$ between these formulae illustrates the simple relation between proper, or "physical," lengths and comoving lengths. Think of space having a grid imprinted on it, which expands as the universe expands. The coordinates on the grid are comoving coordinates. The comoving separation between galaxies that freely expand with the universe never changes. But the physical distance between the galaxies, the proper length, does grow with time as $a = (1+z)^{-1}$. The galaxies themselves, which are gravitationally bound, do not get larger with time. They maintain a fixed proper or physical size. For example, we measure a comoving distance from us to the Coma Cluster of galaxies of roughly $80h^{-1}\,\mathrm{Mpc}$. What was the comoving distance in the past? It was the same. What was the proper distance in the past? The universe was smaller then, so the distance would have been $80(1+z)^{-1}h^{-1}\,\mathrm{Mpc}$. So, how did we measure that distance?

To see how angular sizes change with redshift, use the approximate formula for $D(z)$ and look at how the angular diameter distance curve depends on the cosmology,

$$D_A(z) \approx \frac{c}{H_0}\left(z - \frac{1+q_0}{2}z^2\right)(1+z)^{-1}$$

We see that, as long as $q_0 > -1$, there will always be a maximum in this distance. If we observe galaxies at sequentially larger distance, they get smaller and then begin to get larger again. [Draw a rough sketch] This doesn't mean that objects start getting closer! It means that the angular size starts increasing again! For $\Omega_{matter} = 1$ and $\Omega_{vac} = 0$, $q_0 = 1/2$ and this maximum would occur at $z = 4/3$, which is not very far. For $\Omega_{matter} = 0.3$, $\Omega_{vac} = 0.7$, $q_0 = -0.55$ and the maximum is at $z = 4.4$.

For most models, there is a large range of z over which D_A, and therefore $\theta(z)$, does not change very much. This means that, over that range of z, objects don't change their angular size even though they get further away.

13.3 Observed flux, surface brightness, luminosity distance

What is the received flux from an object that isotropically emits flux with luminosity L (energy per unit time per unit frequency) at redshift z? The photons spread out over an area at our distance of

$$A = 4\pi D^2(z)$$

The redshift affects the photon flux in two ways:

- energies decrease by $(1+z)$ (longer wavelength, shorter freq)
- arrival rates decrease by $(1+z)$ (more time between photons)

Thus, the "bolometric" (integrated over all frequency) flux that we observe is

$$S_{tot} = \frac{L}{4\pi D^2(z)(1+z)^2}$$

This leads us to define a "luminosity distance"

$$D_L(z) = D(z)(1+z)$$

so that

$$S_{tot} = \frac{L}{4\pi D_L^2(z)}$$

How does the surface brightness vary with redshift? Suppose that the source of this flux has intrinsic surface brightness (flux per unit solid angle emitted by unit area of the source)

$$I_0 = \frac{L}{A}$$

Following our previous discussion about proper lengths and angles, the solid angle on the sky covered by an object with proper area dA is

$$d\Omega = \frac{dA(1+z)^2}{D^2(z)}$$

(Ω here means solid angle with units of steradians, not the density parameter). Combining this equation with the equation above for S, we find that the total observed surface brightness of the source is

$$I_{tot} = \frac{S_{tot}}{\Omega} = \frac{L}{A(1+z)^4} = \frac{I_0}{(1+z)^4}$$

Note this carefully! This is a very strong dependence on redshift! In a Euclidean universe that does not expand, surface brightness is constant. But in cosmology, the surface brightness drops dramatically at large redshift. This makes distant galaxies extremely difficult to detect, because the night sky – the seemingly empty regions between the stars – is much brighter than the surface brightness of these galaxies. Depending on the wavelength of the observation, this foreground brightness is due to light refracted and emitted by our own atmosphere, light reflected and emitted by dust in the solar system, and light reflected and emitted by gas and dust in the galaxy.

It is also common to measure the "monochromatic" emission from sources, which is simply the flux per unit frequency $S_\nu = dS/d\nu$ and the corresponding monochromatic surface brightness I_ν. When we measure flux only over a small band $d\nu$ at frequency ν_0, the redshift affects the flux in two more ways (the 0's in the subscripts here can be read as denoting what we observe at redshift $z = 0$):

- the observed ν_0 is smaller than the emitted $\nu_0 = \nu_e/(1+z)$, $\lambda_0 = \lambda_e(1+z)$
- the bandwidth $d\nu_{obs} = d\nu_{emit}/(1+z)$

thus the observed monochromatic flux (where the instrinsic flux is $L_\nu = dL/d\nu_e$) and surface brightness are

$$S_\nu(\nu_0) = \frac{L_\nu(\nu_0[1+z])}{4\pi D^2(z)(1+z)} = \frac{L_\nu(\nu_0[1+z])(1+z)}{4\pi D_L^2(z)}$$

and

$$I_\nu^{obs}(\nu_0) = \frac{I_\nu^{emit}(\nu_0[1+z])}{(1+z)^3}$$

Often, we observe objects at fixed observed frequency over a range of redshifts, thus the "rest frame" frequencies of the received photons are different. We then want to correct the observed fluxes to what they would be if we were able to measure their flux at the same observed frequency. This is know as the "K-correction." To make this correction, we have to know (or guess) the spectrum of the object. We'll return to this later when we talk about galaxies.

13.4 Comoving volume

How much volume is there within an interval of redshift? From the metric

$$ds^2 = -c^2dt^2 + a^2(t)[dr^2 + S_k^2(r)d\psi^2]$$

we see that the comoving volume element is simply (today, when $a = 1$)

$$dV = 4\pi D^2(z)dr = 4\pi D^2(z)\frac{cdz}{H(z)}$$

recalling that we defined $D(z) = S_k(z)$.

A classical cosmological test is to use the number of objects per unit redshift to infer the change comoving volume and thus measure the cosmological parameters. The primary problem with this idea (there have been lots of attempts!) is that the objects themselves evolve with time. Examples: galaxies, quasars...