GALERKIN APPROXIMATIONS OF DELAY DIFFERENTIAL EQUATIONS

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Abstract.

1. Introduction.

2. Preliminaries.

2.1. The type of DDE. We are interested in approximating the solution to the following autonomous scalar DDE of dimension 1:

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= ax(t) + bx(t-\tau) + F(x(t-\tau)), && t > 0 \\ x(t) &= \varphi(t), && t \in [-\tau, 0] \end{aligned} \tag{2.1}$$

for $\varphi \in L^2([-\tau, 0]; \mathbb{R})$, $a, b \in \mathbb{R}$, and where $F : \mathbb{R} \to \mathbb{R}$ is Lipschitz with constant L. It is appropriate to formulate this problem into the Hilbert space

$$\mathcal{H} := L^2([-\tau, 0); \mathbb{R}) \times \mathbb{R}, \tag{2.2}$$

where the inner product is defined for (f_1, γ_1), $(f_2, \gamma_2) \in \mathcal{H}$, as:

$$\langle (f_1, \gamma_1), (f_2, \gamma_2) \rangle_{\mathcal{H}} := \frac{1}{\tau} \int_{-\tau}^{0} f_1(\theta) f_2(\theta) \, \mathrm{d}\theta + \gamma_1 \gamma_2. \tag{2.3}$$

Let us denote by x_t the time evolution of the history segments of a solution to (2.1), i.e.,

$$x_t(\theta) := x(t + \theta), \qquad t \geq 0, \qquad \theta \in [-\tau, 0]. \tag{2.4}$$

Then by introducing

$$u(t) := (x_t, x_t(0)), \tag{2.5}$$

we can rewrite (2.1) as the following abstract ODE on $\mathcal{H}$:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathcal{A}u + \mathcal{F}(u). \tag{2.6}$$

The linear operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is defined by

$$[\mathcal{A}\Psi](\theta) := \begin{cases} \dfrac{\mathrm{d}^+ \Psi^D}{\mathrm{d}\theta}, & \theta \in [-\tau, 0), \\ a\Psi^S + b\Psi^D(-\tau), & \theta = 0, \end{cases} \tag{2.7}$$

for any $\Psi = (\Psi^D, \Psi^S)$ that lives in the domain, $D(\mathcal{A})$, defined as

$$D(\mathcal{A}) := \left\{ \Psi \in \mathcal{H} : \Psi^D \in H^1([-\tau, 0); \mathbb{R}^d), \lim_{\theta \to 0^-} \Psi^D(\theta) = \Psi^S \right\}. \tag{2.8}$$

The nonlinear operator $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is defined by

$$[\mathcal{F}(\Psi)](\theta) := \begin{cases} 0, & \theta \in [-\tau, 0), \\ F\left(\Psi^D(-\tau)\right), & \theta = 0, \end{cases} \qquad \forall\, \Psi = (\Psi^D, \Psi^S) \in \mathcal{H}. \tag{2.9}$$

2.2. Properties and Basic Results of Koornwinder Polynomials. From [2, Eq. (2.1)], the sequence of Koornwinder polynomials $\{K_n\}$ can be built from the Legendre polynomials L_n by

$$K_n(s) := -(1+s)\frac{d}{\mathrm{d}s}L_n(s) + (n^2+n+1)L_n(s), \quad s \in [-1,1],\ n \in \mathbb{N}_0. \tag{2.10}$$

Furthermore, we reproduce from [1, Prop. 3.1] some simple properties that $\{K_n\}$ satisfy.

Proposition 2.1. *The polynomial K_n defined in (2.10) is of degree n and admits the following expansion in terms of the Legendre polynomials:*

$$K_n(s) = -\sum_{j=0}^{n-1}(2j+1)L_j(s) + (n^2+1)L_n(s), \qquad n \in \mathbb{N}_0; \tag{2.11}$$

and the following normalization property holds:

$$K_n(1) = 1, \qquad n \in \mathbb{N}_0. \tag{2.12}$$

Moreover, the sequence given by

$$\{\mathcal{K}_n := (K_n, K_n(1)) : n \in \mathbb{N}_0\} \tag{2.13}$$

forms an orthogonal basis of the product space

$$\mathcal{E} := L^2([-1,1);\mathbb{R}) \times \mathbb{R}, \tag{2.14}$$

where $\mathcal{E}$ is endowed with the following inner product:

$$\langle (f,a),(g,b)\rangle_{\mathcal{E}} = \frac{1}{2}\int_{-1}^{1} f(s)g(s)\,\mathrm{d}s + ab, \quad (f,a),(g,b) \in \mathcal{E}. \tag{2.15}$$

Moreover $\left\{\frac{\mathcal{K}_n}{\|\mathcal{K}_n\|_{\mathcal{E}}}\right\}$ forms a Hilbert basis of $\mathcal{E}$ where the norm $\|\mathcal{K}_n\|_{\mathcal{E}}$ of $\mathcal{K}_n$ induced by $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ possesses the following analytic expression:

$$\|\mathcal{K}_n\|_{\mathcal{E}} = \sqrt{\frac{(n^2+1)((n+1)^2+1)}{2n+1}}, \qquad n \in \mathbb{N}_0. \tag{2.16}$$

Suppose that Π_N is the N-dimensional standard projection into $\mathrm{span}\{\mathcal{K}_n : n \le N\} \subset \mathcal{E}$. It will be relevant to discuss when we have convergence of $[\Pi_N u]^D$ for $u \in \mathcal{E}$. In particular, we will focus on uniform convergence. We define for $f \in L^2([-1,1],\mathbb{R})$ the following:

$$a_n(f) := \frac{2n+1}{2}\int_{-1}^{1} f(x)L_n(x)\,\mathrm{d}x. \tag{2.17}$$

Proposition 2.2. *Let $f \in \mathcal{C}^2([-1,1];\mathbb{R})$ and denote $\psi = (f, f(0)) \in \mathcal{E}$. Then the series*

$$[\Pi_N\psi]^D = \sum_{n=0}^{N}\frac{\langle\psi,\mathcal{K}_n\rangle_{\mathcal{E}}}{\|\mathcal{K}_n\|_{\mathcal{E}}^2}K_n \tag{2.18}$$

converges uniformly to f.

Proof. It is easy to show based on (2.11) we have for $\theta \in [-1,1]$

$$
\begin{aligned}
|K_n(\theta)| &\le (n^2+1)|L_n(\theta)| + \sum_{j=0}^{n-1}(2j+1)|L_j(\theta)| \\
&\le (n^2+1) + \sum_{j=0}^{n-1}(2j+1) \\
&= 2n^2+1,
\end{aligned}
\tag{2.19}
$$

i.e., $\|K_n\|_\infty \le 2n^2+1$.

By the definition of $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ and the Koornwinder polynomials, we have that for $n \in \mathbb{N}_0$

$$
\begin{aligned}
\langle \psi, \mathcal{K}_n\rangle_{\mathcal{E}} &= \frac{1}{2}\int_{-1}^{1} f(x)K_n(x)\,\mathrm{d}x + f(1) \\
&= \frac{1}{2}\left[-\int_{-1}^{1} f(x)(1+x)L_n'(x)\,\mathrm{d}x + (n^2+n+1)\int_{-1}^{1} f(x)L_n(x)\,\mathrm{d}x\right] + f(1).
\end{aligned}
\tag{2.20}
$$

If we use integration by parts, we find that

$$
-\int_{-1}^{1} f(x)(1+x)L_n'(x)\,\mathrm{d}x = -2f(1) + \int_{-1}^{1} f'(x)(1+x)L_n(x)\,\mathrm{d}x + \int_{-1}^{1} f(x)L_n(x)\,\mathrm{d}x. \tag{2.21}
$$

Applying (2.21) to (2.20) gives that

$$
\begin{aligned}
\langle \psi, \mathcal{K}_n\rangle_{\mathcal{E}} &= \frac{1}{2}\int_{-1}^{1} f'(x)(1+x)L_n(x)\,\mathrm{d}x + \frac{n^2+n+2}{2}\int_{-1}^{1} f(x)L_n(x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-1}^{1} f'(x)(1+x)L_n(x)\,\mathrm{d}x + \frac{n^2+n+2}{2n+1}a_n(f).
\end{aligned}
\tag{2.22}
$$

We can also note that by applying the Hoälder inequality we get

$$
\begin{aligned}
\left|\int_{-1}^{1} f'(x)(1+x)L_n(x)\,\mathrm{d}x\right| &\le \|f'\|_\infty \left(\int_{-1}^{1}(1+x)\,\mathrm{d}x\right)^{1/2}\|L_n\|_{L^2} \\
&= \frac{4\|f'\|_\infty}{\sqrt{6n+3}}.
\end{aligned}
\tag{2.23}
$$

Furthermore, from [4, Thm. 2.1] we have

$$
|a_n(f)| \le \frac{V_1}{n-\frac{1}{2}}\sqrt{\frac{\pi}{2n}}, \tag{2.24}
$$

where $V_1 := \int_{-1}^{1} \frac{f''(x)}{\sqrt{1-x^2}}\,\mathrm{d}x < \infty$. Thus,

$$
|\langle \psi, \mathcal{K}_n\rangle_{\mathcal{H}}| \le \frac{2\|f'\|_\infty}{\sqrt{6n+3}} + V_1\sqrt{2\pi}\frac{n^2+n+2}{\sqrt{n}(4n^2+1)}, \tag{2.25}
$$

and so

$$
\begin{aligned}
\frac{|\langle \psi, \mathcal{K}_n \rangle_{\mathcal{H}}|}{\|\mathcal{K}_n\|_{\mathcal{H}}^2} \|K_n\|_\infty &\le \left[\frac{2\|f'\|_\infty}{\sqrt{6n+3}} + V_1 \sqrt{2\pi} \frac{n^2+n+2}{\sqrt{n}(4n^2+1)} \right] \times \left[\frac{(2n+1)(2n^2+1)}{(n^2+1)((n+1)^2+1)} \right] \\
&= O\left(\frac{1}{n^{3/2}} \right).
\end{aligned} \tag{2.26}
$$

By the Weierstrass M-test, the series (2.18) converges uniformly.

Note also that (2.18) is simply the functional part of the Koornwinder expansion of ψ in $\mathcal{H}$. So the series converges in $L^2([-1,1];\mathbb{R})$ to $\psi^D = f$. Therefore, since the series converges uniformly, it must converge uniformly to f. ∎

It will also be necessary to prove certain properties of the series of Koornwinder polynomials

$$
S_N(x) := \sum_{n=0}^{N} \frac{K_n}{\|\mathcal{K}_n\|_{\mathcal{E}}^2}, \quad N \in \mathbb{N}_0,\ x \in [-1,1]. \tag{2.27}
$$

If we were to denote $\psi = (0,1) \in L^2([-1,1)) \times \mathbb{R}$, then S_N would simply be the functional part of $\Pi_N \psi$. The following lemma allows us to express S_N in terms of Legendre Polynomials.

Lemma 2.1. *The functions S_N defined in (2.27) can be expressed as*

$$
S_N(x) = \frac{1}{(N+1)^2+1} \sum_{n=0}^{N} (2n+1) L_n, \quad x \in [-1,1]. \tag{2.28}
$$

Proof. Using (2.11), we can show that for $m \le N \in \mathbb{N}_0$

$$
\begin{aligned}
\int_{-1}^{1} S_N(x) L_m(x)\,\mathrm{d}x &= \sum_{n=0}^{N} \frac{1}{\|\mathcal{K}_n\|_{\mathcal{E}}^2} \int_{-1}^{1} K_n(x) L_m(x)\,\mathrm{d}x \\
&= \|L_m\|_{L^2([-1,1])}^2 \left[(m^2+1) \frac{1}{\|\mathcal{K}_m\|_{\mathcal{E}}^2} - (2m+1) \sum_{k=m+1}^{N} \frac{1}{\|\mathcal{K}_k\|_{\mathcal{E}}^2} \right],
\end{aligned} \tag{2.29}
$$

and so

$$
S_N(x) = \sum_{n=0}^{N} \left[\frac{n^2+1}{\|\mathcal{K}_n\|_{\mathcal{E}}^2} - (2n+1) \sum_{m=n+1}^{N} \frac{1}{\|\mathcal{K}_m\|_{\mathcal{E}}^2} \right] L_n(x). \tag{2.30}
$$

It is easy to show that

$$
\begin{aligned}
\sum_{n=0}^{N} \frac{1}{\|\mathcal{K}_n\|_{\mathcal{E}}^2} &= \sum_{n=0}^{N} \frac{2n+1}{(n^2+1)((n+1)^2+1)} \\
&= \sum_{n=0}^{N} \left[\frac{1}{n^2+1} - \frac{1}{(n+1)^2+1} \right] \\
&= 1 - \frac{1}{(N+1)^2+1}
\end{aligned} \tag{2.31}
$$

and

$$\sum_{m=n+1}^{N} \frac{1}{\|\mathcal{K}_m\|_{\mathcal{E}}^2} = \sum_{m=0}^{N} \frac{1}{\|\mathcal{K}_m\|_{\mathcal{E}}^2} - \sum_{m=0}^{n} \frac{1}{\|\mathcal{K}_m\|_{\mathcal{E}}^2} \tag{2.32}$$
$$= \frac{1}{(n+1)^2+1} - \frac{1}{(N+1)^2+1}.$$

Applying (2.32) to (2.30) gives

$$\begin{aligned} S_N(x) &= \sum_{n=0}^{N} \left[\frac{n^2+1}{\|\mathcal{K}_n\|_{\mathcal{E}}^2} - (2n+1) \left(\frac{1}{(n+1)^2+1} - \frac{1}{(N+1)^2+1} \right) \right] L_n(x) \\ &= \sum_{n=0}^{N} \left[\frac{2n+1}{(n+1)^2+1} - \frac{2n+1}{(n+1)^2+1} + \frac{2n+1}{(N+1)^2+1} \right] L_n(x) \\ &= \sum_{n=0}^{N} \frac{2n+1}{(N+1)^2+1} L_n(x). \end{aligned} \tag{2.33}$$

∎

Now that we have this expression, we can prove the properties of S_N that will be useful when showing the main result.

Proposition 2.3. *For the functions S_N defined in (2.27), we have that*

$$|S_N(x)| < 1, \quad \forall N \in \mathbb{N}_0,\ \forall x \in [-1,1]. \tag{2.34}$$

Furthermore,

$$\lim_{N\to\infty} S_N(x) = 0, \quad \forall x \in (-1,1). \tag{2.35}$$

Proof. It is known that

$$|L_n(x)| \leq 1, \quad \forall x \in [-1,1],\ \forall n \in \mathbb{N}_0. \tag{2.36}$$

Thus for $x \in [-1,1]$ and $N \in \mathbb{N}_0$

$$\begin{aligned} |S_N(x)| &\leq \frac{1}{(N+1)^2+1} \sum_{n=0}^{N} (2n+1)|L_n(x)| \\ &\leq \frac{1}{(N+1)^2+1} \sum_{n=0}^{N} (2n+1) \\ &= \frac{N^2+1}{(N+1)^2+1} \\ &< 1. \end{aligned} \tag{2.37}$$

From [3, Thm. 61], we also have that for $n \geq 1$ and $x \in (-1,1)$

$$|L_n(x)| < \sqrt{\frac{\pi}{2n(1-x^2)}}. \tag{2.38}$$

Then for $x \in (-1,1)$ and $N \in \mathbb{N}_0$

$$
\begin{aligned}
|S_N(x)| &\leq \frac{1}{(N+1)^2+1}\left[1+\sum_{n=1}^{N}(2n+1)|L_n(x)|\right] \\
&\leq \frac{1}{(N+1)^2+1}\left[1+3\sum_{n=1}^{N} n\cdot\sqrt{\frac{\pi}{2n(1-x^2)}}\right] \\
&= \frac{1}{(N+1)^2+1}\left[1+3\cdot\sqrt{\frac{\pi}{2(1-x^2)}}\cdot\sum_{n=1}^{N}\sqrt{n}\right].
\end{aligned} \tag{2.39}
$$

We can note that

$$
\begin{aligned}
\sum_{n=1}^{N}\sqrt{n} &\leq \int_1^{N+1}\sqrt{x}\,\mathrm{d}x \\
&= \frac{2}{3}(N+1)^{3/2}-\frac{2}{3}.
\end{aligned} \tag{2.40}
$$

So

$$
|S_N(x)| \leq \frac{1}{(N+1)^2+1}\left[1+\sqrt{\frac{2\pi}{1-x^2}}\left((N+1)^{3/2}-1\right)\right], \tag{2.41}
$$

where the right-hand side converges to 0 as $N\to\infty$ for fixed $x\in(-1,1)$. Thus $S_N(x)\to 0$ as $N\to\infty$ for $x\in(-1,1)$. □

2.3. The Space X. We define the following inner product space with elements in

$$
X := \mathcal{C}([-\tau,0);\mathbb{R})\times\mathbb{R} \tag{2.42}
$$

and the inner product defined by

$$
(\Phi,\Psi)_X := \Phi^S\Psi^S + \frac{1}{\tau}(\Phi^D,\Psi^D)_{L^2([-\tau,0)} + \Phi^D(-\tau)\Psi^D(-\tau), \quad \Phi,\Psi\in X. \tag{2.43}
$$

It is relatively straight-forward to verify that $(\cdot,\cdot)_X$ is symmetric, bilinear, and positive definite and thus is an inner product. We will also make use of the norm $\|\cdot\|_X$ induced from this inner product. Note that X is **not** a Banach space since Cauchy sequences might not converge in X.

3. Uniform Convergence of Galerkin Solutions.

3.1. Pointwise Convergence in X. It will be helpful to prove a lemma.

LEMMA 3.1. *There is $C>0$ such that for any $N\in\mathbb{N}_0$.*

$$
\|T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))\|_X \leq C\|u(s)-u_N(s)\|_X, \tag{3.1}
$$

where $t\in[0,T]$ and $s\in[0,t]$.

Proof. We have that

$$\begin{aligned}\|T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))\|_X^2 &= \|T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))\|_{\mathcal{H}}^2 \\ &\quad + \left|[T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))]^D(-\tau)\right|^2.\end{aligned} \tag{3.2}$$

Note that for the first term on the right-hand side of (3.2), we have that

$$\begin{aligned}\|T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))\|_{\mathcal{H}} &\le Me^{\omega(t-s)}\|\mathcal{F}(u(s))-\mathcal{F}(u_N(s))\|_{\mathcal{H}} \\ &\le Me^{\omega T}\|\mathcal{F}(u(s))-\mathcal{F}(u_N(s))\|_{\mathcal{H}} \\ &= Me^{\omega T}\left|f([u(s)]^D(-\tau))-f([u_N(s)]^D(-\tau))\right| \\ &\le LMe^{\omega T}\left|[u(s)]^D(-\tau)-[u_N(s)]^D(-\tau)\right| \\ &\le LMe^{\omega T}\|u(s)-u_N(s)\|_X.\end{aligned} \tag{3.3}$$

For the second term on the right-hand side of (3.2), we consider first the case when $t-s\ge\tau$. Then

$$\begin{aligned}\left|[T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))]^D(-\tau)\right| &\le \|T_N(t-s-\tau)\Pi(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))\|_{\mathcal{H}} \\ &\le Me^{\omega T}\|\mathcal{F}(u(s))-\mathcal{F}(u_N(s))\|_{\mathcal{H}} \\ &\le LMe^{\omega T}\|u(s)-u_N(s)\|_X.\end{aligned} \tag{3.4}$$

Now consider the case when $t-s<\tau$. So we have that

$$\begin{aligned}\left|[T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))]^D(-\tau)\right| &= \left|[\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))]^D(t-s-\tau)\right| \\ &= \left|f([u(s)]^D(-\tau))-f([u_N(s)]^D(-\tau))\right|\cdot|S_N^\tau(t-s-\tau)| \\ &\le L\left|[u(s)]^D(-\tau)-[u_N(s)]^D(-\tau)\right| \\ &\le L\|u(s)-u_N(s)\|_X.\end{aligned} \tag{3.5}$$

If we define

$$C := \sqrt{2}\cdot\max\{L, LMe^{\omega T}\} \tag{3.6}$$

and apply (3.3), (3.4), and (3.5) to (3.2), then we get that

$$\|T_N(t-s)\Pi_N(\mathcal{F}(u(s))-\mathcal{F}(u_N(s)))\|_X \le C\|u(s)-u_N(s)\|_X. \tag{3.7}$$

∎

We introduce the following definitions:

$$\begin{aligned}r_N(t) &:= \|u(t)-u_N(t)\|_X, \\ \epsilon_N(t) &:= \|T(t)u_0-T_N(t)\Pi_N u_0\|_X, \\ d_N(t,s) &:= \|\left(T(t-s)-T_N(t-s)\Pi_N\right)\mathcal{F}(u(s))\|_X.\end{aligned} \tag{3.8}$$

One can apply the variation-of-constants formula and the above definitions to get that

$$\begin{aligned}r_N(t) &\le \epsilon_N(t)+\int_0^t d_N(t,s)\,\mathrm{d}s+\int_0^t \|T_N(t-s)\Pi_N\big(\mathcal{F}(u(s))-\mathcal{F}(u_N(s))\big)\|_X\,\mathrm{d}s \\ &\le \epsilon_N(t)+\int_0^t d_N(t,s)\,\mathrm{d}s+C\int_0^t r_N(s)\,\mathrm{d}s.\end{aligned} \tag{3.9}$$

Applying Grönwall's inequality to (3.9) gives

$$
\begin{aligned}
r_N(t) &\le \left[\epsilon_N(t) + \int_0^t d_N(t,s)\,\mathrm{d}s\right] + \int_0^t Ce^{C(t-s)}\left[\epsilon_N(s) + \int_0^s d_N(s,r)\,\mathrm{d}r\right]\mathrm{d}s \\
&\le \left[\epsilon_N(t) + \int_0^t d_N(t,s)\,\mathrm{d}s\right] + Ce^{CT}\int_0^t \left[\epsilon_N(s) + \int_0^s d_N(s,r)\,\mathrm{d}r\right]\mathrm{d}s.
\end{aligned} \tag{3.10}
$$

We wish to show that $r_N(t) \to 0$ as $N \to \infty$ for each fixed $t \in [0,T]$. To this end, we show that each term on the right-hand side of (3.10) converges to 0 as $N \to \infty$ and $t \in [0,T]$ fixed. The following propositions will show this.

Proposition 3.1. *For fixed $t \in [0,T]$,*

$$
\epsilon_N(t) \to 0 \text{ and } \int_0^t \epsilon_N(s)\,\mathrm{d}s \to 0 \tag{3.11}
$$

as $N \to \infty$.

Proof. From the definition of the X norm, we have that

$$
\epsilon_N(t)^2 = \|T(t)u_0 - T_N(t)\Pi_N u_0\|_{\mathcal{H}}^2 + |[T(t)u_0]^D(-\tau) - [T_N(t)\Pi_N u_0]^D(-\tau)|^2. \tag{3.12}
$$

The first term on the right-hand side converges uniformly to 0 by the Trotter-Kato theorem. For the second case, we again consider the case when $t \ge \tau$. Here we can apply the Trotter-Kato theorem again to $\|T(t-\tau)u_0 - T_N(t-\tau)\Pi_N u_0\|_{\mathcal{H}}^2$ to get the term converges to zero. When $t < \tau$, the second term becomes

$$
|u_0^D(t-\tau) - [\Pi_N u_0]^D(t-\tau)|^2 \tag{3.13}
$$

which converges to 0 uniformly by Proposition 2.2. This gives that $\epsilon_N(t) \to 0$.

To show the other convergence, note that $\epsilon_N(s)$ converges pointwisely to 0 on $[0,t]$. Furthermore, we may uniformly bound $\epsilon_N(s)$ by again observing the equality (3.12) and applying the uniform bounds on $\|T_N(\cdot)\|_{\mathcal{H}}$ and on $[\Pi_N u_0]^D$. Then by the Bounded Convergence Theorem, we have $\int_0^t \epsilon_N(s)\,\mathrm{d}s \to 0$. ∎

Proposition 3.2. *For fixed $t \in [0,T]$,*

$$
\int_0^t d_N(t,s)\,\mathrm{d}s \to 0 \text{ and } \int_0^t \int_0^s d_N(s,r)\,\mathrm{d}r\,\mathrm{d}s \to 0, \tag{3.14}
$$

as $N \to \infty$.

Proof. We can again apply the definition of $\|\cdot\|_X$ to get that

$$
\begin{aligned}
d_N^2(t,s) = &\|\left(T(t-s) - T_N(t-s)\Pi_N\right)\mathcal{F}(u(s))\|_{\mathcal{H}}^2 \\
&+ |[T(t-s)\mathcal{F}(u(s))]^D(-\tau) - [T_N(t-s)\Pi_N\mathcal{F}(u(s))]^D(-\tau)|^2.
\end{aligned} \tag{3.15}
$$

For fixed t and s, the first term of the right-hand side converges to zero. For $t-s \ge \tau$ the second term will similarly converge to 0. For $t-s < \tau$, the second term will become

$$
|0 - [\Pi_N\mathcal{F}(u(s))]^D(t-s-\tau)| = |f([u(s)]^D(-\tau))| \cdot |S_N(t-s-\tau)|, \tag{3.16}
$$

which converges a.e. to 0 by (2.35). So for fixed t, $d_N(t,s)$ converges a.e. to 0 for $s \in [0,t]$. Furthermore, we can uniformly bound $d_N(t,s)$ by (2.34).Thus by the Bounded Convergence Theorem, we have $\int_0^t d_N(t,s)\,\mathrm{d}s \to 0$ as $N \to \infty$.

The second convergence follows by the observations that $\int_0^{\cdot} d_N(\cdot,r)\,\mathrm{d}r$ converges pointwise to 0 by our earlier work and can uniformly bounded on $[0,t]$. This allows us to apply the Bounded Convergence Theorem to get that $\int_0^t \int_0^s d_N(s,r)\,\mathrm{d}r\,\mathrm{d}s \to 0$ as $N \to \infty$. ∎

We may now state our result.

THEOREM 3.3. *For $t \in [0,T]$,*

$$\lim_{N\to\infty} \|u(t) - u_N(t)\|_X = 0. \tag{3.17}$$

Proof. Apply propositions (3.1) and (3.2) to the inequality in (3.10). ∎

3.2. Uniform Convergence.

LEMMA 3.2. *The following convergences hold:*

$$\lim_{N\to\infty} \int_0^T \left|[u_N(s)]^D(-\tau) - [u(s)]^D(-\tau)\right|^2 \,\mathrm{d}s, \tag{3.18}$$

and

$$\lim_{N\to\infty} \int_0^T \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s)\|_{\mathcal{H}}\,\mathrm{d}s = 0. \tag{3.19}$$

Proof. Note that

$$\int_0^T \left|[u_N(s)]^D(-\tau) - [u(s)]^D(-\tau)\right|^2 \,\mathrm{d}s \le \sum_{k=0}^{m} \int_{-\tau}^{0} \left|[u_N(k\tau)]^D(\theta) - [u(k\tau)]^D(\theta)\right|^2 \,\mathrm{d}\theta, \tag{3.20}$$

for m such that $T - \tau \le m\tau < T$. In other words,

$$\left\|[u_N(\cdot)]^D(-\tau) - [u(\cdot)]^D(-\tau)\right\|^2_{L^2([0,T];\mathbb{R})} \le \sum_{k=0}^{m} \left\|[u_N(k\tau)]^D - [u(k\tau)]^D\right\|^2_{L^2([-\tau,0);\mathbb{R})}. \tag{3.21}$$

It is a simple corollary of Theorem 3.3 that $\|[u_N(t)]^D - [u(t)]^D]\|_{L^2([0,T];\mathbb{R})} \to 0$ as $N \to \infty$ for any $t \in [0,T]$. This gives that the right side of (3.21) converges to 0 as $N \to \infty$, and thus the left side of (3.21) also converges to 0 as $N \to \infty$. This proves (3.18).

To prove the other convergence, note that

$$\begin{aligned} \int_0^T \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s))\|_{\mathcal{H}}\,\mathrm{d}s &= \int_0^T \left|f\left([u_N(s)]^D(-\tau)\right) - f\left([u(s)]^D(-\tau)\right)\right| \,\mathrm{d}s \\ &\le L\int_0^T \left|[u_N(s)]^D(-\tau) - [u(s)]^D(-\tau)\right| \,\mathrm{d}s \\ &= L\|[u_N(\cdot)]^D(-\tau) - [u(\cdot)]^D(-\tau)\|_{L^1([0,T];\mathbb{R})}. \end{aligned} \tag{3.22}$$

Noting that $L^2([0,T];\mathbb{R})$ is continuously embedded in $L^1([0,T];\mathbb{R})$ and applying (3.18) proves that (3.19) holds. ∎

THEOREM 3.4. *The sequence of functions* $\{u_N\}_{N=0}^{\infty}$, *where*

$$u_N : [0,T] \mapsto \mathcal{H}, \qquad N \in \mathbb{N}_0, \tag{3.23}$$

is uniformly equicontinuous.

Proof. Suppose $t_0, t_1 \in [0,T]$ and $t_0 \leq t_1$. Denote $\delta := t_1 - t_0$. Applying the variation-of-constants formula, we have that for $N \in \mathbb{N}_0$

$$\begin{aligned} \|u_N(t_0) - u_N(t_1)\|_{\mathcal{H}} \leq & \underbrace{\|(T_N(t_0) - T_N(t_0+\delta))\Pi_N u_0\|_{\mathcal{H}}}_{\mathrm{I}(\delta,N)} \\ & + \underbrace{\left\| \int_0^{t_0} [T_N(t_0 - s) - T_N(t_0 + \delta - s)]\Pi_N \mathcal{F}(u_N(s))\,\mathrm{d}s \right\|_{\mathcal{H}}}_{\mathrm{II}(\delta,N)} \\ & + \underbrace{\left\| \int_{t_0}^{t_0+\delta} T_N(t_0 + \delta - s)\Pi_N \mathcal{F}(u_N(s))\,\mathrm{d}s \right\|_{\mathcal{H}}}_{\mathrm{III}(\delta,N)} . \end{aligned} \tag{3.24}$$

We show that for each of these terms, the dependence on δ and N can be separated.

I. We have that

$$\begin{aligned} \mathrm{I}(\delta,N) &= \|T_N(t_0)(I - T_N(\delta))\Pi_N u_0\|_{\mathcal{H}} \\ &\leq Me^{\omega t_0}\|(I - T_N(\delta))\Pi_N u_0\|_{\mathcal{H}} \\ &= Me^{\omega t_0}\|(\Pi_N - T_N(\delta)\Pi_N)u_0\|_{\mathcal{H}} \\ &\leq Me^{\omega t_0}\|(I - T_N(\delta)\Pi_N)u_0\|_{\mathcal{H}} \\ &\leq Me^{\omega T}\|(I - T_N(\delta)\Pi_N)u_0\|_{\mathcal{H}} \\ &\leq Me^{\omega T}\left[\|(I - T(\delta))u_0\|_{\mathcal{H}} + \|(T(\delta) - T_N(\delta)\Pi_N)u_0\|_{\mathcal{H}}\right] \\ &\leq Me^{\omega T}\left[\|(I - T(\delta))u_0\|_{\mathcal{H}} + \sup_{t\in[0,T]}\|(T(t) - T_N(t)\Pi_N)u_0\|_{\mathcal{H}}\right]. \end{aligned} \tag{3.25}$$

Now define the following functions:

$$\mathrm{I}^*(\delta) := Me^{\omega T} \times \|(I - T(\delta))u_0\|_{\mathcal{H}} \tag{3.26}$$

and

$$\mathrm{I}^{**}(N) := Me^{\omega T} \times \sup_{t\in[0,T]}\|(T(t) - T_N(t)\Pi_N)u_0\|_{\mathcal{H}} \tag{3.27}$$

Note that $\lim_{\delta\to 0^+}\mathrm{I}^*(\delta) = 0$ by the continuity of $T(t)$ and $\lim_{N\to\infty}\mathrm{I}^{**}(N) = 0$ by the Trotter-Kato theorem.

II. We have that

$$
\begin{aligned}
\mathrm{II}(\delta, N) &\leq \int_0^{t_0} \|(T_N(t_0 - s) - T_N(t_0 + \delta - s))\Pi_N \mathcal{F}(u_N(s))\|_{\mathcal{H}} \,\mathrm{d}s \\
&\leq M e^{\omega T} \int_0^{t_0} \|(I - T_N(\delta)\Pi_N)\mathcal{F}(u_N(s))\|_{\mathcal{H}} \,\mathrm{d}s \\
&\leq M e^{\omega T} \Bigg[\underbrace{\int_0^{t_0} \|(I - T_N(\delta)\Pi_N)\mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s}_{A} \\
&\qquad + \underbrace{\int_0^{t_0} \|(I - T_N(\delta)\Pi_N)(\mathcal{F}(u_N(s)) - \mathcal{F}(u(s)))\|_{\mathcal{H}} \,\mathrm{d}s}_{B} \Bigg].
\end{aligned} \tag{3.28}
$$

From here, we can note that

$$
\begin{aligned}
A &\leq \int_0^T \|(I - T(\delta))\mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s + \int_0^T \|(T(\delta) - T_N(\delta))\mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s \\
&\leq \int_0^T \|(I - T(\delta))\mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s + \int_0^T \sup_{t \in [0,T]} \|(T(t) - T_N(t))\mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s,
\end{aligned} \tag{3.29}
$$

where both of these terms can easily be shown to converge to zero as $\delta \to 0$ and $N \to \infty$, respectively. Namely, we can apply the Lebesgue Dominated Convergence Theorem. Also note that

$$
B \leq (1 + M e^{\omega T}) \int_0^T \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s, \tag{3.30}
$$

where the right-hand side converges to zero as $N \to \infty$ by (3.19). Now we set

$$
\mathrm{II}^*(\delta) := M e^{\omega T} \int_0^T \|(I - T(\delta))\mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s \tag{3.31}
$$

and

$$
\begin{aligned}
\mathrm{II}^{**}(N) := & M e^{\omega T} \Bigg[\int_0^T \sup_{t \in [0,T]} \|(T(t) - T_N(t))\mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s \\
&\qquad + \left(1 + M e^{\omega T}\right) \int_0^T \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s))\|_{\mathcal{H}} \,\mathrm{d}s \Bigg].
\end{aligned} \tag{3.32}
$$

III. We have that

$$
\begin{aligned}
\mathrm{III}(\delta, N) &\leq \int_{t_0}^{t_0+\delta} \|T_N(t_0+\delta-s)\Pi_N \mathcal{F}(u_N(s))\|_{\mathcal{H}}\, \mathrm{d}s \\
&\leq Me^{\omega T} \int_{t_0}^{t_0+\delta} \|\mathcal{F}(u_N(s))\|_{\mathcal{H}}\, \mathrm{d}s \\
&\leq Me^{\omega T} \left[\int_{t_0}^{t_0+\delta} \|\mathcal{F}(u(s))\|_{\mathcal{H}}\, \mathrm{d}s + \int_{t_0}^{t_0+\delta} \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s))\|_{\mathcal{H}}\, \mathrm{d}s \right] \\
&\leq Me^{\omega T} \left[\delta \times \sup_{t\in[0,T]} \|\mathcal{F}(u(t))\|_{\mathcal{H}} + \int_0^T \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s))\|_{\mathcal{H}}\, \mathrm{d}s \right].
\end{aligned} \tag{3.33}
$$

Note that $\sup_{t\in[0,T]} \|\mathcal{F}(u(t))\|_{\mathcal{H}}$ is finite since $\|\mathcal{F}(u(t))\|_{\mathcal{H}}$ is a continuous function. Now let

$$\mathrm{III}^*(\delta) := Me^{\omega T}\delta \times \sup_{t\in[0,T]} \|\mathcal{F}(u(t))\|_{\mathcal{H}} \tag{3.34}$$

and

$$\mathrm{III}^{**}(N) := Me^{\omega T} \int_0^T \|\mathcal{F}(u_N(s)) - \mathcal{F}(u(s))\|_{\mathcal{H}}\, \mathrm{d}s. \tag{3.35}$$

Clearly $\lim_{\delta\to 0^+} \mathrm{III}^*(\delta) = 0$. Also from (3.19) we have that $\lim_{N\to\infty} \mathrm{III}^{**}(N) = 0$. Thus,

$$
\begin{aligned}
\|u_N(t_0) - u_N(t_1)\|_{\mathcal{H}} &\leq \mathrm{I}(\delta, N) + \mathrm{II}(\delta, N) + \mathrm{III}(\delta, N) \\
&\leq [\mathrm{I}^*(\delta) + \mathrm{II}^*(\delta) + \mathrm{III}^*(\delta)] + [\mathrm{I}^{**}(N) + \mathrm{II}^{**}(N) + \mathrm{III}^{**}(N)].
\end{aligned} \tag{3.36}
$$

Let $\epsilon > 0$. We wish to choose $\delta > 0$ such that $\|u_n(t) - u_n(t')\|_{\mathcal{H}} < \epsilon$ for any $n \in \mathbb{N}_0$ and $t, t' \in [0, T]$ with $|t - t'| < \delta$. Choosing δ^* small enough so that $\mathrm{I}^*(\delta^*) + \mathrm{II}^*(\delta^*) + \mathrm{III}^*(\delta^*) < \epsilon/2$ and N large enough such that $\mathrm{I}^{**}(N) + \mathrm{II}^{**}(N) + \mathrm{III}^{**}(N) < \epsilon/2$, we get that

$$\|u_n(t) - u_n(t')\|_{\mathcal{H}} < \epsilon, \tag{3.37}$$

where $|t - t'| < \delta^*$ and $n \geq N$. For each $n \in \mathbb{N}_0$ that are less than N, we pick $\delta_n > 0$ such that $\|u_n(t) - u_n(t')\|_{\mathcal{H}} < \epsilon$ for $|t - t'| < \delta_n$. This is possible since u_n is uniformly continuous on $[0, T]$. Let $\delta = \min\{\delta^*, \delta_0, \dots, \delta_{N-1}\}$. Then δ satisfies the challenge from ϵ. This proves uniform equicontinuity. ∎

THEOREM 3.5. *For $T > 0$, we have that*

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \|u_N(t) - u(t)\|_{\mathcal{H}} = 0. \tag{3.38}$$

Proof. The above result follows directly from Theorem 3.3 and Theorem 3.4. ∎

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