Notes and class materials are allowed, but no outside help is permitted. Show all relevant work, and present solutions clearly.

Name: ______________________________

Question	Points	Score
1	16	
2	16	
3	20	
4	16	
5	16	
6	16	
Total:	100	

1. (16 points) Define the statements $P(n)$, $Q(n)$, and $R(n)$ about a natural number n as follows.

$$\begin{aligned} P(n) &: n \text{ is prime} \\ Q(n) &: n \text{ is even} \\ R(n) &: n \text{ is a perfect square.} \end{aligned}$$

Now consider the following statement:

$$\forall n \in \mathbb{N} \quad \Big(Q(n) \wedge \big(\neg R(n)\big)\Big) \Rightarrow \big(\neg P(n)\big)$$

Write out in words what the above statement means. Find an $n \in \mathbb{N}$ which is a counter-example, and explain why it is a counter-example.

2. (16 points) Prove by induction that

$$\sum_{r=1}^{n} 3^r = \frac{3^{n+1} - 3}{2}$$

3. (20 points) Let $P(n)$ and $Q(n)$ denote statements about a natural number n. Now define the sets A and B as follows:

$$A = \{n \in \mathbb{N} \mid P(n) \text{ is true}\}$$
$$B = \{n \in \mathbb{N} \mid Q(n) \text{ is true}\}.$$

Now prove that

$$A \cup B = \mathbb{N} \qquad \text{if and only if} \qquad \forall n \in \mathbb{N} \quad (\neg P(n)) \Rightarrow Q(n).$$

4. (16 points) Prove that if $4n - 1$ is prime and $n > 1$, then n is not a perfect square.

5. (16 points) Define the statement P as follows:

$$P\text{: There is a natural number } n \text{ such that } 2^{(n^3)} + 1 \text{ is not prime.}$$

In your solution, address the following points:

- Determine whether P is an existential statement or a universal statement;
- Explain how one would prove P (you DO NOT have to write a proof, just explain how one would generally prove this statement); and
- Write out $\neg P$ in words and state whether this is an existential or universal statement.

6. (16 points) Prove that if $g \circ f$ is a bijection, then g is a surjection and f is an injection. Then show that the converse of this statement is not true.