No notes or calculators allowed. If you run out of room for an answer, continue on the next page. Clearly show and explain all your work. Please do not open the exam until instructed to do so. Keep your eyes on your own exam and do not cheat.

Name: _____

Question	Points	Score
1	10	
2	20	
3	16	
4	12	
5	15	
6	15	
7	12	
Total:	100	

1. (10 points) For each of the following statements, mark **T** if the statement is true and **F** if the statement is false. No justification is required for each answer.

(a) _____ The linear systems

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_2 &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 + x_2 &= 2 \\ 0 &= 0 \end{aligned}$$

have the same solution set.

(b) _____ Let A be an $m \times n$ matrix and $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ such that $A\mathbf{u} = A\mathbf{v} = \mathbf{0}$. Then $A(2\mathbf{u} + \mathbf{v}) = \mathbf{0}$.

(c) _____ Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^3$. Suppose $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent and $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. Then $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \text{Span}\{\mathbf{u}, \mathbf{v}\}$.

(d) _____ Suppose A is an $m \times n$ matrix whose columns span $\mathbb{R}^m$. Then $A\mathbf{x} = \mathbf{b}$ has a *unique* solution for all $\mathbf{b}$ in $\mathbb{R}^n$.

(e) _____ The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x_1, x_2) = x_1 - x_2$ is one-to-one.

2. (20 points) Apply elementary row operations to transform the following matrix into reduced echelon form:

$$\begin{bmatrix} 1 & -1 & 2 & -1 \\ -1 & 1 & -1 & 0 \\ 2 & -2 & 10 & -3 \end{bmatrix}.$$

Show your work. (You do *not* need to state the row operation, but your steps should be clear.)

[Additional work for question 2.]

3. (16 points) Determine if the linear system

$$\begin{aligned} x_1 - 2x_2 \qquad &= 1 \\ x_1 - \ \ x_2 - x_3 &= 3 \\ 3x_1 - 7x_2 - x_3 &= 3 \end{aligned}$$

is consistent. Justify your answer.

[Additional work for question 3.]

4. (12 points) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

You may assume $A \sim U$. That is, A and U are row equivalent.

(a) What are the pivot positions of A?

(b) Does $A\mathbf{x} = \mathbf{b}$ have a solution for each $\mathbf{b}$? Justify your answer.

(c) Does $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution? Justify your answer.

[Additional work for question 4.]

5. (15 points) Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

(a) Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly independent? Justify your answer.

(b) Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? Justify your answer. (*Hint:* choose weights $x_1 = 1$, $x_2 = 1$ and $x_3 = -2$.)

(c) Is

$$\mathbf{b} = \begin{bmatrix} -2 \\ 12 \\ 38 \end{bmatrix}$$

in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$? If yes, write $\mathbf{b}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Show your work.

[Additional work for question 5.]

6. (15 points) Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ -3 & -9 & 3 \\ 2 & 6 & -2 \end{bmatrix}.$$

(a) Solve $A\mathbf{x} = \mathbf{0}$ and express the solution in parametric vector form.

(b) Let $\mathbf{p} = (-1, 1, 0)$. Suppose $\mathbf{b} = A\mathbf{p}$. Calculate the value of $\mathbf{b}$.

(c) Suppose $\mathbf{b}$ is the same as in part (b). Based on the results from parts (a) and (b), write the general solution to $A\mathbf{x} = \mathbf{b}$ in parametric vector form.

[Additional work for question 6.]

7. (12 points) Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

(a) Write $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as linear combinations of $\mathbf{u}_1$ and $\mathbf{u}_2$ (*Hint:* it should be possible to do this without solving a linear system).

(b) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation and $T(\mathbf{u}_1) = (-3, -2, 0)$ and $T(\mathbf{u}_2) = (4, 2, 0)$. Determine the values of $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$.

(c) Write the standard matrix for the transformation T.

(d) Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$? Justify your answer.

[Additional work for question 7.]