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A BU THESIS LATEX TEMPLATE

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TREVOR NORTON

B.S., Virginia Polytechnic Institute and State University, 2015
M.S., Virginia Polytechnic Institute and State University, 2018

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Approved by

First Reader

First M. Last, PhD
Professor of Electrical and Computer Engineering

Second Reader

First M. Last
Associate Professor of ...

Third Reader

First M. Last
Assistant Professor of ...

Facilis descensus Averni;
Noctes atque dies patet atri janua Ditis;
Sed revocare gradum, superasque evadere ad auras,
Hoc opus, hic labor est. Virgil (from Don's thesis!)

Acknowledgments

[This is where the acknowledgments go...]

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TREVOR NORTON

Boston University, College of Arts and Sciences, 2022

Major Professors: First M. Last, PhD
Professor of Electrical and Computer Engineering
Secondary appointment

First M. Last, PhD
Professor of computer Science

ABSTRACT

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List of Abbreviations

FPUT	....	Fermi-Pasta-Ulam-Tsingou
mKdV	....	modified Korteweg-De Vries

Chapter 1

Long-Time stability of small FPUT solitary waves

1.1 Introduction

As shown in earlier work, there exists a wave solution of the FPUT lattice whose profile is well approximated by that of the kink solution to the (defocusing) mKdV. We are now interested in studying the stability of this wave solution on the FPUT lattice. The equations of motion on the lattice are given by

$$\ddot{x}_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z}. \tag{1.1}$$

where V is the interaction potential between neighboring particles and $\dot{}$ denotes the derivative with respect to the time $t \in \mathbb{R}$. Equation (1.1) can be rewritten in the strain variables $u_n := x_{n+1} - x_n$ as follows

$$\ddot{u}_n = V'(u_{n+1}) - 2V'(u_n) + V'(u_{n-1}), \quad n \in \mathbb{Z} \tag{1.2}$$

The moving wave solution in eq. (1.1) corresponds to a kink solution in eq. (1.2).

For the case where V is of the form $V(u) = \frac{1}{2}u^2 + \frac{\epsilon^2}{p+1}u^{p+1}$ for $p \geq 2$, the generalized KdV equation given by

$$2\partial_T W + \frac{1}{12}\partial_X^3 W + \partial_X(W^p) = 0, \quad X \in \mathbb{R} \tag{1.3}$$

serves as a modulation equation for solutions of eq. (1.2) (Bambusi and Ponno, 2006;

Friesecke and Pego, 1999; Schneider and Wayne, 2000). That is, for a local solution $W \in C([-\tau_0,\tau_0], H^s(\mathbb{R}))$ of eq. (1.3) there exist positive constants ϵ_0 and C_0 such that, for all $\epsilon \in (0,\epsilon_0)$, when initial data $(u_{\text{in}}, \dot{u}_{\text{in}}) \in \ell^2(\mathbb{R})$ satisfy

$$\|u_{\text{in}} - W(\epsilon\cdot,0)\|_{\ell^2} + \|\dot{u}_{\text{in}} + \epsilon\partial_X W(\epsilon\cdot,0)\|_{\ell^2} \leq \epsilon^{3/2}, \tag{1.4}$$

the unique solution to eq. (1.2) with initial data $(u_{\text{in}}, \dot{u}_{\text{in}})$ belongs to $C^1([-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}]; \ell^2(\mathbb{Z}))$ and satisfies

$$\|u(t) - W(\epsilon(\cdot - t), \epsilon^3 t)\|_{\ell^2(\mathbb{Z})} + \|\dot{u}(t) + \epsilon\partial_X W(\epsilon(\cdot - t), \epsilon^3 t)\|_{\ell^2(\mathbb{Z})} \leq C_0\epsilon^{3/2},$$
$$t \in [-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}]. \tag{1.5}$$

This approximation was extended to longer time scales on the order of $\epsilon^{-3}|\log(\epsilon)|$ by Khan and Pelinovsky in order to deduce the nonlinear metastability of small FPUT solitary waves from the orbital stability of the corresponding KdV solitary waves (Khan and Pelinovsky, 2017).

The potential for the FPUT in our case is given by

$$V(u) = \frac{1}{2}u^2 - \frac{\epsilon^2}{24}u^4 \tag{1.6}$$

and we wish to study the stability of the moving wave solution. The analysis found in (Khan and Pelinovsky, 2017) is no longer directly applicable. In particular, since we are focusing on kink solutions to eq. (1.2), we no longer have functions which decay in space as we approach infinity. We can proceed in an analogous way to (Khan and Pelinovsky, 2017), but we adapt the method to make estimates with bounded functions.

1.2 Setup of Lattice Equations

The scalar second-order differential equation eq. (1.2) with potential V given by eq. (1.6) can be rewritten as the following first-order system:

$$\begin{cases} \dot{u}_n = q_{n+1} - q_n, \\ \dot{q}_n = u_n - u_{n-1} - \dfrac{\epsilon^2}{6}(u_n^3 - u_{n-1}^3), \end{cases} \qquad n \in \mathbb{Z}. \tag{1.7}$$

Since our solution to eq. (1.2) is a kink solution with wave speed given by c where $c^2 = 1 - \frac{\epsilon^2}{12}$, we use the decomposition

$$u_n(t) = W(\epsilon(n-ct), \epsilon^3 t) + \mathcal{U}_n(t), \quad q_n(t) = P_\epsilon(\epsilon(n-ct), \epsilon^3 t) + \mathcal{Q}_n(t), \quad n \in \mathbb{Z} \tag{1.8}$$

where $W(X,T)$ is a suitable solution to

$$2\partial_T W + \frac{1}{12}\partial_X W - \frac{1}{2}W^2\partial_X W + \frac{1}{12}\partial_X^3 W = 0, \quad X \in \mathbb{R} \tag{1.9}$$

and $P_\epsilon(X,T)$ is found by formally satisfying

$$P_\epsilon(X+\epsilon, T) - P_\epsilon(X,T) = -\epsilon c \partial_X W(X,T) + \epsilon^3 \partial_T W(X,T). \tag{1.10}$$

To define P_ϵ, we assume that it has the form

$$P_\epsilon = P^{(0)} + \epsilon P^{(1)} + \epsilon^2 P^{(2)} + \epsilon^3 P^{(3)} \tag{1.11}$$

and plug this into eq. (1.10). After using Taylor series and collecting terms of equal orders of ϵ, we obtain

$$P_\epsilon := -W + \frac{1}{2}\epsilon\partial_X W + \frac{1}{12}\epsilon^2 W^3 - \frac{1}{8}\epsilon^2\partial_X^2 W + \frac{1}{48}\epsilon^3\partial_X^3 W - \frac{1}{8}\epsilon^3 W^2 \partial_X W. \tag{1.12}$$

Plugging eqs. (1.8) and (1.12) into eq. (1.7) gives the following differential equations for the error terms

$$
\begin{cases}
\dot{\mathcal{U}}_n(t) = \mathcal{Q}_{n+1}(t) - \mathcal{Q}_n(t) + \mathrm{Res}_n^{(1)}(t), \\
\dot{\mathcal{Q}}_n(t) = \mathcal{U}_n - \mathcal{U}_{n-1} \\
\qquad\qquad - \frac{\epsilon^2}{2} W^2(\epsilon(n-ct), \epsilon^3 t)\mathcal{U}_n + \frac{\epsilon^2}{2} W^2(\epsilon(n-1-ct), \epsilon^3 t)\mathcal{U}_{n-1} \\
\qquad\qquad + \mathrm{Res}_n^{(2)}(t) + \mathcal{R}_n(W, \mathcal{U}),
\end{cases}
\quad n \in \mathbb{Z} \tag{1.13}
$$

where the residual and nonlinear terms are given by

$$
\begin{aligned}
\mathrm{Res}_n^{(1)}(t) &:= \epsilon c \partial_X W(\epsilon(n-ct), \epsilon^3 t) - \epsilon^3 \partial_T W(\epsilon(n-ct), \epsilon^3 t) \\
&\quad + P_\epsilon(\epsilon(n+1-ct), \epsilon^3 t) - P_\epsilon(\epsilon(n-ct), \epsilon^3 t) \\
\mathrm{Res}_n^{(2)}(t) &:= \epsilon c \partial_X P_\epsilon(\epsilon(n-ct), \epsilon^3 t) - \epsilon^3 \partial_T P_\epsilon(\epsilon(n-ct), \epsilon^3 t) \\
&\quad + W(\epsilon(n-ct), \epsilon^3 t) - W(\epsilon(n-1-ct), \epsilon^3 t) \\
&\quad - \frac{\epsilon^2}{6} \left[W^3(\epsilon(n-ct), \epsilon^3 t) - W^3(\epsilon(n-1-ct), \epsilon^3 t) \right]
\end{aligned} \tag{1.14}
$$

and

$$
\begin{aligned}
\mathcal{R}_n(W, \mathcal{U})(t) := -\frac{\epsilon^2}{6} \Big[& 3W(\epsilon(n-ct), \epsilon^3 t)\mathcal{U}_n^2(t) - 3W(\epsilon(n-1-ct), \epsilon^3 t)\mathcal{U}_{n-1}^2(t) \\
& + \mathcal{U}_n^3(t) - \mathcal{U}_{n-1}^3(t) \Big].
\end{aligned} \tag{1.15}
$$

One may note that we do not know if there are always local solutions to eq. (1.13) for a given set of initial conditions. Although we will be assuming that $(u_n)_{n \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ and $W \in C([-\tau_0, \tau_0]; L^\infty(\mathbb{R}))$, the error terms $\mathcal{U}$ and $\mathcal{Q}$ are assumed to be in $\ell^2(\mathbb{Z})$. Thus the equations for the error terms can be written as an evolution equation over $\ell^2(\mathbb{Z})$. Equation (1.13) is equivalent to

$$
\begin{aligned}
\dot{X}(t) &= \mathcal{A}(t) X + \mathcal{N}(X, t) \\
X(t_0) &= x_0
\end{aligned} \tag{1.16}
$$

where $X(t) = (\mathcal{U}(t), \mathcal{Q}(t))$ and $x_0 = (\mathcal{U}(0), \mathcal{Q}(0))$,

$$\mathcal{A}(t) := \begin{bmatrix} 0 & e^{\partial} - 1 \\ 1 - e^{-\partial} & 0 \end{bmatrix} \begin{bmatrix} 1 - \frac{\epsilon^2}{2} W^2(\epsilon(\cdot - ct), \epsilon^3 t) & 0 \\ 0 & 1 \end{bmatrix} \tag{1.17}$$

and

$$\mathcal{N}(X, t) := \begin{bmatrix} \mathrm{Res}^{(1)}(t) \\ \mathcal{R}(W, \mathcal{U}) + \mathrm{Res}^{(2)}(t) \end{bmatrix}. \tag{1.18}$$

Each linear operator $\mathcal{A}(t)$ is bounded for any $t \in [-\tau_0, \tau_0]$ and $t \mapsto \mathcal{A}(t)$ is continuous in norm. Also, $\mathcal{N}(X, t)$ is locally Lipschitz in X and the Lipschitz constants are uniformly bounded in time. Thus by the results in sections 5 and 6 of (Pazy, 2012), there are C^1 local solutions of eq. (1.16). This shows that local solutions of eq. (1.13) exist in $\ell^2(\mathbb{Z})$ whenever $W \in C([-\tau_0, \tau_0]; L^\infty(\mathbb{R}))$.

1.3 Preparatory Estimates

To make estimates on the error terms $\mathcal{U}$ and $\mathcal{Q}$, we require an appropriate choice of W. Based on the above discussion, W must be at least be a continuous solution in $L^\infty(\mathbb{R})$ space. Further regularity is required to make sense of eq. (1.12) as well as further derivatives of P_ϵ. So W must be bounded but have spatial derivatives that decay at infinity. The solution of eq. (1.9) must then be continuous in the following normed space.

Definition 1. *For $s \geq 1$, the normed space $\mathcal{X}_s$ is the set of functions*

$$\mathcal{X}_s := \{u \in L^\infty(\mathbb{R}) : u'(x) \in H^{s-1}(\mathbb{R})\} \tag{1.19}$$

with a norm defined by

$$\|u\|_{X_s} := \|u\|_{L^\infty(\mathbb{R})} + \|u'\|_{H^{s-1}(\mathbb{R})}. \tag{1.20}$$

In fact, $\mathcal{X}_s$ is a Banach space.

Proposition 1. *For $s \geq 1$, the normed vector space $\mathcal{X}_s$ is a Banach space.*

Proof. Suppose that $\{u_n\} \subset \mathcal{X}_s$ is a Cauchy sequence. Then for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n, m \geq N$

$$\|u_n - u_m\|_{\mathcal{X}_s} = \|u_n - u_m\|_{L^\infty(\mathbb{R})} + \|u_n' - u_m'\|_{H^{s-1}(\mathbb{R})} < \epsilon. \tag{1.21}$$

Clearly, we have that $\{u_n\}$ is a Cauchy sequence in $L^\infty(\mathbb{R})$ and $\{u_n'\}$ is a Cauchy sequence in $H^{s-1}(\mathbb{R})$. Since L^∞ and H^{s-1} are Banach spaces, we have $u \in L^\infty$ and $v \in H^{s-1}$ such that

$$\begin{aligned} u_n &\to u \quad \text{in } L^\infty(\mathbb{R}) \\ u_n' &\to v \quad \text{in } H^{s-1}(\mathbb{R}). \end{aligned} \tag{1.22}$$

To show that $u_n \to u$ in $\mathcal{X}_s$, we must demonstrate that $v = u'$ (where $'$ denotes a distributional derivative which is *a priori* defined). One can show that $H^{s-1}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$: for $w \in H^{s-1}(\mathbb{R})$ we have

$$\begin{aligned} \|w\|_{L^2} &= \|\hat{w}\|_{L^2} \\ &= \left(\int_{\mathbb{R}} |\hat{w}(\xi)|^2 \, d\xi \right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}} |\hat{w}(\xi)|^2 (1 + |\xi|^{2s}) \, d\xi \right)^{1/2} \\ &= \|w\|_{H^{s-1}}. \end{aligned} \tag{1.23}$$

Thus we also have $u_n' \to v$ in $L^2(\mathbb{R})$.

Let $\phi \in C_c^\infty(\mathbb{R})$ be a test function. By L^2 convergence, we have that

$$\int_{\mathbb{R}} v\phi \, dx = \lim_{n\to\infty} \int_{\mathbb{R}} u_n' \phi \, dx. \tag{1.24}$$

Also, by applying the dominated convergence theorem (and possibly taking a subsequence to get almost everywhere pointwise convergence), we have that

$$\int_{\mathbb{R}} u\phi' \, dx = \lim_{n\to\infty} \int_{\mathbb{R}} u_n \phi' \, dx. \tag{1.25}$$

Therefore, we get that

$$\begin{aligned}\int_{\mathbb{R}} v\phi\,dx &= \lim_{n\to\infty}\int_{\mathbb{R}} u_n'\phi\,dx \\ &= \lim_{n\to\infty} -\int_{\mathbb{R}} u_n\phi'\,dx \\ &= -\int_{\mathbb{R}} u\phi'\,dx.\end{aligned} \tag{1.26}$$

Hence, $u' = v$ and $\mathcal{X}_s$ is a Banach space. □

For our case, we will need at least six spatial derivatives of W and so W should belong to $C([-\tau_0, \tau_0]; \mathcal{X}_6)$. Note that the kink solutions of eq. (1.9) are in this space; for instance, if $W(X,T) = \phi(X)$ with

$$\phi(X) = \frac{1}{\sqrt{2}}\tanh\left(\frac{X}{\sqrt{2}}\right), \tag{1.27}$$

then W is a solution to the mKdV equation and $W \in C(\mathbb{R}; \mathcal{X}_s)$ for any $s \geq 1$.

In order to prove long-time stability, we must get estimates of the $\ell^2(\mathbb{Z})$ norms of the residual terms and the nonlinearity. The following lemma proved in (Dumas and Pelinovsky, 2014) will be useful in bounding $\ell^2(\mathbb{Z})$ norms by Sobolev norms of W.

Lemma 1. *There exists $C > 0$ such that for all $X \in H^1(\mathbb{R})$ and $\epsilon \in (0,1)$,*

$$\|x\|_{\ell^2} \leq C\epsilon^{-1/2}\|X\|_{H^1},$$

where $x_n := X(\epsilon n)$, $n \in \mathbb{Z}$.

We can then show the following estimates.

Lemma 2. *Let $W \in C([-\tau_0, \tau_0]; \mathcal{X}_6)$ be a solution of the modified KdV equation eq. (1.9) and $\tau_0 > 0$. Define*

$$\delta := \sup_{\tau\in[-\tau_0,\tau_0]} \|W(\tau)\|_{\mathcal{X}_s}. \tag{1.28}$$

There exists a positive δ-independent constant C such that the residual and nonlinear *terms satisfy*

$$\|\mathrm{Res}^{(1)}(t)\|_{\ell^2} + \|\mathrm{Res}^{(2)}(t)\|_{\ell^2} \leq C\epsilon^{9/2}(\delta + \delta^5) \tag{1.29}$$

and

$$\|\mathcal{R}(W,\mathcal{U})(t)\|_{\ell^2} \leq C\epsilon^2(\delta + \|\mathcal{U}(t)\|_{\ell^2})\|\mathcal{U}(t)\|_{\ell^2}^2 \tag{1.30}$$

for every $t \in [-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}]$ *and* $\epsilon \in (0,1)$.

Proof. Plugging P_ϵ into $\mathrm{Res}^{(1)}(t)$ and using the Taylor remainder theorem, we get that the terms of order ϵ^4 or lower cancel out and we are left with

$$\begin{aligned}\mathrm{Res}^{(1)}(t) = \epsilon^5 \Bigg[\quad & \epsilon\left(c - 1 + \frac{\epsilon^2}{24}\right)\partial_X W(\epsilon(n-ct), \epsilon^3 t) \\ & - \frac{1}{24}\int_0^1 \partial_X^5 W(\epsilon(n-ct+r), \epsilon^3 t)(1-r)^4\, dr \\ & + \frac{1}{12}\int_0^1 \partial_X^5 W(\epsilon(n-ct+r), \epsilon^3 t)(1-r)^3\, dr \\ & - \frac{1}{16}\int_0^1 \partial_X^5 W(\epsilon(n-ct+r), \epsilon^3 t)(1-r)^2\, dr \\ & + \frac{1}{48}\int_0^1 \partial_X^5 W(\epsilon(n-ct+r), \epsilon^3 t)(1-r)\, dr.\Bigg]\end{aligned} \tag{1.31}$$

Each of these terms can be bounded by the $\mathcal{X}_6$ norm. Thus we get that

$$\|\mathrm{Res}^{(1)}(t)\|_{\ell^2} \leq C\epsilon^{9/2}\|W(\cdot, \epsilon t^3)\|_{\mathcal{X}_6} \leq C\epsilon^{9/2}\delta \quad \text{for } t \in [-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}]. \tag{1.32}$$

Plugging in P_ϵ into $\mathrm{Res}^{(2)}(t)$ similarly gives

$$
\begin{aligned}
\mathrm{Res}^{(2)}(t) = \epsilon\left(c-1+\frac{\epsilon^2}{24}\right)&\big(\partial_X P_\epsilon(\epsilon(n-ct),\epsilon^3 t)\big)\\
&-\frac{\epsilon^3}{24}\big(\epsilon^2\partial_X P_2(\epsilon(n-ct),\epsilon^3 t)+\epsilon^3\partial_X P_3(\epsilon(n-ct),\epsilon^3 t)\big)\\
&-\epsilon^3(\epsilon^2\partial_T P_2(\epsilon(n-ct),\epsilon^3 t)+\epsilon^3\partial_T P_3(\epsilon(n-ct),\epsilon^3 t))\\
&+\frac{\epsilon^5}{24}\int_0^1 \partial_X^5 W(\epsilon(n-ct-r),\epsilon^3 t)(r-1)^4\,dr\\
&-\frac{\epsilon^5}{12}\int_0^1 \partial_X^3 (W^3)(\epsilon(n-ct-r),\epsilon^3 t)(r-1)^2\,dr.
\end{aligned}
\tag{1.33}
$$

The terms with order less than ϵ^5 cancel out. The integral terms can easily be bounded by the $\mathcal{X}_6$ norm For the first three terms, we can compute the expansion exactly:

$$
\begin{aligned}
\partial_X P_\epsilon &= -\partial_X W + \frac{\epsilon}{2}\partial_X^2 W + \frac{\epsilon^2}{4}W^2\partial_X W - \frac{\epsilon^2}{8}\partial_X^3 W + \frac{\epsilon^3}{48}\partial_X^4 W - \frac{\epsilon^3}{24}\partial_X^2(W^3)\\
\partial_X P_2 &= \frac{1}{4}W^2\partial_X W - \frac{1}{8}\partial_X^3 W\\
\partial_X P_3 &= \frac{1}{48}\partial_X^4 W - \frac{1}{4}W(\partial_X W)^2 - \frac{1}{8}W^2\partial_X^2 W\\
\partial_T P_2 &= \frac{-1}{96}W^2\partial_X W + \frac{1}{16}W^4\partial_X W - \frac{1}{16}(\partial_X W)^3 - \frac{3}{16}W\partial_X W\partial_X^2 W\\
&\quad + \frac{1}{192}\partial_X^3 W - \frac{1}{24}W^2\partial_X^3 W + \frac{1}{192}\partial_X^5 W\\
\partial_T P_3 &= \frac{1}{192}W^2\partial_X W - \frac{1}{32}W^4\partial_X W + \frac{1}{96}W(\partial_X W)^2 - \frac{1}{16}W^3(\partial_X W)^2\\
&\quad + \frac{1}{16}(\partial_X W)^2\partial_X^2 W + \frac{1}{32}W(\partial_X^2 W)^2 + \frac{1}{192}W^2\partial_X^3 W + \frac{5}{96}W\partial_X W\partial_X^3 W\\
&\quad - \frac{1}{1152}\partial_X^4 W + \frac{1}{92}W^2\partial_X^4 W - \frac{1}{1152}\partial_X^6 W
\end{aligned}
\tag{1.34}
$$

Notice that the L^2 norm of each term can be bounded by a term of the form

$$\|W\|_{L^\infty}^k\|\partial_X W\|_{H^5}^\ell, \qquad \text{where } 1\le k+\ell\le 5. \tag{1.35}$$

The above term can then in turn be bounded by $C(\delta+\delta^5)$. Thus we get that

$$\|\mathrm{Res}^{(2)}(t)\|_{\ell^2} = C\epsilon^{9/2}\left(\delta+\delta^5\right) \tag{1.36}$$

for $t \in [-\tau_0\epsilon^{-3}, \tau\epsilon^{-3}]$.

For the nonlinear term $\mathcal{R}(W,\mathcal{U})$, we immediately get that

$$\begin{aligned}
\|\mathcal{R}(W,\mathcal{U})(t)\|_{\ell^2} &\leq C\epsilon^2 \left[\|W(\epsilon^3 t)\|_{L^\infty}\|\mathcal{U}^2(t)\|_{\ell^2} + \|\mathcal{U}^3(t)\|_{\ell^2}\right] \\
&\leq C\epsilon^2 \left[\|W(\epsilon^3 t)\|_{L^\infty}\|\mathcal{U}(t)\|_{\ell^\infty}\|\mathcal{U}(t)\|_{\ell^2} + \|\mathcal{U}(t)\|_{\ell^\infty}^2\|\mathcal{U}(t)\|_{\ell^2}\right] \qquad (1.37)\\
&\leq C\epsilon^2 \left[\delta\|\mathcal{U}(t)\|_{\ell^2}^2 + \|\mathcal{U}(t)\|_{\ell^2}^3\right].
\end{aligned}$$

□

The main result of this section will be proved using a Grönwall type estimate using an energy function defined by

$$\mathcal{E}(t) := \frac{1}{2}\sum_{n\in\mathbb{Z}} \mathcal{Q}_n^2(t) + \mathcal{U}_n^2(t) - \frac{\epsilon^2}{2}W^2(\epsilon(n-ct), \epsilon^3 t)\mathcal{U}_n^2(t). \tag{1.38}$$

The above will be nonnegative for W fixed and ϵ sufficiently small.

Lemma 3. *Let $W \in C([-\tau_0, \tau_0], \mathcal{X}_6)$ be a solution to the mKdV equation eq. (1.9) and $\tau_0 > 0$. Define $\epsilon_0 > 0$ to be*

$$\epsilon_0 := \min\left\{1, \left(\sup_{\tau\in[-\tau_0,\tau_0]} \|W(\tau)\|_{L_\infty}\right)^{-1}\right\}. \tag{1.39}$$

For every $\epsilon \in (0, \epsilon_0)$ and for every local solution $(\mathcal{U}, \mathcal{Q}) \in C^1([-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}], \ell^2(\mathbb{Z}))$ of eq. (1.13), the energy-type quantity given in eq. (1.38) is coercive with the bound

$$\|\mathcal{Q}(t)\|_{\ell^2}^2 + \|\mathcal{U}(t)\|_{\ell^2}^2 \leq 4\mathcal{E}(t), \quad \textit{for } t \in (-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}). \tag{1.40}$$

Moreover, when δ is given by eq. (1.28), there exists $C > 0$ independent of ϵ and δ such that

$$\left|\frac{d\mathcal{E}}{dt}\right| \leq C\mathcal{E}^{1/2}\left[\epsilon^{9/2}(\delta + \delta^5) + \epsilon^3(\delta^2 + \delta^4)\mathcal{E}^{1/2} + \epsilon^2(\delta + \mathcal{E}^{1/2})\mathcal{E}\right] \tag{1.41}$$

for every $t \in [-\tau_0\epsilon^{-3}, \tau_0\epsilon^{-3}]$ and $\epsilon \in (0, \epsilon_0)$.

Proof. By the choice of ϵ_0, we have for $\epsilon \in (0, \epsilon_0)$ that

$$1 - \frac{\epsilon^2}{2}\|W\|_{L^\infty}^2 \geq \frac{1}{2}. \tag{1.42}$$

Hence

$$\mathcal{E}(t) \geq \frac{1}{2}\|\mathcal{Q}\|_{\ell^2}^2 + \frac{1}{4}\|\mathcal{U}\|_{\ell^2}^2 \geq \frac{1}{4}\|\mathcal{Q}\|_{\ell^2}^2 + \frac{1}{4}\|\mathcal{U}\|_{\ell^2}^2 \tag{1.43}$$

and eq. (1.40) follows.

Taking the time derivative of $\mathcal{E}(t)$ and using that $(\mathcal{U}, \mathcal{Q})$ solve eq. (1.13) gives us

$$\begin{aligned}\frac{d\mathcal{E}}{dt} = \sum_{n\in\mathbb{Z}} \Bigg[& \mathcal{Q}_n(t)\mathcal{R}_n(W,\mathcal{U})(t) + \mathcal{Q}_n(t)\mathrm{Res}^{(2)}(t) \\ & + \mathcal{U}_n(t)\mathrm{Res}^{(1)}(t)\left(1 - \frac{\epsilon^2}{2}W^2(\epsilon(n-ct),\epsilon^3 t)\right) \\ & + \frac{\epsilon^3}{2}W(\epsilon(n-ct),\epsilon^3 t)\mathcal{U}_n^2(t)\left(c\partial_X W(\epsilon(n-ct),\epsilon^3 t) - \epsilon^2\partial_T W(\epsilon(n-ct),\epsilon^3 t)\right)\Bigg]\end{aligned} \tag{1.44}$$

Using the Cauchy-Schwarz inequality, the estimates found in lemma 2, and coercivity of $\mathcal{E}(t)$, we get that

$$\begin{aligned}\left|\frac{d\mathcal{E}}{dt}\right| &\leq \|\mathcal{Q}\|_{\ell^2}\|\mathcal{R}(W,\mathcal{U})\|_{\ell^2} + \|\mathcal{Q}\|_{\ell^2}\|\mathrm{Res}^{(2)}\|_{\ell^2} + \|\mathcal{U}\|_{\ell^2}\|\mathrm{Res}^{(1)}\|_{\ell^2} \\ &\quad + \frac{1}{2}\epsilon^3\|W\|_{L^\infty}\|\mathcal{U}\|_{\ell^2}^2(c\|\partial_X W\|_{L^\infty} + \epsilon^2\|\partial_T W\|_{L^\infty}) \\ &\leq C\mathcal{E}^{1/2}\left[\epsilon^{9/2}(\delta+\delta^5) + \epsilon^3(\delta^2+\delta^4)\mathcal{E}^{1/2} + \epsilon^2(\delta + \mathcal{E}^{1/2})\mathcal{E}\right]\end{aligned} \tag{1.45}$$

□

1.4 Proof of Long-Time Stability

Now with the setup complete, the main result of this section can be shown. The result and proof are analogous to those of (Khan and Pelinovsky, 2017, Thm. 1).

Theorem 1. *Let $W \in C(\mathbb{R};\mathcal{X}_6)$ be a global solution of the mKdV equation eq.* (1.9) *with $\sup_{\tau\in\mathbb{R}}\|W(\tau)\|_{\mathcal{X}_6} \leq \delta$. For fixed $r \in (0,1/2)$, there exist positive constants ϵ_0, C, and K such that for all $\epsilon \in (0,\epsilon_0)$, when initial data $(u_{\mathrm{in}}, q_{\mathrm{in}}) \in \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z})$ satisfy*

$$\|u_{\mathrm{in}} - W(\epsilon\cdot,0)\|_{\ell^2} + \|q_{\mathrm{in}} + \epsilon\partial_X W(\epsilon\cdot,0)\|_{\ell^2} \leq \epsilon^{3/2}, \tag{1.46}$$

the unique solution (u,q) to the FPU equation eq. (1.7) *belongs to*

$$C^1([-t_0(\epsilon),t_0(\epsilon)],\ell^\infty(\mathbb{Z})) \tag{1.47}$$

with $t_0(\epsilon) := rK^{-1}\epsilon^{-3}|\log(\epsilon)|$ *and satisfies*

$$\|u(t)-W(\epsilon(\cdot-ct),\epsilon^3 t)\|_{\ell^2}+\|q(t)+\epsilon\partial_X W(\epsilon(\cdot-ct),\epsilon^3 t)\|_{\ell^2} \leq C\epsilon^{3/2-r}, \quad t \in [-t_0(\epsilon), t_0(\epsilon)]. \tag{1.48}$$

Proof. From the initial conditions satisfying eq. (1.46), we have at least local solutions to the error equations. That is, there is a unique local solution to eq. (1.16) where $(\mathcal{U},\mathcal{Q}) \in C^1((-t_0,t_0);\ell^2(\mathbb{Z})\times\ell^2(\mathbb{Z}))$ for some $t_0 > 0$.

Set $\mathcal{S} := \mathcal{E}^{1/2}$ where $\mathcal{E}$ is defined in eq. (1.38). From the bound on the initial conditions in eq. (1.46), we get that $\mathcal{S}(0) \leq C_0\epsilon^{3/2}$ for some constant $C_0 > 0$ and ϵ_0 chosen by eq. (1.39). For fixed constants $r \in (0,1/2)$, $C > C_0$, and $K > 0$, define the maximal continuation time by

$$T_{C,K,r} := \sup\left\{T_0 \in (0, rK^{-1}\epsilon^{-3}|\log(\epsilon)|] : \mathcal{S}(t) \leq C\epsilon^{3/2-r}, t \in [-T_0,T_0]\right\}. \tag{1.49}$$

We also define the maximal evolution time of the mKdV equation as $\tau_0(\epsilon) = rK^{-1}|\log(\epsilon)|$. The goal is then to pick C and K so that $T_{C,K,r} = \epsilon^{-3}\tau_0(\epsilon)$.

We have that

$$\begin{aligned} \left|\frac{d}{dt}\mathcal{S}(t)\right| &= \frac{1}{2\mathcal{E}^{1/2}}\left|\frac{d}{dt}\mathcal{E}(t)\right| \\ &\leq C_1(\delta+\delta^5)\epsilon^{9/2} + C_2\epsilon^3\left[(\delta^2+\delta^4)+\epsilon^{-1}(\delta+\mathcal{S})\mathcal{S}\right]\mathcal{S} \end{aligned} \tag{1.50}$$

where $C_1, C_2 > 0$ are independent of δ and ϵ. While $|t| \leq T_{C,K,r}$,

$$C_2\left[(\delta^2+\delta^4)+\epsilon^{-1}(\delta+\mathcal{S})\mathcal{S}\right] \leq C_2\left[(\delta^2+\delta^4)+\epsilon^{-1}(\delta+C\epsilon^{3/2-r})C\epsilon^{3/2-r}\right], \tag{1.51}$$

where the right-hand side is continuous in ϵ for $\epsilon \in [0,\epsilon_0]$. Thus the right-hand side can be uniformly bounded by a constant independent of ϵ. Choose $K > 0$ (dependent on C) sufficiently large so that

$$C_2\left[(\delta^2+\delta^4)+\epsilon^{-1}(\delta+C\epsilon^{3/2-r})C\epsilon^{3/2-r}\right] \leq K. \tag{1.52}$$

Hence, we can get that for $t \in [-T_{C,K,r}, T_{C,K,r}]$

$$\begin{aligned}
\frac{d}{dt} e^{-\epsilon^3 K t} \mathcal{S}(t) &= -\epsilon^3 K e^{-\epsilon^3 K t} \mathcal{S} + e^{-\epsilon^3 K t} \frac{d}{dt} \mathcal{S} \\
&\le -\epsilon^3 K e^{-\epsilon^3 K t} \mathcal{S} + e^{-\epsilon^3 K t} C_1 (\delta + \delta^5) \epsilon^{9/2} \\
&\quad + e^{-\epsilon^3 K t} C_2 \epsilon^3 \left[(\delta^2 + \delta^4) + \epsilon^{-1} (\delta + \mathcal{S}) \mathcal{S} \right] \mathcal{S} \\
&\le -\epsilon^3 K e^{-\epsilon^3 K t} \mathcal{S} + e^{-\epsilon^3 K t} C_1 (\delta + \delta^5) \epsilon^{9/2} + \epsilon^3 K e^{-\epsilon^3 K t} \mathcal{S} \\
&= e^{-\epsilon^3 K t} C_1 (\delta + \delta^5) \epsilon^{9/2}.
\end{aligned} \tag{1.53}$$

Integrating gives

$$\begin{aligned}
\mathcal{S}(t) &\le \left(\mathcal{S}(0) + K^{-1} C_1 (\delta + \delta^5) \epsilon^{3/2} \right) e^{\epsilon^3 K t} - \epsilon^{-3} K^{-1} C_1 (\delta + \delta^5) \\
&\le \left(\mathcal{S}(0) + K^{-1} C_1 (\delta + \delta^5) \epsilon^{3/2} \right) e^{\epsilon^3 K t} \\
&\le \left(\mathcal{S}(0) + K^{-1} C_1 (\delta + \delta^5) \epsilon^{3/2} \right) e^{K \tau_0(\epsilon)} \\
&\le \left(C_0 + K^{-1} C_1 (\delta + \delta^5) \epsilon^{3/2} \right) \epsilon^{3/2 - r}
\end{aligned} \tag{1.54}$$

for $t \in [-T_{C,K,r}, T_{C,K,r}]$, where the last line follows in part from the definition of $\tau_0(\epsilon)$. Now choose $C > C_0$ sufficiently large so that

$$C_0 + K^{-1} C_1 (\delta + \delta^5) \le C. \tag{1.55}$$

Note that our earlier choice of K can be enlarged so that eq. (1.52) still holds as well as the above inequality. Therefore, with these choices of C and K, the maximal interval can be extended to $T_{C,K,r} = \epsilon^{-3} \tau_0(\epsilon)$. $\square$

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