Phys 327 HW2

Trevor McCaffrey
Jun 17, 2022

① Griffiths 4.13

a) Normalize the radial function R_{20} and construct the wave function ψ_{200}.

$$R_{20} = \frac{c_0}{2a}\left(1-\frac{r}{2a}\right)e^{-r/2a} \qquad \int_0^\infty |R_{20}|^2 r^2 dr = 1$$

$$\left(\frac{c_0}{2a}\right)^2 \int_{r=0}^{r=\infty} r^2\left(1-\frac{r}{2a}\right)^2 e^{-r/a}\, dr \qquad u=\frac{r}{a} \quad a\,du = \cancel{\tfrac{1}{a}}dr$$

$$\rightarrow \left(\frac{c_0}{2\cancel{a}}\right)^2 a \underbrace{\int_0^\infty \cancel{a^2} u^2\left(1-\frac{u}{2}\right)^2 e^{-u}\, du}_{I_1} = 1$$

$$I_1 = \int_0^\infty u^2\left(1+\frac{u^2}{4}-u\right)e^{-u}\, du = \frac{1}{a}\left(\frac{2}{c_0}\right)^2$$

$$= \int_0^\infty \left(u^2 - u^3 + \frac{u^4}{4}\right)e^{-u}\, du \qquad * \int_0^\infty x^n e^{-ax} = \frac{n!}{a^{n+1}}$$

$$\rightarrow \frac{1}{a}\left(\frac{2}{c_0}\right)^2 = 2 \cancel{-6} + \cancel{\tfrac{1}{4}(24)} = 2$$

$$\left(\frac{c_0}{2}\right)^{\cancel{2}} = \frac{1}{\sqrt{2a}} \longrightarrow \underline{\underline{c_0 = \sqrt{\frac{2}{a}}}}$$

$$\Rightarrow \boxed{R_{20} = \underbrace{\frac{1}{2a}\sqrt{\frac{2}{a}}}_{\frac{1}{\sqrt{2a^3}}}\left(1-\frac{r}{2a}\right)e^{-r/2a}}$$

$$\psi_{200} = R_{20}\, Y_0^0 = R_{20} \frac{1}{\sqrt{4\pi}}$$

$$\longrightarrow \boxed{\psi_{200} = \frac{1}{\sqrt{8\pi a^3}}\left(1 - \frac{r}{2a}\right)e^{-r/2a}}$$

—— // ——

b) Normalize the radial function R_{21} and construct the wave functions ψ_{211}, ψ_{210}, and ψ_{21-1}

$$R_{21} = \frac{c_0}{4a^2}\, r e^{-r/2a} \;;\quad \int_0^\infty |R_{21}|^2 r^2 dr = 1$$

$$1 = \left(\frac{c_0}{4a^2}\right)^2 \int_0^\infty r^4 e^{-r/a} dr$$

$$\longrightarrow \frac{2\cancel{16}a^4}{c_0^2} = \frac{4!}{(1/a)^5} = 2\cancel{4}^{3}a^5$$

$$\frac{2a^4}{c_0^2} = 3a^{\cancel{5}} \longrightarrow c_0^2 = \frac{2}{3a}$$

$$\boxed{R_{21} = \sqrt{\frac{2}{3a}}\frac{1}{4a^2}\, r e^{-r/2a}}$$

$$\psi_{nlm} = R_{nl}\, Y_l^m$$

$$\psi_{211} = R_{21} Y_1^1,\quad \psi_{210} = R_{21} Y_1^0,\quad \psi_{21-1} = R_{21} Y_1^{-1}$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi}$$

$$\longrightarrow \boxed{\psi_{211} = \frac{-1}{8a^{5/2}\sqrt{\pi}}\, r\sin\theta\, e^{i\phi - r/2a}}$$

$$Y_1^{-1} = \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi}$$

$$\longrightarrow \boxed{\psi_{21-1} = \frac{1}{8a^{5/2}\sqrt{\pi}}\, r\sin\theta\, e^{-(i\phi + r/2a)}}$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \rightarrow \psi_{210} = \sqrt{\frac{\cancel{3}(2)}{4\pi(\cancel{3}a)}}\frac{1}{4a^2} r\cos\theta\, e^{-r/2a}$$

$$\rightarrow \boxed{\psi_{210} = \frac{1}{4a^2\sqrt{2\pi a}} r\cos\theta\, e^{-r/2a}}$$

② Griffiths 4.15

a) Find $\langle r \rangle$ and $\langle r^2 \rangle$ for an electron in the ground state of hydrogen.

$$\langle r \rangle = \int \psi_{100}^* \, r \, \psi_{100} \, r^2 \sin\theta \, dr \, d\phi \, d\theta$$

$$\psi_{100} = R_{10} Y_0^0 = \frac{2}{a^{3/2}} e^{-r/a} \frac{1}{\sqrt{4\pi}} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

$$\langle r \rangle = \frac{1}{\pi a^3} \underbrace{\int_0^\infty r^3 e^{-2r/a} dr}_{\frac{3!}{(2/a)^4} = 6\left(\frac{a^4}{16}\right) = \frac{3a^4}{8}} \underbrace{\int_0^{2\pi} d\phi \int_0^\pi \sin\theta \, d\theta}_{4\pi} = \frac{1}{\pi a^3}\left(\frac{3a^4}{8}\right)(4\pi)$$

$$= \boxed{\frac{3}{2} a}$$

$$\langle r^2 \rangle = \frac{4\pi}{\pi a^3} \int_0^\infty r^4 e^{-2r/a} dr = \frac{4}{a^3}\left(\frac{4!}{[2/a]^5}\right) = \frac{4}{a^3}(24)\left(\frac{a^5}{32}\right) = \boxed{3a^2}$$

b) Find $\langle x \rangle$ and $\langle x^2 \rangle$ for an electron in the ground state of hydrogen.

$$\langle x \rangle = \int \psi_{100}^* \, x \, \psi_{100} \, d^3\vec{r} \; ; \; \psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-\sqrt{x^2+y^2+z^2}/a}$$

$$= \frac{1}{\pi a^3} \underbrace{\int_{-\infty}^{\infty} e^{-\sqrt{x^2+y^2+z^2}} x \, dx}_{\text{odd}} \longrightarrow \boxed{0}$$

Symmetry: $\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle \longrightarrow \langle x^2 \rangle = \frac{1}{3}\langle r^2 \rangle = \boxed{a^2}$

c) Find $\langle x^2 \rangle$ for an electron in the state $n=2, l=1, m=1$. Use $x = r\sin\theta\cos\phi$.

$$\psi_{211} = \frac{-1}{8a^2\sqrt{\pi a}} r\sin\theta \, e^{i\phi - r/2a} \quad ; \quad \langle x^2 \rangle = \int \psi_{211}^* x^2 \psi_{211} r^2 \sin\theta \, dr\, d\phi\, d\theta$$

$$\longrightarrow \langle x^2 \rangle = \frac{1}{64 a^5 \pi} \underbrace{\int_0^\infty r^6 e^{-r/a} dr}_{I_r} \underbrace{\int_0^\theta \sin^5\theta \, d\theta}_{I_\theta} \underbrace{\int_0^{2\pi} \cos^2\phi \, d\phi}_{I_\phi}$$

$$I_r = \frac{6!}{(1/a)^7} = 720 a^7$$

$$I_\theta = \int_0^\pi \sin\theta \underbrace{(1-\cos^2\theta)}_{\sin^2\theta}{}^2 \, d\theta \qquad u = \cos\theta \rightarrow u \in (1,-1) \quad -du = +\sin\theta \, d\theta$$

$$= \int_{-1}^{1} (1-u^2)^2 du = 2\int_0^1 1 - 2u^2 + u^4 \, du$$

$$= 2\left[u - \frac{2u^3}{3} + \frac{u^5}{5}\right]_0^1 = 2\left[1 - \frac{2}{3} + \frac{1}{5}\right] = \frac{16}{15}$$

$$I_\phi = \int_0^{2\pi} \cos^2\phi \, d\phi = \frac{1}{2}\int_0^{2\pi} 1 + \cos 2\phi \, d\phi = \frac{1}{2}\left[\phi + \frac{1}{2}\sin 2\phi\right]_0^{2\pi}$$

$$= \frac{1}{2}(2\pi) = \pi$$

$$\longrightarrow \langle x^2 \rangle = \frac{1}{64 \cancel{a^5} \cancel{\pi}} (720 \cancel{a^7}\, a^2) \left(\frac{16}{15}\right)(\cancel{\pi}) = \boxed{12a^2}$$

Griffiths 4.20

③ Consider the earth-sun system as a gravitational analog to the hydrogen atom.

a) What is the potential energy (replacing eq 4.52)? Let $m = M_\oplus$ and $M = M_\odot$

$$U = -\frac{GmM}{r}$$

b) What is the "Bohr radius," a_g for this system?

$$a \equiv \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} \longrightarrow a_g = \frac{1}{G}\frac{1}{m^2 M}\hbar^2 \longrightarrow \boxed{a_g = \frac{\hbar^2}{Gm^2 M}}$$

$$a_g = \frac{\hbar^2}{GM_\oplus^2 M_\odot} = \frac{(1.05\times10^{-34}\,Js)^2}{(6.67\times10^{-11}\frac{m^3}{kg\,s^2})(5.97\times10^{24}kg)^2(2\times10^{30}kg)}$$

$$= \frac{(1.05)^2}{(6.67)(5.97)^2(2)}\times10^{-135}\,\frac{\cancel{N^2}\,m^2\,\cancel{s^2}\,\cancel{kg}\,\cancel{s^2}\,\cancel{m^{-3}}\,\cancel{kg^{-3}}}{\cancel{kg^2}\,\cancel{m^2}\,\cancel{s^{-4}}}$$

$$= 2.3\times10^{-3}\times10^{-135}\,m$$

$$\boxed{a_g = 2.3\times10^{-138}\,m}$$

c) Write down the gravitational "Bohr formula" and, by equating E_n to the classical energy of a planet in a circular orbit of radius r_0, show that $n = \sqrt{r_0/a_g}$. From this, estimate the quantum number of earth.

$$E_n = -\left[\frac{m}{2\hbar^2}\left(mMG\right)^2\right]\frac{1}{n^2} = \frac{E_1}{n^2} = -\frac{m^3M^2G^2}{2\hbar^2n^2}$$

$$+\frac{GmM}{2r_0} = +\frac{m^3M^2G^2}{2\hbar^2n^2}$$

$$\frac{1}{2r_0} = \frac{m^2MG}{2\hbar^2n^2} \qquad \left(a_g = \frac{\hbar^2}{Gm^2M}\right)$$

$$\longrightarrow r_0 = a_g n^2 \implies n = \sqrt{\frac{r_0}{a_g}} \checkmark$$

$$n_\oplus = \sqrt{\frac{1.5\times10^{11}\,m}{2.3\times10^{-138}\,m}} = \sqrt{6.5\times10^{-1}\times10^{149}} = \boxed{2.5\times10^{74}}$$

---

d) Suppose the earth made a transition to the next lower quantum level $(n-1)$. How much energy (in Joules) would be released? What would be the wavelength of the emitted photon/graviton? Express your answer in light years. Is the remarkable answer a coincidence?

$$E_\gamma = -\underbrace{\frac{m^3M^2G^2}{2\hbar^2}}_{A}\left[\frac{1}{n^2} - \frac{1}{(n-1)^2}\right]$$

$$= A\left[\frac{1}{(n-1)^2} - \frac{1}{n^2}\right] \approx \frac{A}{n^2}\left(1 - \frac{2}{n} - 1\right) = \frac{-2A}{n^3}$$

$$\rightarrow E_\gamma = \frac{m^3 M^2 G^2}{2\hbar^2 n^3} = \frac{(5.97\times10^{24}kg)^3(2\times10^{30}kg)^2(6.67\times10^{-11}m^3/kgs^2)^2}{2\,(1.055\times10^{-34}Js)^2(2.5\times10^{74})^3}$$

$$\Longrightarrow \boxed{E_\gamma \approx 2.1\times10^{-41}J}$$

$$E = \frac{hc}{\lambda} \rightarrow \lambda = \frac{hc}{E} = \frac{6.63\times10^{-34}Js\,(3\times10^{8}m/s)}{2.1\times10^{-41}J} = \boxed{9.5\times10^{15}m}$$

$$\boxed{\lambda \approx 1 \text{ light year!}}$$

Yes coincidence?