No notes or calculators allowed. If you run out of room for an answer, continue on the next page. Clearly show and explain all your work. Please do not open the exam until instructed to do so. Keep your eyes on your own exam and do not cheat.

Name: ______________________________

Question	Points	Score
1	10	
2	15	
3	15	
4	14	
5	10	
6	12	
7	12	
8	12	
Total:	100	

If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

1. (10 points) For each of the following statements, mark **T** if the statement is true and **F** if the statement is false. No justification is required for each answer.

(a) ______ Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then there exists $S : \mathbb{R}^2 \to \mathbb{R}^2$ such that $S(T(\mathbf{x})) = T(S(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

(b) ______ If A is an invertible matrix, then

$$\det(A^{-1}) = \frac{1}{\det A}.$$

(c) ______ Suppose

$$H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

Then $\dim H = 3$.

(d) ______ Suppose A is an 2×3 matrix and $\dim \text{Nul}\, A = 1$. Then $\text{rank}\, A = 1$.

(e) ______ Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in the subspace H of $\mathbb{R}^n$. For any $\mathbf{u}$ in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, $\mathbf{u}$ is also in H.

2. (15 points) Find the inverse of

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

if it exists.

[Additional work for question 2.]

3. (15 points) Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & 4 \end{bmatrix}.$$

Compute each of the following matrix sums and products if defined. If an expression is not undefined, explain why.

(a) AB

(b) BA

(c) AC

(d) $B + C$

(e) $2A - C$

[Additional work for question 3.]

4. (14 points) Compute the following determinant:

$$\det \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

[Additional work for question 4.]

5. (10 points) Let
$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ -2 & 2 & 1 \end{bmatrix}.$$
If A is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Otherwise, explain why A cannot be diagonalized.

[Additional work for question 5.]

6. (12 points) Let

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ -2 & 4 & 1 & -1 \\ 3 & -6 & 5 & 8 \\ 1 & -2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The above is a display of A and its reduced echelon form.

(a) Find a basis for $\operatorname{Col} A$.

(b) Find a basis for $\operatorname{Nul} A$.

[Additional work for question 6.]

7. (12 points) Let

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) Write the characteristic polynomial for A.

(b) Show that $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is an eigenvector of A, and determine its corresponding eigenvalue.

(c) Find a basis for the eigenspace corresponding to $\lambda = 3$. Show your work.

[Additional work for question 7.]

8. (12 points) Suppose that A is a 3×3 matrix and that it has eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

with corresponding eigenvalues $\lambda_1 = 0.5$, $\lambda_2 = 0.9$ and $\lambda_3 = 1$, respectively. Denote

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

to be the *ordered* set of eigenvectors. The set $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

(a) First explain why the set $\mathcal{B}$ is linearly independent. Then explain why $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

(b) Let $\mathbf{x}_0 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. Determine $\left[\mathbf{x}_0\right]_{\mathcal{B}}$, the $\mathcal{B}$-coordinate vector of $\mathbf{x}_0$.

(c) Define

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

for $k = 0, 1, 2, \ldots$ with $\mathbf{x}_0$ as given above. Construct an explicit formula for $\mathbf{x}_k$ that does not include A. Use this formula to describe the long-term behavior of $\mathbf{x}_k$ (that is, what does $\mathbf{x}_k$ approach as $k \to \infty$).

[Additional work for question 8.]