BOSTON UNIVERSITY

COLLEGE OF ARTS AND SCIENCES

Dissertation

A BU THESIS LATEX TEMPLATE

by

TREVOR NORTON

B.S., Virginia Polytechnic Institute and State University, 2015
M.S., Virginia Polytechnic Institute and State University, 2018

Submitted in partial fulfillment of the

requirements for the degree of

Doctor of Philosophy

2022

Approved by

First Reader

First M. Last, PhD
Professor of Electrical and Computer Engineering

Second Reader

First M. Last
Associate Professor of ...

Third Reader

First M. Last
Assistant Professor of ...

Facilis descensus Averni;
Noctes atque dies patet atri janua Ditis;
Sed revocare gradum, superasque evadere ad auras,
Hoc opus, hic labor est.   Virgil (from Don's thesis!)

Acknowledgments

[This is where the acknowledgments go...]

A BU THESIS LATEX TEMPLATE

TREVOR NORTON

Boston University, College of Arts and Sciences, 2022

Major Professors: First M. Last, PhD
Professor of Electrical and Computer Engineering
Secondary appointment

First M. Last, PhD
Professor of computer Science

ABSTRACT

[This is where the text for the abstract will go]

Contents

List of Abbreviations

FPUT   . . . .   Fermi-Pasta-Ulam-Tsingou

mKdV   . . . .   modified Korteweg-De Vries

Chapter 1

Long-Time stability of small FPUT solitary waves

1.1 Introduction

As shown in earlier work, there exists a wave solution of the FPUT lattice whose profile is well approximated by that of the kink solution to the (defocusing) mKdV. We are now interested in studying the stability of this wave solution on the FPUT lattice. The equations of motion on the lattice are given by

$$\ddot{x}_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z}. \tag{1.1}$$

where V is the interaction potential between neighboring particles and $\dot{}$ denotes the derivative with respect to the time $t \in \mathbb{R}$. Equation (1.1) can be rewritten in the strain variables $u_n := x_{n+1} - x_n$ as follows

$$\ddot{u}_n = V'(u_{n+1}) - 2V'(u_n) + V'(u_{n-1}), \quad n \in \mathbb{Z} \tag{1.2}$$

The moving wave solution in eq. (1.1) corresponds to a kink solution in eq. (1.2).

For the case where V is of the form $V(u) = \frac{1}{2}u^2 + \frac{\epsilon^2}{p+1}u^{p+1}$ for $p \geq 2$, the generalized KdV equation given by

$$2\partial_T W + \frac{1}{12}\partial_X^3 W + \partial_X(W^p) = 0, \quad X \in \mathbb{R} \tag{1.3}$$

serves as a modulation equation for solutions of eq. (1.2) (Bambusi and Ponno, 2006;

1

Friesecke and Pego, 1999). That is, for a local solution $W \in C([-\tau_0, \tau_0], H^s(\mathbb{R}))$ of eq. (1.3) there exist positive constants ϵ_0 and C_0 such that, for all $\epsilon \in (0, \epsilon_0)$, when initial data $(u_{\text{in}}, \dot{u}_{\text{in}}) \in \ell^2(\mathbb{R})$ satisfy

$$\|u_{\text{in}} - W(\epsilon\cdot, 0)\|_{\ell^2} + \|\dot{u}_{\text{in}} + \epsilon\partial_X W(\epsilon\cdot, 0)\|_{\ell^2} \leq \epsilon^{3/2}, \tag{1.4}$$

the unique solution to eq. (1.2) with initial data $(u_{\text{in}}, \dot{u}_{\text{in}})$ belongs to $C^1([-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}]; \ell^2(\mathbb{Z}))$ and satisfies

$$\|u(t) - W(\epsilon(\cdot - t), \epsilon^3 t)\|_{\ell^2(\mathbb{Z})} + \|\dot{u}(t) + \epsilon\partial_X W(\epsilon(\cdot - t), \epsilon^3 t)\|_{\ell^2(\mathbb{Z})} \leq C_0\epsilon^{3/2},$$
$$t \in [-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}]. \quad (1.5)$$

Furthermore, the approximation can also be extended to include counter-propagating solutions of the KdV in the case where $p = 2$ (Schneider and Wayne, 2000).

The KdV approximation was extended to longer time scales on the order of $\epsilon^{-3}|\log(\epsilon)|$ by Khan and Pelinovsky in order to deduce the nonlinear metastability of small FPUT solitary waves from the orbital stability of the corresponding KdV solitary waves (Khan and Pelinovsky, 2017).

We consider the FPUT with potential

$$V(u) = \frac{1}{2}u^2 - \frac{1}{24}u^4. \tag{1.6}$$

We will introduce an ansatz that solutions of the FPUT with this potential can be well-approximated by counter-propagating solutions of mKdV equations.

1.2 Counter-propagating ansatz

1.3 Setup of Lattice Equations

The scalar second-order differential equation eq. (1.2) with potential V given by eq. (1.6) can be rewritten as the following first-order system:

$$\begin{cases} \dot{u}_n = q_{n+1} - q_n, \\ \dot{q}_n = u_n - u_{n-1} - \dfrac{1}{6}(u_n^3 - u_{n-1}^3), \end{cases} \quad n \in \mathbb{Z}. \tag{1.7}$$

1.4 Preparatory Estimates

To make estimates on the error terms $\mathcal{U}$ and $\mathcal{Q}$, we require an appropriate choice of W. Based on the above discussion, W must be at least be a continuous solution in $L^\infty(\mathbb{R})$ space. Further regularity is required to make sense of **??** as well as further derivatives of P_ϵ. So W must be bounded but have spatial derivatives that decay at infinity. The solution of **??** must then be continuous in the following normed space.

Definition 1. *For $s \geq 1$, the normed space $\mathcal{X}_s$ is the set of functions*

$$\mathcal{X}_s := \{u \in L^\infty(\mathbb{R}) : u'(x) \in H^{s-1}(\mathbb{R})\} \tag{1.8}$$

with a norm defined by

$$\|u\|_{\mathcal{X}_s} := \|u\|_{L^\infty(\mathbb{R})} + \|u'\|_{H^{s-1}(\mathbb{R})}. \tag{1.9}$$

In fact, $\mathcal{X}_s$ is a Banach space.

Proposition 1. *For $s \geq 1$, the normed vector space $\mathcal{X}_s$ is a Banach space.*

Proof. Suppose that $\{u_n\} \subset \mathcal{X}_s$ is a Cauchy sequence. Then for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n, m \geq N$

$$\|u_n - u_m\|_{\mathcal{X}_s} = \|u_n - u_m\|_{L^\infty(\mathbb{R})} + \|u'_n - u'_m\|_{H^{s-1}(\mathbb{R})} < \epsilon. \tag{1.10}$$

Clearly, we have that $\{u_n\}$ is a Cauchy sequence in $L^\infty(\mathbb{R})$ and $\{u'_n\}$ is a Cauchy sequence in $H^{s-1}(\mathbb{R})$. Since L^∞ and H^{s-1} are Banach spaces, we have $u \in L^\infty$ and $v \in H^{s-1}$ such that

$$
\begin{aligned}
u_n &\to u \quad \text{in } L^\infty(\mathbb{R}) \\
u'_n &\to v \quad \text{in } H^{s-1}(\mathbb{R}).
\end{aligned}
\tag{1.11}
$$

To show that $u_n \to u$ in $\mathcal{X}_s$, we must demonstrate that $v = u'$ (where $'$ denotes a distributional derivative which is *a priori* defined). One can show that $H^{s-1}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$: for $w \in H^{s-1}(\mathbb{R})$ we have

$$
\begin{aligned}
\|w\|_{L^2} &= \|\hat{w}\|_{L^2} \\
&= \left(\int_{\mathbb{R}} |\hat{w}(\xi)|^2 \, d\xi \right)^{1/2} \\
&\leq \left(\int_{\mathbb{R}} |\hat{w}(\xi)|^2 (1 + |\xi|^{2s}) \, d\xi \right)^{1/2} \\
&= \|w\|_{H^{s-1}}.
\end{aligned}
\tag{1.12}
$$

Thus we also have $u'_n \to v$ in $L^2(\mathbb{R})$.

Let $\phi \in C_c^\infty(\mathbb{R})$ be a test function. By L^2 convergence, we have that

$$
\int_{\mathbb{R}} v\phi \, dx = \lim_{n \to \infty} \int_{\mathbb{R}} u'_n \phi \, dx.
\tag{1.13}
$$

Also, by applying the dominated convergence theorem (and possibly taking a subsequence to get almost everywhere pointwise convergence), we have that

$$
\int_{\mathbb{R}} u\phi' \, dx = \lim_{n \to \infty} \int_{\mathbb{R}} u_n \phi' \, dx.
\tag{1.14}
$$

Therefore, we get that

$$
\begin{aligned}
\int_{\mathbb{R}} v\phi\, dx &= \lim_{n\to\infty} \int_{\mathbb{R}} u_n' \phi\, dx \\
&= \lim_{n\to\infty} -\int_{\mathbb{R}} u_n \phi'\, dx \\
&= -\int_{\mathbb{R}} u\phi'\, dx.
\end{aligned}
\tag{1.15}
$$

Hence, $u' = v$ and $\mathcal{X}_s$ is a Banach space. $\qquad\square$

For our case, we will need at least six spatial derivatives of W and so W should belong to $C([-\tau_0, \tau_0]; \mathcal{X}_6)$. Note that the kink solutions of **??** are in this space; for instance, if $W(X, T) = \phi(X)$ with

$$
\phi(X) = \frac{1}{\sqrt{2}} \tanh\left(\frac{X}{\sqrt{2}}\right),
\tag{1.16}
$$

then W is a solution to the mKdV equation and $W \in C(\mathbb{R}; \mathcal{X}_s)$ for any $s \geq 1$.

In order to prove long-time stability, we must get estimates of the $\ell^2(\mathbb{Z})$ norms of the residual terms and the nonlinearity. The following lemma proved in (Dumas and Pelinovsky, 2014) will be useful in bounding $\ell^2(\mathbb{Z})$ norms by Sobolev norms of W.

Lemma 1. *There exists $C > 0$ such that for all $X \in H^1(\mathbb{R})$ and $\epsilon \in (0, 1)$,*

$$
\|x\|_{\ell^2} \leq C\epsilon^{-1/2} \|X\|_{H^1},
$$

where $x_n := X(\epsilon n)$, $n \in \mathbb{Z}$.

We can then show the following estimates.

Lemma 2. *Let $W \in C([-\tau_0, \tau_0]; \mathcal{X}_6)$ be a solution of the modified KdV equation **??** and $\tau_0 > 0$. Define*

$$
\delta := \sup_{\tau \in [-\tau_0, \tau_0]} \|W(\tau)\|_{\mathcal{X}_s}.
\tag{1.17}
$$

There exists a positive δ-independent constant C such that the residual and nonlinear terms satisfy

$$
\|\mathrm{Res}^{(1)}(t)\|_{\ell^2} + \|\mathrm{Res}^{(2)}(t)\|_{\ell^2} \leq C\epsilon^{9/2}(\delta + \delta^5)
\tag{1.18}
$$

and

$$\|\mathcal{R}(W,\mathcal{U})(t)\|_{\ell^2} \le C\epsilon^2(\delta + \|\mathcal{U}(t)\|_{\ell^2})\|\mathcal{U}(t)\|_{\ell^2}^2 \tag{1.19}$$

for every $t \in [-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}]$ and $\epsilon \in (0,1)$.

Proof. Plugging P_ϵ into $\mathrm{Res}^{(1)}(t)$ and using the Taylor remainder theorem, we get that the terms of order ϵ^4 or lower cancel out and we are left with

$$
\begin{aligned}
\mathrm{Res}^{(1)}(t) = \epsilon^5 \Bigg[\quad & \epsilon\left(c - 1 + \frac{\epsilon^2}{24}\right)\partial_X W(\epsilon(n-ct), \epsilon^3 t) \\
& -\frac{1}{24}\int_0^1 \partial_X^5 W(\epsilon(n-ct+r), \epsilon^3 t)(1-r)^4\, dr \\
& +\frac{1}{12}\int_0^1 \partial_X^5 W(\epsilon(n-ct+r), \epsilon^3 t)(1-r)^3\, dr \\
& -\frac{1}{16}\int_0^1 \partial_X^5 W(\epsilon(n-ct+r), \epsilon^3 t)(1-r)^2\, dr \\
& +\frac{1}{48}\int_0^1 \partial_X^5 W(\epsilon(n-ct+r), \epsilon^3 t)(1-r)\, dr. \Bigg]
\end{aligned}
\tag{1.20}
$$

Each of these terms can be bounded by the $\mathcal{X}_6$ norm. Thus we get that

$$\|\mathrm{Res}^{(1)}(t)\|_{\ell^2} \le C\epsilon^{9/2}\|W(\cdot, \epsilon t^3)\|_{\mathcal{X}_6} \le C\epsilon^{9/2}\delta \quad \text{for } t \in [-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}]. \tag{1.21}$$

Plugging in P_ϵ into $\mathrm{Res}^{(2)}(t)$ similarly gives

$$
\begin{aligned}
\mathrm{Res}^{(2)}(t) = {} & \epsilon\left(c - 1 + \frac{\epsilon^2}{24}\right)\left(\partial_X P_\epsilon(\epsilon(n-ct), \epsilon^3 t)\right) \\
& -\frac{\epsilon^3}{24}\left(\epsilon^2 \partial_X P_2(\epsilon(n-ct), \epsilon^3 t) + \epsilon^3 \partial_X P_3(\epsilon(n-ct), \epsilon^3 t)\right) \\
& -\epsilon^3\left(\epsilon^2 \partial_T P_2(\epsilon(n-ct), \epsilon^3 t) + \epsilon^3 \partial_T P_3(\epsilon(n-ct), \epsilon^3 t)\right) \\
& +\frac{\epsilon^5}{24}\int_0^1 \partial_X^5 W(\epsilon(n-ct-r), \epsilon^3 t)(r-1)^4\, dr \\
& -\frac{\epsilon^5}{12}\int_0^1 \partial_X^3(W^3)(\epsilon(n-ct-r), \epsilon^3 t)(r-1)^2\, dr.
\end{aligned}
\tag{1.22}
$$

The terms with order less than ϵ^5 cancel out. The integral terms can easily be bounded by the $\mathcal{X}_6$ norm For the first three terms, we can compute the expansion exactly:

$$\partial_X P_\epsilon = -\partial_X W + \frac{\epsilon}{2}\partial_X^2 W + \frac{\epsilon^2}{4}W^2\partial_X W - \frac{\epsilon^2}{8}\partial_X^3 W + \frac{\epsilon^3}{48}\partial_X^4 W - \frac{\epsilon^3}{24}\partial_X^2(W^3)$$

$$\partial_X P_2 = \frac{1}{4}W^2\partial_X W - \frac{1}{8}\partial_X^3 W$$

$$\partial_X P_3 = \frac{1}{48}\partial_X^4 W - \frac{1}{4}W(\partial_X W)^2 - \frac{1}{8}W^2\partial_X^2 W$$

$$\partial_T P_2 = \frac{-1}{96}W^2\partial_X W + \frac{1}{16}W^4\partial_X W - \frac{1}{16}(\partial_X W)^3 - \frac{3}{16}W\partial_X W\partial_X^2 W$$
$$+ \frac{1}{192}\partial_X^3 W - \frac{1}{24}W^2\partial_X^3 W + \frac{1}{192}\partial_X^5 W$$

$$\partial_T P_3 = \frac{1}{192}W^2\partial_X W - \frac{1}{32}W^4\partial_X W + \frac{1}{96}W(\partial_X W)^2 - \frac{1}{16}W^3(\partial_X W)^2$$
$$+ \frac{1}{16}(\partial_X W)^2\partial_X^2 W + \frac{1}{32}W(\partial_X^2 W)^2 + \frac{1}{192}W^2\partial_X^3 W + \frac{5}{96}W\partial_X W\partial_X^3 W$$
$$- \frac{1}{1152}\partial_X^4 W + \frac{1}{92}W^2\partial_X^4 W - \frac{1}{1152}\partial_X^6 W$$

$$(1.23)$$

Notice that the L^2 norm of each term can be bounded by a term of the form

$$\|W\|_{L^\infty}^k \|\partial_X W\|_{H^5}^\ell, \quad \text{where } 1 \leq k + \ell \leq 5. \quad (1.24)$$

The above term can then in turn be bounded by $C(\delta + \delta^5)$. Thus we get that

$$\|\text{Res}^{(2)}(t)\|_{\ell^2} = C\epsilon^{9/2}\left(\delta + \delta^5\right) \quad (1.25)$$

for $t \in [-\tau_0\epsilon^{-3}, \tau\epsilon^{-3}]$.

For the nonlinear term $\mathcal{R}(W, \mathcal{U})$, we immediately get that

$$\|\mathcal{R}(W, \mathcal{U})(t)\|_{\ell^2} \leq C\epsilon^2 \left[\|W(\epsilon^3 t)\|_{L^\infty}\|\mathcal{U}^2(t)\|_{\ell^2} + \|\mathcal{U}^3(t)\|_{\ell^2}\right]$$
$$\leq C\epsilon^2 \left[\|W(\epsilon^3 t)\|_{L^\infty}\|\mathcal{U}(t)\|_{\ell^\infty}\|\mathcal{U}(t)\|_{\ell^2} + \|\mathcal{U}(t)\|_{\ell^\infty}^2\|\mathcal{U}(t)\|_{\ell^2}\right] \quad (1.26)$$
$$\leq C\epsilon^2 \left[\delta\|\mathcal{U}(t)\|_{\ell^2}^2 + \|\mathcal{U}(t)\|_{\ell^2}^3\right].$$

$\square$

The main result of this section will be proved using a Grönwall type estimate

using an energy function defined by

$$\mathcal{E}(t) := \frac{1}{2} \sum_{n \in \mathbb{Z}} \mathcal{Q}_n^2(t) + \mathcal{U}_n^2(t) - \frac{\epsilon^2}{2} W^2(\epsilon(n-ct), \epsilon^3 t) \mathcal{U}_n^2(t). \tag{1.27}$$

The above will be nonnegative for W fixed and ϵ sufficiently small.

Lemma 3. *Let* $W \in C([-\tau_0, \tau_0], \mathcal{X}_6)$ *be a solution to the mKdV equation* **??** *and* $\tau_0 > 0$. *Define* $\epsilon_0 > 0$ *to be*

$$\epsilon_0 := \min \left\{ 1, \left(\sup_{\tau \in [-\tau_0, \tau_0]} \|W(\tau)\|_{L_\infty} \right)^{-1} \right\}. \tag{1.28}$$

For every $\epsilon \in (0, \epsilon_0)$ *and for every local solution* $(\mathcal{U}, \mathcal{Q}) \in C^1([-\tau_0 \epsilon^{-3}, \tau_0 \epsilon^{-3}], \ell^2(\mathbb{Z}))$ *of* **??**, *the energy-type quantity given in eq.* (1.27) *is coercive with the bound*

$$\|\mathcal{Q}(t)\|_{\ell^2}^2 + \|\mathcal{U}(t)\|_{\ell^2}^2 \leq 4\mathcal{E}(t), \quad \text{for } t \in (-\tau_0 \epsilon^{-3}, \tau_0 \epsilon^{-3}). \tag{1.29}$$

Moreover, when δ *is given by eq.* (1.17), *there exists* $C > 0$ *independent of* ϵ *and* δ *such that*

$$\left| \frac{d\mathcal{E}}{dt} \right| \leq C\mathcal{E}^{1/2} \left[\epsilon^{9/2}(\delta + \delta^5) + \epsilon^3(\delta^2 + \delta^4)\mathcal{E}^{1/2} + \epsilon^2(\delta + \mathcal{E}^{1/2})\mathcal{E} \right] \tag{1.30}$$

for every $t \in [-\tau_0 \epsilon^{-3}, \tau_0 \epsilon^{-3}]$ *and* $\epsilon \in (0, \epsilon_0)$.

Proof. By the choice of ϵ_0, we have for $\epsilon \in (0, \epsilon_0)$ that

$$1 - \frac{\epsilon^2}{2} \|W\|_{L^\infty}^2 \geq \frac{1}{2}. \tag{1.31}$$

Hence

$$\mathcal{E}(t) \geq \frac{1}{2} \|\mathcal{Q}\|_{\ell^2}^2 + \frac{1}{4} \|\mathcal{U}\|_{\ell^2}^2 \geq \frac{1}{4} \|\mathcal{Q}\|_{\ell^2}^2 + \frac{1}{4} \|\mathcal{U}\|_{\ell^2}^2 \tag{1.32}$$

and eq. (1.29) follows.

Taking the time derivative of $\mathcal{E}(t)$ and using that $(\mathcal{U},\mathcal{Q})$ solve **??** gives us

$$
\begin{aligned}
\frac{d\mathcal{E}}{dt} = \sum_{n\in\mathbb{Z}} \Bigg[& \mathcal{Q}_n(t)\mathcal{R}_n(W,\mathcal{U})(t) + \mathcal{Q}_n(t)\mathrm{Res}^{(2)}(t) \\
& + \mathcal{U}_n(t)\mathrm{Res}^{(1)}(t)\left(1 - \frac{\epsilon^2}{2}W^2(\epsilon(n-ct),\epsilon^3 t)\right) \\
& + \frac{\epsilon^3}{2}W(\epsilon(n-ct),\epsilon^3 t)\mathcal{U}_n^2(t)\left(c\partial_X W(\epsilon(n-ct),\epsilon^3 t) - \epsilon^2\partial_T W(\epsilon(n-ct),\epsilon^3 t)\right) \Bigg]
\end{aligned}
$$

$$(1.33)$$

Using the Cauchy-Schwarz inequality, the estimates found in lemma 2, and coercivity of $\mathcal{E}(t)$, we get that

$$
\begin{aligned}
\left|\frac{d\mathcal{E}}{dt}\right| \leq\; & \|\mathcal{Q}\|_{\ell^2}\|\mathcal{R}(W,\mathcal{U})\|_{\ell^2} + \|\mathcal{Q}\|_{\ell^2}\|\mathrm{Res}^{(2)}\|_{\ell^2} + \|\mathcal{U}\|_{\ell^2}\|\mathrm{Res}^{(1)}\|_{\ell^2} \\
& + \frac{1}{2}\epsilon^3\|W\|_{L^\infty}\|\mathcal{U}\|_{\ell^2}^2(c\|\partial_X W\|_{L^\infty} + \epsilon^2\|\partial_T W\|_{L^\infty}) \\
\leq\; & C\mathcal{E}^{1/2}\left[\epsilon^{9/2}(\delta + \delta^5) + \epsilon^3(\delta^2 + \delta^4)\mathcal{E}^{1/2} + \epsilon^2(\delta + \mathcal{E}^{1/2})\mathcal{E}\right]
\end{aligned}
$$

$$(1.34)$$

$\square$

1.5 Proof of Long-Time Stability

Now with the setup complete, the main result of this section can be shown. The result and proof are analogous to those of (Khan and Pelinovsky, 2017, Thm. 1).

Theorem 1. *Let $W \in C(\mathbb{R};\mathcal{X}_6)$ be a global solution of the mKdV equation* **??** *with $\sup_{\tau\in\mathbb{R}}\|W(\tau)\|_{\mathcal{X}_6} \leq \delta$. For fixed $r \in (0,1/2)$, there exist positive constants ϵ_0, C, and K such that for all $\epsilon \in (0,\epsilon_0)$, when initial data $(u_{\mathrm{in}},q_{\mathrm{in}}) \in \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z})$ satisfy*

$$\|u_{\mathrm{in}} - W(\epsilon\cdot,0)\|_{\ell^2} + \|q_{\mathrm{in}} + \epsilon\partial_X W(\epsilon\cdot,0)\|_{\ell^2} \leq \epsilon^{3/2}, \tag{1.35}$$

the unique solution (u,q) to the FPU equation eq. (1.7) belongs to

$$C^1([-t_0(\epsilon),t_0(\epsilon)],\ell^\infty(\mathbb{Z})) \tag{1.36}$$

with $t_0(\epsilon) := rK^{-1}\epsilon^{-3}|\log(\epsilon)|$ *and satisfies*

$$\|u(t)-W(\epsilon(\cdot-ct),\epsilon^3 t)\|_{\ell^2}+\|q(t)+\epsilon\partial_X W(\epsilon(\cdot-ct),\epsilon^3 t)\|_{\ell^2} \leq C\epsilon^{3/2-r}, \quad t \in [-t_0(\epsilon),t_0(\epsilon)]. \tag{1.37}$$

Proof. From the initial conditions satisfying eq. (1.35), we have at least local solutions to the error equations. That is, there is a unique local solution to **??** where $(\mathcal{U},\mathcal{Q}) \in C^1((-t_0,t_0);\ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}))$ for some $t_0 > 0$.

Set $\mathcal{S} := \mathcal{E}^{1/2}$ where $\mathcal{E}$ is defined in eq. (1.27). From the bound on the initial conditions in eq. (1.35), we get that $\mathcal{S}(0) \leq C_0\epsilon^{3/2}$ for some constant $C_0 > 0$ and ϵ_0 chosen by eq. (1.28). For fixed constants $r \in (0,1/2)$, $C > C_0$, and $K > 0$, define the maximal continuation time by

$$T_{C,K,r} := \sup\left\{T_0 \in (0,rK^{-1}\epsilon^{-3}|\log(\epsilon)|] : \mathcal{S}(t) \leq C\epsilon^{3/2-r}, t \in [-T_0,T_0]\right\}. \tag{1.38}$$

We also define the maximal evolution time of the mKdV equation as $\tau_0(\epsilon) = rK^{-1}|\log(\epsilon)|$. The goal is then to pick C and K so that $T_{C,K,r} = \epsilon^{-3}\tau_0(\epsilon)$.

We have that

$$\begin{aligned}\left|\frac{d}{dt}\mathcal{S}(t)\right| &= \frac{1}{2\mathcal{E}^{1/2}}\left|\frac{d}{dt}\mathcal{E}(t)\right| \\ &\leq C_1(\delta+\delta^5)\epsilon^{9/2} + C_2\epsilon^3\left[(\delta^2+\delta^4) + \epsilon^{-1}(\delta+\mathcal{S})\mathcal{S}\right]\mathcal{S}\end{aligned} \tag{1.39}$$

where $C_1, C_2 > 0$ are independent of δ and ϵ. While $|t| \leq T_{C,K,r}$,

$$C_2\left[(\delta^2+\delta^4) + \epsilon^{-1}(\delta+\mathcal{S})\mathcal{S}\right] \leq C_2\left[(\delta^2+\delta^4) + \epsilon^{-1}(\delta+C\epsilon^{3/2-r})C\epsilon^{3/2-r}\right], \tag{1.40}$$

where the right-hand side is continuous in ϵ for $\epsilon \in [0,\epsilon_0]$. Thus the right-hand side can be uniformly bounded by a constant independent of ϵ. Choose $K > 0$ (dependent on C) sufficiently large so that

$$C_2\left[(\delta^2+\delta^4) + \epsilon^{-1}(\delta+C\epsilon^{3/2-r})C\epsilon^{3/2-r}\right] \leq K. \tag{1.41}$$

Hence, we can get that for $t \in [-T_{C,K,r}, T_{C,K,r}]$

$$
\begin{aligned}
\frac{d}{dt} e^{-\epsilon^3 Kt} \mathcal{S}(t) &= -\epsilon^3 K e^{-\epsilon^3 Kt} \mathcal{S} + e^{-\epsilon^3 Kt} \frac{d}{dt} \mathcal{S} \\
&\leq -\epsilon^3 K e^{-\epsilon^3 Kt} \mathcal{S} + e^{-\epsilon^3 Kt} C_1 (\delta + \delta^5) \epsilon^{9/2} \\
&\quad + e^{-\epsilon^3 Kt} C_2 \epsilon^3 \left[(\delta^2 + \delta^4) + \epsilon^{-1} (\delta + \mathcal{S}) \mathcal{S} \right] \mathcal{S} \\
&\leq -\epsilon^3 K e^{-\epsilon^3 Kt} \mathcal{S} + e^{-\epsilon^3 Kt} C_1 (\delta + \delta^5) \epsilon^{9/2} + \epsilon^3 K e^{-\epsilon^3 Kt} \mathcal{S} \\
&= e^{-\epsilon^3 Kt} C_1 (\delta + \delta^5) \epsilon^{9/2}.
\end{aligned}
\tag{1.42}
$$

Integrating gives

$$
\begin{aligned}
\mathcal{S}(t) &\leq \left(\mathcal{S}(0) + K^{-1} C_1 (\delta + \delta^5) \epsilon^{3/2} \right) e^{\epsilon^3 Kt} - \epsilon^{-3} K^{-1} C_1 (\delta + \delta^5) \\
&\leq \left(\mathcal{S}(0) + K^{-1} C_1 (\delta + \delta^5) \epsilon^{3/2} \right) e^{\epsilon^3 Kt} \\
&\leq \left(\mathcal{S}(0) + K^{-1} C_1 (\delta + \delta^5) \epsilon^{3/2} \right) e^{K \tau_0(\epsilon)} \\
&\leq \left(C_0 + K^{-1} C_1 (\delta + \delta^5) \epsilon^{3/2} \right) \epsilon^{3/2 - r}
\end{aligned}
\tag{1.43}
$$

for $t \in [-T_{C,K,r}, T_{C,K,r}]$, where the last line follows in part from the definition of $\tau_0(\epsilon)$. Now choose $C > C_0$ sufficiently large so that

$$
C_0 + K^{-1} C_1 (\delta + \delta^5) \leq C.
\tag{1.44}
$$

Note that our earlier choice of K can be enlarged so that eq. (1.41) still holds as well as the above inequality. Therefore, with these choices of C and K, the maximal interval can be extended to $T_{C,K,r} = \epsilon^{-3} \tau_0(\epsilon)$. $\qquad \square$

References

Bambusi, D. and Ponno, A. (2006). On metastability in FPU. *Communications in mathematical physics*, 264(2):539–561.

Dumas, E. and Pelinovsky, D. (2014). Justification of the log–KdV equation in granular chains: The case of precompression. *SIAM Journal on Mathematical Analysis*, 46(6):4075–4103.

Friesecke, G. and Pego, R. L. (1999). Solitary waves on FPU lattices: I. qualitative properties, renormalization and continuum limit. *Nonlinearity*, 12(6):1601.

Khan, A. and Pelinovsky, D. E. (2017). Long-time stability of small FPU solitary waves. *Discrete & Continuous Dynamical Systems*, 37(4):2065.

Pazy, A. (2012). *Semigroups of linear operators and applications to partial differential equations*, volume 44. Springer Science & Business Media.

Schneider, G. and Wayne, C. E. (2000). Counter-propagating waves on fluid surfaces and the continuum limit of the Fermi-Pasta-Ulam model. In *Equadiff 99: (In 2 Volumes)*, pages 390–404. World Scientific.