GALERKIN APPROXIMATIONS OF NONLINEAR DELAY DIFFERENTIAL EQUATIONS WITH MULTIPLE DELAYS

HONGHU LIU AND TREVOR NORTON

Abstract. Create abstract

1. Introduction.

Points to be addressed:

- Can we deal with time-dependent coefficients in the linear part? Treat them as nonlinear term?

2. Preliminaries.

2.1. DDEs covered by the proposed approach. We consider systems of nonlinear DDEs involving multiple discrete or distributed delays, either in the linear term or in the nonlinearity. Such DDEs can be put into the following form:

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} &= \boldsymbol{A}\boldsymbol{x}(t) + \sum_{i=1}^{p}\boldsymbol{B}_i\boldsymbol{x}(t-\tau_i) + \sum_{i=1}^{p}\boldsymbol{C}_i\int_{t-\tau_i}^{t}\boldsymbol{x}(s)\,\mathrm{d}s \\ &\quad + \boldsymbol{F}\left(t, \boldsymbol{x}(t), \boldsymbol{x}(t-\tau_1), \cdots, \boldsymbol{x}(t-\tau_p), \int_{t-\tau_1}^{t}\boldsymbol{x}(s)\,\mathrm{d}s, \cdots, \int_{t-\tau_p}^{t}\boldsymbol{x}(s)\,\mathrm{d}s\right),\end{aligned} \tag{2.1}$$

where the unknown function $\boldsymbol{x}$ is a d-dimensional vector; p is a positive integer, representing the total number of delays; the τ_i's are distinctive positive scalars arranged in ascending order; $\boldsymbol{A}$, $\boldsymbol{B}_i$, and $\boldsymbol{C}_i$ ($1 \leq i \leq p$) are given $d \times d$ matrices; and $\boldsymbol{F}\colon \mathbb{R}^{2+2p} \to \mathbb{R}^d$ is a given continuous vector function.

In order to simplify the presentation, we first articulate our main contribution in a simple setting of a scalar DDE with a single discrete delay $\tau > 0$:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = ax(t) + bx(t-\tau) + F(x(t-\tau)), \tag{2.2}$$

where $a, b \in \mathbb{R}$, and $F : \mathbb{R} \to \mathbb{R}$ is a given scalar function. Results for the general case of (2.1) is provided afterward in Section 4.

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- Explain in a short paragraph the main difficult compared with the case dealt with in [1].
- To cope with the difficulties, we restrict the initial data to C^2 functions. Refer to Section 4 for results about existence and regularity.

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2.2. The Abstract Formulation of the Linear Operator. It is appropriate to reformulate (2.2) into an abstract ordinary differential equation on the Hilbert space

$$\mathcal{H} := L^2([-\tau, 0); \mathbb{R}) \times \mathbb{R}, \tag{2.3}$$

where the inner product is defined for (f_1, γ_1), $(f_2, \gamma_2) \in \mathcal{H}$, as:

$$\langle (f_1, \gamma_1), (f_2, \gamma_2) \rangle_{\mathcal{H}} := \frac{1}{\tau}\int_{-\tau}^{0} f_1(\theta) f_2(\theta)\,\mathrm{d}\theta + \gamma_1\gamma_2. \tag{2.4}$$

However, it is not yet possible to represent F in this space, so we focus on the linear part of (2.2). Define the linear operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ by

$$[\mathcal{A}\Psi](\theta) := \begin{cases} \dfrac{\mathrm{d}^+\Psi^D}{\mathrm{d}\theta}, & \theta \in [-\tau, 0), \\ a\Psi^S + b\Psi^D(-\tau), & \theta = 0, \end{cases} \tag{2.5}$$

for any $\Psi = (\Psi^D, \Psi^S)$ that lives in the domain, $D(\mathcal{A})$, defined as

$$D(\mathcal{A}) := \left\{ \Psi \in \mathcal{H} : \Psi^D \in H^1([-\tau, 0); \mathbb{R}^d), \lim_{\theta \to 0^-} \Psi^D(\theta) = \Psi^S \right\}. \tag{2.6}$$

It is clear that if $x : [-\tau, \infty)$ satisfies the linear DDE

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= ax(t) + bx(t - \tau), \quad t > 0 \\ x(0) &= \alpha, \\ x(t) &= f(t), \qquad\qquad\qquad t \in [-\tau, 0) \end{aligned} \tag{2.7}$$

then $u(t) = (x_t, x_t(0))$, where $x_t(\theta) = x(t + \theta)$ for $\theta \in [-\tau, 0)$, satisfies the linear, abstract ODE

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= \mathcal{A}u, \quad t > 0 \\ u(0) &= u_0, \end{aligned} \tag{2.8}$$

where $u_0 = (f, \alpha)$. From [2, Thm. 2.4.1], the DDE in (2.7) has a solution $x(t)$. Furthermore, if we define $T(t) : \mathcal{H} \to \mathcal{H}$ by

$$T(t)(f, \alpha) := (x_t, x_t(0)), \quad t \geq 0, \tag{2.9}$$

then $T(t)$ is a C_0-semigroup on $\mathcal{H}$ and $\mathcal{A}$ is its infinitesimal generator [2, Thm. 2.4.4; Thm. 2.4.6]. With this, we know that the solution to (2.8) is $T(t)u_0$.

2.3. The Space X. In order for us to make sense of the nonlinear part of (2.2), we look at a certain subset of the space $\mathcal{H}$. We define the following inner product space with elements in

$$X := \mathcal{C}^+([-\tau, 0); \mathbb{R}) \times \mathbb{R} \subseteq \mathcal{H}, \tag{2.10}$$

where $\mathcal{C}^+([-\tau, 0))$ denotes the set of bounded right-continuous functions on the interval $[-\tau, 0)$, and the inner product defined by

$$(\Phi, \Psi)_X := \Phi^S\Psi^S + \frac{1}{\tau}(\Phi^D, \Psi^D)_{L^2([-\tau,0))} + \Phi^D(-\tau)\Psi^D(-\tau), \quad \Phi, \Psi \in X. \tag{2.11}$$

Note that this is defined since if $f \in C^+([-\tau, 0)$ then $f \in L^2([-\tau, 0)$. It is relatively straight-forward to verify that $(\cdot, \cdot)_X$ is symmetric, bilinear, and positive definite and thus an inner product. We will also make use of the norm $\| \cdot \|_X$ induced from this inner product. Note that X is **not** a Banach space since Cauchy sequences might not converge in X.

We can then define $\mathcal{F} : X \to X \subseteq \mathcal{H}$ by

$$[\mathcal{F}(\Psi)](\theta) := \begin{cases} 0, & \theta \in [-\tau, 0), \\ F\left(\Psi^D(-\tau)\right), & \theta = 0, \end{cases} \qquad \forall\, \Psi = (\Psi^D, \Psi^S) \in X. \tag{2.12}$$

From [2, Thm. 2.4.1], if $u_0 \in X$, then the solution $x(t)$ of (2.2) with initial conditions $x_0 = u_0^D$ and $x(0) = u_0^S$ is continuous on $[0, \infty)$. Therefore, x_t is in $\mathcal{C}^+([-\tau, 0))$ and $u(t) \in X$ for any $t \in [0, \infty)$. Now if we set $u(t) = (x_t, x_t(0))$ where x is the solution to (2.2), then u satisfies the following abstract ODE:

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= \mathcal{A}u(t) + \mathcal{F}(u(t)), \\ u(0) &= u_0. \end{aligned} \tag{2.13}$$

From the above, we can derive the variation of constants formula:

$$u(t) = T(t)u_0 + \int_0^t T(t-s)\mathcal{F}(u(s))\,\mathrm{d}s. \tag{2.14}$$

For a derivation c.f. [4, pg. 105].

2.4. Properties and Basic Results of Koornwinder Polynomials. From [3, Eq. (2.1)], the sequence of Koornwinder polynomials $\{K_n\}$ can be built from the Legendre polynomials L_n by

$$K_n(s) := -(1+s)\frac{d}{\mathrm{d}s}L_n(s) + (n^2+n+1)L_n(s), \quad s \in [-1, 1],\ n \in \mathbb{N}_0. \tag{2.15}$$

Furthermore, we reproduce from [1, Prop. 3.1] some simple properties that $\{K_n\}$ satisfy.

Proposition 2.1. *The polynomial K_n defined in (2.15) is of degree n and admits the following expansion in terms of the Legendre polynomials:*

$$K_n(s) = -\sum_{j=0}^{n-1}(2j+1)L_j(s) + (n^2+1)L_n(s), \qquad n \in \mathbb{N}_0; \tag{2.16}$$

and the following normalization property holds:

$$K_n(1) = 1, \qquad n \in \mathbb{N}_0. \tag{2.17}$$

Moreover, the sequence given by

$$\{\mathcal{K}_n := (K_n, K_n(1)) : n \in \mathbb{N}_0\} \tag{2.18}$$

forms an orthogonal basis of the product space

$$\mathcal{E} := L^2([-1, 1); \mathbb{R}) \times \mathbb{R}, \tag{2.19}$$

where $\mathcal{E}$ is endowed with the following inner product:

$$\langle (f, a), (g, b)\rangle_{\mathcal{E}} = \frac{1}{2}\int_{-1}^{1} f(s)g(s)\,\mathrm{d}s + ab, \quad (f, a), (g, b) \in \mathcal{E}. \tag{2.20}$$

Finally, $\left\{\frac{\mathcal{K}_n}{\|\mathcal{K}_n\|_{\mathcal{E}}}\right\}$ *forms a Hilbert basis of* $\mathcal{E}$ *where the norm* $\|\mathcal{K}_n\|_{\mathcal{E}}$ *of* $\mathcal{K}_n$ *induced by* $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ *possesses the following analytic expression:*

$$\|\mathcal{K}_n\|_{\mathcal{E}} = \sqrt{\frac{(n^2+1)((n+1)^2+1)}{2n+1}}, \qquad n \in \mathbb{N}_0. \tag{2.21}$$

Suppose that $\Pi_N^{\mathcal{E}}$ is the N-dimensional standard projection into $\operatorname{span}\{\mathcal{K}_n : n \leq N\} \subset \mathcal{E}$. It will be relevant to discuss when we have convergence of $[\Pi_N^{\mathcal{E}} u]^D$ for $u \in \mathcal{E}$. In particular, we will focus on uniform convergence.

Proposition 2.2. *Let* $f \in \mathcal{C}^2([-1,1];\mathbb{R})$ *and denote* $\psi = (f, f(0)) \in \mathcal{E}$*. Then the series*

$$[\Pi_N^{\mathcal{E}}\psi]^D = \sum_{n=0}^{N} \frac{\langle \psi, \mathcal{K}_n\rangle_{\mathcal{E}}}{\|\mathcal{K}_n\|_{\mathcal{E}}^2} K_n \tag{2.22}$$

converges uniformly to f*.*

See Appendix A for a proof.

It will also be necessary to prove certain properties of the series of Koornwinder polynomials

$$S_N(x) := \sum_{n=0}^{N} \frac{K_n}{\|\mathcal{K}_n\|_{\mathcal{E}}^2}, \quad N \in \mathbb{N}_0,\ x \in [-1,1]. \tag{2.23}$$

If we were to denote $\psi = (0,1) \in L^2([-1,1)) \times \mathbb{R}$, then S_N would simply be the functional part of $\Pi_N \psi$. The following lemma allows us to express S_N in terms of Legendre Polynomials. Now that we have this expression, we can prove the properties of S_N that will be useful when showing the main result.

Lemma 2.1. *The functions* S_N *defined in* (2.23) *can be expressed as*

$$S_N(x) = \frac{1}{(N+1)^2+1} \sum_{n=0}^{N} (2n+1) L_n, \quad x \in [-1,1]. \tag{2.24}$$

Moreover,

$$|S_N(x)| < 1, \quad \forall N \in \mathbb{N}_0,\ \forall x \in [-1,1], \tag{2.25}$$

and

$$\lim_{N\to\infty} S_N(x) = 0, \quad \forall x \in (-1,1). \tag{2.26}$$

See Appendix A for a proof.

Remark 2.1. *It can easily be shown that* $\lim_{N\to\infty} S_N(-1) = 0$ *and* $\lim_{N\to\infty} S_N(1) = 1$*, which both follow from the expression* (2.24) *when evaluated at* ± 1*. However, for our main results we need only that* $S_N \to 0$ *almost everywhere on* $[-1,1]$*. Therefore we omit the proof of this.*

Applying a linear transformation to the orthogonal polynomials on $[-1,1]$ will give us a set of orthogonal polynomials on $[-\tau, 0]$, from which we can construct an orthogonal basis on $\mathcal{H}$. We define a linear transformation $\mathcal{T}$ by

$$\mathcal{T} \colon [-\tau, 0] \to [-1,1], \qquad \theta \mapsto 1 + \frac{2\theta}{\tau}. \tag{2.27}$$

We can now define the polynomial K_n^τ by

$$
\begin{aligned}
(2.28)\qquad & K_n^\tau \colon [-\tau, 0] \to \mathbb{R}, \\
& \theta \mapsto K_n\Big(1 + \frac{2\theta}{\tau}\Big), \qquad n \in \mathbb{N}.
\end{aligned}
$$

Since the sequence $\{\mathcal{K}_n = (K_n, K_n(1)) : n \in \mathbb{N}\}$ forms an orthogonal basis for $\mathcal{E}$ (cf. Proposition 2.1), it follows then that the polynomial sequence $\mathcal{K}_n^\tau := (K_n^\tau, K_n^\tau(0)) : n \in \mathbb{N}\}$ forms an orthogonal basis for the space $\mathcal{H} = L^2([-\tau, 0); \mathbb{R}) \times \mathbb{R}$ endowed with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ defined in (2.4). We define S_N^τ similarly. It can be verified that Lemma 2.1 and Proposition 2.2 are preserved in $\mathcal{H}$.

We are now able to define

$$
(2.29)\qquad \mathcal{H}_N := \operatorname{span}\{\mathcal{K}_0^\tau, \ldots, \mathcal{K}_N^\tau\}.
$$

Let Π_N be the associated orthogonal projector of $\mathcal{H}_N$. By the construction of the orthogonal basis $\{\mathcal{K}_n^\tau\}$, we have that $\mathcal{H}_N \subset D(\mathcal{A})$. The N-dimensional Galerkin approximation of (2.13) is

$$
\begin{aligned}
(2.30)\qquad \frac{\mathrm{d}u_N}{\mathrm{d}t} &= \mathcal{A}_N u_N + \Pi_N \mathcal{F}(u_N), \\
u_N(0) &= \Pi_N u_0,
\end{aligned}
$$

where $\mathcal{A}_N := \Pi_N \mathcal{A} \Pi_N$. The linear operator $\mathcal{A}_N$ on the finite dimensional space $\mathcal{H}_N$ defines the C_0-semigroup $e^{\mathcal{A}_N t}$. This can be extended to a C_0-semigroup on $\mathcal{H}$:

$$
(2.31)\qquad T_N(t)u = e^{\mathcal{A}_N t}\Pi_N u + (I - \Pi_N)u, \; u \in \mathcal{H}.
$$

From (2.30), we derive the variation of constants formula for the Galerkin approximation:

$$
(2.32)\qquad u_N(t) = T_N(t)\Pi_N u_0 + \int_0^t T_N(t-s)\Pi_N \mathcal{F}(u_N(t))\,\mathrm{d}s.
$$

By [1, Lemma 4.3] and the proof of [1, Thm. 4.1], the results about $T(t)$ and $T_N(t)$ hold.

Proposition 2.3. *For $t > 0$ and $N \in \mathbb{N}_0$,*

$$
(2.33)\qquad \|T_N(t)\|_{\mathcal{H}}, \|T(t)\|_{\mathcal{H}} \le M e^{\omega t}.
$$

Also, for any $T > 0$,

$$
(2.34)\qquad \lim_{N\to\infty} \sup_{t\in[0,T]} \|T(t)u - T_N(t)\Pi_N u\|_{\mathcal{H}} = 0, \quad \forall u \in \mathcal{H}.
$$

The proof of (2.34) relies on a version of the Trotter-Kato theorem [4, Thm. 4.5, p. 88].

2.5. Vectorized Koornwinder Polynomials. We now wish to vectorize the polynomials from the previous subsection so that they form an orthogonal basis for

$$
(2.35)\qquad \mathcal{H}^{\overline{\tau}} := L^2([-\tau_1, 0); \mathbb{R}) \times \cdots \times L^2([-\tau_p, 0); \mathbb{R}) \times \mathbb{R}^p,
$$

where $\overline{\tau} = (\tau_1, \tau_2, \cdots, \tau_p)$ with $0 < \tau_1 \le \tau_2 \le \cdots \le \tau_p$. The inner product of this space is given by

$$
(2.36)\qquad \langle \Psi, \Phi \rangle_{\mathcal{H}^{\overline{\tau}}} = \sum_{i=1}^p \frac{1}{\tau_i} \int_{-\tau_i}^0 \Psi_i^D(\theta)\Phi_i^D(\theta)\,\mathrm{d}\theta + \langle \Psi^S, \Phi^S \rangle, \qquad \forall \Psi, \Phi \in \mathcal{H}^{\overline{\tau}}.
$$

The construction will be similar to that in [1, Section 3.3]. For $j \in \mathbb{N}$, let

$$d_j = \left\lfloor \frac{j-1}{p} \right\rfloor \tag{2.37}$$

and let

$$r_j = \begin{cases} \mathrm{mod}(j,p), & \text{if} \quad \mathrm{mod}\ (j,p) \neq 0 \\ p, & \text{otherwise} \end{cases}.$$

Define

$$\mathbf{K}_j^{\overline{\tau}} := (\underbrace{0,\ldots,0,}_{r_j \text{ - 1 entries}} \ K_{d_j}^{\tau_{r_j}}, \ \underbrace{0,\ldots,0}_{p-r_j \text{ entries}} \ ). \tag{2.38}$$

We shall also define

$$\mathbf{K}_j^{\overline{\tau}}(0) := (\underbrace{0,\ldots,0,}_{r_j \text{ - 1 entries}} \ K_{d_j}^{\tau_{r_j}}(0), \ \underbrace{0,\ldots,0}_{p-r_j \text{ entries}} \ ). \tag{2.39}$$

We introduce

$$\mathbb{K}_j^{\overline{\tau}} := (\mathbf{K}_j^{\overline{\tau}}, \mathbf{K}_j^{\overline{\tau}}(0)), \qquad j \in \mathbb{N}. \tag{2.40}$$

One can check that $\{\mathbb{K}_j^{\overline{\tau}} : j \in \mathbb{N}\}$ forms an orthogonal basis for $\mathcal{H}^{\overline{\tau}}$. We define

$$\mathcal{X} := L^2([-\tau_1, 0); \mathbb{R}) \times \cdots \times L^2([-\tau_p, 0); \mathbb{R}) \tag{2.41}$$

and

$$\langle f, g \rangle_{\mathcal{X}} = \sum_{i=1}^{p} \frac{1}{\tau_i} \int_{-\tau_i}^{0} f(\theta) g(\theta) \, \mathrm{d}\theta, \tag{2.42}$$

i.e., $\mathcal{X}$ is the delay part of $\mathcal{H}^{\overline{\tau}}$.

Proposition 2.4. *The following convergence results hold for any $f \in \mathcal{X}$ and $\alpha \in \mathbb{R}^p$:*

$$\begin{aligned}
&\sum_{j=1}^{\infty} \frac{\langle \alpha, \mathbf{K}_j^{\overline{\tau}}(0) \rangle}{\|\mathbb{K}_j^{\overline{\tau}}\|_{\mathcal{H}}} \mathbf{K}_j^{\overline{\tau}} = 0 \qquad \textit{with respect to } \|\cdot\|_{\mathcal{X}}; \\
&\sum_{j=1}^{\infty} \frac{\langle \alpha, \mathbf{K}_j^{\overline{\tau}}(0) \rangle}{\|\mathbb{K}_j^{\overline{\tau}}\|_{\mathcal{H}}} \mathbf{K}_j^{\overline{\tau}}(0) = \alpha; \\
&\sum_{j=1}^{\infty} \frac{\langle \alpha, \mathbf{K}_j^{\overline{\tau}} \rangle_{\mathcal{X}}}{\|\mathbb{K}_j^{\overline{\tau}}\|_{\mathcal{H}}} \mathbf{K}_j^{\overline{\tau}} = f \qquad \textit{with respect to } \|\cdot\|_{\mathcal{X}}; \textit{ and} \\
&\sum_{j=1}^{\infty} \frac{\langle \alpha, \mathbf{K}_j^{\overline{\tau}} \rangle_{\mathcal{X}}}{\|\mathbb{K}_j^{\overline{\tau}}\|_{\mathcal{H}}} \mathbf{K}_j^{\overline{\tau}}(0) = 0.
\end{aligned} \tag{2.43}$$

Proof. Note that $\{\mathbb{K}_j^{\overline{\tau}} : j \in \mathbb{N}\}$ forms an orthogonal basis for $\mathcal{H}$. Then for $\Psi = (\Psi^D, \Psi^S) \in \mathcal{H}$, we have the following decomposition

$$\begin{aligned}\Psi &= \sum_{j=1}^{\infty} \frac{\langle \Psi, \mathbb{K}_j^{\overline{\tau}} \rangle_{\mathcal{H}}}{\|\mathbb{K}_j^{\overline{\tau}}\|_{\mathcal{H}}^2} \mathbb{K}_j^{\overline{\tau}} \\ &= \sum_{j=1}^{\infty} \left(\langle \Psi^D, \mathbf{K}_j^{\overline{\tau}} \rangle_{\mathcal{X}} + \langle \Psi^S, \mathbf{K}_j^{\overline{\tau}}(0) \rangle \right) \frac{\mathbb{K}_j^{\overline{\tau}}}{\|\mathbb{K}_j^{\overline{\tau}}\|_{\mathcal{H}}^2}.\end{aligned} \tag{2.44}$$

If we set $\Psi = (\mathbf{0}, \alpha) \in \mathcal{H}_p$ and equalize the functional and numerical parts of each side of (2.44), we get the first two convergence results. Similarly, setting $\Psi = (f, \mathbf{0})$ and equalizing the functional and numerical parts of (2.44), we get the last two convergence results. ◻

For the case when $\overline{\tau} = (\tau, \tau, \cdots, \tau)$ for $\tau > 0$, we may write $\mathbb{K}_n^{\tau}$ instead of $\mathbb{K}_n^{\overline{\tau}}$. In this case, our construction corresponds exactly to the $\mathbb{K}_n^{\tau}$ described in [1, Section 3.3]

3. Uniform Convergence of Galerkin Solutions.

3.1. Pointwise Convergence in X. From [2, Thm. 2.4.1], if $u_0 \in X$, then the solution $x(t)$ of (2.7) with initial conditions u_0^D and u_0^S is continuous on $[0, \infty)$. This is sufficient to say that $T(t)u_0 = (x_t, x_t(0)) \in X$ and that $T(t)$ maps X into X. Similarly, we have that Π_N maps $X \subseteq \mathcal{H}$ into $\mathcal{H}_N$ and $T_N(t)$ maps $\mathcal{H}_N$ into $\mathcal{H}_N$. Hence $T_N(t)\Pi_N$ maps X into $\mathcal{H}_N$, which is a subset of X. We summarize these results in the following lemma.

Lemma 3.1. *For any $t \geq 0$, the operators $T(t)$ and $T_N(t)\Pi_N$ map X into itself.*

This justifies the later use of the norm $\|\cdot\|_X$ on certain functions. It will also be helpful to prove the following lemma.

Lemma 3.2. *Let u be the solution for (2.13) and u_N the solution for (2.30) for some initial value $u_0 \in X$. There is $C > 0$ such that for any $N \in \mathbb{N}_0$.*

$$\|T_N(t-s)\Pi_N(\mathcal{F}(u(s)) - \mathcal{F}(u_N(s)))\|_X \leq C\|u(s) - u_N(s)\|_X, \tag{3.1}$$

where $t \in [0, T]$ and $s \in [0, t]$.

Proof. We have that

$$\begin{aligned}\|T_N(t-s)\Pi_N(\mathcal{F}(u(s)) - \mathcal{F}(u_N(s)))\|_X^2 &= \|T_N(t-s)\Pi_N(\mathcal{F}(u(s)) - \mathcal{F}(u_N(s)))\|_{\mathcal{H}}^2 \\ &\quad + \left|[T_N(t-s)\Pi_N(\mathcal{F}(u(s)) - \mathcal{F}(u_N(s)))]^D(-\tau)\right|^2.\end{aligned} \tag{3.2}$$

Note that for the first term on the right-hand side of (3.2), we have that

$$\begin{aligned}\|T_N(t-s)\Pi_N(\mathcal{F}(u(s)) - \mathcal{F}(u_N(s)))\|_{\mathcal{H}} &\leq Me^{\omega(t-s)}\|\mathcal{F}(u(s)) - \mathcal{F}(u_N(s))\|_{\mathcal{H}} \\ &\leq Me^{\omega T}\|\mathcal{F}(u(s)) - \mathcal{F}(u_N(s))\|_{\mathcal{H}} \\ &= Me^{\omega T}\left|F([u(s)]^D(-\tau)) - F([u_N(s)]^D(-\tau))\right| \\ &\leq \mathrm{Lip}(F)Me^{\omega T}\left|[u(s)]^D(-\tau) - [u_N(s)]^D(-\tau)\right| \\ &\leq \mathrm{Lip}(F)Me^{\omega T}\|u(s) - u_N(s)\|_X.\end{aligned} \tag{3.3}$$

For the second term on the right-hand side of (3.2), we consider first the case when $t-s \geq \tau$. Then

$$\begin{aligned}
\left|[T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))]^D(-\tau)\right| &\leq \|T_N(t-s-\tau)\Pi(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))\|_{\mathcal{H}} \\
&\leq Me^{\omega T}\|\mathcal{F}(u(s))-\mathcal{F}(u_N(s))\|_{\mathcal{H}} \\
&\leq \operatorname{Lip}(F)Me^{\omega T}\|u(s)-u_N(s)\|_X.
\end{aligned} \tag{3.4}$$

Now consider the case when $t-s<\tau$. So we have that

$$\begin{aligned}
\left|[T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))]^D(-\tau)\right| &= \left|[\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))]^D(t-s-\tau)\right| \\
&= \left|F([u(s)]^D(-\tau))-F([u_N(s)]^D(-\tau))\right|\cdot|S_N^\tau(t-s-\tau)| \\
&\leq \operatorname{Lip}(F)\left|[u(s)]^D(-\tau)-[u_N(s)]^D(-\tau)\right| \\
&\leq \operatorname{Lip}(F)\|u(s)-u_N(s)\|_X.
\end{aligned} \tag{3.5}$$

Note that since $M \geq 1$ and $\omega T \geq 0$, so $\operatorname{Lip}(F) \leq \operatorname{Lip}(F)Me^{\omega T}$. If we define

$$C := \sqrt{2}\cdot\operatorname{Lip}(F)Me^{\omega T} \tag{3.6}$$

and apply (3.3), (3.4), and (3.5) to (3.2), then we get that

$$\|T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))\|_X \leq C\|u(s)-u_N(s)\|_X. \tag{3.7}$$ □

We introduce the following definitions:

$$\begin{aligned}
r_N(t) &:= \|u(t)-u_N(t)\|_X, \\
\epsilon_N(t) &:= \|T(t)u_0 - T_N(t)\Pi_N u_0\|_X, \\
d_N(t,s) &:= \|\left(T(t-s)-T_N(t-s)\Pi_N\right)\mathcal{F}(u(s))\|_X.
\end{aligned} \tag{3.8}$$

One can apply the variation-of-constants formula and the above definitions to get that

$$\begin{aligned}
r_N(t) &\leq \epsilon_N(t) + \int_0^t d_N(t,s)\,\mathrm{d}s + \int_0^t \|T_N(t-s)\Pi_N\big(\mathcal{F}(u(s))-\mathcal{F}(u_N(s))\big)\|_X\,\mathrm{d}s \\
&\leq \epsilon_N(t) + \int_0^t d_N(t,s)\,\mathrm{d}s + C\int_0^t r_N(s)\,\mathrm{d}s.
\end{aligned} \tag{3.9}$$

Applying Grönwall's inequality to (3.9) gives

$$\begin{aligned}
r_N(t) &\leq \left[\epsilon_N(t) + \int_0^t d_N(t,s)\,\mathrm{d}s\right] + \int_0^t Ce^{C(t-s)}\left[\epsilon_N(s) + \int_0^s d_N(s,r)\,\mathrm{d}r\right]\,\mathrm{d}s \\
&\leq \left[\epsilon_N(t) + \int_0^t d_N(t,s)\,\mathrm{d}s\right] + Ce^{CT}\int_0^t\left[\epsilon_N(s) + \int_0^s d_N(s,r)\,\mathrm{d}r\right]\,\mathrm{d}s.
\end{aligned} \tag{3.10}$$

We wish to show that $r_N(t) \to 0$ as $N \to \infty$ for each fixed $t \in [0,T]$. To this end, we show that each term on the right-hand side of (3.10) converges to 0 as $N \to \infty$ and $t \in [0,T]$ fixed. The following propositions will show this.

Proposition 3.1. *For fixed $t \in [0,T]$,*

$$\epsilon_N(t) \to 0 \text{ and } \int_0^t \epsilon_N(s)\,\mathrm{d}s \to 0 \tag{3.11}$$

as $N \to \infty$.

Proof. From the definition of the X norm, we have that

$$\epsilon_N(t)^2 = \|T(t)u_0 - T_N(t)\Pi_N u_0\|_{\mathcal{H}}^2 + |[T(t)u_0]^D(-\tau) - [T_N(t)\Pi_N u_0]^D(-\tau)|^2. \tag{3.12}$$

The first term on the right-hand side converges uniformly to 0 by the Trotter-Kato theorem. For the second case, we again consider the case when $t \geq \tau$. Here we can apply the Trotter-Kato theorem again to $\|T(t-\tau)u_0 - T_N(t-\tau)\Pi_N u_0\|_{\mathcal{H}}^2$ to get the term converges to zero. When $t < \tau$, the second term becomes

$$|u_0^D(t-\tau) - [\Pi_N u_0]^D(t-\tau)|^2 \tag{3.13}$$

which converges to 0 uniformly by Proposition 2.2. This gives that $\epsilon_N(t) \to 0$.

To show the other convergence, note that $\epsilon_N(s)$ converges pointwisely to 0 on $[0,t]$. Furthermore, we may uniformly bound $\epsilon_N(s)$ by again observing the equality (3.12) and applying the uniform bounds on $\|T_N(\cdot)\|_{\mathcal{H}}$ and on $[\Pi_N u_0]^D$. Then by the Bounded Convergence Theorem, we have $\int_0^t \epsilon_N(s)\,\mathrm{d}s \to 0$. ∎

Proposition 3.2. *For fixed $t \in [0,T]$,*

$$\int_0^t d_N(t,s)\,\mathrm{d}s \to 0 \text{ and } \int_0^t \int_0^s d_N(s,r)\,\mathrm{d}r\,\mathrm{d}s \to 0, \tag{3.14}$$

as $N \to \infty$.

Proof. We can again apply the definition of $\|\cdot\|_X$ to get that

$$\begin{aligned} d_N^2(t,s) = \|&\left(T(t-s) - T_N(t-s)\Pi_N\right)\mathcal{F}(u(s))\|_{\mathcal{H}}^2 \\ &+ |[T(t-s)\mathcal{F}(u(s))]^D(-\tau) - [T_N(t-s)\Pi_N \mathcal{F}(u(s))]^D(-\tau)|^2. \end{aligned} \tag{3.15}$$

For fixed t and s, the first term of the right-hand side converges to zero. For $t-s \geq \tau$ the second term will similarly converge to 0. For $t-s < \tau$, the second term will become

$$|0 - [\Pi_N \mathcal{F}(u(s))]^D(t-s-\tau)| = |F([u(s)]^D(-\tau))| \cdot |S_N(t-s-\tau)|\,, \tag{3.16}$$

which converges a.e. to 0 by (2.26). So for fixed t, $d_N(t,s)$ converges a.e. to 0 for $s \in [0,t]$. Furthermore, we can uniformly bound $d_N(t,s)$ by (2.25). Thus by the Bounded Convergence Theorem, we have $\int_0^t d_N(t,s)\,\mathrm{d}s \to 0$ as $N \to \infty$.

The second convergence follows by the observations that $\int_0^{\cdot} d_N(\cdot,r)\,\mathrm{d}r$ converges pointwise to 0 by our earlier work and can uniformly bounded on $[0,t]$. This allows us to apply the Bounded Convergence Theorem to get that $\int_0^t \int_0^s d_N(s,r)\,\mathrm{d}r\,\mathrm{d}s \to 0$ as $N \to \infty$. ∎

We may now state our result.

Theorem 3.3. *For $t \in [0,T]$,*

$$\lim_{N\to\infty} \|u(t) - u_N(t)\|_X = 0. \tag{3.17}$$

Proof. Apply propositions (3.1) and (3.2) to the inequality in (3.10). ∎

3.2. Uniform Convergence.

Lemma 3.3. *The following convergences hold:*

$$\lim_{N\to\infty}\int_0^T \left|[u_N(s)]^D(-\tau)-[u(s)]^D(-\tau)\right|^2 \,\mathrm{d}s = 0, \tag{3.18}$$

and

$$\lim_{N\to\infty}\int_0^T \|\mathcal{F}(u_N(s))-\mathcal{F}(u(s)\|_{\mathcal{H}}\,\mathrm{d}s = 0. \tag{3.19}$$

Proof. Note that

$$\int_0^T \left|[u_N(s)]^D(-\tau)-[u(s)]^D(-\tau)\right|^2 \,\mathrm{d}s \le \sum_{k=0}^{m}\int_{-\tau}^{0}\left|[u_N(k\tau)]^D(\theta)-[u(k\tau)]^D(\theta)\right|^2 \,\mathrm{d}\theta, \tag{3.20}$$

for m such that $T-\tau\le m\tau<T$. In other words,

$$\left\|[u_N(\cdot)]^D(-\tau)-[u(\cdot)]^D(-\tau)\right\|^2_{L^2([0,T];\mathbb{R})} \le \sum_{k=0}^{m}\left\|[u_N(k\tau)]^D-[u(k\tau)]^D\right\|^2_{L^2([-\tau,0);\mathbb{R})}. \tag{3.21}$$

It is a simple corollary of Theorem 3.3 that $\|[u_N(t)]^D-[u(t)]^D]\|_{L^2([0,T];\mathbb{R})}\to 0$ as $N\to\infty$ for any $t\in[0,T]$. This gives that the right side of (3.21) converges to 0 as $N\to\infty$, and thus the left side of (3.21) also converges to 0 as $N\to\infty$. This proves (3.18).

To prove the other convergence, note that

$$\begin{aligned}\int_0^T \|\mathcal{F}(u_N(s))-\mathcal{F}(u(s))\|_{\mathcal{H}}\,\mathrm{d}s &= \int_0^T \left|F\left([u_N(s)]^D(-\tau)\right)-F\left([u(s)]^D(-\tau)\right)\right|\,\mathrm{d}s\\ &\le \mathrm{Lip}(F)\int_0^T\left|[u_N(s)]^D(-\tau)-[u(s)]^D(-\tau)\right|\,\mathrm{d}s\\ &= \mathrm{Lip}(F)\|[u_N(\cdot)]^D(-\tau)-[u(\cdot)]^D(-\tau)\|_{L^1([0,T];\mathbb{R})}.\end{aligned} \tag{3.22}$$

Noting that $L^2([0,T];\mathbb{R})$ is continuously embedded in $L^1([0,T];\mathbb{R})$ and applying (3.18) proves that (3.19) holds. ∎

Theorem 3.4. *The sequence of functions* $\{u_N\}_{N=0}^{\infty}$*, where*

$$u_N:[0,T]\mapsto\mathcal{H},\qquad N\in\mathbb{N}_0, \tag{3.23}$$

is uniformly equicontinuous.

Proof. Suppose $t_0,t_1\in[0,T]$ and $t_0\le t_1$. Denote $\delta:=t_1-t_0$. Applying the variation-of-constants formula, we have that for $N \in \mathbb{N}_0$

$$
\begin{aligned}
\|u_N(t_0) - u_N(t_1)\|_{\mathcal{H}} \leq & \underbrace{\|(T_N(t_0) - T_N(t_0+\delta))\Pi_N u_0\|_{\mathcal{H}}}_{\mathrm{I}(\delta,N)} \\
& + \underbrace{\left\| \int_0^{t_0} [T_N(t_0 - s) - T_N(t_0 + \delta - s)]\Pi_N \mathcal{F}(u_N(s))\,\mathrm{d}s \right\|_{\mathcal{H}}}_{\mathrm{II}(\delta,N)} \\
& + \underbrace{\left\| \int_{t_0}^{t_0+\delta} T_N(t_0 + \delta - s)\Pi_N \mathcal{F}(u_N(s))\,\mathrm{d}s \right\|_{\mathcal{H}}}_{\mathrm{III}(\delta,N)} .
\end{aligned}
\tag{3.24}
$$

We show that for each of these terms, the dependence on δ and N can be separated.

I. We have that

$$
\begin{aligned}
\mathrm{I}(\delta, N) &= \|T_N(t_0)(I - T_N(\delta))\Pi_N u_0\|_{\mathcal{H}} \\
&\leq Me^{\omega t_0}\|(I - T_N(\delta))\Pi_N u_0\|_{\mathcal{H}} \\
&= Me^{\omega t_0}\|(\Pi_N - T_N(\delta)\Pi_N)u_0\|_{\mathcal{H}} \\
&\leq Me^{\omega t_0}\|(I - T_N(\delta)\Pi_N)u_0\|_{\mathcal{H}} \\
&\leq Me^{\omega T}\|(I - T_N(\delta)\Pi_N)u_0\|_{\mathcal{H}} \\
&\leq Me^{\omega T}\left[\|(I - T(\delta))u_0\|_{\mathcal{H}} + \|(T(\delta) - T_N(\delta)\Pi_N)u_0\|_{\mathcal{H}}\right] \\
&\leq Me^{\omega T}\left[\|(I - T(\delta))u_0\|_{\mathcal{H}} + \sup_{t\in[0,T]} \|(T(t) - T_N(t)\Pi_N)u_0\|_{\mathcal{H}}\right].
\end{aligned}
\tag{3.25}
$$

Now define the following functions:

$$
\mathrm{I}^*(\delta) := Me^{\omega T} \times \|(I - T(\delta))u_0\|_{\mathcal{H}} \tag{3.26}
$$

and

$$
\mathrm{I}^{**}(N) := Me^{\omega T} \times \sup_{t\in[0,T]} \|(T(t) - T_N(t)\Pi_N)u_0\|_{\mathcal{H}} \tag{3.27}
$$

Note that $\lim_{\delta\to 0^+} \mathrm{I}^*(\delta) = 0$ by the continuity of $T(t)$ and $\lim_{N\to\infty} \mathrm{I}^{**}(N) = 0$ by the Trotter-Kato theorem.

II. We have that

$$\begin{aligned}
\mathrm{II}(\delta, N) &\leq \int_0^{t_0} \|(T_N(t_0 - s) - T_N(t_0 + \delta - s))\Pi_N \mathcal{F}(u_N(s))\|_{\mathcal{H}} \,\mathrm{d}s \\
&\leq Me^{\omega T} \int_0^{t_0} \|(I - T_N(\delta)\Pi_N)\mathcal{F}(u_N(s))\|_{\mathcal{H}} \,\mathrm{d}s \\
&\leq Me^{\omega T} \Bigg[\underbrace{\int_0^{t_0} \|(I - T_N(\delta)\Pi_N)\mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s}_{A} \\
&\qquad + \underbrace{\int_0^{t_0} \|(I - T_N(\delta)\Pi_N)(\mathcal{F}(u_N(s)) - \mathcal{F}(u(s)))\|_{\mathcal{H}} \,\mathrm{d}s}_{B} \Bigg].
\end{aligned} \tag{3.28}$$

From here, we can note that

$$\begin{aligned}
A &\leq \int_0^T \|(I - T(\delta))\mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s + \int_0^T \|(T(\delta) - T_N(\delta))\mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s \\
&\leq \int_0^T \|(I - T(\delta))\mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s + \int_0^T \sup_{t\in[0,T]} \|(T(t) - T_N(t))\mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s,
\end{aligned} \tag{3.29}$$

where both of these terms can easily be shown to converge to zero as $\delta \to 0$ and $N \to \infty$, respectively. Namely, we can apply the Lebesgue Dominated Convergence Theorem. Also note that

$$B \leq (1 + Me^{\omega T}) \int_0^T \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s, \tag{3.30}$$

where the right-hand side converges to zero as $N \to \infty$ by (3.19). Now we set

$$\mathrm{II}^*(\delta) := Me^{\omega T} \int_0^T \|(I - T(\delta))\mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s \tag{3.31}$$

and

$$\begin{aligned}
\mathrm{II}^{**}(N) :=& Me^{\omega T} \Bigg[\int_0^T \sup_{t\in[0,T]} \|(T(t) - T_N(t))\mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s \\
&\qquad + \left(1 + Me^{\omega T}\right) \int_0^T \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s \Bigg].
\end{aligned} \tag{3.32}$$

III. We have that

$$
\begin{aligned}
\mathrm{III}(\delta, N) &\le \int_{t_0}^{t_0+\delta} \|T_N(t_0+\delta-s)\Pi_N \mathcal{F}(u_N(s))\|_{\mathcal{H}}\, \mathrm{d}s \\
&\le Me^{\omega T} \int_{t_0}^{t_0+\delta} \|\mathcal{F}(u_N(s))\|_{\mathcal{H}}\, \mathrm{d}s \\
&\le Me^{\omega T} \left[\int_{t_0}^{t_0+\delta} \|\mathcal{F}(u(s))\|_{\mathcal{H}}\, \mathrm{d}s + \int_{t_0}^{t_0+\delta} \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s))\|_{\mathcal{H}}\, \mathrm{d}s \right] \\
&\le Me^{\omega T} \left[\delta \times \sup_{t\in[0,T]} \|\mathcal{F}(u(t))\|_{\mathcal{H}} + \int_0^T \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s))\|_{\mathcal{H}}\, \mathrm{d}s \right].
\end{aligned} \tag{3.33}
$$

Note that $\sup_{t\in[0,T]} \|\mathcal{F}(u(t))\|_{\mathcal{H}}$ is finite since $\|\mathcal{F}(u(t))\|_{\mathcal{H}}$ is a continuous function. Now let

$$
\mathrm{III}^*(\delta) := Me^{\omega T} \delta \times \sup_{t\in[0,T]} \|\mathcal{F}(u(t))\|_{\mathcal{H}} \tag{3.34}
$$

and

$$
\mathrm{III}^{**}(N) := Me^{\omega T} \int_0^T \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s))\|_{\mathcal{H}}\, \mathrm{d}s. \tag{3.35}
$$

Clearly $\lim_{\delta\to 0^+} \mathrm{III}^*(\delta) = 0$. Also from (3.19) we have that $\lim_{N\to\infty} \mathrm{III}^{**}(N) = 0$. Thus,

$$
\begin{aligned}
\|u_N(t_0) - u_N(t_1)\|_{\mathcal{H}} &\le \mathrm{I}(\delta, N) + \mathrm{II}(\delta, N) + \mathrm{III}(\delta, N) \\
&\le [\mathrm{I}^*(\delta) + \mathrm{II}^*(\delta) + \mathrm{III}^*(\delta)] + [\mathrm{I}^{**}(N) + \mathrm{II}^{**}(N) + \mathrm{III}^{**}(N)].
\end{aligned} \tag{3.36}
$$

Let $\epsilon > 0$. We wish to choose $\delta > 0$ such that $\|u_n(t) - u_n(t')\|_{\mathcal{H}} < \epsilon$ for any $n \in \mathbb{N}_0$ and $t, t' \in [0, T]$ with $|t - t'| < \delta$. Choosing δ^* small enough so that $\mathrm{I}^*(\delta^*) + \mathrm{II}^*(\delta^*) + \mathrm{III}^*(\delta^*) < \epsilon/2$ and N large enough such that $\mathrm{I}^{**}(N) + \mathrm{II}^{**}(N) + \mathrm{III}^{**}(N) < \epsilon/2$, we get that

$$
\|u_n(t) - u_n(t')\|_{\mathcal{H}} < \epsilon, \tag{3.37}
$$

where $|t - t'| < \delta^*$ and $n \ge N$. For each $n \in \mathbb{N}_0$ that are less than N, we pick $\delta_n > 0$ such that $\|u_n(t) - u_n(t')\|_{\mathcal{H}} < \epsilon$ for $|t - t'| < \delta_n$. This is possible since u_n is uniformly continuous on $[0, T]$. Let $\delta = \min\{\delta^*, \delta_0, \dots, \delta_{N-1}\}$. Then δ satisfies the challenge from ϵ. This proves uniform equicontinuity. ∎

Theorem 3.5. *For $T > 0$, we have that*

$$
\lim_{N\to\infty} \sup_{t\in[0,T]} \|u_N(t) - u(t)\|_{\mathcal{H}} = 0. \tag{3.38}
$$

Proof. The above result follows directly from Theorem 3.3 and Theorem 3.4. ∎

4. Uniform Convergence of Galerkin Solutions: System of DDEs case.

4.1. Multidimensional Case with Single Linear Delay Term. One can show that a similar convergence result holds for Galerkin approximations of

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{x}(t-\tau_p) + \boldsymbol{F}\left(\boldsymbol{x}(t-\tau_1), \cdots, \boldsymbol{x}(t-\tau_p)\right), \tag{4.1}$$

where $\boldsymbol{x}(t)$ is a function from $[-\tau_p, \infty)$ to $\mathbb{R}^d$ and $\boldsymbol{F} : \mathbb{R}^{dp} \to \mathbb{R}^d$ is Lipschitz continuous. Instead of using $\mathcal{K}_n^\tau$, we use the d-dimensional versions $\mathbb{K}_n^\tau$ as introduced in [1, Section 3.3]. The results in Proposition 2.2, (2.25), and (2.26) hold in appropriate ways for $\mathbb{K}_n^\tau$ and can be proven using the one-dimensional case. Then the uniform convergence of the Galerkin approximations can be proven in an analogous way to Theorem 3.5. Namely, we introduce the inner product space

$$X_p := C^+([-\tau_p, 0); \mathbb{R}^d) \times \mathbb{R}^d \tag{4.2}$$

with the inner product

$$(\Phi, \Psi) := (\Phi^S, \Psi^S)_{\mathbb{R}^d} + \frac{1}{\tau}(\Phi^D, \Psi^D)_{L^2([-\tau,0);\mathbb{R}^d)} + \sum_{i=1}^{p}(\Phi^D(-\tau_i), \Psi^D(-\tau_i))_{\mathbb{R}^d}, \quad \Phi, \Psi \in X. \tag{4.3}$$

The above proofs can be edited to compensate for this new inner product. For instance, the line (3.2) would instead become

$$\begin{aligned}\|T_N(t-s)\Pi_N(\mathcal{F}(u(s)) - \mathcal{F}(u_N(s)))\|_X^2 &= \|T_N(t-s)\Pi_N(\mathcal{F}(u(s)) - \mathcal{F}(u_N(s)))\|_{\mathcal{H}}^2 \\ &\quad + \sum_{i=1}^{p} \left|[T_N(t-s)\Pi_N(\mathcal{F}(u(s)) - \mathcal{F}(u_N(s)))]^D(-\tau_i)\right|^2,\end{aligned} \tag{4.4}$$

where we bound each $\left|[T_N(t-s)\Pi_N(\mathcal{F}(u(s)) - \mathcal{F}(u_N(s)))]^D(-\tau_i)\right|$ in a similar way to how the single delay term was bounded.

4.2. One Dimensional Case with Multiple Linear Delay Terms. We now consider the following one-dimensional DDE given by

$$\begin{aligned}&\frac{\mathrm{d}x(t)}{\mathrm{d}t} = ax(t) + \sum_{i=1}^{p} b_i x(t-\tau_i) + F(x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_p)), \quad t > 0 \\ &x(0) = \alpha, \\ &x(t) = \varphi(t), \quad t \in [-\tau_p, 0)\end{aligned} \tag{4.5}$$

where $x(t)$ is a function from $[-\tau_p, \infty)$ to $\mathbb{R}$ and $F : \mathbb{R}^p \to \mathbb{R}$ is Lipschitz continuous. We embed this into the following multidimensional problem:

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\begin{bmatrix} x_1(t-\tau_1) \\ x_2(t-\tau_2) \\ \vdots \\ x_p(t-\tau_p) \end{bmatrix} + \boldsymbol{F}(\boldsymbol{x}_t), & t > 0 \\
\boldsymbol{x}(0) &= [\alpha, \alpha, \cdots, \alpha]^T, \\
x_1(\theta) &= \varphi(\theta), & -\tau_1 \le \theta < 0 \\
x_2(\theta) &= \varphi(\theta), & -\tau_2 \le \theta < 0 \\
&\vdots \\
x_p(\theta) &= \varphi(\theta), & -\tau_p \le \theta < 0
\end{aligned} \tag{4.6}
$$

where $\boldsymbol{A} = a\boldsymbol{I}$ with $\boldsymbol{I}$ the identity matrix, $\boldsymbol{B} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T \begin{bmatrix} b_1 & \cdots & b_p \end{bmatrix}$,

$$
\boldsymbol{F}(\boldsymbol{x}_t) = \begin{bmatrix} F(x_1(t-\tau_1),\, x_2(t-\tau_2),\, \ldots,\, x_p(t-\tau_p)) \\ \vdots \\ F(x_1(t-\tau_1),\, x_2(t-\tau_2),\, \ldots,\, x_p(t-\tau_p)) \end{bmatrix}, \tag{4.7}
$$

and x_i the ith component of $\boldsymbol{x}$. Note that any x_i will satisfy the one-dimensional problem. Denote $\overline{\tau} = (\tau_1, \tau_2, \cdots, \tau_p)$. We may reference

$$
\begin{bmatrix} x_1(t-\tau_1) & x_2(t-\tau_2) & \cdots & x_p(t-\tau_p) \end{bmatrix}^T \tag{4.8}
$$

by an abuse of notation, $\boldsymbol{x}(t - \overline{\tau})$. We will prove results for the more general DDE given by

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{x}(t-\overline{\tau}), & t > 0 \\
\boldsymbol{x}(0) &= \boldsymbol{\gamma}, \\
x_1(\theta) &= f_1(\theta), & -\tau_1 \le \theta < 0 \\
x_2(\theta) &= f_2(\theta), & -\tau_2 \le \theta < 0 \\
&\vdots \\
x_p(\theta) &= f_p(\theta), & -\tau_p \le \theta < 0
\end{aligned} \tag{4.9}
$$

where $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{p\times p}$, $\boldsymbol{\gamma} \in \mathbb{R}^p$ and $f_i \in L^2([-\tau_i, 0); \mathbb{R})$. Namely, we shall show that we can approximate the solution of the above by Galerkin problems using vectorized Koornwinder polynomials. We can reformulate the above DDE to be in the form of an abstract Cauchy problem by defining the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{H}^{\overline{\tau}}$ by

$$
\mathcal{A}\begin{pmatrix} \Psi^D \\ \Psi^S \end{pmatrix} := \begin{pmatrix} \frac{d^+\Psi^D}{d\theta} \\ \boldsymbol{A}\Psi^S + \boldsymbol{B}\Psi^D(-\overline{\tau}) \end{pmatrix} \tag{4.10}
$$

with the domain of $\mathcal{A}$ given by

$$
\mathcal{D}(\mathcal{A}) = \{(\Psi^D, \Psi^S) \in \mathcal{H}^{\overline{\tau}} \,:\, \Psi_i^D \in H^1([-\tau_i, 0); \mathbb{R}),\ \lim_{\theta\to 0^-} \Psi_i^D(\theta) = \Psi_i^S,\ \text{ for } i = 1, \cdots, p\}. \tag{4.11}
$$

We also define the subspace

$$\mathcal{H}_N^{\overline{\tau}} := \operatorname{span}\{\mathbb{K}_1^{\overline{\tau}}, \cdots, \mathbb{K}_{Np}^{\overline{\tau}}\}, \tag{4.12}$$

i.e., it is the subspace of vectorized Koornwinder polynomials with degree less than or equal to N. To get the Np-dimensional Galerkin approximation, we define the following operator

$$\mathcal{A}_N := \Pi_N \mathcal{A} \Pi_N, \tag{4.13}$$

where

$$\Pi_N : \mathcal{H}^{\overline{\tau}} \to \mathcal{H}_N^{\overline{\tau}} \tag{4.14}$$

is the orthogonal projector into $\mathcal{H}_N^{\overline{\tau}}$. Note that $\mathbb{K}_j^{\tau} \in \mathcal{D}(\mathcal{A})$ for each $j \in \mathbb{N}$, so $\mathcal{H}_N^{\overline{\tau}} \subset \mathcal{D}(\mathcal{A})$ and the operator in (4.13) is well-defined. We can extend $e^{\mathcal{A}_N t}$ to a C_0-semigroup $T_N(t)$ on $\mathcal{H}$ as follows:

$$T_N(t)u = e^{\mathcal{A}_N t}\Pi_N u + (I - \Pi_N)u, \qquad u \in \mathcal{H}. \tag{4.15}$$

To apply the results given in [1, Thm. 4.1], we need to prove the necessary assumptions about $\mathcal{A}$ and $\mathcal{A}_N$. That is, $\mathcal{A}$ generates a C_0 semigroup $T(t)$ and

(A1) The following uniform bound is satisfied by $\{T_N(t)\}_{N\geq 0, t\geq 0}$

$$\|T_N(t)\| \leq Me^{\omega t}, \qquad N \geq 0, \qquad t \geq 0 \tag{4.16}$$

where $\|T_N(t)\| = \sup\{\|T_N(t)u\|_{\mathcal{H}^{\overline{\tau}}} \mid \|u\|_{\mathcal{H}^{\overline{\tau}}} = 1, u \in \mathcal{H}^{\overline{\tau}}\}$.

(A2) The following convergence holds:

$$\lim_{N\to\infty} \|\mathcal{A}_N u - \mathcal{A}u\|_{\mathcal{H}^{\overline{\tau}}} = u, \qquad \forall u \in \mathcal{H}^{\overline{\tau}}. \tag{4.17}$$

We first show that $\mathcal{A}$ is an infinitesimal generator of a C_0 semigroup. The proof will be similar to [2, Thm 2.4.6]. We have the following result by slightly altering the proof from [2, Thm 2.4.1]:

THEOREM 4.1. *Consider the DDE (4.9). For every $\boldsymbol{\gamma} \in \mathbb{R}^p$ and for any choices of $f_i \in L^2([-\tau_i, 0); \mathbb{R})$ for each $i = 1, 2, \cdots, p$, there exists a unique function $\boldsymbol{x}(\cdot)$ on $[0, \infty)$ that is absolutely continuous and satisfies (4.9) almost everywhere. This function is called the solution of (4.9), and it satisfies*

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{\gamma} + \int_0^t e^{\boldsymbol{A}(t-s)}\boldsymbol{B}\boldsymbol{x}(s - \overline{\tau})\,\mathrm{d}s \qquad \text{for } t \geq 0. \tag{4.18}$$

We shall show that the following holds:

LEMMA 4.1. *If $x(t)$ is the solution to (4.9), then the following inequalities hold:*

$$|\boldsymbol{x}(t)|^2 \leq C_t \left[|\boldsymbol{\gamma}|^2 + \sum_{i=1}^{p} \frac{1}{\tau_i} \|f_i\|^2_{L^2([-\tau_i,0);\mathbb{R})} \right] \tag{4.19}$$

and

$$\int_0^t |\boldsymbol{x}(s)|^2\,\mathrm{d}s \leq D_t \left[|\boldsymbol{\gamma}|^2 + \sum_{i=1}^{p} \frac{1}{\tau_i} \|f_i\|^2_{L^2([-\tau_i,0);\mathbb{R})} \right], \tag{4.20}$$

where C_t and D_t are constants that depend only on t.

Proof. We know that for $e^{\boldsymbol{A}t}$ there exists $M_0 > 0$ and $\omega > 0$ such that $|e^{\boldsymbol{A}t}| \leq M_0 e^{\omega t}$. Let

$$M := \max\{M_0, |\boldsymbol{B}|\}.$$

Then from (4.18) we have

$$\begin{aligned}
|\boldsymbol{x}(t)| &\leq |\boldsymbol{\gamma}||e^{\boldsymbol{A}t}| + \left| \int_0^t e^{\boldsymbol{A}t} \boldsymbol{B}\boldsymbol{x}(s-\overline{\tau})\,\mathrm{d}s \right| \\
&\leq M_0|\boldsymbol{\gamma}|e^{\omega t} + \int_0^t M_0 e^{\omega(t-s)} |\boldsymbol{B}||\boldsymbol{x}(s-\overline{\tau})|\,\mathrm{d}s \\
&\leq M|\boldsymbol{\gamma}|e^{\omega t} + M^2 \int_0^t e^{\omega(t-s)} |\boldsymbol{x}(s-\overline{\tau})|\,\mathrm{d}s \\
&\leq M|\boldsymbol{\gamma}|e^{\omega t} + M^2 \sum_{i=1}^{p} \int_0^t e^{\omega(t-s)} |x_i(s-\tau_i)|\,\mathrm{d}s \\
&= M|\boldsymbol{\gamma}|e^{\omega t} + M^2 e^{\omega t} \sum_{i=1}^{p} \int_{-\tau_i}^{t-\tau_i} e^{-\omega(\theta+\tau_i)} |x_i(\theta)|\,\mathrm{d}s.
\end{aligned} \tag{4.21}$$

We also have that for $i = 1, 2, \cdots, p$

$$\begin{aligned}
\int_{-\tau_i}^{t-\tau_i} e^{-\omega(\theta+\tau_i)} |x_i(\theta)|\,d\theta &\leq \int_{-\tau_i}^{0} e^{-\omega(\theta+\tau_i)} |x_i(\theta)|\,d\theta + \int_0^t e^{-|a|(\theta+\tau_i)} |x_i(\theta)|\,\mathrm{d}\theta \\
&= \int_{-\tau_i}^{0} e^{-\omega(\theta+\tau_i)} |f_i(\theta)|\,d\theta + \int_0^t e^{-|a|(\theta+\tau_i)} |x_i(\theta)|\,\mathrm{d}\theta \\
&\leq \int_{-\tau_i}^{0} |f_i(\theta)|\,d\theta + \int_0^t e^{-\omega(\theta+\tau_i)} |x_i(\theta)|\,\mathrm{d}\theta \\
&\leq \sqrt{\tau_i}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})} + \int_0^t e^{-\omega(\theta+\tau_i)} |x_i(\theta)|\,\mathrm{d}\theta \\
&\leq \sqrt{\tau_i}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})} + \int_0^t e^{-\omega\theta} |x_i(\theta)|\,\mathrm{d}\theta \\
&\leq \sqrt{\tau_i}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})} + \int_0^t e^{-\omega\theta} |x_i(\theta)|\,\mathrm{d}\theta.
\end{aligned} \tag{4.22}$$

From (4.21) and (4.22) we have

$$\begin{aligned}
|\boldsymbol{x}(t)| &\leq M|\boldsymbol{\gamma}|e^{\omega t} + M^2 e^{\omega t} \sum_{i=1}^{p} \left[\sqrt{\tau_i}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})} + \int_0^t e^{-\omega\theta} |\boldsymbol{x}(\theta)|\,\mathrm{d}\theta \right] \\
&\leq M|\boldsymbol{\gamma}|e^{\omega t} + M^2 e^{\omega t} \sum_{i=1}^{p} \left[\sqrt{\tau_i}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})} \right] + M^2 e^{\omega t} p \int_0^t e^{-\omega\theta} |\boldsymbol{x}(\theta)|\,\mathrm{d}\theta.
\end{aligned} \tag{4.23}$$

Set $\alpha = M|\boldsymbol{\gamma}| + M^2 \sum_{i=1}^{p} \left[\sqrt{\tau_i}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})} \right]$, $\beta = M^2 p$ and $g(t) = e^{-\omega t}|\boldsymbol{x}(t)|$, we get the following inequality

$$g(t) \leq \alpha + \beta \int_0^t g(\theta)\,\mathrm{d}\theta. \tag{4.24}$$

Applying the integral form of Grönwall's inequality yields

$$g(t) \leq \alpha e^{\beta t}. \tag{4.25}$$

Thus

$$\begin{aligned} |\boldsymbol{x}(t)| &\leq e^{(M^2 p+\omega)t}\left[M|\boldsymbol{\gamma}| + M^2 \sum_{i=1}^{p} \sqrt{\tau_i}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})}\right] \\ &\leq e^{(M^2 p+\omega)t}\left[M|\boldsymbol{\gamma}| + M^2 \tau_p \sum_{i=1}^{p} \frac{1}{\sqrt{\tau_i}}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})}\right] \\ &\leq e^{(M^2 p+\omega)t}\max\{M, M^2\tau_p\}\left[|\boldsymbol{\gamma}| + \sum_{i=1}^{p} \frac{1}{\sqrt{\tau_i}}\|f_i\|_{L^2([-\tau_i,0);\mathbb{R})}\right] \end{aligned} \tag{4.26}$$

and squaring both sides yields

$$|\boldsymbol{x}(t)|^2 \leq e^{2(M^2 p+\omega)t}(\max\{M, M^2\tau_p\})^2\left[|\boldsymbol{\gamma}|^2 + \sum_{i=1}^{p} \frac{1}{\tau_i}\|f_i\|^2_{L^2([-\tau_i,0);\mathbb{R})}\right]. \tag{4.27}$$

This gives the first inequality, and integrating gives the second inequality. ∎

The following theorems are proven similarly to [2, Thm 2.4.4] and [2, Thm 2.4.6].

Theorem 4.2. *Let the operator $T(t)$ be defined by*

$$T(t)\begin{pmatrix} f(\cdot) \\ \boldsymbol{\gamma} \end{pmatrix} := \begin{pmatrix} \boldsymbol{x}(t+\cdot) \\ \boldsymbol{x}(t) \end{pmatrix}, \tag{4.28}$$

where $\boldsymbol{x}(\cdot)$ is the solution to (4.9). Then $T(t)$ for $t \geq 0$ satisfies:

i. $T(t) \in \mathcal{L}(\mathcal{H}^{\overline{\tau}})$ for all $t \geq 0$;
ii. $T(t)$ is a C_0-semigroup on $\mathcal{H}^{\overline{\tau}}$.

Theorem 4.3. *Consider the C_0-semigroup defined by (4.28). Its infinitesimal generator is given by (4.10) with domain (4.11).*

The assumption given by **(A2)** is proven nearly identically to [1, Lem. 4.1]. The assumption **(A3)** requires more details but follows a similar argument to those given in [1, Lem. 4.2, Lem. 4.3].

Proposition 4.4. *Let $\mathcal{A}$ be defined such as in (4.10). Then*

$$\langle \mathcal{A}\Psi, \Psi\rangle_{\mathcal{H}} \leq \omega\|\Psi\|^2_{\mathcal{H}}, \quad \forall \Psi \in \mathcal{D}(\mathcal{A}), \tag{4.29}$$

with

$$\omega = \left(\frac{1}{2\tau_1} + |\boldsymbol{A}| + \frac{\tau_p}{2}|\boldsymbol{B}|^2\right). \tag{4.30}$$

Proof. Let $\Psi \in \mathcal{D}(\mathcal{A})$. By the definition of $\mathcal{A}$, we have

$$\langle \mathcal{A}\Psi, \Psi\rangle_{\mathcal{H}} = \underbrace{\sum_{i=1}^{p} \frac{1}{\tau_i}\int_{-\tau_i}^{0} \frac{\mathrm{d}^+\Psi_i^D}{\mathrm{d}\theta}(\theta)\Psi_i^D(\theta)\,\mathrm{d}\theta}_{(1)} + \underbrace{\langle \boldsymbol{A}\Psi^S, \Psi^S\rangle}_{(2)} + \underbrace{\langle \boldsymbol{B}\Psi^D(-\tau), \Psi^S\rangle}_{(3)}. \tag{4.31}$$

(1) For $i = 1, 2, \ldots, p$, we have

$$\begin{aligned}\frac{1}{\tau_i}\int_{-\tau_i}^{0}\frac{\mathrm{d}^+\Psi_i^D}{\mathrm{d}\theta}(\theta)\Psi_i^D(\theta)\,\mathrm{d}\theta &= \frac{1}{2\tau_i}\left((\Psi_i^S)^2 - (\Psi_i^D(-\tau_i))^2\right) \\ &\le \frac{1}{2\tau_1}(\Psi_i^S)^2 - \frac{1}{2\tau_p}(\Psi_i^D(-\tau_i))^2.\end{aligned} \tag{4.32}$$

So

$$\sum_{i=1}^{p}\frac{1}{\tau_i}\int_{-\tau_i}^{0}\frac{\mathrm{d}^+\Psi_i^D}{\mathrm{d}\theta}(\theta)\Psi_i^D(\theta)\,\mathrm{d}\theta \le \frac{1}{\tau_1}|\Psi^S|^2 - \frac{1}{2\tau_p}|\Psi^D(-\tau)|^2. \tag{4.33}$$

(2) We have

$$\langle \boldsymbol{A}\Psi^S, \Psi^S\rangle \le |\boldsymbol{A}||\Psi^S|^2. \tag{4.34}$$

(3) We have

$$\begin{aligned}\langle \boldsymbol{B}\Psi^D(-\tau), \Psi^S\rangle &\le |\boldsymbol{B}||\Psi^D(-\tau)||\Psi^S| \\ &= \left(\frac{1}{\sqrt{\tau_p}}|\Psi^D(-\tau)|\right)\left(\sqrt{\tau_p}|\boldsymbol{B}||\Psi^S|\right) \\ &\le \frac{|\Psi^D(-\tau)|^2}{2\tau_p} + \frac{\tau_p|\boldsymbol{B}|^2|\Psi^S|^2}{2}.\end{aligned} \tag{4.35}$$

Thus from (1), (2), and (3), we have that

$$\begin{aligned}\langle \mathcal{A}\Psi, \Psi\rangle_{\mathcal{H}} &= \left(\frac{1}{2\tau_1} + |\boldsymbol{A}| + \frac{\tau_p}{2}|\boldsymbol{B}|^2\right)|\Psi^S|^2 \\ &\le \left(\frac{1}{2\tau_1} + |\boldsymbol{A}| + \frac{\tau_p}{2}|\boldsymbol{B}|^2\right)\|\Psi\|_{\mathcal{H}}^2,\end{aligned} \tag{4.36}$$

as desired. ∎

We now prove the following statement:

Proposition 4.5. *Let $\mathcal{A}$ be defined as in (4.10). Then, the linear semigroups $T(t)$ and $T_N(t)$ generated respectively by $\mathcal{A}$ and $\mathcal{A}_N$ defined in (4.13), satisfy*

$$\|T(t)\| \le e^{\omega t} \quad \text{and} \quad \|T_N(t)\| \le e^{\omega t}, \qquad t \ge 0, \tag{4.37}$$

with ω given by (4.30).

Proof. We have that $T(t)$ is a C_0-semigroup with infinitesimal generator $\mathcal{A}$. By [4, Thm. 2.4 c) p.5], it follows that $T(t)u_0 \in \mathcal{D}(\mathcal{A})$ for all $u_0 \in \mathcal{D}(\mathcal{A})$ and that

$$\frac{\mathrm{d}}{\mathrm{d}t}T(t)u_0 = \mathcal{A}T(t)u_0, \qquad \forall\, u_0 \in \mathcal{A},\ t \ge 0. \tag{4.38}$$

Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|T(t)u_0\|_{\mathcal{H}}^2 &= 2\langle \mathcal{A}T(t)u_0, T(t)u_0\rangle_{\mathcal{H}} \\ &\le 2\omega\|T(t)u_0\|_{\mathcal{H}}^2,\end{aligned} \tag{4.39}$$

for any $u_0 \in \mathcal{D}(\mathcal{A})$. Applying Gronwall's inequality and taking the square root of both sides gives

$$\|T(t)u_0\|_{\mathcal{H}} \leq e^{\omega t}\|u_0\|_{\mathcal{H}}, \tag{4.40}$$

for $u_0 \in \mathcal{D}(\mathcal{A})$. For $x \in \mathcal{H}$, since $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$ we can pick $\{u_n\}_{n=1}^{\infty} \subset \mathcal{D}(\mathcal{A})$ where $u_n \to x$ in $\mathcal{H}$. Thus

$$\begin{aligned} \|T(t)x\|_{\mathcal{H}} &= \|T(t)(x - u_n + u_n)\|_{\mathcal{H}} \\ &\leq \|T(t)\|_{\mathcal{H}} \cdot \|x - u_n\|_{\mathcal{H}} + e^{\omega t}\|u_n\|, \end{aligned} \tag{4.41}$$

where the first term on the right goes to 0 and the second term on the right goes to $e^{\omega t}\|x\|_{\mathcal{H}}$ as $n \to \infty$. Thus the inequality holds for all $x \in \mathcal{H}$ and

$$\|T(t)\|_{\mathcal{H}} \leq e^{\omega t}, \qquad t \geq 0. \tag{4.42}$$

For the estimate on T_N, we first note that for $u_0 \in \mathcal{H}$

$$\begin{aligned} \|T_N(t)u_0\|_{\mathcal{H}}^2 &= \left\langle e^{\mathcal{A}_N t}\Pi_N u_0 + (I - \Pi_N)u_0, e^{\mathcal{A}_N t}\Pi_N u_0 + (I - \Pi_N)u_0 \right\rangle_{\mathcal{H}} \\ &= \left\langle e^{\mathcal{A}_N t}\Pi_N u_0, e^{\mathcal{A}_N t}\Pi_N u_0 \right\rangle_{\mathcal{H}} + \left\langle (I - \Pi_N)u_0, (I - \Pi_N)u_0 \right\rangle_{\mathcal{H}} \\ &= \|e^{\mathcal{A}_N t}\Pi_N u_0\|_{\mathcal{H}}^2 + \|(I - \Pi_N)u_0\|_{\mathcal{H}}^2. \end{aligned} \tag{4.43}$$

Also note for $\varphi, \psi \in \mathcal{H}$ that

$$\begin{aligned} \langle \Pi_N \varphi, \psi \rangle_{\mathcal{H}} &= \langle \varphi, \Pi_N \psi \rangle_{\mathcal{H}} - \langle (I - \Pi_N)\varphi, \Pi_N \psi \rangle_{\mathcal{H}} + \langle \Pi_N \varphi, (I - \Pi_N)\psi \rangle_{\mathcal{H}} \\ &= \langle \varphi, \Pi_N \psi \rangle_{\mathcal{H}}, \end{aligned} \tag{4.44}$$

where $(I - \Pi_N)\varphi$ and $(I - \Pi_N)\psi$ are orthogonal to the space $\mathcal{H}_N$ and the inner products $\langle (I - \Pi_N)\varphi, \Pi_N \psi \rangle_{\mathcal{H}}$ and $\langle \Pi_N \varphi, (I - \Pi_N)\psi \rangle_{\mathcal{H}}$ evaluate to 0. We can thus justify moving Π_N between terms in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Therefore

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|T_N(t)u_0\|_{\mathcal{H}}^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\|e^{\mathcal{A}_N t}\Pi_N u_0\|_{\mathcal{H}}^2 \\ &= 2\langle \mathcal{A}_N e^{\mathcal{A}_N t}\Pi_N u_0, e^{\mathcal{A}_N t}\Pi_N u_0 \rangle_{\mathcal{H}} \\ &= 2\langle \Pi_N \mathcal{A} \Pi_N e^{\mathcal{A}_N t}\Pi_N u_0, e^{\mathcal{A}_N t}\Pi_N u_0 \rangle_{\mathcal{H}} \\ &= 2\langle \mathcal{A} \Pi_N e^{\mathcal{A}_N t}\Pi_N u_0, \Pi_N e^{\mathcal{A}_N t}\Pi_N u_0 \rangle_{\mathcal{H}} \\ &\leq 2\omega \|e^{\mathcal{A}_N t}\Pi_N u_0\|_{\mathcal{H}}^2 \\ &\leq 2\omega \left(\|e^{\mathcal{A}_N t}\Pi_N u_0\|_{\mathcal{H}}^2 + \|(I - \Pi_N)u_0\|_{\mathcal{H}}^2 \right) \\ &= \|T_N(t)u_0\|_{\mathcal{H}}^2 \end{aligned} \tag{4.45}$$

Again applying Gronwall's inequality and taking the square root of each side gives

$$\|T(t)u_0\|_{\mathcal{H}} \leq e^{\omega t}\|u_0\|_{\mathcal{H}}, \qquad u_0 \in \mathcal{H}, \tag{4.46}$$

which implies the desired inequality. ∎

Thus we have **(A2)**. Then the solution of ...

4.3. Multidimensional Case with Multiple Linear Delay Terms. Appendix A. Proofs of preparatory Lemmas.

Proof of Proposition 2.2. We define for $f \in L^2([-1,1],\mathbb{R})$ the following:

$$a_n(f) := \frac{2n+1}{2}\int_{-1}^{1} f(x)L_n(x)\,\mathrm{d}x. \tag{A.1}$$

It is easy to show based on (2.16) we have for $\theta \in [-1,1]$

$$\begin{aligned} |K_n(\theta)| &\leq (n^2+1)|L_n(\theta)| + \sum_{j=0}^{n-1}(2j+1)|L_j(\theta)| \\ &\leq (n^2+1) + \sum_{j=0}^{n-1}(2j+1) \\ &= 2n^2+1, \end{aligned} \tag{A.2}$$

i.e., $\|K_n\|_\infty \leq 2n^2+1$.

By the definition of $\langle \cdot,\cdot\rangle_{\mathcal{E}}$ and the Koornwinder polynomials, we have that for $n \in \mathbb{N}_0$

$$\begin{aligned} \langle \psi, \mathcal{K}_n\rangle_{\mathcal{E}} &= \frac{1}{2}\int_{-1}^{1} f(x)K_n(x)\,\mathrm{d}x + f(1) \\ &= \frac{1}{2}\left[-\int_{-1}^{1} f(x)(1+x)L_n'(x)\,\mathrm{d}x + (n^2+n+1)\int_{-1}^{1} f(x)L_n(x)\,\mathrm{d}x\right] + f(1). \end{aligned} \tag{A.3}$$

If we use integration by parts, we find that

$$-\int_{-1}^{1} f(x)(1+x)L_n'(x)\,\mathrm{d}x = -2f(1) + \int_{-1}^{1} f'(x)(1+x)L_n(x)\,\mathrm{d}x + \int_{-1}^{1} f(x)L_n(x)\,\mathrm{d}x. \tag{A.4}$$

Applying (A.4) to (A.3) gives that

$$\begin{aligned} \langle \psi, \mathcal{K}_n\rangle_{\mathcal{E}} &= \frac{1}{2}\int_{-1}^{1} f'(x)(1+x)L_n(x)\,\mathrm{d}x + \frac{n^2+n+2}{2}\int_{-1}^{1} f(x)L_n(x)\,\mathrm{d}x \\ &= \frac{1}{2}\int_{-1}^{1} f'(x)(1+x)L_n(x)\,\mathrm{d}x + \frac{n^2+n+2}{2n+1}a_n(f). \end{aligned} \tag{A.5}$$

We can also note that by applying the Hölder inequality we get

$$\begin{aligned} \left|\int_{-1}^{1} f'(x)(1+x)L_n(x)\,\mathrm{d}x\right| &\leq \|f'\|_\infty \left(\int_{-1}^{1}(1+x)\,\mathrm{d}x\right)^{1/2} \|L_n\|_{L^2} \\ &= \frac{4\|f'\|_\infty}{\sqrt{6n+3}}. \end{aligned} \tag{A.6}$$

Furthermore, from [6, Thm. 2.1] we have

$$|a_n(f)| \leq \frac{V_1}{n-\frac{1}{2}}\sqrt{\frac{\pi}{2n}}, \tag{A.7}$$

where $V_1 := \int_{-1}^{1} \frac{f''(x)}{\sqrt{1-x^2}}\,\mathrm{d}x < \infty$. Thus,

$$|\langle \psi, \mathcal{K}_n \rangle_{\mathcal{H}}| \leq \frac{2\|f'\|_\infty}{\sqrt{6n+3}} + V_1\sqrt{2\pi}\frac{n^2+n+2}{\sqrt{n}(4n^2+1)}, \tag{A.8}$$

and so

$$\begin{aligned} \frac{|\langle \psi, \mathcal{K}_n \rangle_{\mathcal{H}}|}{\|\mathcal{K}_n\|_{\mathcal{H}}^2}\|K_n\|_\infty &\leq \left[\frac{2\|f'\|_\infty}{\sqrt{6n+3}} + V_1\sqrt{2\pi}\frac{n^2+n+2}{\sqrt{n}(4n^2+1)}\right] \times \left[\frac{(2n+1)(2n^2+1)}{(n^2+1)((n+1)^2+1)}\right] \\ &= O\left(\frac{1}{n^{3/2}}\right). \end{aligned} \tag{A.9}$$

By the Weierstrass M-test, the series (2.22) converges uniformly.

Note also that (2.22) is simply the functional part of the Koornwinder expansion of ψ in $\mathcal{H}$. So the series converges in $L^2([-1,1];\mathbb{R})$ to $\psi^D = f$. Therefore, since the series converges uniformly, it must converge uniformly to f. ◻

Proof of Lemma 2.1. Using (2.16), we can show that for $m \leq N \in \mathbb{N}_0$

$$\begin{aligned} \int_{-1}^{1} S_N(x)L_m(x)\,\mathrm{d}x &= \sum_{n=0}^{N} \frac{1}{\|\mathcal{K}_n\|_{\mathcal{E}}^2}\int_{-1}^{1} K_n(x)L_m(x)\,\mathrm{d}x \\ &= \|L_m\|_{L^2([-1,1])}^2\left[(m^2+1)\frac{1}{\|\mathcal{K}_m\|_{\mathcal{E}}^2} - (2m+1)\sum_{k=m+1}^{N}\frac{1}{\|\mathcal{K}_k\|_{\mathcal{E}}^2}\right], \end{aligned} \tag{A.10}$$

and so

$$S_N(x) = \sum_{n=0}^{N}\left[\frac{n^2+1}{\|\mathcal{K}_n\|_{\mathcal{E}}^2} - (2n+1)\sum_{m=n+1}^{N}\frac{1}{\|\mathcal{K}_m\|_{\mathcal{E}}^2}\right]L_n(x). \tag{A.11}$$

It is easy to show that

$$\begin{aligned} \sum_{n=0}^{N}\frac{1}{\|\mathcal{K}_n\|_{\mathcal{E}}^2} &= \sum_{n=0}^{N}\frac{2n+1}{(n^2+1)((n+1)^2+1)} \\ &= \sum_{n=0}^{N}\left[\frac{1}{n^2+1} - \frac{1}{(n+1)^2+1}\right] \\ &= 1 - \frac{1}{(N+1)^2+1} \end{aligned} \tag{A.12}$$

and

$$\sum_{m=n+1}^{N}\frac{1}{\|\mathcal{K}_m\|_{\mathcal{E}}^2} = \sum_{m=0}^{N}\frac{1}{\|\mathcal{K}_m\|_{\mathcal{E}}^2} - \sum_{m=0}^{n}\frac{1}{\|\mathcal{K}_m\|_{\mathcal{E}}^2} = \frac{1}{(n+1)^2+1} - \frac{1}{(N+1)^2+1}. \tag{A.13}$$

Applying (A.13) to (A.11) gives

$$
\begin{aligned}
S_N(x) &= \sum_{n=0}^{N} \left[\frac{n^2+1}{\|\mathcal{K}_n\|_{\mathcal{E}}^2} - (2n+1)\left(\frac{1}{(n+1)^2+1} - \frac{1}{(N+1)^2+1} \right) \right] L_n(x) \\
&= \sum_{n=0}^{N} \left[\frac{2n+1}{(n+1)^2+1} - \frac{2n+1}{(n+1)^2+1} + \frac{2n+1}{(N+1)^2+1} \right] L_n(x) \qquad \text{(A.14)} \\
&= \sum_{n=0}^{N} \frac{2n+1}{(N+1)^2+1} L_n(x).
\end{aligned}
$$

It is known that

$$
|L_n(x)| \leq 1, \quad \forall x \in [-1,1],\ \forall n \in \mathbb{N}_0. \qquad \text{(A.15)}
$$

Thus for $x \in [-1,1]$ and $N \in \mathbb{N}_0$

$$
\begin{aligned}
|S_N(x)| &\leq \frac{1}{(N+1)^2+1} \sum_{n=0}^{N} (2n+1)|L_n(x)| \\
&\leq \frac{1}{(N+1)^2+1} \sum_{n=0}^{N} (2n+1) \qquad \text{(A.16)} \\
&= \frac{N^2+1}{(N+1)^2+1} \\
&< 1.
\end{aligned}
$$

From [5, Thm. 61], we also have that for $n \geq 1$ and $x \in (-1,1)$

$$
|L_n(x)| < \sqrt{\frac{\pi}{2n(1-x^2)}}. \qquad \text{(A.17)}
$$

Then for $x \in (-1,1)$ and $N \in \mathbb{N}_0$

$$
\begin{aligned}
|S_N(x)| &\leq \frac{1}{(N+1)^2+1} \left[1 + \sum_{n=1}^{N} (2n+1)|L_n(x)| \right] \\
&\leq \frac{1}{(N+1)^2+1} \left[1 + 3 \sum_{n=1}^{N} n \cdot \sqrt{\frac{\pi}{2n(1-x^2)}} \right] \qquad \text{(A.18)} \\
&= \frac{1}{(N+1)^2+1} \left[1 + 3 \cdot \sqrt{\frac{\pi}{2(1-x^2)}} \cdot \sum_{n=1}^{N} \sqrt{n} \right].
\end{aligned}
$$

We can note that

$$
\sum_{n=1}^{N} \sqrt{n} \leq \int_1^{N+1} \sqrt{x}\, \mathrm{d}x = \frac{2}{3}(N+1)^{3/2} - \frac{2}{3}. \qquad \text{(A.19)}
$$

So

$$|S_N(x)| \leq \frac{1}{(N+1)^2+1}\left[1+\sqrt{\frac{2\pi}{1-x^2}}\Big((N+1)^{3/2}-1\Big)\right], \tag{A.20}$$

where the right-hand side converges to 0 as $N \to \infty$ for fixed $x \in (-1,1)$. Thus $S_N(x) \to 0$ as $N \to \infty$ for $x \in (-1,1)$. ∎

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