No notes or calculators allowed. If you run out of room for an answer, continue on the next page. Clearly show and explain all your work. Please do not open the exam until instructed to do so. Keep your eyes on your own exam and do not cheat.

Name: ____________________

Question	Points	Score
1	8	
2	12	
3	12	
4	10	
5	10	
6	10	
7	10	
8	9	
9	9	
10	10	
Total:	100	

If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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1. (8 points) For each of the following statements, mark **T** if the statement is true and **F** if the statement is false. No justification is required for each answer.

(a) ______ If A is an $m \times n$ matrix, then $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$ if A has a pivot position in each row.

(b) ______ For all 3×3 matrices A, we have $\det(2A) = 2\det(A)$.

(c) ______ Suppose

$$W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Then $\dim W = 4$.

(d) ______ Suppose a 2×2 matrix has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of $\mathbb{R}^2$.

(e) ______ If B is a 3×2 matrix, then both $\text{Col}\, B$ and $\text{Nul}\, B$ are subspaces of $\mathbb{R}^3$.

(f) ______ Suppose $\mathcal{B}$ is a basis for a vector space V with dimension n. The coordinate mapping $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

(g) ______ Let S be a set that spans the finite-dimensional vector space V. If $V \neq \{\mathbf{0}\}$, then some subset of S is a basis for V.

(h) ______ If V is a vector space and $\dim V = 5$, then any set S with 6 or more vectors spans V.

2. (12 points) Determine if the linear system

$$\begin{aligned} x_1 - x_2 + 2x_3 &= -1 \\ 2x_1 - x_2 + 3x_3 &= 0 \\ x_1 + x_3 &= 2 \end{aligned}$$

is consistent. Justify your answer.

[Additional work for question 2.]

3. (12 points) Find the inverse of

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

if it exists.

[Additional work for question 3.]

4. (10 points) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

(a) Calculate $\det A$. What does this value tell us about the number of solutions to the matrix equation $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b}$ in $\mathbb{R}^3$?

(b) Solve the matrix equation $A\mathbf{x} = \mathbf{e}_1$ for $\mathbf{x}$. What is the significance of this vector? (*Hint*: multiply on the left-hand side of the matrix equation by A^{-1}.)

[Additional work for question 4.]

5. (10 points) Let

$$W = \text{Span}\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

Use the Gram-Schmidt process to produce an orthogonal basis for W.

[Additional work for question 5.]

6. (10 points) Let

$$\mathbf{u} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad W = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Compute the following expressions:

(a) $\mathbf{u} \cdot \mathbf{v}$

(b) $\text{dist}\,(\mathbf{u}, \mathbf{v})$

(c) $\|\mathbf{v}\|$

(d) $\text{proj}_W\, \mathbf{u}$

[Additional work for question 6.]

7. (10 points) Let $T : \mathbb{P}_2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \\ \mathbf{p}(2) \end{bmatrix}.$$

Let $\mathcal{B} = \{1, t, t^2\}$ and $\mathcal{C} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard bases for $\mathbb{P}_2$ and $\mathbb{R}^3$, respectively.

(a) Write the matrix for T relative to bases $\mathcal{B}$ and $\mathcal{C}$.

(b) Describe the kernel of T.

[Additional work for question 7.]

8. (9 points) Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & -2 & 3 \end{bmatrix}.$$

If A is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Otherwise, explain why A cannot be diagonalized.

[Additional work for question 8.]

9. (9 points) Let

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 3 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ 2 \end{bmatrix}.$$

(a) Describe the column space of A. That is, what do the vectors in $\operatorname{Col} A$ look like? Use this to explain why $A\mathbf{x} = \mathbf{b}$ is an inconsistent system.

(b) Write out the normal equations for the least-squares problem $A\mathbf{x} = \mathbf{b}$. You should compute the explicit values for the matrix and vector. **You do not need to solve this system.**

(c) The solution to the least-squares problem is $\hat{\mathbf{x}} = (5, 1)$. Show that $A\hat{\mathbf{x}} = \operatorname{proj}_{\operatorname{Col} A}\mathbf{b}$ by computing $A\hat{\mathbf{x}}$ and finding the projection of $\mathbf{b}$ onto $\operatorname{Col} A$. You may assume that the set $\{\mathbf{e}_1, \mathbf{e}_2\}$ in $\mathbb{R}^3$ is an orthonormal basis for $\operatorname{Col} A$.

[Additional work for question 9.]

10. (10 points) Let

$$\mathbf{p}_1(t) = 1 - t^2 + t^4, \quad \mathbf{p}_2(t) = 2t + t^2 + t^3 - t^4, \quad \mathbf{p}_3(t) = -2 + 4t + 4t^2 + 2t^3 - 4t^4$$

be vectors in $\mathbb{P}_4$ and let $\mathcal{B} = \{1,\ t,\ t^2,\ t^3,\ t^4\}$ denote the standard basis for this space.

(a) Write $[\mathbf{p}_i]_{\mathcal{B}}$ for $i = 1, 2, 3$.

(b) Determine if $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly independent. Justify your answer.

[Additional work for question 10.]