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Facilis descensus Averni;
Noctes atque dies patet atri janua Ditis;
Sed revocare gradum, superasque evadere ad auras,
Hoc opus, hic labor est. Virgil (from Don's thesis!)

Acknowledgments

[This is where the acknowledgments go...]

A BU THESIS LATEX TEMPLATE

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ABSTRACT

[This is where the text for the abstract will go]

Contents

List of Abbreviations

FPUT	....	Fermi-Pasta-Ulam-Tsingou
mKdV	....	modified Korteweg-De Vries

Chapter 1

Long-Time stability of small FPUT solitary waves

1.1 Introduction

As shown in earlier work, there exists a wave solution of the FPUT lattice whose profile is well approximated by that of the kink solution to the (defocusing) mKdV. We are now interested in studying the stability of this wave solution on the FPUT lattice. The equations of motion on the lattice are given by

$$\ddot{x}_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z}. \tag{1.1}$$

where V is the interaction potential between neighboring particles and $\dot{}$ denotes the derivative with respect to the time $t \in \mathbb{R}$. Equation (1.1) can be rewritten in the strain variables $u_n := x_{n+1} - x_n$ as follows

$$\ddot{u}_n = V'(u_{n+1}) - 2V'(u_n) + V'(u_{n-1}), \quad n \in \mathbb{Z} \tag{1.2}$$

The moving wave solution in eq. (1.1) corresponds to a kink solution in eq. (1.2).

For the case where V is of the form $V(u) = \frac{1}{2}u^2 + \frac{\epsilon^2}{p+1}u^{p+1}$ for $p \geq 2$, the generalized KdV equation given by

$$2\partial_T W + \frac{1}{12}\partial_X^3 W + \partial_X(W^p) = 0, \quad X \in \mathbb{R} \tag{1.3}$$

serves as a modulation equation for solutions of eq. (1.2) (Bambusi and Ponno, 2006;

Friesecke and Pego, 1999). That is, for a local solution $W \in C([-\tau_0, \tau_0], H^s(\mathbb{R}))$ of eq. (1.3) there exist positive constants ϵ_0 and C_0 such that, for all $\epsilon \in (0, \epsilon_0)$, when initial data $(u_{\text{in}}, \dot{u}_{\text{in}}) \in \ell^2(\mathbb{R})$ satisfy

$$\|u_{\text{in}} - W(\epsilon \cdot, 0)\|_{\ell^2} + \|\dot{u}_{\text{in}} + \epsilon \partial_X W(\epsilon \cdot, 0)\|_{\ell^2} \leq \epsilon^{3/2}, \tag{1.4}$$

the unique solution to eq. (1.2) with initial data $(u_{\text{in}}, \dot{u}_{\text{in}})$ belongs to $C^1([-\tau_0 \epsilon^{-3}, \tau_0 \epsilon^{-3}]; \ell^2(\mathbb{Z}))$ and satisfies

$$\|u(t) - W(\epsilon(\cdot - t), \epsilon^3 t)\|_{\ell^2(\mathbb{Z})} + \|\dot{u}(t) + \epsilon \partial_X W(\epsilon(\cdot - t), \epsilon^3 t)\|_{\ell^2(\mathbb{Z})} \leq C_0 \epsilon^{3/2},$$
$$t \in [-\tau_0 \epsilon^{-3}, \tau_0 \epsilon^{-3}]. \tag{1.5}$$

Furthermore, the approximation can also be extended to include counter-propagating solutions of the KdV in the case where $p = 2$ (Schneider and Wayne, 2000).

The KdV approximation was extended to longer time scales on the order of $\epsilon^{-3}|\log(\epsilon)|$ by Khan and Pelinovsky in order to deduce the nonlinear metastability of small FPUT solitary waves from the orbital stability of the corresponding KdV solitary waves (Khan and Pelinovsky, 2017).

We consider the FPUT with potential

$$V(u) = \frac{1}{2}u^2 - \frac{1}{24}u^4. \tag{1.6}$$

We will introduce an ansatz that solutions of the FPUT with this potential can be well-approximated by counter-propagating solutions of mKdV equations.

1.2 Counter-Propagating Waves Ansatz

We make the assumption that solutions of eq. (1.2) can be expressed as a sum of two counter-propagating small-amplitude waves, i.e.,

$$u_n(t) \approx \epsilon f(\epsilon(n+t), \epsilon^3 t) + \epsilon g(\epsilon(n-ct), \epsilon^3 t) + \epsilon^3 \phi(\epsilon n, \epsilon t) \tag{1.7}$$

where we allow f to have a fixed non-zero limit, f_∞, at positive infinity and ϕ captures the interaction effects between f and g. The wave speed of g is given by

$$c = c(\epsilon, f_\infty) = 1 - \frac{\epsilon^2 f_\infty^2}{4}. \tag{1.8}$$

Plugging in the ansatz in eq. (1.7) back into eq. (1.2) and doing formal calculations gives that the approximation holds up to order ϵ^6 terms if the the functions f and g satisfy

$$2\partial_2 f = -\frac{1}{6}\partial_1(f^3) + \frac{1}{12}\partial_1^3 f \tag{1.9}$$

and

$$-2\partial_2 g = -\frac{1}{6}\partial_1(g^3 + 3f_\infty g^2) + \frac{1}{12}\partial_1^3 g, \tag{1.10}$$

and ϕ satisfies

$$\begin{aligned}\partial_2^2\phi(\xi,\tau) = \partial_1^2\phi(\xi,\tau) - \frac{1}{6}\partial_1^2\big[&3(f^2(\xi+\tau,\epsilon^2\tau) - f_\infty^2)g(\xi - c\tau, \epsilon^2\tau) \\ &+ 3(f(\xi+\tau,\epsilon^2\tau) - f_\infty)g^2(\xi - c\tau,\epsilon^2\tau)\big]\end{aligned} \tag{1.11}$$

Note that eq. (1.9) is the defocusing mKdV equation and eq. (1.10) is a type of generalized KdV equation.

A natural choice of function space for g is a Sobolev space like $H^k(\mathbb{R})$. However, for f, we want to allow the possibility of the function approaching a non-zero limit at positive and negative infinity while also having sufficient regularity.

Definition 1. *For $k \in \mathbb{N}$, let $\mathcal{X}^k(\mathbb{R})$ be the Banach space*

$$\mathcal{X}^k(\mathbb{R}) := \{f \in L^\infty(\mathbb{R}) \mid f' \in H^{k-1}(\mathbb{R})\} \tag{1.12}$$

with norm

$$\|f\|_{\mathcal{X}^k(\mathbb{R})} := \|f\|_{L^\infty(\mathbb{R})} + \|f'\|_{H^{k-1}(\mathbb{R})}. \tag{1.13}$$

Then $\mathcal{X}^k$ is the set of L^∞ functions which are k times weakly differentiable and whose derivatives are in L^2. That this is a Banach space follows from the Banach space isomorphism

$$\mathcal{X}^k(\mathbb{R}) \cong L^\infty(\mathbb{R}) \cap \dot{H}^1(R) \cap \dot{H}^k(\mathbb{R}), \tag{1.14}$$

where $\dot{H}^k(\mathbb{R})$ denotes the homogeneous Sobolev spaces. For convenience, we let $\mathcal{X}^0(\mathbb{R})$ denote $L^\infty(\mathbb{R})$

The space $\mathcal{X}^k$ is a natural one for f, and allows f to be kink solutions of eq. (1.9). We also have the following inequalities for products of functions in $\mathcal{X}^k$ and H^k that will be useful.

Lemma 1. *For non-negative integers k, there is a $C > 0$ such that*

$$\|fg\|_{H^k} \leq C\|f\|_{\mathcal{X}^k}\|g\|_{H^k} \tag{1.15}$$

for any $f \in \mathcal{X}^k(\mathbb{R})$ and $g \in H^k(\mathbb{R})$.

Lemma 2. *For non-negative integers k, there is a $C > 0$ such that*

$$\|fg\|_{\mathcal{X}^k} \leq C\|f\|_{\mathcal{X}^k}\|h\|_{\mathcal{X}^k} \tag{1.16}$$

for any $f, g \in \mathcal{X}^k(\mathbb{R})$.

See appendix A for proofs.

However, for our main result, we require that ϕ, the term which captures the interaction effects, remains uniformly bounded for all time. Intuitively, if f and g localized, the inhomogeneous term in eq. (1.11) will quickly go to zero and ϕ will no longer experience growth in time. Thus we require that f and g quickly decay to their respective limits at infinity. This is enforced by assuming the functions belong to appropriate weighted Banach spaces.

A suitable choice of space for g is the weighted Sobolev spaces $H_n^k(\mathbb{R})$. Here, H_n^k for $k, n \in \mathbb{N} \cup \{0\}$

$$H_n^k(\mathbb{R}) := \{g \in H^k(\mathbb{R}) \mid g\langle\cdot\rangle^n \in H^k\} \tag{1.17}$$

where $\langle x \rangle = \sqrt{1+x^2}$. The norm on this space is

$$\|g\|_{H_n^k(\mathbb{R})} := \|g\langle\cdot\rangle^n\|_{H^k(\mathbb{R})}. \tag{1.18}$$

This space has the useful property that if $g \in H_n^k$, then its Fourier transform, $\hat{g}$, is in H_k^n and

$$c\|\hat{g}\|_{H_k^n} \leq \|g\|_{H_n^k} \leq C\|\hat{g}\|_{H_k^n} \tag{1.19}$$

for $c, C > 0$ and independent on g.

We want an analogous space for f, but allowing for non-zero limits at infinity. Let $\langle\cdot\rangle_+ : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\langle x \rangle_+ = \begin{cases} \langle x \rangle, & x > 1 \\ 1, & x < 0 \end{cases} \tag{1.20}$$

and $\langle\cdot\rangle_+$ continued smoothly between 0 and 1 such that it is always greater than or equal to 1. Thus $\langle\cdot\rangle_+$ is a function that only acts like $\langle\cdot\rangle$ for positive numbers. The function $\langle\cdot\rangle_-$ is similarly defined but for the negative numbers.

Definition 2. *Define $\mathcal{X}_{n^+}^k(\mathbb{R})$ to be the Banach space of functions where*

$$\mathcal{X}_{n^+}^k(\mathbb{R}) := \{f \in \mathcal{X}^k(\mathbb{R}) \mid \lim_{x\to\infty} f(x) = f_\infty \text{ and } (f - f_\infty)\langle\cdot\rangle_+^n \in \mathcal{X}^k(\mathbb{R})\} \tag{1.21}$$

with norm given by

$$\|f\|_{\mathcal{X}^k_{n^+}(\mathbb{R})} := |f_\infty| + \|(f - f_\infty)\langle\cdot\rangle_+^n\|_{\mathcal{X}^k(\mathbb{R})} \tag{1.22}$$

Similarly,

$$\mathcal{X}^k_{n^-}(\mathbb{R}) := \{f \in \mathcal{X}^k(\mathbb{R}) \mid \lim_{x\to-\infty} f(x) = f_{-\infty} \text{ and } (f - f_{-\infty})\langle\cdot\rangle_-^n \in \mathcal{X}^k(\mathbb{R})\} \tag{1.23}$$

and

$$\|f\|_{\mathcal{X}^k_{n^-}(\mathbb{R})} := |f_{-\infty}| + \|(f - f_{-\infty})\langle\cdot\rangle_-^n\|_{\mathcal{X}^k(\mathbb{R})} \tag{1.24}$$

Define $\mathcal{X}^k_n(\mathbb{R})$ to be the intersection of these Banach spaces. That is,

$$\mathcal{X}^k_n(\mathbb{R}) := \mathcal{X}^k_{n^+}(\mathbb{R}) \cap \mathcal{X}^k_{n^-}(\mathbb{R}), \quad \|f\|_{\mathcal{X}^k_n(\mathbb{R})} := \|f\|_{\mathcal{X}^k_{n^+}(\mathbb{R})} + \|f\|_{\mathcal{X}^k_{n^-}(\mathbb{R})}. \tag{1.25}$$

That $\mathcal{X}^k_{n^\pm}$ are Banach spaces follows from the fact that there exists a linear isomorphism between the Banach space $\mathbb{R} \times \mathcal{X}^k$ and these spaces, which is given by

$$(\alpha, f) \mapsto \alpha + f\langle\cdot\rangle_\pm^{-n}. \tag{1.26}$$

The definitions above are used to prove that ϕ remains bounded for all time. The idea behind the proof is similar to that of (Schneider and Wayne, 2000, Lemma 3.1); if f and g are localized solutions, then the interaction terms of eq. (1.11) will decay quickly and so ϕ will remain bounded. This decay can be quantified by the following lemma.

Lemma 3. *For each $k \geq 0$ and $c \geq c_0 > 0$, there exists $C > 0$ depending only on k such that*

$$\left\| \frac{1}{\langle\cdot+\tau\rangle_+^2 \langle\cdot - c\tau\rangle^2} \right\|_{C^k} \leq C \sup_{x\in\mathbb{R}} \frac{1}{\langle x+\tau\rangle_+^2 \langle x - c\tau\rangle^2}. \tag{1.27}$$

Furthermore,

$$\int_0^\infty \sup_{x\in\mathbb{R}} \frac{1}{\langle x+\tau\rangle_+^2 \langle x - c\tau\rangle^2}\, d\tau < \infty. \tag{1.28}$$

See appendix A for proof.

Proposition 1. *Fix $T_0 > 0$ and suppose that $f \in C([0,T_0], \mathcal{X}_2^{k+1}(\mathbb{R}))$ and $g \in C([0,T_0], H_2^{k+1}(\mathbb{R}))$, with $k > 2$ an integer. Also, suppose that $f(X,T) \to f_\infty$ as $X \to \infty$ for any $T \in [0,T_0]$. Then there exists a constant $C > 0$ such that*

$$\sup_{t\in[0,T_0/\epsilon^3]} \|\phi(\cdot,\epsilon t)\|_{H^k} \leq C\left(\sup_{t\in[0,T_0/\epsilon^3]}\left\{\|f(\cdot,\epsilon^3 t)\|_{\mathcal{X}_2^{k+1}}, \|g(\cdot,\epsilon^3 t)\|_{H_2^{k+1}}\right\}\right)^3 \tag{1.29}$$

and

$$\sup_{t\in[0,T_0/\epsilon^3]} \|\psi(\cdot,\epsilon t)\|_{H^{k-1}} \leq C\left(\sup_{t\in[0,T_0/\epsilon^3]}\left\{\|f(\cdot,\epsilon^3 t)\|_{\mathcal{X}_2^{k+1}}, \|g(\cdot,\epsilon^3 t)\|_{H_2^{k+1}}\right\}\right)^3, \tag{1.30}$$

where $\psi = \partial_2\phi$.

Proof. Set $\partial_2\phi = \psi$. Taking the Fourier transform $\mathcal{F}$ on both sides of eq. (1.11) and writing the ODE as a first order system, we get that

$$\begin{aligned}\partial_2\begin{bmatrix}\hat{\phi}(k,\tau)\\ \hat{\psi}(k,\tau)\end{bmatrix} &= \begin{bmatrix}\hat{\psi}(k,\tau)\\ -k^2\hat{\phi}(k,\tau)\end{bmatrix}\\ &+\begin{bmatrix}0\\ \frac{1}{2}k^2\mathcal{F}[(f^2(\cdot+\tau),\epsilon^2\tau)-f_\infty^2)g(\cdot-c\tau,\epsilon^2\tau)+(f(\cdot+\tau,\epsilon^2\tau)-f_\infty)g^2(\cdot-c\tau,\epsilon^2\tau)](k)\end{bmatrix}.\end{aligned} \tag{1.31}$$

The semigroup generated by the linear part can be computed explicitly. Putting the solution into variation of constants form with initial conditions set to zero gives

$$\begin{aligned}\hat{\phi}(k,T) &= \frac{1}{2}\int_0^T k\sin(k(T-\tau))\times\\ &\mathcal{F}[(f^2(\cdot+\tau),\epsilon^2\tau)-f_\infty^2)g(\cdot-c\tau,\epsilon^2\tau)+(f(\cdot+\tau,\epsilon^2\tau)-f_\infty)g^2(\cdot-c\tau,\epsilon^2\tau)](k)\,d\tau\end{aligned} \tag{1.32}$$

and

$$\begin{aligned}\hat{\psi}(k,T) &= \frac{1}{2}\int_0^T k^2\cos(k(T-\tau))\times\\ &\mathcal{F}[(f^2(\cdot+\tau),\epsilon^2\tau)-f_\infty^2)g(\cdot-c\tau,\epsilon^2\tau)+(f(\cdot+\tau,\epsilon^2\tau)-f_\infty)g^2(\cdot-c\tau,\epsilon^2\tau)](k)\,d\tau\end{aligned} \tag{1.33}$$

Hence we can get that

$$
\begin{aligned}
&\|\phi(\cdot,T)\|_{H^k}\\
&\quad\le C\|\hat{\phi}(\cdot,T)\|_{H^0_k}\\
&\quad\le C\int_0^T \|\partial_1((f^2(\cdot+\tau)-f_\infty^2)g(\cdot-c\tau))\|_{H^k}+\|\partial_1((f(\cdot+\tau)-f_\infty)g^2(\cdot-c\tau))\|_{H^k}\,\mathrm{d}\tau\\
&\quad\le C\int_0^T \|f(\cdot+\tau)\partial_1 f(\cdot+\tau)g(\cdot-c\tau)\|_{H^k}+\|(f^2(\cdot+\tau)-f_\infty^2)\partial_1 g(\cdot-c\tau)\|_{H^k}\\
&\qquad+\|\partial_1 f(\cdot+\tau)g^2(\cdot-c\tau)\|_{H^k}+\|(f(\cdot+\tau)-f_\infty)\partial_1 g(\cdot-c\tau)\|_{H^k}\,\mathrm{d}\tau\\
&\quad\le C\int_0^T \sup_{x\in\mathbb{R}}\frac{1}{\langle x+\tau\rangle_+^2\langle x-c\tau\rangle^2}\times\left(\|f\|^2_{\mathcal{X}_2^{k+1}}\|g\|_{H_2^{k+1}}+\|f\|_{\mathcal{X}_2^{k+1}}\|g\|^2_{H_2^{k+1}}\right)\mathrm{d}\tau,
\end{aligned}
$$

whence eq. (1.29) follows. The proof for eq. (1.30) is analogous. □

1.3 Setup of Lattice Equations

The scalar second-order differential equation eq. (1.2) with potential V given by eq. (1.6) can be rewritten as the following first-order system:

$$
\begin{cases}
\dot{u}_n = q_{n+1}-q_n, \\
\dot{q}_n = u_n-u_{n-1}-\frac{1}{6}(u_n^3-u_{n-1}^3),
\end{cases}
\qquad n\in\mathbb{Z}. \tag{1.34}
$$

We will now introduce the traveling wave ansatz for the system in eq. (1.34), but we first must assume certain regularity and decay of f and g.

Assumption 1. *Let f and g be solutions of eqs.* (1.9) *and* (1.10)*, respectively. Assume that*

$$f\in C_b(\mathbb{R},\mathcal{X}_2^6(\mathbb{R}))\quad \textit{and}\quad g\in C_b(\mathbb{R},H_2^6(\mathbb{R})).$$

Furthermore, assume that f has fixed limits in its spatial variable at $\pm\infty$ given by $f_{\pm\infty}$.

The traveling wave ansatz for u_n and q_n is then given by

$$u_n(t) = \epsilon f(\epsilon(n+t), \epsilon^3 t) + \epsilon g(\epsilon(n-ct), \epsilon^3 t) + \epsilon^3 \phi(\epsilon n, \epsilon t) + \mathcal{U}_n(t) \tag{1.35}$$

and

$$q_n(t) = \epsilon F(\epsilon(n+t), \epsilon^3 t) + \epsilon G(\epsilon(n-ct), \epsilon^3 t) + \epsilon^3 \Phi(\epsilon n, \epsilon t) - \epsilon F_{-\infty} + \mathcal{Q}_n(t). \tag{1.36}$$

The wave speed c is again given by eq. (1.8). The functions F, G, an Φ are chosen to minimize the remainder terms from plugging in the ansatz back into eq. (1.34) and are given explicitly below:

$$F := f - \frac{\epsilon}{2}\partial_1 f + \frac{\epsilon^2}{8}\partial_1^2 f - \frac{\epsilon^2}{12} f^3 - \frac{\epsilon^3}{48}\partial_1^3 f + \frac{\epsilon^3}{8} f^2 \partial_1 f \tag{1.37}$$

$$G := -g + \frac{\epsilon}{2}\partial_1 g + \frac{\epsilon^2 L^2}{4} g + \frac{\epsilon^2}{12}(g^3 + 3Lg^2) - \frac{\epsilon^2}{8}\partial_1^2 g + \frac{\epsilon^3}{48}\partial_1^3 g - \frac{\epsilon^3}{24}\partial_1(g^3 + 3Lg^2) - \frac{\epsilon^3 L^2}{8}\partial_1 g \tag{1.38}$$

$$\Phi := \partial_1^{-1}\psi - \frac{\epsilon}{2}\psi. \tag{1.39}$$

Here $\psi = \partial_2 \phi$ and ∂_1^{-1} is defined as a Fourier multiplier. That $\partial_1^{-1}\psi$ is well-defined and in $H^5(\mathbb{R})$ follows from eq. (1.33). Assumption 1 implies that F has fixed limits in its spatial variable at $\pm\infty$ given by $F_{\pm\infty} = f_{\pm\infty} - \frac{\epsilon^2}{12} f_{\pm\infty}^3$.

We want $\mathcal{U}(t)$ and $\mathcal{Q}(t)$ to be elements of $\ell^2(\mathbb{Z})$ (at least locally in time). However, to satisfy $\mathcal{Q}(0) \in \ell^2(\mathbb{Z})$ and $\dot{u}_n(0) = q_{n+1}(0) - q_n(0)$, a compatibility condition must hold.

Assumption 2. *Assume that*

$$\sum_{n=-\infty}^{\infty} \dot{u}_n(0) = \epsilon F_{+\infty} - \epsilon F_{-\infty}.$$

Note that if this did not hold, then $\mathcal{Q}_n(0) \not\to 0$ as $n \to \infty$ and $\mathcal{Q}(0) \notin \ell^2(\mathbb{Z})$.

1.4 Preparatory Estimates

To make estimates on the error terms $\mathcal{U}$ and $\mathcal{Q}$, we require an appropriate choice of W. Based on the above discussion, W must be at least be a continuous solution in $L^\infty(\mathbb{R})$ space. Further regularity is required to make sense of **??** as well as further derivatives of P_ϵ. So W must be bounded but have spatial derivatives that decay at infinity. The solution of **??** must then be continuous in the following normed space.

Definition 3. *For $s \geq 1$, the normed space $\mathcal{X}_s$ is the set of functions*

$$\mathcal{X}_s := \{u \in L^\infty(\mathbb{R}) : u'(x) \in H^{s-1}(\mathbb{R})\} \tag{1.40}$$

with a norm defined by

$$\|u\|_{X_s} := \|u\|_{L^\infty(\mathbb{R})} + \|u'\|_{H^{s-1}(\mathbb{R})}. \tag{1.41}$$

In fact, $\mathcal{X}_s$ is a Banach space.

Proposition 2. *For $s \geq 1$, the normed vector space $\mathcal{X}_s$ is a Banach space.*

Proof. Suppose that $\{u_n\} \subset \mathcal{X}_s$ is a Cauchy sequence. Then for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n, m \geq N$

$$\|u_n - u_m\|_{\mathcal{X}_s} = \|u_n - u_m\|_{L^\infty(\mathbb{R})} + \|u_n' - u_m'\|_{H^{s-1}(\mathbb{R})} < \epsilon. \tag{1.42}$$

Clearly, we have that $\{u_n\}$ is a Cauchy sequence in $L^\infty(\mathbb{R})$ and $\{u_n'\}$ is a Cauchy sequence in $H^{s-1}(\mathbb{R})$. Since L^∞ and H^{s-1} are Banach spaces, we have $u \in L^\infty$ and $v \in H^{s-1}$ such that

$$\begin{aligned} u_n &\to u \quad \text{in } L^\infty(\mathbb{R}) \\ u_n' &\to v \quad \text{in } H^{s-1}(\mathbb{R}). \end{aligned} \tag{1.43}$$

To show that $u_n \to u$ in $\mathcal{X}_s$, we must demonstrate that $v = u'$ (where $'$ denotes a distributional derivative which is *a priori* defined). One can show that $H^{s-1}(\mathbb{R}) \hookrightarrow$

$L^2(\mathbb{R})$: for $w \in H^{s-1}(\mathbb{R})$ we have

$$\begin{aligned}\|w\|_{L^2} &= \|\hat{w}\|_{L^2} \\ &= \left(\int_{\mathbb{R}} |\hat{w}(\xi)|^2 \, d\xi\right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}} |\hat{w}(\xi)|^2 (1 + |\xi|^{2s}) \, d\xi\right)^{1/2} \\ &= \|w\|_{H^{s-1}}.\end{aligned} \tag{1.44}$$

Thus we also have $u_n' \to v$ in $L^2(\mathbb{R})$.

Let $\phi \in C_c^\infty(\mathbb{R})$ be a test function. By L^2 convergence, we have that

$$\int_{\mathbb{R}} v\phi \, dx = \lim_{n\to\infty} \int_{\mathbb{R}} u_n' \phi \, dx. \tag{1.45}$$

Also, by applying the dominated convergence theorem (and possibly taking a subsequence to get almost everywhere pointwise convergence), we have that

$$\int_{\mathbb{R}} u\phi' \, dx = \lim_{n\to\infty} \int_{\mathbb{R}} u_n \phi' \, dx. \tag{1.46}$$

Therefore, we get that

$$\begin{aligned}\int_{\mathbb{R}} v\phi \, dx &= \lim_{n\to\infty} \int_{\mathbb{R}} u_n' \phi \, dx \\ &= \lim_{n\to\infty} - \int_{\mathbb{R}} u_n \phi' \, dx \\ &= - \int_{\mathbb{R}} u\phi' \, dx.\end{aligned} \tag{1.47}$$

Hence, $u' = v$ and $\mathcal{X}_s$ is a Banach space. □

For our case, we will need at least six spatial derivatives of W and so W should belong to $C([-\tau_0, \tau_0]; \mathcal{X}_6)$. Note that the kink solutions of **??** are in this space; for instance, if $W(X,T) = \phi(X)$ with

$$\phi(X) = \frac{1}{\sqrt{2}} \tanh\left(\frac{X}{\sqrt{2}}\right), \tag{1.48}$$

then W is a solution to the mKdV equation and $W \in C(\mathbb{R}; \mathcal{X}_s)$ for any $s \geq 1$.

In order to prove long-time stability, we must get estimates of the $\ell^2(\mathbb{Z})$ norms of the residual terms and the nonlinearity. The following lemma proved in (Dumas and Pelinovsky, 2014) will be useful in bounding $\ell^2(\mathbb{Z})$ norms by Sobolev norms of W.

Lemma 4. *There exists $C > 0$ such that for all $X \in H^1(\mathbb{R})$ and $\epsilon \in (0,1)$,*

$$\|x\|_{\ell^2} \leq C\epsilon^{-1/2}\|X\|_{H^1},$$

where $x_n := X(\epsilon n)$, $n \in \mathbb{Z}$.

We can then show the following estimates.

Lemma 5. *Let $W \in C([-\tau_0, \tau_0]; \mathcal{X}_6)$ be a solution of the modified KdV equation* **??** *and $\tau_0 > 0$. Define*

$$\delta := \sup_{\tau \in [-\tau_0, \tau_0]} \|W(\tau)\|_{\mathcal{X}_s}. \tag{1.49}$$

There exists a positive δ-independent constant C such that the residual and nonlinear terms satisfy

$$\|\mathrm{Res}^{(1)}(t)\|_{\ell^2} + \|\mathrm{Res}^{(2)}(t)\|_{\ell^2} \leq C\epsilon^{9/2}(\delta + \delta^5) \tag{1.50}$$

and

$$\|\mathcal{R}(W, \mathcal{U})(t)\|_{\ell^2} \leq C\epsilon^2(\delta + \|\mathcal{U}(t)\|_{\ell^2})\|\mathcal{U}(t)\|_{\ell^2}^2 \tag{1.51}$$

for every $t \in [-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}]$ and $\epsilon \in (0,1)$.

Proof. Plugging P_ϵ into $\mathrm{Res}^{(1)}(t)$ and using the Taylor remainder theorem, we get that the terms of order ϵ^4 or lower cancel out and we are left with

$$\begin{aligned}
\mathrm{Res}^{(1)}(t) = \epsilon^5 \Bigg[\quad & \epsilon \left(c - 1 + \frac{\epsilon^2}{24}\right) \partial_X W(\epsilon(n-ct), \epsilon^3 t) \\
& - \frac{1}{24} \int_0^1 \partial_X^5 W(\epsilon(n-ct+r), \epsilon^3 t)(1-r)^4 \, dr \\
& + \frac{1}{12} \int_0^1 \partial_X^5 W(\epsilon(n-ct+r), \epsilon^3 t)(1-r)^3 \, dr \\
& - \frac{1}{16} \int_0^1 \partial_X^5 W(\epsilon(n-ct+r), \epsilon^3 t)(1-r)^2 \, dr \\
& + \frac{1}{48} \int_0^1 \partial_X^5 W(\epsilon(n-ct+r), \epsilon^3 t)(1-r) \, dr. \Bigg]
\end{aligned} \tag{1.52}$$

Each of these terms can be bounded by the $\mathcal{X}_6$ norm. Thus we get that

$$\|\mathrm{Res}^{(1)}(t)\|_{\ell^2} \leq C\epsilon^{9/2} \|W(\cdot, \epsilon t^3)\|_{\mathcal{X}_6} \leq C\epsilon^{9/2}\delta \quad \text{ for } t \in [-\tau_0 \epsilon^{-3}, \tau_0 \epsilon^{-3}]. \tag{1.53}$$

Plugging in P_ϵ into $\mathrm{Res}^{(2)}(t)$ similarly gives

$$\begin{aligned}
\mathrm{Res}^{(2)}(t) = & \, \epsilon \left(c - 1 + \frac{\epsilon^2}{24}\right) \big(\partial_X P_\epsilon(\epsilon(n-ct), \epsilon^3 t)\big) \\
& - \frac{\epsilon^3}{24} \big(\epsilon^2 \partial_X P_2(\epsilon(n-ct), \epsilon^3 t) + \epsilon^3 \partial_X P_3(\epsilon(n-ct), \epsilon^3 t)\big) \\
& - \epsilon^3 (\epsilon^2 \partial_T P_2(\epsilon(n-ct), \epsilon^3 t) + \epsilon^3 \partial_T P_3(\epsilon(n-ct), \epsilon^3 t)) \\
& + \frac{\epsilon^5}{24} \int_0^1 \partial_X^5 W(\epsilon(n-ct-r), \epsilon^3 t)(r-1)^4 \, dr \\
& - \frac{\epsilon^5}{12} \int_0^1 \partial_X^3 (W^3)(\epsilon(n-ct-r), \epsilon^3 t)(r-1)^2 \, dr.
\end{aligned} \tag{1.54}$$

The terms with order less than ϵ^5 cancel out. The integral terms can easily be bounded by the $\mathcal{X}_6$ norm For the first three terms, we can compute the expansion exactly:

$$
\begin{aligned}
\partial_X P_\epsilon &= -\partial_X W + \frac{\epsilon}{2}\partial_X^2 W + \frac{\epsilon^2}{4} W^2 \partial_X W - \frac{\epsilon^2}{8}\partial_X^3 W + \frac{\epsilon^3}{48}\partial_X^4 W - \frac{\epsilon^3}{24}\partial_X^2 (W^3) \\
\partial_X P_2 &= \frac{1}{4} W^2 \partial_X W - \frac{1}{8}\partial_X^3 W \\
\partial_X P_3 &= \frac{1}{48}\partial_X^4 W - \frac{1}{4} W(\partial_X W)^2 - \frac{1}{8} W^2 \partial_X^2 W \\
\partial_T P_2 &= \frac{-1}{96} W^2 \partial_X W + \frac{1}{16} W^4 \partial_X W - \frac{1}{16}(\partial_X W)^3 - \frac{3}{16} W \partial_X W \partial_X^2 W \\
&\quad + \frac{1}{192}\partial_X^3 W - \frac{1}{24} W^2 \partial_X^3 W + \frac{1}{192}\partial_X^5 W \\
\partial_T P_3 &= \frac{1}{192} W^2 \partial_X W - \frac{1}{32} W^4 \partial_X W + \frac{1}{96} W(\partial_X W)^2 - \frac{1}{16} W^3 (\partial_X W)^2 \\
&\quad + \frac{1}{16}(\partial_X W)^2 \partial_X^2 W + \frac{1}{32} W(\partial_X^2 W)^2 + \frac{1}{192} W^2 \partial_X^3 W + \frac{5}{96} W \partial_X W \partial_X^3 W \\
&\quad - \frac{1}{1152}\partial_X^4 W + \frac{1}{92} W^2 \partial_X^4 W - \frac{1}{1152}\partial_X^6 W
\end{aligned}
\tag{1.55}
$$

Notice that the L^2 norm of each term can be bounded by a term of the form

$$
\|W\|_{L^\infty}^k \|\partial_X W\|_{H^5}^\ell, \qquad \text{where } 1 \le k + \ell \le 5. \tag{1.56}
$$

The above term can then in turn be bounded by $C(\delta + \delta^5)$. Thus we get that

$$
\|\mathrm{Res}^{(2)}(t)\|_{\ell^2} = C\epsilon^{9/2}\left(\delta + \delta^5\right) \tag{1.57}
$$

for $t \in [-\tau_0 \epsilon^{-3}, \tau\epsilon^{-3}]$.

For the nonlinear term $\mathcal{R}(W,\mathcal{U})$, we immediately get that

$$
\begin{aligned}
\|\mathcal{R}(W,\mathcal{U})(t)\|_{\ell^2} &\le C\epsilon^2 \left[\|W(\epsilon^3 t)\|_{L^\infty} \|\mathcal{U}^2(t)\|_{\ell^2} + \|\mathcal{U}^3(t)\|_{\ell^2}\right] \\
&\le C\epsilon^2 \left[\|W(\epsilon^3 t)\|_{L^\infty} \|\mathcal{U}(t)\|_{\ell^\infty} \|\mathcal{U}(t)\|_{\ell^2} + \|\mathcal{U}(t)\|_{\ell^\infty}^2 \|\mathcal{U}(t)\|_{\ell^2}\right] \\
&\le C\epsilon^2 \left[\delta \|\mathcal{U}(t)\|_{\ell^2}^2 + \|\mathcal{U}(t)\|_{\ell^2}^3\right].
\end{aligned}
\tag{1.58}
$$

$\square$

The main result of this section will be proved using a Grönwall type estimate using an energy function defined by

$$\mathcal{E}(t) := \frac{1}{2}\sum_{n\in\mathbb{Z}} \mathcal{Q}_n^2(t) + \mathcal{U}_n^2(t) - \frac{\epsilon^2}{2} W^2(\epsilon(n-ct), \epsilon^3 t)\mathcal{U}_n^2(t). \tag{1.59}$$

The above will be nonnegative for W fixed and ϵ sufficiently small.

Lemma 6. *Let $W \in C([-\tau_0, \tau_0], \mathcal{X}_6)$ be a solution to the mKdV equation* **??** *and $\tau_0 > 0$. Define $\epsilon_0 > 0$ to be*

$$\epsilon_0 := \min\left\{1, \left(\sup_{\tau\in[-\tau_0,\tau_0]} \|W(\tau)\|_{L_\infty}\right)^{-1}\right\}. \tag{1.60}$$

For every $\epsilon \in (0, \epsilon_0)$ and for every local solution $(\mathcal{U}, \mathcal{Q}) \in C^1([-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}], \ell^2(\mathbb{Z}))$ of **??***, the energy-type quantity given in eq. (1.59) is coercive with the bound*

$$\|\mathcal{Q}(t)\|_{\ell^2}^2 + \|\mathcal{U}(t)\|_{\ell^2}^2 \leq 4\mathcal{E}(t), \quad \text{for } t \in (-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}). \tag{1.61}$$

Moreover, when δ is given by eq. (1.49), there exists $C > 0$ independent of ϵ and δ such that

$$\left|\frac{d\mathcal{E}}{dt}\right| \leq C\mathcal{E}^{1/2}\left[\epsilon^{9/2}(\delta + \delta^5) + \epsilon^3(\delta^2 + \delta^4)\mathcal{E}^{1/2} + \epsilon^2(\delta + \mathcal{E}^{1/2})\mathcal{E}\right] \tag{1.62}$$

for every $t \in [-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}]$ and $\epsilon \in (0, \epsilon_0)$.

Proof. By the choice of ϵ_0, we have for $\epsilon \in (0, \epsilon_0)$ that

$$1 - \frac{\epsilon^2}{2}\|W\|_{L^\infty}^2 \geq \frac{1}{2}. \tag{1.63}$$

Hence

$$\mathcal{E}(t) \geq \frac{1}{2}\|\mathcal{Q}\|_{\ell^2}^2 + \frac{1}{4}\|\mathcal{U}\|_{\ell^2}^2 \geq \frac{1}{4}\|\mathcal{Q}\|_{\ell^2}^2 + \frac{1}{4}\|\mathcal{U}\|_{\ell^2}^2 \tag{1.64}$$

and eq. (1.61) follows.

Taking the time derivative of $\mathcal{E}(t)$ and using that $(\mathcal{U}, \mathcal{Q})$ solve **??** gives us

$$\begin{aligned}\frac{d\mathcal{E}}{dt} = \sum_{n\in\mathbb{Z}} \Bigg[&\mathcal{Q}_n(t)\mathcal{R}_n(W,\mathcal{U})(t) + \mathcal{Q}_n(t)\mathrm{Res}^{(2)}(t) \\ &+ \mathcal{U}_n(t)\mathrm{Res}^{(1)}(t)\left(1 - \frac{\epsilon^2}{2}W^2(\epsilon(n-ct), \epsilon^3 t)\right) \\ &+ \frac{\epsilon^3}{2}W(\epsilon(n-ct),\epsilon^3 t)\mathcal{U}_n^2(t)\left(c\partial_X W(\epsilon(n-ct),\epsilon^3 t) - \epsilon^2\partial_T W(\epsilon(n-ct),\epsilon^3 t)\right)\Bigg]\end{aligned} \tag{1.65}$$

Using the Cauchy-Schwarz inequality, the estimates found in lemma 5, and coercivity of $\mathcal{E}(t)$, we get that

$$\begin{aligned}\left|\frac{d\mathcal{E}}{dt}\right| &\leq \|\mathcal{Q}\|_{\ell^2}\|\mathcal{R}(W,\mathcal{U})\|_{\ell^2} + \|\mathcal{Q}\|_{\ell^2}\|\mathrm{Res}^{(2)}\|_{\ell^2} + \|\mathcal{U}\|_{\ell^2}\|\mathrm{Res}^{(1)}\|_{\ell^2} \\ &\quad + \frac{1}{2}\epsilon^3\|W\|_{L^\infty}\|\mathcal{U}\|_{\ell^2}^2(c\|\partial_X W\|_{L^\infty} + \epsilon^2\|\partial_T W\|_{L^\infty}) \\ &\leq C\mathcal{E}^{1/2}\left[\epsilon^{9/2}(\delta+\delta^5) + \epsilon^3(\delta^2+\delta^4)\mathcal{E}^{1/2} + \epsilon^2(\delta + \mathcal{E}^{1/2})\mathcal{E}\right]\end{aligned} \tag{1.66}$$

□

1.5 Proof of Long-Time Stability

Now with the setup complete, the main result of this section can be shown. The result and proof are analogous to those of (Khan and Pelinovsky, 2017, Thm. 1).

Theorem 1. *Let $W \in C(\mathbb{R}; \mathcal{X}_6)$ be a global solution of the mKdV equation* **??** *with $\sup_{\tau\in\mathbb{R}} \|W(\tau)\|_{\mathcal{X}_6} \leq \delta$. For fixed $r \in (0, 1/2)$, there exist positive constants ϵ_0, C, and K such that for all $\epsilon \in (0, \epsilon_0)$, when initial data $(u_{\mathrm{in}}, q_{\mathrm{in}}) \in \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z})$ satisfy*

$$\|u_{\mathrm{in}} - W(\epsilon\cdot, 0)\|_{\ell^2} + \|q_{\mathrm{in}} + \epsilon\partial_X W(\epsilon\cdot, 0)\|_{\ell^2} \leq \epsilon^{3/2}, \tag{1.67}$$

the unique solution (u, q) to the FPU equation eq. (1.34) belongs to

$$C^1([-t_0(\epsilon), t_0(\epsilon)], \ell^\infty(\mathbb{Z})) \tag{1.68}$$

with $t_0(\epsilon) := rK^{-1}\epsilon^{-3}|\log(\epsilon)|$ *and satisfies*

$$\|u(t)-W(\epsilon(\cdot-ct),\epsilon^3 t)\|_{\ell^2}+\|q(t)+\epsilon\partial_X W(\epsilon(\cdot-ct),\epsilon^3 t)\|_{\ell^2} \leq C\epsilon^{3/2-r}, \quad t\in[-t_0(\epsilon),t_0(\epsilon)]. \tag{1.69}$$

Proof. From the initial conditions satisfying eq. (1.67), we have at least local solutions to the error equations. That is, there is a unique local solution to **??** where $(\mathcal{U},\mathcal{Q}) \in C^1((-t_0,t_0);\ell^2(\mathbb{Z})\times\ell^2(\mathbb{Z}))$ for some $t_0>0$.

Set $\mathcal{S} := \mathcal{E}^{1/2}$ where $\mathcal{E}$ is defined in eq. (1.59). From the bound on the initial conditions in eq. (1.67), we get that $\mathcal{S}(0) \leq C_0\epsilon^{3/2}$ for some constant $C_0>0$ and ϵ_0 chosen by eq. (1.60). For fixed constants $r\in(0,1/2)$, $C>C_0$, and $K>0$, define the maximal continuation time by

$$T_{C,K,r} := \sup\left\{T_0 \in (0, rK^{-1}\epsilon^{-3}|\log(\epsilon)|] : \mathcal{S}(t) \leq C\epsilon^{3/2-r}, t\in[-T_0,T_0]\right\}. \tag{1.70}$$

We also define the maximal evolution time of the mKdV equation as $\tau_0(\epsilon) = rK^{-1}|\log(\epsilon)|$. The goal is then to pick C and K so that $T_{C,K,r} = \epsilon^{-3}\tau_0(\epsilon)$.

We have that

$$\begin{aligned}\left|\frac{d}{dt}\mathcal{S}(t)\right| &= \frac{1}{2\mathcal{E}^{1/2}}\left|\frac{d}{dt}\mathcal{E}(t)\right| \\ &\leq C_1(\delta+\delta^5)\epsilon^{9/2} + C_2\epsilon^3\left[(\delta^2+\delta^4)+\epsilon^{-1}(\delta+\mathcal{S})\mathcal{S}\right]\mathcal{S}\end{aligned} \tag{1.71}$$

where $C_1, C_2 > 0$ are independent of δ and ϵ. While $|t| \leq T_{C,K,r}$,

$$C_2\left[(\delta^2+\delta^4)+\epsilon^{-1}(\delta+\mathcal{S})\mathcal{S}\right] \leq C_2\left[(\delta^2+\delta^4)+\epsilon^{-1}(\delta+C\epsilon^{3/2-r})C\epsilon^{3/2-r}\right], \tag{1.72}$$

where the right-hand side is continuous in ϵ for $\epsilon\in[0,\epsilon_0]$. Thus the right-hand side can be uniformly bounded by a constant independent of ϵ. Choose $K>0$ (dependent on C) sufficiently large so that

$$C_2\left[(\delta^2+\delta^4)+\epsilon^{-1}(\delta+C\epsilon^{3/2-r})C\epsilon^{3/2-r}\right] \leq K. \tag{1.73}$$

Hence, we can get that for $t \in [-T_{C,K,r}, T_{C,K,r}]$

$$
\begin{aligned}
\frac{d}{dt} e^{-\epsilon^3 K t} \mathcal{S}(t) &= -\epsilon^3 K e^{-\epsilon^3 K t} \mathcal{S} + e^{-\epsilon^3 K t} \frac{d}{dt} \mathcal{S} \\
&\leq -\epsilon^3 K e^{-\epsilon^3 K t} \mathcal{S} + e^{-\epsilon^3 K t} C_1 (\delta + \delta^5) \epsilon^{9/2} \\
&\qquad + e^{-\epsilon^3 K t} C_2 \epsilon^3 \left[(\delta^2 + \delta^4) + \epsilon^{-1} (\delta + \mathcal{S}) \mathcal{S} \right] \mathcal{S} \\
&\leq -\epsilon^3 K e^{-\epsilon^3 K t} \mathcal{S} + e^{-\epsilon^3 K t} C_1 (\delta + \delta^5) \epsilon^{9/2} + \epsilon^3 K e^{-\epsilon^3 K t} \mathcal{S} \\
&= e^{-\epsilon^3 K t} C_1 (\delta + \delta^5) \epsilon^{9/2}.
\end{aligned}
\tag{1.74}
$$

Integrating gives

$$
\begin{aligned}
\mathcal{S}(t) &\leq \left(\mathcal{S}(0) + K^{-1} C_1 (\delta + \delta^5) \epsilon^{3/2} \right) e^{\epsilon^3 K t} - \epsilon^{-3} K^{-1} C_1 (\delta + \delta^5) \\
&\leq \left(\mathcal{S}(0) + K^{-1} C_1 (\delta + \delta^5) \epsilon^{3/2} \right) e^{\epsilon^3 K t} \\
&\leq \left(\mathcal{S}(0) + K^{-1} C_1 (\delta + \delta^5) \epsilon^{3/2} \right) e^{K \tau_0(\epsilon)} \\
&\leq \left(C_0 + K^{-1} C_1 (\delta + \delta^5) \epsilon^{3/2} \right) \epsilon^{3/2 - r}
\end{aligned}
\tag{1.75}
$$

for $t \in [-T_{C,K,r}, T_{C,K,r}]$, where the last line follows in part from the definition of $\tau_0(\epsilon)$. Now choose $C > C_0$ sufficiently large so that

$$
C_0 + K^{-1} C_1 (\delta + \delta^5) \leq C. \tag{1.76}
$$

Note that our earlier choice of K can be enlarged so that eq. (1.73) still holds as well as the above inequality. Therefore, with these choices of C and K, the maximal interval can be extended to $T_{C,K,r} = \epsilon^{-3} \tau_0(\epsilon)$. □

Appendix A

Proofs of lemmas

Proof. The result follows from induction on k.

For $k = 0$, we have

$$\|fg\|_{H^0} \leq \|f\|_{L^\infty}\|g\|_{H^0}. \tag{A.1}$$

Assuming eq. (1.15) holds for $k \geq 0$, we have that

$$\begin{aligned}\|fg\|_{H^{k+1}} &\leq C\left(\|fg\|_{H^k} + \|\partial^{k+1}(fg)\|_{L^2}\right)\\ &\leq C\left(\|f\|_{\mathcal{X}^k}\|g\|_{H^k} + \|\partial^{k+1}(fg)\|_{L^2}\right),\end{aligned}$$

where the second term can be bounded by

$$\begin{aligned}\|\partial^{k+1}(fg)\|_{L^2} &\leq \|\partial^k(\partial^1 fg)\|_{L^2} + \|\partial^k(f\partial^1 g)\|_{L^2}\\ &\leq \|\partial^1 fg\|_{H^k} + \|f\partial^1 g\|_{H^k}\\ &\leq \|\partial^1 f\|_{H^k}\|g\|_{H^k} + \|f\|_{\mathcal{X}^k}\|\partial^1 g\|_{H^k}\\ &\leq \|f\|_{\mathcal{X}^{k+1}}\|g\|_{H^{k+1}} + \|f\|_{\mathcal{X}^{k+1}}\|g\|_{H^{k+1}}\\ &= 2\|f\|_{\mathcal{X}^{k+1}}\|g\|_{H^{k+1}}.\end{aligned}$$

This completes the induction. $\square$

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