Notes and class materials are allowed, but no outside help is permitted. Show all relevant work, and present solutions clearly.

Name: ____________________

Question	Points	Score
1	20	
2	16	
3	15	
4	15	
5	16	
6	18	
Total:	100	

1. (20 points) Define a relation $\sim$ on $\mathbb{R}$ as follows:

$$x \sim y \qquad \text{means that} \qquad x - y \text{ is an integer.}$$

   (i) Verify that $\sim$ is an equivalence relation.

   (ii) Let $[x]$ denote the equivalence class containing $x \in \mathbb{R}$ under the relation $\sim$. Explain what the equivalence classes look like.

   (iii) Define $\oplus$ on the equivalence classes by

$$[x] \oplus [y] = [x + y].$$

   Prove that the result of this operation does not depend on the representatives of the equivalence classes.

2. (16 points) Let C_n be the set of all n-cycles in S_n. For instance, if $n = 4$, then the permutation given in cycle notation by

$$(2\,4\,3\,1)$$

is an element of C_4. Recall that any element in a cycle can be placed at the start as long as the cyclic order is preserved; for example, $(2\,4\,3\,1) = (4\,3\,1\,2)$. Describe a bijection between S_n and $\mathbb{N}_n \times C_n$. Use this to conclude that $|C_n| = n!/n$. *Hint:* Note that an n-cycle can represent different rearrangements of $12\ldots n$ depending on which element of the cycle we choose to be at the start.

3. (15 points) We have two boxes, labelled "A" and "B", which are filled with red and green marbles. Box A has 5 red marbles and 5 green marbles, and box B has 3 red marbles and 7 green marbles. Suppose that we carry out an experiment where we toss a die and then, based on the value of the roll, select a marble from one of the boxes. If the value of the die roll is 1 or 2, we choose a random marble from box A. If the value of the die is 3, 4, 5, or 6, we choose a random marble from box B. If the experiment ends with choosing a green marble, what is the probability that we pulled the marble out of box A? Show all relevant work.

4. (15 points) A set X is finite if it is empty or if there is a bijective correspondence between X and $\mathbb{N}_n$ for some $n \in \mathbb{N}$. We say a set X is infinite if it is not finite. An alternative definition for an infinite set would be to say that X is infinite if there is an injection $f : X \to X$ that is not a surjection. To show this new definition could work, prove the following statement:

   Let X be a finite set. Then if $f : X \to X$ is not a surjection, it is not an injection.

5. (16 points) A new startup has hired five new software engineers: Alice, Beth, Carl, Dorothy, and Ed. There are five computers – labeled 0 to 4 – for the employees to use. However, the computers have varying levels of qualities, with computer 4 being the best. Thus the company wants to create a schedule for when each employee will be assigned a computer depending on the day. The company also wants each employee to use each computer once during the week. Since Alice needs to run tests on the code on Friday, she is assigned computer 4 that day.

	Monday	Tuesday	Wednesday	Thursday	Friday
Computer 0					
Computer 1					
Computer 2					
Computer 3					
Computer 4					Alice

Describe a way to fill in the rest of the schedule so that no name is repeated more than once in each column and row. In addition, write out the first row of your proposed scheduled (you do not need to write out the whole schedule).

6. (18 points) Suppose Andrew wants to rotate the tires of his car so that the wear on the four tires is evenly spread. In addition, Andrew wants to makes sure after rotating the tires in the same way enough times, ever possible arrangement of the tires is eventually obtained. That is, if we label the tires of the car $1, 2, 3, 4$, then Andrew wants a permutation π in S_4 such that any rearrangement of 1234 is effected by π^k for some value $k \geq 0$ (here π^k is the k-fold composition of π).

   (i) Recall that *derangements* in $\mathbb{N}_4$ are permutations σ such that $\sigma(i) \neq i$ for any i in $\mathbb{N}_4$. Explain why the permutation that Andrew wants must be a derangement.

   (ii) List all the derangements in $\mathbb{N}_4$ in cycle notation (with disjoint cycles). *Hint:* Three of the derangements have type $[2^2]$ and six of them have type $[4^1]$.

   (iii) Based on the answer given in (ii), why can Andrew not find the permutation π that he wants?