Readings: Chs 8.2-3, 9.0-5, 10.1-5

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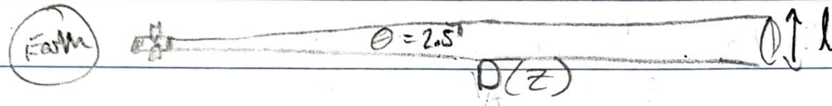
Phys 431 HW2

① FoV of HST

a) What is the proper size in Mpc of the HST field of view at redshifts 0.5, 1, 2, or 4 for $\Omega_{matter} = 1$, $\Omega_\Lambda = 0$, and for $\Omega_{matter} = 0.3$, $\Omega_\Lambda = 0.7$. The HST images areas ~2.5 arcmin across.



$$2.5\ \text{arcmin} \times \frac{1\ \text{degree}}{60\ \text{arcmin}} \times \frac{\pi\ \text{radians}}{180\ \text{degrees}} = 7.27\times10^{-4}\ \text{radians}$$

$$h \equiv \frac{H_0}{100\ \text{km/s/Mpc}} = 0.68$$

$$l = \frac{D(z)\,\theta}{1+z}$$

$$l \approx \frac{c}{H_0}\left(z - \frac{1+q_0}{2}z^2\right)(1+z)^{-1}\theta = 4412\ \text{Mpc}\left(z - \frac{1+q_0}{2}z^2\right)(1+z)^{-1}\theta$$

i) For $\Omega_m = 1, \Omega_\Lambda = 0$: $q_0 = 1/2$

ii) For $\Omega_m = 0.3, \Omega_\Lambda = 0.7$: $q_0 = -0.55$

$\Omega_m = 1, \Omega_\Lambda = 0$: Mattig's Formula

$$z = 0.5:\ l \approx \frac{\overset{6000\,\text{Mpc}}{6\times10^5\ \text{km/s}}}{100\ \text{km/s/Mpc}}\left(\frac{H_0}{100\ \text{km/s/Mpc}}\right)^{-1}_{h^{-1}} \frac{0.5+4\sqrt{1+0.5}}{(1+0.5)^2}\left(7.27\times10^{-4}\ \text{radians}\right)$$

$$\rightarrow \boxed{l = 0.53\,h^{-1}\ \text{Mpc}}$$

$z = 1$: $0.64\,h^{-1}$ Mpc, $z = 2$: $0.61\,h^{-1}$ Mpc, $z = 4$: $0.48\,h^{-1}$ Mpc

$\Omega_m = 0.3,\ \Omega_\Lambda = 0.7,\ q_0 = -0.55$

$$l \approx \cancel{\frac{3\times10^5 \text{km/s}}{100\,\text{km/s/Mpc}}} \left(\cancel{\frac{H_0}{100\,\text{km/s/Mpc}}}\right)^{-1} \left(z - \frac{1-0.55}{2} z^2\right)(1+z)^{-1}(7.27\times10^{-4})$$

$$\boxed{\begin{array}{ll} z=0.5:\ 0.65\,h^{-1}\,\text{Mpc} & z=2:\ 0.80\,h^{-1}\,\text{Mpc} \\ z=1:\ 0.85\,h^{-1}\,\text{Mpc} & z=4:\ 0.17\,h^{-1}\,\text{Mpc} \end{array}}$$

b) What is the comoving size of the field at z=1 for both cosmologies?

$$l = \theta D(z)$$

$\Rightarrow \Omega_\Lambda = 0,\ \Omega_m = 1:$

$$l = 0.64\,h^{-1}\,\text{Mpc}\,(1+1)$$

$$\rightarrow \boxed{l = 1.28\,h^{-1}\,\text{Mpc}}$$

$\Omega_\Lambda = 0.7,\ \Omega_m = 0.3$

$$l = 0.85\,h^{-1}\,\text{Mpc}\,(2) = \boxed{1.70\,h^{-1}\,\text{Mpc}}$$

Problem 2

```
[1]: import numpy as np
     import matplotlib.pyplot as plt
```

2. Angular Sizes of Galaxies

Using HST, the Hubble Deep Field observation (two weeks of pointing at the same place on the sky!) reveals galaxies that range in redshift from $z = 0.2$ to $z = 4$ and in angular size from 0.1 to 5 arcseconds. Leave your answers to this problem in terms of the Hubble constant parameter h.

(a) For each of the two cosmologies in problem 1, make plots of $\theta(z)$ (the angular size on the sky in arcseconds) for several choices of lgal (the proper physical size) in the range $l = 0.5\,\text{kpc}$ to $l = 10\,\text{kpc}$ (recall that one parsec is 3 light years). Discuss the relationship between angular size and proper size of the galaxies. How is this complicated by cosmology?

For $\Omega_\Lambda = 0, \Omega_m = 1$ use Mattig's Formula to derive:

$$\theta(z, l_{\text{gal}}) = \frac{l_{\text{gal}}}{6 \times 10^6\,\text{kpc}} \frac{(1+z)^2}{z - \sqrt{1+z} + 1} h$$

This returns in radians; multiply by

$$\frac{180\,\text{degrees}}{\pi\,\text{radians}} \times \frac{60\,\text{arcmin}}{1\,\text{degree}} \times \frac{60\,\text{arcsec}}{1\,\text{arcmin}} = 206265\,\text{arcsec/radian}$$

to convert to arcseconds.

```
[2]: def theta_matteronly(z, lgal):
         return (lgal/(6.e6)) * ((1+z)**2) / (z-np.sqrt(1+z)+1) * 206265
```

For $\Omega_\Lambda = 0.7, \Omega_m = 0.3$ use Pen's Approximation to derive:

$$\theta(z, l_{\text{gal}}) = \frac{l_{\text{gal}}}{3 \times 10^6\,\text{kpc}} [z - \frac{1+q_0}{2} z^2]^{-1} (1+z) h$$

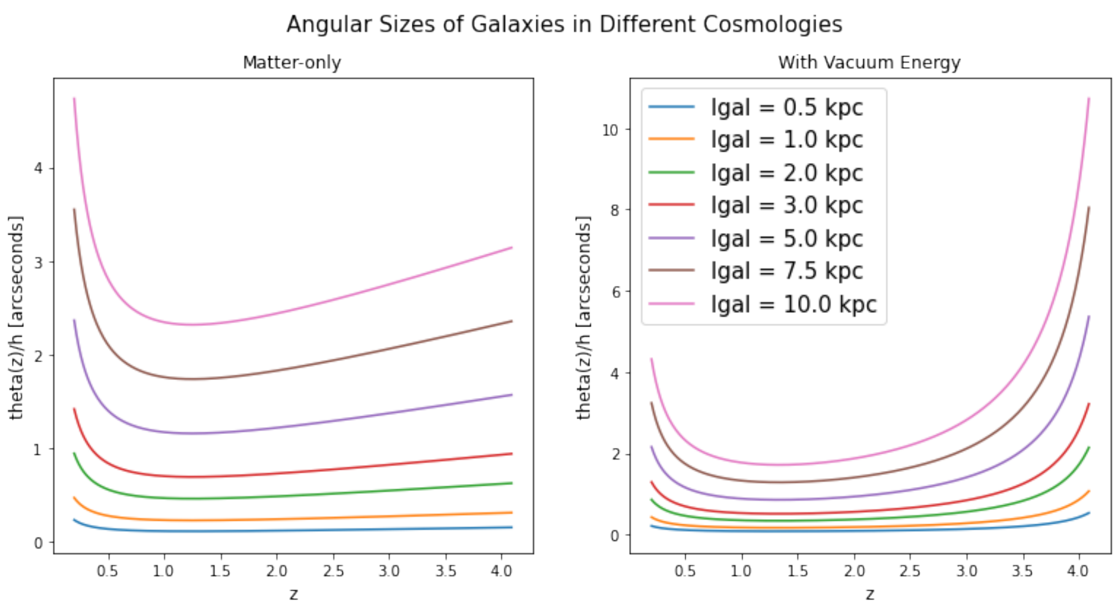
where $q_0 = -0.55$ is the decceleration parameter.

```
[3]: def theta_vacuum(z, lgal):
         return (lgal/(3.e6)) * (z - (1-0.55)/2*z**2)**(-1) * (1+z) * 206265
```

[5]:
```python
lgal = [0.5, 1, 2, 3, 5, 7.5, 10]
zrange = np.arange(0.2, 4+.1, 0.01)

fig, [ax1,ax2] = plt.subplots(1, 2, figsize=(13,6))
for l in lgal:
    ax1.plot(zrange, theta_matteronly(zrange, l), label="lgal = %.1f kpc"%l)
    ax2.plot(zrange, theta_vacuum(zrange, l), label="lgal = %.1f kpc"%l)
ax2.legend(loc="best", prop={"size":15})
ax1.set_ylabel("theta(z)/h [arcseconds]", fontsize=12)
ax2.set_ylabel("theta(z)/h [arcseconds]", fontsize=12)
ax1.set_xlabel("z", fontsize=12)
ax2.set_xlabel("z", fontsize=12)
plt.suptitle("Angular Sizes of Galaxies in Different Cosmologies", fontsize=15)
ax1.set_title("Matter-only", fontsize=12)
ax2.set_title("With Vacuum Energy", fontsize=12)
```

[5]:
```
Text(0.5, 1.0, 'With Vacuum Energy')
```



In both cases, the angular size of galaxies falls off with redshift, plateaus for a bit, then begins to increase again. The increase at higher redshift is much steeper in the case of vacuum energy.

Cosmology complicates observations in this regard because two galaxies of the same proper length at very different distances from us appear on the sky with the same angular size. In the case with vacuum energy, the galaxy at a farther distance may even have a larger angular size.

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b) What measurements or other information about these galaxies would allow us to disentangle this ambiguity?

Having the redshift of each galaxy would allow you determine the proper length as a function of angular size on the size, depending on your choice of cosmological model.

③ <u>Blackbody Radiation</u>

An object emits blackbody radiation of temperature T in its own rest frame. We see it at redshift z and subtending a solid angle Ω.

a) What flux do we measure? The phase-space density of photons is invariant:

$$I_\nu / \nu^3 = \text{constant}$$

where I_ν is the surface brightness $\left(\frac{\text{flux}}{\Omega \nu}\right)$

$$\frac{S_\nu}{\Omega \nu^3} = \text{constant}$$

$$\frac{S_{tot}}{\Omega \nu^4} = \text{constant}$$

$$\frac{dS}{d\nu} = c\nu^3 d\nu$$

$$\rightarrow S = \frac{c\nu^4}{4} d\nu$$

$$\frac{S_{tot,obs}}{\Omega_{obs} \nu_{obs}^4} = \frac{S_{tot,e}}{\Omega_e \nu_e^4}$$

$$\nu_e = \nu_o (1+z)$$

$$\frac{S_{tot,obs}}{\Omega_{obs}} = \frac{\sigma T^4}{\Omega_e (1+z)^4}$$

we have $d\Omega = \frac{dA (1+z)^2}{D^2(z)}$

→ Cosmology increases "rest frame Ω" by $(1+z)^2$

$$\Longrightarrow \boxed{S_{obs} = \frac{\sigma T^4}{(1+z)^2}}$$

b) If the redshift is due to doppler motion, the only effect felt is the change in frequency, so the extra $(1+z)^2$ factor drops out and:

$$\boxed{S_{obs} = \frac{\sigma T^4}{(1+z)^4}}$$

① Uncertainty in the distance scale.
- Suppose that a series of four different standard candles are used to step out along the Cosmic distance ladder from our Galaxy to distances far enough to accurately measure the true expansion rate.
- Each standard candle has an uncertainty of $\Delta M = 0.2$ in its intrinsic magnitude
* Show that, by varying the calibration of each standard candle, it is possible for the Hubble constant to differ from its nominal value by a factor of ~ 0.7 to ~ 1.4.

Recall that $m = -2.5 \log f + \text{constant}$

$$\log f = \frac{m + \text{constant}}{-2.5} \rightarrow f = 10^{\frac{m}{-2.5}} 10^{\frac{\text{constant}}{-2.5}}$$

$$\rightarrow f = A\, 10^{-m/2.5}$$

$L = 4\pi D_L^2 f$; To determine cosmological parameters, we know L, and measure f to determine D_L.

$$f(m) = 10^{-m/2.5}$$

$\Delta f \leq 0.18 f$

$$\rightarrow \Delta f = \frac{\partial f}{\partial m}\Delta m = \left| -\frac{\ln(10)}{2.5} 10^{-m/2.5} \Delta m \right|$$

So, an error Δm in magnitude corresponds to an error of $\Delta f = \frac{\ln 10}{2.5} f \Delta m$ in flux.

For 4 different standard candles, we step out a rung in the cosmic distance ladder via

$$d_L(4) = d_L(1)\left(\frac{d_L(2)}{d_L(1)}\right)\left(\frac{d_L(3)}{d_L(2)}\right)\left(\frac{d_L(4)}{d_L(3)}\right)$$

$$d_L(4) = d_L(1)\left(\frac{f(1)}{f(2)}\right)^{1/2}\left(\frac{f(2)}{f(3)}\right)^{1/2}\left(\frac{f(3)}{f(4)}\right)^{1/2}$$

From Δf above, we may expect f to be as high or low as $f \pm \Delta f = f \pm \frac{\ln(10)}{2.5}(0.2) f = f \pm 0.18 f$

Varying individual fluxes:

$$d_L(4) = d_L(1)\left(\frac{0.82 f(1)}{1.18 f(2)}\right)^{1/2}\left(\frac{0.82 f(2)}{1.18 f(3)}\right)^{1/2}\left(\frac{0.82 f(3)}{1.18 f(4)}\right)^{1/2}$$

$$\rightarrow d_L(4) = d_L(1)\left(\frac{f_1}{f_2}\right)^{1/2}\left(\frac{f_2}{f_3}\right)^{1/2}\left(\frac{f_3}{f_4}\right)^{1/2} \times 0.6$$

or: $$d_L(4) = d_L(1)\left(f_1/f_2\right)^{1/2}\left(f_2/f_3\right)^{1/2}\left(f_3/f_4\right)^{1/2} * \left(\frac{1.18}{0.82}\right)^{3/2}$$

$= 1.7$

Thus the Hubble constant may differ from a factor of ~ 0.6 to ~ 1.7.

⑤ Limits on the parallax method

From Earth, optical telescopes can resolve differences in angle of at best 0.5 arcseconds. With this resolution limit, to what distance can we use trigonometric parallax to measure distances? What if we could measure difference in

$$\frac{1AU}{d} = p \longrightarrow d(pc) = \frac{1}{p(arcsec)}$$

For $0.5''$: $d = \frac{1}{0.5} = \boxed{2pc}$

For $0.01''$: $d = \frac{1}{0.01} = \boxed{100pc}$

⑥ Peculiar velocities and the Hubble constant.

- The observed recession velocity of galaxies is the sum of their Hubble flow velocity $v = Hr$, and the line-of-sight component of their "peculiar" velocity.
- Suppose we estimate H_0 by measuring both the distance and redshift for a single object

* How does the error in the Hubble constant depend on the rms peculiar velocity of this type of object?
* At what distance can we get 5% accuracy in H_0 if $V_{pec} = 200$ km/s?

$$V_{cosmo} = Hr$$

$$\rightarrow cz - V_{pec} = Hr$$

$$\Rightarrow H = \frac{cz - V_{pec}}{r}$$

Thus the error in H is $\boxed{\Delta H = \frac{1}{r} \Delta V_{pec}}$, if we take $\Delta z = \Delta r = 0$

Not given enough to determine distance for 5% accuracy. Need Δv and an estimate for H_0

$$\frac{\Delta H}{H} = \frac{\Delta V_{pec}}{H r} = 0.05$$

$$\rightarrow \boxed{r = \frac{20 \Delta V_{pec}}{H_0}}$$

In [1]:

```python
import numpy as np
import matplotlib.pyplot as plt
```

7. Age of the universe and constraints on cosmological parameters

Make a 2D figure that shows the dependence of the age of the universe on Ω_{matter} and Ω_Λ. Indicate the bounds on this 2D space that are set by age estimates of the Solar system, white dwarfs, and globular clusters. How does the region of Ω_{matter}, Ω_Λ that is allowed by these constraints compare with the region favored by other observations? (See your notes from class, figure 3.5 in the text, and the "Cosmic Triangle" article.) Use the equation

$$t_0 = \frac{2}{3} t_H (0.7\Omega_M + 0.3 - 0.3\Omega_\Lambda)^{-0.3}$$

$$t_0 = 6.52 h^{-1} \text{Gyr} \, (0.7\Omega_M + 0.3 - 0.3\Omega_\Lambda)^{-0.3}$$

which relates the age of the universe now to the Hubble time ($t_H = 1/H$) and the cosmological parameters Ω_M and Ω_Λ.

Note that this equation is only valid if $\Omega_M > \frac{3}{7}(\Omega_\Lambda - 1)$

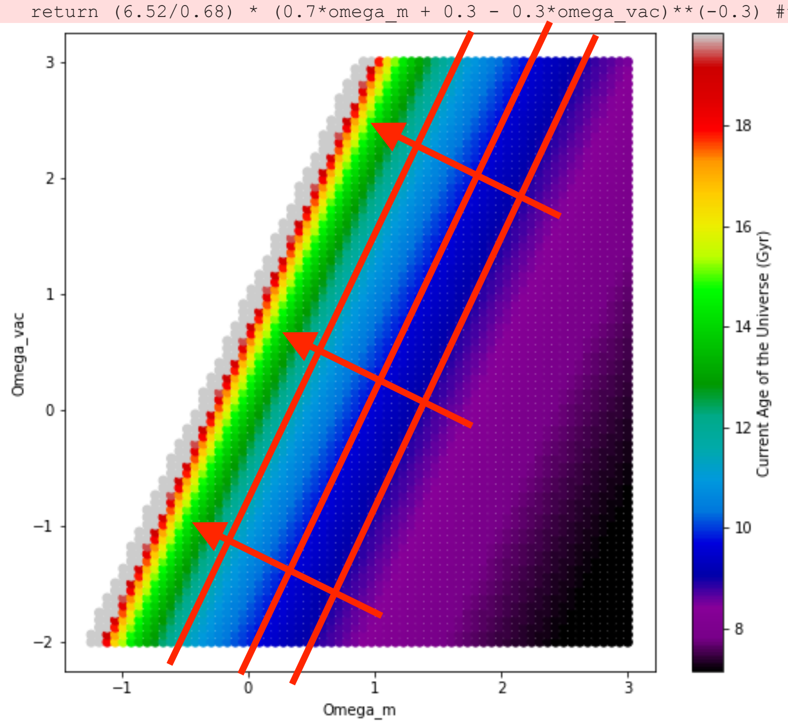
In [2]:

```python
def t0(omega_m, omega_vac):
    return (6.52/0.68) * (0.7*omega_m + 0.3 - 0.3*omega_vac)**(-0.3) #take H0=68km/s/Mpc
```

In [3]:

```python
fig = plt.figure(figsize=(9,8))

om_m   = np.linspace(-2, 3, 10)
om_vac = np.linspace(-2, 3, 10)

#Create mesh grid for plotting/colorbar
xmin, xmax = om_m.min(), om_m.max()
ymin, ymax = om_vac.min(), om_vac.max()
X, Y = np.mgrid[xmin:xmax:80j, ymin:ymax:80j]
positions = np.vstack([X.ravel(), Y.ravel()])
t0list = t0(positions[0,:], positions[1,:])

plt.scatter(positions[0,:], positions[1,:], c=t0list, cmap="nipy_spectral", \
           vmin=np.percentile(t0list[~np.isnan(t0list)], 2), vmax=np.percentile(t0list[~np.isnan(t0list)], 96))

cbar = plt.colorbar()
cbar.ax.set_ylabel('Current Age of the Universe (Gyr)')
plt.xlabel("Omega_m")
plt.ylabel("Omega_vac")
plt.show()
```

```
<ipython-input-2-2c511908b9c7>:2: RuntimeWarning: divide by zero encountered in power
  return (6.52/0.68) * (0.7*omega_m + 0.3 - 0.3*omega_vac)**(-0.3) #take H0=68km/s/Mpc
<ipython-input-2-2c511908b9c7>:2: RuntimeWarning: invalid value encountered in power
  return (6.52/0.68) * (0.7*omega_m + 0.3 - 0.3*omega_vac)**(-0.3) #take H0=68km/s/Mpc
```



Lines from right to left illustrate lower bounds from White Dwarfs, nuclear cosmochronology, and Globular clusters

In [ ]: