Exercise 1.6

Recall that the **trace** of a square matrix is the sum of its diagonal entries.

Thus: $tr\left[(a_{ij})_{i,j=1}^{n}\right] = a_{11} + a_{22} + \dots + a_{nn} = \sum_{j=1}^{n} a_{jj}$

a) Show that if $A \in \mathbb{F}^{n \times m}$ and $B \in \mathbb{F}^{m \times n}$, then $tr(AB) = tr(BA)$

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nm} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \ddots & \\ \vdots & & & \\ b_{m1} & & & b_{mn} \end{pmatrix}$

$\rightarrow AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \dots a_{1m}b_{m1} & a_{11}b_{12} + a_{12}b_{22} + \dots + a_{1m}b_{m2} & \cdots a_{11}b_{1n} + a_{12}b_{2n} + \dots \\ a_{21}b_{11} + a_{22}b_{21} + \dots a_{2m}b_{m1} & a_{21}b_{12} + a_{22}b_{22} + \dots a_{2m}b_{m2} & \vdots \\ \vdots & \ddots & \vdots \\ a_{n1}b_{11} + a_{n2}b_{21} + \dots a_{nm}b_{m1} & \cdots & a_{n1}b_{1n} + a_{n2}b_{2n} + \dots + a_{nm}b_{mn} \end{pmatrix}$

$\rightarrow tr(AB) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ji} b_{ij}$

$BA = \begin{pmatrix} b_{11}a_{11} + b_{22}a_{21} + \dots + b_{1n}a_{n1} & & \\ & b_{21}a_{12} + b_{22}a_{22} + \dots + b_{2n}a_{n2} & \\ & & \ddots \\ & & b_{m1}a_{1m} + b_{m2}a_{2m} + \dots + b_{mn}a_{nm} \end{pmatrix}$

$\rightarrow tr(BA) = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} a_{ji} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ji} b_{ij}$ ✓

$\Longleftrightarrow \boxed{tr(AB) = tr(BA)}$ ✓

b) Give an example of matrices $A, B, C \in \mathbb{F}^{n\times n}$ so that $tr(ABC) \neq tr(BAC)$

Let $A, B, C \in \mathbb{F}^{2\times 2}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$$ABC = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$$ABC = \begin{pmatrix} [a_{11}b_{11} + a_{12}b_{21}]c_{11} + [a_{11}b_{12} + a_{12}b_{22}]c_{21} & [a_{11}b_{11} + a_{12}b_{21}]c_{12} + [a_{11}b_{12} + a_{12}b_{22}]c_{22} \\ [a_{21}b_{11} + a_{22}b_{21}]c_{11} + [a_{21}b_{21} + a_{22}b_{22}]c_{21} & [a_{21}b_{11} + a_{22}b_{21}]c_{12} + [a_{21}b_{21} + a_{22}b_{22}]c_{22} \end{pmatrix}$$

$$BAC = \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$$BAC = \begin{pmatrix} [b_{11}a_{11} + b_{12}a_{21}]c_{11} + [b_{11}a_{12} + b_{12}a_{22}]c_{21} & [b_{11}a_{11} + b_{12}a_{21}]c_{12} + [b_{11}a_{12} + b_{12}a_{22}]c_{22} \\ [b_{21}a_{11} + b_{22}a_{21}]c_{11} + [b_{21}a_{12} + b_{22}a_{22}]c_{21} & [b_{21}a_{11} + b_{22}a_{21}]c_{12} + [b_{21}a_{12} + b_{22}a_{22}]c_{22} \end{pmatrix}$$

Want: $tr(BAC) \neq tr(ABC)$

$$[\cancel{b_{11}a_{11}} + b_{12}a_{21}]c_{11} + [b_{11}a_{12} + b_{12}a_{22}]c_{21}$$
$$+ [b_{21}a_{11} + b_{22}a_{21}]c_{12} + [b_{21}a_{12} + \cancel{b_{22}a_{22}}]c_{22}$$

$$= [\cancel{a_{11}b_{11}} + a_{12}b_{21}]c_{11} + [a_{11}b_{12} + a_{12}b_{22}]c_{21}$$
$$[a_{21}b_{11} + a_{22}b_{21}]c_{12} + [a_{21}b_{21} + \cancel{a_{22}b_{22}}]c_{22}$$

$$\rightarrow c_{11} b_{12} a_{21} + c_{12}\left[b_{21}a_{11} + b_{22}a_{21} - a_{21}b_{11} - a_{22}b_{21}\right]$$
$$= c_{22}a_{21}b_{21} + c_{21}\left[a_{11}b_{12} + a_{12}b_{22} - b_{11}a_{12} - b_{12}a_{22}\right]$$

Set $c_{12} = c_{21} = 0$

$$\rightarrow c_{11} b_{12} a_{21} = c_{22} a_{21} b_{21}$$

Choice: $a_{21} = b_{12} = b_{21} = c_{11} = 1$, $c_{22} = -1$

$$\Rightarrow \boxed{A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

Check: $AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow ABC = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \mathrm{tr}(ABC) = -1$

$BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow BAC = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \mathrm{tr}(BAC) = 1$

$$\mathrm{tr}(ABC) \neq \mathrm{tr}(BAC)$$