Calculators and notes are not permitted during this exam. Show all relevant work, and present solutions clearly.

Name: ______________________________

Question	Points	Score
1	5	
2	5	
3	3	
4	3	
5	3	
6	3	
7	3	
8	20	
9	20	
10	15	
11	20	
Total:	100	

The following questions have **multiple options**. For each problem, select all correct options by marking the circle (e.g. ⊗). Make sure answers clearly marked and free from smudges. You do not need to show your work.

1. (5 points) Let

$$f(x) = \frac{5x^3 - 2x + 1}{x^4 + 1}.$$

Which of the following is true (select all options that apply)?

- √ **The graph of $y = f(x)$ has a horizontal asymptote.**
- ○ The graph of $y = f(x)$ has a vertical asymptote.
- ○ The graph of $y = f(x)$ has a slant asymptote.
- ○ $\lim_{x\to\infty} f(x) = \infty$.
- √ $\lim_{x\to\infty} f(x) = 0$**.**

Solution: Since the order of the denominator is greater than the order of the numerator, $f(x)$ goes to zero as x gets large, and so $\lim_{x\to\infty} f(x) = 0$. As a result, the graph of $y = f(x)$ has a horizontal asymptote at $y = 0$. Since the denominator is strictly greater than 1, there is no vertical asymptote. There is no slant asymptote since the degree of the denominator 3, which is not equal to $4 + 1$.

2. (5 points) Which of the following are continuous functions on $(-\infty, \infty)$ (select all options that apply)?

- √ $10x^5 - 10$
- √ $\cos(\ln(x^2 + 1))$
- √ $h(x) = \begin{cases} 5x - 2, & x < 1 \\ x^2 + 2, & x \geq 1 \end{cases}$
- ○ $g(x) = \begin{cases} x^2 + 1, & x \neq 0 \\ -1, & x = 0 \end{cases}$
- ○ $\dfrac{1}{x - 3}$

Solution: $10x^5 - 10$ is a polynomial and thus continuous on the entire real line. The second option, $\cos(\ln(x^2 + 1))$, is continuous since it is a composition of continuous functions (in particular, $\ln(x^2 + 1)$ is continuous everywhere since $x^2 + 1$ is continuous and greater than 0 and $\ln(y)$ is continuous on $(0, \infty)$). The function $h(x)$

is obviously continuous everywhere except perhaps $x = 1$. It is easily verified that $\lim_{x\to 1^+} h(x) = \lim_{x\to 1^-} h(x) = 3 = h(1)$. The function g is discontinuous at $x = 0$ since $\lim_{x\to 0} g(x) = 1 \neq g(0) = -1$. The last option is not continuous at $x = 3$ since it is not defined there (nor does it have a limit as $x \to 3$).

The following problems are **True or False**. For each problem, mark **T** if the statement is true or **F** if the statement is false. You do not need to show your work.

3. (3 points) __**F**__ If $f(x) = -(x-1)^{-1}$, then $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = -\infty$.

4. (3 points) __**F**__ The function $(x+2)^{1/3}$ is continuous and differentiable at each point in $(-\infty, \infty)$.

5. (3 points) __**T**__ If $\lim_{t\to 5} g(t) = 2$, then $\lim_{t\to 5^+} g(t) = \lim_{t\to 5^-} g(t) = 2$.

6. (3 points) __**F**__ If $h(x)$ is defined at $x = 2$ and $\lim_{x\to 2} h(x)$ exists, then h is continuous at $x = 2$.

7. (3 points) __**T**__ If f is a continuous function on $[-1, 1]$ where $f(-1) = -2$, $f(0) = 1$, and $f(1) = -1$, then there are at least two solutions to the equation $f(x) = 0$ on the interval $[-1, 1]$.

Solution: For the problem 3, $\lim_{x\to 1^-} f(x) = +\infty$. For problem 4, $(x+2)^{1/3}$ is not differentiable at $x = -2$ (it has a vertical tangent line at this point). For problem 5, this follows exactly from the Theorem in Chapter 2. For problem 6, one needs that $\lim_{x\to 2} h(x) = h(2)$ to conclude that h is continuous at $x = 2$. For problem 7, this follows from applying the intermediate value theorem on both $[-1, 0]$ and $[0, 1]$; thus we get that there are at least two solutions.

8. (20 points) For each of the following limits, either compute its value or state that it does not exist. Show all work.

   (a) $\lim_{x\to -1} \sqrt{5x^2+1}$

   (b) $\lim_{x\to 1} \dfrac{1-x^2}{x^2-8x+7}$

   (c) $\lim_{x\to 2} \dfrac{\sqrt{2x}-2}{x-2}$

   (d) $\lim_{x\to 0} x^2\cos(\ln(x^2))$ (Hint: Use the squeeze theorem!)

Solution:

(a)

$$\begin{aligned}
\lim_{x\to -1} \sqrt{5x^2+1} &= \sqrt{\lim_{x\to -1} 5x^2+1} \\
&= \sqrt{5(-1)^2+1} \\
&= \sqrt{6}
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to 1} \frac{1-x^2}{x^2-8x+7} &= \lim_{x\to 1} \frac{-(x-1)(x+1)}{(x-7)(x-1)} \\
&= \lim_{x\to 1} \frac{-(x+1)}{x-7} \\
&= \frac{-2}{-6} \\
&= \frac{1}{3}
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{2x}-2}{x-2} &= \lim_{x\to 2} \frac{\sqrt{2x}-2}{x-2} \cdot \frac{\sqrt{2x}+2}{\sqrt{2x}+2} \\
&= \lim_{x\to 2} \frac{2x-4}{(x-2)(\sqrt{2x}+2)} \\
&= \lim_{x\to 2} \frac{2}{\sqrt{2x}+2} \\
&= \frac{2}{\lim_{x\to 2}(\sqrt{2x}+2)} \\
&= \frac{2}{\sqrt{2\cdot 2}+2} \\
&= \frac{1}{2}
\end{aligned}$$

(d) Note that since

$$-1 \leq \cos(\ln(x^2)) \leq 1$$

we have

$$-x^2 \leq x^2 \cos(\ln(x^2)) \leq x^2.$$

Since $\lim_{x\to 0} x^2 = \lim_{x\to 0} -x^2 = 0$, by the squeeze theorem we have

$$\lim_{x\to 0} x^2 \cos(\ln(x^2)) = 0.$$

9. (20 points) Compute the following derivatives. Show all work.

(a) $\dfrac{d}{du}\left(2\sqrt{u}+9\sin(u)-e^{\sqrt{2}}\right)$

(b) $\dfrac{d}{d\theta}(\theta^3\cos(\theta))$

(c) $\dfrac{d}{dt}\left(e^{-t^2}\right)$

(d) $\dfrac{d}{dx}\left(\dfrac{x^2}{1+x}\right)$

Solution:

(a)

$$\frac{d}{du}\left(2\sqrt{u}+9\sin(u)-e^{\sqrt{2}}\right)=2\cdot\frac{1}{2}u^{-1/2}+9\cos(u)+0=\frac{1}{\sqrt{u}}+9\cos(u)$$

(b)

$$\frac{d}{d\theta}(\theta^3\cos(\theta))=3\theta^2\cos(\theta)-\theta^3\sin(\theta)$$

(c)

$$\frac{d}{dt}\left(e^{-t^2}\right)=e^{-t^2}(-2t)=-2te^{-t^2}$$

(d)

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^2}{1+x}\right)&=\frac{(1+x)(2x)-(x^2)(1)}{(1+x)^2}\\&=\frac{x^2+2x}{(1+x)^2}\end{aligned}$$

10. (15 points) A ball is launched upward into the air and its trajectory is measured for 2 seconds. Let

$$s = f(t) = -10t^2 + 20t \quad \text{for } 0 \leq t \leq 2$$

be the height of the ball in meters t seconds after it is thrown.

(a) Use the definition of the derivative and evaluate the limit to confirm that the velocity of the ball for $0 < t < 2$ is $v = \frac{ds}{dt} = -20t + 20$. Show all work.

(b) Evaluate s and v at $t = 0$. Describe what is happening physically with the ball at this point. Make sure to include units in your answer.

(c) At what point on the graph $s = f(t)$ is the tangent line horizontal? Describe what is happening with the ball at this point.

Solution:

(a)

$$\begin{aligned} f'(t) &= \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \\ &= \lim_{h\to 0} \frac{-10(t+h)^2 + 20(t+h) + 10t^2 - 20t}{h} \\ &= \lim_{h\to 0} \frac{-20th - 10h^2 + 20h}{h} \\ &= \lim_{h\to 0} (-20t - 10h + 20) \\ &= -20t + 20 \end{aligned}$$

(b) $s = f(0) = 0$ and $v = f'(0) = 20$. This means that the when the ball was initially launched it was at the ground level and had a velocity of 20 meters per second.

(c) The tangent line is horizontal when $f'(t) = 0$, and solving $-20t + 20 = 0$ for t we get one solution: $t = 1$. This implies that 1 second after the ball is launched, it reaches it maximum height of $f(1) = 10$ meters and in the next moment begins to fall.

11. (20 points) Let $P(t)$ denote the size of a population at a certain time t. We say a population *grows exponentially* if there is a real value $k > 0$ such that

$$\frac{dP}{dt} = kP. \tag{1}$$

Suppose we are interested in the number of E. coli bacteria in a petri dish. Let P_1 represent the number of E. coli in a petri dish in million and t is the number of hours after the start of our observation. Assume that

$$P_1(t) = 100e^{0.02t} \quad \text{for } t \geq 0. \tag{2}$$

In particular, we have that $P_1(0) = 100$ and so there are 100 *million* bacteria in the petri dish at the start of our observation

(a) Compute $\frac{dP_1}{dt}$ for $t > 0$ and specify the units of the derivative (Hint: use the chain rule).

(b) Confirm that this population grows exponentially (as defined above) and specify the value of k for which eq. (1) holds.

(c) Now suppose that the number of E. coli is modeled by

$$P_2(t) = \frac{1000}{1 + 9e^{-0.02t}}.$$

Evaluate $\lim_{t\to\infty} P_2(t)$, showing all work.

(d) Given your answer in (c), interpret the long-term behavior of this population. What are some possible causes for this behavior?

Solution:

(a) Compute the derivative:

$$\frac{dP_1}{dt} = 100 \cdot (0.02e^{0.02t}) = 2e^{0.02t}.$$

The units of the derivative is millions (of bacteria) per hour.

(b) Plugging in P_1 into eq. (1), we get that

$$2e^{0.02t} = k \cdot 100e^{0.02t},$$

which holds for $t \geq 0$ if $k = 0.02$. Thus P_1 grows exponentially and the value of k is 0.02.

(c)

$$\begin{aligned}
\lim_{t\to\infty} \frac{100}{1+9e^{-0.02t}} &= \frac{100}{1+9\lim_{t\to\infty} e^{-0.02t}} \\
&= \frac{1000}{1+9\cdot 0} \\
&= 1000
\end{aligned}$$

(d) The result in (c) implies that the population eventually approaches a total of 1 billion. The cause could be some physical constraint on the growth of the bacteria: no more room to grow; not enough food; or other factors.