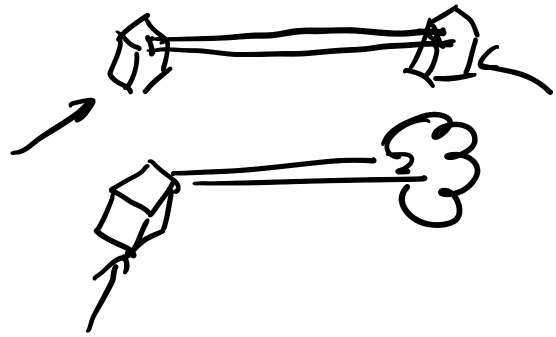
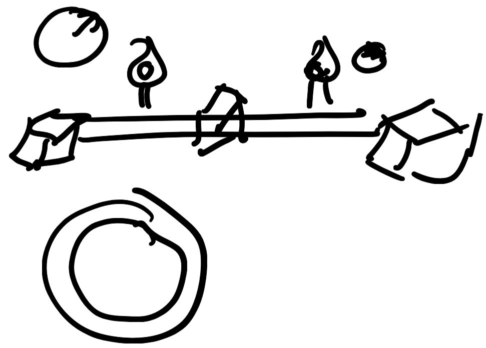
Notes 6



HEAT EQN
$u_t = u_{xx}$



INFINITE BAR

FOURIER TRANSFORM

$f(x)$; , $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} f(x)\,dx$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega)\, e^{ix\omega}\, d\omega = \check{g}(x)$$

INVERSE FOURIER TRANSFORM.

$\check{\hat{f}}(x) = f(x)$. FACT:

"TABLE OF FOURIER TRANSFORMS"

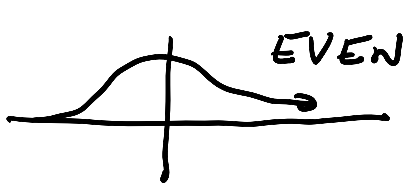
① $f(x) = \begin{cases} 1 & |x| \le a \\ 0 & \text{OTHERWISE} \end{cases}$ $\quad \hat{f}(\omega) = \frac{\sin(a\omega)}{\omega}$

② $f(x) = e^{-|x|}$ $\quad \hat{f}(\omega) = \frac{2}{1+\omega^2}$

③ $e^{-a^2x^2} = f(x)$ $\quad \hat{f}(\omega) = \frac{1}{\sqrt{2}\cdot a} e^{-\frac{\omega^2}{4a^2}}$,

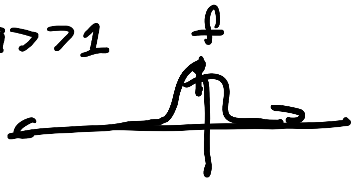
TRICKY! $\int_{-\infty}^{\infty} e^{-x^2} dx$ (USE POLAR COORD'S)

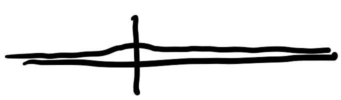
DIFFERENTIAL EQN FOR $\hat{f}(\omega)$



EVEN

$a >> 1$





CHECK: $f(x) = e^{-a^2x^2}$; $\check{\hat{f}}(x) = f(x)$;

$$g(\omega) = (\hat{f})(\omega) = \frac{1}{\sqrt{2}\,a} e^{-\omega^2/4a^2}$$

$$(\check{g})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\omega} \frac{1}{\sqrt{2}\,a} e^{\frac{-\omega^2}{4a^2}} \, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2}\cdot a} \int_{-\infty}^{\infty} e^{-ix\omega} e^{-\omega^2\left(\frac{1}{4a^2}\right)} \, d\omega$$

$$\left(b^2 = \frac{1}{4a^2}\right) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2}\cdot a} \int_{-\infty}^{\infty} e^{-ix\omega} e^{-\omega^2 b^2} \, d\omega$$

$$= \frac{1}{\sqrt{2}\cdot a} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} e^{-\omega^2 b^2} \, d\omega}$$

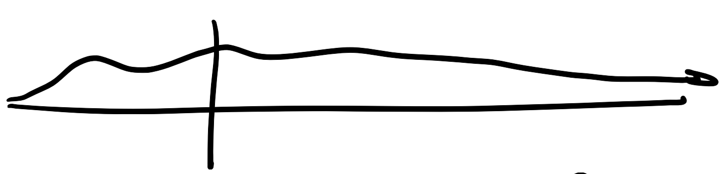
$$= \frac{1}{\sqrt{2}\,a} \frac{1}{\sqrt{2}\,b} e^{-x^2/4b^2} \qquad b = \frac{1}{2a}.$$

$$= \frac{1}{\sqrt{2}\cdot a} \frac{1}{\sqrt{2}\,\frac{1}{2a}} e^{-\frac{x^2}{4}\cdot 4a^2}$$

$$= e^{-a^2x^2} \quad ✓$$

SQUARE NORM OF A FUNCTION.

"NORMS"



¿ HOW BIG IS A FUNCTION?

① MAX $|f(x)|$ ② $\int_{-\infty}^{\infty} |f(x)|\, dx$ ← $\|f\|_1$.

MAX IS BIG.

$\|f\|_2$ IS SMALL.

③ $\sqrt{\int_{-\infty}^{\infty} |f(x)|^2\, dx} = \|f\|_2 =$ SQUARE NORM OF f.

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \|f\|_2^2.$$

FACT: $\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega = \int_{-\infty}^{\infty} |f(x)|^2\, dx$

$$\|\hat{f}\|_2^2 = \|f\|_2^2; \quad \|\hat{f}\|_2 = \|f\|_2$$

EX: CHECK $f(x) = e^{-|x|}$; $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{2}{1+\omega^2}$

$$\int |f(x)|^2 dx = \int |\hat{f}(\omega)|^2 d\omega$$

CHECK FOR GAUSSIAN.

EX: CHECK $f(x) = e^{-a^2x^2}$.

$\wedge$ and $\frac{d}{dx}$: $\widehat{f'}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} f'(x)\, dx$

$$= f(x) e^{-ix\omega} \Big|_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (e^{-ix\omega})' f(x)\, dx$$

ASSUME $|f(x)| \to 0$ AS $x \to \pm\infty$.

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-i\omega) e^{-ix\omega} f(x)\, dx$$

$$= +i\omega \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} f(x)\,dx$$

$$= i\omega\, \hat{f}(\omega). \overset{!}{=} \widehat{f'}(\omega)$$

<u>EX</u>: $\widehat{f''}(\omega) = -w^2 \hat{f}(w).$

FOURIER TRANSFORM +
HEAT EQUATION.

$$u_t(x,t) = u_{xx}(x,t); \quad u(x,0) = f(x).$$

FOURIER TRANSFORM FOR THE <u>X-VARIABLE</u>

$$\hat{u}_t(w,t) = \widehat{u_{xx}}(w,t) \quad ???.$$

$$u(x,t); \qquad \hat{u}(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\omega} u(x,t)\,dx$$

$$\boxed{\hat{u}_t(\omega,t) = -\omega^2\, \hat{u}(\omega,t)}$$

THINK OF ω AS BEING FIXED. !!!

$$\hat{u}(w,t) = c(\omega)\, e^{-w^2 t}$$

$$u(x,0) = f(x); \quad \hat{u}(\omega,0) = \hat{f}(\omega)$$

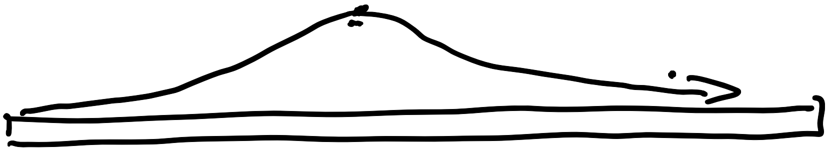
$$\hat{u}(\omega,t) = c(\omega) = \hat{f}(\omega).$$

$$\hat{u}(w,t) = \hat{f}(w)\, e^{-w^2 t}$$

$$\boxed{u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw}\, e^{-w^2 t}\, \hat{f}(w)\, dw}$$

!!!!!

SOLVE FOR IMPORTANT SPECIAL CASE: $e^{-ax^2} = f(x) = u(x,0)$



IN FACT, IT IS GAUSSIAN FOR ALL TIME.

$$f(x) = e^{-a^2x^2} \qquad \hat{f}(w) = \frac{1}{\sqrt{2}\, a}\, e^{-\frac{w^2}{4a^2}}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw}\, e^{-w^2 t}\, \frac{1}{\sqrt{2}\, a}\, e^{-\frac{w^2}{4a^2}}\, dw$$

$$= \frac{1}{\sqrt{2}\, a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw}\, e^{-\left(t + \frac{1}{4a^2}\right) w^2}\, dw$$

$$\left\{ \left(t + \frac{1}{4a^2}\right) = b^2 \right\}$$

$$\frac{1}{\sqrt{2}\, a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixw}\, e^{-b^2 w^2}\, dw$$

!!!

$$\overset{!!}{=} \frac{1}{\sqrt{2}\,a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\omega} e^{-b^2\omega^2} \, d\omega$$

$$= \frac{1}{\sqrt{2}\,a} \frac{1}{\sqrt{2}\,b} e^{-x^2/4b^2}$$

$$= \frac{1}{2ab} e^{-x^2/4b^2}$$

$$b^2 = \left(t + \frac{1}{4a^2}\right)$$

$$\frac{4a^2t+1}{4a^2} = b^2 ; \quad b = \frac{\sqrt{4a^2t+1}}{2a}$$

$$\frac{1}{2a \frac{\sqrt{4a^2t+1}}{2a}} e^{-\frac{x^2}{4} \frac{4a^2}{4a^2t+1}}$$

$$= \boxed{\frac{1}{\sqrt{4a^2t+1}} \cdot e^{-\frac{x^2 \cdot a^2}{4a^2t+1}}} = u(x,t).$$

$u_t = u_{xx}$ | $u(x,0) = f(x)$

CHECK:

EX: | $\checkmark \quad = e^{-a^2x^2}.$