# GALERKIN APPROXIMATIONS OF DELAY DIFFERENTIAL EQUATIONS

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**Abstract.**

## 1. Introduction.

## 2. Preliminaries.

**2.1. The type of DDE.** We are interested in approximating the solution to the following DDE:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= ax(t) + bx(t-\tau) + F(x(t-\tau)), \quad && t > 0 \\ x(t) &= \varphi(t), && t \in [-\tau, 0] \end{aligned} \tag{2.1}$$

for $\varphi \in L^2([-\tau, 0]; \mathbb{R})$ and where $F : \mathbb{R} \to \mathbb{R}$ is Lipschitz with constant $L$. It is appropriate to formulate this problem into the space $\mathcal{H} := L^2([-\tau, 0); \mathbb{R}) \times \mathbb{R}$.

**2.2. Properties and Basic Results of Koornwinder Polynomials.** From [2, Eq. (2.1)], the sequence of Koornwinder polynomials $\{K_n\}$ can be built from the Legendre polynomials $L_n$ by

$$K_n(s) := -(1+s)\frac{d}{\mathrm{d}s}L_n(s) + (n^2+n+1)L_n(s), \quad s \in [-1,1], \ n \in \mathbb{N}_0. \tag{2.2}$$

Furthermore, we reproduce from [1, Prop. 3.1] some simple properties that $\{K_n\}$ satisfy.

Proposition 2.1. *The polynomial $K_n$ defined in (2.2) is of degree $n$ and admits the following expansion in terms of the Legendre polynomials:*

$$K_n(s) = -\sum_{j=0}^{n-1}(2j+1)L_j(s) + (n^2+1)L_n(s), \qquad n \in \mathbb{N}_0; \tag{2.3}$$

*and the following normalization property holds:*

$$K_n(1) = 1, \qquad n \in \mathbb{N}_0. \tag{2.4}$$

*Moreover, the sequence given by*

$$\{\mathcal{K}_n := (K_n, K_n(1)) : n \in \mathbb{N}_0\} \tag{2.5}$$

*forms an orthogonal basis of the product space*

$$\mathcal{E} := L^2([-1,1); \mathbb{R}) \times \mathbb{R}, \tag{2.6}$$

*where $\mathcal{E}$ is endowed with the following inner product:*

$$\langle (f,a),(g,b)\rangle_{\mathcal{E}} = \frac{1}{2}\int_{-1}^{1} f(s)g(s)\,\mathrm{d}s + ab, \quad (f,a),(g,b) \in \mathcal{E}. \tag{2.7}$$

*Moreover* $\left\{\frac{\mathcal{K}_n}{\|\mathcal{K}_n\|_{\mathcal{E}}}\right\}$ *forms a Hilbert basis of* $\mathcal{E}$ *where the norm* $\|\mathcal{K}_n\|_{\mathcal{E}}$ *of* $\mathcal{K}_n$ *induced by* $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ *possesses the following analytic expression:*

$$\|\mathcal{K}_n\|_{\mathcal{E}} = \sqrt{\frac{(n^2+1)((n+1)^2+1)}{2n+1}}, \qquad n \in \mathbb{N}_0. \tag{2.8}$$

Suppose that $\Pi_N$ is the $N$-dimensional standard projection into $\operatorname{span}\{\mathcal{K}_n : n \leq N\} \subset \mathcal{E}$. It will be relevant to discuss when we have convergence of $[\Pi_N u]^D$ for $u \in \mathcal{E}$. In particular, we will focus on uniform convergence. We define for $f \in L^2([-1,1],\mathbb{R})$ the following:

$$a_n(f) := \frac{2n+1}{2}\int_{-1}^{1} f(x)L_n(x)\,\mathrm{d}x. \tag{2.9}$$

PROPOSITION 2.2. *Let* $f \in \mathcal{C}^2([-1,1];\mathbb{R})$ *and denote* $\psi = (f, f(0)) \in \mathcal{E}$. *Then the series*

$$[\Pi_N\psi]^D = \sum_{n=0}^{N} \frac{\langle\psi,\mathcal{K}_n\rangle_{\mathcal{E}}}{\|\mathcal{K}_n\|_{\mathcal{E}}^2} K_n \tag{2.10}$$

*converges uniformly to* $f$.

*Proof.* It is easy to show based on (2.3) we have for $\theta \in [-1,1]$

$$\begin{aligned} |K_n(\theta)| &\leq (n^2+1)|L_n(\theta)| + \sum_{j=0}^{n-1}(2j+1)|L_j(\theta)| \\ &\leq (n^2+1) + \sum_{j=0}^{n-1}(2j+1) \\ &= 2n^2+1, \end{aligned} \tag{2.11}$$

i.e., $\|K_n\|_\infty \leq 2n^2+1$.

By the definition of $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ and the Koornwinder polynomials, we have that for $n \in \mathbb{N}_0$

$$\begin{aligned} \langle\psi,\mathcal{K}_n\rangle_{\mathcal{E}} &= \frac{1}{2}\int_{-1}^{1} f(x)K_n(x)\,\mathrm{d}x + f(1) \\ &= \frac{1}{2}\left[-\int_{-1}^{1} f(x)(1+x)L_n'(x)\,\mathrm{d}x + (n^2+n+1)\int_{-1}^{1} f(x)L_n(x)\,\mathrm{d}x\right] + f(1). \end{aligned} \tag{2.12}$$

If we use integration by parts, we find that

$$-\int_{-1}^{1} f(x)(1+x)L_n'(x)\,\mathrm{d}x = -2f(1) + \int_{-1}^{1} f'(x)(1+x)L_n(x)\,\mathrm{d}x + \int_{-1}^{1} f(x)L_n(x)\,\mathrm{d}x. \tag{2.13}$$

Applying (2.13) to (2.12) gives that

$$\begin{aligned} \langle\psi,\mathcal{K}_n\rangle_{\mathcal{E}} &= \frac{1}{2}\int_{-1}^{1} f'(x)(1+x)L_n(x)\,\mathrm{d}x + \frac{n^2+n+2}{2}\int_{-1}^{1} f(x)L_n(x)\,\mathrm{d}x \\ &= \frac{1}{2}\int_{-1}^{1} f'(x)(1+x)L_n(x)\,\mathrm{d}x + \frac{n^2+n+2}{2n+1}a_n(f). \end{aligned} \tag{2.14}$$

We can also note that by applying the Hoïder inequality we get

$$\left|\int_{-1}^{1} f'(x)(1+x)L_n(x)\,\mathrm{d}x\right| \le \|f'\|_\infty \left(\int_{-1}^{1}(1+x)\,\mathrm{d}x\right)^{1/2} \|L_n\|_{L^2} = \frac{4\|f'\|_\infty}{\sqrt{6n+3}}. \tag{2.15}$$

Furthermore, from [4, Thm. 2.1] we have

$$|a_n(f)| \le \frac{V_1}{n-\frac{1}{2}}\sqrt{\frac{\pi}{2n}}, \tag{2.16}$$

where $V_1 := \int_{-1}^{1} \frac{f''(x)}{\sqrt{1-x^2}}\,\mathrm{d}x < \infty$. Thus,

$$|\langle\psi,\mathcal{K}_n\rangle_{\mathcal{H}}| \le \frac{2\|f'\|_\infty}{\sqrt{6n+3}} + V_1\sqrt{2\pi}\frac{n^2+n+2}{\sqrt{n}(4n^2+1)}, \tag{2.17}$$

and so

$$\begin{aligned}\frac{|\langle\psi,\mathcal{K}_n\rangle_{\mathcal{H}}|}{\|\mathcal{K}_n\|_{\mathcal{H}}^2}\|K_n\|_\infty &\le \left[\frac{2\|f'\|_\infty}{\sqrt{6n+3}} + V_1\sqrt{2\pi}\frac{n^2+n+2}{\sqrt{n}(4n^2+1)}\right] \times \left[\frac{(2n+1)(2n^2+1)}{(n^2+1)((n+1)^2+1)}\right]\\ &= O\left(\frac{1}{n^{3/2}}\right).\end{aligned} \tag{2.18}$$

By the Weierstrass M-test, the series (2.10) converges uniformly.

Note also that (2.10) is simply the functional part of the Koornwinder expansion of $\psi$ in $\mathcal{H}$. So the series converges in $L^2([-1,1];\mathbb{R})$ to $\psi^D = f$. Therefore, since the series converges uniformly, it must converge uniformly to $f$. ∎

It will also be necessary to prove certain properties of the series of Koornwinder polynomials

$$S_N(x) := \sum_{n=0}^{N} \frac{K_n}{\|\mathcal{K}_n\|_{\mathcal{E}}^2}, \quad N \in \mathbb{N}_0,\ x \in [-1,1]. \tag{2.19}$$

If we were to denote $\psi = (0,1) \in L^2([-1,1)) \times \mathbb{R}$, then $S_N$ would simply be the functional part of $\Pi_N\psi$. The following lemma allows us to express $S_N$ in terms of Legendre Polynomials.

Lemma 2.1. *The functions $S_N$ defined in (2.19) can be expressed as*

$$S_N(x) = \frac{1}{(N+1)^2+1}\sum_{n=0}^{N}(2n+1)L_n, \quad x \in [-1,1]. \tag{2.20}$$

*Proof.* Using (2.3), we can show that for $m \le N \in \mathbb{N}_0$

$$\begin{aligned}\int_{-1}^{1} S_N(x)L_m(x)\,\mathrm{d}x &= \sum_{n=0}^{N}\frac{1}{\|\mathcal{K}_n\|_{\mathcal{E}}^2}\int_{-1}^{1}K_n(x)L_m(x)\,\mathrm{d}x\\ &= \|L_m\|_{L^2([-1,1])}^2\left[(m^2+1)\frac{1}{\|\mathcal{K}_m\|_{\mathcal{E}}^2} - (2m+1)\sum_{k=m+1}^{N}\frac{1}{\|\mathcal{K}_k\|_{\mathcal{E}}^2}\right],\end{aligned} \tag{2.21}$$

and so

$$S_N(x) = \sum_{n=0}^{N} \left[ \frac{n^2+1}{\|\mathcal{K}_n\|_{\mathcal{E}}^2} - (2n+1) \sum_{m=n+1}^{N} \frac{1}{\|\mathcal{K}_m\|_{\mathcal{E}}^2} \right] L_n(x). \tag{2.22}$$

It is easy to show that

$$\begin{aligned} \sum_{n=0}^{N} \frac{1}{\|\mathcal{K}_n\|_{\mathcal{E}}^2} &= \sum_{n=0}^{N} \frac{2n+1}{(n^2+1)((n+1)^2+1)} \\ &= \sum_{n=0}^{N} \left[ \frac{1}{n^2+1} - \frac{1}{(n+1)^2+1} \right] \\ &= 1 - \frac{1}{(N+1)^2+1} \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} \sum_{m=n+1}^{N} \frac{1}{\|\mathcal{K}_m\|_{\mathcal{E}}^2} &= \sum_{m=0}^{N} \frac{1}{\|\mathcal{K}_m\|_{\mathcal{E}}^2} - \sum_{m=0}^{n} \frac{1}{\|\mathcal{K}_m\|_{\mathcal{E}}^2} \\ &= \frac{1}{(n+1)^2+1} - \frac{1}{(N+1)^2+1}. \end{aligned} \tag{2.24}$$

Applying (2.24) to (2.22) gives

$$\begin{aligned} S_N(x) &= \sum_{n=0}^{N} \left[ \frac{n^2+1}{\|\mathcal{K}_n\|_{\mathcal{E}}^2} - (2n+1) \left( \frac{1}{(n+1)^2+1} - \frac{1}{(N+1)^2+1} \right) \right] L_n(x) \\ &= \sum_{n=0}^{N} \left[ \frac{2n+1}{(n+1)^2+1} - \frac{2n+1}{(n+1)^2+1} + \frac{2n+1}{(N+1)^2+1} \right] L_n(x) \\ &= \sum_{n=0}^{N} \frac{2n+1}{(N+1)^2+1} L_n(x). \end{aligned} \tag{2.25}$$

$\square$

Now that we have this expression, we can prove the properties of $S_N$ that will be useful when showing the main result.

Proposition 2.3. *For the functions $S_N$ defined in (2.19), we have that*

$$|S_N(x)| < 1, \quad \forall N \in \mathbb{N}_0, \ \forall x \in [-1,1]. \tag{2.26}$$

*Furthermore,*

$$\lim_{N\to\infty} S_N(x) = 0, \quad \forall x \in (-1,1). \tag{2.27}$$

*Proof.* It is known that

$$|L_n(x)| \le 1, \quad \forall x \in [-1,1], \ \forall n \in \mathbb{N}_0. \tag{2.28}$$

Thus for $x \in [-1,1]$ and $N \in \mathbb{N}_0$

$$
\begin{aligned}
|S_N(x)| &\le \frac{1}{(N+1)^2+1} \sum_{n=0}^{N} (2n+1)|L_n(x)| \\
&\le \frac{1}{(N+1)^2+1} \sum_{n=0}^{N} (2n+1) \\
&= \frac{N^2+1}{(N+1)^2+1} \\
&< 1.
\end{aligned} \tag{2.29}
$$

From [3, Thm. 61], we also have that for $n \ge 1$ and $x \in (-1,1)$

$$
|L_n(x)| < \sqrt{\frac{\pi}{2n(1-x^2)}}. \tag{2.30}
$$

Then for $x \in (-1,1)$ and $N \in \mathbb{N}_0$

$$
\begin{aligned}
|S_N(x)| &\le \frac{1}{(N+1)^2+1} \left[1 + \sum_{n=1}^{N} (2n+1)|L_n(x)|\right] \\
&\le \frac{1}{(N+1)^2+1} \left[1 + 3\sum_{n=1}^{N} n \cdot \sqrt{\frac{\pi}{2n(1-x^2)}}\right] \\
&= \frac{1}{(N+1)^2+1} \left[1 + 3 \cdot \sqrt{\frac{\pi}{2(1-x^2)}} \cdot \sum_{n=1}^{N} \sqrt{n}\right].
\end{aligned} \tag{2.31}
$$

We can note that

$$
\begin{aligned}
\sum_{n=1}^{N} \sqrt{n} &\le \int_1^{N+1} \sqrt{x}\, \mathrm{d}x \\
&= \frac{2}{3}(N+1)^{3/2} - \frac{2}{3}.
\end{aligned} \tag{2.32}
$$

So

$$
|S_N(x)| \le \frac{1}{(N+1)^2+1} \left[1 + \sqrt{\frac{2\pi}{1-x^2}} \left((N+1)^{3/2} - 1\right)\right], \tag{2.33}
$$

where the right-hand side converges to 0 as $N \to \infty$ for fixed $x \in (-1,1)$. Thus $S_N(x) \to 0$ as $N \to \infty$ for $x \in (-1,1)$. ∎

**2.3. The Space $X$.** We define the following inner product space with elements in

$$
X := \mathcal{C}([-\tau,0);\mathbb{R}) \times \mathbb{R} \tag{2.34}
$$

and the inner product defined by

$$
(\Phi,\Psi)_X := \Phi^S \Psi^S + \frac{1}{\tau}(\Phi^D,\Psi^D)_{L^2([-\tau,0)} + \Phi^D(-\tau)\Psi^D(-\tau), \quad \Phi,\Psi \in X. \tag{2.35}
$$

It is relatively straight-forward to verify that $(\cdot,\cdot)_X$ is symmetric, bilinear, and positive definite and thus is an inner product. We will also make use of the norm $\|\cdot\|_X$ induced from this inner product. Note that $X$ is **not** a Banach space since Cauchy sequences might not converge in $X$.

## 3. Uniform Convergence of Galerkin Solutions.

**3.1. Pointwise Convergence in $X$.** It will be helpful to prove a lemma.

LEMMA 3.1. *There is $C>0$ such that for any $N \in \mathbb{N}_0$.*

$$\|T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))\|_X \le C\|u(s)-u_N(s)\|_X, \tag{3.1}$$

*where $t\in[0,T]$ and $s\in[0,t]$.*

*Proof.* We have that

$$\begin{aligned}\|T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))\|_X^2 &= \|T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))\|_{\mathcal{H}}^2 \\ &\quad + \left|[T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))]^D(-\tau)\right|^2.\end{aligned} \tag{3.2}$$

Note that for the first term on the right-hand side of (3.2), we have that

$$\begin{aligned}\|T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))\|_{\mathcal{H}} &\le Me^{\omega(t-s)}\|\mathcal{F}(u(s))-\mathcal{F}(u_N(s))\|_{\mathcal{H}} \\ &\le Me^{\omega T}\|\mathcal{F}(u(s))-\mathcal{F}(u_N(s))\|_{\mathcal{H}} \\ &= Me^{\omega T}\left|f([u(s)]^D(-\tau))-f([u_N(s)]^D(-\tau))\right| \\ &\le LMe^{\omega T}\left|[u(s)]^D(-\tau)-[u_N(s)]^D(-\tau)\right| \\ &\le LMe^{\omega T}\|u(s)-u_N(s)\|_X.\end{aligned} \tag{3.3}$$

For the second term on the right-hand side of (3.2), we consider first the case when $t-s\ge\tau$. Then

$$\begin{aligned}\left|[T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))]^D(-\tau)\right| &\le \|T_N(t-s-\tau)\Pi(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))\|_{\mathcal{H}} \\ &\le Me^{\omega T}\|\mathcal{F}(u(s))-\mathcal{F}(u_N(s))\|_{\mathcal{H}} \\ &\le LMe^{\omega T}\|u(s)-u_N(s)\|_X.\end{aligned} \tag{3.4}$$

Now consider the case when $t-s<\tau$. So we have that

$$\begin{aligned}\left|[T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))]^D(-\tau)\right| &= \left|[\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))]^D(t-s-\tau)\right| \\ &= \left|f([u(s)]^D(-\tau))-f([u_N(s)]^D(-\tau))\right|\cdot|S_N^\tau(t-s-\tau)| \\ &\le L\left|[u(s)]^D(-\tau)-[u_N(s)]^D(-\tau)\right| \\ &\le L\|u(s)-u_N(s)\|_X.\end{aligned} \tag{3.5}$$

If we define

$$C := \sqrt{2}\cdot\max\{L, LMe^{\omega T}\} \tag{3.6}$$

and apply (3.3), (3.4), and (3.5) to (3.2), then we get that

$$\|T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))\|_X \le C\|u(s)-u_N(s)\|_X. \tag{3.7}$$

∎

We introduce the following definitions:

$$
\begin{aligned}
r_N(t) &:= \|u(t) - u_N(t)\|_X, \\
\epsilon_N(t) &:= \|T(t)u_0 - T_N(t)\Pi_N u_0\|_X, \\
d_N(t,s) &:= \|\left(T(t-s) - T_N(t-s)\Pi_N\right)\mathcal{F}(u(s))\|_X.
\end{aligned}
\tag{3.8}
$$

One can apply the variation-of-constants formula and the above definitions to get that

$$
\begin{aligned}
r_N(t) &\le \epsilon_N(t) + \int_0^t d_N(t,s)\,\mathrm{d}s + \int_0^t \|T_N(t-s)\Pi_N\big(\mathcal{F}(u(s)) - \mathcal{F}(u_N(s))\big)\|_X\,\mathrm{d}s \\
&\le \epsilon_N(t) + \int_0^t d_N(t,s)\,\mathrm{d}s + C\int_0^t r_N(s)\,\mathrm{d}s.
\end{aligned}
\tag{3.9}
$$

Applying Grönwall's inequality to (3.9) gives

$$
\begin{aligned}
r_N(t) &\le \left[\epsilon_N(t) + \int_0^t d_N(t,s)\,\mathrm{d}s\right] + \int_0^t Ce^{C(t-s)}\left[\epsilon_N(s) + \int_0^s d_N(s,r)\,\mathrm{d}r\right]\mathrm{d}s \\
&\le \left[\epsilon_N(t) + \int_0^t d_N(t,s)\,\mathrm{d}s\right] + Ce^{CT}\int_0^t \left[\epsilon_N(s) + \int_0^s d_N(s,r)\,\mathrm{d}r\right]\mathrm{d}s.
\end{aligned}
\tag{3.10}
$$

We wish to show that $r_N(t) \to 0$ as $N \to \infty$ for each fixed $t \in [0,T]$. To this end, we show that each term on the right-hand side of (3.10) converges to 0 as $N \to \infty$ and $t \in [0,T]$ fixed. The following propositions will show this.

Proposition 3.1. *For fixed $t \in [0,T]$,*

$$
\epsilon_N(t) \to 0 \text{ and } \int_0^t \epsilon_N(s)\,\mathrm{d}s \to 0
\tag{3.11}
$$

*as $N \to \infty$.*

*Proof.* From the definition of the $X$ norm, we have that

$$
\epsilon_N(t)^2 = \|T(t)u_0 - T_N(t)\Pi_N u_0\|_{\mathcal{H}}^2 + |[T(t)u_0]^D(-\tau) - [T_N(t)\Pi_N u_0]^D(-\tau)|^2.
\tag{3.12}
$$

The first term on the right-hand side converges uniformly to 0 by the Trotter-Kato theorem. For the second case, we again consider the case when $t \ge \tau$. Here we can apply the Trotter-Kato theorem again to $\|T(t-\tau)u_0 - T_N(t-\tau)\Pi_N u_0\|_{\mathcal{H}}^2$ to get the term converges to zero. When $t < \tau$, the second term becomes

$$
|u_0^D(t-\tau) - [\Pi_N u_0]^D(t-\tau)|^2
\tag{3.13}
$$

which converges to 0 uniformly by Proposition 2.2. This gives that $\epsilon_N(t) \to 0$.

To show the other convergence, note that $\epsilon_N(s)$ converges pointwisely to 0 on $[0,t]$. Furthermore, we may uniformly bound $\epsilon_N(s)$ by again observing the equality (3.12) and applying the uniform bounds on $\|T_N(\cdot)\|_{\mathcal{H}}$ and on $[\Pi_N u_0]^D$. Then by the Bounded Convergence Theorem, we have $\int_0^t \epsilon_N(s)\,\mathrm{d}s \to 0$. ∎

Proposition 3.2. *For fixed $t \in [0,T]$,*

$$\int_0^t d_N(t,s)\,\mathrm{d}s \to 0 \text{ and } \int_0^t \int_0^s d_N(s,r)\,\mathrm{d}r\,\mathrm{d}s \to 0, \tag{3.14}$$

*as $N \to \infty$.*

*Proof.* We can again apply the definition of $\|\cdot\|_X$ to get that

$$\begin{aligned} d_N^2(t,s) = &\,\| \left(T(t-s) - T_N(t-s)\Pi_N\right) \mathcal{F}(u(s))\|_{\mathcal{H}}^2 \\ &+ |[T(t-s)\mathcal{F}(u(s))]^D(-\tau) - [T_N(t-s)\Pi_N\mathcal{F}(u(s))]^D(-\tau)|^2. \end{aligned} \tag{3.15}$$

For fixed $t$ and $s$, the first term of the right-hand side converges to zero. For $t-s \geq \tau$ the second term will similarly converge to 0. For $t-s < \tau$, the second term will become

$$|0 - [\Pi_N\mathcal{F}(u(s))]^D(t-s-\tau)| = |f([u(s)]^D(-\tau))| \cdot |S_N(t-s-\tau)|\,, \tag{3.16}$$

which converges a.e. to 0 by (2.27). So for fixed $t$, $d_N(t,s)$ converges a.e. to 0 for $s \in [0,t]$. Furthermore, we can uniformly bound $d_N(t,s)$ by (2.26).Thus by the Bounded Convergence Theorem, we have $\int_0^t d_N(t,s)\,\mathrm{d}s \to 0$ as $N \to \infty$.

The second convergence follows by the observations that $\int_0^\cdot d_N(\cdot,r)\,\mathrm{d}r$ converges pointwise to 0 by our earlier work and can uniformly bounded on $[0,t]$. This allows us to apply the Bounded Convergence Theorem to get that $\int_0^t\int_0^s d_N(s,r)\,\mathrm{d}r\,\mathrm{d}s \to 0$ as $N \to \infty$. ∎

We may now state our final result.

Theorem 3.3. *For $t \in [0,T]$,*

$$\lim_{N\to\infty} \|u(t) - u_N(t)\|_X = 0. \tag{3.17}$$

*Proof.* Apply propositions (3.1) and (3.2) to the inequality in (**??**). ∎

### 3.2. Uniform Convergence.

Lemma 3.2. *The following convergences hold:*

$$\lim_{N\to\infty} [u_N(\cdot)]^D(-\tau) = [u(\cdot)]^D(-\tau) \text{ with respect to } L^2([0,T];\mathbb{R}), \tag{3.18}$$

*and*

$$\lim_{N\to\infty} \mathcal{F}(u_N(\cdot)) = \mathcal{F}(u(\cdot)) \text{ with respect to } L^1([0,T];\mathcal{H}). \tag{3.19}$$

*Proof.* Note that

$$\int_0^T \left|[u_N(s)]^D(-\tau) - [u(s)]^D(-\tau)\right|^2 \mathrm{d}s \leq \sum_{k=0}^m \int_{-\tau}^0 \left|[u_N(k\tau)]^D(\theta) - [u(k\tau)]^D(\theta)\right|^2 \mathrm{d}\theta, \tag{3.20}$$

for $m$ such that $T - \tau \leq m\tau < T$. In other words,

$$\left\|[u_N(\cdot)]^D(-\tau) - [u(\cdot)]^D(-\tau)\right\|^2_{L^2([0,T];\mathbb{R})} \leq \sum_{k=0}^m \left\|[u_N(k\tau)]^D - [u(k\tau)]^D\right\|^2_{L^2([-\tau,0);\mathbb{R})}. \tag{3.21}$$

It has been shown that $\|[u_N(t)]^D - [u(t)]^D\|_{L^2([0,T];\mathbb{R})} \to 0$ as $N \to \infty$ for any $t \in [0,T]$. This gives that the right side of (3.21) converges to 0 as $N \to \infty$, and thus the left side of (3.21) also converges to 0 as $N \to \infty$. This proves (3.18).

To prove the other convergence, note that

$$
\begin{aligned}
\int_0^T \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s &= \int_0^T \left| f\left([u_N(s)]^D(-\tau)\right) - f\left([u(s)]^D(-\tau)\right)\right| \,\mathrm{d}s \\
&\le L \int_0^T \left|[u_N(s)]^D(-\tau) - [u(s)]^D(-\tau)\right| \,\mathrm{d}s \\
&= L\|[u_N(\cdot)]^D(-\tau) - [u(\cdot)]^D(-\tau)\|_{L^1([0,T];\mathbb{R})}.
\end{aligned} \tag{3.22}
$$

Noting that $L^2([0,T];\mathbb{R})$ is continuously embedded in $L^1([0,T];\mathbb{R})$ and applying (3.18) proves that (3.19) holds. ∎

THEOREM 3.4. *The sequence of functions $\{u_N\}_{N=0}^{\infty}$, where*

$$
u_N : [0,T] \mapsto \mathcal{H}, \qquad N \in \mathbb{N}_0, \tag{3.23}
$$

*is uniformly equicontinuous.*

*Proof.* Suppose $t_0, t_1 \in [0,T]$ and $t_0 \le t_1$. Denote $\delta := t_1 - t_0$. Applying the variation-of-constants formula, we have that for $N \in \mathbb{N}_0$

$$
\begin{aligned}
\|u_N(t_0) - u_N(t_1)\|_{\mathcal{H}} \le\ & \underbrace{\|(T_N(t_0) - T_N(t_0+\delta))\Pi_N u_0\|_{\mathcal{H}}}_{\mathrm{I}(\delta,N)} \\
&+ \underbrace{\left\|\int_0^{t_0} [T_N(t_0 - s) - T_N(t_0+\delta-s)]\Pi_N \mathcal{F}(u_N(s))\,\mathrm{d}s\right\|_{\mathcal{H}}}_{\mathrm{II}(\delta,N)} \\
&+ \underbrace{\left\|\int_{t_0}^{t_0+\delta} T_N(t_0+\delta-s)\Pi_N \mathcal{F}(u_N(s))\,\mathrm{d}s\right\|_{\mathcal{H}}}_{\mathrm{III}(\delta,N)}.
\end{aligned} \tag{3.24}
$$

We show that for each of these terms, the dependence on $\delta$ and $N$ can be separated.

**I**. We have that

$$
\begin{aligned}
\mathrm{I}(\delta,N) &= \|T_N(t_0)(I - T_N(\delta))\Pi_N u_0\|_{\mathcal{H}} \\
&\le Me^{\omega t_0}\|(I - T_N(\delta))\Pi_N u_0\|_{\mathcal{H}} \\
&= Me^{\omega t_0}\|(\Pi_N - T_N(\delta)\Pi_N)u_0\|_{\mathcal{H}} \\
&\le Me^{\omega t_0}\|(I - T_N(\delta)\Pi_N)u_0\|_{\mathcal{H}} \\
&\le Me^{\omega T}\|(I - T_N(\delta)\Pi_N)u_0\|_{\mathcal{H}} \\
&\le Me^{\omega T}\left[\|(I - T(\delta))u_0\|_{\mathcal{H}} + \|(T(\delta) - T_N(\delta)\Pi_N)u_0\|_{\mathcal{H}}\right] \\
&\le Me^{\omega T}\left[\|(I - T(\delta))u_0\|_{\mathcal{H}} + \sup_{t\in[0,T]}\|(T(t) - T_N(t)\Pi_N)u_0\|_{\mathcal{H}}\right].
\end{aligned} \tag{3.25}
$$

Now define the following functions:

$$\mathrm{I}^*(\delta) := Me^{\omega T} \times \|(I - T(\delta))u_0\|_{\mathcal{H}} \tag{3.26}$$

and

$$\mathrm{I}^{**}(N) := Me^{\omega T} \times \sup_{t\in[0,T]} \|(T(t) - T_N(t)\Pi_N)u_0\|_{\mathcal{H}} \tag{3.27}$$

Note that $\lim_{\delta\to 0^+} \mathrm{I}^*(\delta) = 0$ by the continuity of $T(t)$ and $\lim_{N\to\infty} \mathrm{I}^{**}(N) = 0$ by the Trotter-Kato theorem.

**II**. We have that

$$\begin{aligned}
\mathrm{II}(\delta, N) &\leq \int_0^{t_0} \|(T_N(t_0 - s) - T_N(t_0 + \delta - s))\Pi_N \mathcal{F}(u_N(s))\|_{\mathcal{H}}\, \mathrm{d}s \\
&\leq Me^{\omega T} \int_0^{t_0} \|(I - T_N(\delta)\Pi_N)\mathcal{F}(u_N(s))\|_{\mathcal{H}}\, \mathrm{d}s \\
&\leq Me^{\omega T} \Bigg[ \underbrace{\int_0^{t_0} \|(I - T_N(\delta)\Pi_N)\mathcal{F}(u(s))\|_{\mathcal{H}}\, \mathrm{d}s}_{A} \\
&\qquad + \underbrace{\int_0^{t_0} \|(I - T_N(\delta)\Pi_N)(\mathcal{F}(u_N(s)) - \mathcal{F}(u(s)))\|_{\mathcal{H}}\, \mathrm{d}s}_{B} \Bigg].
\end{aligned} \tag{3.28}$$

From here, we can note that

$$\begin{aligned}
A &\leq \int_0^T \|(I - T(\delta))\mathcal{F}(u(s))\|_{\mathcal{H}}\, \mathrm{d}s + \int_0^T \|(T(\delta) - T_N(\delta))\mathcal{F}(u(s))\|_{\mathcal{H}}\, \mathrm{d}s \\
&\leq \int_0^T \|(I - T(\delta))\mathcal{F}(u(s))\|_{\mathcal{H}}\, \mathrm{d}s + \int_0^T \sup_{t\in[0,T]} \|(T(t) - T_N(t))\mathcal{F}(u(s))\|_{\mathcal{H}}\, \mathrm{d}s,
\end{aligned} \tag{3.29}$$

where both of these terms can easily be shown to converge to zero as $\delta \to 0$ and $N \to \infty$, respectively. Namely, we can apply the Lebesgue Dominated Convergence Theorem. Also note that

$$B \leq (1 + Me^{\omega T}) \int_0^T \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s))\|_{\mathcal{H}}\, \mathrm{d}s, \tag{3.30}$$

where the right-hand side converges to zero as $N \to \infty$ by (3.19). Now we set

$$\mathrm{II}^*(\delta) := Me^{\omega T} \int_0^T \|(I - T(\delta))\mathcal{F}(u(s))\|_{\mathcal{H}}\, \mathrm{d}s \tag{3.31}$$

and

$$\mathrm{II}^{**}(N) := Me^{\omega T}\left[\int_0^T \sup_{t\in[0,T]} \|(T(t)-T_N(t))\mathcal{F}(u(s))\|_{\mathcal{H}}\,\mathrm{d}s \right. \\ \left. + \left(1+Me^{\omega T}\right)\int_0^T \|\mathcal{F}(u_N(s))-\mathcal{F}(u(s))\|_{\mathcal{H}}\,\mathrm{d}s\right]. \tag{3.32}$$

**III**. We have that

$$\begin{aligned} \mathrm{III}(\delta,N) &\le \int_{t_0}^{t_0+\delta} \|T_N(t_0+\delta-s)\Pi_N\mathcal{F}(u_N(s))\|_{\mathcal{H}}\ \mathrm{d}s \\ &\le Me^{\omega T}\int_{t_0}^{t_0+\delta} \|\mathcal{F}(u_N(s))\|_{\mathcal{H}}\ \mathrm{d}s \\ &\le Me^{\omega T}\left[\int_{t_0}^{t_0+\delta}\|\mathcal{F}(u(s))\|_{\mathcal{H}}\,\mathrm{d}s + \int_{t_0}^{t_0+\delta}\|\mathcal{F}(u_N(s))-\mathcal{F}(u(s))\|_{\mathcal{H}}\,\mathrm{d}s\right] \\ &\le Me^{\omega T}\left[\delta\times\sup_{t\in[0,T]}\|\mathcal{F}(u(t))\|_{\mathcal{H}} + \int_0^T \|\mathcal{F}(u_N(s))-\mathcal{F}(u(s))\|_{\mathcal{H}}\,\mathrm{d}s\right]. \end{aligned} \tag{3.33}$$

Note that $\sup_{t\in[0,T]}\|\mathcal{F}(u(t))\|_{\mathcal{H}}$ is finite since $\|\mathcal{F}(u(t))\|_{\mathcal{H}}$ is a continuous function. Now let

$$\mathrm{III}^*(\delta) := Me^{\omega T}\delta\times\sup_{t\in[0,T]}\|\mathcal{F}(u(t))\|_{\mathcal{H}} \tag{3.34}$$

and

$$\mathrm{III}^{**}(N) := Me^{\omega T}\int_0^T \|\mathcal{F}(u_N(s))-\mathcal{F}(u(s))\|_{\mathcal{H}}\,\mathrm{d}s. \tag{3.35}$$

Clearly $\lim_{\delta\to 0^+}\mathrm{III}^*(\delta)=0$. Also from (3.19) we have that $\lim_{N\to\infty}\mathrm{III}^{**}(N)=0$. Thus,

$$\begin{aligned} \|u_N(t_0)-u_N(t_1)\|_{\mathcal{H}} &\le \mathrm{I}(\delta,N)+\mathrm{II}(\delta,N)+\mathrm{III}(\delta,N) \\ &\le [\mathrm{I}^*(\delta)+\mathrm{II}^*(\delta)+\mathrm{III}^*(\delta)] + [\mathrm{I}^{**}(N)+\mathrm{II}^{**}(N)+\mathrm{III}^{**}(N)]. \end{aligned} \tag{3.36}$$

Let $\epsilon>0$. We wish to choose $\delta>0$ such that $\|u_n(t)-u_n(t')\|_{\mathcal{H}}<\epsilon$ for any $n\in\mathbb{N}_0$ and $t,t'\in[0,T]$ with $|t-t'|<\delta$. Choosing $\delta^*$ small enough so that $\mathrm{I}^*(\delta^*)+\mathrm{II}^*(\delta^*)+\mathrm{III}^*(\delta^*)<\epsilon/2$ and $N$ large enough such that $\mathrm{I}^{**}(N)+\mathrm{II}^{**}(N)+\mathrm{III}^{**}(N)<\epsilon/2$, we get that

$$\|u_n(t)-u_n(t')\|_{\mathcal{H}}<\epsilon, \tag{3.37}$$

where $|t-t'|<\delta^*$ and $n\ge N$. For each $n\in\mathbb{N}_0$ that are less than $N$, we pick $\delta_n>0$ such that $\|u_n(t)-u_n(t')\|_{\mathcal{H}}<\epsilon$ for $|t-t'|<\delta_n$. This is possible since $u_n$ is uniformly continuous on $[0,T]$. Let $\delta=\min\{\delta^*,\delta_0,\ldots,\delta_{N-1}\}$. Then $\delta$ satisfies the challenge from $\epsilon$. This proves uniform equicontinuity. ∎

THEOREM 3.5. *For $T > 0$, we have that*

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \|u_N(t) - u(t)\|_{\mathcal{H}} = 0. \tag{3.38}$$

*Proof.* The above result follows directly from Theorem 3.3 and Theorem 3.4. ∎

## 4. Uniform Equicontinuity of Galerkin Solutions.

### 4.1. Initial Lemmas.

LEMMA 4.1. *The following convergences hold:*

$$\lim_{N\to\infty} [u_N(\cdot)]^D(-\tau) = [u(\cdot)]^D(-\tau) \text{ with respect to } L^2([0,T];\mathbb{R}), \tag{4.1}$$

*and*

$$\lim_{N\to\infty} \mathcal{F}(u_N(\cdot)) = \mathcal{F}(u(\cdot)) \text{ with respect to } L^1([0,T];\mathcal{H}). \tag{4.2}$$

*Proof.* Note that

$$\int_0^T \left|[u_N(s)]^D(-\tau) - [u(s)]^D(-\tau)\right|^2 \mathrm{d}s \le \sum_{k=0}^{m} \int_{-\tau}^{0} \left|[u_N(k\tau)]^D(\theta) - [u(k\tau)]^D(\theta)\right|^2 \mathrm{d}\theta. \tag{4.3}$$

In other words,

$$\left\|[u_N(\cdot)]^D(-\tau) - [u(\cdot)]^D(-\tau)\right\|^2_{L^2([0,T];\mathbb{R})} \le \sum_{k=0}^{m} \left\|[u_N(k\tau)]^D - [u(k\tau)]^D\right\|^2_{L^2([0,T];\mathbb{R})}. \tag{4.4}$$

It has been shown that $\|[u_N(t)]^D - [u(t)]^D\|_{L^2([0,T];\mathbb{R})} \to 0$ as $N \to \infty$ for any $t \in [0,T]$. This gives that the right side of (4.4) converges to 0 as $N \to \infty$, and thus the left side of (4.4) also converges to 0 as $N \to \infty$. This proves (4.1).

To prove the other convergence, note that

$$\begin{aligned}
\int_0^T \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s &= \int_0^T \left|f\left([u_N(s)]^D(-\tau)\right) - f\left([u(s)]^D(-\tau)\right)\right| \mathrm{d}s \\
&\le L \int_0^T \left|[u_N(s)]^D(-\tau) - [u(s)]^D(-\tau)\right| \mathrm{d}s \\
&= L\|[u_N(\cdot)]^D(-\tau) - [u(\cdot)]^D(-\tau)\|_{L^1([0,T];\mathbb{R})}.
\end{aligned} \tag{4.5}$$

Noting that $L^2([0,T];\mathbb{R})$ is continuously embedded in $L^1([0,T];\mathbb{R})$ and applying (4.1) proves that (4.2) holds. ∎

### 4.2. Uniform Equicontinuity.

THEOREM 4.1. *The sequence of functions $\{u_N\}_{N=0}^{\infty}$, where*

$$u_N : [0,T] \mapsto \mathcal{H}, \qquad N \in \mathbb{N}_0, \tag{4.6}$$

*is uniformly equicontinuous.*

*Proof.* Suppose $t_0, t_1 \in [0,T]$ and $t_0 \le t_1$. Denote $\delta := t_1 - t_0$. Applying the variation-of-constants formula, we have that for $N \in \mathbb{N}_0$

$$
\begin{aligned}
\|u_N(t_0) - u_N(t_1)\|_{\mathcal{H}} \le &\underbrace{\|(T_N(t_0) - T_N(t_0+\delta))\Pi_N u_0\|_{\mathcal{H}}}_{\mathrm{I}(\delta,N)} \\
&+ \underbrace{\left\| \int_0^{t_0} [T_N(t_0 - s) - T_N(t_0 + \delta - s)]\Pi_N \mathcal{F}(u_N(s))\,\mathrm{d}s \right\|_{\mathcal{H}}}_{\mathrm{II}(\delta,N)} \\
&+ \underbrace{\left\| \int_{t_0}^{t_0+\delta} T_N(t_0 + \delta - s)\Pi_N \mathcal{F}(u_N(s))\,\mathrm{d}s \right\|_{\mathcal{H}}}_{\mathrm{III}(\delta,N)}.
\end{aligned} \tag{4.7}
$$

We show that for each of these terms, the dependence on $\delta$ and $N$ can be separated.

**I**. We have that

$$
\begin{aligned}
\mathrm{I}(\delta, N) &= \|T_N(t_0)(I - T_N(\delta))\Pi_N u_0\|_{\mathcal{H}} \\
&\le Me^{\omega t_0}\|(I - T_N(\delta))\Pi_N u_0\|_{\mathcal{H}} \\
&= Me^{\omega t_0}\|(\Pi_N - T_N(\delta)\Pi_N)u_0\|_{\mathcal{H}} \\
&\le Me^{\omega t_0}\|(I - T_N(\delta)\Pi_N)u_0\|_{\mathcal{H}} \\
&\le Me^{\omega T}\|(I - T_N(\delta)\Pi_N)u_0\|_{\mathcal{H}} \\
&\le Me^{\omega T}\left[\|(I - T(\delta))u_0\|_{\mathcal{H}} + \|(T(\delta) - T_N(\delta)\Pi_N)u_0\|_{\mathcal{H}}\right] \\
&\le Me^{\omega T}\left[\|(I - T(\delta))u_0\|_{\mathcal{H}} + \sup_{t\in[0,T]} \|(T(t) - T_N(t)\Pi_N)u_0\|_{\mathcal{H}}\right].
\end{aligned} \tag{4.8}
$$

Now define the following functions:

$$
\mathrm{I}^*(\delta) := Me^{\omega T} \times \|(I - T(\delta))u_0\|_{\mathcal{H}} \tag{4.9}
$$

and

$$
\mathrm{I}^{**}(N) := Me^{\omega T} \times \sup_{t\in[0,T]} \|(T(t) - T_N(t)\Pi_N)u_0\|_{\mathcal{H}} \tag{4.10}
$$

Note that $\lim_{\delta\to 0^+} \mathrm{I}^*(\delta) = 0$ by the continuity of $T(t)$ and $\lim_{N\to\infty} \mathrm{I}^{**}(N) = 0$ by the Trotter-Kato theorem.

**II**. We have that

$$
\begin{aligned}
\mathrm{II}(\delta, N) &\le \int_0^{t_0} \|(T_N(t_0-s) - T_N(t_0+\delta-s))\Pi_N \mathcal{F}(u_N(s))\|_{\mathcal{H}} \, \mathrm{d}s \\
&\le Me^{\omega T} \int_0^{t_0} \|(I - T_N(\delta)\Pi_N)\mathcal{F}(u_N(s))\|_{\mathcal{H}} \, \mathrm{d}s \\
&\le Me^{\omega T} \Bigg[ \underbrace{\int_0^{t_0} \|(I - T_N(\delta)\Pi_N)\mathcal{F}(u(s))\|_{\mathcal{H}} \, \mathrm{d}s}_{A} \\
&\qquad + \underbrace{\int_0^{t_0} \|(I - T_N(\delta)\Pi_N)(\mathcal{F}(u_N(s)) - \mathcal{F}(u(s)))\|_{\mathcal{H}} \, \mathrm{d}s}_{B} \Bigg].
\end{aligned}
\tag{4.11}
$$

From here, we can note that

$$
\begin{aligned}
A &\le \int_0^T \|(I - T(\delta))\mathcal{F}(u(s))\|_{\mathcal{H}} \, \mathrm{d}s + \int_0^T \|(T(\delta) + T_N(\delta))\mathcal{F}(u(s))\|_{\mathcal{H}} \, \mathrm{d}s \\
&\le \int_0^T \|(I - T(\delta))\mathcal{F}(u(s))\|_{\mathcal{H}} \, \mathrm{d}s + \int_0^T \sup_{t\in[0,T]} \|(T(t) + T_N(t))\mathcal{F}(u(s))\|_{\mathcal{H}} \, \mathrm{d}s,
\end{aligned}
\tag{4.12}
$$

where both of these terms can easily be shown to converge to zero as $\delta \to 0$ and $N \to \infty$, respectively. Namely, we can apply the Lebesgue Dominated Convergence Theorem. Also note that

$$
B \le (1 + Me^{\omega T}) \int_0^T \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s))\|_{\mathcal{H}} \, \mathrm{d}s,
\tag{4.13}
$$

where the right-hand side converges to zero as $N \to \infty$ by (4.2). Now we set

$$
\mathrm{II}^*(\delta) := Me^{\omega T} \int_0^T \|(I - T(\delta))\mathcal{F}(u(s))\|_{\mathcal{H}} \, \mathrm{d}s
\tag{4.14}
$$

and

$$
\begin{aligned}
\mathrm{II}^{**}(N) :=& Me^{\omega T} \Bigg[ \int_0^T \sup_{t\in[0,T]} \|(T(t) + T_N(t))\mathcal{F}(u(s))\|_{\mathcal{H}} \, \mathrm{d}s \\
&\qquad + \left(1 + Me^{\omega T}\right) \int_0^T \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s))\|_{\mathcal{H}} \, \mathrm{d}s \Bigg].
\end{aligned}
\tag{4.15}
$$

**III**. We have that

$$
\begin{aligned}
\mathrm{III}(\delta,N) &\le \int_{t_0}^{t_0+\delta} \|T_N(t_0+\delta-s)\Pi_N\mathcal{F}(u_N(s))\|_{\mathcal{H}}\,\mathrm{d}s \\
&\le Me^{\omega T}\int_{t_0}^{t_0+\delta} \|\mathcal{F}(u_N(s))\|_{\mathcal{H}}\,\mathrm{d}s \\
&\le Me^{\omega T}\left[\int_{t_0}^{t_0+\delta} \|\mathcal{F}(u(s))\|_{\mathcal{H}}\,\mathrm{d}s + \int_{t_0}^{t_0+\delta} \|\mathcal{F}(u_N(s))-\mathcal{F}(u(s))\|_{\mathcal{H}}\,\mathrm{d}s\right] \\
&\le Me^{\omega T}\left[\delta\times\sup_{t\in[0,T]}\|\mathcal{F}(u(t))\|_{\mathcal{H}} + \int_0^T \|\mathcal{F}(u_N(s))-\mathcal{F}(u(s))\|_{\mathcal{H}}\,\mathrm{d}s\right].
\end{aligned}
\tag{4.16}
$$

Note that $\sup_{t\in[0,T]}\|\mathcal{F}(u(t))\|_{\mathcal{H}}$ is finite since $\|\mathcal{F}(u(t))\|_{\mathcal{H}}$ is a continuous function. Now let

$$
\mathrm{III}^*(\delta) := Me^{\omega T}\delta\times\sup_{t\in[0,T]}\|\mathcal{F}(u(t))\|_{\mathcal{H}} \tag{4.17}
$$

and

$$
\mathrm{III}^{**}(N) := Me^{\omega T}\int_0^T \|\mathcal{F}(u_N(s))-\mathcal{F}(u(s))\|_{\mathcal{H}}\,\mathrm{d}s. \tag{4.18}
$$

Clearly $\lim_{\delta\to0^+}\mathrm{III}^*(\delta)=0$. Also from (4.2) we have that $\lim_{N\to\infty}\mathrm{III}^{**}(N)=0$. Thus,

$$
\begin{aligned}
\|u_N(t_0)-u_N(t_1)\|_{\mathcal{H}} &\le \mathrm{I}(\delta,N)+\mathrm{II}(\delta,N)+\mathrm{III}(\delta,N) \\
&\le [\mathrm{I}^*(\delta)+\mathrm{II}^*(\delta)+\mathrm{III}^*(\delta)] + [\mathrm{I}^{**}(N)+\mathrm{II}^{**}(N)+\mathrm{III}^{**}(N)].
\end{aligned}
\tag{4.19}
$$

Let $\epsilon>0$. We wish to choose $\delta>0$ such that $\|u_n(t)-u_n(t')\|_{\mathcal{H}}<\epsilon$ for any $n\in\mathbb{N}_0$ and $t,t'\in[0,T]$ with $|t-t'|<\delta$. Choosing $\delta^*$ small enough so that $\mathrm{I}^*(\delta^*)+\mathrm{II}^*(\delta^*)+\mathrm{III}^*(\delta^*)<\epsilon/2$ and $N$ large enough such that $\mathrm{I}^{**}(N)+\mathrm{II}^{**}(N)+\mathrm{III}^{**}(N)<\epsilon/2$, we get that

$$
\|u_n(t)-u_n(t')\|_{\mathcal{H}}<\epsilon, \tag{4.20}
$$

where $|t-t'|<\delta^*$ and $n\ge N$. For each $n\in\mathbb{N}_0$ that are less than $N$, we pick $\delta_n>0$ such that $\|u_n(t)-u_n(t')\|_{\mathcal{H}}<\epsilon$ for $|t-t'|<\delta_n$. This is possible since $u_n$ is uniformly continuous on $[0,T]$. Let $\delta=\min\{\delta^*,\delta_0,\ldots,\delta_{N-1}\}$. Then $\delta$ satisfies the challenge from $\epsilon$. This proves uniform equicontinuity. ∎

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